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A new additive function and the Smarandache divisor product sequences [1]

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Abstract For any positive integer n, we define the arithmetical function $G(n)$ as $G(1) = 0$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $G(n) = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + \cdots + \frac{\alpha_k}{p_k}$. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $G(n)$ in Smarandache divisor product sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and give two sharper asymptotic formulae for them.

Keywords Additive function, Smarandache divisor product sequences, mean value, elementary method, asymptotic formula.

§1. Introduction and results

In elementary number theory, we call an arithmetical function $f(n)$ as an additive function, if for any positive integers m, n with $(m,\ n) = 1$, we have $f(mn) = f(m) + f(n)$. We call $f(n)$ as a complete additive function, if for any positive integers r and s, $f(rs) = f(r) + f(s)$. There are many arithmetical functions satisfying the additive properties. For example, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and logarithmic function $f(n) = \ln n$ are two complete additive functions, $\omega(n) = k$ is an additive function, but not a complete additive function. About the properties of the additive functions, there are many authors had studied it, and obtained a series interesting results, see references [1], [2], [5] and [6].

In this paper, we define a new additive function $G(n)$ as follows: $G(1) = 0$; If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then $G(n) = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + \cdots + \frac{\alpha_k}{p_k}$. It is clear that this function is a complete additive function. In fact if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, then we have $mn = p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$. Therefore, $G(mn) = \frac{\alpha_1+\beta_1}{p_1} + \frac{\alpha_2+\beta_2}{p_2} + \cdots + \frac{\alpha_k+\beta_k}{p_k} = G(m) + G(n)$. So $G(n)$ is a complete additive function. Now we define the Smarandache divisor product sequences $\{p_d(n)\}$ and $\{q_d(n)\}$ as follows: $p_d(n)$ denotes the product of all positive divisors of n; $q_d(n)$ denotes the product of all positive divisors d of n but n. That is,

$$p_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}; \qquad q_d(n) = \prod_{d|n, d<n} d = n^{\frac{d(n)}{2}-1},$$

[1]This work is supported by the Shanghai Innovation Fund (10-0101-07-410) and Young Teacher Scientific Research Special Fund of Shanghai (37-0101-07-704).

where $d(n)$ denotes the Dirichlet divisor function.

The sequences $\{p_d(n)\}$ and $\{q_d(n)\}$ are introduced by Professor F.Smarandache in references [3], [4] and [9], where he asked us to study the various properties of $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, some authors had studied it, and proved some conclusions, see references [7], [8], [10] and [11].

The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $G(p_d(n))$ and $G(q_d(n))$, and give two sharper asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} G(p_d(n)) = B\cdot x\cdot \ln x + (2\gamma\cdot B - D - B)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where $B = \sum_p \frac{1}{p^2}$, $D = \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant, and $\sum_p$ denotes the summation over all primes.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} G(q_d(n)) = B\cdot x\cdot \ln x + (2\gamma\cdot B - 2B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where B and D are defined as same as in Theorem 1.

§2. Two simple lemmas

In this section, we give two simple lemmas, which are necessary in the proof of the theorems. First we have:

Lemma 1. For any real number $x > 1$, we have the asymptotic formula:

$$\sum_{p\leq x} \frac{1}{p} = \ln\ln x + A + O\left(\frac{1}{\ln x}\right),$$

where A be a constant, $\sum_{p\leq x}$ denotes the summation over all primes $p \leq x$.

Proof. See Theorem 4.12 of reference [6].

Lemma 2. For any real number $x > 1$, we have the asymptotic formulae:

(I) $$\sum_{n\leq x} G(n) = B\cdot x + O(\ln\ln x);$$

(II) $$\sum_{n\leq x} \frac{G(n)}{n} = B\cdot\ln x + C + O\left(\frac{\ln\ln x}{x}\right),$$

where $B = \sum_p \frac{1}{p^2}$, $C = \gamma\cdot B - \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant, and $\sum_p$ denotes the summation over all primes.

Proof. For any positive integer $n > 1$, from the definition of $G(n)$ we have

$$G(n) = \sum_{p|n} \frac{1}{p}.$$

So from this formula and Lemma 1 we have

$$\begin{aligned}\sum_{n\le x} G(n) &= \sum_{n\le x}\sum_{p|n}\frac{1}{p} = \sum_{np\le x}\frac{1}{p} = \sum_{p\le x}\frac{1}{p}\sum_{n\le \frac{x}{p}} 1 = \sum_{p\le x}\frac{1}{p}\left[\frac{x}{p}\right] \\ &= x\cdot\sum_{p\le x}\frac{1}{p^2} + O\left(\sum_{p\le x}\frac{1}{p}\right) = B\cdot x + O\left(\ln\ln x\right),\end{aligned}$$

where $B = \sum_{p} \frac{1}{p^2}$ be a constant. This proves (I) of Lemma 2.

Now we prove (II) of Lemma 2, note that the asymptotic formula

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is the Euler constant. So from Lemma 1 and the definition of $G(n)$ we also have

$$\begin{aligned}\sum_{n\le x}\frac{G(n)}{n} &= \sum_{n\le x}\frac{\sum_{p|n}\frac{1}{p}}{n} = \sum_{np\le x}\frac{1}{p^2 n} = \sum_{p\le x}\frac{1}{p^2}\sum_{n\le\frac{x}{p}}\frac{1}{n} \\ &= \sum_{p\le x}\frac{1}{p^2}\left[\ln x - \ln p + \gamma + O\left(\frac{p}{x}\right)\right] \\ &= \sum_{p\le x}\frac{\ln x}{p^2} - \sum_{p\le x}\frac{\ln p}{p^2} + \sum_{p\le x}\frac{1}{p^2}\gamma + O\left(\frac{1}{x}\sum_{p\le x}\frac{1}{p}\right) \\ &= B\cdot\ln x - \sum_{p}\frac{\ln p}{p^2} + \gamma\cdot B + O\left(\frac{\ln\ln x}{x}\right) \\ &= B\cdot\ln x + C + O\left(\frac{\ln\ln x}{x}\right),\end{aligned}$$

where $C = \gamma\cdot B - \sum_{p}\frac{\ln p}{p^2}$ is a constant. This proves (II) of Lemma 2.

§3. Proof of the theorems

Now we use the above Lemmas to complete the proof of the theorems. First we prove Theorem 1. Note that the complete additive properties of $G(n)$ and the definition of $p_d(n)$, from (II) of Lemma 2 and Theorem 3.17 of [6] we have

$$\begin{aligned}
\sum_{n\le x} G\left(p_d(n)\right) &= \sum_{n\le x} G\left(n^{\frac{d(n)}{2}}\right) = \frac{1}{2}\sum_{n\le x} d(n)G(n) = \frac{1}{2}\sum_{mn\le x} G(mn) \\
&= \frac{1}{2}\sum_{mn\le x}\left(G(m)+G(n)\right) = \sum_{mn\le x} G(m) \\
&= \sum_{m\le\sqrt{x}}\sum_{n\le\frac{x}{m}} G(m) + \sum_{n\le\sqrt{x}}\sum_{m\le\frac{x}{n}} G(m) - \left(\sum_{m\le\sqrt{x}} G(m)\right)\left(\sum_{n\le\sqrt{x}} 1\right) \\
&= \sum_{m\le\sqrt{x}} G(m)\left[\frac{x}{m}\right] + \sum_{n\le\sqrt{x}}\left[\frac{B\cdot x}{n} + O\left(\ln\ln x\right)\right] \\
&\quad\; - \left[\sqrt{x}+O(1)\right]\left[B\cdot\sqrt{x}+O(\ln\ln x)\right] \\
&= x\cdot\sum_{m\le\sqrt{x}}\frac{G(m)}{m} + O\left(\sum_{m\le\sqrt{x}} G(m)\right) + B\cdot x\cdot\sum_{n\le\sqrt{x}}\frac{1}{n} \\
&\quad\; -B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= x\cdot\left[\frac{1}{2}B\cdot\ln x + C + O\left(\frac{\ln\ln x}{\sqrt{x}}\right)\right] + B\cdot x\cdot\left[\ln\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)\right] \\
&\quad\; -B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (C+\gamma B - B)\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (2\gamma B - B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right),
\end{aligned}$$

where $B = \sum_p \frac{1}{p^2}$ and $D = \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant. This proves Theorem 1.

From Lemma 2, Theorem 1 and the definition of $q_d(n)$ we can also deduce that

$$\begin{aligned}
\sum_{n\le x} G\left(q_d(n)\right) &= \sum_{n\le x} G\left(n^{\frac{d(n)}{2}-1}\right) = \frac{1}{2}\sum_{n\le x} d(n)G(n) - \sum_{n\le x} G(n) \\
&= B\cdot x\cdot\ln x + (2\gamma B - B - D)\cdot x - B\cdot x + O\left(\sqrt{x}\ln\ln x\right) \\
&= B\cdot x\cdot\ln x + (2\gamma B - 2B - D)\cdot x + O\left(\sqrt{x}\ln\ln x\right).
\end{aligned}$$

This completes the proof of Theorem 2.

§4. Some notes

For any positive integer n and any fixed real number β, we define the general arithmetical function $H(n)$ as $H(1) = 0$. If $n > 1$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $H(n) = \alpha_1\cdot p_1^{\beta}+\alpha_2\cdot p_2^{\beta}+\cdots+\alpha_k\cdot p_k^{\beta}$. It is clear that this function is a complete additive function. If $\beta = 0$, then $H(n) = \Omega(n)$. If $\beta = -1$, then $H(n) = G(n)$. Using our method we can also give some asymptotic formulae for the mean vale of $H\left(p_d(n)\right)$ and $H\left(q_d(n)\right)$.

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The Merrifield−simmons index in $(n, n+1)$−graphs

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Abstract A $(n, n+1)$−graph G is a connected simple graph with n vertices and $n+1$ edges. In this paper, we determine the lower bound for the Merrifield−simmons index in $(n, n+1)$−graphs in terms of the order n, and characterize the $(n, n+1)$−graph with the smallest Merrifield−simmons index.

Keywords $(n, n+1)$−graphs, σ−index, Merrifield−Simmons index.

§1. Introduction

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For any $v \in V$, $N_G(v) = \{u \mid uv \in E(G)\}$ denotes the neighbors of v, and $d_G(v) =\mid N_G(v) \mid$ is the degree of v in G; $N_G[v] = \{v\} \cup N_G(v)$. A leaf is a vertex of degree one and a stem is a vertex adjacent to at least one leaf. Let $E^{'} \subseteq E(G)$, we denote by $G - E^{'}$ the subgraph of G obtained by deleting the edges of $E^{'}$. $W \subseteq V(G)$, $G - W$ denotes the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If a graph G has components $G_1, G_2, \cdots, G_k$, then G is denoted by $\cup_{i=1}^{k} G_i$. P_n denotes the path on n vertices, C_n is the cycle on n vertices, and S_n is the star consisting of one center vertex adjacent to $n-1$ leaves and T_n is a tree on n vertices.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is called independent if no two vertices of S are adjacent in G. The set of independent sets in G is denoted by $I(G)$. The empty set is an independent set. The number of independent sets in G, denoted by σ−index, is called the Merrifield-Simmons index in theoretical chemistry.

The Merrifield-Simmons index $[1-3]$ is one of the topogical indices whose mathematical properties were studied in some detail $[4-10]$ whereas its applicability for QSPR and QSAR was examined to a much lesser extent; in [2] it was shown that σ−index is correlated with the boiling points.

In this paper, we investigate the $Merrifield-Simmons$ index of $(n, n+1)$−graphs, i.e., connected simply graphs with n vertices and $n+1$ edges. We characterize the $(n, n+1)$−graph with the smallest Merrifield− Simmons index.

Let $Q(C_k, v_1, C_m)$ be a graph obtained from two cycles C_k and C_m which have one common vertex; $Q(C_k, v_1, P_l, u_1, C_m)$ is obtained from two cycles C_k and C_m which are connected by one path P_l; $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})$ obtained from two cycles which have one common path. We divide all $(n, n+1)$−graphs with two cycles C_k and C_m of lengths k and m into three classes.

(i) $Q(C_k, v_1, C_m; n)$ is the set with n vertices in which two cycles C_k and C_m have only one common vertex.

(ii) $Q(C_k, v_1, P_l, u_1, C_m; n)$ is the set with n vertices in which two cycles C_k and C_m connected by one path.

(iii) $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}; n)$ is the set with n vertices in which two cycles have one common path.

§2. Some known results

We give with several important lemmas from $[2-10]$ will be helpful to the proofs of our main results, and also give three lemma which will increase the Merrifield-Simmons index.

Lemma 2.1. ([2]) Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then
$\sigma(G) = \prod_{i=1}^{k} \sigma(G_i)$.

Lemma 2.2. ([4]) For any graph G with any $v \in V(G)$, we have
$\sigma(G) = \sigma(G-v) + \sigma(G-[v])$, where $[v] = N_G(v) \bigcup v$.

Lemma 2.3. ([3]) Let T be a tree. Then $F_{n+2} \leq \sigma(T) \leq 2^{n-1}+1$ and $\sigma(T) = F_{n+2}$ if and only if $T \cong P_n$ and $\sigma(T) = 2^{n-1}+1$ if and only if $T \cong S_n$.

Lemma 2.4. ([5]) Let $n = 4m+i (i \in \{1, 2, 3, 4\})$ and $m \geq 2$.Then
$\sigma((P_n, v_2, T)) > \sigma((P_n, v_4, T)) > \cdots > \sigma((P_n, v_{2m+2\rho}, T)) > \cdots > \sigma((P_n, v_{2m+1}, T))$
$> \sigma((P_n, v_3, T)) > \sigma((P_n, v_1, T))$, where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Lemma 2.5. Let $G^{'}$ is obtained from G which attaches a tree T_{r+1} at vertex v and $G^{''}$ is obtained from G which attaches a tree P_{r+1} at vertex v. Then $\sigma(G^{''}) \leq \sigma(G^{'})$ with the equality if and only if $G^{''} \cong G^{'}$.

Proof. If $r = 1$, the result is correct. We presume that the result is correct if $r \leq k$, if $r = k+1$, then $\sigma(G^{'}) = \sigma(G-u) + \sigma(G-[u]) = \sigma(G-u) + \sigma(G-\{u, v\}) \geq \sigma(G^{''})$ with the equality if and only if $G^{''} \cong G^{'}$.
The proof is completed.

Lemma 2.6[11]. Let G_0 be any one of graphs Ψ, then

$\sigma(G_0 - \{u, v\}) > \sigma(G_0 - \{[u], v\}) + \sigma(G_0 - \{[u], [v]\})$.

Lemma 2.7. Let G_1 be a graph obtained from G_0 which attaches a path P_{s+1} at vertex u and a path P_{t+1} at vertex v; G_2, G_3 are obtained from G_0 which attaches a path P_{s+t+1} at vertex u and v, G_0 is any graph. Then

$\sigma(G_2) < \sigma(G_1)$ or $\sigma(G_3) < \sigma(G_1)$.

Proof. If u, v are adjacent in G_0, then
$\sigma(G_1) = F_{s+2}F_{t+2}\sigma(G_0 - \{u, v\}) + F_{s+2}F_{t+1}\sigma(G_0 - [v]) + F_{s+1}F_{t+2}\sigma(G_0 - [u])$
$\sigma(G_2) = F_{s+t+2}\sigma(G_0 - \{u, v\}) + F_{s+t+2}\sigma(G_0 - [v]) + F_{s+t+1}\sigma(G_0 - [u])$
$\sigma(G_3) = F_{s+t+2}\sigma(G_0 - \{u, v\}) + F_{s+t+2}\sigma(G_0 - [u]) + F_{s+t+1}\sigma(G_0 - [v])$
Let $\sigma(G_2) - \sigma(G_3) = F_{s+t}\sigma(G_0 - [v]) - F_{s+t}\sigma(G_0 - [u]) \geq 0$ and
$\sigma(G_0 - [v]) = \sigma(G_0 - [u]) + \triangle(\triangle \geq 0)$. Then

$\sigma(G_1) - \sigma(G_3)$
$= (F_{s+2}F_{t+2} - F_{s+t+2})\sigma(G_0 - \{u, v\}) + (F_{s+2}F_{t+1} + F_{s+1}F_{t+2} - F_{s+t+3})\sigma(G_0 - [u]) + (F_{s+2}F_{t+1} - F_{s+t+1})\triangle$
$\geq (F_{s+2}F_{t+2} - F_{s+t+2} + F_{s+2}F_{t+1} + F_{s+1}F_{t+2} - F_{s+t+3})\sigma(G_0 - [u]) + (F_{s+2}F_{t+1} - F_{s+t+1})\triangle$
$\frac{1}{5}(L_{s+t+1} - (-1)^s L_{t-s-1})\triangle \geq 0.$
If $\sigma(G_2) - \sigma(G_3) \leq 0$, the proof is as the same as above. If u, v are not adjacent in G_0 and u, v are connected by one path, then

$$\sigma(G_1) = F_{s+2}F_{t+2}\sigma(G_0 - \{u, v\}) + F_{s+2}F_{t+1}\sigma(G_0 - \{u, [v]\}) + F_{s+1}F_{t+2}\sigma(G_0 - \{[u], v\}) + F_{s+1}F_{t+1}\sigma(G_0 - \{[u], [v]\})$$

$$\sigma(G_2) = F_{s+t+2}\sigma(G_0 - \{u, v\}) + F_{s+t+2}\sigma(G_0 - \{u, [v]\}) + F_{s+t+1}\sigma(G_0 - \{[u], v\}) + F_{s+t+1}\sigma(G_0 - \{[u], [v]\})$$

$$\sigma(G_3) = F_{s+t+2}\sigma(G_0 - \{u, v\}) + F_{s+t+2}\sigma(G_0 - \{[u], v\}) + F_{s+t+1}\sigma(G_0 - \{u, [v]\}) + F_{s+t+1}\sigma(G_0 - \{[u], [v]\})$$

Let $\sigma(G_2) - \sigma(G_3) = F_{s+t}\sigma(G_0 - [v]) - F_{s+t}\sigma(G_0 - [u]) \geq 0$ and
$\sigma(G_0 - [v]) = \sigma(G_0 - [u]) + \triangle(\triangle \geq 0)$. Then

$$\begin{aligned}\sigma(G_1) - \sigma(G_3) &\geq F_{s+2}F_{t+2}\sigma(G_0 - \{[u], v\}) + F_{s+2}F_{t+2}\sigma(G_0 - \{[u], [v]\}) \\ &\quad (F_{s+1}F_{t+2} + F_{s+2}F_{t+1})\sigma(G_0 - \{[u], v\}) + F_{s+1}F_{t+1}\sigma(G_0 - \{[u], [v]\}) + F_{s+2}F_{t+1}\triangle \\ &\quad -[F_{s+t+2}\sigma(G_0 - \{u, v\}) + F_{s+t+3}\sigma(G_0 - \{[u], v\}) + F_{s+t+1}\triangle \\ &\quad +F_{s+t+1}\sigma(G_0 - \{[u], [v]\}) \\ &= (F_{s+2}F_{t+2} + F_{s+1}F_{t+2} + F_{s+2}F_{t+1} - F_{s+t+4})\sigma(G_0 - \{[u], v\}) \\ &\quad +(F_{s+2}F_{t+1} - F_{s+t+1})\triangle + (F_{s+2}F_{t+2} - F_{s+1}F_{t+1} - F_{s+t+1})\sigma(G_0 - \{[u], [v]\}) \\ &= (F_{s+2}F_{t+1} - F_{s+t+1})\triangle + F_sF_t(\sigma(G_0 - \{[u], v\}) - \sigma(G_0 - \{[u], [v]\})) \geq 0.\end{aligned}$$

If u, v are not adjacent in G_0 and u, v are connected by one tree, then
$\sigma(G_1) - \sigma(G_3)$
$= F_sF_t(\sigma(G_0 - \{u, v\}) - \sigma(G_0 - \{[u], v\}) - \sigma(G_0 - \{[u], [v]\})) + (F_{s+2}F_{t+1} - F_{s+t+1})\triangle$
From lemma 2.6, we have $\sigma(G_1) - \sigma(G_3) \geq 0$. The proof is completed.

From lemma 2.5 and lemma 2.7, we know all $(n, n+1)-$graphs with the smallest $\sigma-$index belong to the follow three classes.

(i) $Q(C_k, v_1, C_m)$ and P_h have one common vertex.

(ii) $Q(C_k, v, P_l, u_1, C_m)$ and P_h have one common vertex.

(iii) $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})$ and P_h have one common vertex.

§3. The $(n, n+1)-$graph with the smallest Merrifield-Simmons index in $Q(C_k, v_1, C_m; n)$

In this section, we will find the $(n, n + 1)-$graph with the smallest Merrifield-Simmons index in $Q(C_k, v_1, C_m; n)$. and give some good results on orders of $\sigma-$index.

Definition 3.1. Let $Q(C_k, v_1, C_m)$ be a graph with two cycles C_k and C_m which have common vertex v_1 and $Q(C_k, v_1, C_m, v_s, P_{r+1})$ be obtained from $Q(C_k, v_1, C_m)$ and P_{r+1} which have one common vertex v_s as shown Picture 3.1.

Lemma 3.1. ([6]) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ and by definition of *Fibonacci* number F_n and *Lucas* number L_n, we know $F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}}$, $L_n = \alpha^n + \beta^n$, $F_n \cdot F_m = \frac{1}{5}(L_{n+m} - (-1)^n \cdot L_{m-n})$.

Lemma 3.2. Let $m = 4j + i, i \in \{1, 2, 3, 4\}$ and $j \geq 2$.Then
$\sigma(Q(P_m, v_1, C_k)) < \sigma(Q(P_m, v_3, C_k)) < \cdots < \sigma(Q(P_m, v_{2j+1}, C_k)) < \sigma(Q(P_m, v_{2j+2\rho}, C_k))$
$< \cdots < \sigma(Q(P_m, v_4, C_k)) < \sigma(Q(P_m, v_2, C_k))$, where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. Let $1 \leq s \leq \lceil \frac{m+1}{2} \rceil$ and from lemma 2.2 and lemma 2.3, we know
$\sigma(Q(P_m, v_s, C_k)) = F_{s+1}F_{m-s+2}F_{k+1} + F_sF_{m-s+1}F_{k-1}$.
From lemma 3.1, we know
$\sigma(Q(P_m, v_s, C_k)) = \frac{1}{5}[(L_{m+3} + (-1)^s L_{m-2s+1})F_{k+1} + (L_{m+1} + (-1)^{s+1} L_{m-2s-1})F_{k-1}]$
$= \frac{1}{5}[(L_{m+3}F_{k+1} + L_{m+1}F_{k-1}) + (-1)^s L_{m-2s+1}F_{k-2}]$.
From above, we know that the result is correct.

Theorem 3.1. Let vertex v_s is any one of C_{k-k_1+1} which is one subgraph of $Q(C_{k_1}, v_1, C_{k-k_1+1}, v_s, P_{n-k+1})$ and $k - k_1 + 1 = 4m + i(i \in \{1, 2, 3, 4\})$ and $m \geq 2$, then
$\sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_1, P_{n-k+1})) > \sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_3, P_{n-k+1}))$
$> \cdots > \sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_{2m+1}, P_{n-k+1})) > \sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_{2m+2\rho}, P_{n-k+1}))$
$> \cdots > \sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_4, P_{n-k+1})) > \sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_2, P_{n-k+1}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. $\sigma(\sigma Q(C_{k_1}, v_1, C_{k-k_1+1}, v_s, P_{n-k+1}))$
$= F_{n-k+2}(F_{k_1+1}F_sF_{k-k_1-s+3} + F_{k_1-1}F_{s-1}F_{k-k_1-s+2})$
$+F_{n-k+1}(F_{k_1+1}F_{s-1}F_{k-k_1-s+2} + F_{k_1-1}F_{s-2}F_{k-k_1-s+1})$.
From lemma 2.2, lemma 2.3 and lemma 3.1, we know
$\sigma(\sigma Q(C_{k_1}, v_1, C_{k-k_1+1}, v_s, P_{n-k+1}))$
$= \frac{1}{5}\{F_{n-k+2}[F_{k_1+1}(L_{k-k_1+3} + (-1)^{s+1}L_{k-k_1-2s+3}) + F_{k_1-1}(L_{k-k_1+1} + (-1)^s L_{k-k_1-2s+3})]$
$+F_{n-k+1}[F_{k_1+1}(L_{k-k_1+1} + (-1)^s L_{k-k_1-2s+3}) + F_{k_1-1}(L_{k-k_1-1} + (-1)^{s-1}L_{k-k_1-2s+3})]\}$.
$= \frac{1}{5}\{F_{n-k+1}[F_{k_1+1}(L_{k-k_1+3} + L_{k-k_1+1}) + F_{k_1-1}(L_{k-k_1+1} + L_{k-k_1-1})]$
$+F_{n-k}[F_{k_1+1}(L_{k-k_1+3} + (-1)^{s+1}L_{k-k_1-2s+3}) + F_{k_1-1}(L_{k-k_1+1} + (-1)^s L_{k-k_1-2s+3})]\}$.
$= \frac{1}{5}\{[F_{n-k+2}(F_{k_1+1}L_{k-k_1+3} + F_{k_1-1}L_{k-k_1+1}) + F_{n-k+1}(F_{k_1+1}L_{k-k_1+1} + F_{k_1-1}L_{k-k_1-1})]$
$+(-1)^{s+1}L_{k-k_1-2s+3}F_{k_1}\}$.
From above, we know that the result is correct.

Theorem 3.2. Let $k = 4m + i(i \in \{1, 2, 3, 4\})$ and $m \geq 2$, we have
$\sigma(Q(C_4, v_1, C_{k+3}, v_2, P_{n-k+1})) > \sigma(Q(C_6, v_1, C_{k+5}, v_2, P_{n-k+1}))$
$> \cdots > \sigma(Q(C_{2m+2\rho}, v_1, C_{k-2m-2\rho+1}, v_2, P_{n-k+1})) > \sigma(Q(C_{2m+1}, v_1, C_{k-2m}, v_2, P_{n-k+1}))$
$> \cdots > \sigma(Q(C_5, v_1, C_{k-4}, v_2, P_{n-k+1})) > \sigma(Q(C_3, v_1, C_{k-2}, v_2, P_{n-k+1}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. From theorem 3.1, we know $Q(C_{k_1}, v_1, C_{k-k_1+1}, v_s, P_{n-k+1})$ with the smallest σ−index if $s = 2$ and
$\sigma(Q(C_{k_1}, v_1, C_{k-k_1+1}, v_2, P_{n-k+1}))$
$= F_{n-k+2}(F_{k_1+1}F_{k-k_1+1} + F_{k_1-1}F_{k-k_1}) + F_{n-k+1}F_{k_1+1}F_{k-k_1}$.
$= \frac{1}{5}\{F_{n-k+2}[(L_{k+2} + (-1)^{k_1}L_{k-2k_1}) + (L_{k-1} + (-1)^{k_1}L_{k-2k_1+1})]$
$+F_{n-k+1}(L_{k+1} + (-1)^{k_1}L_{k-2k_1-1})\}$.
From above, we known that the result is correct.

Theorem 3.3. Let $k = 4m + i (i \in \{1, 2, 3, 4\})$ and $m \geq 2$, then
$\sigma(Q(C_3, v_1, C_6, v_2, P_{n-5})) > \sigma(Q(C_3, v_1, C_8, v_2, P_{n-7}))$
$> \cdots > \sigma(Q(C_3, v_1, C_{2m+2\rho}, v_2, P_{n-2m-2\rho+1})) > \sigma(Q(C_3, v_1, C_{2m+1}, v_2, P_{n-2m}))$
$> \cdots > \sigma(Q(C_3, v_1, C_5, v_2, P_{n-6})) > \sigma(Q(C_3, v_1, C_3, v_2, P_{n-4}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof.
$$\begin{aligned}\sigma(Q(C_3, v_1, C_{k-2}, v_2, P_{n-k+1}))&= F_{n-k+4}F_{k-1} + F_{n-k+2}F_{k-3}\\&= \tfrac{1}{5}[(L_{n+3} + (-1)^k L_{n-2k+5}) + (L_{n-1} + (-1)^k L_{n-2k+5})]\\&= \tfrac{1}{5}[(L_{n+3} + L_{n-1}) + (-1)^k * 2 * L_{n-2k+5}].\end{aligned}$$
From above, we know that the result is correct.

Corollary 1. If two cycles of $(n, n+1)-$graphs have one common vertex, $Q(C_3, v_1, C_3, v_2, P_{n-4})$ is a graph with the smallest $\sigma-$index.

§4. The graph with the smallest Merrifield-Simmons index in $Q(C_k, v_1, P_l, u_1, C_m; n)$

In this section, we will find the $(n, n+1)-$ graphs with the smallest Merrifield-Simmons index in $Q(C_k, v_1, P_l, u_1, C_m; n)$.

Lemma 4.1. Let $h > 0$ and G be $Q(C_k, v_1, P_l, u_1, C_m)$ which attaches a path P_{h+1} at vertex u and G_1 is $Q(C_k, v_1, P_{l+h}, u_1, C_m)$, where u is any vertex of P_l. Then $\sigma(G) \geq \sigma(G_1)$.

Proof. From Lemma 2.2 and lemma 2.3, we know
$$\begin{aligned}\sigma(G) = &(F_{h+2}F_{s+1}F_{l-s} + F_{h+1}F_sF_{l-s-1})F_{k+1}F_{m+1}\\&+(F_{h+2}F_{s+1}F_{l-s-1} + F_{h+1}F_sF_{l-s-2})F_{k+1}F_{m-1}\\&+(F_{h+2}F_sF_{l-s} + F_{h+1}F_{s-1}F_{l-s-1})F_{k-1}F_{m+1}\\&+(F_{h+2}F_sF_{l-s-1} + F_{h+1}F_{s-1}F_{l-s-2})F_{k-1}F_{m-1}\end{aligned}$$
$$\sigma(G_1) = F_{l+h}F_{k+1}F_{m+1} + F_{l+h-1}F_{k+1}F_{m-1} + F_{l+h-1}F_{k-1}F_{m+1} + F_{l+h-2}F_{k-1}F_{m-1}$$
$$\begin{aligned}&\sigma(G) - \sigma(G_1)\\&= (F_{h+2}F_{s+1}F_{l-s} + F_{h+1}F_sF_{l-s-1} - F_{l+h})F_{k+1}F_{m+1}\\&\quad+(F_{h+2}F_{s+1}F_{l-s-1} + F_{h+1}F_sF_{l-s-2} - F_{l+h-1})F_{k+1}F_{m-1}\\&\quad+(F_{h+2}F_sF_{l-s} + F_{h+1}F_{s-1}F_{l-s-1} - F_{l+h-1})F_{k-1}F_{m+1}\\&\quad+(F_{h+2}F_sF_{l-s-1} + F_{h+1}F_{s-1}F_{l-s-2} - F_{l+h-2})F_{k-1}F_{m-1}\\&= F_hF_{s-1}F_{l-s-2}F_{k+1}F_{m+1} + F_hF_{s-1}F_{l-s-3}F_{k+1}F_{m-1} + F_hF_{s-2}F_{l-s-2}F_{k-1}F_{m+1}\\&\quad+F_hF_{s-2}F_{l-s-3}F_{k-1}F_{m-1} > 0.\end{aligned}$$
The proof is completed.

Lemma 4.2. Let $k = 4j + i (i \in \{1, 2, 3, 4\})$ and $j \geq 2$, we have
$\sigma(Q(C_k, v_1, P_l, u_1, C_m, v_1, P_h)) > \sigma(Q(C_k, v_1, P_l, u_1, C_m, v_3, P_h))$
$> \cdots > \sigma(Q(C_k, v_1, P_l, u_1, C_m, v_{2j+1}, P_h)) > \sigma(Q(C_k, v_1, P_l, u_1, C_m, v_{2j+2\rho}, P_h))$
$> \cdots > \sigma(Q(C_k, v_1, P_l, u_1, C_m, v_4, P_h)) > \sigma(Q(C_k, v_1, P_l, u_1, C_m, v_2, P_h))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

From lemma 2.2 and lemma2.3, we know
$\sigma(Q(C_k, v_1, P_l, u_1, C_m, v_s, P_h))$

$= F_{h+2}F_{m+1}F_lF_{k-s+1}F_s + F_{h+2}F_{m+1}F_{l-1}F_{k-s}F_{s-1} + F_{h+2}F_{m-1}F_{l-1}F_{k-s+1}F_s$
$+F_{h+2}F_{m-1}F_{l-2}F_{k-s}F_{s-1} + F_{h+1}F_{m+1}F_lF_{k-s}F_{s-1} + F_{h+1}F_{m+1}F_{l-1}F_{k-s-1}F_{s-2}$
$+F_{h+1}F_{m-1}F_{l-1}F_{k-s}F_{s-1} + F_{h+1}F_{m-1}F_{l-2}F_{k-s-1}F_{s-2}$
$= a + (-1)^{s+1} * b * L_{k-2s+1}$(where a, b are positive constant).

From above, we know that the result is correct.

Lemma 4.3. Let G_6 is $Q(C_k, v_1, P_{l+h}, u_1, C_m)$ and G_5 be obtained from $Q(C_k, v_1, P_l, u_1, C_m)$ which attach a path P_{h+1} at vertex where v_s is a vertex of C_k, we have $\sigma(G_6) < \sigma(G_5)$.

Proof. From lemma 2.2 and lemma 2.3, we know

$\sigma(G_5) = F_{h+2}F_{m+1}F_{l+k-1} + F_{h+2}F_{m-1}F_{l+k-2} + F_{h+1}F_{m+1}F_{k-1}F_l + F_{h+1}F_{m-1}F_{l-1}F_{k-1}$.

$\sigma(G_6) = F_{k+1}F_{m+1}F_{l+h} + F_{k+1}F_{m-1}F_{l+h-1} + F_{k-1}F_{m+1}F_{l+h-1} + F_{k-1}F_{m-1}F_{l+h-2}$.

$\sigma(G_5) - \sigma(G_6)$
$= (F_{h+2}F_{l+k-1} + F_{h+1}F_{k-1}F_l - F_{k+1}F_{l+h} - F_{k-1}F_{l+h-1})F_{m+1}$
$+(F_{h+2}F_{l+k-2} + F_{h+1}F_{k-1}F_{l-1} - F_{k+1}F_{l+h-1} - F_{k-1}F_{l+h-2})F_{m-1}$
$> (F_{h+2}F_{l+k-1} + F_{h+1}F_{l+k-2} - F_{k+1}F_{l+h+1})F_{m-1}$,
$= \frac{1}{5}(L_{h+l+k+1} + L_{h+l+k-1} - L_{h+k+l} + (-1)^kL_{l+h-k})F_{m-1} > 0$,

The proof is completed.

Lemma 4.4. For $Q(C_k, v_1, P_{n-k+2}, u_1, c_{k-k_1})$, we have

$\sigma(C_k, v_1, P_{n-k+2}, u_1, C_{k-k_1})$
$= F_{k-k_1+1}F_{k_1+1}F_{n-k+2} + F_{k-k_1+1}F_{k_1-1}F_{n-k+1} + F_{k-k_1-1}F_{k_1+1}F_{n-k+1} + F_{k-k_1-1}F_{k_1-1}F_{n-k}$.

Proof. From lemma 2.1 and lemma 2.3, it is proved easily.

Lemma 4.5. For $Q(C_k, v_1, P_{n-k+2}, u_1, C_{k-k_1})$, we have

$$\begin{aligned}
&\sigma(Q(C_k, v_1, P_{n-k+2}, u_1, C_{k-k_1}))\\
&= \frac{1}{5}\{(F_{n-k+2}L_{k+2} + 2F_{n-k+1}L_k + F_{n-k}L_{k-2}) + [(-1)^{k_1}(F_{n-k+2}L_{k-2_{k_1}}\\
&+F_{n-k+1}L_{k-2k_1+2} + F_{n-k+1}L_{k-2k_1-2} + F_{n-k}L_{k-2k_1})]\}.
\end{aligned}$$

Proof. From lemma 3.1 and Lemma 4.4, it is proved easily.

Theorem 4.1. Let $\lfloor \frac{k}{2} \rfloor = 4m + i, i \in \{1, 2, 3, 4\}$ and $m \geq 2$, then

$\sigma((C_4, v_1, P_{n-k+2}, u_1, C_{k-4})) > \sigma((C_6, v_1, P_{n-k+2}, u_1, C_{k-6}))$
$> \ldots > \sigma((C_{2m+2\rho}, v_1, P_{n-k+2}, u_1, C_{k-2m+2\rho})) > \sigma((C_{2m+1}, v_1, P_{n-k+2}, u_1, C_{k-2m-1}))$
$> \ldots > \sigma((C_5, v_1, P_{n-k+2}, u_1, C_{k-5})) > \sigma((C_3, v_1, P_{n-k+2}, u_1, C_{k-3}))$,

where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. By lemma 4.5, it is proved easily .

Corollary 1. $(C_3, v_1, P_{n-4}, u_1, C_3)$ be the $(n, n+1)$−graph with the smallest σ−index in $Q(C_k, v_1, P_l, u_1, C_m; n)$.

Proof. $\sigma((C_3, v_1, P_{n-k+2}, u_1, C_{k-3}))$
$= 3F_{k-2}F_{n-k+2} + F_{k-2}F_{n-k+1} + 3F_{k-4}F_{n-k+1} + F_{k-4}F_{n-k}$
$= \frac{1}{5}[3(L_{n+2} + (-1)^{k-1}L_{n-2k+4}) + (L_{n-1} + (-1)^{k-1}L_{n-2k+3}) + 3(L_{n-3} + (-1)^{k-1}L_{n-2k+5})$
$+(L_{n-4} + (-1)^{k-1}L_{n-2k+4})]$,

From above, we know that the result is correct.

§5. The graph with the smallest Merrifield-Simmons index in $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}; n)$

In this section, we will find the $(n, n+1)-$graph with the smallest Merrifield-Simmons index in $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}; n)$.

Lemma 5.1. For $Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1})$, we have

$$\begin{aligned}
&\sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1}))\\
&= F_{l_1+\triangle x}F_{l_1}(F_sF_{k-2l_1-\triangle x-s+5}F_{r+2} + F_{s-1}F_{k-2l_1-\triangle x-s+4}F_{r+1})\\
&\quad +F_{l_1+\triangle x-1}F_{l_1-1}(F_{s-1}F_{k-2l_1-\triangle x-s+5}F_{r+2} + F_{s-2}F_{k-2l_1-\triangle x-s+4}F_{r+1})\\
&\quad +F_{l_1+\triangle x-1}F_{l_1-1}(F_sF_{k-2l_1-\triangle x-s+4}F_{r+2} + F_{s-1}F_{k-2l_1-\triangle x-s+3}F_{r+1})\\
&\quad +F_{l_1+\triangle x-2}F_{l_1-2}(F_{s-1}F_{k-2l_1-\triangle x-s+4}F_{r+2} + F_{s-2}F_{k-2l_1-\triangle x-s+3}F_{r+1}).
\end{aligned}$$

Proof. By lemma 2.2 and lemma 2.3, it is proved easily.

Lemma 5.2. For $Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1})$, we have

$$\begin{aligned}
&\sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1}))\\
&= \frac{1}{5}\{[F_{l_1+\triangle x}F_{l_1}(L_{k-2l_1-\triangle x+5} + L_{k-2l_1-\triangle x+3} + (-1)^{s+1}L_{k-2l_1-\triangle x-2s+5}F_r)]\\
&\quad +[F_{l_1+\triangle x-1}F_{l_1-1}(L_{k-2l_1-\triangle x+4} + L_{k-2l_1-\triangle x+2} + (-1)^{s}L_{k-2l_1-\triangle x-2s+6}F_r)]\\
&\quad +[F_{l_1+\triangle x-1}F_{l_1-1}(L_{k-2l_1-\triangle x+4} + L_{k-2l_1-\triangle x+2} + (-1)^{s+1}L_{k-2l_1-\triangle x-2s+4}F_r)]\\
&\quad +[F_{l_1+\triangle x-2}F_{l_1-2}(L_{k-2l_1-\triangle x+3} + L_{k-2l_1-\triangle x+1} + (-1)^{s}L_{k-2l_1-\triangle x-2s+5}F_r)].
\end{aligned}$$

Proof. By lemma 3.1 and lemma 5.1, it is proved easily.

Theorem 5.1. Let v_s is a vertex of $P_{k-2l_1-\triangle x+4}$ which is a subgraph of $Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1})$ and $k - 2l_1 - \triangle x + 4 = 4m + i,\ i \in \{1, 2, 3, 4\}$ and $m \geq 2$, then

$$\begin{aligned}
&\sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_1, P_{r+1}))\\
&> \sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_3, P_{r+1}))\\
&> \cdots > \sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_{2m+1}, P_{r+1}))\\
&> \sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_{2m+2\rho}, P_{r+1}))\\
&> \cdots > \sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_4, P_{r+1}))\\
&> \sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_2, P_{r+1})).
\end{aligned}$$

Proof. From lemma 5.5, we know

$$\begin{aligned}
&\sigma(Q(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4}), v_s, P_{r+1}))\\
&= \frac{1}{5}\{[F_{l_1+\triangle x}F_{l_1}(L_{k-2l_1-\triangle x+5} + L_{k-2l_1-\triangle x+3} + (-1)^{s+1}L_{k-2l_1-\triangle x-2s+5}F_r)]\\
&\quad +[F_{l_1+\triangle x-1}F_{l_1-1}(L_{k-2l_1-\triangle x+4} + L_{k-2l_1-\triangle x+2} + (-1)^{s}L_{k-2l_1-\triangle x-2s+6}F_r)]\\
&\quad +[F_{l_1+\triangle x-1}F_{l_1-1}(L_{k-2l_1-\triangle x+4} + L_{k-2l_1-\triangle x+2} + (-1)^{s+1}L_{k-2l_1-\triangle x-2s+4}F_r)]\\
&\quad +[F_{l_1+\triangle x-2}F_{l_1-2}(L_{k-2l_1-\triangle x+3} + L_{k-2l_1-\triangle x+1} + (-1)^{s}L_{k-2l_1-\triangle x-2s+5}F_r)].\\
&= \frac{1}{5}\{[F_{l_1+\triangle x}F_{l_1}(L_{k-2l_1-\triangle x+5} + L_{k-2l_1-\triangle x+3})\\
&\quad +[F_{l_1+\triangle x-1}F_{l_1-1}(L_{k-2l_1-\triangle x+5} + L_{k-2l_1-\triangle x+3})\\
&\quad +[F_{l_1+\triangle x-2}F_{l_1-2}(L_{k-2l_1-\triangle x+3} + L_{k-2l_1-\triangle x+1})\\
&\quad +(-1)^{s+1}L_{k-2l_1-\triangle x-2s+5}F_r(F_{l_1+\triangle x-1}F_{l_1-2} + F_{l_1+\triangle x-2}F_{l_1-1})\}.
\end{aligned}$$

From above, we know that the result is correct.

Lemma 5.3. Let G_8 is $Q(P_{l_1}, P_{l_2}, P_{l_3} + h)$ and G_7 is obtained from $Q(P_{l_1}, P_{l_2}, P_{l_3})$ which attaches a path P_{h+1} at vertex u of P_{l_3} and $h > 0$; Then $\sigma(G_7) > \sigma(G_8)$.

Proof. $\sigma(G_7) = F_{l_1+2}F_{l_2+2}F_{l_3+h+2} + F_{l_1+1}F_{l_2+1}F_{l_3+h+1} + F_{l_1+1}F_{l_2+1}F_{l_3+1}F_{h+2} + F_{l_1}F_{l_2}F_{l_3}F_{h+2}$

$\sigma(G_8) = F_{l_1+2}F_{l_2+2}F_{l_3+h+2} + F_{l_1+1}F_{l_2+1}F_{l_3+h+1} + F_{l_1+1}F_{l_2+1}F_{l_3+h+1} + F_{l_1}F_{l_2}F_{l_3+h}$

$$\begin{aligned}\sigma(G_7) - \sigma(G_8) &= F_{l_1+1}F_{l_2+1}(F_{h+2}F_{l_3+1} - F_{l_3+h+1}) + F_{l_1+1}F_{l_2+1}(F_{h+2}F_{l_3} - F_{l_3+h})\\ &= F_{l_1+1}F_{l_2+1}(F_{h+2}F_{l_3+3} - F_{l_3+h+2})\\ &= F_{l_1+1}F_{l_2+1}F_hF_{l_3} > 0.\end{aligned}$$

The proof is completed.

Lemma 5.4. For $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})$, we have

$$\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) = F_{l_1+\triangle x}F_{l_1}F_{k-2l_1-\triangle x+4} + 2F_{l_1+\triangle x-1}F_{l_1-1}F_{k-2l_1-\triangle x+3} + F_{l_1+\triangle x-2}F_{l_1-2}F_{k-2l_1-\triangle x+2}.$$

Proof. By lemma 2.2 and 2.3, it is proved easily.

Lemma 5.5. For $Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})$, we have

If $l_1 \geq 4$, then

$$\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) = \tfrac{1}{5}[(F_{l_1}L_{k-l_1+4} + 2F_{l_1-1}L_{k-l_1+2} + F_{l_1-2}L_{k-l_1}) + (-1)^{l_1+\triangle x+1}L_{k-3l_1-2\triangle x+4}F_{l_1-4}].$$

If $l_1 = 2$, then

$$\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) = \sigma(Q(P_2, P_{2+\triangle x}, P_{k-\triangle x})) = \tfrac{1}{5}[(L_{k+2} + 2L_k) + (-1)^{\triangle x}L_{k-2\triangle x-2}].$$

If $l_1 = 3$, then

$$\sigma(Q(P_3, P_{3+\triangle x}, P_{k-\triangle x-2})) = \frac{1}{5}[(2L_{k+1} + 2L_{k-1} + L_{k-3}) + (-1)^{\triangle x+4}L_{k-2\triangle x-5}].$$

Proof. By lemma 2.2, lemma 2.3 and lemma 3.1, it is proved easily.

