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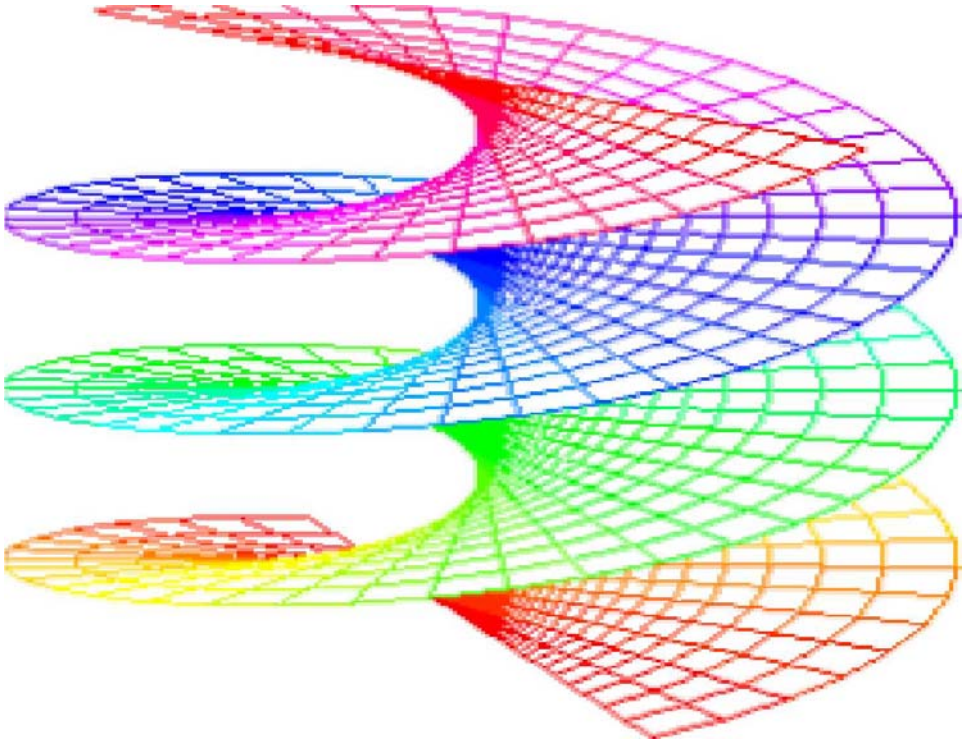
Scientia Magna

International Book Series

Vol. 3, No. 3



2007

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Scientia Magna
- international book series (Vol. 3, No. 3) -

High American Press

2007

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ISBN 13: 978-1-59973-045-5
Printed in USA

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Some faces of Smarandache semigroups' concept in transformation semigroups' approach

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Abstract In the following text, the main aim is to distinguish some relations between Smaradche semigroups and (topological) transformation semigroups areas. We will see that a transformation group is not distal if and only if its enveloping semigroup is a Smarandache semigroup. Moreover we will find a classifying of minimal right ideals of the enveloping semigroup of a transformation semigroup.

Keywords $A-$minimal set, distal, Smarandache semigroup, transformation semigroup.

§1. Preliminaries

By a transformation semigroup (group) (X, S, π) (or simply (X, S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup (group) S with identity e and a continuous map $\pi : X \times S \to X$ $(\pi(x, s) = xs\ (\forall x \in X, \forall s \in S))$ such that:

1) $\forall x \in X, \quad xe = x$,

2) $\forall x \in X, \quad \forall s, t \in S, \quad x(st) = (xs)t$.

In the transformation semigroup (X, S) we have the following definitions:

1. For each $s \in S$, define the continuous map $\pi^s : X \to X$ by $x\pi^s = xs$ $(\forall x \in X)$, then E(X,S)(or simply E(X)) is the closure of $\{\pi^s |\ s \in S\}$ in X^X with pointwise convergence, moreover it is called the enveloping semigroup (or Ellis semigroup) of (X, S). We used to write s instead of π^s $(s \in S)$. E(X,S) has a semigroup structure [1, Chapter 3], a nonempty subset K of E(X,S) is called a right ideal if KE(X,S)$\subseteq K$, and it is called a minimal right ideal if none of the right ideals of E(X,S) be a proper subset of K. For each $p\in$E(X), L_p : E(X)$\to$ E(X) with $L_p(q) = pq$ $(q \in$ E(X)) is continuous.

2. Let $a \in X$, A be a nonempty subset of X and K be a closed right ideal of E(X,S)[2, Definition 1]:

1) We say K is an $a-$minimal set if: $aK = a$E(X, S), K does not have any proper subset like L, such that L is a closed right ideal of E(X, S) and $aL = a$E(X, S). The set of all $a-$minimal sets is denoted by $\mathrm{M}_{(X,S)}(a)$.

2) We say K is an $A-\overline{\text{minimal}}$ set if: $\forall b \in A\ bK = b\mathrm{E}(X,S)$, K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ and $bL = b\mathrm{E}(X,S)$ for all $b \in A$. The set of all $A-\overline{\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}_{(X,S)}(A)$.

3) We say K is an $A-\overline{\overline{\text{minimal}}}$ set if: $AK = A\mathrm{E}(X,S)$, K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ and $AL = A\mathrm{E}(X,S)$.

The set of all $A-\overline{\overline{\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$. $\overline{\mathrm{M}}_{(X,S)}(A)$ and $\mathrm{M}_{(X,S)}(a)$ are nonempty.

3. Let $a \in X$, A be a nonempty subset of X [2, Definition 13]:

1) (X,S) is called distal if $\mathrm{E}(X,S)$ is a minimal right ideal.

2) (X,S) is called $a-$distal if $\mathrm{M}_{(X,S)}(a) = \{\mathrm{E}(X,S)\}$.

3) (X,S) is called $A^{(\overline{\mathrm{M}})}$distal if $\overline{\mathrm{M}}_{(X,S)}(A) = \{\mathrm{E}(X,S)\}$.

4) (X,S) is called $A^{(\overline{\overline{\mathrm{M}}})}$distal if $\overline{\overline{\mathrm{M}}}_{(X,S)}(A) = \{\mathrm{E}(X,S)\}$.

4. Let $a \in X$, A be a nonempty subset of X and C be a nonempty subset of $\mathrm{E}(X,S)$, we introduce the following notations $\mathrm{F}(a,C) = \{p \in C | ap = p\}$, $\mathrm{F}(A,C) = \{p \in C | \forall b \in A\ bp = p\}$, $\overline{\mathrm{F}}(A,C) = \{p \in C |\ Ap = A\}$, $\mathrm{J}(C) = \{p \in C |\ p^2 = p\}$ (the set of all idempotents of C), $\overline{\mathrm{M}}(X,S) = \{\emptyset \neq B \subseteq X |\ \forall K \in \overline{\mathrm{M}}_{(X,S)}(B)\ \mathrm{J}(\mathrm{F}(B,K)) \neq \emptyset\}$, and $\overline{\overline{\mathrm{M}}}(X,S) = \{\emptyset \neq B \subseteq X |\ \overline{\overline{\mathrm{M}}}_{(X,S)}(B) \neq \emptyset \wedge (\forall K \in \overline{\overline{\mathrm{M}}}_{(X,S)}(B)\ \mathrm{J}(\mathrm{F}(B,K)) \neq \emptyset)\}$.

In the following text in the transformation semigroup (X,S) suppose S acts effective on X, i.e., for each $s,\ t \in S$ if $s \neq t$, then there exists $x \in X$ such that $xs \neq xt$. Moreover consider S is a Smarandache semigroup if S has a semigroup structure which does not provides a group structure on S, although S has at least a proper subset with more than one element, which carries a group structure induced by the action considered on S [3, Chapter 4].

§2. Proof of the theorems

Theorem 1. Transformation group (X,G) (with $|G| > 1$) is not distal if and only if $\mathrm{E}(X,G)$ is a Smarandache semigroup.

Proof. If $\mathrm{E}(X,G)$ is a Smarandache semigroup if and only if it is not a group (since $G \subseteq \mathrm{E}(X,G)$), and by [1, Proposition 5.3], equivalently (X,G) is not distal.

Theorem 2. In the transformation semigroup (X,S), if $a \in A \subseteq X$, and S is a Smarandache semigroup, then we have:

1. $\mathrm{E}(X,S)$ is a Smarandache semigroup if and only if (X,S) is not distal;

2. If $\mathrm{F}(a,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $a-$distal;

3. If $A \in \overline{\mathrm{M}}(X)$ and $\mathrm{F}(A,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $A^{\overline{\mathrm{M}}}$distal;

4. If $A \in \overline{\overline{\mathrm{M}}}(X)$ and $\overline{\mathrm{F}}(A,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $A^{\overline{\overline{\mathrm{M}}}}$distal.

Proof.

1. If (X,S) is not distal, then $\mathrm{E}(X)$ is not a group (see [1, Proposition 5.3]), S is a Smarandache semigroup thus there exists a group $A \subseteq S \subseteq \mathrm{E}(X,S)$, such that $|A| > 1$.

2, 3, 4. Use [2, Theorem 18] and a similar method described for (1).

Lemma 3. In the transformation semigroup (X,S) if I a minimal right ideal of $\mathrm{E}(X)$ (resp. βS), then at least one of the following statements occurs:

1) I is a Smarandache semigroup;

2) I is a group (or equivalently $|\mathrm{J}(I)| = 1$);

3) $I = \mathrm{J}(I)$.

Proof. In the transformation semigroup (X, S) if I is a minimal right ideal of $\mathrm{E}(X)$ (resp. βS), then $\{Iv : v \in \mathrm{J}(I)\}$ is a partition of I to some of its isomorphic subgroups, and for each $v \in J(I)$, Iv is a group with identity v (see [1, Proposition 3.5]).

Theorem 4. In the transformation semigroup (X, S), at least one of the following statements occurs:

1) For each I minimal right ideal of $\mathrm{E}(X)$, I is a Smarandache semigroup;

2) For each I minimal right ideal of $\mathrm{E}(X)$, I is a group (or equivalently $|\mathrm{J}(I)| = 1$);

3) For each I minimal right ideal of $\mathrm{E}(X)$, $I = \mathrm{J}(I)$.

Proof. We distinguish the following steps:

Step 1. If I and J are minimal right ideals of $\mathrm{E}(X, S)$ and $v \in \mathrm{J}(J)$, then $L_v|_I : I \to J$ is bijective, since where exists $u \in \mathrm{J}(I)$ with $uv = u$ and $vu = v$ ([1, Proposition 3.6], thus $L_u|J \circ L_v|_I = \mathrm{id}_I$.

Step 2. If I and J are minimal right ideals of $\mathrm{E}(X, S)$ and I is a Smarandache semigroup, then there exists $H \subseteq I$ such that H is a group with more than one element, suppose u be the identity element of H, then Iu is a subgroup of I with identity u and $Iu \neq u$, choose $v \in \mathrm{J}(J)$ such that $uv = u$ and $vu = v$, $vIu(= L_v(Iu))$ is a subgroup of J with identity v, moreover $|vIu| = |Iu| \geq |H| > 1$ and since $L_v|_I : I \to J$ is bijective and $Iu \neq I$, thus $vIu \neq J$, therefore J is a Smarandache semigroup.

Step 3. If there exists minimal right ideal I of $\mathrm{E}(X)$ such that I is a group, it means $|\mathrm{J}(I)| = 1$, since for each minimal right ideal J of $\mathrm{E}(X)$, we have $|\mathrm{J}(J)| = |\mathrm{J}(I)|$, thus J has a unique idemppotent element say v. $\{Jv\} = \{Jw : w \in \mathrm{J}(J)\}$ is a partition of J to some of its isomorphic subgroups, thus $Jv = J$ is a group.

Step 4. If none of minimal right ideals of $\mathrm{E}(X)$ satisfies items 1 and 2 in Lemma 3, then by Lemma 3, all of them will satisfy item 3 in Lemma 3.

Using Lemma 3, will complete the proof.

Example 5. Let $X = \{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\} \cup \{-1, -2\}$ with the induced topology of $\mathbf{R}$; $S = \{\varphi^i\psi^j : i, j \in \mathbf{N} \cup \{0\}\}$ with discrete topology, where $\psi : X \to X$ and $\varphi : X \to X$ such that:

$$x\psi = \begin{cases} -1 & x = -2 \\ -2 & x = -1 \\ 0 & x \in X - \{-1, -2\} \end{cases}, \quad x\varphi = \begin{cases} \frac{1}{n+1} & x = \frac{1}{n}, n \in \mathbf{N} \\ x & x \in \{0, -1, -2\} \end{cases},$$

and $\psi^0 = \varphi^0 = \mathrm{id}_X$, then S is a Smarandache semigroup(under the composition of maps) since $A = \{\psi, \mathrm{id}_X\}$ is a subgroup of S, moreover in the transformation group (X, S), $\mathrm{E}(X) = S$.

Example 6. If $X = [0, 1]$ with induced topology of $\mathbf{R}$, and $G := \{f : [0, 1] \to [0, 1]$ is a homeomorphism$\}$ with discrete topology, then $\mathrm{E}(X, G)$ is a Smarandache semigroup since it contains G and it is not a group, note that if:

$$xp := \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases},$$

References

[1] Ellis, R., Lectures on topological dynamics, W. A. Benjamin, New York, 1969.

[2] Sabbaghan, M., Ayatollah Zadeh Shirazi, F., $A-$minimal sets and related topics in transformation semigroups (I), International Journal of Mathematics and Mathematical Sciences, **25**(2001), No. 10, 637-654.

[3] Vasantha Kandasamy, W. B., Smarandache Semigroups, American Research Press, Rehoboth, 2002.

Scientia Magna
Vol. 3 (2007), No. 3, 5-8

On the F.Smarandache LCM function[1]

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. The main purpose of this paper is using the elementary methods to study the value distribution properties of the function $SL(n)$, and give an interesting asymptotic formula for it.

Keywords F.Smarandache LCM function, value distribution, asymptotic formula.

§1. Introduction

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $SL(17) = 17$, $SL(18) = 9$, $SL(20) = 5$, $\cdots\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained many interesting results, see reference [2], [3], [4] and [5]. For example, Murthy [2] showed that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [3] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

[1]This work partly supported by the N.S.F.C.(10671155).

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

Ge Jian [5] studied the value distribution of $[SL(n) - S(n)]^2$, and proved that

$$\sum_{n\leq x}[SL(n) - S(n)]^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}} \cdot \sum_{i=1}^{k}\frac{c_i}{\ln^i x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are constants. The main purpose of this paper is using the elementary methods to study the value distribution properties of $SL(n)$, and prove an interesting asymptotic formula. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\in N\\ SL(n)\leq x}} 1 = 2^{\frac{x}{\ln x}\left[1+O\left(\frac{\ln\ln x}{\ln x}\right)\right]},$$

where N denotes the set of all positive integers.

From this Theorem we may immediately deduce the following:

Corollary. For any real number $x > 1$, let $\pi(x)$ denotes the number of all primes $p \leq x$, then we have the limit formula

$$\lim_{x\longrightarrow\infty}\left[\sum_{\substack{n\in N\\ SL(n)\leq x}} 1\right]^{\frac{1}{\pi(x)}} = 2.$$

§2. Proof of the theorem

In this section, we shall prove our theorem directly. Let x be any real number with $x > 2$, then for any prime $p \leq x$, there exists one and only one positive integer $\alpha(p)$ such that

$$p^{\alpha(p)} \leq x < p^{\alpha(p)+1}.$$

From the properties of $SL(n)$ and [2] we know that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{2}$$

Let $m = \prod_{p\leq x} p^{\alpha(p)}$. Then for any integer $d|m$, we have $SL(d) \leq x$. For any positive integers u and v with $(u, v) = 1$, if $SL(u) \leq x$, $SL(v) \leq x$, then $SL(uv) \leq x$. On the other hand, for any $SL(n) \leq x$, from the definition of $SL(n)$ we also have $n|m$. So from these and the properties of the Dirichlet divisor function $d(n)$ we have

$$\sum_{\substack{n\in N\\ SL(n)\le x}} 1 = \sum_{d|m} 1 = \prod_{p\le x}(1+\alpha(p)) = e^{\sum_{p\le x}\ln(1+\alpha(p))}. \tag{3}$$

From the definition of $\alpha(p)$ we have $\alpha(p) \le \frac{\ln x}{\ln p} < \alpha(p)+1$ or

$$\alpha(p) = \left[\frac{\ln x}{\ln p}\right]. \tag{4}$$

Therefore, from (4) we may immediately get

$$\begin{aligned}\sum_{p\le x}\ln(1+\alpha(p)) &= \sum_{p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right)\\ &= \sum_{p\le \frac{x}{\ln^2 x}}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) + \sum_{\frac{x}{\ln^2 x}<p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right). \end{aligned}\tag{5}$$

Now we estimate each term in (5). It is clear that

$$\sum_{p\le \frac{x}{\ln^2 x}}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) \ll \sum_{p\le \frac{x}{\ln^2 x}}\ln\ln x \ll \frac{x\ln\ln x}{\ln^3 x}. \tag{6}$$

If $\dfrac{x}{\ln^2 x} < p \le x$, then $1 \le \dfrac{\ln x}{\ln p} < 1 + \dfrac{2\ln\ln x}{\ln x - 2\ln\ln x}$. So from the famous Prime Theorem

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\ln(1+y)\sim y, \quad \texttt{as} \quad y\longrightarrow 0,$$

we have

$$\begin{aligned}\sum_{\frac{x}{\ln^2 x}<p\le x}\ln\left(1+\left[\frac{\ln x}{\ln p}\right]\right) &= \sum_{\frac{x}{\ln^2 x}<p\le x}\ln 2 + O\left(\sum_{\frac{x}{\ln^2 x}<p\le x}\frac{\ln\ln x}{\ln x}\right)\\ &= \ln 2\cdot\frac{x}{\ln x} + O\left(\frac{x\ln\ln x}{\ln^2 x}\right). \end{aligned}\tag{7}$$

Combining (3), (5), (6) and (7) we may immediately obtain

$$\sum_{\substack{n\in N\\ SL(n)\le x}} 1 = 2^{\frac{x}{\ln x}\left[1+O\left(\frac{\ln\ln x}{\ln x}\right)\right]},$$

where N denotes the set of all positive integers. This completes the proof of Theorem.

The corollary follows from

$$\left[\sum_{\substack{n\in N\\ SL(n)\leq x}} 1\right]^{\frac{1}{\pi(x)}} = 2^{1+O\left(\frac{\ln\ln x}{\ln x}\right)} = 2+O\left(\frac{\ln\ln x}{\ln x}\right)$$

as $x \longrightarrow \infty$.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[4] Lv Zhongtian, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No.1, 22-25.

[5] Jian Ge, Mean value of F.Smarandache LCM function, Scientia Magna, **3**(2007), No. 2, 109-112.

[6] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

[7] Wu Qibin, A conjecture involving the F.Smarandache LCM function, Scientia Magna, **3**(2007), No. 1, 26-28.

[8] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[9] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

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Vol. 3 (2007), No. 3, 9-13

An extension of Davenport's theorem

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Abstract In this paper, we give an extension of Davenport's theorem.

Keywords Davenport's theorem, Wronskian.

§1. Introduction

In 1965, Davenport discovered a relation among polynomials as follows.

Theorem 1.1. Let $f,\ g$ be polynomials over $C[x]$ such that $f^2 - g^3 \neq 0$, then we have

$$\frac{1}{2}\deg g \leq \deg(f^2 - g^3) - 1.$$

There were some results which have a connection with this theorem. For example, in [2, Theorem 1.5], Hu-Yang derived an inequality for k functions $f_1^{l_1},\ \cdots,\ f_k^{l_k}$ but they considered the case the functions $f_1^{l_1},\ \cdots,\ f_k^{l_k}$ have no common zeros. In this paper we generalize Davenport's theorem as the following.

Theorem 1.2. Let F an algebraically closed field of characteristic 0. Given polynomials $f_1,\ \cdots,\ f_k (k \geq 2)$ with coefficients in F and positive integers l_j $(1 \leq j \leq k)$ such that $l_1 \leq l_2 \leq \cdots \leq l_k$ and $\sum_{j=1}^{k} l_j \leq kl_1 + k(k-1)$. Suppose that, $f_1^{l_1},\ \cdots,\ f_k^{l_k}$ are linearly independent over F, then we have

$$\left\{1 - \sum_{j=1}^{k} \frac{k-1}{l_j}\right\} \max_{1\leq j\leq k} \deg\left(f_j^{l_j}\right) \leq \deg\left(\sum_{j=1}^{k} f_j^{l_j}\right) - \frac{k(k-1)}{2}. \tag{1}$$

For the case $k = 2,\ l_1 = 2,\ l_2 = 3,\ f_1 = f,\ f_2 = -g$, from Theorem 1.2 we obtain Davenport's Theorem.

§2. Proof of the main theorem

Let f is a rational function, we write f in the form:

$$f = \frac{f_1}{f_2},$$

where $f_1,\ f_2$ are polynomial functions are non-zero and relatively prime on $F[x]$. The degree of f, denoted by $\deg f$, is define to be $\deg f_1 - \deg f_2$.

Let $a \in F$, we write f in the form:

$$f = (x-a)^l \frac{g_1}{g_2},$$

and $g_1(a)g_2(a) \neq 0$, then l is called the order of f at a and is denoted by μ_f^a. We have the following easily proved properties of μ_f^a.

Lemma 2.1. Let $f,\ g$ be two polynomials and $a \in F$, we have

a) $\mu_{f+g}^a \geq \min(\mu_f^a, \mu_g^a)$,

b) $\mu_{fg}^a = \mu_f^a + \mu_g^a$,

c) $\mu_{\frac{f}{g}}^a = \mu_f^a - \mu_g^a$.

Lemma 2.2. Let φ be a the rational function on F and let the derivatives order k of φ satisfy the following $\varphi^{(k)} \not\equiv 0$. Then

$$\mu_{\varphi^{(k)}}^\alpha \geq -\min(\mu_\varphi^\alpha, k) + \mu_\varphi^\alpha.$$

Proof. Let $\varphi(x) = (x-\alpha)^m \frac{f(x)}{g(x)}$, where $f(x),\ g(x)$ are relatively prime and $f(\alpha)g(\alpha) \neq 0$. Then, we have

$$\varphi^{'}(x) = (x-\alpha)^{m-1} \frac{mf(x)g(x) + (x-\alpha)(f'(x)g(x) - f(x)g^{'}(x))}{g^2(x)}.$$

By $\mu_g^\alpha = 0$, we have

$$\mu_{\varphi'}^\alpha \geq m-1.$$

Therefore

$$\mu_{\varphi'}^\alpha \geq -1 + \mu_\varphi^\alpha.$$

From this, we obtain

$$\mu_{\varphi^{(k)}}^\alpha \geq -k + \mu_\varphi^\alpha,$$

and we have

$$\mu_{\varphi^{(k)}}^\alpha \geq -\min(\mu_\varphi^\alpha, k) + \mu_\varphi^\alpha.$$

Let f is a polynomial in $F[x]$ and $f = (x-a_1)^{l_1} \cdots (x-a_k)^{l_k}$, where $a_1,\ \cdots,\ a_k$ are distinct zeros of f and $l_1,\ \cdots,\ l_k$ are integers, for $n \in N$, we set $g = (x-a_1)^{\min(n,\, l_1)} \cdots (x-a_k)^{\min(n,\, l_k)}$, we denote $r_n(f) = \deg g$.

Lemma 2.3. Let $f_0,\ \cdots,\ f_{n+1}$ be $(n+2)$ be polynomials in $F[x]$ have no common zeros and $g_0,\ \cdots,\ g_{n+1}$ be polynomials such that $f_0g_0,\ \cdots,\ f_ng_n$ be linearly independent over F, and

$$f_0g_0 + \cdots + f_ng_n = f_{n+1}g_{n+1}.$$

Then

$$\max_{0\leq j\leq n+1} \deg(f_jg_j) \leq \sum_{j=0}^{n+1} r_n(f_j) + \sum_{j=0}^{n+1} \deg(g_j) - \frac{n(n+1)}{2}.$$

Proof. Since $f_0g_0,\ \cdots,\ f_ng_n$ are linearly independent, then the Wronskian W of $f_0g_0,\ \cdots,\ f_ng_n$ is not vanish. We set

$$P=\frac{W(f_0g_0,\ \cdots,\ f_ng_n)}{f_0g_0\cdots f_ng_n},$$

$$Q=\frac{f_0g_0\cdots f_{n+1}g_{n+1}}{W(f_0g_0,\ \cdots,\ f_ng_n)}.$$

Hence we have

$$f_{n+1}g_{n+1}=PQ.$$

Thus,

$$\deg f_{n+1}g_{n+1}=\deg\frac{f_0\cdots f_{n+1}}{W(f_0g_0,\ \cdots,\ f_ng_n)}+\sum_{j=0}^{n+1}\deg g_j+\deg Q. \tag{2}$$

We first prove

$$\deg\frac{f_0\cdots f_{n+1}}{W(f_0g_0,\ \cdots,\ f_ng_n)}\le n\sum_{i=0}^{n+1}r(f_i).$$

Suppose that, α is a zero of $f_0f_1\cdots f_{n+1}$, by the hypothesis, there exists $\nu,0\le\nu\le n+1$, such that $f_\nu\neq 0$. By the hypothesis, $f_0g_0+\cdots+f_ng_n=f_{n+1}g_{n+1}$, we have

$$\begin{aligned}\mu^\alpha_{\frac{f_0\cdots f_{n+1}}{W(f_0g_0,\cdots,f_ng_n)}} &= \mu^\alpha_{\frac{f_0\cdots f_{\nu-1}f_{\nu+1}\cdots f_{n+1}}{W(f_0g_0,\cdots,f_{\nu-1}g_{\nu-1},f_{\nu+1}g_{\nu+1},\cdots,f_{n+1}g_{n+1})}}\\ &= \sum_{j=0}^{n+1}\mu^\alpha_{f_j}-\mu^\alpha_{W(f_0g_0,\cdots,f_{\nu-1}g_{\nu-1},f_{\nu+1}g_{\nu+1},\cdots,f_{n+1}g_{n+1})},\end{aligned}$$

$W(f_0g_0,\ \cdots,\ f_{\nu-1}g_{\nu-1},\ f_{\nu+1}g_{\nu+1},\cdots,\ f_{n+1}g_{n+1})$ is the sum of following terms

$$\delta f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})'\cdots(f_{\alpha_n}g_{\alpha_n})^{(n)},$$

Where $\alpha_i\in\{0,\ \cdots,\ n+1\}\backslash\{\nu\},\delta=\pm1$. By using Lemma 2.2, we have

$$\begin{aligned}&\mu^\alpha_{f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})'\cdots(f_{\alpha_n}g_{\alpha_n})^{(n)}}\\ \ge&\sum_{f_j(\alpha)=0}\left(\mu^\alpha_{f_{\alpha_j}g_{\alpha_j}}-\min\{n,\mu^\alpha_{f_jg_j}\}\right)\\ =&\sum_{f_j(\alpha)=0}\left(\mu^\alpha_{f_{\alpha_j}}+\mu^\alpha_{g_{\alpha_j}}-\min\{n,\mu^\alpha_{f_{\alpha_j}}+\mu^\alpha_{g_{\alpha_j}}\}\right)\\ \ge&\sum_{f_j(\alpha)=0}\left(\mu^\alpha_{f_{\alpha_j}}+\mu^\alpha_{g_{\alpha_j}}-\min\{n,\mu^\alpha_{f_{\alpha_j}}\}-\min\{n,\mu^\alpha_{g_{\alpha_j}}\}\right)\\ \ge&\sum_{j=0}^{n+1}\mu^\alpha_{f_j}-\sum_{0\le j\le n+1,f_j(a)=0}\min\{n,\mu^\alpha_{f_j}\}.\end{aligned}$$

By Lemma 2.1, we have

$$\mu^\alpha_{W(f_0g_0,\cdots,f_{\nu-1}g_{\nu-1},f_{\nu+1}g_{\nu+1},\cdots,f_{n+1}g_{n+1})}\ge\sum_{j=0}^{n+1}\mu^\alpha_{f_j}-\left(\sum_{0\le j\le n+1,f_j(a)=0}\min\{n,\mu^\alpha_{f_j}\}\right).$$

Hence,

$$\mu^{\alpha}_{\frac{f_0\cdots f_{n+1}}{W(f_0g_0,\cdots,f_ng_n)}} \leq \sum_{0\leq j\leq n+1, f_j(a)=0} \min\{n, \mu^{\alpha}_{f_j}\}.$$

By the definition of degree of a rational function, we have,

$$\deg \frac{f_0\cdots f_{n+1}}{W(f_0g_0,\cdots,f_ng_n)} \leq \sum_{i=0}^{n+1} r_n(f_i). \tag{3}$$

Next, we will prove that

$$\deg P \leq -\frac{n(n+1)}{2}.$$

Here, we have a summa of following terms

$$\delta\frac{(f_{\beta_1}g_{\beta_1})'\cdots(f_{\beta_n}g_{\beta_n})^{(n)}}{f_{\beta_1}g_{\beta_1}\cdots f_{\beta_n}g_{\beta_n}}.$$

We have

$$\begin{aligned}\deg\left(\frac{(f_{\beta_1}g_{\beta_1})'\cdots(f_{\beta_n}g_{\beta_n})^{(n)}}{f_{\beta_1}g_{\beta_1}\cdots f_{\beta_n}g_{\beta_n}}\right) &= \deg\left(\frac{f_{(\beta_1}g_{\beta_1})'}{f_{\beta_1}g_{\beta_1}}\right)+\cdots+\deg\left(\frac{(f_{\beta_n}g_{\beta_n})^{(n)}}{f_{\beta_n}g_{\beta_n}}\right)\\ &= -(1+2+\cdots+n) = -\frac{n(n+1)}{2}.\end{aligned}$$

Therefore,

$$\deg P \leq -\frac{n(n+1)}{2}. \tag{4}$$

From (2), (4), (5), we have

$$\deg f_{n+1} \leq \sum_{j=0}^{n+1} r_n(f_j) + \sum_{j=0}^{n+1}\deg(g_j) - \frac{n(n+1)}{2}.$$

Similar arguments apply to the polynomial $f_0,\ f_1,\ \cdots, f_n$, we have

$$\max_{0\leq j\leq n+1}\deg(f_jg_j) \leq \sum_{j=0}^{n+1} r_n(f_j) + \sum_{j=0}^{n+1}\deg(g_j) - \frac{n(n+1)}{2}.$$

Proof of the theorem 1.2.

We set $f_0 = f_1^{l_1}+\cdots+f_k^{l_k}$ and $h = (f_1,\ \cdots,\ f_k)$ thus there exists polynomials $g_0,\ g_1,\ \cdots,\ g_k$ such that $f_0 = g_0h^{l_1},\ f_1 = g_1h,\cdots,\ f_k = g_kh$ and

$$g_0 = g_1^{l_1} + g_2^{l_2}h^{l_2-l_1} + \cdots + g_k^{l_k}h^{l_k-l_1}.$$

We set for simplicity $\max\limits_{1\leq j\leq k}\deg f_j^{l_j} = f_\alpha^{l_\alpha}$. Lemma 2.3 implies

$$\begin{aligned}\deg g_\alpha^{l_\alpha}h^{l_\alpha-l_1} &\leq r_{k-1}(g_0) + \sum_{j=1}^{k} r_{k-1}\left(g_j^{l_j}\right) + \left(\sum_{j=1}^{k} l_j - kl_1\right)\deg h - \frac{k(k-1)}{2}\\ &\leq \deg g_0 + (k-1)\sum_{j=1}^{k}\deg g_j + \left(\sum_{j=1}^{k} l_j - kl_1\right)\deg h - \frac{k(k-1)}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\deg g_\alpha^{l_\alpha} h^{l_\alpha} &\leq \deg g_0 h^{l_1} + (k-1)\sum_{j=1}^{k} \deg g_j h + \deg h^{l_1} - \deg h^{l_1} - k(k-1)\deg h \\ &\quad + \left(\sum_{j=1}^{k} l_j - kl_1\right)\deg h - \frac{k(k-1)}{2} \\ &\leq \deg f_0 + (k-1)\sum_{j=1}^{k}\deg f_j + \left(\sum_{j=1}^{k} l_j - kl_1 - k(k-1)\right)\deg h - \frac{k(k-1)}{2} \\ &= \deg f_0 + \sum_{j=1}^{k}\frac{k-1}{l_j}\deg f_j^{l_j} + \left(\sum_{j=1}^{k} l_j - k(k-1) - kl_1\right)\deg h - \frac{k(k-1)}{2}.\end{aligned}$$

Thus,

$$\left\{1 - \sum_{j=1}^{k}\frac{k-1}{l_j}\right\}\deg f_\alpha^{l_\alpha} \leq \deg f_0 + \left(\sum_{j=1}^{k} l_j - kl_1 - k(k-1)\right)\deg h - \frac{k(k-1)}{2}.$$

By the hypothesis, we have $\sum_{j=1}^{k} l_j - kl_1 - k(k-1) \leq 0$. From this, it follows that

$$\left\{1 - \sum_{j=1}^{k}\frac{k-1}{l_j}\right\}\max_{1\leq j\leq k}\deg\left(f_j^{l_j}\right) \leq \deg\left(\sum_{j=1}^{k} f_j^{l_j}\right) - \frac{k(k-1)}{2}.$$

Theorem 1.2 is proved.

