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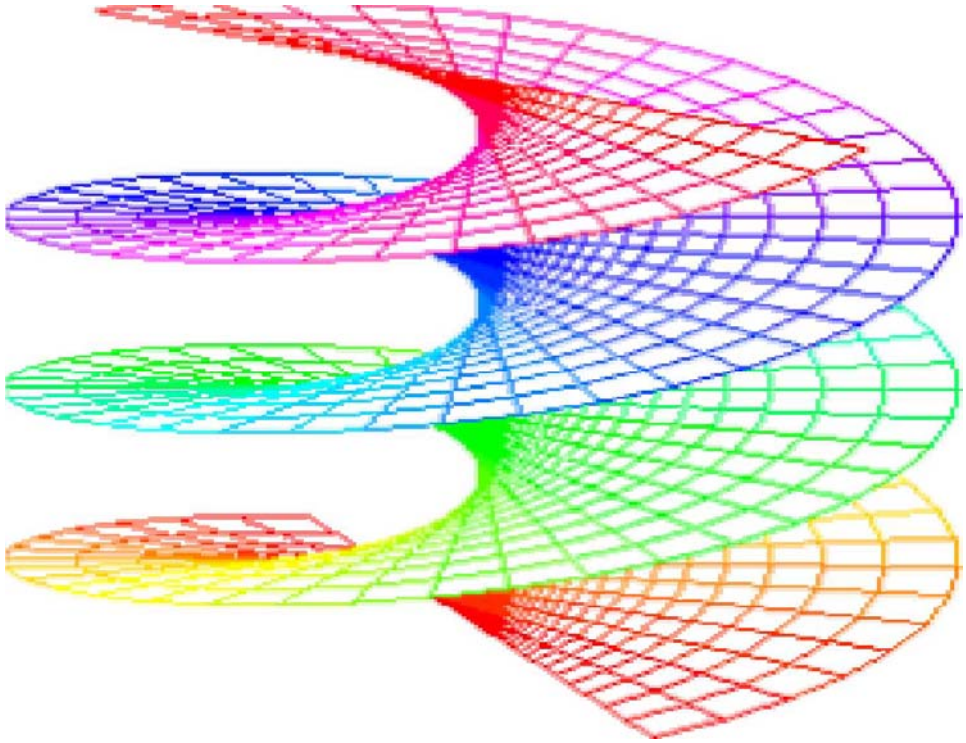

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An holomorphic study of the Smarandache concept in loops

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Abstract If two loops are isomorphic, then it is shown that their holomorphs are also isomorphic. Conversely, it is shown that if their holomorphs are isomorphic, then the loops are isotopic. It is shown that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop. This statement is also shown to be true for some weak Smarandache loops (inverse property, weak inverse property) but false for others (conjugacy closed, Bol, central, extra, Burn, A-, homogeneous) except if their holomorphs are nuclear or central. A necessary and sufficient condition for the Nuclear-holomorph of a Smarandache Bol loop to be a Smarandache Bruck loop is shown. Whence, it is found also to be a Smarandache Kikkawa loop if in addition the loop is a Smarandache A-loop with a centrum holomorph. Under this same necessary and sufficient condition, the Central-holomorph of a Smarandache A-loop is shown to be a Smarandache K-loop.

Keywords Holomorph of loops; Smarandache loops.

§1. Introduction

The study of Smarandache loops was initiated by W.B. Vasantha Kandasamy in 2002. In her book [19], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [16], [3], [5], [8], [9] and [19]. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [20], she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. But in this paper, Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops will be introduced and studied relative to the holomorphs of loops. Interestingly, Adeniran [1] and Robinson [17], Oyebo [15], Chiboka and Solarin [6], Bruck [2], Bruck and Paige [4], Robinson [18], Huthnance [11] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and Bruck loops.

In this study, if two loops are isomorphic then it is shown that their holomorphs are also isomorphic. Conversely, it is shown that if their holomorphs are isomorphic, then the loops are isotopic.

It will be shown that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop. This statement is also shown to be true for some weak Smarandache loops

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(inverse property, weak inverse property) but false for others (conjugacy closed, Bol, central, extra, Burn, A-, homogeneous) except if their holomorphs are nuclear or central. A necessary and sufficient condition for the Nuclear-holomorph of a Smarandache Bol loop to be a Smarandache Bruck loop is shown. Whence, it is found also to be a Smarandache Kikkawa loop if in addition the loop is a Smarandache A-loop with a centrum holomorph. Under this same necessary and sufficient condition, the Central-holomorph of a Smarandache A-loop is shown to be a Smarandache K-loop.

§2. Definitions and Notations

Let $(L, \cdot)$ be a loop. Let $Aum(L, \cdot)$ be the automorphism group of $(L, \cdot)$, and the set $H = (L, \cdot) \times Aum(L, \cdot)$. If we define '$\circ$' on H such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \; \forall \; (\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [2] and is called the Holomorph of $(L, \cdot)$.

The nucleus of $(L, \cdot)$ is denoted by $N(L, \cdot) = N(L)$, its centrum by $C(L, \cdot) = C(L)$ and center by $Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot) = Z(L)$. For the meaning of these three sets, readers should check earlier citations on loop theory.

If in L, $x^{-1} \cdot x\alpha \in N(L)$ or $x\alpha \cdot x^{-1} \in N(L) \; \forall \; x \in L$ and $\alpha \in Aum(L, \cdot)$, $(H, \circ)$ is called a Nuclear-holomorph of L, if $x^{-1} \cdot x\alpha \in C(L)$ or $x\alpha \cdot x^{-1} \in C(L) \; \forall \; x \in L$ and $\alpha \in Aum(L, \cdot)$, $(H, \circ)$ is called a Centrum-holomorph of L hence a Central-holomorph if $x^{-1} \cdot x\alpha \in Z(L)$ or $x\alpha \cdot x^{-1} \in Z(L) \; \forall \; x \in L$ and $\alpha \in Aum(L, \cdot)$.

For the definitions of automorphic inverse property loop (AIPL), anti-automorphic inverse property loop (AAIPL), weak inverse property loop (WIPL), inverse property loop (IPL), Bol loop, Moufang loop, central loop, extra loop, A-loop, conjugacy closed loop (CC-loop) and G-loop, readers can check earlier references on loop theory.

Here, a K-loop is an A-loop with the AIP, a Bruck loop is a Bol loop with the AIP, a Burn loop is Bol loop with the conjugacy closed property, an homogeneous loop is an A-loop with the IP and a Kikkawa loop is an A-loop with the IP and AIP.

Definition 2.1. *A loop is called a Smarandache inverse property loop (SIPL) if it has at least a non-trivial subloop with the IP.*

A loop is called a Smarandache weak inverse property loop (SWIPL) if it has at least a non-trivial subloop with the WIP.

A loop is called a Smarandache G-loop (SG-loop) if it has at least a non-trivial subloop that is a G-loop.

A loop is called a Smarandache CC-loop (SCCL) if it has at least a non-trivial subloop that is a CC-loop.

A loop is called a Smarandache Bol-loop (SBL) if it has at least a non-trivial subloop that is a Bol-loop.

A loop is called a Smarandache central-loop (SCL) if it has at least a non-trivial subloop that is a central-loop.

A loop is called a Smarandache extra-loop (SEL) if it has at least a non-trivial subloop that is a extra-loop.

A loop is called a Smarandache A-loop (SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache K-loop (SKL) if it has at least a non-trivial subloop that is a K-loop.

A loop is called a Smarandache Moufang-loop (SML) if it has at least a non-trivial subloop that is a Moufang-loop.

A loop is called a Smarandache Bruck-loop (SBRL) if it has at least a non-trivial subloop that is a Bruck-loop.

A loop is called a Smarandache Kikkawa-loop (SKWL) if it has at least a non-trivial subloop that is a Kikkawa-loop.

A loop is called a Smarandache Burn-loop (SBNL) if it has at least a non-trivial subloop that is a Burn-loop.

A loop is called a Smarandache homogeneous-loop (SHL) if it has at least a non-trivial subloop that is a homogeneous-loop.

§3. Main Results

Holomorph of Smarandache Loops

Theorem 3.1. Let $(L, \cdot)$ be a Smarandanche loop with subgroup $(S, \cdot)$. The holomorph H_S of S is a group.

Theorem 3.2. A loop is a Smarandache loop if and only if its holomorph is a Smarandache loop.

Proof. Let L be a Smarandache loop with subgroup S. By Theorem 3.1, $(H_S, \circ)$ is a group where $H_S = Aum(S, \cdot) \times (S, \cdot)$. Clearly, $H_S \not\subset H(L, \cdot)$. So, let us replace $Aum(S, \cdot)$ in H_S by $A(S, \cdot) = \{\alpha \in Aum(L, \cdot) \ : \ s\alpha \in S \ \forall \ s \in S\}$, the group of Smarandache loop automorphisms on S as defined in [19]. $A(S, \cdot) \leq Aum(L, \cdot)$ hence, $H_S = A(S, \cdot) \times (S, \cdot)$ remains a group. In fact, $(H_S, \circ) \subset (H, \circ)$ and $(H_S, \circ) \leq (H, \circ)$. Thence, the holomorph of a Smarandache loop is a Smarandache loop.

To prove the converse, recall that $H(L) = Aum(L) \times L$. If $H(L)$ is a Smarandache loop then $\exists \ S_H \subset H(L) \ \ni \ S_H \leq H(L)$. $S_H \subset H(L) \Rightarrow \ \exists \ Bum(L) \subset Aum(L)$ and $B \subset L \ \ni \ S_H = Bum(L) \times B$. Let us choose $Bum(L) = \{\alpha \in Aum(L) \ : \ b\alpha \in B \ \forall \ b \in B\}$, this is the Smarandache loop automorphisms on B. So, $(S_H, \circ) = (Bum(L) \times B, \circ)$ is expected to be a group.

Thus, $(\alpha, x) \circ [(\beta, y) \circ (\gamma, z)] = [(\alpha, x) \circ (\beta, y)] \circ (\gamma, z) \ \forall \ x, y, z \in B, \ \alpha, \beta, \gamma \in Bum(L) \Leftrightarrow x\beta\gamma \cdot (y\gamma \cdot z) = (x\beta\gamma \cdot y\gamma) \cdot z \Leftrightarrow x' \cdot (y' \cdot z) = (x' \cdot y') \cdot z \ \forall \ x', y', z \in B$. So, $(B, \cdot)$ must be a group. Hence, L is a Smarandache loop.

Remark 3.1. It must be noted that if $Aum(L, \cdot) = A(S, \cdot)$, then S is a characteristic subloop.

Theorem 3.3. Let L and L' be loops. $L \cong L'$ implies $H(L) \cong H(L')$.

Proof. If $L \cong L'$ then $\exists$ a bijection $\alpha \ : \ L \to L' \ \ni \ (\alpha, \alpha, \alpha) \ : \ L \to \ L'$ is an isotopism. According to [16], if two loops are isotopic, then their groups of autotopism are

isomorphic. The automorphism group is one of such since it is a form of autotopism. Thus ; $Aum(L) \cong Aum(L') \Rightarrow H(L) = Aum(L) \times L \cong Aum(L') \times L' = H(L')$.

Theorem 3.4. Let $(L, \oplus)$ and $(L', \otimes)$ be loops. $H(L) \cong H(L') \Leftrightarrow x\delta \otimes y\gamma = (x\beta \oplus y)\delta \ \forall \ x, y \in L, \ \beta \in Aum(L)$ and some $\delta, \gamma \in Sym(L')$. Hence, $\gamma\mathcal{L}_{e\delta} = \delta$, $\delta\mathcal{R}_{e\gamma} = \beta\delta$ where e is the identity element in L and $\mathcal{L}_x$, $\mathcal{R}_x$ are respectively the left and right translations mappings of $x \in L'$.

Proof. Let $H(L, \oplus) = (H, \circ)$ and $H(L', \otimes) = (H, \odot)$. $H(L) \cong H(L') \Leftrightarrow \ \exists \ \phi \ : \ H(L) \rightarrow H(L') \ \ni \ [(\alpha, x) \circ (\beta, y)]\phi = (\alpha, x)\phi \odot (\beta, y)\phi$. Define $(\alpha, x)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \ \forall \ (\alpha, x) \in (H, \circ)$ and where $\psi \ : \ L \rightarrow L'$ is a bijection.
$H(L) \cong H(L') \Leftrightarrow (\alpha\beta, x\beta \oplus y)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \odot (\psi^{-1}\beta\psi, y\psi^{-1}\beta\psi) \Leftrightarrow (\psi^{-1}\alpha\beta\psi, (x\beta \oplus y)\psi^{-1}\alpha\beta\psi) = (\psi^{-1}\alpha\beta\psi, x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi) \Leftrightarrow (x\beta \oplus y)\psi^{-1}\alpha\beta\psi = x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi \Leftrightarrow x\delta \otimes y\gamma = (x\beta \oplus y)\delta$ where $\delta = \psi^{-1}\alpha\beta\psi$, $\gamma = \psi^{-1}\beta\psi$.

Furthermore, $\gamma\mathcal{L}_{x\delta} = L_{x\beta}\delta$ and $\delta\mathcal{R}_{y\gamma} = \beta R_y\delta \ \forall \ x, y \in L$. Thus, with $x = y = e$, $\gamma\mathcal{L}_{e\delta} = \delta$ and $\delta\mathcal{R}_{e\gamma} = \beta\delta$.

Corollary 3.1. Let L and L' be loops. $H(L) \cong H(L')$ implies L and L' are isotopic under a triple of the form (δ, I, δ).

Proof. In Theorem 3.4, let $\beta = I$, then $\gamma = I$. The conclusion follows immediately.

Remark 3.2. By Theorem 3.3 and Corollary 3.1, any two distinct isomorphic loops are non-trivialy isotopic.

Corollary 3.2. Let L be a Smarandache loop. If L is isomorphic to L', then $\{H(L), H(L')\}$ and $\{L, L'\}$ are both systems of isomorphic Smarandache loops.

Proof. This follows from Theorem 3.2, Theorem 3.3, Corollary 3.1 and the obvious fact that the Smarandache loop property in loops is isomorphic invariant.

Remark 3.3. The fact in Corollary 3.2 that $H(L)$ and $H(L')$ are isomorphic Smarandache loops could be a clue to solve one of the problems posed in [20]. The problem required us to prove or disprove that every Smarandache loop has a Smarandache loop isomorph.

Smarandache Inverse Properties

Theorem 3.5. Let L be a loop with holomorph $H(L)$. L is an IP-SIPL if and only if $H(L)$ is an IP-SIPL.

Proof. In an IPL, every subloop is an IPL. So if L is an IPL, then it is an IP-SIPL. From [2], it can be stated that L is an IPL if and only if $H(L)$ is an IPL. Hence, $H(L)$ is an IP-SIPL. Conversely assuming that $H(L)$ is an IP-SIPL and using the same argument L is an IP-SIPL.

Theorem 3.6. Let L be a loop with holomorph $H(L)$. L is a WIP-SWIPL if and only if $H(L)$ is a WIP-SWIPL.

Proof. In a WIPL, every subloop is a WIPL. So if L is a WIPL, then it is a WIP-SWIPL. From [11], it can be stated that L is a WIPL if and only if $H(L)$ is a WIPL. Hence, $H(L)$ is a WIP-SWIPL. Conversely assuming that $H(L)$ is a WIP-SWIPL and using the same argument L is a WIP-SWIPL.

Smarandache G-Loops

Theorem 3.7. Every G-loop is a SG-loop.

Proof. As shown in [Lemma 2.2, [7]], every subloop in a G-loop is a G-loop. Hence, the claim follows.

Corollary 3.3. CC-loops are SG-loops.

Proof. In [10], CC-loops were shown to be G-loops. Hence, the result follows by Theorem 3.7.

Theorem 3.8. Let G be a CC-loop with normal subloop H. G/H is a SG-loop.

Proof. According to [Theorem 2.1, [7]], G/H is a G-loop. Hence, by Theorem 3.7, the result follows.

Smarandache Conjugacy closed Loops

Theorem 3.9. Every SCCL is a SG-loop.

Proof. If a loop L is a SCCL, then there exist a subloop H of L that is a CC-loop. CC-loops are G-loops, hence, H is a G-loop which implies L is a SG-loop.

Theorem 3.10 Every CC-loop is a SCCL.

Proof. By the definition of CC-loop in [13], [12] and [14], every subloop of a CC-loop is a CC-loop. Hence, the conclusion follows.

Remark 3.4. The fact in Corollary 3.3 that CC-loops are SG-loops can be seen from Theorem 3.9 and Theorem 3.10.

Theorem 3.11. Let L be a loop with Nuclear-holomorph $H(L)$. L is an IP-CC-SIP-SCCL if and only if $H(L)$ is an IP-CC-SIP-SCCL.

Proof. If L is an IP-CCL, then by Theorem 3.5, $H(L)$ is an IP-SIPL and hence by [Theorem 2.1, [6]] and Theorem 3.10, $H(L)$ is an IP-CC-SIP-SCCL. The converse is true by assuming that $H(L)$ is an IP-CC-SIP-SCCL and using the same reasoning.

Smarandache : Bol loops, central loops, extra loops and Burn loops

Theorem 3.12. Let L be a loop with Nuclear-holomorph $H(L)$. L is a Bol-SBL if and only if $H(L)$ is a Bol-SBL.

Proof. If L is a Bol-loop, then by [17] and [1], $H(L)$ is a Bol-loop. According to [Theorem 6, [20]], every Bol-loop is a SBL. Hence, $H(L)$ is a Bol-SBL. The Converse is true by using the same argument.

Theorem 3.13. Let L be a loop with Nuclear-holomorph $H(L)$. L is a central-SCL if and only if $H(L)$ is a central-SCL.

Proof. If L is a central-loop, then by [15], $H(L)$ is a central-loop. Every central-loop is a SCL. Hence, $H(L)$ is a central-SCL. The Converse is true by using the same argument.

Theorem 3.14. Let L be a loop with Nuclear-holomorph $H(L)$. L is a extra-SEL if and only if $H(L)$ is an extra-SEL.

Proof. If L is a extra-loop, then by [18], $H(L)$ is a extra-loop. Every extra-loop is a SEL. Hence, $H(L)$ is a extra-SEL. The Converse is true by using the same argument.

Corollary 3.4. Let L be a loop with Nuclear-holomorph $H(L)$. L is a IP-Burn-SIP-SBNL if and only if $H(L)$ is an IP-Burn-SIP-SBNL.

Proof. This follows by combining Theorem 3.11 and Theorem 3.12.

Smarandache : A-loops, homogeneous loops

Theorem 3.15. Every A-loop is a SAL.

Proof. According to [Theorem 2.2, [4]], every subloop of an A-loop is an A-loop. Hence, the conclusion follows.

Theorem 3.16. Let L be a loop with Central-holomorph $H(L)$. L is an A-SAL if and only if $H(L)$ is an A-SAL.

Proof. If L is an A-loop, then by [Theorem 5.3, [4]], $H(L)$ is a A-loop. By Theorem 3.15, every A-loop is a SAL. Hence, $H(L)$ is an A-SAL. The Converse is true by using the same argument.

Corollary 3.5. Let L be a loop with Central-holomorph $H(L)$. L is an homogeneous-SHL if and only if $H(L)$ is an homogeneous-SHL.

Proof. This can be seen by combining Theorem 3.5 and Theorem 3.16.

Smarandache : K-loops, Bruck-loops and Kikkawa-loops

Theorem 3.17. Let $(L,\cdot)$ be a loop with holomorph $H(L)$. $H(L)$ is an AIPL if and only if $x\beta^{-1}J \cdot yJ = (x \cdot y\alpha^{-1})J \ \forall \ x,y \in L$ and $\alpha\beta = \beta\alpha \ \forall \ \alpha,\beta \in Aum(L,\cdot)$. Hence, $xJ \cdot yJ = (z \cdot w)J$, $xJ \cdot yJ = (x \cdot w)J$, $xJ \cdot yJ = (y \cdot w)J$, $xJ \cdot yJ = (z \cdot x)J$, $xJ \cdot yJ = (z \cdot y)J$, $xJ \cdot yJ = (x \cdot y)J$, $xJ \cdot yJ = (y \cdot x)J \ \forall \ x,y,z,w \in S$.

Proof. $H(L)$ is an AIPL $\Leftrightarrow$ $\forall \ (\alpha,x),(\beta,y) \in H(L)$, $[(\alpha,x) \circ (\beta,y)]^{-1} = (\alpha,x)^{-1} \circ (\beta,y)^{-1} \Leftrightarrow (\alpha\beta, x\beta\cdot y)^{-1} = (\alpha^{-1},(x\alpha^{-1})^{-1}) \circ (\beta^{-1},(y\beta^{-1})^{-1}) \Leftrightarrow ((\alpha\beta)^{-1},[(x\beta\cdot y)(\alpha\beta)^{-1}]^{-1}) = (\alpha^{-1}\beta^{-1},(x\alpha^{-1})^{-1}\beta^{-1}\cdot(y\beta^{-1})^{-1}) \Leftrightarrow \alpha\beta = \beta\alpha \ \forall \ \alpha,\beta \in Aum(L,\cdot)$ and $(x(\beta\alpha)^{-1})^{-1}\cdot(y\beta^{-1})^{-1} = [x\alpha^{-1} \cdot y(\alpha\beta)^{-1}]^{-1} \Leftrightarrow Aum(L,\cdot)$ is abelian and $(x(\beta\alpha)^{-1})J \cdot y\beta^{-1}J = [x\alpha^{-1} \cdot y(\alpha\beta)^{-1}]J \Leftrightarrow Aum(L,\cdot)$ is abelian and $(x\alpha^{-1}\beta^{-1})J \cdot y\beta^{-1}J = [x\alpha^{-1} \cdot y\beta^{-1}\alpha^{-1}]J \Leftrightarrow Aum(L,\cdot)$ is abelian and $(x(\beta\alpha)^{-1})J \cdot y\beta^{-1}J = [x\alpha^{-1} \cdot y(\alpha\beta)^{-1}]J \Leftrightarrow Aum(L,\cdot)$ is abelian and $x'\beta^{-1}J \cdot y'J = (x' \cdot y'\alpha^{-1})J$ where $x' = x\alpha^{-1}, \ y' = y\beta$.

What follows can be deduced from the last proof.

Theorem 3.18. Let $(L,\cdot)$ be a Bol-SBL with Nuclear-holomorph $H(L)$. $H(L)$ is a Bruck-SBRL if and only if $x\beta^{-1}J \cdot yJ = (x \cdot y\alpha^{-1})J \ \forall \ x,y \in L$ and $\alpha\beta = \beta\alpha \ \forall \ \alpha,\beta \in Aum(L,\cdot)$. Hence,

1. L is a Moufang-SML and a Bruck-SBRL.
2. $H(L)$ is a Moufang-SML.
3. if L is also an A-SAL with Centrum-holomorph $H(L)$ then L is a Kikkawa-SKWL and so is $H(L)$.

Proof. By Theorem 3.12, $H(L)$ is a Bol-SBL. So by Theorem 3.17, $H(L)$ is a Bruck-SBRL $\Leftrightarrow Aum(L,\cdot)$ is abelian and $x\beta^{-1}J \cdot yJ = (x \cdot y\alpha^{-1})J \ \forall \ x,y \in L$.

1. From Theorem 3.17, L is a Bruck-SBRL. From Theorem 3.17, L is an AAIPL, hence L is a Moufang loop since it is a Bol-loop thus L is a Moufang-SML.

2. L is an AAIPL implies $H(L)$ is an AAIPL hence a Moufang loop. Thus, $H(L)$ is a Moufang-SML.

3. If L is also a A-SAL with Centrum-holomorph, then by Theorem 3.5, L and $H(L)$ are both Kikkawa-Smarandache Kikkawa-loops.

Theorem 3.19. Let $(L, \cdot)$ be a SAL with an A-subloop S and Central-holomorph $H(L)$. $H(L)$ is a SKL if and only if $x\beta^{-1}J \cdot yJ = (x \cdot y\alpha^{-1})J \; \forall \; x, y \in S$ and $\alpha\beta = \beta\alpha \; \forall \; \alpha, \beta \in A(S, \cdot)$. Hence, L is a SKL.

Proof. By Theorem 3.16, $H(L)$ is a SAL with A-subloop $H_S = A(S, \cdot) \times (S, \cdot)$. So $H(L)$ is a SKL if and only if H_S is a K-loop $\Leftrightarrow$ $A(S, \cdot)$ is abelian and $x\beta^{-1}J \cdot yJ = (x \cdot y\alpha^{-1})J \; \forall \; x, y \in S \; , \alpha, \beta \in A(S, \cdot)$ by Theorem 3.17. Following Theorem 3.17, S is an AIPL hence a K-loop which makes L to be a SKL.

References

[1] J. O. Adeniran, On holomorphic theory of a class of left Bol loops, Scientific Annals of Al.I.Cuza. Univ. (to appear).

[2] R. H. Bruck, Contributions to the theory of loops, Trans. Amer. Soc. **55**(1994), 245–354.

[3] R. H. Bruck, A survey of binary systems, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, pp. 185.

[4] R. H. Bruck and L. J. Paige, Loops whose inner mappings are automorphisms, The annuals of Mathematics, **63**(1956), No. 2, 308-323.

[5] O. Chein, H. O. Pflugfelder and J. D. H. Smith, Quasigroups and Loops : Theory and Applications, Heldermann Verlag, 1990, pp. 568.

[6] V. O. Chiboka and A. R. T. Solarin, Holomorphs of conjugacy closed loops, Scientific Annals of Al.I.Cuza. Univ. **37**(1991), No.3, 277-284.

[7] V. O. Chiboka and A. R. T. Solarin, Autotopism characterization of G-loops, Scientific Annals of Al.I.Cuza. Univ. **39**(1993), No. 1, 19-26.

[8] J. Dene and A. D. Keedwell, Latin squares and their applications, the English University press Lts, 1974, pp. 549.

[9] E. G. Goodaire, E. Jespers and C. P. Milies, Alternative Loop Rings, NHMS(184), Elsevier, 1996, pp. 387.

[10] E. G. Goodaire and D. A. Robinson, A class of loops which are isomorphic to all their loop isotopes, Can. J. Math. **34**(1982), 662-672.

[11] E. D. Huthnance Jr., A theory of generalised Moufang loops, Ph.D. thesis, Georgia Institute of Technology, 1968.

[12] M. K. Kinyon, K. Kunen and J. D. Phillips, Diassociativity in conjugacy closed loops, Comm. Alg. **32**(2004), 767-786.

[13] M. K. Kinyon, K. Kunen, Power-Associative, conjugacy closed loops, J. Alg. (to appear).

[14] K. Kunen (2000), The structure of conjugacy closed loops, Trans. Amer. Math. Soc. **352**(2000), 2889-2911.

[15] Y. T. Oyebo and J. O. Adeniran, An holomorphic study of Central loops, Pre-print.

[16] H. O. Pflugfelder, Quasigroups and Loops : Introduction, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 1990, pp. 147.

[17] D. A. Robinson, Bol loops, Ph. D thesis, University of Wisconsin, Madison, Wisconsin, 1964.

[18] D. A. Robinson, Holomorphic theory of extra loops, Publ. Math. Debrecen **18**(1971), 59-64.

[19] W. B. Vasantha Kandasamy, Smarandache Loops, Department of Mathematics, Indian Institute of Technology, Madras, India, 2002, pp. 128.

[20] W. B. Vasantha Kandasamy, Smarandache Loops, Smarandache notions journal, **13**(2002), 252-258.

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Some arithmetical properties of primitive numbers of power p[1]

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Abstract The main purpose of this paper is to study the arithmetical properties of the primitive numbers of power p by using the elementary method, and give some interesting identities and asymptotic formulae.

Keywords Primitive numbers of power p; Smarandache function; Asymptotic formula.

§1. Introduction

Let p be a fixed prime and n be a positive integer. The primitive numbers of power p, denoted as $S_p(n)$, is defined as following:

$$S_p(n) = \min\{m : m \in N, p^n | m!\}.$$

In problem 47,48 and 49 of [1], Professor F.Smarandache asked us to study the properties of the primitive numbers sequences $\{S_p(n)\}(n = 1, 2, \cdots)$. It is clear that $\{S_p(n)\}(n = 1, 2, \cdots)$ is the sequence of multiples of prime p and each number being repeated as many times as its exponent of power p is. What's more, there is a very close relationship between this sequence and the famous Smarandache function $S(n)$, where

$$S(n) = \min\{m : m \in N, n | m!\}.$$

Many scholars have studied the properties of $S(n)$, see [2], [3], [4], [5] and [6]. It is easily to show that $S(p) = p$ and $S(n) < n$ except for the cases $n = 4$ and $n = p$. Hence, the following relationship formula is obviously:

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n} \right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x. However, it seems no one has given some nontrivial properties about the primitive number sequences before. In this paper, we studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, we shall prove the following conclusions:

Theorem 1. For any prime p and complex number s, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

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where $\zeta(s)$ is the Riemann zeta-function.

Specially, taking $s = 2, 4$ and $p = 2, 3, 5$, we have the

Corollary. The following identities hold:

$$\sum_{n=1}^{\infty}\frac{1}{S_2^2(n)}=\frac{\pi^2}{18};\qquad \sum_{n=1}^{\infty}\frac{1}{S_3^2(n)}=\frac{\pi^2}{48};\qquad \sum_{n=1}^{\infty}\frac{1}{S_5^2(n)}=\frac{\pi^2}{144};$$

$$\sum_{n=1}^{\infty}\frac{1}{S_2^4(n)}=\frac{\pi^4}{1350};\qquad \sum_{n=1}^{\infty}\frac{1}{S_3^4(n)}=\frac{\pi^4}{7200};\qquad \sum_{n=1}^{\infty}\frac{1}{S_5^4(n)}=\frac{\pi^4}{56160}.$$

Theorem 2. Let p be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1\\ S_p(n)\leq x}}^{\infty}\frac{1}{S_p(n)}=\frac{1}{p-1}\left(\ln x+\gamma+\frac{p\ln p}{p-1}\right)+O(x^{-\frac{1}{2}+\epsilon}),$$

where γ is the Euler constant, ϵ denotes any fixed positive number.

Theorem 3. Let k be any positive integer. Then for any prime p and real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1\\ S_p(n)\leq x}}^{\infty}S_p^k(n)=\frac{x^{k+1}}{(k+1)(p-1)}+O(x^{k+\frac{1}{2}+\epsilon}).$$

§2. Proof of the theorems

To complete the proof of the theorems, we need a simple Lemma.

Lemma. Let b, T are two positive numbers, then we have

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{a^s}{s}ds=\begin{cases}1+O\left(a^b\min\left(1,\frac{1}{T\ln a}\right)\right), & if\ a>1;\\ O\left(a^b\min\left(1,\frac{1}{T\ln a}\right)\right), & if\ 0<a<1;\\ \frac{1}{2}+O\left(\frac{b}{T}\right), & if\ a=1.\end{cases}$$

Proof. See Lemma 6.5.1 of [8].

Now we prove the theorems. First, we prove Theorem 1. Let $m = S_p(n)$, if $p^{\alpha}\|m$, then the same number m will repeat α times in the sequence $S_p(n)$ $(n = 1, 2, \cdots)$. Noting that $S_p(n)$ $(n = 1, 2, \cdots)$ is the sequence of multiples of prime p, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} &= \sum_{\substack{m=1\\ p^{\alpha}\|m}}^{\infty}\frac{\alpha}{m^s}=\sum_{p^{\alpha}}\sum_{\substack{m=1\\ (m,p)=1}}^{\infty}\frac{\alpha}{p^{\alpha s}m^s}\\ &= \sum_{\alpha=1}^{\infty}\frac{\alpha}{p^{\alpha s}}\zeta(s)\left(1-\frac{1}{p^s}\right)=\left(1-\frac{1}{p^s}\right)\zeta(s)\sum_{\alpha=1}^{\infty}\frac{\alpha}{p^{\alpha s}}.\end{aligned}$$

Since

$$\left(1-\frac{1}{p^s}\right)\sum_{\alpha=1}^{\infty}\frac{\alpha}{p^{\alpha s}}=\frac{1}{p^s}+\sum_{\alpha=1}^{\infty}\frac{1}{p^{(\alpha+1)s}}=\frac{1}{p^s}+\frac{1}{p^s}\left(\frac{1}{p^s-1}\right)=\frac{1}{p^s-1},$$

we have the identity

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)}=\frac{\zeta(s)}{p^s-1}.$$

This completes the proof of Theorem 1.