Theorem 5.2. For $Q(P_{l_1}, P_{l_1+\triangle x}, P_{n-2l_1-\triangle x+4})$ with n vertices and $n - 3l_1 + 4 = 4m + i$ and $i \in \{1, 2, 3, 4\}$, we have

(1) If $l_1 \geq 4$ and l_1 is even, then

$\sigma(Q(P_{l_1}, P_{l_1+1}, P_{n-2l_1+3})) > \sigma(Q(P_{l_1}, P_{l_1+3}, P_{n-2l_1+1}))$
$> \cdots > \sigma(Q(P_{l_1}, P_{l_1+2m+1}, P_{n-2l_1-2m+2})) > \sigma(Q(P_{l_1}, P_{l_1+2m+3\rho}, P_{n-2l_1-2m-2\rho+4}))$
$> \sigma(Q(P_{l_1}, P_{l_1+2}, P_{n-2l_1+2})) > \sigma(Q(P_{l_1}, P_{l_1}, P_{n-2l_1+4}))$.

(2) If $l_1 \geq 4$ and l_1 is odd, then

$\sigma(Q(P_{l_1}, P_{l_1}, P_{n-2l_1+4})) > \sigma(Q(P_{l_1}, P_{l_1+2}, P_{n-2l_1+2}))$
$> \cdots > \sigma(Q(P_{l_1}, P_{l_1+2m+2\rho}, P_{n-2l_1-2m-2\rho+4})) > \sigma(Q(P_{l_1}, P_{l_1+2m+1}, P_{n-2l_1-2m+3}))$
$> \sigma(Q(P_{l_1}, P_{l_1+1}, P_{n-2l_1+3}))$.

(3) If $l_1 = 2$, then

$\sigma(Q(P_2, P_4, P_{n-2})) > \sigma(Q(P_2, P_6, P_{n-4}))$
$> \cdots > \sigma(Q(P_2, P_{2m+2\rho+2}, P_{n-2m-2\rho})) > \sigma(Q(P_2, P_{2m+3}, P_{n-2m-1}))$
$> \cdots > \sigma(Q(P_2, P_5, P_{n-3})) > \sigma(Q(P_2, P_3, P_{n-1}))$.

(4) If $l_1 = 3$, then

$\sigma(Q(P_3, P_3, P_{n-2})) > \sigma(Q(P_2, P_5, P_{n-3}))$
$> \cdots > \sigma(Q(P_3, P_{2m+2\rho+3}, P_{n-2m-2\rho-2})) > \sigma(Q(P_2, P_{2m+4}, P_{n-2m-2}))$
$> \cdots > \sigma(Q(P_3, P_6, P_{n-5})) > \sigma(Q(P_3, P_4, P_{n-3}))$.

where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. By lemma 5.4, it is proved easily.

By reckon, the follow unequality is correct.

$\sigma((C_3, v_1, P_{n-4}, u_1, C_3)) = 5F_{n-2}$.

$\sigma(Q(P_2, P_3, P_{n-1})) = 2F_n$.

$\sigma(Q(P_4, P_4, P_{n-4})) = 3F_{n-1} + F_{n-3}$.

$\sigma(Q(P_3, P_4, P_{n-3})) = 3F_{n-1} + F_{n-3}$.

$\sigma(Q(P_6, P_6, P_{n-8})) = 64F_{n-8} + 50F_{n-9} + 9F_{n-10}$.

$\sigma(Q(P_5, P_6, P_{n-7})) = 2F_{n-3} + 18F_{n-5}$.

According to theorem 5.2 and lemma 2.1 and lemma 3.1, the follow unequality is correct.

Corollary 1. $\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) > \sigma(Q(P_2, P_3, P_{n-1}))$
or $\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) > \sigma(Q(P_4, P_4, P_{n-4}) = \sigma(P_3, P_4, P_{n-3}))$
or $\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) > \sigma(Q(P_{l_1}, P_{l_1+1}, P_{n-2l_1+3}))$
or $\sigma(Q(P_{l_1}, P_{l_1+\triangle x}, P_{k-2l_1-\triangle x+4})) > \sigma(Q(P_6, P_6, P_{n-8}))$ and
$\sigma(Q(P_2, P_3, P_{n-1})) < \sigma(Q(P_4, P_4, P_{n-4}))$,$\sigma(Q(P_2, P_3, P_{n-1})) < \sigma(Q(P_6, P_6, P_{n-8}))$ and
$\sigma(Q(P_2, P_3, P_{n-1})) < \sigma(Q(P_{l_1}, P_{l_1+1}, P_{n-2l_1+3}))$.

Corollary 2. The $(n, n+1)-$graph with the smallest $\sigma-$ index is $Q(C_3, v_1, P_{n-4}, u_1, C_3)$.

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Parameters estimation for a mixture of Inverse Weibull distributions from censored data [1]

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Abstract In this paper a failure model of the mixed Inverse Weibull distributions(MIWD) is considered and its estimators for all unknown parameters based on type-II and type-I censored data are obtained by means of EM-Algorithm. Some simulations suggest that EM algorithm is effective to our model.

Keywords mixed Inverse Weibull distribution, failure model, EM-algorithm, censored data.

§1. Introduction

Mixture models play a important role in many applicable fields, such as medicine, psychology research, cluster analysis, life testing and reliability analysis and so on. Mixture distributions have been considered extensively by many authors, see Mclachlan and Peel (2000), Seidel, Mosler and Alker (2000), Bauwens, Hafner and Rombouts (2007) and so on. Sultan, Ismail and Al-Moisheer (2007) discussed the properties and parameters estimation of the mixture model of two Inverse Weibull distributions. In this paper we discuss some properties and parameters estimation of mixed Inverse Weibull distribution(MIWD) from censored data. The mixture of Inverse Weibull distributions has its pdf as

$$f(x,\eta)=\sum_{i=1}^{m}p_i f_i(x,\eta_i). \tag{1}$$

The pdf of the (i)th component is given by

$$f_i(x,\eta_i)=\lambda_i\alpha_i^{-\lambda_i}x^{-(\lambda_i+1)}e^{-(\alpha_i x)^{-\lambda_i}},$$

where $\eta=(p_1,\cdots,p_{m-1},\alpha_1,\cdots,\alpha_m,\lambda_1,\cdots,\lambda_m),\eta_i=(\alpha_i,\lambda_i),0<p_i<1,i=1,\cdots,m-1$ $p_m=1-\sum_{i=1}^{m-1}p_i\alpha_i,\lambda_i>0i=1,2,\cdots,m$ $x>0$, there are $3m-1$ parameters in all. The remainder of this paper has the following organization. In section 2, we discuss some properties of the MIWD given in (1). In section 3, we consider parameters estimation of the MIWD given in (1) under censored data by mean of EM-Algorithm. In section 4, some simulations are carried out to illustrate the estimation technique in section 3. In the last section, we draw some conclusion about this paper.

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§2. Some important properties

Sultan, Ismail and Al-Moisheer (2007) discussed some properties of the mixture of Inverse Weibull distributions with two components, such as expectation, variance and failure function and so on. In following discussion, some properties of the MIWD with the finite components are given.

If X follows the pdf of the MIWD given in (1),we have the following results:

(1) $EX^k = \sum_{i=1}^{m} \frac{p_i}{\alpha_i^k} \Gamma(1 - \frac{k}{\lambda_i}), \quad k < min\{\lambda_1, \cdots, \lambda_m\},$

where $\Gamma(\cdot)$ is the *gamma* function.

(2) The mode and the median of the model (1) are obtained by solving the following nonlinear equation with respect to t

$$\sum_{i=1}^{m} p_i \lambda_i \alpha_i^{-\lambda_i} t^{-(\lambda_i+2)} e^{-(\alpha_i t)^{-\lambda_i}} [-(\lambda_i + 1) + \lambda_i \alpha_i^{-\lambda_i} t^{-\lambda_i}] = 0,$$

$$\sum_{i=1}^{m} p_i e^{-(\alpha_i t)^{-\lambda_i}} = 0.5.$$

(3) The reliability function and the failure function are given by

$$R(t) = 1 - \sum_{i=1}^{m} p_i e^{-(\alpha_i t)^{-\lambda_i}},$$

$$r(t) = \frac{\sum_{i=1}^{m} p_i [\lambda_i \alpha_i^{-\lambda_i} t^{-(\lambda_i+1)} e^{-(\alpha_i t)^{-\lambda_i}}]}{1 - \sum_{i=1}^{m} p_i e^{-(\alpha_i t)^{-\lambda_i}}}.$$

(4) Sultan, Ismail and Al-Moisheer (2007) had proven the class of all finite mixing distributions relative to the Inverse Weibull distributions is identifiable. In this paper, we suppose $\lambda_1 < \cdots < \lambda_m$ as the identifying restriction of model(1).

§3. Parameters estimation via EM-Algorithm

The maximum likelihood estimation (MLE) of the failure model (1) for mixed Inverse Weibull distribution is hard to obtain because of complexity of its likelihood function. EM-Algorithm is to transform the computation of MLE to maximize a so-called Q function. Censored data has two censored mechanism including type-I samples and type-II samples. We only discuss type-II samples detailedly in the following paper.

§3.1 For the type-II censored samples

By carrying failure experiments using n independent samples from model (1) at the same time till we obtain r type-II censored samples and their lives are denoted by $x_1 \cdots x_r$ respectively. Obviously, $x_1 \cdots x_r$ are the complete observation data. The remaining $n-r$ observations denoted by $x_{r+1}, \cdots, x_n$ are censored data and we have $x_{r+1} = \cdots = x_n = x_r$.

Suppose $X = (x_1, x_2, \cdots, x_n)$ are n independent observations of model (1), and denote

$$\eta = (p_1, \cdots, p_{m-1}, \alpha_1, \cdots, \alpha_m, \lambda_1, \cdots, \lambda_m),$$

$$f_{ij} = \lambda_i \alpha_i^{-\lambda_i} x_j^{-(\lambda_i+1)} e^{-(\alpha_i x_j)^{-\lambda_i}} \quad f_j = \sum_{i=1}^{m} p_i f_{ij} \ , i = 1, \cdots, m, j = 1, \cdots, r,$$

$$s_{ij} = 1 - e^{-(\alpha_i x_j)^{-\lambda_i}} \quad s_j = \sum_{i=1}^{m} p_i s_{ij} \ , i = 1, \cdots, m, j = r+1, \cdots, n,$$

where s_x is survival function of life sample. if x_j follows the pdf of model (1), a indicator vector of x_j is given by $I_j = (I_{1j}, I_{2j}, \cdots, I_{mj})$, where I_{ij} is dichotomous variable taking the value 1 if x_j comes from the (i)th mixture component, and 0 otherwise. In addition, $I_j = (I_{1j}, I_{2j}, \cdots, I_{mj})$ follows the multiple distribution. But we don't know which component the observation x_j comes form in fact. Namely, I_j is not observable, here we can deem it as the missing data in EM algorithm. we denote $I = (I_1, \cdots, I_n)$.

For the complete observation data x_j, the joint density of x_j and I_j is given by $g(x_j, I_j|\eta) = \prod_{i=1}^{m}(p_i f_{ij})^{I_{ij}}$. Given x_j and η, the conditional density of I_{ij} is

$$P(I_{ij} = 1|x_j, \eta) = \frac{p_i f_{ij}}{f_j} i = 1, 2, \cdots, m \quad p_m = 1 - \sum_{i=1}^{m-1} p_i.$$

For the censored data x_j, the joint density of x_j and I_j is given by $g(x_j, I_j|\eta) = \prod_{i=1}^{m}(p_i s_{ij})^{I_{ij}}$. Given x_j and η, the conditional density of I_{ij} is

$$P(I_{ij} = 1|x_j, \eta) = \frac{p_i s_{ij}}{s_j} i = 1, 2, \cdots, m \quad p_m = 1 - \sum_{i=1}^{m-1} p_i.$$

For given initial value $\eta^{(0)}$ of the unknown parameter vector, we can obtain parameter estimators of model (1) based on EM algorithm via the following two steps.
E step: Given the $(t-1)$th iteration value $\eta^{(t-1)}$ of η, the Q function of the (t)th iteration is

$$\begin{aligned} Q(\eta|\eta^{(t-1)}) &= E_{\eta^{(t-1)}}[L\ (\eta|X, I)\ ] \\ &= \sum_{j=1}^{r}\sum_{i=1}^{m} c_{ij}^{(t-1)} log(p_i f_{ij}) + \sum_{j=r+1}^{n}\sum_{i=1}^{m} d_{ij}^{(t-1)} log(p_i s_{ij}) \end{aligned}$$

where

$$\eta^{(t-1)} = (p_1^{(t-1)}, \cdots, p_{m-1}^{(t-1)}, \alpha_1^{(t-1)}, \cdots, \alpha_m^{(t-1)}, \lambda_1^{(t-1)}, \cdots, \lambda_m^{(t-1)}),$$

$$f_{ij}^{(t-1)} = f_{ij}(\eta^{(t-1)}) f_j^{(t-1)} = f_j(\eta^{(t-1)}) s_{ij}^{(t-1)} = s_{ij}(\eta^{(t-1)}),\ s_j^{(t-1)} = s_j(\eta^{(t-1)}),$$

$$c_{ij}^{(t-1)} = \frac{p_i^{(t-1)} f_{ij}^{(t-1)}}{f_j^{(t-1)}}, d_{ij}^{(t-1)} = \frac{p_i^{(t-1)} s_{ij}^{(t-1)}}{s_j^{(t-1)}} \quad p_m^{(t-1)} = 1 - \sum_{i=1}^{m-1} p_i^{(t-1)}.$$

M step: We maximize numerically $Q(\eta|\eta^{(t-1)})$ with respect to η to update estimates of the parameters, denoted by $\eta^{(t)}$. First, we have the following results

$$\frac{\partial Q}{\partial p_l} = \sum_{j=1}^{r}(\frac{c_{lj}^{(t-1)}}{p_l} - \frac{c_{mj}^{(t-1)}}{1 - \sum_{i=1}^{m-1} p_i}) + \sum_{j=r+1}^{n}(\frac{d_{lj}^{(t-1)}}{p_l} - \frac{d_{mj}^{(t-1)}}{1 - \sum_{i=1}^{m-1} p_i}),\ l = 1, \cdots, m-1, \quad (2)$$

$$\frac{\partial Q}{\partial \alpha_k} = \sum_{j=1}^{r} c_{kj}^{(t-1)}[-\frac{\lambda_k}{\alpha_k} + (\lambda_k x_j)(\alpha_k x_j)^{-\lambda_k - 1}] + \sum_{j=r+1}^{n} d_{kj}^{(t-1)} \frac{(\lambda_k x_j)(\alpha_k x_j)^{-\lambda_k - 1}}{1 - e^{(\alpha_k x_j)^{-\lambda_k}}},\ k = 1, \cdots, m, \quad (3)$$

$\frac{\partial Q}{\partial \lambda_k} = \sum_{j=1}^{r} c_{kj}^{(t-1)}[\frac{1}{\lambda_k} - log(\alpha_k x_j)(1-(\alpha_k x_j)^{-\lambda_k})] + \sum_{j=r+1}^{n} d_{kj}^{(t-1)} \frac{(\alpha_k x_j)^{-\lambda_k} log(\alpha_k x_j)}{1-e^{(\alpha_k x_j)^{-\lambda_k}}}, \quad (4)$
$k = 1, \cdots, m.$

Form (2), we obtain

$$p_l[\sum_{j=1}^{r} c_{mj}^{(t-1)} + \sum_{j=r+1}^{n} d_{mj}^{(t-1)}] + (\sum_{i=1}^{m-1} p_i)[\sum_{j=1}^{r} c_{lj}^{(t-1)} + \sum_{j=r+1}^{n} d_{lj}^{(t-1)}] = \sum_{j=1}^{r} c_{lj}^{(t-1)} + \sum_{j=r+1}^{n} d_{lj}^{(t-1)},$$

where $l = 1, \cdots, m-1$. From the above equation, we know the the (t)th iteration value in the M-step with respect to parameters $p_l, \cdots, p_{m-1}$ is the solutions of the nonlinear equation denoted by $AP = b$, where P, A, b are given by

$$P = (p_1, p_2, \cdots, p_{m-1})^T,$$

$$A = (a_{ts}) \; a_{ts} = \begin{cases} [\sum_{j=1}^{r} c_{tj}^{(t-1)} + \sum_{j=r+1}^{n} d_{tj}^{(t-1)}] + [\sum_{j=1}^{r} c_{mj}^{(t-1)}) + \sum_{j=r+1}^{n} d_{mj}^{(t-1)}] & t = s \\ \sum_{j=1}^{r} c_{tj}^{(t-1)} + \sum_{j=r+1}^{n} d_{tj}^{(t-1)} & t \neq s \end{cases},$$

$$b = (\sum_{j=1}^{r} c_{1j}^{(t-1)} + \sum_{j=r+1}^{n} d_{1j}^{(t-1)}, \sum_{j=1}^{r} c_{2j}^{(t-1)} + \sum_{j=r+1}^{n} d_{2j}^{(t-1)}, \cdots, \sum_{j=1}^{r} c_{m-1,j}^{(t-1)} + \sum_{j=r+1}^{n} d_{m-1,j}^{(t-1)})^T.$$

Because of $\sum_{j=1}^{r} c_{tj}^{(t-1)} + \sum_{j=r+1}^{n} d_{tj}^{(t-1)} > 0,\ t = 1, \cdots, m-1$，we can obtain that $rank(A) = m-1$，namely, A is a reversible matrix. Thus, the only solution of parameter vector P of the (t)th iteration in the M-step is given by

$$P^{(t)} = (p_1^{(t)}, p_2^{(t)}, \cdots, p_{m-1}^{(t)})^T = A^{-1}b. \quad (5)$$

Form (3), we have

$$\alpha_k = [\frac{\sum_{j=1}^{r} c_{kj}^{(t-1)}}{\sum_{j=1}^{r} c_{kj}^{(t-1)} \cdot x_j^{-\lambda_k} + \sum_{j=r+1}^{n} d_{kj}^{(t-1)} \frac{x_j^{-\lambda_k}}{1-e^{(\alpha_k x_j)^{-\lambda_k}}}}]^{-1/\lambda_k},\ k = 1, 2, \cdots, m. \quad (6)$$

Form (4), we have

$$\lambda_k = \frac{\sum_{j=1}^{r} c_{kj}^{(t-1)}}{\sum_{j=1}^{r} c_{kj}^{(t-1)} log(\alpha_k x_j) \cdot [1-(\alpha_k x_j)^{-\lambda_k}] - \sum_{j=r+1}^{n} d_{kj}^{(t-1)} log(\alpha_k x_j) \cdot \frac{(\alpha_k x_j)^{-\lambda_k}}{1-e^{(\alpha_k x_j)^{-\lambda_k}}}},\ k = 1, 2, \cdots, m. \quad (7)$$

we can obtain the (t)th iteration value of α_k and λ_k denoted by $\alpha_k^{(t)}$ and $\lambda_k^{(t)}$ for $k = 1, 2, \cdots, m$ in the M-step if we choose $\lambda_k = \lambda_k^{(t-1)}, \alpha_k = \alpha_k^{(t-1)}$ in formula (6) and $\lambda_k = \lambda_k^{(t-1)}, \alpha_k = \alpha_k^{(t)}$ in formula (7), see Seidel, Mosler and Alker (2000).

We can update $\eta^{(t-1)}$ as $\eta^{(t)} = (p_1^{(t)}, \cdots, p_{m-1}^{(t)}, \alpha_1^{(t)}, \cdots, \alpha_m^{(t)}, \lambda_1^{(t)}, \cdots, \lambda_m^{(t)})$ by repeating E step and M step till $|log(\eta^{(t)}) - log(\eta^{(t-1)})| < n \cdot 10^{-a} (a \in N)$, see Sultan, Ismail和Al-Moisheer (2007).

§3.2 For the type-I censored samples

By carrying failure experiment using n independent samples from model (1), we obtain r type-I censored samples till given time T, and we can get parameters estimation by just replacing $x_{r+1} = \cdots = x_n = T$ in the place of the type-II censored samples.

§3.3 For the complete samples

By carrying failure experiment using n independent samples from model (1), we obtain n complete observation data till all products become invalid, and we can get parameters estimation by only letting $r = n$ in the place of the type-II censored samples.

§4. Simulation

In this section, we calculate the estimates of unknown parameters of model (1) by using EM algorithm in a Monte Carlo simulation. We only compare parameters estimates of the two type-II censored data with the complete data for a mixture of two Inverse Weibull distributions. We carry out repeated experiments 1000 times under samples of sizes $n = 25, 50, 75$ for each of choice the vector of the unknown parameters. If we denote parameters estimates of the (k)th experiment are $\eta^k = (p_1^k, \alpha_1^k, \alpha_2^k, \lambda_1^k, \lambda_2^k)$, the final means and mean square errors(mse) of the estimates are respectively given by $mean_j = \frac{1}{1000}\sum_{k=1}^{1000}\eta_j^k mse_j = \frac{1}{1000}\sum_{k=1}^{1000}(\eta_j^k - mean_j)^2$, for $j = 1, 2, \cdots, m$, where η_j is the (j)th coordinate of the unknown parameters vector η. The computation results are presented in Tables 1, Tables 2, Tables 3 and Tables 4.

Table 1: Means of parameters estimation based on EM-Algorithm with two type-II censored data

$\eta = (p_1, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$	n	*mean*				
		$\hat{p_1}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\lambda_1}$	$\hat{\lambda_2}$
$(0.7, 2.5, 1, 2, 3)$	25	$5.988E-1$	2.672	1.095	2.729	3.572
	50	$6.320E-1$	2.609	1.084	2.368	3.186
	75	$6.446E-1$	2.592	1.075	2.244	3.023
$(0.3, 1, 2, 2, 3)$	25	$3.467E-1$	1.038	2.036	3.442	4.133
	50	$3.199E-1$	1.041	2.003	2.404	3.410
	75	$3.109E-1$	1.055	1.993	2.199	3.232

Table 2: Mse of parameters estimation based on EM-Algorithm with two type-II censored data

$\eta = (p_1, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$	n	*mse*				
		$\hat{p_1}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\lambda_1}$	$\hat{\lambda_2}$
$(0.7, 2.5, 1, 2, 3)$	25	$3.748E-2$	$6.860E-1$	$1.266E-1$	2.366	3.346
	50	$1.497E-2$	$2.135E-1$	$4.349E-2$	$9.414E-1$	1.451
	75	$1.014E-2$	$1.324E-1$	$2.424E-2$	$4.943E-1$	$7.064E-1$
$(0.3, 1, 2, 2, 3)$	25	$1.804E-2$	$9.441E-2$	$1.851E-1$	6.979	8.371
	50	$8.159E-3$	$3.768E-2$	$5.772E-2$	1.288	1.790
	75	$5.493E-3$	$1.951E-2$	$2.081E-2$	1.126	$9.985E-1$

From Table 1 and 2, we see EM algorithm is effective to the estimation of the unknown parameters of the model (1) under type-II censored data, and the mean square errors of most of estimated parameters decrease as n increases. But we also find that the estimation algorithm will be of no effect if there are too much censored data and few samples in our simulations.

Table 3: Means of parameters estimation based on EM-Algorithm under the complete data

$\eta=(p_1,\alpha_1,\alpha_2,\lambda_1,\lambda_2)$	n	*mean*				
		$\hat{p_1}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\lambda_1}$	$\hat{\lambda_2}$
$(0.7, 2.5, 1, 2, 3)$	25	$6.005E-1$	2.674	1.094	2.711	3.499
	50	$6.306E-1$	2.615	1.080	2.367	3.157
	75	$6.428E-1$	2.598	1.072	2.246	3.049
$(0.3, 1, 2, 2, 3)$	25	$3.551E-1$	1.051	2.044	2.945	4.373
	50	$3.253E-1$	1.056	2.007	2.307	3.514
	75	$3.145E-1$	1.056	1.997	2.177	3.267

Table 4: Mse of parameters estimation based on EM-Algorithm under the complete data

$\eta=(p_1,\alpha_1,\alpha_2,\lambda_1,\lambda_2)$	n	*mse*				
		$\hat{p_1}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\lambda_1}$	$\hat{\lambda_2}$
$(0.7, 2.5, 1, 2, 3)$	25	$3.155E-2$	$5.653E-1$	$1.052E-1$	2.205	2.850
	50	$1.459E-2$	$2.094E-1$	$4.192E-2$	$8.366E-1$	1.363
	75	$9.370E-3$	$1.154E-1$	$2.169E-2$	$4.087E-1$	$7.236E-1$
$(0.3, 1, 2, 2, 3)$	25	$1.983E-2$	$6.912E-2$	$1.232E-1$	4.657	$1.197E1$
	50	$9.753E-3$	$3.375E-2$	$5.309E-2$	$7.824E-1$	3.876
	75	$5.760E-3$	$1.950E-2$	$2.175E-2$	$4.156E-1$	1.184

From Table 3 and 4, we see EM algorithm is very effective to the estimation of the unknown parameters of the model (1) under the complete data, and the mean square errors(mse) of most of estimated parameters decrease as n increases. In addition, we see that the estimation algorithm has better effect from the view of mse than of the type-II censored data at the same samples. Therefore, we should do the complete life observations in life experiment as much as posssible.

§5. Conclusion

We discuss the estimation of the unknown parameters of the mixture of Inverse Weibull distributions denoted by model (1) by mean of EM-Algorithm from the type-II censored data and type-I censored data, and some Monte Carlo simulations are carried out to investigate the performance of the estimation technique in this paper.

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Signed Domination in Relative Character Graphs

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Abstract This paper deals with the author's initiation of signed domination functions in Relative character Graphs (RC-graphs). Given any finite group G and a subgroup H, one can construct a finite, simple, undirected graph $\Gamma(G, H)$. We introduce two canonical functions f and f^* from the vertex set $V = IrrG$ into the set $\{-1, 1\}$ satisfying the neighborhood condition. Then f or f^* is called a signed domination function. Those RC-graphs for which f or f^* is a domination function is called an RC-signed dominated graph.
In this paper we prove that all abelian groups (and subgroups), groups giving rise to K_n, the Frobenius groups with certain special properties are RC-signed dominated. We also study those RC-graphs Γ for which a comparison can be made between a minimum dominating RC-graph number $\gamma_R(\Gamma)$ and the signed domination number $\gamma_S(\Gamma)$ (In some cases these number coincide). Finally certain bounds for are $\gamma_R(\Gamma(G, H)) + \gamma_R(\overline{\Gamma}(G, H))$ obtained in line with similar results for arbitrary graphs obtained by Haas and Wexler.

Keywords Irreducible characters, RC-graphs, signed domination function.

§1. Introduction

Around the year 2000, T. Gnanaseelan, a Ph.D. scholar of Prof. A.V. Jeyakumar, M.K. University constructed a finite, simple, undirected graph for any finite group G and any subgroup H of G, and called it the Relative Character Graph (shortened as RC-graphs, later [1]). Prof. S. Donkin of the Queen Mary College, London wrote in his communication to us that "this construction is new and interesting". Also many later students of Prof. A.V. Jeyakumar obtained various other results and ramifications of the original construction.

Definition 1. The vertex set V of $\Gamma(G, H)$ is the set of all irreducible (complex) characters of G and given any subgroup H of G, two vertices α and β are adjacent precisely when their restrictions α_H and β_H to H contain atleast one common irreducible character of H. Evidently $\Gamma(G, H)$ is a finite, simple undirected graph. (For details, see [4]).

During the year 2005, Prof. A.V. Jeyakumar presented some of these results with some more additions on domination at the GDDSA, Yadava College, Madurai [5] and the author some more works at the National Conference held at SRM University (2007) [7].

The present work is the author's attempt to connect RC-graphs and the concept of 'signed domination'. The background for this new venture is the famous three author book [2], the paper of Haas and Wexler [3], and that of S.B. Rao [6].

§2. Domination Functions in RC-Graphs

Before going into signed domination, we will first revisit Domination Theory of RC-graphs in the framework of dominating functions. First recall the following for an arbitrary finite, simple graph $\Gamma = (V, E)$.

Given any $v \in V$, the closed neighbourhood $N[v]$ of v is $\{u : uv \in E\} \cup \{v\}$. A function $f : V \to \{0, 1\}$ is a dominating function if $f[v] = \sum_{x \in N[v]} f(x) \geq 1$ for all $v \in V$.

The weight of f, denoted by $f(\Gamma)$ is the sum $\sum_{v \in V} f(v)$.

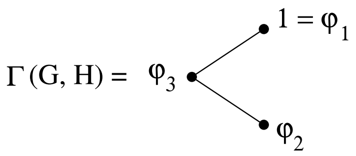
The domination number $\gamma(\Gamma)$ is the minimum weight of all dominating functions on Γ.

From now let $\Gamma(G, H)$ denote the RC-graph of a finite group G and a subgroup H. We shall associate a function $F : V \to [0, 1]$ where $V = IrrG$ as follows: Let $\varphi \in IrrG$. Let $\varphi_H = \Sigma e_i \theta_i$, $e_i \geq 0$ then $\{\theta_i\} \subset IrrH$. Define

$$F(\varphi) = \begin{cases} 0 & \text{if } m(\varphi) = \Sigma e_i \text{ is even} \\ 1 & \text{if } m(\varphi) = \Sigma e_i \text{ is odd} \end{cases}$$

Note that this definition is canonical and the 'best possible' (atleast for the moment) in the framework of RC-graphs.

An easy example is the following: $G = S_3$, $H = \langle (12) \rangle$.



Now $\varphi_{1_H} = 1_H$, $\varphi_{2_H} = 1_H$, $\varphi_{3_H} = 1_H + \theta$, θ the other non-trivial character of H.

Then $m(\varphi_1) = 1$, $m(\varphi_2) = 1$, $m(\varphi_3) = 2$. This in turn gives $F(\varphi_1) = F(\varphi_2) = 1$, $F(\varphi_3) = 0$. It is clear that $F[v] \geq 1$ for all $v \in V$. Hence F is a dominating function.

We immediately encounter a problem. Given any RC-graph $\Gamma(G, H)$, does the function F satisfy the basic condition $F[v] \geq 1$ for all $v \in V$. The answer, to this question is No, in general. The following example shows that there are groups and subgroups for which F is not a dominating function.

Let G be the dihedral group of order 2^m for a large m and H the (non-normal) subgroup of order 2. There are only 4 linear characters and the rest are all irreducibles of degree 2_i for some $i > 0$. $\Gamma(G, H)$ is connected, not a tree and we can always find a $\varphi \in V$ such that deg $\varphi = 2^n$, $n > 0$ and is adjacent to some ψ_i of degree 2^i, $i > 0$.

Since H is cyclic of order 2, all these φ and ψ_i must break into linear characters which shows that $m(\varphi)$ and $m(\psi_i)$ are even. Hence $F(\varphi)$, proving that F is not a dominating function.

Definition 2. A group G is said to be RC-dominated if F is a dominating function for $\Gamma(G, H)$ and F is called RC-signed dominating function.

We shall now study pairs (G, H) such that F is an RC-dominating function for $\Gamma(G, H)$.

Theorem 1. Let G be Abelian and H any subgroup of G. Then F is an RC-dominating function for $\Gamma(G, H)$.

Proof of Theorem 1. Let G be Abelian of order g and H, a subgroup of order k. then we know that (see [1])

1. $\Gamma(G,H)$ is a graph with g vertices and has k connected components.
2. Each component is the complete graph $K_{g/k}$.
   Since every $\varphi \in IrrG$ has degree 1, $m(\varphi) = 1$ for all φ. Thus $F(\varphi) = 1$ and hence $F[\varphi] \geq 1$ for all φ. Hence F is an RC-dominating function.

Theorem 2. If (G,H) is a pair such that $\Gamma(G,H)$ is complete, then F is an RC-dominating function.

Proof of Theorem 2. Since any two $\varphi, \psi \in V$ are adjacent, in particular, φ and 1_G are adjacent for all $\varphi \neq 1_G$. Since $F(1_G) = 1$, clearly $F[\varphi] \geq 1$ for all $\varphi \in V$.

Hence F is an RC-dominating function. The above situations can occur when
1. $H = (1)$.
2. G is simple and H is cyclic of order 2. (This follows from considering Eigenvalues of $\rho(x)$ as ρ runs through all the representations of G. for details, see [1]).
3. Some other situations like $G = A_5$ and H is cyclic of order 3.

This example may be viewed in the linear groups perspective. If $G = PSL(2,11)$ and B is the Borel subgroup of order 55, we can prove $\Gamma(G,B)$ is RC-dominating. Also we can consider $G = PSL(2,13)$ and B the corresponding Boral subgroup of order 78. Again $\Gamma(G,B)$ is RC-dominating. In all these cases the right action of G on G/H is doubly transitive.

Theorem 3. If (G,H) is a pair such that $\Gamma(G,H)$ is a tree, then F is RC-dominating function.

Proof of Theorem 3. In this case the graph is a star and $G = NH$ is Frobenius with N, elementary Abelian of order p^m for some m (p a prime) and $O(H) = p^m - 1$.

(Recall that G is Frobenius if there exists a non-trivial subgroup H such that $H \cap H^x = (1)$ for all $x \notin H$. Then $N = \{G - \bigcup_{x \notin H} H^x\} \cup \{1\}$ is a normal subgroup and $G = NH$ is a semidirect product).

We first recall the following properties for G:

1. Since N is Abelian, deg $\theta = 1$ for all $\theta \in IrrH$.
2. The irreducible characters of G can be partitioned as $A \cup B$ where $A = \{\varphi \in IrrG | Ker\varphi \supset N\}$ and $B = \{\varphi \in IrrG | Ker\varphi \not\supset N\}$.

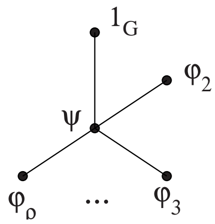
Every element φ of A comes as 'pull-backs' of irreducibles β of H. Now since the graph is a star, $|B| = 1$ and the single element $\psi \in B$.

Let $A = \{\varphi_1 = 1_G, \varphi_2, \ldots, \varphi_r\}$ and $B = \{\psi\}$. $\Gamma(G,H)$ is of the form

Then $\varphi_{i_H} = B_1 \in IrrH$ and hence $F(\varphi_I) = 1 \ \forall\ i$. Since $\Gamma(G,H)$ is a star, $F[\varphi_i] \geq 1$ (whatever be $F(\psi)$) and also $F[\psi] \geq 1$. Hence F is an RC-dominating function.

Theorem 4. If G is a non-abelian simple group and H is an abelian subgroup, then F is an RC-dominating function.

Proof of Theorem 4. First note that for any φ incident with 1_G, $F[\varphi] \geq 1$ since $F(1_G) = 1$.



Since G is simple, $CoreH = (1)$ and $\Gamma(G,H)$ is connected. By a criterion for connectivity (see [1]), we know that φ, ψ belong to the some connected component if and only if $\varphi \subset \psi\chi^s$, for some $s \geq 1$, where $\chi = 1_H^G$, the character induced from 1_H to G. Since H is Abelian, every irreducible character of H is linear and hence $m(\varphi) = \deg \varphi$ for any $\varphi \in IrrG$. If $\deg \varphi =$ odd, then $F[\varphi] \geq 1$. Let $\deg \varphi =$ even. We proceed by induction. If $s = 1$, $\varphi \subset \psi\chi$. If $\deg \psi$ is odd, we are through. Let $\deg \psi$ be even. Assume by induction that wherever $\varphi \subset \psi\chi s - 1$ $\deg \psi$ is even.

Now suppose $\varphi \subset \psi\chi_s$. Then $\varphi \subset \psi\chi^{s-1} \cdot \chi$ which implies there exists $\eta \subset \psi\chi^{s-1}$ such that η and ψ are adjacent. By induction assumption $\deg \eta$ is even; and $\phi \subset \psi\chi$. Let $\deg \psi$ be even. We continue this process until we get the following result: every non-linear irreducible character of G has even degree, ie., degree divisible by 2. By a result of Thomson, G has a normal 2-complement, and in particular, G is not simple, contradiction.

Hence for any φ, nonlinear, there exist $\psi \in I(\theta)$ (the induced cover of θ) such that $F(\psi) = 1$ (where $\varphi_H = r\theta + \ldots$. Thus $F[\varphi] \geq 1$ and hence F is *RC*-dominating for $\Gamma(G,H)$).

§3. Domination Numbers for *RC*-graphs

There are two numbers involved in any *RC*-graphs

1. the usual domination number $\gamma(\Gamma)$ (which is the weight of a minimum dominating function)

2. the weight $= \Sigma F(\varphi)$ of the *RC*-dominating function F we can denote the latter by the notation $\gamma_G(\Gamma(G,H))$.

It is clear that, whenever F is an *RC*-dominating function, then weight $\gamma_F(\Gamma) \geq \gamma(\Gamma)$.

Supported by several available examples, we propose the following question:

Conjecture 1. For all pairs (G,H) for which F is an *RC*-dominating function, $\gamma_G(\Gamma(G,H)) > \gamma(\Gamma(G,H))$.

§4. Signed Domination for *RC*-graphs

We shall now turn over attention to the concept of signed domination for *RC*-graphs:

First recall that for any graph Γ, a function $f : V \to \{-1, 1\}$ is a signed dominating function if $\sum_{v \in N[v]} f(v) \geq 1$ for all $v \in V$. Any graph Γ is signed dominated if we assign $f(v) = 1$ for all $v \in V$. The minimum weight $w(f)$ as f varies over all signed dominating functions for Γ is called the signed dominating number of Γ and is denoted by $\gamma_s(\Gamma)$.

Now consider any RC-graph $\Gamma(G,H)$. We define two special functions f and f^* from the vertex set V of $\Gamma(G,H)$ into $\{-1,1\}$ as follows:

$$f(\varphi)=\begin{cases}1 & \text{if } m(\varphi) \text{ is even}\\ -1 & \text{if } m(\varphi) \text{ is odd}\end{cases}$$

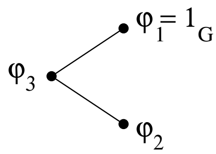
where as before, $m(\varphi)=\Sigma e_i$ if $\varphi_H=\Sigma e_i\theta_i$, $e_i>0$ and $\theta_i\in IrrH$.

$$f^*(\varphi)=\begin{cases}-1 & \text{if } m(\varphi) \text{ is even}\\ 1 & \text{if } m(\varphi) \text{ is odd}\end{cases}$$

Definition 3. An RC-graph $\Gamma(G,H)$ is RC-signed dominated if either $f[v]\geq 1$ or $f^*[v]\geq 1$ for all $v\in V$.

We immediately see from the following simple example that not all graphs $\Gamma(G,H)$ are RC-signed dominated.

Let $G=S_3$ and $H=\{(1),(12)\}$. Then $\Gamma(G,H)$ is the following graph:



deg φ_1 = deg $\varphi_2=1$, deg $\varphi_3=2$. If $\{\theta_1,\theta_2\}=IrrH$, then $\phi_{1_H}=1_H$, $\phi_{2_H}=\theta_2$, $\phi_{3_H}=\theta_1+\theta_2$.

Hence $m(\varphi_1)=m(\varphi_2)=1, m(\varphi_3)=2$.

We see that neither of the assignments f nor f^* gives an RC-signed domination.

The rest of this paper is devoted to the study of RC-signed dominated graphs $\Gamma(G,H)$.

Theorem 5. Let $\Gamma(G,H)$ be complete. Let r = number of φ_i of even degree and s = number of φ_i with odd degree. Then if $r\neq s$, $\Gamma(G,H)$ is RC-signed dominated.

Proof of Theorem 5. Let $\varphi_1(=1_G),\varphi_2,\ldots,\varphi_r$ be the elements of V. Then $m(\varphi_{1H})=1$, and since the graph is complete $f[\varphi_i]=f(\varphi_i)+f(\varphi_1)+\cdots+f(\varphi_{i-1})+f(\varphi_{i+1})+\cdots+f(\varphi_r)$. Hence $f[\varphi_i]$ is a constant for every i, and the same is true for f^* also.

If $r>s$, then $f[\varphi_i]=r-s$ for all i and if $r<s$, $f^*[\varphi_i]=s-r$ and both are ≥ 1.

Hence, as long as $r\neq s$, is RC-signed dominated.

Remark 1. The condition $r\neq s$ is necessary in fact, for $G=PSL(2,11)$, $|IrrG|=8$, the parity of degree being equal.

For $H=(1)$ and for any φ, $f[\varphi]=f^*[\varphi]=0$. Therefore the graph is not RC-signed dominated.

Theorem 6. Let $G=NH$ be a Frobenius group such that H is abelian, of even order. Let A and B denote respectively the irreducible characters of G whose kernels contain N and whose kernels do not contain N. Put $|A|=a$ and $|B|=b$. If $b>a$, then G is RC-signed dominated.

Proof of Theorem 6. From the character theory of Frobenius groups, we know that $|A|=|IrrH|$ and $|B|=t/h$ where $t+1=|IrrN|$ and $h=O(H)$.

Since $O(H) = [G : N]$ is even, it is known that N is also Abelian. Also $a = |A| = |IrrH| = O(H) = h$ and $b = (O(N) - 1)/h$.

Now every β_i of $IrrH$ can be 'pulled back' to get an irreducible φ_i of G, deg $\varphi_i = 1$ for all i and $A = \{\varphi_1' = 1_G, \varphi_2, \ldots, \varphi_r\}$; On the other hand every element $\psi_j \in B$, is of the form $\psi_j = Ind_N^G \theta_j$, $o_j \in IrrN$ and deg $\psi_j = [G : N]$ deg $o_j = O(H)$ (since deg $o_j = 1$ for all j)

Therefore as we have seen already $Res_H \psi_j = Res_H(Ind_N^G \theta_j)$, by Mackey's subgroup theorem, and using deg $\varphi_j = 1$.

Since each irreducible β_i of H occurs with multiplicity e_i of $\beta_i =$ deg ψ_i, we have $m(\psi_j) = \Sigma e_i$ fixed. Also $m(\varphi_i) = 1$ for all i.

Case 1: If $m(\psi_j) = \Sigma e_i$ (fixed) is odd, then $f^*[\chi] \geq 1$ for all $\chi \in IrrG$.

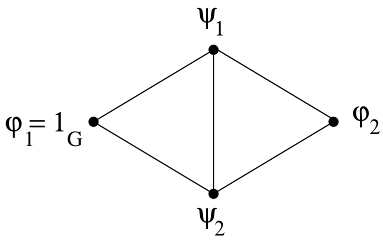
Here $f^*(\varphi_i) = +1$ for all $\varphi_i \in A$ and $f^*(\psi_j) = 1$ for all $\psi_j \in B$.

Case 2: Let $b > a$ (> 1). Let $m(\psi_j) = \Sigma e_i$ be even.

Now take $f : V \to \{-1, 1\}$. Then (since $m(\varphi_i) = 1$, odd),
$f(\varphi_i) = -1$ for all $\varphi_i \in A$ and
$f(\psi_j) = 1$ for all $\psi_j \in B$. Since $b > a$, it follows that
$f[\varphi_i] = -1 + b \geq 1$ and
$f[\psi_j] = b - a \geq 1$.

Hence $\Gamma(G, H)$ is RC-signed dominated.

Remark 2. The condition $b > a$ is necessary, as may be seen by the following example: $G = D_{10} = C_5 \cdot C_2 = N_H \cdot \Gamma(G, H)$ is



Here $|A| = 2$ and $|B| = 2$.

Neither f nor f^* is an RC-signed dominating function, because, $m(\varphi_i) = 1$ and $m(\psi_i) = 2$.

For $f, f[1_G] = -1$ and for f^*, $f^*[\psi_1] = 0$.

Conjecture 2. If G is a non-Abelian simple group and H is any subgroup of G, then $\Gamma(G, H)$ is RC-signed dominated.

§5. RC-Signed Domination Number for RC-graphs

For those graphs for which for which f or f^* is as an RC-signed dominating function, we define the sum as the RC-signed domination number.

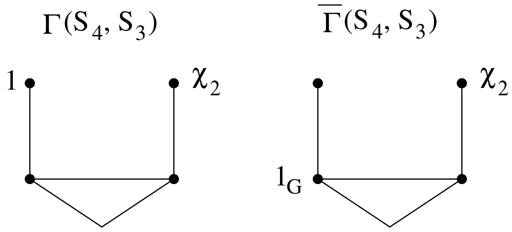
Classically, the complement of a simple, undirected graph Γ plays an important role in determining certain upper bounds. We will first briefly recapitulate some results already obtained on RC-graphs to facilitate our further study.

Proposition 1. [1] Let $G = NH$ be a semidirect product with N normal. Then $\Gamma(G, H) = \overline{\Gamma}(G, N)$ if and only if G is Frobenius with kernal N and complement H.

Proposition 2. [4] Let n and q denote respectively the number of vertices and edges of $\Gamma(G,H)$. Suppose is not a tree and the right action of G on G/H is doubly transitive. If further $q \leq n-1C_2$, then $\overline{\Gamma}(G,H)$ is connected.

Proposition 3. [4] Let $Core_G H = \underset{x\in G}{\cap} H^x = (1)$ and the right action of G on G/H be doubly transitive. Then $\overline{\Gamma}(G,H)$ is connected if and only if diam $\Gamma(G,H) \geq 3$.

Of course, it is easy to see that $\overline{\Gamma}(G,H)$ need not be another RC-graph $\Gamma(G,K)$, K a subgroup. This may be quickly seen from the following example:



(The second graph, though identical with the first one graph - theoretically, is not $\Gamma(S_4,K)$ for any subgroup K of S_4).

In this context we state the following result of Gnanaseelan and propose a problem.