References

[1] Daventport, H., On $f^3(t) - g^2(t)$, Norske Vid. Selsk. Forh. (Trondheim), **38**(1965), 86-87.

[2] P.C. Hu and C.C.Yang, Notes on a generalized *abc*-conjecture over function fields, Ann. Math. Blaise Pascal, **8**(2001), No. 1, 61-71.

[3] Leonid N. Vaserstein and Ethel R. Wheland, Vanishing polynomial sums, Communications in Algebra, **31**(2003), No. 2, 751-772.

[4] Mason, R. C., Equations over function fields, Lecture Notes in Math., Springer, **1068** (1984), 149-157.

Scientia Magna
Vol. 3 (2007), No. 3, 14-17

On the hybrid power mean of the character sums and the general Kloosterman sums[1]

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Abstract The main purpose of this paper is using the estimate for character sums and the analytic method to study the hybrid power mean of the character sums and Kloosterman sums, and give an interesting mean value formula for it.

Keywords Character sums, Kloosterman sums, hybrid power mean formula.

§1. Introduction

Let $q > 2$ be an integer, χ denotes a Dirichlet character modulo q. For any integers m and n, we define the general Kloosterman sums $S(m, n, \chi; q)$ as follows:

$$S(m, n, \chi; q) = \sum_{a=1}^{q}{}' \chi(a) e\left(\frac{ma + n\overline{a}}{q}\right),$$

where $\sum'$ denotes the summation over all a such that $(a, q) = 1$, $a\overline{a} \equiv 1 \bmod q$, χ denotes a Dirichlet character mod q and $e(y) = e^{2\pi iy}$.

This summation is very important, because it is a generalization of the classical Kloosterman sums $S(m, n, \chi_0; q) = S(m, n; q)$, where χ_0 is the principal character mod q. The various properties of $S(m, n; q)$ were investigated by many authors. Perhaps the most famous property of $S(m, n; q)$ is the estimate (see [1] and [2]):

$$|S(m, n; q)| \leq d(q) q^{\frac{1}{2}} (m, n, q)^{\frac{1}{2}}, \tag{1}$$

where $d(q)$ is the divisor function, (m, n, q) denotes the greatest common divisor of m, n and q. If q be a prime p, then S.Chowla [3]and A.V.Malyshev [4] also proved a similar result for $S(m, n, \chi; p)$. On the other hand, H.D.Kloosterman [6] studied the fourth power mean of $S(a, 1; p)$, and proved the identity

$$\sum_{a=1}^{p-1} S^4(a, 1; p) = 2p^3 - 3p^2 - p - 1.$$

This identity can also be found in H.Iwaniec [7]. H.Salié [8] and H.Davenport (independently) obtained the estimate

$$\sum_{a=1}^{p-1} S^6(a, 1; p) \ll p^4.$$

[1]This work is supported by the N.S.F. (10671155) of P.R.China.

The main purpose of this paper is using the estimate for character sums and the analytic method to study the asymptotic properties of the hybrid power mean of the character sums and the Kloosterman sums:

$$\sum_{\chi \mod q}\left|\sum_{n\le N}\chi(n)\right|^{2k}\cdot\left|\sum_{a=1}^{q}{}'\chi(a)e\left(\frac{ma+n\overline{a}}{q}\right)\right|^{2h}, \tag{2}$$

and give an interesting mean value formula for it. About the estimate for character sums, the classical result due to Pólya and Vinogradov (See Theorem 8.21 of [9]) is that the estimates

$$\sum_{n=N+1}^{N+H}\chi(n)\ll q^{\frac{1}{2}}\log q$$

hold for all non-principal character χ modulo q. H.L.Montgomery and R.C.Vaughan [10] proved that for any positive integer k,

$$\sum_{\substack{\chi \mod q\\ \chi\neq\chi_0}}\max_{1\le y\le q}\left|\sum_{n\le y}\chi(n)\right|^{2k}\ll\phi(q)\cdot q^k.$$

But for the asymptotic properties of (2), it seems that none had studied it yet, at least we have not seen such a paper before. However, the problem is very interesting and important, because it can help us to obtain some important information about the upper bound estimate for character sums and the general Kloosterman sums.

In this paper, we shall study the asymptotic properties of (2), and use the analytic method to prove the following two conclusions:

Theorem. Let $q>2$ be an integer, N be a fixed real number with $1<N\le\sqrt{q}$. Then for any integers m and n with $(mn,q)=1$, we have the asymptotic formula

$$\sum_{\chi \mod q}\left|\sum_{n\le N}\chi(n)\right|^{2}\cdot|S(m,n,\chi;q)|^2=\phi^2(q)\cdot N+O\left(q^{\frac{3}{2}}\cdot N^2\cdot d^2(q)\right),$$

where $\phi(q)$ is the Euler function, $d(q)$ denotes the Dirichlet divisor function.

Taking $q=p$ be a prime, then from our Theorem we may immediately deduce the following:

Corollary. Let $p>2$ be a prime, m and n are two integers with $(mn,p)=1$. Then for any real number N with $p^{\epsilon}<N<p^{\frac{1}{2}-\epsilon}$, we have the asymptotic formula

$$\sum_{\chi \mod p}\left|\sum_{n\le N}\chi(n)\right|^{2}\cdot|S(m,n,\chi;p)|^2\sim N\cdot p^2,\quad p\to+\infty,$$

where ϵ be any fixed positive number.

Using our method we can also give a similar asymptotic formula for the hybrid power mean of

$$\sum_{\chi \mod q}\left|\sum_{n\le N}\chi(n)\right|^{4}\cdot\left|\sum_{a=1}^{q}\chi(a)e\left(\frac{ma+n\overline{a}}{q}\right)\right|^{2},$$

but in this case, the constant is very complicate. So we have not give the conclusion in this paper.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. In fact from the properties of character sums we have

$$\begin{aligned} |\ S(m,n,\chi;q)\ |^2 &= \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \chi(a\bar{b}) e\left(\frac{m(a-b)+n(\overline{a}-\overline{b})}{q}\right) \\ &= \sum_{a=1}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb(a-1)+n\bar{b}(\overline{a}-1)}{q}\right) \\ &= \phi(q) + \sum_{a=2}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb(a-1)+n\bar{b}(\overline{a}-1)}{q}\right). \end{aligned} \tag{3}$$

Now from (3) and the orthogonality relation for character sums we have

$$\begin{aligned} &\sum_{\chi \bmod q} \left|\sum_{n\le N} \chi(n)\right|^2 \cdot |\ S(m,n,\chi;q)\ |^2 \\ =\ &\sum_{\chi \bmod q} \sum_{u\le N} \sum_{v\le N} \chi(u\overline{v}) \left[\phi(q) + \sum_{a=2}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb(a-1)+n\bar{b}(\overline{a}-1)}{q}\right)\right] \\ =\ &\phi(q) \cdot \sum_{\chi \bmod q} \sum_{u\le N} \sum_{v\le N} \chi(u\overline{v}) \\ &+ \sum_{u\le N} \sum_{v\le N} \sum_{a=2}^{q}{}' \sum_{b=1}^{q}{}' \sum_{\chi \bmod q} \chi(u\overline{v}a) e\left(\frac{mb(a-1)+n\bar{b}(\overline{a}-1)}{q}\right) \\ =\ &\phi^2(q)\cdot N + \phi(q) \sum_{\substack{u\le N \\ u\ne v}}{}' \sum_{v\le N}{}' \sum_{b=1}^{q}{}' e\left(\frac{mb(v\overline{u}-1)+n\bar{b}(u\overline{v}-1)}{q}\right). \end{aligned} \tag{4}$$

Applying the estimate (1) in (4) we have

$$\begin{aligned} &\sum_{\chi \bmod q} \left|\sum_{n\le N} \chi(n)\right|^2 \cdot |\ S(m,n,\chi;q)\ |^2 \\ =\ &\phi^2(q)\cdot N + O\left(\phi(q) \sum_{\substack{u\le N \\ u\ne v}}{}' \sum_{v\le N}{}' \sqrt{q}\ (u\overline{v}-1,q)^{\frac{1}{2}} d(q)\right) \\ =\ &\phi^2(q)\cdot N + O\left(q^{\frac{3}{2}} \cdot d(q) \cdot \sum_{\substack{u\le N \\ u\ne v}}{}' \sum_{v\le N}{}' \sum_{\substack{d\mid q \\ d\mid m-n}} d^{\frac{1}{2}}\right) \\ =\ &\phi^2(q)\cdot N + O\left(q^{\frac{3}{2}} \cdot N^2 d^2(q)\right). \end{aligned}$$

This completes the proof of the theorem.

References

[1] T.Estermann, On Kloostermann's sum, Mathematica, **8**(1961), 83-86.

[2] J. H. H. Chalk and R. A. Smith, On Bombieri's estimate for exponential sums, Acta Arith., **18**(1971), 191-212.

[3] S.Chowla, On Kloosterman's sum, Norkse Vid., Selbsk.Fak.Frondheim, **40**(1967), 70-72.

[4] A.V.Malyshev, A generalization of Kloosterman sums and their estimates, Vestnik Leningrad Univ., **15**(1960), 59-75.

[5] A.Weil, Uber die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann., **168**(1967), 149-156.

[6] H.D.Kloosterman, On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$, Acta Math., **49**(1926), 407-464.

[7] H.Iwaniec, Topics in Classical Automorphic Forms, Graduate Studies in mathematics, **17**(1997), 61-63.

[8] H.Salié, Uber die Kloostermanschen Summen $S(u, v; q)$, Math.Z., **34**(1931), 91-109.

[9] T.M.Apostol, Introduction to analytic number theory, Springer-Verlag, New York, 1976.

[10] H. L. Montgomery and R. C. Vaughan, Mean values of character sums, Can. J. Math., **31**(1979), 476-487.

Scientia Magna
Vol. 3 (2007), No. 3, 18-25

An equation involving the square sum of natural numbers and Smarandache primitive function

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Abstract For any positive integer n, let $S_p(n)$ denotes the Smarandache primitive function. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$, and give all positive integer solutions for this equation.

Keywords Square sum, Smarandache primitive function, equation, solutions.

§1. Introduction and Results

Let p be a prime, n be any positive integer. The Smarandache primitive function $S_p(n)$ is defined as the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_2(1) = 2, S_2(2) = S_2(3) = 4, S_2(4) = 6, \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu [2] have studied it, and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n, we have

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

Li Jie [3] studied the solvability of the equation $S_p(1) + S_p(2) + \cdots + S_p(n) = S_p\left(\frac{n(n+1)}{2}\right)$, and gave all its positive integer solutions. But it seems that no one knows the relationship between the square sum of natural numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right),$$

and give all positive integer solutions for it.That is, we will prove the following:

Theorem. Let p be a prime, n be any positive integer. Then the equation

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right) \tag{1}$$

has finite solutions.

(i) If $p = 2, 3$ or 5, then the positive integer solutions of the equation (1) are $n = 1, 2$;

(ii) If $p = 7$, then the positive integer solutions of the equation (1) are $n = 1, 2, 3, 4, 5$;

(iii) If $p \geq 11$, then the positive integer solutions of the equation (1) are $n = 1,\ 2,\ \cdots,\ n_p$, where $n_p \geq 1$ is a positive integer, and $n_p = \left[\sqrt[3]{\frac{3p}{2} + \sqrt{\frac{9p^2}{4} - \frac{1}{1728}}} + \sqrt[3]{\frac{3p}{2} - \sqrt{\frac{9p^2}{4} - \frac{1}{1728}}} - \frac{1}{2}\right]$, $[x]$ denotes the biggest integer $\leq x$.

§2. Several lemmas

To complete the proof of the theorem, we need the following several simple lemmas.

Lemma 1. Let p be a prime, n be any positive integer, $S_p(n)$denote the Smarandache primitive function, then we have

$$S_p(k) \begin{cases} = pk, & \text{if } \ k \leq p, \\ < pk, & \text{if } \ k > p. \end{cases}$$

Proof. (See reference [4]).

Lemma 2. Let p be a prime, n be any positive integer, if n and p satisfying $p^\alpha \parallel n!$, then

$$\alpha = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

Proof. (See reference [5]).

Lemma 3. Let p be a prime, n be any positive integer. If $n > [\sqrt{p}]$, then there must exist a positive integer m_k with $1 \leq m_k \leq k^2 (k = 1, 2, \cdots, n)$ such that

$$S_p(1^2) = m_1 p, S_p(2^2) = m_2 p, \cdots, S_p(n^2) = m_n p,$$

and

$$k^2 \leq \sum_{i=1}^{\infty} \left[\frac{m_k p}{p^i}\right].$$

Proof. From the definition of $S_p(n)$, Lemma 1 and Lemma 2, we can easily get the conclusions of Lemma 3.

§3. Proof of the theorem

In this section, we will complete the proof of Theorem. We discuss the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$ in the following cases:

(I) If $p = 2$, then the equation (1) is $S_2(1^2)+S_2(2^2)+\cdots+S_2(n^2) = S_2\left(\frac{n(n+1)(2n+1)}{6}\right)$.

(a) If $n = 1$, $S_2(1^2) = 2 = S_2\left(\frac{1\times(1+1)\times(2+1)}{6}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_2(1^2) + S_2(2^2) = 2 + 3 \times 2 = 8 = S_2\left(\frac{2\times(2+1)\times(2\times 2+1)}{6}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_2(1^2) + S_2(2^2) + S_2(2^2) = 2 + 3 \times 2 + 6 \times 2 = 20$, but $S_2\left(\frac{3\times(3+1)\times(2\times 3+1)}{6}\right) = S_2(14) = 16$, so $n = 3$ is not a solution of the equation (1).

(d) If $n > 3$, then from Lemma 3 we know that there must exist a positive integer m_k with $1 \le m_k \le k^2 (k = 1, 2, \cdots, n)$ such that

$$S_2(1^2) = 2m_1, S_2(2^2) = 2m_2, \ldots, S_2(n^2) = 2m_n.$$

So we have $S_2(1^2) + S_2(2^2) + \cdots + S_2(n^2) = 2(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1$, $m_2 = 3$, $m_3 = 6$, then from Lemma 3, we have

$$\begin{aligned}
& \sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n)-1}{2^i}\right] \\
= & \sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right] \\
= & m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right] \\
= & m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{2^i}\right] \\
\ge & \left(m_1+\sum_{i=1}^{\infty}\left[\frac{m_1}{2^i}\right]\right)+\left(m_2+\sum_{i=1}^{\infty}\left[\frac{m_2-1}{2^i}\right]\right)+\left((m_3-1)+\sum_{i=1}^{\infty}\left[\frac{m_3}{2^i}\right]\right) \\
& +\left(m_4+\sum_{i=1}^{\infty}\left[\frac{m_4}{2^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{2^i}\right]\right) \\
\ge & 1^2+2^2+3^2+\sum_{i=1}^{\infty}\left[\frac{2m_4}{2^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{2m_n}{2^i}\right] \\
\ge & 1^2+2^2+3^2+4^2+\cdots+n^2 \\
= & \frac{n(n+1)(2n+1)}{6},
\end{aligned}$$

then from Lemma 2, we can get

$$2^{\frac{n(n+1)(2n+1)}{6}} \mid (2(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
S_2\left(\frac{n(n+1)(2n+1)}{6}\right) & \le 2(m_1+m_2+\cdots+m_n)-1 \\
& < 2(m_1+m_2+\cdots+m_n) \\
& = S_2(1^2)+S_2(2^2)+\cdots+S_2(n^2).
\end{aligned}$$

So there is no solutions for the equation (1) in this case.

Hence, if $p = 2$, the equation (1) has only two solutions, they are $n = 1,\ 2$.

If $p = 3$ or 5 using the same method we can easily get if and only if $n = 1,\ 2$ are all solutions of the equation (1).

(II) If $p = 7$, then the equation (1) is

$$S_7(1^2) + S_7(2^2) + \cdots + S_7(n^2) = S_7\left(\frac{n(n+1)(2n+1)}{6}\right).$$

(a) If $n = 1$, $S_7(1^2) = 7 = S_7\left(\frac{1\times(1+1)\times(2+1)}{6}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_7(1^2) + S_7(2^2) = 7 + 4\times 7 = 35 = S_7\left(\frac{2\times(2+1)\times(2\times 2+1)}{6}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_7(1^2) + S_7(2^2) + S_7(3^2) = 35 + 8\times 7 = 91 = S_7\left(\frac{3\times(3+1)\times(2\times 3+1)}{6}\right)$, so $n = 3$ is a solution of the equation (1).

(d) If $n = 4$, $S_7(1^2) + S_7(2^2) + S_7(3^2) + S_7(4^2) = 91 + 14\times 7 = 189 = S_7\left(\frac{4\times(4+1)\times(2\times 4+1)}{6}\right)$, so $n = 4$ is a solution of the equation (1).

(e) If $n = 5$, $S_7(1^2) + S_7(2^2) + S_7(3^2) + S_7(4^2) + S_7(5^2) = 189 + 22\times 7 = 343 = S_7\left(\frac{5\times(5+1)\times(2\times 5+1)}{6}\right)$, so $n = 5$ is a solution of the equation (1).

(f) If $n = 6$, $S_7(1^2) + S_7(2^2) + S_7(3^2) + S_7(4^2) + S_7(5^2) + S_7(6^2) = 343 + 32\times 7 = 567$, but $S_7\left(\frac{6\times(6+1)\times(2\times 6+1)}{6}\right) = S_7(91) = 79\times 7 = 553 < 567$, so $n = 6$ is not a solution of the equation (1).

(g) If $n = 7$, $S_7(1^2)+S_7(2^2)+S_7(3^2)+S_7(4^2)+S_7(5^2)+S_7(6^2)S_7(7^2) = 567+43\times 7 = 868$, but $S_7\left(\frac{7\times(7+1)\times(2\times 7+1)}{6}\right) = S_7(140) = 122\times 7 = 854 < 868$, so $n = 7$ is not a solution of the equation (1).

(h) If $n \geq 8$, from Lemma 3 we know there must be a positive integers m_k with $1 \leq m_k \leq k^2 (k = 1,\ 2,\ \cdots,\ n)$ such that

$$S_7(1^2) = 7m_1, S_7(2^2) = 7m_2, \cdots, S_7(n^2) = 7m_n.$$

Then we have $S_7(1^2) + S_7(2^2) + \cdots + S_7(n^2) = 7(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1,\ m_2 = 4,\ m_3 = 8,\ m_4 = 14,\ m_5 = 22,\ m_6 =$

32, $m_7 = 43$, $m_8 = 56$, from Lemma 3, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n)-1}{7^i}\right]\\
=&\sum_{i=1}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n-1)+6}{7^i}\right]\\
=&\ m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n-1)+6}{7^i}\right]\\
=&\ m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{7^i}\right]\\
\geq&\ \left(m_1+\sum_{i=1}^{\infty}\left[\frac{m_1}{7^i}\right]\right)+\left(m_2+\sum_{i=1}^{\infty}\left[\frac{m_2}{7^i}\right]\right)+\left(m_3+\sum_{i=1}^{\infty}\left[\frac{m_3-1}{7^i}\right]\right)+\cdots\\
&+\left(m_7+\sum_{i=1}^{\infty}\left[\frac{m_7}{7^i}\right]\right)+\left((m_8-1)+\sum_{i=1}^{\infty}\left[\frac{m_8}{7^i}\right]\right)+\left(m_9+\sum_{i=1}^{\infty}\left[\frac{m_9}{7^i}\right]\right)+\cdots\\
&+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{7^i}\right]\right)\\
\geq&\ 1^2+2^2+\cdots+8^2+\sum_{i=1}^{\infty}\left[\frac{7m_9}{7^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{7m_n}{7^i}\right]\\
\geq&\ 1^2+2^2+\cdots+n^2\\
=&\ \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

From Lemma 2, we may immediately get

$$7^{\frac{n(n+1)(2n+1)}{6}} \mid (7(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
S_7\left(\frac{n(n+1)(2n+1)}{6}\right) &\leq 7(m_1+m_2+\cdots+m_n)-1\\
&< 7(m_1+m_2+\cdots+m_n)\\
&= S_7(1^2)+S_7(2^2)+\cdots+S_7(n^2).
\end{aligned}$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p = 7$, the equation (1) has only five solutions, they are $n = 1,\ 2,\ 3,\ 4,\ 5$.

(III) If $p \geq 11$ we will discuss the problem in the following cases:

(a) If $\dfrac{n(n+1)(2n+1)}{6} \leq p$, solving this equation we can get $1 \leq n \leq n_p$, and

$$n_p = \left[\sqrt[3]{\frac{3p}{2}+\sqrt{\frac{9p^2}{4}-\frac{1}{1728}}}+\sqrt[3]{\frac{3p}{2}-\sqrt{\frac{9p^2}{4}-\frac{1}{1728}}}-\frac{1}{2}\right],$$

where $[x]$ denotes the biggest integer $\leq x$, then we have

$$S_p\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n(n+1)(2n+1)}{6}p.$$

Noting that $n_p \leq [\sqrt{p}] < p$, so if $1 \leq n \leq n_p$, then $n^2 \leq p$, from Lemma 1 we have

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = 1^2p + 2^2p + \cdots + n^2p = \frac{n(n+1)(2n+1)}{6}p.$$

Combining above two formulas, we may easily get $n = 1,\ 2,\ \cdots,\ n_p$ are the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$.

(*b*) If $n_p < n \leq [\sqrt{p}]$, that is $\frac{n(n+1)(2n+1)}{6} > p$ and $n^2 \leq p$, so from Lemma 1 we have

$$S_p\left(\frac{n(n+1)(2n+1)}{6}\right) < \frac{n(n+1)(2n+1)}{6}p,$$

but

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = 1^2p + 2^2p + \cdots + n^2p = \frac{n(n+1)(2n+1)}{6}p.$$

Hence the equation (1) has no solution in this case.

(*c*) Let $[\sqrt{p}] = t$, if $n = [\sqrt{p}] + 1 = t + 1$, that is $n^2 > p$, $t \geq 3$. Then

$$\begin{aligned} S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) &= S_p(1^2) + S_p(2^2) + \cdots + S_p(t^2) + S_p((t+1)^2) \\ &= 1^2p + 2^2p + \cdots + t^2p + (t^2 + 2t)p \\ &= \frac{2t^3 + 9t^2 + 13t}{6}p. \end{aligned}$$

If $p = 11$, that is $n = t + 1 = 4$, so we have $S_{11}(1^2) + S_{11}(2^2) + S_{11}(3^2) + S_{11}(4^2) = \frac{2 \times 3^3 + 9 \times 3^2 + 13 \times 3}{6} \times 11 = 319$, but $S_{11}(\frac{n(n+1)(2n+1)}{6}) = S_{11}(30) = 308 < 319$. So there is no solution for the equation (1) in this case.

If $p = 13$, that is $n = t + 1 = 4$, then we have $S_{13}(1^2) + S_{13}(2^2) + S_{13}(3^2) + S_{13}(4^2) = \frac{2 \times 3^3 + 9 \times 3^2 + 13 \times 3}{6} \times 13 = 377$, but $S_{13}(\frac{n(n+1)(2n+1)}{6}) = S_{13}(30) = 364 < 377$. So there is no solution for the equation (1) in this case.

If $p \geq 17$, that is $t = [\sqrt{p}] \geq 4$, and $n = t + 1 \geq 5$, notice that

$$\begin{aligned} & \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}p}{p^i} \right] \\ = & \frac{2t^3 + 9t^2 + 13t - 6}{6} + \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}}{p^i} \right] \\ \geq & \frac{2t^3 + 9t^2 + 13t}{6} - 1 + \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}}{p^i} \right] \\ \geq & \frac{2t^3 + 9t^2 + 13t}{6} \\ = & \frac{(t+1)(t+2)(2t+3)}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} S_p\left(\frac{n(n+1)(2n+1)}{6}\right) &= S_p\left(\frac{(t+1)(t+2)(2t+3)}{6}\right) \\ &\leq \frac{2t^3+9t^2+13t-6}{6}p \\ &< \frac{2t^3+9t^2+13t}{6}p \\ &= S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2). \end{aligned}$$

So $n=[\sqrt{p}]+1$ is not a solutions of the equation (1).

If $n \geq [\sqrt{p}]+2 = t+2$, that is $n^2 > p$, $t \geq 3$. Then from Lemma 3, we know that there must exist a positive integers m_k with $1 \leq m_k \leq k^2 (k=1,2,\cdots,n)$ such that

$$S_p(1^2)=m_1p, S_p(2^2)=m_2p, \cdots, S_p(n^2)=m_np,$$

then we have

$$S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2)=(m_1+m_2+\cdots+m_n)p.$$

On the other hand, notice that $m_1=1^2,\ m_2=2^2,\ \cdots,\ m_t=t^2$, from Lemma 3, we have

$$\begin{aligned} &\sum_{i=1}^{\infty}\left[\frac{(m_1+m_2+\cdots+m_n)p-1}{p^i}\right] \\ =\ &\sum_{i=1}^{\infty}\left[\frac{p(m_1+m_2+\cdots+m_n-1)+p-1}{p^i}\right] \\ =\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{p(m_1+m_2+\cdots+m_n-1)+p-1}{p^i}\right] \\ \geq\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{\frac{t(t+1)(2t+1)}{6}p+p-1}{p^i}\right]+ \\ &\sum_{i=1}^{\infty}\left[\frac{m_{t+1}+m_{t+2}+\cdots+m_n-1}{p^i}\right] \\ \geq\ &m_1+m_2+\cdots+m_n+\sum_{i=1}^{\infty}\left[\frac{m_{t+1}+m_{t+2}+\cdots+m_n-1}{p^i}\right] \\ \geq\ &m_1+m_2+\cdots+m_t+\left(m_{t+1}+\sum_{i=1}^{\infty}\left[\frac{m_1}{p^i}\right]\right)+\left(m_{t+2}+\sum_{i=1}^{\infty}\left[\frac{m_{t+2}}{p^i}\right]\right)+\ldots \\ &+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{p^i}\right]\right) \\ \geq\ &\sum_{i=1}^{\infty}\left[\frac{m_1p}{p^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_2p}{p^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{m_np}{p^i}\right] \\ \geq\ &1^2+2^2+\cdots+n^2 \\ =\ &\frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Then from Lemma 2, we can get

$$p^{\frac{n(n+1)(2n+1)}{6}} \mid ((m_1+m_2+\cdots+m_n)p-1)!.$$

Therefore,

$$\begin{aligned} S_p\left(\frac{n(n+1)(2n+1)}{6}\right) &\leq (m_1+m_2+\cdots+m_n)p-1 \\ &< (m_1+m_2+\cdots+m_n)p \\ &= S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2). \end{aligned}$$

From the above, we can deduce that if $p \geq 11$ and $n \geq [\sqrt{p}]+2$, then the equation (1) has no solution.

Now the theorem follows from (I), (II) and (III).

References

[1] Smarandache F, Only problems, not Solutions, Chicago, Xiquan Publ., House, 1993.

[2] Zhang W. P. and Liu D. S., Primitive number of power p and its asymptotic property, Smarandache Notions Journal, No. 13, 2002, 173-175.

[3] Li J., An equation involving the Smarandache primitive function., Acta Mathematica Sinica(Chinese Series), No. 13, 2007, 333-336.

[4] Mark F. and Patrick M., Bounding the Smarandache function, Smarandache Notions Journal, No. 13, 2002, 37-42.

[5] Pan Chengdong and Pan Chengbiao, Elementary number Theory, Beijing, Beijing University Press, 2003.

Scientia Magna
Vol. 3 (2007), No. 3, 26-29

An identity involving the function $e_p(n)$

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Abstract The main purpose of this paper is to study the relationship between the Riemann zeta-function and an infinite series involving the Smarandache function $e_p(n)$ by using the elementary method, and give an interesting identity.

Keywords Riemann zeta-function, infinite series, identity.

§1. Introduction and Results

Let p be any fixed prime, n be any positive integer, $e_p(n)$ denotes the largest exponent of power p in n. That is, $e_p(n) = m$, if $p^m \mid n$ and $p^{m+1} \nmid n$. In problem 68 of [1], Professor F.Smarandache asked us to study the properties of the sequence $\{e_p(n)\}$. About the elementary properties of this function, many scholars have studied it (see reference [2]-[7]), and got some useful results. For examples, Liu Yanni [2] studied the mean value properties of $e_p(b_k(n))$, where $b_k(n)$ denotes the k-th free part of n, and obtained an interesting mean value formula for it. That is, let p be a prime, k be any fixed positive integer, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} e_p(b_k(n)) = \left(\frac{p^k - p}{(p^k - p)(p-1)} - \frac{k-1}{p^k - 1}\right) x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number.

Wang Xiaoying [3] studied the mean value properties of $\sum_{n\leq x}((n+1)^m - n^m)e_p(n)$, and proved the following conclusion:

Let p be a prime, $m \geq 1$ be any integer, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}((n+1)^m - n^m)e_p(n) = \frac{1}{p-1}\frac{m}{m+1}x + O\left(x^{1-\frac{1}{m}}\right).$$

Gao Nan [4] and [5] also studied the mean value properties of the sequences $p^{e_q(n)}$ and $p^{e_q(b(n))}$, got two interesting asymptotic formulas:

$$\sum_{n\leq x} p^{e_q(n)} = \begin{cases} \frac{q-1}{q-p}x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if } q > p; \\ \frac{p-1}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma - 1) + \frac{p+1}{2p}\right)x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if } q = p. \end{cases}$$

and

$$\sum_{n\leq x} p^{e_q(b(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where ϵ is any fixed positive number, γ is the Euler constant.

Lv Chuan [6] used elementary and analytic methods to study the asymptotic properties of $\sum_{n\leq x} e_p(n)\varphi(n)$ and obtain an interesting asymptotic formula:

$$\sum_{n\leq x} e_p(n)\varphi(n) = \frac{3p}{(p+1)\pi^2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Ren Ganglian [7] studied the properties of the sequence $e_p(n)$ and give some sharper asymptotic formulas for the mean value $\sum_{n\leq x} e_p^k(n)$.

Especially in [8], Xu Zhefeng studied the elementary properties of the primitive numbers of power p, and got an useful result. That is, for any prime p and complex number s, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s-1}.$$

In this paper, we shall use the elementary methods to study the relationship between the Riemann zeta-function and an infinite series involving $e_p(n)$, and obtain an interesting identity. That is, we shall prove the following conclusion:

Theorem. For any prime p and complex number s with $Re(s) > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s-1},$$

where $\zeta(s)$ is the Riemann zeta-function.

From this theorem, we can see that $\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)}$ denote the same Dirichlet series. Of course, we can also obtain some relationship between $\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)}$, that is, we have the following conclusion:

Corollary. For any prime p, we have

$$e_p(m) = \sum_{\substack{n\in N \\ S_P(n)=m}} 1.$$

§2. Proof of the theorem

In this section, we shall use elementary methods to complete the proof of the theorem.

