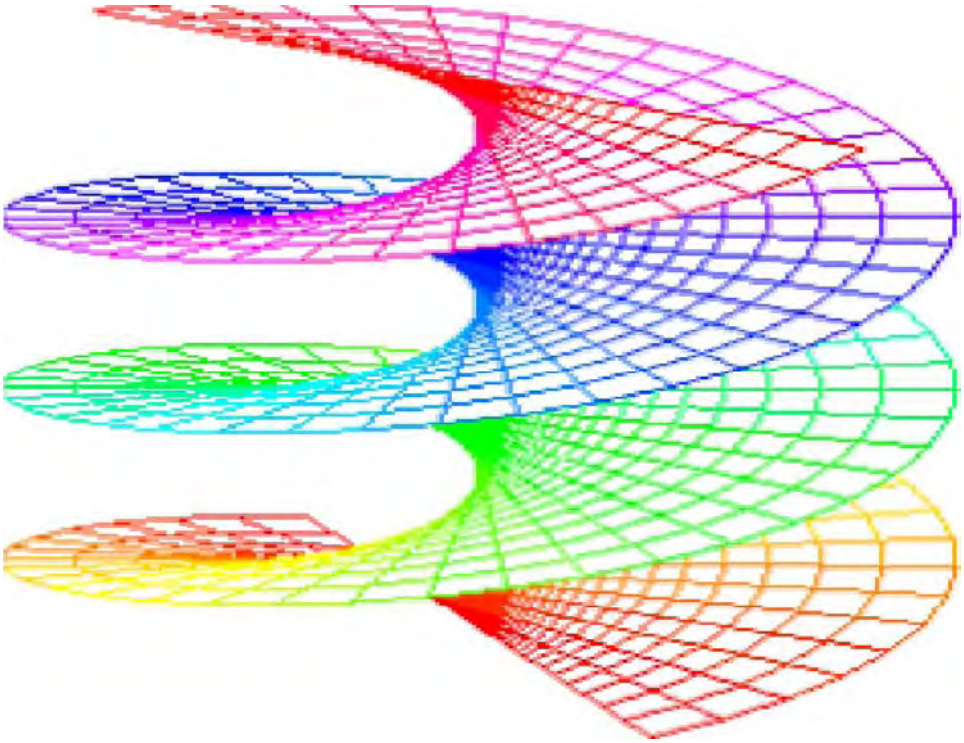
ZHANG WENPENG
W. B. VASANTHA KANDASAMY
editors

Scientia Magna

book series

Vol. 1, No. 2



Hexis
Phoenix
2005

Editors:

Zhang Wenpeng
Department of Mathematics
Northwest University
Xi'an, Shaanxi, P. R. China

W.B.Vasantha Kandasamy
Department of Mathematics
Indian Institute of Technology
IIT Madras, Chennai - 600 036
Tamil Nadu, India

Scientia Magna
- book series (Vol. 1, No. 2) -

Hexis
Phoenix
2005

This book can be ordered in a paper bound reprint from:

Books on Demand
ProQuest Information & Learning
(University of Microfilm International)
300 N. Zeeb Road
P.O. Box 1346, Ann Arbor
MI 48106-1346, USA
Tel.: 1-800-521-0600 (Customer Service)
http://wwwlib.umi.com/bod/basic

ISBN: 1-59973-002-2

Standard Address Number: 297-5092
Printed in the United States of America

Information for Authors

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word XP (or lower), WordPerfect 7.0 (or lower), LaTeX and PDF 6.0 or lower.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 x 11 inches (21,6 x 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writing with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an abstract, followed by the keywords.

All manuscripts are subject to anonymous review by three independent reviewers.

Every letter will be answered.

Each author will receive a free copy of the journal.

Contributing to Scientia Magna book series

Authors of papers in science (mathematics, physics, philosophy, psychology, sociology, linguistics) should submit manuscripts to the main editor:

Prof. Dr. Zhang Wenpeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China, E-mail: wpzhang@nwu.edu.cn and another copy to:

L. Cuciuc, 200 College Road, UNM-Gallup, NM 87301, USA,
E-mail: research@gallup.unm.edu.

Associate Editors

Dr. W. B. Vasantha Kandasamy, Department of Mathematics, Indian Institute of Technology, IIT Madras, Chennai - 600 036, Tamil Nadu, India.

Dr. Larissa Borissova and Dmitri Rabounski, Sirenevi boulevard 69-1-65, Moscow 105484, Russia.

Dr. Liu Huaning, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China, E-mail: hnliu@nwu.edu.cn.

Prof. Yi Yuan, Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China, E-mail: yuanyi@mail.xjtu.edu.cn.

Dr. Xu Zhefeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China, E-mail: zfxu@nwu.edu.cn.

Dr. Zhang Tianping, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi, P.R.China, E-mail: tianpzhang@eyou.com.

Contents

Scientia Magna
Vol. 1 (2005), No. 2, 1-26

Remarks on some of the Smarandache's problem. Part 2

Mladen V. Vassilev[†], Missana and Krassimir T. Atanassov[‡]
[†]5,V.Hugo Str., Sofia-1124, Bulgaria
e-mail:missana@abv.bg
[‡]CLBME-Bulg. Academy of Sci., P.O.Box 12, Sofia-1113, Bulgaria,
e-mail:krat@bas.bg

To Dr. Florentin Smarandache
for his 50th birthday

0. In 1999, the second author of this remarks published a book over 30 of Smarandache's problems in area of elementary number theory (see [1, 2]). After this, we worked over new 20 problems that we collected in our book [28]. These books contain Smarandache's problems, described in [10, 16]. The present paper contains some of the results from [28].

In [16] Florentin Smarandache formulated 105 unsolved problems, while in [10] C.Dumitresu and V. Seleacu formulated 140 unsolved problems of his. The second book contains almost all the problems from [16], but now each problem has unique number and by this reason in [1, 28] and here the authors use the numeration of the problems from [10].

In the text below the following notations are used.
$\mathcal{N}$ - the set of all natural numbers (i.e., the set of all positive integers);
$[x]$ - "floor function" (or also so called "bracket function") - the greatest integer which is not greater than the real non-negative number x;
ζ - Riemann's Zeta-function;
Γ - Euler's Gamma-function;
π - the prime counting function, i.e., $\pi(n)$ denotes the number of prime p such that $p \leq n$;
$]x[$ - the largest natural number strongly smaller than the real (positive) number x;
$\lceil x \rceil$ - the inferior integer part of x, i.e, the smallest integer greater than or equal to x.

For an arbitrary increasing sequence of natural number $C \equiv \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of C, which are not greater than n. When $n < c_1$ we put $\pi_C(n) = 0$.

1. The results in this section are taken from [8].

The second problem from [10] (see also 16-th problem from [16]) is the following:
Smarandache circular sequence:

$$\underbrace{1}_{1}, \underbrace{12, 21}_{2}, \underbrace{123, 231, 312}_{3}, \underbrace{1234, 2341, 3412, 4123}_{4},$$

$$\underbrace{12345, 23451, 34512, 45123, 51234}_{5},$$

$$\underbrace{123456, 234561, 345612, 456123, 561234, 612345}_{6}, \cdots$$

Let $f(n)$ be the n-th member of the above sequence. We shall prove the following

Theorem 1.1. For each natural number n:

$$f(n) = \overline{s(s+1)\dots k12\dots(s-1)},$$

where

$$k \equiv k(n) =]\frac{\sqrt{8n+1}-1}{2}[$$

and

$$s \equiv s(n) = n - \frac{k(k+1)}{2}.$$

2. The results in this section are taken from [25].

The eight problem from [10] (see also 16-th problem from [16]) is the following:

Smarandache mobile periodicals (I):

$$\begin{array}{cccccccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\dots
\end{array}$$

$$
\begin{array}{cccccccccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\ldots \\
\ldots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\ldots \\
\ldots & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\ldots \\
\ldots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\ldots \\
 & & & & . & & & . & & & . & & & \\
 & & & & . & & & . & & & . & & &
\end{array}
$$

This sequence has the form

$$\underbrace{1,111,11011,111,1}_{3},\underbrace{1,111,11011,1100011,11011,111,1}_{7},$$

$$\underbrace{1,111,11011,1100011,110000011,1100011,11011,111,1}_{9},\ldots$$

All digits from the above table generate an infinite matrix A. We described the elements of A.

Let us take a Cartesian coordinate system C with origin in the point containing element "1" in the topmost (i.e., the first) row of A. We assume that this row belongs to the ordinate axis of C (see Fig. 1) and that the points to the right of the origin have positive ordinates.

The above digits generate an infinite sequence of squares, located in the half-plane (determined by C) where the abscissa of the points are nonnegative. Their diameters have the form

$$"110\ldots011".$$

Exactly one of the diameters of each of considered square lies on the abscissa of C. It can be seen (and proved, e.g.,by induction) that the s-th square, denoted by $G_s(s = 0, 1, 2, \ldots)$ has a diameter with length $2s + 4$ and the same square has a highest vertex with coordinates $\langle s^2 + 3s, 0\rangle$ in C and a lowest vertex with coordinates $\langle s^2 + 5s + 4, 0\rangle$ in C.

Let us denote by $a_{k,i}$ an element of A with coordinates $\langle k, i\rangle$ in C.

First, we determine the minimal nonnegative s for which the inequality

$$s^2 + 5s + 4 \geq k$$

holds. We denote it by $s(k)$. Directly it is seen the following

Lemma 2.1 The number $s(k)$ admits the explicit representation:

$$s(k) = \begin{cases} 0, & \text{if} \quad 0 \leq k \leq 4 \\ \left[\frac{\sqrt{4k+9}-5}{2}\right], & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is} \\ & \text{a square of an integer} \\ \left[\frac{\sqrt{4k+9}-5}{2}\right]+1, & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is} \\ & \text{not a square of an integer} \end{cases} \tag{2.1}$$

and the inequality

$$(s(k))^2 + 3s(k) \leq k \leq (s(k))^2 + 5s(k) + 4 \tag{2.2}$$

hold.