Proposition 4. [1] Let $G = NH$ be a semidirect product with N normal. Then $\overline{\Gamma}(G,H) = \Gamma(G,N)$ if and only if G is Frobenius with kernal N and complement H.

Problem 1. Find all groups G which possess a pair of subgroups H and K such that $\overline{\Gamma}(G,H) = \Gamma(G,K)$. Therefore while attempting to study complements of graphs, which are RC-signed dominated, we will not bother to check whether the complement graph is also an RC-graph. The following facts are quite handy for RC-graphs, which depend, on the results of Haas and Wexler [3].

Lemma 1. If Γ is a graph with $\gamma_S(\Gamma) = n$, then every vertex $v \in \Gamma$ is either isolated, an end vertex or adjacent to an end vertex.

Theorem 7. For any graph Γ, $\gamma_S(\Gamma) + \gamma_S(\overline{\Gamma}) \geq -n-2+\sqrt{8n+1}$.

From Lemma 1, we immediately get the following proposition:

Proposition 4. Let $\Gamma(G,H)$ be connected which is RC-signed dominated and $\gamma_R(\Gamma(G,H) = |V|$. Then $\Gamma(G,H)$ is a star.

Proof of Proposition 4. Since $\gamma_R(\Gamma) = |V| = \gamma_S(\Gamma)$, by lemma 1 every vertex is isolated or an end vertex or adjacent to an end vertex. Hence G is Frobenius, with complement H. Otherwise one can prove that the graph is either a triangle or of the form K_r with vertices sticking to every vertex of K_r. The first case gives $\Gamma(S_3,(1))$ which is not RC-signed dominated. The Second case leads only to a star.

Theorem 8. If $\Gamma(G,H)$ is complete and satisfies conditions of Theorem 5 then $\gamma_R(\Gamma(G,H)) = |r-s|$.

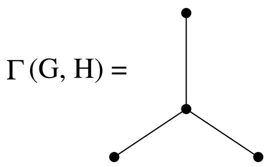
Proof of Theorem 8. From Theorem 5, we get $\Gamma(G,H)$ is RC-signed dominated.

Now $\gamma_R(\Gamma(G,H)) =$ positive number $= |r-s|$.

Theorem 9. Let $G = NH$ be Froberius with N abelian. Then $\gamma_R(\Gamma(G,H)) =$ either $b+a$ or $b-a$.

Proof. Use notations and the proof of Theorem 5. The result follows immediately.

Example 1. $G = A_4C_3$ Here $\gamma_R(\Gamma) = b + a = 1 + 3 = 4$.



Example 2. $G = D_{18}$.

Then $\Gamma(G, H)$ is the usual Frobenius Graph.

Here $\gamma_R(\Gamma) = 4 - 2 = 2$.

In example 1, $\gamma_s(\Gamma) = 4$ and $\gamma_s(\overline{\Gamma}) = 4$ as well so that $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 8 \geq -6 + \sqrt{33}$, the upper bound of Hass et al.

In example 1, $\gamma_s(\Gamma) = 2 \geq -1$

In this case $\gamma_R(\Gamma) + \gamma_R(\overline{\Gamma}) = 4 - 2 = \gamma_s(\Gamma)$.

Conclusion

We raise the following questions

1. To know all groups G and subgroups H such that $\Gamma(G, H)$ is i) RC-dominated ii) RC-signed dominated would be interesting.

2. Within the framework of RC-graph, are there functions which give parameters much closer to $\gamma(\Gamma)$ and $\gamma_s(\Gamma)$ than the ones described above?

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A Note on Smarandache Mukti-Squares

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Abstract In this note, we will introduced Smarandache Mukti-Squares which are non-associative structures. Arun S. Muktibodh in [2] has defined and studied Smarandache Mukti-Squares (SMS) and gave some open problems. We will extend some SMS to Latin squares that satisfies in problems propounded by Muktibodh [2].

Keywords Quasigroups, Smarandache quasigroups, latin square, non-associative structure

§1. Introduction

Smarandache theory is one of the new branches in mathematics that were first defined and studied by Prof. F. Smarandache and W. B. Kandasamy. Study of algebraic structers and Smarandache algebraic structers is one of the interesting problmes in Smarandache theory. In this note, we will introduced Smarandache Mukti-Squares (SMS) which are non-associative algebraic structures. Arun S. Muktibodh in [2] define and study Smarandache Mukti-Squares (SMS) and gave some open problems. We will extend some SMS to Latin squares that satisfies in problems propounded by Muktibodh [2]. First we recall some definitions and theorems.

Definition 1.1. An $n \times n$ array containing symbols from some alphabet of size m with $m \geq n$ is called a square of order n.

Definition 1.2. A Latin Square of order n is an n by n array containing symbols from some alphabet of size n, arrenged so that each symbol appears exactly once in each row and exactly once in each column.

Definition 1.3. If a Latin square L contains a Latin square S properly, then S is called a Sub Latin square.

Definition 1.4. An square in which :

1. No element in the first row is repeated,
2. No element in the first column is repeated,
3. Elements in first row and first column have similar arrange,

is called a Mukti-Square.

Definition 1.5. If a square contains a Latin Square properly the square is called a Smarandache Mukti-Square or SMS.

Example 1. The following are examples of Mukti-Squares of order 3 with alphabets $\{0, 1, 2, 3, 4\}$.

0	1	2
1	2	0
2	0	1

and

0	1	2
1	2	3
2	3	4

and

0	1	2
1	3	4
2	4	3

Also maybe in other rows or columns we have elemnts that repeated. for example:

0	1	2	3
1	2	0	3
2	0	2	3

is an Mukti-Square with element 3 that repeated in third column.

In this note we consider Mukti-Squares that have no repeated elements in rows and columns, and therefore they have Latin squares structures. Hence we see that because of a latin square is equvalent to an quasigroup and vise versa, in quasigroup language, we have only 6 Mukti-Square for quasigroups with alphabet $\{0, 1, 2\}$. Also for quasigroups constructed by useing all elements of alphabet $\{0, 1, 2, 3\}$ we have 96 such Mukti-Square.

§2. Orthogonal Smarandache Mukti-Squares

Two SMS are said to be orthogonal if the Latin squares contained in them are orthogonal. Therefore for two SMS

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

and

$$\begin{matrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{matrix}$$

are orthogonal and we have

$$\begin{matrix} (1,2) & (2,1) & (3,4) & (4,3) \\ (2,1) & (1,2) & (4,3) & (3,4) \\ (3,4) & (4,3) & (2,1) & (1,2) \\ (4,3) & (3,4) & (1,2) & (2,1) \end{matrix}$$

that is an SMS too.

§3. Mukti-Squares of order 2

We investigate some SMS that have one element, without repeatetion, outside of his alphabet.
For SMS with alphabet $A = \{0, 1, 2\}$, let we have

$$\begin{matrix} 0 & 1 \\ 1 & a \end{matrix}$$

with $a \notin A$, and only in one place. Then we can extend this SMS to:

$$\begin{matrix} 0 & 1 & 2 & a \\ 1 & a & 0 & 2 \\ 2 & 0 & a & 1 \\ a & 2 & 1 & 0 \end{matrix}$$

For SMS with two element outside the alphabet of SMS, for example:

$$\begin{matrix} a & 2 \\ 2 & b \end{matrix}$$

with alphabet A, we can contruct the following SMS:

$$\begin{matrix} b & 2 & 1 & a \\ 2 & a & b & 1 \\ 1 & b & a & 2 \\ a & 1 & 2 & b. \end{matrix}$$

For SMS with three element outside the alphabet of SMS:

$$\begin{matrix} 0 & a \\ b & c \end{matrix}$$

we have:

$$\begin{matrix} a & 1 & 0 & b & c & 2 \\ 1 & 0 & a & c & 2 & b \\ 0 & b & c & 2 & a & 1 \\ b & c & 2 & a & 1 & 0 \\ c & 2 & b & 1 & 0 & a \\ 2 & a & 1 & 0 & b & c \end{matrix}$$

Finally for SMS with four element outside of SMS alphabet:

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

we can extend to the following form:

$$\begin{matrix} 0 & 1 & a & b \\ 1 & 0 & c & d \\ a & b & 0 & 1 \\ c & d & 1 & 0 \end{matrix}$$

It is obvius that for SMS with one element outside of alphabet that repeated in SMS, for example

$$\begin{matrix} 1 & a \\ a & 1, \end{matrix}$$

construction of Latin square is simpler. Also this construction is simpler for two and more element with repeatation.

§4. SMS of order 3

In this section at first we recall Theorem 3.3 of [2].

Theorem 4.1. A Latin square of order 3 does not posses an SMS.

proof. (See [2], Theorem 3.3).

Now let

2	1	0
1	0	2
0	2	1

be a Latin square. Therefore we see that

2	1
1	0

and

1	0
0	2

and

0	2
2	1

SMS's that are in Latin squre.

Theorem 4.2. It is not possible to extend an SMS of order 3 to a Latin square if the Latin square contained in the SMS has two elements outside the alphabet of the SMS.

proof. (See [2], Theorem 4.3).

It has been practically tried out but could not construct the Latin square.

Contrexample:

Let in alphabet $\{0, 1, 2\}$

0	1	2
1	3	4
2	4	3

be an SMS with $\{3, 4\}$ outside of alphabet A. Then

4	0	1	2	3
0	1	3	4	2
1	2	4	3	0
2	3	0	1	4
3	4	2	0	1

is a Latin square. Also

3	0	1	2	4
2	1	3	4	0
0	2	4	3	1
4	3	0	1	2
1	4	2	0	3

can be considered for this SMS.

§5. Answer to Problems and some open problems

In [2] we have some open problems, that we answer to them in this section.
1. Can we extend an SMS of order 3 when all the elements of the Latin square contained in the SMS are outside the alphabet of the SMS?

Example 5.1. Let by alphabet $A = \{0, 1, 2\}$ we have an SMS, such that $\{3, 4\} \notin A$. Then we have the following SMS:

0	1	2
1	3	4
2	4	3

Therefore we can construct the Latin square as:

4	0	1	2	3
0	1	3	4	2
1	2	4	3	0
2	3	0	1	4
3	4	2	0	1

Also we have the following Latin square:

3	0	1	2	4
2	1	3	4	0
0	2	4	3	1
4	3	0	1	2
1	4	2	0	3

that can be considered for this SMS.
Now, let we have an SMS of order 3 such that one element of the Latin square contained in

SMS is outside of alphbet $A = \{0, 1, 2\}$ of SMS. For example :

$$\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{array}$$

with $3 \notin A$. Then this SMS can be extended to the following Latin square of order 6 :

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 0 & 5 & 3 & 4 \\ 2 & 0 & 3 & 4 & 5 & 1 \\ 3 & 5 & 4 & 0 & 1 & 2 \\ 4 & 3 & 5 & 1 & 2 & 0 \\ 5 & 4 & 1 & 2 & 0 & 3. \end{array}$$

In other case if $3 \notin A$ appear in array a_{22} of an SMS. That is

$$\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 1. \end{array}$$

Then this SMS can be extended to the following Latin square of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 0 & 5 & 2 & 4 \\ 2 & 0 & 1 & 4 & 5 & 3 \\ 3 & 5 & 4 & 0 & 1 & 2 \\ 4 & 2 & 5 & 1 & 3 & 0 \\ 5 & 4 & 3 & 2 & 0 & 1. \end{array}$$

Next, let we consider an SMS of the form

$$\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{array}$$

where $3, 4 \notin \{0, 1, 2\}$. Then this SMS can be extended to the following Latin square:

0	1	2	3	4	5
1	2	0	5	4	3
2	3	4	5	0	1
3	5	4	0	1	2
4	5	3	1	2	0
5	0	1	2	3	4.

2. Can we extend an SMS of order 4 when one element of the Latin square contained in the SMS is outside the alphabet of the SMS?
Let us consider the following SMS of order 4, with one element outside of alphabet $\{1, 2, 3, 4\}$:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	a.

Then we can extend this SMS to the following Latin square:

1	2	3	4	a	5	6	7
2	3	4	1	6	7	a	5
3	4	1	2	7	a	5	6
4	1	2	a	5	6	7	3
a	6	7	5	1	2	3	4
5	7	a	6	2	3	4	1
6	a	5	7	3	4	1	2
7	5	6	3	4	1	2	a

3. Can we extend an SMS of order 4 when two element of the latin square contained in the SMS are out side the alphabet of SMS?
To answer this quastion we refree the reader to our last construction. It is easily constructable.

At the end we consider the following quastions for next works:
1. Can we construct an algorithm for caculate all SMS of order 3 with alphabet containing n element?
2. Can we construct an algorithm for caculate all SMS of order 4 with alphabet containing n element?
3. Can we write an computer programme for calculating all SMS of order 3?
4. Can we write an computer programme for calculating all SMS of order 4?
5. Can we found applications of SMS in cryptography?
6. Can we found application of SMS in Matroid Theory?

7. what is the interpretation of SMS in geometry, probablity, combinatorial theory or other sciences?

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Generalized Fuzzy BF-Algebras

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Abstract By two relations belonging to $(\in)$ and quasi-coincidence (q) between fuzzy points and fuzzy sets, we define the concept of (α, β)-fuzzy subalgebras where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. We state and prove some theorems in (α, β)-fuzzy BF-algebras.

Keywords BF-algebra, (α, β)-fuzzy subalgebra, fuzzy point.

§1. Introduction

Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [11], J. Neggers and H. S. Kim introduced the notion of B-algebras, which is a generalization of BCK-algebra. In [10], Y. B. Jun , E. H. Roh , and H. S. Kim introduced BH-algebras, which are a generalization of $BCK/BCI/B$-algebras. Recently, Andrzej Walendziak defined a BF-algebra [14].

In 1980, P. M. Pu and Y. M. Liu [12], introduced the idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is used to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $\{\in, \in \vee q\}$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In this note we introduced the notion of (α, β)-fuzzy BF-algebras. We state and prove some theorems discussed in (α, β)-fuzzy BF-subalgebras and level subalgebras.

§2. Preliminary

Definition 2.1. [14] A BF-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $0 * (x * y) = (y * x)$,

for all $x, y \in X$.

Example 2.2. [14] (a) Let $\mathbf{R}$ be the set of real numbers and let $A = (\mathbf{R}; *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then A is a BF-algebra.

(b) Let $A = [0; \infty)$. Define the binary operation $*$ on A as follows: $x * y = |x - y|$, for all $x, y \in A$. Then $(A; *, 0)$ is a BF-algebra.

Proposition 2.3. [14] Let X be a BF-algebra. Then for any x and y in X, the following hold:

(a) $0 * (0 * x) = x$ for all $x \in A$;

(b) if $0 * x = 0 * y$, then x = y for any $x, y \in A$;

(c) if $x * y = 0$, then $y * x = 0$ for any $x, y \in A$.

Definition 2.4. [14] A non-empty subset S of a BF-algebra X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \longrightarrow Y$ of BF-algebras is called a BF-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [15]).

We now review some fuzzy logic concepts (see [2] and [15]).

Let X be a set. A fuzzy set A on X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$.

Let $f : X \longrightarrow Y$ be a function and BF a fuzzy set of Y with membership function μ_B. The inverse image of BF, denoted by $f^{-1}(B)$, is the fuzzy set of X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set of X with membership function μ_A. Then the image of A, denoted by $f(A)$, is the fuzzy set of Y such that

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy set μ of a set X of the form

$$\mu(y) := \begin{cases} t & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

where $t \in (0, 1]$ is called a fuzzy point with support x and value t and is denoted by x_t.

Consider a fuzzy point x_t, a fuzzy set μ on a set X and $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$, we define $x_t \alpha \mu$ as follow:

(i) $x_t \in \mu$ (resp. $x_t q\mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$) and in this case we said that x_t belong to (resp. quasi-coincident with) fuzzy set μ.

(ii) $x_t \in \vee q\mu$ (resp. $x_t \in \wedge q\mu$) means that $x_t \in \mu$ or $x_t q\mu$ (resp. $x_t \in \mu$ and $x_t q\mu$).

Definition 2.5. [2] Let μ be a fuzzy set of a BF-algebra X. Then μ is called a fuzzy BF-algebra (subalgebra) of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

Example 2.6. [2] Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a BF-algebra, but is not a $BCH/BCI/BCK$-algebra.

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = 0.7$, $\mu(1) = 0.1$ and $\mu(2) = 0.3$. Then μ is a fuzzy BF-subalgebra of X.

Definition 2.7. [2] Let μ be a fuzzy set of X. Then the upper level set $U(\mu; \lambda)$ of X is defined as following :

$$U(\mu; \lambda) = \{x \in X \mid \mu(x) \geq \lambda\}.$$

Definition 2.8. Let $f : X \longrightarrow Y$ be a function. A fuzzy set μ of X is said to be f-invariant, if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$, for all $x, y \in X$.

§3. (α, β)-fuzzy BF-algebras

From now on X is a BF-algebra and α, $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$ unless otherwise specified. By $x_t \overline{\alpha} \mu$ we mean that $x_t \alpha \mu$ does not hold.

Theorem 3.1. Let μ be a fuzzy set of X. Then μ is a fuzzy BF-algebra if and only if

$$x_{t_1}, y_{t_2} \in \mu \Rightarrow (x * y)_{\min(t_1, t_2)} \in \mu, \tag{1}$$

for all $x, y \in X$ and $t_1, t_2 \in [0, 1]$.

Proof. Assume that μ is a fuzzy BF-algebra. Let $x, y \in X$ and $x_{t_1}, y_{t_2} \in \mu$, for $t_1, t_2 \in [0,1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$, by hypothesis we can conclude that

$$\mu(x * y) \geq \min(\mu(x), \mu(y)) \geq \min(t_1, t_2).$$

Hence $(x * y)_{\min(t_1,t_2)} \in \mu$.

Conversely, Since $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$ for all $x, y \in X$, then $(x * y)_{\min(\mu(x),\mu(y))} \in \mu$. Therefore $\mu(x * y) \geq \min(\mu(x), \mu(y))$.

Note that if μ is a fuzzy set of X defined by $\mu(x) \leq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \in \wedge q\mu\}$ is empty.

Definition 3.2. A fuzzy set μ of X is said to be an (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge q$, if it satisfies the following condition:

$$x_{t_1}\alpha\mu, y_{t_2}\alpha\mu \Rightarrow (x * y)_{\min(t_1,t_2)}\beta\mu$$

for all t_1, $t_2 \in (0,1]$.

Proposition 3.3. μ is an $(\in, \in)$-fuzzy subalgebra of X if and only if for all $t \in [0,1]$, the nonempty level set $U(\mu; t)$ is a subalgebra of X.

Proof. The proof follows from Theorem 3.1.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

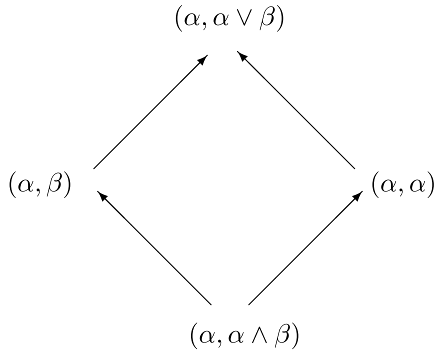
Then $(X, *, 0)$ is a BF-algebra. Let μ be a fuzzy set in X defined $\mu(0) = 0.2$, $\mu(1) = 0.7$ and $\mu(2) = \mu(3) = 0.3$. Then μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X. But

(1) μ is not an $(\in, \in)$-fuzzy subalgebra of X since $1_{0.62} \in \mu$ and $1_{0.66} \in \mu$, but $(1 * 1)_{\min(0.62,0.66)} = 0_{0.62}\overline{\in}\mu$.

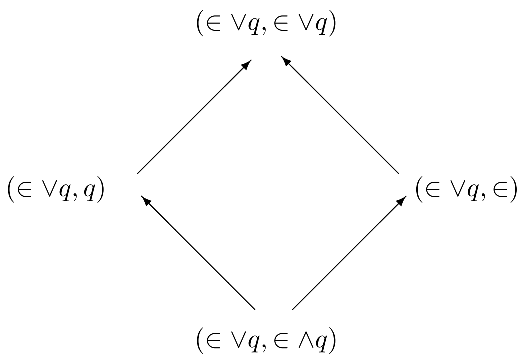
(2) μ is not a $(q, \in \vee q)$-fuzzy subalgebra of X since $1_{0.41}q\mu$ and $2_{0.77}q\mu$, but $(1*2)_{\min(0.41,0.77)} = 3_{0.41}\overline{\in \vee q}\mu$.

(3) μ is not an $(\in \vee q, \in \vee q)$-fuzzy subalgebra of X since $1_{0.5} \in \vee q\mu$ and $3_{0.8} \in \vee q\mu$, but $(1 * 3)_{\min(0.5,0.8)} = 0_{0.5}\overline{\in \vee q}\mu$.

Theorem 3.5. Let μ be a fuzzy set. Then the following diagram shows the relationship between (α, β)-fuzzy subalgebras of X, where α, β are one of $\in$ and q.



and also we have



Proposition 3.6. If μ is a nonzero (α, β)-fuzzy subalgebra of X, then $\mu(0) > 0$.

Proof. Assume that $\mu(0) = 0$. Since μ is non-zero, then there exists $x \in X$ such that $\mu(x) = t > 0$. Thus $x_t \alpha \mu$ for $\alpha = \in$ or $\alpha = \in \vee q$, but $(x * x)_{\min(t,t)} = 0_t \overline{\beta} \mu$. This is a contradiction. Also $x_1 \alpha \mu$ where $\alpha = q$, since $\mu(x) + 1 = t + 1 > 1$. But $(x * x)_{\min(1,1)} = 0_1 \overline{\beta} \mu$, which is a contradiction. Hence $\mu(0) > 0$.

For a fuzzy set μ in X, we denote the support μ by, $X_0 := \{x \in X \mid \mu(x) > 0\}$.

Proposition 3.7. If μ is a nonzero $(\in, \in \vee q)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Suppose that $\mu(x * y) = 0$, then $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$, but $\mu(x*y) = 0 < \min(\mu(x), \mu(y))$ and $\mu(x*y) + \min(\mu(x), \mu(y)) \leq 1$, i.e $(x * y)_{\min(\mu(x),\mu(y))} \overline{\in \vee q} \mu$, which is a contradiction . Hence $x * y \in X_0$. Therefore X_0 is a subalgebra of X.

Proposition 3.8. If μ is a nonzero $(q, \in \vee q)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x)+1 > 1$ and $\mu(y)+1 > 1$ imply that $x_1 q\mu$ and $y_1 q\mu$. If $\mu(x*y) = 0$, then $\mu(x*y) < 1 = \min(1, 1)$ and $\mu(x*y)+\min(1, 1) \leq 1$. Thus $(x*y)_{\min(1,1)} \overline{\in \vee q} \mu$, which is a contradiction. It follows that $\mu(x*y) > 0$ and so $x*y \in X_0$.

Theorem 3.9. Let μ be a nonempty (α, β)-fuzzy subalgebra, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Then X_0 is a subalgebra of X.

Proof. The proof follows from Theorem 3.5 and Propositions 3.7 and 3.8.

Theorem 3.10. Any non-zero (q, q)-fuzzy subalgebra of X is constant on X_0.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra of X. On the contrary, assume that μ is not constant on X_0. Then there exists $y \in X_0$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Suppose that $t_y < t_0$ and so $1 - t_0 < 1 - t_y < 1$. Thus there exists $t_1, t_2 \in (0, 1)$ such that $1 - t_0 < t_1 < 1 - t_y < t_2 < 1$. Then $\mu(0) + t_1 = t_0 + t_1 > 1$ and $\mu(y) + t_2 = t_y + t_2 > 1$. So $0_{t_1} q\mu$ and $y_{t_2} q\mu$. Since

$$\mu(y * 0) + \min(t_1, t_2) = \mu(y) + t_1 = t_y + t_1 < 1,$$

we get that $(y * 0)_{\min(t_1,t_2)} \overline{q} \mu$, which is a contradiction. Now let $t_y > t_0$ and $t_0 \neq 1$. Then $\mu(y) + (1 - t_0) = t_y + 1 - t_0 > 1$, i.e $y_{1-t_0} q\mu$. Since

$$\mu(y * y) + (1 - t_0) = \mu(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

then we get that $(y*y)_{\min(1-t_0,1-t_0)} \overline{q} \mu$, which is a contradiction. Therefore μ is constant on X_0.

Theorem 3.11. μ is a non-zero (q, q)-fuzzy subalgebra if and only if there exists subalgebra S of X such that

$$\mu(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

for some $t \in (0, 1]$.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra. Then by Proposition 3.6 and Theorems 3.9 and 3.10 we have $\mu(0) > 0$, X_0 is a subalgebra of X and

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & otherwise \end{cases}$$

Conversely, let $x_{t_1} q\mu$ and $y_{t_2} q\mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$ imply that $\mu(x) \neq 0$ and $\mu(y) \neq 0$. Thus $x, y \in S$ and so $x * y \in S$. Hence $\mu(x * y) + \min(t_1, t_2) =$

$t + \min(t_1, t_2) > 1$. Therefore μ is a (q, q)-fuzzy subalgebra of X.

Theorem 3.12. μ is a non-zero (q, q)-fuzzy subalgebra of X if and only if $U(\mu; \mu(0)) = X_0$ and for all $t \in [0, 1]$, the nonempty level set $U(\mu; t)$ is a subalgebra of X.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra. Then by Theorem 3.11 we have

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$$

So it is easy to check that $U(\mu; \mu(0)) = X_0$. Let $x, y \in U(\mu; t)$, for $t \in [0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. If $t = 0$, then it is clear that $x * y \in U(\mu; 0)$. Now let $t \in (0, 1]$. Then $x, y \in X_0$ and so $x * y \in X_0$. Hence $\mu(x * y) = \mu(0) \geq t$. Therefore $U(\mu; t)$ is a subalgebra of X.
Conversely, since $U(\mu; \mu(0)) = X_0$ and $0 \in U(\mu; \mu(0))$, then X_0 is a subalgebra of X. Also $U(\mu; \mu(0)) = X_0$ and $X \neq \emptyset$ imply that μ is non-zero. Now let $x \in X_0$. Then $\mu(x) \geq \mu(0)$ and $\mu(x) > 0$. Since $U(\mu; \mu(x)) \neq \emptyset$, so $U(\mu; \mu(x))$ is a subalgebra of X. Then $0 \in U(\mu; \mu(x))$ imply that $\mu(0) \geq \mu(x)$. Hence $\mu(x) = \mu(0)$, for all $x \in X_0$ i.e

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore by Theorem 3.11 μ is a (q, q)-fuzzy subalgebra of X.

The following example shows that the condition " $U(\mu; \mu(0)) = X_0$" is necessary.

Example 3.13. Let $X = \{0, 1, 2, 3\}$ be BF-algebra in Example 3.3. Define fuzzy set μ on X by

$$\mu(0) = 0.6, \quad \mu(1) = \mu(2) = \mu(3) = 0.3$$

Then $X_0 = X$, $U(\mu; \mu(0)) = \{0\} \neq X_0$ and also

$$U(\mu; t) = \begin{cases} X & \text{if } 0 \leq t \leq 0.3 \\ \{0\} & \text{if } 0.3 < t \leq 0.6 \\ \emptyset & \text{if } t > 0.6 \end{cases}$$

is a subalgebra of X, while by Theorem 3.11, μ is not a (q, q)-fuzzy subalgebra.

Theorem 3.14. Every (q, q)-fuzzy subalgebra is an $(\in, \in)$-fuzzy subalgebra.

Proof. The proof follows from Theorem 3.12 and Proposition 3.3.

Note that in Example 3.13 μ is an $(\in, \in)$-fuzzy subalgebra, while it is not a (q, q)-fuzzy subalgebra. So the converse of the above theorem is not true in general.

Theorem 3.15. If μ is a non-zero fuzzy set of X. Then there exists subalgebra S of X such that $\mu = \chi_S$ if and only if μ is an (α, β)-fuzzy subalgebra of X, where (α, β) is one of the following forms:

(i) $(\in, q)$, (ii) $(\in, \in \wedge q)$,

(iii) $(q, \in)$, (iv) $(q, \in \wedge q)$,

(v) $(\in \vee q, q)$, (vi) $(\in \vee q, \in \wedge q)$,

(vii) $(\in \vee q, \in)$.

Proof. Let $\mu = \chi_S$. We show that μ is $(\in, \in \wedge q)$-fuzzy subalgebra. Let $x_{t_1} \in \mu$ and $x_{t_2} \in \mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$ imply that $x, y \in S$. Thus $x * y \in S$, i.e $\mu(x * y) = 1$. Therefore $\mu(x * y) \geq \min(t_1, t_2)$ and $\mu(x * y) + \min(t_1, t_2) > 1$, i.e $(x * y)_{\min(t_1,t_2)} \in \wedge q\mu$. Similar to above argument, we can see that μ is an (α, β)-fuzzy subalgebra of X, where (α, β) is one of the above forms.

Conversely, we show that $\mu = \chi_{X_0}$. Suppose that there exists $x \in X_0$ such that $\mu(x) < 1$. Let $\alpha = \in$, choose $t \in (0, 1]$ such that $t < \min(1 - \mu(x), \mu(x), \mu(0))$. Then $x_t\alpha\mu$ and $0_t\alpha\mu$, but $(x * 0)_{\min(t,t)} = x_t\overline{\beta}\mu$, where $\beta = q$ or $\beta = \in \wedge q$. Which is a contradiction. If $\alpha = q$, then $x_1\alpha\mu$ and $0_1\alpha\mu$, while $(x * 0)_{\min(1,1)} = x_1\overline{\beta}\mu$ where $\beta = \in$ or $\beta = \in \wedge q$, which is a contradiction. Now let $\alpha = \in \vee q$ and choose $t \in (0, 1]$ such that $x_t \in \mu$ but $x_t\overline{q}\mu$. Then $x_t\alpha\mu$ and $0_1\alpha\mu$ but $(x * 0)_{\min(t,1)} = x_t\overline{\beta}\mu$ for $\beta = q$ or $\beta = \in \wedge q$, which is a contradiction. Finally we have $x_1 \in \vee q\mu$ and $0_1 \in \vee q\mu$ but $(x*0)_{\min(1,1)} = x_1\overline{\in}\mu$, which is a contradiction. Therefore $\mu = \chi_{X_0}$.

Theorem 3.16. Let S be a subalgebra of X and let μ be a fuzzy set of X such that

(a) $\mu(x) = 0$ for all $x \in X \backslash S$,

(b) $\mu(x) \geq 0.5$ for all $x \in S$.

Then μ is a $(q, \in \vee q)$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}q\mu$ and $y_{t_2}q\mu$. Then we get that $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$. We can conclude that $x * y \in S$, since in otherwise $x \in X\backslash S$ or $y \in X\backslash S$ and therefore $t_1 > 1$ or $t_2 > 1$ which is a contradiction. If $\min(t_1, t_2) > 0.5$, then $\mu(x*y)+\min(t_1, t_2) > 1$ and so $(x*y)_{\min(t_1,t_2)}q\mu$. If $\min(t_1, t_2) \leq 0.5$, then $\mu(x*y) \geq \min(t_1, t_2)$ and thus $(x * y)_{\min(t_1,t_2)} \in \mu$. Hence $(x * y)_{\min(t_1,t_2)} \in \vee q\mu$.

Theorem 3.17. Let μ be a $(q, \in \vee q)$-fuzzy subalgebra of X such that μ is not constant on the set X_0. Then there exists $x \in X$ such that $\mu(x) \geq 0.5$. Moreover, $\mu(x) \geq 0.5$ for all $x \in X_0$.

Proof. Assume that $\mu(x) < 0.5$ for all $x \in X$. Since μ is not constant on X_0, then there exists $x \in X_0$ such that $t_x = \mu(x) \neq \mu(0) = t_0$. Let $t_0 < t_x$. Choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. It follows that $x_\delta q\mu$, $\mu(x * x) = \mu(0) = t_0 < \delta = \min(\delta, \delta)$ and $\mu(x*x)+\min(\delta, \delta) = \mu(0)+\delta = t_0+\delta < 1$. Thus $(x*x)_{\min(\delta,\delta)}\overline{\in \vee q}\mu$, which is a contradiction. Now, if $t_x < t_0$ then we can choose $\delta > 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. Thus $0_\delta q\mu$ and $x_1q\mu$, but $(x * 0)_{\min(1,\delta)} = x_\delta\overline{\in \vee q}\mu$, because $\mu(x) < 0.5 < \delta$ and $\mu(x) + \delta = t_x + \delta < 1$, which is a contradiction. Hence $\mu(x) \geq 0.5$ for some $x \in X$. Now we show that $\mu(0) \geq 0.5$. On the contrary, assume that $\mu(0) = t_0 < 0.5$. Since there exists $x \in X$ such that $\mu(x) = t_x \geq 0.5$, it follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0+t_1 < 1 < t_x+t_1$. Then $\mu(x)+t_1 = t_x+t_1 > 1$, and so $x_tq\mu$. Thus we can conclude that

$$\mu(x * x) + \min(t_1, t_1) = \mu(0) + t_1 = t_0 + t_1 < 1,$$

and

$$\mu(x*x)=\mu(0)=t_0<t_1=\min(t_1,t_1).$$

Therefore $(x*x)_{\min(t_1,t_1)}\overline{\in\vee q}\mu$, which is a contradiction. Thus $\mu(0)\geq 0.5$. Finally we prove that $\mu(x)\geq 0.5$ for all $x\in X_0$. On the contrary, let $x\in X_0$ and $t_x=\mu(x)<0.5$. Consider $0<t<0.5$ such that $t_x+t<0.5$. Then $\mu(x)+1=t_x+1>1$ and $\mu(0)+(0.5+t)>1$, imply that $x_1q\mu$ and $0_{0.5+t}q\mu$. But $(x*0)_{\min(1,0.5+t)}=x_{0.5+t}\overline{\in\vee q}\mu$, since $\mu(x*0)=\mu(x)<0.5+t$ and $\mu(x)+0.5+t=t_x+0.5+t<0.5+0.5=1$. Which is a contradiction. Therefore $\mu(x)\geq 0.5$ for all $x\in X_0$.

Theorem 3.18. Let μ be a non-zero fuzzy set of X. Then μ is a $(q,\in\vee q)$-fuzzy subalgebra of X if and only if there exists subalgebra S of X such that

$$\mu(x)=\begin{cases} a & \text{if } x\in S\\ 0 & \text{otherwise}\end{cases} \quad or \quad \mu(x)=\begin{cases} \geq 0.5 & \text{if } x\in S\\ 0 & \text{otherwise}\end{cases}$$

for some $a\in(0,1]$

Proof. Let μ be a $(q,\in\vee q)$-fuzzy subalgebra of X. If μ is constant on X_0, then

$$\mu(x)=\begin{cases} \mu(0) & \text{if } x\in X_0\\ 0 & \text{otherwise}\end{cases}$$

If μ is not constant on X_0, then by Theorem 3.17 we have

$$\mu(x)=\begin{cases} \geq 0.5 & \text{if } x\in X_0\\ 0 & \text{otherwise}\end{cases}$$

Conversely, the proof follows from Theorems 3.11, 3.5 and 3.16.

Theorem 3.19. Let μ be a non-zero $(q,\in\vee q)$-fuzzy subalgebra of X. Then the nonempty level set $U(\mu;t)$ is a subalgebra of X, for all $t\in[0,0.5]$.

Proof. If μ is constant on X_0, then by Theorem 3.11, μ is a (q,q)-fuzzy subalgebra. Thus by Theorem 3.12 we have the nonempty level set $U(\mu;t)$ is a subalgebra of X, for $t\in[0,1]$. If μ is not constant on X_0, then by Theorem 3.17, we have $\mu(x)=\begin{cases} \geq 0.5 & \text{if } x\in X_0\\ 0 & \text{otherwise}\end{cases}$.
Now we show that the nonempty level set $U(\mu;t)$ is a subalgebra of X for $t\in[0,0.5]$. If $t=0$, then it is clear that $U(\mu;t)$ is a subalgebra of X. Now let $t\in(0,0.5]$ and $x,y\in U(\mu;t)$. Then $\mu(x),\mu(y)\geq t>0$ imply that $x,y\in X_0$. Thus $x*y\in X_0$ and so $\mu(x*y)\geq 0.5\geq t$. Therefore $x*y\in U(\mu;t)$.

Theorem 3.20. Let μ be a non-zero fuzzy set of X, $U(\mu;0.5)=X_0$ and the nonempty level set $U(\mu;t)$ is a subalgebra of X, for all $t\in[0,1]$. Then μ is a $(q,\in\vee q)$-fuzzy subalgebra of X.

Proof. Since $\mu\neq 0$ we get that $X_0\neq\emptyset$. Thus by hypothesis we have $U(\mu;0.5)\neq\emptyset$ and so X_0 is a subalgebra of X. Also $\mu(x)\geq 0.5$, for all $x\in X_0$ and $\mu(x)=0$, if $x\notin X_0$. Therefore by Theorem 3.16, μ is a $(q, \in \vee q)$-fuzzy subalgebra of X.

Theorem 3.21. A fuzzy set μ of X is an $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$, for all $x, y \in X$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X and $x, y \in X$. If $\mu(x)$ or $\mu(y) = 0$, then $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$. Now let $\mu(x)$ and $\mu(y) \neq 0$. If $\min(\mu(x), \mu(y)) < 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. Since, assume that $\mu(x * y) < \min(\mu(x), \mu(y))$, then there exists $t > 0$ such that $\mu(x * y) < t < \min(\mu(x), \mu(y))$. Thus $x_t \in \mu$ and $y_t \in \mu$ but $(x * y)_{\min(t,t)} = (x*y)_t \overline{\in \vee q} \mu$, since $\mu(x*y) < t$ and $\mu(x*y)+t < 1 < 2t < 1$, which is a contradiction. Hence if $\min(\mu(x), \mu(y)) < 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. If $\min(\mu(x), \mu(y)) \geq 0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$. So we can get that

$$(x * y)_{\min(0.5,0.5)} = (x * y)_{0.5} \in \vee q\mu.$$

Then $\mu(x * y) > 0.5$. Consequently, $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$. for all $x, y \in X$.

Conversely, let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. So $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Then by hypothesis we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t_1, t_2, 0.5)$. If $\min(t_1, t_2) \leq 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. If $\min(t_1, t_2) > 0.5$, then $\mu(x * y) \geq 0.5$. Thus $\mu(x * y) + \min(t_1, t_2) > 1$. Therefore $(x * y)_{\min(t_1,t_2)} \in \vee q\mu$.

Theorem 3.22. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X.

(i) If there exists $x \in X$ such that $\mu(x) \geq 0.5$, then $\mu(0) \geq 0.5$.

(ii) If $\mu(0) < 0.5$, then μ is an $(\in, \in)$-fuzzy subalgebra of X.

Proof. (i) Let $\mu(x) \geq 0.5$. Then by hypothesis we have $\mu(0) = \mu(x*x) \geq \min(\mu(x), \mu(x), 0.5) = 0.5$.

(ii) Let $\mu(0) < 0.5$. Then by (i) $\mu(x) < 0.5$, for all $x \in X$. Now let $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Thus $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t_1, t_2, 0.5) = \min(t_1, t_2)$. Therefore $(x * y)_{\min(t_1,t_2)} \in \mu$.

Lemma 3.23. Let μ be a non-zero $(\in, \in \vee q)$ fuzzy subalgebra of X. Let $x, y \in X$ such that $\mu(x) < \mu(y)$. Then

$$\mu(x * y) = \begin{cases} \mu(x) & \text{if } \mu(y) < 0.5 \text{ or } \mu(x) < 0.5 \leq \mu(y) \\ \geq 0.5 & \text{if } \mu(x) \geq 0.5 \end{cases}$$

Proof. Let $\mu(y) < 0.5$. Then we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) = \mu(x)$. Also

$$\mu(x) = \mu((x * y) * (0 * y)) \geq \min\{\mu(x * y), \mu(0 * y), 0.5\} \qquad (1)$$

Now we show that $\mu(0*y) \geq \mu(y)$. Since $\mu(y) < 0.5$, then $\mu(0) = \mu(y*y) \geq \min\{\mu(y), \mu(y), 0.5\} = \mu(y)$. Thus $\mu(0 * y) \geq \min\{\mu(0), \mu(y), 0.5\} = \mu(y)$. Hence (1) and hypothesis imply that $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$. Since $\mu(x) < \mu(y)$, then $\mu(x) \geq \mu(x*y)$. Therefore $\mu(x*y) = \mu(x)$. Now let $\mu(x) < 0.5 \leq \mu(y)$. Then similar to above argument $\mu(x * y) \geq \mu(x)$ and $\mu(x) \geq \min\{\mu(x * y), \mu(0 * y), 0.5\}$. Since $\mu(y) \geq 0.5$, then by Theorem 3.22(i), $\mu(0) \geq 0.5$. Thus $\mu(0 * y) \geq \min\{\mu(0), \mu(y), 0.5\} = 0.5$. So by hypothesis we get that $\mu(x) \geq \min\{\mu(x * y), 0.5\}$. Thus $\mu(x) < 0.5$ imply that $\mu(x) \geq \mu(x * y)$. Therefore $\mu(x * y) = \mu(x)$. Let $\mu(x) \geq 0.5$. Then

$\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) = 0.5$.

Theorem 3.24. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X. Then for all $t \in [0, 0.5]$, the nonempty level set $U(\mu; t)$ is a subalgebra of X. Conversely, if the nonempty level set μ is a subalgebra of X, for all $t \in [0, 1]$, then μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X. If $t = 0$, then $U(\mu; t)$ is a subalgebra of X. Now let $U(\mu; t) \neq \emptyset$, $0 < t \leq 0.5$ and $x, y \in U(\mu; t)$. Then $\mu(x), \mu(y) \geq t$. Thus by hypothesis we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t, 0.5) \geq t$. Therefore $U(\mu; t)$ is a subalgebra of X.

Conversely, let $x, y \in X$. Then we have

$$\mu(x), \mu(y) \geq \min(\mu(x), \mu(y), 0.5) = t_0$$

Hence $x, y \in U(\mu; t_0)$, for $t_0 \in [0, 1]$ and so $x * y \in U(\mu; t_0)$. Therefore $\mu(x * y) \geq t_0 = \min(\mu(x), \mu(y), 0.5)$, i.e μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Theorem 3.25. Let S be a subset of X. The characteristic function χ_S of S is an $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if S is a subalgebra of X.

Proof. Let X_S be an $(\in, \in \vee q)$-fuzzy subalgebra of X and $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. Hence $(x * y)_1 = (x * y)_{\min(1,1)} \in \vee q \chi_S$, which implies that $\chi_S(x * y) > 0$. Thus $x * y \in S$. Therefore S is a subalgebra of X.

Conversely, if S is a subalgebra of X, then χ_S is an $(\in, \in)$-fuzzy subalgebra of X. So by Theorem 3.5 we get that μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.$\Box$

Lemma 3.26. Let $f : X \to Y$ be a BF-homomorphism and G be a fuzzy set of Y with membership function μ_G. Then $x_t \alpha \mu_{f^{-1}(G)} \Leftrightarrow f(x)_t \alpha \mu_G$, for all $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

Proof. Let $\alpha = \in$. Then

$$x_t \alpha \mu_{f^{-1}(G)} \Leftrightarrow \mu_{f^{-1}(G)}(x) \geq t \Leftrightarrow \mu_G(f(x)) \geq t \Leftrightarrow (f(x))_t \alpha \mu_G$$

The proof of the other cases is similar to above argument.

Theorem 3.27. Let $f : X \to Y$ be a BF-homomorphism and G be a fuzzy set of Y with membership function μ_G.

(i) If G is an (α, β)-fuzzy subalgebra of Y, then $f^{-1}(G)$ is an (α, β)-fuzzy subalgebra of X,

(ii) Let f be epimorphism. If $f^{-1}(G)$ is an (α, β)-fuzzy subalgebra of X, then G is an (α, β)-fuzzy subalgebra of Y.

Proof. (i) Let $x_t \alpha \mu_{f^{-1}(G)}$ and $y_r \alpha \mu_{f^{-1}(G)}$, for $t, r \in (0, 1]$. Then by Lemma 3.26, we get that $(f(x))_t \alpha \mu_G$ and $(f(y))_r \alpha \mu_G$. Hence by hypothesis $(f(x) * f(y))_{\min(t,r)} \beta \mu_G$. Then $(f(x * y))_{\min(t,r)} \beta \mu_G$ and so $(x * y)_{\min(t,r)} \beta \mu_{f^{-1}(G)}$.

(ii) Let $x, y \in Y$. Then by hypothesis there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Assume that $x_t \alpha \mu_G$ and $y_r \alpha \mu_G$, then $(f(x'))_t \alpha \mu_G$ and $(f(y'))_r \alpha \mu_G$. Thus $x'_t \alpha \mu_{f^{-1}(G)}$ and $y'_r \alpha \mu_{f^{-1}(G)}$ and therefore $(x' * y')_{\min(t,r)} \beta \mu_{f^{-1}(G)}$. So

$$(f(x' * y'))_{\min(t,r)} \beta \mu_G \Rightarrow (f(x') * f(y'))_{\min(t,r)} \beta \mu_G \Rightarrow (x * y)_{\min(t,r)} \beta \mu_G.$$

Theorem 3.28. Let $f : X \to Y$ be a BF-homomorphism and H be an $(\in, \in \vee q)$-fuzzy subalgebra of X with membership function μ_H. If μ_H is an f-invariant, then $f(H)$ is an $(\in, \in \vee q)$-fuzzy subalgebra of Y.