Now we prove Theorem 2 and Theorem 3. Let $x\geq 1$ be any real number. If we set $a=\frac{x}{S_p(n)}$ in the lemma, then we can write

$$\begin{aligned}&\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}\frac{x^s}{S_p^{s-k}(n)s}ds\\ =&\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}S_p^k(n)+O\left(\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}\frac{x^b}{S_p^{b-k}(n)}\min\left(1,\frac{1}{T\ln\left(\frac{x}{S_p(n)}\right)}\right)\right),\end{aligned}\tag{1}$$

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\sum_{\substack{n=1\\S_p(n)>x}}^{\infty}\frac{x^s}{S_p^{s-k}(n)s}ds=O\left(\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}\frac{x^b}{S_p^{b-k}(n)}\min\left(1,\frac{1}{T\ln\left(\frac{x}{S_p(n)}\right)}\right)\right),\tag{2}$$

where k is any integer. Combining (1) and (2), we find

$$\begin{aligned}&\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{x^s}{s}\sum_{n=1}^{\infty}\frac{1}{S_p^{s-k}(n)}ds\\ =&\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}S_p^k(n)+O\left(\sum_{n=1}^{\infty}\frac{x^b}{S_p^{b-k}(n)}\min\left(1,\frac{1}{T\ln\left(\frac{x}{S_p(n)}\right)}\right)\right).\end{aligned}\tag{3}$$

Then from Theorem 1, we can get

$$\sum_{\substack{n=1\\S_p(n)\leq x}}^{\infty}S_p^k(n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{\zeta(s-k)x^s}{(p^{s-k}-1)s}ds+O\left(x^b\min\left(1,\frac{1}{T\ln\left(\frac{x}{S_p(n)}\right)}\right)\right)\tag{4}$$

Now we calculate the first term in the right side of (4).

When $k=-1$, taking $b=\frac{1}{2}$ and $T=x$, we move the integral line from $s=\frac{1}{2}+iT$ to $s=-\frac{1}{2}+iT$. This time, the function

$$f(s)=\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}$$

have a second order pole point at $s=0$. Its residue is $\frac{1}{p-1}\left(\ln x+\gamma-\frac{p\ln p}{p-1}\right)$. Hence, we can write

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds\\ =&\frac{1}{p-1}\left(\ln x+\gamma-\frac{p\ln p}{p-1}\right)+\frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{-\frac{1}{2}-iT}+\int_{-\frac{1}{2}-iT}^{-\frac{1}{2}+iT}+\int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT}\right)\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds.\end{aligned}\tag{5}$$

We can easily get the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{-\frac{1}{2}-iT}+\int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT}\right)\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds\right|\\ \ll&\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{\zeta(\sigma+1+iT)x^{\frac{1}{2}}}{(p^{\sigma+1+iT}-1)T}\right|d\sigma\ll\frac{x^{\frac{1}{2}}}{T}=x^{-\frac{1}{2}},\end{aligned}\tag{6}$$

and

$$\left|\frac{1}{2\pi i}\int_{-\frac{1}{2}-iT}^{-\frac{1}{2}+iT}\frac{\zeta(s+1)x^s}{(p^{s+1}-1)s}ds\right|\ll\int_0^T\left|\frac{\zeta(\frac{1}{2}+it)x^{-\frac{1}{2}}}{(p^{\frac{1}{2}+it}-1)(\frac{1}{2}+t)}\right|dt\ll x^{-\frac{1}{2}+\epsilon}.\tag{7}$$

Combining (4), (5), (6) and (7), we have

$$\sum_{\substack{n=1\\S_p(n)\le x}}^{\infty}\frac{1}{S_p(n)}=\frac{1}{p-1}\left(\ln x+\gamma+\frac{p\ln p}{p-1}\right)+O(x^{-\frac{1}{2}+\epsilon}).$$

This is the result of Theorem 2.

When $k\ge 1$, taking $b=k+\frac{3}{2}$ and $T=x$, we move the integral line of (4) from $s=k+\frac{3}{2}$ to $s=k+\frac{1}{2}$. Now the function

$$g(s)=\frac{\zeta(s-k)x^s}{(p^{s-k}-1)s}$$

have a simple pole point at $s=k+1$ with residue $\frac{x^{k+1}}{(p-1)(k+1)}$. Using the same method we can also get

$$\sum_{\substack{n=1\\S_p(n)\le x}}^{\infty}S_p^k(n)=\frac{x^{k+1}}{(k+1)(p-1)}+O(x^{k+\frac{1}{2}+\epsilon}).$$

This completes the proofs of the theorems.

References

[1] F.Smaradache, Only problems, not solutions, Xiquan Publishing House, Chicago, 1993.

[2] P.Erdös, Problem 6674, Amer. Math. Monthly, **98**(1991), pp. 965.

[3] Ashbacher, C., Some Properties of the Smarandache-Kurepa and Smarandache-Wagstaff Functions, Mathematics and Informatics Quarterly, **7**(1997), pp. 114-116.

[4] Begay, A., Smarandache Ceil Functions, Bulletin of Pure and Applied Sciences, **16E**(1997), pp.227-229.

[5] Mark Farris and Patrick Mitchell, Bounding the Smarandache function, Smarandache Notions Journal, **13**(2002), pp. 37-42.

[6] Kevin Ford, The normal behavior of the Smarandache function, Smarandache Notions Journal, **10**(1999), pp. 81-86.

[7] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[8] Pan Chengdong and Pan Chengbiao, Element of the analytic Number Theory, Science Press, Beijing, 1991, pp.96.

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Smarandache Quasigroups

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Abstract In this paper, we have introduced Smarandache quasigroups which are Smarandache non-associative structures. W.B.Kandasamy [2] has studied Smarandache groupoids and Smarandache semigroups etc. Substructure of Smarandache quasigroups are also studied.

Keywords Quasigroup; Smarandache Quasigroup.

1. Introduction

W.B.Kandasamy has already defined and studied Smarandache groupoids, Smarandache semigroups etc. A quasigroup is a groupoid whose composition table is LATIN SQUARE. We define Smarandache quasigroup as a quasigroup which contains a group properly.

2. Preliminaries

Definition 2.1. A groupoid S such that for all a, $b \in S$ there exist unique x, $y \in S$ such that $ax = b$ and $ya = b$ is called a quasigroup.

Thus a quasigroup does not have an identity element and it is also non-associative.

Example 2.1. Here is a quasigroup that is not a loop.

*	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

We note that the definition of quasigroup Q forces it to have a property that every element of Q appears exactly once in every row and column of its operation tables. Such a table is called a LATIN SQUARE. Thus, quasigroup is precisely a groupoid whose multiplication table is a LATIN SQUARE.

Definition 2.2. If a quasigroup $(Q, *)$ contains a group $(G, *)$ properly then the quasigroup is said to be Smarandache quasigroup.

A Smarandache quasigroup is also denoted by S-quasigroup.

Example 2.2. Let Q be a quasigroup defined by the following table;

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is a group *w.r.t.* $*$ which is a proper subset of Q. Therefore Q is a Smarandache quasigroup.

Definition 2.3. A quasigroup Q is idempotent if every element x in Q satisfies $x * x = x$.

Theorem 2.1. If a quasigroup contains a Smarandache quasigroup then the quasigroup is a Smarandache quasigroup.

Proof. Follows from definition of Smarandache quasigroup.

Example 2.3. $(Q, *)$ defined by the following table is a quasigroup.

$*$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$(Q, *)$ is an idempotent quasigroup.

Definition 2.4. An element x in a quasigroup Q is called idempotent if $x.x = x$.

Consider a quasigroup;

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Here 2 is an idempotent element.

Example 2.4. The smallest quasigroup which is neither a group nor a loop is a quasigroup of order 3 as given by the following table;

$*$	q_1	q_2	q_3
q_1	q_1	q_2	q_3
q_2	q_3	q_1	q_2
q_3	q_2	q_3	q_1

3. A new class of Quasigroups

V.B.Kandasamy [2] has defined a new class of groupoids as follows;

Definition 3.1. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$. For a, $b \in Z_n$ define a binary operation $*$ on Z_n as: $a * b = ta + ub \ (mod\ n)$ where t, u are two distinct element in $Z_n \setminus \{0\}$ and $(t, u) = 1$. Here $+$ is the usual addition of two integers and ta means the product of two integers t and a. We denote this groupoid by $Z_n(t, u)$.

Theorem 3.1. Let $Z_n(t, u)$ be a groupoid. If $n = t + u$ where both t and u are primes then $Z_n(t, u)$ is a quasigroup.

Proof. When t and u are primes every row and column in the composition table will have distinct n element. As a result $Z_n(t, u)$ is a quasigroup.

Corollary 3.1. If $Z_p(t, u)$ is a groupoid and $t + u = p$, $(t, u) = 1$ then $Z_p(t, u)$ is a quasigroup.

Proof. Follows from the theorem.

Example 3.1. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Let $t = 2$ and $u = 3$. Then $5 = 2 + 3$, $(2, 3) = 1$ and the composition table is:

*	0	1	2	3	4
0	0	3	1	4	2
1	2	0	3	1	4
2	4	2	0	3	1
3	1	4	2	0	3
4	3	1	4	2	0

Thus $Z_5(2, 3)$ is a quasigroup.

Definition 3.2. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $t = u$. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.2. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Then $Z_5(3, 3)$ is a quasigroup as given by the following table:

*	0	1	2	3	4
0	0	3	1	4	2
1	3	1	4	2	0
2	1	4	2	0	3
3	4	2	0	3	1
4	2	0	3	1	4

Definition 3.3. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $t = 1$ and $u = n - 1$. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.3. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(1, 7)$ is a quasigroup as given by the following table:

*	0	1	2	3	4	5	6	7
0	0	7	6	5	4	3	2	1
1	1	0	7	6	5	4	3	2
2	2	1	0	7	6	5	4	3
3	3	2	1	0	7	6	5	4
4	4	3	2	1	0	7	6	5
5	5	4	3	2	1	0	7	6
6	6	5	4	3	2	1	0	7
7	7	6	5	4	3	2	1	0

Definition 3.4. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $(t, u) = 1$, $t + u = n$ and $|t - u|$ is a minimum. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.4. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(3, 5)$ is a quasigroup as given by the following table:

*	0	1	2	3	4	5	6	7
0	0	5	2	7	4	1	6	3
1	3	0	5	2	7	4	1	6
2	6	3	0	5	2	7	4	1
3	1	6	3	0	5	2	7	4
4	4	1	6	3	0	5	2	7
5	7	4	1	6	3	0	5	2
6	2	7	4	1	6	3	0	5
7	5	2	7	4	1	6	3	0

Definition 3.5. Let $(Q, *)$ be a quasigroup. A proper subset V of Q is called a subquaisgroup of Q if V itself is a quasigroup under $*$.

Definition 3.6. Let Q be a quasigroup. A subquaisgroup V of Q is said to be normal subquaisgroup of Q if:

1. $aV = Va$
2. $(Vx)y = V(xy)$
3. $y(xV) = (yx)V$

for all a, x, $y \in Q$.

Example 3.5. Let Q be a quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	4	3	2	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	2	1	4	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Here $V = \{1, 2, 3, 4\}$ is a normal subquasigroup of Q.

Definition 3.7. A subquasigroup is said to be simple if it has no proper nontrivial normal subgroup.

4. Substructures of Smarandache Quasigroups

Definition 4.1. Let $(Q, *)$ be a Smarandache quasigroup. A nonempty subset H of Q is said to be a Smarandache subquasigroup if H contains a proper subset K such that k is a group under $*$.

Example 4.1. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Consider $S = \{1, 2, 3, 4\}$ then S is a subquasigroup which contains a group $G = \{1, 2\}$. Therefore S is a Smarandache subquasigroup.

Example 4.2. There do exist Smarandache quasigroup which do not posses any Smarandache subquasigroup. Consider the quasigroup Q defined by the following table:

*	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Clearly, Q is Smarandache quasigroup as it contains a group $G = \{2\}$. But there is no subquasigroup, not to talk of Smarandache subquasigroup.

Definition 4.2. Let Q be a S-quasigroup. If $A \subset Q$ is a proper subset of Q and A is a subgroup which can not be contained in any proper subquasigroup of Q we say A is the largest subgroup of Q.

Example 4.3. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Clearly, $A = \{1, 2, 3, 4\}$ is the largest subgroup of Q.

Definition 4.3. Let Q be a S-quasigroup. If A is a proper subset of Q which is subquasigroup of Q and A contains the largest group of Q then we say A to be the Smarandache hyper subquasigroup of Q.

Example 4.4. Let Q be a quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	4	3	6	5	8	7
2	2	1	3	4	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Here $A = \{1, 2, 3, 4\}$ is the subquasigroup of Q which contains the largest group $\{1, 2\}$ of Q. A is a Smarandache hyper subquasigroup of Q.

Definition 4.4. Let Q be a finite S-quasigroup. If the order of every subgroup of Q divides the order of the S-quasigroup Q then we say Q is a Smarandache Lagrange quasigroup.

Example 4.5. In the above example 4.4, Q is a S-quasigroup whose only subgroup are $\{1\}$ and $\{1, 2\}$. Clearly, order of these subgroups divide the order of the quasigroup Q. Thus Q is the Smarandache Lagrange quasigroup.

Definition 4.5. Let Q be a finite S-quasigroup. p is the prime such that p divides the order of Q. If there exist a subgroup A of Q of order p or p^l, $(l > 1)$ we say Q has a Smarandache p-Sylow subgroup.

Example 4.6. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Consider $A = \{1, 2, 3, 4\}$ then A is a subgroup of Q whose order 2^2 divides order of Q. Therefore Q has a Smarandache 2-Sylow subgroup.

Definition 4.6. Let Q be a finite S-quasigroup. An element $a \in A$, $a \subset Q$ (A a proper subset of Q and A is the subgroup under the operation of Q) is said to be a Smarandache Cauchy element of Q if $a^r = 1$, $(r > 1)$ and 1 is the unit element of A and r divides the order of Q otherwise a is not a Smarandache Cauchy element of Q.

Definition 4.7. Let Q be a finite S-quasigroup if every element in every subgroup of Q is a Smarandache Cauchy element of Q then we say that Q is a Smarandache Cauchy quasigroup.

Example 4.6. In the above example 4.6 there are three subgroup of Q. They are $\{1\}$, $\{1, 2\}$ and $\{1, 2, 3, 4\}$. Each element in each subgroup is a Smarandache Cauchy element as $1^2 = 2^2 = 3^2 = 4^2 = 1$ in each respective subgroup. Thus Q is a Smarandache Cauchy group.

References

[1] R. H. Bruck, A survey of binary system, Springer-Verlag, New York, 1958.

[2] W. B. Kandasamy, Smarandache Semigroup, American Research Press, 2002.

[3] Robinson Derek J. S., A course in the theory of Groups, Springer-verlag, New York, 1996.

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Two Classes of Smarandache Determinants

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Abstract In this paper we solve some conjectures concerned the Smarandache cyclic determinants and the Smarandache bisymmetric determinants.

Keywords Smarandache cyclic determinant; Smarandache bisymmetric determinants; Cyclic convolution; Elimination.

§1. Smarandache cyclic determinants

For any positive integer n, then $n \times n$ determinant

$$\begin{vmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n-1 & n & \cdots & n-3 & n-2 \\ n & 1 & \cdots & n-2 & n-1 \end{vmatrix} \tag{1}$$

is called the nth Smarandache cyclic natural determinant, and denote by $SCND(n)$. in[1], Murthy given the following conjecture.

Conjecture 1.1. $SCND(n) = (-1)^{[n/2]}n^{n-1}(n+1)/2$, where $[n/2]$ is the interger part of the $n/2$.

Let a, d be complex numbers. The $n \times n$ determinant

$$\begin{vmatrix} a & a+d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & \cdots & a+(n-1)d & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a+(n-2)d & a+(n-1)d & \cdots & a+(n-4)d & a+(n-3)d \\ a+(n-1)d & a & \cdots & a+(n-3)d & a+(n-2)d \end{vmatrix} \tag{2}$$

is called the nth Smarandache cyclic arithmetic determinant with parameters (a, d), and denoted by $SCAD(n; a, d)$. In this respect, Murthy [1] give the following conjecture.

Conjecture 1.2. $SCAD(n; a, d) = (-1)^{[n/2]}(nd)^{n-1}(a+(n-1)d)/2$.

In this section we shall show that Conjecture 1.1 is true and Conjecture 1.2 is false. We now prove the following two results.

Theorem 1.1. For any positive integer n,

$$SCND(n) = (-1)^{[n/2]} n^{n-1}(n+1)/2. \tag{3}$$

Theorem 1.2. For any positive integer n and any complex numbers a, d,

$$SCAD(n;a,d) = \begin{cases} a & \text{if } n = 1 \\ (-1)^{[n/2]}(nd)^{n-1}(a+(n-1)d)/2 & \text{if } n > 1 \end{cases} \tag{4}$$

The proofs of our theorems depend on a well known result concerned cyclic determinants. Let $a_1, a_2, \cdots, a_n$ be complex numbers. Then the $n \times n$ determinant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_n & a_1 \end{vmatrix} \tag{5}$$

is called the n-th cyclic determinant with parameters $(a_1, a_2, \cdots, a_n)$, and denoted by $CD(a_1, a_2, \cdots, a_n)$. Then we have

Lemma 1.1.

$$CD(a_1, a_2, \cdots, a_n) = \prod_{x^n=1} (a_1 + a_2 x + \cdots + a_n x^n)$$

where the product $\prod_{x^n=1}$ means x through over all complex numbers with $x^n = 1$.

Proof of Theorem 1.1. We see from (1) and (5) that

$$SCND(n) = (-1)^r CD(1, 2, \cdots, n), \tag{6}$$

Where

$$r = \begin{cases} \frac{n}{2} - 1, & \text{if n is even} \\ \frac{n-1}{2}, & \text{if n is odd} \end{cases} \tag{7}$$

By Lemma1.1, we get

$$D(1, 2, \cdots, n) = \prod_{x^n=1} (1 + 2x + \cdots + nx^{n-1}) \tag{8}$$

Notice that if $x^n = 1$, then

$$(1 + 2x + \cdots + nx^{n-1})(1 - x) = 1 + x + \cdots + x^{n-1} - n \tag{9}$$

$$= \begin{cases} 0 & \text{if } x = 1, \\ -n & \text{if } x \neq 1. \end{cases} \tag{10}$$

By (8) and (10), we obtain

$$D(1,2,\cdots,n)\prod_{\substack{x^n=1\\ x\neq 1}}(1-x)=(-1)^{n-1}n^n\left(\frac{n+1}{2}\right), \tag{11}$$

where the product $\prod_{\substack{x^n=1\\ x\neq 1}}$ means x through over all complex numbers satisfying $x^n=1$ and $x\neq 1$. Since

$$\prod_{\substack{x^n=1\\ x\neq 1}}(1-x)=n \tag{12}$$

we get from (11) that

$$D(1,2,\cdots,n)=(-1)^{n-1}n^{n-1}(n+1)/2. \tag{13}$$

Further, by (7), we get

$$r+n-1\equiv[\frac{n}{2}]\quad(\mod 2). \tag{14}$$

Thus, by (6), (13) and (14), we obtain (3). The theorem is proved.

Proof of Theorem 1.2. By (2), if $n=1$, then (4) holds. We may therefore assume that $n>1$. We see from (2) and (5) that

$$SCAD(n;a,d)=(-1)^rCD(a,a+d,\cdots,a+(n-1)d), \tag{15}$$

where r satisfies (7). By Lemme 1.1, we get

$$CD(a,a+d,\cdots,a+(n-1)d)=\prod_{x^n=1}\left(a+(a+d)x+\cdots+(a+(n-1)d)x^{n-1}\right). \tag{16}$$

Notice that if $x^n=1$, then

$$\begin{aligned}&(a+(a+d)x+\cdots+(a+(n-1)d)x^{n-1})(1-x)\\ =\;& a+dx+\cdots+dx^{n-1}-(a+(n-1)d) \qquad (17)\\ =\;&\begin{cases}0 & \text{if } x=1,\\ -nd & \text{if } x\neq 1.\end{cases} \qquad (18)\end{aligned}$$

Hence, by (16) and (18), we get

$$CD(a,a+d,\cdots,a+(n-1)d)\prod_{\substack{x^n=1\\ x\neq 1}}(1-x)=(-1)^{n-1}(nd)^{n-1}\left(na+\frac{n(n-1)}{2}d\right) \tag{19}$$

Thus, by (12), (14), (16) and (19), we obtain (4). The theorem is proved.

§2. Smarandache bisymmetric determinants

For any positive integer n, the $n \times n$ determinant

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n-2 & n-1 & n & \cdots & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix} \tag{20}$$

is called the nth Smarandache bisymmetric natural determinant, and denoted by SBND(n). In [1], Murthy given the following conjecture.

Conjecture 2.1. $SBND(n) = (-1)^{[n/2]}2^{n-3}n(n+1)$.

Let a, d be complex numbers, then the $n \times n$ determinant

$$\begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-2)d & a+(n-1)d & a+(n-2)d \\ a+2d & a+3d & a+4d & \cdots & a+(n-1)d & a+(n-2)d & a+(n-3)d \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a+(n-2)d & a+(n-1)d & a+(n-2)d & \cdots & a+3d & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & a+(n-3)d & \cdots & a+2d & a+d & a \end{vmatrix} \tag{21}$$

is called the nth Smarandache bisymmetric arithmetic determinant with parameters (a, d), and denoted by $SBND(n; a, d)$. In this respect, Murthy [1] given the following conjecture.

Conjecture 2.2. $SBAD(n; a, d) = (-1)^{[n/2]}2^{n-3}d^{n-1}(a + (n-1)d)$.

Unfortunately, we shall show that both Conjecture 2.1 and 2.2 are false. In this paper we will prove the following two results.

Theorem 2.1. For any positive integer n,

$$SBND(n) = (-1)^{n(n-1)/2}2^{n-2}(n+1) \tag{22}$$

Theorem 2.2. For any positive integer n and any complex numbers a, d,

$$SBAD(n; a, d) = (-1)^{n(n-1)/2}2^{n-2}d^{n-1}(2a + (n-1)d). \tag{23}$$

Proof of Theorem 2.1. Let $R(m)(m = 1, 2, \cdots, n)$ denote the mth row of $SBND(n)$.

We first successively add $-R(i)$ to $R(i+1)$ for $i = n-1, n-2, \cdots, 1$. Then, by (20), we get

$$SBND(n) = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & -1 & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} \tag{24}$$

Let $C(m)(m = 1, 2, \cdots, n)$ be the mth column of the determinant in (24). Next, we successively add $C(1)$ to $C(j)$ for $j = 2, \cdots, n$. Then we get

$$SBND(n) = \begin{vmatrix} 1 & 3 & 4 & \cdots & n-1 & n & n+1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 2 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = (-1)^{n(n-1)} 2^{n-2}(n+1). \tag{25}$$

Thus, the theorem if proved.

Proof of Theorem 2.2. Let $R(m)(m = 1, 2, \cdots, n)$ denote the mth row of $SBAD(n; a, d)$. We first successively add $-R(i)$ to $R(i+1)$ for $i = n-1, n-2, \cdots, 1$. Then, by (21), we get

$$SBAD(n; a, d) = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ d & d & d & \cdots & d & d & -d \\ d & d & d & \cdots & d & -d & -d \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d & d & d & \cdots & -d & -d & -d \\ d & d & -d & \cdots & -d & -d & -d \\ d & -d & -d & \cdots & -d & -d & -d \end{vmatrix} \tag{26}$$

Let $C(m)(m = 1, 2, \cdots, n)$ denote the mth column of the determinant in (26). Next, we

successively add $C(1)$ to $C(j)$ for $j = 2, \cdots, n$. Then we get

$$SBAD(n; a, d) = \begin{vmatrix} a & 2a+d & 2a+2d & \cdots & 2a+(n-2)d & 2a+(n-1)d \\ d & 2d & 2d & \cdots & 2d & 0 \\ d & 2d & 2d & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d & 2d & 2d & \cdots & 0 & 0 \\ d & 2d & 0 & \cdots & 0 & 0 \\ d & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} \quad (27)$$

$$= (-1)^{n(n-1)/2} 2^{n-2} d^{n-1} (2a + (n-1)d) \quad (28)$$

Thus, the theorem is proved.

References

[1] A. Murthy, Smarandache determinant sequences, Smarandache Notions Journal, **12**(2001), 275-278.

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On a Subvariety of $\overset{+}{\mathbf{S}}\ell$

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Abstract For any band variety $\mathbf{V}$, $\dot{\mathbf{V}} \cap \overset{+}{\mathbf{S}}\ell$ denotes the variety consisting of the semirings whose additive reduct is a semilattice and whose multiplicative reduct belongs to $\mathbf{V}$. We describe a subvariety $\mathbf{D} \bigvee (\dot{\mathbf{R}}\mathbf{eB} \cap \overset{+}{\mathbf{S}}\ell)$ of $\overset{+}{\mathbf{S}}\ell$ and prove that a semiring S belongs to $\mathbf{D} \bigvee (\dot{\mathbf{R}}\mathbf{eB} \cap \overset{+}{\mathbf{S}}\ell)$ if and only if S is a subdirect product of distributive lattice and a member in $\dot{\mathbf{R}}\mathbf{eB} \cap \overset{+}{\mathbf{S}}\ell$.

Keywords Idempotent semirings; Subdirect product; Variety; Congruence.

§1. Introduction

A semigroup $(S, \cdot)$ is said to be a *band* , it means that S satisfies the identity $x^2 \approx x$, we denote the class of all bands by $\mathbf{I}$. Suppose S is a band. If S satisfy the identity $xyx \approx x$, then S is called a rectangular band,we denote by $\mathbf{ReB}$ the class of all rectangular bands. If S satisfy the identity $axyz \approx ayxz$,then S is said to be *normal band*. We denote the class of all normal bands by $\mathbf{NB}$. Especially,we denote the class of all bands S in which S satisfies the identity $xy \approx yx$, i.e.,S is a semilattice by $\mathbf{S}\ell$. We can easily obtain from well-known Birkoff theorem that above classes of bands are subvarieties of band variety $\mathbf{I}$.

A *semiring* is an algebra $(S, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that both the reducts $(S, +)$ and $(S, \cdot)$ are semigroups and such that the distributive laws $x(y + z) \approx xy + xz$ and $(x + y)z \approx xz + yz$ hold. The semiring is said to be an *idempotent semiring* if the two reducts are *bands*, that is, semirings where every element is an idempotent.

Suppose $\mathbf{V}$ is a subvariety of $\mathbf{I}$,we denote by $\dot{\mathbf{V}}$ the variety of all idempotent semirings S in which the multiplicative reduct $(S, \cdot)$ of S belongs to $\mathbf{V}$. The variety consisting of all the idempotent semirings with commutative addition will be denoted by $\overset{+}{\mathbf{S}}\ell$. The additive reducts of the members of $\overset{+}{\mathbf{S}}\ell$ are semilattices. $\dot{\mathbf{V}} \cap \overset{+}{\mathbf{S}}\ell$ denotes the variety consisting of the semirings whose additive reduct is a semilattice and whose multiplicative reduct belongs to $\mathbf{V}$. In particularly, if $\mathbf{V}$ is the subvariety $\mathbf{S}\ell$, then $\dot{\mathbf{S}}\ell \bigcap \overset{+}{\mathbf{S}}\ell$ denotes the variety of all idempotent semirings in which both the additive reduct and multiplicative reduct are semilattice. We note that $\mathbf{Bi} = \dot{\mathbf{S}}\ell \bigcap \overset{+}{\mathbf{S}}\ell$,we call the member of $\mathbf{Bi}$ bi-semilattice. Let S be a bi-semilattice, if S

satisfies the additional identity $x + xy \approx x$ and $x + yz \approx (x + y)(x + z)$, then S is said to be a distributive lattice. We denote by $\mathbf{D}$ the class of all distributive lattices. Clearly, $\mathbf{D}$ is a subvariety of $\mathbf{Bi}$.

In [3], Professor F.Pastijn and the second author have described the lattice $L(\overset{+}{\mathbf{S}}\ell)$ of subvarieties of $\overset{+}{\mathbf{S}}\ell$. In this paper, we mainly describe a certain subvariety of $\overset{+}{\mathbf{S}}\ell$ in view of semiring congruences and get some interesting results. we prove that if a semiring S belongs to $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ if and only if S a subdirect product of a distributive lattice and a member in $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$.

§2. Characterization of $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$

In this section,we mainly study the subvariety $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ and get some interesting results.

We denote by $\mathbf{Con}(S)$ the set of all congruences on semiring S. Let $\mathbf{A}$ is a class of semirings and $\rho \in \mathbf{Con}(S)$,if $S/\rho \in \mathbf{A}$,then ρ is called an $\mathbf{A}$-congruence. If there exist another congruences θ such that $S/\theta \in \mathbf{A}$ and $\rho \subseteq \theta$,then ρ is called the least $\mathbf{A}$-congruence on S. So we have the following lemma

Lemma 2.1. Let semiring S belong to $\dot{\mathbf{N}}\mathbf{B}\cap\overset{+}{\mathbf{S}}\ell$, define the relations μ on S by

$$x\mu y \Longleftrightarrow (\exists a \in S)\ xay = yax,$$

then μ is the least $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$ congruence on S.

Proof. By lemma IV.5.5 in [2],we know that μ is the congruence on multiplicative reduct $(S, \cdot)$ of semiring S, here we need show μ is the congruence on the additive reduct $(S, +)$ of S. Assume that a, $b \in S$ and $a\mu b$. By the definition we have that $\exists x \in S$ such that $axb = bxa$, for any $c \in S$,we have

$$\begin{aligned}
(a + c)axb(b + c) &= (axb + caxb)(b + c)\\
&= axb + axbc + caxb + caxbc\\
&= bxa + bxac + cbxa + cbxac\\
&= b(bxa)(a + c) + c(bxa)(a + c)\\
&= (b + c)bxa(a + c)\\
&= (b + c)axb(a + c).
\end{aligned}$$

Thus $(a+c)\mu(b+c)$, further, $(S, +)$ is a semilattice, hence μ is a congruence on $(S, +)$, thus μ is a semiring congruence. Since $xx(xyx) = (xyx)xx$, it follows that $x\mu xyx$, that is,μ is $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$-congruence on S. If θ is any $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$-congruence and $x\mu y$, then there exists $a \in S$ such that $xay = yax$, thus $xay\theta yax$, which implies $x\theta y$. Consequently, $\mu \subseteq \theta$. Therefore, we obtain that μ is the least $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$-congruence on S. □

From [3],we know that $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ is the proper subvariety of $\mathbf{N}\cap\overset{+}{\mathbf{S}}\ell$ which is determined by the additional identity

$$x + xyx \approx x \tag{1}$$

So $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ also satisfy the additional identity $x + xyx \approx x$. It obvious that both $\mathbf{D}$ and $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$ satisfy the additional identity $xy + yzx \approx yx + xzy$,therefore, the subvariety $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ of $\overset{+}{\mathbf{S}}\ell$ satisfy the additional identity

$$xy + yzx \approx yx + xzy. \tag{2}$$

Green's relation play an very important role in studying the theory of semigroup, some authors have studied semirngs from the Green's relation of additive and multiplicative reducts and have gotten many beautiful results. In this paper, we denote $\dot{\mathcal{D}}$ the Green-$\mathcal{D}$ relation of multiplicative reduct of a semiring. We have the main result of this section.

Theorem 2.2. *S is a semiring, then $S \in \mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell) \Longleftrightarrow$ S is a subdirect product of a distributive lattice and a member in $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$.*

Proof. $\Longleftarrow$. It is trivial.

$\Longrightarrow$. It is obvious that $\mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$ is a subvariety of $\dot{\mathbf{N}}\mathbf{B}\cap\overset{+}{\mathbf{S}}\ell$. From lemma 3.2 in [4] and lemma 2.1, we have that both $\dot{\mathcal{D}}$ and $\dot{\mathcal{D}}\cap\mu$ are semiring congruences on S. Assume a, $b \in S$ and $a(\dot{\mathcal{D}}\cap\mu)b$. By the definition of μ, there exists c in S such that $acb = bca$, therefore $cacb = cbca$. From $\dot{\mathcal{D}}$ is a congruence we have $ac\dot{\mathcal{D}}bc$, thus $ac = bc$. Similarly, we have $ca = cb$. Since $S \in \mathbf{D}\bigvee(\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell)$, from above we know that S satisfies the additional identity (2). We have

$$ab + bca = ba + acb. \tag{3}$$

By multiplying b on the right to the (3), we obtain $ab + bcab = b + acb = b + bcb$. From $(S, \cdot)$ is a normal band we have $ab + bcab = ab + b \cdot c \cdot ab \cdot b = ab + bcb$, thus $ab + bcb = b + bcb$.

By multiplying a on the left to the (3),we have

$$\begin{aligned} a + acb &= ab + abca \\ &= ab + a \cdot ab \cdot c \cdot a \\ &= ab + aca \quad ((S, \cdot) \textit{ is a normal band and } a\dot{\mathcal{D}}b) \\ &= ab + bcb \quad (ac = bc,\ ca = cb), \end{aligned}$$

thus $a + acb = ab + bcb = ab + bcab = b + bcb$, i.e.,$a + aca = b + bcb$. By (2), we obtain $a = b$, therefore, $(\dot{\mathcal{D}}\cap\mu)$ is a equality relation on S. By lemma 3.2 in [4] and (1), quotient semiring $(S/\dot{\mathcal{D}}, +, \cdot)$ is a distributive lattice. By lemma 2.1 we know that quotient semiring $(S/\mu, +, \cdot)$ is a member in $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$, thus S is a subdirect product of a distributive lattice and a member in $\dot{\mathbf{R}}\mathbf{eB}\cap\overset{+}{\mathbf{S}}\ell$. □

References

[1] J. M. Howie, Fundamentals of Semigroup Theory, Oxford Science Publications, Oxford, 1995.

[2] M. Ptrich, and N.R.Reilly, Completely Regular Semigroups, Wiley, New York, 1999.