Second, we introduce the integer $\delta(k)$ and $\varepsilon(k)$ by

$$\delta(k) \equiv k - (s(k))^2 - 3s(k), \tag{2.3}$$

$$\varepsilon(k) \equiv (s(k))^2 + 5s(k) + 4 - k. \tag{2.4}$$

From (2.2) we have $\delta(k) \geq 0$ and $\varepsilon(k) \geq 0$. Let P_k be the infinite strip orthogonal to the abscissa of C and lying between the straight lines passing through those vertices of the square $G_{s(k)}$ lying on the abscissa of C. Then $\delta(k)$ and $\varepsilon(k)$ characterize the location of point with coordinate $\langle k, i\rangle$ in C in strip P_k. Namely, the following assertion is true.

Proposition 2.1. The elements $a_{k,i}$ of the infinite matrix A are described as follows:
if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \end{cases}$$

if $k \geq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \geq |i| + 2, \\ 1, & \text{if } |i| \leq \varepsilon(k) \leq |i| + 1 \end{cases}$$

where here and below $s(k)$ is given by (2.1), $\delta(k)$ and $\varepsilon(k)$ are given by (2.3) and (2.4), respectively.

Below, we propose another description of elements of A, which can be proved (e.g., by induction) using the same considerations.

$$a_{k,i} = \begin{cases} 1, & \text{if } \langle k, i\rangle \in \\ & \{\langle (s(k))^2 + 3s(k), 0\rangle, \langle (s(k))^2 + 5s(k) + 4, 0\rangle\} \\ & \bigcup\{\langle (s(k))^2 + 3s(k) + j, -j\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, -j + 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j - 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j\rangle : 1 \leq j \leq s(k) + 2\} \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j + 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j - 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j\rangle : \\ & 1 \leq j \leq s(k) + 1\} \\ 0, & \text{otherwise} \end{cases}$$

Similar representations are possible for the ninth, tenth and eleventh problems. In [28] we introduce eight modifications of these problems, giving formulae for their (k, i)-th members $a_{k,i}$.

Essentially more interesting is Problem 103 from [10]:

Smarandache numerical carpet:
has the general form

$$
\begin{array}{ccccccccccccccc}
 & & & & & & . & & & & & & & & \\
 & & & & & & . & & & & & & & & \\
 & & & & & & . & & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & a & 1 & & & & & & & \\
 & & & & 1 & a & b & a & 1 & & & & & & \\
 & & & 1 & a & b & c & b & a & 1 & & & & & \\
 & & 1 & a & b & c & d & c & b & a & 1 & & & & \\
 & 1 & a & b & c & d & e & d & c & b & a & 1 & & & \\
1 & a & b & c & d & e & f & e & d & c & b & a & 1 & & \\
 & 1 & a & b & c & d & e & d & c & b & a & 1 & & & \\
 & & 1 & a & b & c & d & c & b & a & 1 & & & & \\
 & & & 1 & a & b & c & b & a & 1 & & & & & \\
 & & & & 1 & a & b & a & 1 & & & & & & \\
 & & & & & 1 & a & 1 & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & . & & & & & & & & \\
 & & & & & & . & & & & & & & & \\
 & & & & & & . & & & & & & & &
\end{array}
$$

On the border of level 0, the elements are equal to "1";
they form a rhomb.
Next, on the border of level 1, the elements are equal to "a";
where "a" is the sum of all elements of the previous border;
the "a"s form a rhomb too inside the previous one.
Next again, on the border of level 2, the elements are equal to "b";

where "b" is the sum of all elements of the previous border;

the "b"s form a rhomb too inside the previous one.

And so on ...

The above square, that Smarandache named *"rhomb"*, corresponds to the square from our construction for the case of $s = 6$, if we begin to count from $s = 1$, instead of $s = 0$. In [10] a particular solution of the Problem 103 is given, but there a complete solution is not introduced. We will give a solution below firstly for the case of Problem 103 and then for a more general case.

It can be easily seen that the number of the elements of the s-th square side is $s + 2$ and therefore the number of the elements from the contour of this square is just equal to $4s + 4$.

The s-th square can be represented as a set of sub-squares, each one included in the next. Let us number them inwards, so that the outmost (boundary) square is the first one. As it is written in Problem 103, all of its elements are equal to 1. Hence, the value of the elements of the subsequent (second) square will be (using also the notation from problem 103):

$$a_1 = a = (s + 2) + (s + 1) + (s + 1) + s = 4(s + 1);$$

the value of the elements of the third square will be

$$a_2 = b = a(4(s - 1) + 4 + 1) = 4(s + 1)(4s + 1);$$

the value of the elements of the fourth square will be

$$a_3 = c = b(4(s - 2) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3);$$

the value of the elements of the fifth square will be

$$a_4 = d = c(4(s - 3) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3)(4s - 7);$$

etc.,where the square, corresponding to the initial square *(rhomb)*, from Problem 103 has the form

$$\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & 1 & a_1 & \cdot & \cdot & \cdot & a_1 & 1 & \\
 & & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & 1 \\
1 & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_3 & a_2 & a_1 & 1 \\
 & & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & 1 \\
 & & & 1 & a_1 & \cdot & \cdot & \cdot & a_1 & 1 & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & 1 & & & & & &
\end{array}$$

It can be proved by induction that the elements of this square that stay on t-th place are given by the formula

$$a_t = 4(s+1)\prod_{i=0}^{t-2}(4s+1-4i).$$

If we would like to generalize the above problem, we can construct, e.g., the following extension:

$$\begin{array}{ccccccccccc}
 & & & & x & & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & x & a_1 & \cdot & \cdot & \cdot & a_1 & x & \\
 & & x & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & x \\
x & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_3 & a_2 & a_1 & x \\
 & & x & a_1 & a_2 & \cdot & \cdot & \cdot & a_2 & a_1 & x \\
 & & & x & a_1 & \cdot & \cdot & \cdot & a_1 & x & \\
 & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & x & & & & & &
\end{array}$$

where x is given number. Then we obtain

$$a_1 = 4(s+1)x$$

$$a_2 = 4(s+1)(4s+1)x$$

$$a_3 = 4(s+1)(4s+1)(4s-3)x$$

$$a_4 = 4(s+1)(4s+1)(4s-3)(4s-7)x$$

etc. and for $t \geq 1$

$$a_t = 4(s+1)\prod_{i=0}^{t-2}(4s+1-4i)x.$$

where it assumed that

$$\prod_{i=0}^{-1}\cdot = 1.$$

3. The results in this section are taken from [21].

The 15-th Smarandache's problem from [10] is the following: *"Smarandache's simple numbers":*

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27,$$

$$29, 31, 33, \ldots$$

A number n is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to n. Generally speaking, n has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where p and q are distinct primes".

Let us denote: by S - the sequence of all Smarandache's simple numbers and by s_n - the n-th term of S; by $\mathcal{P}$ - the sequence of all primes and by p_n - the n-th term of $\mathcal{P}$; by $\mathcal{P}^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $\mathcal{P}^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $\mathcal{PQ}$ - the sequence $\{p.q\}_{p,q} \in \mathcal{P}$, where $p < q$.

In the present section we find $\pi_S(n)$ in an explicit form and using this, we find the n-th term of S in explicit form, too.

First, we note that instead of $\pi_P(n)$ we use the notation $\pi(n)$.

Hence

$$\pi_{\mathcal{P}^2}(n) = \pi(\sqrt{n}), \pi_{\mathcal{P}^3}(n) = \pi(\sqrt[3]{n}),$$

Thus, using the definition of S, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt[3]{n}) + \pi_{\mathcal{PQ}}(n) \tag{4.1}$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in [18]. Other explicit formulae for $\pi(n)$ are given in [14]. One of them is known as Mináč's formula. It is given below

$$\pi(n) = \sum_{k=2}^{n} [\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]]. \tag{4.2}$$

Therefore, the problem of finding of explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt[3]{n})$ is solved successfully. It remains only to express $\pi_{\mathcal{PQ}}(n)$ in an explicit form.

Let $k \in \{1, 2, \ldots, \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_k q$, which $p \in \mathcal{P}$, $q > p_k$ for which $p_k.q \leq n$. The quality of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(\frac{n}{p_k}) - k. \tag{4.3}$$

When $k = 1, 2, \ldots, \pi(\sqrt{n})$, the number $p_k.q$, as defined above, describe all numbers of the kind $p.q$, with $p, q \in \mathcal{P}$, $p < q, p.q < n$. But the quantity of the last numbers is equal to $\pi_{\mathcal{PQ}}(n)$. Hence

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})} (\pi(\frac{n}{p_k}) - k), \tag{4.4}$$

because of (4.3). The equality (4.4), after a simple computation yields the formula

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) - \frac{\pi(\sqrt{n})(\pi(\sqrt{n})+1)}{2}. \tag{4.5}$$

In [20] the identity

$$\sum_{k=1}^{\pi(b)} \pi(\frac{n}{p_k}) = \pi(\frac{n}{b}).\pi(b) + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\frac{n}{b})} \pi(\frac{n}{p_{\pi(\frac{n}{b})+k}}) \tag{4.6}$$

is proved, under the condition $b > 2$ (b is a real number). When $\pi(\frac{n}{2}) = \pi(\frac{n}{b})$, the right hand-side of (4.6) is reduced to $\pi(\frac{n}{b}).\pi(b)$. In the case $b = \sqrt{n}$ and $n \geq 4$ equality (4.6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})} \pi(\frac{n}{p_k}) = (\pi(\sqrt{n}))^2 + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})} \pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{4.7}$$

If we compare (4.5) with (4.7) we obtain for $n \geq 4$

$$\pi_{\mathcal{PQ}}(n) = \frac{\pi(\sqrt{n})(\pi(\sqrt{n})-1)}{2} + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})} \pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{4.8}$$

Thus, we have two different explicit representations for $\pi_{\mathcal{PQ}}(n)$. These are formulae (4.5) and (4.8). We note that the right hand side of (4.8) reduces to $\frac{\pi(\sqrt{n})(\pi(\sqrt{n})-1)}{2}$, when $\pi(\frac{n}{2}) = \pi(\sqrt{n})$.

Finally, we observe that (4.1) gives an explicit representation for $\pi_S(n)$, since we may use formula (4.2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (4.5), or (4.8) for $\pi_{\mathcal{PQ}}(n)$.

The following assertion solves the problem for finding of the explicit representation of s_n.