Proof. Let y_1 and $y_2 \in Y$. If $f^{-1}(y_1)$ or $f^{-1}(y_2) = \emptyset$, then $\mu_{f(H)}(y_1 * y_2) \geq \min(\mu_{f(H)}(y_1), \mu_{f(H)}(y_2), 0.5)$. Now let $f^{-1}(y_1)$ and $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus by hypothesis we have

$$\begin{aligned}
\mu_{f(H)}(y_1 * y_2) &= \sup_{t \in f^{-1}(y_1 * y_2)} \mu_H(t) \\
&= \sup_{t \in f^{-1}(f(x_1 * x_2))} \mu_H(t) \\
&= \mu_H(x_1 * x_2) \qquad \text{since } \mu_H \text{ is an } f\text{-invariant} \\
&\geq \min(\mu_H(x_1), \mu_H(x_2), 0.5) \\
&= \min(\sup_{t \in f^{-1}(y_1)} \mu_H(t), \sup_{t \in f^{-1}(y_2)} \mu_H(t), 0.5) \\
&= \min(\mu_{f(H)}(y_1), \mu_{f(H)}(y_2), 0.5)
\end{aligned}$$

So by Theorem 3.21, $f(H)$ is an $(\in, \in \vee q)$-fuzzy subalgebra of Y.

Theorem 3.29. Let $f : X \to Y$ be a BF-homomorphism.
(i) If S is a subalgebra of X, then $f(S)$ is a subalgebra of Y,
(ii) If S' is a subalgebra of Y, then $f^{-1}(S')$ is a subalgebra of X.
Proof. The proof is easy.

Theorem 3.30. Let $f : X \to Y$ be a BF-homomorphism. If H is a non-zero (q, q)-fuzzy subalgebra of X with membership function μ_H, then $f(H)$ is a non-zero (q, q)-fuzzy subalgebra of Y.

Proof. Let H be a non-zero (q, q)-fuzzy subalgebra of X. Then by Theorem 3.10, we have

$$\mu_H(x) = \begin{cases} \mu_H(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}.$$

Now we show that $\mu_{f(H)}(y) = \begin{cases} \mu_H(0) & \text{if } y \in f(X_0) \\ 0 & \text{otherwise} \end{cases}$.
Let $y \in Y$. If $y \in f(X_0)$, then there exist $x \in X_0$ such that $f(x) = y$. Thus $\mu_{f(H)}(y) = \sup_{t \in f^{-1}(y)} \mu_H(t) = \mu_H(0)$. If $y \notin f(X_0)$, then it is clear that $\mu_{f(H)}(y) = 0$. Since X_0 is subalgebra of X, then $f(X_0)$ is a subalgebra of Y. Therefore by Theorem 3.11, $f(H)$ is a non-zero

(q, q)-fuzzy subalgebra of Y.

Theorem 3.31. Let $f : X \to Y$ be a BF-homomorphism. If H is an (α, β)-fuzzy subalgebra of X with membership function μ_H, then $f(H)$ is an (α, β)-fuzzy subalgebra of Y, where (α, β) is one of the following form

(i) $(\in, q)$, (ii) $(\in, \in \wedge q)$,

(iii) $(q, \in)$, (iv) $(q, \in \wedge q)$,

(v) $(\in \vee q, q)$, (vi) $(\in \vee q, \in \wedge q)$,

(vii) $(\in \vee q, \in)$, (viii) $(q, \in \vee q)$.

Proof. The proof is similar to the proof of Theorem 3.30, by using of Theorems 3.15 and 3.18.

Theorem 3.32. Let $f : X \to Y$ be a BF-homomorphism and H be an $(\in, \in)$-fuzzy subalgebra of X with membership function μ_H. If μ_H is an f-invariant, then $f(H)$ is an $(\in, \in)$-fuzzy subalgebra of Y.

Proof. Let $z_t \in \mu_{f(H)}$ and $y_r \in \mu_{f(H)}$, where $t, r \in (0, 1]$. Then $\mu_{f(H)}(z) \geq t$ and $\mu_{f(H)}(y) \geq r$. Thus $f^{-1}(z), f^{-1}(y) \neq \emptyset$ imply that there exists $x_1, x_2 \in X$ such that $f(x_1) = z$ and $f(x_2) = y$. since μ_H is an f-invariant, then $\mu_{f(H)}(z) \geq t$ and $\mu_{f(H)}(y) \geq r$ imply that $\mu_H(x_1) \geq t$ and $\mu_H(x_2) \geq r$. So by hypothesis we have

$$\begin{aligned}\mu_{f(H)}(z * y) &= \sup_{t \in f^{-1}(z*y)} \mu_H(t)\\ &= \sup_{t \in f^{-1}(f(x_1 * x_2))} \mu_H(t)\\ &= \mu_H(x_1 * x_2)\\ &\geq \min(t, r)\end{aligned}$$

Therefore $(z * y)_{\min(t,r)} \in \mu_{f(H)}$, i.e $f(H)$ is an $(\in, \in)$-fuzzy subalgebra of Y.

Theorem 3.33. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q)$-fuzzy subalgebra of X. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Proof. By Theorem 3.21 we have, for all $i \in \Lambda$

$$\mu_i(x * y) \geq \min(\mu_i(x), \mu_i(y), 0.5)$$

Therefore

$$\begin{aligned}\mu(x * y) = \inf_{i \in \Lambda} \mu_i(x * y) &\geq \inf_{i \in \Lambda} \min(\mu_i(x), \mu_i(y), 0.5)\\ &= \min(\inf_{i \in \Lambda} \mu_i(x), \inf_{i \in \Lambda} \mu_i(y), 0.5)\end{aligned}$$

$$= \min(\mu(x), \mu(y), 0.5)$$

Therefore by Theorem 3.21, μ is an $(\in, \in \vee q)$-fuzzy subalgebra.

Theorem 3.34. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in)$-fuzzy subalgebra of X. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in)$-fuzzy subalgebra of X.

Proof. Let $x_t \in \mu$ and $y_r \in \mu$, $t, r \in (0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Thus for all $i \in \Lambda$, $\mu_i(x) \geq t$ and $\mu_i(y) \geq r$ imply that $\mu_i(x * y) \geq \min(t, r)$. Therefore $\mu(x * y) \geq \min(t, r)$ i.e $(x * y)_{\min(t,r)} \in \mu$.

Theorem 3.35. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of (α, β)-fuzzy subalgebra of X. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an (α, β)-fuzzy subalgebra of X, where (α, β) is one of the following form

(i) $(\in, q)$, (ii) $(\in, \wedge q)$,
(iii) $(q, \in)$,(iv) $(q, \in \wedge q)$,
(v) $(\in \vee q, q)$, (vi) $(\in \vee q, \in \wedge q)$,
(vii) $(\in \vee q, \in)$, (viii) $(q, \in \vee q)$,
(ix) (q, q).

Proof. We prove theorem for (q, q)-fuzzy subalgebra. The proof of the other cases is similar, by using Theorems 3.15 and 3.18.
If there exists $i \in \Lambda$ such that $\mu_i = 0$, then $\mu = 0$. So μ is a (q, q)-fuzzy subalgebra. Let $\mu_i \neq 0$ for all $i \in \Lambda$. Then by Theorem 3.10 we have $\mu_i(x) = \begin{cases} \mu_i(0) & \text{if } x \in X_0^i \\ 0 & \text{otherwise} \end{cases}$, for all $i \in \Lambda$.

So it is clear that $\mu(x) = \begin{cases} \mu(0) & \text{if } x \in \bigcap_{i\in\Lambda} X_0^i \\ 0 & \text{otherwise} \end{cases}$.

Since $\bigcap_{i\in\Lambda} X_0^i$ is a subalgebra of X, then by Theorem 3.11 μ is a (q, q)-fuzzy subalgebra of X.

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1-Equidomination cover of a graph

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Abstract A set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex v in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. A decomposition of a graph G is a collection Ψ of edge disjoint subgraphs $G_1, G_2, \cdots, G_n$ of G such that every edge of G is in exactly one G_i. If $\gamma(G_i) = 1$ for each i, then Ψ is called an 1-equidomination *cover* of a graph G. The minimum cardinality of an 1-equidomination cover of G is called the 1-equidomination covering number of a graph G and is denoted by $\gamma_e^{(1)}(G)$. In this paper we initiate a study on this parameter.

Keywords 1-equidomination cover, 1-equidomination covering number, minimum 1-equidomination cover.

§1. Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. For terms not defined here we refer to Harary [5].

The theory of domination is one of the fastest growing areas in Graph theory, which has been investigated by Berge [1], Cockayne and Hedetniemi [4], and Walikar *et al* [10]. A set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex v in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A comprehensive study of domination is given in [7,8].

Another important area in Graph theory is decomposition of graphs. A decomposition of a graph G is a collection Ψ of edge disjoint subgraphs $G_1, G_2, \cdots, G_n$ of G such that every edge of G is in exactly one G_i. If each G_i is isomorphic to a subgraph H of G, then Ψ is called a H-decomposition. Various types of decompositions have been studied by several authors by imposing conditions on G_i in the decomposition. Some such decompositions are path covering [6], line clique covering [3] and star decomposition [9].

Using these two concepts, we introduce the concept of 1-equidomination cover of a graph which is motivated by the concepts of line clique cover and star decomposition of a graph. The concepts of line clique cover and line clique covering number were introduced by S. A. Choudum *et al* [3]. A line clique cover of a graph G is a collection of cliques in G that cover all the edges of G and the minimum cardinality of a line clique cover is called the line clique covering number $\theta_1(G)$. The concepts of star decomposition and star number were introduced by V. R. Kulli *et al* [9]. A star decomposition of a graph G is a collection of stars in G that cover all the edges of G and the minimum cardinality of a star decomposition is called the star number $s(G)$.

We observe that the domination number of clique and star is one and so each of line clique cover and star decomposition of a graph is a decomposition of G in which every member has domination number one. This motivates us to define the more general concept of 1-equidomination cover of a graph which is a decomposition of G in which every member has domination number one and the minimum cardinality of such a decomposition is called an 1-equidomination covering number of a graph G.

In this paper we initiate a study on this parameter. We need the following definition and theorems.

Definition 1.1. A k-factor of G is a k-regular spanning subgraph of G, and G is k-factorable if there are edge disjoint k-factors $G_1, G_2, \cdots, G_n$ such that $G = G_1 \cup G_2 \cup, \cdots, \cup G_n$.

Theorem 1.2.[5] Every regular bipartite graph is 1-factorable.

Theorem 1.3.[5] If G is bipartite then $\beta_1(G) = \alpha_0(G)$.

Theorem 1.4.[5] A graph G has a 1-factor if and only if $\beta_1(G) = p/2$.

Theorem 1.5.[2] For any connected graph G, if q is even, then G has a P_3-decomposition.

Theorem 1.6.[9] The star numbers of some graphs are given as follows.

(i) $s(P_p) = \left\lceil \frac{p-1}{2} \right\rceil$, $p \geq 2$.

(ii) $s(C_p) = \left\lceil \frac{p}{2} \right\rceil$, $p \geq 4$.

(iii) $s(K_{m,n}) = n$ if $m \geq n$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

§2. Main Results

Definition 2.1. An 1-equidomination cover of a graph G is a collection $\Psi = \{G_1, G_2, \cdots, G_n\}$ of subgraphs of G such that

(i) Each G_i is connected

(ii) Every edge of G is in exactly one G_i and

(iii) $\gamma(G_i) = 1, 1 \leq i \leq n$.

It is clear that for any graph G, $\Psi = E(G)$ is an 1-equidomination cover of G.

Definition 2.2. The minimum cardinality of an 1-equidomination cover of G is called the 1-equidomination covering number of G and is denoted by $\gamma_e^{(1)}(G)$. An 1-equidomination cover Ψ of G such that $|\Psi| = \gamma_e^{(1)}(G)$ is called a minimum 1-equidomination cover of G.

We now proceed to obtain some bounds for $\gamma_e^{(1)}(G)$.

Theorem 2.3. For any connected graph G, we have $1 \leq \gamma_e^{(1)}(G) \leq q$. Further $\gamma_e^{(1)}(G) = 1$ if and only if $\gamma(G) = 1$ and $\gamma_e^{(1)}(G) = q$ if and only if $G \cong K_2$.

Proof. The inequalities are trivial. Obviously $\gamma_e^{(1)}(G) = 1$ if and only if $\gamma(G) = 1$.

Suppose $\gamma_e^{(1)}(G) = q$. Now, if $G \neq K_2$, then G has a path P on three vertices, because G is connected. Hence $\Psi = \{P\} \cup \{E(G) - E(P)\}$ is an 1-equidomination cover of G so that $\gamma_e^{(1)}(G) \leq |\Psi| = q - 1$, which is a contradiction. Thus $G \cong K_2$. Also clearly $\gamma_e^{(1)}(K_2) = 1 = q$. This completes the proof.

Corollary 2.4.

(i) For the complete graph K_p, $\gamma_e^{(1)}(K_p) = 1$.

(ii) For the wheel W_p on p vertices, we have $\gamma_e^{(1)}(W_p) = 1$.

Proof. Follows from Theorem 2.3.

Theorem 2.5. For any connected graph G, we have $\gamma_e^{(1)}(G) \leq \lceil q/2 \rceil$.

Proof. If q is even, then it follows from Theorem 1.5 that G has a P_3- decomposition. Since a P_3-decomposition is an 1-equidomination cover of G, we have $\gamma_e^{(1)}(G) \leq q/2$. Suppose q is odd. If there exists an edge e which is not a bridge, let $H = G - e$. If not, then G is a tree. Now, let $H = G - v$, where v is a pendant vertex and let e be the edge incident at v.

Now, in either of the cases, H is connected with even number of edges and hence by Theorem 1.5, H has a P_3-decomposition, say ψ. Hence $\psi \cup \{e\}$ is an 1-equidomination cover of G so that $\gamma_e^{(1)}(G) \leq |\psi| = \frac{q-1}{2} + 1 = \lceil q/2 \rceil$. This completes the proof.

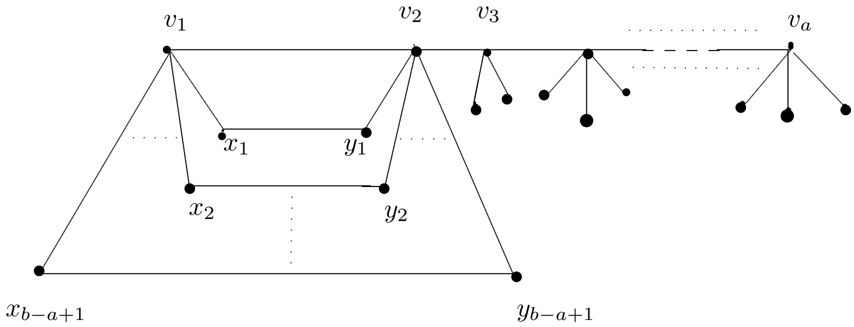
Remark 2.6. The bound given in Theorem 2.5 is sharp. For the cycle C_p, $\gamma_e^{(1)}(C_p) = \lceil p/2 \rceil = \lceil q/2 \rceil$.

In the following theorem we establish a relation between $\gamma(G)$ and $\gamma_e^{(1)}(G)$.

Theorem 2.7. For any connected graph G, we have $\gamma(G) \leq \gamma_e^{(1)}(G)$. Further, if a and b are two positive integers with $1 < a \leq b$, then there exists a connected graph G such that $\gamma(G) = a$ and $\gamma_e^{(1)}(G) = b$.

Proof. Let $\Psi = \{G_1, G_2, \cdots, G_n\}$ be a minimum 1-equidomination cover of G. Let v_i, $1 \leq i \leq n$, be the vertex which dominates all the vertices in G_i. Then $\{v_1, v_2, \cdots, v_n\}$ is a dominating set of G so that $\gamma(G) \leq n = \gamma_e^{(1)}(G)$.

Now, suppose a and b are two integers with $1 < a \leq b$. We now construct a graph G as follows. Consider a path $P = (v_1, v_2, \cdots, v_a)$. Attach $b - a + 1$ pendant vertices, say $x_1, x_2, \cdots, x_{b-a+1}$ at v_1, attach $b-a+1$ pendant vertices, say $y_1, y_2, \cdots, y_{b-a+1}$ at v_2 and then attach at least one pendant vertex at each v_i where $3 \leq i \leq a$. Now join the vertices x_j and y_j by an edge for all j, where $1 \leq j \leq b-a+1$. Then $\gamma(G) = a$. We now prove that $\gamma_e^{(1)}(G) = b$.



Let $G_i = (v_1, x_i, y_i), 1 \leq i \leq b - a + 1$

$G_{b-a+2} = \langle N\,[v_2] \rangle$ and

$H_i = \langle N\,[v_{i+2}] \rangle - v_{i+1}v_{i+2}, 1 \leq i \leq a - 2.$

Then $\Psi = \{G_1, G_2, \cdots, G_{b-a+2}, H_1, H_2, \cdots, H_{a-2}\}$ is an 1-equidomination cover of G and hence $\gamma_e^{(1)}(G) \leq |\psi| = b$. Now let ψ be any minimum 1-equidomination cover of G. Now it is clear that no two of the edges in $\{x_iy_i : 1 \leq i \leq b - a + 1\} \cup \{v_1v_2\}$ lie on the same member of ψ. Also any two non-adjacent pendant edges of G lie on the different members of ψ. Hence it follows that $|\psi| \geq b - a + 2 + a - 2$ so that $\gamma_e^{(1)}(G) \geq b$. Thus $\gamma_e^{(1)}(G) = b$. This completes the proof.

Theorem 2.8. If $\gamma_e^{(1)}(G) = \gamma(G)$, then every member of any minimum 1-equidomination cover ψ contains at least one vertex which does not lie on any other member of ψ.

Proof. Suppose $\gamma_e^{(1)}(G) = \gamma(G)$. Let $\Psi = \{G_1, G_2, \cdots, G_\gamma\}$ be a minimum 1-equidomination cover of G and let v_i be a vertex in G_i which dominates all the vertices of G_i. We now claim that each G_i has a vertex u_i such that $u_i \notin G_j$ for $j \neq i$. Suppose not. Assume without loss of generality that every vertex of G_1 lies in some G_i where $i > 1$. Then $S = \{v_2, v_3, \cdots, v_\gamma\}$ is a dominating set of G, which is a contradiction. Hence each G_i has a vertex u_i such that $u_i \notin G_j$ for $j \neq i$. This completes the proof.

Remark 2.9. The converse of Theorem 2.8 is not true. For the cycle C_6, every member of any minimum 1-equidomination cover ψ contains at least one vertex which does not lie in any other member of ψ. However, $\gamma_e^{(1)}(C_6) = 3$ and $\gamma(C_6) = 2$.

The following theorem gives a relation between $\gamma_e^{(1)}(G)$ and $\alpha_0(G)$.

Theorem 2.10. For a graph G, $\gamma_e^{(1)}(G) \leq \alpha_0(G)$, where $\alpha_0(G)$ is the vertex covering number of G.

Proof. Let $S = \{v_1, v_2, \cdots, v_{\alpha_0}\}$ be a minimum vertex cover of G. Let G_1 be the subgraph of G consisting of the vertex v_1 and the edges incident with v_1. Having defined G_i, let G_{i+1} be the subgraph of G consisting of the vertex v_{i+1} together with the edges incident at v_{i+1} and edge disjoint from $G_1, G_2, \cdots, G_i$. Then $\psi = \{G_1, G_2, \cdots, G_{\alpha_0}\}$ is an 1-equidomination cover of G and hence $\gamma_e^{(1)}(G) \leq \alpha_0(G)$. This completes the proof.

Corollary 2.11. If G is a graph having no triangles, then $\gamma_e^{(1)}(G) = \alpha_0(G)$.

Proof. Let $\Psi = \{G_1, G_2, \cdots, G_n\}$ be a minimum 1-equidomination cover of G. Let v_i,

$1 \leq i \leq n$, be the vertex which dominates all the vertices in G_i. Then $\{v_1, v_2, \cdots, v_n\}$ is a vertex cover of G so that $\alpha_0(G) \leq n = \gamma_e^{(1)}(G)$. Hence it follows from Theorem 2.10 that $\gamma_e^{(1)}(G) = \alpha_0(G)$. This completes the proof.

As a consequence of Theorem 1.3 and Corollary 2.11, we have

Corollary 2.12. If G is bipartite, then $\gamma_e^{(1)}(G) = \alpha_0(G) = \beta_1(G)$.

Corollary 2.13. If G is a bipartite graph of order p having a 1-factor, then $\gamma_e^{(1)}(G) = p/2$.

Proof. Follows from Theorem 1.4 and Corollary 2.12.

The n-cube Q_n is the graph whose vertices are the ordered n-tuples of 0's and 1's, two points being joined if and only if they differ in exactly one coordinate. Then Q_n is a bipartite graph with 2^n vertices. Further, Q_n has a 1-factor and hence as a consequence of Corollary 2.13, we have

Corollary 2.14. $\gamma_e^{(1)}(Q_n) = 2^{n-1}$.

Now, the following problem naturally arises.

Problem 2.15. Characterize graphs for which $\gamma_e^{(1)}(G) = p/2$.

In the following Theorems, we relate the 1-equidomination number with the star number and the line clique covering number of a graph.

Theorem 2.16. For any connected graph G, $\gamma_e^{(1)}(G) \leq s(G)$, where $s(G)$ is the star number of G. Further equality holds for triangle-free graphs.

Proof. As every star decomposition of a graph G is an 1-equidomination cover of G, we have $\gamma_e^{(1)}(G) \leq s(G)$.

Now, suppose G is a triangle-free graph. Let $\Psi = \{G_1, G_2, \cdots, G_n\}$ be any 1-equidomination cover of G. since $\gamma_e^{(1)}(G_i) = 1$ for each i, $1 \leq i \leq n$ and G has no triangles, it follows that G_i is a star and hence every 1-equidomination cover of G is a star decomposition of G so that $\gamma_e^{(1)}(G) \geq s(G)$. Thus $\gamma_e^{(1)}(G) = s(G)$. This completes the proof.

Corollary 2.17.

(i) $\gamma_e^{(1)}(P_p) = \left\lceil \frac{p-1}{2} \right\rceil, p \geq 2$.

(ii) $\gamma_e^{(1)}(C_p) = \left\lceil \frac{p}{2} \right\rceil, p \geq 4$.

(iii)$\gamma_e^{(1)}(K_{m,n}) = n$ if $m \geq n$.

Proof. Follows from the Theorem 2.16 and Theorem 1.6.

Thus there is an infinite family of graphs for which $\gamma_e^{(1)}(G) = s(G)$. This leads to the following problem.

Problem 2.18. Characterize graphs for which $\gamma_e^{(1)}(G) = s(G)$.

Theorem 2.19. For any connected graph G, we have $\gamma_e^{(1)}(G) \leq \theta_1(G)$. Further equality holds if and only if $G \cong K_p$.

Proof. Since every line clique cover of a graph G is an 1-equidomination cover of G, it follows that $\gamma_e^{(1)}(G) \leq \theta_1(G)$. Now suppose $\gamma_e^{(1)}(G) = \theta_1(G)$. Let $\Psi = \{G_1, G_2, \cdots, G_n\}$ be a minimum line clique cover of G. Now, if G is not complete then $n > 1$ and there exists a vertex v such that $v \in V(G_1) \cup V(G_2)$ and hence $\{G_1 \cup G_2, \cdots, G_n\}$ is an 1-equidomination cover of

G, which is a contradiction. Thus G is complete. The converse is obvious. This completes the proof.

Conclusion

We conclude this paper by posing the following problems for further investigation.

[(i)] Characterize graphs for which $\gamma_e^{(1)}(G) = \lceil q/2 \rceil$.

[(ii)] Characterize graphs for which $\gamma_e^{(1)}(G) = \gamma(G)$.

[(iii)] Characterize graphs for which $\gamma_e^{(1)}(G) = \alpha_0(G)$.

Further the concept of 1-equidomination cover of a graph can be generalized to the concept of k-equidomination cover of G which is defined to be a decomposition of G in which every member has domination number k and we shall present this in the subsequent papers.

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Green∼ relations and the natural partial orders on U-semiabundant semigroups[1]

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Abstract In this paper, we mainly discuss the Smarandache relation on U-semiabundant semigroups.

Keywords U-semiabundant semigroups, Green∼ relations, Natural partial orders.

§1. Introduction

In generalizing regular semigroups, a generalized Green relation $\widetilde{\mathcal{L}}^U$ was introduced by M. V. Lawson [6] on a semigroup S as follows: Let E be the set of all idempotents of S and U be a subset of E. For any $a, b \in S$, define

$$(a,b) \in \widetilde{\mathcal{L}}^U \quad \text{if and only if} \quad (\forall\ e \in U) \quad (\ ae = a \Leftrightarrow be = b\ );$$

$$(a,b) \in \widetilde{\mathcal{R}}^U \quad \text{if and only if} \quad (\forall\ e \in U) \quad (\ ea = a \Leftrightarrow eb = b\ ).$$

It is clear that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}^U$.

It is easy to verify that if S is an abundant semigroup and $U = E(S)$ then $\mathcal{L}^* = \widetilde{\mathcal{L}}^U$, $\mathcal{R}^* = \widetilde{\mathcal{R}}^U$; if S is a regular semigroup and $U = E(S)$ then $\mathcal{L} = \widetilde{\mathcal{L}}^U$, $\mathcal{R} = \widetilde{\mathcal{R}}^U$.

Recall that a semigroup S is called U-semiabundant if each $\widetilde{\mathcal{L}}^U$-class and each $\widetilde{\mathcal{R}}^U$-class contains an element from U. It is clear that regular semigroups and abundant semigroups are all U-semiabundant semigroups.

The natural partial order on a regular semigroup was first studied by Nambooripad [9] in 1980. Later on, M. V. Lawson [7] in 1987 first introduced the natural partial order on an abundant semigroup. The partial orders on various kinds of semigroups have been investigated by many authors, for example, H. Mitsch [4], Sussman [10], Abian [1] and Burgess[2]. In [3], we have introduced the natural partial order on U-semiabundant semigroups and described the properties of such semigroups by using the natural partial order. In this paper, we will mainly discuss the Smarandache relation between Green∼ relation and Natural partial orders.

We first cite some basic notions which will be used in this paper. Suppose that e, f are elements of $E(S)$. The preorders ω^r and ω^l are defined as follows:

$$e\omega^r f \Leftrightarrow fe = e \quad \text{and} \quad e\omega^l f \Leftrightarrow ef = e.$$

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In addition, $\omega = \omega^r \cap \omega^l$, the usual ordering on $E(S)$.

We use $\mathcal{D}_E$ to denote the relation $(\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$. Assume that (S, U) is an U-semiabundant semigroup. It will be said that U is closed under basic products if $e, f \in U$ and $(e, f) \in \mathcal{D}_E$ then $ef \in U$.

Recall in [8] that an U-semiabundant semigroup $S(U)$ is called reduced if $\omega^r = \omega^l$ on U. A reduced U-semiabundant semigroup $S(U)$ is idempotent connected(IC) if it satisfies the two equations

IC_l: For any $f \in \omega(x^*) \cap U$, $xf = (xf)^+x$;

IC_r: For any $e \in \omega(x^+) \cap U$, $ex = x(ex)^*$.

For terminologies and notations not given in this paper, the reader is referred to Howie [5].

§2. Green∼ relations and the natural partial orders

It is well-known that the Green∼ relations play an important role in the study of U-semiabundant semigroups, similar to that of Green's relations for regular semigroups. It is natural to find the similar properties of Green∼ relations as Green's relations. This is the main aim of this section. To begin with, we verify the following fact which enables the study to get under way.

Lemma 2.1. Let S be a semigroup and $a, b \in S$. Then:

(i)[6] $b \in \widetilde{R}^U(a)$ if and only if there exist $a_0, \cdots, a_n \in S^1$ and $x_1, \cdots, x_n \in S$ such that $a = a_0, b = a_n$ 且$(a_i, a_{i-1}x_i) \in \widetilde{\mathcal{R}}^U$, for $i = 1, 2, \cdots n$.

(ii)[6] $b \in \widetilde{L}^U(a)$ if and only if there exist $a_0, \cdots, a_n \in S^1$ and $x_1, \cdots, x_n \in S$ such that $a = a_0, b = a_n$ 且$(a_i, x_i a_{i-1}) \in \widetilde{\mathcal{L}}^U$, for $i = 1, 2, \cdots n$.

(iii) $b \in \widetilde{J}^U(a)$ if and only if there exist $a_0, \cdots, a_n \in S^1$ and $x_1, \cdots, x_n, y_1, \cdots, y_n \in S^1$ such that $a = a_0, b = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \widetilde{\mathcal{D}}^U$, for $i = 1, 2, \cdots n$.

Proof. The proof of (i) and (ii) can refer to[6]. We only need to proof (iii).

Construct set

$$I = \{x \in S | \text{there exsit} x_1, \cdots, x_n, y_1, \cdots, y_n \in S^1 \text{such that} a = a_0, x = a_n$$

$$\text{and} (a_i, x_i a_{i-1} y_i) \in \widetilde{\mathcal{D}}^U, i = 1, 2, \cdots n\}.$$

$\Leftarrow$: By assumption we know $b \in I$. This means that there exist $a_0, \cdots, a_n \in S$ and $x_1, \cdots, x_n, y_1, \cdots, y_n \in S^1$ such that $a = a_0, b = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \widetilde{\mathcal{D}}^U$, for $i = 1, 2, \cdots n$. So, there exist $s_1, \cdots, s_n, t_1, \cdots, t_n \in S$ such that $a_i \widetilde{\mathcal{L}}^U s_1 \widetilde{\mathcal{R}}^U t_1 \widetilde{\mathcal{L}}^U s_2 \widetilde{\mathcal{R}}^U t_2 \cdots s_n \widetilde{\mathcal{R}}^U t_n \widetilde{\mathcal{L}}^U x_i a_{i-1} y_i$. Because $\widetilde{J}^U(a)$ is a U-admissible ideal and $a_0 = a \in \widetilde{J}^U(a)$, we have $x_1 a_0 y_1 \in \widetilde{J}^U(a)$, $a_1 \in \widetilde{J}^U(a)$ Similarly, we can reduce that $a_2, a_3, \cdots, a_n \in \widetilde{J}^U(a)$. Therefore, $b = a_n \in \widetilde{J}^U(a)$.

$\Rightarrow$: We need to proof that $\widetilde{J}^U(a) \subseteq I$. For this purpose, only need to proof that I is a U-admissible ideal including element a. Clearly, $a \in I$. Next we will verify that I is a U-admissible ideal. If $b \in I$, then for all $s \in S, bs \in I$. In fact there exist $a_0 = b, a_1 = bs$ and $x_1 = 1, y_1 = s$ such that $(bs, 1 \cdot b \cdot s) \in \widetilde{\mathcal{D}}^U$. Similarly, $sb \in I$. On the other hand, if $b \in I$, then $\widetilde{R}^U_b \subseteq I$. Since for all $x \in \widetilde{R}^U_b, (x, 1 \cdot b \cdot 1) \in \widetilde{\mathcal{R}}^U \subseteq \widetilde{\mathcal{D}}^U$, that is, $x \in I$. Similarly, $\widetilde{L}^U_b \subseteq I$.

Corollary 2.1. Let S be a semigroup and $a, b \in S$. Then:

(i) $(a, b) \in \widetilde{\mathcal{L}}^U$ if and only if $\widetilde{L}^U(a) = \widetilde{L}^U(b)$;

(ii) $(a,b)\in\widetilde{\mathcal{R}}^U$ if and only if $\widetilde{R}^U(a)=\widetilde{R}^U(b)$.

Lemma 2.2. Let $S(U)$ be a U-semiabundant semigroup, U is closed under basic products, $\widetilde{\mathcal{R}}^U$ is a left congruence. If $a\widetilde{\leqslant}_r b$ for all $a,b\in S$, then for all $x\in\widetilde{R}^U_b$, there exists $y\in\widetilde{R}^U_a$ such that $y\widetilde{\leqslant}_r x$.

Proof. Suppose that $a\widetilde{\leqslant}_r b$. Then, by Theorem 2.6[3], there exists $a^+\in\widetilde{R}^U_a\cap U$ such that $a^+\omega b^+$ and $a=a^+b$. For all $x\in\widetilde{R}^U_b$, it is easy to see that $(x,b^+)\in\widetilde{\mathcal{R}}^U$. Because $\widetilde{\mathcal{R}}^U$ is a left congruence, we have that $(a^+x,a^+b^+)=(a^+x,a^+)\in\widetilde{\mathcal{R}}^U$. On the other hand, $\widetilde{R}^U_{a^+x}=\widetilde{R}^U_{a^+}=\widetilde{R}^U_{b^+a^+}\leqslant\widetilde{R}^U_{b^+}=\widetilde{R}^U_x$. By Theorem 2.4[3], we can get that $a^+x\widetilde{\leqslant}_r x$.

Lemma 2.3. Let $S(U)$ be a U-semiabundant semigroup, U is closed under basic products, $\widetilde{\mathcal{R}}^U$ is a left congruence. If $(a,bx)\in\widetilde{\mathcal{R}}^U$ for all $a,b,x\in S$, then there exists $c\in\widetilde{R}^U_a$ such that $c\widetilde{\leqslant}_r b$.

Proof. Assume that $(a,bx)\in\widetilde{\mathcal{R}}^U$. Then, for all $b^+\in\widetilde{R}^U_b\cap U$, Since $b^+bx=bx$, we know that $b^+a=a$. $b^+a^+=a^+$, that is $(a^+,b^+)\in\omega^r\subseteq\mathcal{D}^E$. Because U is closed under basic products, we have that $a^+b^+\in U$. It is easy to verify that $(a^+b^+,b^+)\in\omega$. By Theorem 2.4[3], we can get that $a^+b^+\widetilde{\leqslant}_r b^+$. Obviously, $(a^+b^+,a^+)\in\mathcal{R}$. By Corollary 2.3[3] we know that $(a^+b^+,a^+)\in\widetilde{\mathcal{R}}^U$. So $(a^+b^+,a)\in\widetilde{\mathcal{R}}^U$. By Lemma 2.2, for $b\in\widetilde{R}^U_{b^+}$, there exists $c\in\widetilde{R}^U_{a^+b^+}=\widetilde{R}^U_a$ such that $c\widetilde{\leqslant}_r b$.

Basing on these lemmas above, we now give a characterization for the principle U-admissible right ideal:

Theorem 2.1. Let $S(U)$ be a U-semiabundant semigroup, U is closed under basic products, $\widetilde{\mathcal{R}}^U$ is a left congruence on $S(U)$. Then for all $a,b\in S$, the following statements are equivalent:

(i) $a\in\widetilde{\mathcal{R}}^U(b)$;

(ii)There exists $c\in\widetilde{R}^U_a$ such that $c\widetilde{\leqslant}_r b$.

Proof. (i)$\Rightarrow$ (ii): Let $a\in\widetilde{R}^U(b)$. By Lemma 2.1, we know that there are $a_0,\cdots,a_n\in S^1$ and $x_1,\cdots,x_n\in S$ such that $b=a_0, a=a_n$ $(a_i,a_{i-1}x_i)\in\mathcal{R}^U$, for $i=1,2,\cdots n$. So, by Lemma 2.3, there exists $c_i\in\widetilde{R}^U_{a_i}$ such that $c_i\widetilde{\leqslant}_r a_{i-1}$. By Lemma 2.2, there exists $c'_i\in\widetilde{R}^U_{c_i}(\widetilde{R}^U_{a_i})$ such that $c'_2\widetilde{\leqslant}_r c_1, c'_{i+1}\widetilde{\leqslant}_r c'_i, i=2,3,\cdots,n-1$. Hence, there exists $c'_n\in\widetilde{R}^U_{a_n}=\widetilde{R}^U_a$ such that $c'_n\widetilde{\leqslant}_r c'_{n-1}\widetilde{\leqslant}_r\cdots\widetilde{\leqslant}_r c'_2\widetilde{\leqslant}_r c_1\widetilde{\leqslant}_r a_0=b$.

(ii)$\Rightarrow$ (i): If there exists $c\in\widetilde{R}^U_a$ such that $c\widetilde{\leqslant}_r b$. By Theorem 2.4[3], $\widetilde{R}^U_c\leqslant\widetilde{R}^U_b$, that is, $\widetilde{R}^U(c)\subseteq\widetilde{R}^U(b)$. By Corollary 2.1(ii), we know that $\widetilde{R}^U(a)=\widetilde{R}^U(c)$. So we have that $a\in\widetilde{R}^U(b)$.

The dual result for the principle U-admissible left ideal may be similarly proved.

Corollary 2.2. Let $S(U)$ be a U-semiabundant semigroup, U is closed under basic products, $\widetilde{\mathcal{R}}^U$ is a left congruence. The following statements are equivalent:

(i) $\widetilde{\mathcal{R}}^U=(\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r)\cap({}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U)$;

(ii) $\widetilde{\mathcal{L}}^U=(\widetilde{\mathcal{L}}^U\circ\widetilde{\leqslant}_l)\cap({}_l\widetilde{\geqslant}\circ\widetilde{\mathcal{L}}^U)$.

Proof. We need only proof (i). Similarly, we can proof (ii).

For all $a,b\in S(U)$, let $(a,b)\in\widetilde{\mathcal{R}}^U$. By Corollary 2.1(ii) we can get that $\widetilde{R}^U(a)=\widetilde{R}^U(b)$. So $a\in\widetilde{R}^U(b)$ and $b\in\widetilde{R}^U(a)$. By Theorem 2.1 we know that $(a,b)\in\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r$ and $(b,a)\in\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r$, that is, $(a,b)\in{}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U$. Hence $(a,b)\in(\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r)\cap({}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U)$. Therefore, $\widetilde{\mathcal{R}}^U\subseteq(\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r)\cap({}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U)$.

Conversely, let $(a,b)\in(\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r)\cap({}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U)$. then $(a,b)\in\widetilde{\mathcal{R}}^U\circ\widetilde{\leqslant}_r$ and $(a,b)\in{}_r\widetilde{\geqslant}\circ\widetilde{\mathcal{R}}^U$. By

Theorem 2.1 we know that $a \in \widetilde{R}^U(b)$ and $b \in \widetilde{R}^U(a)$. So $\widetilde{R}^U(a) \subseteq \widetilde{R}^U(b)$ and $\widetilde{R}^U(b) \subseteq \widetilde{R}^U(a)$. This means that $\widetilde{R}^U(a) = \widetilde{R}^U(b)$. By Corollary 2.1 we can get that $(a, b) \in \widetilde{\mathcal{R}}^U$. Therefore, $(\widetilde{\mathcal{R}}^U \circ \widetilde{\leqslant}_r) \cap ({}_r\widetilde{\geqslant} \circ \widetilde{\mathcal{R}}^U) \subseteq \widetilde{\mathcal{R}}^U$.

Lemma 2.4. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. If $b\widetilde{\leqslant}a$ for all $a, b \in S(U)$, then there exists $y \in \widetilde{D}_b^U$ such that $y\widetilde{\leqslant}x$ for all $x \in \widetilde{D}_a^U$.

Proof. By the conditions and Lemma 3.1[3] we know that $\widetilde{\leqslant}_r = \widetilde{\leqslant}_l = \widetilde{\leqslant}$. Let $a, b \in S(U)$, $b\widetilde{\leqslant}a$. By $x \in \widetilde{D}_a^U$ we can obtain that , there exist $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n \in S(U)$ such that $a = x_1, x = y_n$ and $x_1\widetilde{\mathcal{R}}^U y_1 \widetilde{\mathcal{L}}^U x_2 \widetilde{\mathcal{R}}^U y_2 \cdots \widetilde{\mathcal{L}}^U x_n \widetilde{\mathcal{R}}^U y_n$. By Lemma 2.2, there exists $y_1' \in \widetilde{R}_b^U$ such that $y_1'\widetilde{\leqslant}y_1$. So, By the dual of Lemma 2.2 we known that, there exists $x_2' \in \widetilde{L}_{y_1'}^U$ such that $x_2'\widetilde{\leqslant}x_2$. Repeatedly this argument, there exist $x_3', \cdots, x_n'$ and $y_2', \cdots, y_n' \in S(U)$ such that $b\widetilde{\mathcal{R}}^U y_1' \widetilde{\mathcal{L}}^U x_2' \widetilde{\mathcal{R}}^U y_2' \widetilde{\mathcal{L}}^U x_3' \cdots \widetilde{\mathcal{L}}^U x_n' \widetilde{\mathcal{R}}^U y_n' y_n' \widetilde{\leqslant} y_n = x$. we have proved that for all $x \in \widetilde{D}_a^U$, there exists $y_n' \in \widetilde{D}_b^U$ such that $y_n'\widetilde{\leqslant}x$.

Lemma 2.5. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. If $(a, xby) \in \widetilde{\mathcal{D}}^U$ for all $a, b \in S(U)$, and $x, y \in S(U)^1$, then there exists $c \in \widetilde{D}_a^U$ such that $c\widetilde{\leqslant}b$.

Proof. By the conditions and Lemma 3.1[3], we know that $\widetilde{\leqslant}_r = \widetilde{\leqslant}_l = \widetilde{\leqslant}$. By Lemma 2.1(i), $xby \in \widetilde{R}^U(xb)$. By Theorem 2.1, there exists $u \in \widetilde{R}_{xby}^U$ such that $u\widetilde{\leqslant}_r xb$. So $u\widetilde{\leqslant}xb$. On the other hand, By Lemma 2.1(ii), we can obtain that $xb \in \widetilde{L}^U(b)$. By the dual of Theorem 2.1 we know that there exists $v \in \widetilde{L}_{xb}^U$ such that $v\widetilde{\leqslant}_l b$. So $v\widetilde{\leqslant}b$. By applying the dual of Lemma 2.2 to the fact $u\widetilde{\leqslant}xb$ we can get that there exists $w \in \widetilde{L}_u^U$ such that $w\widetilde{\leqslant}v$. Hence $w\widetilde{\leqslant}b$. In fact, $w\widetilde{\mathcal{L}}^U u \widetilde{\mathcal{R}}^U xby$. So, $(w, xby) \in \widetilde{\mathcal{D}}^U$. Since $(a, xby) \in \widetilde{\mathcal{D}}^U$, we have that $(w, a) \in \widetilde{\mathcal{D}}^U$. We have proved that there exists $w \in \widetilde{D}_a^U$ such that $w\widetilde{\leqslant}b$.

Now we arrive at the structure of the principle U-admissible ideal of the reduced *IC* $U-$semiabundant semigroup:

Theorem 2.2. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. Then for all $a, b \in S(U)$, the following statements are equivalent:

(i) $a \in \widetilde{J}^U(b)$;

(ii) There exists $c \in \widetilde{D}_a^U$ such that $c\widetilde{\leqslant}b$.

Proof. By Lemma 2.1(iii), (ii)⇒(i) is obvious. We need only prove (i)⇒(ii).

Suppose that $a \in \widetilde{J}^U(b)$. Then by Lemma 2.1(iii), there exist $a_0, a_1, \cdots, a_n \in S(U), x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n \in S^1(U)$ such that $b = a_0, a = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \widetilde{\mathcal{D}}^U, i = 1, 2, \cdots, n$. By Lemma 2.5, we know that there exists $c_i \in \widetilde{D}_{a_i}^U$ such that $c_i\widetilde{\leqslant}a_{i-1}, i = 1, 2, \cdots, n$. Therefore, by Lemma 2.4, there exists $c_i' \in \widetilde{D}_{c_i}^U(\widetilde{D}_{a_i}^U)$ such that $c_2'\widetilde{\leqslant}c_1, c_i'\widetilde{\leqslant}c_{i-1}', i = 3, 4, \cdots, n$. So, $c_n'\widetilde{\leqslant}c_{n-1}'\widetilde{\leqslant} \cdots \widetilde{\leqslant}c_2'\widetilde{\leqslant}c_1\widetilde{\leqslant}a_0 = b. Also, c_n' \in \widetilde{D}_{a_n}^U = \widetilde{D}_a^U$. This means that there exists $c_n' \in \widetilde{D}_a^U$ such that $c_n'\widetilde{\leqslant}b$.

Proposition 2.1. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. Then $\widetilde{\mathcal{J}}^U = (\widetilde{\mathcal{D}}^U \circ \widetilde{\leqslant}) \cap (\widetilde{\geqslant} \circ \widetilde{\mathcal{D}}^U)$.

Proof. Let $a, b \in S(U)$, and $(a, b) \in \widetilde{\mathcal{J}}^U$. Then by the definition of $\widetilde{\mathcal{J}}^U$ we can get that $\widetilde{J}^U(a) = \widetilde{J}^U(b)$. By Theorem 2.2 we can easily see that $(a, b) \in (\widetilde{\mathcal{D}}^U \circ \widetilde{\leqslant}) \cap (\widetilde{\geqslant} \circ \widetilde{\mathcal{D}}^U)$.

Conversely, if $(a, b) \in (\widetilde{\mathcal{D}}^U \circ \widetilde{\leqslant}) \cap (\widetilde{\geqslant} \circ \widetilde{\mathcal{D}}^U)$. then $(a, b) \in \widetilde{\mathcal{D}}^U \circ \widetilde{\leqslant}$ and $(a, b) \in \widetilde{\geqslant} \circ \widetilde{\mathcal{D}}^U$.

By Theorem 2.2 we know that $a \in \widetilde{J}^U(b)$ and $b \in \widetilde{J}^U(a)$. Therefore, $\widetilde{J}^U(a) \subseteq \widetilde{J}^U(b)$ and $\widetilde{J}^U(b) \subseteq \widetilde{J}^U(a)$, that is, $\widetilde{J}^U(a) = \widetilde{J}^U(b)$. By the definition of $\widetilde{\mathcal{J}}^U$ we know that $(a,b) \in \widetilde{\mathcal{J}}^U$. It follows that $(\widetilde{\mathcal{D}}^U \circ \widetilde{\leqslant}) \cap (\widetilde{\geqslant} \circ \widetilde{\mathcal{D}}^U) \subseteq \widetilde{\mathcal{J}}^U$.