Let $m = e_p(n)$, if $p^m \mid\mid n$, then we can write $n = p^m n_1$, where $(n_1, p) = 1$. Noting that, $e_p(n)$ is the largest exponent of power p, so we have

$$\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s} = \sum_{m=1}^{\infty} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{m}{(p^m n_1)^s} = \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \sum_{\substack{n_1=1 \\ p \nmid n_1}}^{\infty} \frac{1}{n_1^s} = \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\sum_{n_1=1}^{\infty} \frac{1}{n_1^s} - \sum_{\substack{n_1=1 \\ p \mid n_1}}^{\infty} \frac{1}{n_1^s} \right), \quad (1)$$

let $n_1 = pn_2$, then

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\sum_{n_1=1}^{\infty} \frac{1}{n_1^s} - \sum_{\substack{n_1=1 \\ p \mid n_1}}^{\infty} \frac{1}{n_1^s} \right) &= \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\zeta(s) - \sum_{n_2=1}^{\infty} \frac{1}{p^s n_2^s} \right) \\ &= \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\zeta(s) - \zeta(s) \frac{1}{p^s} \right) \\ &= \zeta(s) \left(1 - \frac{1}{p^s} \right) \sum_{m=1}^{\infty} \frac{m}{p^{ms}}. \end{aligned}$$

Since

$$\sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{m+1}{p^{(m+1)s}},$$

$$\frac{1}{p^s} \cdot \sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \sum_{m=1}^{\infty} \frac{m}{p^{(m+1)s}},$$

then

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{m}{p^{ms}} - \frac{1}{p^s} \cdot \sum_{m=1}^{\infty} \frac{m}{p^{ms}} &= \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{m+1}{p^{(m+1)s}} - \sum_{m=1}^{\infty} \frac{m}{p^{(m+1)s}} \\ &= \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{1}{p^{(m+1)s}} = \sum_{m=1}^{\infty} \frac{1}{p^{ms}}. \end{aligned}$$

That is,

$$\left(1 - \frac{1}{p^s} \right) \sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \sum_{m=1}^{\infty} \frac{1}{p^{ms}} = \frac{1}{p^s} \frac{1}{1 - \frac{1}{p^s}},$$

so

$$\sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \frac{1}{p^s \left(1 - \frac{1}{p^s} \right)^2}. \quad (2)$$

Now, combining (1) and (2), we have the following identity

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{e_p(n)}{n^s} &= \sum_{m=1}^{\infty}\frac{m}{p^{ms}}\left(\sum_{n_1=1}^{\infty}\frac{1}{n_1^s}-\sum_{\substack{n_1=1\\ p|n_1}}^{\infty}\frac{1}{n_1^s}\right)\\ &= \zeta(s)\left(1-\frac{1}{p^s}\right)\sum_{m=1}^{\infty}\frac{m}{p^{ms}}\\ &= \zeta(s)\left(1-\frac{1}{p^s}\right)\frac{1}{p^s\left(1-\frac{1}{p^s}\right)^2}=\frac{\zeta(s)}{p^s-1}.\end{aligned}$$

This completes the proof of Theorem.

Then, noting the definition and properties of $S_p(n)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)}=\sum_{m=1}^{\infty}\frac{1}{(pm)^s}\sum_{\substack{n\in N\\ S_P(n)=mp}}1, \tag{3}$$

and we also have

$$\sum_{n=1}^{\infty}\frac{e_p(n)}{n^s}=\sum_{m=1}^{\infty}\frac{e_p(mp)}{(mp)^s},$$

therefore, from the definition of $e_p(n)$, we can easily get

$$\sum_{m=1}^{\infty}\frac{e_p(mp)}{(mp)^s}=\sum_{m=1}^{\infty}\frac{1}{(pm)^s}\sum_{\substack{n\in N\\ S_P(n)=mp}}1. \tag{4}$$

Combining (3) and (4), it is clear that

$$e_p(m)=\sum_{\substack{n\in N\\ S_P(n)=m}}1.$$

This completes the proof of Corollary.

References

[1] F.Smarandache, Only Problems, not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Liu Yanni and Gao Peng, Mean value of a new arithmetic function, Scientia Magna, **1**(2005), No. 1, 187-189.

[3] Wang Xiaoying, On the mean value of an arithmetical function, Hexis, 2004, 77-79.

[4] Gao Nan, A number theoretic function and its mean value, Hexis, 2004, 49-52.

[5] Gao Nan, A Hybrid number theoretic function and its mean value, Hexis, 2004, 107-109.

[6] Lv Chuan, On the mean value of an arithmetical function, Hexis, 2004, 89-92.

[7] Ren Ganglian, A number theoretic function and its mean value, Scientia Magna, **1**(2005), No. 1, 163-166.

[8] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

Scientia Magna
Vol. 3 (2007), No. 3, 30-35

A note on the Smarandache inversion sequence

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Abstract In a recent paper, Muneer [1] introduced the Smarandache inversion sequence. In this paper, we study some properties of the Smarandache inversion sequence. Moreover, we find the necessary and sufficient condition such that $[SI(n)]^2 + [SI(n+1)]^2$ is a perfect square.

Keywords Smarandache reverse sequence, Smarandache inversion, perfect square.

§1. Introduction

The Smarandache reverse sequence is (see, for example, Ashbacher [2])

$$1, 21, 321, 4321, 54321, \cdots,$$

and in general, the $n-th$ term of the sequence is

$$S(n) = n(n-1)\cdots 321.$$

In connection with the Smarandache reverse sequence, Muneer [1] introduced the concept of the Smarandache inversion sequence, $SI(n)$, defined as follows :

Definition 1.1. The value of the Smarandache inversion of (positive) integers in a number is the number of order relations of the form $i > j$ (where i and j are digits of the positive integers of the number under consideration), with $SI(0) = 0$, $SI(1) = 0$.

More specifically, for the Smarandache reverse sequence number

$$S(n) = n(n-1)\cdots 321,$$

the following order relations hold :

$$\begin{aligned}
&(A-1)n > n-1 > \cdots > 3 > 2 > 1,\\
&(A-2)n-1 > n-2 > \cdots > 3 > 2 > 1,\\
&\cdots\\
&(A-(n-1))2 > 1.
\end{aligned}$$

Note that, the number of order relations in $(A-1)$ is $n-1$, that in $(A-2)$ is $n-2$, and so on, and finally, the number of order relation in $(A-(n-1))$ is 1. We thus have the following result :

Lemma 1.1. $SI(n) = \dfrac{n(n-1)}{2}$ for any integer $n \geq 1$.

Proof. $SI(n) = (n-1) + (n-2) + \cdots + 1 = \dfrac{n(n-1)}{2}$.

Lemma 1.2. For any integer $n \geq 1$, $\sum_{i=1}^{n} SI(1) = \dfrac{n(n^2-1)}{6}$.

Proof. Using Lemma 1.1,

$$\begin{aligned}\sum_{i=1}^{n} SI(1) &= \sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right) \\ &= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] = \frac{n(n^2-1)}{6}.\end{aligned}$$

Muneer [1] also derived the following results.

Lemma 1.3. $SI(n+1) + SI(n) = n^2$ for any integer $n \geq 1$.

Lemma 1.4. $SI(n+1) - SI(n) = n$ for any integer $n \geq 1$.

Proof. Since

$$SI(n+1) = \frac{n(n+1)}{2} = \frac{n(n-1)}{2} + n = SI(n) + n,$$

we get the desired result.

Lemma 1.5. $[SI(n+1)]^2 - [SI(n)]^2 = n^3$ for any integer $n \geq 1$.

Proof. Using Lemma 1.3 and Lemma 1.4,

$$[SI(n+1)]^2 - [SI(n)]^2 = [SI(n+1) + SI(n)][SI(n+1) - SI(n)] = (n^2)(n) = n^3.$$

Lemma 1.6. $SI(n+1)SI(n-1) + SI(n) = \left(\dfrac{n(n-1)}{2}\right)^2$ for any integer $n \geq 1$.

We also have the following recurrence relation.

Lemma 1.7. $SI(n+1) - SI(n-1) = 2n - 1$ for any integer $n \geq 1$.

Proof. Using Lemma 1.4,

$$\begin{aligned}SI(n+1) - SI(n-1) &= [SI(n+1)SI(n)] + [SI(n) - SI(n-1)] \\ &= n + (n-1) = 2n - 1.\end{aligned}$$

Muneer [1] also considered the equation

$$[SI(n)]^2 + [SI(n+1)]^2 = k^2 \tag{1}$$

for some integers $n \geq 1$, $k \geq 1$, and found two solutions, namely, $n = 7$ and $n = 8$.

In this note, we derive a necessary and sufficient condition such that (1) is satisfied. This is given in the next section.

§2. Main Results

We consider the equation

$$[SI(n)]^2 + [SI(n+1)]^2 = k^2 \tag{2}$$

for some integers $n \geq 1$, $k \geq 1$. By definition,

$$[SI(n)]^2 + [SI(n+1)]^2 = \left(\frac{n(n-1)}{2}\right)^2 + \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{2}n^2(n^2+1)$$

We thus arrive at the following result.

Lemma 2.1. The equation (2) has a solution (for n and k) if and only if $\frac{1}{2}(n^2+1)$ is a perfect square.

Lemma 2.2. The Diophantine equation

$$\frac{1}{2}(n^2+1) = k^2 \tag{3}$$

has a solution (for n and k) if and only if there is an integer $m \geq 1$ such that $m^2 + (m+1)^2$ is a perfect square, and in that case, $n = 2m+1$, $k^2 = m^2 + (m+1)^2$.

Proof. We consider the equation (3) in the equivalent form

$$n^2 + 1 = 2k^2, \tag{4}$$

which shows that n must be odd; so let

$$n = 2m + 1. \tag{5}$$

for some integer $m \geq 1$. Then, from (4),

$$(2m+1)^2 + 1 = 2k^2,$$

that is, $(4m^2 + 4m + 1) + 1 = 2k^2$, that is, $m^2 + (m+1)^2 = k^2$.

Searching for all consecutive integers upto 1500, we found only four pairs of consecutive integers whose sums of squares are perfect squares. These are

$$(1)32 + 42 = 52, \tag{6}$$

$$(2)202 + 212 = 292, \tag{7}$$

$$(3)1192 + 1202 = 1692, \tag{8}$$

$$(4)6962 + 6972 = 9852. \tag{9}$$

The first two give respectively the solutions

(a) $[SI(7)]^2 + [SI(8)]^2 = 35^2$,

(b) $[SI(41)]^2 + [SI(42)]^2 = 1189^2$,

which were found by Muneer [1], while the other two give respectively the solutions

(c) $[SI(239)]^2 + [SI(240)]^2 = 40391^2$,

(d) $[SI(1393)]^2 + [SI(1394)]^2 = 1372105^2$.

The following lemma, giving the general solution of the Diophantine equation $x^2+y^2 = z^2$, is a well-known result (see, for example, Hardy and Wright [3]).

Lemma 2.3. The most general (integer) solution of the Diophantine equation $x^2+y^2 = z^2$ is

$$x = 2ab, \quad y = a^2 - b^2, \quad z = a^2 + b^2, \tag{10}$$

where $x > 0$, $y > 0$, $z > 0$ are integers with $(x, y) = 1$ and x is even, and a and b are of opposite parity with $(a, b) = 1$.

Lemma 2.4. The problem of solving the Diophantine equation

$$m^2 + (m+1)^2 = k^2, \tag{11}$$

is equivalent to the problem of solving the Diophantine equations

$$x^2 - 2y^2 = 1.$$

Proof. By Lemma 2.3, the general solution of the Diophantine equation

$$(m+1)^2 + m^2 = k^2$$

has one of the following two forms :

(a) $m = 2ab$, $m + 1 = a^2 - b^2$, $k = a^2 + b^2$ for some integers $a, b \geq 1$ with $(a, b) = 1$;

(b) $m = a^2 - b^2$, $m + 1 = 2ab$, $k = a^2 + b^2$ for some integers $a, b \geq 1$ with $(a, b) = 1$.

In case (a),

$$1 = (m+1) - m = (a^2 - b^2)^2 - 2ab = (a-b)^2 - 2ab^2,$$

which leads to the Diophantine equation $x^2 - 2y^2 = 1$.

In case (b),

$$-1 = m - (m+1) = (a^2 - b^2)^2 - 2ab = (a-b)^2 - 2ab^2,$$

leading to the Diophantine equation $x^2 - 2y^2 = -1$.

The general solutions of the Diophantine equations $x^2 - 2y^2 = \pm 1$ are given in the following lemma (see, for example, Hardy and Wright [3]).

Lemma 2.5. All solutions of the Diophantine equation

$$x^2 - 2y^2 = 1$$

are given by

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2n}, \tag{12}$$

$n \geq 0$ is an integer; and all solutions of the Diophantine equation

$$x^2 - 2y^2 = -1,$$

are given by

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2n+1}, \tag{13}$$

$n \geq 0$ is an integer.

Remark 2.1. Lemma 2.5 shows that the Diophantine equation $m^2 + (m+1)^2 = k^2$ has infinite number of solutions. The first four solutions of the Diophantine equation (11) are given in (6 - 9). It may be mentioned here that the first and third solutions can be obtained from

(12) corresponding to $n = 1$ and $n = 2$ respectively, while the second and the fourth solutions can be obtained from (13) corresponding to $n = 0$ and $n = 1$ respectively. The fifth solution may be obtained from (12) with $n = 3$ as follows :

$$x + \sqrt{2}y = (1 + \sqrt{2})^6 = 99 + 70\sqrt{2} \ \Rightarrow \ x = 99, \quad y = 70.$$

Therefore,

$$a - b = 99, \quad b = 70 \ \Rightarrow \ a = 169, b = 70,$$

and finally,

$$m = 2ab = 23660, \quad m + 1 = a^2 - b^2 = 23661.$$

Corresponding to this, we get the following solution to (2) :

$$[SI(47321)]^2 + [SI(47322)]^2 = 1583407981^2.$$

§3. Some Observations

In [1], Muneer has found three relations connecting four consecutive Smarandache inversion functions. These are as follows :

(1) $SI(6) + SI(7) + SI(8) + SI(9) = 10^2$,

(2) $SI(40) + SI(41) + SI(42) + SI(43) = 58^2$,

(3) $SI(238) + SI(239) + SI(240) + SI(241) = 338^2$.

Searching for more such relations upto $n = 1500$, we got a fourth one :

(4) $SI(1392) + SI(1393) + SI(1394) + SI(1395) = 1970^2$.

Since

$$SI(n-1) + SI(n) + SI(n+1) + SI(n+2) = (n-1)^2 + (n+1)^2,$$

the problem of finding four consecutive Smarandache inversion functions whose sum is a perfect square reduces to the problem of solving the Diophantine equation

$$m^2 + (m+2)^2 = k^2.$$

In this respect, we have the following result.

Lemma 3.1. If m_0, $m_0 + 1$ and $k_0 = \sqrt{m_0^2 + (m_0+1)^2}$ is a solution of the Diophantine equation

$$m^2 + (m+1)^2 = k^2, \tag{14}$$

then $2m_0$, $2(m_0 + 1)$ and $l_0 = 2\sqrt{m_0^2 + (m_0+1)^2}$ is a solution of the Diophantine equation

$$m^2 + (m+2)^2 = l^2, \tag{15}$$

and conversely.

Proof. First, let m_0, $m_0 + 1$ and $k_0 = \sqrt{m_0^2 + (m_0+1)^2}$ be a solution of (14), so that

$$m_0^2 + (m_0+1)^2 = k_0^2, \tag{16}$$

Multiplying throughout of (1) by 4, we get

$$(2m_0)^2 + [2(m_0+1)]^2 = (2k_0)^2,$$

so that $2m_0$, $2(m_0+1)$ and $l_0 = 2k_0$ is a solution of (15).

Conversely, let m_0, m_0+2 and $l_0 = \sqrt{m_0^2 + (m_0+2)^2}$ be a solution of (15). Note that, m_0 and m_0+2 are of the same parity. Now, both m_0 and m_0+2 cannot be odd, for otherwise,

$$m_0 \equiv 1(mod 2), \quad m_0 + 2 \equiv 1(mod 2) \;\Rightarrow\; l_0^2(mod 4),$$

which is impossible. Thus, both m_0 and m_0+2 must be even. It, therefore, follows that $\frac{m_0}{2}$, $\frac{m_0}{2}+1$ and $k_0 = \frac{l_0}{2}$ is a solution of (14).

References

[1] Muneer, J. K., Smarandache inversion sequence, Scientia Magna, **4**(2006), No. 2, 1-14.

[2] Charles Ashbacher, Pluckings from the tree of Smarandache sequences and functions, American Research Press, 1998.

[3] Hardy, G.H. and Wright, E.M., An introduction to the theory of numbers, Oxford Science Publications, Clarendon Press, Oxford, UK, 5th Edition 2002.

Scientia Magna
Vol. 3 (2007), No. 3, 36-39

Some Dirichlet series involving special sequences

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Abstract For any fixed positive integer $k > 1$ and any positive integer n, let $a_k(n)$ and $b_k(n)$ denote the inferior and superior k-power part of n respectively. That is, $a_k(n)$ denotes the largest k-power less than or equal to n, and $b_k(n)$ denotes the smallest k-power greater than or equal to n. In this paper, we study the properties of the sequences $a_k(n)$ and $b_k(n)$, and give four interesting identities for them.

Keywords Inferior and superior k-power, Riemann zeta-function, multiplicative function.

§1. Introduction and Results

For any fixed positive integer $k > 1$ and any positive integer n, let $a_k(n)$ and $b_k(n)$ denote the inferior and superior k-power part of n respectively. That is, $a_k(n)$ denotes the largest k-power less than or equal to n, and $b_k(n)$ denotes the smallest k-power greater than or equal to n. For example, let $k = 2$, then $a_2(1) = a_2(2) = a_2(3) = 1$, $a_2(4) = a_2(5) = a_2(6) = a_2(7) = 4, \cdots$, $b_2(1) = 1$, $b_2(2) = b_2(3) = b_2(4) = 4$, $b_2(5) = b_2(6) = b_2(7) = b_2(8) = 8, \cdots$. In problem 40 and 41 of [1], Professor F.Smarandache asks us to study the properties of the sequences $a_k(n)$ and $b_k(n)$. About these problems, Professor Wenpeng Zhang [3] gave two interesting asymptotic formulas of the cure part of a positive integer . These are

$$\sum_{n\leq x} d(a_3(n)) = \frac{2}{9\pi^4}Axln^3x + Bxln^2x + Cxlnx + Dx + O(x^{\frac{5}{6}+\varepsilon}),$$

$$\sum_{n\leq x} d(b_3(n)) = \frac{2}{9\pi^4}Axln^3x + Bxln^2x + Cxlnx + Dx + O(x^{\frac{5}{6}+\varepsilon}),$$

where A, B, C and D are constants; $d(n)$ denotes the Dirichlet divisor function; ε is any fixed positive number. And Jianfeng Zheng [4] gave two asymptotic formulas of the k-power part of a positive integer. These are

$$\sum_{n\leq x} d(a_k(n)) = \frac{1}{kk!}(\frac{6}{k\pi^2})^{k-1}A_0xln^kx + A_1xln^{k-1}x + \cdots + A_{k-1}xlnx + A_kx + O(x^{1-\frac{1}{2k}+\varepsilon}),$$

$$\sum_{n\leq x} d(b_k(n)) = \frac{1}{kk!}(\frac{6}{k\pi^2})^{k-1}A_0xln^kx + A_1xln^{k-1}x + \cdots + A_{k-1}xlnx + A_kx + O(x^{1-\frac{1}{2k}+\varepsilon}),$$

where $A_0, A_1, \cdots, A_k$ are constants, especially when k equals to 2, $A_0 = 1$.

In this paper, we shall use the elementary method to prove the following:

Theorem 1. For any complex number s with $Res > 2$, we have

$$\sum_{n=1}^{\infty}\frac{\sigma(a_k(n))}{(a_k(n))^s}=\sum_{i=0}^{k-1}C_k^i\zeta(ks-i)\zeta(ks-k-i)\prod_p\left[1+p^{-(ks-i)}\frac{p-p^k}{1-p}\right],$$

where $\zeta(s)$ denotes the Riemann zeta-function, and $\sigma(n)$ denotes the Dirichlet divisor function.

Theorem 2. For any complex number s with $Res > 2$, we have

$$\sum_{n=1}^{\infty}\frac{\sigma(b_k(n))}{(b_k(n))^s}=\sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\zeta(ks-i)\zeta(ks-k-i)\prod_p\left[1+p^{-(ks-i)}\frac{p-p^k}{1-p}\right].$$

Theorem 3. For any complex number s with $Res > 2$, we have

$$\sum_{n=1}^{\infty}\frac{\varphi(a_k(n))}{(a_k(n))^s}=\sum_{i=0}^{k-1}C_k^i\frac{\zeta(ks-k-i)}{\zeta(ks-k-i+1)},$$

where $\varphi(n)$ denotes the Euler function.

Theorem 4. For any complex number s with $Res > 2$, we have

$$\sum_{n=1}^{\infty}\frac{\varphi(b_k(n))}{(b_k(n))^s}=\sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\frac{\zeta(ks-k-i)}{\zeta(ks-k-i+1)}.$$

From the above Theorems, we may immediately deduce the following:

Corollary. For any complex number s with $Res > 2$, we have

$$\sum_{n=1}^{\infty}\frac{\sigma(a_2(n))}{(a_2(n))^s}=2\frac{\zeta(2s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(4s-4)}+\frac{\zeta(2s)\zeta(2s-1)\zeta(2s-2)}{\zeta(4s-2)};$$

$$\sum_{n=1}^{\infty}\frac{\sigma(b_2(n))}{(b_2(n))^s}=2\frac{\zeta(2s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(4s-4)}-\frac{\zeta(2s)\zeta(2s-1)\zeta(2s-2)}{\zeta(4s-2)};$$

$$\sum_{n=1}^{\infty}\frac{\varphi(a_2(n))}{(a_2(n))^s}=2\frac{\zeta(2s-3)}{\zeta(2s-2)}+\frac{\zeta(2s-2)}{\zeta(2s-1)};$$

$$\sum_{n=1}^{\infty}\frac{\varphi(b_2(n))}{(b_2(n))^s}=2\frac{\zeta(2s-3)}{\zeta(2s-2)}-\frac{\zeta(2s-2)}{\zeta(2s-1)}.$$

§2. A simple lemma

To complete the proof of the theorems, we need the following:

Lemma. Assume $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ converges absolutely for $Res > \sigma_a$. If $f(n)$ is a multiplicative function, then we have

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_p\left[1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\right],\ (Res>\sigma_a).$$

Proof. (See Theorem 11.7 of [2]).

§3. Proof of the theorems

Now we complete the proof of the theorems. First we prove Theorem 1. Let $D_k(n) = \sigma(n^k)$, then $D_k(n)$ is a multiplicative function, and

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sigma(a_k(n))}{(a_k(n))^s} &= \sum_{1^k\le n<2^k}\frac{\sigma(1^k)}{1^{ks}}+\sum_{2^k\le n<3^k}\frac{\sigma(2^k)}{2^{ks}}+\sum_{3^k\le n<4^k}\frac{\sigma(3^k)}{3^{ks}}+\cdots+\\
&\quad \sum_{N^k\le n<(N+1)^k}\frac{\sigma(N^k)}{N^{ks}}+\cdots\\
&= \sum_{N=1}^{\infty}\frac{[(N+1)^k-N^k]\sigma(N^k)}{N^{ks}}=\sum_{N=1}^{\infty}\frac{\left(\sum_{i=0}^{k-1}C_k^iN^i\right)\sigma(N^k)}{N^{ks}}\\
&= \sum_{i=0}^{k-1}C_k^i\sum_{N=1}^{\infty}\frac{\sigma(N^k)}{N^{ks-i}}=\sum_{i=0}^{k-1}C_k^i\sum_{N=1}^{\infty}\frac{D_k(N)}{N^{ks-i}}\\
&= \sum_{i=0}^{k-1}C_k^i\prod_p\left[1+\frac{D_k(p)}{p^{ks-i}}+\frac{D_k(p^2)}{p^{2(ks-i)}}+\cdots+\frac{D_k(p^n)}{p^{n(ks-i)}}+\cdots\right]\\
&= \sum_{i=0}^{k-1}C_k^i\prod_p\left[1+\sum_{n=1}^{\infty}p^{-n(ks-i)}\frac{1-p^{kn+1}}{1-p}\right]\\
&= \sum_{i=0}^{k-1}C_k^i\prod_p\frac{1+p^{-(ks-i)}(p+p^2+\cdots+p^{k-1})}{(1-p^{-(ks-i)})(1-p^{-(ks-k-i)})}\\
&= \sum_{i=0}^{k-1}C_k^i\zeta(ks-i)\zeta(ks-k-i)\prod_p\left[1+p^{-(ks-i)}\frac{p-p^k}{1-p}\right].
\end{aligned}$$

It is clear that the right hand side of the above is convergent if $Res > 2$, and divergent if $Res \le 2$. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. By the method of proving Theorem 1 we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sigma(b_k(n))}{(b_k(n))^s} &= \frac{\sigma(1^k)}{1^{ks}}+\sum_{1^k<n\le 2^k}\frac{\sigma(2^k)}{2^{ks}}+\sum_{2^k<n\le 3^k}\frac{\sigma(3^k)}{3^{ks}}+\cdots+\sum_{(N-1)^k<n\le N^k}\frac{\sigma(N^k)}{N^{ks}}+\cdots\\
&= \sum_{N=1}^{\infty}\frac{[N^k-(N-1)^k]\sigma(N^k)}{N^{ks}}\\
&= \sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\sum_{N=1}^{\infty}\frac{\sigma(N^k)}{N^{ks-i}}=\sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\sum_{N=1}^{\infty}\frac{D_k(N)}{N^{ks-i}}\\
&= \sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\prod_p\left[1+\sum_{n=1}^{\infty}p^{-n(ks-i)}\frac{1-p^{kn+1}}{1-p}\right]\\
&= \sum_{i=0}^{k-1}(-1)^{k-i+1}C_k^i\zeta(ks-i)\zeta(ks-k-i)\prod_p\left[1+p^{-(ks-i)}\frac{p-p^k}{1-p}\right].
\end{aligned}$$

The right hand side of the above is convergent if $Res > 2$, and divergent if $Res \leq 2$. This complete the proof of Theorem 2.

Using the methods of proving Theorem 1 and Theorem 2 we can also prove Theorem 3 and Theorem 4.

References

[1] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Tom M.Apostol, Introduction to analytical number theory, Spring-Verlag, New York, 1976.

[3] Wenpeng Zhang, On the Cube Part Sequence of a Positive Integer, Journal of Xianyang Teachers' College, **18**(2003), 5-7.

[4] Jianfeng Zheng, On the inferior and superior k-power part of a positive integer and divisor function, Smarandache Notions Journal, **14**(2004), 88-91.

Scientia Magna
Vol. 3 (2007), No. 3, 40-43

An asymptotic formula for $S_k(n!)$

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Abstract For any fixed positive integer k and any positive integer n, the famous Smarandache ceil function $S_k(n)$ is defined as $S_k(n) = \min\{m \in N : \ n \mid m^k\}$. The main purpose of this paper is using the elementary method to study the asymptotic properties of $S_k(n!)$, and give an interesting asymptotic formula for it.

Keywords F.Smarandache ceil function, asymptotic formula, elementary method.

§1. Introduction and results

For any fixed positive integer k and any positive integer n, the famous Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n) = \min\{m \in N : \ n \mid m^k\}.$$

For exemple, the first few values of $S_3(n)$ are: $S_3(1) = 1,\ S_3(2) = 2,\ S_3(3) = 3,\ S_3(4) = 2,\ S_3(5) = 5,\ S_3(6) = 6,\ S_3(7) = 7,\ S_3(8) = 2,\ S_3(9) = 3,\ S_3(10) = 10,\ S_3(11) = 11,\ S_3(12) = 6,\ S_3(13) = 13,\ S_3(14) = 14,\ S_3(15) = 15,\ S_3(16) = 4,\ S_3(17) = 17,\ S_3(18) = 6,\ S_3(19) = 19,\ S_3(20) = 10, \cdots$.

This function was introduced by Professor F. Smarandache. In reference [1], Ibstedt presented that $S_k(n)$ is a multiplicative function. That is,

$$(\forall a, b \in N)\ (a, b) = 1 \Longrightarrow S_k(a \cdot b) = S_k(a) \cdot S_k(b).$$

But $S_k(n)$ is not a completely multiplicative function. In fact, $(2, 4) = 2$, if $k = 3$, $S_3(2) = 2,\ S_3(4) = 2,\ S_3(2 \cdot 4) = S_3(8) = 2 \neq S_3(2) \cdot S_3(4) = 4$; If $k = 5$, $S_5(2) = 2,\ S_5(4) = 2,\ S_5(2 \cdot 4) = S_5(8) = 2 \neq S_5(2) \cdot S_5(4) = 4$; If $k = 2$, $S_2(3) = 3,\ S_2(12) = 6,\ S_2(3 \cdot 12) = S_2(36) = 6 \neq S_2(3) \cdot S_2(12) = 18$; If $k = 4$, $S_4(3) = 3,\ S_4(12) = 6,\ S_4(3 \cdot 12) = S_4(36) = 6 \neq S_4(3) \cdot S_4(12) = 18$. So $S_k(n)$ is not a completely multiplicative function.

On the other hand, it is easily to show that $S_k(p^\alpha) = p^{\lceil \frac{\alpha}{k} \rceil}$, where p be a prime and $\lceil x \rceil$ denotes the least integer greater than x. So if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime power decomposition of n, then the following identity is obviously:

$$S_k(n) = S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{k} \rceil}.$$

About the other elementary properties of $S_k(n)$, many people had studied it, and obtained some interesting results. For example, Xu Zhefeng [2] studied the mean value properties of the

compound function involving Ω and $S_k(n)$, and proved that for any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n\leq x} \Omega(S_k(n)) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$, γ is the Euler constant, and $\sum_p$ denotes the sum over all the primes.

Sabin Tabirca and Tatiana Tabirca [3] proved that

$$n = p_1p_2\cdots p_s \Longleftrightarrow S_k(n) = n.$$

Ren Dongmei [4] studied the mean value properties of the composite function involving $d(n)$ and $S_k(n)$, she proved that

$$\sum_{n\leq x} d\left(S_k(n)\right) = \frac{6\zeta(k)x\ln x}{\pi^2}\prod_p\left(1-\frac{1}{p^k+p^{k-1}}\right) + Cx + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where k is a given positive integer with $k \geq 2$, $\zeta(s)$ is the Riemann zeta-function, C is a computable constant, and ϵ is any fixed positive number.

He Xiaolin and Guo Jinbao [5] proved that for any real number $x \geq 2$ and any fixed positive integer $k \geq 2$,

$$\sum_{n\leq x} \sigma_\alpha\left(S_k(n)\right) = \frac{6x^{\alpha+1}\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\pi^2}R(\alpha+1) + O(x^{\alpha+\frac{1}{2}+\epsilon}),$$

where $\alpha > 0, \sigma_a(n) = \sum_{d|n} d^\alpha$, $\zeta(s)$ is the Riemann zeta-function, ϵ be any fixed positive number, and $R(\alpha+1) = \prod_p\left(1-\frac{1}{p^{k(\alpha+1)-\alpha}-p^{(k-1)(\alpha+1)}}\right)$.

In this paper, we use the elementary method to study the asymptotic properties of $S_k(n!)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let k be a fixed positive integer, then for any integer $n \geq 3$, we have the asymptotic formula

$$\ln(S_k(n!)) = \frac{n\ln n}{k} + O(n).$$

§2. Some lemmas

To complete the proof of the theorem, we need the following two simple lemmas.

Lemma 1. For any positive integer $n \geq 2$, we have

$$\pi(n) = \frac{n}{\ln n} + O\left(\frac{n}{\ln^2 n}\right),$$

where $\pi(n)$ denotes the number of the prime up to n.

Proof. See reference [6].

Lemma 2. Let $n \geq 2$ be any positive integer, then we have the asymptotic formula,

$$\sum_{p \leq n} \frac{\ln p}{p} = \ln n + O(1).$$

Proof. See reference Theorem 4.10 of [6].