Theorem 4.1. The n-th term s_n of S admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\theta(n)} [\frac{1}{1+[\frac{\pi_S(n)}{n}]}]; \tag{4.9}$$

$$s_n = -2\sum_{k=0}^{\theta(n)} \zeta(-2[\frac{\pi_S(n)}{n}]); \tag{4.10}$$

$$s_n = \sum_{k=0}^{\theta(n)} \frac{1}{\Gamma(1-[\frac{\pi_S(n)}{n}])}, \tag{4.11}$$

where

$$\theta(n) \equiv [\frac{n^2+3n+4}{4}],\ n = 1, 2, \ldots$$

We note that (4.9)-(4.11) are representations using, respectively, "floor function", Riemann's Zeta-function and Euler's Gamma-function. Also, we note that in (4.9)-(4.11) $\pi_S(k)$ is given by (4.1), $\pi(k)$ is given by (4.2) (or by others formulae like (4.2)) and $\pi_{\mathcal{PQ}}(n)$ is given by (4.5), or by (4.8). Therefore, formulae (4.9)-(4.11) are explicit.

4. The results in this section are taken from [6].

The 17-th problem from [10] (see also the 22-nd problem from [16]) is the following:

Smarandache's digital products:

$$\underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{0,1,2,3,4,5,6,7,8,9},$$

$$\underbrace{0,2,4,6,8,10,12,14,16,18}, \underbrace{0,3,6,9,12,15,18,21,24,27},$$

$$\underbrace{0,4,8,12,16,20,24,28,32,36}, \underbrace{0,5,10,15,20,25\ldots}$$

($d_p(n)$is the product of digits.)

Let the fixed natural number n have the form $n = \overline{a_1 a_2 \ldots a_k}$, where $a_1, a_2, \ldots, a_k \in \{0, 1, \ldots, 9\}$ and $a_1 \geq 1$. Therefore,

$$n = \sum_{i=1}^{k} a_i 10^{i-1}.$$

Hence, $k = [\log_{10} n] + 1$ and

$$a_1(n) \equiv a_1 = [\frac{n}{10^{k-1}}],$$

$$a_2(n) \equiv a_2 = [\frac{n - a_1 10^{k-1}}{10^{k-2}}],$$

$$a_3(n) \equiv a_3 = [\frac{n - a_1 10^{k-1} - a_2 10^{k-2}}{10^{k-3}}],$$

$$\ldots$$

$$a_{[\log_{10}(n)]}(n) \equiv a_{k-1} = [\frac{n - a_1 10^{k-1} - \ldots - a_{k-2} 10^2}{10}],$$

$$a_{[\log_{10}(n)]+1}(n) \equiv a_k = n - a_1 10^{k-1} - \ldots - a_{k-1} 10.$$

Obviously, $k, a_1, a_2, \ldots, a_k$ are functions only of n. Therefore,

$$d_p(n) = \prod_{i=1}^{[\log_{10}(n)]+1} a_i(n).$$

5. The results in this section are taken from [4, 27].

The 20-th problem from [10] is the following (see also Problem 25 from [16]):
Smarandache devisor products:

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19,$$

$$8000, 441, 484, 23, 331776, 125, 676, 729, 21952, 29, 810000, 31, 32768,$$

$$1089, 1156, 1225, 10077696, 37, 1444, 1521, 2560000, 41, \ldots$$

($P_d(n)$ is the product of all positive divisors of n.)

The 21-st problem from [10] is the following (see also Problem 26 from [16]):
Smarandache proper devisor products:

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1,$$

$$13824, 5, 26, 27, 784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39,$$

$$64000, 1, \ldots$$

($p_d(n)$ is the product of all positive divisors of n but n.)

Let us denote by $\tau(n)$ the number of all devisors of n. It is well-known (see, e.g., [13]) that

$$P_d(n) = \sqrt{n^{\tau(n)}} \tag{6.1}$$

and of course, we have

$$p_d(n) = \frac{P_d(n)}{n}. \tag{6.2}$$

But (6.1) is not a good formula for $P_d(n)$, because it depends on function τ and to express $\tau(n)$ we need the prime number factorization of n.

Below, we give other representations of $P_d(n)$ and $p_d(n)$, which do not use the prime number factorization of n.

Proposition 6.1. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{6.3}$$

holds.

Here and further the symbols

$$\prod_{k/n} \cdot \quad \text{and} \quad \sum_{k/n} \cdot$$

mean the product and the sum, respectively, of all divisors of n.

Let

$$\theta(n,k) \equiv [\frac{n}{k}] - [\frac{n-1}{k}]$$
$$= \begin{cases} 1, & \text{if} \quad k \quad \text{is a divisor of} \quad n \\ 0, & \text{otherwise} \end{cases}$$

The following assertion is obtained as a corollary of (6.2) and (6.3).

Proposition 6.2. For $n \geq 1$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{[\frac{n}{k}]-[\frac{n-1}{k}]}$$

holds.

For $n = 1$ we have

$$p_d(1) = 1.$$

Proposition 6.3. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{6.5}$$

holds, where here and further we assume that $0! = 1$.

Now (6.2) and (6.5) yield.

Proposition 6.4. For $n \geq 2$ representation

$$p_d(n) = \prod_{k=2}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!}$$

holds.

Another type of representation of $p_d(n)$ is the following

Proposition 6.5. For $n \geq 3$ representation

$$p_d(n) = \prod_{k=1}^{n-2} (k!)^{\theta(n,k)-\theta(n,k+1)},$$

where $\theta(n,k)$ is given by (6.4).

Further, we need the following

Theorem 6.1. [22] For $n \geq 2$ the identity

$$\prod_{k=2}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n-1}(k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.6}$$

holds.

Now, we shall deduce some formulae for

$$\prod_{k=1}^{n} P_d(k) \quad \text{and} \quad \prod_{k=1}^{n} p_d(k).$$

Proposition 6.6. Let f be an arbitrary arithmetic function. then the identity

$$\prod_{k=1}^{n}(P_d(k))^{f(k)} = \prod_{k=1}^{n} k^{\rho(n,k)} \tag{6.7}$$

holds, where

$$\rho(n,k) = \sum_{s=1}^{[\frac{n}{k}]} f(ks).$$

Now we need the following

Lemma 6.1. For $n \geq 1$ the identity

$$\prod_{k=1}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n} k^{[\frac{n}{k}]}$$

holds.

Proposition 6.7. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n}[\frac{n}{k}]! \tag{6.8}$$

holds. As a corollary from (6.2) and (6.8), we also obtain

Proposition 6.8. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=2}^{n}[\frac{n}{k}]! \tag{6.9}$$

holds.

From (6.6) and (6.9), we obtain

Proposition 6.9. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=1}^{n-1}(k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.10}$$

holds.

As a corollary from (6.10) we obtain, because of (6.2)

Proposition 6.10. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{6.10}$$

holds.

Now, we return to (6.7) and suppose that

$$f(k) > 0 \ (k = 1, 2, \ldots).$$

Then after some simple computations we obtain

Proposition 6.11. For $n \geq 1$ representation

$$P_d(k) = \prod_{k=1}^{n} k^{\sigma(n,k)} \tag{6.11}$$

holds, where

$$\sigma(n,k) = \frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)}.$$

For $n \geq 2$ representation

$$p_d(k) = \prod_{k=1}^{n-1} k^{\sigma(n,k)} \tag{6.12}$$

holds.

Note that although f is an arbitrary arithmetic function, the situation with (6.11) and (6.12) is like the case $f(x) \equiv 1$, because

$$\frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)} = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

Finally, we use (6.7) to obtain some new inequalities, involving $P_d(k)$ and $p_d(k)$ for $k = 1, 2, \ldots, n$.

Putting

$$F(n) = \sum_{k=1}^{n} f(k)$$

we rewrite (6.7) as

$$\prod_{k=1}^{n} (P_d(k))^{\frac{f(k)}{F(n)}} = \prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(F(n))}. \tag{6.13}$$

Then we use the well-known Jensen's inequality

$$\sum_{k=1}^{n} \alpha_k x_k \geq \prod_{k=1}^{n} x_k^{\alpha_k},$$

that is valid for arbitrary positive numbers x_k, $\alpha_k (k = 1, 2, \ldots, n)$ such that

$$\sum_{k=1}^{n} \alpha_k = 1,$$

for the case:

$$x_k = P_d(k),$$
$$\alpha_k = \frac{f(k)}{F(n)}.$$

Thus we obtain from (6.13) inequality

$$\sum_{k=1}^{n} f(k).P_d(k) \geq (\sum_{k=1}^{n} f(k)).\prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(\sum_{s=1}^{n} f(s))}. \tag{6.14}$$

If $f(x) \equiv 1$, then (6.14) yields the inequality

$$\frac{1}{n}\sum_{k=1}^{n} P_d(k) \geq \prod_{k=1}^{n} (\sqrt[n]{k})^{[\frac{n}{k}]}.$$

If we put in (6.14)

$$f(k) = \frac{g(k)}{k}$$

for $k = 1, 2, \ldots, n$, then we obtain

$$\sum_{k=1}^{n} g(k).p_d(k) \geq (\sum_{k=1}^{n} \frac{g(k)}{k}).\prod_{k=1}^{n} (\sqrt[k]{k})^{(\sum_{s=1}^{[\frac{n}{k}]} \frac{g(ks)}{s})/(\sum_{s=1}^{n} \frac{g(s)}{s})}. \tag{6.15}$$

because of (6.2).

Let $g(x) \equiv 1$. Then (6.15) yields the very interesting inequality

$$(\frac{1}{H_n}\sum_{k=1}^{n} p_d(k))^{H_n} \geq \prod_{k=1}^{n} (\sqrt[k]{k})^{H_{[\frac{n}{k}]}},$$

where H_m denotes the m-th partial sum of the harmonic series, i.e.,

$$H_m = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m}.$$

All of the above inequalities become equalities if and only if $n = 1$.

6. The results in this section are taken from [29].

The 25-th and the 26-th problems from [10] (see also the 30-th and the 31-st problems from [16]) are the following:

Smarandache's cube free sieve:

$$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26,$$

$$28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 49, 50,$$

$$51, 52, 53, 55, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 73, \ldots$$

Definition: from the set of natural numbers (except 0 and 1):
- take off all multiples of 2^3 *(i.e. 8,16,24,32,40,...)*

- take off all multiples of 3^3
- take off all multiples of 5^3
. . . and so on (take off all multiples of all cubic primes).

Smarandache's m-power free sieve:
Definition: from the set of natural numbers (except 0 and 1) take off all multiples of 2^m*, afterwards all multiples of* 3^m *. . . and so on(take off all multiples of all m-power primes,* $m \geq 2$*). (One obtains all m-power free numbers.)*

Here we introduce the solution for both of these problems.