It is well-known that in a general semigroup $\widetilde{\mathcal{D}}^U \neq \widetilde{\mathcal{J}}^U$. Naturally, it is interesting to consider that in what semigroup $\widetilde{\mathcal{D}}^U = \widetilde{\mathcal{J}}^U$? In the next proposition we give the necessary condition for the case.

Proposition 2.2. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. If $\widetilde{D}^U_a \neq \widetilde{J}^U_a$ for all $a \in S(U)$, then in $\widetilde{D}^U_a$ there exist infinite elements $a_1, a_2, \cdots$ such that $a \widetilde{>} a_1 \widetilde{>} a_2 \widetilde{>} \cdots$.

Proof. Let $\widetilde{D}^U_a \neq \widetilde{J}^U_a$. Then there exists $b \in \widetilde{J}^U_a$ such that $b \notin \widetilde{D}^U_a$. Obviously, $b \in \widetilde{J}^U(a)$. By Theorem 2.2, there exists $c_1 \in \widetilde{D}^U_b$ such that $c_1 \widetilde{\leqslant} a$. We may claim that $a \neq c_1$. Otherwise, $a = c_1 \in \widetilde{D}^U_b$ and $b \notin \widetilde{D}^U_a$ a contradiction! So, $a \widetilde{>} c_1$. Since $c_1 \in \widetilde{D}^U_b \subseteq \widetilde{J}^U_b = \widetilde{J}^U_a$, we have that $a \in \widetilde{J}^U(c_1)$. By Theorem 2.2, there exists $a_1 \in \widetilde{D}^U_a$ such that $a_1 \widetilde{\leqslant} c_1$. So, $a \widetilde{>} a_1$. In fact, $\widetilde{D}^U_{a_1} \neq \widetilde{J}^U_{a_1}$. Repeating the proceeding above, we may obtain that there exists $c_2 \in \widetilde{J}^U_{a_1}$ such that $c_2 \widetilde{<} a_1$. Since $a_1 \in \widetilde{J}^U(c_2)$, by Lemma 2.5 we know that $a_2 \in \widetilde{D}^U_{a_1}(\widetilde{D}^U_a)$ such that $a_2 \widetilde{\leqslant} c_2$. So $a \widetilde{>} a_1 \widetilde{>} a_2$. Repeating the proceeding above, we may obtain infinite elements $a_3, a_4, \cdots$ in $\widetilde{D}^U_a$ such that $a \widetilde{>} a_1 \widetilde{>} a_2 \widetilde{>} \cdots$.

If $S(U)$ is *IC*-abundant semigroup satisfying $U = E(S)$, then $\mathcal{L}^* = \widetilde{\mathcal{L}}^U, \mathcal{R}^* = \widetilde{\mathcal{R}}^U$. We can obtain the following conclusion:

Corollary 2.3 [11] Let S be an *IC* abundant semigroup. If $D^*_a \neq J^*_a$ for all $a \in S$, then there exist infinite elements $a_1, a_2, \cdots$ in D^*_a such that $a > a_1 > a_2 > \cdots$.

Especially, for regular $S(U)$, if $U = E(S)$, $\mathcal{L} = \widetilde{\mathcal{L}}^U, \mathcal{R} = \widetilde{\mathcal{R}}^U$, then we have that:

Corollary 2.4 [11] Let S be a regular. If $D_a \neq J_a$ for all $a \in S$, then there exist infinite elements $a_1, a_2, \cdots$ in D_a such that $a > a_1 > a_2 > \cdots$.

Now, recall a definition in reference [5]. A semigroup S is called to satisfy the condition min_L or min_R, if the partial order set $S/\mathcal{L}$ or $S/\mathcal{R}$ satisfies the minimal condition.

Proposition 2.3. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. If $S(U)$ satisfies min_L or min_R, then $\widetilde{\mathcal{D}}^U = \widetilde{\mathcal{J}}^U$.

Proof. We need only prove the case for min_L. Similarly, we can prove that min_R.

Assume that $\widetilde{\mathcal{D}}^U \neq \widetilde{\mathcal{J}}^U$. Then there exists $a \in S(U)$ such that $\widetilde{D}^U_a \neq \widetilde{J}^U_a$. By Proposition 2.2, there exist infinite elements $a_1, a_2, \cdots$ in $\widetilde{D}^U_a$ such that $a \widetilde{>} a_1 \widetilde{>} a_2 \widetilde{>} \cdots$. In fact, if $x \widetilde{\leqslant} y$, then there exist $e, f \in U$ such that $x = ey = yf$. Therefore, $L_x = L_{ey} \leqslant L_y$. Denote $a_0 = a$, we have that $L_{a_{i-1}} \geqslant L_{a_i}, i = 1, 2, \cdots$. Now we will prove that $L_{a_{i-1}} \neq L_{a_i}$. If $L_{a_{i-1}} = L_{a_i}$, then $(a_{i-1}, a_i) \in \mathcal{L} \subseteq \mathcal{L}^*$. So $(a_{i-1}, a_i) \in \widetilde{\leqslant} \cap \mathcal{L}^* = 1_{S(U)}$. This contradicts $a_{i-1} \widetilde{>} a_i$! Hence, $L_a > L_{a_1} > L_{a_2} > \cdots > L_{a_n} > \cdots$. This contradicts the hypothesis! Thus $\widetilde{\mathcal{D}}^U = \widetilde{\mathcal{J}}^U$.

As an immediate consequence of Proposition 2.3, we have:

Corollary 2.5. Let $S(U)$ be a reduced *IC* U-semiabundant semigroup, U is closed under basic products and $S(U)$ satisfies congruence condition. Then $\widetilde{\mathcal{D}}^U = \widetilde{\mathcal{J}}^U$.

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Chaos synchronization of the Sprott N System with parameter[1]

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Abstract In this paper, the chaotic characteristic of the Sprott N system with parameter is studied in theoretical analysis and numerical simulations, and abundant dynamical behavior is presented. When the parameter $\beta = 1.9$, the system exist chaotic attractor. The abundance dynamical behavior of the system is presented by the global bifurcation graph and the Lyapunov exponent. In order to realize the full state hybrid projective synchronization (FSHPS) of the system, Smarandache controller is design. Numerical simulations show that the controller works well.

Keywords Chaos, Bifurcation, The Sprott N system with parameter, FSHPS.

§1. Introduction

Since synchronization of chaotic systems was first introduced by Fujisaka and Yamada [1] and Pecora and Carroll [2]. Due to the importance and applications of coupled systems, ranging from chemical oscillators, coupled neurons, coupled circuits to mechanical oscillators, various synchronization schemes have been proposed by many scientists from different research fields [3-10]. Recently, a new type of chaotic synchronization-full state hybrid projective synchronization(FSHPS) in continuous-time chaotic and hyper-chaotic systems based on the Lyapunov's direct method is presented and investigated by wen[11],many notable results and a series of important applications to security communication regarding FSHPS has been presented in Refs [12-14].

We organize this paper as follows. In Section 2, the chaotic characteristic of the Sprott N autonomous system with parameter is studied by theoretical analysis and numerical simulation. In section 3, the scheme of full state hybrid projective synchronization(FSHPS) is given.A proper Smarandache controller is designed and the synchronization of the system is achieved under it.

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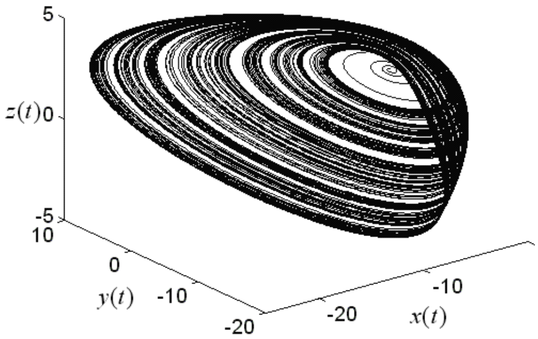

Fig. 1. The chaotic attractor in three-dimensional phase space

§2. Dynamical behavior

A series of three dimensional autonomous systems is presented by J.C.Sprott in 1994 [15], in this paper, the N system of those is taken as example. The governing equations of the Sprott N system are:

$$\begin{cases} \dot{x} = -2y \\ \dot{y} = x + z^2 \\ \dot{z} = 1 + y - 2z, \end{cases} \tag{1}$$

The equilibrium of the system (1) is P(-0.25,0.0.5).The Lyapunov exponents of the system are LEs=(0.076,0,-2.076),which shows the system is chaotic. In order to get abundance dynamical behavior of the system, the parameter β is leaded to the system. The governing equations of the Sprott N system with parameter β can be described as

$$\begin{cases} \dot{x_1} = -2x_2 \\ \dot{x_2} = x_1 + x_3^2 \\ \dot{x_3} = 1 + x_2 - \beta x_3, \end{cases} \tag{2}$$

where $x = (x_1, x_2, x_3)$ is the state variable.The initial conditions are $(x_1(0), x_2(0), x_3(0))$=(1,5,2), when the parameter β=1.9, the system exits a chaotic attractor. The chaotic attractor in three-dimensional phase space is illustrated in Fig 1.

For this system, bifurcation can easily be detected by examining graphs of abs(z)versus the control parameter β. The dynamical behavior of the system (2) can be characterized with its Lyapunov exponents which are computed numerically. The bifurcation diagram and the Lyapunov exponents spectrum are showed in Fig 2.

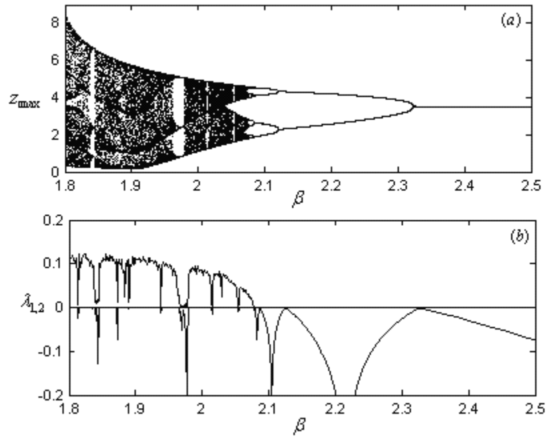

Fig. 2. The nonlinear dynamic behavior of system (2)

§3. Chaos synchronization base on FSHPS

We recall a class of automomous chaotic flows in the form of

$$\dot{x}(t) = F(x), \tag{3}$$

$x = (x_1, x_2, \cdots, x_n)^T$ is the state vector, and $F(x) = (F_1(x), F_2(x), \cdots, F_n(x))^T$ is continuous nonlinear vector function.

We take (3) as the drive system and the response system is given by

$$\dot{y}(t) = F(y) + u, \tag{4}$$

$y = (y_1, y_2, \cdots, y_n)^T$ is the state vector, and $F(y) = (F_1(y), F_2(y), \cdots, F_n(y))^T$ is continuous nonlinear vector function.$u = u(x, y) = (u_1(x, y), u_2(x, y), \cdots, u_n(x, y))^T$ is the controller to be determined for the purpose of full state hybrid projective synchronization.Let the vector error state be $e(t) = y(t) - \alpha x(t)$.Thus, the error dynamical system between the drive system (3) and the response system (4) is

$$\dot{e}(t) = \dot{y}(t) - \alpha\dot{x}(t) = \hat{F}(x, y) + u, \tag{5}$$

where $\hat{F}(x, y) = F(y) - \alpha F(x) = (F_1(y) - \alpha_1 F_1(x), F_2(y) - \alpha_2 F_2(x), \cdots, F_n(y) - \alpha_n F_n(x))^T$.

In the following, we will give a simple principle to select suitable feedback controller u such that the two chaotic or hyper-chaotic systems are FSHPS. If the Lyapunov function candidate V is take as:

$$V = \frac{1}{2}e^T Pe, \tag{6}$$

where P is a positive definite constant matrix,obviously, V is positive define. One way choose as the corresponding identity matrix in most case. The time derivative of V along the trajectory of the error dynnmical system is as follows

$$\dot{V} = e^T P(u + \hat{F}), \tag{7}$$

Suppose that we can select an appropriate controller u such that $\dot{V}$ is negative definite. Then, based on the Lyapunov's direct method, the FSHPS of chaotic or hyper-chaotic flows is synchronization under nonlinear controller u .

In order to observe the FSHPS of system (2), we define the response system of (2) as follows

$$\begin{cases} \dot{y_1} = -2y_2 + u_1 \\ \dot{y_2} = y_1 + y_3^2 + u_2 \\ \dot{y_3} = 1 + y_2 - \beta y_3 + u_3, \end{cases} \tag{8}$$

where $u = (u_1, u_2, u_3)^T$ is the nonlinear controller to be designed for FSHPS of two Sprott N chaotic systems with two significantly different initial conditions.

Define the FSHPS error signal as $e(t) = y(t) - \alpha x(t)$,i.e.,

$$\begin{cases} e_1(t) = y_1 - \alpha_1 x_1 \\ e_2(t) = y_2 - \alpha_2 x_2 \\ e_3(t) = y_3 - \alpha_3 x_3, \end{cases} \tag{9}$$

where $\alpha = diag(\alpha_1, \alpha_2, \alpha_3)$,and α_1, α_2 and $\alpha_3$are different desired in advance scaling factors for FSHPS. The error dynamical system can be written as

$$\begin{cases} \dot{e_1}(t) = e_2 + (\alpha_2 - \alpha_1)x_2 + u_1 \\ \dot{e_2}(t) = -e_1 + e_2 e_3 + \alpha_3 x_3 e_2 + \alpha_2 x_2 e_3 + (\alpha_2 - \alpha_1)x_1 + (\alpha_2\alpha_3 - \alpha_2)x_2 x_3 + u_2 \\ \dot{e_3}(t) = 1 - \alpha_3 - e_2^2 - 2\alpha_2 x_2 e_2 + (\alpha_3 - \alpha_2)x_2^2 + u_3, \end{cases} \tag{10}$$

The goal of control is to find a controller $u = (u_1, u_2, u_3)^T$ for system (10) such that system (2) and (8) are in FSHPS. We now choose the control functions u_1,u_2 and u_3 as follows

$$\begin{cases} u_1 = -2e_2 - (\alpha_2 - \alpha_1)x_2 \\ u_2 = -e_2 e_3 - \alpha_3 x_3 e_2 - \alpha_2 x_2 e_3 - (\alpha_2 - \alpha_1)x_1 - (\alpha_2\alpha_3 - \alpha_2)x_2 x_3 \\ u_3 = \alpha_3 - 1 + e_2^2 + 2\alpha_2 x_2 e_2 - (\alpha_3 - \alpha_2)x_2^2 - e_3^2, \end{cases} \tag{11}$$

If the Lyapunov function candidate is taken as:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{12}$$

The time derivative of V along the trajectory of the error dynnmical system (10) is as follows

$$\dot{V} = -(e_1^2 + e_2^2 + e_3^2), \tag{13}$$

Since V is a positive definite function and $\dot{V}$ is a negative definite function, according to the Lyapunov's direct method, the error variables become zero as time tends to infinity, i.e.,

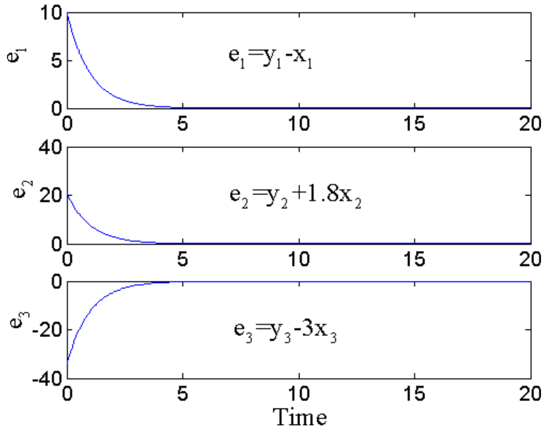

Fig. 3. The time evolution of the errors with the scaling factors $\alpha_1 = 1, \alpha_2 = -1.8, \alpha_3 = 3$

$\lim_{t\to\infty} \parallel y_i - \alpha_i x_i \parallel = 0, i = 1, 2, 3$.This means that the two Sprott A systems are in FSHPS under the controller (11).

For the numerical simulations, fourth-order Runge-Kutta method is used to solve the systems of differential equations (2) and (8). The initial states of the drive system and response system are $(x_1(0), x_2(0), x_3(0) = (1, 5, 2)$ and $(y_1(0), y_2(0), y_3(0) = (11, 15, 12)$ The state errors between two Sprott systems are shown in Fig.3. Obviously, the synchronization errors converge asymptotically to zero and two systems are indeed achieved chaos synchronization.

§4. Conclusion and discussion

In the paper, the problem synchronization of the Sprott N system with parameter is investigated. An effective full state hybrid projective synchronization (FSHPS) controller and analytic expression of the controller for the system are designed. Because of the complete synchronization, anti-synchronization, projective synchronization are all included in FSHPS, our results contain and extend most existing works. But there are exist many interesting and difficult problems left our for in-depth study about this new type of synchronization behavior, therefore, further research into FSHPS and its application is still important and insightful, although it is not in the category of generalized synchronization.

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On the equivalence of some iteration schemes with their errors

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Abstract We show that the convergence of modified Mann iteration, modified Mann iteration with errors, modified Ishikawa iteration, modified Ishikawa iteration with errors, modified Noor iteration, modified Noor iteration with errors, modified Smarandacheian multistep iteration and Smarandacheian multistep iteration with errors are equivalent for uniformly Lipschitzian strongly successively pseudocontractive maps in an arbitrary real Banach space. The results generalise and extend the results of several authors, including Rhoades and Soltuz [20, 21] and Rafiq [19].

Keywords modified Mann-Ishikawa iterations (with errors), modified Noor- multistep iteration (with errors), uniformly Lipschitzian maps, strongly successively pseudocontractive maps.

§1. Introduction

Let X be a real Banach space, and K a non-empty subset of X, T a self mapping of K and $F(T)$, $D(T)$ and I are the set of fixed points, domain of T and identity operator respectively. Let J denote the normalised duality mapping from X to 2^{X^*} defined by

$$J(x) = \{f \in X^* :< x, f >= \|x\|^2, \|f\| = \|x\|\} \ \forall \ x \ \in \ X$$

where X^* denotes the dual space of X and $< \cdot, \cdot >$ denotes the generalised duality pairing.

A map $T : K \to K$ is said to be strongly succesively pseudocontractive if there exists $k \in (0, 1)$ and $n \in N^+$ such that

$$\|x - y\| \leq \|x - y + r[(I - T^n - kI)x - (I - T^n - kI)y\| \tag{1.1}$$

$\forall \ x, \ y \ \in K$ and $r > 0$.

Equivalently, T is strongly successively pseudocontractive if there exists $k \in (0, 1)$ such that

$$< T^n x - T^n y, j(x - y) >\leq (1 - k)\|x - y\|^2,$$

for all $x, \ y \in K$, $n \in N^+$ and $j(x - y) \in J(x - y)$ (see [4]).
T is said to be strongly pseudocontractive if T^n is replaced by T in (1.1). An example of a

successively strongly pseudocontractive map that is not strongly pseudocontractive is in [12, Example 1.2].
T is said to be Uniformly Lipschitzian if there exists some constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.2}$$

$\forall x, y \in K$ and $n \in N^+$.

If T^n is replaced by T in (1.2), then T is said to be a Lipschitzian map.
T is said to be strongly accretive if there exists $k \in (0, 1)$, such that

$$< Tx - Ty, j(x - y) >\geq k\|x - y\|^2 \tag{1.3}$$

$\forall\ x,\ y\ \in K$ and $j(x - y)\ \in\ J(x - y)$
T is said to be accretive if there exists $j(x - y) \in J(x - y)$ such that

$$< Tx - Ty, j(x - y) >\geq 0 \tag{1.4}$$

$\forall\ x, y\ \in K$ and $j(x - y) \in J(x - y)$

It is well known that a map T is strongly pseudocontractive map if and only if $(1 - T)$ is a strongly accretive map.

The procedures of approximating the fixed points of pseudocontractive maps include Mann [13], Ishikawa [8], Noor [17] and multistep iteration process [24]. Also see ([1-25]).

The Mann iteration scheme was introduced in 1953 [13] to obtain a fixed point for many functions for which Banach principle fails. In 1974, Ishikawa [8], introduced another iteration scheme sometimes referred to as two-step iteration scheme. Noor [17] introduced a three-step iterative scheme and used it to approximate solution of variational problems in Hilbert spaces. Noor, Rassias and Huang [18] extend the procedure to solving non-linear equations in Banach spaces. Golwinski and Le Tallec [5] used the scheme to approximate solutions of the elastoviscoplasticity problem, liquid crystal theory and eigen computation.

The modified Mann iteration with errors is defined as

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= (1 - b_n)x_n + b_n T^n_{x_n} + c_n(s_n - x_n), \quad n \geq 1 \end{aligned} \tag{1.5}$$

where $\{s_n\}$ is a bounded sequence in K and $\{a_n\}$, $\{b_n\}$ and $\{a_n\}$ are sequences in [0, 1) such that $a_n + b_n + c_n = 1\ \forall\ n\ \in\ N$

Observe that (1.5) is equivalent to

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n s_n, \quad n \geq 1$$

Remark 1.
1. If T^n is replaced by T in (1.5), we obtain the modified Mann iteration with errors in the

sense of Xu [25]. If in addition, $c_n = 1$, then (1.5) is called the Mann iteration with errors in the sense of Liu [10].
2. If T^n is replaced by T in (1.5), and $c_n = 0$, then (1.5) is called the Mann iteration.

In 2004, Rhoades and Soltuz [21] introduced the modified multistep iteration as follows,

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1}, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n, p \geq 2 \end{aligned} \tag{1.6}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ..., p-1)$ in (0, 1) satisfying certain conditions.

Remark 2.
1. If T^n is replaced by T, the modified multistep iteration (1.6) is referred to as a multistep iteration.
2. If $p = 3$, (1.6) becomes the modified Noor or three step iteration procedure and if in addition, T^n is replaced by T, it is called Noor or three step iteration.
3. If $p = 2$, (1.6) becomes the modified Ishikawa iteration procedure and if in addition, T^n is replaced by T, it is called Ishikawa iteration.

The modified multistep iteration with errors introduced by Liu and Kang [11] is defined by

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + w_n \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + w_n^1, \quad i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + w_n^{p-1}, \quad p \geq 2 \end{aligned} \tag{1.7}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ..., p-1)$ are in [0, 1) and the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ..., p-1)$ are convergent sequences in K, all satisfy certain conditions.

Remark 3.
If T^n is replaced by T, the modified multistep iteration with errors (1.8) reduces to the Noor and the Ishikawa iteration with errors respectively when $p = 3$ and 2. If in addition, $w_n = w_n^i = 0, (i = 1, 2, ...)$, $\forall n \in N$, then (1.8) reduces to Noor and Ishikawa iterations (without errors) respectively.

The Ishikawa and Mann iteration with errors of (1.7) was introduced by Liu [11]. Several papers have been written using this version of iteration procedure with errors. For example, see [6, 7, 9, 14, 15].

However, it should be noted that the iteration process with errors (1.7) is not satisfactory. The errors can occur in a random way. The condition then imposed on the error terms which say that they tend to zero as n tends to infinity are therefore unreasonable (see [3]). This informed the introduction of a better modified iterative processes with errors by Xu [23].

The Xu's modified multistep with errors is defined as follows:

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + c_n(w_n - u_n), \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + c_n^i(w_n^i - u_n), i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + c_n^{p-1}(w_n^{p-1} - u_n),\ p \geq 2, \end{aligned} \tag{1.8}$$

Respectively, where the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ..., p-1)$ are bounded sequences and $\{b_n\}$, $\{b_n^i\}, (i = 1, ..., p-1)$ in [0, 1) satisfy certain conditions $n \in N$.

Observe that the modified multistep iteration with errors (1.8) is equivalent to

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= a_n u_n + b_n T^n v_n^1 + c_n w_n, \\ v_n^i &= a_n^i u_n + b_n^i T^n v_n^{i+1} + c_n^i w_n^i, i = 1, ..., p-2 \\ v_n^{p-1} &= a_n^{p-1} u_n + b_n^{p-1} T^n u_n + c_n^{p-1} w_n^{p-1},\ p \geq 2, n \geq 1 \end{aligned}$$

where the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ..., p-1)$ are bounded sequences in K, and $\{a_n\}$, $\{a_n\}^i$, $\{b_n\}$, $\{b_n^i\}, (i = 1, ..., p-1)$ are in [0, 1) satisfying
$a_n + b_n + c_n = a_n^i + b_n^i + c_n^i = 1$, $i = 1, 2, ..., p-1$.

Rhoades and Soltuz ([20-21]) proved the equivalence of (modified) Mann- Ishikawa and multistep iteration for the strongly (successively) pseudocontractive maps (both Liptchitzian and non-Liptschitzian) with the assumption that K is bounded in a real Banach space. In [6], Huang et al. recently generalise the results of [20-21] to multistep iteration with errors in the sense of Liu (1.7).

In this paper, we show that the modified Mann, Ishikawa, Noor and Smarandacheian multistep iteration with errors (1.8) (using the more satisfactory definition of Xu [23] and that these iterations without errors are all equivalent for (uniformly) Lipchitzian strongly (successively) pseudocontractive maps in an arbitrary real Banach space without assuming boundedness of T in any form.

The results generalise and extend the results of several authors, including those in [5-6], [9-16] and [18-25].

The following Lemma is needed for our results.

Lemma [9]. If X is a real Banach space and $\{\alpha_n\}$ a non-negative sequence which satisfies the following inequality,

$$\alpha_{n+1} \leq (1-\lambda_n)\alpha_n + \delta_n + \gamma_n$$

where $\lambda_n \in (0,1)$, $\gamma \geq 0\ \forall\ n \in N$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

§2. Main results

Theorem 1. Let X be an arbitrary Banach space, K a non-empty closed convex subset of X and T a strongly successively pseudo contractive and uniformly Lipschitzian self map of K with constant $L \geq 1$. Suppose that T has a fixed point $x^* \in F(T)$. Let $x_1,\ u_1 \in K$ and define $\{x_n\}$ and $\{u_n\}$ by (1.5) and (1.8) with $\{b_n\}$, $\{c_n\}$, $\{b_n^i\}$ and $\{c_n^i\}$, $i = 1, ..., p-1$, $\forall\ n \in N$ as sequences in [0,1) satisfying

$$\lim_{n\to\infty} b_n \quad = 0 = \quad \lim_{n\to\infty} b_n^1,$$

and

$$\sum_{n=0}^{\infty} b_n = \infty, \lim_{n\to\infty} c_n = 0, c_n = o(b_n) \tag{2.1}$$

Then the following are equivalent:
(i) The modified Mann iteration with errors (1.5) converges strongly to $x^* \in F(T)$.
(ii) The modified Smarandacheian multi-step iteration with errors (1.8) converges strongly to $x^* \in F(T)$.

Proof. The existence of a fixed point x^* follows from ([4, corollary 1]) which holds in an arbitrary Banach space and the uniqueness follows since T is a strongly successively pseudo contractive and uniformly Lipschitzian map.
(ii) implies (i): It is obvious by setting $b_n^i = 0 = c_n^i$ in (1.8), $(i = 1, ..., p-1)$, $\forall\ n \in N$.
(i) implies (ii): From (1.8), we have

$$\begin{aligned}
u_n &= u_{n+1} + b_n u_n - b_n T^n v_n^1 - c_n(w_n - u_n) \\
&= u_{n+1} + 2b_n u_{n+1} - 2b_n u_{n+1} - b_n T^n u_{n+1} + b_n T^n u_{n+1} \\
&\quad - b_n k u_{n+1} + b_n k u_{n+1} + b_n u_n - b_n T^n v_n^1 - c_n(w_n - u_n) \\
&= (1 + b_n)u_{n+1} + b_n(I - T^n - kI)u_{n+1} - (2-k)b_n[(1-b_n)u_n \\
&\quad + b_n(T^n u_{n+1} - T^n v_n^1) + b_n u_n - c_n(w_n - u_n) \\
&= (1 + b_n)u_{n+1} + b_n(I - T^n - kI)u_{n+1} - (2-k)b_n u_{n+1} \\
&\quad + b_n T^n v_n^1 + c_n(w_n - u_n)] + b_n(T^n u_{n+1} - T^n v_n^1) \\
&\quad + b_n u_n - c_n(w_n - u_n) \\
&= (1 + b_n)u_{n+1} + b_n(I - T^n - kI)u_{n+1} - (1-k)b_n u_n \\
&\quad + (2-k)b_n^2(u_n - T^n v_n^1) + b_n(T^n u_{n+1} - T^n v_n^1) \\
&\quad - c_n[1 + (2-k)b_n](w_n - u_n)
\end{aligned} \tag{2.2}$$

From (1.5), we have

$$\begin{aligned}
x_n &= x_{n+1} + b_n x_n - b_n T^n x_n - c_n(s_n - x_n) \\
&= x_{n+1} + 2b_n x_{n+1} - 2b_n x_{n+1} + b_n T^n x_{n+1} \\
&\quad -b_n T^n x_{n+1} - kb_n x_{n+1} \\
&\quad +kb_n x_{n+1} + b_n x_n - b_n T^n x_n - c_n(s_n - x_n) \\
&= (1+b_n)x_{n+1} + b_n(I - T^n - kI)x_{n+1} - (2-k)b_n x_{n+1} + b_n(T^n x_{n+1} \\
&\quad -T^n x_n) + b_n x_n - c_n(s_n - x_n) \\
&= (1+b_n)x_{n+1} + b_n(I - T^n - kI)x_{n+1} \\
&\quad -(2-k)b_n[(1-b_n)x_n + b_n T^n x_n + c_n(s_n - x_n)] \\
&\quad +b_n x_n + b_n(T^n x_{n+1} - T^n x_n) - c_n(s_n - x_n) \\
&= (1+b_n)x_{n+1} + b_n(I - T^n - kI)x_{n+1} - (1-k)b_n x_n \\
&\quad +(2-k)b_n^2(x_n - T^n x_n) - c_n(1 + (2-k)b_n)(s_n - x_n) \\
&\quad +b_n(T^n x_{n+1} - T^n x_n) \qquad (2.3)
\end{aligned}$$

Subtracting (2.2) from (2.3), we have

$$\begin{aligned}
x_n - u_n &= (1+b_n)(x_{n+1} - u_{n+1}) + b_n[(I - T^n - kI)x_{n+1} \\
&\quad -(I - T^n - kI)u_{n+1}] - (1-k)b_n(x_n - u_n) \\
&\quad +(2-k)b_n^2(x_n - u_n - T^n x_n + T^n v_n^1) \\
&\quad -c_n(1 + (2-k)b_n)[s_n - x_n - w_n + u_n] \\
&\quad +b_n[T^n x_{n+1} - T^n x_n - T^n u_{n+1} + T^n v_n^1] \qquad (2.4)
\end{aligned}$$

Rewriting (2.4), we have

$$\begin{aligned}
&(1+b_n)(x_{n+1} - u_{n+1}) + b_n[(I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1}] \\
&= (x_n - u_n) + (1-k)b_n(x_n - u_n) \\
&\quad -(2-k)b_n^2(x_n - u_n - T^n x_n + T^n v_n^1) \\
&\quad +c_n(1 + (2-k)b_n)[s_n - x_n - w_n + u_n] \\
&\quad -b_n[T^n x_{n+1} - T^n x_n - T^n u_{n+1} + T^n v_n^1] \qquad (2.5)
\end{aligned}$$

But

$$\begin{aligned}
&(1+b_n)(x_{n+1} - u_{n+1})b_n((I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1}) \\
= \;& (1+b_n)[(x_{n+1} - u_{n+1}) + \frac{b_n}{1+b_n}((I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1})] \qquad (2.6)
\end{aligned}$$

Suppose $x = x_{n+1}$ and $y = u_{n+1}$ in (1.1) and using (2.6), we obtain

$$(1+b_n)||x_{n+1} - u_{n+1}||$$
$$\leq (1+b_n)[||x_{n+1} - u_{n+1}|| + \tfrac{b_n}{1+b_n}((I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1})] \qquad (2.7)$$

In view of (2.6) and (2.7), (2.5) becomes

$$
\begin{aligned}
&(1+b_n)\|x_{n+1}-u_{n+1}\| \\
&\le (1+(1-k)b_n)\|x_n-u_n\| + (2-k)b_n^2\|x_n-T^nx_n\| \\
&+ (2-k)b_n^2\|u_n-T^nv_n^1\| + b_n\|T^nx_{n+1}-T^nx_n\| + b_n\|T^nu_{n+1}-T^nv_n^1\| \\
&+ c_n(1+(2-k)b_n)\|s_n-x_n\| + c_n(1+(2-k)b_n)\|w_n-u_n\| \\
&\le (1+(1-k)b_n) + +(2-k)b_n^2\|x_n-T^nx_n\| \\
&+ (2-k)b_n^2\|u_n-T^nv_n^1\| + Lb_n\|x_{n+1}-x_n\| + b_nL\|u_{n+1}-v_n^1\| \\
&+ c_n(1+(2-k)b_n)\|s_n-x_n\| + c_n(1+(2-k)b_n)\|w_n-u_n\| \qquad (2.8)
\end{aligned}
$$

We now evaluate $||u_n - T^nv_n^1||$, $||x_{n+1}-x_n||$ and $||u_{n+1}-v_n^1||$.

$$
\begin{aligned}
||x_{n+1}-x_n|| &= ||(1-b_n)x_n + b_nT^nx_n + c_n(s_n-x_n) - x_n|| \\
&\le b_n||x_n-T^nx_n|| + c_n||s_n-x_n|| \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
||u_n-v_n^1|| &= ||(u_n-(1-b_n^1)u_n - b_n^1T^nv_n^2 - c_n^1(w_n^1-u_n))|| \\
&\le b_n^1||u_n-T^nv_n^2|| + c_n^1||w_n^1-u_n|| \qquad (2.10)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||u_n-v_n^i|| &\le b_n^i||u_n-T^nv_n^{i+1}|| + c_n^i||w_n^i-u_n|| \qquad (2.11)\\
||u_n-v_n^{p-1}|| &\le b_n^{p-1}||u_n-T^nu_n^p|| + c_n^{p-1}||w_n^{p-1}-u_n|| \qquad (2.12)\\
||u_{n+1}-v_n^1|| &= ||(1-b_n)u_n + b_nT^nv_n^1 + c_n(w_n-u_n) - v_n^1|| \\
&\le ||u_n-v_n^1|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\le b_n^1||u_n-T^nv_n^2|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\quad +c_n^1||w_n^1-u_n|| \qquad (2.13)\\
||u_{n+1}-v_n^2|| &= ||(1-b_n)u_n + b_nT^nv_n^1 + c_n(w_n-u_n) - v_n^2|| \\
&\le ||u_n-v_n^2|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\le b_n^2||u_n-Tv_n^3|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\quad +c_n^2||w_n^2-u_n|| \qquad (2.14)\\
||u_{n+1}-v_n^i|| &= ||(1-b_n)u_n + b_nT^nv_n^1 + c_n(w_n-u_n) - v_n^i|| \\
&\le ||u_n-v_n^i|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\le b_n^i||u_n-T^nv_n^{i+1}|| + b_n||u_n-T^nv_n^1|| + c_n||w_n-u_n|| \\
&\quad +c_n^i||w_n^i-u_n|| \qquad (2.15)\\
||u_n-T^nv_n^2|| &\le ||x_n-u_n|| + ||x_n-T^nx_n|| + L||x_n-v_n^2|| \qquad (2.16)\\
||u_n-T^nv_n^i|| &\le ||x_n-u_n|| + ||x_n-T^nx_n|| + L||x_n-v_n^i|| \qquad (2.17)
\end{aligned}
$$

In view of (2.16) and (2.17), we have

$$\begin{aligned}
||u_n - T^n v_n^1|| &= ||u_n - x_n + x_n - T^n x_n + T^n x_n - T^n v_n^1|| \\
&\leq ||u_n - x_n|| + ||x_n - T^n x_n|| + L||x_n - v_n^1|| \\
&= ||u_n - x_n|| + ||x_n - T^n x_n|| \\
&\quad + L||x_n - (1 - b_n^1)u_n - b_n^1 T^n v_n^2 - c_n^1(w_n^1 - u_n)|| \\
&\leq (1 + L)||x_n - u_n|| + ||x_n - T^n x_n|| + b_n^1 L||x_n - T^n x_n|| \\
&\quad + b_n^1 L||u_n - x_n|| + b_n^1 L^2||x_n - v_n^2|| + c_n^1 L(w_n^1 - u_n)|| \\
&= (1 + L + b_n^1 L)||x_n - u_n|| + (1 + b_n^1 L)||x_n - T^n x_n|| \\
&\quad + b_n^1 L^2||x_n - (1 - b_n^2)u_n - b_n^2 T^n v_n^3 - c_n^2(w_n^2 - u_n)|| \\
&\quad + c_n^1 L||w_n^1 - u_n|| \\
&\leq (1 + L + b_n^1 L + b_n^1 L^2)||x_n - u_n|| \\
&\quad + (1 + b_n^1 L)||x_n - T^n x_n|| + b_n^1 b_n^2 L^2||u_n - T^n v_n^3|| \\
&\quad + c_n^1 L(w_n^1 - u_n)|| + b_n^1 c_n^2 L^2||w_n^2 - u_n|| \\
&\leq (1 + L + b_n^1 L + b_n^1 L^2)||x_n - u_n|| + (1 + b_n^1 L)||x_n - T^n x_n|| \\
&\quad + b_n^1 b_n^2 L^2||x_n - u_n|| + b_n^1 b_n^2 L^2||x_n - T^n x_n|| \\
&\quad + b_n^1 b_n^2 L^3||x_n - v_n^3|| + c_n^1 L||w_n^1 - u_n|| + b_n^1 c_n^2 L^2||w_n^2 - u_n|| \\
&\leq (1 + L + b_n^1 L + 2b_n^1 L^2)||x_n - u_n|| \\
&\quad + (1 + b_n^1 L + b_n^1 L^2)||x_n - T^n x_n|| \\
&\quad + b_n^1 L^3||x_n - v_n^3|| + c_n^1 L||w_n^1 - u_n|| + c_n^2 L^2||w_n^2 - u_n|| \\
&\leq (1 + L + 3b_n^1 L^2)||x_n - u_n|| + (1 + 2b_n^1 L^2)||x_n - T^n x_n|| + \\
&\quad b_n^1 L^3||x_n - v_n^3|| + c_n^1 L||w_n^1 - u_n|| + c_n^2 L^2||w_n^2 - u_n|| \qquad (2.18)
\end{aligned}$$

Continuing in this way, we have

$$\begin{aligned}
||u_n - T^n v_n^1|| &\leq (1 + L + (p-1)b_n^1 L^{p-2})||x_n - u_n|| \\
&\quad + (1 + (p-2)b_n^1 L^{p-2})||x_n - T^n x_n|| \\
&\quad + b_n^1 L^{p-1}||x_n - v_n^{p-1}|| + \sum_{i=1}^{p-2} c_n^i L^i||w_n^i - u_n|| \qquad (2.19)
\end{aligned}$$

But, from (1.8),

$$\begin{aligned}
||x_n - v_n^{p-1}|| &\leq ||x_n - u_n|| + b_n^{p-1}||u_n - T^n u_n|| \\
&\quad + c_n^{p-1}||w_n^{p-1} - u_n|| \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
||u_n - T^n u_n|| &\leq ||x_n - u_n|| + ||x_n - T^n x_n|| + L||x_n - u_n|| \\
&= (1 + L)||x_n - u_n|| + ||x_n - T^n x_n|| \qquad (2.21)
\end{aligned}$$

Substituting (2.21) in (2.20), we obtain

$$\begin{aligned}
||x_n - v_n^{p-1}|| &\leq (1 + b_n^{p-1} + b_n^{p-1}L)||x_n - u_n|| \\
&+c_n^{p-1}||w_n^{p-1} - u_n|| + b_n^{p-1}||x_n - T^n x_n|| \qquad (2.22) \\
||u_n - T^n v_n^1|| &= (1 + L + (p-1)b_n^1 L^{p-2})||x_n - u_n|| \\
&+(1 + (p-2)b_n^1 L^{p-2})||x_n - T^n x_n|| \\
&+b_n^1 L^{p-1}(1 + b_n^{p-1} + b_n^{p-1}L)||x_n - u_n|| \\
&+b_n^1 c_n^{p-1} L^{p-1}\|w_n^{p-1} - u_n\| + b_n^1 L^{p-1} b_n^{p-1}||x_n - T^n x_n|| \\
&+\sum_{i=1}^{p-2} c_n^i L^i ||w_n^i - u_n|| \\
&\leq (1 + L + (p+2)b_n^1 L^p)||x_n - u_n|| \\
&+(1 + (p-1)b_n^1 L^{p-1})||x_n - T^n x_n|| \\
&\sum_{i=1}^{p-2} c_n^i L^i ||w_n^i - u_n|| + c_n^{p-1} L^{p-1}\|w_n^{p-1} - u_n\| \\
&= (1 + L + (p+2)b_n^1 L^p)||x_n - u_n|| \\
&+(1 + (p-1))b_n^1 L^{p-1}||x_n - T^n x_n|| \\
&+\sum_{i=1}^{p-1} c_n^i L^i ||w_n^i - u_n|| \qquad (2.23)
\end{aligned}$$

Similarly, using (1.8), we have

$$\begin{aligned}
\|u_n &- T^n v_n^2\| \leq \|u_n - x_n\| + \|x_n - T^n x_n\| + L\|x_n - v^2\| \\
&\leq (1 + L)\|x_n - u_n\| + \|x_n - T^n x_n\| + L b_n^2\|u_n - x_n\| \\
&+ L b_n^2\|x_n - T^n x_n\| + L^2 b_n^2\|x_n - v_n^3\| + L c_n^2\|w_n^2 - u_n\| \\
&= (1 + L + L b_n^2)\|x_n - u_n\| + (1 + L b_n^2)\|x_n - T^n x_n\| \\
&+ L^2 b_n^2\|x_n - v_n^3\| + L c_n^2\|w_n^2 - u_n\| \\
&= (1 + L + L b_n^2)\|x_n - u_n\| + (1 + L b_n^2)\|x_n - T^n x_n\| \\
&+ L^2 b_n^2\|x_n - (1 - b_n^3)u_n - b_n^3 T^n v_n^4 - c_n^3(w_n^3 - u_n)\| \\
&+ L c_n^2||w_n^2 - u_n\| \\
&\leq (1 + L + L b_n^2 + L^2 b_n^2)\|x_n - u_n\| + (1 + L b_n^2)\|x_n - T^n x_n\| \\
&+ L^2 b_n^2 b_n^3\|u_n - x_n + x_n - T^n x_n + T^n x_n - T^n v_n^4\| \\
&+ L^2 b_n^2 c_n^3\|w_n^3 - u_n\| + L c_n^2\|w_n^2 - u_n\| \\
&\leq (1 + L + L b_n^2 + 2L^2 b_n^2)\|x_n - u_n\| \\
&+ (1 + 2L^2 b_n^2)\|x_n - T^n x_n\| + b_n^2 L^3\|x_n - v_n^4\| \\
&+ L c_n^2\|w_n^2 - u_n\| + L^2 c_n^3\|w_n^3 - u_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1+L+2b_n^2L^2+b_n^2L^3+b_n^2L)\|x_n-u_n\| \\
&+ (1+2b_n^2L^2+b_n^2L^3)\|x_n-T^nx_n\| \\
&+ b_n^2b_n^4L^3\|u_n-T^nv_n^5\|+L^3c_n^4\|w_n^4-u_n\|+L^2c_n^3\|w_n^3-u_n\| \\
&+ Lc_n^2\|w_n^2-u_n\| \\
&\leq (1+L+(p+1)b_nL^{p-1})\|x_n-u_n\|+(1+3b_n^2L^3)\|x_n-T^nx_n\| \\
&+ b_n^2L^3\|x_n-v_n^5\|+L^3c_n^4\|w_n^4-u_n\|+L^2c_n^3\|w_n^3-u_n\| \\
&+ Lc_n^2\|w_n^2-u_n\| \qquad (2.24)
\end{aligned}$$

Generalising (2.24), we have

$$\begin{aligned}
\|u_n-T^nv_n^2\| &\leq (1+L+(p-1)b_n^2L^{p-3})\|x_n-u_n\| \\
&+ (1+(p-3)b_n^2L^{p-3}\|x_n-T^nx_n\| \\
&+ b_n^2L^{p-3}\|x_n-v_n^{p-1}\|+\sum_{i=1}^{p-2}c_n^iL^i\|w_n^i-u_n\| \quad (p\geq 3) \qquad (2.25)
\end{aligned}$$

Using (2.22), (2.25) becomes

$$\begin{aligned}
\|u_n-T^nv_n^2\| &\leq (1+L+(p-1)b_n^2L^{p-3})\|x_n-u_n\| \\
&+ (1+(p-3)b_n^2L^{p-3}\|x_n-T^nx_n\| \\
&+ b_n^2L^{p-3}(1+b_n^{p-1}+b_n^{p-1}L)\|x_n-u_n\| \\
&+ b_n^2L^{p-3}b_n^{p-1}\|x_n-T^nx_n\| \\
&+ b_n^2L^{p-3}c_n^{p-1}\|w_n^{p-1}-u_n\|+\sum_{i=1}^{p-2}c_n^iL^i\|w_n^i-u_n\| \\
&\leq (1+L+(p+2)b_n^2L^{p-2})\|x_n-u_n\| \\
&+ (1+(p-2)b_n^2L^{p-2})\|x_n-T^nx_n\| \\
&+ c_n^{p-1}L^{p-1}\|w_n^{p-1}-u_n\|+\sum_{i=2}^{p-2}c_n^iL^i\|w_n^i-u_n\| \\
&\leq (1+L+(p+2)b_n^2L^{p-2})\|x_n-u_n\| \\
&+ (1+(p-2)b_n^2L^{p-2})\|x_n-T^nx_n\| \\
&+ \sum_{i=2}^{p-1}c_n^iL^i\|w_n^i-u_n\| \qquad (2.26)
\end{aligned}$$