§3. Proof of the theorem

In this section, we will complete the proof of our theorem. In fact, for any positive integer $n \geq 1$, let

$$n! = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

then from the properties of $S_k(n)$, we have

$$\begin{aligned} \ln(S_k(n!)) &= \ln\left(p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \cdots p_r^{\lceil \frac{\alpha r}{k} \rceil}\right) \\ &= \lceil \frac{\alpha_1}{k} \rceil \ln p_1 + \lceil \frac{\alpha_2}{k} \rceil \ln p_2 + \cdots + \lceil \frac{\alpha_r}{k} \rceil \ln p_r. \end{aligned}$$

It is clear that (see reference [6] Theorem 3.14),

$$\alpha_i = \sum_{j=1}^{\infty} \left[\frac{n}{p_i^j}\right], \quad i = 1, 2, \cdots r.$$

Noting that, if $p^j > n$, then $\left[\dfrac{n}{p^j}\right] = 0, \sum\limits_{n \leq x} \ln p = O(x)$, so from Lemma 1 we can write

$$\begin{aligned} \ln(S_k(n!)) &= \sum_{p \leq n} \left[\frac{1}{k} \sum_{j \leq \frac{\ln n}{\ln p}} \left[\frac{n}{p_i^j}\right] \right] \ln p \\ &= \sum_{p \leq n} \left[\frac{1}{k} \sum_{j=1}^{[\frac{\ln n}{\ln p}]} \frac{n}{p^j} + O\left(\frac{\ln n}{\ln p}\right) \right] \ln p \\ &= \sum_{p \leq n} \left(\frac{1}{k} \sum_{j=1}^{[\frac{\ln n}{\ln p}]} \frac{n}{p^j} + O\left(\frac{\ln n}{\ln p}\right) \right) \ln p + O(n) \\ &= \sum_{p \leq n} \left(\frac{1}{k} \sum_{j=1}^{[\frac{\ln n}{\ln p}]} \frac{n}{p^j} \right) \ln p + O(n) \\ &= \frac{n}{k} \left(\sum_{p \leq n} \frac{1}{p-1} - \sum_{p \leq n} \frac{1}{p^{[\frac{\ln n}{\ln p}]}(p-1)} \right) \ln p + O(n) \\ &= \frac{n}{k} \left(\sum_{p \leq n} \frac{1}{p-1} \right) \ln p + O(n) \\ &= \frac{n}{k} \left(\sum_{p \leq n} \frac{1}{p} + \sum_{p \leq n} \frac{1}{p(p-1)} \right) \ln p + O(n). \end{aligned}$$

Noting that,

$$\sum_{p\leq n}\frac{1}{p(p-1)}=\sum_{p}\frac{1}{p(p-1)}-\sum_{p>n}\frac{1}{p(p-1)},$$

but

$$\sum_{p}\frac{\ln p}{p(p-1)}<\sum_{n=2}^{\infty}\frac{\ln n}{n(n-1)}=O(1),$$

$$\sum_{p>n}\frac{\ln p}{p(p-1)}<\sum_{p>n}\frac{\ln n}{p(p-1)}=O\left(\frac{\ln n}{n}\right),$$

so

$$\ln\left(S_k(n!)\right)=\frac{n}{k}\sum_{p\leq n}\frac{\ln p}{p}+O(n).$$

Combining Lemma 2 and the above formula we may immediately get the asymptotic formula

$$\ln\left(S_k(n!)\right)=\frac{n\ln n}{k}+O(n).$$

This completes the proof of the theorem.

References

[1] Ibstedt and Surfining, on the ocean of number-a few Smarandache Notions and Similar Topics, Erhus University Press, New Mexico, USA.

[2] Xu Zhefeng, On the Smarandache ceil function and the number of prime factors, Research on Smarandache problems in number theory, Hexis, 2004, 73-76.

[3] Sabin Tabirca and Tatiana Tabirca, Some new results concerning the Smarandache ceil function, Smarandache Notions Journal, **13**(2002), 30-36.

[4] Ren Dongmei, On the Smarandache ceil function and the dirichlet divisor function, Research on Smarandache problems in number theory, Hexis, **2**(2005), 51-54.

[5] He Xiaolin and Guo Jinbao, Some asymptotic properties involving the Smarandanche ceil function, Research on Smarandache problems in number theory, Hexis, 2004, 133-137.

[6] Tom. M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[7] F.Smarandache, Only Problems, Not Solution, Chicago, Xiquan Publising House, 1993.

[8] Pan Chengdong and Pan Chengbiao, Elementary Number Theory, Beijing University Press, Beijing, 2003.

Scientia Magna
Vol. 3 (2007), No. 3, 44-51

A fuzzy relaxed approach for multi-objective transportation problem

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Abstract In this paper, a fuzzy relaxed approach for multi-objective transportation problem is presented by using γ -operator which allow some degree of compensation between aggregated membership functions. It is shown that solutions obtained by this way are efficient. As the solvability of the resulting problem is not generally guaranteed , we turn the resulting problem to new single objective linear programming by inequalities, and the optimal compromise solution is obtained. Simultaneously we avoid the computation of parameter γ. Numerical examples are given to demonstrate the efficiency and practicality of the proposed method.

Keywords Fuzzy programming, relaxed, multi-objective transportation problem, γ-operator.

§1. Introduction

The application of fuzzy set theory to human decision making has received considerable attention by a number of authors since the pioneering work of Bellman and Zadeh. We present in this paper an approach related to Zimmermann's and using the γ-operator is to combine fuzzy objective in the sense of the 'fuzzy OR'. It is shown that solutions obtained by this way are efficient. As the solvability of the resulting problem is not generally guaranteed and the computation of optimal compensation degree γ is difficulty (Li(2002)), a fuzzy relaxed approach is proposed for multi-objective transportation problem . In this approach, we turn the resulting problem to new single objective linear programming by inequalities, the optimal compromise solution is obtained. Simultaneously the compensation degree γ of γ -operator is avoided. Illustrative numerical examples demonstrated the efficiency and practicality of the proposed approach.

§2. Multi-objective linear transportation problem

Consider a transportation situation having m origins (or sources) $O_i(i = 1, 2, \ldots, m)$ and n destinations $D_j(j = 1, 2, \ldots, n)$. Let a_i be the quantity of a homogeneous product which we want to transport to n destinations D_j to satisfy the demand for b_j units of the product there. A penalty C_{ij}^k is associated with transportation of a unit of the product from source i to destination j for k-th penalty criterion. The penalty could represent transportation cost,

delivery time, quantity of goods delivered, under-used capacity, etc.. A variable x_{ij} represents the unknown quantity to be transported from origin O_i to destination D_j.

A transportation problem can be represented as a multi-objective linear programming problem:

$$\min Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}^k x_{ij} (k = 1, 2, \ldots, K). \tag{1}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n, \tag{3}$$

$$x_{ij} \geq 0,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n, \tag{4}$$

where $a_i > 0$ for all $i, b_j > 0$ for all $j, C_{ij}^k \geq 0$ for all (i, j), and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (balanced condition).

Notice that the balanced condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is both a necessary and sufficient condition for the existence of a feasible solution of the (2)-(4).

The multi-objective transportation problem is considered as a vector-minimum problem. For each objective Z_k, two values U_k and L_k is the upper bound and lower bound of Z_k, respectively. L_k is the aspired level of achievement for k- th objective, U_k the highest acceptable level of achievement for k-th objective, $d_k = U_k - L_k$, the degradation allowance or leeway, for k-th objective. U_k and L_k can be determined as following for each objective Z_k, respectively,

$$U_k = \frac{1}{K}(Z_{1k} + Z_{2k} + \ldots + Z_{Kk}),\ L_k = Z_{kk},\ k = 1, 2, \ldots, K,$$

$Z_{ij} = Z_j(x^{(i)}),\ (i, j = 1, 2, \ldots, K,),\ x^{(i)}(i = 1, 2, \ldots, K)$ is the isolated optimal solution of the k-th objective function.

§3. Fuzzy relaxed approach to multi-objective transportation problem using γ-operator

Let X denote the feasible region satisfying all of the constraints of the problem (1). Moreover, we have the following concept.

Definition.(Pareto optimal solution) $x_1 \in X$ is said to be a Pareto optimal solution if and only if there does not exist another $x_2 \in X$ such that $Z_k(x_1) \geq Z_k(x_2)$ for all $k = 1, 2, \ldots, K$ and $Z_k(x_1) > Z_k(x_2)$ for at least one k.

The membership function for the k-th objective function Z_k is defined as

$$\mu_k(x) = \begin{cases} 1 & \text{if } Z_k \le L_k, \\ 1 - \frac{Z_k - L_k}{U_k - L_k} & \text{if } L_k < Z_k < U_k, \\ 0 & \text{if } Z_k \ge U_k. \end{cases} \tag{5}$$

Let $F = \bigcap_{k=1}^{K} (\text{support } \mu_k) \neq \phi$, where support $\mu_k = \{x | \mu_k(x) > 0\}$. The decision maker wishes to obtain a solution satisfying $L_k \le Z_k \le U_k$. Zimmermann's fuzzy programming appears to be an ideal approach for obtaining the optimal compromise solution to a multi-objective transportation problem. However, due to the ease of computation, the aggregate operator used in Zimmermann's fuzzy programming is the 'min' operator, which does not guarantee a non-dominated solution. Usually we have two methods to overcome this shortcoming. Firstly, constructing a new compensatory operator which comprehensives the non-compensatory operator 'min' used to represent the 'logical AND' and the compensation operator 'max' used to represent the 'logical OR', for example, M.K.Luhandjula (1982) presented the fuzzy approach for the linear maximum problem using the min-bound sum operator; Secondly, constructing new operators which are named the 'fuzzy AND' and 'fuzzy OR'.

In this paper, we define the fuzzy decision as the following Werners-compensatory operator, which represents 'fuzzy OR'.

$$D = \{(x, \mu_D(x)) | x \in F\},\ \mu_D = \gamma \min_{k \in \{1,2,\ldots,K\}} \mu_k + (1-\gamma)\frac{1}{K} \min \left\{ 1, \sum_{k=1}^{K} \mu_k \right\}.$$

Using this operator to solve problem (1)-(4), we have the following programming

$$\max \left[\gamma \min_{k \in \{1,2,\ldots,K\}} \mu_k + (1-\gamma)\frac{1}{K} \min \left\{ 1, \sum_{k=1}^{K} \mu_k \right\} \right], \tag{6}$$

s.t. $x \in F$. Although the solution obtained by (6) is efficient for (1), it is not always easy to solve (6) by available methods. Assume $\alpha = \min_{k \in \{1,2,\ldots,K\}} \mu_k,\ \beta = \min \left\{ 1, \sum_{k=1}^{K} \mu_k \right\}$, then we turn the (6) into the following

$$\max (1-\gamma)\alpha + \gamma \frac{1}{K}\beta. \text{ s.t. } \mu_k \le \alpha,$$

where $k = 1,\ 2,\ \ldots,\ K,\ \sum_{k=1}^{K} \mu_k \ge \beta,\ \beta \le 1,\ x \in X$.

Lemma 1. Suppose L_k and U_k finite and $L_k < U_k$. If x is solution of (6), then x is efficient for (1).

Proof. Suppose x optimal for (6) and non-efficient for (1). Then there exists $x' \in X$ so that $Z_k(x') \le Z_k(x),\ k = 1, 2, \ldots, K$ and $Z_k(x') < Z_k(x)$ for at least one k. We have

$$\frac{Z_k(x') - U_k}{L_k - U_k} \ge \frac{Z_k(x) - U_k}{L_k - U_k},\ k = 1, 2, \ldots, K,$$

$$\frac{Z_k(x') - U_k}{L_k - U_k} > \frac{Z_k(x) - U_k}{L_k - U_k},\ k = 1, 2, \ldots, K,$$

for at least one k.

As $0 < \gamma < 1$, we have

$$\gamma \min_{k\in\{1,2,\ldots,K\}} \frac{Z_k(x') - U_k}{L_k - U_k} \geq \gamma \min_{k\in\{1,2,\ldots,K\}} \frac{Z_k(x) - U_k}{L_k - U_k},$$

and

$$(1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \frac{Z_k(x') - U_k}{L_k - U_k}\right\} > (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \frac{Z_k(x) - U_k}{L_k - U_k}\right\}.$$

It results that

$$\gamma \min_{k\in\{1,2,\ldots,K\}} \frac{Z_k(x') - U_k}{L_k - U_k} + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \frac{Z_k(x') - U_k}{L_k - U_k}\right\}$$

$$> \gamma \min_{k\in\{1,2,\ldots,K\}} \frac{Z_k(x) - U_k}{L_k - U_k} + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \frac{Z_k(x) - U_k}{L_k - U_k}\right\}.$$

This contradicts the fact that optimal for (6).

Lemma 2. If (x^*, α^*, β^*) is optimal for (7), then

$$\alpha^* = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*),\ \beta^* = \min\left\{1, \sum_{k=1}^{K} \mu_k(x^*)\right\}.$$

Proof. Suppose (x^*, α^*, β^*) is optimal for (7) and $\alpha^* \neq \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*)$, Let $\alpha' = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*)$ then $\alpha' > \alpha^*$. (x^*, α^*, β^*) verifies the constraints of (7) and $\gamma\alpha^* + (1-\gamma)\frac{1}{K}\beta^* < \gamma\alpha' + (1-\gamma)\frac{1}{K}\beta^*$, this contradicts the fact that (x^*, α^*, β^*) is optimal for (7). $\beta^* = \min\left\{1, \sum_{k=1}^{K} \mu_k(x^*)\right\}$ can be proved in the same way.

Proposition. x^* is optimal for (6) if and only if (x^*, α^*, β^*), where

$$\alpha^* = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*),\ \beta^* = \min\left\{1, \sum_{k=1}^{K} \mu_k(x^*)\right\}$$

is the solution to the problem (7).

Proof. (a) Necessity. If x^* is optimal for (6), then

$$\gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*) + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \mu_k(x^*)\right\} \tag{7}$$

$$\geq \gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x) + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K} \mu_k(x)\right\}. \tag{8}$$

Suppose that (x^*, α^*, β^*) with

$$\alpha^* = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*),\ \beta^* = \min\left\{1, \sum_{k=1}^{K} \mu_k(x^*)\right\}$$

is not optimal for (7). There exists $(x', \alpha', \beta') \neq (x^*, \alpha^*, \beta^*)$, so that

$$\mu_k(x') \leq \alpha', \ k = 1, 2, \ldots, K,$$

$$\sum_{k=1}^{K} \mu_k(x') \geq \beta', \ \beta' \leq 1, \ x' \in F,$$

$$\gamma\alpha^* + (1-\gamma)\frac{1}{K}\beta^* < \gamma\alpha' + (1-\gamma)\frac{1}{K}\beta'.$$

It results that

$$\begin{aligned} & \gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x') + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K}\mu_k(x')\right\} \\ \geq & \ \gamma\alpha' + (1-\gamma)\frac{1}{K}\beta' \\ > & \ \gamma\alpha^* + (1-\gamma)\frac{1}{K}\beta^* \\ = & \ \gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*) + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K}\mu_k(x^*)\right\}, \end{aligned}$$

which contradicts (8).

(b) Sufficiency. If (x^*, α^*, β^*) is optimal for (7), by the Lemma 1 we have

$$\alpha^* = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*), \ \beta^* = \min\left\{1, \sum_{k=1}^{K}\mu_k(x^*)\right\}.$$

Suppose x^* is not optimal for (6), there exists x' so that

$$\begin{aligned} & \gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x') + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K}\mu_k(x')\right\} \\ > & \ \gamma \min_{k\in\{1,2,\ldots,K\}} \mu_k(x^*) + (1-\gamma)\frac{1}{K}\min\left\{1, \sum_{k=1}^{K}\mu_k(x^*)\right\} \\ = & \ \gamma\alpha^* + (1-\gamma)\frac{1}{K}\beta^*. \end{aligned}$$

Let

$$\alpha' = \min_{k\in\{1,2,\ldots,K\}} \mu_k(x'), \ \beta' = \min\left\{1, \sum_{k=1}^{K}\mu_k(x')\right\},$$

(x', α', β') verifies the constraints of (7) and

$$\gamma\alpha' + (1-\gamma)\frac{1}{K}\beta' > \gamma\alpha^* + (1-\gamma)\frac{1}{K}\beta^*.$$

This contradicts the fact that (x^*, α^*, β^*) is optimal for (7).

Although the problem (7) has above properties, it is difficulty to solve (7) and find the compensation degree γ of γ-operator. So, we use the following inequalities

$$\frac{\beta}{K} = (1-\gamma)\frac{\beta}{K} + \gamma\frac{1}{K}\beta \leq (1-\gamma)\alpha + \gamma\frac{1}{K}\beta,$$

a fuzzy relaxed approach using the γ -operator is proposed. In order to solve (7), we only solve the following single objective linear programming

$$\max \frac{\beta}{K},\ \text{s.t.}\ \sum_{k=1}^{K} \mu_k \geq \beta,\ \beta \leq 1,\ x \in X. \tag{9}$$

We obtain the optimal solution $y = (\beta^1, x^1)$ for (10), where

$$x^1 = (x^1_{11}, x^1_{12}, \ldots, x^1_{1n}, x^1_{21}, \ldots, x^1_{2n}, \ldots, x^1_{m1}, \ldots, x^1_{mn}),$$

The optimal solution objective values $Z^1 = (Z^1_1, Z^1_2, \ldots, Z^1_K)$ with respect to x^1. Let

$$\alpha' = \min_{k \in \{1,2,\ldots,K\}} \mu_k(x'),\ \beta' = \min \left\{ 1, \sum_{k=1}^{K} \mu_k(x') \right\},$$

(x', α', β') is a solution of the problem (7). By Proposition, the optimal compromise solution x^* and objective values for (1) is $x^* = x'$ and $Z^* = Z'$, respectively.

Algorithm.

Step 1. Solve the multi-objective transportation problem as a single objective transportation problem by taking one of the objectives at one time.

Step 2. From the results of Step1, find the lower bound L_k and upper bound U_k values for the objective Z_k.The membership function for the objective function Z_k is defined by (5).

Step 3. Applying Werners-compensatory operator proposed by (6) for the multi-objective transportation, we get the following programming problem (7).

Step 4. Solve the problem (8). If the solution to above (8) does not exist, $U_k := U_k + \frac{1}{4}(U_k - L_k)$, go to step 3. The optimal solution and objective values of the problem(8) is denoted by x^1 and Z^1, respectively. The optimal compromise solution x^* and objective values for (1) is $x^* = x^1$ and $Z^* = Z^1$, respectively , for the original problem (1) are obtained.

§4. Numerical example

$$\min Z_1 = x_{11} + 4x_{12} + 5x_{13} + 4x_{21} + 6.5x_{22} + 5x_{23}, \tag{10}$$

$$\min Z_2 = 2x_{11} + 3x_{12} + 4x_{13} + x_{21} + 9x_{22} + 6x_{23},$$

$$\min Z_3 = 4x_{11} + 2x_{12} + 10x_{13} + 3x_{21} + 8x_{22} + x_{23},$$

$$\text{s.t.}\ \sum_{j=1}^{3} x_{1j} = 11,\ \sum_{j=1}^{3} x_{2j} = 15,$$

$$\sum_{i=1}^{2} x_{i1} = 7,\ \sum_{i=1}^{2} x_{i2} = 10,\ \sum_{i=1}^{2} x_{i3} = 9,$$

$$x_{ij} \geq 0,\ i = 1, 2;\ j = 1, 2, 3.$$

Applying the fuzzy relaxed approach proposed, We obtain the optimal compromise solution and optimal objective value asked on MATLAB 7.0 platform

$$x^* = (1.5554, 9.2798, 0.1648, 5.4446, 0.7202, 8.8352),$$

$$Z^* = (Z_1^*, Z_2^*, Z_3^*) = (110.1343, 96.5471, 57.3598).$$

But applying fuzzy programming approach of A.K.Bit s[8] and U_k values in Rakesh (2000): $U_k = \max\{Z_{1k}, Z_{2k}, \ldots, Z_{kk}\}$, we get:

$$x^* = (2.8140, 8.1860, 0, 4.1860, 1.8140, 9),$$

$$Z^* = (Z_1^*, Z_2^*, Z_3^*) = (109.0930, 104.6977, 63.6977).$$

Applying Algorithm (Rakesh et al (1997)), we obtain the following solution:

$$x^* = (2.3448, 8.6552, 0, 4.6552, 1.3448, 9),$$

$$Z^* = (Z_1^*, Z_2^*, Z_3^*) = (109.3275, 101.4136, 60.4136).$$

Applying Algorithm (Rakesh et al (1997)), but in Rakesh (1987) $U_k = \max\{Z_{1k}, Z_{2k}, \ldots, Z_{Kk}\}$, we get :

$$x^* = (2.8138, 8.1862, 0, 4.1862, 1.8138, 9),$$

$$Z^* = (Z_1^*, Z_2^*, Z_3^*) = (109.0930, 104.6977, 63.6977).$$

Above results show that the solutions obtained is nearer to the ideal solution (107,89,51) than the solutions (109.3275, 101.4136, 60.4136), (109.0930, 104.6977, 63.6977) and (109.0931, 104.6966, 63.6966). Hence, we have obtained the better optimal compromise solution by the fuzzy relaxed algorithm with a new upper U_k.

§5. Concluding remarks

In this paper, we suggested a fuzzy relaxed approach for the multi-objective transportation problem by using γ-operator which allow some degree of compensation between aggregated membership functions. As the solvability of the resulting problem is not generally guaranteed, we turn the resulting problem to the single objective linear programming by inequalities, the optimal compromise objective values are obtained. Simultaneously the compensation degree γ of γ-operator is obtained.

References

[1] M.K.Luhandjula, Mathematical programming in the presence of fuzzy quantities and random variables, The Journal of Fuzzy Mathematics, **11**(2003), 27-38.

[2] H.J.Zimmermann and P.Zysno, Latent connectives in human decision making, Fuzzy Sets and Systems, **4**(1980), 37-51.

[3] Rakesh Verma, M.P.Biswal and A.Biswas, Fuzzy programming technique to solve multi-objective transportation problems with some non-linear membership functions, Fuzzy Sets and Systems, **91**(1997), 37-43.

[4] Li Rongjun, Theory and application of fuzzy multi-criterion decision making, Science Press, 2002.

[5] Jiafu Tang and Dingwei Wang, Fuzzy optimization theory and methodology, a survey, Control Theory and Applications, **17**(2000), 159-164.

[6] Bit AK, Biswal MP, Alam SS, Fuzzy programming approach to multi-criteria decision making transportation problem, Fuzzy Sets and Systems, **50**(1992), 135-141.

[7] Mohamed S. Osman, Omar M .Saad and Ali A, Yahia, A fuzzy interactive approach to identify the best compromise solution for multi-objective programming problems, The Journal of Fuzzy Mathematics, **8**(2000), No. 3, 619-627 .

[8] Shuping Gao, IAlgorithms of multiobjectiv Transportation problem with Uncertain coefficients, Far.East J.Math.Sci, **32**(2005), 96-106.

[9] Bit AK, Biswal MP and Alam SS, Fuzzy programming approach to multicriteria decision making transporation problem, Fuzzy Sets and Systems, **50**(1992), 35-41.

[10] Lushu Li, K.K.Lai, A fuzzy approach to the multi-objective transportation problem, Computer and Operation Research, **27**(2000), 43-57.

[11] Shuping Gao and Sanyang Liu, Two-phase fuzzy algorithms for multi-objective transportation problem, Journal of Fuzzy Mathematics, **12**(2004), No. 1, 147-155.

Scientia Magna
Vol. 3 (2007), No. 3, 52-55

An infinity series involving the Smarandache-type function

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Abstract In this paper, we using the elementary method to study the convergent property of one class Dirichlet series involving a special sequences, and give several interesting identities for it.

Keywords Riemann zeta-function, infinity series, identity.

§1. Introduction and Results

For any positive integer n and $m \geq 2$, the Smarandache-type function $B_m(n)$ is defined as the largest m-th power dividing n. That is,

$$B_m(n) = \max\{x^m : x^m \mid n\}(\forall n \in N^*).$$

For example, $B_2(1) = 1$, $B_2(2) = 1$, $B_2(3) = 1$, $B_2(4) = 2$, $B_2(5) = 1$, $B_2(6) = 1$, $B_2(7) = 1$, $B_2(8) = 2$, $B_2(9) = 3$, $\cdots$. This function was first introduced by Professor Smarandache. In [1], Henry Bottomley presented that $B_m(n)$ is a multiplicative function. That is,

$$(\forall a, b \in N)(a, b) = 1 \Rightarrow B_m(a \cdot b) = B_m(a) \cdot B_m(b).$$

It is easily to show that $B_m(p^\alpha) = p^{im}$, $\alpha = im + l$, $(i \geq 0, 0 \leq l < m)$, where p is a prime. So, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime powers decomposition of n, then the following identity is obviously:

$$B_m(n) = B_m(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{i_1 m} p_2^{i_2 m} \cdots p_r^{i_r m},$$

where $\alpha_j = i_j m + l_j$ $(i_j \geq 0, 0 \leq l_j < m)$.

Similarly, for any positive integer n and any fixed positive integer m, we define another Smarandache function $C_m(n)$ as following:

$$C_m(n) = \max\{x \in N : x^m \mid n\}(\forall n \in N^*).$$

Obviously, $C_m(n)$ is also a multiplicative function.

From the definition of $B_m(n)$ and $C_m(n)$, we may immediately get

$$B_m(n) = C_m(n)^m.$$

Now let k be a fixed positive integer, for any positive integer n, we define the arithmetical function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : d \mid n, (d,k) = 1\}.$$

For example, $\delta_2(1) = 1$, $\delta_2(2) = 1$, $\delta_2(3) = 1$, $\delta_2(4) = 1$, $\delta_3(6) = 2, \cdots$. About the elementary properties of this function, many scholars have studied it and got some useful results (see reference [2], [3]). In reference [2], Xu Zhefeng studied the divisibility of $\delta_k(n)$ by $\varphi(n)$, and proved that $\varphi(n) \mid \delta_k(n)$ if and only if $n = 2^\alpha 3^\beta$, where $\alpha > 0$, $\beta \geq 0$, $\alpha, \beta \in N$. In reference [3], Liu Yanni and Gao Peng studied the mean value properties of $\delta_k(b_m(n))$, and obtained an interesting mean value formula for it. That is, they proved the following conclusion:

Let k and m are two fixed positive integers. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} \delta_k(b_m(n)) = \frac{x^2}{2} \frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m + 1}{p^{m-1}(p+1)} + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p|k}$ denotes the product over all different prime divisors of k.

Let $\mathcal{A}$ denotes the set of all positive integers n satisfying the equation $B_m(n) = \delta_k(n)$. That is, $\mathcal{A} = \{n \in N, B_m(n) = \delta_k(n)\}$. In this paper, we using the elementary method to study the convergent property of the Dirichlet series involving the set $\mathcal{A}$, and give several interesting identities for it. That is, we shall prove the following conclusions:

Theorem 1. Let $m \geq 2$ be a fixed positive integer. Then for any real number $s > 1$, we have the identity:

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \zeta(ms) \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}}.$$

Theorem 2. For any complex number s with $Re(s) > 2$, we have the identity:

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{B_m(n)}{n^s} = \zeta(ms-1) \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

§2. Proof of the theorems

Now we complete the proof of our Theorems. First we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } n \in \mathcal{A}, \\ 0, & \text{if } otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}<\sum_{n=1}^{\infty}\frac{1}{n^s},$$

and $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is convergent if $s>1$, thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}$ is also convergent if $s>1$. Now we find the set $\mathcal{A}$. From the definition of $B_m(n)$ and $\delta_k(n)$ we know that $B_m(n)$ and $\delta_k(n)$ both are multiplicative functions. So in order to find all solutions of the equation $B_m(n)=\delta_k(n)$, we only discuss the case $n=p^\alpha$. Let $\alpha=im+l$, where $i\geq 0$, $0\leq l<m$, then $B_m(n)=p^{im}$. If $n=p^\alpha$, $(p,k)=1$, now $\delta_k(n)=p^\alpha$, then the equation $B_m(n)=\delta_k(n)$ has solution if and only if $\alpha=im$, $i\geq 0$. If $n=p^\alpha$, $p\mid k$, now $\delta_k(n)=1$, then the equation $B_m(n)=\delta_k(n)$ has solution if and only if $\alpha=l$, $0\leq l<m$.

Thus, by the Euler product formula, we have

$$\begin{aligned}
\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{1}{n^s} &= \prod_{p}\left(1+\frac{a(p)}{p^s}+\frac{a(p^2)}{p^{2s}}+\cdots+\frac{a(p^{m-1})}{p^{(m-1)s}}+\cdots\right)\\
&= \prod_{p|k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\prod_{p\dagger k}\left(1+\frac{1}{p^{ms}}+\frac{1}{p^{2ms}}+\cdots\right)\\
&= \prod_{p|k}\frac{1-\frac{1}{p^{ms}}}{1-\frac{1}{p^s}}\prod_{p\dagger k}\frac{1}{1-\frac{1}{p^{ms}}}\\
&= \prod_{p}\frac{1}{1-\frac{1}{p^{ms}}}\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}\\
&= \zeta(ms)\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}.
\end{aligned}$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let $s=\sigma+it$ be a complex number. Note that $B_m(n)\ll n$, so it is clear that $\sum_{n=1}^{\infty}\frac{B_m(n)}{n^s}$ is an absolutely convergent series for $Re(s)>2$, by

the Euler product formula and the definition of $B_m(n)$ we get

$$\begin{aligned}
\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{B_m(n)}{n^s} &= \prod_p\left(1+\frac{B_m(p)}{p^s}+\frac{B_m(p^2)}{p^{2s}}+\cdots\right)\\
&= \prod_{p|k}\left(1+\frac{B_m(p)}{p^s}+\frac{B_m(p^2)}{p^{2s}}+\cdots+\frac{B_m(p^{m-1})}{p^{(m-1)s}}\right)\\
&\quad \prod_{p\dagger k}\left(1+\frac{B_m(p^m)}{p^{ms}}+\frac{B_m(p^{2m})}{p^{2ms}}+\cdots\right)\\
&= \prod_{p|k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\prod_{p\dagger k}\left(1+\frac{p^m}{p^{ms}}+\frac{p^{2m}}{p^{2ms}}+\cdots\right)\\
&= \prod_{p|k}\frac{1-\frac{1}{p^{ms}}}{1-\frac{1}{p^s}}\prod_{p\dagger k}\frac{1}{1-\frac{1}{p^{ms-1}}}\\
&= \prod_p\frac{1}{1-\frac{1}{p^{ms-1}}}\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}\\
&= \zeta(ms-1)\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}.
\end{aligned}$$

This completes the proof of Theorem 2.

References

[1] Henry Bottomley, Some Smarandache-type Multiplicative Functions, Smarandache Notions, **13**(2002), 134-139.

[2] Zhefeng Xu, On the function $\varphi(n)$ and $\delta_k(n)$, Research On Smarandache Problems In Number Theory, Hexis, 2005, 9-11.

[3] Liu Yanni and Gao Peng, On the m-power free part of an integer, Scientia Magna, **1**(2005), No. 1, 203-206.

[4] Zhang Pei, An Equation involving the function $\delta_k(n)$, Scientia Magna, **2**(2006) No. 4.

[5] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[6] Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Beijing, Science Press, 1997.

[7] Yang Hai and Fu Ruiqing, On the K-power Part Residue Function, Scientia Magna Journal, **1**(2005), No. 1.

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On the Lorentz matrix in terms of Infeld-van der Waerden symbols

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Abstract We exhibit expressions, in terms of Pauli matrices which directly generate Lorentz transformations in special relativity.

Keywords Pauli matrices, Lorentz transformations.

In space time an event is represented by $(x^j) = (ct, x, y, z)$, $j = 0, \ldots, 3$, with the metric $(g_{jr}) = Diag(1, -1, -1, -1)$. If it is necessary to employ another frame of reference, then the new coordinates $\tilde{x}^r$ are connected with x^j via the linear transformation:

$$\tilde{x}^j = L^j_r x^r, \tag{1}$$

where the Lorentz matrix $\underset{\sim}{L}$ verifies the restriction:

$$L^j_a g_{rj} L^r_b = g_{ab}, \tag{2}$$

because the Minkowskian line element must remain invariant under $\underset{\sim}{L}$, that is, $\tilde{x}^r \tilde{x}_r = x^r x_r$.

From (2) we see that $\underset{\sim}{L}$ has six degrees of freedom, which permits to work with four complex numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma = 1$, then the components of homogeneous Lorentz transformation $\underset{\sim}{L}$ can be written in the form [1]-[4]:

$$\begin{array}{ll}
L^0_0 = \frac{1}{2}(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^*), & L^0_1 = \frac{1}{2}(\alpha^*\beta + \gamma^*\delta) + c.c., \\
L^0_2 = \frac{i}{2}(\alpha^*\beta + \gamma^*\delta) + c.c., & L^0_3 = \frac{1}{2}(\alpha\alpha^* - \beta\beta^* + \gamma\gamma^* - \delta\delta^*), \\
L^1_0 = \frac{1}{2}(\alpha^*\gamma + \beta^*\delta) + c.c., & L^1_1 = \frac{1}{2}(\alpha^*\delta + \beta\gamma^*) + c.c., \\
L^1_2 = \frac{i}{2}(\alpha^*\delta + \beta\gamma^*) + c.c., & L^1_3 = \frac{1}{2}(\alpha^*\gamma - \beta^*\delta) + c.c., \\
L^2_0 = \frac{i}{2}(\alpha\gamma^* - \beta^*\delta) + c.c., & L^2_1 = \frac{i}{2}(\alpha\delta^* + \beta\gamma^*) + c.c., \\
L^2_2 = \frac{1}{2}(\alpha^*\delta - \beta^*\gamma) + c.c., & L^2_3 = \frac{i}{2}(\alpha\gamma^* + \beta^*\delta) + c.c., \\
L^3_0 = \frac{1}{2}(\alpha\alpha^* + \beta\beta^* - \gamma\gamma^* - \delta\delta^*), & L^3_1 = \frac{1}{2}(\alpha^*\beta - \gamma^*\delta) + c.c., \\
L^3_2 = \frac{i}{2}(\alpha^*\beta - \gamma^*\delta) + c.c., & L^3_3 = \frac{1}{2}(\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \delta\delta^*),
\end{array} \tag{3}$$

where c.c. means the complex conjugate of all the previous terms. Therefore, any 2×2 complex matrix [5]-[8]:

$$\underset{\sim}{U} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \; Det\underset{\sim}{U} = \alpha\delta - \beta\gamma = 1, \tag{4}$$

generates one Lorentz matrix through (3).