For every natural number m we denote the increasing sequence $a_1^{(m)}, a_2^{(m)}, a_3^{(m)}, \ldots$ of all m-power free numbers by $\overline{m}$. Then we have

$$\emptyset \equiv \overline{1} \subset \overline{2} \subset \ldots \subset \overline{(m-1)} \subset \overline{m} \subset \overline{(m+1)} \subset \ldots$$

Also, for $m \geq 2$ we have

$$\overline{m} = \bigcup_{k=1}^{m-1} (\overline{2})^k$$

where

$$(\overline{2})^k = \{x | (\exists x_1, \ldots, x_k \in \overline{2})(x = x_1.x_2 \ldots x_k)\}$$

for each natural number $k \geq 1$.

Let us consider $\overline{m}$ as an infinite sequence for $m = 2, 3, \ldots$. Then $\overline{2}$ is a subsequence of $\overline{m}$. Therefore, the inequality

$$a_n^{(m)} \leq a_n^{(2)}$$

holds for $n = 1, 2, 3, \ldots$.

Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$ be the sequence of all primes. It is obvious that this sequence is a subsequence of $\overline{2}$. Hence the inequality

$$a_n^{(2)} \leq p_n$$

holds for $n = 1, 2, 3, \ldots$. But it is well-known that

$$p_n \leq \theta(n) \equiv [\frac{n^2 + 3n + 4}{4}]$$

(see [12]). Therefore, for any $m \geq 2$ and $n = 1, 2, 3, \ldots$ we have

$$a_n^{(m)} \leq a_n^{(2)} \leq \theta(n).$$

Hence, there exits $\lambda(n)$ such that $\lambda(n) \leq \theta(n)$ and inequality:

$$a_n^{(m)} \leq a_n^{(2)} \leq \lambda(n).$$

holds. In particular, it is possible to use $\theta(n)$ instead of $\lambda(n)$.

In [28] we find the following explicit formulae for $a_n^{(m)}$ when $m \geq 2$ is fixed:

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} [\frac{1}{1 + [\frac{\pi_{\overline{m}}(k)}{n}]}]; \tag{7.1}$$

$$a_n^{(m)} = -2\sum_{k=0}^{\lambda(n)} \zeta(-2[\frac{\pi_{\overline{m}}(k)}{n}]); \tag{7.2}$$

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} \frac{1}{\Gamma(1-[\frac{\pi_{\overline{m}}(k)}{n}])}. \tag{7.3}$$

Thus, the 26-th Smarandache's problem is solved and for $m = 3$ the 25-th Smarandache's problem is solved, too.

The following problems are interesting.

Problem 7.1. Does there exist a constant $C > 1$, such that $\lambda(n) \leq C.n$?

Problem 7.2. Is $C \leq 2$?

Below we give the main explicit representation of function $\pi_{\overline{m}}(n)$, that takes part in formulae (7.1) - (7.3). In this way we find the main explicit representation for $a_n^{(m)}$, that is based on formulae (7.1) - (7.3), too.

Theorem 7.1. Function $\pi_{\overline{m}}(n)$ allows representation

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s \in \overline{2} \bigcap \{2,3,\ldots,[\sqrt[m]{n}]\}} (-1)^{\omega(s)}.[\frac{n}{s^m}],$$

where $\omega(s)$ denotes the number of all different prime divisors of s.

7. The results in this section are taken from [24].

The 28-th problem from [10] (see also the 94-th problem from [16]) is the following:
Smarandache odd sieve:

$$7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 83,$$

$$85, 91, 93, 97, \ldots$$

(All odd numbers that are not equal to the difference of two primes).
A sieve is to get this sequence:
- subtract 2 from all prime numbers and obtain a temporary sequence;
- choose all odd numbers that do not belong to the temporary one.

We find an explicit form of the n-th term of the above sequence, that will be denoted by $C = \{C_n\}_{n=1}^{\infty}$ below.

Firstly, we shall note that the above definition of C can be interpreted to the following equivalent form as follows, having in mind that every odd number is a difference of two prime numbers if and only if it is a difference of a prime number and 2:

Smarandache's odd sieve contains exactly these odd numbers that cannot be represented as a difference of a prime and 2.

We rewrite the last definition to the following equivalent form, too:

Smarandache's odd sieve contains exactly these odd numbers that are represented as a difference of a composite odd number and 2.

We find an explicit form of the n-th term of the above sequence, using the third definition of it. Initially, we use the following two assertions.

Lemma 8.1. For every natural number $n \geq 1$, C_{n+1} is exactly one of the numbers: $u \equiv C_n + 2, v \equiv C_n + 4$ or $w \equiv C_n + 6$.

Corollary 8.1. For every natural number $n \geq 1$:

$$C_{n+1} \leq C_n + 6.$$

Corollary 8.2. For every natural number $n \geq 1$:

$$C_n \leq 6n + 1. \tag{8.1}$$

Now, we return to the Smarandache's problem.

In [18] the following three universal explicit formulae are introduced, using numbers $\pi_C(k)$ $(k = 0, 1, 2, \ldots)$, that can be used to represent numbers C_n:

$$C_n = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = -2 \sum_{k=0}^{\infty} \zeta(-2[\frac{\pi_C(k)}{n}]),$$

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}.$$

For the present case, having in mind (8.1), we substitute symbol ∞ with $6n + 1$ in sum $\sum_{k=0}^{\infty}$ for C_n and we obtain the following sums:

$$C_n = \sum_{k=0}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{8.2}$$

$$C_n = -2 \sum_{k=0}^{6n+1} \zeta(-2[\frac{\pi_C(k)}{n}]), \tag{8.3}$$

$$C_n = \sum_{k=0}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{8.4}$$

We must explain why $\pi_C(n)$ $(n = 1, 2, 3, \ldots)$ is represented in an explicit form. It can be directly seen that the number of the odd numbers, that are not bigger than n, is exactly equal to

$$\alpha(n) = n - [\frac{n}{2}], \tag{8.5}$$

because the number of the even numbers that are not greater than n is exactly equal to $[\frac{n}{2}]$.

Let us denote by $\beta(n)$ the number of all odd numbers not bigger than n, that can be represented as a difference of two primes. According to the second form of the above given definition, $\beta(n)$ coincides with the number of all odd numbers m such that $m \leq n$ and m has the form $m = p - 2$, where p is an odd prime number. Therefore, we must study all odd prime numbers, because of the inequality $m \leq n$. The number of these prime numbers is exactly $\pi(n + 2) - 1$. therefore,

$$\beta(n) = \pi(n + 2) - 1. \tag{8.6}$$

Omitting from the number of all odd numbers that are not greater than n the quantity of those numbers that are a difference of two primes, we find exactly the quantity of these odd numbers that are not greater than n and that are not a difference of two prime numbers, i.e., $\pi_C(n)$. Therefore, the equality

$$\pi_C(n) = \alpha(n) - \beta(n)$$

holds and from (8.5) and (8.6) we obtain:

$$\pi_C(n) = (n - [\frac{n}{2}]) - (\pi(n+2) - 1) = n + 1 - [\frac{n}{2}] - \pi(n+2).$$

But $\pi(n+2)$ can be represented in an explicit form, e.g., by Miná̈c's formula and therefore, we obtain that the explicit form of $\pi_C(N)$ is

$$\pi_C(N) = N + 1 - [\frac{N}{2}] - \sum_{k=2}^{N+2}[\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]], \tag{8.7}$$

where $N \geq 1$ is a fixed natural number.

It is possible to put $[\frac{N+3}{2}]$ instead of $N + 1 - [\frac{N}{2}]$ into (8.7).

Now, using each of the formulae (8.2) - (8.4), we obtain C_n in an explicit form, using (8.7).

It can be checked directly that

$$C_1 = 7, C_2 = 13, C_3 = 19, C_4 = 23, C_5 = 25, C_6 = 31,$$

$$C_7 = 33, \ldots$$

and

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = \pi_C(3) = \pi_C(4) = \pi_C(5) = \pi_C(6) = 0.$$

Therefore from (8.2) - (8.4) we have the following explicit formulae for C_n

$$C_n = 7 + \sum_{k=7}^{6n+1}[\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = 7 + -2\sum_{k=7}^{6n+1}\zeta(-2.[\frac{\pi_C(k)}{n}]),$$

$$C_n = 7 + \sum_{k=7}^{6n+1}\frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])},$$

where $\pi_C(k)$ is given by (8.7).

8. The results in this section are taken from [7, 26].

The 46-th Smarandache's problem from [10] is the following:

Smarandache's prime additive complements;

$$1, 0, 0, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 5, 4, 3, 2, 1,$$

$$0, 1, 0, 5, 4, 3, 2, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0 \ldots$$

(For each n to find the smallest k such that $n+k$ is prime.)

Remarks: Smarandache asked if it is possible to get as large as we want but finite decreasing $k, k-1, k-2, \ldots, 2, 1, 0$ (odd k) sequence included in the previous sequence - i.e., for any even integer are there two primes those difference is equal to it? He conjectured the answer is negative.

Obviously, the members of the above sequence are differences between first prime number that is greater or equal to the current natural number n and the same n. It is well-known that the number of primes smaller than or equal to n is $\pi(n)$. Therefore, the prime number smaller than or equal to n is $p_{\pi(n)}$. Hence, the prime number that is greater than or equal to n is the next prime number, i.e., $p_{\pi(n)+1}$. Finally, the n-th member of the above sequence will be equal to

$$\begin{cases} p_{\pi(n)+1} - n, & \text{if } n \text{ is not a prime number} \\ 0, & \text{otherwise} \end{cases}$$

We shall note that in [3] the following new formula p_n for every natural number n is given:

$$p_n = \sum_{i=0}^{\theta(n)} sg(n - \pi(i)),$$

where $\theta(n) = [\frac{n^2+3n+4}{4}]$ and

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Let us denote by a_n the n-th term of the above sequence. Next, we propose a way for obtaining an explicit formula for a_n $(n = 1, 2, 3, \ldots)$. Extending the below results, we give an answer to the Smarandache's question from his own remark in [10]. At the end, we propose a generalization of Problem 46 and present a proof of an assertion related to Smarandache's conjecture for Problem 46.

Proposition 9.1. a_n admits the representation

$$a_n = p_{\pi(n-1)+1} - n, \tag{9.1}$$

where $n = 1, 2, 3, \ldots$, π is the prime counting function and p_k is the k-th term of prime number sequence.