Substituting (2.23) and (2.26) in (2.13), we obtain

$$\begin{aligned}
\|u_{n+1}-v_n^1\| &\leq b_n^1[1+L+(p+2)b_n^2L^{p-2}]\|x_n-u_n\| \\
&+ b_n^1[1+(p-2)b_n^2L^{p-2}]\|x_n-T^nx_n\| \\
&+ \sum_{i=2}^{p-1}c_n^iL^i\|w_n^i-u_n\|
\end{aligned}$$

$$\begin{aligned}
&+ \quad b_n[1+L+(p+2)b_n^1L^p]\|x_n-u_n\| \\
&+ \quad b_n[(1+(p-1))b_n^1L^{p-1}]\|x_n-T^nx_n\| \\
&+ \quad \sum_{i=1}^{p-1} c_n^iL^i\|w_n^i-u_n\| \qquad (2.27)
\end{aligned}$$

Substituting (2.9), (2.23) and (2.27) in (2.8), we have

$$\begin{aligned}
(1+b_n)\|x_{n+1}-u_{n+1}\| \quad \leq \quad & [1+(1-k)b_n\|x_n-u_n\|+(2-k)b_n^2\|x_n-T^nx_n\| \\
+ \quad & (2-k)b_n^2[1+L+(p+2)b_n^1L^p]\|x_n-u_n\| \\
+ \quad & (2-k)b_n^2[1+(p-1)b_n^1L^{p-1}]\|x_n-T^nx_n\| \\
+ \quad & (2-k)b_n^2\sum_{i=1}^{p-1} c_n^iL^i\|w_n^i-u_n\| \\
+ \quad & Lb_n^2\|x_n-T^nx_n\|+Lb_nc_n\|s_n-x_n\| \\
+ \quad & b_nb_n^1L[1+L+(p+1)b_n^2L^{p-2}]\|x_n-u_n\| \\
+ \quad & b_nb_n^1L[1+L+(p-2)b_n^2L^{p-2}]\|x_n-T^nx_n\| \\
+ \quad & b_nL\sum_{i=2}^{p-1} c_n^iL^i\|w_n^i-u_n\| \\
+ \quad & (b_n)^2L[1+L+(p+2)b_n^1L^p]\|x_n-u_n\| \\
+ \quad & (b_n)^2L[1+(p-1)b_n^1L^{p-1}]\|x_n-T^nx_n\| \\
+ \quad & b_nL\sum_{i=1}^{p-1} c_n^iL^i\|w_n^i-u_n\| \\
+ \quad & c_n(1+(2-k)b_n)\|s_n-x_n\| \\
+ \quad & c_n(1+(2-k)b_n)\|w_n-u_n\|
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| \quad \leq \quad & \frac{1}{1+b_n}[1+(1-k)b_n+(2-k)b_n(1+L+(p+2)b_n^1L^p \\
+ \quad & b_nL(2+2L+(p+1)b_n^2L^{p-2}+(p+2)b_n^1L^p]\|x_n-u_n\| \\
+ \quad & [(2-k)b_n(2+(p-1)pb_n^1L^{p-1}+b_nL[2+(p-2)b_n^2L^{p-2} \\
+ \quad & (p-1)b_n^1L^{p-1})]\|x_n-T^nx_n\|+((2-k)b_n^2+b_nL)\sum_{i=1}^{p-1} c_n^iL^i\|w_n^i-u_n\|] \\
+ \quad & b_n(c_n(1+(2-k)))(\|s_n-x_n\|+\|w_n-u_n\|) \qquad (2.28)
\end{aligned}$$

Note that $(1+b_n)^{-1}=1-b_n+b_n^2$.

$$\|x_{n+1}-u_{n+1}\| \quad \leq \quad A_n\|x_n-u_n\|+B_n\|x_n-T^nx_n\|+\gamma_n \qquad (2.29)$$

where

$$\begin{aligned} A_n &= [1+(1-k)b_n][(1-b_n+b_n^2]+b_n[(2-k)(1+L+(p+2)b_n^1L^p \\ &+ b_nL(2+2L+(p+1)b_n^2L^{p-2}+(p+2)b_n^1L^p] \\ B_n &= b_n[(2-k)(2+(p-1)b_n^1L^{p-1}+2L+(p-2)b_n^2L^{p-2}+(p-1)b_n^1L^{p-1})] \\ \gamma_n &= ((2-k)b_n^2+b_nL)\sum_{i=1}^{p-1}c_n^iL^i\|w_n^i-u_n\|]+b_n(c_n(1+(2-k)))\|w_n-u_n\| \\ &+b_n(c_n(1+(2-k))\|s_n-x_n\| \end{aligned}$$

Note that

$$\begin{aligned} [1+(1-k)b_n](1-b_n+b_n^2) &= 1-kb_n+kb_n^2+(1-k)b_n^3 \\ &\leq 1-kb_n+kb_n^2+(1-k)b_n^2 \\ &= 1-kb_n+b_n^2 \end{aligned}$$

Therefore, $A_n \leq 1-kb_n+b_n(M_1+M_2+M_3)$ where
$M_1=(2-k)(1+L+(p+2)b_n^1L^p)$, $M_2=2+2L+(p+1)b_n^2L^{p-2}$, $M_3=(p+2)b_n^1L^p$

Since by assumption, $b_n,\ b_n^1 \to 0$, there exists an integer N such that

$$b_n(M_1+M_2+M_3) \quad \leq \quad k(1-k)\forall n \geq N$$

Thus,
$A_n \leq 1-kb_n+k(1-k)b_n = 1-k^2b_n$
Hence, $\lambda_n = k^2b_n \subset (0,1)$ Substituting A_n in (2.29), we have
$\|x_{n+1}-u_{n+1}\| \leq (1-k^2b_n)\|x_n-u_n\|+B_n\|x_n-T^nx_n\|+\gamma_n$
By assumption, we have $\lim_{n\to\infty}\|x_n-x^*\|=0$ $(x^*\in F(T))$. Since T is uniformly Lipschitzian, we have

$$\begin{aligned} 0 &\leq \|x_n-T^nx_n\| \leq \|T^nx_n-T^nx^*\|+\|x_n-x^*\| \\ &\leq (1+L)\|x_n-x^*\| \to 0 \ as\ n\to\infty \end{aligned}$$

So we have $lim_{n\to\infty}\|x_n-T^nx_n\|=0$
Hence, all the assumptions of our Lemma are satisfied. Hence, we have $\lim_{n\to\infty}\|x_n-u_n\|=0$.
Then $\|u_n-x^*\| \leq \|x_n-u_n\|+\|x_n-x^*\| \to 0, (n\to\infty)$

Corollary 1. Let $X, K, L, T, \{a_n\}, \{b_n\}, \{c_n\}, \{a_n^i\}, \{b_n^i\}\{c_n^i\}, \{w_n\}, \{w_n^i\}$, $i=1,...,p-1 (p\geq 2)$, be as in Theorem 1, then for any initial points $u_1, x_1 \in K$, the following statements are equivalent: 1. (Modified) Mann iteration with errors (1.5) converges strongly to the unique fixed point of T.
2. (Modified) Ishikawa iteration with errors (if $p=2$ in (1.8)), converges strongly to the unique fixed point of T.
3. (Modified) Noor iteration with errors (if $p=3$ in (1.8)), converges strongly to the unique fixed point of T.
4. (Modified) Smarandacheian multi-step iteration with errors (1.8) converges strongly to the

unique fixed point of T.

Corollary 2. Let $X, K, L, T, \{a_n\}, \{b_n\}, \{c_n\}, \{a_n^i\}, \{b_n^i\}\{c_n^i\}, , \{w_n\}, \{w_n^i\}$, $i = 1, ..., p-1 (p \geq 2)$, be as in Theorem 1, then for any initial points $u_1, x_1 \in K$, the following statements are equivalent:

1. (Modified) Mann iteration converges strongly to the unique fixed point of T.
2. (Modified) Ishikawa iteration converges strongly to the unique fixed point of T.
3. (Modified) Noor iteration converges strongly to the unique fixed point of T.
4. (Modified) Smarandacheian multi-step Noor iteration converges strongly to the unique fixed point of T

. Proof. If $s_n = w_n = w_n^i = 0$ for each $i = 1, ..., p-1$ in Corollary 1, the result follows.

In view of Corollary 1 and Corollary 2, we have the following theorem.

Theorem 2. Let X be an arbitrary Banach space, K a non-empty closed convex subset of X and T a strongly successively pseudocontractive and uniformly Lipschitzian self map of K with constant $L \geq 1$. Suppose that T has a fixed point $x^* \in F(T)$. Let $x_1, u_1 \in K$ and define $\{x_n\}$ and $\{u_n\}$ by (1.7) and (1.6) respectively, with $\{s_n\}$, $\{w_n\}$, $\{w_n^i\}$, $i = 1, ..., p-1$ bounded sequences in K and $\{b_n\}$, $\{b_n^i\}$, $i = 1, ..., p-1$ $\forall n \in N$ as sequences in [0, 1) satisfying
$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} b_n^i, i = 1, ..., p-1, n \geq 0$
and $\sum_{n=0}^{\infty} b_n = \infty, \sum_{n=0}^{\infty} c_n = 0, c_n^i = o(b_n^i$.
Then, the following are equivalent:

1. The (modified) Mann iteration converges strongly to $x^* \in F(T)$.
2. The (modified) Mann iteration with errors converges strongly to $x^* \in F(T)$.
3. The (modified) Ishikawa iteration converges strongly to $x^* \in F(T)$.
4. The (modified) Ishikawa iteration with errors converges strongly to $x^* \in F(T)$.
5. The (modified) Noor iteration converges strongly to $x^* \in F(T)$.
6. The (modified) Noor iteration with errors converges strongly to $x^* \in F(T)$.
7. The (modified) Smarandacheian multi-step iteration with errors converges strongly to $x^* \in F(T)$.
8. The (modified) Smarandacheian multi-step iteration converges strongly to $x^* \in F(T)$.

Remark 5.

Our results generalise and extend the results of [1-25] in the following way:

1. Theorem 5 of [21] and Theorem 4 of [20] are special cases of our Theorems 1 and 2. in that error terms are not considered in [21] and [20].

3. Our results extend the equivalence of convergence of modified Mann and Ishikawa, and Noor iteration to the more generalised (modified) Smarandacheian multi-step iterations with errors for the uniformly Lipschitzian strongly (successively) pseudocontractive operators in arbitrary real Banach space. The assumption of boundedness of subset K in X or boundedness of range of the operator T is not necessary.

4. Our results generalises the results of [6] and [7] in the sense that we used a more general and acceptable iterations with errors in the sense of Xu [23], of which the iteration with errors due

to Liu [10] used in [6] and [7] are special cases.
Consequently, our theorems and corollaries improves and generalise all the recent results in [18, 26] and their references.

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Smarandache $\nu-$Connected spaces

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Abstract In this paper Smarandache $\nu-$connectedness and Smarandache locally $\nu-$connectedness in topological space are introduced, obtained some of its basic properties and interrelations are verified with other types of connectedness.

Keywords Smarandache $\nu-$connected and Smarandache locally $\nu-$connected spaces

§1. Introduction

After the introduction of semi open sets by Norman Levine various authors have turned their attentions to this concept and it becomes the primary aim of many mathematicians to examine and explore how far the basic concepts and theorems remain true if one replaces open set by semi open set. The concept of semi connectedness and locally semi connectedness are introduced by Das and J. P. Sarkar and H. Dasgupta in their papers. Keeping this in mind we here introduce the concepts of connectedness using $\nu-$open sets in topological spaces. Throughout the paper a space X means a topological space (X, τ). The class of $\nu-$open sets is denoted by $\nu - O(X, \tau)$ respectively. The interior, closure, $\nu-$interior, $\nu-$closure are defined by A^o, A^-, νA^o, νA^- respectively. In section 2 we discuss the basic definitions and results used in this paper. In section 3 we discuss about Smarandache $\nu-$connectedness and $\nu-$components and in section 4 we discuss locally Smarandache $\nu-$connectedness in the topological space and obtain their basic properties.

§2. Preliminaries

A subset A of a topological space (X, τ) is said to be regularly open if $A = ((A)^-)^o$, semi open(regularly semi open or $\nu-$open) if there exists an open(regularly open) set O such that $O \subset A \subset (O)^-$ and $\nu-$closed if its complement is $\nu-$open. The intersection of all $\nu-$closed sets containing A is called $\nu-$closure of A, denoted by $\nu(A)^-$. The class of all $\nu-$closed sets are denoted by $\nu-$CL(X, τ). The union of all $\nu-$open sets contained in A is called the $\nu-$interior of A, denoted by $\nu(A)^o$. A function f: $(X, \tau) \to (Y, \sigma)$ is said to be $\nu-$continuous if the inverse image of any open set in Y is a $\nu-$open set in X; said to be $\nu-$irresolute if the inverse image of any $\nu-$open set in Y is a $\nu-$open set in X and is said to be $\nu-$open if the image of every $\nu-$open set is $\nu-$open. f is said to be $\nu-$homeomorphism if f is bijective, $\nu-$irresolute and $\nu-$open. Let x be a point of (X, τ) and V be a subset of X, then V is said to be $\nu-$neighbourhood of x if there exists a $\nu-$open set U of X such that $x \in U \subset V$. $x \in X$ is said to be $\nu-$limit point of U iff for each $\nu-$open set V containing x, $V \cap (U - \{x\}) \neq \phi$. The set of all $\nu-$limit points of U is called $\nu-$derived set of U and is denoted by $D_\nu(U)$. union and intersection of $\nu-$open sets is not open whereas union of regular and $\nu-$open set is $\nu-$open.

Note 1. Clearly every regularly open set is $\nu-$open and every $\nu-$open set is semi-open but the reverse implications do not holds good. that is, RO(X)$\subset \nu - O(X) \subset$SO(X).

Theorem 2.1. (i)If $B \subset$X such that $A \subset B \subset (A)^-$ then B is $\nu-$open iff A is $\nu-$open.
(ii)If A and R are regularly open and S is $\nu-$open such that $R \subset S \subset (R)^-$. Then $A \cap$R $= \phi$ $\Rightarrow A \cap S = \phi$.

Theorem 2.2. (i) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is $\nu-$open in X iff A is $\nu-$open in $\tau_{/Y}$.
(ii)Let $Y \subseteq X$ and $A \in \nu - O(Y, \tau_{/Y})$ then $A \in \nu - O(X, \tau)$ iff Y is $\nu-$open in X.

Theorem 2.3. An almost continuous and almost open map is $\nu-$irresolute.

Example 1. Identity map is $\nu-$irresolute.

§3.$\nu-$Connectedness.

Definition 3.01. A topological space is said to be Smarandache $\nu-$connected if it cannot be represented by the union of two non-empty disjoint $\nu-$open sets.

Note 2. Every Smarandache $\nu-$connected space is connected but the converse is not true in general is shown by the following example.

Example 2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$; then (X, τ) is connected but not $\nu-$connected

Note 3. Every Smarandache $\nu-$connected space is r-connected but the converse is not true in general is shown by the following example.

Example 3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ then (X, τ) is r-connected but not Smarandache $\nu-$connected

similary one can show that every semi connected space is Smarandache $\nu-$connected but the converse is not true in general.

Theorem 3.01. Let (X, τ) and (Y, σ) be two topological spaces. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\nu-$open and $\nu-$continuous mapping, then the inverse image of each $\nu-$open set in Y is $\nu-$open in X

Corollary 3. Let (X, τ) and (Y, σ) be two topological spaces. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an r-open and r-continuous mapping, then the inverse image of each $\nu-$open set in Y is $\nu-$open in X

Theorem 3.02. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\nu-$continuous mapping, and (X, τ) is Smarandache $\nu-$connected space, then (Y, σ) is also $\nu-$connected.

Corollary 4. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a r-continuous mapping, and (X, τ) is Smarandache $\nu-$connected space, then (Y, σ) is also Smarandache $\nu-$connected.

Theorem 3.03. Let (X, τ) be a topological space and
(i) A be $\nu-$open. Then A is Smarandache $\nu-$connected if and only if $(A, \tau/A)$ is Smarandache $\nu-$connected
(ii) A be r-open. Then A is Smarandache $\nu-$connected if and only if $(A, \tau/A)$ is Smarandache $\nu-$connected

Lemma 3.01. If A and B are two subsets of a topological space (X, τ) such that $A \subset B$ then $\nu(A)^- \subset \nu(B)^-$

Lemma 3.02. If A is $\nu-$connected and $A \subset C \cup D$ where C and D are $\nu-$separated, then either $A \subset C$ or $A \subset D$.

Proof. Write $A = (A \cap C) \cup (A \cap D)$. Then by lemma 3.01, we have $(A \cap C) \cap (\nu(A)^- \cap \nu(D)^-) \subset C \cap \nu(D)^-$. Since C and D are $\nu-$separated, $C \cap \nu(D)^- = \phi$ and so $(A \cap C) \cap (\nu(A)^- \cap \nu(D)^-) = \phi$. Similar argument shows that $(\nu(A)^- \cap \nu(C)^-) \cap (A \cap D) = \phi$. So if both $(A \cap C) \neq \phi$ and $(A \cap D) \neq \phi$, then A is not Smarandache $\nu-$connected, which is a contradiction for A is Smarandache $\nu-$connected. Therefore either $(A \cap C) = \phi$ or $(A \cap D) = \phi$, which in turn implies that either $A \subset C$ or $A \subset D$.

Lemma 3.03. The union of any family of Smarandache $\nu-$connected sets having non-empty intersection is a Smarandache $\nu-$connected set.

Proof. If $E = \cup E_\alpha$ is not $\nu-$connected where each E_α is Smarandache $\nu-$connected, then $E = A \cup B$, where A and B are $\nu-$separated sets. Let $x \in \cap E_\alpha$ be any point, then $x \in E_\alpha$ for each E_α and so $x \in E$ which implies that $x \in A \cup B$ in turn implies that either $x \in A$ or $x \in B$.

Without loss of generality assume $x \in A$. Since $x \in E_\alpha$, $A \cap E_\alpha \neq \phi$ for every α. By lemma 3.02, either each $E_\alpha \subset A$ or each $E_\alpha \subset B$. Since A and B are disjoint we must have each $E_\alpha \subset A$ and hence each $E \subset A$ which gives that $B = \phi$.

Lemma 3.4. If A is Smarandache $\nu-$connected and $A \subset B \subset \nu(A)^-$, then B is Smarandache $\nu-$connected set.

Proof. If E is not $\nu-$connected, then $E = A \cup B$, where A and B are $\nu-$separated sets. By lemma 3.02 either $E \subset A$ or $E \subset B$. If $E \subset A$, then $\nu(E)^- \subset \nu(A)^-$ and so $\nu(E)^- \cap B \subset \nu(A)^- \cap B = \phi$. On the other hand $B \subset E \subset \nu(E)^-$ and so $\nu(E)^- \cap B$. Thus we have $B = \phi$, which is a contradiction. Hence the Lemma.

Corollary 5. If A is Smarandache $\nu-$connected and $A \subset B \subset (A)^-$, then B is Smarandache $\nu-$connected set.

Lemma 3.5. If $f\colon (X,\tau) \to (Y,\sigma)$ is $\nu-$open and $\nu-$continuous, $A \subset X$ is $\nu-$open. Then if A is $\nu-$connected, f(A) is also Smarandache $\nu-$connected.

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ is open and $\nu-$continuous, $A \subset X$ be open. Since A is $\nu-$connected and open in (X,τ), then $(A,\tau_{/A})$ is also $\nu-$connected (by Th. 3.03). Now $f_{/A} : (A,\tau_{/A}) \to (f(A),\sigma_{f(A)})$ is onto and $\nu-$continuous and so by theorem 3.02 $f(A)$ is also $\nu-$connected in $(f(A),\sigma_{f(A)})$. Now for f is open, f(A) is open in (Y,σ) and so by theorem 3.03, f(A) is Smarandache $\nu-$connected in (Y,σ)

We have the following corollaries from the above theorem

Corollary 6. If $f\colon (X,\tau) \to (Y,\sigma)$ is r-open and r-continuous, $A \subset X$ is r-open. Then if A is $\nu-$connected, then f(A) is also Smarandache $\nu-$connected.

Corollary 7. If $f\colon (X,\tau) \to (Y,\sigma)$ is $\nu-$open and $\nu-$continuous, $A \subset X$ is r-open. Then if A is connected, then f(A) is also Smarandache $\nu-$connected.

Definition 3.02. Let (X,τ) be a topological space and $x \in X$. The $\nu-$component of x, denoted by ν C(x), is the union of Smarandache $\nu-$connected subsets of X containing x.

Further if $E \subset X$ and if $x \in E$, then the union of all $\nu-$connected set containing x and contained in E is called the $\nu-$component of E corresponding to x. By the term that C is a $\nu-$component of E, we mean that C is $\nu-$component of E corresponding to some point of E.

Lemma 3.06. Show that $\nu C(x)$ is Smarandache $\nu-$connected for any $x \in X$

Proof. As the union of any family of Smarandache $\nu-$connected sets having a non-empty intersection is a Smarandache $\nu-$connected set, it follows that $\nu C(x)$ is Smarandache $\nu-$connected

Theorem 3.04. In a Topological space (X,τ),

(i) Each $\nu-$component ν(x) is a maximal Smarandache $\nu-$connected set in X.

(ii) The set of all distinct $\nu-$components of points of X form a partition of X and (iii) Each $\nu(x)$ is $\nu-$closed in X

Proof. (i) follows from the definition 3.02

(ii) Let x and y be any two distinct points and $\nu C(x)$ and $\nu C(y)$ be two r-components of x and y respectively. If $\nu C(x) \cap \nu C(y) \neq \phi$, then by lemma 3.03, $\nu C(x) \cup \nu C(y)$ is Smarandache $\nu-$connected. But $\nu C(x) \subset \nu C(x) \cup \nu C(y)$ which contradicts the maximality of $\nu C(x)$.

Let $x \in X$ be any point, then $x \in \nu C(x)$ implies $\cup\{x\} \subset \cup \nu C(x)$ for all $x \in X$ which implies $X \subset \cup \nu C(x) \subset X$. Therefore $\cup \nu C(x) = X$

(iii) Let $x \in X$ be any point, then $(\nu C(x))^-$ is a $\nu-$connected set containing x. But νC(x) is the maximal Smarandache $\nu-$connected set containing x, therefore $(\nu C(x))^- \subset \nu C(x)$. Hence $\nu C(x)$ is $\nu-$closed in X.

§4. Locally $\nu-$connectedness

Definition 4.01. A topological space (X, τ) is called
(i) Smarandache locally $\nu-$connected at $x \in X$ iff for every $\nu-$open set U containing x, there exists a Smarandache $\nu-$connected open set C such that $x \in C \subset U$.
(ii) Smarandache locally $\nu-$connected iff it is Smarandache locally $\nu-$connected at each $x \in X$.

Remark 3. Every Smarandache locally $\nu-$connected topological space is Smarandache locally connected but converse is not true in general.

Remark 4. Smarandache local $\nu-$connectedness does not imply Smarandache $\nu-$connectedness as shown by the following example.

Example 4. X = {a, b, c} and $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ then (X, τ) is Smarandache locally $\nu-$connected but not Smarandache $\nu-$connected.

Remark 5. Smarandache $\nu-$connectedness does not imply Smarandache local $\nu-$connectedness in general.

Theorem 4.01. A topological space (X, τ) is Smarandache locally $\nu-$connected iff the $\nu-$components of $\nu-$open sets are open sets.

Theorem 4.02. If $f : (X, \tau) \to (Y, \sigma)$ is a $\nu-$continuous open and onto mapping, and (X, τ) is Smarandache locally $\nu-$connected space, then (Y, σ) is also locally Smarandache $\nu-$connected.

Proof. Let U be any $\nu-$open subset of Y and C be any $\nu-$component of U, then $f^{-1}(U)$ is $\nu-$open in X. Let A be any $\nu-$component of $f^{-1}(U)$. Since X is locally $\nu-$connected and $f^{-1}(U)$ is $\nu-$open, A is open by theorem 4.01. Also f(A) is $\nu-$connected subset of Y and since C is $\nu-$component of U, it follows that either $f(A) \subset C$ or $f(U) \cap C = \phi$. Thus $f^{-1}(C)$ is the union of collection of $\nu-$components of $f^{-1}(U)$ and so $f^{-1}(C)$ is open. As f is open and onto, $C = f \circ f^{-1}(C)$ is open in Y. Thus any $\nu-$component of $\nu-$open set in Y is open in Y and hence by above theorem Y is Smarandache locally $\nu-$connected.

Corollary 8. If $f\colon (X,\tau) \to (Y,\sigma)$ is a r-continuous r-open and onto mapping, and (X,τ) is Smarandache locally $\nu-$connected space, then (Y,σ) is also Smarandache locally $\nu-$connected.

Proof. Immediate consequence of the above theorem.

Note 4. semi connectedness need not imply and implied by locally semi connectedness. Similarly a Smarandache $\nu-$connected space need not imply and implied by Smarandache locally $\nu-$connected in general.

Theorem 4.03. A topological space (X,τ) is Smarandache locally $\nu-$connected iff given any $x \in X$ and a $\nu-$open set U containing x, there exists an open set C containing x such that C is contained in a single $\nu-$component of U.

Proof. Let X be Smarandache locally $\nu-$connected, $x \in X$ and U be a $\nu-$open set containing x. Let A be a $\nu-$ component of U containing x. Since X is Smarandache locally $\nu-$connected and U is $\nu-$open, there is a Smarandache $\nu-$connected set C such that $x \in C \subset U$. By theorem 3.01, A is the maximal Smarandache $\nu-$connected set containing x and so $x \in C \subset A \subset U$. Since $\nu-$components are disjoint sets, it follows that C is not contained in any other $\nu-$component of U.

Conversely, suppose that given any point $x \in X$ and any $\nu-$open set U containing x, there exists an open set C containing x which is contained in a single $\nu-$component F of U. Then $x \in C \subset F \subset U$. Let $y \in F$, then $y \in U$. Thus there is an open set O such that $y \in O$ and O is contained in a single $\nu-$component of U. As the $\nu-$components are disjoint sets and $y \in F, y \in O \subset F$. Thus F is open. Thus for every $x \in X$ and for every $\nu-$open set U containing x, there exists a Smarandache $\nu-$connected open set F such that $x \in F \subset U$. Thus (X,τ) is Smarandache locally $\nu-$connected at x. Since $x \in X$ is arbitrary, (X,τ) is Smarandache locally $\nu-$connected.

Remark 6.

Connected $\Leftarrow$	semi-connected
$\Downarrow$	$\Downarrow$
r-Connected $\Leftarrow$	$\nu-$Connected.

none is reversible

Example 5. FOR X = {a, b, c, d}; $\tau_1 = \{\phi, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, X\}$
$\tau_2 = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $\tau_3 = \{\phi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, X\}$

(X,τ_1) is both r-connected and Smarandache $\nu-$connected; (X,τ_2) is r-connected but not Smarandache $\nu-$connected and (X,τ_3) is neither r-connected and nor Smarandache $\nu-$connected

Conclusion.

In this paper we defined new type of connectedness using $\nu-$open sets and studied their interrelations with other connectedness.

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Smarandache #-rpp semigroups whose idempotents satisfy permutation identities

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Abstract The aim of this paper is to study Smarandache #-rpp semigroups whose idempotents satisfy permutation identities. After some properties are obtained, the weak spined product structure of such semigroups is established.

Keywords Smarandache #-rpp semigroups, normal band, weak spined product.

§1. Introduction

Similar to rpp rings, a semigroup S is called an rpp semigroup if for any $a \in S, aS^1$ regarded as a right S^1 system is projective. In the study of the structure of rpp semigroups, Fountain[1] considered a Green-like right congruence relation $\mathcal{L}^*$ on a semigroup S defined by $(a, b \in S)a\mathcal{L}^*b$ if and only if $ax = ay \Leftrightarrow bx = by$ for all $x, y \in S^1$. Dually, we can define the left congruence relation $\mathcal{R}^*$ on a semigroup S. It can be observed that for $a, b \in S, a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ when S is a regular semigroup. Also, we can easily see that a semigroup S is an rpp semigroup if and only if each $\mathcal{L}^*$-class of S contains at least one idempotent. Later on, Fountain[2]called a semigroup S an abundant semigroup if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of contain at least one idempotent. An important subclass of the class of rpp semigroups is the class of C-rpp semigroups. We call an rpp semigroup S a C-rpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-rpp semigroup if and only if S is a strong semilattice of left cancellative monoids (see [1]). Because a Clifford semigroup can always be expressed as a strong semilattice of groups, we see immediately that the concept of C-rpp semigroups is a proper generalization of Clifford semigroups. Guo-Shum-Zhu [3] called an rpp semigroup S a strongly rpp semigroup if every L_a^* contains a unique idempotent $a^+ \in L_a^* \cap E(S)$ such that $a^+a = a$ holds, where $E(S)$ is the set of all idempotents of S. They then called a strongly rpp semigroup S a left C-rpp semigroup if $\mathcal{L}^*$ is a congruence on S and $eS \subseteq Se$ holds for any $e \in E(S$. It is noticed that the set $E(S)$ of idempotents of a left C-rpp semigroup S forms a left regular band, that is,$ef = efe$ for $e, f \in E(S)$. Because of this crucial observation, we can describe the left C-rpp semigroup by using the left regular band and the C-rpp semigroup.The structure of left C- rpp semigroups and abundant semigroups has been investigated by many authors (see[4-12], In particular, it was proved in [3] that if S is a strongly rpp semigroup whose set of idempotents $E(S)$ forms a left regular band, then S is a left C-rpp semigroup if and only if S is a semilattice of direct products of a left zero band and a left cancellative monoid, that is, the left C-rpp semigroup S is expressible as a semilattice of left cancellative strips.

Let S be a semigroup, A a subset of S and let

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

a non-identity permutation on n objects. Then a semigroup A is said to satisfy the permutation identity determined by σ (in short, to satisfy a permutation identity if there is no ambiguity) If $(\forall x_1, x_2, \cdots, x_n \in A) x_1x_2\cdots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$, Where $x_1x_2\cdots x_n$ is the product of $x_1, x_2, \cdots, x_n$ in S. If $A = S$, then S is called a PI- semigruop. Guo[13] investigated abundant semigroups whose idempotents satisfy permutation identities,and the quasi-spined product structure of such semigroups was established. In particular, the structure of PI-abundant semigroups was obtained. Later,Guo[14] again discussed strongly rpp semigroups whose idempotents satisfy permutation identities,Du-He[15]obtained the structure of eventually strongly rpp semigroups whose idempotents satisfy permutation identities.

By modifying Green's star relations, Kong-Shum[16] have introduced a new set of Green's #-relations on a semigroup and by using these new Green's relations, they were able to give a description for a wider class of abundant semigroups, namely, the class of #- abundant semigroups(see[16]). As a generalization of rpp semigroups whose idempotents satisfy permutation identities, the aim of this paper is to investigate Smarandache #-rpp semigroups whose idempotents satisfy permutation identities,that is,PI- #-rpp.

For terminology and notations not give in this paper, the reader is referred to references[17,18].

§2.Preliminaries

We first recall that the Green's #- relations defined in [16].

$$a\mathcal{L}^{\#}b \text{ if and only if for all } e, f \in E(S^1), ae = af \Leftrightarrow be = bf,$$
$$a\mathcal{R}^{\#}b \text{ if and only if for all } e, f \in E(S^1), ea = fa \Leftrightarrow eb = fb.$$

We easily check that the relations $\mathcal{L}^{\#}$ and $\mathcal{R}^{\#}$ are equivalent relation. However, $\mathcal{L}^{\#}$ is not a right compatible (that is,right congruence),$\mathcal{R}^{\#}$ is not a left compatible (that is,left congruence), and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^{\#}, \mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^{\#}$. A semigroup S is right #-abundant if each $\mathcal{L}^{\#}$- class of S contains at least one idempotent,write as #- rpp. we can define left # -abundant semigroups dually, write as #-lpp. A semigroup S is called #-abundant if it is both right # -abundant and left #- abundant. Abundant semigroups S is proper subclass of #-abundant semigroups(see[16]), and if a, b are regular elements of S , then $a\mathcal{L}^{\#}b$ if and only if $a\mathcal{L}b$ (see[16]).

If there is no special indication of $\mathcal{L}^{\#}$ relation on S, we always suppose $\mathcal{L}^{\#}$ is a right congruence on S, and always suppose that S is a Smarandache #-rpp semigroup whose idempotents satisfy permutation identities,that is,PI- #-rpp.

Lemma 2.1. **[16]** For any $e \in E(S), a \in S$, the following conditions are equivalent:
(1)$(e, a) \in \mathcal{L}^{\#}$;
(2)$a = ae$ and $ag = ah \Leftrightarrow eg = eh(\forall g, h \in E(S^1))$.

A band B is that a semigroup in which every element is an idempotent. We call a band B a [left,right]normal band if B satisfies the identity $(abc = acb, abc = bac)abcd = acbd$.

Lemma 2.2. **[14]** The following statements are equivalent for a band B:

(1)B is normal;

(2)B is a strong semilattice of rectangular bands;

(3)$\mathcal{L}$ and $\mathcal{R}$ are a left normal band congruence and a right normal band congruence on, respectively.

It is well known that any band is a semilattice of rectangular bands. If $B = \cup_{\alpha\in Y} B_\alpha$ is the semilattice decomposition of a band B into rectangular bands B_α with $\alpha \in Y$, then we shall write $B_\alpha = E(e)$ for $e \in B_\alpha$ and $B_\alpha \geq B_\beta$ when $\alpha \geq \beta$ on the indexed semilattice Y. Next, we always assume that S is a Smarandache #-rpp semigroup satisfying the permutation identity: $x_1x_2\cdots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$. We denote idempotents in the $\mathcal{L}^{\#}$- class of a by $a^{\#}$ for every $a \in S$.

Lemma 2.3. **[17]** $E(S)$ is a normal band.

Lemma 2.4. Let $a \in S, e, f \in E(S)$. Then

(1)$efa = efa^{\#}$;

(2)$eaf = eaef$.

Proof. Suppose that i is the minimum positive number such that $\sigma(i) \neq i$. Obviously $i < \sigma(i)$.

(1)Take $x_j = e$ when $1 \leq j < i, x_j = f$, if $1 \leq j < i, x_j = f, i \leq j < \sigma(i), \sigma(i) = a$ otherwise $\sigma(i) = a^{\#}$. Then $e(x_1x_2\cdots x_n)a^{\#} = efa$. On the other hand, by Lemma 2.4, $e(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}) = eafa^{\#}$, and hence, $efa = eafa^{\#}$.

(2)Now, assume that $x_j = e$ when $1 \leq j < \sigma(i), x_{\sigma(i)} = a$ otherwise $x_j = f$, then $e(x_1x_2\cdots x_n)f = eaf$, and $ee(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)})f = eaef$ or $eafef$. Hence, by Lemma 2.3, we can infer that $eafef = eaa^{\#}fef = eaa^{\#}eff = eaef$. Thus, we get $eaf = eaef$

Now suppose that $E(S) = [Y, E_\alpha, \Psi_{\alpha,\beta}]$ is the strong semilattice of rectangular bands E_α. If $e \in E_\alpha$, we will write the rectangular band E_α by $E(e)$. Also if $E_\alpha E_\beta \subseteq E_\beta$, then we can write $E_\beta \leq E_\alpha$.

Lemma 2.5. For every $a, b \in S$ and $f \in E(S)$. If $a = bf$,then $E(a^{\#}) \leq E(f)$.

Proof. If $a = bf$, then $a = af$. By the definition of $\mathcal{L}^{\#}$, we have $a^{\#} = a^{\#}f$, and so $E(a^{\#}) \leq E(f)$.

§3.The structure of Smarandache #-rpp semigroups satisfying permutation identities

In this section, we will give the concept of weak spined product of semigroups, and the structure of Smarandache #-rpp semigroups satisfying permutation identities is obtained.

We now define a relation ε on S as follows:

$$a\varepsilon b \text{ if and only if for some } f \in E(b^{\#}), a = bf,$$

where $a, b \in S$.

Lemma 3.1.(1) ε is a congruence on S preserving $\mathcal{L}^{\#}$-class;
(2)$\varepsilon \cap \mathcal{L}^{\#} = \imath_S$ (the identical mapping on S).

Proof. First of all,we prove that ε is an equivalence relation. Obviously, $a\varepsilon a$ since $a = aa^{\#}$ for all $a \in S$. Hence ε is reflexive.

Let $a, b \in S$ with $a\varepsilon b$. Then for some $f \in E(b^{\#}), a = bf$. By Lemma 2.5,$E(a^{\#}) < E(f) = E(b^{\#})$. It follows that $a^{\#}b^{\#} \in E(a^{\#})$, and $a\varepsilon b$ means that $E(b^{\#}) < E(b^{\#})$. Since

$$a(b^{\#}b^{\#}) = ab^{\#} = bfb^{\#} = bb^{\#}fb^{\#} = bb^{\#} = b.$$

We have $b\varepsilon a$, and hence ε satisfies the symmetric relation. On the other hand, by the above proof, $b\varepsilon a$ means that $E(b^{\#}) \leq E(a^{\#})$. Therefore,$E(b^{\#}) = E(a^{\#})$.

Next,we show that ε is transitive. We let $a, b, c \in S$ with $a\varepsilon b, b\varepsilon c$. Then we have $E(a^{\#}) = E(b^{\#}) = E(c^{\#})$. By the definition of ε, there exist $e, f \in E(c^{\#})$ such that $a = be, b = cf$. Thus, $a = efe$. Notice that $fe \in E(c^{\#})$,we get $a\varepsilon c$. So ε is indeed an equivalence relation on S.

Following we show that ε is both left and right compatible. Now let $a, b, c \in S$ and $a\varepsilon b$. Then there exists $f \in E(b^{\#})$ such that $a = bf$ .Obviously,$ca = cbf = cb(cb)^{\#}f$. By Lemma2.5, $E(ca^{\#}) \leq E((ca)^{\#})$ and $E((ca)^{\#}) \leq E(f)$, and hence $(ca)^{\#}b^{\#} \in E((ca)^{\#})$ but $a = bf$, this implies that $b = ab^{\#}$. We have $cb = cab^{\#} = ca(ca)^{\#}b^{\#}$. Hence $ca\varepsilon cb$. By lemma 2.4, we have

$$ac = bfc = bb^{\#}fc = bb^{\#}cfc = bc(fc^{\#}) = bc(bc)^{\#}fc^{\#}.$$

We can deduce that$E((ac)^{\#}) \leq E((bc)^{\#})$ and $E((ac)^{\#}) \leq E(fc^{\#})$ by lemma 2.5. A similar argument for $b = ab^{\#}$, we can infer that $E((bc)^{\#}) \leq E((ac)^{\#})$. Hence $E((ac)^{\#}) = E((bc)^{\#})$. This means that $(bc)^{\#}fc^{\#} \in E((bc)^{\#})$. Therefore,$ac\varepsilon bc$, and hence ε is a congruence on S.

Finally, we prove that ε preserves $\mathcal{L}^{\#}$- class. We need to prove that if $a\mathcal{L}^{\#}b$ then $(ag)\varepsilon = (ah)\varepsilon$ implies $(bg)\varepsilon = (bh)\varepsilon$, where $g, h \in E(S^1)$ and $g\varepsilon, h\varepsilon \in (S/\varepsilon)^1$. Since $(ag)\varepsilon(ah)$, by the definition of ε, we have $ah = agf$ for some $f \in E((ag)^{\#})$. Therefore, we obtain $bh = bgf$ since $a\mathcal{L}^{\#}b$. Since $\mathcal{L}^{\#}$ is a right congruence on S, we have $ag\mathcal{L}^{\#}bg$ for $g \in E(S^1)$ so that $E((ag)^{\#}) = E((bg)^{\#})$ and thereby $f \in E((ag)^{\#}) = E((bg)^{\#})$. Thus, by the definition of ε, $bh = bgf$ allows $(bg, bh) \in \varepsilon$, that is, $(bg)\varepsilon = (bh)\varepsilon$. According to this result and its dual and the definition of $\mathcal{L}^{\#}$, we conclude that $a\varepsilon\mathcal{L}^{\#(S/\varepsilon)}b\varepsilon$.

(2) Let $(a, b) \in \varepsilon \cap \mathcal{L}^{\#}$. Then $a = bf$ for some $f \in E(b^{\#})$. Since $a\mathcal{L}^{\#}b$,we get $a^{\#}\mathcal{L}^{\#}b^{\#}$. So $a = ab^{\#} = bfb^{\#} = bb^{\#}fb^{\#} = bb^{\#} = b$. Hence $\varepsilon \cap \mathcal{L}^{\#} = \imath_S$.

Lemma 3.2. $E(S/\varepsilon)$ is a left normal band.

Proof. Since S is a PI-semigroup, we easily check that S/ε is a PI-semigroup. Notice that $a\varepsilon \in E(S/\varepsilon)$ implies $a \in E(S)$ and $\varepsilon \cap (E(S) \times E(S)) = \mathcal{R}$. So $E(S/\varepsilon) = E/\mathcal{R}$. Thus $E(S/\varepsilon)$ is a left normal band.

Lemma 3.3. If $E(S)$ is a left normal band, then S satisfies the identity $abc = acb$.

Proof. Let i have the same neaning as that in the proof of lemma 2.4. For every $a, b, c \in S$, take $x_i = b, x_{\sigma(i)} = c$ and $x_j = a^{\#}$ otherwise then $a^{\#}(x_1x_2\cdots x_n)c^{\#} = a^{\#}ba^{\#}ca^{\#}c^{\#}, a^{\#}bca^{\#}c^{\#}$ $a^{\#}ba^{\#}c$ or $a^{\#}bc$. By hypothesis and Lemma 2.4,we have

$$a^{\#}ba^{\#}ca^{\#}c^{\#} = a^{\#}ba^{\#}c = a^{\#}bb^{\#}a^{\#}c = a^{\#}bb^{\#}a^{\#}b^{\#}c = a^{\#}ba^{\#}b^{\#}c = a^{\#}bb^{\#}c = a^{\#}bcc^{\#} = a^{\#}bc.$$

Thus, we obtain that $a^{\#}(x_1x_2\cdots x_n)c^{\#} = a^{\#}ca^{\#}ba^{\#}c^{\#}, a^{\#}cba^{\#}c^{\#}$ or $a^{\#}cbc^{\#}$. In other word, we have

$$\begin{aligned} a^{\#}ca^{\#}bc^{\#} &= a^{\#}ca^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}a^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}a^{\#}cba^{\#}c^{\#} \\ &= a^{\#}c^{\#}ac^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}ba^{\#}c^{\#} = a^{\#}cba^{\#}c^{\#} \\ &= a^{\#}cbc^{\#} = a^{\#}cc^{\#}b(b^{\#}c^{\#}b^{\#}) = a^{\#}cc^{\#}bc^{\#}b^{\#} \\ &= a^{\#}cc^{\#}bb^{\#} = a^{\#}cb. \end{aligned}$$

Hence $a^{\#}(x_{\sigma(1)})x_{\sigma(2)}\cdots x_{\sigma(n)} = a^{\#}ca^{\#}, ba^{\#}c^{\#}, a^{\#}cba^{\#}c^{\#}$ or $a^{\#}cbc^{\#}$. So we obtain $a^{\#}bc = a^{\#}cb$, and hence we have $abc = aa^{\#}bc = aa^{\#}cb = acb$.

Theorem 3.4. Let S be a Smarandache #-rpp semigroup with set $E(S)$ of idempotents. Denote by λ_a the inner left translation of determined by $a \in S$. Then the following statements are equivalent:

(1) S satisfies permutation identities;

(2) S satisfies the identity ;

(3) For all $e \in E(S)$,eSe is a commutative semigroup and λ_e is a homomorphism;

(4) For all $e \in E(S)$,eS satisfies the identity:$abc = bac$ and λ_e is a homomorphism.

Proof. (1) $\Rightarrow$ (2). Let S be a PI-Smarandache #-rpp semigroup. For every $a, b, c, d \in S$, by Lemma 3.1-3.3,$(adcd)\varepsilon = (acbd)\varepsilon$. Then for some $f \in E((acbd)^{\#}), abcd = acbdf$. Furthermore, we also have

$$\begin{aligned} abcd &= abcdd^{\#}acbdfd^{\#} = acbd(acbd)^{\#}fd^{\#} \\ &= acbd(acbd)^{\#}f(acbd)^{\#}d^{\#} \\ &= acbd(acbd)^{\#}d^{\#} = acbd. \end{aligned}$$

by Lemma 2.3.

(2) $\Rightarrow$ (3). Assume that (2) holds. Let $e \in E(S)$. Then for every $a, b \in eSe$. we have $a = ea = ae$ and $b = eb = be$. Hence $ab = eabe = ebae = eab$. This means that eSe is a commutative semigroup. On the other hand, since $\lambda_e(a)\lambda_e(b) = eaeb = eeab = eab = \lambda_e(ab)$ is a homomorphism of into itself. Therefore (3) holds.

(3) $\Rightarrow$ (4). Suppose that (3) holds. It remains to prove the first part. For all $a, b, c \in eS$, we get $a = ea, b = eb$ and $c = ec$. Since λ_e is a homomorphism, We have

$$abc = eaebec = (eae)(ebe)c = (ebe)(eae)(ebe)c = (eb)(ea)(ec) = bac.$$

(4) $\Rightarrow$ (2). Let $a, b, c, d \in S$. Then

$$abcd = a(a^{\#}bcd) = a(a^{\#}b)(a^{\#}c)(a^{\#}d) = a(a^{\#}c)(a^{\#}b)(a^{\#}d) = a(a^{\#}cbd) = acbd.$$

(2) $\Rightarrow$ (1). This part is trivial.