The following relations, which are not explicitly in the literature, give us directly all the components (3):

$$\begin{array}{ll} L^u_v = -\frac{1}{2}U^{ar}\sigma^u_{aj}\sigma_{v\,br}U^{t\,bj}, & u,v=1,2,3, \\ L^u_0 = \frac{1}{2}\sigma^u_{jr}Q^{jr}, & u=0,\ldots,3, \\ L^0_v = -\frac{1}{2}\sigma_{v\,jk}R^{jk}, & v=1,2,3, \end{array} \tag{5}$$

such that:

$$\underset{\sim}{U}^t = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix},\ \underset{\sim}{Q} = \underset{\sim}{U}\underset{\sim}{U}^t,\ \underset{\sim}{R} = \underset{\sim}{U}^t\underset{\sim}{U}, \tag{6}$$

with the Infeld-van der Waerden symbols [9]:

$$\begin{array}{ll} (\sigma^0_{ab}) = (\sigma_{0\,ab}) = \underset{\sim}{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (\sigma^1_{ab}) = (-\sigma_{1\,ab}) = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (\sigma^2_{ab}) = (-\sigma_{2\,ab}) = -\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & (\sigma^3_{ab}) = (-\sigma_{3\,ab}) = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{array} \tag{7}$$

where $\sigma_j,\ j=x,y,z$ are the known Pauli matrices [6]-[8], [10].

The expressions (5) are originals and they show explicitly a direct relationship between $\underset{\sim}{L}$ and $\underset{\sim}{U}$, which may be useful in applications of spinorial calculus [9] in Minkowski spacetime.

References

[1] J. Aharoni, The special theory of relativity, Clarendon Press, Oxford, 1959.

[2] J.L. Synge, Relativity: the special theory, North-Holland Pub., Amsterdam, 1965.

[3] P.J. Greenberg and J.P. Knaver, Stud. Appl. Math., **53**(1974), 165.

[4] J. López-Bonilla, J. Morales and G. Ovando, Bull. Allahabad Math. Soc., **17**(2002), 53-58.

[5] J.L. Synge, Comm. Dublin Inst. Adv. Stud. Ser. A, No. 21, 1972.

[6] L.H. Ryder, Quantum field theory, Cambridge Univ. Press (1985).

[7] J. López-Bonilla, J. Morales and G. Ovando, Indian J. Theor. Phys., **52**(2004), 91-96.

[8] M. Acevedo, J. López-Bonilla and M. Sánchez, Apeiron, **12**(2005), No. 4, 371-384.

[9] G.F. Torres del Castillo, 3-D spinors, spin-weighted functions and their applications, Birkhser, Boston, 2003.

[10] W. Pauli, Z. Physik, **37**(1926), 263.

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Vol. 3 (2007), No. 3, 58-62

On the mean value of the Smarandache LCM function

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Abstract For any positive integer n, the F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them.

Keywords F.Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [2] and [3].

For example, Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

Jianbin Chen [5] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x}(SL(n) - P(n))^2 = \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

The main purpose of this paper is using the elementary methods to study the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them. That is, we shall prove the following two conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} P(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $P(n)$ denotes the largest prime divisor of n, and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} p(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, b_i $(i = 1, 2, \cdots, k)$ are computable constants and $b_1 = \dfrac{1}{3}$.

Whether there exist an asymptotic formula for $\sum\limits_{n\leq x} \dfrac{p(n)}{SL(n)}$ and $\sum\limits_{n\leq x} \dfrac{P(n)}{SL(n)}$ is an open problem.

§2. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof the theorems.

First we prove Theorem 1. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}. \tag{1}$$

Now we consider the summation

$$\sum_{n\leq x} P(n)SL(n). \tag{2}$$

We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $P(n) \geq \sqrt{n}$ and $n = m \cdot P(n)$, $m < P(n)$;

B: $n^{\frac{1}{3}} < P(n) \leq \sqrt{n}$ and $n = m \cdot P^2(n)$, $m < n^{\frac{1}{3}}$;

C: $n^{\frac{1}{3}} < p_1 < P(n) \leq \sqrt{n}$ and $n = m \cdot p_1 \cdot P(n)$, where p_1 is a prime;

D: $P(n) \leq n^{\frac{1}{3}}$.

It is clear that if $n \in A$, then from (1) we know that $SL(n) = P(n)$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n\in A}P(n)SL(n) &= \sum_{n\in A}P(n)^2=\sum_{\substack{m\cdot p\le x\\ m<p}}p^2=\sum_{m\le\sqrt{x}}\sum_{m<p<\frac{x}{m}}p^2\\ &= \sum_{m\le\sqrt{x}}\left[\frac{x^2}{m^2}\pi\left(\frac{x}{m}\right)-\int_m^{\frac{x}{m}}2y\pi(y)dy+O\left(m^3\right)\right]\\ &= \sum_{m\le\sqrt{x}}\left(\frac{x^3}{m^3}\sum_{i=1}^{k}\frac{b_i}{\ln^i\frac{x}{m}}+O\left(\frac{x^3}{m^3\cdot\ln^{k+1}\frac{x}{m}}\right)\right)\\ &= \zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),\end{aligned}\tag{3}$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1=\dfrac{1}{3}$, b_i $(i=2,3,\cdots,k)$ are computable constants.

Similarly, if $n\in C$, then we also have $SL(n)=P(n)$. So

$$\begin{aligned}\sum_{n\in C}P(n)SL(n) &= \sum_{\substack{mp_1p\le x\\ m<p_1<p}}p^2=\sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\sum_{p_1<p\le\frac{x}{p_1m}}p^2\\ &= \sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\left[\frac{x^2}{p_1^2m^2}\pi\left(\frac{x}{p_1m}\right)-p_1^2\pi(p_1)-\int_{p_1}^{\frac{x}{p_1m}}2y\pi(y)dy\right]\\ &= \sum_{m\le x^{\frac{1}{3}}}\frac{x^3}{m^3}\sum_{i=1}^{k}\frac{d_i(m)}{\ln^i\frac{x}{m}}+O\left(\frac{x^3}{m^3\ln^{k+1}\frac{x}{m}}\right)\\ &= x^3\cdot\sum_{i=1}^{k}\frac{h_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),\end{aligned}\tag{4}$$

where h_i $(i=1,2,\cdots,k)$ are computable constants.

Now we estimate the error terms in set B. Using the same method of proving (3),we have

$$\begin{aligned}\sum_{n\in B}P(n)SL(n) &= \sum_{m\cdot p^2\le x}p^3=\sum_{m\le x^{\frac{1}{3}}}\sum_{m<p<\sqrt{\frac{x}{m}}}p^3\\ &= \zeta(2)\cdot x^2\cdot\sum_{i=1}^{k}\frac{e_i}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right)\\ &= O(x^2).\end{aligned}\tag{5}$$

Finally, we estimate the error terms in set D. For any integer $n\in D$, let $SL(n)=p^\alpha$. We

assume that $\alpha \geq 1$. This time note that $P(n) \leq n^{\frac{1}{3}}$, we have

$$\begin{aligned}\sum_{n\in D} P(n)SL(n) &= \sum_{\substack{mp^\alpha \leq x \\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{\alpha+1} \ll \sum_{\substack{mp^\alpha \leq x \\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{2\alpha} \\ &\ll \sum_{\substack{p^\alpha \leq x \\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{2\alpha} \sum_{m\leq \frac{x}{p^\alpha}} 1 \\ &\ll x \sum_{\substack{p^\alpha \leq x \\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^\alpha \ll x^{\frac{7}{3}}. \end{aligned} \tag{6}$$

Combining (2), (3), (4), (5) and (6) we may immediately obtain the asymptotic formula

$$\begin{aligned}\sum_{n\leq x} P(n)SL(n) &= \sum_{n\in A} P(n)SL(n) + \sum_{n\in B} P(n)SL(n) \\ &+ \sum_{n\in C} P(n)SL(n) + \sum_{n\in D} P(n)SL(n) \\ &= x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right).\end{aligned}$$

where $P(n)$ denotes the largest prime divisor of n, and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

This proves Theorem 1.

Now we prove Theorem 2. We separate all integer n in the interval $[1, x]$ into four subsets $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ as follows: $\bar{A}$: $n = 1$; $\bar{B}$: $n = p^\alpha$, $\alpha \geq 1$; $\bar{C}$: $n = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_i \geq 1$, $(i = 1, 2)$; $\bar{D}$: $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, $\alpha_i \geq 1$, $(i = 1, 2, \cdots, s)$, $s \geq 3$. $p(n)$ denotes the smallest prime divisor of n, $p(1) = 0$ and $SL(1) = 1$. Then we have

$$\sum_{n\leq x} p(n)SL(n) = \sum_{n\in \bar{B}} p(n)SL(n) + \sum_{n\in \bar{C}} p(n)SL(n) + \sum_{n\in \bar{D}} p(n)SL(n). \tag{7}$$

Obviously if $n \in \bar{B}$, then from (1) we know that $SL(n) = p^\alpha$. Therefore,

$$\begin{aligned}\sum_{n\in \bar{B}} p(n)SL(n) &= \sum_{p^\alpha \leq x} pp^\alpha = \sum_{p\leq x} p^2 + \sum_{\substack{p^\alpha \leq x \\ \alpha\geq 2}} p^{\alpha+1} \\ &= x^2\pi(x) - \int_{\frac{3}{2}}^{x} 2y\pi(y)dy + O\left(\sum_{2\leq\alpha\leq \ln x} \sum_{p\leq x^{\frac{1}{\alpha}}} p^{2\alpha}\right) \\ &= x^2\pi(x) - \int_{\frac{3}{2}}^{x} 2y\pi(y)dy + O\left(x^{\frac{5}{2}}\right) \\ &= x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \end{aligned} \tag{8}$$

where $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

If $n \in \bar{C}$, then $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1 < p_2$, and $SL(n) \geq \sqrt{n}$, so we have

$$\begin{aligned}\sum_{n\in\bar{C}} p(n)SL(n) &= \sum_{p_1^{\alpha_1}p_2^{\alpha_2}\leq x} SL(p_1^{\alpha_1}p_2^{\alpha_2})p_1 \\ &= \sum_{p_1^{\alpha_1}\leq\sqrt{x}}\sum_{p_2^{\alpha_2}\leq\frac{x}{p_1^{\alpha_1}}} p_2^{\alpha_2}p_1 + \sum_{p_2^{\alpha_2}\leq\sqrt{x}}\sum_{p_1^{\alpha_1}\leq\frac{x}{p_2^{\alpha_2}}} p_1^{\alpha_1}p_1 \ll x^{\frac{11}{4}}. \qquad (9)\end{aligned}$$

Finally, we estimate the error terms in set $\bar{D}$, this time, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $s \geq 3$. Therefore, $n^{\frac{1}{3}} \leq SL(n) \leq \sqrt{n}$, and $p(n) \leq n^{\frac{1}{3}}$, so we have

$$\sum_{n\in\bar{D}} p(n)SL(n) \ll \sum_{n\leq x} n^{\frac{1}{3}} n^{\frac{1}{2}} \ll x^{\frac{11}{6}}. \qquad (10)$$

Combining (1), (7), (8), (9) and (10) we may immediately obtain the asymptotic formula

$$\sum_{n\leq x} p(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, and $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

This completes the proof of Theorem 2.

References

[1] F.Smarandache, Only Problems, not solutions, Chicago, Xiquan Publ. House, 1993.

[2] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[4] Lv Zhongtian, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No. 1, 22-25.

[5] Jianbin Chen, Value distribution of the F.Smarandache LCM function, Scientia Magna, **3**(2007), No. 2, 15-18.

[6] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[7] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

Scientia Magna
Vol. 3 (2007), No. 3, 63-69

Ishikawa iterative approximation of fixed points for multi-valued $\phi-$strongly pseudo-contract mappings[1]

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Abstract In the uniformly smooth Banach spaces, we discuss Ishikawa iterative approximation problem of fixed points for multi-valued $\phi-$strongly pseudo-contract mappings. Firstly we proved the strong convergence theorem of Ishikawa iterative sequence with error approximation multi-valued $\phi-$strongly pseudo-contract mappings, then the theorem of Ishikawa iterative with error convergence strongly to the unique solutions of the equation $f \in Tx$ and $f \in Tx + x$ for multi-valued $\phi-$strongly accretive are proved. Our results extend and generalize the results in reference [1].

Keywords $\phi-$strongly pseudo-contract, $\phi-$strongly accretive, fixed point, iterative sequence.

§1. Introduction and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*. $F(T)$ denote the set of all fixed points of T. The normalized duality mapping $J : E \to 2^{X^*}$ is defined by: $J(x) = \{f \in X^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|x\| = \|f\|\}$. Where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

It is well known that if X is uniformly smooth, then J is uniformly continuous on bounded subset of X. We denote the single-valued normalized duality mapping by j.

Definition 1. Let X be an arbitrary real Banach space and K be a nonempty subset in X. Suppose that $T : K \to 2^K$ is a multi-valued mapping.

1) T is said to be strongly accretive. If $\forall x, y \in K$ and $\forall \xi \in Tx$, $\forall \eta \in Ty$, there exist a $j(x-y) \in J(x-y)$ such that $\langle \xi - \eta, j(x-y)\rangle \geq k\|x-y\|^2$. Where $k > 0$ is a constant. Without loss of generality we assume $k \in (0, 1)$.

2) T is said to be $\phi-$strongly accretive. If $\forall x, y \in K$and $\forall \xi \in Tx$, $\forall \eta \in Ty$, such that $\langle \xi - \eta, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|$. Here exists a $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$.

3) T is said to be strongly pseudo-contractive. If $I - T$ is strongly accretive. Where I denotes the identity mapping.

[1]Foundation item: The project supported by the Shanxi Natural Science Foundation (2005A21).

4) T is said to be $\phi-$strongly pseudo-contractive. If $I-T$ is $\phi-$strongly accretive. Where I denotes the identity mapping. This implies that $\forall x, y \in K$ and $\forall \xi \in Tx$, $\forall \eta \in Ty$ such that $\langle \xi - \eta, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$. Here exists a $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$.

Definition 2. (1) Let X be an arbitrary real Banach space and K be a nonempty subset of X. Suppose that $T : K \to 2^K$ is a multi-valued mapping. $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequence in[0,1], then for any given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T x_n + v_n (n \geq 0). \end{cases} \tag{1}$$

is called the Ishikawa iterative sequence with error for T. Where $\{u_n\}$, $\{v_n\}$ are two bounded sequence in K.

(2) If $\beta_n = \alpha_n (\forall n \in N)$ in (1), the sequence $\{x_n\}$ defined by

$$x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T x_n + u_n (n \geq 0) \tag{2}$$

is called the Mann iterative sequence with errors for T. Where $\{u_n\}$ is bounded sequence in K.

Recently, the questions of Ishikawa iterative approximation of fixed points for multi-valued $\phi-$strongly pseudo-contract mappings have been studied by many authors [1-8]. Zhang[1]proved the strong convergence theorem which generalizes some results [3-8].

The purpose of this paper is to extend Theorem 2.1 in [1] in the following ways:

(1) The assumption that K is bounded have be removed.

(2) The assumption of the uniformly continuity of have be removed.

(3) The methods of proof have be improved.

And we investigate the problem of Ishikawa iterative sequence approximating to the unique solution of the equation $f \in Tx$ and $f \in Tx+x$ for multi-valued $\phi-$strongly accretive mapping.

We need the following lemmas.

Lemma 1.[2] Let X be real Banach space. For $\forall x, y \in X$, there exist a $j(x-y) \in J(x-y)$ such that $\|x-y\|^2 \leq \|x\|^2 + 2\langle y, j(x-y)\rangle$. Where $J : X \to 2^{X^*}$ is the normalized duality mapping.

Lemma 2. Let X be real Banach space and K be a nonempty subset of X. $T : X \to 2^X$ is a multi-valued $\phi-$strongly accretive mapping. $\forall f \in K$, the mapping $S : K \to 2^K$ defined by $Sx = f - Tx + x$. If $\forall x, y \in K$and $\forall \xi \in Tx$, $\forall \eta \in Ty$, here exists a $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ such that $\langle \xi - \eta, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|$, such as the mapping $S : K \to 2^K$ is a multi-valued $\phi-$strongly pseudo-contractive.

Proof. $T : K \to 2^K$ is a multi-valued $\phi-$ strongly pseudo-contractive for $\forall \xi \in Sx, \forall \eta \in Sy$, here exists $\xi_1 \in Tx$, $\eta_1 \in Ty$, $j(x-y) \in J(x-y)$ such that $\langle \xi - \eta, j(x-y)\rangle = \langle (f - \xi_1 + x) - (f - \eta_1 + y), j(x-y)\rangle = \langle x-y, j(x-y)\rangle - \langle \xi_1 - \eta_1, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$. Hence $S : K \to 2^K$ is a multi-valued $\phi-$strongly pseudo-contractive.

The following results extend the theorem 2.1 in [1].

§2. Main Results

Theorem 1. Suppose that X is an uniformly smooth real Banach space and K be a

nonempty closed convex subset of X. $T : K \to 2^K$ is a multi-valued $\phi-$strongly pseudo-contractive and the range $R(T)$ is bounded. Let $\{u_n\}$, $\{v_n\}$ are two bounded sequence in K, $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in [0,1] satisfying the conditions: (1) $\alpha_n \to 0$, $\beta_n \to 0 (n \to \infty)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$; (2) $\|u_n\| = o(\alpha_n)$,$\|v_n\| \to 0 (n \to \infty)$. If $F(T) = \{x \in K : x \in Tx\} \neq \emptyset$, then $\forall x_0 \in K$, the Ishikawa iterative sequence with errors $\{x_n\}$ defined by (1) strongly converges to the unique fixed point of T in K.

Proof. Let $q \in F(T) \neq \emptyset$, then $F(T) = \{q\}$. If $p \in F(T)$, then $p \in Tp$. Moreover there exist some $j(x-y) \in J(x-y)$ such that $\|p-q\|^2 = \langle p-q, j(p-q)\rangle \leq \|p-q\|^2 - \phi(\|p-q\|)\|p-q\|$. It follows that $\phi(\|p-q\|)\|p-q\| \leq 0$. From the definition 1, we know that $p = q$. It implies that $F(T) = \{q\}$.

Let $\{x_n\}$, $\{y_n\}$ is the Ishikawa iterative sequence defined by (1), there exists $\xi_n \in Tx_n$ and $\eta_n \in Ty_n$, such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\eta_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n\xi_n + v_n. \end{cases} \tag{3}$$

By using Lemma 1, we have,

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(\eta_n - q) - u_n\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle \eta_n - q, j(x_{n+1} - q)\rangle + 2\langle u_n, j(x_{n+1} - q)\rangle \\ &= (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle \eta_n - q, j(x_{n+1} - q) - j(y_n - q)\rangle \\ &\quad +2\alpha_n\langle \eta_n - q, j(y_n - q)\rangle + 2\langle u_n, j(x_{n+1} - q)\rangle. \end{aligned} \tag{4}$$

Because $\|u_n\| = o(\alpha_n)$, hence there exists $\delta_n \to 0 (n \to \infty)$ such that $\|u_n\| = \alpha_n\delta_n (\forall n > 0)$. Let $0 < \delta_n < 1 (\forall n > 0)$. The range $R(T)$ is bounded. Let $M_1 = \sup_{\xi \in R(T)} \|\xi - q\| + \|x_0 - q\| + 1$. First we proof:

$$\|x_n - q\| \leq M_1 (\forall n \geq 0). \tag{5}$$

If $n = 0$, it is easily seen that (5) holds. Suppose that for n (5) holds. For $n+1$, we obtain

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)(x_n - q) + \alpha_n(\eta_n - q) - u_n\| \\ &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n[\|\eta_n - q\| + \delta_n] \leq M_1. \end{aligned}$$

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)x_n + \beta_n\xi_n + v_n - x_n\| \\ &\leq \beta_n\|x_n - \xi_n\| + \|v_n\| \\ &\leq \beta_n[\|x_n - q\| + \|\xi_n - q\|] + \|v_n\| \\ &\leq 2\beta_n M_1 + \|v_n\|. \end{aligned}$$

Thus

$$\|y_n - x_n\| \to 0 (n \to \infty). \tag{6}$$

Since $\|y_n - q\| \leq \|y_n - x_n\| + \|x_n - q\|$, then the sequence $\{y_n - q\}_{n=1}^{\infty}$ is bounded in K. Now let $M = M_1 + \sup_{n \geq 0} \|y_n - q\|$. We have

$$\begin{aligned} \|(x_{n+1} - q) - (y_n - q)\| &= \| - \alpha_n(x_n - q) + \alpha_n(\eta_n - q) - u_n + \beta_n(x_n - q) - \beta_n(\xi_n - q) - v_n\| \\ &\leq 2[\alpha_n + \beta_n]M + \|u_n\| + \|v_n\| \to 0 (n \to \infty). \end{aligned}$$

Because X is uniformly smooth, then j is uniformly continuous on bounded subset of X, we obtain $\|j(x_{n+1}-q)-j(y_n-q)\| \to 0 (n \to \infty)$. Let $e_n = \langle \eta_n - q, j(x_{n+1}-q)-j(y_n-q)\rangle$. We have $e_n \to 0 (n \to \infty)$. Since T is $\phi-$ strongly pseudo-contractive mapping. Then it follows form (4) that

$$\|x_{n+1}-q\|^2 \le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n e_n + 2\alpha_n\|y_n-q\|^2$$
$$-2\alpha_n\phi(\|y_n-q\|)\|y_n-q\| + 2\|u_n\|\cdot\|x_{n+1}-q\|. \tag{7}$$

$$\begin{aligned}\|y_n-q\|^2 &= \|(1-\beta_n)x_n+\beta_n\xi_n+v_n-q\|^2\\ &= \|(x_n-q)-\beta_n(x_n-q)+\beta_n(\xi_n-q)+v_n\|^2\\ &\le (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n\langle\xi_n-q, j(y_n-q)\rangle + 2\langle v_n, j(y_n-q)\rangle\\ &\le (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n M^2 + 2\|v_n\|M. \end{aligned}\tag{8}$$

Substituting (8) into (7), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n e_n + 2\alpha_n(1-\beta_n)^2\|x_n-q\|^2\\ &+ 2\alpha_n(2\beta_n M^2 + 2\|v_n\|M) - 2\alpha_n\phi(\|y_n-q\|)\|y_n-q\| + 2\|u_n\|\cdot\|x_{n+1}-q\|\end{aligned}$$
$$\le \|x_n-q\|^2 - \alpha_n\phi(\|y_n-q\|)\|y_n-q\| + \alpha_n[\lambda_n - \phi(\|y_n-q\|)\|y_n-q\|], \tag{9}$$

where $\lambda_n = (\alpha_n + 2\beta_n^2)M^2 + (4\|v_n\| + 2\delta_n)M + 2e_n$.

Since

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0,$$

and as $n \to \infty$, $\|v_n\| \to 0$, $\delta_n \to 0$, $e_n \to 0$.

We obtain $\lambda_n \to 0 (n \to \infty)$. We show that

$$\inf_{n\ge 0}\|y_n - q\| = 0.$$

If not assume that

$$\inf_{n\ge 0}\|y_n - q\| = \delta > 0,$$

that for every $n \ge 0$, $\|y_n - q\| \ge \delta > 0$, hence

$$\phi(\|y_n-q\|)\|y_n-q\| \ge \phi(\delta)\delta > 0.$$

Since

$$\lim_{n\to\infty}\lambda_n = 0,$$

we see that, there exists some fixed $N_0 \in N$, such that

$$\lambda_n < \phi(\delta)\delta,$$

for all $n > N_0$.

From (9), for every $n > N_0$, we have

$$\|x_{n+1}-q\|^2 \le \|x_n-q\|^2 - \alpha_n\phi(\delta)\delta.$$

Leads to

$$\phi(\delta)\delta \sum_{n=N_0}^{\infty} \alpha_n \leq \|x_{N_0} - q\|^2,$$

which is in contradiction with the assumption that

$$\sum_{n=0}^{\infty} \alpha_n = \infty.$$

Hence there exists a subsequence $\|y_{n_i} - q\|_{i=1}^{\infty}$ of $\|y_n - q\|_{n=1}^{\infty}$, such that $\|y_{n_i} - q\| \to 0 (i \to \infty)$. Using (6), we have

$$\|x_{n_i} - q\| \to 0 (i \to \infty).$$

We have

$$\forall \varepsilon > 0, \exists i_0 \in N, \forall i \geq i_0, \|x_{n_i} - q\| < \varepsilon. \tag{10}$$

Suppose that $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$, $\|u_n\| \to 0 (n \to \infty)$, $\|v_n\| \to 0 (n \to \infty)$ and $\lim\limits_{n\to\infty} \lambda_n = 0$, then there exists some $N_1 \in N (N_1 > N_0)$. $\forall n > N_1$, such that $0 \leq \alpha_n < \frac{\varepsilon}{16M}$, $0 \leq \beta_n < \frac{\varepsilon}{16M}$, $\|u_n\| < \frac{\varepsilon}{8}$, $\|v_n\| < \frac{\varepsilon}{8}$, $0 \leq \lambda_n < \phi(\frac{\varepsilon}{2})\varepsilon$.

Now we want to prove that

$$\|x_{n_i+k} - q\| < \varepsilon. \tag{11}$$

as $n_i \geq N_1$ for all $K \in N$ and $\forall i \geq i_0$. Then $k = 1$, if $\|x_{n_i+1} - q\| \geq \varepsilon$, then $\forall n_i \geq N_1$,

$\|x_{n_i+1} - q\| = \|(x_{n_i} - q) + \alpha_{n_i}(\eta_{n_i} - x_{n_i}) + u_{n_i}\| \leq \|x_{n_i} - q\| + \alpha_{n_i}\|\eta_{n_i} - x_{n_i}\| + \|u_{n_i}\|$.

Hence $\|x_{n_i} - q\| \geq \|x_{n_i+1} - q\| - \alpha_{n_i}\|\eta_{n_i} - x_{n_i}\| - \|u_{n_i}\| > \varepsilon - 2M\frac{\varepsilon}{16M} - \frac{\varepsilon}{8} = \frac{3}{4}\varepsilon > 0$,

$$\begin{aligned} \|y_{n_i} - q\| &= \|(1-\beta_{n_i})x_{n_i} + \beta_{n_i}\xi_{n_i} + v_{n_i} - q\| \\ &\geq \|x_{n_i} - q\| - \beta_{n_i}\|\xi_{n_i} - x_{n_i}\| - \|v_{n_i}\| \\ &\geq \frac{3}{4}\varepsilon - 2M\frac{\varepsilon}{16M} - \frac{\varepsilon}{8} = \frac{1}{2}\varepsilon > 0. \end{aligned}$$

Using (9), (10) and definition 1, we have,

$$\begin{aligned} \varepsilon^2 &\leq \|x_{n_i+1} - q\|^2 \leq \|x_{n_i} - q\|^2 + \lambda_{n_i}\alpha_{n_i} - 2\alpha_{n_i}\phi(\|y_{n_i} - q\|)\|y_{n_i} - q\| \\ &\leq \varepsilon^2 + \alpha_{n_i}\phi(\frac{\varepsilon}{2})\varepsilon - 2\alpha_{n_i}\phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2} = \varepsilon^2. \end{aligned}$$

This is a contradiction. Suppose that $\|x_{n_i+k} - q\| < \varepsilon$, we show that $\|x_{n_i+k+1} - q\| < \varepsilon$. If not, assume that $\|x_{n_i+k+1} - q\| \geq \varepsilon$, then

$$\|x_{n_i+k} - q\| \geq \|x_{n_i+k+1} - q\| - \alpha_{n_i+k}\|\eta_{n_i+k} - x_{n_i+k}\| - \|u_{n_i+k}\| > \varepsilon - 2M\frac{\varepsilon}{16M} - \frac{\varepsilon}{8} = \frac{3}{4}\varepsilon > 0$$

$$\|y_{n_i+k} - q\| \geq \|x_{n_i+k} - q\| - \beta_{n_i+k}\|\xi_{n_i+k} - x_{n_i+k}\| - \|v_{n_i+k}\| > \frac{3}{4}\varepsilon - 2M\frac{\varepsilon}{16M} - \frac{\varepsilon}{8} = \frac{1}{2}\varepsilon > 0$$

$$\begin{aligned} \varepsilon^2 &\leq \|x_{n_{i+1}+k+1} - q\|^2 \leq \|x_{n_i+k} - q\|^2 + \lambda_{n_i+k}\alpha_{n_i+k} - 2\alpha_{n_i+k}\phi(\|y_{n_i+k} - q\|)\|y_{n_i+k} - q\| \\ &\leq \varepsilon^2 + \alpha_{n_i+k}\phi(\frac{\varepsilon}{2})\varepsilon - 2\alpha_{n_ik}\phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2} = \varepsilon^2, \end{aligned}$$

which is a contradiction. So we can deduce that $\lim\limits_{n\to\infty} \|x_n - q\| = 0$.

This completes the proof.

§3. Ishikawa iterative process for solution of the equation

Theorem 2. Suppose that X is an uniformly smooth real Banach space and K be a nonempty convex subset in X. $T : K \to 2^K$ is a multi-valued $\phi-$strongly accretive mapping, where $\{\alpha_n\}$,$\{\beta_n\}$ are two real sequences in [0, 1], and $\{u_n\}$,$\{v_n\}$ are two bounded sequences in K satisfying the condition:

(1) $\alpha_n \to 0$,$\beta_n \to 0 (n \to \infty)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $\|u_n\| = \circ(\alpha_n)$,$\|v_n\| \to 0 (n \to \infty)$. $\forall f \in K$, for every $x \in K$, the mapping $S : K \to 2^K$ defined by $Sx = f - Tx + x$. The range $R(S)$ is bounded. If $F(S) \neq \emptyset$, then for $\forall x_0 \in K$, the Ishikawa iterative sequence with errors $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n \in (1-\beta_n)x_n + \beta_n S x_n + v_n, \end{cases}$$

strongly converges to the unique solution of equations $f \in Tx$.

Proof. Let $q \in F(S) \neq \phi$, then $q \in S(q) = f - Tq + q$. We have $f \in Tq$. Since T is a multi-valued ϕ-strongly accretive mapping. By lemma 2, the mapping $S : K \to 2^K$ defined by $Sx = f - Tx + x$ is a multi-valued ϕ-strongly pseudo-contractive. Since the range $R(S)$ is bounded, by theorem 1, $\{x_n\}$ strongly converges to the unique fixed point q of S in K, then the sequence $\{x_n\}$ strongly converges to the unique solution q of equations $f \in Tx$. This completes the proof.

Corollary. Suppose that X is an uniformly smooth real Banach space and K be a nonempty convex subset in X. $T : K \to 2^K$ is a multi-valued $\phi-$strongly accretive mapping, where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in[0,1], and $\{u_n\}$, $\{v_n\}$ are two bounded sequence in K satisfying the conditions:

(1) $\alpha_n \to 0$,$\beta_n \to 0 (n \to \infty)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $\|u_n\| = \circ(\alpha_n)$, $\|v_n\| \to 0 (n \to \infty)$. $\forall f \in K$, for every $x \in K$, the mapping $S : K \to 2^K$ defined by $Sx = f - Tx$. The range $R(S)$ is bounded. If $F(S) \neq \emptyset$, then $\forall x_0 \in K$, the Ishikawa iterative sequence with errors $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n \in (1-\beta_n)x_n + \beta_n S x_n + v_n, \end{cases}$$

strongly converges to the unique solution of equations $f \in Tx + x$.