It is clear that (9.1) gives an explicit representation for a_n since several explicit formulae for $\pi(k)$ and p_k are known (see, e.g. [14]).

Let us define

$$n(m) = m! + 2.$$

Then all numbers

$$n(m), n(m)+1, n(m)+2, \ldots, n(m)+m-2$$

are composite. Hence

$$a_{n(m)} \geq m - 1.$$

This proves the Smarandache's conjecture, since m may grow up to infinity. Therefore $\{a_n\}_{n=1}^{\infty}$ is unbounded sequence.

Now, we shall generalize Problem 46.

Let

$$\mathbf{c} \equiv c_1, c_2, c_3, \ldots$$

be a strictly increasing sequence of positive integers.

Definition. Sequence

$$\mathbf{b} \equiv b_1, b_2, b_3, \ldots$$

is called c-additive complement of $\mathbf{c}$ if and only if b_n is the smallest non-negative integer, such that $n + b_n$ is a term of $\mathbf{c}$.

The following assertion generalizes Proposition 1.

Proposition 9.2. b_n admits the representation

$$b_n = c_{\pi_c(n-1)+1} - n,$$

where $n = 1, 2, 3, \ldots$, $\pi_c(n)$ is the counting function of $\mathbf{c}$, i.e., $\pi_c(n)$ equals to the quantity of c_m, $m = 1, 2, 3, \ldots$, such that $c_m \leq n$.

Let

$$d_n \equiv c_{n+1} - c_n \ (n = 1, 2, 3, \ldots).$$

The following assertion is related to the Smarandache's conjecture from Problem 46.

Proposition 9.3. If $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, then $\{b_n\}_{n=1}^{\infty}$ is unbounded sequence, too.

Open Problem. Formulate necessary conditions for the sequence $\{b_n\}_{n=1}^{\infty}$ to be unbounded.

9. The results in this section are taken from [23].

Solving of the Diophantine equation

$$2x^2 - 3y^2 = 5 \tag{10.1}$$

i.e.,

$$2x^2 - 3y^2 - 5 = 0$$

was put as an open Problem 78 by F. Smarandache in [16]. In [28] this problem is solved completely. Also, we consider here the Diophantine equation

$$l^2 - 6m^2 = -5,$$

i.e.,

$$l^2 - 6m^2 + 5 = 0$$

and the Pellian equation

$$u^2 - 6v^2 = 1,$$

i.e.,

$$u^2 - 6v^2 - 1 = 0.$$

In [28] we introduce a generalization of the Smarandache's problem 78 from [16].

If we consider the Diophantine equation

$$2x^2 - 3y^2 = p, \tag{10.2}$$

where $p \neq 2$ is a prime number, then using [13], Chapter VII, exercise 2 and the same method as in the case of (10.1), we obtain the following result.

Theorem 10.1. (1) The necessary and sufficient condition for solvability of (10.2) is:

$$p \equiv 5(\text{mod}24) \text{ or } p \equiv 23(\text{mod}24) \tag{10.3}$$

(2) if (10.3) is valid, then there exist exactly one solution $< x, y > \in \mathcal{N}^2$ of (10.2) such that the inequalities

$$x < \sqrt{\frac{3}{2}.p}$$

and

$$y < \sqrt{\frac{3}{2}.p}$$

holds. Every other solution $< x, y > \in \mathcal{N}^2$ of (10.2) has the form:

$$x = l + 3m$$

$$y = l + 2m,$$

where $< l, m > \in \mathcal{N}^2$ is a solution of the Diophantine equation

$$l^2 - 6m^2 = -p.$$

The problem how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [13].

10. The results in this section are taken from [9]. In [15, 17] F. Smarandache formulates the following four problems:

Problem 1. *Let p be an integer ≥ 3. Then:*

p is a prime if and only if

$$(p-3)! \textit{ is congruent to } \frac{p-1}{2}(\text{mod}p).$$

Problem 2. *Let p be an integer ≥ 4. Then:*

p is a prime if and only if

$$(p-4)! \textit{ is congruent to } (-1)^{\lceil \frac{p}{3} \rceil + 1} \lceil \frac{p+1}{6} \rceil (\text{mod}p). \tag{11.1}$$

Problem 3. *Let p be an integer ≥ 5. Then:*

p is a prime if and only if

$$(p-5)! \text{ is congruent to } rh + \frac{r^2-1}{24} (\text{mod} p), \tag{11.2}$$

with $h = \lceil \frac{p}{24} \rceil$ *and* $r = p - 24$.

Problem 4. *Let* $p = (k-1)!h + 1$ *be a positive integer* $k > 5$, h *natural number. Then:*

p *is a prime if and only if*

$$(p-k)! \text{ is congruent to } (-1)^t h (\text{mod} p). \tag{11.3}$$

with $t = h + \lceil \frac{p}{h} \rceil + 1$.

Everywhere above $\lceil x \rceil$ means the inferior integer part of x, i.e., the smallest integer greater than or equal to x.

In [28] we discussed these four problems.

Problem 1. Admits the following representation:
Let $p \geq 3$ *be an odd number. Then:*

$$p \text{ is a prime if and only if } (p-3)! \equiv \frac{p-1}{2} (mod p).$$

Different than Smarandache's proof of this assertion is given in [28].

Problem 2. Is false, because, for example, if $p = 7$, then (11.1) obtains the form

$$6 \equiv (-1)^4 2 (\text{mod} 7),$$

where

$$6 = (7-4)!$$

and

$$(-1)^4 2 = (-1)^{\lceil \frac{7}{3} \rceil + 1} \lceil \frac{8}{6} \rceil,$$

i.e.,

$$6 \equiv 2 (\text{mod} 7),$$

which is impossible.

Problem 3. Can be modified, having in mind that from $r = p - 24h$ it follows:

$$rh + \frac{r^2-1}{24} = (p-24h).h + \frac{p^2 - 48ph + 24^2h^2 - 1}{24}$$
$$= ph - 24h^2 + \frac{p^2-1}{24} - 2ph + 24h^2 = \frac{p^2-1}{24} - ph,$$

i.e., (11.2) has the form

p *is a prime if and only if*

$$(p-5)! \text{ is congruent to } \frac{p^2-1}{24} (\text{mod} p),$$

Different than the Smarandache's proof of this assertion is given in [28].

Problem 4. Also can be simplified, because

$$t = h + \lceil \frac{p}{h} \rceil + 1$$

$$= h + \lceil \frac{(k-1)!h+1}{h} \rceil + 1$$

$$= h + (k-1)! + 1 + 1 = h + (k-1)! + 2,$$

i.e.,

$$(-1)^t = (-1)^h,$$

because for $k > 2$: $(k-1)! + 2$ is an even number. Therefore, (11.3) obtains the form

p is a prime if and only if

$$(p-k)! \text{ is congruent to } (-1)^h h(\mathrm{mod}p),$$

Let us assume that (11.4) is valid. We use again the congruences

$$(p-1) \equiv -1(\mathrm{mod}p)$$

$$(p-2) \equiv -2(\mathrm{mod}p)$$

$$\dots$$

$$(p-(k-1)) \equiv -(k-1)(\mathrm{mod}p)$$

and obtain the next form of (11.4)

p is a prime if and only if

$$(p-1)! \equiv (-1)^h.(-1)^{k-1}.(k-1)!.h(\mathrm{mod}p)$$

or

p is a prime if and only if

$$(p-1)! \equiv (-1)^{h+k-1}.(p-1)(\mathrm{mod}p).$$

But the last congruence is not valid, because, e.g., for $k = 5, h = 3, p = 73 = (5-1)! + 1$[1] holds

$$72! \equiv (-1)^9.72(\mathrm{mod}73),$$ [2]

i.e.,

$$72! \equiv 1(\mathrm{mod}73),$$

[1] In [28] there is a misprint: 3! instead of 3.

[2] In [28] there is a misprint: $(-1)^9$ instead of $(-1)^7$.

while from Wilson's Theorem follows that

$$72! \equiv -1(\mathrm{mod}73).$$

11. The results in this section are taken from [5].

In [17] F. Smarandache discussed the following particular cases of the well-known characteristic functions (see, e.g., [11, 30]).

12.1) Prime function: $P : N \to \{0, 1\}$, with

$$P(n) = \begin{cases} 0, & \text{if } n \text{ is a prime} \\ 1, & \text{otherwise} \end{cases}$$

More generally: $P_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$P_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0, & \text{if } n_1, n_2, \ldots, n_k \text{ are all prime numbers} \\ 1, & \text{otherwise} \end{cases}$$

12.2) Coprime function is defined similarly: $C_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0, & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\ 1, & \text{otherwise} \end{cases}$$

In [28] we formulate and prove four assertions related to these functions.

Proposition 12.1. For each $k, n_1, n_2, \ldots, n_k$ natural numbers:

$$P_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k}(1 - P(n_i)).$$

Proposition 12.2. For each $k, n_1, n_2, \ldots, n_k$ natural numbers:

$$C_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k} \prod_{j=i+1}^{k} (1 - C_2(n_i, n_j)).$$

Proposition 12.3. For each natural number n:

$$C_{\pi(n)+P(n)}(p_1, p_2, \ldots, p_{\pi(n)+P(n)-1}, n) = P(n).$$

Proposition 12.4. For each natural number n:

$$P(n) = 1 - \prod_{i=1}^{\pi(n)+P(n)-1} (1 - C_2(p_i, n)).$$

Corollary 12.1. For each natural number $k, n_1, n_2, \ldots, n_k$:

$$P_k(n_1, n_2, \ldots, n_k) = 1 - \prod_{i=1}^{k} \prod_{j=1}^{\pi(n_i)+P(n_i)-1} (1 - C_2(p_j, n_i)).$$

References

[1] K.Atanassov, On some of the Smarandache's problems, American Research Press, Lupton, 1999.

[2] K.Atanassov, Remarks on some of the Smarandache's problems. Part 1, Smarandache Notions Journal, Spring, **12**(2001), 82-98.

[3] K.Atanassov, A new formula for the n-th prime number. Comptes Rendus de l'Academie Bulgare des Sciences,**7-8-9**(2001).

[4] K.Atanassov, On the 20-th and the 21-st Smarandache's Problems. Smarandache Notions Journal, Spring ,**1-2-3**(2001), 111-113.