[3] F. Pastijn and X.Z.Zhao, Varieties of idempotent semirings with a commutative addition, Algebra Universal, to appear.

[4] X. Z. Zhao, Y. Q. Guo, and K. P. Shum, $\mathcal{D}$-subvariety of idempotent semirings, Algebra Colloquium, **9** (2002), 15-28.

[5] F. Pastijn, Idempotent distributive semirings II, Semigroup Forum, **26** (1983), 153-166.

[6] F. Pastijn, and Y. Q. Guo, The lattice of idempotent distributive semiring varieties, Science in China (Ser. A), **48**(1999), No. 8, 785-804.

[7] F. Pastijn, and A. Romanowska, Idempotent distributive semirings, Acta Sci. Math. (Szeged), **44**(1982), 239-253.

[8] F. Pastijn, and X. Z. Zhao, Green's $\mathcal{D}$-relation for the multiplicative reduct of an idempotent semiring, Arch. Math. (Brno), **36**(2000), 77-93.

[9] M. Petrich, Lectures in Semigroups, Wiley, London, 1977.

[10] T. Saitô, Ordered idempotent semigroups, J. Math. Soc. Japan, **14**(1962), No. 2, 150-169.

[11] X. Z. Zhao, Idempotent semirings with a commutative additive reduct, Semigroup Forum, **64**(2002), 289-296.

[12] X. Z. Zhao, K. P. Shum, and Y. Q. Guo, $\mathcal{L}$-subvariety of idempotent semirings, Algebra Universal, **46**(2001), 75-96.

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A note on q-nanlogue of Sándor's functions

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Abstract The additive analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals have been recently studied by J. Sándor. In this note, we obtain q-analogues of Sándor's theorems [6].

Keywords q-gamma function; Pseudo-Smarandache function; Smarandache-simple function; Asymtotic formula.

Dedicated to Sun-Yi Park on 90th birthday

§1. Introduction

The additive analogues of Smarandache functions S and S_* have been introduced by Sándor [5] as follows:

$$S(x) = \min\{m \in N : x \leq m!\}, \qquad x \in (1, \infty),$$

and

$$S_*(x) = \max\{m \in N : m! \leq x\}, \qquad x \in [1, \infty),$$

He has studied many important properties of S_* relating to continuity, differentiability and Riemann integrability and also p roved the following theorems:

Theorem 1.1.

$$S_* \sim \frac{\log x}{\log \log x} \qquad (x \to \infty).$$

Theorem 1.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^{\alpha}},$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

In [1], Adiga and Kim have obtained generalizations of Theorems 1.1 and 1.2 by the use of Euler's gamma function. Recently Adiga-Kim-Somashekara-Fathima [2] have established a q-analogues of these results on employing analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals as follows:

$$Z(x) = \min\left\{m \in N : x \leq \frac{m(m+1)}{2}\right\}, \qquad x \in (0, \infty),$$

$$Z_*(x) = \max\left\{m \in N : \frac{m(m+1)}{2} \leq x\right\}, \qquad x \in [1, \infty),$$

$$P(x) = \min\{m \in N : p^x \leq m!\}, \quad p > 1, \quad x \in (0, \infty),$$

$$P_*(x) = \max\{m \in N : m! \leq p^x\}, \quad p > 1, \quad x \in [1, \infty).$$

He has also proved the following theorems:

Theorem 1.3.

$$Z_* \sim \frac{1}{2}\sqrt{8x+1} \qquad (x \to \infty).$$

Theorem 1.4. The series

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^{\alpha}},$$

is convergent for $\alpha > 2$ and divergent for $\alpha \leq 2$. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^{\alpha}},$$

is convergent for all $\alpha > 0$.

Theorem 1.5.

$$\log P_*(x) \sim \log x \quad (x \to \infty),$$

Theorem 1.6. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\log\log n}{\log P_*(n)}\right)^{\alpha}$$

is convergent for all $\alpha > 1$ and divergent for $\alpha \leq 1$.

The main purpose of this note is to obtain q-analogues of Sándor's Theorems 1.3 and 1.5. In what follows, we make u se of the following notations and definitions. F. H. Jackson defined a q-analogues of the gamma function which extends the q-factorial

$$(n!)_q = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}), \quad \text{cf} \quad [3],$$

which becomes the ordinary factorial as $q \to 1$. He defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}}(1-q)^{1-x}, \quad 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(1-q)^{1-x}q^{\binom{x}{2}}, q > 1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary gamma function.

§2. Main Theorems

We now defined the q-analogues of Z and Z_* as follows:

$$Z_q(x) = \min\left\{\frac{1-q^m}{1-q} : x \le \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right\}, \quad m \in N, \quad x \in (0,\infty),$$

and

$$Z_q^*(x) = \max\left\{\frac{1-q^m}{1-q} : \frac{\Gamma_q(m+2)}{2\Gamma_q(m)} \le x\right\}, \quad m \in N, \quad x \in \left[\frac{\Gamma_q(m+2)}{2\Gamma_q(1)}, \infty\right),$$

where $0 < q < 1$. Clearly, $Z_q(x) \to Z(x)$ and $Z_q^*(x) \to Z_*(x)$ as $q \to 1^-$. From the definitions of Z_q and Z_q^*, it is clear that

$$Z_q(x) = \left\{\begin{array}{ll} 1, & \text{if } x \in \left(0, \frac{\Gamma_q(3)}{2\Gamma_q(1)}\right] \\ \frac{1-q^m}{1-q}, & \text{if} x \in \left(\frac{\Gamma_q(m+1)}{2\Gamma_q(m-1)}, \frac{\Gamma_q(m+2)}{2\Gamma_q(m)}\right], m \ge 2, \end{array}\right] \tag{1}$$

and

$$Z_q^* = \frac{1-q^m}{1-q} \quad \text{if} \quad x \in \left[\frac{\Gamma_q(m+2)}{2\Gamma_q(m)}, \frac{\Gamma_q(m+3)}{2\Gamma_q(m+1)}\right). \tag{2}$$

Since

$$\frac{1-q^{m-1}}{1-q} \le \frac{1-q^m}{1-q} = \frac{1-q^{m-1}}{1-q} + q^{m-1} \le \frac{1-q^{m-1}}{1-q} + 1,$$

(1) and (2) imply that for $x > \frac{\Gamma_q(3)}{2\Gamma_q(1)}$,

$$Z_q^* \le Z_q \le Z_q^* + 1.$$

Hence it suffices to study the function Z_q^*. We now prove our main theorems.

Theorem 2.1. If $0 < q < 1$, then

$$\frac{\sqrt{1+8xq}-(1+2q)}{2q^2} < Z_q^* \le \frac{\sqrt{1+8xq}-1}{2q}, \quad x \ge \frac{\Gamma_q(3)}{2\Gamma_q(1)}.$$

Proof. If

$$\frac{\Gamma_q(k+2)}{2\Gamma_q(k)} \le x < \frac{\Gamma_q(k+3)}{2\Gamma_q(k+1)}, \tag{3}$$

then

$$Z_q^* = \frac{1-q^k}{1-q}$$

and

$$(1-q^k)(1-q^{k+1}) - 2x(1-q)^2 \leq 0 < (1-q^{k+1})(1-q^{k+2}) - 2x(1-q)^2. \tag{4}$$

Consider the functions f and g defined by

$$f(y) = (1-y)(1-yq) - 2x(1-q)^2$$

and

$$g(y) = (1-yq)(1-yq^2) - 2x(1-q)^2.$$

Note that f is monotonically decreasing for $y \leq \frac{1+q}{2q}$ and g is strictly decreasing for $y \leq \frac{1+q}{2q^2}$. Also $f(y_1) = 0 = g(y_2)$ where

$$y_1 = \frac{(1+q) - (1-q)\sqrt{1+8xq}}{2q},$$

$$y_2 = \frac{(q+q^2) - q(1-q)\sqrt{1+8xq}}{2q^3}.$$

Since $y_1 \leq \frac{1+q}{2q}$, $y_2 \leq \frac{1+q}{2q^2}$ and $q^k < \frac{1+q}{2q} < \frac{1+q}{2q^2}$, from (4), it follows that

$$f(q^k) \leq f(y_1) = 0 = g(y_2) < g(q^k).$$

Thus $y_1 < q^k < y_2$ and hence

$$\frac{1-y_2}{1-q} < \frac{1-q^k}{1-q} < \frac{1-y_1}{1-q}.$$

i.e.

$$\frac{\sqrt{1+8xq} - (1+2q)}{2q^2} < Z_q^* \leq \frac{\sqrt{1+8xq} - 1}{2q}.$$

This completes the proof.

Remark. Letting $q - 1^-$ in the above theorem, we obtain Sándor's Theorem 1.3.

We define the q-analogues of P and P_* as follows:

$$P_q(x) = \min\{m \in N : p^x \leq \Gamma_q(m+1)\}, \quad p > 1, \quad x \in (0, \infty),$$

and

$$P_q^*(x) = \max\{m \in N : \Gamma_q(m+1) \leq p^x\}, \quad p > 1, \quad x \in [1, \infty),$$

where $0 < q < 1$. Clearly, $P_q(x) \to P(x)$ and $P_q^* \to P_*(x)$ as $q \to 1^-$. From the definitions of P_q and P_q^*, we have

$$P_q^*(x) \leq P_q(x) \leq P_q^*(x) + 1.$$

Hence it is enough to study the function P_q^*.

Theorem 2.2. If $0 < q < 1$, then

$$P_*(x) \sim \frac{x \log p}{\log\left(\frac{1}{1-q}\right)} \qquad (x \to \infty).$$

Proof. If $\Gamma_q(n+1) \leq p^x < \Gamma_q(n+2)$, then

$$P_q^*(x) = n$$

and

$$\log \Gamma_q(n+1) \leq \log p^x < \log \Gamma_q(n+2). \tag{5}$$

But by the q-analogue of Stirling's formula established by Moak [4], we have

$$\log \Gamma_q(n+1) \sim \left(n + \frac{1}{2}\right) \log \left(\frac{q^{n+1}}{q-1}\right) \sim n \log \left(\frac{1}{1-q}\right). \tag{6}$$

Dividing (5) throughout by $n \log \left(\frac{1}{1-q}\right)$, we obtain

$$\frac{\log \Gamma_q(n+1)}{n \log \left(\frac{1}{1-q}\right)} \leq \frac{x \log p}{P_q^*(x) \log \left(\frac{1}{1-q}\right)} < \frac{\log \Gamma_q(n+2)}{n \log \left(\frac{1}{1-q}\right)}. \tag{7}$$

Using (6) and (7), we deduce

$$\lim_{x \to \infty} \frac{x \log p}{P_q^*(x) \log \left(\frac{1}{1-q}\right)} = 1.$$

This completes the proof.

References

[1] C. Adiga and T. Kim, On a generalization of Sándor's function, Proc. Jangjeon Math. Soc., **5**(2002), 121-124.

[2] C. Adiga, T. Kim, D. D. Somashekara and N. Fathima, On a q-analogue of Sándor's function, J. Inequal. Pure and Appl. Math., **4**(2003), 1-5.

[3] T. Kim, Non-archimedean q-integrals with multiple Changhee q-Bernoulli polynomials, Russian J. Math. Phys., **10**(2003), 91-98.

[4] D. S. Moak, The q-analogue of Stirling's formula, Rocky Mountain J. Math., **14**(1984), 403-413.

[5] J. Sándor, On an additive analogue of the function S, Note Numb. Th. Discr. Math., **7**(2001), 91-95.

[6] J. Sándor, On an additive analogue of certain arithmetic function, J. Smarandache Notions, **14**(2004), 128-133.

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On the mean value of the F.Smarandache simple divisor function

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Abstract In this paper, we introduce a new arithmetic function $\tau_{sp}(n)$ which we called the simple divisor function. The main purpose of this paper is to study the asymptotic properties of the mean value of $\tau_{sp}(n)$ by using the elementary methods, and obtain an interesting asymptotic formula for it.

Keywords Smarandache simple number divisor; Simple divisor function; Asymptotic formula.

§1. Introduction

A positive integer n is called simple number if the product of its all proper divisors is less than or equal to n. In problem 23 of [1], Professor F.Smarandache asked us to study the properties of the sequence of the simple numbers. About this problem, many scholars have studied it before. For example, in [2], Liu Hongyan and Zhang Wenpeng studied the mean value properties of $1/n$ and $1/\phi(n)$ (where n is a simple number), and obtained two asymptotic formulae for them. For convenient, let $\mathcal{A}$ denotes the set of all simple numbers, they proved that

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}}\frac{1}{n}=(\ln\ln x)^2+B_1\ln\ln x+B_2+O\left(\frac{\ln\ln x}{\ln x}\right)$$

and

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}}\frac{1}{\phi(n)}=(\ln\ln x)^2+C_1\ln\ln x+C_2+O\left(\frac{\ln\ln x}{\ln x}\right),$$

where B_1, B_2, C_1, C_2 are constants, and $\phi(n)$ is the Euler function.

For $n>1$, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. If one of the divisor d of n satisfing $\tau(d)\leq 4$ (where $\tau(n)$ denotes the numbers of all divisors of n), then we call d as a simple number divisor. In this paper, we introduce a new arithmetic function

$$\tau_{sp}(n)=\sum_{\substack{d|n\\ \tau(d)\leq 4}}1,$$

which we called the simple divisor function. The main purpose of this paper is to study the asymptotic property of the mean value of $\tau_{sp}(n)$ by using the elementary methods, and obtain an interesting asymptotic formula for it. That is, we will prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x}\tau_{sp}(n) = \frac{1}{2}x(\log\log x)^2 + \frac{1}{2}(a+1)x\log\log x + \left(\frac{b+A}{2} + B + C\right)x + O\left(\frac{x\log\log x}{\log x}\right),$$

where a and b are two computable constants, $A = \gamma + \sum_p(\log(1-1/p)+1/p)$, γ is the Euler constant, $B = \sum_p \frac{1}{p^2}$ and $C = \sum_p \frac{1}{p^3}$.

§2. Two Lemmas

Before the proof of Theorem, two useful Lemmas will be introduced which we will use subsequently.

Lemma 1. For any real number $x \geq 1$, we have the asymptotic formula

$$(a) \qquad \sum_{n\leq x}\omega(n) = x\log\log x + Ax + O\left(\frac{x}{\log x}\right),$$

$$(b) \qquad \sum_{n\leq x}\omega^2(n) = x(\log\log x)^2 + ax\log\log x + bx + O\left(\frac{x\log\log x}{\log x}\right),$$

where $A = \gamma + \sum_p(\log(1-1/p)+1/p)$, γ is the Euler constant, a and b are two computable constants.

Proof. See references [3] and [4].

Lemma 2. For any positive integer $n \geq 1$, we have

$$\tau_{sp}(n) = \frac{1}{2}\omega^2(n) + \frac{1}{2}\omega(n) + \sum_{p^2|n}1 + \sum_{p^3|n}1,$$

where $\omega(n)$ denotes the number of all different prime divisors of n, $\sum_{p^2|n}1$ denotes the number of all primes such that $p^2 \mid n$, $\sum_{p^3|n}1$ denotes the number of all primes such that $p^3 \mid n$.

Proof. Let $n > 1$, we can write $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then from the definition of $\tau_{sp}(n)$ we know that there are only four kinds of divisors d such that the number of the divisors of d less or equal to 4. That is $p \mid n$, $p_ip_j \mid n$, $p^2 \mid n$ and $p^3 \mid n$, where $p_i \neq p_j$.

Hence, we have

$$\begin{aligned}\tau_{sp}(n) &= \sum_{p|n}1 + \sum_{\substack{p_ip_j|n\\ p_i\neq p_j}}1 + \sum_{p^2|n}1 + \sum_{p^3|n}1\\ &= \omega(n) + \frac{1}{2}\omega(n)(\omega(n)-1) + \sum_{p^2|n}1 + \sum_{p^3|n}1\\ &= \frac{1}{2}\omega^2(n) + \frac{1}{2}\omega(n) + \sum_{p^2|n}1 + \sum_{p^3|n}1\end{aligned}$$

This proves Lemma 2.

§3. Proof of the theorem

Now we completes the proof of Theorem. From the definition of the simple divisor function, Lemma 1 and Lemma 2, we can write

$$\begin{aligned}
\sum_{n\le x}\tau_{sp}(n) &= \sum_{n\le x}\left(\frac{1}{2}\omega^2(n)+\frac{1}{2}\omega(n)+\sum_{p^2|n}1+\sum_{p^3|n}1\right)\\
&= \frac{1}{2}\sum_{n\le x}\omega^2(n)+\frac{1}{2}\sum_{n\le x}\omega(n)+\sum_{n\le x}\sum_{p^2|n}1+\sum_{n\le x}\sum_{p^3|n}1\\
&= \frac{1}{2}\left(x(\log\log x)^2+ax\log\log x+bx+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ \frac{1}{2}\left(x\log\log x+Ax+O\left(\frac{x}{\log x}\right)\right)+\sum_{p\le x}\left[\frac{x}{p^2}\right]+\sum_{p\le x}\left[\frac{x}{p^3}\right]\\
&= \frac{1}{2}\left(x(\log\log x)^2+(a+1)x\log\log x+(b+A)x+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ x\sum_{p\le x}\frac{1}{p^2}+O\left(\frac{x}{\log x}\right)+x\sum_{p\le x}\frac{1}{p^3}+O\left(\frac{x}{\log x}\right)\\
&= \frac{1}{2}\left(x(\log\log x)^2+(a+1)x\log\log x+(b+A)x+O\left(\frac{x\log\log x}{\log x}\right)\right)\\
&+ (B+C)x+O\left(\frac{x}{\log x}\right)\\
&= \frac{1}{2}x(\log\log x)^2+\frac{1}{2}(a+1)x\log\log x+\left(\frac{b+A}{2}+B+C\right)x+O\left(\frac{x\log\log x}{\log x}\right).
\end{aligned}$$

where $B=\sum_p\frac{1}{p^2}$ and $C=\sum_p\frac{1}{p^3}$.

This completes the proof of Theorem.

References

[1] F. Smarandache, Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993, pp. 23.

[2] Liu Hongyan and Zhang Wenpeng, On the simple numbers and the mean value properties, Smarandache Notions Journal, **14**(2004), pp. 171.

[3] G.H.Hardy and S.Ramanujan, The normal number of prime factors of a number n, Quart. J. Math., **48**(1917), pp. 76-92.

[4] H.N.Shapiro, Introduction to the theory of numbers, John Wiley and Sons, 1983, pp. 347.

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A discussion on a number theoretic function

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Abstract Given a positive integer n, we define the function $\delta_k(n)$ as follow:

$$\delta_k(n) = \max\{d : \ d|n, (d,k) = 1\},$$

$a_m(n)$ denotes the m-th power free part of n (if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}\cdots p_q^{\alpha_q}$, where $\alpha_i < m$ $(i = 1, 2\cdots, s), \alpha_j \geq m$ $(j = s+1, \cdots, q)$, then $a_m(n) = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$). In this paper, we study the number of the solutions of the equations $\delta_k(n) = a_m(n)$, and use the analytic method to obtain several interesting asymptotic formulas for it.

Keywords Mean value; Asymptotic formula; Perron formula; Riemann zeta function.

§1. Introduction and results

For any positive integer n, we define the function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : \ d|n, (d,k) = 1\},$$

$a_m(n)$ denotes the m-th power free part of n (if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}\cdots p_q^{\alpha_q}$, where $\alpha_i < m$ $(i = 1, 2\cdots, s), \alpha_j \geq m$ $(j = s+1, \cdots, q)$, then $a_m(n) = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$). Let A denotes the set of all solutions of the equation $\delta_k(n) = a_m(n)$. In this paper, we study the asymptotic properties of the set A, and use the analytic method to obtain several interesting asymptotic formulas for it. That is, we shall prove the following conclusions:

Theorem 1. For any complex number s with $Re(s) > 1$, we have the identity:

$$\sum_{n\in A}\frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ms)}\prod_{p|k}\frac{p^{ms} - p^{(m-1)s} + 1}{p^{ms} - 1},$$

where $\prod_{p|k}$ denotes the product over all different prime divisors of k, $\zeta(s)$ is the Riemann zeta function.

Theorem 2. For any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \frac{1}{\zeta(m)}x\prod_{p|k}\frac{p^m - p^{m-1} + 1}{p^m - 1} + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where ε is any fixed positive number, $m \geq 2$ is a positive integer.

Corollary. Let B denote the set of all solutions of the equation $\delta_k(n) = a_2(n)$. Then we have the identity:

$$\sum_{n\in B}\frac{1}{n^2} = \frac{15}{\pi^2}\prod_{p|k}\frac{p^6+1}{(p^4-1)(p^2+1)}.$$

§2. Proof of the theorems

Now we complete the proof of the theorems.

Let positive number $n = n_1u = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}\cdots p_q^{\alpha_q}$, where $n_1 = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ does not contain the $m-th$ power part of n, $u = p_{s+1}^{\alpha_{s+1}}\cdots p_q^{\alpha_q}$ is a m-full number. Obviously, $\alpha_i < m(i = 1, 2\cdots s)$ and $\alpha_j \geq m(j = s+1\cdots q)$. From the definitions of functions $\delta_k(n)$ and $a_m(n)$ we know that both of them are multiplicative functions. That is,

$$\delta_k(n) = \delta_k(n_1)\delta_k(u)$$

$$a_m(n) = a_m(n_1)a_m(u) = n_1,$$

According to the discussion above, the solutions of the equation $\delta_k(n) = a_m(n)$ transfer itself to the solutions of equation $\delta_k(n_1)\delta_k(u) = n_1$.

Noting that: If $\alpha_j \geq m$, $p_j|k$ $(j = s+1, \cdots, q)$, we have $\delta_k(u) = 1$. At this time, if $\alpha_i < m$, $p_i\dagger k$, $(i = 1, 2, \cdots, s)$,we get $\delta_k(n_1) = n_1$. This shows the existing of equation $\delta_k(n_1)\delta_k(u) = n_1$. Define arithmetic function $b(n)$ as follows:

$$b(n) = \begin{cases} 1 & \text{if } n \text{ is the solution of equation } \delta_k(n) = a_m(n); \\ & \text{if } n > 1 \end{cases} \tag{1}$$

Now we have

$$\sum_{n\in A}\frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

From the definition of $b(n)$, we know it is a multiplicative function. Then from the Euler product formula(see [1]), we can write:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{b(n)}{n^s} &= \prod_{p}\left(1+\frac{b(p)}{p^s}+\frac{b(p^2)}{p^{2s}}+\cdots\right) \\
&= \prod_{p\dagger k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\prod_{p|k}\left(1+\frac{1}{p^{ms}}+\frac{1}{p^{(m+1)s}}+\cdots\right) \\
&= \prod_{p\dagger k}\frac{1-\frac{1}{p^{ms}}}{1-\frac{1}{p^s}}\prod_{p|k}\left(1+\frac{1}{p^{ms}}\cdot\frac{1}{1-\frac{1}{p^s}}\right) \\
&= \prod_{p}\frac{1-\frac{1}{p^{ms}}}{1-\frac{1}{p^s}}\prod_{p|k}\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{ms}}}\cdot\left(1+\frac{1}{p^{ms}}\cdot\frac{1}{1-\frac{1}{p^s}}\right) \\
&= \frac{\zeta(s)}{\zeta(ms)}\prod_{p|k}\frac{p^{ms}-p^{(m-1)s}+1}{p^{ms}-1}
\end{aligned}$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. From the definition of the function $b(n)$ we can write:

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \sum_{n\leq x} b(n)$$

Let Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

from Theorem 1, we know that

$$f(s) = \frac{\zeta(s)}{\zeta(ms)} \prod_{p|k} \frac{p^{ms} - p^{(m-1)s} + 1}{p^{ms} - 1},$$

and this function has a simple pole point at $s = 1$ with residue

$$\frac{1}{\zeta(m)} \prod_{p|k} \frac{p^m - p^{m-1} + 1}{p^m - 1},$$

because

$$|b(n)| \leq 1, \quad |\sum_{n=1}^{\infty} \frac{b(n)}{n^\sigma}| \leq \zeta(\sigma)$$

where $\sigma > 1$ is the real part of s, so by Perron's formula(see [2]), we have:

$$\begin{aligned}\sum_{n\leq x} \frac{a(n)}{n^{s_0}} &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B\left(b+\sigma_0\right)}{T}\right) \\ &+ O\left(x^{1-\sigma_0} H(2x) \min\left(1, \frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0} H(2x) \min\left(1, \frac{x}{T\|x\|}\right)\right),\end{aligned}$$

where N is the nearest integer to x, and $\|x\| = |x - N|$.

Taking $a(n) = b(n)$, $s_0 = 0$, $b = 2$, $T = x^{\frac{3}{2}}$, $H(x) = 1$, $B(\sigma) = \zeta(\sigma)$ in above, then we have:

$$\sum_{n\leq x} b(n) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\zeta(s)}{\zeta(ms)} R(s) \frac{x^s}{s} ds + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where

$$R(s) = \prod_{p|k} \frac{p^{ms} - p^{(m-1)s} + 1}{p^{ms} - 1},$$

we move the integral line from $s = 2 \pm iT$ to $s = \frac{1}{2} \pm iT$, this time the function

$$\frac{\zeta(s)}{\zeta(ms)} \cdot \frac{x^s}{s} R(s)$$

has a simple pole point at $s = 1$ with residue $\frac{R(1)}{\zeta(m)} x$, so we have

$$\frac{1}{2\pi i} \left(\int_{2-iT}^{2+iT} + \int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{2-iT} \right) \frac{\zeta(s)}{\zeta(ms)} \frac{x^s}{s} R(s) ds = \frac{R(1)}{\zeta(m)} x,$$

Now taking $T = x^{\frac{3}{2}}$, we have the estimate

$$|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\frac{\zeta(s)}{\zeta(ms)}\frac{x^s}{s}R(s)ds| \ll \frac{x^2}{T} = x^{\frac{1}{2}}$$

and

$$|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta(s)}{\zeta(ms)}\frac{x^s}{s}R(s)ds| \ll x^{\frac{1}{2}+\varepsilon}.$$

Note that

$$R(1) = \prod_{p|k}\frac{p^m - p^{m-1} + 1}{p^m - 1}$$

we get

$$\sum_{n\leq x} b(n) = \frac{1}{\zeta(m)}x\prod_{p|k}\frac{p^m - p^{m-1} + 1}{p^m - 1} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

That is

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \frac{1}{\zeta(m)}x\prod_{p|k}\frac{p^m - p^{m-1} + 1}{p^m - 1} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 2.

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[2] Pan Chengdong and Pan Chengbiao, Element of the analytic Number Theory, Science Press, Beijing, 1991, pp. 96.

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On the mean value of the Smarandache ceil function[1]

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Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\}, \qquad \forall n \in N^*.$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it.

Keywords Smarandache ceil function; Mean value; Asymptotic formula.

§1. Introduction

For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\}, \qquad \forall n \in N^*.$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots$. This was introduced by Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function. For example, Ibstedt [2] [3] studied this function both theoretically and computationally, and got the following conclusions:

$$(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b), \qquad a, b \in N^*.$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = S_k\left(p_1^{\alpha_1}\right) \cdots S_k\left(p_r^{\alpha_r}\right).$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{4}+\epsilon}),$$

where A_1 and A_2 are two computable constants, ϵ is any fixed positive integer.

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§2. Proof of the theorem

To complete the proof of the theorem, we need the following Lemma, which is called the Perron's formula (See reference [4]):

Lemma. Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $s = \sigma + it$, convergent absolutely for $\sigma > \sigma_a$, and that there exist a positive increasing function $H(u)$ and a function $B(u)$ such that

$$a(n) \leq H(n), \qquad n = 1, 2, \cdots,$$

and

$$\sum_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \leq B(\sigma), \qquad \sigma > \sigma_a.$$

Then for any $s_0 = \sigma_0 + it_0$, $b_0 > \sigma_a$, $b_0 \geq b > 0$, $b_0 \geq \sigma_0 + b > \sigma_a$, $T \geq 1$ and $x \geq 1$, x not to be an integer, we have

$$\begin{aligned} \sum_{n \leq x} a(n)n^{-s_0} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s_0+s)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\ + \quad & O\left(x^{1-\sigma_0}H(2x)\min\left(1, \frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0}H(N)\min\left(1, \frac{x}{T\,||\,x\,||}\right)\right), \end{aligned}$$

where N is the nearest integer to x, $||\, x \,|| = |N - x|$.

Now we complete the proof of the theorem. Let $s = \sigma + it$ be a complex number and

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{S_2(n)n^s}.$$

Note that $|\frac{1}{S_2(n)}| \leq \frac{1}{\sqrt{n}}$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$, by Euler product formula [5] and the definition of $S_2(n)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{1}{S_2(p)p^s} + \frac{1}{S_2(p^2)p^{2s}} + \frac{1}{S_2(p^3)p^{3s}}\right. \\ &\quad \left. + \frac{1}{S_2(p^4)p^{4s}} + \cdots + \frac{1}{S_2(p^{2k})p^{2ks}} + \frac{1}{S_2(p^{2k+1})p^{(2k+1)s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^{s+1}} + \frac{1}{p^{2s+1}} + \frac{1}{p^{3s+2}} + \frac{1}{p^{4s+2}} + \cdots + \frac{1}{p^{2ks+k}} + \frac{1}{p^{(2k+1)s+k+1}}\right. \\ &\quad \left. + \frac{1}{p^{2(k+1)s+k+1}} + \frac{1}{p^{(2(k+2)+1)s+k+2}} + \cdots\right) \\ &= \prod_p \frac{1}{1-\frac{1}{p^{2s+1}}}\left(1 + \frac{1}{p^{s+1}}\right) \\ &= \frac{\zeta(2s+1)\zeta(s+1)}{\zeta(2s+2)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_p$ denotes the product over all primes.

Taking

$$H(x) = 1; \qquad B(\sigma) = \frac{2}{2\sigma - 1}, \qquad \sigma > \frac{1}{2};$$

$s_0 = 0$; $b = 1$; $T = x^{\frac{5}{4}}$ in the above Lemma we may get

$$\sum_{n \le x} \frac{1}{S_2(n)} = \frac{1}{2i\pi} \int_{1-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} f(s) \frac{x^s}{s} ds + O(x^{-\frac{1}{4}+\varepsilon}).$$

To estimate the main term, we move the integral line in the above formula from $s = 1 \pm ix^{\frac{5}{4}}$ to $s = -\frac{1}{4} \pm ix^{\frac{5}{4}}$. This time, the function $f(s)\frac{x^s}{s}$ have a third order pole point at $s = 0$ with residue

$$\frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2,$$

where A_1 and A_2 are two computable constants.

Hence, we have

$$\begin{aligned} & \frac{1}{2\pi i} \left(\int_{1-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} + \int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}+ix^{\frac{5}{4}}}^{-\frac{1}{4}-ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1-ix^{\frac{5}{4}}} \right) \frac{\zeta(2s+1)\zeta(s+1)x^s}{\zeta(2s+2)s} ds \\ = & \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2. \end{aligned}$$

We can easily get the estimate

$$\left| \frac{1}{2\pi i} \left(\int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}+i\frac{5}{4}}^{-\frac{1}{4}-i\frac{5}{4}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1-ix^{\frac{5}{4}}} \right) \frac{\zeta(2s+1)\zeta(s+1)x^s}{\zeta(2s+2)s} ds \right| \ll x^{-\frac{1}{4}+\epsilon}.$$

From above we may immediately get the asymptotic formula:

$$\sum_{n \le x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{4}+\epsilon}).$$

This completes the proof of the theorem.

References

[1] F. Smarandache, Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993, pp. 42.

[2] Ibstedt, Surfing on the ocean of numbers-a few Smarandache notions and similar topics, Erhus University press, New Mexico, 1997.

[3] Ibstedt, Computational Aspects of Number Sequences, American Research Press, Lupton, 1999.

[4] Pan Chengdong and Pan Chengbiao, Goldbach conjecture, Science Press, Beijing, 1992, pp. 145.

[5] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

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Some identities involving the Smarandache ceil function

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Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\} \, (\forall n \in N^*) \, .$$

In this paper, we use the elementary methods to study the arithmetical properties of $S_k(n)$, and give some identities involving the Smarandache ceil function.

Keywords Smarandache ceil function; Arithmetical properties; Identity.

§1. Introduction

For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\} \, (\forall n \in N^*) \, .$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots\cdots$. $S_3(1) = 1$, $S_3(2) = 2$, $S_3(3) = 3$, $S_3(4) = 2$, $S_3(5) = 5$, $S_3(6) = 6$, $S_3(7) = 7$, $S_3(8) = 2$, $\cdots\cdots$.

The dual function of $S_k(n)$ is defined as

$$\overline{S_k}(n) = \max\{x \in N : x^k \mid n\} \, (\forall n \in N^*) \, .$$

For example, $\overline{S_2}(1) = 1$, $\overline{S_2}(2) = 1$, $\overline{S_2}(3) = 1$, $\overline{S_2}(4) = 2$, $\cdots$. For any primes p and q with $p \neq q$, $\overline{S_2}(p^2) = p$, $\overline{S_2}(p^{2m+1}) = p^m$ and $\overline{S_2}(p^m q^n) = \overline{S_2}(p^m)\overline{S_2}(q^n)$.