Let S be a Smarandache #-rpp semigroup whose idempotents form a subsemigroup $E(S)$. Let Y be the structure semilattice of $E(S)$ such that $E(S) = \cup_{\alpha\in Y} E_\alpha$ is structure decomposition of $E(S)$. Now let B be a right normal band and $B = \cup_{\alpha\in Y} B_\alpha$ is a semilattice composition of the right zero band B_α. For $a \in S$, if $a^{\#} \in E_\alpha$ we denote $a^\triangle = \alpha$. Take $M = \{(a,s) \in S \times B | x \in B_{a^\triangle}$. Define a multiplication "$\circ$ " on M as follows:

$$(a,x) \circ (b,y) = (ab, y\varphi_{b^\triangle,(ab)^\triangle}), i.e. = (ab, zy),$$

when $z \in B_(ab)^\triangle$. Notice that $ab = abb^{\#}$, we have $(ab)^{\#} = (ab)^{\#}b^{\#}$. It follows that $(ab)^\triangle = (ab)^\triangle b^\triangle$. This means that $(ab)^\triangle \leq b^\triangle$(" $\triangle$ " is the natural order). Accordingly, $y\varphi_{b^\triangle,(ab)^\triangle} \in B_{(ab)^\triangle}$. So M is well defined and with respect to "$\circ$ ", M is closed.

Lemma 3.5. M is a semigroup.

Proof. Because with respect to "$\circ$ ", M is closed. We only need to show that "$\circ$ " satisfies the associative law. Let $(a,x),(b,y),(c,z) \in M$. Then by the above statement, we can show that $(abc)^\triangle \leq (bc)^\triangle \leq c^\triangle$. Thus

$$\begin{aligned}((a,x)\circ(b,y))\circ(c,z) &= (ab, y\varphi_{b^\triangle,(ab)^\triangle})\circ(c,z) = (abc, z\varphi_{c^\triangle,(abc)^\triangle})\\ &= (a,x)\circ(bc, z\varphi_{c^\triangle,(bc)^\triangle}) = (a,x)\circ((b,y)\circ(c,z)).\end{aligned}$$

So " $\circ$" is associative. Hence M is indeed a semigroup.

Definition 3.6. We call $(M,\circ)$ above the weak–spined product of S and B , and denote it by $WS(S,B)$.

Lemma 3.7. If S satisfies the identity $abc = acb$, the $WS(S,B)$ satisfies the identity $abcd = acbd$.

Proof. Let $(a.i),(b,j),(c,k),(d,l) \in WS(S,B)$. Then

$$(a.i)\circ(b,j)\circ(c,k)\circ(d,l) = (abcd, l\varphi_{d^\triangle,(abcd)^\triangle}) = (abcd, l\varphi_{d^\triangle,(acbd)^\triangle}) = (a.i)\circ(c,k)\circ(b,j)\circ(d,l).$$

Hence $WS(S,B)$ satisfies the identity.

Theorem 3.8. A Smarandache #-rpp semigroup is a PI-Smarandache #-rpp semigroup if and only if it is isomorphic to the weak spined product of a Smarandache #-rpp semigroup satisfying the identity $abc = acb$ and a right normal band.

Proof. By Lemma 3.7, it suffices to prove the "only if "part. Suppose that S is a PI-Smarandache #-rpp semigroup with normal band $E(S)$. Then by Lemma 3.2 and Lemma 3.3, S/ε is a Smarandache #-rpp semigroup satisfying the identity $abc = aeb$. Let Y be the structure decomposition of $E(S)$. By Lemma 2.2, we have $E(S)/\mathcal{L} = \cup_{\alpha\in Y} E_\alpha/\mathcal{L}$ that is a right normal band. Notice that ε is idempotent pure and $\varepsilon \cap (E(S) \times E(S)) = \mathcal{R}$, we can easily know that $E(S/\varepsilon) = E(S)/\mathcal{R} = \cup_{\alpha\in Y} E_\alpha/\mathcal{R}$. Thus we can consider the weak spined product $WS(S/\varepsilon, E(S)/\mathcal{L})$.

Define

$$\theta : S \to WS(S/\varepsilon, E(S)/\mathcal{L}), a \mapsto (ae, \overline{a^{\#}}),$$

where $\overline{a^{\#}}$ is the $\mathcal{L}$ - class of containing $a^{\#}$. In order to prove the theorem, we need only to show that θ is an isomorphism. By Lemma 3.1, θ is well defined and injective. Take any element $(x, \overline{e}) \in WS(S/\varepsilon, E(S)/\mathcal{L})(e \in E(S))$, since $x \in S/\varepsilon$, there exists $s \in S$ such that $x = s\varepsilon$. By the definition of $WS(S/\varepsilon, E(S)/\mathcal{L}), s^{\#}\mathcal{D}^E a$. This means that θ is onto. For all $s,t \in S$, since $st = (st)t^{\#}$, by the definition of $\mathcal{L}^{\#}$, we have $(st)^{\#} = (st)^{\#}t^{\#}$. Furthermore, by Lemma 3.1, we can get $\overline{(st)^{\#}} \in B((st)\varepsilon)^\triangle$. Then $\theta(st) = ((st)\varepsilon, \overline{(st)^{\#}}) = ((st)\varepsilon, \overline{(st)^{\#}t^{\#}}) = (s\varepsilon, \overline{s^{\#}})(t\varepsilon, \overline{t^{\#}}) = \theta(s)\theta(t)$. To sum up,$\theta$ is an isomorphism of S onto $WS(S/\varepsilon, E(S)/\mathcal{L})$. The proof is completed.

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Smarandache U-liberal semigroup structure

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Abstract In this paper, Smarandache U-liberal semigroup structure is given. It is shown that a semigroup S is Smarandache U-liberal semigroup if and only if it is a strong semilattice of some rectangular monoids. Consequently, some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups are generalized and extended.

Keywords Smarandache U-liberal semigroup, $U-$semiabundant semigroups, normal band, rectangular monoid, strong semilattice.

§1. Introduction and preliminaries

In order to generalize regular semigroups, new Green's relations, namely, the Green's $*$-relations on a semigroup S have been introduced in [1] and [2] as follows:

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*, \qquad \mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

In [3], Fountain investigated a class of semigroups called abundant semigroups in which each $\mathcal{L}^*-$class and each $\mathcal{R}^*-$class of S contain at least an idempotent. Actually, the class of regular semigroups are properly contained in the class of abundant semigroups.

In 1980, El-Qallali generalized the Green's $*$- relations to the Green's $\sim -$relations on a semigroup S in [4] as follows:

$$\tilde{\mathcal{L}} = \{(a, b) \in S \times S : (\forall e \in E(S))\ ae = a \Leftrightarrow be = b\},$$

$$\tilde{\mathcal{R}} = \{(a, b) \in S \times S : (\forall e \in E(S))\ ea = a \Leftrightarrow eb = b\},$$

$$\tilde{\mathcal{H}} = \tilde{\mathcal{L}} \cap \tilde{\mathcal{R}}, \qquad \tilde{\mathcal{D}} = \tilde{\mathcal{L}} \vee \tilde{\mathcal{R}}.$$

In his thesis, El-Qallali obtained and studied a much bigger class of semigroups, called semi-abundant semigroup.

After that, many authors study this class of semigroups, and obtain a lot of interesting conclusions and results(see [5],[6],[7] etc.).

In recent years, some scholars have observed that one can pay special attention to a subset U of $E(S)$ instead of the whole set $E(S)$ of a semiabundant semigroup S. In particular, Lawson in [8] noticed that if U is a subset of $E(S)$ of a semiabundant semigroup S then U is perhaps good enough to provide sufficient information for the whole semigroup S. The semigroup S is usually denoted by $S(U)$ and the equivalence relations on $S(U)$ with respect to $U \subseteq E(S)$ can be given by

$$\tilde{\mathcal{L}}^U = \{(a,b) \in S \times S | U_a^r = U_b^r\},$$
$$\tilde{\mathcal{R}}^U = \{(a,b) \in S \times S | U_a^l = U_b^l\},$$
$$\tilde{\mathcal{H}}^U = \tilde{\mathcal{L}}^U \cap \tilde{\mathcal{R}}^U,$$
$$\tilde{\mathcal{Q}}^U = \{(a,b) \in S \times S | U_a = U_b\},$$

where $U_a^l = \{u \in U | ua = a\}$, $U_a^r = \{u \in U | au = a\}$ and $U_a = U_a^l \cap U_a^r = \{u \in U | ua = a = au\}$ for any $a \in S$.

A semigroup $S(U)$ is said to be a $U-$ semiabundant semigroup if every $\tilde{\mathcal{L}}^U$ and every $\tilde{\mathcal{R}}^U-$ class of $S(U)$ contain at least one element of U respectively. A semigroup $S(U)$ is said to be a $U-$ semi-superabundant semigroup if every $\tilde{\mathcal{H}}^U$ of $S(U)$ contains at least one element of U. In this case, the unique element in $\tilde{\mathcal{H}}_a^U \cap U$ is denoted by $a_U^\diamond$. On the other hand, a semigroup $S(U)$ is called by He in [7] a $U-$ liberal semigroup if every $\tilde{\mathcal{Q}}^U-$ class of S contains an element of U. It is routine to check that a $\tilde{\mathcal{Q}}^U-$ class contains at most one element of U. Denote the unique element in $\tilde{\mathcal{Q}}_a^U \cap U$, if it exists, by a_U° . The structure of Smarandache $U-$ liberal semigroups has also been recently investigated by He in [7].

For a Smarandache $U-$ liberal semigroup $S(U)$, we call the following condition the Ehresmann type condition, in brevity, the ET-condition:

$$(\forall a, b \in S)(ab)_U^0 \mathcal{D}(U) a_U^0 b_U^0,$$

where

$$\mathcal{D}(U) = \{(e,f) \in U \times U | (\exists g \in U) e \mathcal{R} g \mathcal{L} f\}.$$

A Smarandache $U-$ liberal semigroup $S(U)$ is called an orthodox $U-$ liberal semigroup if U is a subsemigroup of $S(U)$ and the ET-condition holds on $S(U)$.

In general, unlike the usual Green's relations on a semigroup S, $\tilde{\mathcal{L}}^U$ is not necessarily a right congruence on S and $\tilde{\mathcal{R}}^U$ is not necessarily a left congruence on S (see [8]).

We say that a semigroup $S(U)$ satisfies the (CR) condition if $\tilde{\mathcal{L}}^U$ is a right congruence on S and that $S(U)$ satisfies the (CL) condition if $\tilde{\mathcal{R}}^U$ is a left congruence on S. If the semigroup $S(U)$ satisfies both the (CR) and (CL) condition, then we say $S(U)$ satisfies the (C) condition.

The studies on the structures of semigroups play an important role in the research of the algebraic theories of semigroups. From [7], it is known that a $U-$semi-superabundant semigroup $S(U)$ is an orthodox $U-$liberal semigroup for some $U \subseteq E(S)$ if and only if it is a semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha)]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha \in Y} U_\alpha$ is a subsemigroup of S. Meanwhile, notice that a normal band is a strong semilattice of some rectangular bands. Naturally, we will quote such a question: whether will a normal orthodox U-liberal semigroup $S(U)$ be a strong semilattice of some rectangular monoids?

In this paper, we will consider the question quoted above. Consequently, we show that a semigroup $S(U)$ is a normal orthodox Smarandache $U-$liberal semigroup if and only if it is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a normal band of $S(U)$. Consequently, some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups are generalized and extended.

For notations and terminologies not mentioned in this paper, the reader is referred to [7],[9],[10].

§2. Normal Orthodox $U-$liberal Semigroups

In this section, we will give a construction of normal orthodox $U-$liberal semigroups.

Firstly, we recall the following lemmas.

Lemma 2.1. [7] Let $\mathcal{F}$ be one of Green's relations $\mathcal{L}, \mathcal{R}$ or $\mathcal{H}$ and $\tilde{\mathcal{F}}^U$ the corresponding Green $\sim -$ relations on the semigroup S. Then, for any $a, b \in S$, we have

(i) $\mathcal{F} \subseteq \tilde{\mathcal{F}}^U$and for $a, b \in Reg_U(S)$, $a, b \in \tilde{\mathcal{F}}^U$ if and only if $a, b \in \mathcal{F}$, where $Reg_U(S) = \{a \in S|(\exists e, f \in U)e\mathcal{L}a\mathcal{R}f\}$;

(ii)$\tilde{\mathcal{H}}^U \subseteq \tilde{\mathcal{Q}}^U$ and $\tilde{Q}^U_a$ contains at most one element in U;

(iii) If $S(U)$ is a $U-$ semi-superabundant semigroup, then $S(U)$ is a Smarandache $U-$ liberal semigroup with $\tilde{\mathcal{Q}}^U = \tilde{\mathcal{H}}^U$.

Lemma 2.2. [7] The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a Smarandache $U-$ liberal semigroup for some $U \subseteq E(S)$ and U itself is a rectangular band;

(ii) $S(U)$ is an orthodox $U-$ liberal semigroup such that U is a rectangular band;

(iii) S is isomorphic to a rectangular monoid.

Lemma 2.3. [7] The following statements are equivalent for a semigroup S:

(i) $S(U)$ is an orthodox U-liberal semigroup for some $U \subseteq E(S)$;

(ii) $S = [Y; S_\alpha(U_\alpha)]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a subsemigroup of S;

(iii)$S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U \subseteq E(S)$ and U is a subsemigroup of S.

Now, we will give our main theorem.

Theorem 2.4. The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a normal orthodox U-liberal semigroup for some normal band $U \subseteq E(S)$;

(ii) $S(U)$ is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a normal band of S;

(iii)$S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U \subseteq E(S)$ and U is a normal band of S.

Proof. $(i) \Rightarrow (ii)$

Assume that $S(U)$ is a normal orthodaox $U-$liberal semigroup for some normal band $U \subseteq E(S)$. Note that $U \subseteq E(S)$ is a normal band, we will have $U = [Y; U_\alpha, f_{\alpha,\beta}]$ or $U = [Y; I_\alpha \times \Lambda_\alpha; \varphi_{\alpha,\beta}, \psi_{\alpha,\beta}]$, where $(i,j)f_{\alpha,\beta} = (i\varphi_{\alpha,\beta}, j\psi_{\alpha,\beta}), (i,j) \in I_\alpha \times \Lambda_\alpha$.

For any $\alpha \in Y$, we form the set $S_\alpha = \{x \in S | x^0_U \in U_\alpha\}$. Since $S(U)$ satisfies the ET-condition, for all $x \in S_\alpha$, $y \in S_\beta$, we have $(xy)^0_U \mathcal{D}(U) x^0_U y^0_U$. This leads to $xy \in S_{\alpha\beta}$ and hence $S(U) = [Y; S_\alpha(U_\alpha)]$.

Notice that every semigroup $S_\alpha(U_\alpha)$ is a Smarandache $U_\alpha-$ liberal semigroup and U_α is a rectangular band. By Lemma 2.2, $S_\alpha(U_\alpha)$ is isomorphic to a rectangular monoid. For convenience, we denote $S_\alpha(U_\alpha) = I_\alpha \times T_{U_\alpha} \times \Lambda_\alpha$.

Now, define a mapping,

$$\Phi_{\alpha,\beta} : S_\alpha(U_\alpha) \to S_\beta(U_\beta),$$

$$(i_\alpha, u_\alpha, \lambda_\alpha) \to (i_\alpha\varphi_{\alpha,\beta}, 1_{U_\beta}, \lambda_\alpha\psi_{\alpha,\beta})(i_\alpha, u_\alpha, \lambda_\alpha) = (i_\alpha\varphi_{\alpha,\beta} i_\alpha, 1_{U_\beta} u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha).$$

In the following, we will prove that S is a strong semilattice of $S_\alpha(U_\alpha)$, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$.

Firstly, $\Phi_{\alpha,\beta}$ is a homomorphism.

For any $x = (i_\alpha, u_\alpha, \lambda_\alpha)$, $y = (j_\alpha, v_\alpha, \mu_\alpha) \in S_\alpha (\forall \alpha \geq \beta)$,

$$\begin{aligned}
(xy)\Phi_{\alpha,\beta} &= [(i_\alpha, u_\alpha, \lambda_\alpha)(j_\alpha, v_\alpha, \mu_\alpha)]\Phi_{\alpha,\beta} \\
&= (i_\alpha j_\alpha, u_\alpha v_\alpha, \lambda_\alpha \mu_\alpha)\Phi_{\alpha,\beta} \\
&= (i_\alpha, u_\alpha v_\alpha, \mu_\alpha)\Phi_{\alpha,\beta} \\
&= (i_\alpha\varphi_{\alpha,\beta} i_\alpha, 1_{U_\beta} u_\alpha v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha),
\end{aligned}$$

$$\begin{aligned}
x\Phi_{\alpha,\beta} y\Phi_{\alpha,\beta} &= (i_\alpha, u_\alpha, \lambda_\alpha)\Phi_{\alpha,\beta}(j_\alpha, v_\alpha, \mu_\alpha)\Phi_{\alpha,\beta} \\
&= (i_\alpha\varphi_{\alpha,\beta} i_\alpha, 1_{U_\beta} u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)(j_\alpha\varphi_{\alpha,\beta} j_\alpha, 1_{U_\beta} v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \\
&= (i_\alpha\varphi_{\alpha,\beta} i_\alpha j_\alpha\varphi_{\alpha,\beta} j_\alpha, 1_{U_\beta} u_\alpha 1_{U_\beta} v_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \\
&= (i_\alpha\varphi_{\alpha,\beta} i_\alpha, 1_{U_\beta} u_\alpha v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \quad (\text{ since } U_\alpha \text{ is normal}).
\end{aligned}$$

Thus, $(xy)\Phi_{\alpha,\beta} = x\Phi_{\alpha,\beta} y\Phi_{\alpha,\beta}$.

Secondly, $\Phi_{\alpha,\alpha}$ is an identity mapping.

For any $x = (i_\alpha, u_\alpha, \lambda_\alpha) \in S_\alpha$, $x\Phi_{\alpha,\alpha} = (i_\alpha\varphi_{\alpha,\alpha} i_\alpha, 1_{U_\alpha} u_\alpha, \lambda_\alpha\psi_{\alpha,\alpha}\lambda_\alpha) = (i_\alpha, u_\alpha, \lambda_\alpha) = x$. Hence, $\Phi_{\alpha,\alpha}$ is an identity mapping.

Thirdly, notice that for any $x = (i_\alpha, 1_{U_\alpha}, \lambda_\alpha) \in E(S_\alpha) = I_\alpha \times 1_{U_\alpha} \times \Lambda_\alpha$, $y = (i_\beta, 1_{U_\beta}, \lambda_\beta) \in E(S_\beta) = I_\beta \times 1_{U_\beta} \times \Lambda_\beta$, $xy \in E(S_{\alpha\beta}) = I_{\alpha\beta} \times 1_{U_{\alpha\beta}} \times \Lambda_{\alpha\beta}$, we can get $1_{U_\alpha} 1_{U_\beta} = 1_{U_{\alpha\beta}}$. Especially, when $\alpha \geq \beta$, we have $1_{U_\alpha} 1_{U_\beta} = 1_{U_{\alpha\beta}} = 1_{U_\beta}$. Now, for any $\alpha, \beta, \gamma \in Y (\alpha \geq \beta \geq \gamma)$ and any $x = (i_\alpha, u_\alpha, \lambda_\alpha) \in S_\alpha(U_\alpha)$, we will have

$$(x\Phi_{\alpha,\beta})\Phi_{\beta,\gamma} = (i_\alpha, u_\alpha, \lambda_\alpha)\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = (i_\alpha\varphi_{\alpha,\beta} i_\alpha, 1_{U_\beta} u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\Phi_{\beta,\gamma}$$

$$\begin{aligned}
&= ((i_\alpha\varphi_{\alpha,\beta} i_\alpha)\varphi_{\alpha,\beta}(i_\alpha\varphi_{\alpha,\beta} i_\alpha), 1_{U_\gamma}(1_{U_\beta} u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\
&= (((i_\alpha\varphi_{\alpha,\beta})(i_\alpha\varphi_{\alpha,\beta} i_\alpha))\varphi_{\alpha,\beta}(i_\alpha\varphi_{\alpha,\beta} i_\alpha), 1_{U_\gamma}(1_{U_\beta} u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\
&= (i_\alpha\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta} i_\alpha)\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta} i_\alpha), 1_{U_\gamma}(1_{U_\beta} u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\
&= (i_\alpha\varphi_{\alpha,\gamma}(i_\alpha\varphi_{\alpha,\beta} i_\alpha)\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta}) i_\alpha, 1_{U_\gamma}(1_{U_\beta} u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\
&= (i_\alpha\varphi_{\alpha,\gamma} i_\alpha, 1_{U_\gamma} u_\alpha, \lambda_\alpha\psi_{\alpha,\gamma}\lambda_\alpha) \quad (\text{since } I_\alpha \text{ is a left zero band}) \\
&= x\Phi_{\alpha,\gamma}.
\end{aligned}$$

Finally, for any $\alpha, \beta \in Y$, $a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$, since $a_\alpha b_\beta \in S_{\alpha\beta} \triangleq S_\gamma$, we have

$$\begin{aligned} a_\alpha b_\beta &= (i_\alpha, u_\alpha, \lambda_\alpha)(i_\beta, u_\beta, \lambda_\beta) \\ &= (i_\alpha i_\beta, u_\alpha u_\beta, \lambda_\alpha \lambda_\beta) \\ &= (i_\alpha \varphi_{\alpha,\gamma} i_\alpha i_\beta \varphi_{\beta,\gamma} i_\beta, 1_{U_\gamma} u_\alpha 1_{U_\gamma} u_\beta, \lambda_\alpha \psi_{\alpha,\gamma} \lambda_\alpha \lambda_\beta \psi_{\beta,\gamma} \lambda_\beta) \\ &= (i_\alpha \varphi_{\alpha,\gamma} i_\alpha, 1_{U_\gamma} u_\alpha, \lambda_\alpha \psi_{\alpha,\gamma} \lambda_\alpha)(i_\beta \varphi_{\beta,\gamma} i_\beta, 1_{U_\gamma} u_\beta, \lambda_\beta \psi_{\beta,\gamma} \lambda_\beta) \\ &= (a_\alpha \Phi_{\alpha,\gamma})(b_\beta \Phi_{\beta,\gamma}). \end{aligned}$$

Thus, summing up the above discussions, $S(U)$ is isomorphic to a strong semilattice of rectangular monoids $S_\alpha(U_\alpha)$, that is, $S(U) = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$.

$(ii) \Rightarrow (iii)$ The proof is similar with the corresponding $(ii) \Rightarrow (iii)$ of Lemma 2.3.

For any $\alpha \in Y$, assume that $S_\alpha(U_\alpha) = I_\alpha \times T_\alpha \times \Lambda_\alpha$. Then, it is not hard to see that for any $(i, x, \lambda) \in S_\alpha$,$(j, y, \mu) \in S_\beta$, $(i, x, \lambda)\tilde{\mathcal{L}}^U(j, y, \mu)$ if and only if $\alpha = \beta$ and $\lambda = \mu \in \Lambda_\alpha$. On the other hand, if $(i, x, \lambda)\tilde{\mathcal{L}}^U(j, y, \lambda)$ for some $\lambda \in \Lambda_\alpha$, then for all $(k, z, \nu) \in S_\gamma(\nu \in Y)$, we have

$$(i, 1_{T_\alpha}, \lambda)(k, z, \nu) = (k', z', \nu')(\in S_{\alpha\gamma}),$$

$$(i, x, \lambda)(k', 1_{T_{\alpha\gamma}}, \nu') = (i', x', \lambda'),$$

$$(j, y, \lambda)(k', 1_{T_{\alpha\gamma}}, \nu') = (j', y', \lambda''),$$

Consequently, by using the above relations, we derive that
$(i, x, \lambda)(k, z, \nu) = (i, x, \lambda)(i, 1_{T_\alpha}, \lambda)(k, z, \nu) = (i, x, \lambda)(k', z', \nu') = (i, x, \lambda)(k', 1_{T_{\alpha\gamma}}, \nu')(k', z', \nu)'$
$= (i', x', \lambda')(k', z', \nu')= (i', x'z', \lambda')$;
$(j, y, \lambda)(k, z, \nu) = (j, y, \lambda)(i, 1_{T_\alpha}, \lambda)(k, z, \nu) = (j, y, \lambda)(k', z', \nu') = (j, y, \lambda)(k', 1_{T_{\alpha\gamma}}, \nu')(k', z', \nu)'$
$= (j', y', \lambda'')(k', z', \nu')= (j', y'z', \nu')$.
Thereby, we obtain that $(i, x, \lambda)(k, z, \nu)\tilde{\mathcal{L}}^U(j, y, \lambda)(k, z, \nu)$ so that $\tilde{\mathcal{L}}^U$ is a right congruence on S.

Similarly, we can show that $(i, x, \lambda)\tilde{\mathcal{R}}^U(j, y, \lambda)$ if and only if $i = j \in I_\alpha$ for some $\alpha \in Y$,and so $\tilde{\mathcal{R}}^U$ is a left congruence on S.

Hence, together with Lemma 2.3, $S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U \subseteq E(S)$. Note that U is a normal band of S, (iii) holds.

$(iii) \Rightarrow (i)$ The proof is similar with the corresponding $(iii) \Rightarrow (i)$ of Lemma 2.3.

Assume that (iii)holds. Then, by Lemma 2.1, $S(U)$ is Smarandache $U-$ liberal semigroup and for all $a \in S(U)$, $a_U^\circ = a_U^\diamond$. Since $S(U)$ satisfies the (C) condition, we have, for all $a, b \in S(U)$,

$$(ab)_U^\diamond \tilde{\mathcal{R}}^U ab \tilde{\mathcal{R}}^U ab_U^\diamond \tilde{\mathcal{R}}^U (ab_U^\diamond)_U^\diamond \tilde{\mathcal{L}}^U ab_U^\diamond \tilde{\mathcal{L}}^U a_U^\diamond b_U^\diamond.$$

This leads to $(ab)_U^\diamond \tilde{\mathcal{R}}^U (ab_U^\diamond)_U^\diamond \tilde{\mathcal{L}}^U a_U^\diamond b_U^\diamond$. By Lemma 2.1 (i), we will get $(ab)_U^\diamond \mathcal{R} (ab_U^\diamond)_U^\diamond \mathcal{L} a_U^\diamond b_U^\diamond$. Consequently, $(ab)_U^\circ = (ab)_U^\diamond \mathcal{D} a_U^\diamond b_U^\diamond = a_U^\circ b_U^\circ$ holds. This shows that $S(U)$ satisfies the ET-condition. Note that U is a normal band of S, (i) holds.

Now, if we let $U = E(S)$ in Theorem 2.4, then we immediately have the following corollary.

Corollary 2.5. The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a normal orthodox $E(S)$-liberal semigroup;

(ii) $S(U)$ is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(E(S_\alpha)); \Phi_{\alpha,\beta}]$, where $S_\alpha(E(S_\alpha))$ is a rectangular monoid for every $\alpha \in Y$ and $E(S)$ is a normal band of S.

(iii)$S(U)$ is a semi-superabundant semigroup satisfying the (C) condition, and $E(S)$ is a normal band of S.

In the above corollary, if we restrict the semigroup S to the abundant or regular semigroups, then it is not hard for us to get

Corollary 2.6. The following statements are equivalent for a semigroup S:

(i) S is a normal orthocrypto semigroup;

(ii)S is a strong semilattice of rectangular cancellative monoids, i.e., $S = [Y; S_\alpha; \Phi_{\alpha,\beta}]$, where $S_\alpha = I_\alpha \times T_\alpha \times \Lambda_\alpha$, and I_α is a left zero band, Λ_α is a right zero band, T_α is a cancellative monoid for every $\alpha \in Y$.

Corollary 2.7. The following statements are equivalent for a semigroup S:

(i) S is a normal orthocryptogroup;

(ii)S is a strong semilattice of rectangular groups.

Hence, our main result generalizes and extends some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups.

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How to sperate traffic networks for the fraction regulation[1]

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Abstract In a traffic network with fixed demands, to obtain the system optimum flow pattern, the flow need to be regulated, but usually the regulation cost is expensive and it is difficult to implement an optimal assignment of routes. The efficient optimal regulation has to be connected with all characteristics of the studied networks, including the link latency functions and network configurations. For the simple traffic network with cut-edges, a strategy to sperate the traffic network for the fraction regulation is given, and the complex degree for the proposed strategy is also analyzed. For the simple traffic network with no cut-edge, the definition of the priority regulation coefficient is given, then based on this definition, a strategy to sperate the traffic network for the fraction regulation is given, finally, the regulation effectiveness ratio for the proposed strategy is analyzed. For the traffic network with loops and no cut-edge, based on the characters of link latency functions in the loops, a strategy to sperate the traffic network for the fraction regulation is given, and the regulation effectiveness ratio for the proposed strategy is also analyzed, we call this strategy Smarandache strategy.

Keywords system optimum, optimal regulation, traffic network.

§1. Introduction

A fundamental problem arising in the management of large-scale traffic and communication networks is that of routing traffic to optimize network performance. One problem of this type is the following: given the rate of traffic between each pair of nodes in a network, find an assignment of traffic to paths so that the sum of all travel times (the total latency) is minimized(called as system optimum). However, in the absence of network regulation, uses act in a purely selfish manner. In this case, network uses are free to act according to their own interests, without regard to overall network performance, and the routes chosen by uses form a Nash equilibrium (called as user equilibrium). To discuss optimizing network performance, the two fundamental questions are deserved to be considered. The first question is how much does

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network performance suffer from this lack of regulation in the absence of network regulation? The second question is how to regulate the network flow to optimize network performance?

For the first question, there have been many results in recent years (see for example [2]-[10]). For the second question, it is difficult to impose optimal or near-optimal routing strategies on the traffic in a network. The efficient optimal regulation has to be connected with all characteristics of the studied networks, including the link latency functions and network configurations. Roughgarden[1] shows that if a fraction of the network traffic is centrally controlled and is carefully routed by a network manager, then the corresponding induced equilibrium is less inefficient then a flow at Nash equilibrium. However, only the restricted setting of networks of parallel links is considered in Roughgarden[1]. Recently, the feasibility and efficiency of fraction optimal regulation according to networks configuration are analyzed by Xu and Shi[7].

Motivated by the recent results in this field, in this paper, we discuss the stratedge to sperate traffic networks for the fraction regulation. For the simple traffic network with cut-edges, a strategy to sperate the traffic network for the fraction regulation is given, and the complex degree for the proposed strategy is also analyzed. For the simple traffic network with no cut-edge, the definition of the priority regulation coefficient is given, then based on this definition, a strategy to sperate the traffic network for the fraction regulation is given, finally, the regulation effectiveness ratio for the proposed strategy is analyzed. For the traffic network with loops and no cut-edge, based on the character of link latency functions in the loops, a strategy to sperate the traffic network for the fraction regulation is given, and the regulation effectiveness ratio for the proposed strategy is also analyzed, we call this strategy Smarandache strategy.

§2. Preliminaries

We consider a traffic network $G = (N, A)$ with vertex set N, edge set A, and source-destination vertex pairs $\{r, s\}$. We denote the $r \to s$ path by k, the demand amount from r to s by d_{rs}, the flow amount in the edge a by ν_a, the travel time (or latency) in the edge a by $\tau_a(\nu_a)$, the flow amount between (r, s) by f_k^{rs}. We denote the vector $d = (..., d_{rs}, ...)^T$, the vector $\nu = (..., \nu_a, ...)^T$. Let $\delta_{ak}^{rs} = 1$, if a in the path k between (r, s), otherwise, $\delta_{ak}^{rs} = 0$.

The system optimum feasible flow in a network is a special case of the following non-linear program (denoted by SO, see [10])

$$\min_{\nu\in\Omega} C(\nu) = \sum_{a\in A} \nu_a t_a(\nu_a)$$

$$\sum_k f_k^{rs} = d_{rs} \qquad (\forall r, s)$$

$$f_k^{rs} \geq 0 \qquad (\forall k, r, s)$$

$$\nu_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{ak}^{rs} \qquad (\forall a) \tag{2.1}$$

where Ω is the feasible set.

Based on the definition of cut-edge in graph theory (see [6]), we give the following definition:

Definition 2.1. For a connected traffic network $G = (N, A)$, if the edge is deleted which leads the number of connection parts to increase, then we called this edge as a cut-edge.

§3. The simple traffic network with cut-edges

Lemma 3.1 [6]. In a simple network with no loop, $a \in A$ is a cut-edge if and only if a does'nt belong to any simple round path in the network.

Strategy 3.1.

Step 1 Set the adjacent matrix M for the traffic network G. Let $j = 1, k = 1$.

Step 2 i=1.

Step 3 If $m(i,j) = 1$, then $m(i,j) = 0$; $m(j,i) = 0$; $r(k) = j; j = i$;
for $m = 1, 2, \cdots, k-1$, if $r(k) = r(m)$ and $m > 1$, then $M(r(2), r(1)) = 1$; $M(r(1), r(2))) = 1; \cdots;$ $M(r(m), r(m-1)) = 1$; $M(r(m-1), r(m)) = 1$; $j = r(m)$; $i = j+1$; $k = 1$; go to Step 3, otherwise, $k = k+1$; go to Step 2.

Step 4 If $i < n$, then $i = i+1$; go to step 3,
Otherwise, for $\bar{i} = 1, 2, \cdots, m$, if $j = r(\bar{i})$ and $\bar{i} > 1$,
then $i = j+1$; $j = r(\bar{i}-1)$; $k = 1$; go to step 3,
else $j = j+1$; $k = 1$; go to Step 2.

Step 5 If $M = 0$, then G is a simple traffic network with no cut-edge. Otherwise, any $m(i,j) = 0$ means the edge from the vertex a_i to the vertex a_j is a cut-edge of G.

Step 6 By the cut-edge set $A = \{a_1, a_2, \cdots, a_m\}$, sperate the traffic network, i. e.,

$$G = G_1 \bigcup G_2 \bigcup \cdots \bigcup G_{m+1} \bigcup \{a_1\} \bigcup \{a_2\} \bigcup \cdots \bigcup \{a_m\},$$

where $G_i \bigcap G_j = \emptyset, \quad i \neq j$.

Remark 3.1. Based on the results in Xu and Shi[7], we can easily know the efficiency and feasibility of Strategy 3.1.

Theorem 3.1. Given the traffic network G, the complex degree for the strategy 3.1 is at most $O(n^3)$, where n is the number of vertexes in N.

Proof. From the strategy 3.1, we know the following three points.

1. Search a possible round path for any vertex $x \to x$, this procedure at most repeat n times;
2. At most search n vertexes to look for any round path;
3. At most search n vertexes to look for the next vertex in the round path.

From the above-mentioned three points, we can conclude that the complex degree for the strategy 3.1 is $O(n^3)$.

Example 3.1. In the traffic network (see fig. 3.1), how to sperate it for the fraction regulation?

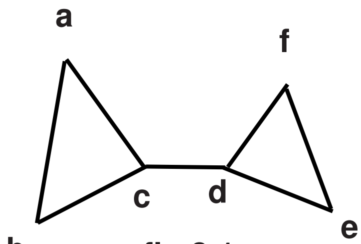

fig 3.1

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the last adjacent matrix , we know the cut-edge set is $\{\overline{cd}\}$. Finally, according the step 5 in strategy 3.1, by $\overline{cd}$, we can sperate the traffic network(see fig 4.1) to two parts for the fraction regulation.

Strategy 3.2.

(1)For any vertex x in the network G, set $d(x) = 0$, $Q = \emptyset, \bar{T} = \emptyset, C = \emptyset$.
(2)Choose any vertex v, add v to the rear of Q.
(3)If $Q = \emptyset$,then quit.
(4)Take the front vertex x in Q, $Q = Q - x$.
(5)If $L(x) = \emptyset$, then go to (3).
(6)Take the vertex y in $L(x)$, $L(x) = L(x) - y$.
(7)If $d(y) = 0$, then begin
add (x, y) to T;$f(y) = x$;$d(y) = d(x) + 1$; add y to the rear of Q; go to (5);
end
(8)If $d(x) < d(y)$, then
begin
add (x, y) to $\bar{T}$; $\bar{f}(y) = x$; go to (5);
end
(9) Add (x, y) to C, go to (5).
(10) Set $i = 0$.

(11) Set $TN[i] = \emptyset, CV = \emptyset$.
(12) For $k = 1, 2, \cdots, m$, if $d(v_k) = i$, then $TN[i] = TN[i] + v_k$. If $i < n$, then $i = i + 1$, go to (11).
(13) For $k = 1, 2, \cdots, n$, if $TN[k]$ has only a vertex x and there exists a unique vertex y, such that $f(y) = x$, then (x, y) is a cut-edge, and $CV = CV + x$, else if $TN[k]$ have more than a vertex and for any two adjacent vertexes $x, y \in TN[k]$, there exists a unique edge $(x, y) \in \bar{T}$, then for $x \in TN[k]$, if there exists a vertex $y \in TN[k+1]$, such that $(x, y) \in T$, then $CV = CV + x$, else if for $x \in TN[k], y \in TN[k+1]$, there exists a path which is connected by the edges in C and T, then for any $x \in TN[k]$, if there exists a vertex $y \in TN[k+1]$, such that $(x, y) \in T$, then $CV = CV + x$.
(14) For any $x, y \in CV$, if there exists a unique edge (x, y), which connects the vertexes x and y, then (x, y) is a cut-edge.
(15)By the cut-edge set $A = \{a_1, a_2, \cdots, a_m\}$, sperate the traffic network, i. e.,

$$G = G_1 \bigcup G_2 \bigcup \cdots \bigcup G_{m+1} \bigcup \{a_1\} \bigcup \{a_2\} \bigcup \cdots \bigcup \{a_m\},$$

where $G_i \bigcap G_j = \emptyset, \quad i \neq j$.

Lemma 3.1. [8] In a simple network with no loop, if $TN[i] = \{v\}$, then the vertex v is a cut-vertex in the network.

Remark 3.2. From the Lemma 3.1, we can easily know the efficiency and feasibility of Strategy 3.2.

§4. The simple traffic network with no cut-edge

Definition 4.1. $s_a = \frac{\tau_a(\widetilde{v}_a)\widetilde{v}_a}{c(\widetilde{V})}$ is called as the priority regulation coefficient, where $\widetilde{V} = \{\widetilde{v}_a | a \in A\}$ is the system optimal flow distribution.

Obviously, if the priority regulation coefficient of a is larger than other edges, then the regulation on a should be done first. Let c_t be the admissible total regulation cost, $c(a_i)$ be the regulation cost for the edge a_i, RA be the regulated edge set.

Strategy 4.1.
Step 1 Set $A = \{a_1, a_2, \cdots, a_n\}$. Compute the system optimum flow distribution $\widetilde{V}$, and the the priority regulation coefficients for all edges. Rearrange the elements in A such that $s_{a_1} \geq s_{a_2} \geq s_{a_n}$.
Step 2 Set $i = 1$, $c = 0$, $RA = \emptyset$.
Step 3 Set $c = c + c(a_i)$. If $c < c_t$, then take corresponding measures to regulate edge a_i, and $RA = RA + \{a_i\}$.
Step 4 If $c < c_t$ and $i < n$, then $i = i + 1$, go to step 3. Otherwise, output RA.

Theorem 4.1. Supposed that all the link latency functions are linear, for the strategy 4.1, the regulation effectiveness ratio

$$k \le \frac{4-(s_1+s_2+\cdots+s_m)}{3}, \tag{4.1}$$

where m is the number of regulated edges.

Proof.

$$\begin{aligned} k &= \frac{\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\overline{v}_a)\overline{v}_a}{\sum_{a\in A}\tau_a(\widetilde{v}_a)\widetilde{v}_a} \\ &= \frac{\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\overline{v}_a)\overline{v}_a}{\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a}, \end{aligned} \tag{4.2}$$

where $\overline{V},\widetilde{V}$ is the flow distribution of user equilibrium and system optimum, respectively. From Theorem 4.5 in Roughgarden [1], we have

$$\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\overline{v}_a)\overline{v}_a \le \frac{4}{3}\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a \tag{4.3}$$

From (4.2) and (4.3), we have

$$k \le \frac{\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\frac{4}{3}\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a}{\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a} \tag{4.4}$$

From the definition 3.1, we have

$$\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i} = (s_1+s_2+\cdots+s_m)(\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i}+\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a) \tag{4.5}$$

From (4.5), we have

$$\sum_{i=1}^{m}\tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i} = \frac{(s_1+s_2+\cdots+s_m)\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}}\tau_a(\widetilde{v}_a)\widetilde{v}_a}{1-(s_1+s_2+\cdots+s_m)} \tag{4.6}$$

From (4.4) and (4.6), we have

$$k \le \frac{4-(s_1+s_2+\cdots+s_m)}{3}.$$

The proof is completed.

If some link latency functions are nonlinear, then Theorem 4.1 can not be used, we can use the following theorem.

Theorem 4.2. Supposed the constant $\alpha \ge 1$ satisfy

$$x\tau_a(x) \le \alpha\int_0^x \tau_a(t)dt$$

for all edges a and all positive real number x. Then for the strategy 4.1, the regulation effectiveness ratio

$$k \le \alpha + (1-\alpha)(s_1+s_2+\cdots+s_m), \tag{4.7}$$

where m is the number of regulated edges.
Proof. From Corollary 2.7 in the Roughgarden [1], we have

$$\sum_{a\in A-\{a_1,a_2,\cdots,a_m\}} \tau_a(\overline{v}_a)\overline{v}_a \leq \alpha \sum_{a\in A-\{a_1,a_2,\cdots,a_m\}} \tau_a(\widetilde{v}_a)\widetilde{v}_a \tag{4.8}$$

From (4.2) and (4.8), we have

$$k \leq \frac{\sum_{i=1}^{m} \tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i} + \alpha \sum_{a\in A-\{a_1,a_2,\cdots,a_m\}} \tau_a(\widetilde{v}_a)\widetilde{v}_a}{\sum_{i=1}^{m} \tau_{a_i}(\widetilde{v}_{a_i})\widetilde{v}_{a_i} + \sum_{a\in A-\{a_1,a_2,\cdots,a_m\}} \tau_a(\widetilde{v}_a)\widetilde{v}_a} \tag{4.9}$$

From (4.6) and (4.9), we have

$$k \leq \alpha + (1-\alpha)(s_1 + s_2 + \cdots + s_m).$$

The proof is completed.

Example 4.1. In the traffic network(see fig. 4.1), supposed $\tau_{ab}(v) = v$, $\tau_{ac}(v) =$

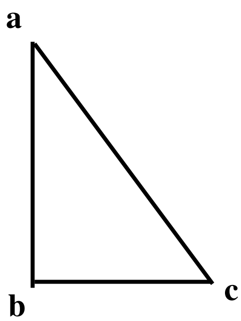

fig 4.1

$2v$, $\tau_{bc}(v) = 4v, d_{ac} = 7, v_{ac} = 4$ and $c_t = 18$.
From Lemma 2.4 in Roughgarden [1], we know the system optimum flow distribution $\widetilde{V}$ satisfy

$$\begin{cases} \widetilde{v}_{ab} + 4\widetilde{v}_{bc} = 2\widetilde{v}_{ac} \\ \widetilde{v}_{ab} = \widetilde{v}_{bc} \\ \widetilde{v}_{ab} + \widetilde{v}_{ac} = 1 \end{cases}$$

Thus, we obtain

$$\widetilde{v}_{ab} = 2, \quad \widetilde{v}_{ac} = 5, \quad \widetilde{v}_{bc} = 2.$$

Then,

$$s_{ab} = 2/35, \quad s_{ac} = 5/7, \quad s_{bc} = 8/35.$$

Since $50 - 32 = 18$, only the edge $\overline{ac}$ should be regulated.

§5. The traffic network with loops and no cut-edge

Let $L = \{L_1, L_2, \cdots, L_m\}$ be the loop sets. In this section, we assume

$$\tau_a(v_a) = \begin{cases} n_a + m_a v_a, & a \in A - (L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m), \\ \tau_a, & a \in L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m. \end{cases}$$

Obviously, for small and middle-scale cities, loop roads satisfy this assumption.

Lemma 5.1. If $\widetilde{V}$ is the system optimal flow, then for any $i \in \{1, 2, \cdots, m\}$, any $r, s \in L_i$, and any path k from vertex s to vertex t, we have

$$\sum_{a \in k} (\tau_a(\widetilde{v}_a)\widetilde{v}_a)' = \tau_{rs}. \tag{5.1}$$

where τ_{rs} is the total latency from r to s in loop L_i.

Proof. From Lemma 2.4 in Roughgarden[1], we have

$$\sum_{a \in k} (\tau_a(\widetilde{v}_a)\widetilde{v}_a)' = \sum_{a \in k'} (\tau_a(\widetilde{v}_a)\widetilde{v}_a)', \tag{5.2}$$

where k' is a path in the L_i from r to s.
Considering the links latency functions in loops are all constant numbers, which is only related to the length of the edge, we have

$$\sum_{a \in k'} (\tau_a(\widetilde{v}_a)\widetilde{v}_a)' = \tau_{rs}. \tag{5.3}$$

From (5.2) and (5.3), we have

$$\sum_{a \in k} (\tau_a(\widetilde{v}_a)\widetilde{v}_a)' = \tau_{rs}.$$

The proof is completed.

Strategy 5.1.
Step 1. Solve the linear equation system

$$B^T B v = B^T b, \tag{5.4}$$

where

$$b = [\cdots, \tau_{rs} - \sum_{a \in k} n_a, \cdots]^T,$$

$$B = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & 2m_a & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix},$$

$$v = [\cdots, v_a, \cdots]^T.$$

Step 2. According to the solution of (5.4), regulate the flow of the edge in the region between two adjacent loops.