[5] K.Atanassov, On four prime and coprime functions, Smarandache Notions Journal, Spring ,**1-2-3**(2001), 122-125.

[6] K.Atanassov, On the 17-th Smarandache's Problem, Smarandache Notions Journal, Spring ,**1-2-3**(2002), 124-125.

[7] K.Atanassov, On the 46-th Smarandache's Problem, Smarandache Notions Journal, Spring ,**1-2-3**(2002), 126-127.

[8] K.Atanassov, On the second Smarandache's Problem, Notes on Number Theory and Discrete Mathematics, **3**(2003), 46-48.

[9] K.Atanassov, On four Smarandache's Problems, Notes on Number Theory and Discrete Mathematics,**1-6**(2005).

[10] Dumitrescu C., V.Seleacu, Some Sotions and Questions in Number Theory, Erhus Univ. Press, Glendale, 1994.

[11] Grauert H., Lieb I., Fischer W, Differential- und Integralrechnung, Springer-Verlag, Berlin, 1967.

[12] Mitrinović, D., M. Popadić, Inequalities in Number Theory, Niś, Univ. of Niś, 1978.

[13] Nagell T., Introduction to Number Theory, John Wiley & Sons, Inc., New York, 1950.

[14] Ribenboim P. The New Book of Prime Number Records, Springer, New York, 1995.

[15] F.Smarandache, Criteria for a number to be prime, Gazeta Matematica, Vol. XXXVI, No. 2, 49-52, Bucharest, 1981 (in Romanian).

[16] F.Smarandache, Only Problems, Not Solutions!, Xiquan Publ. House, Chicago, 1993.

[17] F.Smarandache, Collected Papers, Vol. II, Kishinev University Press, Kishinev, 1997.

[18] Vassilev-Missana M., Three formulae for **n**-th prime and six for **n**-th term for twin primes, Notes on Number Theory and Discrete Mathematics, **1**(2001), 15-20.

[19] Vassilev-Missana M., Some explicit formulae for the composite numbers. Notes on Number Theory and Discrete Mathematics, **2** (2001).

[20] Vassilev-Missana M., On one remarkable identity related to function $\pi(x)$, Notes on Number Theory and Discrete Mathematics, **4** (2001), 129-136.

[21] Vassilev-Missana M., On 15-th Smarandache's problem. Notes on Number Theory and Discrete Mathematics, **2** (2003) 42-45.

[22] Vassilev-Missana M., Some representations concerning the product of divisors of n. Notes on Number Theory and Discrete Mathematics, **2** (2004), 54-56.

[23] Vassilev M., Atanassov K., Note on the Diophantine equation $2x^2 - 3y^2 = p$, Smarandache Notions Journal, **1-2-3** (2000) 64-68.

[24] Vassilev-Missana M., K.Atanassov, On 28-th Smarandache's Problem. Notes on Number Theory and Discrete Mathematics, **2** (2001), 61-64.

[25] Vassilev-Missana M., K.Atanassov, On five Smarandache's problems. Notes on Number Theory and Discrete Mathematics, Vol. 10, **2** (2004), 34-53.

[26] Vassilev-Missana, M., K. Atanassov, Remarks on the 46-th Smarandache's Problem. Notes on Number Theory and Discrete Mathematics, **3** (2004), 84-88.

[27] Vassilev-Missana, M., K. Atanassov, On two Smarandache's problems, Notes on Number Theory and Discrete Mathematics, **4** (2004), 106-112.

[28] Vassilev-Missana, M., K. Atanassov, Some Smarandache problems, Hexis, Phoenix, 2004.

[29] Vassilev P., Vassilev-Missana M., K.Atanassov, On 25-th and 26-st Smarandache's Problems. Notes on Number Theory and Discrete Mathematics, **4** (2003), 99-104.

[30] Yosida K., Function Analysys, Springer-Verlag, Berlin, 1965.

Scientia Magna
Vol. 1 (2005), No. 2, 27-35

Smarandache groupoids

W.B.Vasantha Kandasamy
Department of Mathematics, Indian Institute of Technology, Madras
Chennai-600 036, India
vasantak@md3.vsnl.net.in

Abstract
In this paper, we study the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache-Bol groupoids and Strong Bol groupoids and obtain many interesting results about them.

Keywords Smarandache groupoid; Smarandache subgroupoid; Smarandache ideal of a Smarandache groupoid; Smarandache semi-normal groupoid; Smarandache normal groupoid; Smarandache semi conjugate subgroupoid; Smarandache Bol groupoid; Smarandache Moufang groupoid.

Definition [1]: A groupoid $(G, *)$ is a non-empty set, closed with respect to an operation $*$ (in general $*$ need not to be associative).

Definition 1: A *Smarandache groupoid* G is a groupoid which has a proper subset $S \subset G$ which is a semigroup under the operation of G.

Example 1: Let $(G, *)$ be a groupoid on modulo 6 integers. $G = \{0, 1, 2, 3, 4, 5\}$ is given by the following table:

$*$	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Clearly $S_1 = \{0, 3\}, S_2 = \{1, 4\}$ and $S_3 = \{2, 5\}$ are semigroups of G. So $(G, *)$ is a Smarandache groupoid.

Example 2: Let $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of integers modulo 10. Define an operation $*$ on G by choosing a pair $(1, 5)$ such that $a * b = 1a + 5b (\text{mod} \quad 10)$ for all $a, b \in G$.

The groupoid is given by the following table.

$*$	0	1	2	3	4	5	6	7	8	9
0	0	5	0	5	0	5	0	5	0	5
1	1	6	1	6	1	6	1	6	1	6
2	2	7	2	7	2	7	2	7	2	7
3	3	8	3	8	3	8	3	8	3	8
4	4	9	4	9	4	9	4	9	4	9
5	5	0	5	0	5	0	5	0	5	0
6	6	1	6	1	6	1	6	1	6	1
7	7	2	7	2	7	2	7	2	7	2
8	8	3	8	3	8	3	8	3	8	3
9	9	4	9	4	9	4	9	4	9	4

Clearly $S_1 = \{0, 5\}$, $S_2 = \{1, 6\}$, $S_3 = \{2, 7\}$, $S_4 = \{3, 8\}$ and $S_5 = \{4, 9\}$ are semigroupoids under the operation $*$. Thus $\{G, *, (1, 5)\}$ is a Smarandache groupoid.

Theorem 2. Let $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$. Define $*$ on Z_{2p} for $a, b \in Z_{2p}$ by $a*b = 1a+pb($ mod $2p)$. $\{Z_{2p}, *, (1, p)\}$ is a Smarandache groupoid.

Proof. Under the operation $*$ defined on Z_{2p} we see $S_1 = \{0, p\}$, $S_2 = \{1, p+1\}$, $S_3 = \{2, p+2\}$,$\cdots$,$S_p = \{p-1, 2p-1\}$ are semigroup under the operation $*$. Hence $\{Z_{2p}, *, (1, p)\}$ is a Smarandache groupoid.

Example 3: Take $Z_6 = \{0, 1, 2, 3, 4, 5\}$. $(2, 5) = (m, n)$. For $a, b \in Z_6$ define $a * b = ma + nb(\text{mod } 6)$. The groupoid is given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	2	1	0	5	4	3
2	4	3	2	1	0	5
3	0	5	4	3	2	1
4	2	1	0	5	4	3
5	4	3	2	1	0	5

Every singleton is an idempotent semigroup of Z_6.

Theorem 3. Let $Z_{2p} = \{0, 1, 2, \cdots, p-1\}$. Define $*$ on Z_{2p} by $a * b = 2a + (2p-1)b(\text{mod } 2p)$ for $a, b \in Z_{2p}$. Then $\{Z_{2p}, *, (2, 2p-1)\}$ is a Smarandache groupoid.

Proof. Under the operation $*$ defined on Z_{2p} we see that every element of Z_{2p} is idempotent, therefore every element forms a singleton semigroup. Hence the claim.

Example 4: Consider $Z_6 = \{Z_6, *, (4, 5)\}$ given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

$\{3\}$ is a semigroup. Hence $*$ is a Smarandache groupoid. It is easily verified that Z_6 is a Smarandache groupoid as $\{Z_6, *, (4,5)\}$ has an idempotent semigroup $\{3\}$ under $*$.

Theorem 4. Let $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$ be the set of integers modulo $2p$. Define $*$ on $a, b \in Z_{2p}$ by $a(2p-2) + b(2p-1) (\text{mod } 2p)$. Then $\{Z_{2p}, *, (2p-2, 2p-1)\}$ is a Smarandache groupoid.

Proof. $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$. Take $(2p-2, 2p-1) = 1$ from Z_{2p}. For $a, b \in Z_p$ define $a*b = a(2p-2)+b(2p-1) (\text{mod } 2p)$. Clearly for $a = b = p$ we have $(2p-2)p+(2p-1)p = p (\text{mod } 2p)$. Hence $\{p\}$ is an idempotent semigroup of Z_{2p}. So $\{Z_{2p}, *, (2p-2, 2p-1)\}$ is a Smarandache groupoid.

Definition 5: Let $(G, *)$ be a Smarandache groupoid. A non-empty subset H of G is said to be a Smarandache groupoid if H contains a proper subset $K \subset H$ such that K is a semigroup under the operation $*$.

Theorem 6. Not every subgroupoid of a Smarandache groupoid S is in general a Smarandache subgroupoid of S.

Proof. By an example.

Let $Z_6 = \{0, 1, 2, 3, 4, 5\} (\text{mod } 6)$. Take $(t, u) = (4, 5) = 1$. For $a, b \in Z_6$ define $*$ on Z_6 by $a * b = at + bu (\text{mod } 6)$ given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly $\{Z_6, *, (4,5)\}$ is a Smarandache groupoid for it contains $\{0, 3\}$ as a semigroup. But this groupoid has the following subgroupoids: $A_1 = \{0, 2, 4\}$ and $A_2 = \{1, 3, 5\}$. A_1 has no non-trivial semigroup($\{0\}$ is a trivial semigroup). But A_2 has a non-trivial semigroup, viz.$\{3\}$. Hence the claim.

Theorem 7. If a groupoid contains a Smarandache subgroupoid, then the groupoid is a Smarandache groupoid.

Proof. Let G be a groupoid and $H \subset G$ be a Smarandache subgroupoid, that is H contains a proper subset $P \subset H$ such that P is a semigroup. So $P \subset G$ and P is a semigroup. Hence G is a Smarandache groupoid.