These functions were introduced by F.Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function and its dual. For example, Ibstedt [2] and [3] studied these functions both theoretically and computationally, and got the following conclusions:

$$(\forall a, b \in N^*)(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b),$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} . \cdots . p_r^{\alpha_r}) = S_k(p_1^{\alpha_1}) . \cdots . S_k(p_r^{\alpha_r}).$$

Ding Liping [4] studied the mean value properties of the Smarandache ceil function, and obtained a sharp asymptotic formula for it. That is, she proved the following conclusion:

Let real number $x \geq 2$, then for any fixed positive integer $k \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} S_k(n) = \frac{x^2}{2}\zeta(2k-1)\prod_p \left[1 - \frac{1}{p(p+1)}\left(1 + \frac{1}{p^{2k-3}}\right)\right] + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all prime p, and ϵ denotes any fixed positive number.

Lu Yaming [7] studied the hybrid mean value involving $\overline{S_k}(n)$ and $d(n)$, and obtained the following asymptotic formula:

$$\sum_{n \leq x} d(\overline{S_k}(n)) = \zeta(k)x + \zeta\left(\frac{1}{k}\right) + O\left(x^{\frac{1}{k+1}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $d(n)$ is the Dirichlet divisor function.

In this paper, we use the elementary methods to study the arithmetical properties of Smarandache ceil function and its dual, and give some interesting identities involving these functions. That is, we shall prove the following:

Theorem 1. For any real number $\alpha > 1$ and integer $k \geq 2$, we have the identity:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_k^{\alpha}(n)} = \frac{2^{\alpha} - k - 1}{2^{\alpha} + k - 1}\prod_p \left(1 + \frac{k}{p^{\alpha} - 1}\right),$$

where $\prod_p$ denotes the product over all prime p.

Theorem 2. For any real number $\alpha > 1$ and integer $k \geq 2$, we also have the identities:

$$\sum_{n=1}^{\infty} \frac{\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha - 1)}{\zeta(k\alpha)}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha - 1)}{\zeta(k\alpha)}\left[\frac{(2^{\alpha} - 1)(2^{k\alpha - 1} - 1)}{2^{\alpha - 2}(2^{k\alpha} - 1)} - 1\right],$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $k = 2$, $\alpha = 2$ and 4, from our Theorems we may immediately deduce the following:

Corollary 1. Let $S_k(n)$ denotes the Smarandache ceil function, then we have the identities:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^2(n)} = \frac{1}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^4(n)} = \frac{91}{102}.$$

Corollary 2. Let $\overline{S_k}(n)$ denotes the dual function of the Smarandache ceil function, then we have the identities:

$$\sum_{n=1}^{\infty} \frac{\overline{S_2}(n)}{n^2} = \frac{15}{\pi^2} \cdot \zeta(3)$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\overline{S_2}(n)}{n^2}=\frac{6}{\pi^2}\cdot\zeta(3).$$

§2. Proof of the theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1. For any real number α with $\alpha>1$, let

$$f(\alpha)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{S_k^{\alpha}(n)}.$$

Then from the multiplicative property of $S_k(n)$ we may get

$$\begin{aligned} f(\alpha) &= \sum_{n=1}^{\infty}\frac{1}{S_k^{\alpha}(2n-1)}-\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{S_k^{\alpha}(2n-1)S_k^{\alpha}(2^t)} \\ &= \left(\sum_{n=1}^{\infty}\frac{1}{S_k^{\alpha}(2n-1)}\right)\left(1-\sum_{t=1}^{\infty}\frac{1}{S_k^{\alpha}(2^t)}\right). \end{aligned} \tag{1}$$

For any prime p, note that $S_k(p)=p$, $S_k(p^2)=p$, $\cdots$, $S_k(p^k)=p$, $S_k(p^{k+1})=p^2$, $S_k(p^{tk+r})=p^{t+1}$ for any integers $t\geq 0$ and $1\leq r\leq k$. So from the Euler product formula [6] we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty}\frac{1}{S_k^{\alpha}(2n-1)}\right) &= \prod_{\substack{p\\ p\neq 2}}\left(1+\frac{1}{S_k^{\alpha}(p)}+\frac{1}{S_k^{\alpha}(p^2)}+\cdots+\frac{1}{S_k^{\alpha}(p^k)}+\cdots\right) \\ &= \prod_{\substack{p\\ p\neq 2}}\left(1+\frac{k}{p^{\alpha}}+\frac{k}{p^{2\alpha}}+\cdots+\frac{k}{p^{n\alpha}}+\frac{k}{p^{(n+1)\alpha}}+\cdots\right) \\ &= \prod_{\substack{p\\ p\neq 2}}\left(1+\frac{k}{p^{\alpha}-1}\right) \\ &= \frac{2^{\alpha}-1}{2^{\alpha}+k-1}\prod_{p}\left(1+\frac{k}{p^{\alpha}-1}\right). \end{aligned} \tag{2}$$

Similarly, we also have

$$1-\sum_{t=1}^{\infty}\frac{1}{S_k^{\alpha}(2^t)}=1-\frac{k}{2^{\alpha}-1}. \tag{3}$$

Combining (1), (2) and (3) we may immediately get the identity

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{S_k^{\alpha}(n)}=\frac{2^{\alpha}-k-1}{2^{\alpha}+k-1}\prod_{p}\left(1+\frac{k}{p^{\alpha}-1}\right).$$

This proves Theorem 1.

Now we prove Theorem 2. For any real number α with $\alpha > 1$ and integer $k \geq 2$, let

$$g(\alpha) = \sum_{n=1}^{\infty} \frac{\overline{S_k}(n)}{n^{\alpha}}.$$

Then from the multiplicative property of $\overline{S_k}(n)$ and the Euler product formula [6] we have

$$\begin{aligned}
g(\alpha) &= \prod_p \left(1 + \frac{\overline{S_k}(p)}{p^{\alpha}} + \frac{\overline{S_k}(p^2)}{p^{2\alpha}} + \frac{\overline{S_k}(p^3)}{p^{3\alpha}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^{\alpha}} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(k-1)\alpha}} + \frac{p}{p^{k\alpha}} + \cdots + \frac{p}{p^{(2k-1)\alpha}} + \cdots\right) \\
&= \prod_p \left(\frac{1-\frac{1}{p^{k\alpha}}}{1-\frac{1}{p^{\alpha}}} + \frac{p}{p^{k\alpha}}\frac{1-\frac{1}{p^{k\alpha}}}{1-\frac{1}{p^{\alpha}}} + \frac{p^2}{p^{2k\alpha}}\frac{1-\frac{1}{p^{k\alpha}}}{1-\frac{1}{p^{\alpha}}} + \cdots\right) \\
&= \prod_p \left(\frac{1-\frac{1}{p^{k\alpha}}}{1-\frac{1}{p^{\alpha}}}\right) \prod_p \left(1 + \frac{p}{p^{k\alpha}} + \frac{p^2}{p^{2k\alpha}} + \cdots\right) \\
&= \prod_p \left(\frac{1-\frac{1}{p^{k\alpha}}}{1-\frac{1}{p^{\alpha}}}\right) \prod_p \frac{1}{1-\frac{1}{p^{k\alpha-1}}} \\
&= \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)}.
\end{aligned}$$

This proves the first formula of Theorem 2.

Similarly, we can also get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{S_k}(n)}{n^{\alpha}} &= \sum_{n=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}} - \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}} \frac{\overline{S_k}(2^t)}{2^{t\alpha}} \\
&= \left(\sum_{n=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}}\right)\left(1 - \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}\right) \\
&= \left(1 - \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}\right) \prod_{\substack{p \\ p\neq 2}} \left(1 + \frac{\overline{S_k}(p)}{p^{\alpha}} + \frac{\overline{S_k}(p^2)}{p^{2\alpha}} + \frac{\overline{S_k}(p^3)}{p^{3\alpha}} + \cdots\right) \\
&= \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)} \frac{1 - \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}}{1 + \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}} \\
&= \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)} \left[\frac{(2^{\alpha}-1)(2^{k\alpha-1}-1)}{2^{\alpha-2}(2^{k\alpha}-1)} - 1\right].
\end{aligned}$$

This completes the proof of Theorem 2.

Taking $k = 2$, then from Theorem 1 we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^{\alpha}(n)} = \frac{2^{\alpha}-3}{2^{\alpha}+1} \prod_p \left(1 + \frac{2}{p^{\alpha}-1}\right) = \frac{2^{\alpha}-3}{2^{\alpha}+1} \prod_p \frac{p^{\alpha}+1}{p^{\alpha}-1} = \frac{2^{\alpha}-3}{2^{\alpha}+1} \frac{\zeta^2(\alpha)}{\zeta(2\alpha)}, \tag{4}$$

where $\zeta(\alpha)$ is the Riemann zeta-function.

Note that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(8) = \pi^8/9450$, from (4) we may immediately deduce Corollary 1.

Corollary 2 follows from Theorem 2 with $k = 2$.

References

[1] F. Smarandache, Only problems, Not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Ibstedt, Surfing on the Ocean of Numbers-A Few Smarandache Notions and Similar Topics, Erhus University press, New Mexico, 1997.

[3] Ibstedt, Computational Aspects of Number Sequences, American Research Press, Lupton, 1999.

[4] S.Tabirca and T. Tabirca, Some new results concerning the Smarandache ceil function, Smarandache Notions Journal, **13**(2002), 30-36.

[5] Ding Liping, On the mean value of Smarandache ceil function, Scientia Magna, **1**(2005), No. 2, 74-77.

[6] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[7] Lu Yaming, On a dual function of the Smarandache ceil function, Research on Smarandache problems in number theory (edited by Zhang Wenpeng, Li Junzhuang and Liu Duansen), Hexis, Phoenix AZ, 2005, 55-57.

[8] Ji Yongqiang, An equation involving the Smarandache ceil function, Scientia Magna, **1**(2005), No. 2, 149-151.

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The Structure of Principal Filters on Po-semigroups[1]

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Abstract The structure of principal filters on any po-semigroup S is investigated by using the relation $\mathcal{N}$ which is the smallest complete semilattice congruence on S. In particular, we prove that for any po-semigroup S, $\mathcal{N}$ is the equality relation on S if and only if S is a semilattice, and $\mathcal{N}$ is the universal relation on S if and only if S is the only principal filter. We also investigate the complete semilattice congruence classes of S.

Keywords Principal filters; Complete semilattice congruence; Po-semigroups.

A po-semigroup is a semigroup S with a partial ordered " $\leq$ " such that

$$(\forall a, b, c \in S)\ a \leq b \Rightarrow ac \leq bc \text{ and } ca \leq cb.$$

Various kinds of po-semigroups have been widely studied by many authors (see [1-5]). In [8], the authors have proved that every principal filter of any po-semigroup S can be uniquely expressed by the $\mathcal{N}$-classes of S. In this paper, we will consider a structure of principal filter on po-semigroups. By using the relation $\mathcal{N}$ which is the smallest complete semilattice congruence on any po-semigroup S, we will observe that $\mathcal{N}$ on any po-semigroup S is the equality relation if and only if S is a semilattice and $\mathcal{N}$ is the universal relation if and only if S is the only principal filter.

We first recall some basic notions and terminologies from [2] and [7].

Suppose that S is a po-semigroup and T a subsemigroup of S. For a non-empty subset H of T, we use $(H]_T$ and $[H)_T$ to denote the following subsets of S, respectively,

$$(H]_T = \{x \in T \mid (\exists y \in H) x \leq y\},$$

$$[H)_T = \{x \in T \mid (\exists y \in H) x \geq y\}.$$

In particular, $(H]_T$ is denoted by $(H]$ and $[H)_T$ by $[H)$ when $T = S$. A filter F of a po-semigroup S is a subsemigroup of S satisfying the following conditions

(i) $a, b \in S, ab \in F$ implies that $a \in F$ and $b \in F$,

(ii) $[F) \subseteq F$.

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For every $a \in S$ there is a unique smallest filter of S containing a, denoted by $N(a)$, which is called the principal filter generated by a. The relation $\mathcal{N}$ on a po-semigroup S is defined by the rule that for any $x, y \in S$, $x\mathcal{N}y$ if and only if $N(x) = N(y)$. A congruence σ on a po-semigroup S is a semilattice congruence if for any $a, b \in S$, $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$. A semilattice congruence σ on S is called a complete semilattice congruence if for any $a, b \in S, a \leq b$ implies $(a, ab) \in \sigma$.

According to [3], $\mathcal{N}$ on a po-semigroup S is the smallest complete semilattice congruence on S. Of course, $S/\mathcal{N}$ is a semilattice Y. We also denote by $\leq_Y$ the natural partial order on the semilattice $Y(= S/\mathcal{N})$.

Lemma 1. ([8]) Let S be a po-semigroup and $a \in S$. Then N_a is a semiprime ideal of $N(a)$.

Lemma 2. ([8]) Let S be a po-semigroup and a in S, then $N(a) = \bigcup\{N_b : N_b \geq_Y N_a\} = \{b \in S : N_b \geq_Y N_a\}$.

Theorem 3. Let S be a po-semigroup. Then the following are equivalent:

(i) S is a semilattice;

(ii) For every $a \in S$, $N(a) = [a)$;

(iii) $\mathcal{N}$ is the equality relation on S.

Proof. (i)⇒ (ii). Let S be a semilattice. For any $a \in S$ and $x, y \in [a)$, we have $x \geq a, y \geq a$. This implies that $xy \geq a^2 = a$ and $xy \in [a)$. Hence, $[a)$ is a subsemigroup of S.

To prove that $[a)$ is a filter containing a, we suppose that $b, c \in S$ such that $bc \in [a)$. Then we have $bc \geq a$ and $abc = bca = a$. Hence,

$$ab = ba = abcb = abc = a, \quad ac = ca = abcc = abc = a,$$

and so $b \geq a, c \geq a$. This shows that $b \in [a), c \in [a)$. Since $[a) \subseteq [a)$ always holds, $[a)$ is a filter containing a, as required.

Let T be a filter containing a. By the definition of filters, we have $[T) \subseteq T$. Since $a \in T$, then $[a) \subseteq [T) \subseteq T$. Consequently, $[a)$ is the smallest filter containing a and then $N(a) = [a)$.

(ii) ⇒ (iii). Suppose that $a\mathcal{N}b$ for $a, b \in S$. Then $[a) = N(a) = N(b) = [b)$. Since $a \in [a) = [b)$ and $b \in [b) = [a)$, we have that $a \geq b, b \geq a$ and so $a = b$. This implies that $\mathcal{N} = 1_S$.

(iii) ⇒ (i). For any $a, b \in S$, we have $N_a = \{a\}, N_b = \{b\}$. Since N_a and N_b are both semilattice congruence classes of S, it is easy to see that $N_aN_a \subseteq N_a$ and $N_aN_b = N_bN_a$. Clearly, $a^2 = a, ab = ba$. This shows that S is a semilattice as required.

Moreover, the partial order on S is the natural order of semilattice. In fact, $N_a \leq_Y N_b$ if and only if $N_aN_b = N_bN_a = N_a$. Since $\mathcal{N} = 1_S$, we have that $a \leq b$ if and only if $ab = ba = a$. □

Theorem 4. Let S be a po-semigroup. Then the following are equivalent:

(i) $\mathcal{N}$ is the universal relation on S;

(ii) S has only one filter and $N(a) = S$, for any $a \in S$;

(iii) S has only one complete semilattice congruence on S.

Proof. Since $\mathcal{N}$ is the smallest complete semilattice congruence on S, it is trivial that (i) $\Leftrightarrow$(iii).

(i) $\Rightarrow$ (ii). Since $\mathcal{N}$ is the universal relation on S which means that for every $a \in S, N_a = S$, we have that $N_a \subseteq N(a) \subseteq S$ by Lemma 1. Hence $N(a) = S$ as required.

(ii) $\Rightarrow$ (i). For any $a, b \in S$, we have $N(a) = S = N(b)$. This shows that $a\mathcal{N}b$ and $\mathcal{N} = \omega_S$, as required. □

Theorem 5. Let σ be a complete semilattice congruence on a po-semigroup S and Y the semilattice S/σ. Then for any $\alpha \in Y$, we have

(i) S_α is the union of some $\mathcal{N}$-classes;

(ii) The set $T = \bigcup\{S_\beta : \beta \geq_Y \alpha, \beta \in Y\}$ is a filter;

(iii) For any $a \in S_\alpha$, $N(a) = T$ if and only if σ is the smallest complete semilattice congruence on S.

Proof. (i) Since $\mathcal{N}$ is the smallest complete semilattice congruence on S, we have that $(a, b) \in \mathcal{N} \subseteq \sigma$ for every $a \in S_\alpha$ and $b \in N_a$. It is clear that S_α is a semilattice congruence class of S and so $b \in S_\alpha$. We have proved that $N_a \subseteq S_\alpha$. Consequently, $\bigcup\limits_{a \in S_\alpha} N_a \subseteq S_\alpha$. Clearly, $S_\alpha \subseteq \bigcup\limits_{a \in S_\alpha} N_a$. Hence we have $S_\alpha = \bigcup\limits_{a \in S_\alpha} N_a$. This is exactly the union of some $\mathcal{N}$-classes.

(ii) To see that T is a filter, we first prove that T is a subsemigroup of S. Since $\emptyset \neq S_\alpha \subseteq T$, T is not empty. For any $x, y \in T$, we have β and γ in Y such that $x \in S_\beta, y \in S_\gamma$, $\beta \geq_Y \alpha$ and $\gamma \geq_Y \alpha$. This implies that $xy \in S_\beta S_\gamma \subseteq S_{\beta\gamma}$ and $\beta\gamma \geq_Y \alpha$. Hence, $xy \in T$ and T is a subsemigroup of S.

Suppose that $xy \in T$ and $x, y \in S$, we have β, γ and δ in Y such that $x \in S_\gamma, y \in S_\delta$, $xy \in S_\beta$ and $\beta \geq_Y \alpha$. This implies that $xy \in S_\gamma S_\delta \subseteq S_{\gamma\delta}$ and $\gamma\delta = \beta \geq_Y \alpha$. Since Y is a semilattice, it is easy to see that $\gamma \geq_Y \alpha$ and $\delta \geq_Y \alpha$. Thus, we have $x \in T$ and $y \in T$.

For any $x \in [T)$, there exists an element β in Y such that $x \in S_\beta$ and an element y in S_γ such that $x \geq y$, where $\gamma \in Y$ and $\gamma \geq_Y \alpha$. This shows that $xy \in S_\beta S_\gamma \subseteq S_{\beta\gamma}$. Since σ is a complete semilattice congruence, we can see that $(xy, y) \in \sigma$. From $y \in S_\gamma$, we immediately have $xy \in S_\gamma$. Then we have $\beta\gamma = \gamma$ and so $\beta \geq_Y \gamma \geq_Y \alpha$ in Y. Hence, $x \in T$ and $[T) \subseteq T$ as required. We have shown that T is a filter.

(iii) If σ is the smallest complete semilattice congruence on S, we have $\sigma = \mathcal{N}$ and S_α is a $\mathcal{N}$-class for every $\alpha \in Y$. Then we have $S_\alpha = N_a$ for any $a \in S_\alpha$. T is the union of all the $\mathcal{N}$-classes which are greater than N_a. This is exactly the set $\bigcup\{N_b : N_b \geq_Y N_a\}$. By Lemma 2, $N(a) = \bigcup\{S_\beta : \beta \geq_Y \alpha, \beta \in Y\}$.

Conversely, suppose that $(a, b) \in \sigma$ and $a \in S_\alpha$, then we have $b \in S_\alpha$. Since $N(a) = \bigcup\{S_\beta : \beta \geq_Y \alpha, \beta \in Y\}$ for any $a \in S_\alpha$, we now have $N(a) = N(b)$ and $(a, b) \in \mathcal{N}$, then $\sigma \subseteq \mathcal{N}$. We have known that $\mathcal{N}$ is the smallest complete semilattice congruence on S, so $\sigma = \mathcal{N}$ and σ is the smallest complete semilattice congruence on S. □

The following Corollary is a direct result of Theorem 5.

Corollary 6. Let σ be a complete semilattice congruence on a po-semigroup S and Y the semilattice S/σ. For any $\alpha \in Y$ and $a \in S_\alpha$, $N(a) \subseteq \bigcup\{S_\beta : \beta \geq_Y \alpha, \beta \in Y\}$.

Corollary 7. Let σ be a complete semilattice congruence on a po-semigroup S and Y the semilattice S/σ. If there exists a maximal element α in Y such that S_α has no proper subsemigroups, we have $N(a) = S_\alpha$ for any $a \in S_\alpha$.

Proof. Suppose that α is a maximal element in Y. By Corollary 6, we have $N(a) \subseteq \bigcup\{S_\beta : \beta \geq_Y \alpha, \beta \in Y\} = S_\alpha$ for any $a \in S_\alpha$. This shows that $N(a)$ is a subsemigroup of S_α. From $a \in S_\alpha$ we know that S_α is not empty. Since S_α has no proper subsemigroups, we have $N(a) = S_\alpha$. □

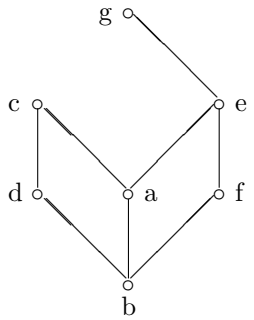
Remark. Suppose that α is a maximal element in Y. If S_α is finite, there must exists an idempotent e such that $\{e\}$ is a subsemigroup of S_α. If e is a maximal element in S_α and $ab = e$ if and only if $a = b = e$, we know that$\{e\}$ is a filter. If e isn't a maximal element in S_α and S_α has no proper subsemigroups except $\{e\}$, we have $S_\alpha = N(a)$ for every $a \in S_\alpha$.

Example. The set $S = \{a, b, c, d, e, f, g\}$ with the multiplication " $*$ " and the order " $\leq$ " below is a po-semigroup.

$*$	a	b	c	d	e	f	g
a	a	b	a	b	a	b	a
b	b	b	b	b	b	b	b
c	a	b	c	d	a	b	a
d	b	b	d	d	b	b	b
e	a	b	a	b	e	f	e
f	b	b	b	b	f	f	f
g	a	b	a	b	e	f	g

$$\leq := 1_S \bigcup\{(a,c),(a,e),(a,g),(b,a),(b,c),(b,d),$$
$$(b,e),(b,f),(b,g),(d,c),(e,g),(f,e),(f,g)\}.$$

The Hasse diagram of S is shown below.



We now define a complete semilattice congruence σ on S as follow:

$$\sigma := 1_S \bigcup\{(a,b),(b,a),(c,d),(d,c),(e,f),(f,e)\}.$$

Then $S/\sigma = \{\{a,b\},\{c,d\},\{e,f\},\{g\}\}$. If we denote $S_\alpha = \{a,b\}$, $S_\beta = \{c,d\}$, $S_\gamma = \{e,f\}$, $S_\delta = \{g\}$, the order on semilattice $Y = S/\sigma$ is shown below.

$$\begin{aligned} \leq_Y \quad = \quad & \{(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma),(\delta,\delta), \\ & (\alpha,\beta),(\alpha,\gamma),(\alpha,\delta),(\gamma,\delta)\}. \end{aligned}$$

From the Cayley table above, we know that S is a semilattice. By Theorem 3, we can easily see that $N(a) = \{a,c,e,g\}$, $N(b) = \{a,b,c,d,e,f,g\}$, $N(c) = \{c\}$, $N(d) = \{c,d\}$, $N(e) = \{e,g\}$, $N(f) = \{e,f,g\}$, $N(g) = \{g\}$ and $\mathcal{N} = 1_S$. By Corollary 6, we can see that $N(a) \subseteq S_\alpha \bigcup S_\beta \bigcup S_\gamma \bigcup S_\delta$, $N(b) \subseteq S_\alpha \bigcup S_\beta \bigcup S_\gamma \bigcup S_\delta$, $N(c) \subseteq S_\beta$, $N(d) \subseteq S_\beta$, $N(e) \subseteq S_\gamma \bigcup S_\delta$, $N(f) \subseteq S_\gamma \bigcup S_\delta$ and $N(g) = S_\delta$ by Corollary 7.

References

[1] Kehayopulu N., On weakly commutative poe-semigroups, Semigroup Forum, **34**(1987), 367-370.

[2] Kehayopulu N., On weakly commutative poe-semigroups, Semigroup Forum, **41**(1990), 373-376.

[3] Kehayopulu N., Remark on ordered semigroups. Math. Japonica, **36**(1990), No. 6, 1061-1063.

[4] Z.L.Gao and K.P.Shum, On cyclic commutative po-semigroups, PU. M. A, **8**(1997), No. 2-3-4, 261-273.

[5] Y.L.Cao and X.Z.Xu, Nil-extensions of simple po-semigroups, Communications in Algebra, **28**(2000), No.5, 2477-2496.

[6] J.M.Howie, An introduction to semigroup theory, Academic Press, London, 1976.

[7] J.M.Howie, Fundamentals of semigroup theory, Oxford University Press, New York, 1995.

[8] Ren X.M, Yan J.Z and K.P.Shum, On Principal Filters of Po-semigroups, to appear.

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F.Smarandache additive k-th power complements[1]

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Abstract For any given natural number $k \geq 2$ and any positive integer n, we call $a_k(n)$ as a k-th power complement number of n if $a_k(n)$ denotes the smallest non-negative integer such that $n + a_k(n)$ is a perfect k-th power. In this paper, we studied the convergent property of the series $\sum_{n=1}^{\infty} \frac{1}{(n+a_k(n))^{\alpha}}$ by using the elementary methods, and obtained an interesting identity for it.

Keywords Smarandache additive k-th power complements; Identity.

§1. Introduction

For any given natural number $k \geq 2$ and any positive integer n, we call $b_k(n)$ as the Smarandache k-th power complement number of n if $b_k(n)$ denotes the smallest positive integer such that $nb_k(n)$ is a perfect k-th power, if $k = 2$, then we called $b_2(n)$ as the square complements of n. In problem 27 of [1], Professor F.Smarandache ask us to study the properties of $b_2(n)$. About this problem, some authors have studied it, for example, Liu Hongyan and Gou Su [2] used the elementary method to study the mean value properties of $b_2(n)$ and $\frac{1}{b_2(n)}$. Zhang Hongli and Wang Yang [3] studied the mean value of $\tau(b_2(n))$, and obtained an asymptotic formula by using the analytic method.

Similarly, we define the additive k-th power complements $a_k(n)$ of n as follows: $a_k(n)$ is the smallest non-negative integer such that $a_k(n) + n$ is a complete k-th power. That is

$$a_k(n) = \min\{l \mid n + l = m^k, l \geq 0, m \in N^+\}.$$

If $k = 2$, we call $a_2(n)$ as the additive square complements of n which is defined as the smallest positive integer l such that $n + l$ is a perfect square. For example, $a_2(1) = 0$, $a_2(2) = 2$, $a_2(3) = 1$, $a_2(5) = 4$, $a_2(6) = 3$, $a_2(7) = 2$, $\cdots$.

About this problem, many scholars have studied it and obtained some interesting results. For example, Xu Zhefeng [4] proved the following asymptotic formula:

$$\sum_{n \leq x} a_k(n) = \frac{k^2}{4k-2} x^{2-1/k} + O(x^{2-2/k}), \quad x \geq 3.$$

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Yi Yuan [5] studied the mean value properties of $d(n+a(n))$, and got the following conclusion:

$$\sum_{n\leq x} d(n+a(n)) = \frac{3}{4\pi^2}x\ln^2 x + A_1 x\ln x + A_2 x + O(x^{\frac{3}{4}+\epsilon}),$$

where $d(n)$ is the Dirichlet divisor function, A_1 and A_2 are computable constants, ϵ denotes any fixed positive number.

In this paper, we studied the convergent property of the series $\sum_{n=1}^{\infty}\frac{1}{(n+a_k(n))^{\alpha}}$ by using the elementary methods, and obtained an interesting identity for it. That is, we will prove the following:

Theorem. Let $k\geq 2$ be an integer, then for any real number $\alpha\leq 1$, the infinity series

$$\sum_{n=1}^{\infty}\frac{1}{(n+a_k(n))^{\alpha}}$$

is divergent, it is convergent if $\alpha>1$, and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n+a_k(n))^{\alpha}} &= \binom{k}{1}\zeta(k\alpha-k+1)-\binom{k}{2}\zeta(k\alpha-k+2)+\binom{k}{3}\zeta(k\alpha-k+3)\\ &\quad -\cdots-(-1)^{k-1}\binom{k}{1}\zeta(k\alpha-1)-(-1)^k\zeta(k\alpha),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $k=2$, $\alpha=2$ and $k=3$, $\alpha=3$ in our Theorem, we may immediately deduce the following:

Corollary. For additive square complements and additive cubic complements, we have the identities

$$\sum_{n=1}^{\infty}\frac{1}{(n+a_2(n))^2} = 2\zeta(3)-\zeta(4)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(n+a_3(n))^3} = 3\zeta(7)-3\zeta(8)+\zeta(9).$$

§2. Proof of the theorem

In this section, we will complete the proof of Theorem. For any positive integer $n\geq 1$, there must be a positive integer m such that

$$(m-1)^k < n \leq m^k.$$

So the number of all such n is $m^k-(m-1)^k$ which satisfying $a_k(n)+n=m^k$, then from the definition of $a_k(n)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n+a_k(n))^{\alpha}} &= \sum_{m=1}^{\infty}\frac{m^k-(m-1)^k}{m^{k\alpha}}\\ &= \sum_{m=1}^{\infty}\frac{\binom{k}{1}m^{k-1}-\binom{k}{2}m^{k-2}+\binom{k}{3}m^{k-3}-\cdots-(-1)^{k-1}\binom{k}{1}m-(-1)^k}{m^{k\alpha}}\\ &= \binom{k}{1}\zeta(k\alpha-k+1)-\binom{k}{2}\zeta(k\alpha-k+2)+\binom{k}{3}\zeta(k\alpha-k+3)\\ &\quad -\cdots-(-1)^{k-1}\binom{k}{1}\zeta(k\alpha-1)-(-1)^k\zeta(k\alpha).\end{aligned}$$

This completes the proof of Theorem.

References

[1] F. Smarandache. Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993, pp. 23.

[2] Liu H.Y and Gou S. On a problem of F.Smarandache, Journal of Yanan University (Natural Science Edition), **20**(2001), 5-6.

[3] Zhang H.L. and Wang Y. On the mean value of divisor function for square complements, Basic Science Journal of Textile University, **15**(2002), 44-46.

[4] Xu Zhefeng. On the additive k-th power complements, Research on Smarandache problems in number theory (Collected papers), Hexis, 2004, pp. 13-16.

[5] Yi Yuan. On the asymptotic property of divisor function for additive complement, Research on Smarandache problems in number theory (Collected papers), Hexis, 2004, pp. 65-68.

[6] Tom M.Apostol. Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

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The Smarandache Reverse Auto Correlated Sequences of Natural Numbers

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Abstract In this paper we give an explicit formula for the n times Smarandache reverse auto correlated sequence of natural numbers.

Keywords Smarandache reverse auto correlated sequence, natural number.

Let $A = \{a(m)\}_{m=1}^{\infty}$ be a sequence. If the sequence $B = \{b(m)\}_{m=1}^{\infty}$ satisfying

$$b(m) = \sum_{k=1}^{m} a(k)a(m-k+1), m \geq 1, \tag{1}$$

then B is called the Smarandache reverse auto correlated sequence of A, and denoted by $SRACS(A)$. Further, for any positive integer n, let $SRACS(n, A)$ denote the n times Smarandache reverse auto correlated sequence of A. Then we have $SRACS(1, A) = SRACS(A)$, $SRACS(2, A) = SRACS(SRACS(A))$ and

$$SRACS(n, A) = SRACS(SRACS(n-1, A)), n \geq 1. \tag{2}$$

Recentely, Muthy [1] proposed the following conjecture:

Conjecture. For any positive integer n, if $a(m) = m$ $(m \geq 1)$ and $SRACS(n, A) = B = \{b(m)\}_{m=1}^{\infty}$, then

$$b(m) = \binom{2^{n+1} + m - 1}{2^{n+1} - 1}, m \geq 1 \tag{3}$$

In this paper we completely verify the above-mentioned conjecture as follows.

Theorem. For any positive integer n, if $a(m) = m$ $(m \geq 1)$ and $SRACS(n, A) = B = \{b(m)\}_{m=1}^{\infty}$, then $b(m)$ $(m \geq 1)$ satisfy (3).

