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Some remarks on Fuzzy Bairs spaces

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Abstract In continuation of earlier work [14], we further investigate several characterizations of Fuzzy Baire spaces.

Keywords Fuzzy dense, Fuzzy nowhere dense, Fuzzy first category, Fuzzy second category, totally Fuzzy second category, Fuzzy F_σ-set, Fuzzy G_δ-set, Fuzzy nodec, Fuzzy regular, Fuzzy Baire spaces.

§1. Introduction

The concepts of Fuzzy sets and Fuzzy set operations were first introduced by L. A. Zadeh in his classical paper [15] in the year 1965. Thereafter the paper of C. L. Chang [4] in 1968 paved the way for the subsequent tremendous growth of the numerous Fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in Fuzzy setting and thus a modern theory of Fuzzy topology has been developed. X. Tang [10] used a slightly changed version of Chang's Fuzzy topological spaces to model spatial objects for GIS data bases and Structured Query Language (SQL) for GIS. The concepts of Baire spaces have been studied extensively in classical topology in [5], [6], [8] and [9]. The concept of Baire spaces in Fuzzy setting was introduced and studied by the authors in [14]. In this paper we study several characterizations of Fuzzy Baire spaces.

§2. Definition and properties

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a Fuzzy topological space due to Chang.

Definition 2.1. Let λ and μ be any two Fuzzy sets in (X, T). Then we define
$\lambda \vee \mu$: X $\rightarrow [0,1]$ as follows : $(\lambda \vee \mu)(x) = \max\ \{\lambda(\mathrm{x}),\mu(\mathrm{x})\}$,
$\lambda \wedge \mu$: X $\rightarrow [0,1]$ as follows : $(\lambda \wedge \mu)(x) = \min\ \{\lambda(\mathrm{x}),\mu(\mathrm{x})\}$.

Definition 2.2. Let (X, T) be a Fuzzy topological space and λ be any Fuzzy set in (X, T). We define $int(\lambda) = \vee\{\mu/\mu \leq \lambda,\ \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu,\ 1 - \mu \in T\}$.

For a Fuzzy set λ in a Fuzzy topological space (X, T), it is easy to see that $1 - cl(\lambda) = int(1-\lambda)$ and $1 - int(\lambda) = cl(1-\lambda)$ [2].

Definition 2.3.[12] A Fuzzy set λ in a Fuzzy topological space (X, T) is called Fuzzy dense if there exists no Fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.4.[13] A Fuzzy set λ in a Fuzzy topological space (X, T) is called Fuzzy nowhere dense if there exists no non-zero Fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.5.[3] A Fuzzy set λ in a Fuzzy topological space (X, T) is called a Fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $1 - \lambda_i \in T$ for $i \in I$.

Definition 2.6.[3] A Fuzzy set λ in a Fuzzy topological space (X, T) is called a Fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$ for $i \in I$.

Definition 2.7.[12] A Fuzzy set λ in a Fuzzy topological space (X, T) is called Fuzzy first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are Fuzzy nowhere dense sets in (X, T). Any other Fuzzy set in (X, T) is said to be of Fuzzy second category.

Definition 2.8.[14] Let λ be a Fuzzy first category set in (X, T). Then $1 - \lambda$ is called a Fuzzy residual set in (X, T).

Definition 2.9.[12] A Fuzzy topological space (X, T) is called Fuzzy first category if $1 = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are Fuzzy nowhere dense sets in (X, T). A topological space which is not of Fuzzy first category, is said to be a Fuzzy second category space.

Definition 2.10.[1] Let (X, T) be a Fuzzy topological space. Suppose $A \subset X$ and $T_A = \{\mu/_A : \mu \in T\}$. Then (A, T_A) is called a Fuzzy subspace of (X, T). In short we shall denote (A, T_A) by A. The Fuzzy subspace A is said to be a Fuzzy open subspace if its characteristic function χ_A is Fuzzy open in (X, T).

Lemma 2.1.[2] For a family of $\{\lambda_\alpha\}$ of Fuzzy sets of a Fuzzy topological space (X, T), $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case A is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.

§3. Fuzzy Baire spaces

Definition 3.1.[14] Let (X, T) be a Fuzzy topological space. Then (X, T) is called a Fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where λ_i's are Fuzzy nowhere dense sets in (X, T).

Definition 3.2. A Fuzzy topological space (X, T) is called a Fuzzy nodec space if every non-zero Fuzzy nowhere dense set is Fuzzy closed in (X, T). That is, if λ is a Fuzzy nowhere dense set in (X, T), then $1 - \lambda \in T$.

Definition 3.3.[7] A Fuzzy space X is called a Fuzzy regular space iff each Fuzzy open set λ of X is a union of Fuzzy open sets λ_α's of X such that $cl(\lambda_\alpha) \leq \lambda$ for each α.

Definition 3.4. A Fuzzy topological space (X, T) is called a totally Fuzzy second category if every non-zero Fuzzy closed set λ is a Fuzzy second category set in (X, T).

Theorem 3.1.[14] Let (X, T) be a Fuzzy topological space. Then the followings are equivalent:

(1) (X, T) is a Fuzzy Baire space,

(2) $int(\lambda) = 0$ for every Fuzzy first category set λ in (X, T),

(3) $cl(\mu) = 1$ for every Fuzzy residual set μ in (X, T).

Proposition 3.1. If the Fuzzy topological space (X, T) is a Fuzzy Baire space, then no non-zero Fuzzy open set is a Fuzzy first category set in (X, T).

Proof. Suppose λ is a non-zero Fuzzy open set in (X, T) such that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are Fuzzy nowhere dense sets in (X, T). Then $int(\lambda) = int(\vee_{i=1}^{\infty}(\lambda_i))$. Since λ is Fuzzy open, $int(\lambda) = \lambda$. Hence $int(\vee_{i=1}^{\infty}(\lambda_i)) = \lambda \neq 0$. But this is a contradiction to (X, T) being a Fuzzy Baire space, in which $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$ where λ_i's are Fuzzy nowhere dense sets in (X, T). Therefore $\lambda \neq \vee_{i=1}^{\infty}(\lambda_i)$. Hence no non-zero Fuzzy open set in a Fuzzy Baire space is a Fuzzy first category set.

Theorem 3.2.[14] If $cl(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$, where λ_i's are Fuzzy dense and Fuzzy open sets in (X, T), then (X, T) is a Fuzzy Baire space.

Proposition 3.2. Let (X, T) be a Fuzzy Baire space. If $A \subseteq X$ is such that χ_A (the characteristic function of $A \subseteq X$) is Fuzzy open in (X, T), then the Fuzzy subspace (A, T_A) is a Fuzzy Baire space.

Proof. Let $\lambda_i (i \in N)$ be Fuzzy open and Fuzzy dense sets in (A, T_A). Now λ_i is a Fuzzy open set in (A, T_A), implies that there exists a Fuzzy open set μ_i in (X, T) such that $\mu_i/_A = \lambda_i$. That is $(\mu_i \wedge \chi_A) = \lambda_i$. Since μ_i and χ_A are Fuzzy open in (X, T), λ_i is a Fuzzy open set in (X, T). Now $cl_A(\lambda_i) = 1_A$ implies that $cl_X(\lambda_i)/_A = 1/_A$. Hence $cl_X(\lambda_i) = 1$. Now λ_i's are Fuzzy open and Fuzzy dense in (X, T) and since (X, T) is a Fuzzy Baire space, $cl(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$. Now $cl(\wedge_{i=1}^{\infty}(\lambda_i))/_A = 1/_A$ in (X, T) implies that $cl_A(\wedge_{i=1}^{\infty}(\lambda_i)) = 1_A$, where λ_i's are Fuzzy open and Fuzzy dense sets in (A, T_A). Therefore (A, T_A) is a Fuzzy Baire space.

Proposition 3.3. If (X, T) is a Fuzzy nodec space, then (X, T) is not a Fuzzy Baire space.

Proof. Let λ_i be a Fuzzy nowhere dense set in a Fuzzy nodec space (X, T). Then λ_i is Fuzzy closed, that is, $cl(\lambda_i) = \lambda_i$. Now $\vee_{i=1}^{\infty} cl(\lambda_i) = \vee_{i=1}^{\infty}(\lambda_i)$ and $\vee_{i=1}^{\infty}(\lambda_i)$ is a Fuzzy first category set in (X, T). Hence $\vee_{i=1}^{\infty} cl(\lambda_i)$ is a Fuzzy first category set in (X, T). Now $int(\vee_{i=1}^{\infty} cl(\lambda_i)) > \vee_{i=1}^{\infty}(intcl(\lambda_i)) = 0$. (Since λ_i is a Fuzzy nowhere dense set, $intcl(\lambda_i) = 0$.) Hence $int(\vee_{i=1}^{\infty} cl(\lambda_i)) \neq 0$. Therefore (X, T) is not a Fuzzy Baire space.

Proposition 3.4. Let (X, T) be a Fuzzy topological space. Then (X, T) is of Fuzzy second category space if and only if $\wedge_{i=1}^{\infty}(\lambda_i) \neq 0$, where λ_i's are Fuzzy open and Fuzzy dense sets in (X, T).

Proof. Let (X, T) be a Fuzzy second category space. Suppose that $\wedge_{i=1}^{\infty}(\lambda_i) = 0$, where $\lambda_i \in T$ and $cl(\lambda_i) = 1$ then $1 - (\wedge_{i=1}^{\infty}(\lambda_i)) = 1 - 0 = 1$ that is

$$\vee_{i=1}^{\infty}(1 - \lambda_i) = 1. \tag{1}$$

Since $\lambda_i \in T$, $1 - \lambda_i$ is Fuzzy closed and hence

$$cl(1 - \lambda_i) = 1 - \lambda_i. \tag{2}$$

Now $cl(\lambda_i) = 1$ implies that $1 - cl(\lambda_i) = 0$ and hence

$$int(1 - \lambda_i) = 0. \tag{3}$$

Then from (2) and (3) we get $intcl(1 - \lambda_i) = 0$. This means that $1 - \lambda_i$ is a Fuzzy nowhere dense set in (X, T). Hence from (1), we have $\vee_{i=1}^{\infty}(1 - \lambda_i) = 1$, where $(1 - \lambda_i)$'s are Fuzzy nowhere dense sets in (X, T). This implies that (X, T) must be a Fuzzy first category space, but this is a contradiction to (X, T) being a Fuzzy second category space. Hence $\wedge_{i=1}^{\infty}(\lambda_i) \neq 0$, where $\lambda_i \in T$ and $cl(\lambda_i) = 1$.

Conversely, suppose that $\wedge_{i=1}^{\infty}(\lambda_i) \neq 0$ where λ_i's are Fuzzy open and Fuzzy dense sets in (X, T). Assume that the Fuzzy topological space (X, T) is not a Fuzzy second category space. Then $\vee_{i=1}^{\infty}\lambda_i = 1$, where λ_i's are Fuzzy nowhere dense sets in (X, T). Then $1 - (\vee_{i=1}^{\infty}(\lambda_i)) = 0$, which implies that

$$\wedge_{i=1}^{\infty}(1 - \lambda_i) = 0. \tag{4}$$

Now $\vee_{i=1}^{\infty}\lambda_i \leq \vee_{i=1}^{\infty}cl(\lambda_i)$ implies that $1 - (\vee_{i=1}^{\infty}\lambda_i \geq 1 - (\vee_{i=1}^{\infty}cl(\lambda_i))$ then $\wedge_{i=1}^{\infty}(1 - \lambda_i) \geq \wedge_{i=1}^{\infty}(1 - cl(\lambda_i))$. From (4) we have $\wedge_{i=1}^{\infty}(1 - cl(\lambda_i)) = 0$. Since λ_i is a Fuzzy nowhere dense set, $1 - cl(\lambda_i)$ is a Fuzzy dense set in (X, T). Hence we have $\wedge_{i=1}^{\infty}(1 - cl(\lambda_i)) = 0$ where $1 - cl(\lambda_i) \in T$ and $1 - cl(\lambda_i)$ is a Fuzzy dense set in (X, T). But this is a contradiction to the hypothesis. Hence (X, T) must be a Fuzzy second category space.

Proposition 3.5. If (X, T) is a Fuzzy Baire space, then every non-zero Fuzzy residual set λ in (X, T) contains a Fuzzy G_δ set η in (X, T) such that $cl(\eta) \neq 1$.

Proof. Let λ be a Fuzzy residual set in (X, T). Then $1 - \lambda$ is a Fuzzy first category set in (X, T) and hence $1 - \lambda = \vee_{i=1}^{\infty}(\mu_i)$, where μ_i's are Fuzzy nowhere dense sets in (X, T). Now $1 - cl(\mu_i)$ is a Fuzzy open set in (X, T) and $\eta = \wedge_{i=1}^{\infty}1 - cl(\mu_i)$ is a Fuzzy G_δ set in (X, T). But $\wedge_{i=1}^{\infty}(1 - cl(\mu_i)) = 1 - \vee_{i=1}^{\infty}cl(\mu_i) < 1 - \vee_{i=1}^{\infty}(\mu_i) < 1 - (1 - \lambda) = \lambda$. Hence we have $\eta < \lambda$. Then $cl(\eta) < cl(\lambda)$. Since (X, T) is a Fuzzy Baire space, $cl(\lambda) = 1$. Hence $cl(\lambda) < 1$ implies that $cl(\eta) \neq 1$.

Proposition 3.6. If λ is a Fuzzy first category set in a Fuzzy Baire space (X, T), then there is a non-zero Fuzzy F_σ-set δ in (X, T) such that $\lambda < \delta$ and $int(\delta) \neq 0$.

Proof. Let λ be a Fuzzy first category set in (X, T). Then $1 - \lambda$ is a Fuzzy residual set in (X, T). Then by proposition 3.5, there is a Fuzzy G_δ set η in (X, T) such that $\eta < 1 - \lambda$ and $cl(\eta) \neq 1$. Then $\lambda < 1 - \eta$ and $1 - cl(\eta) \neq 0$. Hence we have $\lambda < 1 - \eta$ and $int(1 - \eta) \neq 0$. Since η is a Fuzzy G_δ set, $1 - \eta$ is a Fuzzy F_σ set in (X, T). Let $\delta = 1 - \eta$, hence if λ is a Fuzzy first category set in (X, T), then there is a Fuzzy F_σ set δ in (X, T) such that $\lambda < \delta$ and $int(\delta) \neq 0$.

Proposition 3.7. If (X, T) is a Fuzzy Baire space and if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, then there exists at least one Fuzzy set λ_i such that $intcl(\lambda_i) \neq 0$.

Proof. Suppose $intcl(\lambda_i) = 0$ for all $i \in N$, then λ_i's are Fuzzy nowhere dense sets in (X, T). Then $\vee_{i=1}^{\infty}(\lambda_i) = 1$ implies that $int(\vee_{i=1}^{\infty}(\lambda_i)) = int(1) = 1 \neq 0$, a contradiction to (X, T) being a Fuzzy Baire space in which $int(\vee_{i=1}^{\infty}(\lambda_i) = 0$. Hence $intcl(\lambda_i) \neq 0$, for at least one $i \in N$.

The following guaranties the existence of non-dense, Fuzzy G_δ sets in a Fuzzy Baire spaces.

Proposition 3.8. If (X, T) is a Fuzzy Baire space, then there exist Fuzzy G_δ-sets μ_k in (X, T) such that $cl(\mu_k) \neq 1$.

Proof. Let λ_j be a Fuzzy first category set in (X, T). Then $\lambda_j = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are Fuzzy nowhere dense sets in (X, T). Now $1 - cl(\lambda_i)$ is a Fuzzy open set in (X, T) and $\mu_k = \wedge_{i=1}^{\infty}(1 - cl(\lambda_i))$ is a Fuzzy G_δ-set in (X, T). But $\wedge_{i=1}^{\infty}(1 - cl(\lambda_i)) = 1 - (\vee_{i=1}^{\infty}cl(\lambda_i)) < 1 - (\vee_{i=1}^{\infty}(\lambda_i)) = 1 - \lambda_j$. Hence there is a Fuzzy G_δ-set μ_k in (X, T) such that $\mu_k < 1 - \lambda_j$ which implies that $cl(\mu_k) < cl(1 - \lambda_j) = 1 - int(\lambda_j) = 1 - 0 = 1$, (since (X, T) is a Fuzzy Baire space, $int(\lambda_j) = 0$). Therefore $cl(\mu_k) \neq 1$.

Proposition 3.9. If $\lambda \leq \mu$ and μ is a Fuzzy nowhere dense set in a Fuzzy topological space (X, T), then λ is also a Fuzzy nowhere dense set in (X, T).

Proof. Now $\lambda \leq \mu$ implies that $intcl(\lambda) \leq intcl(\mu)$. Now μ is a Fuzzy nowhere dense set implies that $intcl(\mu) = 0$. Then $intcl(\lambda) = 0$. Hence λ is a Fuzzy nowhere dense set in (X, T).

Proposition 3.10. If (X, T) is a totally Fuzzy second category, Fuzzy regular space, then (X, T) is a Fuzzy Baire space.

Proof. Let (X, T) be a totally Fuzzy second category, Fuzzy regular space and λ_i be Fuzzy open and Fuzzy dense sets in (X, T). Let $\lambda = \wedge_{i=1}^{\infty} \lambda_i$. Then $1 - \lambda = 1 - \wedge_{i=1}^{\infty} \lambda_i = \vee_{i=1}^{\infty} (1 - \lambda_i)$. Since λ_i is Fuzzy open and Fuzzy dense in (X, T), $1 - \lambda_i$ is a Fuzzy nowhere dense set in (X, T) for each $i \in N$. Hence $1 - \lambda$ is a Fuzzy first category set in (X, T). Now we claim that $cl(\lambda) = 1$. Suppose $cl(\lambda) \neq 1$. Then there exists a non-zero Fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. Hence $1 - \lambda > 1 - \mu > 0$. Since μ is Fuzzy closed in (X, T), $1 - \mu$ is a Fuzzy open set in (X, T). Since (X, T) is Fuzzy regular and $1 - \mu$ is Fuzzy open, there exist Fuzzy open sets δ_j in (X, T) such that $1 - \mu = \vee_{i=1}^{\infty} (\delta_j)$ and $cl(\delta_j) \leq 1 - \mu$ for each j. Now $cl(\wedge_j (\delta_j)) \leq \wedge cl(\delta_j) \leq \wedge (1 - \mu)$, that is $cl(\wedge_j (\delta_j)) \leq 1 - \mu \leq 1 - \lambda$, that is $cl(\wedge_j (\delta_j)) < 1 - \lambda$. Since $1 - \lambda$ is a Fuzzy first category set by proposition 3.9 $cl(\wedge_j (\delta_j))$ is a Fuzzy first category set in (X, T). Now $cl(\wedge (\delta_j))$ is a Fuzzy closed set in a totally Fuzzy second category space, then $cl(\wedge_i (\delta_j))$ is not a Fuzzy first category set in (X, T), which is a contradiction. Hence our assumption that $cl(\lambda) \neq 1$ does not hold. Therefore $cl(\lambda) = 1$, hence by theorem 3.2 (X, T) is a Fuzzy Baire space.

Proposition 3.11. Let (X, T) be a totally Fuzzy second category space. Then no non-zero Fuzzy closed set is a Fuzzy first category set in (X, T).

Proof. Let λ be a non-zero Fuzzy closed set in (X, T). Assume that λ is a Fuzzy first category set. Then it is not a Fuzzy second category set in (X, T), which is a contradiction to (X, T) being a totally Fuzzy second category space. Therefore no non-zero Fuzzy closed set is a Fuzzy first category set in (X, T).

Definition 3.5.[11] A Fuzzy topological space (X, T) is called a Fuzzy P-space if countable intersection of Fuzzy open sets in (X, T) is Fuzzy open. That is, every non-zero Fuzzy G_δ set in (X, T) is Fuzzy open in (X, T).

Proposition 3.12. If the Fuzzy topological space (X, T) is a Fuzzy Baire P-space, and if λ is a Fuzzy first category set in (X, T) then $int(\lambda) = 0$ and $cl(\lambda) \neq 1$.

Proof. Let λ be a Fuzzy first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty} (\lambda_i)$ where λ_i's are Fuzzy nowhere dense sets in (X, T). Since (X, T) is a Fuzzy Baire space, $int(\vee_{i=1}^{\infty} (\lambda_i)) = 0$. That is, $int(\lambda) = 0$.

Suppose that $cl(\lambda) = 1$. Now $1 - cl(\lambda_i)$ is a non-zero Fuzzy open set in (X, T) (since λ_i is a Fuzzy nowhere dense, $cl(\lambda_i) \neq 1$.) Let $\mu = \wedge_{i=1}^{\infty} (1 - cl(\lambda_i))$, then μ is a non-zero Fuzzy G_δ set in (X, T). Since (X, T) is a Fuzzy P-space, μ is Fuzzy open in (X, T). Hence $\mu = int(\mu)$. Now $\mu = \wedge_{i=1}^{\infty} 1 - cl(\lambda_i) = 1 - \vee_{i=1}^{\infty} cl(\lambda_i) \leq 1 - \vee_{i=1}^{\infty} (\lambda_i) = 1 - \lambda$, that is, $\mu \leq 1 - \lambda$. Then $int(\mu) \leq int(1 - \lambda) = 1 - cl(\lambda) = 1 - 1 = 0$, which implies that $int(\mu) = 0$, which implies that $\mu = 0$, a contradiction to μ being a non-zero Fuzzy G_δ set in (X, T). Hence our assumption that $cl(\lambda) = 1$ does not hold. Hence $cl(\lambda) \neq 1$, $int(\lambda) = 0$ and $cl(\lambda) \neq 1$ for any Fuzzy first category set in a Fuzzy Baire P-space.

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A view on ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected spaces

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Abstract In this paper, a new class of intuitionistic Fuzzy smooth quasi uniform topological space called ordered intuitionistic Fuzzy smooth quasi uniform topological space is introduced. Tietze extention theorem for ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected spaces has been discussed besides providing several other propositions.

Keywords Intuitionistic Fuzzy smooth quasi uniform open set, ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected space, lower (r, s) intuitionistic Fuzzy quasi uni -form continuous function, upper (r, s) intuitionistic Fuzzy quasi uniform continuous function and ordered (r, s) intuitionistic Fuzzy quasi uniform continuous function.
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§1. Introduction

The concept of Fuzzy set was introduced by Zadeh [12]. Since then the concept has invaded nearly all branches of Mathematics. Chang [2] introduced and developed the theory of Fuzzy topological spaces and since then various notions in classical topology have been extended to Fuzzy topological spaces. Fuzzy sets have applications in many fields such as information [6] and control [5]. Atanassov [1] generalised Fuzzy sets to intuitionistic Fuzzy sets. Cocker [3] introduced the notions of an intuitionistic Fuzzy topological space. Young Chan Kim and Seok Jong Lee [10,11] have discussed some properties of Fuzzy quasi uniform space. Tomasz Kubiak [7,8] studied L-Fuzzy normal spaces and Tietze extension Theorem and extending continuous L-Real functions. G. Thangaraj and G. Balasubramanian [9] discussed On Fuzzy pre-basically disconnected spaces. In this paper, a new class of intuitionistic Fuzzy smooth quasi uniform topological spaces called ordered intuitionistic Fuzzy smooth quasi uniform topological spaces is introduced. Tietze extention theorem for ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected spaces has been discussed besides providing several other propositions.

§2. Preliminaries

Definition 2.1.[1] Let X be a non empty fixed set and I the closed interval [0,1]. An intuitionistic Fuzzy set(IFS). A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the function $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every Fuzzy set A on a nonempty set X is an IFS of the following form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ for the intuitionistic Fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x) : x \in X\rangle\}$. For a given non empty set X, denote the family of all intuitionistic Fuzzy sets in X by the symbol ζ^X.

Definition 2.2.[1] Let X be a nonempty set and the $IFSs$ A and B in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$.

Definition 2.3.[1] The $IFSs$ $0_\sim$ and $1_\sim$ are defined by $0_\sim=\{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim=\{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.4.[3] An intuitionistic Fuzzy topology (IFT) in Coker's sense on a non-empty set X is a family τ of $IFSs$ in X satisfying the following axioms.

(T_1) $0_\sim$,$1_\sim \in \tau$;

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;

(T_3) $\cup G_i \in \tau$ for arbitrary family $\{G_i / i \in I\} \subseteq \tau$.

In this paper by (X, τ) or simply by X we will denote the Cocker's intuitionistic Fuzzy topological space ($IFTS$). Each $IFSs$ in τ is called an intuitionistic Fuzzy open set($IFOS$) in X. The complement $\overline{A}$ of an $IFOS$ A in X is called an intuitionistic Fuzzy closed set ($IFCS$) in X.

Definition 2.5.[4] Let a and b be two real numbers in [0,1] satisfying the inequality $a+b \leq 1$. Then the pair $\langle a, b\rangle$ is called an intuitionistic Fuzzy pair. Let $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$ be any two intuitionistic Fuzzy pairs. Then define

(i) $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$,

(ii) $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$,

(iii) If $\{\langle a_i, b_i / i \in J\rangle\}$ is a family of intuitionistic Fuzzy pairs, then $\vee\langle a_i, b_i\rangle = \langle \vee a_i, \wedge b_i\rangle$ and $\wedge\langle a_i, b_i\rangle = \langle \wedge a_i, \vee b_i\rangle$,

(iv) The complement of an intuitionistic Fuzzy pair $\langle a, b\rangle$ is the intuitionistic Fuzzy pair defined by $\overline{\langle a, b\rangle} = \langle b, a\rangle$,

(v) $1^\sim = \langle 1, 0\rangle$ and $0^\sim = \langle 0, 1\rangle$.

Definition 2.6.[9] Let (X, T) be any Fuzzy topological space. (X, T) is called Fuzzy basically disconnected if the closure of every Fuzzy open F_σ is Fuzzy open.

Definition 2.7.[11] A function $\mathcal{U} : \Omega_X \to L$ is said to be an L-Fuzzy quasi-uniformity on X if it satisfies the following conditions.

(i) $\mathcal{U}(f_1 \sqcap f_2) \geq \mathcal{U}(f_1) \wedge \mathcal{U}(f_2)$ for $f_1, f_2 \in \Omega_X$,

(ii) For $f \in \Omega_X$ we have $\vee\{\mathcal{U}(f_1)/f_1 \circ f_1 \leq f\} \geq \mathcal{U}(f)$,

(iii) If $f_1 \geq f$ then $\mathcal{U}(f_1) \geq \mathcal{U}(f)$,

(iv) There exists $f \in \Omega_X$ such that $\mathcal{U}(f) = 1$.

Then the pair $(X, \mathcal{U})$ is said to be an L-Fuzzy quasi uniform space.

§3. Ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected spaces

Definition 3.1. Let Ω_X denotes the family of all intuitionistic Fuzzy functions $f : \zeta^X \to \zeta^X$ with the following properties.

(i) $f(0_\sim) = 0_\sim$,

(ii) $A \subseteq f(A)$ for every $A \in \zeta^X$,

(iii) $f(\cup A_i) = \cup f(A_i)$ for every $A_i \in \zeta^X, i \in J$.

For $f \in \Omega_X$, the function $f^{-1} \in \Omega_X$ is defined by $f^{-1}(A) = \cap\{B/f(\overline{B}) \subseteq \overline{A}\}$.

For $f, g \in \Omega_X$, we define, for all $A \in \zeta^X$, $f \sqcap g(A) = \cap\{f(A_1) \cup g(A_2)/A_1 \cup A_2 = A\}$, $(f \circ g)(A) = f(g(A))$.

Definition 3.2. Let $(X, \mathcal{U})$ be an intuitionistic Fuzzy quasi uniform space. Define, for each $r \in (0, 1] = I_0$, $s \in [0, 1) = I_1$ with $r + s \leq 1$ and $A \in \zeta^X$, $(r, s)IFQI_{\mathcal{U}}(A) = \cup\{B/f(B) \subseteq A$ for some $f \in \Omega_X$ with $\mathcal{U}(f) > \langle r, s\rangle\}$.

Definition 3.3. Let $(X, \mathcal{U})$ be an intuitionistic Fuzzy quasi uniform space. Then the function $T_{\mathcal{U}} : \zeta^X \to I_0 \times I_1$ is defined by $T_{\mathcal{U}}(A) = \cup\{\langle r, s\rangle/(r, s)IFQI_{\mathcal{U}}(A) = A$, $r \in I_0$, $s \in I_1$ with $r + s \leq 1\}$. Then the pair $(X, T_{\mathcal{U}})$ is called an intuitionistic Fuzzy smooth quasi uniform topological space. The members of $(X, T_{\mathcal{U}})$ are called an intuitionistic Fuzzy smooth quasi uniform open set.

Note 3.1. The complement of an intuitionistic Fuzzy smooth quasi uniform open set is an intuitionistic Fuzzy smooth quasi uniform closed set.

Definition 3.4. Let $(X, T_{\mathcal{U}})$ be an intuitionistic Fuzzy smooth quasi uniform topological space and A be an intuitionistic Fuzzy set. Then the intuitionistic Fuzzy smooth quasi uniform interior of A is denoted and defined by $IFSQint_{\mathcal{U}}(A) = \cup\{B/B \subseteq A$ and B is an intuitionistic Fuzzy smooth quasi uniform open set where $r \in I_0$, $s \in I_1$ with $r + s \leq 1\}$.

Definition 3.5. Let $(X, T_{\mathcal{U}})$ be an intuitionistic Fuzzy smooth quasi uniform topological space and A be an intuitionistic Fuzzy set. Then the intuitionistic Fuzzy smooth quasi uniform closure of A is denoted and defined by $IFSQcl_{\mathcal{U}}(A) = \cap\{B/B \supseteq A$ and B is an intuitionistic Fuzzy smooth quasi uniform closed set where $r \in I_0$, $s \in I_1$ with $r + s \leq 1$ $\}$.

Definition 3.6. Let $(X, T_{\mathcal{U}})$ be an intuitionistic Fuzzy smooth quasi uniform topological space and A be an intuitionistic Fuzzy set. Then A is said to be an intuitionistic Fuzzy smooth quasi uniform G_δ set if $A = \cap_{i=1}^{\infty} A_i$ where each A_i is an intuitionistic Fuzzy smooth quasi uniform open set, where $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. The complement of an intuitionistic Fuzzy smooth quasi uniform G_δ set is an intuitionistic Fuzzy smooth quasi uniform F_σ set.

Note 3.2. Every intuitionistic Fuzzy smooth quasi uniform open set is an intuitionistic Fuzzy smooth quasi uniform G_δ set and every intuitionistic Fuzzy smooth quasi uniform closed set is an intuitionistic Fuzzy smooth quasi uniform F_σ set.

Definition 3.7. Let $(X, T_{\mathcal{U}})$ be an intuitionistic Fuzzy smooth quasi uniform topological space and A be any intuitionistic Fuzzy set in $(X, T_{\mathcal{U}})$. Then A is said to be

(i) increasing intuitionistic Fuzzy set if $x \leq y$ implies $A(x) \leq A(y)$. That is, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$;

(ii) decreasing intuitionistic Fuzzy set if $x \leq y$ implies $A(x) \geq A(y)$. That is, $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$.

Definition 3.8. Let X be an ordered set. $T_{\mathcal{U}}$ is an intuitionistic Fuzzy smooth quasi uniform topology defined on X. Then $(X, T_{\mathcal{U}}, \leq)$ is said to be an ordered intuitionistic Fuzzy smooth quasi uniform topological space.

Definition 3.9. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space and A be any intuitionistic Fuzzy set in $(X, T_{\mathcal{U}}, \leq)$. Then we define

(i) $IFSQI_{\mathcal{U}}(A) =$ Intuitionistic Fuzzy smooth quasi uniform increasing closure of $A =$ The smallest intuitionistic Fuzzy smooth quasi uniform increasing closed set containing in A.

(ii) $IFSQD_{\mathcal{U}}(A) =$ Intuitionistic Fuzzy smooth quasi uniform decreasing regular closure of $A =$ The smallest intuitionistic Fuzzy smooth quasi uniform decreasing closed set containing in A.

(iii) $IFSQI^0_{\mathcal{U}}(A) =$ Intuitionistic Fuzzy smooth quasi uniform increasing interior of $A =$ The greatest intuitionistic Fuzzy smooth quasi uniform increasing open set contained in A.

(iv) $IFSQD^0_{\mathcal{U}}(A) =$ Intuitionistic Fuzzy smooth quasi uniform decreasing interior of $A =$ The greatest intuitionistic Fuzzy smooth quasi uniform decreasing open set contained in A.

Proposition 3.1. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space. Then for any two intuitionistic Fuzzy sets A and B in $(X, T_{\mathcal{U}}, \leq)$ the following are valid.

(i) $\overline{IFSQI_{\mathcal{U}}(A)} = IFSQD^0_{\mathcal{U}}(\overline{A})$,

(ii) $\overline{IFSQD_{\mathcal{U}}(A)} = IFSQI^0_{\mathcal{U}}(\overline{A})$,

(iii) $\overline{IFSQI^0_{\mathcal{U}}(A)} = IFSQD_{\mathcal{U}}(\overline{A})$,

(iv) $\overline{IFSQD^0_{\mathcal{U}}(A)} = IFSQI_{\mathcal{U}}(\overline{A})$.

Proof. Since $IFSQI_{\mathcal{U}}(A)$ is an intuitionistic Fuzzy smooth quasi uniform increasing closed set containing A, $\overline{IFSQI_{\mathcal{U}}(A)}$ is an intuitionistic Fuzzy smooth quasi uniform decreasing open set such that $\overline{IFSQI_{\mathcal{U}}(A)} \subseteq \overline{A}$. Let B be another intuitionistic Fuzzy smooth quasi uniform decreasing open set such that $B \subseteq \overline{A}$. Then $\overline{B}$ is an intuitionistic Fuzzy smooth quasi uniform increasing closed set such that $\overline{B} \supseteq A$. It follows that $IFSQI_{\mathcal{U}}(A) \subseteq \overline{B}$. That is, $B \subseteq \overline{IFSQI_{\mathcal{U}}(A)}$. Thus, $\overline{IFSQI_{\mathcal{U}}(A)}$ is the largest intuitionistic Fuzzy smooth quasi uniform decreasing open set such that $\overline{IFSQI_{\mathcal{U}}(A)} \subseteq \overline{A}$. That is, $\overline{IFSQI_{\mathcal{U}}(A)} = IFSQD^0_{\mathcal{U}}(\overline{A})$. The proof of (2), (3) and (4) are similar to (1).

Definition 3.10. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space.

(i) An intuitionistic Fuzzy set A in $(X, T_{\mathcal{U}}, \leq)$ which is both intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open and intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) F_σ is defined by intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open F_σ.

(ii) An intuitionistic Fuzzy set A in $(X, T_{\mathcal{U}}, \leq)$ which is both intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) closed and intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) G_δ is defined by intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) closed G_δ.

(iii) An intuitionistic Fuzzy set A in $(X, T_{\mathcal{U}}, \leq)$ which is both intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open F_σ and intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) closed G_δ is defined by intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) closed open G_δ F_σ.

Definition 3.11. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space. Let A be any intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set in $(X, T_{\mathcal{U}}, \leq)$. If $IFSQI_{\mathcal{U}}(A)$ is an intuitionistic Fuzzy smooth quasi uniform increasing open set in $(X, T_{\mathcal{U}}, \leq)$, then $(X, T_{\mathcal{U}}, \leq)$ is said to be upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space. Similarly we can define lower intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

Definition 3.12. An ordered intuitionistic Fuzzy smooth quasi uniform topological space $(X, T_{\mathcal{U}}, \leq)$ is said to be ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected space if it is both upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space and lower intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

Proposition 3.2. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space. Then the following statements are equivalent:

(i) $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space,

(ii) For each intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set A, then $IFSQD^0_{\mathcal{U}}(A)$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed,

(iii) For each intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set A, we have $IFSQD_{\mathcal{U}}(IFQD^0_{\mathcal{U}}(\overline{(A)})) = \overline{IFSQI_{\mathcal{U}}(A)}$,

(iv) For each intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set A and intuitionistic Fuzzy smooth quasi uniform decreasing set B in $(X, T_{\mathcal{U}}, \leq)$ with $IFSQI_{\mathcal{U}}(A) = \overline{B}$, we have, $IFSQD_{\mathcal{U}}(B) = \overline{IFSQI_{\mathcal{U}}(A)}$.

Proof. (i) $\Rightarrow$ (ii) Let A be any intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set. Then $\overline{A}$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set and so by assumption (1), $IFSQI_{\mathcal{U}}(\overline{A})$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set. That is, $IFSQD^0_{\mathcal{U}}(A)$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed.

(ii) $\Rightarrow$ (iii) Let A be any intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set. Then $\overline{A}$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set. Then by (2), $IFSQD^0_{\mathcal{U}}(\overline{A})$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set. Now, $IFSQD_{\mathcal{U}}(IFSQD^0_{\mathcal{U}}(\overline{A})) = IFSQD^0_{\mathcal{U}}(\overline{A}) = \overline{IFSQI_{\mathcal{U}}(A)}$.

(iii) $\Rightarrow$ (iv) Let A be an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set and B be an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set such that $IFSQI_{\mathcal{U}}(A) = \overline{B}$. By (3), $IFSQD_{\mathcal{U}}(\overline{IFSQI_{\mathcal{U}}(A)}) = \overline{IFSQI_{\mathcal{U}}(A)}$. $IFSQD_{\mathcal{U}}(B) = \overline{IFSQI_{\mathcal{U}}(A)}$.

(iv) $\Rightarrow$ (i) Let A be an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set. Put $B = \overline{IFSQI_{\mathcal{U}}(A)}$. Clearly, B is an intuitionistic Fuzzy smooth quasi uniform decreasing set. By (4) it follows that $IFSQD_{\mathcal{U}}(B) = \overline{IFSQI_{\mathcal{U}}(A)}$. That is, $\overline{IFSQI_{\mathcal{U}}(A)}$ is an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set. Hence $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

Proposition 3.3. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space. Then $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space if and only if for each A and B are intuitionistic Fuzzy smooth quasi uniform decreasing closed open G_δ F_σ such that $A \subseteq B$ we have, $IFSQD_{\mathcal{U}}(A) \subseteq IFSQD^0_{\mathcal{U}}(A)$.

Proof. Suppose $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space and let A be an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set and B be an intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set such that $A \subseteq B$. Then by (2) of Proposition 3.2, $IFSQD^0_{\mathcal{U}}(A)$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed set. Also, since A is an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set and $A \subseteq B$, it follows that $A \subseteq IFSQD^0_{\mathcal{U}}(B)$. This implies that $IFSQD_{\mathcal{U}}(A) \subseteq IFSQD^0_{\mathcal{U}}(B)$.

Conversely, let B be any intuitionistic Fuzzy smooth quasi uniform decreasing closed open G_δ F_σ set. Then by Definition 3.4, $IFSQD^0_{\mathcal{U}}(B)$ is an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set and it is also clear that $IFSQD^0_{\mathcal{U}}(B) \subseteq B$. Therefore by assumption, $IFSQD_{\mathcal{U}}(IFSQD^0_{\mathcal{U}}(B)) \subseteq IFSQD^0_{\mathcal{U}}(B)$. This implies that $IFSQD^0_{\mathcal{U}}(B)$ is an intuitionistic Fuzzy smooth quasi uniform decreasing closed set. Hence by (2) of Proposition 3.2, it follows that $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

Remark 3.1. Let $(X, T_{\mathcal{U}}, \leq)$ be an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space. Let $\{A_i, \overline{B_i}/i \in N\}$ be collection such that A_i's are intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ sets and B_i are intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ sets. Let A and $\overline{B}$ be an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set and intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set respectively. If $A_i \subseteq A \subseteq B_j$ and $A_i \subseteq B \subseteq B_j$ for all $i, j \in N$, then there exists an intuitionistic Fuzzy smooth quasi uniform decreasing closed open $G_\delta F_\sigma$ set C such that $IFSQD_{\mathcal{U}}(A_i) \subseteq C \subseteq IFSQD^0_{\mathcal{U}}(B_j)$ for all $i, j \in N$.

Proof. By Proposition 3.3, $IFSQD_{\mathcal{U}}(A_i) \subseteq IFSQD_{\mathcal{U}}(A) \cap IFSQD^0_{\mathcal{U}}(B) \subseteq IFSQD^0_{\mathcal{U}}(B_j)$ for all $i, j \in N$. Letting $C = IFSQD_{\mathcal{U}}(A) \cap IFSQD^0_{\mathcal{U}}(B)$ in the above, we have C is an intuitionistic Fuzzy smooth quasi uniform decreasing closed open $G_\delta F_\sigma$ set satisfying the required conditions.