Remark 5.1. Since $\tau_a(v_a) = n_a + m_a v_a$, thus $(v_a \tau_a(v_a))' = \tau_a(v_a) + v_a \tau_a'(v_a) = n_a + 2m_a v_a$. From Lemma 5.1, we know that in order to obtain the system optimal flow, it is necessary to solve the linear equation system $Bv = b$. However, it is difficult to know whether B is singular or not. But $B^T B$ is obviously singular, so there exists a solution V^* for the linear equation system $B^T Bv = B^T b$, which is called the linear least square solution for the $Bv = b$.

Theorem 5.1. For the Strategy 4.1, the regulation effectiveness ratio

$$k = \frac{\sum_{a \in A-(L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m)} v_a^* \tau_a(v_a^*) + \sum_{i=1}^{m} \sum_{a \in L_i} \tau_a v_a}{c(\widetilde{V})}$$

where $\widetilde{V}$ is the system optimum flow, v_a is the flow amount in edge a of loop L_i.

Proof. From the strategy 5.1 and the link latency functions character in the loops, we can prove this theorem immediately.

§6. Conclusion

In this paper, some strategy to sperate the traffic network for fraction regulation to obtain the network system optimum are discussed, which can be used to do the fraction optimal regulation of traffic networks. Furthermore, the effectiveness of the proposed strategies are also analyzed for the linear latency functions. However, for more complicated latency functions, the effectiveness of the proposed strategies and some new strategies are to be further studied.

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The generalized reflexive solutions of matrix equation $AXB + CYD = F$ and its approximation problem[1]

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Abstract Given generalized reflection matrices R, S, i.e., $R^* = R$, $R^2 = I$, $S^* = S$, $S^2 = I$, a complex matrix A is said to be generalized reflexive (or anti-reflexive), if $RAS = A$ (or $RAS = -A$). In this paper, Smarandache iterative algorithm is proposed to solve matrix equation $AXB + CYD = F$ with generalized reflexive matrix dual (X,Y). For any initial iterative matrix pair (X_1, Y_1), we show that a solution of this equation can be obtained within finite iteration steps in the absence of roundoff errors. Some numerical examples illustrate the feasibility and efficiency of this algorithm.

Keywords Matrix equation, generalized reflexive matrices, Smarandache iterative algorithm, least-norm solution, optimal approximation.

§1. Introduction

Let $C_k^{m\times n}$ be the set of all $m \times n$ complex matrices with rank k, and $UC^{n\times n}$ the set of all unitary matrices in $C^{n\times n}$. A^*, $\mathscr{R}(A)$ and $tr(A)$ denote the conjugate transpose, column space and trace of A, respectively. For matrices $A = (a_1, a_2, \ldots, a_n)$, where $a_i \in C^m$, and $B \in C^{m\times n}$, let $vec(A) = (a_1^*, a_2^*, \ldots, a_n^*)^*$, and $A \otimes B$ be the Kronecker product of A and B. Moreover, define $\langle A, B\rangle = tr(B^*A)$ as the inner product of matrices A and B, which generates the Frobenius norm, i.e., $\|A\| = \sqrt{< A, A >} = \sqrt{tr(A^*A)}$.

Definition 1.1. Let $R \in C^{m\times m}$, $S \in C^{n\times n}$ be generalized reflection matrices, i.e., $R^* = R, R^2 = I$, $S^* = S, S^2 = I$, then a matrix $A \in C^{m\times n}$ is said to be generalized reflexive (or anti-reflexive) matrix, if $RAS = A$ (or $RAS = -A$).

The set of all $m \times n$ generalized reflexive (or anti-reflexive) matrices with respect to matrix dual (R, S) is denoted by $GRC^{m\times n}$ $(GARC^{m\times n})$.

Remark 1.1. (a) In this paper, let R, S be generalized reflection matrices as in Definition

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1.1. (b) For $E \in GRC^{m\times n}$, $F \in GARC^{m\times n}$, then $tr(F^*E) = 0$, we say matrices E and F are orthogonal each other.

The following lemma derived from [1] (also [2]), indicates the special structure properties of generalized reflection matrices.

Lemma 1.1. For given generalized reflection matrices R, S, if

$$P_1 := \frac{I+R}{2} \in C_r^{m\times m},\ Q_1 := \frac{I+S}{2} \in C_s^{n\times n},$$

then there exist unitary matrices $U \in UC^{m\times m}, V \in UC^{n\times n}$ such that

$$R = U\begin{bmatrix} I_r & 0 \\ 0 & -I_{m-r}\end{bmatrix}U^*, \quad S = V\begin{bmatrix} I_s & 0 \\ 0 & -I_{n-s}\end{bmatrix}V^*. \tag{1}$$

Based on the characteristics of generalized reflection matrices, we have

Lemma 1.2. Assume matrices R, S as in (1), then $A \in GRC^{n\times n}$ if and only if

$$A = U\begin{bmatrix} A_1 & 0 \\ 0 & A_2\end{bmatrix}V^*, \forall\, A_1 \in C^{r\times s},\ A_2 \in C^{(m-r)\times(n-s)}. \tag{2}$$

Proof. Partition U^*AV relative to (1) as $U^*AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$, where $A_{11} \in C^{r\times s}$. It follows from Definition 1.1 that

$$\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r}\end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} I_s & 0 \\ 0 & -I_{n-s}\end{bmatrix} = \pm\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}.$$

Comparing the two sides with blocks, we get (2) holds.

From Lemma 1.2, we can easily get a generalized reflexive matrix by choosing different A_i $(i = 1, 2)$ for given R and S.

In this paper, we consider the following two problems:

Problem I. Given generalized reflection matrices $R \in C^{m\times m}$, $S \in C^{n\times n}$, and A, $C \in C^{p\times m}$, B, $D \in C^{n\times q}$, $F \in C^{p\times q}$, find X, $Y \in GRC^{m\times n}$, such that

$$AXB + CYD = F. \tag{3}$$

Denote $S_E = \{(X,Y)|\, AXB + CYD = F,\ X,\ Y \in GRC^{m\times n}\}$.

Problem II. Assume S_E is nonempty. For given matrices X_0, $Y_0 \in C^{m\times n}$, find $(\hat{X}, \hat{Y}) \in S_E$ such that

$$\|\hat{X} - X_0\|^2 + \|\hat{Y} - Y_0\|^2 = \min_{(X,Y)\in S_E}\{\|X - X_0\|^2 + \|Y - Y_0\|^2\}.$$

The matrix equation (3) is the generalized Sylvester equation, which has been widely investigated. Such as, the special form of that $AX - YB = C$, has been studied in [3, 4], and the necessary and sufficient conditions for its consistency and general solution, were given by g-inverse. For the matrix equation (3), the solvability conditions and general solution have been derived in [5-9] by applying singular value decomposition (SVD), generalized SVD (GSVD) or canonical correlation decomposition (CCD), respectively. Especially, the symmetric solution of matrix equation $AXA^T + BYB^T = C$ has been represented in [10] by GSVD. In addition, motivated by the Classical Conjugate Gradient, Peng [11] has established an iterative algorithm (similar algorithms see [12-17]). Comparing with the matrix equation decomposition methods, the iterative algorithm is more convenient in practical applications.

However, for matrix equation (3) with generalized reflexive X, Y, it is very difficult to meet. Therefore, motivated by the iterative method mentioned in [11], in this paper, we try to find the generalized reflexive solution of this matrix equation by constructing iterative algorithm.

Problem II is the optimal approximation problem, which occurs frequently in experimental design[18]. Here, the given matrices X_0, Y_0 may be obtained from experiments, but they are not necessary to be needed forms, and may not satisfy minimum residual requirements. But the nearness matrices $\hat{X}$, $\hat{Y}$ satisfy the needed forms and the minimum residual restrictions (see, e.g, [8-17]).

This paper is organized as follows: In section 2, an algorithm will be constructed for solving Problem I, we call it Smarandache iterative algorithm, and the feasibility of this algorithm will be proved. In section 3, we will show that the solution of Problem II can be obtained by the least-norm solution of matrix equation $A\check{X}B + C\check{Y}D = \check{F}$. In section 4, some numerical examples will be given to illustrate our results.

§2. The iterative method for Problem I

In this section, we will first introduce the iterative method of Problem I, after that give the analysis of the feasibility about this method in the form of lemmas.

The iterative algorithm of Problem I can be expressed as follows:

Algorithm 2.1.

Step 1: Input matrices $R \in C^{m\times m}$, $S \in C^{n\times n}$, and A, $C \in C^{p\times m}$, B, $D \in C^{n\times q}$ $F \in C^{p\times q}$. Choosing arbitrary $X_1, Y_1 \in GRC^{m\times n}$.

Step 2: Compute

$R_1 = F - AX_1B - CY_1D$,

$P_1 = A^*R_1B^*, \quad Q_1 = \frac{1}{2}(P_1 + RP_1S)$,

$J_1 = C^*R_1D^*, \quad L_1 = \frac{1}{2}(J_1 + RJ_1S)$,

$k := 1$.

Step 3: Compute

$X_{k+1} = X_k + \frac{\|R_k\|^2}{\|Q_k\|^2+\|L_k\|^2}Q_k$,

$Y_{k+1} = Y_k + \frac{\|R_k\|^2}{\|Q_k\|^2+\|L_k\|^2}L_k$.

Step 4: Compute

$$R_{k+1} = F - AX_{k+1}B - CY_{k+1}D$$
$$= R_k - \frac{\|R_k\|^2}{\|Q_k\|^2+\|L_k\|^2}(AQ_kB + CL_kD),$$
$$P_{k+1} = A^*R_{k+1}B^*, \quad J_{k+1} = C^*R_{k+1}D^*,$$
$$Q_{k+1} = \frac{1}{2}(P_{k+1} + RP_{k+1}S) - \frac{tr(P_{k+1}^*Q_k)+tr(J_{k+1}^*L_k)}{\|Q_k\|^2+\|L_k\|^2}Q_k,$$
$$L_{k+1} = \frac{1}{2}(J_{k+1} + RJ_{k+1}S) - \frac{tr(P_{k+1}^*Q_k)+tr(J_{k+1}^*L_k)}{\|Q_k\|^2+\|L_k\|^2}L_k.$$

Step 5: If $R_k = 0$, stop; otherwise go to step 3.

From Algorithm 2.1, we can see that $X_i,\ Y_i,\ Q_i,\ L_i \in GRC^{m\times n}$ $(i = 1, 2, \cdots)$.

Lemma 2.1. For $R_i,\ P_i,\ Q_i,\ J_i,\ L_i$ $(i = 1, 2, \dots)$, generated by Algorithm 2.1, we have

$$tr(R_{i+1}^*R_1) = tr(R_i^*R_1) - \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2}[tr(Q_i^*P_1) + tr(L_i^*J_1)], \tag{4}$$

when $j > 1$,

$$tr(R_{i+1}^*R_j) = tr(R_i^*R_j) - \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2}[tr(Q_i^*Q_j) + tr(L_i^*L_j)]$$
$$- \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2} \times \frac{tr(P_j^*Q_{j-1}) + tr(J_j^*L_{j-1})}{\|Q_{j-1}\|^2 + \|L_{j-1}\|^2}[tr(Q_i^*Q_{j-1}) + tr(L_i^*L_{j-1})]. \tag{5}$$

Proof. From Algorithm 2.1, noting that $RQ_iS = Q_i$, $RL_iS = L_i$, we obtain

$$tr(R_{i+1}^*R_j) = tr(R_i^*R_j) - \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2}tr[(AQ_iB + CL_iD)^*R_j)]$$
$$= tr(R_i^*R_j) - \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2}tr(Q_i^*A^*R_jB^* + L_i^*C^*R_jD^*)$$
$$= tr(R_i^*R_j) - \frac{\|R_i\|^2}{\|Q_i\|^2 + \|L_i\|^2}tr(Q_i^*P_j + L_i^*J_j). \tag{6}$$

Letting $j = 1$ in (6) implies that (4) holds.

Furthermore, when $j > 1$,

$$tr(Q_i^*P_j + L_i^*J_j) = tr\left(Q_i^*\frac{P_j+RP_jS}{2}\right) + tr\left(Q_i^*\frac{P_j - RP_jS}{2}\right)$$
$$+ tr\left(L_i^*\frac{J_j + RJ_jS}{2}\right) + tr\left(L_i^*\frac{J_j - RJ_jS}{2}\right)$$
$$= tr(Q_i^*Q_j) + \frac{tr(P_j^*Q_{j-1}) + tr(J_j^*L_{j-1})}{\|Q_{j-1}\|^2 + \|L_{j-1}\|^2}tr(Q_i^*Q_{j-1})$$
$$+ tr(L_i^*L_j) + \frac{tr(P_j^*Q_{j-1}) + tr(J_j^*L_{j-1})}{\|Q_{j-1}\|^2 + \|L_{j-1}\|^2}tr(L_i^*L_{j-1})$$
$$= tr(Q_i^*Q_j) + tr(L_i^*L_j)$$
$$+ \frac{tr(P_j^*Q_{j-1}) + tr(J_j^*L_{j-1})}{\|Q_{j-1}\|^2 + \|L_{j-1}\|^2}[tr(Q_i^*Q_{j-1}) + tr(L_i^*L_{j-1})]. \tag{7}$$

Submitting (7) into (6), which deduce (5). The proof is completed.

Lemma 2.2. The sequences $\{R_i\}$, $\{Q_i\}$, $\{L_i\}$ in Algorithm 2.1, satisfy that

$$tr(R_i^*R_j)=0,\ tr(Q_i^*Q_j)+tr(L_i^*L_j)=0,\ \ i,j=1,2,\cdots,k\,(k\geq 2),\ i\neq j. \tag{8}$$

Proof. We prove the conclusions by induction when $i>j$, the others are analogous.

When $k=2$, from Lemma 2.1 and the proof of equality (7), we have

$$\begin{aligned}tr(R_2^*R_1)&=tr(R_1^*R_1)-\frac{\|R_1\|^2}{\|Q_1\|^2+\|L_1\|^2}[tr(Q_1^*P_1)+tr(L_1^*J_1)]\\&=tr(R_1^*R_1)-\frac{\|R_1\|^2}{\|Q_1\|^2+\|L_1\|^2}\ [tr(Q_1^*Q_1)+\ tr(L_1^*L_1)]\\&=\|R_1\|^2-\frac{\|R_1\|^2}{\|Q_1\|^2+\|L_1\|^2}\left(\|Q_1\|^2+\|L_1\|^2\right)\\&=0,\end{aligned}$$

and

$$\begin{aligned}tr(Q_2^*Q_1)+tr(L_2^*L_1)&=\frac{1}{2}tr[(P_2+RP_2S)^*\,Q_1]-\ \frac{tr(P_2^*Q_1)+tr(J_2^*L_1)}{\|Q_1\|^2+\|L_1\|^2}tr(Q_1^*Q_1)\\&\quad+\frac{1}{2}tr[(J_2+RJ_2S)^*\,L_1]-\ \frac{tr(P_2^*Q_1)+tr(J_2^*L_1)}{\|Q_1\|^2+\|L_1\|^2}tr(J_1^TJ_1)\\&=\ tr(P_2^*Q_1)+tr(J_2^*L_1)-\ \frac{tr(P_2^*Q_1)+tr(J_2^*L_1)}{\|Q_1\|^2+\|L_1\|^2}\left(\|Q_1\|^2+\|L_1\|^2\right)\\&=0.\end{aligned} \tag{9}$$

Assume (8) holds for $k=t$, that is, $tr(R_t^*R_j)=0$, $tr(Q_t^*Q_j)+tr(L_t^*L_j)=0$, $j=1,2,\cdots,t-1$. Similar to the proof of (9), we get by Lemma 2.1 that $tr(R_{t+1}^*R_t)=0$, $tr(Q_{t+1}^*Q_t)+tr(L_{t+1}^*L_t)=0$.

Next, we prove that $tr(R_{t+1}^*R_j)=0$, $tr(Q_{t+1}^*Q_j)+tr(L_{t+1}^*L_j)=0$ hold. In fact, when $j=1$, noting that (4) and the proof of (7), then

$$\begin{aligned}tr(R_{t+1}^*R_1)&=tr(R_t^*R_1)-\ \frac{\|R_t\|^2}{\|Q_t\|^2+\|L_t\|^2}[tr(Q_t^*P_1)+tr(L_t^*J_1)]\\&=-\ \frac{\|R_t\|^2}{\|Q_t\|^2+\|L_t\|^2}[tr(Q_t^*Q_1)+tr(L_t^*L_1)]\\&=0.\end{aligned}$$

Connecting with (9) and the assumptions, yields to

$$\begin{aligned}tr(Q_{t+1}^*Q_1)+tr(L_{t+1}^*L_1)&=\ tr(P_t^*Q_1)+tr(J_t^*L_1)\\&\quad-\frac{tr(P_{t+1}^*Q_t)+tr(J_{t+1}^*L_t)}{\|Q_t\|^2+\|L_t\|^2}\ [tr(Q_t^*Q_1)+tr(L_t^*L_1)]\\&=\ tr(P_t^*Q_1)+tr(J_t^*L_1)\\&=\ tr(R_t^*AQ_1B)+tr(R_t^*CL_1D)\\&=\frac{\|Q_1\|^2+\|L_1\|^2}{\|R_1\|^2}tr[R_t^*(R_1-R_2)]\end{aligned}$$

$$= 0. \tag{10}$$

Moreover, when $2 \leq j \leq t-1$, Lemma 2.1 and the assumptions imply that $tr(R^*_{t+1}R_j) = 0$. Similar to the proof of (10), we get $tr(Q^*_{t+1}Q_j) + tr(L^*_{t+1}L_j) = 0$. Hence, we complete the proof.

Remark 2.1. From Lemma 2.2, we know that, R_i $(i = 1, 2, \cdots, pq)$ can be regarded as an orthogonal basis of matrix space $R^{p\times q}$ for their orthogonality each other. If $R_i \neq 0$, $(i = 1, 2, \cdots, pq)$, we can compute R_{pq+1}, then there must be $tr(R^*_{pq+1}R_i) = 0$, that is, (X_{pq+1}, Y_{pq+1}) consists of a solution pair of Problem I, Hence, if matrix equation (3) is consistent, then any solution of which can be obtained by Algorithm 2.1 at most $pq + 1$ iteration steps.

Lemma 2.3. Suppose that $(\bar{X}, \bar{Y})$ is an arbitrary solution pair of matrix equation (3), then R_k, X_k, Y_k, Q_k, L_k generated by the iterative algorithm satisfy that

$$tr[(\bar{X} - X_k)^*Q_k] + tr[(\bar{Y} - Y_k)^*L_k] = \|R_k\|^2,\ k = 1, 2, \cdots. \tag{11}$$

Proof. We prove it by induction. If $k = 1$, from Algorithm 2.1 and Lemma 2.2, noting that $R(\bar{X} - X_i)S = \bar{X} - X_i$, we get

$$\begin{aligned}
& tr[(\bar{X} - X_1)^*Q_1] + tr[(\bar{Y} - Y_1)^*L_1] \\
&= \frac{1}{2}\{tr[(\bar{X} - X_1)^*(P_1 + RP_1S)] + tr[(\bar{Y} - Y_1)^*(J_1 + RJ_1S)]\} \\
&= tr[(\bar{X} - X_1)^*P_1] + tr[(\bar{Y} - Y_1)^*J_1] \\
&= tr[(\bar{X} - X_1)^*A^*R_1B^*] + tr[(\bar{Y} - Y_1)^*C^*R_1D^*] \\
&= tr[(A(\bar{X} - X_1)B)^*R_1] + tr[(C(\bar{Y} - Y_1)D)^*R_1] \\
&= tr[(F - AX_1B - CY_1D)^*R_1] \\
&= \|R_1\|^2.
\end{aligned} \tag{12}$$

Assume (11) holds for $k = t$, then

$$\begin{aligned}
& tr[(\bar{X} - X_{t+1})^*Q_t] + tr[(\bar{Y} - Y_{t+1})^*L_t] \\
&= tr[(\bar{X} - X_t)^*Q_t] - \frac{\|R_t\|^2}{\|Q_t\|^2 + \|L_t\|^2} tr(Q^*_tQ_t) \\
&+ tr[(\bar{Y} - Y_t)^*L_t] - \frac{\|R_t\|^2}{\|Q_t\|^2 + \|L_t\|^2} tr(L^*_tL_t) \\
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|Q_t\|^2 + \|L_t\|^2}[tr(Q^*_tQ_t) + tr(L^*_tL_t)] \\
&= 0,
\end{aligned}$$

which deduces that

$$\begin{aligned}
& tr[(\bar{X} - X_{t+1})^*Q_{t+1}] + tr[(\bar{Y} - Y_{t+1})^*L_{t+1}] \\
&= \tfrac{1}{2}\left\{tr[(\bar{X} - X_{t+1})^*(P_{t+1} + RP_{t+1}S)] + tr[(\bar{Y} - Y_{t+1})^*(J_{t+1} + RJ_{t+1}S)]\right\}
\end{aligned}$$

$$-\frac{tr(P_{t+1}^*Q_t)+tr(J_{t+1}^*L_t)}{\|Q_t\|^2+\|L_t\|^2}tr[(\bar{X}-X_{t+1})^*Q_t+(\bar{Y}-Y_{t+1})^*L_t]$$
$$=tr[(\bar{X}-X_{t+1})^*A^*R_{t+1}B^*]+tr[(\bar{Y}-Y_{t+1})^*C^TR_{t+1}D^*]$$
$$=\|R_{s+1}\|^2.$$

That is, the equality (11) holds for any positive integers.

Remark 2.2. Lemma 2.3 reveals that the matrix equation (3) is consistent, i.e.,$R_i = 0$, if and only if $Q_i = 0$, $L_i = 0$. In other words, if there exist a positive number t such that $R_t \neq 0$ but $Q_t = 0$, $L_t = 0$, then matrix equation (3) is inconsistent. Therefore, the solvability of Problem I can be determined automatically by Algorithm 2.1.

Based on the previous analysis, we have the following conclusion, whose proof is omitted.

Theorem 2.1. Suppose that Problem I is consistent, for any initial matrices $X_1, Y_1 \in GRC^{m\times n}$, the solution of equation (3) can be obtained within finite iteration steps in the absence of roundoff errors.

The following lemma recited from [11] is essential for gaining the least-norm solution of Problem I.

Lemma 2.4. If the consistent linear equations $My = b$ has a solution $\tilde{y} \in R(M^*)$, then $\tilde{y}$ is the unique least-norm solution.

Theorem 2.2. Suppose that Problem I is consistent. If choose initial iterative matrices $X_1 = A^*HB^* + RA^*HB^*S$, and $Y_1 = C^*HD^* + RC^*HD^*S$, where arbitrary $H \in C^{p\times q}$, or especially, $X_1, Y_1 = 0$, then the solution generated by Algorithm 2.1 is the unique least-norm solution of Problem I.

Proof. Algorithm 2.1 and Theorem 2.1 imply that, if let $X_1 = A^*HB^* + RA^*HB^*S$, $Y_1 = C^*HD^* + RC^*HD^*S$, where arbitrary $H \in C^{p\times q}$, we can obtain a solution pair $(\tilde{X}, \tilde{Y})$ of Problem I, which have the forms of $X^* = A^*GB^* + RA^*GB^*S$, $Y^* = C^*GD^* + RC^*GD^*S$. So it is enough to show that $(\tilde{X}, \tilde{Y})$ is the least-norm solution pair of matrix equation (3).

Considering the following matrix equations

$$\begin{cases} AXB + CYD = F, \\ ARXSB + CRYSD = F. \end{cases} \tag{13}$$

It is clear that the solvability of (13) is equivalent to that of matrix equation (3). What's more, the two equations have same solutions in $GRC^{m\times n}$.

Denote $vec(\tilde{X}) = \tilde{x}$, $vec(\tilde{Y}) = \tilde{y}$, $vec(X) = x$, $vec(Y) = y$, $vec(F) = f$, $vec(G) = g$, then the matrix equations (13) can be changed equivalently into

$$\begin{pmatrix} B^*\otimes A & D^*\otimes C \\ (B^*S)\otimes(AR) & (D^*S)\otimes(CR) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ f \end{pmatrix}. \tag{14}$$

In addition, the iterative solution pair $(\tilde{X}, \tilde{Y})$ can be rewritten as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} B\otimes A^* & (SB)\otimes(RA^*) \\ D\otimes C^* & (SD)\otimes(RC^*) \end{pmatrix} \begin{pmatrix} g \\ g \end{pmatrix}$$
$$\in \mathscr{R}\left(\begin{pmatrix} B^*\otimes A & D^*\otimes C \\ (B^*S)\otimes(AR) & (D^*S)\otimes(CR) \end{pmatrix}^*\right).$$

It follows from Lemma 2.4 that $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ is the least-norm solution of the linear systems (14). Making use of the property of *vec* operator, $(\tilde{X}, \tilde{Y})$ is the least-norm solution pair of matrix equations (13), which means that it is also the least-norm solution of Problem I.

§3. The solution of Problem II

When matrix equation (3) is consistent, i.e., its solution set S_E is nonempty. It is easy to verify that S_E is a closed convex set in matrix inner product space $GRC^{m\times n}$, resorting to the optimal approximation theorem, we know that the optimal approximation solution of Problem II is unique.

Without loss of generality, we can assume the given matrices $X_0, Y_0 \in GRC^{m\times n}$, in Problem II, because of the orthogonality between generalized reflexive matrices and generalized anti-reflexive matrices. In fact, for $X, Y \in GRC^{m\times n}$, we have

$$\begin{aligned}\|X-X_0\|^2+\|Y-Y_0\|^2 &= \left\|X-\tfrac{X_0+RX_0S}{2}-\tfrac{X_0-RX_0S}{2}\right\|^2+\left\|Y-\tfrac{Y_0+RY_0S}{2}-\tfrac{Y_0-RY_0S}{2}\right\|^2 \\ &= \left\|X-\tfrac{X_0+RX_0S}{2}\right\|^2+\left\|\tfrac{X_0-RX_0S}{2}\right\|^2+\left\|Y-\tfrac{Y_0+RY_0S}{2}\right\|^2+\left\|\tfrac{Y_0-RY_0S}{2}\right\|^2.\end{aligned}$$

Writing $\check{X}=X-X_0$, $\check{Y}=Y-Y_0$, $\check{F}=F-AX_0B-CY_0D$, then Problem II is equivalent to find the least-norm solution $(\tilde{X}', \tilde{Y}')\in S_E$ of the following matrix equation

$$A\check{X}B+C\check{Y}D=\check{F}. \tag{15}$$

Theorem 2.2 implies that, if let initial iteration matrices $\check{X}_1 = A^*\check{H}B^*+RA^*\check{H}B^*S$, $\check{Y}_1 = C^*\check{H}D^*+RC^*\check{H}D^*S$, where arbitrary $\check{H}\in C^{p\times q}$, or especially, let $\check{X}_1$, $\check{Y}_1=0$, we can obtain the unique least-norm solution $(\tilde{X}', \tilde{Y}')$ of matrix equation (15) by Algorithm 2.1. Furthermore, the optimal approximation solution pair $(\hat{X}, \hat{Y})$ can be obtained by $(\hat{X}, \hat{Y}) = (\tilde{X}'+X_0,\ \tilde{Y}'+Y_0)$.

§4. Numerical examples

In this section, we make some numerical tests in real field to verify our conclusions. All these work will be finished by MATLAB software. However, because of the roundoff errors, R_i (i=1,2,$\cdots$) will unequal to zero in the iterative process. Therefore, for arbitrary positive number ε small enough, e.g., $\varepsilon = 1.0e-010$, the iteration stops whenever $\|R_k\| < \varepsilon$, and (X_k, Y_k) is regarded as a solution pair of the matrix equation.

Example 4.1. Input matrices A, B, C, D, F, R, S as follows:

$$A=\begin{pmatrix} 5 & -3 & 0 & 3 & 0 & 2 & 8\\ 0 & -4 & -6 & 4 & -6 & 0 & -4\\ -6 & 0 & 7 & 0 & 7 & 3 & 1\\ 0 & 5 & -3 & -5 & -3 & 0 & 3\\ 4 & -7 & 0 & 7 & 0 & -8 & -3\\ -1 & 0 & -6 & 0 & -5 & 9 & 0\\ 0 & -3 & 8 & 3 & -7 & 0 & 1 \end{pmatrix},\ C=\begin{pmatrix} 4 & -5 & -4 & 1 & 4 & 4 & 2\\ -1 & 2 & 1 & 3 & 0 & 0 & -1\\ 3 & -4 & -3 & 0 & 1 & 5 & 3\\ 0 & 3 & 0 & -2 & -5 & -1 & 6\\ -5 & 1 & 5 & -4 & -6 & 7 & 8\\ 0 & -2 & 0 & -7 & 3 & 0 & -4\\ -7 & 1 & 7 & -4 & -6 & -1 & 5 \end{pmatrix},$$

$$B=\begin{pmatrix} -3 & 5 & -5 & -2 & 5 & -12\\ 0 & 4 & 9 & 9 & 4 & -6\\ 6 & -1 & 7 & 0 & -1 & 3\\ -2 & 4 & 0 & 5 & 4 & 5\\ -1 & -6 & -2 & 0 & -6 & 2\\ 0 & -9 & 1 & 1 & -9 & 2 \end{pmatrix},\quad D=\begin{pmatrix} -2 & 2 & -1 & 14 & 1 & -1\\ -5 & 5 & 0 & 12 & -4 & 4\\ 0 & 0 & -2 & 8 & 0 & 0\\ -8 & 8 & -4 & 14 & 1 & -1\\ 1 & -1 & 7 & -10 & -2 & 2\\ -8 & 8 & 5 & 0 & 6 & -6 \end{pmatrix},$$

$$F=\begin{pmatrix} -82 & -523 & -81 & -1213 & -459 & 845\\ -136 & -44 & 520 & 648 & -201 & -951\\ -466 & 583 & -153 & -187 & 653 & 178\\ 188 & -187 & -913 & -473 & 386 & -649\\ 562 & -677 & 297 & 129 & -755 & 618\\ 82 & -298 & 422 & -89 & -402 & -408\\ 907 & -1781 & -617 & -633 & -1875 & 1664 \end{pmatrix},$$

$$R=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ S=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1). The solutions of Problem I

If given initial iterative matrices

$$X_1=\begin{pmatrix} 4 & 4 & -4 & 0 & 0 & 0\\ -4 & -4 & 8 & 0 & 0 & 0\\ -4 & -8 & 8 & 0 & 0 & 0\\ 0 & 0 & 0 & 4 & -4 & -8\\ 0 & 0 & 0 & 4 & -4 & -4\\ 0 & 0 & 0 & -8 & 8 & 12\\ 8 & 4 & -8 & 0 & 0 & 0 \end{pmatrix},\quad Y_1=\begin{pmatrix} -4 & 8 & 8 & 0 & 0 & 0\\ 8 & -8 & 4 & 0 & 0 & 0\\ -8 & 4 & -4 & 0 & 0 & 0\\ 0 & 0 & 0 & -8 & 4 & 4\\ 0 & 0 & 0 & 8 & 4 & 8\\ 0 & 0 & 0 & 4 & 8 & 8\\ -4 & -4 & 8 & 0 & 0 & 0 \end{pmatrix},$$

Then, by Algorithm 2.1 and iteration 145 steps, we obtain a solution pair of Problem I:

$$X_{145} = \begin{pmatrix} -5.0000 & -4.0000 & 6.0000 & 0 & 0 & 0 \\ 7.5451 & 2.5191 & -4.8199 & 0 & 0 & 0 \\ -6.0000 & -8.0000 & 3.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.7347 & 1.0241 & 3.2550 \\ 0 & 0 & 0 & 8.0000 & -9.0000 & 2.0000 \\ 0 & 0 & 0 & 3.0000 & -3.0000 & 4.0000 \\ -2.0000 & -8.0000 & -4.0000 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_{145} = \begin{pmatrix} -8.0000 & 1.5000 & 5.0000 & 0 & 0 & 0 \\ -2.0000 & -4.0000 & 6.0000 & 0 & 0 & 0 \\ -4.0000 & 10.5000 & -1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.0000 & -5.0000 & 7.0000 \\ 0 & 0 & 0 & -2.0000 & 6.0000 & 5.0000 \\ 0 & 0 & 0 & 4.0000 & -3.0000 & 8.0000 \\ -4.0000 & -6.0000 & 3.0000 & 0 & 0 & 0 \end{pmatrix},$$

and $\|R_{145}\| = 8.4440e - 011 < \varepsilon$, and $\|(X_{145}, Y_{145})\| = 48.6449$.

Especially, choosing $X_1 = 0$, $Y_1 = 0$, from Theorem 2.2, the least-norm solution pair is

$$X_{143} = \begin{pmatrix} -5.0000 & -4.0000 & 6.0000 & 0 & 0 & 0 \\ 7.2734 & 2.7588 & -4.9097 & 0 & 0 & 0 \\ -6.0000 & -8.0000 & 3.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.3685 & 0.0150 & 4.1249 \\ 0 & 0 & 0 & 8.0000 & -9.0000 & 2.0000 \\ 0 & 0 & 0 & 3.0000 & -3.0000 & 4.0000 \\ -2.0000 & -8.0000 & -4.0000 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_{143} = \begin{pmatrix} -2.0000 & -4.5000 & 3.0000 & 0 & 0 & 0 \\ -2.0000 & -4.0000 & 6.0000 & 0 & 0 & 0 \\ 2.0000 & 4.5000 & -3.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.0000 & -5.0000 & 7.0000 \\ 0 & 0 & 0 & -2.0000 & 6.0000 & 5.0000 \\ 0 & 0 & 0 & 4.0000 & -3.0000 & 8.0000 \\ -4.0000 & -6.0000 & 3.0000 & 0 & 0 & 0 \end{pmatrix},$$

and $\|R_{143}\| = 8.7803e - 011 < \varepsilon$, and $\|(X_{143}, Y_{143})\| = 45.1825$.

(2). The solution of Problem II

Suppose that the given matrices

$$X_0 = \begin{pmatrix} 5 & 4 & -6 & 0 & 0 & 0 \\ -7 & -3 & 5 & 0 & 0 & 0 \\ 6 & 8 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 6 & -5 \\ 0 & 0 & 0 & -8 & 9 & -2 \\ 0 & 0 & 0 & -3 & 3 & -4 \\ 2 & 8 & -9 & 0 & 0 & 0 \end{pmatrix}, Y_0 = \begin{pmatrix} -9 & -3 & 4 & 0 & 0 & 0 \\ -7 & -4 & 6 & 0 & 0 & 0 \\ -5 & 6 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -5 & 7 \\ 0 & 0 & 0 & 2 & 6 & 5 \\ 0 & 0 & 0 & 4 & -3 & 8 \\ -8 & -6 & 9 & 0 & 0 & 0 \end{pmatrix},$$

Compute $\check{F} = AX_0B + CY_0D$, let initial iterative matrices $X_1 = Y_1 = 0$, then we can obtain the least-norm solution $(\tilde{X}', \tilde{Y}')$ of the new matrix equation (15). Hence, the unique solution $(\hat{X}, \hat{Y})$ of Problem II is

$$\hat{X} = \tilde{X}^* + X_0 = \begin{pmatrix} -5.0000 & -4.0000 & 6.0000 & 0 & 0 & 0 \\ 8.2161 & 1.9271 & -4.5982 & 0 & 0 & 0 \\ -6.0000 & -8.0000 & 3.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.6390 & 3.5152 & 1.1073 \\ 0 & 0 & 0 & 8.0000 & -9.0000 & 2.0000 \\ 0 & 0 & 0 & 3.0000 & -3.0000 & 4.0000 \\ -2.0000 & -8.0000 & -4.0000 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{Y} = \tilde{Y}^* + Y_0 = \begin{pmatrix} -9 & -3 & 4 & 0 & 0 & 0 \\ -2 & -4 & 6 & 0 & 0 & 0 \\ -5 & 6 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -5 & 7 \\ 0 & 0 & 0 & -2 & 6 & 5 \\ 0 & 0 & 0 & 4 & -3 & 8 \\ -4 & -6 & 3 & 0 & 0 & 0 \end{pmatrix}.$$

and $\|R_{145}\| = 8.0904e - 011 < \varepsilon$, $\|(\hat{X}, \hat{Y})\| = 61.7436$.

Example 4.2. Given matrices A, B, C, D, F, R, S as following,

$$A = \begin{pmatrix} 3 & -3 & 0 & 9 \\ -3 & -9 & -12 & 12 \\ 9 & 3 & 12 & 6 \end{pmatrix}, B = \begin{pmatrix} 6 & -3 & 0 & 3 \\ -9 & 0 & 3 & -3 \\ 0 & -6 & 12 & 3 \\ 3 & -6 & 3 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & -2 & 0 & 6 \\ -2 & -6 & -8 & 8 \\ 6 & 2 & 8 & 4 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & -2 & 0 & 2 \\ -6 & 0 & 2 & -2 \\ 0 & -4 & 8 & 2 \\ 2 & -4 & 2 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} 100 & 15 & -110 & 10 \\ 120 & 120 & -360 & 30 \\ 80 & -90 & 140 & -10 \end{pmatrix}.$$

By the iterative algorithm in this paper, we discuss the solvability of Problem I. Just as the statements in Example 4.1, for any positive number ε, however small enough, e.g., $\varepsilon = 1.0e-005$, whenever $\|R_k\| < \varepsilon$ or $\|R_k\| > \varepsilon$, $\|Q_k\| < \varepsilon$, and $\|L_k\| < \varepsilon$, stop the iteration.

Let X_1, $Y_1 = 0 \in R^{4\times 4}$, by Algorithm 2.1, we obtain

$$\|R_6\| = 1.0574e+004 > \varepsilon,\ \|Q_6\| = 2.6216e-007 < \varepsilon,\ \|L_6\| = 1.1548e-007 < \varepsilon.$$

Therefore, from Theorem 2.2, we know that matrix equation $AXB + CYD = F$ is inconsistent.

§5. Conclusions

In this paper, we have established an iterative algorithm, i.e., Algorithm 2.1, for solving the matrix equation $AXB + CYD = F$ over generalized reflexive matrix dual (X, Y). By this algorithm, the consistence of the equation can be determined automatically. In particular, we can also obtain its least-norm solution by choosing special initial iterative matrices, which appears in Theorem 2.2. In addition, the optimal approximation solution of problem II can be gained by the least-norm solution of a new (but similar to (3)) matrix equation, which is included in section 3. Finally, some numerical examples show that the iterative algorithm is efficient. Certainly, if we make only some trivial changes, for instance, $Q_1 = \frac{1}{2}(P_1 - RP_1S)$, $L_1 = \frac{1}{2}(J_1 - RJ_1S)$, then the generalized anti-reflexive solution of matrix equation (3) can be obtained. Moreover, if let $R = S$ in Algorithm 2.1, then the iterative solution is the generalized centrosymmetric solution.

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On the Smarandache function and the divisor product sequences

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Abstract Let n be any positive integer, $P_d(n)$ denotes the product of all positive divisors of n. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of a new arithmetical function $S\left(P_d(n)\right)$, and give an interesting asymptotic formula for it.

Keywords Smarandach function, Divisor product sequences, Composite function, mean value, Asymptotic formula.

§1. Introduction

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divide $m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$. And the Smarandache divisor product sequences $\{P_d(n)\}$ is defined as the product of all positive divisors of n. That is, $P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}$, where $d(n)$ is the Dirichlet divisor function. For examples, $P_d(1) = 1, P_d(2) = 2, P_d(3) = 3, P_d(4) = 8, \cdots$. In problem 25 of reference [1], Professor F.Smarandache asked us to study the properties of the function $S(n)$ and the sequence $\{P_d(n)\}$. About these problems, many scholars had studied them, and obtained a series interesting results, see references [2], [3], [4], [5] and [6]. But at present, none had studied the mean value properties of the composite function $S\left(P_d(n)\right)$, at least we have not seen any related papers before. In this paper, we shall use the elementary methods to study the mean value properties of $S(P_d(n))$, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any fixed positive integer k and any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} S(P_d(n)) = \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, \ 3, \ \cdots, \ k)$ are computable constants.

§2. Some simple lemmas

To complete the proof of the theorem, we need the following several simple lemmas. First we have

Lemma 1. For any positive integer α, we have the estimate

$$S(p^\alpha) \leq \alpha p.$$

Especially, when $\alpha \leq p$, we have $S(p^\alpha) = \alpha p$, where p is a prime.

Proof. See reference [3].

Lemma 2. For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, then we have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

Lemma 3. Let $P(n)$ denotes the greatest prime divisor of n, if $P(n) > \sqrt{n}$, then we have $S(n) = P(n)$.

Proof. The proof of Lemma 2 and Lemma 3 can be found in reference [4].

§3. Proof of the theorem

In this section , we shall use the above lemmas to complete the proof of our theorem. For any positive integer n, it is clear that from the definition of $P_d(n)$ we have

$$P_d^2(n) = \left(\prod_{r|n} r\right) \cdot \left(\prod_{r|n} \frac{n}{r}\right) = n^{\sum\limits_{r|n} 1} = n^{d(n)}.$$

So we have the identity $P_d(n) = n^{\frac{d(n)}{2}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. First we separate all integers n in the interval $[1,\ x]$ into two subsets A and B as follows:

$$A = \{n: \ n \leq x,\ P(n) \leq \sqrt{n}\}, \quad B = \{n: \ n \leq x,\ P(n) > \sqrt{n}\}.$$

If $n \in A$, then from Lemma 1 and Lemma 2, and note that $P_d(n) = n^{\frac{d(n)}{2}}$ we have

$$P_d(n) = n^{\frac{d(n)}{2}} = p_1^{\frac{\alpha_1 d(n)}{2}} p_2^{\frac{\alpha_2 d(n)}{2}} \cdots p_k^{\frac{\alpha_k d(n)}{2}}.$$

Therefore,

$$\begin{aligned} S\left(P_d(n)\right) &= S\left(p_1^{\frac{\alpha_1 d(n)}{2}} p_2^{\frac{\alpha_2 d(n)}{2}} \cdots p_k^{\frac{\alpha_k d(n)}{2}}\right) = \max_{1\leq i\leq k}\left\{S\left(p_i^{\frac{\alpha_i d(n)}{2}}\right)\right\} \\ &\leq \max_{1\leq i\leq k}\left\{\frac{\alpha_i d(n)}{2} p_i\right\} \leq \frac{d(n)}{2}\sqrt{n}\ln n. \end{aligned}$$

From reference [10] we know that

$$\sum_{n\le x} d(n) = x\ln x + O(x).$$

So we have the estimate

$$\sum_{n\in A} S\left(P_d(n)\right) \le \sum_{n\in A} \frac{d(n)}{2}\sqrt{n}\ln n \ll \sum_{n\le x} d(n)\sqrt{x}\ln x \ll x^{\frac{3}{2}}\ln^2 x. \tag{1}$$

If $n\in B$, let $n = n_1 p$, where $n_1 < \sqrt{n} < p$. It is clear that $d(n_1) < \sqrt{n} < p$ and $d(n) = 2d(n_1)$. So from Lemma 3 we have

$$\begin{aligned}\sum_{n\in B} S\left(P_d(n)\right) &= \sum_{\substack{n_1 p\le x\\ n_1<p}} S\left((n_1p)^{\frac{d(n_1p)}{2}}\right) = \sum_{\substack{n_1 p\le x\\ n_1<p}} S\left(p^{\frac{d(n_1p)}{2}}\right)\\ &= \sum_{n\le\sqrt{x}}\sum_{n<p\le\frac{x}{n}} d(n)p = \sum_{n\le\sqrt{x}} d(n)\sum_{n<p\le\frac{x}{n}} p\\ &= \sum_{n\le\sqrt{x}} d(n)\sum_{p\le\frac{x}{n}} p + O\left(\sum_{n\le\sqrt{x}} d(n)\cdot\frac{n}{\ln n}\right)\\ &= \sum_{n\le\sqrt{x}} d(n)\sum_{p\le\frac{x}{n}} p + O(x). \end{aligned}\tag{2}$$

From the Abel's summation formula (see Theorem 4.2 of [10]) and the Prime Theorem (see Theorem 3.2 of [11]) we have

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2,\ \cdots,\ k)$ are computable constants and $a_1 = 1$. We have

$$\begin{aligned}\sum_{p\le\frac{x}{n}} p &= \frac{x}{n}\pi\left(\frac{x}{n}\right) - \int_2^{\frac{x}{n}}\pi(y)dy\\ &= \frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2\ln^i n}{n^2\ln^2 x} + O\left(\frac{x^2}{n^2\ln^{k+1} x}\right),\end{aligned}\tag{3}$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Note that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\sum_{n=1}^{\infty}\frac{d(n)}{n^2} = \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2 = \frac{\pi^4}{36}, \tag{4}$$

from (2), (3) and (4) we obtain

$$\begin{aligned}\sum_{n\in B} S\left(P_d(n)\right) &= \frac{x^2}{2\ln x}\sum_{n\le\sqrt{x}}\frac{d(n)}{n^2}+\sum_{n\le\sqrt{x}}\sum_{i=2}^{k}\frac{c_i\cdot x^2 d(n)\ln^i n}{n^2\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right)\\ &= \frac{\pi^4}{72}\frac{x^2}{\ln x}+\sum_{i=2}^{k} b_i\cdot\frac{x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),\end{aligned} \tag{5}$$

where b_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants.

Now combining (1) and (5) we may immediately get the asymptotic formula

$$\begin{aligned}\sum_{n\le x} S\left(P_d(n)\right) &= \sum_{n\in A} S\left(P_d(n)\right)+\sum_{n\in B} S\left(P_d(n)\right)\\ &= \frac{\pi^4}{72}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k} b_i\cdot\frac{x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),\end{aligned}$$

where b_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants. This completes the proof of Theorem.

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