Proof. For a fixed sequence $A = \{a(m)\}_{m=1}^{\infty}$, let

$$f(A; x) = a(1) + a(2)x + a(3)x^2 + \cdots = \sum_{m=1}^{\infty} a(m)x^{m-1}. \tag{4}$$

Further, let $B = \{b(m)\}_{m=1}^{\infty}$ be the Smarandache reverse auto correlated sequence of A, and let

$$g(A; x) = b(1) + b(2)x + b(3)x^2 + \cdots = \sum_{m=1}^{\infty} b(m)x^{m-1}. \tag{5}$$

Then, by the definition of multiplication of power series (see [2]), we see from (1), (4) and (5) that

$$g(A;x) = (f(A;x))^2. \tag{6}$$

Furthermore, for a fixed positive integer n, if $SRACS(n,A) = B = \{b(m)\}_{m=1}^{\infty}$, and

$$g(n,A;x) = b(1) + b(2)x + b(3)x^2 + \cdots = \sum_{m=1}^{\infty} b(m)x^{m-1}, \tag{7}$$

then from (2) and (6) we obtain

$$g(n,A;x) = (f(A;x))^{2^n}. \tag{8}$$

If $a(m) = m$ for $m \geq 1$, then we get

$$f(A;x) = 1 + 2x + 3x^2 + \cdots = \sum_{m=1}^{\infty} mx^{m-1} = (1-x)^{-2}, \tag{9}$$

by (4). Therefore, by (8), if $SRACS(n,A) = B = \{b(m)\}_{m=1}^{\infty}$ and $g(n,A;x)$ satisfies (7), then from (9) we obtain

$$g(n,A;x) = (1-x)^{-2^{n+1}} = \sum_{m=1}^{\infty} \binom{2^{n+1}+m-1}{2^{n+1}-1} x^{m-1}, \tag{10}$$

Thus, by (7) and (10), we get (3). The theorem is proved.

References

[1] A.Murthy, Smarandache reverse auto correlated sequences and some Fibonacci derived Smarandache sequences, Smarandache Notions J., **12**(2001), 279-282.

[2] I. Niven, Formal power series, Amer. Math. Monthly, **76**(1969), 871-889.

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Smarandache Partitions

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Abstract I study Smarandache numbers partitions, and the partitions set of these numbers. This study conducted by Computer Algebra System namely, Maple 8.

Keywords Smarandache numbers s(n); Partitions P(n); Partitions sets; Smarandache numbers partitions P(s(n)).

§1.1 The procedure

Using the following procedure, we can verify the number of unrestricted partitions of the Smarandache numbers n is denoted by $P(s(n))$. With the Maple (V. 8)[see, 2] definitions.

S:= proc(n::nonnegint)
option remember;
local i, j, $fact$:
$fact$:=1:
for i from 2 while irem ($fact$, n)<> 0 do
$fact$:= $fact$ *i:
od :
return $i-1$:
end proc:
b:= proc(n::nonnegint)
option remember;
with (combstruct):
count (Partition(n));
end proc:

This procedure can verify the number of partitions, very fast, for example, it can verify the number of partitions of 200 in 0.2 second, while George Andrews said that "Actual enumeration of the $P(200) = 3972999029388$ would certainly take more than a lifetime, [1, p 150]."

Below the first 100 Smarandache numbers verifying by the above procedure:

§1.2 Partition counting of Smarandache numbers

By using the above procedure, we can got the first 100 partitions of Smarandache numbers as follows:

$P(s(1)) = 1$ $P(s(2)) = 2$ $P(s(3)) = 3$ $P(s(4)) = 5$
$P(s(5)) = 7$ $P(s(6)) = 3$ $P(s(7)) = 15$ $P(s(8)) = 5$
$P(s(9)) = 11$ $P(s(10)) = 7$ $P(s(11)) = 56$ $P(s(12)) = 5$
$P(s(13)) = 101$ $P(s(14)) = 15$ $P(s(15)) = 7$ $P(s(16)) = 11$
$P(s(17)) = 297$ $P(s(18)) = 11$ $P(s(19)) = 490$ $P(s(20)) = 7$
$P(s(21)) = 15$ $P(s(22)) = 56$ $P(s(23)) = 1255$ $P(s(24)) = 5$
$P(s(25)) = 42$ $P(s(26)) = 101$ $P(s(27)) = 30$ $P(s(28)) = 15$
$P(s(29)) = 4565$ $P(s(30)) = 7$ $P(s(31)) = 6842$ $P(s(32)) = 22$
$P(s(33)) = 56$ $P(s(35)) = 297$ $P(s(35)) = 15$ $P(s(36)) = 11$
$P(s(37)) = 21637$ $P(s(38)) = 490$ $P(s(39)) = 101$ $P(s(40)) = 7$
$P(s(41)) = 44583$ $P(s(42)) = 15$ $P(s(43)) = 63261$ $P(s(44)) = 56$
$P(s(45)) = 11$ $P(s(46)) = 1255$ $P(s(47)) = 124754$ $P(s(48)) = 11$
$P(s(49)) = 135$ $P(s(50)) = 42$ $P(s(51)) = 297$ $P(s(52)) = 101$
$P(s(53)) = 329931$ $P(s(54)) = 30$ $P(s(55)) = 56$ $P(s(56)) = 15$
$P(s(57)) = 490$ $P(s(58)) = 4565$ $P(s(59)) = 831820$ $P(s(60)) = 7$
$P(s(61)) = 1121505$ $P(s(62)) = 6842$ $P(s(63)) = 15$ $P(s(64)) = 22$
$P(s(65)) = 101$ $P(s(66)) = 56$ $P(s(67)) = 2679689$ $P(s(68)) = 297$
$P(s(69)) = 1255$ $P(s(70)) = 15$ $P(s(71)) = 4697205$ $P(s(72)) = 11$
$P(s(73)) = 6185689$ $P(s(74)) = 21637$ $P(s(75)) = 42$ $P(s(76)) = 490$
$P(s(77)) = 56$ $P(s(78)) = 101$ $P(s(79)) = 13848650$ $P(s(80)) = 11$
$P(s(81)) = 30$ $P(s(82)) = 44583$ $P(s(83)) = 23338469$ $P(s(84)) = 15$
$P(s(85)) = 297$ $P(s(86)) = 63261$ $P(s(87)) = 4565$ $P(s(88)) = 56$
$P(s(89)) = 49995925$ $P(s(90)) = 11$ $P(s(91)) = 101$ $P(s(92)) = 1255$
$P(s(93)) = 6842$ $P(s(94)) = 124754$ $P(s(95)) = 490$ $P(s(96)) = 22$
$P(s(97)) = 133230930$ $P(s(98)) = 135$ $P(s(99)) = 56$ $P(s(100)) = 42$

We can not (without lose of generality)that: $P(s(4)) = P(s(8)) = P(s(12)) = P(s(24))$, this is because $s(4) = s(8) = s(12) = s(24) = 4$, and so on.

§2.1 The procedure of partitions sets

Now, the following procedure, we can verify the unrestricted partitions of the Smarandache numbers. With the Maple (V. 8) definitions.

S:= proc (n::nonnegint)
option remember;
local i, j, $fact$:

$fact := 1$:
for i from 2 while irem ($fact$, n)<> 0 do
$fact$:= $fact$ *i:
od :
return $i - 1$:
end proc:
b:= proc (n::nonnegint)
option remember;
with (combstruct):
allstructs (Partition(n));
end proc:

§2.2 Partition Sets of Smarandache numbers

By using the above procedure, we can got the first 15 partition sets of Smarandache numbers as follows:

PartitionSetOf $(s(1)) = [[1]]$,
PartitionSetOf $(s(2)) = [[1,1],[2]]$,
PartitionSetOf $(s(3)) = [[[1,1,1],[1,2],[3]]$,
PartitionSetOf $(s(4)) = [[1,1,1,1],[1,1,2],[2,2],[1,3],[4]]$,
PartitionSetOf $(s(5)) = \{[\ [1,1,1,1,1],[1,1,1,2],[1,2,2],[1,1,3],[2,3],[1,4],[5]\ ]\}$,
PartitionSetOf $(s(6)) = \{[\ [1,1,1],[1,2],[3]\ ]\}$,
PartitionSetOf $(s(7)) = \{[\ [1,1,1,1,1,1,1],[1,1,1,1,1,2],[1,1,1,2,2],[1,2,2,2],[1,1,1,1,3],$
$[1,1,2,3],[2,2,3],[1,3,3],[1,1,1,1,4],[1,2,4],[3,4],[1,1,5],[2,5],[1,6],[7]\ ]\}$,
PartitionSetOf $(s(8)) = \{[\ [1,1,1,1],[1,1,2],[2,2],[1,3],[4]\ ]\}$,
PartitionSetOf $(s(9)) = \{[\ [1,1,1,1,1,1],[1,1,1,1,2],[1,1,2,2],[2,2,2],[1,1,1,3],[1,2,3],$
$[3,3],[1,1,4],[2,4],[1,5],[6]\ ]\}$,
PartitionSetOf $(s(10)) = \{[\ [1,1,1,1,1],[1,1,1,2],[1,2,2],[1,1,3],[2,3],[1,4],[5]\ ]\}$,
PartitionSetOf $(s(11)) = \{[\ [1,1,1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1,2,2],$
$[1,1,1,1,1,2,2,2],[1,1,1,2,2,2,2,2],[1,2,2,2,2,2,2],[1,1,1,1,1,1,1,1,1,3],[1,1,1,1,1,1,1,2,3],$
$[1,1,1,1,2,2,3],[1,1,2,2,2,2,3],[2,2,2,2,2,3],[1,1,1,1,1,3,3],[1,1,1,2,3,3],[1,2,2,3,3,],$
$[1,1,3,3,3],[2,3,3,3],[1,1,1,1,1,1,1,4],[1,1,1,1,1,2,4],[1,1,1,2,2,4],[1,2,2,2,4],$
$[1,1,1,1,3,4],[1,1,2,3,4],[2,2,3,4],[1,3,3,4],[1,1,1,4,4],[1,2,4,4],[3,4,4],[1,1,1,1,1,5],$
$[1,1,1,1,2,5],[1,1,2,2,5],[2,2,2,5],[1,1,1,3,5],[1,2,3,5],[3,3,5],[1,1,4,5],[2,4,5],[1,5,5,],$
$[1,1,1,1,1,6],[1,1,1,2,6],[1,2,2,6],[1,1,3,6],[2,3,6],[1,4,6],[5,6],[1,1,1,1,7],[1,1,2,7],[2,2,7],$
$[1,3,7],[4,7],[1,1,1,8],[1,2,8],[3,8],[1,1,9],[2,9],[1,10],[11]\ ]\}$,
PartitionSetOf $(s(12)) = \{[\ [1,1,1,1],[1,1,2],[2,2],[1,3],[4]\ ]\}$,
PartitionSetOf $(s(13)) = \{[\ [1,1,1,1,1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1,1,1,2],$
$[1,1,1,1,1,1,1,1,1,2,2],[1,1,1,1,1,1,1,2,2,2,2],[1,1,1,1,1,2,2,2,2,2],[1,1,1,2,2,2,2,2],$
$[1,2,2,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,3],[1,1,1,1,1,1,1,1,2,3],[1,1,1,1,1,1,1,2,2,3],$
$[1,1,1,1,2,2,2,3],[1,1,2,2,2,2,3],[2,2,2,2,2,3],[1,1,1,1,1,1,1,3,3],[1,1,1,1,1,2,3,3],$
$[1,1,1,2,2,3,3],[1,2,2,2,3,3],[1,1,1,1,3,3,3],[1,1,2,3,3,3],[2,2,3,3,3],[1,3,3,3,3],$

$[1,1,1,1,1,1,1,1,1,4],[1,1,1,1,1,1,1,1,2,4],[1,1,1,1,1,1,2,2,4],[1,1,1,1,2,2,2,4],[1,2,2,2,2,2,4]$,
$[1,1,1,1,1,1,1,3,4],[1,1,1,1,1,2,3,4],[1,1,2,2,2,3,4],[2,2,2,3,4],[1,1,1,3,3,4],[1,2,3,3,4]$,
$[3,3,3,4],[1,1,1,1,1,1,4,4],[1,1,1,2,4,4],[1,2,2,2,4,4],[1,1,3,4,4],[2,3,4,4],[1,4,4,4]$,
$[1,1,1,1,1,1,1,1,1,5],[1,1,1,1,1,1,1,2,5],[1,1,1,1,2,2,5],[1,1,2,2,2,5],[2,2,2,2,5]$,
$[1,1,1,1,1,1,3,5],[1,1,1,2,3,5],[1,2,2,2,3,5],[1,1,3,3,3,5][2,3,3,5],[1,1,1,1,4,5],[1,1,2,4,5]$,
$[2,2,4,5],[1,3,4,5],[4,4,5],[1,1,1,5,5],[1,2,5,5],[3,5,5]\ ],[1,1,1,1,1,1,1,1,6],[1,1,1,1,1,1,2,6]$,
$[1,1,1,2,2,6],[1,2,2,2,2,6],[1,1,1,1,1,3,6],[1,1,2,3,3,6],[2,2,3,6,],[1,3,3,6],[1,1,1,4,6]$,
$[1,2,4,6],[3,4,6],[1,1,5,6],[2,5,6],[1,6,6],[1,1,1,1,1,1,1,7],[1,1,1,1,2,7],[1,1,2,2,7],,$
$[2,2,2,7],[1,1,1,3,7][1,2,3,7],[3,3,7],[1,1,4,7],[2,4,7],[1,5,7],[6,7],[1,1,1,1,1,1,8]$,
$[1,1,1,2,8],[1,2,2,8],[1,1,3,8],[2,3,8],[1,4,8],[5,8],[1,1,1,1,9],[1,1,2,9],[2,2,9],[1,3,9]$,
$[4,9],[1,1,1,10],[1,2,10],[3,10],[1,1,11],[2,11],[1,12],[13]]\}$,

PartitionSetOf $(s(14)) = \{[\ [1,1,1,1,1,1,1],[1,1,1,1,1,1,2],[1,1,1,2,2],[1,2,2,2]$,
$[1,1,1,1,3],[1,1,2,3],[2,2,3],[1,3,3],[1,1,1,4],[1,2,4],[3,4],[1,1,5],[2,5],[1,6],[7]\ ]\}$,

PartitionSetOf $(s(15)) = \{[\ [1,1,1,1,1],[1,1,1,2],[1,2,2],[1,1,3],[2,3],[1,4],[5]\ ]\}$,

We can not (without lose of generality) that: Partitions of $P(s(4)) = P(s(8)) = P(s(12)) = P(s(24))$, this is because all of them have the same Smarandache numbers and the same partitions sets, and so on.

References

[1] G. E. Andrews, Number Theory, Drover, New York , 1971.

[2] M.B. Monagan etc., Maple 7, Programming Guide, Waterloo Maple Inc., 2001, pp. 36.

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On Algebraic Multi-Group Spaces

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Abstract A Smarandache multi-space is a union of $n, n \geq 2$ spaces $A_1, A_2, \cdots, A_n$ with some additional conditions. Combining classical groups with Smarandache multi-spaces, the conception of multi-group spaces is introduced in this paper, which is a generalization of the classical algebraic structures, such as the group, the filed, the body, $\cdots$, etc.. Similar to groups, some characteristics of multi-group spaces are obtained in this paper.

Keywords multi-space; group; multi-group space; Jordan-Hölder theorem.

§1. Introduction

The notion of multi-spaces is introduced by Smarandache in [5] under his idea of hybrid mathematics: *combining different fields into a unifying field* ([6]). Today, this idea is widely accepted by the world of sciences. For mathematics, definite or exact solution under a given condition is not the only object for mathematician. New creation power has emerged. New era for mathematics has come now.

A Smarandache multi-space is defined by

Definition 1.1 *For any integer $i, 1 \leq i \leq n$, let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

The conception of multi-group spaces is a generalization of the classical algebraic structures, such as the group, the filed, the body, $\cdots$, etc., which is defined as follows.

Definition 1.2 *Let $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ be a complete multi-space with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations "$\times$" and "$\circ$" , $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation "$\circ$" provided their operation results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group space.

Remark: The following special cases convince us that the multi-group spaces are generalization of the group, the field and the body, $\cdots$, etc..

(i) If $n = 1$, then $\widetilde{G} = (G_1; \times_1)$ is just a group.

(ii) If $n = 2, G_1 = G_2 = \widetilde{G}$, Then $\widetilde{G}$ is a body. If $(G_1; \times_1)$ and $(G_2; \times_2)$ are commutative groups, then $\widetilde{G}$ is a field.

Notice that in [7]*and*[8] various bispaces, such as bigroup, bisemigroup, biquasigroup, biloop, bigroupoid, biring, bisemiring, bivector, bisemivector, binear-ring, $\cdots$, etc., consider two operations on two different sets are introduced.

§2. Characteristics of multi-group spaces

For a multi-group space $\widetilde{G}$ and a subset $\widetilde{G_1} \subset \widetilde{G}$, if $\widetilde{G_1}$ is also a multi-group space under a subset $O(\widetilde{G_1}), O(\widetilde{G_1}) \subset O(\widetilde{G})$, then $\widetilde{G}$ is called a *multi-group subspace*, denoted by $\widetilde{G_1} \preceq \widetilde{G}$. We have the following criterion for the multi-group subspaces.

Theorem 2.1. *For a multi-group space $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$, a subset $\widetilde{G_1} \subset \widetilde{G}$ is a multi-group subspace if and only if for any integer $k, 1 \leq k \leq n$, $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$.*

Proof. If $\widetilde{G_1}$ is a multi-group space with the operation set $O(\widetilde{G_1}) = \{\times_{i_j} | 1 \leq j \leq s\} \subset O(\widetilde{G})$, then

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i) = \bigcup_{j=1}^{s} G'_{i_j}$$

where $G'_{i_j} \preceq G_{i_j}$ and $(G_{i_j}; \times_{i_j})$ is a group. Whence, if $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then there exist an integer $l, k = i_l$ such that $\widetilde{G_1} \bigcap G_k = G'_{i_l}$, i.e., $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$.

Now if for any integer k, $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$, let N denote the index set k with $\widetilde{G_1} \bigcap G_k \neq \emptyset$. Then

$$\widetilde{G_1} = \bigcup_{j \in N} (\widetilde{G_1} \bigcap G_j)$$

and $(\widetilde{G_1} \bigcap G_j, \times_j)$ is a group. Since $\widetilde{G_1} \subset \widetilde{G}$ and $O(\widetilde{G_1}) \subset O(\widetilde{G})$, the associative law and distribute law are also true for the $\widetilde{G_1}$. Therefore, $\widetilde{G_1}$ is a multi-group subspace of $\widetilde{G}$.

For a finite multi-group subspace, we get the following criterion.

Theorem 2.2. *Let $\widetilde{G}$ be a finite multi-group space with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. A subset $\widetilde{G_1}$ of $\widetilde{G}$ is a multi-group subspace under an operation subset $O(\widetilde{G_1}) \subset O(\widetilde{G})$ if and only if for each operation "$\times$" in $O(\widetilde{G_1})$, $(\widetilde{G_1}; \times)$ is complete.*

Proof. Notice that for a multi-group space $\widetilde{G}$, its each multi-group subspace $\widetilde{G_1}$ is complete.

Now if $\widetilde{G_1}$ is a complete set under each operation "$\times_i$" in $O(\widetilde{G_1})$, we know that $(\widetilde{G_1}\bigcap G_i;\times_i)$ is a group (see also [9]) or an empty set. Whence, we get that

$$\widetilde{G_1} = \bigcup_{i=1}^{n}(\widetilde{G_1}\bigcap G_i).$$

Therefore, $\widetilde{G_1}$ is a multi-group subspace of $\widetilde{G}$ under the operation set $O(\widetilde{G_1})$.

For a multi-group subspace $\widetilde{H}$ of the multi-group space $\widetilde{G}$, $g \in \widetilde{G}$, define

$$g\widetilde{H} = \{g \times h | h \in \widetilde{H}, \times \in O(\widetilde{H})\}.$$

Then for $\forall x, y \in \widetilde{G}$,

$$x\widetilde{H}\bigcap y\widetilde{H} = \emptyset \ \text{ or } \ x\widetilde{H} = y\widetilde{H}.$$

In fact, if $x\widetilde{H}\bigcap y\widetilde{H} \neq \emptyset$, let $z \in x\widetilde{H}\bigcap y\widetilde{H}$, then there exist elements $h_1, h_2 \in \widetilde{H}$ and operations "$\times_i$" and "$\times_j$" such that

$$z = x \times_i h_1 = y \times_j h_2.$$

Since $\widetilde{H}$ is a multi-group subspace, $(\widetilde{H}\bigcap G_i;\times_i)$ is a subgroup. Whence, there exists an inverse element h_1^{-1} in $(\widetilde{H}\bigcap G_i;\times_i)$. We get that

$$x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.$$

That is,

$$x = y \times_j h_2 \times_i h_1^{-1}.$$

Whence,

$$x\widetilde{H} \subseteq y\widetilde{H}.$$

Similarly, we can also get that

$$x\widetilde{H} \supseteq y\widetilde{H}.$$

Therefore, we get that

$$x\widetilde{H} = y\widetilde{H}.$$

Denote the union of two sets A and B by $A\bigoplus B$ if $A\bigcap B = \emptyset$. Then we get the following result by the previous proof.

Theorem 2.3. *For any multi-group subspace $\widetilde{H}$ of a multi-group space $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x\in T} x\widetilde{H}.$$

For the case of finite groups, since there is only one binary operation "$\times$" and $|x\widetilde{H}| = |y\widetilde{H}|$ for any $x, y \in \widetilde{G}$, we get the following corollary, which is just the Lagrange theorem for finite groups.

Corollary 2.1. (Lagrange theorem) *For any finite group G, if H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

For a multi-group space $\widetilde{G}$ and $g \in \widetilde{G}$, denote by $\overrightarrow{O(g)}$ all the binary operations associative with g and by $\widetilde{G}(\times)$ the elements associative with the binary operation "$\times$". For a multi-group subspace $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\times)$, if $\forall h \in \widetilde{H}$,

$$g \times h \times g^{-1} \in \widetilde{H},$$

then call $\widetilde{H}$ a *normal multi-group subspace* of $\widetilde{G}$, denoted by $\widetilde{H} \triangleleft \widetilde{G}$. If $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$, similar to the normal subgroup of a group, it can be shown that $g \times \widetilde{H} = \widetilde{H} \times g$, where $g \in \widetilde{G}(\times)$. We have the following result.

Theorem 2.4. *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a multi-group space with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. Then a multi-group subspace $\widetilde{H}$ of $\widetilde{G}$ is normal if and only if for any integer $i, 1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or $\widetilde{H} \bigcap G_i = \emptyset$.*

Proof. We have known that

$$\widetilde{H} = \bigcup_{i=1}^{n} (\widetilde{H} \bigcap G_i).$$

If for any integer $i, 1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$, then we know that for $\forall g \in G_i, 1 \leq i \leq n$,

$$g \times_i (\widetilde{H} \bigcap G_i) \times_i g^{-1} = \widetilde{H} \bigcap G_i.$$

Whence, for $\forall \circ \in O(\widetilde{H})$ and $\forall g \in \overrightarrow{\widetilde{G}(\circ)}$,

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}.$$

That is, $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$.

Now if $\widetilde{H}$ is a normal multi-group subspace of $\widetilde{G}$, then by definition, we know that for $\forall \circ \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\circ)$,

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}.$$

Not loss of generality, we assume that $\circ = \times_k$, then we get that

$$g \times_k (\widetilde{H} \bigcap G_k) \times_k g^{-1} = \widetilde{H} \bigcap G_k.$$

Therefore, $(\widetilde{H} \bigcap G_k; \times_k)$ is a normal subgroup of $(G_k, \times_k)$. For the operation "$\circ$" is chosen arbitrarily, we know that for any integer i, $1 \leq i \leq n$, $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or an empty set.

For a multi-group space $\widetilde{G}$ with an operation set $O(\widetilde{G}) = \{\times_i |\ 1 \leq i \leq n\}$, an order of operations in $O(\widetilde{G})$ is said an *oriented operation sequence*, denoted by $\overrightarrow{O}(\widetilde{G})$. For example, if $O(\widetilde{G}) = \{\times_1, \times_2\times_3\}$, then $\times_1 \succ \times_2 \succ \times_3$ is an oriented operation sequence and $\times_2 \succ \times_1 \succ \times_3$ is another.

For an oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, we construct a series of normal multi-group subspaces

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_m = \{1_{\times_n}\}$$

by the following programming.

STEP 1: *Construct a series*

$$\widetilde{G} \rhd \widetilde{G}_{11} \rhd \widetilde{G}_{12} \rhd \cdots \rhd \widetilde{G}_{1l_1}$$

under the operation "$\times_1$" .

STEP 2: *If a series*

$$\widetilde{G}_{(k-1)l_1} \rhd \widetilde{G}_{k1} \rhd \widetilde{G}_{k2} \rhd \cdots \rhd \widetilde{G}_{kl_k}$$

has be constructed under the operation "$\times_k$" *and* $\widetilde{G}_{kl_k} \neq \{1_{\times_n}\}$, *then construct a series*

$$\widetilde{G}_{kl_1} \rhd \widetilde{G}_{(k+1)1} \rhd \widetilde{G}_{(k+1)2} \rhd \cdots \rhd \widetilde{G}_{(k+1)l_{k+1}}$$

under the operation "$\times_{k+1}$" .

This programming is terminated until the series

$$\widetilde{G}_{(n-1)l_1} \rhd \widetilde{G}_{n1} \rhd \widetilde{G}_{n2} \rhd \cdots \rhd \widetilde{G}_{nl_n} = \{1_{\times_n}\}$$

has be constructed under the operation "$\times_n$" .

The number m is called the length of the series of normal multi-group subspaces. For a series

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_n = \{1_{\times_n}\}$$

of normal multi-group subspaces, if for any integer $k, s, 1 \leq k \leq n, 1 \leq s \leq l_k$, there exists a normal multi-group subspace $\widetilde{H}$ such that

$$\widetilde{G}_{ks} \rhd \widetilde{H} \rhd \widetilde{G}_{k(s+1)},$$

then $\widetilde{H} = \widetilde{G}_{ks}$ or $\widetilde{H} = \widetilde{G}_{k(s+1)}$, we call this series is *maximal*. For a maximal series of finite normal multi-group subspaces, we have the following result.

Theorem 2.5. *For a finite multi-group space* $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ *and an oriented operation sequence* $\overrightarrow{O}(\widetilde{G})$, *the length of maximal series of normal multi-group subspaces is a constant, only dependent on* $\widetilde{G}$ *itself.*

Proof. The proof is by the induction on the integer n.

For $n = 1$, the maximal series of normal multi-group subspaces is just a composition series of a finite group. By Jordan-Hölder theorem (see [1] or [3]), we know the length of a composition series is a constant, only dependent on $\widetilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume the assertion is true for cases of $n \leq k$. We prove it is also true in the case of $n = k + 1$. Not loss of generality, assume the order of binary operations in $\overrightarrow{O}(\widetilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s = \{1_{\times_1}\}.$$

By Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Theorem 3.6, we know a maximal series of normal multi-group subspace of $\widetilde{G}$ gotten by the STEP 1 under the operation "$\times_1$" is

$$\widetilde{G} \triangleright \widetilde{G} \setminus (G_1 \setminus G_2) \triangleright \widetilde{G} \setminus (G_1 \setminus G_3) \triangleright \cdots \triangleright \widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\}).$$

Notice that $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is still a multi-group space with less or equal to k operations. By the induction assumption, we know the length of its maximal series of normal multi-group subspaces is only dependent on $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$, is a constant. Therefore, the length of a maximal series of normal multi-group subspaces is also a constant, only dependent on $\widetilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal multi-group subspaces of $\widetilde{G}$ is a constant under an oriented operations $\overrightarrow{O}(\widetilde{G})$, only dependent on $\widetilde{G}$ itself.

As a special case, we get the following corollary.

Corollary 2.2. (Jordan-Hölder theorem) *For a finite group G, the length of the composition series is a constant, only dependent on G.*

§3. Open Problems on Multi-group Spaces

Problem 3.1 *Establish a decomposition theory for multi-group spaces.*

In group theory, we know the following decomposition results([1] and [3]) for a group.

Let G be a finite Ω-group. Then G can be uniquely decomposed as a direct product of finite non-decomposition Ω-subgroups.

Each finite Abelian group is a direct product of its Sylow p-subgroups.

Then Problem 3.1 can be restated as follows.

Whether can we establish a decomposition theory for multi-group spaces similar to above two results in group theory, especially, for finite multi-group spaces?

Problem 3.2 *Define the conception of simple multi-group spaces for multi-group spaces. For finite multi-group spaces, whether can we find all simple multi-group spaces?*

For finite groups, we know that there are four simple group classes ([9]):

Class 1: the cyclic groups of prime order;

Class 2: the alternating groups $A_n, n \geq 5$;

Class 3: the 16 groups of Lie types;

Class 4: the 26 sporadic simple groups.

Problem 2.3 *Determine the structure properties of a multi-group space generated by finite elements.*

For a subset A of a multi-group space $\widetilde{G}$, define its spanning set by

$$\langle A\rangle = \{a \circ b | a, b \in A \text{ and } \circ \in O(\widetilde{G})\}.$$

If there exists a subset $A \subset \widetilde{G}$ such that $\widetilde{G} = \langle A\rangle$, then call $\widetilde{G}$ is generated by A. Call $\widetilde{G}$ is *finitely generated* if there exist a finite set A such that $\widetilde{G} = \langle A\rangle$. Then Problem 2.3 can be restated by

Can we establish a finite generated multi-group theory similar to the finite generated group theory?

References

[1] G.Birkhoff and S.Mac Lane, A Survey of Modern Algebra, Macmillan Publishing Co., Inc, 1977.

[2] Daniel Deleanu, A Dictionary of Smarandache Mathematics, Buxton University Press, London, New York, 2004.

[3] Lingzhao Nie and Shishun Ding, Introduction to Algebra, Higher Education Publishing Press, 1994.

[4] L.F.Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries, American Research Press, 2005.

[5] Smarandache F., Mixed noneuclidean geometries, eprint arXiv: math/0010119, 10/2000.

[6] Smarandache F., A Unifying Field in Logics, Neturosophic Probability, Set, and Logic, American research Press, Rehoboth, 1999.

[7] W.B.Vasantha Kandasamy, Bialgebraic structures and Smarandache bialgebraic structures, American Research Press, 2003.

[8] W.B.Vasantha Kandasamy and F.Smarandache, Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock, 2004.

[9] Mingyao Xu, Introduction to Group Theory (I)(II), Science Publish Press, Beijing ,1999.

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The Smarandache P and S persistence of a prime

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In [1], Sloane has defined the multiplicative persistence of a number in the following manner. Let's N be any n-digits number with $N = x_1x_2x_3\cdots x_n$ in base 10. Multiplying together the digits of that number $(x_1 \cdot x_2 \cdot \cdots \cdot x_n)$, another number N' results. If this process is iterated, eventually a single digit number will be produced. The number of steps to reach a single digit number is referred to as the persistence of the original number N. Here is an example:

$$679 \to 378 \to 168 \to 48 \to 32 \to 6.$$

In this case, the persistence of 679 is 5.

Of course, that concept can be extended to any base b. In [1], Sloane conjectured that, in base 10, there is a number c such that no number has persistence greater than c. According to a computer search no number smaller than 10^{50} with persistence greater than 11 has been found. In [2], Hinden defined in a similar way the additive persistence of a number where, instead of multiplication, the addition of the digits of a number is considered. For example, the additive persistence of 679 is equal to 2.

$$679 \to 22 \to 4.$$

Following the same spirit, in this article we introduce two new concepts: the Smarandache P-persistence and the Smarandache S-persistence of a prime number. Let X be any n-digits prime number and suppose that $X = x_1x_2x_3\cdots x_n$ in base 10. If we multiply together the digits of that prime $(x_1 \cdot x_2 \cdot \cdots \cdot x_n)$ and add them to the original prime $(X + x_1 \cdot x_2 \cdot \cdots \cdot x_n)$ a new number results, which may be a prime. If it is a prime then the process will be iterated otherwise not. The number of steps required to X to collapse in a composite number is called the Smarandache P-persistence of prime X. As an example, let's calculate the Smarandache P-persistence of the primes 43 and 23:

$$43 \to 55;$$

$$23 \to 29 \to 47 \to 75,$$

which is 1 and 3, respectively. Of course, the Smarandache P-persistence minus 1 is equal to the number of primes that we can generate starting with the original prime X. Before proceeding, we must highlight that there will be a class of primes with an infinite Smarandache P-persistence; that is, primes that will never collapse in a composite number. Let's give an

example:

$$61 \to 67 \to 109 \to 109 \to 109 \cdots .$$

In this case, being the product of the digits of the prime 109 always zero, the prime 61 will never reach a composite number. In this article, we shall not consider that class of primes since it is not interesting. The following table gives the smallest multidigit primes with Smarandache P-persistence less than or equal to 8:

Smarandache P-persistence	Prime
1	11
2	29
3	23
4	347
5	293
6	239
7	57487
8	486193

By looking in a greater detail at the above table, we can see that, for example, the second term of the sequence (29) is implicitly inside the chain generated by the prime 23. In fact:

$$29 \to 47 \to 75$$

$$23 \to 29 \to 47 \to 75$$

We can slightly modify the above table in order to avoid any prime that implicitly is inside other terms of the sequence.