Proposition 3.3. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected space. Let $\{A_q\}_{q \in Q}$ and $\{B_q\}_{q \in Q}$ be monotone increasing collections of an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ sets and intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ sets of $(X, T_{\mathcal{U}}, \leq)$ respectively. Suppose that $A_{q_1} \subseteq B_{q_2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{C_q\}_{q \in Q}$ of an intuitionistic Fuzzy smooth quasi uniform decreasing closed open $G_\delta F_\sigma$ sets of $(X, T_{\mathcal{U}}, \leq)$ such that $IFSQD_{\mathcal{U}}(A_{q_1}) \subseteq C_{q_2}$ and $C_{q_1} \subseteq IFSQD^0_{\mathcal{U}}(B_{q_2})$ whenever $q_1 < q_2$.

Proof. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions).

For every $n \geq 2$, we shall define inductively a collection $\{C_{q_i}/1 \leq i < n\} \subset \zeta^X$ such that

$$IFSQD_{\mathcal{U}}(A_q) \subseteq C_{q_i}, \text{ if } q < q_i; \; C_{q_i} \subseteq IFSQD^0_{\mathcal{U}}(B_q), \text{ if} q_i < q, \text{ for all } i < n. \qquad (S_n)$$

By Proposition 3.3, the countable collections $\{ISFQD_{\mathcal{U}}(A_q)\}$ and $\{IFSQD^0_{\mathcal{U}}(B_q)\}$ satisfying $IFSQD_{\mathcal{U}}(A_{q_1}) \subseteq IFSQD^0_{\mathcal{U}}(B_{q_2})$ if $q_1 < q_2$. By Remark 3.1, there exists an intuitionistic Fuzzy smooth quasi uniform decreasing closed open $G_\delta F_\sigma$ set D_1 such that

$$IFSQD_{\mathcal{U}}(A_{q_1}) \subseteq D_1 \subseteq IFSQD^0_{\mathcal{U}}(B_{q_2}).$$

Letting $C_{q_1} = D_1$, we get (S_2). Assume that intuitionistic Fuzzy sets C_{q_i} are already defined for $i < n$ and satisfy (S_n). Define $E = \cup\{C_{q_i}/i < n, q_i < q_n\} \cup A_{q_n}$ and $F = \cap\{C_{q_j}/j < n, q_j > q_n\} \cap B_{q_n}$. Then $IFSQD_{\mathcal{U}}(C_{q_i}) \subseteq IFSQD_{\mathcal{U}}(E) \subseteq IFSQD^0_{\mathcal{U}}(C_{q_j})$ and $IFSQD_{\mathcal{U}}(C_{q_i}) \subseteq IFSQD^0_{\mathcal{U}}(F) \subseteq IFSQD^0_{\mathcal{U}}(C_{q_j})$ whenever $q_i < q_n < q_j (i, j < n)$, as well as $A_q \subseteq IFSQD_{\mathcal{U}}(E) \subseteq B_{q'}$ and $A_q \subseteq IFSQD^0_{\mathcal{U}}(F) \subseteq B_{q'}$ whenever $q < q_n < q^{'}$. This shows that the countable collection $\{C_{q_i}/i < n, q_i < q_n\} \cup \{A_q | q < q_n\}$ and $\{C_{q_j}/j < n, q_j > q_n\} \cup \{B_q | q > q_n\}$ together with E and F fulfil the conditions of Remark 3.1. Hence, there exists an intuitionistic Fuzzy smooth quasi uniform decreasing closed open $G_\delta F_\sigma$ set D_n such that $IFSQD_{\mathcal{U}}(D_n) \subseteq B_q$, if $q_n < q$; $A_q \subseteq IFSQD^0_{\mathcal{U}}(D_n)$, if $q < q_n$; $IFSQD_{\mathcal{U}}(C_{q_i}) \subseteq IFSQD^0_{\mathcal{U}}(D_n)$ if $q_i < q_n$ $IFSQ, D_{\mathcal{U}}(D_n) \subseteq IFSQD^0_{\mathcal{U}}(C_{q_i})$ if $q_n < q_j$, where $1 \leq i, j \leq n-1$. Letting $C_{q_n} = D_n$ we obtain an intuitionistic Fuzzy sets $C_{q_1}, C_{q_2}, C_{q_3}, \ldots, C_{q_n}$ that satisfy $(\mathbf{S_{n+1}})$. Therefore, the collection $\{C_{q_i}/i = 1, 2, \ldots\}$ has the required property.

Definition 3.13. Let $(X, T_{\mathcal{U}}, \leq)$ and $(Y, S_{\mathcal{V}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological spaces and $f : (X, T, \leq) \to (Y, S, \leq)$ be an intuitionistic Fuzzy function. Then f is said to be an (r, s) intuitionistic Fuzzy quasi uniform increasing (decreasing) continuous function if for any intuitionistic Fuzzy smooth quasi uniform open (closed) set A in $(Y, S_{\mathcal{V}}, \leq)$, $f^{-1}(A)$ is an intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open F_σ (closed G_δ) set in $(X, T_{\mathcal{U}}, \leq)$.

If f is both (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function and (r, s) intuitionistic Fuzzy quasi uniform decreasing continuous function then it is called ordered (r, s) intuitionistic Fuzzy quasi uniform continuous function.

§4. Tietze extention theorem for ordered intuitionistic Fuzzy smooth quasi uniform basically disconnected space

An intuitionistic Fuzzy real line $\mathbb{R}_I(I)$ is the set of all monotone decreasing intuitionistic Fuzzy set $A \in \zeta^{\mathbb{R}}$ satisfying

$$\cup\{A(t) : t \in \mathbb{R}\} = 0^{\sim},$$
$$\cap\{A(t) : t \in \mathbb{R}\} = 1^{\sim}.$$

After the identification of intuitionistic Fuzzy sets $A, B \in \mathbb{R}_I(I)$ if and only if $A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \mathbb{R}$ where

$$A(t-) = \cap\{A(s) : s < t\} \text{ and}$$
$$A(t+) = \cup\{A(s) : s > t\}.$$

The natural intuitionistic Fuzzy topology on $\mathbb{R}_I(I)$ is generated from the basis $\{L_t^I,\ R_t^I : t \in \mathbb{R}\}$ where $L_t^I,\ R_t^I$ are function from $\mathbb{R}_I(I) \to \mathbb{I}_I(I)$ are given by $L_t^I[A] = \overline{A(t-)}$ and $R_t^I[A] = A(t+)$.

The intuitionistic Fuzzy unit interval $\mathbb{I}_I(I)$ is a subset of $\mathbb{R}_I(I)$ such that $[A] \in \mathbb{I}_I(I)$ if the member and non member of A are defined by

$$\mu_A(t) = \begin{cases} 0, & \text{if } t \geq 1; \\ 1, & \text{if } t \leq 0; \end{cases}$$

and

$$\gamma_A(t) = \begin{cases} 1, & \text{if } t \geq 0; \\ 0, & \text{if } t \leq 1; \end{cases}$$

respectively.

Definition 4.1. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space and $f : X \to \mathbb{R}_I(I)$ be an intuitionistic Fuzzy function. Then f is said to be lower (r, s) intuitionistic Fuzzy quasi uniform continuous function if $f^{-1}(R_t^I)$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set or intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set, for $t \in \mathbb{R}$.

Definition 4.2. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space and $f : X \to \mathbb{R}_I(I)$ be an intuitionistic Fuzzy function. Then f is said to be upper (r, s) intuitionistic Fuzzy quasi uniform continuous function if $f^{-1}(L_t^I)$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set or intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ set, for $t \in \mathbb{R}$.

Note 4.1. Let X be a non empty set and $A \in \zeta^X$. Then $A^\sim = \langle \mu_A(x), \gamma_A(x) \rangle$ for every $x \in X$.

Proposition 4.1. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space, $A \in \zeta^X$ and $f : X \to \mathbb{R}_I(I)$ be such that

$$f(x)(t) = \begin{cases} 1^\sim, & \text{if } \ t < 0; \\ A^\sim, & \text{if } \ 0 \leq t \leq 1; \\ 0^\sim, & \text{if } \ t > 1, \end{cases}$$

and for all $x \in X$. Then f is lower (upper) (r, s) intuitionistic Fuzzy quasi uniform continuous function if and only if A is an intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open F_σ (closed G_δ) set.

Proof. It suffices to observe that

$$f^{-1}(R_t^I) = \begin{cases} 1_\sim, & \text{if } \ t < 0; \\ A, & \text{if } \ 0 \leq t \leq 1; \\ 0_\sim, & \text{if } \ t > 1, \end{cases}$$

and

$$f^{-1}(\overline{L_t^I}) = \begin{cases} lr1_\sim, & \text{if } \ t < 0; \\ A, & \text{if } \ 0 \leq t \leq 1; \\ 0_\sim, & \text{if } \ t > 1. \end{cases}$$

Thus proved.

Definition 4.3. let X be any non empty set. An intuitionistic Fuzzy* characteristic function of an intuitionistic Fuzzy set A in X is a map $\Psi_A : X \to \mathbb{I}_I(I)$ defined by $\Psi_A(x) = A^{\sim}$ for each $x \in X$.

Proposition 4.2. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space, $A \in \zeta^X$. Then Ψ_A is lower (upper) (r, s) intuitionistic Fuzzy quasi uniform continuous function if and only if A is an intuitionistic Fuzzy smooth quasi uniform increasing (decreasing) open F_σ (closed G_δ) set.

Proof. Proof is similar to Proposition 4.1.

Proposition 4.3. Let $(X, T_{\mathcal{U}}, \leq)$ be an ordered intuitionistic Fuzzy smooth quasi uniform topological space. Then the following are equivalent:

(i) $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

(ii) If $g, h : X \to \mathbb{R}_I(I)$, g is an lower (r, s) intuitionistic Fuzzy quasi uniform continuous function, h is an upper (r, s) intuitionistic Fuzzy quasi uniform continuous function and $g \subseteq h$, then there exists an (r, s) intuitionistic Fuzzy quasi uniform continuous function $f : (X, T_{\mathcal{U}}, \leq) \to \mathbb{R}_I(I)$ such that $g \subseteq f \subseteq h$.

(iii) If $\overline{A}$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set and B is an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set such that $B \subseteq A$, then there exists an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function $f : (X, T, \leq) \to \mathbb{R}_I(I)$ such that $B \subseteq f^{-1}(\overline{L_1^I}) \subseteq f^{-1}(R_0^I) \subseteq A$.

Proof. (i) $\Rightarrow$ (ii) Define $A_r = h^{-1}(L_r^I)$ and $B_r = g^{-1}(\overline{R_r^I})$, for all $r \in Q$ (Q is the set of all rationals). Clearly, $\{A_r\}_{r \in Q}$ and $\{B_r\}_{r \in Q}$ are monotone increasing families of an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ sets and intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ sets of $(X, T_{\mathcal{U}}, \leq)$. Moreover $A_r \subseteq B_s$ if $r < s$. By Proposition 3.4, there exists a monotone increasing family $\{C_r\}_{r \in Q}$ of an intuitionistic Fuzzy smooth quasi uniform decreasing closed open G_δ F_σ sets of $(X, T_{\mathcal{U}}, \leq)$ such that $IFSQD_{\mathcal{U}}(A_r) \subseteq C_s$ and $C_r \subseteq IFSQD_{\mathcal{U}}^0(B_s)$ whenever $r < s$ $(r, s \in Q)$. Letting $V_t = \bigcap_{r<t} \overline{C_r}$ for $t \in \mathbb{R}$, we define a monotone decreasing family $\{V_t \mid t \in \mathbb{R}\} \subseteq \zeta^X$. Moreover we have $IFSQI_{\mathcal{U}}(V_t) \subseteq IFSQI_{\mathcal{U}}^0(V_s)$ whenever $s < t$. We have,

$$
\begin{aligned}
\bigcup_{t \in \mathbb{R}} V_t &= \bigcup_{t \in \mathbb{R}} \bigcap_{r<t} \overline{C_r} \\
&\supseteq \bigcup_{t \in \mathbb{R}} \bigcap_{r<t} \overline{B_r} \\
&= \bigcup_{t \in \mathbb{R}} \bigcap_{r<t} g^{-1}(R_r^I) \\
&= \bigcup_{t \in \mathbb{R}} g^{-1}(\overline{L_t^I}) \\
&= g^{-1}\Big(\bigcup_{t \in \mathbb{R}} \overline{L_t^I}\Big) \\
&= 1_{\sim}
\end{aligned}
$$

Similarly, $\bigcap_{t\in\mathbb{R}} V_t = 0_\sim$. Now define a function $f : (X, T_{\mathcal{U}}, \leq) \to \mathbb{R}_I(I)$ possessing required conditions. Let $f(x)(t) = V_t(x)$, for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion, it follows that f is well defined. To prove f is an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function. Observe that $\bigcup_{s>t} V_s = \bigcup_{s>t} IFSQI^0_{\mathcal{U}}(V_s)$ and $\bigcap_{s<t} V_s = \bigcap_{s<t} IFSQI_{\mathcal{U}}(V_s)$. Then $f^{-1}(R^I_t) = \bigcup_{s>t} V_s = \bigcup_{s>t} IFSQI^0_{\mathcal{U}}(V_s)$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set and $f^{-1}(\overline{L^I_t}) = \bigcap_{s<t} V_s = \bigcap_{s<t} IFSQI_{\mathcal{U}}(V_s)$ is an intuitionistic Fuzzy smooth quasi uniform increasing closed G_δ set. Therefore, f is an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is, $g^{-1}(\overline{L^I_t}) \subseteq f^{-1}(\overline{L^I_t}) \subseteq h^{-1}(\overline{L^I_t})$ and $g^{-1}(R^I_t) \subseteq f^{-1}(R^I_t) \subseteq h^{-1}(R^I_t)$ for each $t \in \mathbb{R}$.

We have,

$$
\begin{aligned}
g^{-1}(\overline{L^I_t}) &= \bigcap_{s<t} g^{-1}(\overline{L^I_s}) \\
&= \bigcap_{s<t} \bigcap_{r<s} g^{-1}(R^I_r) \\
&= \bigcap_{s<t} \bigcap_{r<s} \overline{B_r} \\
&\subseteq \bigcap_{s<t} \bigcap_{r<s} \overline{C_r} \\
&= \bigcap_{s<t} V_s \\
&= f^{-1}(\overline{L^I_t}),
\end{aligned}
$$

and

$$
\begin{aligned}
f^{-1}(\overline{L^I_t}) &= \bigcap_{s<t} V_s \\
&= \bigcap_{s<t} \bigcap_{r<s} \overline{C_r} \\
&\subseteq \bigcap_{s<t} \bigcap_{r<s} \overline{A_r} \\
&= \bigcap_{s<t} \bigcap_{r<s} h^{-1}(\overline{L^I_r}) \\
&= \bigcap_{s<t} h^{-1}(\overline{L^I_s}) \\
&= h^{-1}(\overline{L^I_t})
\end{aligned}
$$

Similarly,

$$\begin{aligned}
g^{-1}(R_t^I) &= \bigcup_{s>t} g^{-1}(R_s^I) \\
&= \bigcup_{s>t} \bigcup_{r>s} g^{-1}(R_r^I) \\
&= \bigcup_{s>t} \bigcup_{r>s} \overline{B_r} \\
&\subseteq \bigcup_{s>t} \bigcap_{r<s} \overline{C_r} \\
&= \bigcup_{s>t} V_s \\
&= f^{-1}(R_t^I),
\end{aligned}$$

and

$$\begin{aligned}
f^{-1}(R_t^I) &= \bigcup_{s>t} V_s \\
&= \bigcup_{s>t} \bigcap_{r<s} \overline{C_r} \\
&\subseteq \bigcup_{s>t} \bigcup_{r>s} \overline{A_r} \\
&= \bigcup_{s>t} \bigcup_{r>s} h^{-1}(\overline{L_r^I}) \\
&= \bigcup_{s>t} h^{-1}(R_s^I) \\
&= h^{-1}(R_t^I).
\end{aligned}$$

Hence, the condition (ii) is proved.

(ii) $\Rightarrow$ (iii) $\overline{A}$ is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set and B is an intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ set such that $B \subseteq A$. Then, $\Psi_B \subseteq \Psi_A$, Ψ_B and Ψ_A lower and upper (r, s) intuitionistic Fuzzy quasi uniform continuous function respectively. Hence by (2), there exists an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function $f : (X, T_{\mathcal{U}}, \leq) \to \mathbb{I}_I(I)$ such that $\Psi_B \subseteq f \subseteq \Psi_A$. Clearly, $f(x) \in [0, 1]$ for all $x \in X$ and $B = \Psi_B^{-1}(\overline{L_1^I}) \subseteq f^{-1}(\overline{L_1^I}) \subseteq f^{-1}(R_0^I) \subseteq \Psi_A^{-1}(R_0^I) = A$. Therefore, $B \subseteq f^{-1}(\overline{L_1^I}) \subseteq f^{-1}(R_0^I) \subseteq A$.

(iii) $\Rightarrow$ (i) Since $f^{-1}(\overline{L_1^I})$ and $f^{-1}(R_0^I)$ are intuitionistic Fuzzy smooth quasi uniform decreasing closed G_δ and intuitionistic Fuzzy smooth quasi uniform decreasing open F_σ sets by Proposition 3.3, $(X, T_{\mathcal{U}}, \leq)$ is an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space.

Note 4.2. Let X be a non empty set and $A \subset X$. Then an intuitionistic Fuzzy set χ_A^* is of the form $\langle x, \chi_A(x), 1 - \chi_A(x)\rangle$ where

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Proposition 4.4. Let $(X, T_{\mathcal{U}}, \leq)$ be an upper intuitionistic Fuzzy smooth quasi uniform basically disconnected space. Let $A \subset X$ be such that χ_A^* is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ in $(X, T_{\mathcal{U}}, \leq)$. Let $f : (A, T_{\mathcal{U}}/A) \to \mathbb{I}_I(I)$ be an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function . Then f has an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous extension over $(X, T_{\mathcal{U}}, \leq)$.

Proof. Let $g, h : X \to \mathbb{I}_I(I)$ be such that $g = f = h$ on A and $g(x) = \langle 0, 1\rangle = 0^{\sim}$, $h(x) = \langle 1, 0\rangle = 1^{\sim}$ if $x \notin A$. For every $t \in \mathbb{R}$, We have,

$$g^{-1}(R_t^I) = \begin{cases} B_t \cap \chi_A^*, & \text{if } t \geq 0 \ ; \\ 1_{\sim}, & \text{if } t < 0, \end{cases}$$

where B_t is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ such that $B_t/A = f^{-1}(R_t^I)$ and

$$h^{-1}(L_t^I) = \begin{cases} D_t \cap \chi_A^*, & \text{if } t \leq 1 \ ; \\ 1_{\sim}, & \text{if } t > 1, \end{cases}$$

where D_t is an intuitionistic Fuzzy smooth quasi uniform increasing open F_σ set such that $D_t/A = f^{-1}(L_t^I)$. Thus, g is an lower (r, s) intuitionistic Fuzzy quasi uniform continuous function and h is an upper (r, s) intuitionistic Fuzzy quasi uniform continuous function with $g \subseteq h$. By Proposition 4.3, there is an (r, s) intuitionistic Fuzzy quasi uniform increasing continuous function $F : X \to \mathbb{I}_I(I)$ such that $g \subseteq F \subseteq h$. Hence $F \equiv f$ on A.

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Computing the number of integral points in 4-dimensional ball

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Abstract In recent years, open problem that found the number of integral points on a polytope in high dimension space are appeared. There are many reasons for considering structures in higher dimension, some of them practical and some are aesthetic. Our main purpose is to introduce a procedure in which it makes the operation of computing the factoring of $N = p.q$ as easier as the direct computation fast, therefore, two approaches are working on for finding the number of integral points make benefit from the concept of the Ehrhart polynomial and its application on integral points on a polytope. Polytopes which are taken is the cube, and a map is making between a ball and a polytope in four dimension, then discuss the relation between the number of integral points on a cube from dimension one to n dimension. We found a relation between the radius of the ball, the edge of the cube and the dimension together with Pascal triangle. Two different methods are used, but in this paper we present only one of them and the other we are working on.

Keywords Polytope, lattice points.
2000 Mathematics Subject Classification: 52B20, 11P21

§1. Introduction and preliminaries

A wide variety of pure and applied mathematics involve the problem of counting the number of integral points (lattice points) inside a region in space. Applications in pure mathematics are number theory, toric Hilbert functions, Kostant's partition function in representation theory and Ehrhart polynomial in combinatorics while the applied are: cryptography, integer programming, statistical contingency and mass spectroscope analysis. Perhaps the most basic case is when the region is a polytope (a convex bounded polyhedron).

[5] shows that every arrangement of spheres (and hence every central arrangemen of hyperplanes) is combinatorially equivalent to some convex polytope, [9] proved that there is a relation between the number of lattice point on a sphere and the volume of it. In [14], although a four dimensional Euclidean geometry with time as the fourth dimension was already known since Galileo Galilei's time, it was Einstein who showed that the fourth dimension, time, is essentially different from the other three dimensions. Therefore, his early creations were unrealistic. And yet, real $4D$-objects have to exist, if the relativistic geometry is real. What do they look like? The difficult factorization problem for $n = p \cdot q$ with p and q large primes, presented as follows:

For an integer number $n = p \cdot q$ consider the 4-dimensional convex body $B(N) = \{x \in R^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \le N\}$, thus if we know that $N = p \cdot q$, and $B(N)$ denotes the number of lattice points in $B(N)$.

The fast factorization of n is based on fast computing of $B(N)$. And the application for this problem relates to RSA cryptosystems. Many optimization techniques involve a substep that counts the number of lattice points in a set S, that can be described by a set of linear constraints, i.e. S is the intersection of z^d and a rational polyhedron [11]. The problem of counting the number of elements in S is therefore equivalent to count the number of integral points in a polytope which implies that the count is finite (since the polytope is bounded polyhedron). Different algorithms are used to find the number of lattice points since 1980 dates, all of them depend on the concept of integer programming for more see [2,3].

Some of the basic definitions needed to consolidate results are given as follows:

Definition 1.1.[10] Let $Ax \le b$ where $A \in R^{m\times d}$ is a given real matrix, and $b \in R^m$ is a known real vector. A set $P = \{x \in R^d : Ax \le b\}$ is said to be a polyhedron. Every bounded polyhedron is said to be a polytope.

Definition 1.2.[4] Let $P \subset R^d$ be a lattice polytope, for a positive integer t, $tP = \{tX : X \in P\}$.

Definition 1.3.[13] Let $P \subset R^d$ be a lattice d-polytope. A map $L : N \longrightarrow N$ is defined by $L(P,t) = card(tP \cap Z^d)$, where $card$ means the cardinality of $(tP \cap Z^d)$ and N is the set of natural numbers. It is seen that $L(P,t)$ can be represented as, $L(P,t) = 1 + \Sigma c_i t^i$, this polynomial is said to be the Ehrhart polynomial of a lattice d-polytope P.

Theorem 1.1.[13](Pick's theorem) For $d = 2$, $P \subset R^d$ and P is an integral polyhedron. The famous formula, states that: The number of integral points in an integral polyhedron is equal to the area of the polyhedron plus half the number of integral points on the boundary of the polyhedron plus one, $|P \bigcap Z^2| = area(P) + |\partial P \bigcap Z^2|/2 + 1$.

This formula is useful because it is much more efficient than the direct enumeration of integral points in a polyhedron. The area of P is computed by triangulating the polyhedron. Furthermore, the boundary P is a union of finitely many straight-line intervals, and counting integral points in intervals.

Theorem 1.2.[1](Ehrhart's theorem) Let P be a convex lattice polygon and let t be a positive integer, the following equality always holds. $|P \bigcap Z^2| = area(P)t^2 + |\partial P \bigcap Z^2|t/2 + 1$.

Theorem 1.3.[1](Ehrhart - Macdonald reciprocity) Let P be a d-polytope in R^d with integer vertices, let $L(P,t)$ be the number of integer points in tP, and $L(P_o,t)$ be the number of integer points in the relative interior of tP. Then let $L(P,t)$ and $L(P_o,t)$ are polynomial functions of m of degree d satisfy $L(P,0) = 1$ and $L(P_o,t)$ are polynomial functions of t of degree d that satisfy $L(P,0) = 1$ and $L(P_o,t) = (-1)^d L(P,-t)$.

Theorem 1.4.[8](Jacobi 1829) The number of representations of N as a sum of four squares equates 8 times the sum of all divisors of N that are not divisible by 4.

§2. The proposed method

The proposed method is given in this section is to give a procedure for computing the number of integral points in 4-dimensional ball which is depending on the Ehrhart polynomials of a polytope and its properties.

Procedure 2.1. In this procedure we cover a ball in four dimension by a cube with edges a, and make use of the Ehrhart polynomial for the cube in 4-dimension. Approximately computing the number of integral points depend on the Ehrhart polynomials of the cube. First imagine a circle putting in first quadrant in a square with the same center with dimension two and get a general formula for the number of integral points include the radius of the circle and the edge of the cube which as follows:

In dimension two, let $a =$ the edge of the square, $r =$ radius of the circle.
Ncube=number of integral points on a cube.
Ncircle=number of integral points on a circle.
Now if $a = 2$ then $r = 1$ and Ncube=1.
If $a = 3$ then $r = 3/2$ and Ncube=4.

Combinatorialy the number of integral points on a circle is computed which is similar to the number of integral points on a cube. Continue in this computation until we reach to the general formula as follow:

From the general formula of the Ehrhart polynomial for a cube, which is $L(P,t) = (t+1)^n$ we have the number of integral points in a cube is $(a-1)^2$, where a is the edge of the square. We didn't stop at this point but we want to of our computation and try to compute using Ehrhart polynomial for the square and then number of integral points by putting 1 in the Ehrhart polynomial as follows using theorem 2.1

$$|P \bigcap Z^2| = area(P)t^2 + |\partial P \bigcap Z^2|t/2 + 1,$$

$$L(P,t) = 4t^2 + 4t + 1.$$

The number of integral points is 9.

The number that entirely in P, can be found by using

$$L(P_0,t) = (-1)^d L(P,-t) = (-1)^2[4{-}1^2 + 4{-}1 + 1] = 1,$$

and so on. For dimension 3, we put a ball in a cube also we get a general formula as we are obtained it in dimension two, and the results are compared with the Ehrhart polynomial.

$$L(P,t) = (t+1)^d, L(P_0,t) = (t-1)^d,$$

$$L(P_0,2t) = (2t-1)^3, L(P_0,2) = 1,$$

$$L(P_0,3t) = (3t-1)^3, L(P_0,3) = 8,$$

$$L(P_0,4t) = (4t-1)^4, L(P_0,4) = 27,$$

$$L(P_0,nt) = (nt-1)^3 = \text{number of lattice points in a sphere.}$$

For dimension four, the general formula

$$L(P_0, t) = (t-1)^d,$$

$$L(P_0, nt) = (nt-1)^4.$$

Table 1. Number of lattice points in dimension 2

n	a	r	Ncube	Ncircle
1	2	1	1	1
	3	3/2	4	4
	4	2	9	9
	5	5/2	16	16
	6	3	25	25
	7	7/2	36	36
	8	4	49	49
	9	9/2	64	64

Table 2. Number of lattice points in dimension 3

n	a	r	Ncube	Ncircle
1	2	1	1	1
	3	3/2	8	8
	4	2	27	27
	5	5/2	64	64
	6	3	5^3	5^3
	7	7/2	6^3	6^3
	8	4	7^3	7^3
	9	9/2	8^3	8^3

Table 3. Number of lattice points in dimension 4

n	a	r	Ncube	Ncircle
1	2	1	1	1
	3	3/2	2^4	2^4
	4	2	3^4	3^4
	5	5/2	4^4	4^4
	6	3	5^4	5^4
	7	7/2	6^4	6^4
	8	4	7^4	7^4
	9	9/2	8^4	8^4

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Turán Type inequalities for (p, q)-Gamma function

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Abstract The aim of this paper is to establish new Turán-type inequalities involving the (p, q)-polygamma functions. As an application, when $p \to \infty$, $q \to 1$, we obtain some results from [14] and [15].

Keywords (p, q)-Gamma function, (p, q)-psi function.
2000 Mathematics Subject Classification: 33B15, 26A48.

§1. Introduction and preliminaries

The inequalities of the type

$$f_n(x)f_{n+2}(x) - f_{n+1}^2(x) \leq 0$$

have many applications in pure mathematics as in other branches of science. They are named by Karlin and Szegő in [8], Turán-type inequalities because the first of these type of inequalities was introduced by Turán in [18]. More precisely, he used some results of Szegő in [17] to prove the previous inequality for $x \in (-1, 1)$, where f_n is the Legendre polynomial of degree n. This classical result has been extended in many directions, as ultraspherical polynomials, Lagguere and Hermite polynomials, or Bessel functions, and so forth. Many results of Turán-type have been established on the zeros of special functions.

Recently, W. T. Sulaiman in [15] proved some Turán-type inequalities for some q-special functions as well as the polygamma functions, by using the following inequality:

Lemma 1.1. Let $a \in R_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then

$$\Big(\int_0^a g(x) f^{\frac{m+n}{2}} d_q x\Big)^2 \leq \Big(\int_0^a g(x) f^m d_q x\Big)\Big(\int_0^a g(x) f^n d_q x\Big) \tag{1}$$

Let's give some definitions for gamma and polygamma function.

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \dfrac{d}{dx}\ln\Gamma(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [2]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt = -\gamma - \frac{1}{x} + \sum_{n\geq 1}\frac{x}{n(n+x)}, \tag{2}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant.

Euler gave another equivalent definition for the $\Gamma(x)$ (see [12,13])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})},\ x > 0, \tag{3}$$

where p is positive integer, and

$$\Gamma(x) = \lim_{p\to\infty}\Gamma_p(x). \tag{4}$$

The following representations are valid:

$$\Gamma_p(x) = \int_0^p \left(1 - \frac{t}{p}\right)^p t^{x-1}dt,$$

$$\psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}(1 - e^{-(p+1)t})}{1 - e^{-t}}dt,$$

$$\psi_p^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{t^m e^{-xt}}{1 - e^{-t}}(1 - e^{-pt})dt.$$

Jackson defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0 < q < 1, \tag{5}$$

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}},\ q > 1, \tag{6}$$

where $(a;q)_\infty = \prod_{j\geq 0}(1 - aq^j)$.

The q-gamma function has the following integral representation

$$\Gamma_q(t) = \int_0^\infty x^{t-1}E_q^{-qx}d_qx,$$

where $E_q^x = \sum_{j=0}^\infty q^{\frac{j(j-1)}{2}}\frac{x^j}{[j]!} = (1 + (1-q)x)_q^\infty$, which is the q-analogue of the classical exponential function.

It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$.

Definition 1.1. For $x > 0,\ p \in N$ and for $q \in (0,1)$,

$$\Gamma_{p,q}(x) = \frac{[p]_q^x[p]_q!}{[x]_q[x+1]_q\cdots[x+p]_q}\ , \tag{7}$$

where $[p]_q = \frac{1-q^p}{1-q}$.

The (p, q)-analogue of the psi function is defined as the logarithmic derivative of the (p, q)-gamma function, and has the following series representation and integral representation:

$$\psi_{(p,q)}(x) = -\ln[p]_q - \log q \sum_{k=0}^{p} \frac{q^{x+k}}{1-q^{x+k}}, \tag{8}$$

$$\psi_{(p,q)}(x) = -\ln[p]_q - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t), \tag{9}$$

$$\psi_{(p,q)}^{(n)}(x) = (-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t). \tag{10}$$

where $\gamma_q(t)$ is a discrete measure with positive masses-$\log q$ at the positive points-$k \log q, k = 1, 2, \cdots$ i.e.

$$\gamma_q(t) = -\log q \sum_{k=1}^{\infty} \delta(t + k \log q), \quad 0 < q < 1.$$

In this paper, we give an extension of the main result of W. T. Sulaiman [15], V. Krasniqi etc. [13] and C. Mortici [14].

§2. Main results

Theorem 2.1. For $n = 1, 2, 3, \cdots$, let $\psi_{(p,q),n} = \psi_{(p,q)}^{(n)}$ be the n-th derivative of the function $\psi_{(p,q)}$. Then

$$\psi_{(p,q),\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) \le \psi_{(p,q),m}^{\frac{1}{s}}(x)\psi_{(p,q),n}^{\frac{1}{l}}(y), \tag{11}$$

where $\frac{m+n}{2}$ is an integer, $s > 1$, $\frac{1}{s}+\frac{1}{l}=1$.

Proof. Let m and n be two integers of the same parity. From (10), it follows that:

$$\begin{aligned}
\psi_{(p,q),\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{l}\right) &= (-1)^{\frac{m}{s}+\frac{n}{l}+1} \int_0^{\infty} \frac{t^{\frac{m}{s}+\frac{n}{l}} e^{-\left(\frac{x}{s}+\frac{y}{l}\right)t}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t) \\
&= (-1)^{\frac{m+1}{s}}(-1)^{\frac{n+1}{l}} \int_0^{\infty} \frac{t^{\frac{m}{s}} e^{-\left(\frac{x}{s}\right)t}}{\left(1-e^{-t}\right)^{\frac{1}{s}}} \\
&\quad \cdot (1-e^{-(p+1)t})^{\frac{1}{s}} \frac{t^{\frac{n}{l}} e^{-\left(\frac{y}{l}\right)t}}{\left(1-e^{-t}\right)^{\frac{1}{l}}}(1-e^{-(p+1)t})^{\frac{1}{l}} d\gamma_q(t) \\
&\le \left[(-1)^{m+1} \int_0^{\infty} \frac{t^m e^{-xt}}{\left(1-e^{-t}\right)}(1-e^{-(p+1)t})d\gamma_q(t)\right]^{\frac{1}{s}} \\
&\quad \cdot \left[(-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-yt}}{\left(1-e^{-t}\right)}(1-e^{-(p+1)t})d\gamma_q(t)\right]^{\frac{1}{l}} \\
&= \psi_{(p,q),m}^{\frac{1}{s}}(x)\psi_{(p,q),n}^{\frac{1}{l}}(y).
\end{aligned}$$

Remark 2.1. Let p tends to ∞, then we obtain Theorem 2.2 from [15]. On putting $y = x$ then we obtain generalization of Theorem 2.1 from [15].

Another type via Minkowski's inequality is the following:

Theorem 2.2. For $n = 1, 2, 3, \cdots$, let $\psi_{(p,q),n} = \psi^{(n)}_{(p,q)}$ the n-th derivative of the function $\psi_{(p,q)}$. Then

$$\Big(\psi_{(p,q),m}(x) + \psi_{(p,q),n}(y)\Big)^{\frac{1}{p}} \leq \psi^{\frac{1}{p}}_{(p,q),m}(x) + \psi^{\frac{1}{p}}_{(p,q),n}(y), \tag{12}$$

where $\frac{m+n}{2}$ is an integer, $p \geq 1$.

Proof.

Since

$$(a+b)^p \geq a^p + b^p,\ a, b \geq 0,\ p \geq 1,$$

$$\begin{aligned}
\Big(\psi_{(p,q),m}(x) + \psi_{(p,q),n}(y)\Big)^{\frac{1}{p}} &= \Bigg[(-1)^{m+1}\int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)d\gamma_q(t) \\
&\quad + (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&= \Bigg[\int_0^\infty \Bigg[\Bigg[(-1)^{\frac{m+1}{p}}\frac{t^{\frac{m}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p + \\
&\quad + \Bigg[(-1)^{\frac{n+1}{p}}\frac{t^{\frac{n}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p\Bigg]d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&\leq \Bigg[\int_0^\infty \Bigg[(-1)^{\frac{m+1}{p}}\frac{t^{\frac{m}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}} + \\
&\quad + (-1)^{\frac{n+1}{p}}\frac{t^{\frac{n}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&\leq (-1)^{\frac{m+1}{p}}\Bigg[\int_0^\infty \Bigg[\frac{t^{\frac{m}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p d\gamma_q(t)\Bigg]^{\frac{1}{p}} + \\
&\quad + (-1)^{\frac{n+1}{p}}\Bigg[\int_0^\infty \Bigg[\frac{t^{\frac{n}{p}}e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&= (-1)^{\frac{m+1}{p}}\Bigg[\int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&\quad + (-1)^{\frac{n+1}{p}}\Bigg[\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)d\gamma_q(t)\Bigg]^{\frac{1}{p}} \\
&= \psi^{\frac{1}{p}}_{(p,q),m}(x) + \psi^{\frac{1}{p}}_{(p,q),n}(y)
\end{aligned}$$

Remark 2.2. Let p tends to ∞, then we obtain generalization of Theorem 2.3 from [15].

Theorem 2.3. For every $x>0$ and integers $n\geq 1$, we have:

1. If n is odd, then $\left(\exp\psi^{(n)}_{(p,q)}(x)\right)^2 \geq \exp\psi^{(n+1)}_{(p,q)}(x)\exp\psi^{(n-1)}_{(p,q)}(x)$;

2. If n is even, then $\left(\exp\psi^{(n)}_{(p,q)}(x)\right)^2 \leq \exp\psi^{(n+1)}_{(p,q)}(x)\exp\psi^{(n-1)}_{(p,q)}(x)$.

Proof. We use (10) to estimate the expression

$$\begin{aligned}\psi^{(n)}_{(p,q)}(x)-\frac{\psi^{(n+1)}_{(p,q)}(x)+\psi^{(n-1)}_{(p,q)}(x)}{2} &= (-1)^{n+1}\Big(\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t)\\ &\quad+\frac{1}{2}\int_0^\infty \frac{t^{n+1}e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t)\\ &\quad+\frac{1}{2}\int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t)\Big)\\ &= (-1)^{n+1}\Big(\int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}(t+1)^2(1-e^{-(p+1)t})d\gamma_q(t)\Big).\end{aligned}$$

Now, the conclusion follows by exponentiating the inequality

$$\psi^{(n)}_{(p,q)}(x)\geq(\leq)\frac{\psi^{(n+1)}_{(p,q)}(x)+\psi^{(n-1)}_{(p,q)}(x)}{2}$$

as n is odd, respectively even.

Remark 2.3. Let p tends to ∞, q tends to 1, then we obtain generalization of Theorem 3.3 from [14].

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Circulant determinant sequences with binomial coefficients

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Abstract In this note we introduce the concept of circulant determinant sequences with binomial coefficients and derive formulas for the n-th term of the given sequences as well as the sum of the first n terms.

Keywords Circulant matrices with binomial coefficients, determinant sequence, eigenvalues, sum of the first n terms of a sequence.
Mathematics Subject Classification: 11B25, 11B83, 15B36.

§1. Introduction

In [1], Murthy introduced the concept of the Smarandache Cyclic Determinant Natural Sequence, the Smarandache Cyclic Arithmetic Determinant Sequence, the Smarandache Bisymmetric Determinant Natural Sequence, and the Smarandache Bisymmetric Arithmetic Determinant Sequence.

Circulant matrices are either right-circulant or left circulant. Hence, in particular, Smarandache Cyclic Determinant Natural Sequence and Smarandache Cyclic Arithmetic Determinant Sequence are examples of left-circulant determinant sequences. In general, a right-circulant determinant sequence, which we denote by $\{M_n^+\}_{n\in\mathbb{N}}$, has an n-th term of the form

$$M_n^+ = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{vmatrix}.$$

That is,

$$\{M_n^+\} = \left\{ |c_0|, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_0 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{vmatrix}, \cdots \right\}.$$

Similarly, a left-circulant determinant sequence, which we denote by $\{M_n^-\}_{n\in\mathbb{N}}$, is the sequence of the form

$$\{M_n^-\} = \left\{ |c_0|, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_0 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_0 \\ c_2 & c_1 & c_0 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & c_0 \\ c_2 & c_3 & c_0 & c_1 \\ c_3 & c_0 & c_1 & c_2 \end{vmatrix}, \cdots \right\}.$$

In this note, we present two new examples of circulant determinant sequences. The first is the right-circulant determinant sequence with binomial coefficients and the second is the left-circulant determinant sequence with binomial coefficients. We also derive the formulas for the n-th term of the two sequences. Also, we determine the sum of the first n terms of each of the two sequences.

§2. Main results

In this section we provide a formal definition of the two circulant determinant sequences with binomial coefficients and derive the formula for their respective n-th term.

Definition 2.1. The right-circulant determinant sequence with binomial coefficients, denoted by $\{R_n\}$, is the sequence of the form

$$\{R_n\} = \left\{ |1|, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \end{vmatrix}, \cdots \right\}.$$

In can be seen easily from the above definition that the circulant matrix $\{R_n\} = |c_{ij}|$ where $c_{ij} \equiv \begin{pmatrix} n-1 \\ j-i \end{pmatrix} \pmod{n}$ for all $i, j = 1, 2, \cdots, n$.