Proof. Let $T^{'} = I + T$, it follows $f \in Tx + x$ and $f \in T^{'}x$. By the theorem 2, the sequence $\{x_n\}$ strongly converges to the unique solution of equations $f \in T^{'}x$. The sequence $\{x_n\}$ strongly converges to the unique solution of equations $f \in Tx + x$.

This completes the proof.

References

[1] Zhang S. S., Ishikawa iterative approximations of fixed points and solutions for multi-valued ϕ-strongly accretive and multi-valued ϕ-strongly pseudo-contractive mappings, Journal of mathematical research and exposition, **22**(2002), No. 3, 447-453.

[2] Chang S. S., Some problems and results in the study of nonlinear analysis, Nonlinear Anal TMA, **30**(1997), No. 7, 4197-4208.

[3] Chidume C. E., Approximation of fixed points of strongly pscudo-contractive mapping, Proc. Amer. Math. Soc., **120**(1994), 545-551.

[4] Tan K. K., Xu H K. Iterative approximation of Lipschitz strongly accretive operators in Banach space, J. Math. Anal. Appl., **99**(1993), No. 178, 9-21 .

[5] Osilike M. O., Ishikawa solution of nonlinear equations of the ϕ-strongly accretive type, J. Math. Anal. Appl., **200**(1996), 259-271.

[6] Zhou H. Y., Some convergence theorems for Ishikawa iteration sequences of certain nonlinear operators in uniformly smooth Banach spaces, Acta. Appl. Math., **3**(1989), No. 32, 183-196.

[7] Chang S. S., On Chidume s open question and approximate salution of multived strongly accretive mapping equations in Banach spaces, J. Math. Anal. Appl., **216**(1997), 94-111.

[8] Dend L., Ding X. P., Iterative approximations of Lipschitz strictly pseudo contractive mappings in uniformly smooth Banach spaces, Nonlinear, Anal. TMA., **2430**(1995), 981-987.

Scientia Magna
Vol. 3 (2007), No. 3, 70-73

On the divisibility of the Smarandache combinatorial sequence

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Abstract In this paper, we study the divisibility of the Smaradache combinatorial sequence of degree r, and prove that there has only the consecutive terms 1, 2, 3 of the Smarandache combinatorial sequence of degree 3 are pairwise coprime.

Keywords Smarandache combinatorial sequences, divisibility, congruence.

§1. Introduction and Results

Let r be an integer with $r > 1$. We define the Smarandache combinatorial sequence of degree r as $SCS(r) = \{a(r,n)\}_{n=1}^{\infty}$, where $a(r,n) = n$ $(n = 1, 2, \cdots, r)$, and $a(r,n)$ $(n > r)$ is the sum of all the products of the previous terms of the sequence taking r terms at a time. In [1], Murthy asked that how many of the consecutive terms of $SCS(r)$ are pairwise coprime. Le Maohua [2] study the divisibility of the Smaradache combinatorial sequence of degree two, and obtained the following conclusion:

- For any positive integer n , we have $a(2, n+1) \equiv 0 (\mathrm{mod}\ a(2,n))$.
- There has only the consecutive terms 1, 2 of $SCS(2)$ are pairwise coprime.

In this paper, we use the elementary method to study the divisibility of the Smaradache combinatorial sequence of degree r and prove the following results:

Theorem. Let $a(r,n) = b(n)$, if $n > r$, then we have

$$b(n) = \frac{1}{r!}\left\{\left(\sum_{i=0}^{n-1} b(i)\right)^r - \sum_{I} \frac{r!}{r_1! r_2! \cdots r_{n-1}!} b^{r_1}(1) b^{r_2}(2) \cdots b^{r_{n-1}}(n-1)\right\},$$

where

$$I = \left\{(r_1, r_2, \cdots, r_{n-1}) | \sum_{i=1}^{n-1} r_i = r,\ \ 0 \le r_i \le r\right\},$$

any r components of the vector $(r_1, r_2, \cdots, r_{n-1})$ can't equate to 1 at the same time.

Corollary 1. Let n, r are two positive integers, if $n > r$, then we have

$$a(r, n+1) \equiv 0 (\mathrm{mod}\ a(r,n)).$$

By the above corollary 1, we may immediately obtain the following corollary.

Corollary 2. There has only the consecutive terms 1, 2, 3 of $SCS(3)$ are pairwise coprime.

§2. A simple lemma

To complete the proof of the theorem, we need the following simple lemma.

Lemma. Let $r,\ r_1,\ r_2\cdots r_n$ are positive integers and $b_1,\ b_2,\cdots,b_n$ are arbitrary real numbers, we have

$$(b_1+b_2+\cdots+b_n)^r=\sum_{\substack{r_1+r_2+\cdots+r_n=r\\0\leq r_i\leq r}}\frac{r!}{r_1!r_2!\cdots r_n!}b_1^{r_1}b_2^{r_2}\cdots b_n^{r_n},\quad (i=1,2,\cdots,n). \tag{1}$$

Proof. We use induction on n.

The formula (1) is true for $n=1$. If $n=2$, then we have

$$\begin{aligned}\text{The right hand of (1)} &= b_1^r+\frac{r!}{(r-1)!}b_1^{r-1}b_2+\frac{r!}{(r-2)!2!}b_1^{r-1}b_2^2+\cdots+b_2^r\\ &= \sum_{i=0}^{r}\mathbf{C}_r^ib_1^{r-i}b_2^i\\ &= (b_1+b_2)^r=\text{the left hand of (1)}\end{aligned}$$

Therefore, the formula (1) is true for $n=2$. Assume that the lemma is true for $n\leq k$, if $n=k+1$, then we have

$$\sum_{\substack{r_1+r_2+\cdots+r_{k+1}=\ r\\0\leq r_i\leq r}}\frac{r!}{r_1!r_2!\cdots r_{k+1}!}\ b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}b_{k+1}^{r_{k+1}}$$

$$\begin{aligned}=&\sum_{\substack{r_1+r_2+\cdots+r_k=r\\0\leq r_i\leq r}}\frac{r!}{r_1!r_2!\cdots r_k!}\ b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}\\ &+\mathbf{C}_r^1\sum_{\substack{r_1+r_2+\cdots+r_k=r-1\\0\leq r_i\leq r-1}}\frac{(r-1)!}{r_1!r_2!\cdots r_k!}b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}b_{k+1}\\ &+\mathbf{C}_r^2\sum_{\substack{r_1+r_2+r_k=r-2\\0\leq r_i\leq r-2}}\frac{(r-2)!}{r_1!r_2!\cdots r_k!}b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}b_{k+1}^2+\cdots\\ &+\mathbf{C}_r^{r-1}\sum_{\substack{r_1+r_2+\cdots+r_k=1\\0\leq r_i\leq 1}}\frac{1}{r_1!r_2!\cdots r_k!}b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}b_{k+1}^{r-1}+b_{k+1}^r\\ =&\ (b_1+b_2+\cdots+b_k)^r+\mathbf{C}_r^1(b_1+b_2+\cdots+b_k)^{r-1}b_{k+1}\\ &+\mathbf{C}_r^2(b_1+b_2+\cdots+b_k)^{r-2}b_{k+1}^2+\cdots\\ &+\mathbf{C}_r^{r-1}(b_1+b_2+\cdots+b_k)b_{k+1}^{r-1}+b_{k+1}^r\\ =&\sum_{i=0}^{r}\mathbf{C}_r^i(b_1+b_2+\cdots+b_k)^{r-i}b_{k+1}^i\\ =&\ (b_1+b_2+\cdots+b_{k+1})^r.\end{aligned}$$

Thus, the lemma is proved.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. Let $a(r,n) = b(n)$, then from the definition of $SCS(r)$ we know that $b(1) = 1, b(2) = 2, \cdots, b(r) = r$. If $n > r$, then we get from (1) that

$$\left(\sum_{i=0}^{n-1} b(i)\right)^r = \sum_{\substack{r_1+r_2+\cdots+r_{n-1}=r \\ 0\le r_i\le r}} \frac{r!}{r_1!r_2!\cdots r_{n-1}!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_{n-1}}(n-1). \tag{2}$$

By the definition of $SCS(r)$ and formula (2), we have

$$b(n) = \frac{1}{r!}\left\{\left(\sum_{i=0}^{n-1} b(i)\right)^r - \sum_I \frac{r!}{r_1!r_2!\cdots r_{n-1}!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_{n-1}}(n-1)\right\}, \tag{3}$$

where

$$I = \left\{(r_1, r_2, \cdots, r_{n-1}) | \sum_{i=1}^{n-1} r_i = r,\ 0 \le r_i \le r\right\},$$

any r components of the vector $(r_1, r_2, \cdots, r_{n-1})$ can't equate to 1 at the same time.

Thus, the theorem is proved.

From the formula (2), we also have

$$b(n+1) = \frac{1}{r!}\left\{\left(\sum_{i=0}^{n} b(i)\right)^r - \sum_J \frac{r!}{r_1!r_2!\cdots r_n!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_n}(n)\right\}, \tag{4}$$

where

$$J = \left\{(r_1, r_2, \cdots, r_n) | \sum_{i=1}^{n} r_i = r,\ 0 \le r_i \le r\right\},$$

any r components of the vector $(r_1, r_2, \cdots, r_n)$ can't equate to 1 at the same time.

Since

$$\begin{aligned}
& b(n+1) - b(n) \\
= & \frac{1}{r!}\left[\left(\sum_{i=0}^{n} b(i)\right)^r - \sum_J \frac{r!}{r_1!r_2!\cdots r_n!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_n}(n) - \left(\sum_{i=0}^{n-1} b(i)\right)^r\right] \\
& + \frac{1}{r!}\sum_I \frac{r!}{r_1!r_2!\cdots r_{n-1}!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_{n-1}}(n-1) \\
= & \frac{b(n)}{(r-1)!}\left[\left(\sum_{i=0}^{n-1} b(i)\right)^{r-1} - \sum_K \frac{(r-1)!}{r_1!r_2!\cdots r_{n-1}!} b^{r_1}(1)b^{r_2}(2)\cdots b^{r_{n-1}}(n-1)\right],
\end{aligned}$$

where

$$K = \left\{(r_1, r_2, \cdots, r_{n-1}) | \sum_{i=1}^{n-1} r_i = r-1, 0 \le r_i \le r-1\right\},$$

any $r-1$ components of the vector $(r_1, r_2, \cdots, r_{n-1})$ can't equate to 1 at the same time.

Hence

$$b(n+1) - b(n) = Cb(n),$$

where $C = a(r-1,n)$ is a constant.

Using the properties of congruence, we have

$$a(r,n+1) \equiv 0(\text{mod } a(r,n)).$$

Thus, the corollary 1 is proved .

References

[1] A.Murthy, Some new Smarandache sequences, functions and partitions, Smarandache Nations, **11**(2000), 179-183.

[2] Le Maohua, The divisibility of the Smarandache combinatorial sequence of degree two, Smarandache Notions, **14**(2004), 153-155.

[3] Tom M.Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

Scientia Magna
Vol. 3 (2007), No. 3, 74-77

On the Smarandache prime part

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Abstract For any positive integer n, the Smarandache Superior Prime Part $P_p(n)$ is the smallest prime number greater than or equal to n; For any positive integer $n \geq 2$, the Smarandache Inferior Prime Part $p_p(n)$ is the largest prime number less than or equal to n. The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of $\frac{S_n}{I_n}$, and give an interesting asymptotic formula for it, where $I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n$ and $S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n$.

Keywords Smarandache superior prime part, Smarandache inferior prime part, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer $n \geq 1$, the Smarandache Superior Prime Part $P_p(n)$ is defined as the smallest prime number greater than or equal to n. For example, the first few values of $P_p(n)$ are $P_p(1) = 2$, $P_p(2) = 2$, $P_p(3) = 3$, $P_p(4) = 5$, $P_p(5) = 5$, $P_p(6) = 7$, $P_p(7) = 7$, $P_P(8) = 11$, $P_p(9) = 11$, $P_p(10) = 11$, $P_p(11) = 11$, $P_p(12) = 13$, $P_p(13) = 13$, $P_p(14) = 17$, $P_p(15) = 17$, $\cdots$. For any positive integer $n \geq 2$, we also define the Smarandache Inferior Prime Part $p_p(n)$ as the largest prime number less than or equal to n. Its first few values are $p_p(2) = 2$, $p_p(3) = 3$, $p_p(4) = 3$, $p_p(5) = 5$, $p_p(6) = 5$, $p_p(7) = 7$, $p_p(8) = 7$, $p_p(9) = 7$, $p_p(10) = 7$, $p_p(11) = 11$, $\cdots$. In the book "Only problems, Not solutions" (see reference [1], problems 39), Professor F.Smarandache asked us to study the properties of the sequences $\{P_p(n)\}$ and $\{p_p(n)\}$. About these problems, it seems that none had studied them, at least we have not seen related results before. But these problems are very interesting and important, because there are close relationship between the Smarandache prime part and the prime distribution problem. Now we define

$$I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n,$$

and

$$S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n.$$

In problem 10 of reference [2], Kenichiro Kashihara asked us to determine:

(A). If $\lim_{n\to\infty} (S_n - I_n)$ converges or diverges. If it converges, find the limit.

(B). If $\lim_{n\to\infty} \frac{S_n}{I_n}$ converges or diverges. If it converges, find the limit.

For the problem (A), we can not make any progress at present. But for the problem (B), we have solved it completely. In fact we shall obtain a sharper result.

In this paper, we use the elementary and analytic methods to study the asymptotic properties of $\frac{S_n}{I_n}$, and give a shaper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{3}}\right).$$

From this theorem we may immediately deduce the following:

Corollary. The limit S_n/I_n converges as $n \longrightarrow \infty$, and

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$

This solved the problem B of reference [2].

§2. Some lemmas

In order to complete the proof of the theorem, we need the following several lemmas.

First we have

Lemma 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{p_{n+1}\le x} (p_{n+1} - p_n)^2 \ll x^{\frac{23}{18}+\varepsilon},$$

where p_n denotes the n-th prime, ε denotes any fixed positive number.

Proof. This is a famous result due to D.R.Heath Brown [3] and [4].

Lemma 2. Let x be a positive real number large enough, then there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

Proof. For any real number x large enough, let P_n denotes the largest prime with $P_n \le x$.

Then from Lemma 1, we may immediately deduce that

$$(P_n - P_{n-1})^2 \ll x^{\frac{23}{18}+\varepsilon},$$

or

$$P_n - P_{n-1} \ll x^{\frac{2}{3}}.$$

So there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

This proves Lemma 2.

Lemma 3. For any real number $x > 1$, we have the asymptotic formulas

$$\sum_{n\le x} P_p(n) = \frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right),$$

and

$$\sum_{n\le x} p_p(n) = \frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right).$$

Proof. We only prove first asymptotic formula, similarly we can deduce the second one.

Let P_k denotes the k-th prime. Then from the definition of $P_p(n)$ we know that for any fixed prime P_r, there exist $P_{r+1} - P_r$ positive integer n such that $P_p(n) = P_r$.

So we have

$$\begin{aligned}\sum_{n\le x} P_p(n) &= \sum_{P_{n+1}\le x} P_n\cdot(P_{n+1}-P_n)\\ &= \frac{1}{2}\sum_{P_{n+1}\le x}\left(P_{n+1}^2-P_n^2\right)-\frac{1}{2}\sum_{P_{n+1}\le x}(P_{n+1}-P_n)^2\\ &= \frac{1}{2}P^2(x)-2-\frac{1}{2}\sum_{P_{n+1}\le x}(P_{n+1}-P_n)^2, \qquad (1)\end{aligned}$$

where $P(x)$ denotes the largest prime such that $P(x)\le x$.

From Lemma 2, we know that

$$P(x) = x + O\left(x^{\frac{2}{3}}\right). \qquad (2)$$

Now from (1), (2) and Lemma 1, we may immediately deduce that

$$\sum_{n\le x} P_p(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{5}{3}}\right) + O\left(x^{\frac{23}{18}+\varepsilon}\right) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{5}{3}}\right).$$

This proves the first asymptotic formula of Lemma 3.

The second asymptotic formula follows from Lemma 1, Lemma 2 and the identity

$$\begin{aligned}\sum_{n\le x} p_p(n) &= \sum_{P_n\le x} P_n\cdot(P_n-P_{n-1})\\ &= \frac{1}{2}\sum_{P_n\le x}\left(P_n^2-P_{n-1}^2\right)+\frac{1}{2}\sum_{p_n\le x}(P_n-P_{n-1})^2\\ &= \frac{1}{2}P^2(x)+\frac{1}{2}\sum_{P_n\le x}(P_n-P_{n-1})^2.\end{aligned}$$

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. In fact for any positive integer $n > 1$, from Lemma 3 and the definition of I_n and S_n we have

$$I_n = \{p_p(2)+p_p(3)+\cdots+p_p(n)\}/n = \frac{1}{n}\left[\frac{1}{2}n^2+O\left(n^{\frac{5}{3}}\right)\right] = \frac{1}{2}n+O\left(n^{\frac{2}{3}}\right), \qquad (3)$$

and

$$S_n = \{P_p(2)+P_p(3)+\cdots+P_p(n)\}/n = \frac{1}{n}\left[\frac{1}{2}n^2+O\left(n^{\frac{5}{3}}\right)\right] = \frac{1}{2}n+O\left(n^{\frac{2}{3}}\right). \qquad (4)$$

Combining (3) and (4), we have

$$\frac{S_n}{I_n} = \frac{\frac{1}{2}n+O\left(n^{\frac{2}{3}}\right)}{\frac{1}{2}n+O\left(n^{\frac{2}{3}}\right)} = 1+O\left(n^{\frac{-1}{3}}\right).$$

This completes the proof of Theorem.

The corollary follows from our Theorem as $n \longrightarrow \infty$.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[3] D.R.Heath-Brown, The differences between consecutive primes, Journal of London Math. Soc., **18**(1978), No. 2, 7-13.

[4] D.R.Heath-Brown, The differences between consecutive primes, Journal of London Math. Soc., **19**(1979), No. 2, 207-220.

[5] D.R.Heath-Brown and H.Iwaniec, On the difference of consecutive primes, Invent. Math., **55**(1979), 49-69.

[6] I.Balacenoiu and V.Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), 192-201.

[7] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[8] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[9] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[10] Lv Zhongtian, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No. 1, 22-25.

[11] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[12] Zhang Wenpeng, Elementary Number Theory, Shaanxi Normal University Press, Shaanxi, 2007.

Scientia Magna
Vol. 3(2007), No. 3, 78-81

The existence of solutions to the Direchlet problem

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Abstract Using Schauder fixed point theorem,the existence of solutions to the Dirichlet problem $\Delta u = f(x,u)$, $u|_{\partial\Omega} = 0$ is obtained.

Keywords Dirichlet problem, Schauder fixed point theorem, similar Hartman condition.

§1. Introduction

Consider the Dirichlet problem

$$(P) \qquad \Delta u = f(x,u), x \in R; u = 0, x \in \partial R.$$

Where Ω is a bounded neighborhood of R^n with smooth boundary $f : \overline{\Omega} \times R \to R$ is a caratheodory mapping. On research of this problem, most articles always exert some given conditions on nonlinear term f, then obtain the existence of solutions with calculus of variations, see [1],[2],[3]. In [4], the author exerted Hartman conditions on nonlinear term f studied the case $n = 1$(also see [5]), where Schauder fixed point theorem is used and the existence of solutions for problem (P) is obtained. In this paper, we generalize the method of [4] into more ordinary problem (P) for the case $n \geq 1$).

§2. The preparing knowledge and the lemma

Assume $|\cdot|_P$ is the norm of $L^P(\Omega)$ space and $\|\cdot\|$ is the norm of $H_0^1(\Omega)$ space, which can be induced from the following definition of inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx, u, v \in H_0^1(\Omega).$$

Suppose λ_1 is the first characteristic value of $-\triangle$, then the inequality of Poicare

$$\int_\Omega u^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla u|^2 dx, u \in H_0^1(\Omega). \tag{1}$$

holds.

Lemma. $\forall h \in L^2(\Omega)$, there is a unique solution $s(h)$ to the problem

$$(P_0) \qquad \Delta u - u = h(x), u|_{\partial\Omega} = 0.$$

Where the mapping $S : L^2(\Omega) \to H_0^1(\Omega)$ is continuous S maps the bounded set of $L^2(\Omega)$ into the bounded set of $H_0^1(\Omega)$.

Proof. Obviously, the solutions of problem (P_0) are equal to the critical points of functional $J_h(u)$, where

$$J_h(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + \frac{u^2}{2} + h(x)u(x)]dx, u \in H_0^1(\Omega). \tag{2}$$

Thus we can easily know that J_h is a strictly convex and lower semi-continuous coercive functional. We can also conclude that the unique minimum point u_0 is a weak solution of problem (P_0) and every minimal sequence $(V_k)_{k\in N}$ of functional $J_h(u)$ strongly converges to u_0, where $u \in H_0^1(\Omega)$. Define mapping S as following:

$$S : L^2(\Omega) \to H_0^1(\Omega), h \mapsto S(h) = u_0.$$

If $(h_k)_{k\in N} \in L^2(\Omega)$ is a sequence converging to h and $u_k = S(h_k)$ is a minimum point of J_{h_k} , then $(u_k)_{k\in N}$ is bounded in $H_0^1(\Omega)$ space and we have

$$\begin{aligned} J_h(u_0) \le J_h(u_k) &= J_{h_k}(u_k) + \int_\Omega (h - h_k)u_k dx \\ &\le J_{h_k}(u_0) + \int_\Omega (h - h_k)u_k dx \\ &= J_h(u_0) + \int_\Omega (h - h_k)(u_k - u_0)dx. \end{aligned}$$

So $(u_k)_{k\in N}$ is also a minimal sequence of J_h and strongly converges to u_0 , where $u_k \in H_0^1(\Omega)$. Thus the mapping $S : L^2(\Omega) \to H_0^1(\Omega)$ is continuous.

Finally, if $|h|_2 \le R$, where R is a positive number, then from $J_h(S(h)) \le J_h(0) = 0$ and $u_0 = S(h)$, we can obtain

$$\int_\Omega \frac{|\nabla u|^2}{2}dx + \int_\Omega \frac{u_0^2}{2}dx \le |h|_2|u|_2 \le R|u_0|_2 \le CR\|u_0\|.$$

So the norm $\|u_0\|$ is bounded. The proof is complete.

§3. Main results

Theorem. If

$$|f(x,t)| \le C_1 + C_2|t|, \forall x \in \overline{\Omega}, \forall t \in R, \tag{3}$$

where $C_1, C_2, R > 0$, such that

$$\int_\Omega f(x,u) \cdot u dx \ge 0, \tag{4}$$

for all $u \in L^2(\Omega)$ and $|u|_2 = R$, then the problem (P) has at least a solution u such that $|u|_2 \leq R$.

Proof. Define a mapping $P_R : L^2(\Omega) \to L^2(\Omega)$, where $P_R(u) = u$ when $|u|_2 \leq R$ and $P_R(u) = \frac{Ru}{u_2}$ when $|u|_2 > R$. We can easily conclude that the mapping P_R is continuous and bounded in $L^2(\Omega)$ space. Consider the problem

$$\Delta u - u = f(x, P_R(u)) - P_R(u) = f_R(x, u),$$

where $u|_{\partial\Omega} = 0$. Obviously, problem (P_1) is equal to problem (P) when $|u|_2 \leq R$. From the lemma, we know that the Dirichlet problem $\Delta u - u = h(x)$ has a unique solution $S(h)$ for every $h \in L^2(\Omega)$, where $u|_{\partial\Omega} = 0$. So seeking for solutions of the problem (P_1) is equal to solving this fixed point problem

$$u = S \circ N_{f_R}(u), \tag{5}$$

where $h \in L^2(\Omega)$ and N_{f_R} is a Nemisky operator concerned with f_R . Continuously applying the lemma and (3) we can prove that $S \circ N_{f_R}$is a completely continuous operator from $L^2(\Omega)$ to $L^2(\Omega)$ and $S \circ N_{f_R}$ is bounded in $L^2(\Omega)$ space. Using Schauder fixed point theorem we know that u is a solution of problem (P_1). If $|u|_2 \leq R$, then u is a solution of the problem (P). Now we prove $|u|_2 \leq R$ with the reduction to absurdity.

Suppose $|u|_2 > R$, then we have

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &= \int_\Omega (-\Delta u \cdot u) dx \\ &= \int_\Omega \left[-u^2 - f(x, P_R(u)) \cdot u + \frac{Ru^2}{|u|_2} \right] dx. \end{aligned} \tag{6}$$

Further, we have

$$\int_\Omega \left(-u^2 + \frac{Ru^2}{|u|_2} \right) dx \leq 0,$$

and

$$-\int_\Omega f(x, P_R(u)) \cdot u dx \leq 0.$$

Then

$$\int_\Omega \left[-u^2 - f(x, P_R(u)) \cdot u + \frac{Ru^2}{|u|_2} \right] dx \leq 0.$$

This contradicts the fact

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_1 |u|_2^2 \geq \lambda_1 R^2 > 0,$$

Thus $|u|_2 \leq R$. The proof is complete.

Conclusion. If we modify condition (4) of the theorem to $f(x, u) \cdot u \geq 0$, where $u \in L^2(\Omega)$ and $|u|_2 = R$, then the problem (P) has at least a solution such that $|u|_2 \leq R$.

References

[1] N.P.Cac, On an elliptic boundary value problem at double resonance, J. math. Appl., **132**(1988), 473-483.

[2] Rabinowitzp.H., Minimax methods in critical point theory with applications to differential equations, (BMS Regional Conf. ser. in Math,) American Mathematical Society, Providence, Rhode Island, **65**(1986).

[3] C. L. Tang and S.Q. Liu, Existence and multiplicity of solutions for a class of semi-linear elliptic equations. J. Math. Amal. Appl., **236**(1999), 399-414.

[4] J. Mawhin, Some boundary value problems for Hartman-type perturbations of the ordinary vector P-Laplacian, Nonlinear Anal. **40**(2001), 497-503.

[5] P. h. Hartman, On boundary value problems for systems of ordinary nonlinear second order differential equations Trans. Amer. Math. Soc., **96**(1960), 493-509.

Scientia Magna
Vol. 3 (2007), No. 3, 82-84

On the fifth-kind Chebyshev polynomials

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Abstract We show that $\frac{[1-T_{n+1}(x)]}{(1-x)}$, where $T_m(x)$ is a first-kind Chebyshev polynomial, is a polynomial of degree n in x.

Keywords Chebyshev polynomials, Fejér Kernel.

Here we consider the first-kind Chebyshev polynomials $T_m(x)$, $x \in [-1,1]$, defined by [1-3],

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1,\ T_5(x) = 16x^5 - 20x^3 + 5x,\ \cdots \tag{1}$$

then,

$$\frac{(1-T_1)}{(1-x)} = 1, \qquad \frac{(1-T_2)}{(1-x)} = 2(1+x),$$
$$\frac{(1-T_3)}{(1-x)} = (2x+1)^2, \qquad \frac{(1-T_4)}{(1-x)} = 8x^2(1+x), \tag{2}$$
$$\frac{(1-T_5)}{(1-x)} = 16x^4 + 16x^3 - 4x^2 - 4x + 1, \cdots$$

that is,

$$\widetilde{W}_{n-1}(x) \equiv \frac{[1-T_n(x)]}{1-x}, \qquad n = 1, 2, \cdots \tag{3}$$

are polynomials of degree $(n-1)$ in x, in fact, (1) can be written [4] in terms of the Gauss hypergeometric function,

$$T_m(x) = F_1\left(-m,\ m;\ \frac{1}{2};\ \frac{1-x}{2}\right)$$
$$= 1 - m^2(1-x) + \frac{m^2(m^2-1)}{6}(1-x)^2 - \frac{m^2(m^2-1)(m^2-4)}{90}(1-x)^3 + \cdots, \tag{4}$$

thus it is evident that $[1 - T_m(x)]$ accepts to $(1 - x)$ as factor, which is showed in (2). From (3) and (4) we can obtain the expression,

$$\widetilde{W}_{n-1}(x) = n \sum_{r=1}^{n} \frac{2^r}{n+r} \begin{pmatrix} n+r \\ 2r \end{pmatrix} (x-1)^{r-1}, \qquad n = 1,\ 2,\ \cdots \tag{5}$$

where we have explicitly to $\widetilde{W}_m$ as a polynomial of degree m in x,

$$\begin{aligned} &\widetilde{W}_0 = 1, \quad \widetilde{W}_1 = 2x + 2, \quad \widetilde{W}_2 = 4x^2 + 4x + x, \\ &\widetilde{W}_3 = 8x^3 + 8x^2, \quad \widetilde{W}_4 = 16x^4 + 16x^3 - 4x^2 - 4x + 1, \cdots \end{aligned} \tag{6}$$

On the other hand, we know that [5], [6]:

$$T_n(\cos\theta) = \cos(n\theta), \qquad x = \cos(\theta) \tag{7}$$

therefore,

$$\begin{aligned} T_n &= 1 - 2\sin^2(n\tfrac{\theta}{2}) = 1 - 2\sin^2(\tfrac{\theta}{2})\tfrac{\sin^2(n\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})}, \\ &= 1 - (1 - cos\theta)2\pi n \underset{F}{K_n}(\theta) = 1 - (1-x)2\pi n \underset{F}{K_n}, \end{aligned} \tag{8}$$

being $\underset{F}{K_n}$ the Fejér Kernel [4]-[8],

$$\underset{F}{K_n}(\theta) = \frac{1}{2\pi n} \frac{\sin^2(n\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})}, \qquad n = 0, 1, 2, \cdots \tag{9}$$

of great importance in Fourier series. Then (3) and (8) imply the interesting relationship,

$$\widetilde{W}_{m-1}(x) = 2\pi m \underset{F}{K_m}(\theta), \qquad x = cos\theta, \tag{10}$$

that is, the Fejér Kernel generates to $\widetilde{W}_n$ which are new in the literature and that we name fifth-kind Chebyshev polynomials.

In [2]-[9], we find the Chebyshev polynomials of fourth-kind,

$$W_n(x) \equiv \frac{\sin(n + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})} = 2\pi \underset{D}{K_n}(\theta), \qquad x = \cos\theta, \tag{11}$$

where appears the Dirichlet Kernel $\underset{D}{K_n}$ [4]-[6], [8]. Thus (11) makes very natural the existence of (10), and then each Kernel has its corresponding Chebyshev- like polynomials.

We also have the Chebyshev polynomials of the second and third kinds [1], [2], [9],

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos(\frac{\theta}{2})}, \tag{12}$$

however, it is important to note the fundamental character of our polynomials $\widetilde{W}_m$ because they generate the four kinds,

$$T_m(x) = 1 + (x-1)\widetilde{W}_{m-1}(x), \qquad W_m(x) = \widetilde{W}_m(x) - \widetilde{W}_{m-1}(x),$$

$$2(1+x)U_m(x) = \widetilde{W}_{m+1}(x) - \widetilde{W}_{m-1}(x), \tag{13}$$

$$(1+x)V_m(x) = 2 + (x-1)[\widetilde{W}_m(x) + \widetilde{W}_{m-1}(x)].$$

In other work we will investigate themes as recurrence, generating function, interpolation properties, Rodrigues formula etc., for the Chebyshev polynomials $\widetilde{W}_n$ introduced in this paper.

References

[1] L. Boyadjiev and R. Scherer, Kuwait J. Sci. Eng., **28**(2001), No. 2, 227-240.

[2] J.C. Mason and D.C. Handscomb, Chebyshev polynomials, Chapman & Hall-CRC Press, London, 2002.

[3] J.H. Caltenco, J. López-Bonilla, M.Martínez and R. Peña., J. Bangladesh Acad. Sci., **26**(2002), 115-118.

[4] C. Lanczos, Applied analysis, Dover, New York, 1988.

[5] C. Lanczos, Discourse on Fourier series, Oliver & Boyd, Edinburgh, 1966.

[6] J. LLópez-Bonilla, S. Vidal B. and J. Yalja M., Rev. Mate. Teor. Aplic., **14**(2007), No. 1, 101-104.

[7] L. Fejér, Math. Annalen, **58**(1904), 51-69.

[8] R. Rodríguez del Río and E. Zuazua, Cubo Mat. Educ., **5**(2003), No. 2, 185-224.

[9] W. Gautschi, J. Compt. Appl. Math., **43**(1992), 19-35.