Smarandache P-persistence	Prime
1	11
2	163
3	23
4	563
5	1451
6	239
7	57487
8	486193

Now, for example, the prime 163 will generate a chain that isn't already inside any other chain generated by the primes listed in the above table. What about primes with Smarandache P-persistence greater than 8? Is the above sequence infinite? We will try to give an answer

to the above question by using a statistical approach. Let's indicate with L the Smarandache P-persistence of a prime. Thanks to an u-basic code the occurrrencies of L for different values of N have been calculated. Here an example for $N = 10^7$ and $N = 10^8$:

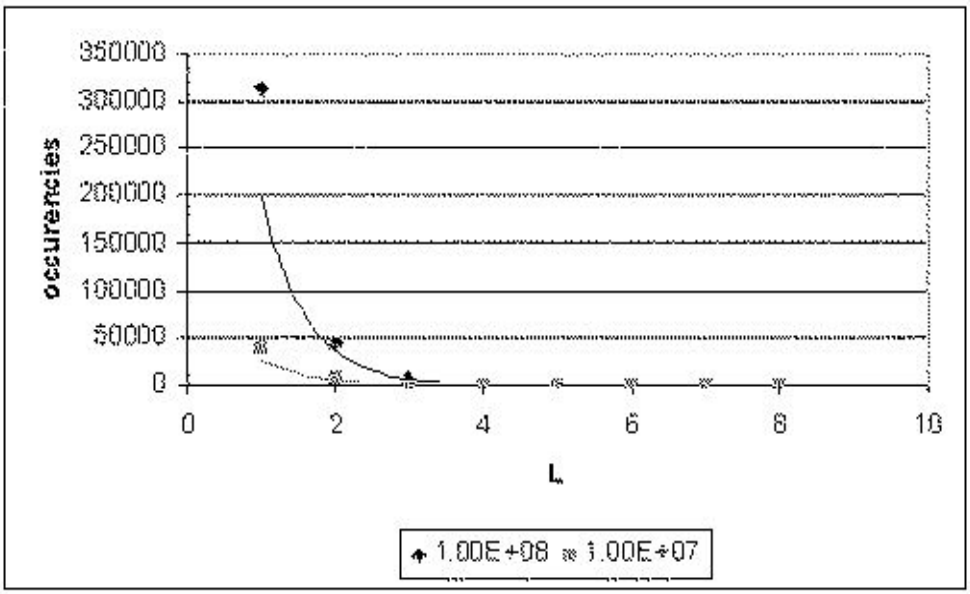

Figure 1. Plot of the occurrencies for each P-persistence at two different values of N.

The interpolating function for that family of curves is given by:

$$a(N) \cdot e^{-b(N) \cdot L}$$

where $a(n)$ and $b(n)$ are two function of N. To determine the behaviour of those two functions, the values obtained interpolating the histogram of occurencies for different N have been used:

N	a	b
$1.00E+04$	2238.8	1.3131
$1.00E+05$	17408	1.4329
$1.00E+06$	121216	1.5339
$1.00E+07$	$1.00E+06$	1.6991
$1.00E+08$	$1.00E+07$	1.968

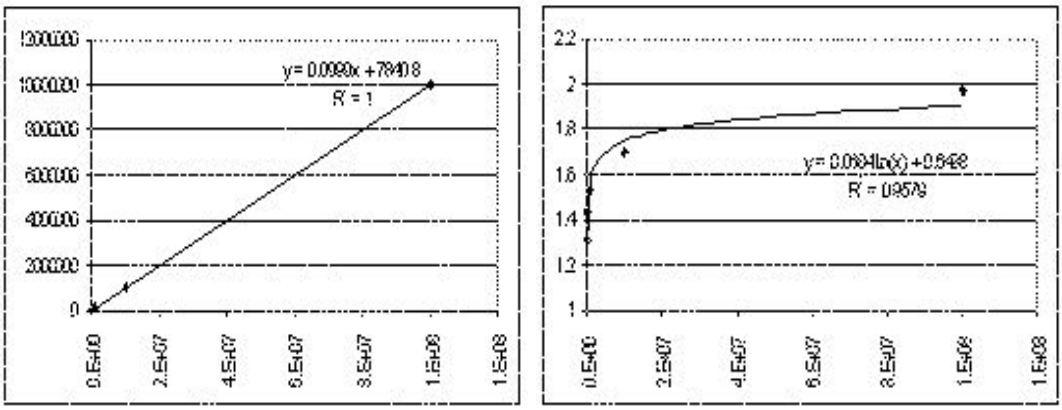

Figure 2. Plot of the two functions $a(N)$ and $b(N)$ versus N

According to those data we can see that :

$$a(N) \approx k \cdot N \qquad b(N) \approx h \cdot \ln(N) + c$$

where k, h and c are constants (see Figure 2).

So the probability that $L \geq M$ (where M is any integer) for a fixed N is given by:

$$P(L \geq M) \approx \frac{\int_M^\infty kN \cdot e^{-(h \ln N + c) \cdot L} dL}{\int_0^\infty kN \cdot e^{-(h \ln N + c) \cdot L} dL} = e^{-(h \cdot \ln N + c) \cdot M}$$

and the counting function of the primes with Smarandache P-persistence $L = M$ below N is given by $N \cdot P(L = M)$. In Figure 3, the plot of counting function versus N for 4 different L values is reported. As we can see, for $L < 15$ and $L \geq 15$ there is a breaking in the behaviour of the occurrencies. For $L \geq 15$, the number of primes is very very small (less than 1) regardless the value of N and it becomes even smaller as N increases. The experimental data seem to support that L cannot take any value and that most likely the maximum value should be $L = 14$ or close to it. So the following conjecture can be posed:

Conjecture 1. There is an integer M such that no prime has a Smarandache P-persistence greater than M. In other words the maximum value of Smarandache P-persistence is finite

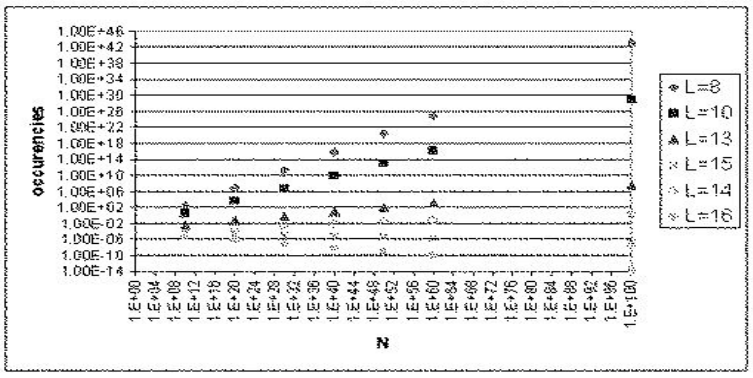

Figure 3. Counting function for the P-persistence for difference values of N

Following a similar argumentation the Smarandache S-persistence of a prime can be defined. In particular it is the number of steps before a prime number collapse to a composite number considering the sum of the digits instead of the product as done above. For example let's calculate the Smarandache S-persistence of the prime 277:

$$277 \to 293 \to 307 \to 317 \to 328.$$

In this case we have a Smarandache S-persistence equal to 4. The sequence of the smallest multidigit prime with Smarandache S-persistence equal to $1, 2, 3, 4 \cdots$ has been found by Rivera [3]. Anyway no prime has been found with the Smarandache S-persistence greater than 8 up to $N = 18038439735$. Moreover by following the same statistical approach used above for the Smarandache P-persistence the author has found a result similar to that obtained for the Smarandache P-persistence(see [3] for details). Since the statistical approach applied to the Smarandache P and S persistence gives the same result (counting function always smaller than 1 for $L \geq 15$) we can be confident enough to pose the following conjecture:

Conjecture 2. The maximum value of the Smarandache P and S persistence is the same.

References

[1] N.Sloane, The persistence of a number, Recr. Math. J., Spring, **2**(1973).

[2] H.J.Hinden, The Additive Persistence of a Number, Recr. Math. J., **7**(1974), 134-135.

[3] C.Rivera, Puzzle 163: P+SOD(P), http://www.primepuzzles.net/puzzles/puzz-163.htm.

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On the solutions of an equation involving the Smarandache function

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Abstract Let n be any positive integer, the Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!\}$. In this paper, we discussed the solutions of the following equation involving the Smarandache function: $S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k)$, and proved that the equation has infinity positive integer solutions.

Keywords Smarandache function, equation, positive integer solutions.

§1. Introduction

For any positive integer n, the Smarandache function $S(n)$ is defined as follows:

$$S(n) = \min\{m : n|m!\}.$$

From this definition we know that $S(n) = \max\limits_{1\le i\le r}\{S(p_i^{\alpha_i})\}$, if n has the prime powers factorization: $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Of course, this function has many arithmetical properties, and they are studied by many people (see references [1], [4] and [5]).

In this paper, we shall use the elementary methods to study the solvability of the equation

$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k),$$

and prove that it has infinity positive integer solutions for any positive integer k. That is, we shall prove the following main conclusion:

Theorem. For any integer $k \ge 1$, the equation

$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k) \tag{1}$$

has infinity positive integer solutions.

§2. Proof of the theorem

In this section, we shall give the proof of the theorem in two ways, the first proof of the theorem is based on the following:

Lemma 1. For any positive integer m, there exist positive integers $a_1^{(m)}, a_2^{(m)}, \cdots, a_m^{(m)}$ which are independent of x, satisfying

$$x^m = (x-1)(x-2)\cdots(x-m) + \sum_{l=1}^{m-1} a_l^{(m)}(x-1)(x-2)\cdots(x-m+l) + a_m^{(m)}, \tag{2}$$

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where x is an arbitrary real number.

Proof. We use induction to prove this Lemma. It is clear that the lemma holds if $m = 1$. That is, $x = (x-1)+1$ holds for any real number x, so we get

$$a_1^{(1)} = 1.$$

Now we assume that the lemma holds for $m = k$ $(k \geq 1)$, then for $m = k+1$, we have

$$\begin{aligned}
x^{k+1} &= x(x-1)(x-2)\cdots(x-k) + \sum_{l=1}^{k-1} a_l^{(k)} x(x-1)(x-2)\cdots(x-k+l) + a_k^{(k)} x \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1)(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-1} a_l^{(k)} (x-1)(x-2)\cdots(x-k+l)(x-k+l-1) + \\
&\quad + \sum_{l=1}^{k-1} a_l^{(k)} (k-l+1)(x-1)(x-2)\cdots(x-k+l) + a_k^{(k)}(x-1) + a_k^{(k)} \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1+a_1^{(k)})(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-2} a_{l+1}^{(k)} (x-1)(x-2)\cdots(x-k+l) + \\
&\quad + \sum_{l=1}^{k-2} a_l^{(k)} (k-l+1)(x-1)(x-2)\cdots(x-k+l) + (2a_{k-1}^{(k)} + a_k^{(k)})(x-1) + a_k^{(k)} \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1+a_1^{(k)})(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-2} (a_{l+1}^{(k)} + a_l^{(k)}(k-l+1))(x-1)(x-2)\cdots(x-k+l) + \\
&\quad + (2a_{k-1}^{(k)} + a_k^{(k)})(x-1) + a_k^{(k)}
\end{aligned}$$

so we can take

$$a_1^{(k+1)} = k+1+a_1^{(k)}, \tag{3}$$

$$a_l^{(k+1)} = a_l^{(k)} + a_{l-1}^{(k)}(k-l+2), \ (2 \leq l \leq k), \tag{4}$$

$$a_{k+1}^{(k+1)} = a_k^{(k)}, \tag{5}$$

and it is obvious from the inductive assumption and (3), (4), (5) that $a_1^{(k+1)}, a_2^{(k+1)}, \cdots, a_{k+1}^{(k+1)}$ are positive integers which are independent of x, and so the lemma holds for $m = k+1$. This completes the proof of Lemma 1.

Now we complete the proof of the theorem. From Lemma 1 we know that for any positive integer k, there exist positive integers $a_1, a_2, \cdots, a_{k-1}$ such that

$$p^{k-1} = (p-1)(p-2)\cdots(p-k+1) + \sum_{l=1}^{k-2} a_l (p-1)(p-2)\cdots(p-k+l+1) + a_{k-1}.$$

Hence

$$p^k = p(p-1)(p-2)\cdots(p-k+1) + \sum_{l=1}^{k-2} a_l p(p-1)(p-2)\cdots(p-k+l+1) + a_{k-1}p. \tag{6}$$

Note that $a_1, a_2, \cdots, a_{k-1}$ are independent of p and p is a prime large enough, from the definition of $S(n)$ we have

$$S(p^k) = kp,$$

$$S(p(p-1)(p-2)\cdots(p-k+1)) = p,$$

$$S(a_l p(p-1)(p-2)\cdots(p-k+l+1)) = p,\ (1 \le l \le k-2)$$

$$S(a_{k-1}p) = p.$$

From these equations and (6) we know that $m_1 = p(p-1)(p-2)\cdots(p-k+1)$, $m_{l+1} = a_l p(p-1)(p-2)\cdots(p-k+l+1)$ $(1 \le l \le k-2)$, $m_k = a_{k-1}p$ is a solution of (1), and (1) has infinity positive integer solutions because p is arbitrary.

The second proof of the theorem is based on the Vinogradov's three-primes theorem which we describle as the following:

Lemma 2. Every odd integer bigger than c can be expressed as sum of three odd primes, where c is a constant large enough.

Proof. (see §20.2 and §20.3 of [2]).

Lemma 3. Let odd integer $k \ge 3$, then any sufficiently large odd integer n can be expressed as sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k. \tag{7}$$

Proof. We will prove this lemma by induction. From Lemma 2 we know that it is true for $k = 3$. If it is true for odd integer k, then we will prove that it is also true for $k+2$. In fact, from Lemma 2 we know that every sufficient large odd integer n can be expressed as

$$n = p^{(1)} + p^{(2)} + p^{(3)},$$

and we can assume that $p^{(1)}$ is also sufficiently large and then satisfying

$$p^{(1)} = p_1 + p_2 + \cdots + p_k,$$

so we have

$$n = p_1 + p_2 + \cdots + p_k + p^{(2)} + p^{(3)}.$$

This means that n can be expressed as sum of $k+2$ odd primes, and Lemma 3 follows from the induction.

Now we give the second proof of the theorem. From Lemma 3 we know that for any odd integer $k \ge 3$, every sufficient large prime p can be expressed as

$$p = p_1 + p_2 + \cdots + p_k.$$

So we have

$$S(p) = S(p_1) + S(p_2) + \cdots + S(p_k).$$

This means that the theorem is true for odd integer $k \ge 3$.

If $k \ge 4$ is even, then for every sufficiently large prime p, $p-2$ is odd, and by Lemma 3, we have

$$p - 2 = p_1 + p_2 + \cdots + p_{k-1},$$

so

$$p = 2 + p_1 + p_2 + \cdots + p_{k-1},$$

or

$$S(p) = S(2) + S(p_1) + S(p_2) + \cdots + S(p_{k-1}).$$

This means that the theorem is true for even integer $k \geq 4$.

At last, for any prime $p \geq 3$, we have

$$S(p^2) = S(p^2 - p) + S(p),$$

so the theorem is also true for $k = 2$.

This completes the second proof of Theorem.

References

[1] Li Hailong and Zhao Xiaopeng, On the Smarandache function and the k-th roots of a positive integer, Research on Smarandache Problems in Number Theory, Hexis, 2004, 119-122.

[2] Pan Chengdong and Pan Chengbiao, Element of the analytic number theory, Science Press, Beijing, 1997.

[3] F. Smarandache, Only Problems, Not Solutions, Xiquan Publ. House, Chicago, 1993.

[4] Wang Yongxing, On the Smarandache function, Research on Smarandache Problems in Number Theory (Vol, II), Hexis, 2005, 103-106.

[5] Yang Cundian and Liu Duansen, On the mean value of a new arithmetical function, Research on Smarandache Problems in Number Theory (Vol, II), Hexis, 2005, 75-77.

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A Random Distribution Experiment

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§1. Introduction

In one of my many experiments with numbers I found an interesting property which I will describe in everyday terms before going into a more formal analysis. Imagine a circular putting green on a golf course. A golfer wants to practice putting from the edge of the green. He therefore drops a large number of golf balls on the very edge of the green. He then stands on the edge of the green and is struck by the thought What might be the average distance from here to all these golf balls? We measure with the radius of the green as unit and consider the diameter of a golf ball as negligible. The amazing result is that: the average distance is $A \approx \frac{4}{\pi}$. Of course, this result was not obtained by experimentation but through formal treatment of a related problem [1]. But its similarity with the famous Buffon's needle experiment [2] makes it interesting to compare the estimation of obtained by simulating the two experiments. This will be done at the end of the article.

§2. A problem and its solution

From a fixed point $A(a, 0)$ of a circle C, what is the average distance of all points on the circumference of C?

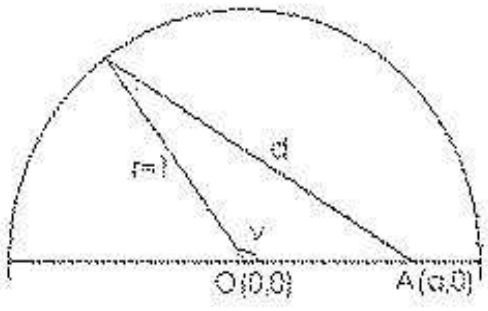

Figure 1.

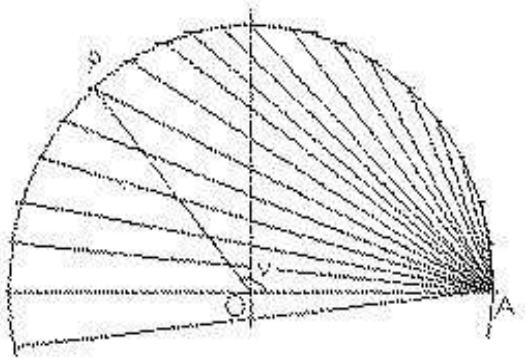

Figure 2.

With notations in figure 1 we have (with $r = 1$)

$$d = (a^2 + 1 - 2a\cos\gamma)^{\frac{1}{2}}. \tag{1}$$

To begin with let's consider discrete points on the periphery by dividing the circumference into n equal arcs, each occupying the angle $\alpha = \frac{2\pi}{n}$. Let d_k be the kth arc.

$$d_k = \left(a^2 + 1 - 2a\cos\frac{2k\pi}{n}\right)^{\frac{1}{2}}. \tag{2}$$

For the average distance we have

$$A_n = \frac{1}{n}\sum_{k=1}^{n} d_k. \tag{3}$$

The result of implementing (3) for $a = \{0, 0.1, \cdots, 0.9, 1.0\}$ with $n = 1024$ is shown in table 1. The convergence of the average with increasing values of n for $a = 1$ corresponding to figure 2 is shown in table 2 and figure 3. As is seen A_n converges rapidly. Five correct decimals are obtained already for $n = 1024$.

a	0.00	0.10	0.20	0.30	0.40	0.50
A_{1024}	1.00000	1.00250	1.01003	1.02263	1.04042	1.06354

a	0.60	0.70	0.80	0.90	1.00
A_{1024}	1.09224	1.12683	1.16781	1.21600	1.27324

Table 1.

n	4	8	16	32	64	128	256	1024	2048
A_n	1.20711	1.25683	1.26915	1.27222	1.27298	1.27318	1.27322	1.27324	1.27324

Table 2.

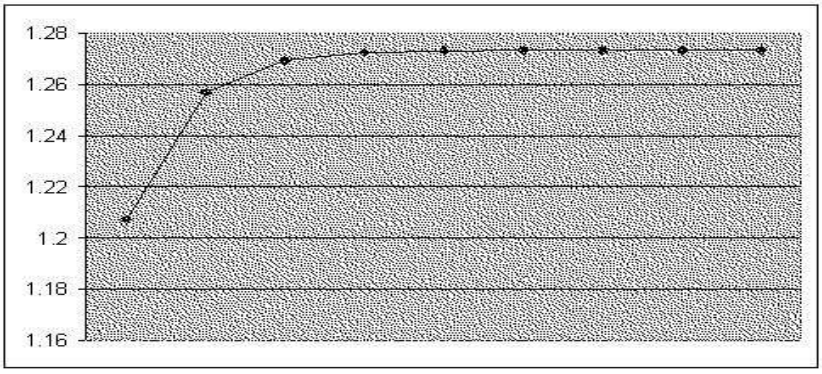

Figure 3. Average distances corresponding to $n = 4, 8, 16, 32, \cdots, 2048$.

For a=1, which corresponds to figure 2, formula (2) takes the form

$$d_k = 2\sin\frac{k\pi}{n}. \tag{4}$$

Consequently, we have the following expression for the average

$$A_n = \frac{2}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n}. \tag{5}$$

We can now leave our discrete points and obtain a general expression for the average $\overline{A}$

$$\overline{A} = \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n} \tag{6}$$

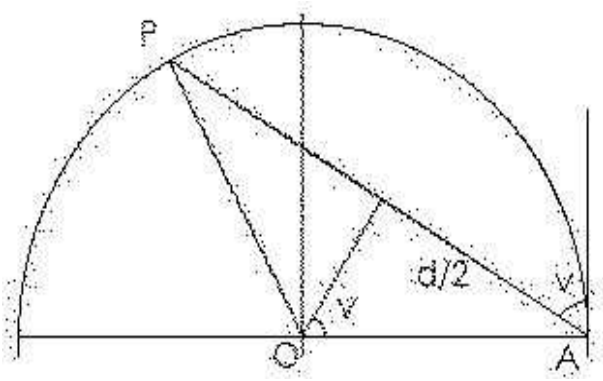

Figure 4.

As P moves along the periphery of the semi-circle in figure 4 the angle v moves from 0 to π. If we choose v randomly in the interval $0 \le v \le \pi$ what value do we expect for $AP = d$? This expectation value E corresponds to the classical average which deals with a discrete random variable. In our case we have $d = 2\sin v$. E is calculated from

$$E = \frac{2\int_0^\pi d\sin v dv}{\int_0^\pi dv} = \frac{4}{\pi} \approx 1.2732. \tag{7}$$

From (6) and (7) we now obtain the interesting result

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n} = \frac{2}{\pi}. \tag{8}$$

This result has been obtained through a geometric consideration and a simple integration. It would be interesting to compare this with a proof of this formula by analytical means.

§3. Analytical proof of (8)

Consider the complex expression

$$S = \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)$$

to which we apply Euler's formula

$$S = \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} e^{\frac{ik\pi}{n}}. \tag{9}$$

Adding the geometric series we get

$$S = \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) = \frac{e^{\frac{i\pi}{n}}(e^{i\pi}-1)}{n(e^{\frac{i\pi}{n}}-1)} = \frac{2}{n(e^{\frac{-i\pi}{n}}-1)} \tag{10}$$

It remains to consider $\lim_{n\to\infty} n(e^{\frac{-i\pi}{n}}-1)$. We apply Euler's formula again.

$$\lim_{n\to\infty} n(e^{\frac{-i\pi}{n}}-1) = \lim_{n\to\infty} n\left(\cos\frac{\pi}{n} - 1 - i\sin\frac{\pi}{n}\right) \tag{11}$$

We only need to the first two terms of each power series of the trigonometric functions to obtain our result

$$\lim_{n\to\infty} n(e^{\frac{-i\pi}{n}}-1) = \lim_{n\to\infty} n\left(1 - \frac{\pi^2}{n^2 2!} + \cdots - 1 - i\left(\frac{\pi}{n} - \frac{\pi^3}{n^3 3!} + \cdots\right)\right) = 0 - i\pi.$$

Inserting this and taking the limit as $n \to \infty$ in (10) we get

$$\lim_{n\to\infty} S = \frac{2}{0 - i\pi} = \frac{2(0+i\pi)}{0 - i^2\pi^2} = \frac{2i}{\pi} \tag{12}$$

and as it is the imaginary part we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n} = \frac{2}{\pi}. \tag{8}$$

Comparing this with the simple way in which the result was found earlier it's like using a sledge hammer to kill a mosquito.

§4. Simulation experiments

A frame work in which we can compare simulation experiments for Buffon's needle experiment, which will be described below, and our golf ball experiments will be set up. I will henceforth refer to the two cases as Needles and Golf Balls respectively. The same random number generator is used and randomized in the same way in the two experiments. Programs are written in UBASIC. Ten experiments were carried out in each case. In each experiment 1000000 needles respectively 1000000 golf balls were dropped.

4.1 Buffon's needle experiment and simulation of π

A needle of length L is tossed at random onto a plane ruled with parallel lines a distance d apart. $L \leq d$. If the needle is tossed a sufficiently large number of times then the ratio A between the number of times the needle intersects a line and the total number of tosses will be described by

$$A = \frac{2L}{\pi d} \qquad \text{or} \qquad \pi = \frac{2L}{Ad} \tag{12}$$

Refer to figure 5. The problem is invariant on horizontal translations $P \to Q$ of the type $x' = x \pm nd$, where d is the distance between the lines parallel to the Y-axis and n is any integer. The problem is also invariant on vertical translations $(Q \to R)$ in the system OXY. Without affecting the problem we can therefore perform the translations $P \to Q \to R(r, 0)$ so that $0 \le r < d$ with conservation of the angle v which lies in the interval $-\pi/2 \le v < \pi/2$. No restriction is imposed on the problem by putting the length of the needle $L = 1$ and consequently $d \ge 1$.

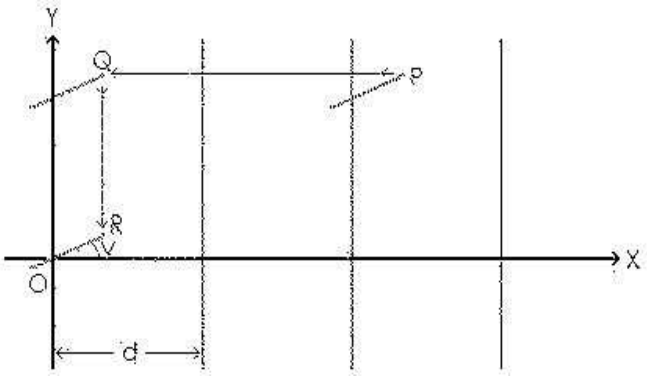

Figure 5.

The needle intersects the Y-axis iff

$$r \le \delta \cos v. \tag{13}$$

Our random variables are r and v. For each toss of the needle we have $r = d \times rnd$ and $v = \pi \times rnd - \frac{\pi}{2}$ where rnd are computer generated random numbers in the interval $0 \le rnd < 1$. In the program below the number of times M the needles intersects the $Y - axis$ is counted for N tosses $A = M/N$ is calculated and π is calculated from (12). The approximate value for π is denoted G in the $UBASIC$ program listed below.

```
10    'simbuf, 040821
20    open "simbuf.dat" for create as #1
30    D = 1.2 : L = 1 : N = 1000000
40    for J = 1 to 10
50    pause:randomize
60    M = 0
70    for I = 1 to N
80    V = pi(1) * rnd - pi(0.5)
90    X = D * rnd
100    if X < L * cos(V) then inc M
110    next
120    A = M/N
130    G = 2 * L/(D * A)
140    print using(10,5), G:print #1 using(10,5), G
150    H = H + G
160    next
```

```
170     P = H/10
180     print:print using(10, 5), P; pi(1); P − pi(1)
190     print #1:print #1 using(10, 5), P; pi(1); P − pi(1)
200     close #1:end
```

This program was implemented 10 times with the following result:

3.1465, 3.1382, 3.1438, 3.1452, 3.1444, 3.1450, 3.1397, 3.1424, 3.1412 and 3.1440.

The average value 3.1430 is an approximation of π which differs with only 0.0014 from the value of π.

4.2 The golf ball experiment and simulation of π

From figure 4 we have $d = 2sinv$. Calculation is made for a random distribution of 1000000 balls. The estimated value for π is denoted G in the self-explanatory $UBASIC$ program listed below.

```
10      'simgolf, 040821
20      open "simgolf.dat" for create as #1
30      M = 10
40      for J = 1 to M
50      pause:S = 0
60      randomize
70      N = 1000000
80      for I = 1 to N
90      V = 2 ∗ pi(1) ∗ rnd
100      if S = S + 2 ∗ sin(V/2)
110      next
120      Ave = S/N
130      G = 4/Ave
140      print #1 using(10, 5), G
150      print using(10, 5), G
160      H = H + G
170      next
180      P = H/M
190      print:print using(10, 5), P; pi(1); P − pi(1)
200      print #1:print #1 using(10, 5), P; pi(1); P − pi(1)
210      end
```

I ran this program 10 times with the following results:

3.1431, 3.1421, 3.1429, 3.1412, 3.1422, 3.1413, 3.1422, 3.1399, 3.1404 and 3.1407.

The average value 3.14162 is correct to 4 decimals, fairly good approximation for π. The difference from the true value (to 5 decimals) is only 0.00003.

§5. Needles and Golf Balls - Comparison

The simulation programs used earlier were adapted for multiple runs, 100 in each case. Each run consists of tossing the needle and the golf ball 10000 times each. The average and

standard deviation were calculated in each case. In addition the frequency of simulation values of π for each interval of length 0.01 was represented in staple diagrams in figures 6 and 7.

Figure 6 shows the result for needles: Average simulation result for $\pi \approx 3.1435$. Standard deviation = 0.0287. The difference between the simulation result and π (to four decimals) is 0.0019.

Figure 7 shows the result for golf balls: Average simulation result for $\pi \approx 3.1413$. Standard deviation = 0.0148. The difference between the simulation result and π (to four decimals) is 0.0003.

At first sight it is surprising that d in (12) can be chosen arbitrarily as long as it is not smaller that L. The explanation is that the larger we choose d the larger will be the standard deviation, i.e. the trend curve in figure 6 will be flattened and a very large number of tosses will be required before "the needle shows its preference for π". The simulation program was executed for $d = 1.15$ which by experimentation was found a reasonable choice.

The golf ball experiment is much better behaved. For the same number of tosses the average is much closer to π with a trend curve that is closer to the average.

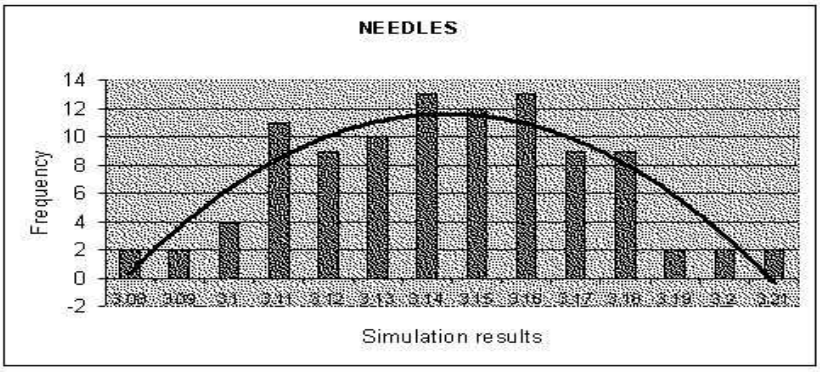

Figure 6.

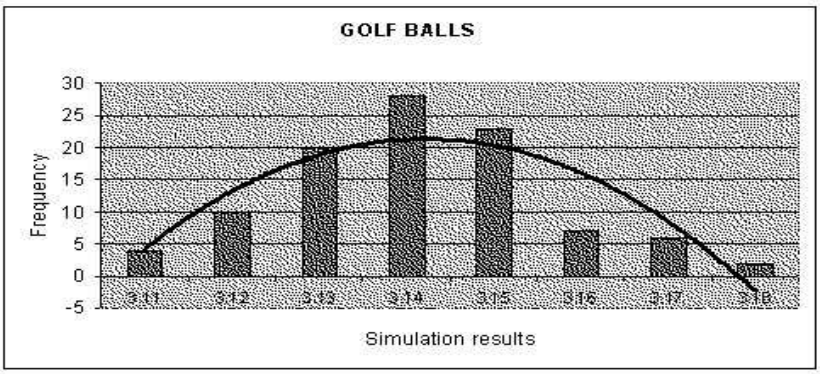

Figure 7.

References

[1] Probability Theory, Y. A. Rozanov, Dover Publications, Inc. New York, 1969.

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On Multi-Metric Spaces

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Abstract A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining Smarandache multi-spaces with classical metric spaces, the conception of multi-metric spaces is introduced. Some characteristics of multi-metric spaces are obtained and the Banach's fixed-point theorem is generalized in this paper.

Keywords metric; multi-space; multi-metric space; Banach theorem.

§1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field* ([7]), which is defined as follows.

Definition 1.1 *For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

As we known, a set M associative a function $\rho : M \times M \to R^+ = \{x \mid x \in R, x \geq 0\}$ is called a *metric space* if for $\forall x, y, z \in M$, the following conditions for the metric function ρ hold:

(1) (*definiteness*) $\rho(x, y) = 0$ if and only if $x = y$;
(*ii*) (*symmetry*) $\rho(x, y) = \rho(y, x)$;
(*iii*) (*triangle inequality*) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

By combining Smarandache multi-spaces with classical metric spaces, a new kind of spaces called multi-metric spaces is found, which is defined in the following.

Definition 1.2 *A multi-metric space is a union $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ such that each M_i is a space with metric ρ_i for $\forall i, 1 \leq i \leq m$.*

When we say *a multi-metric space* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, it means that a multi-metric space with metrics $\rho_1, \rho_2, \cdots, \rho_m$ such that (M_i, ρ_i) is a metric space for any integer $i, 1 \leq i \leq m$. For a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, $x \in \widetilde{M}$ and a positive number R, a *R-disk* $B(x, R)$ in $\widetilde{M}$ is defined by

$$B(x,R) = \{\ y \mid \text{there exists an integer } k, 1 \leq k \leq m \text{ such that } \rho_k(y,x)\ < R, y \in \widetilde{M}\}$$

The main purpose of this paper is to find some characteristics of multi-metric spaces. For terminology and notations not defined here can be seen in [1] – [2], [4] for terminologies in the metric space and in [3], [5] – [9] for multi-spaces and logics.