Definition 2.2. The left-circulant determinant sequence with binomial coefficients, denoted by $\{L_n\}$, is the sequence of the form

$$\{L_n\} = \left\{ |1|, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{vmatrix}, \cdots \right\}.$$

Obviously, the circulant matrix $L_n = |c_{ij}|$ where $c_{ij} \equiv \begin{pmatrix} n-1 \\ i-j \end{pmatrix} \pmod{n}$ for all $i, j = 1, 2, \cdots, n$.

We first prove the following lemmas before we proceed to our main results.

Lemma 2.1. The eigenvalues of a right-circulant matrix with binomial coefficients are given by

$$\lambda_0 = 2^{n-1},\ \lambda_m = \left(1 + e^{\frac{2\pi i m}{n}}\right)^{n-1},$$

for $m = 1, 2, \cdots, n-1$.

Proof. Note that the eigenvalue of a circulant matrix is given by

$$\lambda_m = \sum_{m=0}^{n-1} c_k e^{\frac{2\pi i m k}{n}}.$$

So we have,

$$\lambda_m = \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} e^{\frac{2\pi i m k}{n}}.$$

It follows that if $m = 0$, we have

$$\lambda_0 = \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} = 2^{n-1}.$$

For $m = 1, 2, \cdots, n-1$, we use the fact that $(1+x)^{n-1} = \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} x^k$. Hence,

$$\lambda_m = \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} \left(e^{\frac{2\pi i m}{n}}\right)^k = \left(1 + e^{\frac{2\pi i m}{n}}\right)^{n-1}.$$

Lemma 2.2. For any natural odd number n we have,

$$\prod_{m=1}^{n-1} \left(1 + e^{\frac{2\pi i m}{n}}\right) = 1.$$

Proof. Let $\epsilon = e^{\frac{2\pi i}{n}}$ be the n-th root of unity and consider the polynomial $X^n - 1$. It is clear that $1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$ are exactly n distinct roots of $X^n - 1$. Hence, we can express $X^n - 1$ as follows:

$$X^n - 1 = (X-1)(X-\epsilon)(X-\epsilon^2)\cdots(X-\epsilon^{n-1}) = \prod_{m=0}^{n-1}(X - \epsilon^m).$$

But, $X^n - 1 = (X-1)(X^{n-1} + X^{n-2} + \cdots + X^2 + X + 1)$. It follows that,

$$(X-1)\prod_{m=1}^{n-1}(X-\epsilon^m) = (X-1)(X^{n-1} + X^{n-2} + \cdots + X^2 + X + 1).$$

Thus,

$$\prod_{m=1}^{n-1}(X-\epsilon^m) = (X^{n-1} + X^{n-2} + \cdots + X^2 + X + 1).$$

Replacing X by $-X$ and noting that $n-1$ is even, we will obtain

$$\prod_{m=1}^{n-1}(X+\epsilon^m) = (X^{n-1} - X^{n-2} + X^{n-3} - \cdots - X + 1).$$

Letting $X = 1$, we have $\prod_{m=1}^{n-1}(1+\epsilon^m) = 1$. This proves the theorem.

Now we have the following results.

Theorem 2.1. The formula for the n-th term of the right-circulant determinant sequence with binomial coefficients, denoted by R_n, is given by

$$R_n = \left(1 + (-1)^{n-1}\right) 2^{n-2}.$$

Proof. We consider the two possible cases.

Case 1. If n is even, say $n = 2k$ for some $k = 1, 2, \cdots$, we have

$$\begin{aligned} R_{2k} &= \prod_{m=0}^{2k-1} \lambda_m = \prod_{m=0}^{2k-1} \left(1 + e^{\frac{2\pi i m}{2k}}\right)^{2k-1} \\ &= \Big[\left(1+e^0\right)\left(1+e^{\frac{2\pi i}{2k}}\right)\cdots\left(1+e^{\frac{2\pi i k}{2k}}\right)\left(1+e^{\frac{2\pi i (k+1)}{2k}}\right) \\ &\quad \cdots\left(1+e^{\frac{2\pi i(2k-2)}{2k}}\right)\left(1+e^{\frac{2\pi i(2k-1)}{2k}}\right)\Big]^{2k-1} \\ &= 0. \end{aligned}$$

Case 2. If n is odd, say $n = 2k-1$ for some $k = 1, 2, \cdots$, we have, by virtue of Lemma 2.4,

$$R_{2k-1} = \prod_{m=0}^{2k-2} \lambda_m = 2^{2k-2}\left[\prod_{m=1}^{2k-2}\left(e^{\frac{2\pi i m}{2k-1}} + 1\right)\right]^{2k-2} = 2^{2k-2}.$$

Thus, for any natural number n, $R_n = (1 + (-1)^{n-1})2^{n-2}$.

Theorem 2.2. The formula for the n-th partial sum of the sequence $\{R_n\}$, denoted by RS_n, is given by

$$RS_n = \frac{4^{\lfloor \frac{n+1}{2} \rfloor} - 1}{3}.$$

Proof. Let RS_n be the partial sum of the first n terms of the right-circulant determinant sequence with binomial coefficients then

$$\begin{aligned} RS_n &= \sum_{k=1}^{n}(1+(-1)^{k-1})2^{k-2} \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(1+(-1)^{2k-1})2^{2(k-1)} + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor}(1+(-1)^{2(k-1)})2^{2(k-1)-1} \\ &= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor}(2^2)^{k-1} \\ &= \frac{4^{\lfloor \frac{n+1}{2} \rfloor} - 1}{3}. \end{aligned}$$

Remark 2.1. From the previous theorem, we can see that $3|\left(4^{\lfloor\frac{n+1}{2}\rfloor}-1\right)$ for all natural number n.

Theorem 2.3. The formula for the n-th term of the left-circulant determinant sequence with binomial coefficients, denoted by L_n, is given by

$$L_n = (-1)^{\lfloor\frac{n-1}{2}\rfloor}\left(1+(-1)^{n-1}\right)2^{n-2}.$$

Proof. It can be seen easily that $R_n = L_n$ for $n < 3$. Now, for $n \geq 3$ we fixed the first row of R_n and apply the row operation $R_i \leftrightarrow R_{n+2-i}$ for $2 \leq i \leq \lfloor\frac{n+1}{2}\rfloor$ obtaining $L_n = (-1)^{\lfloor\frac{n-1}{2}\rfloor}R_n$. Thus, $L_n = (-1)^{\lfloor\frac{n-1}{2}\rfloor}\left(1+(-1)^{n-1}\right)2^{n-2}$.

Theorem 2.4. The formula for the n-th partial sum of the sequence $\{L_n\}$, denoted by LS_n, is given by

$$LS_n = \frac{1-(-4)^{\lfloor\frac{n+1}{2}\rfloor}}{5}.$$

Proof. Let LS_n be the partial sum of the first n terms of the left-circulant determinant sequence with binomial coefficients then

$$\begin{aligned}
LS_n &= \sum_{k=1}^{n}(-1)^{\lfloor\frac{n-1}{2}\rfloor}(1+(-1)^{k-1})2^{k-2}\\
&= \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}(-1)^{\lfloor\frac{2k-1}{2}\rfloor}(1+(-1)^{2k-1})2^{2(k-1)}\\
&\quad + \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^{\lfloor\frac{2(k-1)}{2}\rfloor}(1+(-1)^{2(k-1)})2^{2(k-1)-1}\\
&= \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(-4)^{k-1}\\
&= \frac{1-(-4)^{\lfloor\frac{n+1}{2}\rfloor}}{5}.
\end{aligned}$$

Remark 2.2. We can see clearly that from the previous theorem $5|\left(1-(-4)^{\lfloor\frac{n+1}{2}\rfloor}\right)$ for all natural number n.

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On the mean value of $\tau_3^{(e)}(n)$ over cube-full numbers[1]

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Abstract Let $\tau_k^{(e)}(n) = \prod\limits_{p_i^{a_i}\|n} d_k(a_i)$. In this paper we study the mean value of $\tau_3^{(e)}(n)$ over cube-full numbers and establish the asymptotic formula for it.

Keywords Exponential divisor function, convolution method, asymptotic formula.
2000 Mathematics Subject Classification: 11N37.

§1. Introduction

The integer $d = \prod\limits_{i=1}^{r} p_i^{b_i}$ is called an exponential divisor of $n = \prod\limits_{i=1}^{r} p_i^{a_i}$ if $b_i|a_i (i = 1, 2, \cdots, r)$, denoted by $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n) = \sum\limits_{d|_e n} 1$, which is firstly studied by M. V. Subbarao [2], J. Wu [4] and L. Tóth [3] improved the mean value for $\tau^{(e)}(n)$ later. And the best result at present belongs to L. Tóth:

$$\sum_{n\le x} (\tau^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\epsilon}),$$

where $r \ge 1$ is an integer, $P_l(t)$ is a polynomial in t of degree l, and

$$A_r = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}\right), \quad u_r = \frac{2^{r+1}-1}{2^{r+2}+1}.$$

For $k \ge 2$, L. Tóth [3] also defined the function $\tau_k^{(e)}(n) := \prod_{p_i^{a_i}\|n} d_k(a_i)$, which is the generalization of $\tau^{(e)}(n)$. He proved that

$$\sum_{n\le x} \tau_k^{(e)}(n) = C_k x + x^{\frac{1}{2}} Q_{k-2}(\log x) + O(x^{\omega_k+\epsilon}),$$

where $Q_l(t)$ is a polynomial in t of degree l, and

$$C_k = \prod_p (1 + \sum_{a=2}^{\infty} \frac{d_k(a) - d_k(a-1)}{p^a}), \quad \omega_k = \frac{2k-1}{4k+1}.$$

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For $k \geq 2,\ r \geq 1$, L. Dong and D. Zhang [6] recently proved that

$$\sum_{n\leq x}(\tau_k^{(e)}(n))^r = A_{k,r}x + x^{\frac{1}{2}}Q_{k^r-2}(\log x) + O(x^{c(k,r)+\epsilon}),$$

where

$$A_{k,r} = \prod_p\left(1+\sum_{a=2}^{\infty}\frac{(d_k(a))^r-(d_k(a-1))^r}{p^a}\right),\quad c(k,r) = \frac{1}{3-\alpha_{k^r-1}}.$$

In this paper, we study $\tau_3^{(e)}(n)$ over cube-full numbers and establish the mean value estimate for it. We have the following result:

Theorem 1.1. Let

$$f_3(n) = \begin{cases} 1, & \text{if } n \text{ is cube-full,} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\sum_{n\leq x}\tau_3^{(e)}(n)f_3(n) = x^{\frac{1}{3}}P(\log x) + x^{\frac{1}{4}}Q(\log x) + x^{\frac{1}{5}}R(\log x) + O(x^{\frac{834809}{4443404}+\epsilon}), \tag{1}$$

where $P(\log x),\ Q(\log x)$ and $R(\log x)$ are polynomials in $\log x$ of degree 2, 5, 2.

Notations: Throughout this paper, ϵ denotes a fixed but sufficiently small positive constant, the devisor function $d(n) = \sum\limits_{n=ab} 1$, $d_k(n) = \sum\limits_{n=m_1\cdots m_k} 1$, and we denote $f(x) \ll g(x)$ or $f(x) = O(g(x))$ for $|f(x)| \leq Cg(x)$.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas:

Lemma 2.1. Suppose s is a complex number with $\Re s > \frac{1}{3}$, then

$$F(s) := \sum_{n=1}^{\infty}\frac{\tau_3^{(e)}(n)f_3(n)}{n^s} = \zeta^3(3s)\zeta^6(4s)\zeta^3(5s)H(s), \tag{2}$$

where $H(s)$ can be written as a Dirichlet series $H(s) = \sum_{n=1}^{\infty}\frac{h(n)}{n^s}$, which is absolutely convergent for $\Re s > \frac{1}{6}$.

Proof. The function $\tau_3^{(e)}(n)$ is multiplicative, so by the Euler product formula, for $\Re s > \frac{1}{3}$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\tau_3^{(e)}(n)f_3(n)}{n^s} &= \prod_p(1+\frac{\tau_3^{(e)}(p)f_3(p)}{p^s}+\frac{\tau_3^{(e)}(p^2)f_3(p^2)}{p^{2s}}+\frac{\tau_3^{(e)}(p^3)f_3(p^3)}{p^{3s}}+\cdots)\\ &= \prod_p(1+\frac{3}{p^{3s}}+\frac{6}{p^{4s}}+\frac{3}{p^{5s}}+\frac{9}{p^{6s}}+\cdots)\\ &= \zeta^3(3s)\prod_p(1+\frac{6}{p^{4s}}+\frac{3}{p^{5s}}+\frac{3}{p^{6s}}+\cdots)\\ &= \zeta^3(3s)\zeta^6(4s)\prod_p(1+\frac{3}{p^{5s}}+\frac{3}{p^{6s}}+\cdots)\\ &= \zeta^3(3s)\zeta^6(4s)\zeta^3(5s)\prod_p(1+\frac{3}{p^{6s}}+\cdots)\\ &= \zeta^3(3s)\zeta^6(4s)\zeta^3(5s)H(s),\end{aligned}$$

where $H(s)=\prod_p(1+\frac{3}{p^{6s}}+\cdots)$. It is easily seen that $H(s)$ can be written as Dirichlet series which is absolutely convergent for $\Re s>\frac{1}{6}$.

Lemma 2.2 Let

$$m(\sigma)=\begin{cases}\frac{64}{31-103\sigma} & \frac{1}{2}\le\sigma\le\frac{5}{8},\\ \frac{10}{5-6\sigma} & \frac{5}{8}\le\sigma\le\frac{35}{54},\\ \frac{19}{6-6\sigma} & \frac{35}{54}\le\sigma\le\frac{41}{60},\\ \frac{2112}{859-948\sigma} & \frac{41}{60}\le\sigma\le\frac{3}{4},\\ \frac{12408}{4537-4890\sigma} & \frac{3}{4}\le\sigma\le\frac{5}{6},\\ \frac{4324}{1031-1044\sigma} & \frac{5}{6}\le\sigma\le\frac{7}{8},\\ \frac{98}{31-32\sigma} & \frac{5}{8}\le\sigma\le 0.91591\cdots,\\ \frac{24\sigma-9}{(4\sigma-1)(1-\sigma)} & 0.91591\cdots\le\sigma\le 1-\epsilon.\end{cases}$$

Then

$$\int_0^T|\zeta(\sigma+it)|^{m(\sigma)}dt\ll T^{1+\epsilon}.$$

Proof. See Theorem 8.4 of Ivic [1].

Lemma 2.3. Let $g(m)$, $h(l)$ be arithmetic functions such that

$$\sum_{m\le x}g(m)=\sum_{j=1}^{J}x^{\alpha_j}P_j(\log x)+O(x^{\alpha}),\quad \sum_{l\le x}|h(l)|=O(x^{\beta}),$$

where $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_J>\alpha>\beta>0$, $P_j(t)(j=1,\ 2,\ \cdots,\ J)$ are polynomials in t. If $f(n)=\sum_{n=ml}g(m)h(l)$, then

$$\sum_{n\le x}f(n)=\sum_{j=1}^{J}x^{\alpha_j}Q_j(\log x)+O(x^{\alpha}),$$

where $Q_j(t)(j=1,\ 2,\ \cdots,\ J)$ are polynomials in t.

Proof. See lemma 2.2 of [5].

Lemma 2.4. Suppose $s = \sigma + it$, for $\zeta(s)$ we have

$$\zeta(s) \ll \begin{cases} (|t|+2)^{\frac{1-\sigma}{3}} \log(|t|+2), & \frac{1}{2} \leq \sigma \leq 1; \\ \log(|t|+2), & 1 < \sigma \leq 2. \end{cases}$$

Proof. We can get the first estimate by $\zeta(\frac{1}{2}+it) \ll (|t|+2)^{\frac{1}{6}}$, $\zeta(1+it) \ll \log(|t|+2)$ and Phragmén-Lindelöf Theorem.

Now we prove our theorem. Let $\zeta^3(3s)\zeta^6(4s)\zeta^3(5s) = \sum_{m=1}^{\infty} \frac{g(m)}{m^s}$. By Perron's formula, we have

$$\sum_{m\leq x} g(m) = \frac{1}{2\pi i} \int_{\frac{1}{3}+\epsilon-iT}^{\frac{1}{3}+\epsilon+iT} \frac{\zeta^3(3s)\zeta^6(4s)\zeta^3(5s)x^s}{s} ds + O(x^\epsilon), \tag{3}$$

Shifting the contour to the segment from $\sigma_0 - iT$ to $\sigma_0 + iT$ ($\frac{3}{16} < \sigma_0 < \frac{1}{5}$), by the residue theorem, we have

$$\sum_{m\leq x} g(m) = x^{\frac{1}{3}} P^{'}(\log x) + x^{\frac{1}{4}} Q^{'}(\log x) + x^{\frac{1}{5}} R^{'}(\log x) + I_1 + I_2 - I_3 + O(x^\epsilon),$$

where $P^{'}(\log x)$, $Q^{'}(\log x)$ and $R^{'}(\log x)$ are polynomials in $\log x$ of degree 2, 5, 2, and

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\zeta^3(3s)\zeta^6(4s)\zeta^3(5s)x^s}{s} ds, \\ I_2 &= \frac{1}{2\pi i} \int_{\sigma_0+iT}^{\frac{1}{3}+\epsilon+iT} \frac{\zeta^3(3s)\zeta^6(4s)\zeta^3(5s)x^s}{s} ds, \\ I_3 &= \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\frac{1}{3}+\epsilon-iT} \frac{\zeta^3(3s)\zeta^6(4s)\zeta^3(5s)x^s}{s} ds. \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} I_2 &\ll \int_{\sigma_0}^{\frac{1}{3}+\epsilon} \frac{\zeta^3(3\sigma+3iT)\zeta^6(4\sigma+4iT)\zeta^3(5\sigma+5iT)x^\sigma}{T} d\sigma \\ &\ll \log^{12} T \left(\int_{\sigma_0}^{\frac{1}{5}} T^{3-16\sigma} x^\sigma d\sigma + \int_{\frac{1}{5}}^{\frac{1}{4}} T^{2-11\sigma} x^\sigma d\sigma + \int_{\frac{1}{4}}^{\frac{1}{3}+\epsilon} T^{-3\sigma} x^\sigma d\sigma \right), \end{aligned}$$

now set $T = x$, so $I_2 \ll x^{\sigma_0+\epsilon}$. We can get $I_3 \ll x^{\sigma_0+\epsilon}$ by similar arguments. Now we go on to bound I_1.

$$\begin{aligned} I_1 &\ll x^{\sigma_0} (\int_0^1 \frac{\zeta^3(3\sigma_0+3it)\zeta^6(4\sigma_0+4it)\zeta^3(5\sigma_0+5it)}{\sqrt{\sigma_0^2+t^2}} dt \\ &+ \int_1^T \frac{\zeta^3(3\sigma_0+3it)\zeta^6(4\sigma_0+4it)\zeta^3(5\sigma_0+5it)}{t} dt) \\ &\ll x^{\sigma_0} (1 + \int_1^T \frac{\zeta^3(3\sigma_0+3it)\zeta^6(4\sigma_0+4it)\zeta^3(5\sigma_0+5it)}{t} dt), \end{aligned}$$

It suffices to prove

$$I_4 = \int_1^T \zeta^3(3\sigma_0 + 3it)\zeta^6(4\sigma_0 + 4it)\zeta^3(5\sigma_0 + 5it)dt \ll T^{1+\epsilon}. \tag{4}$$

Actually, suppose $q_i > 0 (i = 1, 2, 3)$ such that $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$, it follows from Holder's inequality that

$$I_4 \ll \left(\int_1^T |\zeta^3(3\sigma_0 + 3it)|^{q_1} dt\right)^{\frac{1}{q_1}} \left(\int_1^T |\zeta^6(4\sigma_0 + 4it)|^{q_2} dt\right)^{\frac{1}{q_2}} \left(\int_1^T |\zeta^3(5\sigma_0 + 5it)|^{q_3} dt\right)^{\frac{1}{q_3}}.$$

By Lemma 2.2, we take $q_1 = \frac{m(3\sigma_0)}{3}, q_2 = \frac{m(4\sigma_0)}{6}, q_3 = \frac{m(5\sigma_0)}{3}, \sigma_0 = \frac{834809}{4443464} = 0.18787\cdots$ to get (4), then $I_1 \ll x^{\sigma_0+\epsilon}$. So we obtain

$$\sum_{m\le x} g(m) = x^{\frac{1}{3}} P^{'}(\log x) + x^{\frac{1}{4}} Q^{'}(\log x) + x^{\frac{1}{5}} R^{'}(\log x) + O(x^{\frac{834809}{4443464}+\epsilon}).$$

We get from Lemma 2.1 that $\sum_{l\le x} |h(l)| \ll x^{\frac{1}{6}+\epsilon}$, then our theorem 1.1 follows from the Dirichlet convolution and Lemma 2.3.

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Fekete-Szegö inequality for certain classes of close-to-convex functions

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Abstract We introduce some classes of close to convex functions and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$, μ real, for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $|z| < 1$, belonging to these classes.

Keywords Univalent functions, starlike functions, convex functions, close to convex functio -ns, bounded functions.

§1. Introduction and preliminaries

Let A denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit *disc* $E = \{z : |z| < 1\}$. Let S be the class of functions of the form (1) which are analytic univalent in E.

We shall concentrate on the coefficient problem for the class S and certain of its subclasses. In 1916, Bieberbach [3] proved that $|a_2| \leq 2$ for $f(z) \in S$ as a corollary to an elementary area theorem. He conjectured that, for each function $f(z) \in S$, $|a_n| \leq n$; equality holds for the Koebe function $k(z) = z/(1-z)^2$, which maps the unit *disc* E onto the entire complex plane minus the slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. De Branges [5] solved the Bieberbach conjecture in 1984. The contribution of Löwner [10] in proving that $|a_3| \leq 3$ for the class S was huge.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class S. This thought prompted Fekete and Szegö [6] and they used Löwner's method to prove the following well-known result for the class S. If $f(z) \in S$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \tag{2}$$

The inequality (2) plays a very important role in determining estimates of higher coefficients for some subclasses of S (see Chichra [4], Babalola [2]).

Next, we define some subclasses of S and obtain analogous of (2).

We denote by S^* the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ and satisfying the condition

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \;\; z \in E. \tag{3}$$

We denote by K the class of convex univalent functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in A$ which satisfies the condition

$$\Re\left(\frac{(zh'(z))'}{h'(z)}\right) > 0, \;\; z \in E. \tag{4}$$

A function $f(z) \in A$ is said to be close to convex if there exists a function $g(z) \in S^*$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \;\; z \in E. \tag{5}$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [8], who showed that all close to convex functions are univalent. The immediate shoot of C are its following subclasses:

$$C_1 = \left\{ f(z) \in A : \Re\left(\frac{zf'(z)}{h(z)}\right) > 0, \;\; h(z) \in K, \;\; z \in E \right\}, \tag{6}$$

$$C' = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \;\; g(z) \in S^*, \;\; z \in E \right\}, \tag{7}$$

$$C_1' = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, \;\; h(z) \in K, \;\; z \in E \right\}. \tag{8}$$

Abdel Gawad and Thomas [1] investigated the class C_1 and also obtained (2) for $-\infty < \mu \leq 1$ (although this result seems to be doubtful).

Let U be the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, \; z \in E, \tag{9}$$

and satisfying the conditions $w(0) = 0, |w(z)| < 1$. It is known (see [11]) that

$$|d_1| \leq 1, \; |d_2| \leq 1 - |d_1|^2. \tag{10}$$

We shall apply the subordination principle due to Rogosinski [12], which states that if $f(z) \prec F(z)$, then $f(z) = F(w(z))$, $w(z) \in U$ (where $\prec$ stands for subordination).

Hummel [7] proved a conjecture of V. Singh that $|c_3 - c_2^2| \leq \frac{1}{3}$ for the class K. Keogh and Merkes [9] obtained the estimates (2) for the classes S^*, K and C. Estimates (2) for the classes C_1, $C^{'}$ and $C_1^{'}$ have been waiting to be determined for the last 60 years.

§2. Preliminary lemmas

Lemma 2.1. Let $g(z) \in S^*$, then

$$|b_3 - \frac{3\mu}{4}b_2^2| \leq \begin{cases} 3(1-\mu), & \text{if } \mu \leq \frac{2}{3}; \\ 1, & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ 3(\mu-1), & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

This lemma is a direct consequence of the result of Keogh and Merkes [9] which states that for $g(z) \in S^*$,

$$|b_3 - \mu b_2^2| \leq \begin{cases} 3-4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu-3, & \text{if } \mu \geq 1. \end{cases}$$

Lemma 2.2. Let $h(z) \in K$, then

$$|c_3 - \frac{3\mu}{4}c_2^2| \leq \begin{cases} 1-\frac{3}{4}\mu, & \text{if } \mu \leq \frac{8}{9}; \\ \frac{1}{3}, & \text{if } \frac{8}{9} \leq \mu \leq \frac{16}{9}; \\ \frac{3}{4}\mu-1, & \text{if } \mu \geq \frac{16}{9}. \end{cases}$$

This lemma is a direct consequence of a result of Keogh and Merkes [9], which states that for $h(z) \in K$,

$$|c_3 - \mu c_2^2| \leq \begin{cases} 1-\mu, & \text{if } \mu \leq \frac{2}{3}; \\ \frac{1}{3}, & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ \mu-1, & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

Unless mentioned otherwise, throughout the paper we assume the following notations: $w(z) \in U,\ z \in E$.

For $0 < c < 1$, we write $w(z) = z(\frac{c+z}{1+cz})$ so that $\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots$, where $z \in E$.

§3. Main results

Theorem 3.1. Let $f(z) \in C'$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{19}{9} - \frac{9\mu}{4}, & \text{if } \mu \leq \frac{16}{27}; \\ \frac{64}{81\mu} - \frac{5}{9}, & \text{if } \frac{16}{27} \leq \mu \leq \frac{2}{3}; \\ \frac{5}{9} + \frac{(8-9\mu)^2}{81\mu}, & \text{if } \frac{2}{3} \leq \mu \leq \frac{8}{9}; \\ \frac{5}{9} + \frac{(9\mu-8)^2}{16-9\mu}, & \text{if } \frac{8}{9} \leq \mu \leq \frac{32}{27}; \\ \frac{5\mu}{4} - \frac{7}{9}, & \text{if } \frac{32}{27} \leq \mu \leq \frac{4}{3}; \\ \frac{9\mu}{4} - \frac{19}{9}, & \text{if } \mu \geq \frac{4}{3}. \end{cases} \tag{11}$$

These results are sharp.

Proof. By definition of C',

$$\frac{(zf'(z))'}{g'(z)} = \frac{1+w(z)}{1-w(z)},$$

which on expansion yields

$$1 + 4a_2z + 9a_3z^2 + \cdots = (1 + 2b_2z + 3b_3z^2 + \cdots)(1 + 2d_1z + 2(d_2 + d_1^2)z^2 + \cdots).$$

Identifying terms in above expansion,

$$a_2 = \frac{1}{2}(b_2 + d_1), \tag{12}$$

$$a_3 = \frac{b_3}{3} + \frac{4}{9}b_2d_1 + \frac{2}{9}(d_2 + d_1^2). \tag{13}$$

From (12) and (13) and using (10), it is easily established that

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}\Big|b_3 - \frac{3}{4}\mu b_2^2\Big| + \frac{1}{18}|8-9\mu||b_2||d_1| + \frac{1}{36}(8(1-|d_1|^2) + |8-9\mu||d_1|^2), \tag{14}$$

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}\Big|b_3 - \frac{3}{4}\mu b_2^2\Big| + \frac{1}{18}|8-9\mu|xy + \frac{1}{36}(|8-9\mu| - 8)x^2, \tag{15}$$

where $x = |d_1| \leq 1$ and $y = |b_2| \leq 2$.

Case I. Suppose that $\mu \leq \frac{2}{3}$. By Lemma 2.1, (15) can be written as

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + (1-\mu) + \frac{1}{9}(8-9\mu)x - \frac{\mu}{4}x^2 = H_0(x),$$

then

$$H_0^{'}(x) = \frac{1}{9}(8-9\mu) - \frac{\mu}{2}x, \quad H_0^{''}(x) = -\frac{\mu}{2}.$$

Subcase I(i). For $\mu \leq 0$, since $x \geq 0$, we have $H_0'(x) > 0$. $H_0(x)$ is an increasing function in $[0,1]$ and $\max H_0(1) = \dfrac{19}{9} - \dfrac{9\mu}{4}$.

Subcase I(ii). Suppose $0 < \mu \leq \frac{2}{3}$. $H_0^{'}(x) = 0$ when $x = \frac{2(8-9\mu)}{9\mu} = x_0$, and $x_0 > 1$ if and only if $\mu < \frac{16}{27}$, we have $\max H_0(x) = H_0(1) = \frac{19}{9} - \frac{9\mu}{4}$. Combining the above two subcases, we obtain first result of (11).

Subcase I(iii). For $\frac{16}{27} \leq \mu \leq \frac{2}{3}(x_0 < 1)$, since $H_0^{''}(x) < 0$, therefore we have $\max H_0(x) = H_0(x_0) = \frac{64}{81\mu} - \frac{5}{9}$.

Case II. Suppose that $\frac{2}{3} \leq \mu \leq \frac{8}{3}$, then by Lemma 2.1, (15) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9}|8-9\mu|x - \frac{\mu}{4}x^2.$$

Subcase II(i). $\frac{2}{3} < \mu < \frac{8}{9}$. Under the above condition, from (15), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9}(8-9\mu)x - \frac{\mu}{4}x^2 = H_1(x),$$

then

$$H_1^{'}(x) = \frac{1}{9}(8-9\mu) - \frac{\mu}{2}x, H_1^{''}(x) = -\frac{\mu}{2} < 0.$$

$H_1^{'}(x) = 0$ implies that $x = \frac{2(8-9\mu)}{9\mu} = x_1$ and $\max H_1(x) = H_1(x_1) = \frac{5}{9} + \frac{(8-9\mu)^2}{81\mu}$.

Subcase II(ii). For $\frac{8}{9} \leq \mu \leq \frac{32}{27}$, by Lemma 2.1, (15) reduces to

$$|a_3 - \mu a_2^2| \leq \frac{5}{9} + (9\mu - 8)x + \frac{(16-9\mu)}{36}x^2 = H_2(x),$$

then

$$H_2^{'}(x) = (9\mu - 8) - \frac{1}{18}(9\mu - 16)x, H_2^{''}(x) < 0.$$

$H_2^{'}(x)$ vanishes when $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_2 < 1$ and $\max H_2(x) = H_2(x_2) = \frac{5}{9} + \frac{(8-9\mu)^2}{(16-9\mu)}$.

Subcase II(iii). $\frac{32}{27} \leq \mu \leq \frac{4}{3}$. (15) can be expressed as

$$|a_3 - \mu a_2^2| \leq \frac{5}{9} + \frac{1}{9}(9\mu - 8)x - \frac{(16-9\mu)}{36}x^2 = H_3(x),$$

then

$$H_3^{'}(x) = \frac{1}{9}(9\mu - 8) - \frac{1}{18}(16 - 9\mu)x.$$

$H_3^{'}(x) = 0$ yields $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_3 \geq 1$ and $\max H_3(x) = H_3(1) = \frac{5\mu}{4} - \frac{7}{9}$.

Case III. $\mu \geq \frac{4}{3}$. By Lemma 2.1, (15) can be put in the form

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + (\mu - 1) + \frac{1}{9}(9\mu - 8)x - \frac{(16-9\mu)}{36}x^2 = H_4(x),$$

then

$$H_4^{'}(x) = \frac{1}{9}(9\mu - 8) - \frac{1}{18}(16 - 9\mu)x,$$

which vanishes at $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_4 \geq 1$ and therefore $\max H_4(x) = H_4(1) = \frac{9\mu}{4} - \frac{19}{9}$.

The first and second inequalities of (11) coincide at $\mu = \frac{16}{27}$ and each is equal to $\frac{7}{9}$.

The second and third inequalities of (11) coincide at $\mu = \frac{2}{3}$ and each is equal to $\frac{17}{27}$.

The third and fourth inequalities of (11) coincide at $\mu = \frac{8}{9}$ and each is equal to $\frac{5}{9}$.

The fourth and fifth inequalities of (11) coincide at $\mu = \frac{32}{27}$ and each is equal to $\frac{19}{27}$.

The fifth and last inequalities of (11) coincide at $\mu = \frac{4}{3}$ and each is equal to $\frac{8}{9}$.

Results of (11)are sharp for the functions defined by their respective derivatives in order as follows:

$f_1^{'}(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)^2}{(1-t)^4}dt\right)\right]$,

$f_2^{'}(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2ct+2t^2+...)}{(1-t)^3}dt\right)\right]$ where $c = \frac{2(8-9\mu)}{9\mu}$,

$f_3^{'}(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2dt+2t^2+...)}{(1-t)^3}dt\right)\right]$ where $d = \frac{2(8-9\mu)}{9\mu}$,

$f_4^{'}(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2et+2t^2+...)}{(1-t)^3}dt\right)\right]$ where $e = \frac{2(9\mu-8)}{(16-9\mu)}$,

$f_5^{'}(z) = \frac{1}{z}\left[\left(\int_0^z [(1 + \frac{29}{3\sqrt{5}}t)^{\frac{15}{29}}dt]\right)\right]$ where $|t| < \frac{3\sqrt{5}}{29}$,

$f_6^{'}(z) = f_1^{'}(z)$.

The proof of the theorem is complete.

Theorem 3.2. Let $f(z) \in C_1^{\prime}$, then

$$|a_3-\mu a_2^2|\le\begin{cases}1-\mu, & \text{if } \mu\le\frac{4}{9};\\ \frac{16}{81\mu}+\frac{1}{9}, & \text{if } \frac{4}{9}\le\mu\le\frac{8}{9};\\ \frac{1}{3}+\frac{(9\mu-8)^2}{36(16-9\mu)}, & \text{if } \frac{8}{9}\le\mu\le\frac{4}{3};\\ \frac{3\mu}{4}-\frac{5}{9}, & \text{if } \frac{4}{3}\le\mu\le\frac{16}{9};\\ \mu-1, & \text{if } \mu\ge\frac{16}{9}.\end{cases}$$

These results are sharp.

Proof. Proceeding as in Theorem 3.1, we have

$$|a_3-\mu a_2^2|\le\frac{2}{9}+\frac{1}{3}|c_3-\frac{3}{4}\mu c_2^2|+\frac{1}{18}|8-9\mu||c_2||d_1|+\frac{1}{36}(|8-9\mu|-8)|d_1|^2. \tag{16}$$

Case I. Suppose that $\mu\le\frac{8}{9}$. By Lemma 2.2, and putting $x=|d_1|\le 1$ and $y=|c_2|\le 1$,(16) reduces to

$$\begin{aligned}|a_3-\mu a_2^2|&\le\frac{2}{9}+\frac{1}{3}\left(1-\frac{3\mu}{4}\right)+\frac{1}{18}(8-9\mu)xy-\frac{\mu}{4}x^2\\ &=\left(\frac{5}{9}-\frac{\mu}{4}\right)+\frac{1}{18}(8-9\mu)x-\frac{\mu}{4}x^2=H_6(x),\end{aligned}$$

then

$$H_6'(x)=\frac{8-9\mu}{18}-\frac{\mu}{2}x,\quad H_6''(x)=-\frac{\mu}{2}.$$

When $H_6'(x)=0$, we have $8-9\mu=9\mu x=9\mu x_6$.

Subcase I(i). For $\mu\le 0$, since $x\ge 0$ we have $H_6'(x)\ge 0$. Suppose $\mu>0$. Since $x\le 1$, $H_6'(x)\ge 4/9-\mu>0$ if and only if $\mu<4/9$. Then for $\mu<4/9$, we have $H_6(x)\le H_6(1)=1-\mu$.

Subcase I(ii). Suppose that $\frac{4}{9}\le\mu\le\frac{8}{9}$. Then $\max H_6(x)=H_6(x_6)=16/81\mu+1/9$.

Case II. Suppose that $\frac{8}{9}\le\mu\le\frac{16}{9}$. By Lemma 2.2 and (16),

$$|a_3-\mu a_2^2|\le\frac{1}{3}+\frac{1}{18}(9\mu-8)x-\frac{1}{36}(16-9\mu)x^2=H_7(x),$$

then $H_7'(x)=0$ when $x=(9\mu-8)/(16-9\mu)=x_7$, and $H_7''(x)=-(16-9\mu)/18<0$. Since $x_7\le 1$, this is relevant only for $\mu\le\frac{4}{3}$.

Subcase II(i). Suppose that $\frac{8}{9}\le\mu\le\frac{4}{3}$. Then

$$\max H_7(x)=H_7(x_7)=\frac{1}{3}+\frac{(9\mu-8)^2}{36(16-9\mu)}.$$

Subcase II(ii). If $\frac{4}{3}\le\mu\le\frac{16}{9}$, then $H_7'(x)\ge 0$, so $H_7(x)$ is a monotonically increasing function of x and $\max H_7(x)=H_7(1)=3\mu/4-5/9$.

Case III. Suppose that $\mu\ge\frac{16}{9}$. By Lemma 2.2, from (16),

$$|a_3-\mu a_2^2|\le\frac{2}{9}+\frac{1}{3}\left(\frac{3\mu}{4}-1\right)+\frac{1}{18}(9\mu-8)x+\frac{1}{36}(9\mu-16)x^2=H_8(x),$$

we have $H_8'(x) > 0$ and $\max H_8(x) = H_8(1) = \mu - 1$.

This completes the proof.

Extremal function $f_1(z)$ for the first and the last results is defined by $f_1'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)}{(1-t)^2}dt\right)\right]$.

Extremal function $f_2(z)$ for the second bound is defined by $f_2'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t)^2}dt\right)\right]$, where $c = \frac{(8-9\mu)}{9\mu}$.

Extremal function $f_3(z)$ for the third bound is defined by $f_3'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t^2)^2}dt\right)\right]$, where $c = \frac{(9\mu-8)}{16-9\mu}$.

Extremal function $f_4(z)$ for the fourth bound is defined by $f_4'(z) = \frac{1}{z}\left[\left(\int_0^z (1+\frac{19t}{3\sqrt{3}})^{\frac{9}{19}}dt\right)\right]$, where $|t| \le \frac{3\sqrt{3}}{19}$.

Proceeding as in Theorem 3.2 and using elementary calculus, we can easily prove the following theorem.

Theorem 3.3. Let $f(z) \in C_1$. Then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{5}{3} - \frac{9\mu}{4}, & \text{if } \mu \le \frac{2}{9}; \\ \frac{2}{3} + \frac{1}{9\mu}, & \text{if } \frac{2}{9} \le \mu \le \frac{2}{3}; \\ 1 - \frac{\mu}{4} + \frac{(3\mu-2)^2}{12(4-3\mu)}, & \text{if } \frac{2}{3} \le \mu \le \frac{8}{9}; \\ \frac{7}{9} + \frac{(3\mu-2)^2}{12(4-3\mu)}, & \text{if } \frac{8}{9} \le \mu \le \frac{10}{9}; \\ \frac{7}{9} + 2(\mu-1), & \text{if } \frac{10}{9} \le \mu \le \frac{16}{9}; \\ \frac{9\mu}{4} - \frac{5}{3}, & \text{if } \mu \ge \frac{16}{9}. \end{cases}$$

The results are sharp.

Extremal function $f_1(z)$ for the first and the last results is defined by $f_1(z) = \left[\left(\int_0^z \frac{(1+t)}{(1-t)^2}dt\right)\right]$.

Extremal function $f_2(z)$ for the second bound is defined by $f_2(z) = \left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t)}dt\right)\right]$, where $c = \frac{(2-3\mu)}{3\mu}$.

Extremal function $f_3(z)$ for the third and fourth bound is defined by

$$f_3(z) = \left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t)}dt\right)\right],$$

where $c = \frac{(3\mu-2)}{2(4-3\mu)}$.

Extremal function $f_4(z)$ for the fifth bound is defined by

$$f_4(z) = \left[\left(\int_0^z (1+\frac{10\sqrt{2}}{3}t)^{\frac{3}{5}}dt\right)\right],$$

where $|t| \le \frac{3}{10\sqrt{2}}$.

Open problems on Fekete-Szegö inequality for the following classes:

(i) $C_1(A,B) = \left\{f(z) \in A : \frac{zf'(z)}{h(z)} \prec \frac{1+Az}{1+Bz},\ h(z) \in K,\ -1 \le B < A \le 1,\ z \in E\right\}$, (ii)$C^{'}(A,B)=\left\{f(z)\in A:\frac{(zf'(z))'}{g'(z)}\prec\frac{1+Az}{1+Bz},\ g(z)\in S^{*},\ -1\leq B<A\leq 1,\ z\in E\right\}$,

(iii)$C^{''}(A,B)=\left\{f(z)\in A:\frac{(zf'(z))'}{h'(z)}\prec\frac{1+Az}{1+Bz},\ h(z)\in K,\ -1\leq B<A\leq 1,\ z\in E\right\}$.