Scientia Magna
Vol. 3 (2007), No. 3, 85-92

Deficient functions of random Dirichlet series of finite order in the half plane

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Abstract In this paper, we study the random Dirichlet series of general enough in the half-plane, and prove that the random Dirichlet series of finite order almost surely do not have deficient function.

Keywords Finite order, Dirichlet series, random series, deficient function.

§1. Introduction and Results

Many results have been obtained on value distribution of random Dirichlet series, see [1]. Recently, Professor Sun proved that random Taylor series of finite order almost surely (a.s.) do not have deficient function in [2]. But for random Drirchlet series, the corresponding result has not been studied. In this paper, the problem on random Dirichlet series in the right-half plane will be discussed.

Consider Drichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}, \quad (s = x + it), \tag{1}$$

where $a_n \in \mathbf{C}$ and $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \uparrow +\infty$, which satisfy the following conditions

$$\overline{\lim_{n\to\infty}} \frac{\ln |a_n|}{\lambda_n} = 0; \qquad \overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < +\infty. \tag{2}$$

Then the abscissas of convergent and the abscissas of absolute convergent of series (1) are both zero. Thus $f(s)$ defined by (1) is an analytic function in the right-half plane.

Set

$$M(x, f) = \sup\{|f(x + it)|; \quad -\infty < t < +\infty\}.$$

Then the order of Dirichlet series (1) is defined as

$$\overline{\lim_{x\to 0^+}} \frac{\ln^+ \ln^+ M(x, f)}{\ln \frac{1}{x}} = \rho.$$

Suppose also that $\{(\Omega_j, \mathbf{A_j}, P_j)\}$ is a infinite sequence of probability space, $\{(\Omega = \prod_{j=0}^{\infty} \Omega_j,\ \mathbf{A} =$

$\prod_{j=0}^{\infty}\mathbf{A_j}$, $P=\prod_{j=0}^{\infty}P_j)\}$ is the product probability space. In this paper, we shall prove the following theorem:

Theorem. Suppose that random Dirichlet series

$$f_\omega(s)=\sum_{n=0}^{\infty}a_nX_n(\omega_n)e^{-\lambda_n s} \tag{3}$$

satisfies the condition (2) and the following conditions:

1)

$$\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=h>0; \tag{4}$$

2)

$$\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n-\ln^+\ln^+|a_n|}=\rho\in(1,+\infty). \tag{5}$$

3) In the probability space $(\Omega,\mathbf{A},P)$ $(\omega\in\Omega)$,$\{X_n=X_n(\omega)\}$ is a sequence of non-degenerate, independent complex random variables of the same distribution and verifying

$$0<E(|X_n(\omega)|^2)=\sigma^2<+\infty.$$

Then the series (3) almost surely(a.s.) do not have deficient function.

§2. Some lemmas

In order to derive our main result, we need the following lemmas.

Lemma 1.([2]) Under the above hypotheses on $\{X_n(\omega)\}$, we have

1) $\exists u,d\in(0,1)$ such that $\forall c\in\mathbf{C},j\in\mathbf{N}$, we have

$$P_j(|X_j(\omega_j)-c|<2u)\le d<1;$$

2) $\forall\omega\in\Omega$ (a. s.), $\exists N=N(\omega)$ such that when $n>N$, we have$|X_n(\omega)|<n$;

3) For all subsequence $\{n_p\}\to\infty$,

$$\overline{\lim_{p\to\infty}}\sqrt[n_p]{|X_{n_p}(\omega)|}=1.\quad a.s.$$

Lemma 2. Suppose that series (1.1) satisfies (1.2), then

$$\overline{\lim_{x\to0^+}}\frac{\ln^+\ln^+M(x,f)}{\ln\frac{1}{x}}=\rho\Leftrightarrow\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n-\ln^+\ln^+|a_n|}=\rho.\quad(0<\rho<+\infty)$$

Proof. By Theorem 3.3.1 of [1], we have

$$\overline{\lim_{x\to0^+}}\frac{\ln^+\ln^+M(x,f)}{\ln\frac{1}{x}}=\rho\Leftrightarrow\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n}=\rho.$$

On the other hand,

$$\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n}=\frac{\rho}{1+\rho}\Leftrightarrow\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n-\ln^+\ln^+|a_n|}=\rho\quad(0<\rho<+\infty).$$

Therefore

$$\overline{\lim_{x\to0^+}}\frac{\ln^+\ln^+M(x,f)}{\ln\frac{1}{x}}=\rho\Leftrightarrow\overline{\lim_{n\to\infty}}\frac{\ln^+\ln^+|a_n|}{\ln\lambda_n-\ln^+\ln^+|a_n|}=\rho,$$

and the proof is complete.

Lemma 3.([1]) Suppose B is a horizontal half strip in $\{s;Res>0\}$ with width large than $\frac{2\pi}{h}$. For the Dirichlet series given by (1), which satisfies the conditions (2) and (4), we have

$$\overline{\lim_{x\to0^+}}\frac{\ln^+\ln^+M(x,f,B)}{\ln\frac{1}{x}}=\rho,$$

where $M(x,f,B)=\sup\limits_t\{|f(x+it)|:s=x+it\in B\}$.

Set

$$z=H(s)=\frac{s-1}{s+1},\tag{6}$$

$$s=H^{-1}(z)=\frac{z+1}{1-z}.\tag{7}$$

It is well known that the linear fractional transformation (6) maps the right-half plane $Res>0$ onto the open unit disk $\{z:|z|<1\}$, and maps the point $s=1$ onto the origin $z=0$. Its inverse transformation (7) maps $\{z:|z|<1\}$ onto the right-half plane $Res>0$.

Let

$$g(z)=f(H^{-1}(z))=\sum_{n=0}^{\infty}a_ne^{-\lambda_nH^{-1}(z)}.$$

Then $g(z)$ is an analytic function in $\{z:|z|<1\}$.

Lemma 4. For $g(z)=f(H^{-1}(z))=\sum\limits_{n=0}^{\infty}a_ne^{-\lambda_nH^{-1}(z)}$, we have

$$\overline{\lim_{r\to1^-}}\frac{\ln^+\ln^+M(r,g)}{\ln\frac{1}{1-r}}=\rho,$$

where $\rho\in(0,+\infty)$ is the order of $f(s)$ defined by (1).

Proof. Let

$$B=\{s=x+it:x>0,|t|<q,q>\frac{\pi}{h}\}.$$

Without loss of generality, we suppose that x satisfies $0<x<1$. Fix $x\in(0,1)$, since $\frac{\sqrt{(1-x)^2+q^2}}{\sqrt{(1+x)^2+q^2}}<1$, and $0<\frac{1-x+y}{1+x+y}\to1\quad(y\to+\infty)$, then there is $c>0$, such that

$$\frac{\sqrt{(1-x)^2+q^2}}{\sqrt{(1+x)^2+q^2}}<\frac{1-x+c}{1+x+c}<1.$$

Let $r=\frac{1-x+c}{1+x+c}$. Then

$$x=\frac{(1+c)(1-r)}{1+r},\qquad\text{and } x\to0^+\Leftrightarrow r\to1^-.$$

Set

$$D(r) = \{z : |z| < r\}; \quad B(x) = \{s : Res \geq x, |t| < q\}.$$

For any $s \in B(x) - B(1)$, we have

$$|z| = \frac{|s-1|}{|s+1|} = \frac{\sqrt{(1-Res)^2+t^2}}{\sqrt{(1+Res)^2+t^2}} \leq \frac{\sqrt{(1-x)^2+q^2}}{\sqrt{(1+x)^2+q^2}} < r.$$

Thus $z = H(s)$ maps $B(x) - B(1)$ into a subset of $D(r)$. Therefore

$$B(x) - B(1) \subset H^{-1}(D(r)).$$

For $z \in D(r)$, i. e. $|z| < r$,

$$H^{-1}(z) = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-2iImz-|z|^2}{1-2Rez+|z|^2}.$$

Then

$$ReH^{-1}(z) = \frac{1-|z|^2}{1-2Rez+|z|^2} \geq \frac{1-|z|}{1+|z|} > \frac{1-r}{1+r} = \frac{x}{1+c}.$$

Denote $A(x) = \{s : Res \geq x\}$. Then $H^{-1}(D(r)) \subset A(\frac{x}{1+c})$, and

$$B(x) - B(1) \subset H^{-1}(D(r)) \subset A(\frac{x}{1+c}). \tag{8}$$

So

$$M(x, f, B) \leq M(r, g) \leq M(\frac{x}{1+c}, f). \quad (0 < x < 1)$$

Thus

$$\overline{\lim_{x\to 0^+}} \frac{\ln\ln M(x,f,B)}{-\ln x} \leq \overline{\lim_{r\to 1^-}} \frac{\ln\ln M(r,g)}{-\ln \frac{(1+c)(1-r)}{1+r}} \leq \overline{\lim_{x\to 0^+}} \frac{\ln\ln M(\frac{x}{1+c}, f)}{-\ln \frac{x}{1+c} - \ln(1+c)}.$$

Combining Lemma 3, we have

$$\overline{\lim_{r\to 1^-}} \frac{\ln\ln M(r,g)}{\ln \frac{1}{1-r}} = \rho,$$

and the proof is complete.

Lemma 5.([5]) If $g(z)$ is an analytic function in $\{z : |z| < 1\}$, then

$$\overline{\lim_{r\to 1^-}} \frac{\ln\ln M(r,g)}{\ln \frac{1}{1-r}} \geq \overline{\lim_{r\to 1^-}} \frac{\ln T(r,g)}{\ln \frac{1}{1-r}} \geq \overline{\lim_{r\to 1^-}} \frac{\ln\ln M(r,g)}{\ln \frac{1}{1-r}} - 1.$$

According to Lemma 4 and Lemma 5, we have

$$\rho - 1 \leq \rho' = \overline{\lim_{r\to 1^-}} \frac{\ln T(r,g)}{\ln \frac{1}{1-r}} \leq \rho.$$

Lemma 6. For the Dirichlet series $f_1(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$, $f_2(s) = \sum_{n=0}^{\infty} b_n e^{-\lambda_n s}$, $(s = x+it)$, which satisfy the conditions (2), we have

$$f_1(s) \equiv f_2(s) \Leftrightarrow a_n = b_n \quad (n = 0, 1, 2, \cdots),$$

where $\{a_n\}, \{b_n\} \subset \mathbf{C}$.

Proof. We only need to prove the necessity. Let

$$F(s) = f_1(s) - f_2(s) = \sum_{n=0}^{\infty}(a_n - b_n)e^{-\lambda_n s} = \sum_{n=0}^{\infty} c_n e^{-\lambda_n s}. \tag{9}$$

By (2), for $\varepsilon > 0$, when n is sufficiently large,

$$|a_n| < e^{\lambda_n \varepsilon}, \quad |b_n| < e^{\lambda_n \varepsilon}.$$

Then $|c_n| \leq |a_n| + |b_n| < 2e^{\lambda_n \varepsilon}$. When $c_n \neq 0$,

$$\overline{\lim_{n\to\infty}} \frac{\ln |c_n|}{\lambda_n} \leq 0, \qquad \overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < +\infty.$$

By means of Valiron Formula, the abscissa of convergence and the abscissa of absolute convergence are equal to or smaller than zero, and the abscissa of uniformly convergence $x_u \leq 0 < +\infty$.

If the coefficients of the series (9) are not all zero, then $F(s)$ converges absolutely on $x > 0$. The maximum item of $F(s)$ verifies $m(x) > 0$, $(x > 0 \geq x_u)$. By [1], $M(x) \geq m(x) > 0$, which contradicts $F(s) \equiv 0$, and the proof is complete.

§3. Proof of the theorem

By Lemma 1 and Lemma 2, we have that for any $\omega = (\omega_0, \omega_1, \omega_2, \cdots) \in \Omega(a.s.)$, $f_\omega(s)$ defined by (3) is an analytic function of order $\rho \in (1, +\infty)$ in the half-plane.

Set

$$\Psi(\rho) = \{\psi = \sum_{n=0}^{\infty} \beta_n e^{-\lambda_n s} : \overline{\lim_{n\to\infty}} \frac{\ln |\beta_n|}{\lambda_n} = 0, \overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < +\infty, \overline{\lim_{n\to\infty}} \frac{\ln^+ \ln^+ |\beta_n|}{\ln \lambda_n - \ln^+ \ln^+ |\beta_n|} < \rho\},$$

$$g_\omega(z) = f_\omega(H^{-1}(z)) = \sum_{n=0}^{\infty} a_n X_n(\omega_n) e^{-\lambda_n H^{-1}(z)}, \tag{10}$$

$\varphi(z) = \psi(H^{-1}(z))$ and $\Phi(\rho') = \{\varphi = \psi(H^{-1}(z)); \psi \in \Psi(\rho)\}$. Then $\forall \omega = (\omega_0, \omega_1, \omega_2, \cdots) \in \Omega$ (a. s.), $g_\omega(z)$ is an analytic function in unit disc of order $\rho' \in (0, \infty)$. $\Phi(\rho')$ is the set of all analytic functions in unit disc with the order smaller than ρ' and complex numbers.

We now prove that: If the function (10) satisfying the conditions (2), (4) and (5), then almost surely $g_\omega(z)$ do not have deficient function, i. e.

$$P\left\{\omega : \inf\left(\overline{\lim_{r\to 1^-}} \frac{N(r, \frac{1}{g_\omega - \varphi})}{T(r, g_\omega)}; \varphi \in \Phi(\rho')\right) < 1\right\} = 0.$$

Choose $\delta \in (0, \rho/3)$ and $\Delta > 0$. Let

$$\Phi(\rho', \Delta) = \{\varphi = \psi(H^{-1}(z)) : \psi \in \Psi(\rho - 3\delta), \frac{\ln^+ \ln^+ |\beta_n|}{\ln \lambda_n - \ln^+ \ln^+ |\beta_n|} < \rho - 2\delta, n > \Delta\} \tag{11}$$

Choose positive integer p. Let $E = E(p, \Delta, \delta)$ denote a set of ω in probability space in which $g_\omega(z)$ has deficient function $\varphi \in \Phi(\rho', \Delta)$ with deficient number larger than $\frac{1}{p}$. Notice that

$$\bigcup_{\delta>0} \bigcup_{\Delta=1}^{\infty} \bigcup_{p=1}^{\infty} E(p, \Delta, \delta) \subset \Omega$$

is a set in which $g_\omega(z)$ has deficient function. Obviously, we only need to prove that $\forall \varepsilon > 0$, $P(E(p, \Delta, \delta)) < \varepsilon$.

Since

$$\overline{\lim_{n\to\infty}} \frac{\ln^+ \ln^+ |ua_n|}{\ln \lambda_n - \ln^+ \ln^+ |ua_n|} = \rho,$$

we can choose an increasing sequence of positive integer $\{n(t)\}_{t=1}^{\infty}$ such that $n(1) > \Delta$ and

$$\frac{\ln^+ \ln^+ |ua_{n_{(t)}}|}{\ln \lambda_{n_{(t)}} - \ln^+ \ln^+ |ua_{n_{(t)}}|} > \rho - \varepsilon. \tag{12}$$

Choose $N > \dfrac{\ln \frac{\varepsilon}{p+1}}{\ln d}$. Then $(p+1)d^N < \varepsilon$, where u, d are constance in Lemma 1.

Denote

$$\overline{\Omega} = \prod_{j=0}^{n(N)} \Omega_j; \qquad \overline{\overline{\Omega}} = \prod_{j=n(N)+1}^{\infty} \Omega_j.$$

For $\overline{\overline{\omega^*}} = (\omega^*_{n(N)+1}, \omega^*_{n(N)+2}, \cdots) \in \overline{\overline{\Omega}}$, denote $\Gamma = \Gamma(\overline{\overline{\omega^*}})$ as a set verifying: if

$$\overline{\omega'} = (\omega'_0, \omega'_1, \cdots, \omega'_{n(N)}) \in \overline{\Omega},$$

then function $\displaystyle\sum_{n=n(N)+1}^{\infty} a_n X_n(\omega^*_n) e^{-\lambda_n H^{-1}(z)} + \sum_{n=0}^{n(N)} a_n X_n(\omega'_n) e^{-\lambda_n H^{-1}(z)}$ have deficient function $\varphi \in \Phi(\rho', \Delta)$ with deficient number larger than $\dfrac{1}{p}$.

We shall prove for $p+2$ elements,

$$\overline{\omega}(k) = (\omega_0(k), \omega_1(k), \cdots, \omega_{n(N)}(k)) \in \Gamma, \quad k = 1, 2, \cdots, p+2$$

there is $k_1, k_2 \in \{1, 2, \cdots, p+2\}$, $(k_1 \neq k_2)$ such that for all $j \in \{n(1), n(2), \cdots, n(N)\}$, we have

$$|X_j(\omega_j(k_1)) - X_j(\omega_j(k_2))| < 2u.$$

Otherwise, there is $p+2$ functions

$$g_k(z) = \sum_{n=n(N)+1}^{\infty} a_n X_n(\omega^*_n) e^{-\lambda_n H^{-1}(z)} + \sum_{n=0}^{n(N)} a_n X_n(\omega_n(k)) e^{-\lambda_n H^{-1}(z)}, \quad (k = 1, 2, \cdots, p+2).$$

They have deficient function

$$\varphi_k^{\#} = \sum_{n=0}^{\infty} \beta_n(k) e^{-\lambda_n H^{-1}(z)} \in \Phi(\rho', \Delta), \quad (k = 1, 2, \cdots, p+2).$$

Thus function $g = \sum_{n=n(N)+1}^{\infty} a_n X_n(\omega_n^*) e^{-\lambda_n H^{-1}(z)}$ have at least $p+2$ deficient function:

$$\varphi_k = \varphi_k^{\#} - \sum_{n=0}^{n(N)} a_n X_n(\omega_n(k)) e^{-\lambda_n H^{-1}(z)}, \quad (k = 1, 2, \cdots, p+2).$$

And for all $k, l \in \{1, 2, \cdots, p+2\}$, $\quad (k \neq l)$, there is $j = j(k,l) \in \{n(1), n(2), \cdots, n(N)\}$ such that

$$|X_j(\omega_j(k)) - X_j(\omega_j(l))| \geq 2u.$$

Then $|X_j(\omega_j(k))a_j - X_j(\omega_j(l))a_j| \geq 2u|a_j|$. By (12), we have

$$u|a_j| > \exp\{\lambda_j^{\frac{\rho-\varepsilon}{\rho+1-\varepsilon}}\}.$$

By (11),

$$|\beta_j(k)| < \exp\{\lambda_j^{\frac{\rho-2\delta}{\rho+1-2\delta}}\}, \quad |\beta_j(l)| < \exp\{\lambda_j^{\frac{\rho-2\delta}{\rho+1-2\delta}}\}.$$

Then

$$|\beta_j(k) - \beta_j(l)| \leq |\beta_j(k)| + |\beta_j(l)| < 2\exp\{\lambda_j^{\frac{\rho-2\delta}{\rho+1-2\delta}}\}.$$

Choose $\varepsilon : 0 < \varepsilon < 2\delta$. Therefore

$$|\beta_j(k) - \beta_j(l)| < 2\exp\{\lambda_j^{\frac{\rho-2\delta}{\rho+1-2\delta}}\} < 2u|a_j| \leq |X_j(\omega_j(k))a_j - X_j(\omega_j(l))a_j|.$$

By Lemma 6, $\varphi_k \neq \varphi_l$, i. e., the $p+2$ small functions above are different. But by Zhuang Qitai Theorem in [3], g has at most $p+1$ different small function. Thus we arrive at a contradiction.

Let $\overline{P} = \prod_{j=1}^{n(N)} P_j$. When $P(\Gamma) = 0$, $\Gamma \subset \overline{\Omega} = \emptyset$. Or there are $p+1$ elements

$$\overline{\omega}(k) = (\omega_0(k), \omega_1(k), \cdots, \omega_{n(N)}(k)) \in \Gamma, \quad k = 1, 2, \cdots, p+1,$$

such that for any other elements (if there is), $\overline{\omega} = (\omega_0, \omega_1, \cdots, \omega_{n(N)}) \in \Gamma$, we have

$$\overline{\omega} \in \bigcup_{k=1}^{p+1} \{\overline{\omega} : |X_j(\omega_j(k)) - X_j(\omega_j)| < 2u, \quad j = n(1), n(2), \cdots, n(N)\}.$$

By its independence and Lemma 1,

$$\begin{aligned} \overline{P}(\Gamma) &\leq \sum_{k=1}^{p+1} \prod_{t=0}^{n(N)} P_t\{\omega_t \in \Omega_t : \overline{\omega} = (\omega_0, \omega_1, \cdots, \omega_{n(N)}) \in \Gamma\} \\ &\leq \sum_{k=1}^{p+1} \prod_{t=1}^{N} P_{n(t)}\{\omega_{n(t)} \in \Omega_{n(t)} : |X_{n(t)}(\omega_{n(t)}(k)) - X_{n(t)}(\omega_{n(t)})| < 2u\} \\ &\leq (p+1)d^N < \varepsilon. \end{aligned} \tag{13}$$

In other words, if $\omega = (\omega_0, \omega_0, \omega_0, \cdots) = (\overline{\omega}, \overline{\overline{\omega}}) \in E = E(p, \Delta, \delta) \subset \Omega$, then

$$\overline{\omega} = (\omega_0, \omega_1, \omega_2, \cdots, \omega_{n(N)}) \in \Gamma = \Gamma(\overline{\overline{\omega}}). \tag{14}$$

By Fubini-Levi Theorem, see [4], combining (13) and (14) , we have

$$\begin{aligned} P(E) &= E(\mathbf{1}_E) = \lim_{m\to\infty}\int\cdots\int \mathbf{1}_E P_0(d\omega_0)P_1(d\omega_1)\cdots P_m(d\omega_m) \\ &= \lim_{m\to\infty}\int\cdots\int P_{n(N)+1}(d\omega_{n(N)+1})\cdots P_m(d\omega_m)\int\cdots\int \mathbf{1}_E P_0(d\omega_0)\cdots P_{n(N)}(d\omega_{n(N)}) \\ &\le \lim_{m\to\infty}\int\cdots\int P_{n(N)+1}(d\omega_{n(N)+1})\cdots P_m(d\omega_m)\int\cdots\int \mathbf{1}_\Gamma P_0(d\omega_0)\cdots P_{n(N)}(d\omega_{n(N)}) \\ &\le (p+1)d^N \lim_{m\to\infty}\int\cdots\int P_{n(N)+1}(d\omega_{n(N)+1})\cdots P_m(d\omega_m) < \varepsilon. \end{aligned}$$

Thus by $f_\omega(s) = f_\omega(H^{-1}(z))$, we can obtain the conclusion of Theorem, and the proof is completed.

References

[1] Jiarong Yu, Xiaoqing Ding and Fanji Tian, Value distribution of Dirichlet series and random Dirichlet series, Wuhan University Press, Wuhan, 2004.

[2] Daochun Sun, Deficient function of random Taylor series. Acta Math., Scientia, **19**(1999), No. 3, 356-360.

[3] Le Yang, Value distribution and its new study, Science Press, Beijing, 1982.

[4] Kahane J P., Some random seies of functions, Cambridge Universtiy Press, London, 1985.

[5] Nevalinna R., Le théorème de Picard-Borel et la théoric des fonctions meromorphes, Gautier-Villars, Paris, 1929.

[6] Daochun Sun, Random Dirichlet series on the right-half plane, Acta Math., Scientia, **19**(1999), No. 1, 107-112.

[7] Zhongsheng Gao and Dao-chun Sun, Small function of Dirichlet series, Acta Math., Sinica, **46**(2003), No. 2, 397-402.

[8] Daochun Sun and Zhongsheng Gao, Growth order of Dirichlet seires on the right-half plane, Acta Math., Scientia, **22A**(2002), No. 4, 557-563.

Scientia Magna
Vol. 3 (2007), No. 3, 93-96

On the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : n|m^n,\ m \in N\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary methods to study a limit problem related to the Pseudo-Smarandache-Squarefree function $Zw(n)$, and give an interesting limit theorem.

Keywords Pseudo-Smarandache-Squarefree function, mean value, limit.

§1. Introduction and Results

For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : n|m^n,\ m \in N\}$, where N denotes the set of all positive integers. For example, the first few values of $Zw(n)$ are $Zw(1) = 1$, $Zw(2) = 2$, $Zw(3) = 3$, $Zw(4) = 2$, $Zw(5) = 5$, $Zw(6) = 6$, $Zw(7) = 7$, $Zw(8) = 2$, $Zw(9) = 3$, $Zw(10) = 10$, $Zw(11) = 11$, $Zw(12) = 6$, $Zw(13) = 13$, $Zw(14) = 14$, $Zw(15) = 15$, $\cdots$. Obviously, the Pseudo-Smarandache-Squarefree function $Zw(n)$ has the following properties:

(1) If p be a prime, then $Zw(p) = p$.

(2) If m be a square-free number (m¿1, and for any prime p, if $p|m$, then $p^2 \dagger m$), then $Zw(m) = m$.

(3) If p be any prime and $k \geq 1$, then we have $Zw(p^k) = p$.

(4) $Zw(n) \leq n$.

(5) The function $Zw(n)$ is multiplicative. That is, if $(m,\ n) = 1$, then $Zw(mn) = Zw(m)Zw(n)$.

(6) The function $Zw(n)$ is not additive. That is, for some positive integers m and n, $Zw(m+n) \neq Zw(m) + Zw(n)$.

According to the above properties, the $Zw(n)$ function is very similar to the Möbius function:

$$Zw(n) = \begin{cases} n & \text{if } n \text{ is a square free number;} \\ 1 & \text{if and only if } n = 1; \\ \text{Product of distinct prime factors of } n & \text{if } n \text{ is not a square-free number.} \end{cases}$$

On the other hand, we can easily deduce that $\sum_{n=1}^{\infty}\frac{1}{Zw(n)}$ is divergent . In fact for any prime p, we have $Zw(p)=p$. So that $\sum_{n=1}^{\infty}\frac{1}{Zw(n)}>\lim_{x\longrightarrow\infty}\sum_{p\leq x}\frac{1}{p}=+\infty$.

About the other elementary properties of $Zw(n)$, some authors also had studied it, and obtained some interesting results, see references [1], [2], [5], [7] and [8]. Simultaneously, F. Russo [1] proposed some new problems, two of them as follows:

Problem 1: Evaluate limit $\prod_{n=2}^{\infty}\frac{1}{Zw(n)}$.

Problem 2: Evaluate $\lim_{k\to\infty}\frac{Zw(k)}{\theta(k)}$, where $\theta(k)=\sum_{n\leq k}\ln\left(Zw(n)\right)$.

The problem 1 had been solved by Maohua Le [2]. But for the problem 2, it seems that none had studied it yet, at least we have not seen such a paper before. The problem is interesting, because it can help us to obtain some deeply properties of the Pseudo-Smarandache-Squarefree function $Zw(n)$. The main purpose of this paper is using the elementary methods to study this problem, and give an interesting limit theorem for it. That is, we shall prove the following conclusion:

Theorem. For any positive integer $k>1$, let $Zw(n)$ and $\theta(k)$ are defined as the above, then we have the asymptotic formula

$$\frac{Zw(k)}{\theta(k)}=\frac{Zw(k)}{\sum_{n\leq k}\ln\left(Zw(n)\right)}=O\left(\frac{1}{\ln k}\right).$$

From this theorem we may immediately deduce the following:

Corollary. For any positive integer k, we have the limit

$$\lim_{k\to\infty}\frac{Zw(k)}{\theta(k)}=0.$$

It is clear that our corollary solved the problem 2.

§2. Proof of the theorem

To complete the proof of the theorem, we need the following an important Lemma.

Lemma. For any real number $x\geq 1$, we have the asymptotic formula

$$\sum_{n\leq x}|\mu(n)|=\frac{6}{\pi^2}x+O\left(\sqrt{x}\right),$$

where $\mu(n)$ denotes the Möbius function.

Proof. For any real number $x>1$ and positive integer n, from the properties of the Möbius function $\mu(n)$ (See reference [3]):

$$|\mu(n)|=\sum_{d^2|n}\mu(d),$$

and note that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$$

we have

$$\begin{aligned}
\sum_{n\leq x}|\mu(n)| &= \sum_{n\leq x}\sum_{d^2|n}\mu(d)\\
&= \sum_{md^2\leq x}\mu(d)\\
&= \sum_{d\leq\sqrt{x}}\mu(d)\sum_{m\leq\frac{x}{d^2}}1\\
&= \sum_{d\leq\sqrt{x}}\mu(d)\left(\frac{x}{d^2}+O(1)\right)\\
&= x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>\sqrt{x}}\frac{\mu(d)}{d^2}\right)+O\left(\sum_{d\leq\sqrt{x}}|\mu(d)|\right)\\
&= x\left(\frac{6}{\pi^2}+O\left(\frac{1}{\sqrt{x}}\right)\right)+O\left(\sqrt{x}\right)\\
&= \frac{6}{\pi^2}x+O\left(\sqrt{x}\right).
\end{aligned}$$

This proves our Lemma.

Now, we shall use this Lemma to complete the proof our theorem. In fact note that for any square-free number n, $Zw(n)=n$, we have

$$\begin{aligned}
\theta(k) &= \sum_{n\leq k}\ln\left(Zw(n)\right)\\
&\geq \sum_{n\leq k}|\mu(n)|\ln n\\
&\geq \sum_{\sqrt{k}\leq n\leq k}|\mu(n)|\ln(\sqrt{k})\\
&= \frac{1}{2}\ln k\sum_{\sqrt{k}\leq n\leq k}|\mu(n)|\\
&= \frac{1}{2}\ln k\left(\sum_{n\leq k}|\mu(n)|-\sum_{n\leq\sqrt{k}}|\mu(n)|\right). \qquad (1)
\end{aligned}$$

So from Lemma and (1) we have

$$\begin{aligned}
\theta(k) &\geq \frac{1}{2}\ln k\left(\sum_{n\leq k}|\mu(n)|-\sum_{n\leq\sqrt{k}}|\mu(n)|\right)\\
&\geq \frac{1}{2}\ln k\left(\frac{6}{\pi^2}k+O(\sqrt{k})\right)\\
&= \frac{3}{\pi^2}k\cdot\ln k+O(\sqrt{k}\cdot\ln k). \qquad (2)
\end{aligned}$$

Note that $Zw(n) \leq n$, from (2) we may immediately deduce that

$$0 < \frac{Zw(k)}{\theta(k)} \leq \frac{k}{\frac{3}{\pi^2}k \cdot \ln k + O(\sqrt{k} \cdot \ln k)} = O\left(\frac{1}{\ln k}\right)$$

or

$$\frac{Zw(k)}{\theta(k)} = O\left(\frac{1}{\ln k}\right).$$

This completes the proof of Theorem.

The corollary follows from our theorem as $\longrightarrow \infty$.

References

[1] F.Russo, A set of new Smarandache functions, sequences and conjectures in number theory, American Research Press, USA, 2000.

[2] Maohua Le, On the Pseudo-Smarandache-squarefree function, Smarandache Notions Journal, **13**(2002), 229-236.

[3] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[4] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[5] Jozsef Sandor, On a dual of the Pseudo-Smarandache function, Smarandache Notions (Book series), **13**(2002), 16-23.

[6] Xu Zhefeng, On the Value Distribution of the Smarandache Function，Acta Mathematica Sinica (in Chinese), **49**(2006), No.5, 1009-1012.

[7] Maohua Le, Two function equations, Smarandache Notions Journal, **14**(2004), 180-182.

[8] David Gorski, The pseudo-Smarandache functions, Smarandache Notions Journal, **12** (2000), 140-145.

[9] Jozsef Sandor, On additive analogues of certain arithmetic function, Smarandache Notions Journal, **14**(2004), 128-132.

[10] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[11] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

Scientia Magna
Vol. 3 (2007), No. 3, 97-100

On the mean value of the Pseudo-Smarandache function

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Abstract For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer k such that $n \mid \frac{k(k+1)}{2}$. That is, $Z(n) = \min\left\{k : \ n|\frac{k(k+1)}{2}\right\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of $\frac{p(n)}{Z(n)}$, and give a sharper asymptotic formula for it, where $p(n)$ denotes the smallest prime divisor of n.