§2. Characteristics of multi-metric spaces

For metrics on spaces, we have the following result.

Theorem 2.1. *Let $\rho_1, \rho_2, \cdots, \rho_m$ be m metrics on a space M and F a function on $\mathbf{E}^m$ such that the following conditions hold:*

(i) $F(x_1, x_2, \cdots, x_m) \geq F(y_1, y_2, \cdots, y_m)$ *if for* $\forall i, 1 \leq i \leq m$, $x_i \geq y_i$;

(ii) $F(x_1, x_2, \cdots, x_m) = 0$ *only if* $x_1 = x_2 = \cdots = x_m = 0$;

(iii) For two m-tuples $(x_1, x_2, \cdots, x_m)$ *and* $(y_1, y_2, \cdots, y_m)$,

$$F(x_1, x_2, \cdots, x_m) + F(y_1, y_2, \cdots, y_m) \geq F(x_1 + y_1, x_2 + y_2, \cdots, x_m + y_m).$$

Then $F(\rho_1, \rho_2, \cdots, \rho_m)$ is also a metric on M.

Proof. We only need to prove that $F(\rho_1, \rho_2, \cdots, \rho_m)$ satisfies the metric conditions for $\forall x, y, z \in M$.

By *(ii)*, $F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = 0$ only if for any integer i, $\rho_i(x,y) = 0$. Since ρ_i is a metric on M, we know that $x = y$.

For any integer $i, 1 \leq i \leq m$, since ρ_i is a metric on M, we know that $\rho_i(x,y) = \rho_i(y,x)$. Whence,

$$F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = F(\rho_1(y,x), \rho_2(y,x), \cdots, \rho_m(y,x)).$$

Now by *(i)* and *(iii)*, we get that

$$\begin{aligned}
&F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) + F(\rho_1(y,z), \rho_2(y,z), \cdots, \rho_m(y,z)) \\
&\geq F(\rho_1(x,y) + \rho_1(y,z), \rho_2(x,y) + \rho_2(y,z), \cdots, \rho_m(x,y) + \rho_m(y,z)) \\
&\geq F(\rho_1(x,z), \rho_2(x,z), \cdots, \rho_m(x,z)).
\end{aligned}$$

Therefore, $F(\rho_1, \rho_2, \cdots, \rho_m)$ is a metric on M.

Corollary 2.1. If $\rho_1, \rho_2, \cdots, \rho_m$ are m metrics on a space M, then $\rho_1 + \rho_2 + \cdots + \rho_m$ and $\frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + \cdots + \frac{\rho_m}{1+\rho_m}$ are also metrics on M.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is said *convergence to a point* $x, x \in \widetilde{M}$ if for any positive number $\epsilon > 0$, there exist numbers N and $i, 1 \leq i \leq m$ such that if $n \geq N$ then

$$\rho_i(x_n, x)\ <\ \epsilon.$$

If $\{x_n\}$ convergence to a point $x, x \in \widetilde{M}$, we denote it by $\lim\limits_n x_n = x$.

We have a characteristic for convergent sequences in a multi-metric space.

Theorem 2.2. *A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if there exist integers N and $k, 1 \leq k \leq m$ such that the subsequence $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k).*

Proof. If there exist integers N and $k, 1 \leq k \leq m$, such that $\{x_n | n \geq N\}$ is a convergent subsequence in (M_k, ρ_k), then for any positive number $\epsilon > 0$, by definition there exists a positive integer P and a point $x, x \in M_k$ such that

$$\rho_k(x_n, x) \ < \ \epsilon$$

if $n \geq \max\{N, \ P\}$.

Now if $\{x_n\}$ is a convergent sequence in the multi-space $\widetilde{M}$, by definition for any positive number $\epsilon > 0$, there exist a point $x, x \in \widetilde{M}$ and natural numbers $N(\epsilon)$ and $k, 1 \leq k \leq m$ such that if $n \geq N(\epsilon)$, then

$$\rho_k(x_n, x) \ < \ \epsilon,$$

that is, $\{x_n | n \geq N(\epsilon)\} \subset M_k$ and $\{x_n | n \geq N(\epsilon)\}$ is a convergent sequence in (M_k, ρ_k).

Theorem 2.3. *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a multi-metric space. For two sequences $\{x_n\}, \{y_n\}$ in $\widetilde{M}$, if $\lim\limits_n x_n = x_0$, $\lim\limits_n y_n = y_0$ and there is an integer p such that $x_0, y_0 \in M_p$, then $\lim\limits_n \rho_p(x_n, y_n) = \rho_p(x_0, y_0)$.*

Proof. According to Theorem 2.2, there exist integers N_1 and N_2 such that if $n \geq max\{N_1, N_2\}$, then $x_n, y_n \in M_p$. Whence, we have that

$$\rho_p(x_n, y_n) \leq \rho_p(x_n, x_0) + \rho_p(x_0, y_0) + \rho_p(y_n, y_0)$$

and

$$\rho_p(x_0, y_0) \leq \rho_p(x_n, x_0) + \rho_p(x_n, y_n) + \rho_p(y_n, y_0).$$

Therefore,

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ \leq \ \rho_p(x_n, x_0) + \rho_p(y_n, y_0).$$

For any positive number $\epsilon > 0$, since $\lim\limits_n x_n = x_0$ and $\lim\limits_n y_n = y_0$, there exist numbers $N_1(\epsilon), N_1(\epsilon) \geq N_1$ and $N_2(\epsilon), N_2(\epsilon) \geq N_2$ such that $\rho_p(x_n, x_0) \leq \frac{\epsilon}{2}$ if $n \geq N_1(\epsilon)$ and $\rho_p(y_n, y_0) \leq \frac{\epsilon}{2}$ if $n \geq N_2(\epsilon)$. Whence, if $n \geq max\{N_1(\epsilon), N_2(\epsilon)\}$, then

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ < \ \epsilon.$$

Whether a convergent sequence can has more than one limit point? The following result answers this question.

Theorem 2.4. *If $\{x_n\}$ is a convergent sequence in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, then $\{x_n\}$ has only one limit point.*

Proof. According to Theorem 2.2, there exist integers N and $i, 1 \leq i \leq m$ such that $x_n \in M_i$ if $n \geq N$. Now if

$$\lim_n x_n = x_1 \text{ and } \lim_n x_n = x_2,$$

and $n \geq N$, by definition,

$$0 \leq \rho_i(x_1, x_2) \leq \rho_i(x_n, x_1) + \rho_i(x_n, x_2).$$

Whence, we get that $\rho_i(x_1, x_2) = 0$. Therefore, $x_1 = x_2$.

Theorem 2.5. *Any convergent sequence in a multi-metric space is a bounded points set.*

Proof. According to Theorem 2.4, we obtain this result immediately.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is called a *Cauchy sequence* if for any positive number $\epsilon > 0$, there exist integers $N(\epsilon)$ and $s, 1 \leq s \leq m$ such that for any integers $m, n \geq N(\epsilon)$, $\rho_s(x_m, x_n) < \epsilon$.

Theorem 2.6. *A Cauchy sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is convergent if and only if for $\forall k, 1 \leq k \leq m$, $|\{x_n\} \bigcap M_k|$ is finite or infinite but $\{x_n\} \bigcap M_k$ is convergent in (M_k, ρ_k).*

Proof. The necessity of these conditions is by Theorem 2.2.

Now we prove the sufficiency. By definition, there exist integers $s, 1 \leq s \leq m$ and N_1 such that $x_n \in M_s$ if $n \geq N_1$. Whence, if $|\{x_n\} \bigcap M_k|$ is infinite and $\lim_n \{x_n\} \bigcap M_k = x$, then there must be $k = s$. Denoted by $\{x_n\} \bigcap M_k = \{x_{k1}, x_{k2}, \cdots, x_{kn}, \cdots\}$.

For any positive number $\epsilon > 0$, there exists an integer $N_2, N_2 \geq N_1$ such that $\rho_k(x_m, x_n) < \frac{\epsilon}{2}$ and $\rho_k(x_{kn}, x) < \frac{\epsilon}{2}$ if $m, n \geq N_2$. According to Theorem 4.7, we get that

$$\rho_k(x_n, x) \leq \rho_k(x_n, x_{kn}) + \rho_k(x_{kn}, x) < \epsilon$$

if $n \geq N_2$. Whence, $\lim_n x_n = x$.

A multi-metric space $\widetilde{M}$ is said *completed* if every Cauchy sequence in this space is convergent. For a completed multi-metric space, we obtain two important results similar to the metric space theory in classical mathematics.

Theorem 2.7. *Let $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ be a completed multi-metric space. For a ϵ-disk sequence $\{B(\epsilon_n, x_n)\}$, where $\epsilon_n > 0$ for $n = 1, 2, 3, \cdots$, the following conditions hold:*

(i) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(ii) $\lim_{n \to +\infty} \epsilon_n = 0$.

Then $\bigcap_{n=1}^{+\infty} B(\epsilon_n, x_n)$ only has one point.

Proof. First, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. By the condition (i), we know that if $m \geq n$, then $x_m \in B(\epsilon_m, x_m) \subset B(\epsilon_n, x_n)$. Whence, for $\forall i, 1 \leq i \leq m$, $\rho_i(x_m, x_n) < \epsilon_n$ if $x_m, x_n \in M_i$.

For any positive number ϵ, since $\lim_{n \to +\infty} \epsilon_n = 0$, there exists an integer $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $\epsilon_n < \epsilon$. Therefore, if $x_n \in M_l$, then $\lim x_m = x_n$. Whence, there exists an integer

N such that if $m \geq N$, then $x_m \in M_l$ by Theorem 2.2. Take integers $m, n \geq \max\{N, N(\epsilon)\}$. We know that

$$\rho_l(x_m, x_n) \ < \ \epsilon_n \ < \ \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence.

By the assumption, $\widetilde{M}$ is completed. We know that the sequence $\{x_n\}$ is convergence to a point $x_0, x_0 \in \widetilde{M}$. By conditions (i) and (ii), we have that $\rho_l(x_0, x_n) \ < \epsilon_n$ if we take $m \to +\infty$. Whence, $x_0 \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$.

Now if there a point $y \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$, then there must be $y \in M_l$. We get that

$$0 \leq \rho_l(y, x_0) = \lim_n \rho_l(y, x_n) \leq \lim_{n \to +\infty} \epsilon_n = 0$$

by Theorem 2.3. Therefore, $\rho_l(y, x_0) = 0$. By definition of a metric on a space, we get that $y = x_0$.

Let $\widetilde{M_1}$ and $\widetilde{M_2}$ be two multi-metric spaces and $f : \widetilde{M_1} \to \widetilde{M_2}$ a mapping, $x_0 \in \widetilde{M_1}, f(x_0) = y_0$. For $\forall \epsilon > 0$, if there exists a number δ such that for $forall x \in B(\delta, x_0)$, $f(x) = y \in B(\epsilon, y_0) \subset \widetilde{M_2}$, i.e.,

$$f(B(\delta, x_0)) \subset B(\epsilon, y_0),$$

then we say f is *continuous at point* x_0. If f is connected at every point of $\widetilde{M_1}$, then f is said a *continuous mapping* from $\widetilde{M_1}$ to $\widetilde{M_2}$.

For a continuous mapping f from $\widetilde{M_1}$ to $\widetilde{M_2}$ and a convergent sequence $\{x_n\}$ in $\widetilde{M_1}$, $\lim\limits_n x_n = x_0$, we can prove that

$$\lim_n f(x_n) = f(x_0).$$

For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ and a mapping $T : \widetilde{M} \to \widetilde{M}$, if there is a point $x^* \in \widetilde{M}$ such that $Tx^* = x^*$, then x^* is called a *fixed point* of T. Denoted by $^{\#}\Phi(T)$ the number of all fixed points of a mapping T in $\widetilde{M}$. If there are a constant $\alpha, 1 < \alpha < 1$ and integers $i, j, 1 \leq i, j \leq m$ such that for $\forall x, y \in M_i$, $Tx, Ty \in M_j$ and

$$\rho_j(Tx, Ty) \leq \alpha \rho_i(x, y),$$

then T is called a *contraction* on $\widetilde{M}$.

Theorem 2.8. *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space and T a contraction on $\widetilde{M}$. Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Proof. Choose arbitrary points $x_0, y_0 \in M_1$ and define recursively

$$x_{n+1} = Tx_n, \quad y_{n+1} = Tx_n$$

for $n = 1, 2, 3, \cdots$. By definition, we know that for any integer $n, n \geq 1$, there exists an integer $i, 1 \leq i \leq m$ such that $x_n, y_n \in M_i$. Whence, we inductively get that

$$0 \leq \rho_i(x_n, y_n) \leq \alpha^n \rho_1(x_0, y_0).$$

Notice that $0 < \alpha < 1$, we know that $\lim\limits_{n\to+\infty} \alpha^n = 0$. Therefore, there exists an integer i_0 such that

$$\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0.$$

Therefore, there exists an integer N_1 such that $x_n, y_n \in M_{i_0}$ if $n \geq N_1$. Now if $n \geq N_1$, we have that

$$\begin{aligned}
\rho_{i_0}(x_{n+1}, x_n) &= \rho_{i_0}(Tx_n, Tx_{n-1}) \\
&\leq \alpha\rho_{i_0}(x_n, x_{n-1}) = \alpha\rho_{i_0}(Tx_{n-1}, Tx_{n-2}) \\
&\leq \alpha^2\rho_{i_0}(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^{n-N_1}\rho_{i_0}(x_{N_1+1}, x_{N_1}).
\end{aligned}$$

and generally, for $m \geq n \geq N_1$,

$$\begin{aligned}
\rho_{i_0}(x_m, x_n) &\leq \rho_{i_0}(x_n, x_{n+1}) + \rho_{i_0}(x_{n+1}, x_{n+2}) + \cdots + \rho_{i_0}(x_{n-1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\rho_{i_0}(x_{N_1+1}, x_{N_1}) \\
&\leq \frac{\alpha^n}{1-\alpha}\rho_{i_0}(x_{N_1+1}, x_{N_1}) \to 0 (m, n \to +\infty)
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. Similarly, we can prove $\{y_n\}$ is also a Cauchy sequence.

Because $\widetilde{M}$ is a completed multi-metric space, we have that

$$\lim_n x_n = \lim_n y_n = z^*.$$

We prove z^* is a fixed point of T in $\widetilde{M}$. In fact, by $\rho_{i_0}(\lim\limits_n x_n, \lim\limits_n y_n) = 0$, there exists an integer N such that

$$x_n, y_n, Tx_n, Ty_n \in M_{i_0}$$

if $n \geq N + 1$. Whence, we know that

$$\begin{aligned}
0 \leq \rho_{i_0}(z^*, Tz^*) &\leq \rho_{i_0}(z^*, x_n) + \rho_{i_0}(y_n, Tz^*) + \rho_{i_0}(x_n, y_n) \\
&\leq \rho_{i_0}(z^*, x_n) + \alpha\rho_{i_0}(y_{n-1}, z^*) + \rho_{i_0}(x_n, y_n).
\end{aligned}$$

Notice

$$\lim_{n\to+\infty} \rho_{i_0}(z^*, x_n) = \lim_{n\to+\infty} \rho_{i_0}(y_{n-1}, z^*) = \lim_{n\to+\infty} \rho_{i_0}(x_n, y_n) = 0.$$

We get that $\rho_{i_0}(z^*, Tz^*) = 0$, i.e., $Tz^* = z^*$.

For other chosen points $u_0, v_0 \in M_1$, we can also define recursively

$$u_{n+1} = Tu_n, \quad v_{n+1} = Tv_n$$

and get the limit points $\lim\limits_n u_n = \lim\limits_n v_n = w^* \in M_{i_0}, Tu^* \in M_{i_0}$. Since

$$\rho_{i_0}(z^*, u^*) = \rho_{i_0}(Tz^*, Tu^*) \leq \alpha\rho_{i_0}(z^*, u^*)$$

and $0 < \alpha < 1$, there must be $z^* = u^*$.

Similar consider the points in $M_i, 2 \leq i \leq m$, we get that

$$1 \leq^{\#} \Phi(T) \leq m.$$

Corollary 2.2.(Banach) *Let M be a metric space and T a contraction on M. Then T has just one fixed point.*

§3. Open problems for a multi-metric space

On a classical notion, only one metric maybe considered in a space to ensure the same on all the times and on all the situations. Essentially, this notion is based on an assumption that spaces are homogeneous. In fact, it is not true in general.

Multi-Metric spaces can be used to simplify or beautify geometrical figures and algebraic equations. One example is shown in Fig.1, in where the left elliptic curve is transformed to the right circle by changing the metric along x, y-axes and an elliptic equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to equation

$$x^2 + y^2 = r^2$$

of a circle of radius r.

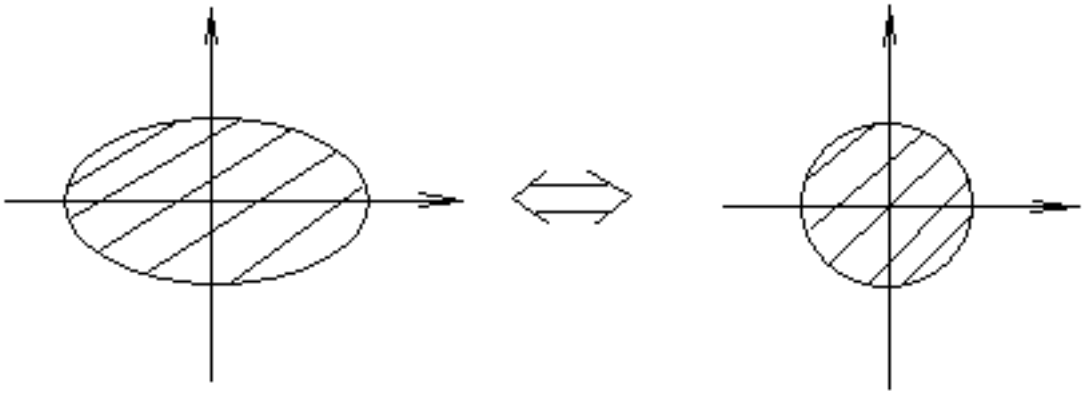

Fig.1

Generally, in a multi-metric space we can simplify a polynomial similar to the approach used in projective geometry. *Whether this approach can be contributed to mathematics with metrics?*

Problem 3.1 *Choose suitable metrics to simplify the equations of surfaces and curves in* $\mathbf{E}^3$.

Problem 3.2 *Choose suitable metrics to simplify the knot problem. Whether can it be used for classifying* 3*-dimensional manifolds?*

Problem 3.3 *Construct multi-metric spaces or non-linear spaces by Banach spaces. Simplify equations or problems to linear problems.*

References

[1] R. Abraham, J.E.Marsden and T.Ratiu, Manifolds, Tensor Analysis and Applications, Addison-Wesley Publishing Company, Inc., 1983.

[2] S.S.Chern and W.H.Chern, Lectures in Differential Geometry, Peking University Press, 2001.

[3] Daniel Deleanu, A Dictionary of Smarandache Mathematics, Buxton University Press, London, New York,2004.

[4] J.M.Lee, Riemannian Manifolds, Springer-Verlag New York, Inc.,1997.

[5] L.F.Mao, Automorphism Groups of Maps, Surfaces and Smarandache Geometries, American Research Press, 2005.

[6] Smarandache F., Mixed noneuclidean geometries, eprint arXiv: math/0010119, 10/2000.

[7] Smarandache F., A Unifying Field in Logics. Neturosophic Probability, Set, and Logic, American research Press, Rehoboth, 1999.

[8] Smarandache F., Neutrosophy, A new Branch of Philosophy, Multi-Valued Logic, **3**(2002) (special issue on Neutrosophy and Neutrosophic Logic), 297-384.

[9] Smarandache F., A Unifying Field in Logic: Neutrosophic Field, Multi-Valued Logic, **3** (2002)(special issue on Neutrosophy and Neutrosophic Logic), 385-438.

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On a equation for the square complements

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Abstract For any positive integer n, let $a(n)$ denotes the square complements of n. That is, $a(n)$ is the smallest positive integer such that $na(n)$ is a perfect square number. In this paper, we use the analytic method to study the number of the solutions of the equation involving the square complements, and obtain its all solutions of this equation.

Keywords Solution; Square complements; Equation.

§1. Introduction and Result

For any positive integer n, the square complements $a(n)$ is defined as the smallest positive integer such that $na(n)$ is a perfect square. For example, $a(1) = 1$, $a(2) = 2$, $a(3) = 3$, $a(4) = 1$, $a(5) = 5 \cdots$. In problem 27 of [1], Professor F.Smarandache asked us to study the properties of $a(n)$. About this problem, some authors had studied it before. For example, Liu Hongyan and Gou Su [2] used the elementary method to study the mean values of $a(n)$ and $\frac{1}{a(n)}$; Yang Haiwen and Guo Jinbao [3] gave some asymptotic formulae, one of which is:

$$\sum_{n\leq x} n^a a^t(n) = \frac{\zeta(2(t+1))}{(a+t+1)\zeta(2)} x^{a+t+1} + O(x^{a+t+\frac{1}{2}}),$$

where a and t are positive integers, $\zeta(s)$ is the Riemann zeta-function.

In this paper, we use the analytic method to study the number of the solution of the equation involving square complements, and give all solutions of the equation. That is, we will prove the following:

Theorem. The equation

$$\sum_{k=1}^{n} a(k) = a\left(\frac{n(n+1)}{2}\right)$$

has only three solutions, they are $n = 1$, 2, 3.

§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. Let x be a real number with $x > 1$ and m be a positive integer with $m > 1$. Then we have the following estimate:

$$\sum_{n\leq x} \frac{1}{n^m} > \zeta(m) - \frac{m}{(m-1)x^{m-1}}.$$

Proof. From the Euler's summation formula, we can easily obtain

$$\begin{aligned}\sum_{n\le x}\frac{1}{n^m} &= \int_1^x \frac{dt}{t^m} - m\int_1^x \frac{t-[t]}{t^{m+1}}dt + 1 - \frac{x-[x]}{x^m}\\ &= \frac{x^{1-m}}{1-m} - \frac{1}{1-m} + 1 - m\int_1^\infty \frac{t-[t]}{t^{m+1}}dt + m\int_x^\infty \frac{t-[t]}{t^{m+1}}dt - \frac{x-[x]}{x^m}\\ &> \zeta(m) - \frac{m}{(m-1)x^{m-1}},\end{aligned}$$

where we have used the identity (see [4])

$$\zeta(m) = 1 - \frac{1}{1-m} - m\int_1^\infty \frac{t-[t]}{t^{m+1}}dt.$$

This completes the proof of Lemma 1.

Lemma 2. For any integer number $n \ge 1$, we have the estimate:

$$\sum_{k=1}^{n} a(k) > \frac{\zeta(4)}{2\zeta(2)}n^2 - \left(\frac{3}{2}\zeta(2) + \frac{1}{8}\right)n^{\frac{3}{2}} - \left(\frac{2}{3\zeta(2)}\right)n,$$

Proof. From the definition of $a(n)$, the Euler's summation formula (see [4]) and the properties of the Möbius function, we can get

$$\begin{aligned}\sum_{k\le n} a(k) &= \sum_{m^2k\le n} k|\mu(k)| = \sum_{m^2d^2h\le n} d^2h\mu(d) = \sum_{m^2d^2\le n} d^2\mu(d)\sum_{h\le \frac{n}{m^2d^2}} h\\ &= \sum_{m^2d^2\le n} d^2\mu(d)\left(\frac{n^2}{2m^4d^4} + \frac{n}{2m^2d^2} - \frac{n}{m^2d^2}\{\frac{n}{m^2d^2}\} - \frac{1}{2}\left(\{\frac{n}{m^2d^2}\} - \{\frac{n}{m^2d^2}\}^2\right)\right)\\ &> \frac{n^2}{2}\sum_{m^2d^2\le n}\frac{\mu(d)}{m^4d^2} - n\sum_{m^2d^2\le n}\frac{1}{m^2} - \frac{1}{2}\sum_{m^2d^2\le n}\frac{d^2}{4}\\ &> \frac{n^2}{2}\sum_{m\le\sqrt{n}}\frac{1}{m^4}\sum_{d\le\frac{\sqrt{n}}{m}}\frac{\mu(d)}{d^2} - \zeta(2)n^{\frac{3}{2}} - \frac{1}{8}n^{\frac{3}{2}}\end{aligned}$$

and

$$\sum_{d\le\frac{\sqrt{n}}{m}}\frac{\mu(d)}{d^2} > \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - \sum_{d>\frac{\sqrt{n}}{m}}\frac{1}{d^2} > \frac{1}{\zeta(2)} - \frac{m}{\sqrt{n}} > \frac{1}{\zeta(2)} - \frac{m^2}{\sqrt{n}} \quad .$$

So from Lemma 1 we have

$$\begin{aligned}\sum_{k\le n} a(k) &> \frac{n^2}{2}\sum_{m\le\sqrt{n}}\frac{1}{m^4}\left(\frac{1}{\zeta(2)} - \frac{m^2}{\sqrt{n}}\right) - \zeta(2)n^{\frac{3}{2}} - \frac{1}{8}n^{\frac{3}{2}}\\ &= \frac{n^2}{2\zeta(2)}\sum_{m\le\sqrt{n}}\frac{1}{m^4} - \frac{n^{\frac{3}{2}}}{2}\sum_{m\le\sqrt{n}}\frac{1}{m^2} - \zeta(2)n^{\frac{3}{2}} - \frac{1}{8}n^{\frac{3}{2}}\\ &> \frac{n^2}{2\zeta(2)}\left(\zeta(4) - \frac{4}{3n}\right) - \frac{3}{2}\zeta(2)n^{\frac{3}{2}} - \frac{1}{8}n^{\frac{3}{2}}\\ &= \frac{\zeta(4)}{2\zeta(2)}n^2 - \left(\frac{3}{2}\zeta(2) + \frac{1}{8}\right)n^{\frac{3}{2}} - \left(\frac{2}{3\zeta(2)}\right)n.\end{aligned}$$

This completes the proof of Lemma 2.

§3. Proof of the theorem

Now we complete the proof of the theorem. We will discuss it into two cases:

(i). If $\frac{n(n+1)}{2}$ is a square free number, then from the definition of $a(n)$ we have

$$a(\frac{n(n+1)}{2}) = \frac{n(n+1)}{2}.$$

Note that $a(n) \leq n$, so we may immediately obtain

$$\sum_{k=1}^{n} a(k) \leq a\left(\frac{n(n+1)}{2}\right),$$

and the equality holds if and only if each $a(n)$ on the left side satisfying $a(n) = n$. So we can easily get three solutions of the equation: $n = 1, 2, 3$ in this case.

(ii). If $\frac{n(n+1)}{2}$ has no square divisor, then from the definition of $a(n)$ we have the following estimate:

$$a(\frac{n(n+1)}{2}) \leq \frac{n(n+1)}{8}.$$

From Lemma 2 and note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, we have

$$\sum_{k=1}^{n} a(k) > \frac{\zeta(4)}{2\zeta(2)}n^2 - \left(\frac{3}{2}\zeta(2) + \frac{1}{8}\right)n^{\frac{3}{2}} - \left(\frac{2}{3\zeta(2)}\right)n > \frac{3}{10}n^2 - 3n^{\frac{3}{2}} - \frac{4}{9}n.$$

We can easily obtain

$$\frac{3}{10}n^2 - 3n^{\frac{3}{2}} - \frac{4}{9}n > \frac{n(n+1)}{8}, \quad \text{if} \quad n > 361.$$

Thus there is no solution for the equation

$$\sum_{k=1}^{n} a(k) = a\left(\frac{n(n+1)}{2}\right), \quad \text{if} \quad n > 361.$$

If n varies from 4 to 361, we can not find any other solutions for the equation.

This completes the proof of the theorem.

Now we give the calculating programm as following:

```
#include<stdio.h>
#include<math.h>
//function
int get(int n)
{int i,m;
float g;
for(i=1;i≤ n;i++)
{g=sqrt(i*n);m=(int)g;
if(g-m==0)return i;
} }
    main()
{int n=361 ;
```

```
int sum;
int i,j;
for(i=1;i≤ n;i++)
{sum=0;
for (j=0;j≤ i;j++) sum=sum+get(j);
if(sum==get(i*(i+1)/2))
printf"%d\n",i);}}
```

References

[1] F. Smarandache, Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993.

[2] Liu Hongyan and Gou Su, On a problem of F. Smarandache, Journal of Yanan University (Natural Science Edition), **20**(2001), 5-6.

[3] Yang Haiwen and Guo Jinbao, Some asymptotic formulas of the square complement, Basic Science Journal of Textile Universities, **17**(2004), No.1, 11-13.

[4] Apostol T. M., Introduction to Analytic Number Theory, Spring-Verlag, New York, 1976.

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An arithmetical function and its mean value[1]

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Abstract A new arithmetical function is introduced, and an interesting asymptotic formula on its mean value is given.
Keywords Arithmetical function; Mean value; Asymptotic formula.

§1. Introduction

For any fixed positive integer n, the new arithmetical function has the following definition:

$$\overline{S}_k(n) = \max\{x \in N \mid x^k \mid n\} \, (\forall n \in N^*) \, .$$

Because

$$\begin{aligned}(\forall a, b \in N^*)(a,b) = 1 \Rightarrow \overline{S}_k(ab) \quad &= \max\{x \in N | x^k \mid a\} \ \cdot \max\{x \in N | x^k \mid b\} \\ &= \overline{S}_k(a) \cdot \overline{S}_k(b),\end{aligned}$$

and

$$\overline{S}_k(p^\alpha) = p^{\lfloor \frac{\alpha}{k} \rfloor},$$

where $\lfloor x \rfloor$ denotes the greatest integer not more than x. Therefore, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime power decomposition of n, then we have

$$\overline{S}_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{\lfloor \frac{\alpha_1}{k} \rfloor} \cdots p_r^{\lfloor \frac{\alpha_r}{k} \rfloor} = \overline{S}_k(p_1^{\alpha_1}) \cdots \overline{S}_k(p_r^{\alpha_r}).$$

So $\overline{S}_k(n)$ is a multiplicative function. There are close relations between this function and the Smarandache ceil function, so we call $\overline{S}_k(n)$ as the dual function of the Smarandache ceil function. In this paper, we study the mean value properties of $\overline{S}_k(n)$, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem 1. Let $x \geq 2$, for any fixed positive integer $k > 2$, we have the asymptotic formula

$$\sum_{n \leq x} \overline{S}_k(n) = \frac{\zeta(k-1)}{\zeta(k)} x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, ϵ denotes any fixed positive number.

Theorem 2. For $k = 2$, we have the asymptotic formula

$$\sum_{n \leq x} \overline{S}_2(n) = x\left(\frac{3}{\pi^2} \ln x + C\right) + O\left(x^{\frac{3}{4}+\epsilon}\right),$$

where C is a computable constant.

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§2. A Lemma

To complete the proof of the theorems, we need the following famous Perron's formula (See Lemma 8.5 of [1]):

Lemma. Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $s = \sigma + it$, converge absolutely for $\sigma > \beta$, and that there exist a positive λ and a positive increasing function $A(s)$ such that

$$\sum_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \ll (\sigma - \beta)^{-1},\ \sigma \to \beta^{+0}$$

and

$$a(n) \ll A(n),\ n = 1, 2, \cdots .$$

Then for any $b > 0$, $b + \sigma > \beta$, and x not to be an integer, we have

$$\begin{aligned}\sum_{n\le x} a(n)n^{-s_0} \quad &= \quad \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s_0+\omega)\frac{x^{\omega}}{\omega}\mathrm{d}\omega + O\left(\frac{x^b}{T(b+\sigma-\beta)^{\lambda}}\right)\\ &\quad +O\left(\frac{A(2x)x^{1-\sigma}\log x}{T\,||\,x\,||}\right),\end{aligned}$$

where $||\ x\ ||$ is the nearest integer to x.

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. For any positive integer $k \ge 2$, let

$$f(s) = \sum_{n=1}^{\infty} \frac{\overline{S}_k(n)}{n^s}.$$

If $\Re(s) > 1$, then from the Euler product formula (See Theorem 11.6 and Theorem 11.7 of [3]), we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{\overline{S}_k(p)}{p^s} + \frac{\overline{S}_k(p^2)}{p^{2s}} + \cdots + \frac{\overline{S}_k(p^k)}{p^{ks}} + \cdots\right)\\ &= \prod_p \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(k-1)s}} + \frac{p}{p^{ks}} + \frac{p}{p^{(k+1)s}} + \cdots + \frac{p}{p^{(2k-1)s}} + \frac{p^2}{p^{2ks}} + \cdots\right)\\ &= \prod_p \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} + \frac{p}{p^{ks}}\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} + \frac{p^2}{p^{2ks}}\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} + \cdots\right)\\ &= \prod_p \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\right)\prod_p \left(1 + \frac{p}{p^{ks}} + \frac{p^2}{p^{2ks}} + \frac{p^3}{p^{3ks}} + \cdots\right)\\ &= \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

So by the Perron's formula with $s_0 = 0$, $b = \frac{3}{2}$ and $T = x$, we have

$$\sum_{n\leq x} \overline{S}_k(n) = \frac{1}{2i\pi}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\frac{x^s}{s}ds + O(x^{\frac{1}{2}+\varepsilon}).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\frac{x^s}{s}ds,$$

we move the integral line from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, if $k > 2$, then the function

$$f(s) = \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\frac{x^s}{s}$$

has a simple pole point at $s = 1$ with residue $\frac{\zeta(k-1)}{\zeta(k)}x$. So we have

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)\zeta(ks-1)x^s}{\zeta(ks)s}ds = \frac{\zeta(k-1)}{\zeta(k)}x.$$

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)\zeta(ks-1)x^s}{\zeta(ks)s}ds \ll x^{\frac{1}{2}+\epsilon},$$

from above we may immediately get the asymptotic formula:

$$\sum_{n\leq x} \overline{S}_k(n) = \frac{\zeta(k-1)}{\zeta(k)}x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Theorem 1.