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On α-generalized regular weakly closed sets in topological spaces

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Abstract In this paper we introduce a new class of sets called α-generalized regular weakly closed sets in topological space and discuss some of the basic properties of α-generalized regular weakly closed sets. This new class of sets that lie between the class of regular weakly closed (briefly rw-closed) sets and the class of generalized pre regular weakly closed (briefly gprw-closed) sets.

Keywords Regular open sets, regular semi-open sets and αgrw-closed sets.
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§1. Introduction

Levine [13] introduced generalized closed (briefly g-closed) sets in 1970. Regular open sets and regular semi-open sets have been introduced and investigated by Stone [22] and Cameron [6] respectively. Benchalli and Wali [4] introduced the concept of regular weakly closed sets in 2007. Sanjay Mishra et.al [20] defined generalized pre regular weakly closed sets in 2012. Many researchers [21,19,23,24,9] during the last decade have introduced and studied the sets like ω-closed sets, mildly generalized closed sets, g^*-closed sets, semi-closed sets and semi-pre closed sets and πg-closed sets.

In this paper, we define and study the properties of α-generalized regular weakly closed sets (αgrw-closed) in topological space which is properly placed between the regular weakly closed sets and generalized pre regular weakly closed sets.

§2. Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is called
i) a preopen set [16] if $A \subseteq int(cl(A))$ and a preclosed set if $cl(int(A)) \subseteq A$,
ii) a semi-open set [12] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$,
iii) an α-open set [18] if $A \subseteq int(cl(int(A)))$ and a α-closed set if $cl(int(cl(A))) \subseteq A$,

iv) a semi-preopen set [2] (β-open [1]) if $A \subseteq cl(int(cl(A)))$ and a semi-preclosed (β-closed [1]) if $int(cl(int(A))) \subseteq A$,
v) regular open set [22] if A = int(cl(A)) and a regular closed set [21] if A = cl(int(A)),
vi) θ-closed set [25] if $A = cl_\theta(A)$, where $cl_\theta(A) = \{x \in X : cl(U) \cap A \neq \phi, U \in \tau$ and $x \in U\}$,
vii) δ-closed set [25] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \in X : int(cl(U)) \cap A \neq \phi, U \in \tau$ and $x \in U\}$,
viii) π-open set [9] if A is a finite union of regular open sets.

The α-closure (resp. semi-closure, semi-preclosure and pre-cosure) of a subset A of X denoted by $\alpha cl(A)$ (resp. $scl(A)$, $spcl(A)$ and $pcl(A)$) is defined to be the intersection of all α-closed sets (resp. semi-closed sets, semi-preclosed sets and pre-closed sets) containing A.

Definition 2.2. A subset A of a topological space (X, τ) is called regular semi-open [6] if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$. The family of all regular semi-open sets of X is denoted by $RSO(X)$.

Definition 2.3. A subset A of a topological space (X, τ) is called
i) a generalized closed set (briefly g-closed) [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
ii) a semi generalized closed set (briefly sg-closed) [5] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open,
iii) a generalized semi closed set (briefly gs-closed) [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
iv) a weakly closed set (briefly ω-closed) [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open,
v) a weakly generalized closed set (briefly wg-closed) [17] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open,
vi) a α-generalized closed set (briefly αg-closed) [14] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
vii) a generalized semi-preclosed set(briefly gsp-closed) [10] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
viii) a generalized preclosed set (briefly gp-closed) [15] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
ix) a regular weakly closed set (briefly rw-closed) [4] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open,
x) a generalized pre regular weakly (briefly gprw-closed) [20] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open,
xi) a mildly generalized closed set (briefly mildly g-closed) [19] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open,
xii) a strongly generalized closed set (briefly g^*-closed) [23] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open,
xiii) a *g-closed set [23] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open,
xiv) a ψ-closed set [24] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open,
xv) a regular weakly generalized closed set (briefly rwg-closed) [17] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular open,
xvi) a θ-generalized closed set (briefly θ-g-closed) [8] if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

xvii) a δ-generalized closed set (briefly δ-g-closed) [7] if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
xviii) a π-generalized closed set (briefly πg-closed) [9] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open.

§3. α-Generalized regular weakly closed sets

Definition 3.1. A subset A of a topological space (X, τ) is called α-generalized regular weakly closed [briefly αgrw-closed] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open in (X, τ). We denote the set of all αgrw-closed sets in (X, τ) by αGRWC(X).

Theorem 3.2. Every ω-closed set is αgrw-closed.

Proof. Let A be ω-closed and $A \subseteq U$ where U is regular semi-open. Since every regular semi-open set is semi-open and $\alpha cl(A) \subseteq cl(A)$, $\alpha cl(A) \subseteq U$. Hence A is αgrw-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.3. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $A = \{c\}$ is αgrw-closed but not ω-closed in (X, τ).

Theorem 3.4. Every rw-closed set is αgrw-closed.

Proof. Let A be rw-closed and $A \subseteq U$ where U is regular semi-open. Then $cl(A) \subseteq U$. Since $\alpha cl(A) \subseteq cl(A)$, $\alpha cl(A) \subseteq U$. Hence A is αgrw-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. In Example 3.3, the set $A = \{c\}$ is αgrw-closed but not rw-closed in (X, τ).

Theorem 3.6. Every α-closed set is αgrw-closed.

Proof. Let A be an α-closed set and $A \subseteq U$ where U is regular semi open. Then $\alpha cl(A) = A \subseteq U$. Hence A is αgrw-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $A = \{a\}$ is αgrw-closed but not α-closed in (X, τ).

Theorem 3.8. Every αgrw-closed set is gprw-closed.

Proof. Let A be an αgrw-closed set and $A \subseteq U$ where U is regular semi open. Since $pcl(A) \subseteq \alpha cl(A)$, $pcl(A) \subseteq U$. Hence A is gprw-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $A = \{b\}$ is gprw-closed but not αgrw-closed in (X, τ).

Theorem 3.10. Every closed set is αgrw-closed.

Proof. Every closed set is rw-closed [4] and by Theorem 3.4, every rw-closed set is αgrw-closed. Hence the proof.

Theorem 3.11 Every regular closed set is αgrw-closed.

Proof. Every regular closed set is rw-closed [4] and by Theorem 3.4, every rw-closed set is αgrw-closed. Hence the proof.

Theorem 3.12. Every θ-closed set is αgrw-closed.

Proof. Every θ-closed set is rw-closed [4] and by Theorem 3.4, every rw-closed set is αgrw-closed. Hence the proof.

Theorem 3.13. Every δ-closed set is αgrw-closed.

Proof. Every δ-closed set is rw-closed [4] and by Theorem 3.4, every rw-closed set is αgrw-closed. Hence the proof.

Theorem 3.14. Every π-closed set is αgrw-closed.

Proof. Every π-closed set is rw-closed [4] and by Theorem 3.4, every rw-closed set is αgrw-closed. Hence the proof.

Remark 3.15. The following example shows that αgrw-closed sets are independent of g-closed sets, wg-closed sets, αg-closed sets, gs-closed sets, sg-closed sets, gsp-closed sets and gp-closed sets.

Example 3.16. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, X\}$. Then

1. Closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, X.
2. αgrw-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{c, d\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
3. g-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{b, c\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
4. wg-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
5. αg-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
6. gs-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
7. sg-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
8. gsp-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
9. gp-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.

Remark 3.17. The following example shows that αgrw-closed sets are independent of g^*-closed sets, mildly g-closed sets, semi closed sets, πg-closed sets, θ-generalized closed sets, δ-generalized closed sets, *g-closed sets, ψ-closed sets and rwg-closed sets.

Example 3.18. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then

1. closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, X.
2. αgrw-closed sets in (X, τ) are $\emptyset$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
3. g^*-closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, X.
4. Mildly g-closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, X.
5. Semi closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, X.
6. πg-closed sets in (X, τ) are $\emptyset$, $\{c\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, X.
7. θg-closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{c, d\}$, $\{b, d\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, X.
8. δg-closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{c, d\}$, $\{b, d\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, X.

9. $^*g$-closed sets in (X, τ) are $\emptyset$, $\{d\}$, $\{c, d\}$, $\{b, d\}$, $\{a, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, X.
10. ψ-closed sets in (X, τ) are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, X.
11. rwg-closed sets in (X, τ) are $\emptyset$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{a, b, c\}, X$.

Theorem 3.19. If A and B are αgrw-closed then $A \cup B$ is αgrw-closed.

Proof. Let A and B be any two αgrw-closed sets. Let $A \cup B \subseteq U$ and U is regular semi open. We have $\alpha cl(A) \subseteq U$ and $\alpha cl(B) \subseteq U$. Thus $\alpha cl(A \cup B) = \alpha cl(A) \cup \alpha cl(B) \subseteq U$. Hence $A \cup B$ is αgrw-closed.

Remark 3.20. The intersection of two αgrw-closed sets of a topological space (X, τ) is generally not αgrw-closed.

Example 3.21. In Example 3.18, Then $\{a, b\}$ and $\{b, c, d\}$ are αgrw-closed sets. But $\{a, b\} \cap \{b, c, d\} = \{b\}$ is not αgrw-closed in (X, τ).

Theorem 3.22. If a subset A of X is αgrw-closed, then $\alpha cl(A) - A$ does not contain any non empty regular semi open sets.

Proof. Suppose that A is αgrw-closed set in X. Let U be a regular semiopen set such that $U \subseteq \alpha cl(A) - A$ and $U \neq \emptyset$. Now $U \subseteq X - A$ which implies $A \subseteq X - U$. Since U is regular semi open, $X - U$ is also regular semiopen in X [11]. Since A is an αgrw-closed set in X, by definition we have $\alpha cl(A) \subseteq X - U$. So $U \subseteq X - \alpha cl(A)$. Also $U \subseteq \alpha cl(A)$. Therefore $U \subseteq \alpha cl(A) \cap (X - \alpha cl(A)) = \emptyset$, which is a contradiction. Hence $\alpha cl(A) - A$ does not contain any non empty regular semiopen set in X.

The converse of the above theorem needs not be true as seen from the following example.

Example 3.23. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $A = \{b\}$. Then $\alpha cl(A) - A = \{a, b\} - \{b\} = \{a\}$ does not contain nonempty regular semiopen set, but A is not an αgrw-closed set in (X, τ).

Theorem 3.24. For an element $x \in X$, the set $X - \{x\}$ is αgrw-closed or regular semi open.

Proof. Suppose $X - \{x\}$ is not regular semi open. Then X is only regular semi open set containing $X - \{x\}$ and also $\alpha cl(X - \{x\}) \subseteq X$. Hence $X - \{x\}$ is αgrw-closed set in X.

Theorem 3.25. If A is regular open and αgrw-closed, then A is α-closed.

Proof. Suppose A is regular open and αgrw-closed. As every regular open set is regular semi open and $A \subseteq A$, we have $\alpha cl(A) \subseteq A$. Also $A \subseteq \alpha cl(A)$. Therefore $\alpha cl(A) = A$. Hence A is α-closed.

Theorem 3.26. If A is an αgrw-closed subset of X such that $A \subseteq B \subseteq \alpha cl(A)$, then B is an αgrw-closed set in X.

Proof. Let A be an αgrw-closed set of X, such that $A \subseteq B \subseteq \alpha cl(A)$. Let $B \subseteq U$ and U be regular semiopen set. Then $A \subseteq U$. Since A is αgrw-closed, we have $\alpha cl(A) \subseteq U$. Now $\alpha cl(B) \subseteq \alpha cl(\alpha cl(A)) = \alpha cl(A) \subseteq U$. Therefore B is αgrw-closed set.

The converse of the above theorem need not be true as seen from the following example.

Example 3.27. In Example 3.18. Let $A = \{c, d\}$ and $B = \{b, c, d\}$ are αgrw-closed sets in (X, τ). Thus any αgrw-closed set need not lie between a αgrw-closed and its α-closure.

Theorem 3.28. Let A be an αgrw-closed in (X, τ). Then A is α-closed if and only if

$\alpha cl(A) - A$ is regular semi open.

Proof. Suppose A is α-closed. Then $\alpha cl(A) = A$ and so $\alpha cl(A) - A = \emptyset$, which is regular semi open in X. Conversely, suppose $\alpha cl(A) - A$ is regular semi open in X. Since A is αgrw-closed, by Theorem 3.22, $\alpha cl(A) - A$ does not contain any non empty regular semi open in X. Then $\alpha cl(A) - A = \emptyset$. Therefore $\alpha cl(A) = A$. Hence A is α-closed.

Theorem 3.29. If A is both open and αg-closed then A is αgrw-closed.

Proof. Let A be an open and αg-closed. Let $A \subseteq U$ and U be regular semiopen. Now $A \subseteq A$ and by hypothesis $\alpha cl(A) - A$. Therefore $\alpha cl(A) \subseteq U$. Hence A is αgrw-closed.

Remark 3.30. If A is both open and αgrw-closed, then A need not be αg-closed as seen from the following example.

Example 3.31. In Example 3.18, the subsets $\{a, b\}$ and $\{a, b, c\}$ are αgrw-closed and open but not αg-closed.

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Some properties and generalizations of generalized m-power matrices[1]

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Abstract In this paper, some properties of generalized m-power matrices are firstly obtained. Also, relative properties of such matrices are generalized to some more general cases, including the generalized m-power transformations.

Keywords Generalized m-power matrix, property, generalization.

§1. Introduction

The m-idempotent matrices and m-unit-ponent matrices are two typical matrices and have many interesting properties (for example, see [1]-[6]).

A matrix $A \in \mathbb{C}^{n\times n}$ is called an m-idempotent (m-unit-ponent) matrix if there exists positive integer m such that $A^m = A(A^m = I)$.

In [1], we define generalized m-power matrices and generalized m-power transformations, and give two equivalent characterizations of generalized m-power matrices which extends the corresponding results about m-idempotent matrices and m-unit-ponent matrices. Also, we generalize the relative results of generalized m-power matrices to the ones of generalized m-power transformations.

Recall by [1] that a matrix $A \in \mathbb{C}^{n\times n}$ is called a generalized m-power matrix if it satisfies that $\prod_{i=1}^{m}(A + \lambda_i I) = O$, where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are the pairwise different complex numbers.

In this paper, we will also study some properties of generalized m-power matrices. Further, we will generalize relative properties of such matrices to some more general cases, including the generalized m-power transformations.

For notations and terminologies occurred but not mentioned in this paper, readers are referred to [1], [2].

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§2. Properties of generalized m-power matrices

In this section, we will discuss some properties of generalized m-power matrices. First, we introduce a lemma as follow:

Lemma 2.1.[3] Let $f_1(x), f_2(x), \cdots, f_m(x) \in \mathbb{C}[x]$ be pairwisely co-prime and $A \in \mathbb{C}^{n\times n}$. Then

$$\sum_{i=1}^{m} r(f_i(A)) = (m-1)n + r(\prod_{i=1}^{m}(f_i(A))).$$

Theorem 2.1. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n\times n}$. For any positive integer l, k, m, if

$$\prod_{i=1}^{m}(A + \lambda_i I) = O,$$

then

$$r((A + \lambda_1 I)^l) + r(\prod_{i=2}^{m}(A + \lambda_i I)^k) = n.$$

Proof. Take $f_i(x) = x + \lambda_i (i = 1, 2, \cdots, m)$, where $\lambda_i \neq \lambda_j$ when $i \neq j$. Clearly, if $i \neq j$, then

$$(f_i(x), f_j(x)) = (x + \lambda_i, x + \lambda_j) = 1.$$

Also, since

$$(f_1(x), \prod_{i=2}^{m} f_i(x)) = (x + \lambda_1, \prod_{i=2}^{m}(x + \lambda_i)) = 1,$$

we have

$$(f_1^l(x), (\prod_{i=2}^{m} f_i(x))^k) = 1.$$

By Lemma 2.1, we can immediately get

$$r((A + \lambda_1 I)^l) + r(\prod_{i=2}^{m}(A + \lambda_i I)^k) = n.$$

By Theorem 2.1, we can obtain the following corollaries. Consequently, the corresponding results in [4] and [5] are generalized.

Corollary 2.1. Let $A \in \mathbb{C}^{n\times n}$. For any positive integers l, k, m, if $A^m = I$, then

$$r((A - I)^l) + r((A^{m-1} + A^{m-2} + \cdots + A + I)^k) = n.$$

Corollary 2.2. Let $A \in \mathbb{C}^{n\times n}$. For any positive integer l, k, m, if $A^{m+l} = A$, then

$$r(A^l) + r((A^m - I)^k) = n.$$

Corollary 2.3. Let $A \in \mathbb{C}^{n\times n}$ and $A^2 = I$. For any positive integer l, k, then

$$r((A - I)^l) + r((A + I)^k) = n.$$

Corollary 2.4. Let $A \in \mathbb{C}^{n\times n}$ and $A^2 = A$. For any positive integer l, k, then

$$r((A^l)) + r((A - I)^k) = n.$$

§3. Generalizations of generalized m-power matrices

By the definition of generalized m-power matrix in [1], we know that a generalized m-power matrix A is a one which satisfies that $\prod_{i=1}^{m}(A+\lambda_i I)=0$, where $\lambda_1,\lambda_2,\cdots,\lambda_m$ are the pairwise different complex numbers. Note that I is invertible and commutative with A. Thus, naturally, we can consider such question: when we substitute I for B where B is invertible and commutative with A, whether can we get some similar results with the ones of generalized m-power matrices.

In this section, we will consider such question, and study some generalizations of generalized m-power matrices.

Lemma 3.1.[6] Let $A,B\in\mathbb{C}^{n\times n}$ satisfying that B is invertible and $AB=BA$. Assume that $\lambda_1,\lambda_2,\cdots,\lambda_m\in\mathbb{C}$ are the pairwise different complex numbers, then

$$r(\prod_{i=1}^{m}(A+\lambda_i B))=\sum_{i=1}^{m}r(A+\lambda_i B)-(m-1)n.$$

Theorem 3.1. Let $A,B\in\mathbb{C}^{n\times n}$ satisfying that B is invertible and $AB=BA$. Assume that $\lambda_1,\lambda_2,\cdots,\lambda_m$ are the pairwise different complex numbers, then $\prod\limits_{i=1}^{m}(A+\lambda_i B)=0$ if and only if $\sum\limits_{i=1}^{m}r(A+\lambda_i B)=(m-1)n$.

Proof. $\Rightarrow$ Assume that $\prod\limits_{i=1}^{m}(A+\lambda_i B)=0$, by Lemma 3.1 we can get $\sum\limits_{i=1}^{m}r(A+\lambda_i B)=(m-1)n$.

$\Leftarrow$ Assume that $C=B^{-1}A$. Take $f_i(x)=x+\lambda_i(i=1,2,\cdots,m)$, where $\lambda_i\neq\lambda_j$ when $i\neq j$. Clearly, $(f_i(x),f_j(x))=1$ if $i\neq j$. By Lemma 3.1,

$$\sum_{i=1}^{m}r(C+\lambda_i I)=\sum_{i=1}^{m}r(B^{-1}A+\lambda_i I)=(m-1)n+r(\prod_{i=1}^{m}(B^{-1}A+\lambda_i I)).$$

Also since

$$\sum_{i=1}^{m}r(A+\lambda_i B)=\sum_{i=1}^{m}r(B(B^{-1}A+\lambda_i I))=\sum_{i=1}^{m}r(B^{-1}A+\lambda_i I)=(m-1)n,$$

and $AB=BA$, we can get

$$r(\prod_{i=1}^{m}(B^{-1}A+\lambda_i I))=0.$$

Thus,

$$\prod_{i=1}^{m}(A+\lambda_i B)=B^m\prod_{i=1}^{m}(B^{-1}A+\lambda_i I)=0.$$

By the above theorem, we immediately get the following corollary.

Corollary 3.1. Let $A,B\in\mathbb{C}^{n\times n}$ satisfying that B is invertible and $AB=BA$. For any positive m, $A^m=B^m$ if and only if $\sum\limits_{i=1}^{m}r(A-\varepsilon_i B)=(m-1)n$, where $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m$ are the m power unit roots.

Further, we also have the following result.

Lemma 3.2.[1] Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n\times n}$. Then $\prod_{i=1}^{m}(A+\lambda_i I)=0$ if and only if $\sum_{i=1}^{m} r(A+\lambda_i I)=(m-1)n$.

Theorem 3.2. Let $A \in \mathbb{C}^{n\times n}$. Assume that $a_1, a_2, \cdots, a_m \in \mathbb{C}^*$ and $b_1, b_2, \cdots, b_m \in \mathbb{C}$ satisfying $a_i b_j \neq a_j b_i (i,j=1,2,\cdots,m)$. Then $\prod\limits_{i=1}^{m}(a_iA+b_iI)=0$ if and only if $\sum\limits_{i=1}^{m} r(a_iA+b_iI)=(m-1)n$.

Proof. $\Rightarrow$ For any $a_1, a_2, \cdots, a_m \in \mathbb{C}^*$ and $b_1, b_2, \cdots, b_m \in \mathbb{C}$, assume that $\prod\limits_{i=1}^{m}(a_iA+b_iI)=0$. Then we have

$$\prod_{i=1}^{m}[a_i(A+\frac{b_i}{a_i}I)]=0.$$

By Lemma 3.2, we have

$$\sum_{i=1}^{m} r(A+\frac{b_i}{a_i}I)=(m-1)n.$$

Thus,

$$\sum_{i=1}^{m} r(a_iA+b_iI)=\sum_{i=1}^{m} r(A+\frac{b_i}{a_i}I)=(m-1)n.$$

$\Leftarrow$ Take $f_i(x)=a_ix+b_i, g_i(x)=x+\frac{b_i}{a_i}(i=1,2,\cdots,m)$, where $\frac{b_i}{a_i}\neq\frac{b_j}{a_j}$ while $i\neq j$. Then we have

$$(g_i(x),g_j(x))=(x+\frac{b_i}{a_i},x+\frac{b_j}{a_j})=1.$$

By Lemma 3.1, we can get

$$\sum_{i=1}^{m} r(a_iA+b_iI)=\sum_{i=1}^{m} r(A+\frac{b_i}{a_i}I)=(m-1)n+r(\prod_{i=1}^{m}(A+\frac{b_i}{a_i}I)).$$

And by

$$\sum_{i=1}^{m} r(a_iA+b_iI)=(m-1)n,$$

we have

$$\prod_{i=1}^{m}(A+\frac{b_i}{a_i})I=0.$$

Thus,

$$\prod_{i=1}^{m}(a_iA+b_iI)=0.$$

§4. Properties and generalizations of generalized m-power transformations

In this section, analogous with the discussions of the generalized m-power matrices, we will study some properties and generalizations of generalized m-power transformations.

Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Recall by [1] that σ is called a generalized m-power transformation if it satisfies that

$$\prod_{i=1}^{m}(\sigma+\lambda_i\epsilon)=\theta,$$

for pairwise different complex numbers $\lambda_1,\lambda_2,\cdots,\lambda_m$, where ϵ is the identical transformation and θ is the null transformation. Especially, σ is called an m-idempotent (m-unit-ponent, respectively) transformation if it satisfies that $\sigma^m=\sigma(\sigma^m=\epsilon$, respectively.)

Sine the following results and their proofs of generalized m-power transformations are similar with the ones of generalized m-power matrices in Section 2 and 3, we omit them here.

Theorem 4.1. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, ι the identical transformation and θ the null transformation. Assume that $k,l,m\in\mathbb{Z}^+$ and $\lambda_1,\lambda_2,\cdots,\lambda_m$ are the pairwise different complex numbers. If

$$\prod_{i=1}^{m}(\sigma+k_i\iota)=\theta,$$

then

$$\dim Im(\sigma+k_1)^l+\dim Im(\prod_{i=2}^{m}(\sigma+k_i)^k)=n.$$

Corollary 4.1. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, ι the identical transformation and θ the null transformation. For any $k,l,m\in\mathbb{Z}^+$, if $\sigma^m=\iota$, then

$$\dim Im(\sigma-\iota)^l+\dim Im(\sigma^{m-1}+\sigma^{m-2}+\cdots+\sigma+\iota)^k=n.$$

Corollary 4.2. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, ι the identical transformation and θ the null transformation. For any $k,l,m\in\mathbb{Z}^+$, if $\sigma^{m+l}=\sigma$, then $\dim Im(\sigma^l)+\dim Im(\sigma-\iota)^k=n$.

Corollary 4.3. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, ι the identical transformation and θ the null transformation. For any $k,l\in\mathbb{Z}^+$, if $\sigma^2=\iota$, then $\dim Im(\sigma-\iota)^l+\dim Im(\sigma+\iota)^k=n$.

Corollary 4.4. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, ι the transformation identity and θ the null transformation. For any $k,l\in\mathbb{Z}^+$, if $\sigma^2=\sigma$, then $\dim Im(\sigma^l)+\dim Im(\sigma-\iota)^k=n$.

Theorem 4.2. Let V be a n dimensional vector space over a field F, σ a linear transformation on V, τ a invertible linear transformation on V, ι the transformation identity and θ the null transformation. Assume that $\lambda_1,\lambda_2,\cdots,\lambda_m$ are the pairwise different complex numbers and $\sigma\tau=\tau\sigma$. Then $\prod\limits_{i=1}^{m}(\sigma+k_i\tau)=\theta$ if and only if

$$\sum_{i=1}^{m}\dim Im(\sigma+k_i\tau)=(m-1)n.$$

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Filter on generalized topological spaces

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Abstract The aim of this paper is to introduce filter generalized topological spaces and to investigate the relationships between generalized topological spaces and filter generalized topological spaces. For establishment of their relationships, we define some closed sets in these spaces. Basic properties and characterizations related to these sets are also discussed.

Keywords GTS, FGTS, g_μ-closed set, μ^Ω-closed set, μ-F_f-closed set.
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§1. Introduction and preliminaries

Ideals in topological spaces were introduced by Kuratowski [11] and Vaidya-nathswamy [22]. Jankovic and Hamlett [10] defined *-closed sets, then *-perfect and *-dense in itself were obtained by Hayashi [9]. Levine [12] introduced the concept of generalized closed sets in topological spaces and then Noiri and Popa [16] had studied it in detail. The topological ideals and the relationship among I_g-closed sets, g-closed sets and *-closed sets were introduced and studied in [8]. Navaneethakrishnan and Joseph [15] investigated g-closed sets in ideal topological spaces. Ozbakir and Yildirim defined m*-perfect, m*-dense in itself, m*-closed and m-I_g-closed sets in ideal minimal spaces and studied the same in [18].

In this paper we introduce the filter [21] generalized topological spaces. However the pioneer of the notation of generalized topological spaces is Csaszar [1,2,4], then Csaszar [3,5,6,7], Min [14], Noiri and Roy [17,19] and Sarsak [20] have studied it further. The definitions of μ^Ω-closed, μ^Ω-dense in itself, μ^Ω-perfect and μ-F_f-closed are given. We discuss some properties and characterizations of these sets, and determine the relationships between these sets with μ-T_1 spaces.

Definition 1.1.[21] A subcollection $\mathcal{F}$ (not containing the empty set) of $\exp(X)$ is called a filter on X if $\mathcal{F}$ satisfies the following conditions:

1. $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$,
2. $A,\ B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

A concept from monotone function has been introduced by Csaszar [1,2,4] on 1997. Using this notion topology has been reconstructed. The concept is, a map $\gamma : \exp\,(X) \rightarrow \exp\,(X)$

possessing the property monotony (i.e. such that $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$). We denote by $\Gamma(X)$ the collections of all mapping having this property. One of the consequence of the above notion is generalized topological space [4,17,19], its formal definition is:

Definition 1.2. Let X be a non-empty set, and $\mu \subseteq \exp(X)$. μ is called a generalized topology on X if $\phi \in \mu$ and the union of elements of μ belongs to μ.

Let X be a non-empty set and μ be a generalized topology (GT) on X, then (X, μ) is called generalized topological space (GTS).[17,19,14,13,20]

The member of μ is called μ-open set and the complement of μ-open set is called μ-closed set. A GT μ is said to be a quasi-topology on X if $M, N \in \mu$ implies $M \cap N \in \mu$. Again c_μ and i_μ are the notation of μ-closure and μ-interior [17,19,20,13] respectively. These two operators obeyed the following relations:

Lemma 1.1.[17] Let (X, μ) be a GTS and $A \subset X$, then

1. $c_\mu(A) = X - i_\mu(X - A)$,
2. c_μ and i_μ both are idempotent operators.

Definition 1.3.[13] Let (X, μ) be a GTS. A subset A of X is said to be g_μ-closed set if $c_\mu(A) \subseteq M$ whenever $A \subset M$ and $M \in \mu$.

Definition 1.4.[19] Let (X, μ) be a generalized topological space. Then the generalized kernel of $A \subseteq X$ is denoted by g-ker(A) and defined as $g\text{-ker}(A) = \cap\{G \in \mu : A \subseteq G\}$.

Lemma 1.2.[19] Let (X, μ) be a generalized topological space and $A \subseteq X$. Then g-ker(A) $= \{x \in X : c_\mu(\{x\}) \cap A \neq \phi\}$.

If $\mathcal{F}$ is a filter on X, then $(X, \mu, \mathcal{F})$ is called a filter generalized topological space (FGTS).

§2. Local function on FGTS

Definition 2.1. Let $(X, \mu, \mathcal{F})$ be a FGTS. A mapping $\Omega : \exp(X) \to \exp(X)$ is defined as follows: $\Omega(A) \subseteq X$ by $x \in \Omega(A)$ if and only if $x \in M \in \mu$ imply $A \cap U \in \mathcal{F}$. If $M_\mu = \cup\{M : M \in \mu\}$ and $x \notin M_\mu$ then by definition $x \in \Omega(A)$.

The mapping is called the local function associated with the filter $\mathcal{F}$ and generalized topology μ.

Theorem 2.1. Let μ be a GTS on a set X, $\mathcal{F}$, $\mathcal{J}$ filters on X and A, B be subsets of X. The following properties hold:

1. If $A \subseteq B$, then $\Omega(A) \subseteq \Omega(B)$,
2. If $\mathcal{J} \subseteq \mathcal{F}$, then $\Omega(A)(\mathcal{J}) \subseteq \Omega(A)(\mathcal{F})$,
3. $\Omega(A) = c_\mu(\Omega(A)) \subseteq c_\mu(A)$,
4. $\Omega(A) \cup \Omega(B) \subseteq \Omega(A \cup B)$,
5. $\Omega(\Omega(A)) \subseteq \Omega(A)$,
6. $\Omega(A)$ is a μ-closed set.

Proof. (1) Let $A \subseteq B$. Assume that $x \notin \Omega(B)$. Then we have $U \cap B \notin \mathcal{F}$ for some $U \in \mu(x)$. Since $U \cap A \subseteq U \cap B$ and $U \cap B \notin \mathcal{F}$, we obtain $U \cap A \notin \mathcal{F}$ from the definition of filters. Thus, we have $x \notin \Omega(A)$. Hence we have $\Omega(A) \subseteq \Omega(B)$.
(2) Let $\mathcal{J} \subseteq \mathcal{F}$ and $x \in \Omega(A)(\mathcal{J})$. Then we have $U \cap A \in \mathcal{J}$ for every $U \in \mu(x)$. By hypothesis, we obtain $U \cap A \in \mathcal{F}$. So $x \in \Omega(A)(\mathcal{F})$.
(3) We have $\Omega(A) \subseteq c_\mu(\Omega(A))$ in general. Let $x \in c_\mu(\Omega(A))$. Then $\Omega(A) \cap U \neq \phi$, for every $U \in \mu(x)$. Therefore, there exists some $y \in \Omega(A) \cap U$ and $U \in \mu(y)$. Since $y \in \Omega(A)$, $A \cap U \in \mathcal{F}$ and hence $x \in \Omega(A)$. Hence we have $c_\mu(\Omega(A)) \subseteq \Omega(A)$ and $c_\mu(\Omega(A)) = \Omega(A)$. Again, let $x \in c_\mu(\Omega(A)) = \Omega(A)$, then $A \cap U \in \mathcal{F}$ for every $U \in \mu(x)$. This implies $A \cap U \neq \phi$, for every $U \in \mu(x)$. Therefore, $x \in c_\mu(A)$. This proves $\Omega(A) = c_\mu(\Omega(A)) \subseteq c_\mu(A)$.
(4) This follows from (1).
(5) Let $x \in \Omega(\Omega(A))$. Then, for every $U \in \mu(x)$, $U \cap \Omega(A) \in \mathcal{F}$ and hence $U \cap \Omega(A) \neq \phi$. Let $y \in U \cap \Omega(A)$. Then $U \in \mu(y)$ and $y \in \Omega(A)$. Hence we have $U \cap A \in \mathcal{F}$ and $x \in \Omega(A)$. This shows that $\Omega(\Omega(A)) \subseteq \Omega(A)$.
(6) This follows from (3).

Proposition 2.1. Let $(X, \mu, \mathcal{F})$ be a FGTS. If $M \in \mu$, $M \cap A \notin \mathcal{F}$ implies $M \cap \Omega(A) = \phi$. Hence $\Omega(A) = X - M_\mu$ if $A \notin \mathcal{F}$.

Proof. Suppose $x \in M \cap \Omega(A)$, $M \in \mu$ and $x \in \Omega(A)$ would imply $M \cap A \in \mathcal{F}$. Now $A \notin \mathcal{F}$ implies $M \cap A \notin \mathcal{F}$ for every $M \in \mu$ and $x \notin \Omega(A)$ when $x \in M_\mu$, thus $\Omega(A) \subseteq X - M_\mu$ on the other hand we know $X - M_\mu \subseteq \Omega(A)$.

Proposition 2.2. Let $(X, \mu, \mathcal{F})$ be a FGTS. $X = \Omega(X)$ if and only if $\mu - \{\phi\} \subseteq \mathcal{F}$.

Proof. Assume $X = \Omega(X)$. Then $M \in \mu$, $M \neq \phi$ would imply the existence of $x \in M$ and the $x \in \Omega(X)$ would furnish $M \cap X = M \in \mathcal{F}$ so that $\mu - \{\phi\} \subseteq \mathcal{F}$. Conversely, $\mu - \{\phi\} \subseteq \mathcal{F}$ implies $M = M \cap X \in \mathcal{F}$ whenever $x \in M \in \mu$ so that $x \in \Omega(X)$ for $x \in X$. Hence $X = \Omega(X)$.

Proposition 2.3. Let $(X, \mu, \mathcal{F})$ be a FGTS. $M \in \mu$ implies $M \subseteq \Omega(M)$ if and only if $M, N \in \mu$, $M \cap N \notin \mathcal{F}$ implies $M \cap N = \phi$.

Proof. Assume $M \subseteq \Omega(M)$ whenever $M \in \mu$. If $x \in M \cap N$ and $M, N \in \mu$ then $x \in \Omega(M)$, hence $M \cap N \in \mathcal{F}$, consequently $M, N \in \mu$ and $M \cap N \notin \mathcal{F}$ can only hold when $M \cap N = \phi$. Conversely, if the latter statement is true and $x \in M \in \mu$ then $x \in N \in \mu$ implies $M \cap N \neq \phi$, hence $M \cap N \in \mathcal{F}$, so that $x \in \Omega(M)$ therefore, $M \subseteq \Omega(M)$ whenever $M \in \mu$.

Proposition 2.4. Let $(X, \mu, \mathcal{F})$ be a FGTS. If $A \subseteq X$ implies that $\Omega(A \cup \Omega(A)) = \Omega(A)$.

Proof. Let $x \notin \Omega(A)$ implies the existence of $M \in \mu$ such that $x \in M$ and $M \cap A \notin \mathcal{F}$. By Proposition 2.3 $M \cap \Omega(A) = \phi$. Hence $M \cap (A \cup \Omega(A)) = M \cap A \notin \mathcal{F}$. Therefore, $x \notin \Omega(A \cup \Omega(A))$.

Definition 2.2. Let (X, μ) be a GTS with a filter $\mathcal{F}$ on X.

The set operator c^Ω is called a generalized Ω-closure and is defined as $c^\Omega(A) = A \cup \Omega(A)$, for $A \subset X$. We will denote by $\mu^\Omega(\mu; \mathcal{F})$ the generalized structure, generated by c^Ω, that is, $\mu^\Omega(\mu; \mathcal{F}) = \{U \subset X : c^\Omega(X - U) = (X - U)\}$. $\mu^\Omega(\mu; \mathcal{F})$ is called $\Omega\mu$-generalized structure with respect to μ and $\mathcal{F}$ (in short $\Omega\mu$-generalized structure) which is finear than μ.

The element of $\mu^\Omega(\mu; \mathcal{F})$ are called μ^Ω-open and the complement of μ^Ω-open is called μ^Ω-closed.

Theorem 2.2. The set operator c^Ω satisfies the following conditions:

1. $A \subseteq c^\Omega(A)$,

2. $c^{\Omega}(\phi) = \phi$ and $c^{\Omega}(X) = X$,

3. If $A \subseteq B$, then $c^{\Omega}(A) \subseteq c^{\Omega}(B)$,

4. $c^{\Omega}(A) \cup c^{\Omega}(B) \subseteq c^{\Omega}(A \cup B)$,

5. $c^{\Omega}(A \cap B) \subseteq c^{\Omega}(A) \cap c^{\Omega}(B)$.

The proofs are clear from Theorem 2.1 and the definition of c^{Ω}.

Proposition 2.5. Let $(X, \mu, \mathcal{F})$ be a FGTS. Then F is μ^{Ω}-closed if and only if $\Omega(F) \subseteq F$.

Proposition 2.6. Let $(X, \mu, \mathcal{F})$ be a FGTS. Then the following statements are equivalent:

1. $A \subseteq \Omega(A)$,

2. $\Omega(A) = c^{\Omega}(A)$,

3. $c_{\mu}(A) \subseteq \Omega(A)$,

4. $\Omega(A) = c_{\mu}(A)$.

Proof. (1) $\Leftrightarrow$ (2) Since $A \cup \Omega(A) = c^{\Omega}(A)$.
(2) $\Rightarrow$ (3) Given that $\Omega(A) = c^{\Omega}(A) = A \cup \Omega(A)$. That is, $A \subset \Omega(A)$, implies that $c_{\mu}(A) \subset c_{\mu}(\Omega(A)) = \Omega(A)$, by Theorem 2.1.
(3) $\Rightarrow$ (4) Since $\Omega(A) \subseteq c_{\mu}(A)$.
(4) $\Rightarrow$ (1) Since $A \subset c_{\mu}(A)$.

Theorem 2.3. Let $(X, \mu, \mathcal{F})$ be a FGTS. The set $\{M - F : M \in \mu, F \notin \mathcal{F}\}$ constitutes a base $\mathcal{B}$ for $\mu^{\Omega}(\mu; \mathcal{F})$.

Proof. Let $M \in \mu$ and $F \notin \mathcal{F}$ implies $M - F \in \mu^{\Omega}(\mu; \mathcal{F})$, since $H = X - (M - F) = X - (M \cap (X - F)) = (X - M) \cup F$ is μ^{Ω}-closed by $x \notin H$ if and only if $x \in M - F$ hence $x \in M$ and $M \cap H = M \cap ((X - M) \cup F) = M \cap F \notin \mathcal{F}$ so that $x \in \Omega(H)$, thus $\Omega(H) \subseteq H$. Hence $\mathcal{B} \subseteq \mu^{\Omega}(\mu; \mathcal{F})$. If $A \in \mu^{\Omega}(\mu; \mathcal{F})$, then $C = X - A$ is μ^{Ω}-closed, hence $\Omega(C) \subseteq C$. Thus $x \in A$ implies $x \notin C$ and $x \notin \Omega(C)$ so there exists $M \in \mu$ such that $x \in M$ and $F = M \cap C \notin \mathcal{F}$, therefore, $x \in M - F \subseteq X - C = A$. Hence A is the union of sets in $\mathcal{B}$.

Theorem 2.4. Let $(X, \mu, \mathcal{F})$ be a FGTS. Then $\mu^{\Omega}(\mu; \mathcal{F}) \subset \mu^{\Omega}(\mu^{\Omega}(\mu; \mathcal{F}))$.

Proof. It is clear that $\mu^{\Omega}(\mu; \mathcal{F}) \subset \mu^{\Omega}(\mu^{\Omega}(\mu; \mathcal{F}))$.

§3. Generalized closed sets on FGTS

Definition 3.1. A subset A of a FGTS $(X, \mu, \mathcal{F})$ is called μ-F-generalized closed (briefly, μ-F_f-closed) if $\Omega(A) \subset U$ whenever U is μ-open and $A \subset U$. A subset A of a FGTS $(X, \mu, \mathcal{F})$ is called μ-F-generalized open(briefly, μ-F_f-open) if $X - A$ is μ-F_f-closed.