Definition 8:

i) A Smarandache Left Ideal A of the Smarandache Groupoid G satisfies the following conditions:

1. A is a Smarandache subgroupoid. 2. For all $x \in G$, and $x \in A$, $\quad xa \in A$.

ii) Similarly, one defines a Smarandache Right Ideal.

iii) If A is both a Smarandache right and left ideals then A is a Smarandache Ideal. We take $\{0\}$ as a trivial Smarandache ideal.

Example 5: Let $\{Z_6, *, (4,5)\}$ be a Smarandache groupoid. $A = \{1,3,5\}$ is a Smarandache subgroupoid and A is Smarandache left ideal and not a Smarandache right ideal. Easy to verify.

Theorem 9. Let G be a groupoid. An ideal of G in general is not a Smarandache ideal of G even if G is a Smarandache groupoid.

Proof. By an example. Consider the groupoid $G = \{Z_6, *, (2,4)\}$ given by the following table.

$*$	0	1	2	3	4	5
0	0	4	2	0	4	2
1	2	0	4	2	0	4
2	4	2	0	4	2	0
3	0	4	2	0	4	2
4	2	0	4	2	0	4
5	4	2	0	4	2	0

Clearly G is a Smarandache groupoid for $\{0,3\}$ is a semigroup of G. Now, $\{0,4,2\}$ is an ideal of G but is not a Smarandache ideal as $\{0,4,2\}$ is not a Smarandache subgroupoid.

Definition 10: Let G be a Smarandache groupoid and V be a Smarandache subgroupoid of G. We say V is a Smarandache semi-normal subgroupoid if:

1. $aV = X$ for all $a \in G$; 2. $Va = Y$ for all $a \in G$, where either X or Y is a Smarandache subgroupoid of G but X and Y are both subgroupoids.

Example 6: Consider the groupoid $G = \{Z_6, *, (4,5)\}$ given by the table.

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly G is a Smarandache groupoid as $\{3\}$ is a semigroup. Take $A = \{1, 3, 5\}$. A is also a Smarandache subgroupoid. Now $aA = A$ is a Smarandache groupoid. $Aa = \{0, 2, 4\}$. $\{0, 2, 4\}$ is not a Smarandache subgroupoid of G. Hence A is a Smarandache semi-normal subgroupoid.

Definition 11: Let A be a Smarandache groupoid and V be a Smarandache subgroupoid. V is said to be *Smarandache normal subgroupoid* if $aV = X$ and $Va = Y$ where both X and Y are Smarandache subgroupoids of G.

Theorem 12. Every Smarandache normal subgroupoid is a Smarandache semi-normal subgroupoid, and not conversely.

Proof. By the definition 10 and 11, we see every Smarandache normal subgroupoid is Smarandache semi-normal subgroupoid. We prove the converse by an example. In example 6 we see A is a Smarandache semi-normal subgroupoid but not a normal subgroupoid as $Aa = \{0, 2, 4\}$ is only a subgroupoid and not a Smarandache subgroupoid.

Example 7: Let $G = \{Z_8, *, (2, 6)\}$ be a groupoid given by the following table:

$*$	0	1	2	3	4	5	6	7
0	0	6	4	2	0	6	4	2
1	2	0	6	4	2	0	6	4
2	4	2	0	6	4	2	0	6
3	6	4	2	0	6	4	2	0
4	0	6	4	2	0	6	4	2
5	2	0	6	4	2	0	6	4
6	4	2	0	6	4	2	0	6
7	6	4	2	0	6	4	2	0

Clearly G is a Smarandache groupoid for $\{0, 4\}$ is a semigroupoid G. $A = \{0, 2, 4, 6\}$ is a Smarandache subgroupoid. Clearly $Aa = A$ for all $a \in G$. So A is a Smarandache normal subgroupoid of G.

Definition 13: Let G be a Smarandache groupoid H and P be subgroupoids of G, we say H and P are Smarandache semi-conjugate subgroupoids of G if:

1. H and P are Smarandache subgroupoids.
2. $H = xP$ or Px, for some $x \in G$.
3. $P = xH$ or Hx, for some $x \in G$.

Example 8: Consider the groupoid $G = \{Z_{12}, *, (1, 3)\}$ which is given by the following table:

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	3	6	9	0	3	6	9	0	3	6	9
1	1	4	7	10	1	4	7	10	1	4	7	10
2	2	5	8	11	2	5	8	11	2	5	8	11
3	3	6	9	0	3	6	9	0	3	6	9	0
4	4	7	10	1	4	7	10	1	4	7	10	1
5	5	8	11	2	5	8	11	2	5	8	11	2
6	6	9	0	3	6	9	0	3	6	9	0	3
7	7	10	1	4	7	10	1	4	7	10	1	4
8	8	11	2	5	8	11	2	5	8	11	2	5
9	9	0	3	6	9	0	3	6	9	0	3	6
10	10	1	4	7	10	1	4	7	10	1	4	7
11	11	2	5	8	11	2	5	8	11	2	5	8

Clearly G is a Smarandache groupoid for $\{0, 6\}$ is a semigroup of G. Let $A_1 = \{0, 3, 6, 9\}$ and $A_2 = \{2, 5, 8, 11\}$ be two subgroupoids. Clearly A_1 and A_2 are Smarandache subgroups of G as $\{0, 6\}$ and $\{2, 8\}$ are semigroups of A_1 and A_2 respectively.

Now:

$$\begin{aligned} A_1 &= 3\{2, 5, 8, 11\} = 3A_2 \\ &= \{0, 3, 6, 9\} \end{aligned}$$

and similarly:

$$A_2 = 2\{0, 3, 6, 9\} = 2A_1.$$

Hence A_1 and A_2 are conjugate Smarandache subgroupoids of G.

Definition 15: Let $G_1, G_2, G_3, \cdots, G_n$ be n groupoids. We say $G = G_1 \times G_2 \times \cdots \times G_n$ is a *Smarandache direct product of groupoids* if G has a proper subset H of G which is a semigroup under the operations of G. It is important to note that each G_i need not be a Smarandache groupoid for in that case G will be obviously a Smarandache groupoid. Hence we take any set of n groupoids and find the direct product.

Definition 16: Let $(G, *)$ and $(G', \circ)$ be any two Smarandache groupoids. A map ϕ from $(G, *)$ to $(G', \circ)$ is said to be a *Smarandache groupoid homomorphism* if $\phi(a * b) = \phi(a) \circ \phi(b)$ for all $a, b \in A$.

We say the *Smarandache groupoid* homomorphism is an *isomorphism* if ϕ is an isomorphism.

Definition 17: Let G be a Smarandache groupoid. We say G is a *Smarandache commutative groupoid* if there is a proper subset A of G which is a commutative semigroup under the operation of G.

Definition 18: Let G be Smarandache groupoid. We say G is *Smarandache inner commutative groupoid* if every semigroup contained in every Smarandache subgroupoid of G is commutative.

Theorem 19. Every Smarandache inner commutative groupoid G is a Smarandache commutative groupoid and not conversely.

Proof. By the very definition 18 and 19 we see if G is a Smarandache inner commutative groupoid then G is Smarandache commutative groupoid.

To prove the converse we prove it by an example. Let $Z_2 = \{0,1\}$ be integers modulo 2. Consider set of all 2×2 matrices with entries from $Z_2 = (0,1)$ denote it by $M_{2\times 2}$.

$$M_{2\times 2} = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix} \right.$$

$$\left. \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix} \right\}.$$

$M_{2\times 2}$ is made into a groupoid by for $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ in $M_{2\times 2}$.

$$\begin{aligned} A \circ B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \circ \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_3 + a_2b_1(\mathrm{mod}2) & a_1b_4 + a_2b_2(\mathrm{mod}2) \\ a_3b_3 + a_4b_1(\mathrm{mod}2) & a_3b_4 + a_4b_2(\mathrm{mod}2) \end{pmatrix} \end{aligned}$$

Clearly $(M_{2\times 2}, \circ)$ is a Smarandache groupoid for $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

So $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ is a semigroup.

Now consider $A_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ is a Smarandache groupoid but A_1 is non-commutative Smarandache groupoid for A_1 contains a non-commutative semigroupoid S. $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ such that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $(M_{2\times 2}, \circ)$ is a Smarandache commutative groupoid but not Smarandache inner commutative groupoid.

Definition 20: A groupoid G is said to be a $Moufang \quad groupoid$ if for every x, y, z in G we have $(xy)(zx) = (x(yz))x$.

Definition 21: A Smarandache groupoid $(G, *)$ is said to be $Smarandache \quad Moufang \quad groupoid$ if there exists $H \subset G$ such that H is a Smarandache groupoid satisfying the Moufang identity: $(xy)(zx) = (x(yz)x)$ for all x, y, z in H.

Definition 22: Let S be a Smarandache groupoid. If every Smarandache subgroupoid H of S satisfies the Moufang identity for all x, y, z in H then S is a $Smarandache \quad Strong \quad Moufang \quad groupoid$.

Theorem 23. Every Smarandache Strong Moufang groupoid is a Smarandache Moufang groupoidand not conversely.

Proof. Every Strong Smarandache Moufang groupoid is a Smarandache Moufang groupoid. The proof of the converse can be proved by constructing examples.

Definition 24: A groupoid G is said to be a $Bol \quad groupoid$ if $((xy)z)y = x((yz))y$ for all $x, y, z \in G$.

Definition 25: Let G be a groupoid. G is said to be $Smarandache - Bol \quad groupoid$ if G has a subgroupoid H of G such that H is a Smarandache subgroupoid and satisfies the identity $((xy)z)y = x((yz))y$ for all x, y, z in H.

Definition 26: Let G be a groupoid. We say G is $Smarandache \quad Strong \quad Bol \quad groupoid$ if every Smarandache subgroupoid of G is a Bol groupoid.

Theorem 27. Every Smarandache Strong Bol groupoid is a Smarandache Bol groupoid and the converse is not true.

Proof. Obvious.

Theorem 28. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$ be the set of integers modulo n. Let $G = \{Z_n, *, (t, u)\}$ be a Smarandache groupoid. G is a Smarandache Bol groupoid if $t^3 = t(mod \quad n)$ and $u^2 = u(mod \quad n)$.

Proof. Easy to verify.