For $k = 2$, note that $\zeta(2) = \frac{\pi^2}{6}$, $f(s) = \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\frac{x^s}{s}$ has a 2 order pole point at $s = 1$ with residue $x\left(\frac{3}{\pi^2}\ln x + C\right)$, from the above we can also deduce that

$$\sum_{n\leq x} \overline{S}_2(n) = x\left(\frac{3}{\pi^2}\ln x + C\right) + O\left(x^{\frac{3}{4}+\epsilon}\right).$$

This completes the proof of Theorem 2.

References

[1] Pan Chengdong and Pan Chengbiao, Goldbach Conjecture, Science Press, Beijing 1992, pp. 145.

[2] F. Smarandache, Only Problems, Not Solutions, Xiquan Publ. House, Chicago, 1993.

[3] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976, 230-231.

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Some identities involving Bernoulli numbers and Euler numbers

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Abstract The main purpose of this paper is using the elementary method to obtain some interesting identities involving the Bernoulli numbers and the Euler numbers.

Keywords The Bernoulli and the Euler numbers; Identity; Elementary method.

§1. Introduction

Let z be any complex number with $|z| < 2\pi$. The Bernoulli numbers B_n and the Euler numbers E_{2n} $(n = 0, 1, 2, \cdots)$ are defined by the following generated functions (See [1], [2] and [3]):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < \frac{\pi}{2} \tag{1}$$

and

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!}. \tag{2}$$

For example, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = -\frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = -\frac{5}{66}$, $\cdots$, $B_{2n+1} = 0$ for $n \geq 1$, and

$$\sum_{k=0}^{r} \frac{2^{2k} B_{2k}}{(2k)!(2r+1-2k)!} = \frac{1}{(2r)!}$$

holds for any integer $r \geq 1$ (See exercise 16 for chapter 12 of [4]). $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 11385$, $E_{10} = 150521$, $\cdots$, and

$$\sum_{s=0}^{n} (-1)^s \binom{2n}{2s} E_{2s} = 0, \quad n \geq 1.$$

The Bernoulli numbers and the Euler numbers have extensive applications in combinational mathematics and analytic number theory. So there are many scholars have investigated their arithmetical properties. For example, G.Voronoi first proved a very useful congruence for Bernoulli numbers, one of its Corollaries (See [5] Proposition 15.2.3 and its Corollary) is that for any prime $p \equiv 3 \pmod 4$ with $p > 3$, we have

$$2\left(2 - \left(\frac{2}{p}\right)\right) B_m \equiv -\sum_{j=1}^{m-1} \left(\frac{j}{p}\right) (\mathrm{mod} p),$$

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where (x/p) denotes the Legendre symbol and $m = (p+1)/2$. Liu Guodong [6] obtained some identities involving the Bernoulli numbers. That is, for any integers $n \geq 1$ and $k \geq 0$,

$$(a) \qquad \sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2-2^{2j}}{(2k+1)^{2j}} B_{2j} = \frac{(2n+1)2^{2n}}{(2k+1)^{2n+1}} \sum_{s=0}^{k} s^{2n};$$

$$(b) \qquad \sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2-2^{2j}}{(2k+2)^{2j}} B_{2j} = \frac{2n+1}{2^{2n}(k+1)^{2n+1}} \sum_{s=0}^{k} (2s+1)^{2n}.$$

For the Euler numbers, Zhang Wenpeng [3] obtained an important congruence, i.e.,

$$E_{p-1} = \begin{cases} 0 \pmod{p}, & p \equiv 1 \pmod{4}, \\ -2 \pmod{p}, & p \equiv 3 \pmod{4}. \end{cases}$$

where p be a prime.

Liu Guodong [7] proved that for any positive integers n and k,

$$E_{2n} \equiv (-1)^{n+k} 2^{2n+1} \sum_{i=1}^{k} (-1)^i i^{2n} \pmod{(2k+1)^2}.$$

Other results involving the Bernoulli numbers and the Euler numbers can also be found in [8], [9] and [10]. This paper as a note of [6] and [7], we use the elementary method to obtain some other identities for the Bernoulli numbers and the Euler numbers. That is, we shall prove the following:

Theorem 1. For any positive integers n and k, we have the identity

$$\sum_{t=0}^{n} \binom{2n+2}{2t} \left(2-2^{2t}\right) \frac{B_{2t}}{(2k)^{2t}} = \frac{4(n+1)}{(2k)^{2n+2}} \sum_{m=1}^{k} (2m-1)^{2n+1}.$$

Theorem 2. For any positive integers n and k, we have

$$E_{2n} - (2k)^{2n} \sum_{t=0}^{n} (-1)^{n+k-t} \binom{2n}{2t} \frac{E_{2t}}{(2k)^{2t}} = 2 \sum_{m=0}^{k-1} (-1)^{m+n} (2m+1)^{2n}.$$

From Theorem 2 we may immediately deduce the following:

Corollary 1. For any odd prime p, we have the congruence

$$E_{\frac{p^2-1}{4}} \equiv \begin{cases} (-1)^{\frac{p^2-1}{8}} \ 2 \pmod{p}, & p \equiv 3 \pmod{4}; \\ (-1)^{\frac{p^2-1}{8}} \frac{4\sqrt{p}}{\pi} L(1, \chi_2\chi_4) \pmod{p}, & p \equiv 1 \pmod{4}, \end{cases}$$

where χ_2 denotes the Legendre symbol modulo p, χ_4 denotes the non-principal character mod 4, and $L(1, \chi_2\chi_4)$ denotes the Dirichlet L-function corresponding to character $\chi_2\chi_4 \mod 4p$.

This Corollary is interesting, because it shows us some relations between the Euler numbers and the Dirichlet L-function. From Corollary 1 we can also get the following:

Corollary 2. For any prime p with $p \equiv 3 \pmod{4}$, we have the congruence

$$E_{\frac{p^2-1}{4}} \equiv 2 \left(\frac{2}{p}\right) \equiv \begin{cases} 2 \pmod{p}, & \text{if } p \equiv 7 \pmod{8}; \\ -2 \pmod{p}, & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

§2. Some Lemmas

To complete the proof of Theorems, we need the following three simple lemmas. First we have

Lemma 1. For any integer $n \geq 1$, we have have the identities

$$(A) \qquad 2\sum_{m=1}^{n}\sin(2m-1)x = \frac{1-\cos 2nx}{\sin x};$$

$$(B) \qquad 2\sum_{m=0}^{n-1}(-1)^m\cos(2m+1)x = \frac{1-(-1)^n\cos 2nx}{\cos x}.$$

Proof. In fact, this Lemma is the different forms of the exercise 3.2.9 of [11], where is

$$\sum_{m=1}^{n}\frac{\sin(2m-1)x}{\sin x} = \left(\frac{\sin nx}{\sin x}\right)^2.$$

Note that $2\sin^2 nx = 1-\cos 2nx$, from the above we can deduce the formula (A) of Lemma 1.

If we substitute x by $\pi/2-y$ in (A), we may immediately get formula (B).

Lemma 2. For any real number x with $0<|x|<\pi$, we have the identity

$$\frac{1}{\sin x} = \sum_{n=0}^{\infty}(-1)^n\left(2-2^{2n}\right)\frac{B_{2n}}{(2n)!}x^{2n-1}.$$

Proof. (See reference [12]).

Lemma 3. Let p be an odd prime, χ be an even primitive character $\bmod p$. Then we have

$$\sum_{n\leq p/4}\chi(n) = \frac{G(\chi)}{\pi}L(1,\overline{\chi}\chi_4),$$

where $G(\chi)=\sum_{n=1}^{p-1}\chi(n)\,e^{\frac{2\pi in}{p}}$ is the Gauss sums, χ_4 denotes the non-principal character $\bmod 4$, and $L(1,\overline{\chi}\chi_4)$ denotes the Dirichlet L-function corresponding to character $\overline{\chi}\chi_4 \mod 4p$.

Proof. (See Theorem 3.7 of [13]).

§3. Proof of the theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1. Note that

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},$$

from Lemma 2 and (A) of Lemma 1 we have

$$\begin{aligned} & 2\sum_{m=1}^{n}\sum_{s=0}^{\infty}(-1)^s\frac{(2m-1)^{2s+1}}{(2s+1)!}x^{2s+1} \\ = & \left(\sum_{s=0}^{\infty}(-1)^s\left(2-2^{2s}\right)\frac{B_{2s}}{(2s)!}x^{2s-1}\right)\left(1-\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s}}{(2s)!}x^{2s}\right) \\ = & \left(\sum_{s=0}^{\infty}(-1)^s\left(2-2^{2s}\right)\frac{B_{2s}}{(2s)!}x^{2s-1}\right)\left(\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s+2}}{(2s+2)!}x^{2s+2}\right) \\ = & \sum_{s=0}^{\infty}(-1)^s\left(\sum_{t=0}^{s}\left(2-2^{2t}\right)\frac{B_{2t}}{(2t)!}\frac{(2n)^{2s-2t+2}}{(2s-2t+2)!}\right)x^{2s+1}. \end{aligned} \tag{3}$$

Comparing the coefficient of x^{2k+1} on both side of (3), we get

$$2\sum_{m=1}^{n}\frac{(2m-1)^{2k+1}}{(2k+1)!}=\sum_{t=0}^{k}\left(2-2^{2t}\right)\frac{B_{2t}}{(2t)!}\frac{(2n)^{2k-2t+2}}{(2k-2t+2)!}$$

or

$$\sum_{t=0}^{k}\binom{2k+2}{2t}\left(2-2^{2t}\right)\frac{B_{2t}}{(2n)^{2t}}=\frac{4(k+1)}{(2n)^{2k+2}}\sum_{m=1}^{n}(2m-1)^{2k+1}.$$

This proves Theorem 1.

Now we prove Theorem 2. From (2) and (B) of Lemma 1 we have

$$\begin{aligned} & 2\sum_{m=0}^{n-1}(-1)^m\sum_{s=0}^{\infty}(-1)^s\frac{(2m+1)^{2s}}{(2s)!}x^{2s} \\ = & \left(\sum_{s=0}^{\infty}E_{2s}\frac{x^{2s}}{(2s)!}\right)\left(1-(-1)^n\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s}}{(2s)!}x^{2s}\right) \\ = & \sum_{s=0}^{\infty}E_{2s}\frac{x^{2s}}{(2s)!}-(-1)^n\sum_{s=0}^{\infty}\sum_{t=0}^{s}\frac{E_{2t}}{(2t)!}(-1)^{s-t}\frac{(2n)^{2s-2t}}{(2s-2t)!}x^{2s}. \end{aligned} \tag{4}$$

Comparing the coefficient of x^{2k} on both side of (4), we may immediately deduce

$$2\sum_{m=0}^{n-1}(-1)^{m+k}\frac{(2m+1)^{2k}}{(2k)!}=\frac{E_{2k}}{(2k)!}-\sum_{t=0}^{k}(-1)^{n+k-t}\frac{E_{2t}}{(2t)!}\frac{(2n)^{2k-2t}}{(2k-2t)!}$$

or

$$2\sum_{m=0}^{n-1}(-1)^{m+k}(2m+1)^{2k}=E_{2k}-(2n)^{2k}\sum_{t=0}^{k}(-1)^{n+k-t}\binom{2k}{2t}\frac{E_{2t}}{(2n)^{2t}}.$$

This completes the proof of Theorem 2.

To prove Corollary 1, taking $k=p$ and $n=(p^2-1)/8$ in Theorem 2 we may get

$$2E_{2n}+(2p)^{2n}\sum_{t=0}^{n-1}(-1)^{n-t}\binom{2n}{2t}\frac{E_{2t}}{(2p)^{2t}}=2\sum_{m=0}^{p-1}(-1)^{m+n}(2m+1)^{2n}$$

or

$$E_{\frac{p^2-1}{4}} \equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \pmod{p}. \tag{5}$$

For any integer a with $(a,p)=1$, from the Euler's criterion (See Theorem 9.2 of [4]) we know that

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod{p},$$

where $(a/p)=\chi_2(a)$ is the Legendre symbol modulo p.

By this formula we may get

$$a^{\frac{p^2-1}{4}} \equiv \left(\frac{a}{p}\right)^{\frac{p+1}{2}} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \equiv 3 \pmod 4; \\ \left(\frac{a}{p}\right) \pmod{p}, & \text{if } p \equiv 1 \pmod 4. \end{cases} \tag{6}$$

If $p \equiv 3 \pmod 4$, note that $\left(\frac{0}{p}\right)=0$, from (5) and (6) we can get

$$\begin{aligned} E_{\frac{p^2-1}{4}} &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m \left(\frac{2m+1}{p}\right)^2 \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} 2 \pmod{p}. \end{aligned}$$

If $p \equiv 1 \pmod 4$, note that $\left(\frac{-1}{p}\right)=1$ (an even character mod p), $G(\chi_2)=\sqrt{p}$ and $\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)=0$, from (5), (6) and Lemma 3 we may obtain

$$\begin{aligned} E_{\frac{p^2-1}{4}} &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m \left(\frac{2m+1}{p}\right) \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \left[2\sum_{m=0}^{(p-1)/2} \left(\frac{4m+1}{p}\right) - \sum_{m=0}^{p-1} \left(\frac{2m+1}{p}\right)\right] \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} 2 \sum_{m=0}^{(p-1)/2} \left(\frac{m+\overline{4}}{p}\right) \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} 2 \sum_{m=\frac{1-p}{4}}^{(p-1)/4} \left(\frac{m}{p}\right) \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} 4 \sum_{m=1}^{(p-1)/4} \left(\frac{m}{p}\right) \pmod{p} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \frac{4\sqrt{p}}{\pi} L(1,\chi_2\chi_4) \pmod{p}, \end{aligned}$$

where $\overline{a}$ denotes the solution of the congruence $ax \equiv 1 \pmod{p}$ and $\overline{4} = \frac{1-p}{4}$.

This completes the proof of Corollary 1.

Note. Using the exercise 3.2.7 and 3.2.8 of [11], we can also deduce the other identities and congruences involving the Bernoulli numbers and the Euler numbers.

References

[1] E. Lehmer, On Congruences Involving Bernoulli Numbers and the quotients of Fermat and Wilson, Annals of Math. **39**(1938), pp. 350-360.

[2] K. Dilcher, Sums of products of Bernoulli numbers, Journal of Number Theory, Vol. 60, **(1)**(1996), pp. 23-41.

[3] Wenpeng Zhang, Some identities involving the Euler and the factorial numbers, The Fibonacci Quarterly, Vol. 36, **(2)**(1998), pp. 154-157.

[4] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976, pp. 265.

[5] Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Springer-Verlag, New York, 1990.

[6] Guodong Liu and Hui Luo, some identities involving Bernoulli numbers, The Fibonacci Quarterly, Vol. 43, **(3)**(2005), pp. 208-212.

[7] Guodong Liu, On congruences of Euler numbers modulo an square, The Fibonacci Quarterly, Vol. 43,**(2)**(2005), pp. 132-136.

[8] Glenn J. Fox, Congruences relating rational values of Bernoulli and Euler polynomials, The Fibonacci Quarterly,Vol. 39, **(1)**(2001), pp. 50-57.

[9] M. Kanebo, A recurrence formula for the Bernoulli numbers, Proc. Japan Acad. Ser. A. Math. Sci., **71**(1995), pp. 192-193.

[10] H. Momiyama, A new recurrence formula for the Bernoulli numbers, The Fibonacci Quarterly, Vol. 39, **(3)**(2001), pp. 285-288.

[11] M. Ram Murty, Problems in Analytic Number Theory, Springer-Verlag, New York, 2001, pp. 41.

[12] C. Jordan, Calculus of finite differences Introduction to Analytic Number Theory, Springer-Verlag, New York, 1965, pp. 260.

[13] B. C. Berndt, Classical theorems on quadratic residues, Extrait de L'Enseignement mathematique, T.XXII, **(3-4)**(1976), pp.261-304.

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Counterexamples to a conjecture concerning class number of binary quadratic forms

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Abstract Denote the binary quadratic form $ax^2+bxy+cy^2$ by (a,b,c) and its equivalent class by $[a,b,c]$. Let $H(D)=\{[a,b,c] \mid b^2-4ac=D\}$ and $H_4(D)=\{[a,b,c]^4 \mid b^2-4ac=D\}$, denote $h(D)$ and $h_4(D)$ as the order of $H(D)$ and $H_4(D)$ respectively. Sun["Quartic residues and binary qudratic forms", Journal of Number Theory, 2005, 113(1)] conjectured that: Let p and q be the primes of the form $4k+1$ such that $(\frac{p}{q})=1$, where $(\frac{p}{q})$ is the Legendre symbol, then $h_4(-4pq)=h_4(-64pq)=h(-4pq)/8$. In this paper we find some counterexamples to the conjecture, thus disprove it.

Keywords Binary quadratic form, Class number, Algorithms.

§1. Introduction

A binary quadratic form f with discriminant b^2-4ac is a function $f(x,y)=ax^2+bxy+cy^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, which is denoted more briefly by (a,b,c). We say f is primitive if gcd(a,b,c)=1. Two binary quadratic forms (a,b,c) and (a_1,b_1,c_1) are said to be equivalent if there exits an integeral matrix $C=\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ of determinant equal to 1(i.e., with $ru-st=1$) such that

$$g(x,y)=\begin{pmatrix} x & y \end{pmatrix}C'\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}C\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

denoted as $(a,b,c)\sim(a_1,b_1,c_1)$. We denote the equivalent class of (a,b,c) by $[a,b,c]$. Let $H(D)$ be the class group which consists of primitive, integeral quadratic forms of discriminant D, and let $h(D)$ be the corresponding class number. Let $H_4(D)$ be the subgroup of $H(D)$ consisting of the fourth powers of the classes in $H(D)$,i.e., $H_4(D)=\{[a,b,c]^4 \mid b^2-4ac=D\}$ and let $h_4(D)$ be the order of $H_4(D)$.

Z.H.Sun[2] posed several conjectures concerning the relations between $h(D)$ and $h_4(D)$ for some special cases of D, one of them is

Conjecture. [2,conjecture 8.4] Let p and q be primes of the form $4k+1$ such that $(\frac{p}{q})=1$,where $(\frac{p}{q})$ is the Legendre symbol. Then $h_4(-4pq)=h_4(-64pq)=h(-4pq)/8$.

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In this paper we give some counterexamples to above conjecture, thus disprove it. In section 2 we describe two algorithms for counting $h(D)$ and $h_4(D)$ and in section 3 we tabulate some examples which are unfavorable for the conjecture.

§2. Counting $h(D)$ and $h_4(D)$

Let us now consider the problem of computing the class group $H(D)$ for $D < 0$ as to the conjecture. Firstly we need the following:

Definition. [1,Definition 5.3.2] A binary quadratic form (a, b, c) with discriminant $D = b^2 - 4ac < 0$ and $a > 0$ is said to be reduced if either $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

Lemma 2.1 [1,Proposition 5.3.3] In every class of binary quadratic forms with discriminant $D < 0$ and $a > 0$ there exists exactly one reduced form. In particular $h(D)$ is equal to the number of primitive reduced forms of discriminant D.

Now we are ready to describe a procedure to compute $h(D)$ with reduced forms.

Procedure 1. Computing $h(D)$;
{Input a negative integer D, Output the class group $H(D)$ and class number $h(D)$}
Begin Bound$\longleftarrow [\sqrt{-D/3}]$; b $\longleftarrow$ D mod 2;
  $h \longleftarrow 1$; output the form $(1, b, (b^2 - D)/4)$;
  Repeat $q \longleftarrow (b^2 - D)/4$; If $b > 1$ then $a \longleftarrow b$ else $a \longleftarrow 2$; $r \longleftarrow q/a$;
    repeat if if (q mod a=0) and $(a^2 \leq q)$ and (gcd(a,b,r)=1) then
      begin If (a=b) Or $(a^2 = q)$ or (b=0) Then
        Begin $h \longleftarrow h + 1$; output the form $(a, b, q/a)$ End Else
        Begin $h \longleftarrow h + 2$; output the form $(a, b, q/a)$ and $(a, -b, q/a)$ End
      end; $a \longleftarrow a + 1$; $r \longleftarrow q/a$
    until $a^2 > q$; $b \longleftarrow b + 2$
  Until $b >$ *Bound*; output $h = h(D)$
End.

Before counting $h_4(D)$, Let's recall the composition of two binary quadratic forms.

Lemma 2.2 [1,Definition 5.4.6] Let $(a_i, b_i, c_i) \in H(D)(i = 1, 2)$. Set $s = (b_1 + b_2)/2, n = (b_1 - b_2)/2$, and let $u, v, w, d \in \mathbb{Z}$ be such that $ua_1 + va_2 + ws = d =$ gcd (a_1, a_2, s). Then the composition of the two elements as the unique class is $[a_3, b_3, c_3] = [a_1, b_1, c_1][a_2, b_2, c_2]$, where $a_3 = \frac{a_1 a_2}{d^2}, b_3 = b_2 + \frac{2a_2}{d}(v(s - b_2) - wc_2), c_3 = \frac{b_3^2 - D}{4a_3}$.

According to lemma 2.2 one can describe a procedure to compute the fourth powers of the classes in $H(D)$ (i.e. $[a, b, c]^4$), as following:

Procedure 2. Computing $h_4(D)$;
{Input a negative integer D, Output the class subgroup $H_4(D)$ and its order $h_4(D)$}
Begin using Procedure 1 output $H(D)$ and $h(D)$;
  For every $(a, b, c) \in H(D)$ Do
    begin $(a_4, b_4, c_4) \longleftarrow (a, b, c)^4$; output (a_4, b_4, c_4);
      count different elments (a_4, b_4, c_4)'s and output its number $h_4(D)$
    end
End.

§3. Disproof the conjecture

Using above Algorithms, we seek some prime pairs (p, q) on computer and get many counterexamples to the conjecture. We tabulate some of them in table 1 and give an exact example.

Example. Taking $p = 17$, $q = 89$, we have

$H(-4 \cdot 17 \cdot 89) = \{(1, 0, 1513),\ (17, 0, 89),\ (2, 2, 757),\ (37, 4, 41),\ (37, -4, 41),\ (11, 8, 139),$
$(11, -8, 139), (22, 14, 71),\ (22, -14, 71),\ (19, 16, 83),\ (19, -16, 83),$
$(38, 22, 43),\ (29, 26, 58), (38, -22, 43), (29, -26, 58),\ (34, 34, 53)\};$

$H_4(-4 \cdot 17 \cdot 89) = \{(1, 0, 1513)\};$

$H(-64 \cdot 17 \cdot 89) = \{(1, 0, 24208), (16, 0, 1513), (17, 0, 1424), (89, 0, 272).....$
$(167, 144, 176), (167, -144, 176), (164, 148, 181), (164, -148, 181)\};$

$H_4(-64 \cdot 17 \cdot 89) = \{(1, 0, 24208), (16, 0, 1513)\}$, and

$h(-4 \cdot 17 \cdot 89) = 16$; $h_4(-4 \cdot 17 \cdot 89) = 1$;

$h(-64 \cdot 17 \cdot 89) = 64$; $h_4(-64 \cdot 17 \cdot 89) = 2$.

we can see $h_4(-4 \cdot 17 \cdot 89) \neq h(-4 \cdot 17 \cdot 89)/8$ and $h_4(-4 \cdot 17 \cdot 89) \neq h_4(-64 \cdot 17 \cdot 89)$.

Table 1:

(p, q)	$h(-4 \cdot p \cdot q)$	$h_4(-4 \cdot p \cdot q)$	$h_4(-64 \cdot p \cdot q)$	$h(-4 \cdot p \cdot q)/8$
(41, 73)	48	3	6	6
(41, 113)	32	2	4	4
(41, 337)	112	7	14	14
(41, 353)	48	3	6	6
(41, 401)	112	7	14	14
(41, 433)	96	6	12	12
(41, 449)	112	7	14	14
(73,97)	64	4	8	8
(73,137)	160	10	20	20
(73,257)	128	8	16	16
(73,353)	176	11	22	22
(89,257)	48	3	6	6
(89,409)	256	16	32	32
(89,673)	240	15	30	30

References

[1] H. Cohen, A Course in Computational Algebraic Number Theory, Graduate Texts in Mathematics **138**, Springer-Verlag, Berlin, 1996.

[2] Z. H. Sun, Quartic Residues and Binary Quadratic Forms, Journal of Number Theory, **113(1)**(2005), 10-52.

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On Smarandache dual function

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Abstract Smarandache dual function $S^*(n)$ denotes the greatest positive integer m such that $m! \mid n$, where n denotes any positive integer. That is, $S^*(n) = \max\{m : m! \mid n\}$. In this paper, we studied the convergent property of the series $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^\alpha}$ by using the elementary methods, and obtained an interesting identity.

Keywords Smarandache dual function; Series; Identity.

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is,

$$S(n) = \min\{m : n \mid m!\}.$$

It was introduced in [1] by Professor Smarandache, and he also asked us to investigate the properties of $S(n)$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, it is easy to know that $S(n) = \max\{S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), \cdots, S(p_k^{\alpha_k})\}$. So, we can studied the properties of $S(n)$ through $S(p_i^{\alpha_i})$. About the properties of $S(n)$, many scholars have show their interest on it, see [2], [3] and [4]. For example, Farris Mark and Mitchell Patrick [2] studied the bounding of Smarandache function, and they gave an upper and lower bound for $S(p^\alpha)$, i.e.

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Wang Yongxing [3] studied the mean value $\sum_{n \leq x} S(n)$, and obtained an asymptotic formula by using the elementary methods. He proved that:

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Similarly, we introduce another function as following which have close relationship with the Smarandache function. It is the Smarandache dual function $S^*(n)$ which denotes the greatest positive integer m such that $m! \mid n$, where n denotes any positive integer. That is,

$$S^*(n) = \max\{m : m! \mid n\}.$$

About this problem, J.Sandor in [5] conjectured that

$$S^*((2k-1)!(2k+1)!) = q - 1,$$

[1]This work is supported by the N.S.F.(60472068) of P.R.China.

where k is a positive integer, q is the first prime following $2k+1$. This conjecture was proved by Le Maohua [6].

In this paper, we studied the convergent property of the series $\sum_{n=1}^{\infty}\frac{S^*(n)}{n^\alpha}$ by using the elementary methods, and obtained an interesting identity. That is, we will prove the following:

Theorem. *For any real number $\alpha \le 1$, the infinity series*

$$\sum_{n=1}^{\infty}\frac{S^*(n)}{n^\alpha}$$

is divergent, it is convergent if $\alpha > 1$, and

$$\sum_{n=1}^{\infty}\frac{S^*(n)}{n^\alpha} = \zeta(\alpha)\sum_{n=1}^{\infty}\frac{1}{(n!)^\alpha},$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $\alpha = 2$ and $\alpha = 4$ in our Theorem, we may immediately deduce the following:

Corollary. For Smarandache dual function, we have the identities

$$\sum_{n=1}^{\infty}\frac{S^*(n)}{n^2} = \frac{\pi^2}{6}\sum_{n=1}^{\infty}\frac{1}{(n!)^2}$$

and

$$\sum_{n=1}^{\infty}\frac{S^*(n)}{n^4} = \frac{\pi^4}{90}\sum_{n=1}^{\infty}\frac{1}{(n!)^4}.$$

§2. Proof of the theorem

In this section, we will complete the proof of Theorem. For any real number $\alpha \le 1$, note that $S^*(n) \ge 1$, and the series

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$$

is divergent, so the series $\sum_{n=1}^{\infty}\frac{S^*(n)}{n^\alpha}$ is also divergent if $\alpha \le 1$.

For any positive integer $n \ge 1$, there must be a positive integer m such that

$$n = m! \cdot l,$$

where $l \not\equiv 0 (\mathrm{mod}\ m+1)$. So for $\alpha > 1$, from the definition of $S^*(n)$, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{S^*(n)}{n^\alpha} &= \sum_{l=1}^{\infty}\sum_{\substack{m=1\\ m+1\dagger l}}^{\infty}\frac{m}{(m!)^\alpha\cdot l^\alpha}\\
&= \sum_{m=1}^{\infty}\sum_{\substack{l=1\\ m+1\dagger l}}^{\infty}\frac{m}{(m!)^\alpha\cdot l^\alpha}\\
&= \sum_{m=1}^{\infty}\frac{m}{(m!)^\alpha}\left(\sum_{l=1}^{\infty}\frac{1}{l^\alpha}-\sum_{l=1}^{\infty}\frac{1}{(m+1)^\alpha\cdot l^\alpha}\right)\\
&= \sum_{m=1}^{\infty}\frac{m}{(m!)^\alpha}\zeta(\alpha)\left(1-\frac{1}{(m+1)^\alpha}\right)\\
&= \zeta(\alpha)\left(\sum_{m=1}^{\infty}\frac{m}{(m!)^\alpha}-\sum_{m=1}^{\infty}\frac{m}{((m+1)!)^\alpha}\right)\\
&= \zeta(\alpha)\left(1+\sum_{m=1}^{\infty}\frac{1}{((m+1)!)^\alpha}\right)\\
&= \zeta(\alpha)\sum_{m=1}^{\infty}\frac{1}{(m!)^\alpha}.
\end{aligned}$$

This completes the proof of Theorem.

References

[1] F. Smarandache, Only problems, not Solutions, Xiquan Publ House, Chicago, 1993.

[2] Farris Mark and Mitchell Patrick, Bounding the Smarandache function, Smarandache Notions Journal, **13**(2002), pp. 37-42.

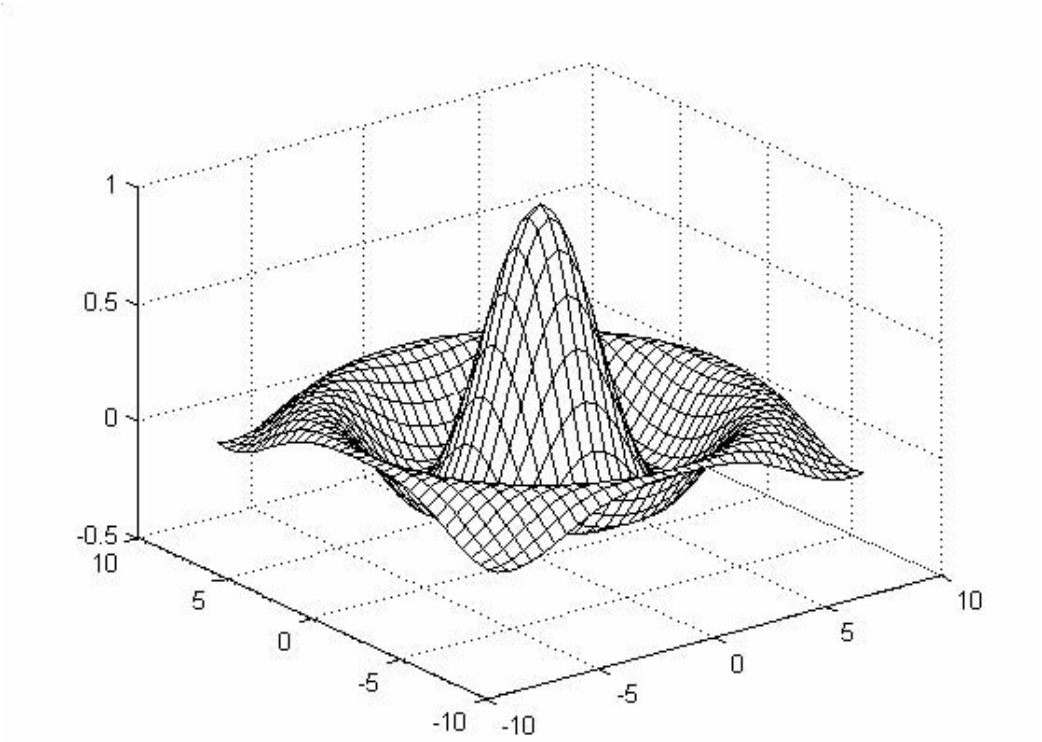
[3] Wang Yongxing, Research on Smarandache problems in number theory, **II**(2005), pp. 103-106.

[4] Yang Cundian and Liu Duansen, On the mean value of a new arithmetical function, Research on Smarandache problems in number theory, **II**(2005), pp. 75-78.

[5] J.Sandor, On certain generalizations of the Smarandache function, Smarandache Notions Journal, **11**(2000), 202-212.

[6] Le Maohua, A conjecture concerning the Smarandache dual function, Smarandache Notions Journal, **14**(2004), 153-155.

[7] Tom M.Apostol, Introduction to Analytic Number Theory, New York: Springer-Verlag. 1976.



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