Theorem 3.1. If $(X, \mu, \mathcal{F})$ is any FGTS $(X, \mu, \mathcal{F})$, then the followings are equivalent:

1. If A is μ-F_f-closed,

2. $c^{\Omega}(A) \subset U$ whenever $A \subset U$ and U is μ-open in X,

3. $c^{\Omega}(A) \subset g\text{-ker}(A)$,

4. $c^{\Omega}(A) - A$ contain no nonempty μ-closed set,

5. $\Omega(A) - A$ contains no nonempty μ-closed set.

Proof. (1)$\Rightarrow$(2). If A is μ-F_f-closed, then $\Omega(A) \subset U$ whenever $A \subset U$ and U is μ-open in X and so $c^{\Omega}(A) = A \cap \Omega(A) \subset U$ whenever $A \subset U$ and U is μ-open in X.
(2)$\Rightarrow$(3). Suppose $x \in c^{\Omega}(A)$ and $x \notin g$-ker(A). Then $c_\mu(\{x\}) \cap A = \phi$ (from Lemma 1.2), implies that $A \subset X - (c_\mu(\{x\}))$. By (2), a contradiction, since $x \in c^{\Omega}(A)$.
(3)$\Rightarrow$(4). Suppose $F \subset (c^{\Omega}(A)) - A$, F is μ-closed and $x \in F$. Since $F \subset (c^{\Omega}(A)) - A$, $F \cap A = \phi$. We have $c_\mu(\{x\}) \cap A = \phi$ because F is μ-closed and $x \in F$. It is a contradiction.
(4)$\Rightarrow$(5). This is obvious from the definition of $c^{\Omega}(A)$.
(5)$\Rightarrow$(1). Let U be a μ-open subset containing A. Now $\Omega(A) \cap (X - U) \subset \Omega(A) - A$. Since $\Omega(A)$ is a μ-closed set and intersection of two μ-closed sets is a μ-closed set, then $\Omega(A) \cap (X - U)$ is a μ-closed set contained in $\Omega(A) - A$. By assumption, $\Omega(A) \cap (X - U) = \phi$. Hence we have $\Omega(A) \subset U$.

From Theorem 3.1(3), it follows that every μ-closed is μ-F_f-closed. Since $\Omega(F) = \phi$ for $F \notin \mathcal{F}$, F is μ-F_f-closed. Since $\Omega(\Omega(A)) \subset \Omega(A)$, from definition, it follows that $\Omega(A)$ is always μ-F_f-closed for every μ-F_f-closed subset A of X.

Theorem 3.2. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. If A is μ-F_f-closed and μ-open then A is μ^{Ω}-closed set.

Proof. It is obvious from definition.

Definition 3.2.[20] A space (X, μ) is called μ-T_1 if any pair of distinct points x and y of X, there exists a μ-open set U of X containing x but not y and a μ-open set V of X containing y but not x.

Theorem 3.3.[20] A GTS (X, μ) is μ-T_1 if and only if the singletone of X are μ-closed.

Theorem 3.4. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subseteq X$. If (X, μ) is a μ-T_1 space, then A is μ^{Ω}-closed if and only if A is μ-F_f-closed.

Proof. It is obvious from the Theorem 3.1(3) and the Theorem 3.2.

Theorem 3.5. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. If A is a μ-F_f-closed set, then the following are equivalent:

1. A is a μ^{Ω}-closed set,

2. $c^{\Omega}(A) - A$ is a μ-closed set,

3. $\Omega(A) - A$ is a μ-closed set.

Proof. (1)$\Rightarrow$(2). If A is μ^{Ω}-closed, then $c^{\Omega}(A) - A = \phi$ and so $c^{\Omega}(A) - A$ is μ-closed.
(2)$\Rightarrow$(3). This follows from the fact that $c^{\Omega}(A) - A = \Omega(A) - A$, it is clear.
(3)$\Rightarrow$(1). If $\Omega(A) - A$ is μ-closed and A is μ-F_f-closed, from Theorem 2.1(5), $\Omega(A) - A = \phi$ and so A is μ^{Ω}-closed.

Lemma 3.1. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. If A is μ^{Ω}-dense in itself, then $\Omega(A) = c_\mu(\Omega(A)) = c_\mu(A) = c^{\Omega}(A)$.

Proof. Let A be μ^{Ω}-dense in itself. Then we have $A \subset \Omega(A)$ and hence $c_\mu(A) \subset c_\mu(\Omega(A))$. We know that $\Omega(A) = c_\mu(\Omega(A)) \subset c_\mu(A)$ from Theorem 2.1(3). In this case $c_\mu(A) = c_\mu(\Omega(A)) = \Omega(A)$. Since $\Omega(A) = c_\mu(A)$, we have $c^{\Omega}(A) = c_\mu(A)$.

It is obvious that every g_μ-closed set [13] is a μ-F_f-closed set but not vice versa. The following Theorem 3.3 shows that for μ^Ω-dense in itself, the concepts g_μ-closedness and μ-F_f-closedness are equivalent.

Theorem 3.6. If $(X, \mu, \mathcal{F})$ is a FGTS and A is μ^Ω-dense in itself, μ-F_f-closed subset of X, then A is g_μ-closed.

Proof. Suppose A is a μ^Ω-dense in itself, μ-F_f-closed subset of X. If U is any μ-open set containing A, then by Theorem 3.1(1), $c^\Omega(A) \subset U$. Since A is μ^Ω-dense in itself, by Lemma 3.1, $c_\mu(A) \subset U$ and so A is g_μ-closed.

Theorem 3.7. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. Then A is μ-F_f-closed if and only if $A = H - N$ where H is μ^Ω-closed and N contains no nonempty μ-closed set.

Proof. If A is μ-F_f-closed, then by Theorem 3.1(4), $N = \Omega(A) - A$ contains no nonempty μ-closed set. If $H = c^\Omega(A)$, then H is μ^Ω-closed such that $H - N = (A \cup \Omega(A)) - (\Omega(A) - A) = (A \cup \Omega(A)) \cap ((X - \Omega(A)) \cup A) = A$. Conversely, suppose $A = H - N$ where H is μ^Ω-closed and N contains no nonempty μ-closed set. Let U be a μ-open set such that $A \subset U$. Then $H - N \subset U$ which implies that $H \cap (X - U) \subset N$. Now $A \subset H$ and $\Omega(H) \subset H$ implies that $\Omega(A) \cap (X - U) \subset \Omega(H) \cap (X - U) \subset H \cap (X - U) \subset N$. By hypothesis, since $\Omega(A) \cap (X - U)$ is μ-closed, $\Omega(A) \cap (X - U) = \phi$ and so $\Omega(A) \subset U$ which implies that A is μ-F_f-closed.

Following theorem gives a property of μ-F_f-closed sets and the Corollary 3.1 follows from Theorem 3.8 and the fact that, if $A \subset B \subset \Omega(A)$, then $\Omega(A) = \Omega(B)$ and B is μ^Ω-dense in itself.

Theorem 3.8. Let $(X, \mu, \mathcal{F})$ be a FGTS. If A and B are subsets of X such that $A \subset B \subset c^\Omega(A)$ and A is μ-F_f-closed, then B is μ-F_f-closed.

Proof. Since A is μ-F_f-closed, $c^\Omega(A) - A$ contains no nonempty μ-closed set. Since $c^\Omega(B) - B \subset c^\Omega(A) - A$, $c^\Omega(B) - B$ contains no nonempty μ-closed set and so by Theorem 3.1(3), B is μ-F_f-closed.

Corollary 3.1. Let $(X, \mu, \mathcal{F})$ be a FGTS. If A and B are subsets of X such that $A \subset B \subset \Omega(A)$ and A is μ-F_f-closed, then A and B is g_μ-closed.

Theorem 3.9. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. Then A is μ-F_f-open if and only if $F \subset i^\Omega(A)$ whenever F is μ-closed and $F \subset A$ (where i^Ω denotes the interior operator of (X, μ^Ω)).

Proof. Suppose A is μ-F_f-open. If F is μ-closed and $F \subset A$, then $X - A \subset X - F$ and so $c^\Omega(X - A) \subset X - F$. Therefore, $F \subset i^\Omega(A)$ (from Lemma 1.1). Conversely, suppose the condition holds. Let U be a μ-open set such that $X - A \subset U$. Then $X - U \subset A$ and so $X - U \subset i^\Omega(A)$ which implies that $c^\Omega(X - A) \subset U$. Therefore, $X - A$ is μ-F_f-closed and so A is μ-F_f-open.

Theorem 3.10. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. If A is μ-F_f-open and $i^\Omega(A) \subset B \subset A$, then B is μ-F_f-open.

Proof. The proof is obvious from above theorem.

The following theorem gives a characterization of μ-F_f-closed sets in terms of μ-F_f-open sets.

Theorem 3.11. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subset X$. Then followings are equivalent:

1. A is μ-F_f-closed,

2. $A \cup (X - \Omega(A))$ is μ-F_f-closed,

3. $\Omega(A) - A$ is μ-F_f-open.

Proof. (1)$\Rightarrow$(2). Suppose A is μ-F_f-closed. If U is any μ-open set such that $(A \cup (X - \Omega(A))) \subset U$, then $X - U \subset X - (A \cup (X - \Omega(A)) = \Omega(A) - A$. Since A is μ-F_f-closed, by Theorem 3.1(4), it follows that $X - U = \phi$ and so $X = U$. Since X is only μ-open set containing $A \cup (X - \Omega(A))$, clearly, $A \cup (X - \Omega(A))$ is μ-F_f-closed.
(2)$\Rightarrow$(1). Suppose $A \cup (X - \Omega(A))$ is μ-F_f-closed. If F is any μ-closed set such that $F \subset \Omega(A) - A$, then $A \cup (X - \Omega(A)) \subset X - F$ and $X - F$ is μ-open. Therefore, $\Omega(A \cup (X - \Omega(A))) \subset X - F$ which implies that $\Omega(A) \cup \Omega(X - \Omega(A)) \subset X - F$ and so $F \subset X - \Omega(A)$. Since $F \subset \Omega(A)$, it follows that $F = \phi$. Hence A is μ-F_f-closed.
The equivalence of (2) and (3) follows from the fact that $X - (\Omega(A) - A) = A \cup (X - \Omega(A))$.

Theorem 3.12. Let $(X, \mu, \mathcal{F})$ be a FGTS. Then every subset of X is μ-F_f-closed if and only if every μ-open set is μ^{Ω}-closed.

Proof. Suppose every subset of X is μ-F_f-closed. If U is μ-open, then U is μ-F_f-closed and so $\Omega(U) \subset U$. Hence U is μ^{Ω}-closed. Conversely, suppose that every μ-open set is μ^{Ω}-closed. If $A \subset X$ and U is a μ-open set such that $A \subset U$, then $\Omega(A) \subset \Omega(U) \subset U$ and so A is μ-F_f-closed.

§4.Generalized open sets on FGTS

Definition 4.1. Let $(X, \mu, \mathcal{F})$ be a FGTS and $A \subseteq X$. Then

1. $A \in \alpha(\mu_{\mathcal{F}})$ if $A \subseteq i_\mu(c^{\Omega}(i_\mu(A)))$,

2. $A \in \sigma(\mu_{\mathcal{F}})$ if $A \subseteq c^{\Omega}(i_\mu(A))$,

3. $A \in \pi(\mu_{\mathcal{F}})$ if $A \subseteq i_\mu(c^{\Omega}(A))$,

4. $A \in \beta(\mu_{\mathcal{F}})$ if $A \subseteq c^{\Omega}(i_\mu(c^{\Omega}(A)))$.

Lemma 4.1. Let $(X, \mu, \mathcal{F})$ be a FGTS, we have the following

1. $\mu \subseteq \alpha(\mu_{\mathcal{F}}) \subseteq \sigma(\mu_{\mathcal{F}}) \subseteq \beta(\mu_{\mathcal{F}})$,

2. $\mu \subseteq \alpha(\mu_{\mathcal{F}}) \subseteq \pi(\mu_{\mathcal{F}}) \subseteq \beta(\mu_{\mathcal{F}})$.

Definition 4.2. Let $(X, \mu, \mathcal{F})$ be a FGTS. Then FGTS is said to be μ-extermally disconnected if $c^{\Omega}(A) \in \mu$ for $A \subseteq X$ and $A \in \mu$.

Theorem 4.1. For a GT μ. Then the following statements are equivalent:

1. $(X, \mu, \mathcal{F})$ is μ-extermally disconnected,

2. $i^{\Omega}(A)$ is μ-closed for each μ-closed set $A \subseteq X$,

3. $c^{\Omega}(i_\mu(A)) \subseteq i_\mu(c^{\Omega}(A))$ for each $A \subseteq X$,

4. $A \in \pi(\mu_{\mathcal{F}})$ for each $A \in \sigma(\mu_{\mathcal{F}})$,

5. $c^{\Omega}(A) \in \mu$ for each $A \in \beta(\mu_{\mathcal{F}})$,

6. $A \in \pi(\mu_{\mathcal{F}})$ for each $A \in \beta(\mu_{\mathcal{F}})$,

7. $A \in \alpha(\mu_{\mathcal{F}})$ if and only if $A \in \sigma(\mu_{\mathcal{F}})$.

Proof. (1) $\Rightarrow$ (2). Let A be a μ-closed set. Then $X - A$ is μ-open. By using (1), $c^{\Omega}(X - A) = X - i^{\Omega}(A) \in \mu$. Thus $i^{\Omega}(A)$ is μ-closed.
(2) $\Rightarrow$ (3). Let $A \subseteq X$. Then $X - i_{\mu}(A)$ is μ-closed and by (2), $i^{\Omega}(X - i_{\mu}(A))$ is μ-closed. Therefore, $c^{\Omega}(i_{\mu}(A))$ is μ-open and hence $c^{\Omega}(i_{\mu}(A)) \subseteq i_{\mu}(c^{\Omega}(A))$.
(3) $\Rightarrow$ (4). Let $A \in \sigma(\mu_{\mathcal{F}})$. By (3), we have $A \subseteq c^{\Omega}(i_{\mu}(A)) \subseteq i_{\mu}(c^{\Omega}(A))$. Thus, $A \in \pi(\mu_{\mathcal{F}})$.
(4) $\Rightarrow$ (5). Let $A \in \beta(\mu_{\mathcal{F}})$. Then $c^{\Omega}(A) = c^{\Omega}(i_{\mu}(c^{\Omega}(A)))$ and $c^{\Omega}(A) \in \sigma(\mu_{\mathcal{F}})$. By (4), $c^{\Omega}(A) \in \pi(\mu_{\mathcal{F}})$. Thus $c^{\Omega}(A) \subseteq i_{\mu}(c^{\Omega}(A))$ and hence $c^{\Omega}(A)$ is μ-open.
(5) $\Rightarrow$ (6). Let $A \in \beta(\mu_{\mathcal{F}})$. By (5), $c^{\Omega}(A) = i_{\mu}(c^{\Omega}(A))$. Thus, $A \subseteq c^{\Omega}(A) \subseteq i_{\mu}(c^{\Omega}(A))$ and hence $A \in \pi(\mu_{\mathcal{F}})$.
(6) $\Rightarrow$ (7). Let $A \in \sigma(\mu_{\mathcal{F}})$, then $A \in \beta(\mu_{\mathcal{F}})$. Then by (6), $A \in \pi(\mu_{\mathcal{F}})$. Since $A \in \sigma(\mu_{\mathcal{F}})$ and $A \in \pi(\mu_{\mathcal{F}})$, then $A \in \alpha(\mu_{\mathcal{F}})$.
(7) $\Rightarrow$ (1). Let A be a μ-open set. Then $c^{\Omega}(A) \in \sigma(\mu_{\mathcal{F}})$ and by using (7) $c^{\Omega}(A) \in \alpha(\mu_{\mathcal{F}})$. Therefore, $c^{\Omega}(A) \subseteq i_{\mu}(c^{\Omega}(i_{\mu}(c^{\Omega}(A)))) = i_{\mu}(c^{\Omega}(A))$ and hence $c^{\Omega}(A) = i_{\mu}(c^{\Omega}(A))$. Hence $c^{\Omega}(A)$ is μ-open and $(X, \mu, \mathcal{F})$ is μ-extermally disconnected.

Proposition 4.1. Let $(X, \mu, \mathcal{F})$ be a FGTS with μ be quasi-topology. If $U \in \mu$, then $U \cap \Omega(A) = U \cap \Omega(U \cap A)$, for any $A \subseteq X$.

Proof. It is obvious from Theorem 2.1.

Theorem 4.2. Let $(X, \mu, \mathcal{F})$ be a FGTS with μ be quasi-topology. Then the following statements are equivalent:

1. $(X, \mu, \mathcal{F})$ is μ-extermally disconnected,

2. $c^{\Omega}(A) \cap c_{\mu}(B) \subseteq c_{\mu}(A \cap B)$ for each $A, B \in \mu$,

3. $c^{\Omega}(A) \cap c_{\mu}(B) = \phi$ for each $A, B \in \mu$ with $A \cap B = \phi$.

Proof. (1) $\Rightarrow$ (2). Let $A, B \in \mu$. Since $c^{\Omega}(A) \in \mu$ and $B \in \mu$, then by Proposition 4.1, $c^{\Omega}(A) \cap c_{\mu}(B) \subseteq c_{\mu}(c^{\Omega}(A) \cap B) \subseteq c_{\mu}(c^{\Omega}(A \cap B)) \subseteq c_{\mu}(A \cap B)$. Thus $c^{\Omega}(A) \cap c_{\mu}(B) \subseteq c_{\mu}(A \cap B)$.
(2) $\Rightarrow$ (3). Let $A, B \in w$ with $A \cap B = \phi$. By using (2), we have $c^{\Omega}(A) \cap c_{\mu}(B) \subseteq c_{\mu}(A \cap B) \subseteq c_{\mu}(\phi) = \phi$. Thus $c^{\Omega}(A) \cap c_{\mu}(B) = \phi$.
(3) $\Rightarrow$ (1). Let $c^{\Omega}(A) \cap c_{\mu}(B) = \phi$ for each $A, B \in \mu$ with $A \cap B = \phi$. Let $F \subseteq X$ be a μ-open set. Since F and $X - c^{\Omega}(F)$ are disjoint μ-open sets, then $c^{\Omega}(F) \cap c_{\mu}(X - c^{\Omega}(F)) = \phi$. This implies that $c^{\Omega}(F) \subseteq i_{\mu}(c^{\Omega}(F))$. Thus, $c^{\Omega}(F)$ is μ-open and hence $(X, \mu, \mathcal{F})$ is μ-extermally disconnected.

Theorem 4.3. The followings are equivalent for FGTS $(X, \mu, \mathcal{F})$:

1. X is μ-extremally disconnected,

2. For any two disjoint μ-open and μ^{Ω}-open sets A and B, respectively, there exists disjoint μ^{Ω}-closed and μ-closed sets M and N, respectively, such that $A \subseteq M$ and $B \subseteq N$.

Proof. (1) $\Rightarrow$ (2). Let X be w-extremally disconnected. Let A and B be two disjoint w-open and μ^{Ω}-open sets, respectively. Then $c^{\Omega}(A)$ and $X - c^{\Omega}(A)$ are disjoint μ^{Ω}-closed and μ-closed sets containing A and B, respectively.
(2) $\Rightarrow$ (1). Let A be an μ-open subset of X. Then, A and $B = X - c^{\Omega}(A)$ are disjoint μ-open and μ^{Ω}-open sets, respectively. This implies that there exists disjoint μ^{Ω}-closed and μ-closed sets M and N, respectively, such that $A \subseteq M$ and $B \subseteq N$. Since $c^{\Omega}(A) \subseteq c^{\Omega}(M) = M \subseteq X - N \subseteq X - B = c^{\Omega}(A)$, then $c^{\Omega}(A) = M$. Since $B \subseteq N \subseteq X - M = B$, then $B = N$. Thus, $c^{\Omega}(A) = X - N$ is μ-open. Hence, X is μ-extremally disconnected.

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On open problems on the connected bicritical graphs[1]

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Abstract In the paper [1], the following problems have been proposed. Is it true that every connected bicritical graph has a minimum dominating set containing any two specified vertices of the graphs? Is it true if G is a connected bicritical graph, then $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number? We disprove the second problem and show the truth of the first problem for a certain family of graphs. Furthermore this family of graphs is characterized with respect to bicriticality, diameter, vertex connectivity and edge connectivity.

Keywords Domination number, bicritical, diameter, connectivity, circulant graph.

§1. Introduction and preliminaries

In this paper, we concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notations on graphs that are not defined here can be found in [7].

We denote the distance between two vertices x and y in G by $d_G(x, y)$. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set S such that G-S is disconnected or has only one vertex. A graph G is k-connected if its connectivity is at least k. A graph is k-edge-connected if every disconnecting set of edges has at least k edges. The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set.

Let $G = (V, E)$ be a graph. A set $S \subset V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S, that is $V = \cup_{s\in S} N[s]$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G and a dominating set of minimum cardinality is called a $\gamma(G)$-set.

A dominating set S is called an independent dominating set of G if no two vertices of S are adjacent. The minimum cardinality among the independent dominating sets of G is the independent domination number $i(G)$.

Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. We define a graph G to be (γ, k)-critical, if $\gamma(G - S) < \gamma(G)$ for any set S of k vertices. Obviously, a (γ, k)-critical graph G has $\gamma(G) \geq 2$. The $(\gamma, 1)$-critical

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graphs are precisely the domination critical graphs introduced by Brighman, Chinn, and Dutton. The $(\gamma, 2)$-critical graphs are precisely the domination bicritical graphs introduced by Brigham, Haynes, Henning, and Rall. We call a graph critical (respectively bicritical) if it is domination critical (respectively, domination bicritical). Further, we call a graph γ-critical (respectively γ-bicritical) if it is domination critical (respectively, bicritical) with domination number γ. For more details see [1-6].

The circulant graph $C_{n+1}\langle 1,4\rangle$ is the graph with vertex set $\{v_0, v_1, \cdots, v_n\}$ and edge set $\{v_i v_{i+j(\mathrm{mod}\ n+1)} | i \in \{0,1,\cdots,n\}\}$ and $j \in \{1,4\}$.

The authors of [1] stated the following Observation and Problems:

Observation 1.1. For a bicritical graph G and $x,\ y \in V(G)$, $\gamma(G)$-2 $\leq \gamma(G$-$\{x,y\}) \leq \gamma(G)$-1.

Problem 1.1.[1] Is it true that every connected bicritical graph has a minimum dominating set containing any two specified vertices of the graphs?

Problem 1.2.[1] Is it true if G is a connected bicritical graph, then $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number?

The Problem 1.2 is rejected by counterexample $G = C_{n+1}\langle 1,4\rangle$ once $n+1 \equiv 4(\mathrm{mod}\ 9)$. We prove that the circulant graphs $G = C_{n+1}\langle 1,4\rangle$ with $n+1 \equiv k(\mathrm{mod}\ 9)$ for $k \in \{3,4,8\}$ are bicritical and $\gamma(G$-$\{x,y\}) = \gamma(G)$-1 for any pair $x,\ y$ in $V(G)$. We answer the question posed in Problem 1, in the affirmative for these graphs.

§2. Preliminary results

We verify domination number of certain graphs to achieve the main results.

Observation 2.1. Any $5k$ vertices such as $\{v_i, v_{i+1}, \cdots, v_{i+5k-1}\}$ from $C_{n+1}\langle 1,4\rangle$, cannot be dominated by any k vertices for which $k \leq n/5$.

Lemma 2.1. Let $G = (C_{n+1}\langle 1,4\rangle)$ be a circulant graph. Then the average of domination number is at most $\frac{9}{2}$.

Proof. Let v_i and v_j be two vertices of a γ-set S where $i < j$ (mod $n+1$) and for any k, $(i < k < j)$, $v_k \notin S$. There are some cases, once $v_i,\ v_j$ have common adjacent vertex or $j - i = l(\mathrm{mod}\ n+1)$ where $l \notin \{1,2,3,4,5,8\}$.

Let v_i and v_j be adjacent or dominate a common vertex, i.e, $l \in \{1,2,3,4,5,8\}$, then they dominate at most 9 vertices.

Suppose that $l \notin \{1,2,3,4,5,8\}$. Let $l = 6$. Then the vertex v_{i+3} is not dominated by v_i and v_j. For dominating v_{i+3} there must be one of v_{i+7} or v_{i-1} in S. Each of v_{i+7} or v_{i-1} dominates at most three new vertices $\{v_{i+3}, v_{i+8}, v_{i+11}\}$ or $\{v_{i-5}, v_{i-2}, v_{i+3}\}$ respectively that have not been dominated by $v_i,\ v_j$. So in this case, 3 vertices $\{v_i, v_j, v_{i-1}\}$ or $\{v_i, v_j, v_{i+7}\}$ dominate 13 vertices.

Let $l = 7$. Then the vertices $v_{i+2},\ v_{i+5}$ are not dominated by v_i and v_j. For dominating $v_{i+2},\ v_{i+5}$ there must be two vertices $v_{i-2},\ v_{i+9}$ in S. Each of v_{i-2} or v_{i+9} dominates four new vertices $\{v_{i-6}, v_{i-3}, v_{i-2}, v_{i+2}\}$ or $\{v_{i+5}, v_{i+9}, v_{i+10}, v_{i+13}\}$ respectively. Hence 4 vertices $\{v_i, v_j, v_{i-2}, v_{i+9}\}$ dominate 18 vertices.

Let $l \notin \{1,2,3,4,5,6,7,8\}$. Then the vertices $\{v_{i+2}, v_{i+3}, v_{i+l-2}, v_{i+l-3}\}$ are not dominated by v_i and v_j. For dominating these vertices the set S must contain the vertices $\{v_{i-2}, v_{i-1}, v_{i+l+1}, v_{i+l+2}\}$. Theses four vertices dominate twelve new vertices $\{v_{i-6}, v_{i-3}, v_{i+2}, v_{i-5}, v_{i-2}, v_{i+3}, v_{i+l-3}, v_{i+l+2}, v_{i+l+5}, v_{i+l-2}, v_{i+l+3}, v_{i+l+6}\}$. Thus six vertices $\{v_i, v_j, v_{i-2}, v_{i-1}, v_{i+l+1}, v_{i+l+2}\}$ dominate 22 vertices. Anyway, the average of domination number is at most $\frac{9}{2}$.

Observation 2.2. Let $G = C_{n+1}\langle 1,4\rangle)$ and $n+1 = 9m+k$ where $0 \leq k \leq 8$ and $S_1 = \{v_{9i}, v_{9i+2} |\ 0 \leq i \leq m-1\}(\mathrm{mod}\ n+1)$. If $k=1$, then S_1 dominates $9m-1$ vertices. Otherwise S_1 dominates $9m$ vertices.

Proof. By Lemma 2.2, S_1 dominates the most vertices in the among sets of vertices with cardinality $|S_1|$. Let $k \neq 1$. Any two vertices $v_{9i},\ v_{9i+2}$ dominates nine vertices $\{v_{9i-4}, v_{9i-1}, v_{9i}, v_{9i+1}, v_{9i+4}, v_{9i-2}, v_{9i+2}, v_{9i+3}, v_{9i+6}\}$ (mod $n+1$) and two vertices $v_{9(m-1)+2}, v_0 = v_{9m+k}$ do not dominate any common vertex. Thus S_1 dominates $9m$ vertices. Now, let $k=1$. Two vertices $v_{9(m-1)+2},\ v_0 = v_{9m+1}$ (mod $n+1$) dominate a common vertex v_{n-3}, so 4 vertices $v_0,\ v_2,\ v_{9(m-1)}$ and $v_{9(m-1)+2}$ dominate 17 vertices. Thus S_1 dominates $9m-1$ vertices.

Observation 2.3. Let $G = C_{n+1}\langle 1,4\rangle$. Let S_1 be same as in the above Observation and V_1 be a subset vertices of $V(G)$ that are dominated by S_1. Then $M = V(G) - V_1$ is as follows.

If $k=0$, then $M=\emptyset$. If $k=1$, then $M=\{v_{n-4}, v_{n-2}\}$. If $k=2$, then $M=\{v_{n-5}, v_{n-2}\}$. If $k=3$, then $M=\{v_{n-6}, v_{n-4}, v_{n-2}\}$. If $k=4$, then $M=\{v_{n-7}, v_{n-5}, v_{n-4}, v_{n-2}\}$. If $k=5$, then $M=\{v_{n-8}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$. If $k=6$, then $M=\{v_{n-9}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$. If $k=7$, then $M=\{v_{n-10}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$. If $k=8$, then $M=\{v_{n-11}, v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$.

Now, we are ready to prove the following main theorem:

Theorem 2.1. $\gamma(C_{n+1}\langle 1,4\rangle) = \begin{cases} 2\lfloor\frac{n+1}{9}\rfloor, & n+1 \equiv 0(\mathrm{mod}\ 9); \\ 2\lfloor\frac{n+1}{9}\rfloor+3, & n+1 \equiv 8(\mathrm{mod}\ 9); \\ 2\lfloor\frac{n+1}{9}\rfloor+1, & n+1 \equiv 1 \text{ or } 2(\mathrm{mod}\ 9); \\ 3, & n+1 = 13; \\ 2\lfloor\frac{n+1}{9}\rfloor+2, & \text{o.w.} \end{cases}$

Proof. Let $G = C_{n+1}\langle 1,4\rangle$ with vertex set $\{v_0, v_1, \cdots, v_n\}$. If $G = C_{13}\langle 1,4\rangle$, obviously the set $S = \{v_0, v_6, v_7\}$ is a γ-set, so $\gamma(C_{13}\langle 1,4\rangle) = 3$.

By Lemma 2.1 and Observations 2.2, 2.3, for $n+1 = 9m+k$ where $0 \leq k \leq 8$ the set $S_1 = \{v_{9i}, v_{9i+2} |\ 0 \leq i \leq m-1\}$ (mod $n+1$) is a minimum dominating set for G-M where M has been specified by Observation 2.4 and $|S_1| = 2\lfloor\frac{n+1}{9}\rfloor$.

If $n+1 \equiv 0(\mathrm{mod}\ 9)$, then $M=\emptyset$ and S_1 is a γ-set of G and $\gamma(G) = 2\lfloor\frac{n+1}{9}\rfloor$.

If $n+1 \equiv 1(\mathrm{mod}\ 9)$, then $M=\{v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-3}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor\frac{n+1}{9}\rfloor+1$.

If $n+1 \equiv 2(\mathrm{mod}\ 9)$, then $M=\{v_{n-5}, v_{n-2}\}$ and $S_1 \cup \{v_{n-6}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor\frac{n+1}{9}\rfloor+1$.

If $n+1 \equiv 3(\mathrm{mod}\ 9)$, then $M=\{v_{n-6}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-4}, v_{n-6}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor\frac{n+1}{9}\rfloor+2$.

If $n+1 \equiv 4(\mathrm{mod}\ 9)$, then $M=\{v_{n-7}, v_{n-5}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-4}, v_{n-6}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor\frac{n+1}{9}\rfloor+2$.

If $n+1 \equiv 5 (\mathrm{mod}\ 9)$, then $M = \{v_{n-8}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-4}, v_{n-6}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor \frac{n+1}{9} \rfloor + 2$.

If $n+1 \equiv 6 (\mathrm{mod}\ 9)$, then $M = \{v_{n-9}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-5}, v_{n-6}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor \frac{n+1}{9} \rfloor + 2$.

If $n+1 \equiv 7 (\mathrm{mod}\ 9)$, then $M = \{v_{n-10}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-6}, v_{n-8}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor \frac{n+1}{9} \rfloor + 2$.

If $n+1 \equiv 8 (\mathrm{mod}\ 9)$, then $M = \{v_{n-11}, v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}\}$ and $S_1 \cup \{v_{n-5}, v_{n-6}, v_{n-7}\}$ is a γ-set of G and $\gamma(G) = 2\lfloor \frac{n+1}{9} \rfloor + 3$.

§3. On the problems 1.1 and 1.2

In this section we discuss on the Problems 1.1 and 1.2.

Note that $\gamma(C_{13}\langle 1,4\rangle) = 3$. It is easy to see that $\gamma(C_{13}\langle 1,4\rangle - \{x,y\}) \geq 3$ for any $x,\ y \in V(C_{n+1}\langle 1,4\rangle)$, because $|V(C_{13}\langle 1,4\rangle - \{x,y\})| = 11$. Hence the graph $C_{13}\langle 1,4\rangle$ is not bicritical.

Theorem 3.1. The graph $C_{n+1}\langle 1,4\rangle$ is bicritical for $n+1 = 9m+4$, $(m \geq 2)$ and $n+1 = 9m+3,\ n+1 = 9m+8$, $(m \geq 1)$.

Proof. Let $G = C_{n+1}\langle 1,4\rangle$ and $V(G) = \{0, 1, \cdots, n\}$. Let S be a γ-set of G as introduced in Theorem 2.5 and U be an arbitrary subset of $V(G)$ with cardinality 2. It is sufficient to prove that $\gamma(G-U) < |S|$ for $U = \{x,y\}$ and $d_G(x,y) \leq \lfloor \frac{9m+k}{2} \rfloor$ where $k = 3,\ 4,\ 8$, for another amounts it is symmetrically proved. Note that without loss of generality, we may assume that $x = 0$ and y is an arbitrary vertex which satisfies in $d_G(0,y) \leq \lfloor \frac{9m+k}{2} \rfloor$ (we provide the proof for $y \in \{1, \cdots, \lfloor \frac{9m+k}{2} \rfloor\}$). Let $n+1 = 9m+3$. First we prove for $n+1 \geq 30$.

Let $F_1 = \{\lfloor \frac{9t+2}{2} \rfloor |\ 3 \leq t \leq m\}$, $F_2 = \{\lfloor \frac{9s+2}{2} \rfloor + 2|\ 3 \leq s \leq m$ and s is odd$\}$ and $F_3 = \{5, 7\}$.

Let $U = \{0, j\}$ where $j \in F_1$. Then $j = 9l+5$ or $j = 9l+10$ for some l. If $j = 9l+5$ we assign $S_0 = \{9i+5, 9i+7|\ 0 \leq i \leq l-1\} \cup \{j-3 = 9l+2\} \cup \{j+3+9k, j+5+9k|\ 0 \leq k \leq m-(l+1)\}$ to $G-U$. If $j = 9l+10$ we assign $S_0 = \{9i+5, 9i+7|\ 0 \leq i \leq l\} \cup \{j-3 = 9l+7\} \cup \{j+3+9k, j+5+9k|\ 0 \leq k \leq m-(l+2)\} \cup \{n-1\}$ to $G-U$.

Let $U = \{0, j\}$ where $j \in F_2$. Then $j = 9l+7$ for some l. We assign $S_0 = \{9k+3, 9k+5|\ 0 \leq k \leq m-1\} \cup \{n-2\}$ to $G-U$.

Let $U = \{0, 5\}$. We assign $S_0 = \{3\} \cup \{9k+10, 9k+12|\ 0 \leq k \leq m-2\} \cup \{10+9(m-1), 9m\}$ to $G-U$.

Let $U = \{0, 7\}$. We assign $S_0 = \{4\} \cup \{9k+10, 9k+12|\ 0 \leq k \leq m-2\} \cup \{9m, 9m+1\}$ to $G-U$.

Finally, let $U = \{0, j\}$ where $j \notin F_1 \cup F_2 \cup F_3$. We assign $S_0 = \{9k+5, 9k+7|\ 0 \leq k \leq m-1\} \cup \{9m+1\}$ to $G-U$.

It is easy to verify that $|S_0| = 2\lfloor \frac{n}{9} \rfloor + 1$ and all vertices of $G-U$ are dominated by S_0. Then, Theorem 2.5 implies that $\gamma(G-U) < \gamma(G)$, hence G is bicritical with $|V(G)| \geq 30$.

Now, we show the truth of the two remaining cases: i.e. $n+1 = 12$ and $n+1 = 21$. For $n+1 = 12$, we consider $\{2, 5, 7\}$ with $U = \{0, j\}$ and $j \in \{1, 3, 4, 6, 8, 9, 10, 11\}$. Also we consider $\{3, 6, 9\}$ with $U = \{0, j\}$ and $j \in \{2, 5, 7\}$.

For $n+1 = 21$, we assign $S_0 = \{5, 7, 14, 16, 19\}$ to $G-U$ with $U = \{0, j\}$ and $j \notin \{5, 7, 14, 16, 19\}$. We consider $S_0 = \{3, 7, 9, 16, 18\}$ for $G-U$ with $U = \{0, j\}$ and $j \in \{5, 14\}$.

Finally we assign $S_0 = \{3, 5, 12, 14, 18\}$ to $G - U$ with $U = \{0, j\}$ and $j \in \{7, 16, 19\}$. Hence $|S_0| = |S| - 1$ for $n + 1 = 9m + 3$.

Let $n + 1 = 9m + 4 \geq 22$. Let $F = \{\lfloor \frac{9r+3}{2} \rfloor,\ \lfloor \frac{9r+3}{2} \rfloor + 2 |\ r$ is odd and $r \geq 1\}$.

Let $U = \{0, j\}$ with $j \in F \cup \{2\}$. We assign $S_0 = \{9k+5, 9k+7 |\ 0 \leq k \leq m-1\} \cup \{n-1\}$ to $G-U$. Finally, let $U = \{0, j\}$ with $j \notin F \cup \{2\}$. We assign the set $S_0 = \{2\} \cup \{9k+6, 9k+8 |\ 0 \leq k \leq m-1\}$ to $G - U$. It is easy to see that $|S_0| = \gamma(C_{n+1}\langle 1, 4\rangle) - 1$ for $n + 1 = 9m + 4$.

Let $n + 1 = 9m + 8$. Let $F = \{\lfloor \frac{9r+7}{2} \rfloor,\ \lfloor \frac{9r+7}{2} \rfloor + 2 |\ r$ is even and $r \geq 0\}$.

Let $U = \{0, j\}$ with $j \in F$. Then $G - U$ is dominated by $S_0 = \{9k + j + 3, 9k + j + 5 |\ 0 \leq k \leq m, (\text{mod } n + 1)\}$. Finally let $j \notin F$, we assign the set $S_0 = \{9k + 3, 9k + 5 |\ 0 \leq k \leq m\}$ to $G - U$. It is easy to see that $2m + 2 = |S_0| = \gamma(C_{n+1}\langle 1, 4\rangle) - 1$ where $n + 1 = 9k + 8$.

Therefore $C_{n+1}\langle 1, 4\rangle$ is bicritical for $n + 1 \in \{9m + 3, 9m + 4, 9m + 8\}$.

As an immediate result we have the following that shows the above bound of Observation A is sharp for circulant bicritical graphs.

Corollary 3.1. Let $G = C_{n+1}\langle 1, 4\rangle$ be bicritical then $\gamma(G - \{x, y\}) = \gamma(G) - 1$ where $x,\ y \in V(G)$.

Theorem 3.2. The graph $C_{n+1}\langle 1, 4\rangle$ with $n \geq 8$ is not bicritical where $n + 1 \equiv l(\text{mod } 9)$ and $l \in \{0, 1, 2, 5, 6, 7\}$.

Proof. Let S be the γ-set of $G = C_{n+1}\langle 1, 4\rangle$ with structures in Lemma 2.1, Observations 2.2, 2.3 and Theorem 2.1. We consider the following.

Let $n + 1 = 9m$. Then $\gamma(G) = 2\frac{n+1}{9} = 2m$. If $U = \{x, y\}$, then $|V(G - U)| = n - 1 = 9(m - 1) + 7$. Since the average domination number is at most $\frac{9}{2}$, hence $G - U$ is dominated by at least $2(m - 1) + 2 = 2m$ vertices. Hence G is not bicritical for $n + 1 = 9m$.

Let $n + 1 = 9m + 1$. Then $\gamma(G) = 2m + 1$. Observation 2.3 implies that the set $S_1 = \{v_{9i}, v_{9i+2} |\ 0 \leq i \leq m - 1\}$ (mod $n + 1$) dominates $9m - 1$ vertices $V(G) - \{v_{n-4}, v_{n-2}\}$ where $d_G(v_{n-4}, v_{n-2}) = 2$ and any other set with $2m$ vertices with structure different from of S_1 dominates less than $9m - 1$ vertices of G. So, if $U = \{v_0, v_1\}$, then $G - U$ is not dominated by a set same as S_1, because $d_G(v_0, v_1) = 1$. Hence G is not bicritical for $n + 1 = 9m + 1$.