Keywords Pseudo-Smarandache function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer k such that $n \mid \frac{k(k+1)}{2}$. That is, $Z(n) = \min\left\{k : \ n|\frac{k(k+1)}{2}, \ n \in N\right\}$, where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, $Z(5) = 4$, $Z(6) = 3$, $Z(7) = 6$, $Z(8) = 15$, $Z(9) = 8$, $Z(10) = 4$, $Z(11) = 10$, $Z(12) = 8$, $Z(13) = 12$, $Z(14) = 7$, $Z(15) = 5$, $\cdots$. About the elementary properties of $Z(n)$, some authors had studied it, and obtained many valuable results. For example, Richard Pinch [3] proved that for any given $L > 0$, there are infinitely many values of n such that

$$\frac{Z(n+1)}{Z(n)} > L.$$

Simultaneously, Maohua Le [4] proved that if n is an even perfect number, then n satisfies

$$S(n) = Z(n).$$

The main purpose of this paper is using the elementary methods to study the mean value properties of $\frac{p(n)}{Z(n)}$, and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be any fixed positive integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}\frac{p(n)}{Z(n)} = \frac{x}{\ln x} + \sum_{i=2}^{k}\frac{a_i x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, and a_i $(i = 2,\ 3,\ \cdots, k)$ are computable constants.

§2. Proof of the theorem

In order to complete the proof of the theorem, we need the following several useful lemmas.

Lemma 1. For any prime $p \geq 3$, we have identity $Z(p) = p - 1$.

Proof. See reference [5].

Lemma 2. For any prime $p \geq 3$ and any $k \in N$, we have $Z\left(p^k\right) = p^k - 1$.

Proof. See reference [5].

Lemma 3. For any positive n, $Z(n) \geq \sqrt{n}$.

Proof. See reference [3].

Now, we shall use these lemmas to complete the proof of our theorem. We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $\Omega(n) = 0$, this time $n = 1$;

B: $\Omega(n) = 1$, then $n = p$, a prime;

C: $\Omega(n) = 2$, then $n = p^2$ or $n = p_1 p_2$, where p_i $(i = 1,\ 2)$ are two different primes with $p_1 < p_2$;

D: $\Omega(n) \geq 3$. This time, $p(n) \leq n^{\frac{1}{3}}$, where $\Omega(n) = \Omega\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}\right) = \alpha_1 + \alpha_2 + \cdots + \alpha_s$. In fact in this case, we have $p^3(n) \leq p^{\Omega(n)}(n) \leq n$ and thus $p(n) \leq n^{\frac{1}{3}}$.

Let $p(n)$ denotes the smallest prime divisor of n, then we have $p(1) = 0$, $Z(1) = 1$ and

$$\sum_{n \in A} \frac{p(n)}{Z(n)} = 0.$$

So we have

$$\sum_{n \leq x} \frac{p(n)}{Z(n)} = \sum_{n \in B} \frac{p(n)}{Z(n)} + \sum_{n \in C} \frac{p(n)}{Z(n)} + \sum_{n \in D} \frac{p(n)}{Z(n)}. \tag{1}$$

From Lemma 1 we know that if $n \in B$, then we have $Z(2) = 3$ and $Z(p) = p - 1$ with $p > 2$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where k be any fixed positive integer, a_i $(i = 1,\ 2,\ \ldots,\ k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n\in B}\frac{p(n)}{Z(n)} &= \sum_{p\le x}\frac{p}{Z(p)}=\frac{2}{3}+\sum_{\substack{p\le x\\ p\ge 3}}\frac{p}{Z(p)}\\ &= \sum_{p\le x}\frac{p}{p-1}+O(1)\\ &= \sum_{p\le x}1+\sum_{p\le x}\frac{1}{p-1}+O(1)\\ &= \frac{x}{\ln x}+\sum_{i=2}^{k}\frac{a_i x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),\end{aligned}\tag{2}$$

where a_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants.

Now we estimate the error terms in set D. From the definition of $\Omega(n)$ we know that $p(n)\le n^{\frac{1}{3}}$ if $n\in D$. From Lemma 3 we know that $Z(n)\ge\sqrt{n}$, so we have the estimate

$$\sum_{n\in D}\frac{p(n)}{Z(n)}\le\sum_{n\le x}\frac{n^{\frac{1}{3}}}{\sqrt{n}}=x^{\frac{5}{6}}.\tag{3}$$

Finally, we estimate the error terms in set C. For any integer $n\in C$, we have $n=p^2$ or $n=p_1p_2$. If $n=p^2$, then from Lemma 2 we have

$$\sum_{p^2\le x}\frac{p}{Z(p^2)}=\frac{2}{Z(4)}+\sum_{p^2\le x}\frac{p}{p^2-1}\ll\ln\ln x.\tag{4}$$

If $n=p_1p_2$, let $Z(p_1p_2)=k$, then from the definition of $Z(n)$ we have $p_1p_2\mid\dfrac{k(k+1)}{2}$.

If $p_1p_2\mid k$, then

$$\sum_{\substack{p_1p_2\le x\\ Z(p_1p_2)=k,\ p_1p_2\mid k}}\frac{p(p_1p_2)}{Z(p_1p_2)}\ll\sum_{p_1\le\sqrt{x}}\sum_{p_1<p_2\le\frac{x}{p_1}}\frac{p_1}{p_1p_2}\ll\sqrt{x}\cdot\ln\ln x.\tag{5}$$

If $p_1p_2\mid k+1$, then we also have the same estimate as in (5).

If $p_1\mid k+1$ and $p_2\mid k$, let $k=tp_1-1$, where $t\in N$, then we have

$$\sum_{p_1p_2\le x}\frac{p(p_1p_2)}{Z(p_1p_2)}\ll\sum_{p_1\le\sqrt{x}}\sum_{t\le x}\frac{p_1}{tp_1-1}+\sqrt{x}\cdot\ln\ln x\ll\sqrt{x}\cdot\ln\ln x.\tag{6}$$

If $p_1\mid k$ and $p_2\mid k+1$, then we can also obtain the same estimate as in (6).

From (4), (5) and (6) we have the estimate

$$\sum_{n\in C}\frac{p(n)}{Z(n)}\ll\sqrt{x}\cdot\ln\ln x.\tag{7}$$

Combining (1), (2), (3) and (7) we may immediately deduce the asymptotic formula

$$\begin{aligned}\sum_{n\le x}\frac{p(n)}{Z(n)} &= \sum_{n\in A}\frac{p(n)}{Z(n)}+\sum_{n\in B}\frac{p(n)}{Z(n)}+\sum_{n\in C}\frac{p(n)}{Z(n)}+\sum_{n\in D}\frac{p(n)}{Z(n)}\\ &= \frac{x}{\ln x}+\sum_{i=2}^{k}\frac{a_i x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),\end{aligned}$$

where a_i $(i = 2,\ 3,\ \cdots, k)$ are computable constants.

This completes the proof of Theorem.

Some notes:

For any real number $x > 1$, whether there exist an asymptotic formula for the mean values

$$\sum_{n\leq x} \frac{P(n)}{Z(n)} \quad \text{and} \quad \sum_{n\leq x} \frac{Z(n)}{P(n)}$$

are two open problems, where $P(n)$ denotes the largest prime divisor of n.

References

[1] Ashbacher, Charles, On numbers that are Pseudo-Smarandache and Smarandache perfect, Smarandache Notions Journal, **14**(2004), 40-41.

[2] Kashihara, Kenichiro, Comments and Topics On Smarandache Notions and Problems, USA, Erhus University Press, Vail, Arizona, 1996.

[3] Richard Pinch, Some properties of the Pseudo-Smarandache function, Scientia Magna, **1**(2005), No. 2, 167-172.

[4] Maohua Le, The function equation $S(n) = Z(n)$, Scientia Magna, **1**(2005), No. 2, 109-110.

[5] A.A.K. Majumdar, A note on the Pseudo-Smarandache function, Scientia Magna, **2**(2006), No. 3, 1-15.

[6] Gioia, Anthony A, The Theory of Numbers-An Introduction, NY, USA, Dover Publication Inc., 2001.

[7] Ibstedt, Henry, Surfing on the Ocean of Numbers-A Few Smarandache Notions and Similar Topics, USA, Erhus University Press, 1997.

[8] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[9] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[10] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

[11] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

Scientia Magna
Vol. 3 (2007), No. 3, 101-103

A class of exact solution of the BBM equation

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Abstract The problem of solving BBM equation can be converted into a nonlinear algebra equations by supposing its exact solution form. In this paper, applying above method, a class of exact solution to BBM equation is obtained.

Keywords BBM equation, traveling wave solution, exact solution.

§1. Introduction

BBM equation

$$u_t + u_x + uu_x + Pu_{xx} = 0 \tag{1}$$

is a mathematical model presented in studying the long wavelength problem of Hydrodynamic in 1970's by Benjamin-Bona-Mahong, Where $P(P < 0)$ is dispersion coefficient,and it is more exact than KdV equations [1]. Recently, a lot of exact solutions of equation (1) are obtained by homogeneous balance method [2], trigonometric function method [3] and so on, where some solutions are the form of polynomial with the function of tangent, cotangent, tanh, coth etc. For the exact solitary wave solution of many non-linear mathematical equations of physics with rational fraction form of hyperbolic function, so we want to get the exact solution of equations (1) in this way.

§2. Introduction of Method

Let

$$u(x,t) = u(\xi) = u(x + Ct) \tag{2}$$

be a traveling wave solution of the equation

$$F(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{3}$$

where C is undetermined coefficient of wave velocity. Substitute (3) into (2), we obtain the following ordinary differential equation with independent variable ξ

$$G(u, u', u'', \cdots) = 0. \tag{4}$$

[1]This research is supported by the SF of education department of Gansu province (0608-04).

Let

$$u(\xi)=\frac{a_{2m}(a^{a\xi})^{2m}+a_{2m-1}(e^{a\xi})^{2m-1}+\cdots+a_0}{b_{2m}(e^{a\xi})^{2m}+b_{2m-1}(e^{a\xi})^{2m-1}+\cdots+b_0} \tag{5}$$

be the solution of equation(4), where m is obtained by homogeneous balance method and $a_i, b_i(i=0,1,\cdots,2m)$ is undetermined coefficients. Substitute (5) into (4) and set the coefficient of $e^{a\xi}$ is zero for any power, then we can obtain a nonlinear algebra equations with undetermined coefficients $a_i, b_i(i=0,1,\cdots,2m), \alpha, C$. The equations can be solved by Mathematica or Maple software. And then the exact solution of equation (2) can be obtained through back substituting the solution of the equations into (5) and setting $\xi=x+Ct$.

§3. The exact solution of the BBM equation

Assume equation (1) has a traveling wave solution of the form (3). Substitute (3) into (2), we get

$$(1+C)u(\xi)+\frac{1}{2}u^2(\xi)+Pu^{''}(\xi)=0. \tag{6}$$

If the solution of equation (6) has the form of (5), then we can obtain $m=1$ by the method of homogeneous balance. So let

$$u(\xi)=\frac{a(e^{a\xi})^2+b(e^{a\xi})+c}{f(e^{a\xi})^2+g(e^{a\xi})+h}, \tag{7}$$

where a, b, c, g, h, α are undetermined coefficients. Substitute (7) into (6) and set the coefficient of $e^{a\xi}$ is zero for any power, we obtain the following nonlinear algebra equations with undetermined coefficients $a, b, c, g, h, \alpha, P, C$.

$$\begin{aligned}
&3ch^2+2Cch^2=0,\\
&2Caf^2+3af^2=0,\\
&3bh^2+2Cbh^2-2PC\alpha^2chg+2PC\alpha^2bh^2+6cph+4Ccgh=0,\\
&6cfh-2PC\alpha cg^2+6bgh+8PC\alpha^2ah^2+4Cbgh-8PC\alpha^2cfh\\
&+4Ccfh+3ah^2+2Ccg^2=0,\\
&6cfg+4Cbgh+6bgh+6PC\alpha^2agh+4Cagh+6agh-12PCc\alpha^2bfh\\
&+4Ccfg+3bg^2+6PC\alpha^2cfg+2Cbg^2=0,\\
&4Cafh-8PC\alpha^2afh+2PC\alpha^2ag^2+6bfg+2Ccf^2+3ag^2+2Cag^2\\
&+6agh+4Cbfg-2PC\alpha^2bfg+3cf^2+8PC\alpha^2cf^2=0,\\
&4Cafg+6afg+2Pc\alpha^2bf^2+3bf^2+2Cbf^2-2PC\alpha^2=0.
\end{aligned} \tag{8}$$

By mathematical software of Maple we obtain the following nine kinds solutions of equations (8).

Case 1: $a=b=g=h=0, C=C, f=f, c=c, a=a, P=-\dfrac{3+2C}{8C\alpha^2}$.

Case 2: $a=c=g=h=0, \alpha=\alpha, C=C, b=b, f=f, P=-\dfrac{3+2C}{2C\alpha^2}$.

Case 3: $a = h = 0, \alpha = \alpha, C = C, f = f, g = g, b = \dfrac{cf}{g}, P = -\dfrac{3+2C}{2C\alpha^2}$.

Case 4: $c = h = 0, \alpha = \alpha, f = f, g = g, P = P, b = b, a = \dfrac{bf}{g}, C = -\dfrac{3}{2}$.

Case 5: $b = f = h = 0, f = f, g = g, a = a, C = c, \alpha = \dfrac{\mu}{6\delta}, P = -\dfrac{3+2C}{2C\alpha^2}$.

Case 6: $b = c = f = g = 0, h = h, a = a, C = C, \alpha = \alpha, P = -\dfrac{3+2C}{2C\alpha^2}$.

Case 7: $c = f = 0, b = b, g = g, h = h, C = C, \alpha = \alpha, P = -\dfrac{3+2C}{2C\alpha^2}, a = \dfrac{bg}{h}$.

Case 8: $P = 0, a = a, b = b, c = c, f = f, g = g, h = h, \alpha = \alpha, C = -\dfrac{3}{2}$.

Case 9: $P = P, c = c, f = f, g = g, h = h, \alpha = \alpha, C = -\dfrac{3}{2}, a = \dfrac{cf}{h}, b = \dfrac{cg}{h}$.

Substituting those solutions into (7) and setting $\xi = x + Ct$, we can get the exact solution of equation (1).

References

[1] Benjamin. T. B., Bona. J. L., Mahony. J. J., Model Equations for Long Waves in Nonlinear Dispersive System, Philos Trans Roy Soc London Ser A, **272**(1972), 47-78.

[2] Wang Mingliang, Exact solutions for a compound KdV-Burgers equation, Physics Letters A, **213**(1996), 279-287.

[3] Yan. C. T. A., Simple Transformation for Nonlinear Waves, Physics Letters A, 1996, 77-84.

Scientia Magna
Vol. 3 (2007), No. 3, 104-107

An equation involving the two Smarandache LCM dual functions

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Abstract For any positive integer n, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime power, the famous F.Smarandache LCM function $SL(n)$ is defined as $SL(n) = \min\{k : k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\} = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the solutions of an equation involving the two Smarandache LCM dual functions, and give the exact expression of the solutions for this equation.

Keywords Smarandache LCM function, dual function, equation, solutions.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. That is,

$$SL(n) = \min\{k :\ k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\}.$$

The first few value of Smarandache LCM function $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $SL(17) = 17$, $SL(18) = 9$, $SL(20) = 5$, $\cdots$. About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1], [2], [3], [4] and [5]. For example, Murthy [2] proved that if n is a prime, then $SL(n) = S(n)$, where $S(n) = \min\{m :\ n|m!,\ m \in N\}$ denotes the F.Smarandache function. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime power, then from [2] we know that

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}.$$

Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n),\quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [3] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

In [6], we define the Smarandache LCM dual function $SL^*(n)$ as follows:

$$SL^*(n) = \max\{k : \ k \in N, \ [1, \ 2, \ \cdots, \ k] \mid n\}.$$

It is easy to calculate that $SL^*(1) = 1$, $SL^*(2) = 2$, $SL^*(3) = 1$, $SL^*(4) = 2$, $SL^*(5) = 1$, $SL^*(6) = 3$, $SL^*(7) = 1$, $SL^*(8) = 2$, $SL^*(9) = 1$, $SL^*(10) = 2$, $\cdots$. Obviously, if n is an odd number, then $SL^*(n) = 1$. If n is an even number, then $SL^*(n) \geq 2$. We also study the mean value distribution property of $SL^*(n)$ and give a sharper asymptotic formula for it, namely for any real number $x > 1$,

$$\sum_{n \leq x} SL^*(n) = c \cdot x + O(\ln^2 x),$$

where $c = \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p-1)}{[1, \ 2, \ \cdots, \ p^\alpha]}$ is a constant.

In [7], Yanrong Xue defined another Smarandache LCM dual function $\overline{SL}(n)$ as follows: $\overline{SL}(1) = 1$, and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime power, then

$$\overline{SL}(n) = \min\{p_1^{\alpha_1}, \ p_2^{\alpha_2}, \ \cdots, \ p_r^{\alpha_r}\},$$

where $p_1 < p_2 < \ \cdots < p_r$ are primes. She solved a conjecture involving function $\overline{SL}(n)$ which is to say that there is no any positive integer $n > 1$ such that $\sum_{d|n} \frac{1}{\overline{SL}(n)}$ is an positive integer.

Both $SL^*(n)$ and $\overline{SL}(n)$ are called the Smarandache LCM dual function of $SL(n)$. But about the relationship between this two dual functions, it seems that none had studied it yet, at least we have not seen such a paper before. Obviously if $n \neq 1$ is an odd number, then $SL^*(n) = 1$ and $\overline{SL}(n) > 1$, so for any odd number $n \neq 1$, $SL^*(n) < \overline{SL}(n)$. But if $n = 6m$ with $2 \nmid m, 3 \nmid m$, then $SL^*(n) \geq 3, \overline{SL}(n) = 2$, so there exists infinite positive integers n such that $SL^*(n) > \overline{SL}(n)$. It is natural to ask whether there exist positive integer n satisfying the equation:

$$SL^*(n) = \overline{SL}(n). \tag{2}$$

In this paper, we use the elementary method to study this problem, and give an exact expression of the solutions for the equation (2). That is, we shall prove the following

Theorem. Every positive integer n satisfying (2) can be expressed as

$$n = 1 \ \ \texttt{or} \ \ n = (F_m - 1) \prod_{2 < p < F_m} p^{\alpha(p)} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where F_m be a Fermat prime, $r \geq 0$ be any integer; $\alpha(p)$ are positive integers such that $p^{\alpha(p)} > F_m$, $2 < p < F_m$; $p_i > F_m$, $\alpha_i \geq 0$, $i = 1, \ 2, \ \cdots, \ r$.

§2. Some useful lemmas

To complete the proof of the theorem, we need the following several lemmas.

Lemma 1. For any positive integer n, there exist a prime p and a positive integer α such that the identity

$$SL^*(n) = p^\alpha - 1.$$

Proof. See reference [3].

Lemma 2. If $2^l + 1$ is prime, then l is a power of 2.

Proof. See reference [5].

Lemma 3. For any prime q and odd prime p, if $p^\alpha - 1 = q^\beta$ for any given integers $\alpha \geq 1,\ \beta \geq 1$, then $q = 2$ and p be a Fermat prime. Namely, there exists a integer m such that $p = 2^{2^m} + 1$.

Proof. Because p is an odd prime, so $q^\beta = p^\alpha - 1$ is an even number, hence $q = 2$. Consequently $p^\alpha - 1 = 2^\beta$, that is $p^\alpha = 2^\beta + 1$. Assume that $p - 1 = 2^l t$, where $2 \nmid t$, then $p = 2^l t + 1$, so

$$p^\alpha = (2^l t + 1)^\alpha = C_\alpha^0 + C_\alpha^1 2^l t + \cdots + C_\alpha^\alpha (2^l t)^\alpha = 2^\beta + 1,$$

hence

$$C_\alpha^1 2^l t + \cdots + C_\alpha^\alpha (2^l t)^\alpha = 2^\beta.$$

Since t divide every term of the left side, so t divide the term of the right, namely $t \mid 2^\beta$, but t is an odd number, hence $t = 1$. That is to say $p = 2^l + 1$, from Lemma 2, there exists an integer m such that $p = 2^{2^m} + 1$. This completes the proof of Lemma 3.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. It is easy versify $n = 1$ is a solution of the equation (2).

(A). If n be an odd number larger than 1.

From the definition of $SL^*(n)$ and $\overline{SL}(n)$ we have $SL^*(n) = 1$, but $\overline{SL}(n) > 1$. Hence $SL^*(n) < \overline{SL}(n)$.

(B). If n be an even number.

Assume that $n = 2^\gamma \cdot s$, where s is an odd number. From Lemma 1, there exists a prime p and a positive integer α such that $SL^*(n) = p^\alpha - 1$. Assume that $\overline{SL}(n) = q^\beta$.

(i) If $p = 2$, then $SL^*(n) = 2^\alpha - 1$. This time, we have $2^\alpha - 1 = q^\beta$, then from the definition of $SL^*(n)$ we deduce that $2^\alpha - 1 \mid n$, at the same time, from the definition of $\overline{SL}(n)$ we also have $2^\alpha - 1 < 2^\gamma$, so $2^\alpha \leq 2^\gamma$, but $2^\gamma \mid n$, which implies $2^\alpha \mid n$, so $SL^*(n) \geq 2^\alpha$. This is a contradiction.

(ii) If $p \geq 3$. Assume that $p^\alpha - 1 = q^\beta$, from Lemma 3 we have $p = F_m = 2^{2^m} + 1$, a Fermat prime. Next we will show that $\alpha = 1$.

If $\alpha \geq 2$, then $p^{\alpha-1} < p^\alpha - 1$. For one hand, from the definition of $SL^*(n)$ and $SL^*(n) = p^\alpha - 1$ we have $p^\alpha - 1 \mid n$, so $p^{\alpha-1} \mid n$, on the other hand, from the definition of $\overline{SL}(n)$ we have $\overline{SL}(n) \leq p^{\alpha-1} < p^\alpha - 1 = SL^*(n)$, this is a contradiction. Hence $\alpha = 1$. Consequently every positive even integer n satisfying (2) can be expressed as

$$n = (F_m - 1) \prod_{2<p<F_m} p^{\alpha(p)} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where F_m be a Fermat prime, $r \geq 0$ be any integer; $\alpha(p)$ are positive integers such that $p^{\alpha(p)} > F_m$, $2 < p < F_m$; $p_i > F_m,\ \alpha_i \geq 0,\ i = 1,\ 2,\ \cdots,\ r$.

Associating (A) and (B), we complete the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] A.Murthy, Some notions on least common multiples, Smarandache Notions Journal, **12**(2001), 307-309.

[3] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14**(2004), 186-188.

[4] Lv Zhongtian, On the F.Smarandache LCM function and its mean value, Scientia Magna, **3**(2007), No. 1, 22-25.

[5] Jian Ge, Mean value of F.Smarandache LCM function, Scientia Magna, **3**(2007), No. 2, 109-112.

[6] Chengliang Tian and Na Yuan, On the Smarandache LCM dual function, Scientia Magna, **3**(2007), No. 2, 25-28.

[7] Yanrong Xue, On a conjecture involving the function $SL^*(n)$, Scientia Magna, **3**(2007), No. 2, 41-43.

[8] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[9] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

Scientia Magna
Vol. 3 (2007), No. 3, 108-109

About Smarandache prime additive complement

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Abstract For any positive integer n, the Smarandache prime additive complement function $SPAC(n)$ is defined as the smallest integer $k \geq 0$ such that $n + k$ is a prime. The main purpose of this paper is using the elementary method to prove that it is possible to have k as large as we want $k,\ k-1,\ k-2,\ \cdots,\ 2,\ 1,\ 0$ included in the sequence $\{SPAC(n)\}$.

Keywords The Smarandache prime additive complement, sequence.

§1. Introduction and Results

For any positive integer n, the famous Smarandache prime additive complement function $SPAC(n)$ is defined as the smallest integer $k \geq 0$ such that $n + k$ is a prime. The first few value of this sequence are:

$$1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 3,\ 2,\ 1,\ 0,\ 1,\ 0,\ 3,\ 2,\ 1,\ 0,\ 1,\ 0,\ 3,\ 2,\ 1,\ 0,\ 5,\ 4,\ 3,\ 2,\ 1,\ 0, \cdots .$$

In the book "Only problems, not solutions", Professor F.Smarandache asked us to study the properties of the sequence $\{SPAC(n)\}$. He also proposed the following problem:

Problem A. If it is possible to have k as large as we want

$$k,\ k-1,\ k-2,\ k-3,\ \cdots,\ 2,\ 1,\ 0 \text{ (odd } k)$$

included in the sequence $\{SPAC(n)\}$.In reference [6], Kenichiro Kashihara proposed another problem as follows:

Problem B. Is it possible to have k as large as we want

$$k,\ k-1,\ k-2,\ k-3,\ \cdots,\ 2,\ 1,\ 0 \text{ (even } k \text{)}$$

included in this sequence.

About these two problems, it seems that none had studied them yet, at least we have not seen such a paper before. The problems are important and interesting, because there are close relationship between the sequence $\{SPAC(n)\}$ and the prime distribution. The main purpose of this paper is using the elementary method to study these two problems, and proved that they are true. That is, we shall prove the following:

Theorem. It is possible to have the positive integer k as large as we want

$$k,\ k-1,\ k-2,\ k-3,\ \cdots,\ 2,\ 1,\ 0$$

included in the sequence $\{SPAC(n)\}$.

§2. Proof of the theorem

In this section, we shall prove our theorem directly. Let k and n are positive integers with $n > k+1$, here k as large as we want. Let P be the smallest prime such that $P > n! + n$. It is clear that $P-1$, $P-2$, $\cdots$, $P-k$, $\cdots$, $n!+n$, $\cdots$, $n!+2$ are all composite numbers. Now we consider $k+1$ positive integers:

$$p-k,\ p-k+1,\ p-k+2,\ \cdots,\ p-1,\ p.$$

The Smarandache prime additive complements of these numbers are

$$SPAC(p-k) = k,\ SPAC(p-k-1) = k-1,\ \cdots,\ SPAC(p-1) = 1,\ SPAC(p) = 0.$$

Note that the numbers $k,\ k-1,\ k-2,\ \cdots,\ 1,\ 0$ are included in the sequence $\{SPAC(n)\}$. So the Problem A and Problem B are true.

This completes the proof of the theorem.

References

[1] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] David Gorski, The Pseudo Smarandache Functions, Smarandache Notions Journal, **12**(2000), 140-145.

[3] Maohua Le, Two function equations, Smarandache Notions Journal, **14**(2004), 180-182.

[4] Jozsef Sandor, On a dual of the Pseudo-Smarandache function, Smarandache Notions (Book series), **13**(2002), 16-23.

[5] Jozsef Sandor, On additive analogues of certain arithmetic function, Smarandache Notions Journal, **14**(2004), 128-132.

[6] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[7] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

[8] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[9] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

Scientia Magna
Vol. 3 (2007), No. 3, 110-111

Some new problems about the Smarandache function and related problems[1]

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$. The main purpose of this paper is to introduce some new unsolved problems involving the Smarandache function and the related functions.

Keywords Smarandache function, F.Smarandache LCM function, F.Smarandache dual function, Pseudo-F.Smarandache function.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$. About the properties of $S(n)$, there are many people had studied it, and obtained a series conclusions, see references [1], [2], [3] and [4]. Here we introduce two unsolved problems about the Smarandache function, they are:

Problem 1. If $n > 1$ and $n \neq 8$, then $\sum\limits_{d|n} \dfrac{1}{S(d)}$ is not a positive integer, where $\sum\limits_{d|n}$ denotes the summation over all positive divisors of n.

Problem 2. Find all positive integer solutions of the equation $\sum\limits_{d|n} S(d) = \phi(n)$, where $\phi(n)$ is the Euler function.

For any positive integer n, the F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n|[1, \ 2, \ \cdots, \ k]$, where $[1, \ 2, \ \cdots, \ k]$ is the smallest common multiple of $1, \ 2, \ \cdots, \ k$. About this function, there are three unsolved problems as follows:

Problem 3. If $n > 1$ and $n \neq 36$, then $\sum\limits_{d|n} \dfrac{1}{SL(d)}$ is not a positive integer.

Problem 4. Find all positive integer solutions of the equation $\sum\limits_{d|n} SL(d) = \phi(n)$.

[1]This work is supported by the N.S.F. (10671155) of P.R.China.

Problem 5. Study the mean value properties of $\ln SL(n)$, and give an asymptotic formula for $\sum_{n\leq x} \ln(SL(n))$.

For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime power, we define function $\bar{S}(n) = \max\{\alpha_1 p_1,\ \alpha_2 p_2, \cdots, \alpha_k p_k\}$, and $\bar{S}(1) = 1$. There are two unsolved problems about this function as follows:

Problem 6. If $n > 1$ and $n \neq 24$, then $\sum_{d|n} \frac{1}{\bar{S}(d)}$ is not a positive integer.

Problem 7. Find all positive integer solutions of the equation $\sum_{d|n} \bar{S}(d) = \phi(n)$.

For any positive integer n, the dual function $S^*(n)$ of the Smarandache function is defined as the largest positive integer m such that $m!|n$. That is, $S^*(n) = \max\{m :\ m \in N,\ m!|n\}$. About this function, there are two unsolved problems as follows:

Problem 8. Find all positive integer solutions of the equation $\sum_{d|n} S^*(d) = \phi(n)$.

Problem 9. Study the calculating problem of the product $\prod_{d|n} S^*(d)$, and give an exact calculating formula for it.

For any positive integer n, the Pseudo-F.Smarandache function $Z(n)$ is defined as the largest positive integer m such that $(1 + 2 + 3 + \cdots + m)\,|n$. That is, $Z(n) = \max\{m :\ m \in N,\ \frac{m(m+1)}{2}|n\}$. For this function, there is an unsolved problem as follows:

Problem 10. Study the mean value properties of $Z(n)$, and give an asymptotic formula for $\sum_{n\leq x} Z(n)$.

References

[1] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Liu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 1, 76-79.

[3] Fu Jing, An equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 4, 83-86.

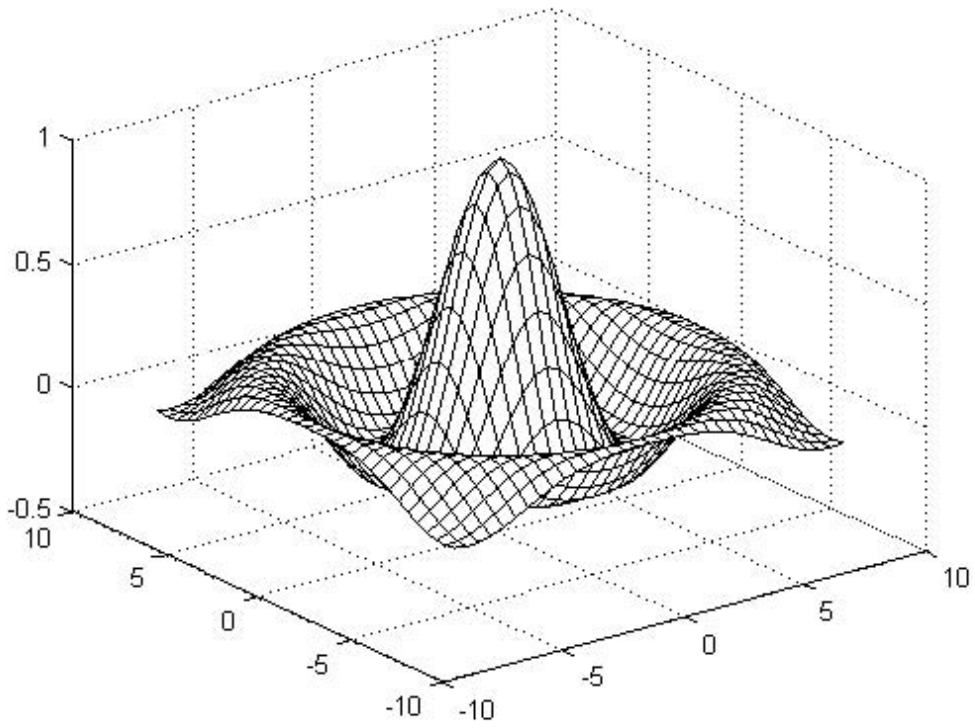
[4] Jozsef Sandor, On additive analogues of certain arithmetical function, Smarandache Notions Journal, **14** (2004), 128-132.

[5] Le Maohua, An equation concerning the Smarandache LCM function, Smarandache Notions Journal, **14** (2004), 186-188.

[6] Jozsef Sandor, On certain inequalities involving the Smarandache function, Scientia Magna, **2**(2006), No. 3, 78-80.

[7] Xu Zhefeng, The mean value of the Smarandache function, Mathematic Sinica，**49**(2006), No. 5, 1009-1012.

[8] Tom M.Apostol, Introduction to analytical number theory, Spring-Verlag, New York, 1976.



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