Let $n + 1 = 9m + 2$. Then $\gamma(G) = 2m + 1$. Observation 2.3, implies that the set $S_1 = \{v_{9i}, v_{9i+2} |\ 0 \leq i \leq m - 1\}(\text{mod } n + 1)$ dominates $9m$ vertices $V(G) - \{v_{n-2}, v_{n-5}\}$ where $d_G(v_{n-5}, v_{n-2}) = 3$ and any other set with $2m$ vertices with different structure of S_1 dominates less than $9m$ vertices of G. So, if $U = \{v_0, v_1\}$, then $G - U$ is not dominated by a set same as S_1, because $d_G(v_0, v_1) = 1$. Hence G is not bicritical for $n + 1 = 9m + 2$.

Finally, let $n+1 = 9m+k$ where $k \in \{5, 6, 7\}$. Then $\gamma(G) = 2m+2$. Observations 2.3 and 2.4 imply that S_1 dominates $9m$ vertices of $G - M$ where M has been specified in Observation 2.4. It is obviously seen that $M - \{v_{n-4}, v_{n-5}\}$ is not dominated by any one vertex. On the other hand any other set with $2m$ vertices with different structure of S_1 dominates less than $9m$ vertices of G. So $2m + 1$ vertices cannot dominate $G - \{x, y\}$ with $d_G(x, y) = 1$. Hence G is not bicritical for $n + 1 = 9m + k$ where $k \in \{5, 6, 7\}$.

Theorem 3.3. Let $G = C_{n+1}\langle 1, 4\rangle$ with $n + 1 \equiv k \ (\text{mod } 9)$, $k \in \{3, 4, 8\}$. Then any pair of vertices are in some $\gamma(G)$-set.

Proof. Let $n + 1 = 9m + k$ and $0 \leq l \leq \lceil \frac{m-1}{2} \rceil$. It is sufficient to show that $\{0, 9l + t\}$ is in a $\gamma(G)$-set for $t \in \{1, 3, 4, 5, 6, 8\}$ and given l, because of Theorem 2.5. We prove the result for $k = 3$ and the two other cases are similarly proved. The $\gamma(G)$-set is $S_1 \cup \{n-4, n-6\}$ for $k = 3$. One can substitute the set $\{n-4, n-6\}$ with one of the sets $\{n, n-2\}$, $\{n-3, n-2\}$, $\{n-7, n-3\}$, $\{n-4, n-2\}$ or $\{n-5, n-3\}$. Since $9l \in S_1$ and $S_1 \cup \{n, n-2\}$ is a γ-set, it is easy to see that $S_1 + 1 \cup \{n+1 = 0, n-1\}$ is a γ-set where $S_1 + 1 = \{s + 1 (\text{mod } n+1) |\ s \in S_1\}$. Thus $\{0, 9l + 1\}$ is in a γ-set.

We also show that $S_1 + 3 \cup \{n-2+3 = 0, n-3+3 = n\}$, $S_1 + 4 \cup \{n-3+4 = 0, n-7+4\}$, $S_1+5 \cup \{n-4+5 = 0, n-2+4\}$, $S_1+6 \cup \{n-5+6 = 0, n-3+6\}$ and $S_1+8 \cup \{n-7+8 = 0, n-3+8\}$ are $\gamma(G)$-sets. Thus $\{0, 9l + t\}$ is in a $\gamma(G)$-set for $t \in \{1, 3, 4, 5, 6, 8\}$ and given l. We may use similar proofs, once $n + 1 \equiv 4$ or $8 (\text{mod } 9)$. Therefore G is a connected bicritical graph so that any two specified vertices are in a γ-set.

We have shown that the answer to the question in Problem 1 is yes for $C_{n+1}\langle 1, 4\rangle$ with $n + 1 \equiv k (\text{mod } 9)$, $k \in \{3, 4, 8\}$.

For providing the rejection of Problem 2, we first see an example.

Example 3.1. Let $G = C_{22}\langle 1, 4\rangle$. Then $\gamma(G) \neq i(G)$. We verify this result as follows.

If a $\gamma(G)$-set contains $\{0, 2\}$, then the set $U = \{5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19\}$ has not been dominated by $\{0, 2\}$.

If a $\gamma(G)$-set contains $\{0, 3\}$, then the set $U = \{5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 3\}$.

If a $\gamma(G)$-set contains $\{0, 5\}$, then the set $U = \{2, 3, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 5\}$.

If a $\gamma(G)$-set contains $\{0, 6\}$, then the set $U = \{3, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 6\}$.

If a $\gamma(G)$-set contains $\{0, 7\}$, then the set $U = \{2, 5, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 7\}$.

If a $\gamma(G)$-set contains $\{0, 8\}$, then the set $U = \{2, 3, 5, 6, 10, 11, 13, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 8\}$.

If a $\gamma(G)$-set contains $\{0, 9\}$, then the set $U = \{2, 3, 6, 7, 11, 12, 14, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 9\}$.

If a $\gamma(G)$-set contains $\{0, 10\}$, then the set $U = \{2, 3, 5, 7, 8, 12, 13, 15, 16, 17, 19, 20\}$ has not been dominated by $\{0, 10\}$.

If a $\gamma(G)$-set contains $\{0, 11\}$, then the set $U = \{2, 3, 5, 6, 8, 9, 13, 14, 16, 17, 19, 20\}$ has not been dominated by $\{0, 11\}$.

Let $\{a, b\}$ be any independent vertices in a $\gamma(G)$-set, then the situation of $\{a, b\}$ will satisfy one of the above cases. It is also easy to see that the set U cannot be dominated by any four independent vertices of U. Thus $\gamma(G) \neq i(G)$. Since $G = C_{22}\langle 1, 4\rangle$ is a connected bicritical graph, so the answer to Problem 1.2 is "no".

In the following we exhibit a family of graphs for which the answer to Problem 1.2 is also "no".

Theorem 3.4. Let $G = C_{n+1}\langle 1, 4\rangle$ with $n + 1 = 9m + 4$ $(m \geq 2)$. Then there is no $\gamma(G)$-set so that $\gamma(G) = i(G)$.

Proof. Theorem 2.1 asserts that S_1 dominates $G - M$ where $M = \{n-7, n-5, n-4, n-2\}$ and $S_1 \cup \{n-4, n-6\}$ is a $\gamma(G)$-set and any two independent vertices in M cannot dominate

M. Hence by choosing S_1 one cannot have a γ-set so that $\gamma(G) = i(G)$. If S is any independent vertex set with cardinality $|S_1|$, then Lemma 2.1 and Observation 2.2 says if S dominates the vertex set V_1 of G then $V(G) \setminus V_1$ cannot be dominated by two independent vertices. Therefore $\gamma(G) \neq i(G)$.

Remark 3.1. The Family of graphs $G = C_{n+1}\langle 1, 4\rangle$ with $n + 1 = 9m + 3$ and $n + 1 = 9m + 8$ are connected bicritical graphs with $\gamma(G) = i(G)$. Because Theorem 2.5, shows that $S_1 \cup \{n-2, n-4\}$ and $S_1 \cup \{n-2, n-5, n-7\}$ are independent γ-sets of $C_{9m+3}\langle 1, 4\rangle$ and $C_{9m+8}\langle 1, 4\rangle$ respectively.

§4. Diameter and vertex (edge) connectivity

Here, the diameter of $C_{n+1}\langle 1, 4\rangle$ is compared with the γ-set.

Theorem 4.1. $diam(C_{n+1}\langle 1,4\rangle) = \begin{cases} \lfloor \lfloor \frac{n+1}{2} \rfloor \div 4 \rfloor + 1, & \text{if } 4 \mid \lfloor \frac{n+1}{2} \rfloor - 1 \text{ or } 4 \mid \lfloor \frac{n+1}{2} \rfloor; \\ \lfloor \lfloor \frac{n+1}{2} \rfloor \div 4 \rfloor + 2, & \text{if } 4 \nmid \lfloor \frac{n+1}{2} \rfloor - 1 \text{ and } 4 \nmid \lfloor \frac{n+1}{2} \rfloor. \end{cases}$

Proof. We determine the diameter of $(C_{n+1}\langle 1, 4\rangle)$ as follows. If $\lfloor \frac{n+1}{2} \rfloor = 4k,\ k \geq 1$ then $diam(C_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k-1}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k-1}) = k + 1$.

If $\lfloor \frac{n+1}{2} \rfloor = 4k+1$ then $diam(C_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k+1}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k+1}) = k+1$.

If $\lfloor \frac{n+1}{2} \rfloor = 4k+2$ then $diam(C_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k+2}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k+2}) = k+2$.

If $\lfloor \frac{n+1}{2} \rfloor = 4k+3$ then $diam(C_{n+1}\langle 1, 4\rangle) = d(v_0, v_{4k+2}) = d(v_0, v_{4k}) + d(v_{4k}, v_{4k+2}) = k+2$.

Theorem 4.2. (i) Let $G = C_{n+1}\langle 1, 4\rangle$. Then $diam(G) < \gamma(G)$ for $n \notin \{9, 13\}$.

(ii) $(\gamma(G) - diam(G)) \to \infty$ once $n \to \infty$.

Proof. For $n = 9$ and $n = 13$ it is clearly $\gamma(G) = diam(G)$. For another n, depending to the relation between $4 \mid \lfloor \frac{n+1}{2} \rfloor$, $4 \nmid \lfloor \frac{n+1}{2} \rfloor$, $4 \mid \lfloor \frac{n+1}{2} \rfloor - 1$ or $4 \nmid \lfloor \frac{n+1}{2} \rfloor - 1$ with $n+1 \equiv k \pmod 9$ in Theorems 2.1 and 3.3, we have the following.

For $n = 8m$, then $n = 72t+r$ where $r \in \{0, 8, 16, 24, 32, 40, 48, 56, 64\}$. An easy calculation shows that $\gamma(G) - diam(G) \in \{7t+2, 7t+3, 7t+4, 7t+5, 7t+6, 7t+7\}$.

For $n = 8m+1$, then $n = 72t+r$ where $r \in \{1, 9, 17, 25, 33, 41, 49, 57, 65\}$. So $\gamma(G) - diam(G) \in \{7t+3, 7t+4, 7t+5, 7t+6, 7t+7\}$.

For $n = 8m+2$, then $n = 72t+r$ where $r \in \{2, 10, 18, 26, 34, 42, 50, 58, 66\}$. So $\gamma(G) - diam(G) \in \{7t+2, 7t+4, 7t+5, 7t+6, 7t+7, 7t+8\}$.

For $n = 8m+3$, $n = 72t+r$ where $r \in \{3, 11, 19, 27, 35, 43, 51, 59, 67\}$. So $\gamma(G) - diam(G) \in \{7t+2, 7t+4, 7t+5, 7t+6, 7t+7\}$.

For $n = 8m+4$, then $n = 72t+r$ where $r \in \{4, 12, 20, 28, 36, 44, 52, 60, 68\}$. So $\gamma(G) - diam(G) \in \{7t+2, 7t+3, 7t+5, 7t+6, 7t+7\}$.

For $n = 8m+5$, then $n = 72t+r$ where $r \in \{5, 13, 21, 29, 37, 45, 53, 61, 69\}$. So $\gamma(G) - diam(G) \in \{7t+2, 7t+3, 7t+4, 7t+6, 7t+7\}$.

For $n = 8m+6$, then $n = 72t+r$ where $r \in \{6, 14, 22, 30, 38, 46, 54, 62, 70\}$. So $\gamma(C_{n+1}\langle 1, 4\rangle) - diam(C_{n+1}\langle 1, 4\rangle) \in \{7t+2, 7t+3, 7t+4, 7t+5, 7t+7\}$.

For $n = 8m+7$, then $n = 72t+r$ where $r \in \{7, 15, 23, 31, 39, 47, 55, 63, 71\}$. So $\gamma(G) - diam(G) \in \{7t+1, 7t+2, 7t+3, 7t+4, 7t+5, 7t+6\}$.

Thus we see that $\gamma(G) > diam(G)$. Hence the result holds.

Since $n \to \infty$ then $t \to \infty$ therefore $\lim(\gamma(G) - diam(G)) \to \infty$ as $n \to \infty$. Hence the desired result follows.

Now, we study the vertex and edge connectivity of $C_{n+1}\langle 1, 4\rangle$. Recall that a classic well-known theorem [3] implies that for any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 4.3. $\kappa(C_{n+1}\langle 1, 4\rangle) = \lambda(C_{n+1}\langle 1, 4\rangle) = 4$.

Proof. Let $G = C_{n+1}\langle 1, 4\rangle$. Since G is 4-regular, then $\kappa(G) \leq \lambda(G) \leq \delta(G) = 4$. Therefore it is sufficient to prove $\kappa(G) \geq 4$. Let U be a subset of $V(G)$ with $|U| \leq 3$. We prove that $G-U$ is connected. Since G has no cut vertex, so $|U| \geq 2$. Consider $u,\ v \in V(G) \setminus U$, the original circular arrangement has a clockwise u, v-path and a counterclockwise u, v-path along the circle. Let A and B be the sets of internal vertices on these two paths. Then $A \cap U \leq 1$ or $B \cap U \leq 1$. Since in G each vertex has edges to the next two vertices in a particular direction, deleting at most one vertex cannot block travel in that direction. Thus we can be find a u, v-path in $G - S$ via the set A or B in which S has at most one vertex. So $\kappa(G) \geq 4$ and then $\kappa(G) = \lambda(G) = 4$.

Theorems 2.1, 3.1 and 4.3 imply that Problem 2 of [1] "if G is a connected bicritical graph, is it true that $\lambda \geq 3$" is partially true.

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Some properties of certain subclasses of univalent integral operators

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Abstract For analytic function of the form $f_i(z) = z + \sum_{n=2} a_n^i z^n$, in the open unit disk, a class $\Gamma_\alpha(\zeta_1, \zeta_2; \gamma)$ is introduced and some properties for $\Gamma_\alpha(\zeta_1, \zeta_2; \gamma)$ of $f_i(z)$ and $\Gamma_\alpha(\zeta_1, \zeta_2, \mu; \gamma)$ of $(f_i(z))^\mu$ in relation to the coefficient bounds, convex combination and convolution were discussed.

Keywords Analytic, univalence, coefficient bound, convolution, convex combination, integral operator.
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§1. Introduction and preliminaries

Let A denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

analytic and normalized with $f(0) = f'(0) - 1 = 0$ in the open disk, $U = \{z \in C : |z| < 1\}$. In [6], Seenivasagan gave a condition of the unvalence of the integral operator

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{1/\alpha} ds\right\}^{1/\beta},$$

where $f_i(z)$ is defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n, \tag{1}$$

while Makinde in [5] gave a condition for the starlikeness for the function:

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{1/\alpha} ds,\ \alpha \in C, \tag{2}$$

where $f_i(z)$ is defined by (1).

Also, Kanas and Ronning [2] introduced the class of function of the form

$$f(z) = (z - w) + \sum_{n=2}^{\infty} a_n^i (z - w)^n,$$

where w is a fixed point in the unit disk normalized with $f(w) = f'(w) - 1 = 0$. We define $f_i(z)$ by

$$f_i(z) = (z - w) + \sum_{n=2}^{\infty} a_n^i (z - w)^n, \tag{3}$$

where w is a fixed point in the unit disk, $|z - w| = (r + d) < 1$ and $F_{w,\alpha}$ is defined by

$$F_{w,\alpha}(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s - w)}{s - w} \right)^{1/\alpha} ds, \ \alpha \in C. \tag{4}$$

Furthermore, Xiao-Feili et al [7] denote $L_1^*(\beta_1, \beta_2, \lambda)$ as a subclass of A such that:

$$L_1^*(\beta_1, \beta_2, \gamma) = \left\{ f \in A : \left| \frac{f'(z) - 1}{\beta_1 f'(z) + \beta_2} \leq \lambda \right| \right\}, \ 0 \leq \beta_1 \leq 1; \ 0 < \beta_2 \leq 1; \ 0 < \lambda \leq 1,$$

for some $\beta_1, \ \beta_2$ and for some real λ. Also, he denoted T to be the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,$$

and $L^*(\beta_1, \beta_2, \lambda)$ denotes the subclass of $L_1^*(\beta_1, \beta_2, \lambda)$ defined by: $L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \bigcap T$ for some real number, $0 \leq \beta_1 \leq 1, \ 0 < \beta_2 \leq 1, \ 0 < \lambda \leq 1$.

The class $L^*(\beta_1, \beta_2, \lambda)$ was studied by Kim and Lee in [3], see also [1,2,7]. Let $F_\alpha(z)$ be defined by (2), then

$$\frac{zF_\alpha''(z)}{F_\alpha'(z)} = \sum_{i=1}^{k} \frac{1}{\alpha} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right).$$

Let $G(z)$ be denoted by

$$G(z) = \sum_{i=1}^{k} \frac{1}{\alpha} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right).$$

We define

$$\Gamma_\alpha(\zeta_1, \zeta_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1 (G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\}, \tag{5}$$

for some complex $\zeta_1, \ \zeta_2, \ \alpha$ and for some real $\gamma, \ 0 \leq |\zeta_1| \leq 1, \ 0 < |\zeta_2| \leq 1, \ |\alpha| \leq 1$ and $0 < \gamma \leq 1$.

Let $f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n$ and $g_i(z) = z + \sum_{n=2}^{\infty} b_n^i z^n$, we define the convolution of $f_i(z)$ and $g_i(z)$ by

$$f_i(z) * g_i(z) = (f_i * g_i)(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n. \tag{6}$$

Furthermore, let $f_i(z)$ be as defined in (1), using binomial expansion, we obtain:

$$(f_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty} \mu a_n^i z^{n+n-1}, \ \mu \geq 1. \tag{7}$$

For a fixed point w in the unit disk, we denote by

$$(f_{iw}(z))^{\mu} = (z-w)^{\mu} + \sum_{n=2}^{\infty} \mu a_n^i (z-w)^{n+n-1}. \tag{8}$$

§2. Main results

Theorem 2.1. Let $f_i(z)$ be as in (1) and $F_\alpha(z)$ be as in (2). Then $f_i(z)$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty} n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]|a_n^i| \leq \gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|, \tag{9}$$

for $0 \leq \zeta_1 \leq 1,\ 0 < \zeta_2 \leq 1,\ 0 < \alpha \leq 1.$

Proof. From (5), we have

$$\begin{aligned}
\left|\frac{G(z)+\frac{1}{\alpha}-1}{\zeta_1(G(z)+\frac{1}{\alpha})+\zeta_2}\right| &= \left|\frac{\sum_{i=1}^{k}\frac{1}{\alpha}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\frac{1}{\alpha}-1}{\zeta_1\left(\sum_{i=1}^{k}\frac{1}{\alpha}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\frac{1}{\alpha}\right)+\zeta_2}\right| \\
&= \left|\frac{\sum_{i=1}^{k}\frac{zf_i'(z)}{\alpha f_i(z)}-1}{\sum_{i=1}^{k}\frac{\zeta_1 zf_i'(z)}{\alpha f_i(z)}+\zeta_2}\right| \\
&= \left|\frac{\frac{\sum_{i=1}^{k} z+\sum_{n=2}^{\infty} na_n^i z^n-(\alpha z+\sum_{n=2}^{\infty}\alpha na_n^i z^n)}{\alpha z+\sum_{n=2}^{\infty}\alpha na_n^i z^n}}{\frac{\zeta_1 z+\sum_{n=2}^{\infty} n\zeta_1 a_n^i z^n+\alpha\zeta_2 z+\sum_{n=2}^{\infty}\alpha\zeta_2 na_n^i z^n}{\alpha z+\sum_{n=2}^{\infty}\alpha a_n^i z^n}}\right| \\
&= \left|\frac{\sum_{i=1}^{k}(1-\alpha+\sum_{n=2}^{\infty}(n-\alpha)a_n^i z^{n-1})}{\sum_{i=1}^{k}(\zeta_1+\alpha\zeta_2+\sum_{n=2}^{\infty}(n\zeta_1+\alpha\zeta_2)a_n^i z^{n-1})}\right| \\
&\leq \frac{|1-\alpha|+\sum_{i=1}^{k}\sum_{n=2}^{\infty}(n-\alpha)|a_n^i|}{|\zeta_1+\alpha\zeta_2|-\sum_{i=1}^{k}\sum_{n=2}^{\infty}(n\zeta_1+\alpha\zeta_2)|a_n^i|}.
\end{aligned}$$

Let $f_i(z)$ satisfies the inequality (9), then $f_i(z) \in \Gamma(\zeta_1, \zeta_2, \gamma)$.

Conversely, let $f_i(z) \in \Gamma(\zeta_1, \zeta_2, \gamma)$, then

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty} n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]|a_n^i| \leq \gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|.$$

Corollary 2.1. If $f_i(z) \in \Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, then we have:

$$|a_n^i| \leq \frac{\gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|}{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]}.$$

Theorem 2.2. Let the function $f_i(z) \in \Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $g_i(z)$ defined by

$$g_i(z) = z + \sum_{n=2}^{\infty} b_n^i z^n$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)f_i(z) + \lambda g_i(z) = z + \sum_{n=2}^{\infty} c_n^i z^n$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, where $c_n^i = (1-\lambda)a_n^i + \lambda b_n^i,\ 0 \le \lambda \le 1$.

Proof. Suppose that each of $f_i(z),\ g_i(z)$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then by (9), we have:

$$\begin{aligned}
\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]\}|c_n^i| &= \sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]\}|(1-\lambda)a_n^i + \lambda b_n^i| \\
&= (1-\lambda)\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]\}|a_n^i| \\
&+ \lambda\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]\}|b_n^i| \\
&= \lambda|\zeta_1+\alpha\zeta_2| - |1-\alpha|,
\end{aligned}$$

which shows that convex combination of $f_i(z),\ g_i(z)$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$.

Theorem 2.3. Let $f_i(z)$ be as in (1) and $F_\alpha(z)$ be as in (2) then the function $C_i(z)$ defined by

$$C_i(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n$$

is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty} n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]|a_n^i b_n^i| \le \gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|,$$

for $0 \le \zeta_1 \le 1,\ 0 < \zeta_2 \le 1,\ 0 < \alpha \le 1,\ \ a_n^i b_n^i \ge 0$.

Proof. The proof of this theorem is similar to that of the Theorem 2.1, thus we omit the proof.

Corollary 2.2. Let $f_i(z)$ be as in (1) and $F_\alpha(z)$ be as in (2) then the function $C_i(z)$ defined by

$$C_i(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n$$

is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, we have:

$$|a_n^i b_n^i| \le \frac{\gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|}{n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]}.$$

Corollary 2.3. Let $f_i(z)$ be as in (1) and $F_\alpha(z)$ be as in (2) then the function $C_i(z)$ defined by

$$C_i(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n$$

is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, we have:

$$|a_n^i| \le \frac{\gamma|\zeta_1+\alpha\zeta_2| - |1-\alpha|}{n|b_n^i|[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]}.$$

Corollary 2.4. Let $f_i(z)$ be as in (1) and $F_\alpha(z)$ be as in (2) and the function $C_i(z)$ defined by

$$C_i(z) = z + \sum_{n=2}^{\infty} a_n^i b_n^i z^n$$

is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, we have:

$$|b_n^i| \leq \frac{\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|}{n|a_n^i|[(1+\gamma\zeta_1) + \alpha(\gamma\zeta_2 - 1)]}.$$

Theorem 2.4. Let the function $C_i(z)$ be in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $\varphi(z)$ be defined by

$$\varphi(z) = z + \sum_{n=2}^{\infty} A_n^i B_n^i z^n$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then the function $H(z)$ defined by

$$H(z) = (1-\lambda)C_i(z) + \lambda\varphi_i(z) = z + \sum_{n=2}^{\infty} C_n^i z^n$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, where

$$C_n^i = (1-\lambda)a_n^i b_n^i + \lambda A_n^i B_n^i, \ 0 \leq \lambda \leq 1.$$

Proof. Following the procedure of the proof of the Theorem 2.2, we obtain the result.

Corollary 2.5. Let $f_i(z)$ be as in (3) and $F_\alpha(z)$ be as in (4). Then $f_i(z)$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if:

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty} n[(1+\gamma\zeta_1) + \alpha(\gamma\zeta_2 - 1)]|a_n^i| \leq \gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|,$$

for $0 \leq \zeta_1 \leq 1, \ 0 < \zeta_2 \leq 1, \ 0 < \alpha \leq 1$.

Corollary 2.6. Let function $f_i(z)$ defined by (3) belong to the class of $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $g_i(z)$ defined by

$$g_i(z) = (z-w) + \sum_{n=2}^{\infty} b_n^i (z-w)^n$$

be in the same class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)f_i(z) + \lambda g_i(z) = z + \sum_{n=2}^{\infty} C_n^i z^n$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, where

$$C_n^i = (1-\lambda)a_n^i + \lambda b_n^i, \ 0 \leq \lambda \leq 1.$$

Corollary 2.7. Let $f_i(z)$ be as in (3) and $F_\alpha(z)$ be as in (4) then the function $C_i(z)$ defined by

$$C_i(z) = (z-w) + \sum_{n=2}^{\infty} a_n^i b_n^i (z-w)^n$$

is in the class $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty} n[(1+\gamma\zeta_1)+\alpha(\gamma\zeta_2-1)]|a_n^i b_n^i| \leq \gamma|\zeta_1+\alpha\zeta_2|-|1-\alpha|,$$

for $0\leq\zeta_1\leq 1,\ 0<\zeta_2\leq 1,\ 0<\alpha\leq 1,\ \ a_n^i b_n^i\geq 0$.

Corollary 2.8. Let the function $C_i(z)$ be in the class $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$ and the function $\varphi_i(z)$ be defined by

$$\varphi_i(z)=(z-w)+\sum_{n=2}^{\infty}A_n^iB_n^i(z-w)^n$$

be in the same $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$. Then the function $H(z)$ defined by

$$H(z)=(1-\lambda)C_i(z)+\lambda\varphi_i(z)=(z-w)+\sum_{n=2}^{\infty}C_n^i(z-w)^n,$$

is also in the class $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$, where

$$C_n^i=(1-\lambda)a_n^ib_n^i+\lambda A_n^iB_n^i,\quad 0\leq\lambda\leq 1.$$

Remark 2.1. The corollary 2.5, 2.6, 2.7 and 2.8 yield Theorem 2.1, 2.2, 2.3 and 2.4 respectively when $w=0$.

Theorem 2.5. Let $(f_i(z))^\mu$ be as in (7) and F_α^μ be defined by

$$F_\alpha^\mu(z)=\int_0^z\prod_{i=1}^{k}\left(\frac{(f_i(s))^\mu}{s}\right)^{1/\alpha}ds,\ \alpha\in C.$$

Then $(f_i(z))^\mu$ is in the class $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}(\mu+n-1)[(\mu+\gamma\mu\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]|a_n^i|\leq\gamma|\mu\zeta_1+\alpha\zeta_2|-|\mu-\alpha|, \tag{10}$$

for $0\leq\zeta_1\leq 1,\ 0<\zeta_2\leq 1,\ 0<\alpha\leq 1$.

Proof. From (5), we have:

$$\begin{aligned}\left|\frac{G(z)+\frac{1}{\alpha}-1}{\zeta_1(G(z)+\frac{1}{\alpha}+\zeta_2)}\right| &= \left|\frac{\sum_{i=1}^k\frac{1}{\alpha}\left(\frac{z((f_i(z))^\mu)'}{(f_i(z))^\mu}-1\right)+\frac{1}{\alpha}-1}{\zeta_1\left(\sum_{i=1}^k\frac{1}{\alpha}\left(\frac{z((f_i(z))^\mu)'(z)}{(f_i(z))^\mu}-1\right)\frac{1}{\alpha}+\zeta_2\right)}\right|\\ &= \left|\frac{\sum_{i=1}^k\frac{z((f_i(z))^\mu)'}{\alpha(f_i(z))^\mu}-1}{\sum_{i=1}^{k}\frac{\zeta_1 z((f_i(z))^\mu)'}{\alpha(f_i(z))^\mu+\zeta_2}}\right|\\ &\leq \frac{|\mu-\alpha|+\sum_{i=1}^k\sum_{n=2}^\infty\mu(\mu+n-1-\alpha)|a_n^i|}{|\mu\zeta_1+\alpha\zeta_2|-\sum_{i=1}^k\sum_{n=2}^\infty\mu(\zeta_1(\mu+n-1)+\alpha\zeta_))|a_n^i|}.\end{aligned}$$

Let $(f_i(z))^\mu$ satisfies the inequality (9), then $(f_i(z))^\mu\in\Gamma(\zeta_1,\zeta_2,\gamma)$.

Coversely, let $(f_i(z))^\mu\in\Gamma(\zeta_1,\zeta_2,\gamma)$, then

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}(\mu+n-1)[(\mu+\gamma\mu\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]|a_n^i|\leq\gamma|\mu\zeta_1+\alpha\zeta_2|-|\mu-\alpha|.$$

Remark 2.2. Theorem 2.5 is a generalisation of the Theorem 2.1.

Corollary 2.9. Let $(f_i(z))^\mu$ be in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, then, we have,

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}|a_n^i| \leq \frac{\gamma|\mu\zeta_1+\alpha\zeta_2|-|\mu-\alpha|}{\mu(\mu+n-1)[(1+\gamma\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]}.$$

Theorem 2.6. Let the function $(f_i(z))^\mu$ be in the class and the $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $(g_i(z))^\mu$ be defined by

$$(g_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty}\mu b_n^i z^{\mu+n-1}$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)(f_i(z))^\mu + \lambda(g_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty}\mu C_n^i z^{\mu+n-1}$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ where $C_n^i = (1-\lambda)a_n^i + \lambda b_n^i, \ \ 0 \leq \lambda \leq 1$.

Proof. Suppose that each of $(f_i(z))^\mu, \ (g_i(z))^\mu$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then using (9) and following the procedure of the proof of the Theorem 2.2, we obtain the result.

Theorem 2.7. Let $(f_i(z))^{w,\mu}$ be as in (7) and F_α^μ be defined by

$$F_\alpha^{w,\mu}(z) = \int_0^z \prod_{i=1}^{k}\left(\frac{(f_i(s))^\mu}{(s-w)}\right)^{1/\alpha} ds, \ \alpha \in C.$$

Then function $C_i(z)$ defined by

$$(C_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty}\mu a_n^i b_n^i z^{\mu+n-1}$$

belongs to the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}(\mu+n-1)[(\mu+\gamma\mu\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]|a_n^i b_n^i| \leq \gamma|\mu\zeta_1+\alpha\zeta_2|-|\mu-\alpha|,$$

for $0 \leq \zeta_1 \leq 1, \ 0 < \zeta_2 \leq 1, \ 0 < \alpha \leq 1$.

The proof follows the sane procedure as that of Theorem 2.5.

Theorem 2.8. Let the function $(C_i(z))^\mu$ be in $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $(\varphi_i(z))^\mu$ defined by

$$(\varphi_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty}\mu A_n^i B_n^i z^{\mu+n-1}$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)(C_i(z))^\mu + \lambda(\varphi_i(z))^\mu = z^\mu + \sum_{n=2}^{\infty}\mu C_n^i z^{\mu+n-1}$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, where $C_n^i = (1-\lambda)a_n^i + \lambda b_n^i, \ \ 0 \leq \lambda \leq 1$.

Proof. The proof follows the sane procedure as that of Theorem 2.6.

Corollary 2.10. Let $(f_i(z))^\mu(z)$ be as in (8) and F^μ_α be defined by

$$F^\mu_\alpha(z) = \int_0^z \prod_{i=1}^k \left(\frac{(f_i(s))^\mu}{(s-w)}\right)^{1/\alpha} ds, \ \alpha \in C.$$

Then $(f_i(z))^\mu(z)$ is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if

$$\sum_{i=1}^k \sum_{n=2}^\infty (\mu+n-1)[(\mu+\gamma\mu\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]|a^i_n b^i_n| \leq \gamma|\mu\zeta_1+\alpha\zeta_2| - |\mu-\alpha|,$$

for $0 \leq \zeta_1 \leq 1, \ 0 < \zeta_2 \leq 1 \ \ 0 < \alpha \leq 1$.

Corollary 2.11. Let the function $(f_i(z))^\mu$ be in $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $(g_i(z))^\mu$ defined by

$$(g_i(z))^\mu = (z-w)^\mu + \sum_{n=2}^\infty \mu b^i_n (z-w)^{\mu+n-1}$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)(f_i(z))^\mu + \lambda(g_i(z))^\mu = (z-w)^\mu + \sum_{n=2}^\infty \mu C^i_n (z-w)^{\mu+n-1}$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ where $C^i_n = (1-\lambda)a^i_n + \lambda b^i_n, \ \ 0 \leq \lambda \leq 1$.

Corollary 2.12. Let $(f_i(z))^\mu(z)$ be as in (8) and F^μ_α be defined by

$$F^\mu_\alpha(z) = \int_0^z \prod_{i=1}^k \left(\frac{(f_i(s))^\mu}{(s-w)}\right)^{1/\alpha} ds, \ \alpha \in C.$$

Then the function $(C_i(z))^\mu(z)$ defined by

$$(C_i(z))^\mu(z) = (z-w)^\mu + \sum_{n=2}^\infty \mu a^i_n b^i_n (z-w)^{\mu+n-1}$$

belongs to the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if

$$\sum_{i=1}^k \sum_{n=2}^\infty (\mu+n-1)[(\mu+\gamma\mu\zeta_1)+\alpha\mu(\gamma\zeta_2-1)]|a^i_n b^i_n| \leq \gamma|\mu\zeta_1+\alpha\zeta_2| - |\mu-\alpha|,$$

for $0 \leq \zeta_1 \leq 1, \ 0 < \zeta_2 \leq 1, \ 0 < \alpha \leq 1$.

Corollary 2.13. Let the function $(C_i(z))^\mu$ be in $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ and the function $(\varphi_i(z))^\mu$ defined by

$$(\varphi_i(z))^\mu = (z-w)^\mu + \sum_{n=2}^\infty \mu A^i_n B^i_n (z-w)^{\mu+n-1}$$

be in the same $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$. Then, the function $h_i(z)$ defined by

$$h_i(z) = (1-\lambda)(C_i(z))^\mu + \lambda(\varphi_i(z))^\mu = (z-w)^\mu + \sum_{n=2}^\infty \mu C^i_n (z-w)^{\mu+n-1}$$

is also in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$, where $C^i_n = (1-\lambda)a^i_n b^i_n + \lambda A^i_n b^i_n, \ \ 0 \leq \lambda \leq 1$.

Remark 2.3. The corollary 2.7, 2.8 yield Theorem 2.3, 2.4 respectively when $w = 0$.

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The semi normed space defined by entire rate sequences

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Abstract In this paper we introduce the sequence spaces $\Gamma_\pi(p,\sigma,q,s)$ $\Lambda_\pi(p,\sigma,q,s)$ and define a semi normed space (X,q), semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Keywords Entire rate sequence, analytic sequence, invariant mean, semi norm.

§1. Introduction and preliminaries

A complex sequence, whose k-th term is x_k, is denoted by $\{x_k\}$ or simply x. Let φ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic rate if $\sup_k \left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ_π. A sequence x is called entire rate sequence if $\lim\limits_{k\to\infty} \left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}} = 0$. The vector space of all entire rate sequences will be denoted by Γ_π. Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1,2,3,\cdots$.

A continuous linear functional φ on Λ_π is said to be an invariant mean or a σ-mean if and only if
(1) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\varphi(e) = 1$ where $e = (1,1,1,\cdots)$ and,
(3) $\varphi(\{x_\sigma(n)\}) = \varphi(\{x_n\})$ for all $x \in \Lambda_\pi$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space C of all real convergent sequences in the sense that $\varphi(x) = \lim x$ for all $x \in C$. Consequently $C \subset V_\sigma$, where V_σ is the set of analytic sequences all of those σ-means are equal.

If $x = (x_n)$, set $Tx = (Tx)^{1/n} = (x_\sigma(n))$. It can be shown that

$$V_\sigma = \{x = (x_n) : \lim_{m\to\infty} t_{mn}(x_n)^{1/n} = L \text{ uniformly in } n,\ L = \sigma - \lim_{n\to\infty}(x_n)^{1/n}\},$$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{1/n}}{m+1}. \tag{1}$$

Given a sequence $x = \{x_k\}$ its n-th section is the sequence $x^{(n)} = \{x_1, x_2, \cdots, x_n, 0, 0, \cdots\}$, $\delta^{(n)} = (0, 0, \cdots, 1, 0, 0, \cdots)$, 1 in the n-th place and zeros elsewhere. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1, 2, \cdots)$ are continuous.

§2. Definition and properties

Definition 2.1. The space consisting of all those sequences x in w such that $\left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right) \to 0$ as $k \to \infty$ is denoted by Γ_π. In other words $\left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)$ is a null sequence. Γ_π is called the space of entire rate sequences. The space Γ_π is a metric space with the metric $d(x, y) = \left\{\sup_k \left(\left|\frac{x_k - y_k}{\pi_k}\right|^{1/k}\right) : k = 1, 2, 3, \cdots\right\}$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_π.

Definition 2.2. The space consisting of all those sequences x in w such that $\left\{\sup_k \left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)\right\} < \infty$ is denoted by Λ_π. In other words $\left\{\sup_k \left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)\right\}$ is a bounded sequence.

Definition 2.3. Let p, q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, then p and q are said to be equivalent.

Lemma 2.1. Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \le Mp(x)$ for all $x \in X$.

Definition 2.4. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in N$.

Definition 2.5. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.1. From the above two definitions, it is clear that a sequence space E is solid implies that E is monotone.

Definition 2.6. A sequence E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup_k p_k = G$. Let $D = \max(1, 2^{G-1})$. Then for $a_k,\ b_k \in C$, the set of complex numbers for all $k \in N$ we have

$$|a_k + b_k|^{1/k} \le D\{|a_k|^{1/k} + |b_k|^{1/k}\}. \tag{2}$$

Let (X, q) be a semi normed space over the field C of complex numbers with the semi norm q. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over X. We define the following sequence spaces:

$$\Lambda_\pi(p,\sigma,q,s)=\Big\{x\in\Lambda(X):\sup_{n,k}k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right)^{p_k}<\infty \text{ uniformly in } n\geq 0, s\geq 0\Big\},$$

$$\Gamma_\pi(p,\sigma,q,s)=\Big\{x\in\Gamma_\pi(X):k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right)^{p_k}\to 0, \text{ as } k\to\infty \text{ uniformly in } n\geq 0, s\geq 0\Big\}.$$

§3. Main results

Theorem 3.1. $\Gamma_\pi(p,\sigma,q,s)$ is a linear space over the set of complex numbers.

The proof is easy, so omitted.

Theorem 3.2. $\Gamma_\pi(p,\sigma,q,s)$ is a paranormed space with

$$g(x)=\left\{\sup_{k\geq 1}k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right), \text{ uniformly in } n>0\right\},$$

where $H=\max\left(1,\sup_k p_k\right)$.

Proof. Clearly $g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence. It can be easily verified that $g(x+y)\leq g(x)+g(y)$. Next $x\to\theta,\lambda$ fixed implies $g(\lambda x)\to 0$. Also $x\to\theta$ and $\lambda\to 0$ implies $g(\lambda x)\to 0$. The case $\lambda\to 0$ and x fixed implies that $g(\lambda x)\to 0$ follows from the following expressions.

$$g(\lambda x)=\left\{\sup_{k\geq 1}k^{-s}q\left(\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right), \text{ uniformly in } n,m\in N\right\},$$

$$g(\lambda x)=\left\{(|\lambda|\,r)^{pm/H}:\sup_{k\geq 1}k^{-s}q\left(\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right)r>0, \text{ uniformly in } n,m\in N\right\},$$

where $r=1/|\lambda|$. Hence $\Gamma_\pi(p,\sigma,q,s)$ is a paranormed space. This completes the proof.

Theorem 3.3. $\Gamma_\pi(p,\sigma,q,s)\cap\Lambda_\pi(p,\sigma,q,s)\subseteq\Gamma_\pi(p,\sigma,q,s)$.

The proof is easy, so omitted.

Theorem 3.4. $\Gamma_\pi(p,\sigma,q,s)\subset\Lambda_\pi(p,\sigma,q,s)$.

The proof is easy, so omitted.

Remark 3.1. Let q_1 and q_2 be two semi norms on X, we have

(i) $\Gamma_\pi(p,\sigma,q_1,s)\cap\Gamma_\pi(p,\sigma,q_2,s)\subseteq\Gamma_\pi(p,\sigma,q_1+q_2,s)$,

(ii) If q_1 is stronger than q_2, then $\Gamma_\pi(p,\sigma,q_1,s)\subseteq\Gamma_\pi(p,\sigma,q_2,s)$,

(iii) If q_1 is equivalent to q_2, then $\Gamma_\pi(p,\sigma,q_1,s)=\Gamma_\pi(p,\sigma,q_2,s)$.

Theorem 3.5.

(i) Let $0\leq p_k\leq r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\Gamma_\pi(r,\sigma,q,s)\subset\Gamma_\pi(p,\sigma,q,s)$,

(ii) $s_1\leq s_2$ implies $\Gamma_\pi(p,\sigma,q,s_1)\subset\Gamma_\pi(p,\sigma,q,s_2)$.

Proof. (i) Let

$$x\in\Gamma_\pi(r,\sigma,q,s), \tag{3}$$

$$k^{-s}\left\{q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}\to 0 \text{ as } k\to\infty. \tag{4}$$

Let $t_k = k^{-s}\left\{q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda > \lambda_k$. Define $u_k = t_k$ $(t_k \ge 1)$, $u_k = 0$ $(t_k < 1)$ and $v_k = 0$ $(t_k \ge 1)$, $v_k = t_k$ $(t_k < 1)$, $t_k = u_k + v_k, t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$u_k^{\lambda_k} \le t_k \quad \text{and} \quad v_k^{\lambda_k} \le v_k^{\lambda}, \tag{5}$$

i.e. $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$ by (5).

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}\right)^{\lambda_k} \le k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k},$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}\right)^{p_k/r_k} \le k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k},$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \le k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k}.$$

$$\text{But } k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k} \to 0 \text{ as } k \to \infty \text{ by (4)},$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \to 0 \text{ as } k \to \infty.$$

Hence

$$x \in \Gamma_\pi(p, \sigma, q, s). \tag{6}$$

From (3) and (6) we get $\Gamma_\pi(r, \sigma, q, s) \subset \Gamma_\pi(p, \sigma, q, s)$. This completes the proof.

(ii) The proof is easy, so omitted.

Theorem 3.6. The space $\Gamma_\pi(p, \sigma, q, s)$ is solid and as such is monotone.

Proof. Let $\left(\frac{x_k}{\pi_k}\right) \in \Gamma_\pi(p, \sigma, q, s)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in N$. Then

$$k^{-s}\left(q\left\{\left|\frac{\alpha_k x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \le k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \text{ for all } k \in N,$$

$$\left(q\left\{\left|\frac{\alpha_k x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \le \left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \text{ for all } k \in N.$$

This completes the proof.

Theorem 3.7. The space $\Gamma_\pi(p, \sigma, q, s)$ are not convergence free in general.

The proof follows from the following example.

Example 3.1. Let $s = 0$; $p_k = 1$ for k even and $p_k = 2$ for k odd. Let $X = C$, $q(x) = |x|$ and $\sigma(n) = n+1$ for all $n \in N$. Then we have $\sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2$ and $\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3$. Therefore, $\sigma^k(n) = (n+k)$ for all $n, k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = (1/k)^k \pi_k$ and $y_k = k^k \pi_k$ for all $k \in N$, i.e. $\left|\frac{x_k}{\pi_k}\right|^{1/k} = 1/k$ and $\left|\frac{y_k}{\pi_k}\right|^{1/k} = k$, for all $k \in N$.

Hence, $\left|\left(\frac{1}{n+k}\right)^{n+k}\right|^{p_k} \to 0$ as $k \to \infty$. Therefore $\left(\frac{x_k}{\pi_k}\right) \in \Gamma_\pi(p,\sigma)$. But $\left|\left(\frac{1}{n+k}\right)^{n+k}\right|^{p_k} r \not\to 0$ as $k \to \infty$. Hence $\left(\frac{y_k}{\pi_k}\right) \notin \Gamma_\pi(p,\sigma)$. Hence the space $\Gamma_\pi(p,\sigma,q,s)$ are not convergence free in general. This completes the proof.

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Timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-space R_1^3

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Abstract In this paper, we introduce the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-space R_1^3 and study their relations between distribution parameters, shape operators, Gaussian curvatures, mean curvatures and q^{th} fundamental forms. Also, an example related to the timelike parallel p_3-equidistant ruled surfaces is given.

Keywords Parallel p_i-equidistant ruled surface, Minkowski space, ruled surface.

§1. Introduction

The kinematic geometry of the infinitesimal positions of a rigid body in spatial motions is not only important, but interesting as well. In a spatial motion, the trajectory of the oriented lines and points embedded in a moving rigid body are generally ruled surfaces and curves, respectively. Thus the spatial geometry of ruled surfaces and curves is important in the study of rational design problems in spatial mechanisms.

A. T. Yang applied some characteristic invariants of ruled surfaces to mechanism theory[14]. In classical differential geometry, timelike ruled surfaces and their distribution parameters in the Minkowski 3-space have been studied extensively [10,11]. Uğurlu studied the geometry of timelike surfaces [12]. In [8], Özyılmaz and Yaylı showed integral invariants of timelike ruled surfaces. On the other hand, M. Tosun at all introduced scalar normal curvature of 2-dimensional timelike ruled surfaces [9].

In 1986, Valeontis described the parallel p-equidistant ruled surfaces in the Euclidean 3-space [13]. Then, Masal and Kuruoğlu defined spacelike parallel p_i-equidistant ruled surfaces and timelike parallel p_i-equidistant ruled surfaces (with a timelike base curve) in the Minkowski 3-space R_1^3 and applied their the shape operators, curvatures [5,6].

This paper is organized as follow: in Section 3 firstly, we define timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-space R_1^3. Later, curvatures, dralls, matrices of shape operators, Gaussian curvatures, mean curvatures and q-th fundamental forms of these surfaces and some relations between them have been found. Finally, an example for the timelike parallel p_3-equidistant ruled surfaces with a spacelike base curve has been given.

§2. Preliminaries

Let R_1^3 be denotes the three-dimensional Minkowski space, i.e. three dimensional vector space R^3 equipped with the flat metric $g = dx_1^2 - dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is rectangular coordinate system of R_1^3 since g is indefinite metric, recall that a vector v in R_1^3 can have one of three casual characters: It can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) > 0$ and null $g(v, v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. v is a unit vector if $g(v, v) = \pm 1$. Furthermore, vectors v and w are said to be orthogonal if $g(v, w) = 0$ [7].

For any vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in R_1^3$, the Lorentzian product $v \wedge w$ of v and w is defined as [1].

$$v \wedge w = (v_3w_2 - v_2w_3, v_3w_1 - v_1w_3, v_2w_1 - v_1w_2).$$

A regular curve $\alpha : I \to R_1^3$, $I \subset R$ in R_1^3 is said to be spacelike, timelike and null curve if the velocity vector $\alpha'(t)$ is a spacelike, timelike and null vector, respectively [3].

Let M be a semi-Riemannian hypersurface in R_1^3, D and N represent Levi-Civita connection and unit normal vector field of M, respectively. For all $X \in \chi(M)$ the transformation

$$S(X) = -D_X N \tag{1}$$

is called a shape operator of M, where $\chi(M)$ is the space of vector fields of M. Then the function is defined as

$$II(X, Y) = \varepsilon g(S(X), Y)N, \quad \text{for all } X,\ Y \in \chi(M), \tag{2}$$

bilinear and symmetric. II is called the shape tensor (or second fundamental form tensor) of M, where $\varepsilon = \langle N, N\rangle$ [7]. Let $S(P)$ be a shape operator of M at point P. Then $K : M \to R$, $K(P) = \det S(P)$ function is called the Gaussian curvature function of M. In this case the value of $K(P)$ is defined to be the Gaussian curvature of M at the point P. Similarly, the function $H : M \to R$, $H(P) = \frac{traceS(P)}{\dim M}$ is called the mean curvature of M at point P.

Let us suppose that α be a curve in M. If

$$S(T) = \lambda T, \tag{3}$$

then the curve α is named curvature line (principal curve) in M, where T is the tangential vector field of α and λ is scalar being not equal to zero. If the following equation holds

$$g(S(T), T) = 0, \tag{4}$$

then α is called a asymptotic curve. If α is a geodesic curve in M, then we have

$$D_T T = 0. \tag{5}$$

For $X_P, Y_P \in T_M(P)$, if $II(X_P, Y_P) = 0$, then X_P, Y_P are called the conjugate vectors. If $II(X_P, X_P) = 0$, then X_P is called the asymptotic direction. The fundamental form I^q, $1 \leq q \leq 3$, on M such that

$$I^q(X, Y) = g\left(S^{q-1}(X), Y\right) \text{ for all } X,\ Y \in \chi(M), \tag{6}$$

is called the q^{th} fundamental form of M. If $P_S(\lambda)$ is the characteristic polynomial of the shape operator of M, then we have

$$P_S(\lambda) = \det(\lambda I - S), \tag{7}$$

where I is an unit matrice and λ is a scalar. If the induced metric on M is Lorentz metric, then M is called the time like surface.

Lemma 2.1. A surface in the 3-dimensional Minkowski space R_1^3 is a timelike surface if and only if a normal vector field of surface is a spacelike vector field [2].

The family of lines with one parameter in R_1^3 is called the ruled surface and each of these lines of this family is named as the rulings of the ruled surface. Thus, the parametrization of the ruled surface is given by $\varphi(t, v) = \alpha(t) + vX(t)$ where α and X are the base curve and unit vector in the direction of the rulings of the ruled surface, respectively. For the striction curve of ruled surface $\varphi(t, v)$, we can write

$$\overline{\alpha} = \alpha - \frac{g(\alpha', X')}{g(X', X')} X. \tag{8}$$

For the drall (distribution parameter) of the ruled surface $\varphi(t, v)$, we can write

$$P_X = -\frac{\det(\alpha', X, X')}{g(X', X')},\ g(X', X') \neq 0. \tag{9}$$

§3. Timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-Space R_1^3

Let $\alpha : I \to R_1^3$, $\alpha(t) = (\alpha_1(t),\ \alpha_2(t),\ \alpha_3(t))$ be a differentiable spacelike curve parameterized by arc-length in the Minkowski 3-space, where I is an open interval in R containing the origin. The tangent vector field of α is denoted by V_1. Let D be the Levi-Civita connection on R_1^3 and $D_{V_1} V_1$ be a timelike vector. If V_1 moves along α, then a timelike ruled surface which is given by the parameterization

$$\varphi(t, v) = \alpha(t) + vV_1(t) \tag{10}$$

can be obtained in the Minkowski 3-space. The timelike ruled surface with a spacelike base curve is denoted by M. $\{V_1, V_2, V_3\}$ is an orthonormal frame field along α in R_1^3, where V_2 is a timelike vector and V_3 is a spacelike vector. If k_1 and k_2 are the natural curvature and torsion of $\alpha(t)$, respectively, then for α the Frenet formulas are given by

$$V_1' = k_1 V_2, \quad V_2' = k_1 V_1 + k_2 V_3, \quad V_3' = k_2 V_2, \tag{11}$$

where "$'$" means derivative with respect to time t. Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have $k_1 = \|\alpha''\| > 0$.

For the timelike ruled surface M given with the parametrization (10), we see

$$\varphi_t = V_1 + v k_1 V_2, \quad \varphi_v = V_1, \quad \varphi_t \wedge \varphi_v = v k_1 V_3.$$

It is obvious that $\varphi_t \wedge \varphi_v \in \chi^{\perp}(M)$. This means that M is really a timelike ruled surface. Therefore we can say that $v \in R$.

The planes corresponding to subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$ and $Sp\{V_3, V_1\}$ along striction curves of timelike ruled surface M are called asymptotic plane, polar plane and central plane, respectively.

Let us suppose that $\alpha^* = \alpha^*(t^*)$ is another differentiable spacelike curve with arc-length and $\{V_1^*, V_2^*, V_3^*\}$ is Frenet frame of this curve in three dimensional Minkowski space R_1^3. Hence, we define timelike ruled surface M^* parametrically as follows

$$\varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1^*(t^*), \quad (t^*, v^*) \in I \times R.$$

Definition 3.1. Let M and M^* be two timelike ruled surfaces with a spacelike base curve with the generators V_1 of M and V_1^* of M^* and p_1, p_2 and p_3 be the distances between the polar planes, central planes and asymptotic planes, respectively. If

(i) the generator vectors of M and M^* are parallel,

(ii) the distances p_i, $1 \leq i \leq 3$ are constant,

then the pair of ruled surfaces M and M^* are called the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3. If $p_i = 0$, then the pair of M and M^* are called the timelike parallel p_i-equivalent ruled surfaces with a spacelike base curve, where the base curves of ruled surfaces M and M^* are the class of C^2. Therefore, the pair of timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve are defined parametrically as

$$\begin{aligned} M &: \varphi(t.v) = \alpha(t) + v V_1(t),\ (t, v) \in I \times R, \\ M^* &: \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*),\ (t^*, v^*) \in I \times R, \end{aligned} \tag{12}$$

where t and t^* are arc-length parameters of curves α and α^*, respectively.
Throughout this paper, M and M^* will be used for the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve. If the striction curve $\gamma = \gamma(t)$ is a base curve of M, then the base curve α^* of M^* can be written as

$$\alpha^* = \gamma + p_1 V_1 + p_2 V_2 + p_3 V_3, \tag{13}$$

where $p_1(t)$, $p_2(t)$, $p_3(t)\,(t \in I)$, is the class of C^2. If γ^* is a striction curve of M^*, then from (8), (11) and (12), it's seen that

$$\gamma^* = \gamma + \left(\frac{p_3 k_2 + p_2'}{-k_1}\right) V_1 + p_2 V_2 + p_3 V_3. \tag{14}$$

Now we consider the Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ of ruled surfaces M and M^*. From definition 3.1 it is obvious that $V_1^*(t^*) = V_1(t)$. Furthermore, from $\frac{dV_i}{dt} = \frac{dV_i^*}{dt^*}\frac{dt^*}{dt}$, $1 \leq i \leq 3$, and the equation (11), we can find $V_2^*(t^*) = V_2(t)$ and $V_3^*(t^*) = V_3(t)$, for $\frac{dt^*}{dt} > 0$. If k_1 and k_1^* are the natural curvatures of base curves of M and M^* and k_2 and k_2^* are the torsions of base curves of M and M^*, then from the Frenet formulas we have $k_i^* = k_i \frac{dt}{dt^*}$, $\quad 1 \leq i \leq 2$. Hence the following Theorem comes into existence.

Theorem 3.1. Let M and M^* be timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3.

(i) The Frenet vectors of timelike parallel p_i-equidistant ruled surfaces M and M^* at the points $\alpha(t)$ and $\alpha^*(t^*)$ are equivalent for $\frac{dt^*}{dt} > 0$.

(ii) If k_1 and k_1^* are the natural curvatures of base curves of M and M^* and k_2 and k_2^* are the torsions of base curves of M and M^*, then we have $k_i^* = k_i \frac{dt}{dt^*}$, $1 \leq i \leq 2$.

(iii) For the distance between the polar planes of the timelike parallel p_i-equidistant ruled surfaces (or the timelike parallel p_i-equivalent ruled surfaces) with a spacelike base curve can be given $p_1 = \frac{p_3 k_2 + p_2'}{-k_1} =$ constant (or $p_1 = \frac{p_3 k_2 + p_2'}{-k_1} = 0$).

(iv) The base curves of M and M^* are the striction curves.

(v) The striction curve of M is an inclined curve if and only if the striction curve of M^* is a inclined curve.

If P_{V_i} and $P_{V_i^*}$ are the i-th dralls of the Frenet vectors at the corresponding points of the base curves of timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve M and M^*, respectively. From (2.9), we find

$$P_{V_1} = 0, \quad P_{V_2} = -\frac{k_2}{k_1^2 + k_2^2}, \quad P_{V_3} = -\frac{1}{k_2}, \quad P_{V_1^*} = 0, \quad P_{V_2^*} = -\frac{k_2^*}{{k_1^*}^2 + {k_2^*}^2}, \quad P_{V_3^*} = -\frac{1}{k_2^*}.$$

So, from theorem 3.1 we obtain

$$P_{V_i^*} = P_{V_i}\frac{dt^*}{dt}, \quad 1 \leq i \leq 3.$$

Hence the following theorem comes into existence.

Theorem 3.2. Let M and M^* be timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3. For i-th dralls of M and M^*, respectively, then we have that $P_{V_i^*} = P_{V_i}\frac{dt^*}{dt}$, $\quad 1 \leq i \leq 3$. Here, we'll study the matrices S and S^* of the shape operators of timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve. From equation (12), we write $\varphi_t = V_1 + vk_1V_2$, $\quad \varphi_v = V_1$. It is clear that $g(\varphi_t, \varphi_v) \neq 0$. From Gram-Schmidt method, we obtain

$$X = \varphi_v = V_1, \quad Y = \varphi_t - \varphi_v = vk_1V_2, \tag{15}$$

where $X, Y \in \chi(M)$ form an orthogonal basis $\{X(\alpha(t)), Y(\alpha(t))\}$ of a tangent space at each point $\alpha(t)$ of M. So, normal vector field and unit normal vector field of M are

$$N = X \wedge Y = -vk_1V_3, \tag{16}$$

and

$$N_0 = \frac{N}{\|N\|} = \begin{cases} -V_3, & \text{for} \quad v > 0; \\ V_3, & \text{for} \quad v < 0, \end{cases} \tag{17}$$

respectively. Similarly, from equation (12), we find

$$X^* = V_1^*, \quad Y^* = v^* k_1^* V_2^*, \tag{18}$$

where $X^*,\ Y^* \in \chi(M^*)$ form an orthogonal basis $\{X^*(\alpha^*(t^*)), Y^*(\alpha^*(t^*))\}$ of a tangent space at each point $\alpha^*(t^*)$ of M^*. We can write the unit normal vector field of M^* as

$$N_0^* = \begin{cases} -V_3^*, & \text{for} \quad v^* > 0, \\ V_3^*, & \text{for} \quad v^* < 0. \end{cases} \tag{19}$$

The shape operator S of M can be written as $S(X) = aX + bY,\ \ S(Y) = cX + dY.$ Therefore, the matrix corresponding to the shape operator is

$$S = \begin{bmatrix} \frac{g(S(X),X)}{g(X,X)} & \frac{g(S(X),Y)}{g(Y,Y)} \\ \frac{g(S(Y),X)}{g(X,X)} & \frac{g(S(Y),Y)}{g(Y,Y)} \end{bmatrix}. \tag{20}$$

From equation (17), there are two special cases for shape operator S ($v > 0$ and $v < 0$). First, let us suppose that $v > 0$. Considering the equations (15), (17), (20) and (1), we find

$$S = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2}{vk_1} \end{bmatrix}. \tag{21}$$

For $v < 0$, considering same equations, we obtain the following result

$$S = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k_2}{vk_1} \end{bmatrix}. \tag{22}$$

Similarly, the shape operator matrices S^* of ruled surface M^* are found to be

$$S^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2^*}{v^* k_1^*} \end{bmatrix}, \qquad (v^* > 0) \tag{23}$$

and

$$S^* = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k_2^*}{v^* k_1^*} \end{bmatrix}, \quad (v^* < 0) \tag{24}$$

From theorem 3.1, for $v = v^*$ we find the following result $S^* = S$. If H and H^* are the mean curvatures of M and M^*, then we obtain

$$H^* = H = \frac{tr\ S}{\dim\ M} = \begin{cases} \frac{k_2}{2vk_1}, & \text{for } v > 0, \\ -\frac{k_2}{2vk_1}, & \text{for } v < 0, \end{cases}$$

where $v = v^*$. From the definition of the principal curve and from the equation $S^* = S$, the principal curve in M is the principal curve in M^*, too. Similarly, from the definitions of the asymptotic curve and the geodesic curve and from the equation $S^* = S$ we say the asymptotic curve and geodesic curve in M is the asymptotic curve and geodesic curve in M^*, too. Hence we can give the following Theorem:

Theorem 3.3. Let M and M^* be timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3.

(i) If S and S^* are the matrices of the shape operators of M and M^*, respectively, then $S^* = S$.

(ii) If the Gaussian curvatures of M and M^* are K and K^*, respectively, the following formula can be obtained: $K^* = K$.

(iii) If the mean curvatures of M and M^* are H and H^*, respectively, then we obtain $H^* = H$.

(iv) The geodesic curves in M are the geodesic curves in M^*, too.

(v) The principal curve (line of curvature) in M is the principal curve in M^*, too.

(vi) The asymptotic curve in M is the asymptotic curve in M^*, too.

Let II and II^* are the shape tensors of M and M^*, respectively. From the equation (2) and from the theorem 3.3, we find $II^*(X,Y) = II(X,Y)$, where $X,\ Y \in \chi(M)$ and $X,\ Y \in \chi(M^*)$, $v = v^*$. From the definitions of the conjugate vectors, the asymptotic directions and the equation $II^*(X,Y) = II(X,Y)$, we say the conjugate vectors and asymptotic directions in M are also the conjugate vectors and asymptotic directions in M^*. Hence the following theorem can be given:

Theorem 3.4. Let M and M^* be timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3.

(i) If II and II^* are the shape tensors of M and M^*, respectively, then we have $II^* = II$.

(ii) The conjugate vectors in M are also the conjugate vectors in M^*.

(iii) Asymptotic directions in M are also the asymptotic directions in M^*.

Let I^q and I^{*q} are the q-th fundamental forms of M and M^*, respectively. From the definition of the fundamental form and the theorem 3.3, we find $I^{*q}(X,Y) = I^q(X,Y), \quad 1 \leq q \leq 3$ where $X,\ Y \in \chi(M)$ and $X,\ Y \in \chi(M^*)$, $v = v^*$.

Let $P_S(\lambda)$ and $P_{S^*}(\lambda)$ be the characteristic polynomials of the shape operators of M and M^*, respectively. From the equation (7) and theorem 3.3, we have $P_{S^*}(\lambda) = P_S(\lambda)$ where $v = v^*$. Hence the following theorem can be given:

Theorem 3.5. Let M and M^* be timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in R_1^3.

(i) If I^q and I^{*q} are the q^{th} fundamental forms of M and M^*, respectively, then the relation between the fundamental forms is found as follows $I^{*q} = I^q, \quad 1 \leq q \leq 3$.

(ii) If $P_S(\lambda)$ and $P_{S^*}(\lambda)$ are the characteristic polynomials of the shape operators of M and M^*, respectively, then we obtain $P_{S^*}(\lambda) = P_S(\lambda)$.

Example 3.1. M and M^* be timelike parallel p_3-equidistant ruled surfaces in three dimensional Minkowski space R_1^3 defined by the following parametric equations,

$$M : \varphi(t,v) = (\sinh t + v\cosh t, \cosh t + v\sinh t, 1)$$

and

$$M^* : \varphi^*(t^*,v^*) = (2\sinh t^* + v^*\cosh t^*, 2\cosh t^* + v^*\sinh t^*, 3)$$

where the curves $\alpha(t) = (\sinh t, \cosh t, 1)$ and $\alpha^*(t^*) = (2\sinh t^*, 2\cosh t^*, 3)$ are spacelike base curves of M and M^*, respectively, (Figure 1).

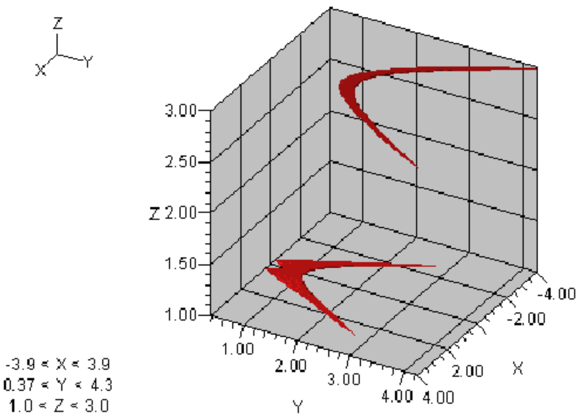

Figure 1

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Basically Disconnectedness In Soft L-Fuzzy $\mathcal{V}$ Spaces With Reference to Soft L-Fuzzy $B\mathcal{V}$ Open Set

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Abstract In this paper, the new concepts of soft L-fuzzy topological space and soft L-fuzzy $\mathcal{V}$ space are introduced. In this connection, the concept of soft L-fuzzy $B\mathcal{V}$ basically disconnected space is studied. Besides giving some interesting properties, some characterizations are studied. Tietze extension theorem for a soft L-fuzzy $B\mathcal{V}$ basically disconnected space is established.

Keywords Soft L-fuzzy $B\mathcal{V}$ basically disconnected space, lower (upper) soft L-fuzzy $B\mathcal{V}$ con -tinuous function, strongly soft L-fuzzy $B\mathcal{V}$ continuous function.

2000 Mathematics subject classification: 54A40, 03E72.

§1. Introduction and preliminaries

The concept of fuzzy set was introduced by Zadeh [12]. Fuzzy sets have applications in many fields such as information [6] and control [5]. The thoery of fuzzy topological spaces was introduced and developed by Chang [3] and since then various notions in classical topology has been extended to fuzzy topological spaces. The concept of fuzzy basically disconnected space was introduced and studied in [7]. The concept of L-fuzzy normal spaces and Tietze extension theorem was introduced and studied in [10]. The concept of soft fuzzy topological space was introduced by Ismail U. Triyaki [8]. J. Tong [9] introduced the concept of B-set in topological space. The concept of fuzzy B-set was introduced by M.K. Uma, E. Roja and G. Balasubramanian [12]. In this paper, the new concepts of soft L-fuzzy topological space and soft L-fuzzy $\mathcal{V}$ space are introduced. In this connection, the concept of soft L-fuzzy $B\mathcal{V}$ basically disconnected space is studied. Besides giving some interesting properties, some characterizations are studied. Tietze extension theorem for a soft L-fuzzy $B\mathcal{V}$ basically disconnected space is established.

Definition 1.1. Let (X, T) be a topological space on fuzzy sets. A fuzzy set λ of (X, T) is said to be

(i) fuzzy t-set if $int\lambda = intcl\lambda$,

(ii) fuzzy B-set if $\lambda = \mu \wedge \gamma$ where μ is fuzzy open and γ is a fuzzy t-set.

Lemma 1.1. For a fuzzy set λ of a fuzzy space X.

(i)$1 - int\lambda = cl(1-\lambda)$,

(ii)$1 - cl\lambda = int(1-\lambda)$.

Definition 1.2. Let X be a non-empty set. A soft fuzzy set(in short, SFS) A have the form $A = (\lambda, M)$ where the function $\lambda : X \to I$ denotes the degree of membership and M is the subset of X. The set of all soft fuzzy set will be denoted by SF(X).

Definition 1.3. The relation $\sqsubseteq$ on $SF(X)$ is given by $(\mu, N) \sqsubseteq (\lambda, M) \Leftrightarrow \mu(x) \leq \lambda(x), \forall x \in X$ and $M \subseteq N$.

Proposition 1.1. If $(\mu_j, N_j) \in SF(X), j \in J$, then the family $\{(\mu_j, N_j)|j \in J\}$ has a meet, i.e. g.l.b. in $(SF(X), \sqsubseteq)$ denoted by $\sqcap_{j\in J}(\mu_j, N_j)$ and given by $\sqcap_{j\in J}(\mu_j, N_j) = (\mu, N)$ where

$$\mu(x) = \wedge_{j\in J}\mu_j(x)\forall x \in X$$

and

$$M = \cap M_j \text{ for } j \in J.$$

Proposition 1.2. If $(\mu_j, N_j) \in SF(X)$, $j \in J$, then the family $\{(\mu_j, N_j)|j \in J\}$ has a join, i.e. l.u.b. in $(SF(X), \sqsubseteq)$ denoted by $\sqcup_{j\in J}(\mu_j, N_j)$ and given by $\sqcup_{j\in J}(\mu_j, N_j) = (\mu, N)$ where

$$\mu(x) = \vee_{j\in J}\mu_j(x), \forall x \in X$$

and

$$M = \cup M_j \text{ for } j \in J.$$

Definition 1.4. Let X be a non-empty set and the soft fuzzy sets A and C are in the form $A = (\lambda, M)$ and $C = (\mu, N)$. Then

(i) $A \sqsubseteq C$ if and only if $\lambda(x) \leq \mu(x)$ and $M \subseteq N$ for $x \in X$,

(ii) $A = C$ if and only if $A \sqsubseteq C$ and $C \sqsubseteq A$,

(iii) $A \sqcap C = (\lambda, M) \sqcap (\mu, N) = (\lambda(x) \wedge \mu(x), M \cap N)$ for all $x \in X$,

(iv) $A \sqcup C = (\lambda, M) \sqcup (\mu, N) = (\lambda(x) \vee \mu(x), M \cup N)$ for all $x \in X$.

Definition 1.5. For $(\mu, N) \in SF(X)$ the soft fuzzy set $(\mu, N)' = (1-\mu, X \setminus N)$ is called the complement of (μ, N).

Remark 1.1. $(1-\mu, X/N) = (1, X) - (\mu, N)$.

Proof. $(1, X) - (\mu, N) = (1, X) \sqcap (\mu, N)' = (1, X) \sqcap (1-\mu, X/N) = (1-\mu, X/N)$.

Definition 1.6. Let S be a set. A set $T \subseteq SF(X)$ is called an SF-topology on X if

SFT1 $(0, \emptyset) \in T$ and $(1, X) \in T$,

SFT2 $(\mu_j, N_j) \in T, j = 1, 2, \cdots, n \Rightarrow \sqcap_{j=1}^{n}(\mu_j, N_j) \in T$,

SFT3 $(\mu_j, N_j) \in T, j \in J \Rightarrow \sqcup_{j\in J}(\mu_j, N_j) \in T$.

As usual, the elements of T are called open, and those of $T' = \{(\mu, N)|(\mu, N)' \in T\}$ closed. If T is an SF-topology on X we call the pair (X, T) an SF-topological space(in short, $SFTS$).

Definition 1.7. The closure of a soft fuzzy set (μ, N) will be denoted by $\overline{(\mu, N)}$. It is given by

$$\overline{(\mu, N)} = \sqcap\{(\nu, L)|(\mu, N) \sqsubseteq (\nu, L) \in T'\}.$$

Likewise the interior is given by

$$(\mu, N)^\circ = \sqcup\{(\nu, L)|(\nu, L) \in T, (\nu, L) \sqsubseteq (\mu, N)\}.$$

Note 1.1. (i) The soft fuzzy closure $\overline{(\mu, N)}$ is denoted by $SFcl(\mu, N)$.

(ii) The soft fuzzy interior $(\mu, N)^{\circ}$ is denoted by $SFint(\mu, N)$.

Proposition 1.3. Let $\varphi : X \to Y$ be a point function.

(i) The mapping $\varphi^{\rightharpoonup}$ from $SF(X)$ to $SF(Y)$ corresponding to the image operator of the difunction (f, F) is given by

$$\varphi^{\rightharpoonup}(\mu, N) = (\nu, L) \text{ where } \nu(y) = sup\{\mu(x)|y = \varphi(x)\}, \text{ and}$$

$$L = \{\varphi(x)|x \in N \text{ and } \nu(\varphi(x)) = \mu(x)\}.$$

(ii) The mapping $\varphi^{\leftharpoonup}$ from $SF(X)$ to $SF(Y)$ corresponding to the inverse image of the difunction (f, F) is given by

$$\varphi^{\leftharpoonup}(\nu, L) = (\nu \circ \varphi, \varphi^{-1}[L]).$$

Definition 1.8. Let (X, T) be a fuzzy topological space and let λ be a fuzzy set in (X, T). λ is called fuzzy G_δ if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$ where each $\lambda_i \in T$, $i \in I$.

Definition 1.9. Let (X, T) be a fuzzy topological space and let λ be a fuzzy set in (X, T). λ is called fuzzy F_σ if $\lambda = \vee_{i=1}^{\infty}\lambda_i$ where each $\bar{\lambda}_i \in T$, $i \in I$.

Definition 1.10. Let (X, T) be any fuzzy topological space. (X, T) is called fuzzy basically disconnected if the closure of every fuzzy open F_σ set is fuzzy open.

Definition 1.11. An intutionistic fuzzy set U of an intutionistic fuzzy topological space (X, T) is said to be an intutionistic fuzzy compact relative to X if for every family $\{U_j : j \in J\}$ of intutionistic fuzzy open sets in X such that $U \subseteq \cup_{j\in J}U_j$, there is a finite subfamily $\{U_j : j = 1, 2, \cdots, n\}$ of intutionistic fuzzy open sets such that $U \subseteq \cup_{j=1}^{n}U_j$.

Definition 1.12. The L-fuzzy real line $\mathbb{R}(L)$ is the set of all monotone dsecreasing elements $\lambda \in L^{\mathbb{R}}$ satisfying $\vee\{\lambda(t) : t \in \mathbb{R}\} = 1$ and $\wedge\{\lambda(t) : t \in \mathbb{R}\} = 0$, after the identification of $\lambda, \mu \in L^{\mathbb{R}}$ iff $\lambda(t+) = \wedge\{\lambda(s) : s < t\}$ and $\lambda(t-) = \vee\{\lambda(s) : s > t\}$. The natural L-fuzzy topology on $\mathbb{R}(L)$ is generated from the basis $\{L_t, R_t : t \in \mathbb{R}\}$, where $L_t[\lambda] = \lambda(t-)'$ and $R_t[\lambda] = \lambda(t+)$. A partial order on $\mathbb{R}(L)$ is defined by $[\lambda] \le [\mu]$ iff $\lambda(t-) \le \mu(t-)$ and $\lambda(t+) \le \mu(t+)$ for all $t \in \mathbb{R}$.

Definition 1.13. The L-fuzzy unit interval $I(L)$ is a subset of $\mathbb{R}(L)$ such that $[\lambda] \in I(L)$ if $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$. It is equipped with the subspace L-fuzzy topology.

§2. Soft L-fuzzy topological space

In this paper, $(L, \sqsubseteq, ')$ stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. A soft L-fuzzy set in X is an element of the set $L \times L$ of all functions from X to $L \times L$ i.e. $(\lambda, M) : X \to L \times L$ be such that $(\lambda, M)(x) = (\lambda(x), M(x)) = (\lambda(x), \chi_M(x))$ for all $x \in X$.

A soft L-fuzzy topology on X is a subset T of $L \times L$ such that

(i) $(0_X, 0_X), (1_X, 1_X) \in T$,

(ii) $(\mu_j, N_j) \in T$, $j = 1, 2, \cdots, n \Rightarrow \sqcap_{j=1}^{n}(\mu_j, N_j) \in T$,

(iii) $(\mu_j, N_j) \in T$, $j \in J \Rightarrow \sqcap_{j\in J}(\mu_j, N_j) \in T$.

A set X with a soft L-fuzzy topology on it is called a soft L-fuzzy topological space. The members of T are called the soft L-fuzzy open sets in the soft L-fuzzy topological space.

A soft L-fuzzy set (λ, M) in X is called a soft L-fuzzy closed if $(\lambda, M)'$ is the soft L-fuzzy open where $(\lambda, M)' = (1 - \lambda, 1 - M) = (1_X, 1_X) - (\lambda, M)$.

If $(\lambda, M), (\mu, N) : X \to L \times L$, we define $(\lambda, M) \sqsubseteq (\mu, N) \Leftrightarrow \lambda(x) \leq \mu(x)$ and $M(x) \leq N(x)$ for all $x \in X$.

A function f from a soft L-fuzzy topological space X to a soft L-fuzzy topological space Y is called soft L-fuzzy continuous if $f^{-1}(\mu, N)$ is soft L-fuzzy open in (X, T), for each soft L-fuzzy open set in (Y, S).

If (X, T) is a soft L-fuzzy topological space and $A \subseteq X$ then (A, T_A) is a soft L-fuzzy topological space which is called a soft L-fuzzy subspace of (X, T) where

$$T_A = \{(\lambda, M)/A : (\lambda, M) \text{ is a soft } L\text{-fuzzy set in } X\}.$$

The soft L-fuzzy real line $\mathbb{R}(L \times L)$ is the set of all monotone decreasing soft L-fuzzy set $(\lambda, M) : \mathbb{R}(L \times L) \to L \times L$ satisfying

$$\sqcup\{(\lambda, M)(t)/t \in \mathbb{R}\} = \sqcup\{(\lambda, \chi_M)(t)/t \in \mathbb{R}\} = (1_X, 1_X),$$

$$\sqcap\{(\lambda, M)(t)/t \in \mathbb{R}\} = \sqcap\{(\lambda, \chi_M)(t)/t \in \mathbb{R}\} = (0_X, 0_X),$$

after the identification of $(\lambda, M), (\mu, N) : \mathbb{R}(L \times L) \to L \times L$ if for every $t \in \mathbb{R}$ iff

$$(\lambda, M)(t-) = (\mu, N)(t-),$$

and

$$(\lambda, M)(t+) = (\mu, N)(t+),$$

where $(\lambda, M)(t-) = \sqcap_{s<t}(\lambda, M)(s)$ and $(\lambda, M)(t+) = \sqcup_{s>t}(\lambda, M)(s)$. The natural soft L-fuzzy topology on $\mathbb{R}(L \times L)$ by taking a sub-basis $\{L_t, R_t/t \in \mathbb{R}\}$ where

$$L_t[\lambda, M] = (\lambda, M)(t-)', \;\; R_t[\lambda, M] = (\lambda, M)(t+).$$

This topology is called the soft L-fuzzy topology for $\mathbb{R}(L \times L)$. $\{L_t/t \in \mathbb{R}\}$ and $\{R_t/t \in \mathbb{R}\}$ are called the left and right hand soft L-fuzzy topology respectively.

A partial order on $\mathbb{R}(L \times L)$ is defined by $[\lambda, M] \sqsubseteq [\mu, N] \Leftrightarrow (\lambda, M)(t-) \sqsubseteq (\mu, N)(t-)$ and $(\lambda, M)(t+) \sqsubseteq (\mu, N)(t+)$ for all $t \in \mathbb{R}$. The soft L-fuzzy unit interval $I(L \times L)$ is a subset of $\mathbb{R}(L \times L)$ such that $[\lambda, M] \in I(L \times L)$ if

$$(\lambda, M)(t) = (1_X, 1_X) \text{ for } t < 0,$$

and

$$(\lambda, M)(t) = (0_X, 0_X) \text{ for } t > 1.$$

It is equipped with the subspace soft L-fuzzy topology.

Definition 2.1. Let (X, T) be soft L-fuzzy topological space. For any soft L-fuzzy set (λ, M) on X, the soft L-fuzzy closure of (λ, M) and the soft L-fuzzy interior of (λ, M) are defined as follows:

$$SLFcl(\lambda, M) = \sqcap\{(\mu, N) : (\lambda, M) \sqsubseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy closed set in } X\},$$

$$SLFint(\lambda, M) = \sqcup\{(\mu, N) : (\lambda, M) \sqsupseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy open set in } X\}.$$

Definition 2.2. Let T be a soft L-fuzzy topology on X. Then (X, T) is called soft L-fuzzy non-compact if $\sqcup_{i\in I}(\lambda_i, M_i) = (1, X)$, (λ_i, M_i) be soft L-fuzzy set in T, $i \in I$, there is a finite subset J of I with $\sqcup_{j\in J}(\lambda_j, M_j) \neq (1, X)$.

Definition 2.3. Let (X, T) be a soft fuzzy L-fuzzy topological space. Let (λ, M) be any soft L-fuzzy set. Then (λ, M) is said to be soft L-fuzzy compact set if every family $\{(\lambda_j, M_j) : j \in J\}$ of soft L-fuzzy open sets in X such that $(\lambda, M) \sqsubseteq \sqcup_{j\in J}(\lambda_j, M_j)$, there is a finite subfamily $i \in I$, there is a finite subfamily $\{(\lambda_j, M_j) : j = 1, 2, \cdots, n\}$ of soft L-fuzzy open sets such that $(\lambda, M) \sqsubseteq \sqcup_{j\in J}(\lambda_j, M_j)$.

Definition 2.4. Let (X, T) be a soft L-fuzzy topological space. Let (λ, M) be any soft L-fuzzy set. Then (λ, M) is said to be a soft L-fuzzy t-open set if $SLFint(\lambda, M) = SLFint(SLFcl(\lambda, M))$.

Definition 2.5. Let (X, T) be a soft L-fuzzy topological space. Let (λ, M) be any soft L-fuzzy set. Then (λ, M) is said to be a soft L-fuzzy B open set (in short, $SLFBOS$) if $(\lambda, M) = (\mu, N) \sqcap (\gamma, L))$ where (μ, N) is a soft L-fuzzy open set and (γ, L) is a soft L-fuzzy t-open set. The complement of soft L-fuzzy B-open set is a soft L-fuzzy B closed set (in short, $SLFBCS$).

§3. Soft L-fuzzy $B\mathcal{V}$ basically disconnected space

Definition 3.1. Let (X, T) be a soft L-fuzzy topological space and a soft L-fuzzy non-compact spaces. Let $\mathcal{C}$ be a collection of all soft L-fuzzy set which are both soft L-fuzzy closed and soft L-fuzzy compact sets in (X, T). Let

$$(\gamma, L)^- = \{(\lambda, M) \in \mathcal{C} : (\lambda, M) \sqcap (\gamma, L) \neq (0_X, 0_X), (\gamma, L) \text{ is a soft } L\text{-fuzzy open set }\},$$

$$(\delta, P)^+ = \{(\lambda, M) \in \mathcal{C} : (\lambda, M) \sqcap (\delta, P) = (0_X, 0_X),$$
$$(\delta, P) \text{ is a soft } L\text{-fuzzy compact set in } (X, T)\}.$$

Then the collection $\mathcal{V} = \{(\lambda, M) : (\lambda, M) \in (\gamma, L)^-\} \sqcup \{(\mu, N) : (\mu, N) \in (\delta, P)^+\}$ is said to be soft L-fuzzy $\mathcal{V}$ structure on (X, T) and the pair $(X, \mathcal{V})$ is said to be soft L-fuzzy $\mathcal{V}$ space.

Notation 3.1. Each member of soft L-fuzzy $\mathcal{V}$ space is a soft L-fuzzy $\mathcal{V}$open set. The complement of soft L-fuzzy $\mathcal{V}$open set is a soft L-fuzzy $\mathcal{V}$closed set.

Definition 3.2. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. For any soft L-fuzzy set (λ, M) on X, the soft L-fuzzy $\mathcal{V}$ closure of (λ, M) and the soft L-fuzzy $\mathcal{V}$ interior of (λ, M) are defined as follows:

$$SLF\mathcal{V}cl(\lambda, M) = \sqcap\{(\mu, N) : (\lambda, M) \sqsubseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy } \mathcal{V} \text{ closed set in } X\},$$

$$SLF\mathcal{V}int(\lambda, M) = \sqcup\{(\mu, N) : (\lambda, M) \sqsupseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy } \mathcal{V} \text{ open set in } X\}.$$

Definition 3.3. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Let (λ, M) be any soft L-fuzzy set in X. Then (λ, M) is said to be a soft L-fuzzy $t\mathcal{V}$ open set if $SLF\mathcal{V}int(\lambda, M) = SLF\mathcal{V}int(SLF\mathcal{V}cl(\lambda, M))$.

Definition 3.4. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Let (λ, M) be any soft L-fuzzy set in X. Then (λ, M) is said to be a soft L-fuzzy $B\mathcal{V}$ open set (in short, $SLFB\mathcal{V}OS$) if $(\lambda, M) = (\mu, N) \sqcap (\gamma, L))$ where (μ, N) is a soft L-fuzzy $\mathcal{V}$ open set and (γ, L) is a soft L-fuzzy $t\mathcal{V}$ open set. The complement of soft L-fuzzy $B\mathcal{V}$ open set is a soft L-fuzzy $B\mathcal{V}$ closed set (in short, $SLFB\mathcal{V}CS$).

Definition 3.5. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. A soft L-fuzzy set (λ, M) is said to be soft L-fuzzy $\mathcal{V}G_\delta$ set (in short, $SLF\mathcal{V}G_\delta$) if $(\lambda, M) = \sqcap_{i=1}^{\infty}(\lambda_i, M_i)$, where each $(\lambda_i, M_i) \in \mathcal{V}$. The complement of soft L-fuzzy $\mathcal{V}G_\delta$ set is said to be soft L-fuzzy $\mathcal{V}F_\sigma$ (in short, $SLF\mathcal{V}F_\sigma$) set.

Remark 3.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. For any soft L-fuzzy set (λ, M),

(i) which is both soft L-fuzzy $B\mathcal{V}$ open and soft L-fuzzy $\mathcal{V}F_\sigma$. Then (λ, M) is said to be soft L-fuzzy $B\mathcal{V}$ open F_σ (in short, $SLFB\mathcal{V}OF_\sigma$).

(ii) which is both soft L-fuzzy $B\mathcal{V}$ closed and soft L-fuzzy $\mathcal{V}G_\delta$. Then (λ, M) is said to be soft L-fuzzy $B\mathcal{V}$ closed G_δ (in short, $SLFB\mathcal{V}CG_\delta$).

(iii) which is both soft L-fuzzy $B\mathcal{V}$ open F_σ and soft L-fuzzy $B\mathcal{V}$ closed G_δ. Then (λ, M) is said to be soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ (in short, $SLFB\mathcal{V}COGF$).

Definition 3.6. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. For any soft L-fuzzy set (λ, M) in X, the soft L-fuzzy $B\mathcal{V}$ closure of (λ, M) and the soft L-fuzzy $B\mathcal{V}$ interior of (λ, M) are defined as follows:

$$SLFB\mathcal{V}cl(\lambda, M) = \sqcap\{(\mu, N) : (\lambda, M) \sqsubseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy } B\mathcal{V} \text{ closed }\},$$

$$SLFB\mathcal{V}int(\lambda, M) = \sqcup\{(\mu, N) : (\lambda, M) \sqsupseteq (\mu, N), (\mu, N) \text{ is a soft } L\text{-fuzzy } B\mathcal{V} \text{ open set }\}.$$

Proposition 3.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. For any soft L-fuzzy set (λ, M) in X, the following statements are valid.

(i) $SLFB\mathcal{V}int(\lambda, M) \sqsubseteq (\lambda, M) \sqsubseteq SLFB\mathcal{V}cl(\lambda, M)$,

(ii) $(SLFB\mathcal{V}int(\lambda, M))' = SLFB\mathcal{V}cl(\lambda, M)'$,

(iii) $(SLFB\mathcal{V}cl(\lambda, M))' = SLFB\mathcal{V}int(\lambda, M)'$.

Definition 3.7. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Then $(X, \mathcal{V})$ is said to be soft L-fuzzy $B\mathcal{V}$ basically disconnected if the soft L-fuzzy $B\mathcal{V}$ closure of every soft L-fuzzy $B\mathcal{V}$ open F_σ set is a soft L-fuzzy $B\mathcal{V}$ open set.

Proposition 3.2. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space, the following conditions are equivalent:

(i) $(X, \mathcal{V})$ is a soft L-fuzzy $B\mathcal{V}$ basically disconnected space,

(ii) For each soft L-fuzzy $B\mathcal{V}$ closed G_δ set (λ, M), $SLFB\mathcal{V}int(\lambda, M)$ is soft L-fuzzy $B\mathcal{V}$ closed,

(iii) For each soft L-fuzzy $B\mathcal{V}$ open F_σ set (λ, M),

$$SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}cl(\lambda, M))' = (1_X, 1_X),$$

(iv) For every pair of soft L-fuzzy $B\mathcal{V}$ open F_σ sets (λ, M) and (μ, N) with $SLFB\mathcal{V}$ $cl(\lambda, M) + (\mu, N) = (1_X, 1_X)$, we have $SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(\mu, N) = (1_X, 1_X)$.

Proof. (i)$\Rightarrow$(ii). Let (λ, M) be any soft L-fuzzy $B\mathcal{V}$ closed G_δ set in X. Then $(\lambda, M)'$ is soft L-fuzzy $B\mathcal{V}$ open F_σ. Now,

$$SLFB\mathcal{V}cl(\lambda, M)^{'} = (SLFB\mathcal{V}int(\lambda, M))^{'}.$$

By (i), $SLFB\mathcal{V}cl(\lambda, M)^{'}$ is soft L-fuzzy $B\mathcal{V}$ open. Then $SLFB\mathcal{V}int(\lambda, M)$ is soft L-fuzzy $B\mathcal{V}$ closed.

(ii)$\Rightarrow$(iii). Let (λ, M) be any soft L-fuzzy $B\mathcal{V}$ open F_σ set. Then

$$\begin{aligned} & SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}cl(\lambda, M))^{'} \quad (1)\\ = \ & SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}int(\lambda, M)^{'}. \end{aligned}$$

Since (λ, M) is a soft L-fuzzy $B\mathcal{V}$ open F_σ set. Now, $(\lambda, M)^{'}$ is a soft L-fuzzy $B\mathcal{V}$ closed G_δ set. Hence by (ii), $SLFB\mathcal{V}int(\lambda, M)^{'}$ is soft L-fuzzy $B\mathcal{V}$ closed. Therefore, by (1)

$$\begin{aligned} & SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}cl(\lambda, M))^{'}\\ = \ & SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}int(\lambda, M)^{'})\\ = \ & SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}int(\lambda, M)^{'}\\ = \ & SLFB\mathcal{V}cl(\lambda, M) + (SLFB\mathcal{V}cl(\lambda, M))^{'}\\ = \ & SLFB\mathcal{V}cl(\lambda, M) + (1_X, 1_X) - SLFB\mathcal{V}cl(\lambda, M)\\ = \ & (1_X, 1_X) \end{aligned}$$

Therefore, $SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}cl(\lambda, M))^{'} = (1_X, 1_X)$.

(iii)$\Rightarrow$(iv). Let (λ, M) and (μ, N) be soft L-fuzzy $B\mathcal{V}$ open F_σ sets with

$$SLFB\mathcal{V}cl(\lambda, M) + (\mu, N) = (1_X, 1_X). \quad (2)$$

By (iii),

$$\begin{aligned} (1_X, 1_X) &= SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(SLFB\mathcal{V}cl(\lambda, M))^{'}\\ &= SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl((1_X, 1_X) - SLFB\mathcal{V}cl(\lambda, M))\\ &= SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(\mu, N). \end{aligned}$$

Therefore, $SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(\mu, N) = (1_X, 1_X)$.

(iv)$\Rightarrow$(i). Let (λ, M) be a soft L-fuzzy $B\mathcal{V}$ open F_σ set. Put $(\mu, N) = (SLFB\mathcal{V}cl(\lambda, M))^{'} = (1_X, 1_X) - SLFBmathcalV\ cl(\lambda, M)$. Then $SLFB\mathcal{V}\ cl(\lambda, M) + (\mu, N) = (1_X, 1_X)$. Therefore by (iv), $SLFB\mathcal{V}cl(\lambda, M) + SLFB\mathcal{V}cl(\mu, N) = (1_X, 1_X)$. This implies that $SLFB\mathcal{V}cl(\lambda, M)$ is soft L-fuzzy $B\mathcal{V}$ open and so $(X, \mathcal{V})$ is soft L-fuzzy $B\mathcal{V}$ basically disconnected.

Proposition 3.3. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Then $(X, \mathcal{V})$ is soft L-fuzzy $B\mathcal{V}$ basically disconnected if and only if for all soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets (λ, M) and (μ, N) such that $(\lambda, M) \sqsubseteq (\mu, N)$, $SLFB\mathcal{V}cl(\lambda, M) \sqsubseteq SLFB\mathcal{V}\ int(\mu, N)$.

Proof. Let (λ, M) and (μ, N) be any soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets with $(\lambda, M) \sqsubseteq (\mu, N)$. By (ii) of Proposition 3.2, $SLFB\mathcal{V}int(\mu, N)$ is soft L-fuzzy $B\mathcal{V}$ closed. Since (λ, M) is soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$, $SLFB\mathcal{V}cl(\lambda, M) \sqsubseteq SLFB\mathcal{V}int(\mu, N)$.

Conversely, let (μ, N) be any soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ then $SLFB\mathcal{V}int(\mu, N)$ is soft L-fuzzy $B\mathcal{V}$ open F_σ in X and $SLFB\mathcal{V}int(\mu, N) \sqsubseteq (\mu, N)$. Therefore by assumption, $SLFB\mathcal{V}cl(SLFB\mathcal{V}int(\mu, N)) \sqsubseteq SLFB\mathcal{V}int(\mu, N)$. This implies that $SLFB\mathcal{V}int(\mu, N)$ is soft L-fuzzy $B\mathcal{V}$ closed G_δ. Hence by (ii) of Proposition 3.2, it follows that $(X, \mathcal{V})$ is soft L-fuzzy $B\mathcal{V}$ basically disconnected.

Remark 3.2. Let $(X, \mathcal{V})$ be a soft L-fuzzy $B\mathcal{V}$ basically disconnected space. Let $\{(\lambda_i, M_i), (\mu_i, N_i)'/i \in \mathbb{N}\}$ be collection such that $(\lambda_i, M_i)'s$ and $(\mu_i, N_i)'s$ are soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets and let (λ, M) and (μ, N) be soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets. If

$$(\lambda_i, M_i) \sqsubseteq (\lambda, M) \sqsubseteq (\mu_j, N_j) \text{ and } (\lambda_i, M_i) \sqsubseteq (\mu, N) \sqsubseteq (\mu_j, N_j),$$

for all $i, j \in \mathbb{N}$, then there exists a soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ set (γ, L) such that $SLFB\mathcal{V}cl\ (\lambda_i, M_i) \sqsubseteq (\gamma, L) \sqsubseteq SLFB\mathcal{V}int(\mu_j, N_j)$ for all $i,\ j \in \mathbb{N}$.

Proof. By Proposition 3.3, $SLFB\mathcal{V}\ cl(\lambda_i, M_i) \sqsubseteq SLFB\mathcal{V}cl(\lambda, M) \sqcap SLFB\mathcal{V}int(\mu, N) \sqsubseteq SLFB\mathcal{V}int(\mu_j, N_j)$ for all $i, j \in \mathbb{N}$. Therefore, $(\gamma, L) = SLFB\mathcal{V}cl(\lambda, M) \sqcap SLFB\mathcal{V}int(\mu,\ N)$ is a soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ set satisfying the required conditions.

Proposition 3.4. Let $(X, \mathcal{V})$ be a soft L-fuzzy $B\mathcal{V}$ basically disconnected space. Let $\{\lambda_l, M_l\}_{l \in Q}$ and $\{\mu_l, N_l\}_{l \in Q}$ be monotone increasing collections of soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$ and suppose that $(\lambda_{q_1}, M_{q_1}) \sqsubseteq (\mu_{q_2}, N_{q_2})$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{\gamma_l, L_l\}_{l \in Q}$ of soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$ such that $SLFB\mathcal{V}cl(\lambda_{q_1}, M_{q_1}) \sqsubseteq (\gamma_{q_2}, L_{q_2})$ and $(\gamma_{q_1}, L_{q_1}) \sqsubseteq SLFB\mathcal{V}int(\mu_{q_2}, N_{q_2})$ whenever $q_1 < q_2$.

Proof. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geq 2$, we shall define inductively a collection $\{(\gamma_{q_i}, L_{q_i})/1 \leq i \leq n\}$ is a subset of $L \times L$ in X such that $SLFB\mathcal{V}cl(\lambda_q, M_q) \sqsubseteq (\gamma_{q_i}, L_{q_i})$ if $q < q_i, (\gamma_{q_i}, L_{q_i}) \sqsubseteq SLFB\mathcal{V}int(\mu_q, N_q)$ if $q_i < q$, for all $i < n \cdots (S_n)$. By Proposition 3.3, the countable collections $\{SLFB\mathcal{V}cl(\lambda_q, M_q)\}$ and $\{SLFB\mathcal{V}int(\mu_q, N_q)\}$ satisfy $SLFB\mathcal{V}cl(\lambda_{q_1}, M_{q_1}) \sqsubseteq SLFB\mathcal{V}int(\mu_{q_2}, N_{q_2})$ if $q_1 < q_2$.

By Remark 3.2, there exists a soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ set (δ_1, P_1) such that

$$SLFB\mathcal{V}cl(\lambda_{q_1}, M_{q_1}) \sqsubseteq (\delta_1, P_1) \sqsubseteq SLFB\mathcal{V}int(\mu_{q_2}, N_{q_2}).$$

Let $(\gamma_{q_1}, L_{q_1}) = (\delta_1, P_1)$, we get (S_2).

Define

$$\Psi = \sqcup\{(\gamma_{q_i}, L_{q_i})/i < n, q_i < q_n\} \sqcup \{(\lambda_{q_n}, M_{q_n})\},$$

and

$$\Phi = \sqcap\{(\gamma_{q_j}, L_{q_j})/j < n, q_j > q_n\} \sqcap \{(\mu_{q_n}, N_{q_n})\}.$$

Then

$$SLFB\mathcal{V}cl(\gamma_{q_i}, L_{q_i}) \sqsubseteq SLFB\mathcal{V}cl(\Psi) \sqsubseteq SLFB\mathcal{V}int(\gamma_{q_j}, L_{q_j}),$$

and

$$SLFB\mathcal{V}cl(\gamma_{q_i}, L_{q_i}) \sqsubseteq SLFB\mathcal{V}cl(\Phi) \sqsubseteq SLFB\mathcal{V}int(\gamma_{q_j}, L_{q_j}),$$

whenever $q_i < q_n < q_j (i, j < n)$ as well as

$$(\lambda_q, M_q) \sqsubseteq SLFB\mathcal{V}cl(\Psi) \sqsubseteq (\mu_{q'}, N_{q'}),$$

and

$$(\lambda_q, M_q) \sqsubseteq SLFB\mathcal{V}int(\Phi) \sqsubseteq (\mu_{q'}, N_{q'}),$$

whenever $q < q_n < q'$. This shows that the countable collections $\{(\gamma_{q_i}, L_{q_i})/i < n, q_i < q_n\} \cup \{(\lambda_q, M_q)|q < q_n\}$ and $\{(\gamma_{q_j}, L_{q_j})/j < n, q_j > q_n\} \cup \{(\mu_q, N_q)|q > q_n\}$ together with Ψ and Φ fulfil the conditions of Remark 3.2. Hence, there exists soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ set (δ_n, P_n) such that $SLFB\mathcal{V}cl(\delta_n, P_n) \sqsubseteq (\mu_q, N_q)$ if $q_n < q$, $(\lambda_q, M_q) \sqsubseteq SLFB\mathcal{V}int(\delta_n, P_n)$ if $q < q_n$, $SLFB\mathcal{V}cl(\gamma_{q_i}, L_{q_i}) \sqsubseteq SLFB\mathcal{V}int(\delta_n, P_n)$ if $q_i < q_n$, $SLFB\mathcal{V}cl(\delta_n, P_n) \sqsubseteq SLFB\mathcal{V}int(\gamma_{q_j}, L_{q_j})$ if $q_n < q_j$ where $1 \leq i, j \leq n-1$. Now, setting $(\gamma_{q_n}, L_{q_n}) = (\delta_n, P_n)$ we obtain the soft L-fuzzy sets $(\gamma_{q_1}, L_{q_1}), (\gamma_{q_2}, L_{q_2}), (\gamma_{q_3}, L_{q_3}), \cdots, (\gamma_{q_n}, L_{q_n})$ that satisfy (S_{n+1}). Therefore, the collection $\{(\gamma_{q_i}, L_{q_i})/i = 1, 2, \cdots\}$ has the required property.

§4. Properties and characterizations of $SLFB\mathcal{V}$ basically disconnected spaces

Definition 4.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. A function $f : X \to \mathbb{R}(L \times L)$ is called lower (upper) soft L-fuzzy $B\mathcal{V}$ continuous if $f^{-1}(R_t)(f^{-1}(L_t))$ is soft L-fuzzy $B\mathcal{V}$ open F_σ (soft L-fuzzy $B\mathcal{V}$ open F_σ / soft L-fuzzy $B\mathcal{V}$ closed G_δ), for each $t \in \mathbb{R}$.

Proposition 4.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. For any soft L-fuzzy set (λ, M) in X and let $f : X \to \mathbb{R}(L \times L)$ be such that

$$f(x)(t) = \begin{cases} (1_X, 1_X), & \text{if} \quad t < 0; \\ (\lambda, M)(x), & \text{if} \quad 0 \leq t \leq 1; \\ (0_X, 0_X), & \text{if} \quad t > 1, \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then f is lower (upper) soft L-fuzzy $B\mathcal{V}$ continuous iff (λ, M) is soft L-fuzzy $B\mathcal{V}$ open F_σ(soft L-fuzzy $B\mathcal{V}$ open F_σ / soft L-fuzzy $B\mathcal{V}$ closed G_δ).

Proof.

$$f^{-1}(R_t) = \begin{cases} (1_X, 1_X), & \text{if} \quad t < 0; \\ (\lambda, M), & \text{if} \quad 0 \leq t < 1; \\ (0_X, 0_X), & \text{if} \quad t > 1, \end{cases}$$

implies that f is lower soft L-fuzzy $B\mathcal{V}$ continuous iff (λ, M) is soft L-fuzzy $B\mathcal{V}$ open F_σ.

$$f^{-1}(L_t) = \begin{cases} (1_X, 1_X), & \text{if} \quad t < 0; \\ (\lambda, M), & \text{if} \quad 0 < t \leq 1; \\ (0_X, 0_X), & \text{if} \quad t > 1, \end{cases}$$

implies that f is upper soft L-fuzzy $B\mathcal{V}$ continuous iff (λ, M) is soft L-fuzzy $B\mathcal{V}$ closed G_δ.

Definition 4.2. The soft L-fuzzy characteristic function of a soft L-fuzzy set (λ, M) in X is a map $\chi_{(\lambda,M)} : X \to L \times L$ defined by

$$\chi_{(\lambda,M)}(x) = (\lambda, M)(x) = (\lambda(x), \chi_M(x)),$$

for each $x \in X$.

Proposition 4.2. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Let (λ, M) be any soft L-fuzzy set in X. Then $\chi_{(\lambda,M)}$ is lower (upper) soft L-fuzzy $B\mathcal{V}$ continuous iff (λ, M) is soft L-fuzzy $B\mathcal{V}$ open F_σ(soft L-fuzzy $B\mathcal{V}$ open F_σ / soft L-fuzzy $B\mathcal{V}$ closed G_δ).

Proof. The proof follows from Definition 4.2 and Proposition 4.1.

Definition 4.3. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. A function $f : (X, \mathcal{V}) \to \mathbb{R}(L \times L)$ is said to be strongly soft L-fuzzy $B\mathcal{V}$ continuous if $f^{-1}(R_t)$ is soft L-fuzzy $B\mathcal{V}$ open F_σ / soft L-fuzzy $B\mathcal{V}$ closed G_δ and $f^{-1}(L_t)$ is soft L-fuzzy $B\mathcal{V}$ open F_σ / soft L-fuzzy $B\mathcal{V}$ closed G_δ set for each $t \in \mathbb{R}$.

Notation 4.1. The collection of all strongly soft L-fuzzy $B\mathcal{V}$ continuous functions in soft L-fuzzy $\mathcal{V}$ space with values $\mathbb{R}(L \times L)$ is denoted by $SC_{B\mathcal{V}}$.

Proposition 4.3. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space. Then the following conditions are equivalent:

(i) $(X, \mathcal{V})$ is a soft L-fuzzy $B\mathcal{V}$ basically disconnected space,

(ii) If $g, h : X \to \mathbb{R}(L \times L)$ where g is lower soft L-fuzzy $B\mathcal{V}$ continuous, h is upper soft L-fuzzy $B\mathcal{V}$ continuous, then there exists $f \in SC_{B\mathcal{V}}(X, \mathcal{V})$ such that $g \sqsubseteq f \sqsubseteq h$,

(iii) If $(\lambda, M)'$, (μ, N) are soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets such that $(\mu, N) \sqsubseteq (\lambda, M)$, then there exists strongly soft L-fuzzy $B\mathcal{V}$ continuous functions $f : X \to \mathbb{R}(L \times L)$ such that $(\mu, N) \sqsubseteq (L_1)'f \sqsubseteq R_0 f \sqsubseteq (\lambda, M)$.

Proof. (i)⇒(ii). Define $(\xi_k, E_k) = L_k h$ and $(\eta_k, C_k) = {R_k}'g$, $k \in Q$. Thus we have two monotone increasing families of soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$. Moreover $(\xi_k, E_k) \sqsubseteq (\eta_s, C_s)$ if $k < s$. By Proposition 3.4, there exists a monotone increasing family $\{(\nu_k, F_k)\}_{k\in Q}$ of soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets of $(X, \mathcal{V})$ sets such that $SLFB\mathcal{V}cl(\xi_k, E_k) \sqsubseteq (\nu_s, F_s)$ and $(\nu_k, F_k) \sqsubseteq SLFB\mathcal{V}int(\eta_s, C_s)$ whenever $k < s$. Letting $(\phi_t, D_t) = \sqcap_{k<t}(\nu_k, F_k)'$ for all $t \in \mathbb{R}$, we define a monotone decreasing family $\{(\phi_t, D_t)/t \in \mathbb{R}\}$ is a subset of $L \times L$. Moreover, we have $SLFB\mathcal{V}cl(\phi_t, D_t) \sqsubseteq SLFB\mathcal{V}int(\phi_s, D_s)$ whenever $s < t$. We have

$$\begin{aligned}
\sqcup_{t\in\mathbb{R}}(\phi_t, D_t) &= \sqcup_{t\in\mathbb{R}} \sqcap_{k<t} (\nu_k, F_k)' \\
&\sqsupseteq \sqcup_{t\in\mathbb{R}} \sqcap_{k<t} (\eta_k, C_k)' \\
&= \sqcup_{t\in\mathbb{R}} \sqcap_{k<t} g^{-1}(R_k) \\
&= g^{-1}(\sqcup_{t\in\mathbb{R}} R_t) \\
&= (1_X, 1_X).
\end{aligned}$$

Similarly, $\sqcap_{t\in\mathbb{R}}(\phi_t, D_t) = (0_X, 0_X)$. Now define a function $f : X \to \mathbb{R}(L \times L)$ possessing the required properties. Let $f(x)(t) = (\phi_t, D_t)(x)$ for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion it follows that f is well defined. To prove f is strongly soft L-fuzzy $B\mathcal{V}$ continuous. Observe that

$$\sqcup_{s>t}(\phi_s, D_s) = \sqcup_{s>t} SLFB\mathcal{V}int(\phi_s, D_s),$$

and

$$\sqcap_{s<t}(\phi_s, D_s) = \sqcap_{s<t} SLFB\mathcal{V}cl(\phi_s, D_s).$$

Then $f^{-1}(R_t) = R_t \circ f = R_t(\phi_t, D_t)(x) = \sqcup_{s>t}(\phi_s, D_s) = \sqcup_{s>t} SLFB\mathcal{V}int(\phi_s, D_s)$ is soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$. And $f^{-1}(L'_t) = \sqcap_{s<t}(\phi_s, D_s) = \sqcap_{s<t} SLFB\mathcal{V}cl(\phi_s, D_s)$ is soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$. Therefore, f is strongly soft L-fuzzy $B\mathcal{V}$ continuous. To conclude the proof it remains to show that $g \sqsubseteq f \sqsubseteq h$. That is, $g^{-1}(L'_t) \sqsubseteq f^{-1}(L'_t) \sqsubseteq h^{-1}(L'_t)$ and $g^{-1}(R_t) \sqsubseteq f^{-1}(R_t) \sqsubseteq h^{-1}(R_t)$ for each $t \in \mathbb{R}$. We have

$$\begin{aligned}
g^{-1}(L'_t) &= \sqcap_{s<t} g^{-1}(L'_s) \\
&= \sqcap_{s<t} \sqcap_{k<s} g^{-1}(R_k) \\
&= \sqcap_{s<t} \sqcap_{k<s} (\eta_k, C_k)' \\
&\sqsubseteq \sqcap_{s<t} \sqcap_{k<s} (\nu_k, F_k)' \\
&= \sqcap_{s<t}(\phi_s, D_s) \\
&= f^{-1}(L'_t), \\
f^{-1}(L'_t) &= \sqcap_{s<t}(\phi_s, D_s) \\
&= \sqcap_{s<t} \sqcap_{k<s} (\nu_k, F_k) \\
&\sqsubseteq \sqcap_{s<t} \sqcap_{k<s} (\xi_k, E_k)' \\
&= \sqcap_{s<t} \sqcap_{k<s} h^{-1}(L'_k) \\
&= \sqcap_{s<t} h^{-1}(L'_s) \\
&= h^{-1}(L'_t).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
g^{-1}(R_t) &= \sqcup_{s>t} g^{-1}(R_s) \\
&= \sqcup_{s>t} \sqcup_{k>s} g^{-1}(R_k) \\
&= \vee_{s>t} \vee_{k>s} (\eta_k, C_k)' \\
&\sqsubseteq \sqcup_{s>t} \sqcap_{k<s} (\nu_k, F_k)' \\
&= \sqcup_{s>t}(\phi_s, D_s) \\
&= f^{-1}(R_t), \\
f^{-1}(R_t) &= \sqcup_{s>t}(\phi_s, D_s) \\
&= \sqcup_{s>t} \sqcap_{k<s} (\nu_k, F_k)' \\
&\sqsubseteq \sqcup_{s>t} \sqcup_{k>s} (\xi_k, E_k)' \\
&= \sqcup_{s>t} \sqcup_{k>s} h^{-1}(L'_k) \\
&= \sqcup_{s>t} h^{-1}(R_s) \\
&= h^{-1}(R_t).
\end{aligned}$$

Thus, (ii) is proved.

(ii)⇒(iii). Suppose that (λ, M) is soft L-fuzzy $B\mathcal{V}$ closed G_δ and (μ, N) is soft L-fuzzy $B\mathcal{V}$ open F_σ such that $(\mu, N) \sqsubseteq (\lambda, M)$. Then $\chi_{(\mu,N)} \sqsubseteq \chi_{(\lambda,M)}$, where $\chi_{(\mu,N)}$, $\chi_{(\lambda,M)}$ are lower and upper soft L-fuzzy $B\mathcal{V}$ continuous functions, respectively. Hence by (ii), there exists a strong soft L-fuzzy $B\mathcal{V}$ continuous function $f : X \to \mathbb{R}(L \times L)$ such that $\chi_{(\mu,N)} \sqsubseteq f \sqsubseteq \chi_{(\lambda,M)}$. Clearly, $f(x) \in \mathbb{R}(L \times L)$ for all $x \in X$ and $(\mu, N) = L'_1\chi_{(\mu,N)} \sqsubseteq L'_1 f \sqsubseteq R_0 f \sqsubseteq R_0\chi_{(\lambda,M)} = (\lambda, M)$. Therefore, $(\mu, N) \sqsubseteq L'_1 f \sqsubseteq R_0 f \sqsubseteq (\lambda, M)$.

(iii)⇒(i). $L'_1 f$ and $R_0 f$ are soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ sets. By Proposition 3.3, $(X, \mathcal{V})$ is a soft L-fuzzy $B\mathcal{V}$ basically disconnected space.

§5. Tietze extension theorem

Definition 5.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $\mathcal{V}$ space and $A \subseteq X$ then $(A, \mathcal{V}/A)$ is a soft L-fuzzy $\mathcal{V}$ space which is called a soft L-fuzzy $\mathcal{V}$ subspace of $(X, \mathcal{V})$ where $\mathcal{V}/A = \{(\lambda, M)/A : (\lambda, M) \in \mathcal{V}\}$.

Remark 5.1. Let X be a non-empty set and let $A \subset X$. Then the characteristic function* of A is a map $\chi^*_A = (\chi_A, \chi_A) : X \to \{(1_X, 1_X), (0_X, 0_X)\}$ is defined by

$$\chi^*_A = \begin{cases} (1_X, 1_X), & \text{if} \quad x \in A; \\ (0_X, 0_X), & \text{if} \quad x \notin A. \end{cases}$$

Proposition 5.1. Let $(X, \mathcal{V})$ be a soft L-fuzzy $B\mathcal{V}$ basically disconnected space and let $A \subset X$ be such that χ^*_A is a soft L-fuzzy $B\mathcal{V}$ open F_σ set in X. Let $f : (A, \mathcal{V}/A) \to I(L \times L)$ be strong soft L-fuzzy $B\mathcal{V}$ continuous. Then f has a strong soft L-fuzzy $B\mathcal{V}$ continuous extension over $(X, \mathcal{V})$.

Proof. Let $g, h : X \to \mathbb{R}(L \times L)$ be such that $g = f = h$ on A and $g(x) = (0_X, 0_X)$, $h(x) = (1_X, 1_X)$ if $x \notin A$, we have

$$R_t g = \begin{cases} (\mu_t, N_t) \sqcap \chi^*_A, & \text{if} \quad t \geq 0; \\ (1_X, 1_X), & \text{if} \quad t < 0, \end{cases}$$

where (μ_t, N_t) is soft L-fuzzy $B\mathcal{V}$ open F_σ and is such that $(\mu_t, N_t)/A = R_t f$ and

$$L_t h = \begin{cases} (\lambda_t, M_t) \sqcap \chi^*_A, & \text{if} \quad t \leq 1; \\ (1_X, 1_X), & \text{if} \quad t > 1, \end{cases}$$

where (λ_t, M_t) is soft L-fuzzy $B\mathcal{V}$ closed open $G_\delta F_\sigma$ and is such that $(\lambda_t, M_t)/A = L_t f$. Thus, g is lower soft L-fuzzy $B\mathcal{V}$ continuous and h is upper soft L-fuzzy $B\mathcal{V}$ continuous with $g \sqsubseteq h$. By Proposition 4.3, there is a strong soft L-fuzzy $B\mathcal{V}$ continuous function $F : X \to I(L \times L)$ such that $g \sqsubseteq F \sqsubseteq h$. Hence $F \equiv f$ on A.

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On right circulant matrices with Perrin sequence

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Abstract In this paper, the eigenvalues, the Euclidean norm and the inverse of right circulant matrices with Perrin sequence were obtained.

Keywords Perrin sequence, right circulant matrix.

§1. Introduction

The Perrinn sequence is a sequence whose terms satisfy the recurrence relation

$$q_n = q_{n-2} + q_{n-3}, \tag{1}$$

with initial values $q_0 = 3,\ q_1 = 0,\ q_2 = 2$. The n-th term of the Perrinn sequence is given by:

$$q_n = r_1^n + r_2^n + r_3^n, \tag{2}$$

where

$$\begin{aligned}
r_1 &= \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}},\\
r_2 &= \frac{-1+i\sqrt{3}}{2}\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \frac{-1-i\sqrt{3}}{2}\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}},\\
r_3 &= \frac{-1-i\sqrt{3}}{2}\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \frac{-1+i\sqrt{3}}{2}\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}.
\end{aligned}$$

The numbers $r_1,\ r_2$ and r_3 are the roots of the equation $x^3 - x - 1 = 0$. Moreover, r_1 is called the Plastic number.

A right circulant matrix with Perrin sequenc is a matrix of the form

$$RCIRC_n(\vec{q}) = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ q_{n-1} & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\ q_{n-2} & q_{n-1} & q_0 & \cdots & q_{n-4} & q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_2 & q_3 & q_4 & \cdots & q_0 & q_1 \\ q_1 & q_2 & q_3 & \cdots & q_{n-1} & q_0 \end{pmatrix}$$

where p_k are the first n terms of the Perrinn sequence.

§2. Preliminary results

To prove the main results, the following lemmas will be used.

Lemma 2.1.

$$\sum_{k=0}^{n-1}(r\omega^{-m})^k = \frac{1-r^n}{1-r\omega^{-m}},$$

where $\omega = e^{2i\pi/n}$.

Proof. Note that $\sum_{k=0}^{n-1}(r\omega^{-m})^k$ is a geometric series with first term 1 and common ratio $r\omega^{-m}$. Using the formula for the sum of a geometric series, we have

$$\begin{aligned}\sum_{k=0}^{n-1}(r\omega^{-m})^k &= \frac{1-r^n\omega^{-mn}}{1-r\omega^{-m}} \\ &= \frac{1-r^n(\cos 2\pi + i\sin 2\pi)}{1-r\omega^{-m}} \\ &= \frac{1-r^n}{1-r\omega^{-m}}.\end{aligned}$$

Lemma 2.2.

$$\sum_{j=1}^{n}\frac{a_j(1-r_j^n)}{1-r_j\omega^{-m}} = \frac{\sum_{j=1}^{n}\left[a_j(1-r_j^n)\prod_{k\neq j}(1-r_k\omega^{-m})\right]}{\prod_{j=1}^{n}(1-r_j\omega^{-m})}. \tag{3}$$

Proof. By combining these fractions we have

$$\begin{aligned}\sum_{j=1}^{n}\frac{a_j(1-r_j^n)}{1-r_j\omega^{-m}} &= \frac{a_1(1-r_1^n)\prod_{k\neq 0}(1-r_k\omega^{-m}) + \ldots + a_n(1-r_n^n)\prod_{k\neq n}(1-r_k\omega^{-m})}{\prod_{j=1}^{n}(1-r_j\omega^{-m})} \\ &= \frac{\sum_{j=1}^{n}\left[a_j(1-r_j^n)\prod_{k\neq j}(1-r_k\omega^{-m})\right]}{\prod_{j=1}^{n}(1-r_j\omega^{-m})}.\end{aligned}$$

§3. Main Results

Theorem 3.1. The eigenvalues of $RCIRC_n(\vec{q})$ are given by

$$\lambda_m = \sum_{k=1}^{3} \frac{(1-r_k^n)}{1-r_k\omega^{-m}},$$

where $m = 0, 1, \cdots, n-1$.

Proof. From [1], the eigenvalues of a right circulant matrix are given by the Discrete Fourier transform

$$\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk}, \tag{4}$$

where c_k are the entries in the first row of the right circulant matrix. Using this formula, the eigenvalues of $RCIRC_n(\vec{q})$ are

$$\begin{aligned}\lambda_m &= \sum_{k=0}^{n-1}\left[r_1^k + r_2^k + r_3^k\right]\omega^{-mk} \\ &= \sum_{k=0}^{n-1} r_1^k\omega^{-mk} + \sum_{k=0}^{n-1} r_2^k\omega^{-mk} + \sum_{k=0}^{n-1} r_3^k\omega^{-mk}.\end{aligned}$$

By Lemma 2.1 we will get the desired equation.

Theorem 3.2. The Eucliden norm of $RCIRC_n(\vec{q})$ is given by

$$\|RCIRC_n(\vec{p})\|_E = \sqrt{n\left[\sum_{k=1}^{3}\frac{(1-r_k^{2n})}{1-r_k^2} + \frac{2(1-r_1^n r_2^n)}{1-r_1r_2} + \frac{2(1-r_2^n r_3^n)}{1-r_2r_3} + \frac{2(1-r_1^n r_3^n)}{1-r_1r_3}\right]}.$$

Proof.

$$\begin{aligned}\|RCIRC_n(\vec{p})\|_E &= \sqrt{n\sum_{k=0}^{n-1}\left[r_1^k + r_2^k + r_3^k\right]^2} \\ &= \sqrt{n\sum_{k=0}^{n-1}\left[r_1^{2k} + r_2^{2k} + r_3^{2k} + 2r_1^k r_2^k + 2r_2^k r_3^k + 2r_1^k r_3^k\right]}.\end{aligned}$$

Note that each term in the summation is from a geometric sequence, so using the formula for sum of geometric sequence, the theorem follows.

Theorem 3.3. The inverse of $RCIRC_n(\vec{p})$ is given by

$$RCIRC_n\left(s_0,\ s_1,\ \ldots, s_{n-1}\right),$$

where

$$\begin{aligned}
s_k &= \frac{1}{n}\sum_{m=0}^{n-1} F(j,k,\omega^{-m})\omega^{mk}, \\
\Omega(j,k,\omega^{-m}) &= \frac{\prod_{j=1}^{3}\rho(j,\omega^{-m})}{\sum_{j=1}^{3}\left[\sigma(j)\prod_{k\neq j}\tau(k,\omega^{-m})\right]}, \\
\rho(j,\omega^{-m}) &= 1-r_j\omega^{-m}, \\
\sigma(j) &= 1-r_j^n, \\
\tau(k,\omega^{-m}) &= 1-r_k\omega^{-m}.
\end{aligned}$$

Proof. The entries of the inverse of a right circulant matrix can be solved using the Inverse Discrete Fourier transform

$$s_k = \frac{1}{n}\sum_{m=0}^{n-1}\lambda_m^{-1}\omega^{mk}, \tag{5}$$

where λ_m are the eigenvalues of the right circulant matrix. Using this equation and Theorem 3.1 we have

$$s_k = \frac{1}{n}\sum_{m=0}^{n-1}\left[\sum_{k=1}^{3}\frac{1-r_k^n}{1-r_k\omega^{-m}}\right]^{-1}\omega^{mk}.$$

By Lemma 2.2 we have

$$\begin{aligned}
s_k &= \frac{1}{n}\sum_{m=0}^{n-1}\left[\frac{\sum_{j=1}^{3}\left[(1-r_j^n)\prod_{k\neq j}(1-r_k\omega^{-m})\right]}{\prod_{j=1}^{3}(1-r_j\omega^{-m})}\right]^{-1}\omega^{mk} \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\left[\frac{\prod_{j=1}^{3}(1-r_j\omega^{-m})}{\sum_{j=1}^{3}\left[(1-r_j^n)\prod_{k\neq j}(1-r_k\omega^{-m})\right]}\right]\omega^{mk} \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\left[\frac{\prod_{j=1}^{3}\rho(j,\omega^{-m})}{\sum_{j=1}^{3}\left[\sigma(j)\prod_{k\neq j}\tau(k,\omega^{-m})\right]}\right]\omega^{mk} \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\Omega(j,k,\omega^{-m})\omega^{mk}.
\end{aligned}$$

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