Example 9: Let $G = \{Z_6, *, (2, 3)\}$ defined by the following table:

$*$	0	1	2	3	4	5
0	0	3	0	3	0	3
1	2	5	2	5	2	5
2	4	1	4	1	4	1
3	0	3	0	3	0	3
4	2	5	2	5	2	5
5	4	1	4	1	4	1

$\{0, 3\}$ is a Smarandache subgroupoid and since $2^3 = 2(mod \quad 6)$ and $3^2 = 3(mod \quad 6)$ we see G is a Smarandache Bol groupoid.

Problem 2: Let $\{0, 1, 2, \cdots, n-1\}$ be the ring of integers modulo n. $G = \{Z_n, *, (t, u)\}$ be a groupoid. Find conditions on n, t and u so that G:

1. is a Smarandache groupoid.
2. has Smarandache semi-normal subgroupoids.
3. has Smarandache normal subgroupoids.
4. is Smarandache commutative.
5. is Smarandache inner commutative.
6. is a Smarandache-Bol groupoid.
7. is a Smarandache Strong Bol groupoid.
8. is a Smarandache-Moufang groupoid.
9. is a Smarandache-Strong-Moufang groupoid.
10. has always a pair of Smarandache conjugate subgroupoid.

References

[1] R.H.Bruck, A Survey of Binary Systems, Springer Verlag, 1958.

[2] Raul Padilla, Smarandache Algebraic Strucures, Bulletin of Pure and Applied Sciences, Delhi, Vol. 17E, **1**(1998), 119-121.

http://www.gallup.unm.edu/~Smarandache/ALG-S-TXT.TXT.

[3] W.B.Vasantha Kandasamy, On ordered groupoids and its groupoid rings, J. of Mathematics and Comp. Sci., Vol. 9, 145-147, 1996.

Scientia Magna
Vol. 1 (2005), No. 2, 36-38

On the primitive numbers of power P and its mean value properties[1]

Ding Liping
Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China,710069

Abstract Let p be a prime, n be any fixed positive integer. $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n. In this paper, we study the mean value properties of $S_p(n)$ for p, and give a sharp asymptotic formula for it.
Keywords Primitive numbers; Mean value; Asymptotic formula.

§1. Introduction

Let p be a prime, n be any fixed positive integer, $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $S_3(5) = 12$, $\cdots$. In problem 49 of book [1], Professor F. Smarandache asks us to study the properties of the sequence $S_p(n)$. About this problem, some asymptotic properties of this sequence have been studied by Zhang Wenpeng and Liu Duansen [2], they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\log p}\log n\right).$$

The problem is interesting because it can help us to calculate the Smarandache function. In this paper, we use the elementary methods to study the mean value properties of $S_p(n)$ for p, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem Let $x \geq 2$, for any fixed positive integer n, we have the asymptotic formula

$$\sum_{p\leq x} S_p(n) = \frac{nx^2}{2\log x} + \sum_{m=1}^{k-1}\frac{na_m x^2}{\log^{m+1} x} + O\left(\frac{nx^2}{\log^{k+1} x}\right),$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants.

§2. Some Lemmas

To complete the proof of the theorem, we need the following:

Lemma Let p be a prime, n be any fixed positive integer. Then we have the estimate

$$(p-1)n \leq S_p(n) \leq np.$$

[1]This work is supported by the N.S.F(10271093) and P.N.S.F of P.R.China

Proof. Let $S_p(n) = k = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$ under the base p. Then from the definition of $S_p(n)$ we know that $p \mid S_p(n)!$ and the $S_p(n)$ denotes the smallest integer satisfy the condition. However, let

$$(np)! = 1 \cdot 2 \cdot 3 \cdots p \cdot (p+1) \cdots 2p \cdot (2p+1) \cdots np = up^l.$$

where $l \geq n$, $p\dagger u$.

So combining this and $p \mid S_p(n)!$ we can easily obtain

$$S_p(n) \leq np. \tag{1}$$

On the other hand, from the definition of $S_p(n)$ we know that $p \mid S_p(n)!$ and $p^n\dagger (S_p(n)-1)!$, so that $\alpha_1 \geq 1$, note that the factorization of $S_p(n)!$ into prime powr is

$$k! = \prod_{q \leq k} q^{\alpha_q(k)}.$$

where $\prod\limits_{q \leq k}$ denotes the product over all prime , and

$$\alpha_q(k) = \sum_{i=1}^{\infty} [\frac{k}{q^i}]$$

because $p \mid S_p(n)!$, so we have

$$n \leq \alpha_p(k) = \sum_{i=1}^{\infty} [\frac{k}{p^i}] = \frac{k}{p-1}$$

or

$$(p-1)n \leq k \tag{2}$$

combining (1) and (2) we immediately get the estimate

$$(p-1)n \leq S_p(n) \leq np.$$

This completes the proof of the lemma.

§3. Proof of the theorem

In this section, we complete the proof of Theorem. Based on the result of lemma

$$(p-1)n \leq S_p(n) \leq np$$

we can easily get

$$\sum_{p \leq x} (p-1)n \leq \sum_{p \leq x} S_p(n) \leq \sum_{p \leq x} np.$$

Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise,} \end{cases}$$

Then from [3] we know that for any positive integer k,

$$\sum_{n\leq x} a(n) = \pi(x) = \frac{x}{logx}(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x}) + O\left(\frac{x}{\log^{k+1} x}\right).$$

By Abel's identity we have

$$\begin{aligned}
& \sum_{p\leq x} p = \sum_{m\leq x} a(m)m \\
= \quad & \pi(x)x - \int_2^x \pi(t)dt \\
= \quad & \frac{x^2}{logx} + \frac{x^2}{logx}\sum_{m=1}^{k-1}\frac{m!}{\log^m x} - \int_2^x \frac{t}{logt}(1 + \sum_{m=1}^{k-1}\frac{m!}{\log^m t})dt + O\left(\frac{x^2}{\log^{k+1} x}\right) \\
= \quad & \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right)
\end{aligned}$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants. From above we have

$$\sum_{p\leq x}(p-1) = \sum_{p\leq x} p - \pi(x) = \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right).$$

Therefore

$$\sum_{p\leq x} S_p(n) = \sum_{p\leq x} k == \frac{x^2}{2\log x} + \sum_{m=1}^{k-1}\frac{a_m x^2}{\log^(m+1)x} + O\left(\frac{x^2}{\log^{k+1} x}\right).$$

This completes the proof of the theorem.

Acknowledgments

The author express his gratitude to his supervisor Professor Zhang Wenpeng for his very helpful and detailed instructions.

References

[1] F.Smarandache, Only problems, Not Solutions, Chicago, Xiquan Publ. House, 1993.

[2] Zhang Wenpeng and Liu Duansen, On the primitive numbers of power P and its asymptotic property, Smarandache Notions Journal, **13**(2002), 173-175.

[3] M.Ram Murty, Problems in analytic number theory, Springer-Verlag, New York, 2001, pp. 36

Scientia Magna
Vol. 1 (2005), No. 2, 39-45

Consecutive, reversed, mirror, and symmetric Smarandache sequence of triangular numbers

Delfim F.M.Torres[†] and Viorica Teca [‡]

[†]Department of Mathematics University of Aveiro,Portugal
email: delfim@mat.ua.pt
[‡]Faculty of Mathematics-Informatics University of Craiova,Romania
email: viorica_teca@yahoo.com

Abstract We use the Maple system to check the investigations of S.S.Gupta regarding the Smarandache consecutive and the reversed Smarandache sequence of triangular numbers [Smarandache Notions Journal,Vol. 14, 2004, pp.366-368]. Furthermore, we extend previous investigations to the mirror and symmetric Smarandache sequences of triangular numbers.

The nth triangular number $t_n, n \in N$, is defined by $t_n = \sum_{i=1}^{n} i = n(n+1)/2$. These numbers were first studied by the Pythagoreans.
The first k terms of the triangular sequence $\{t_n\}_{n=1}^{\infty}$ are easily obtained in Maple:
> t:=n − >n∗(n+1)/2:
> first :=k − > seq(t(n), n=1,· · ·,k):
For example:
> first(20);

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210$$

In this short note we are interested in studying Smarandache sequences of triangular numbers with the help of the Maple system. To define the Smarandache sequences, it is convenient to introduce first the concatenation operation. Given two positive integer numbers n and m, the concatenation operation conc is defined in Maple by the following function:

> conc :=(n, m) − > n∗$10^{length(m)}$+m:

For example,

> conc$(12, 345)$;

$$12345$$

Given a positive integer sequence $\{u_n\}_{n=1}^{\infty}$, we define the corresponding Smarandache Consecutive Sequence $\{scs_n\}_{n=1}^{\infty}$ recursively:

$$scs_1 = u_1,$$

$$scs_n = conc(scs_{n-1}, u_n).$$

In Maple we define:

> scs_n :=(u, n) − > if $n = 1$ then u(1) else conc(scs_n(u, n−1), u(n)) fi:

> scs := (u, n) − > seq(scs_n(u, i), i=1$\cdots$ n):

The standard Smarandache consecutive sequence, introduced by the Romanian mathematician Florentin Smarandache, is obtained when one chooses $u_n = n, \forall n \in N$. The first 10 terms are: > scs(n->n,10);

$$1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 12345678910$$

Another example of a Smarandache consecutive sequence is the Smarandache consecutive sequence of triangular numbers. With our Maple definitions, the first 10 terms of such sequence are obtained with the following command:

> scs(t, 10);

$$1, 13, 136, 13610, 1361015, 136101521, 13610152128,$$

$$1361015212836, 136101521283645, 13610152128364555,$$

Sometimes, it is preferred to display Smarandache sequence in "triangular form".

> show :=L − >map(i − >print(i), L):

> show([scs(t, 10)]):

1

13

136

13610

1361015

136101521

13610152128

1361015212836

136101521283645

13610152128364555

The Reversed Smarandache Sequence (rss) associated with a given sequence $\{u_n\}_{n=1}^{\infty}$, is defined recursively by

$$rss_1 = u_1$$

$$rss_n = conc(u_n, rss_{n-1}).$$

In Maple we propose the following definitions:

> rss_n :=(u, n) − > if $n = 1$ then u(1) else conc(u(n), res_n(u, n−1)) fi:

> rss :=(u, n) − > seq(rss_n(u, i), i=1,$\cdots$ n):

The first terms of the reversed Smarandache sequence of triangular numbers are now easily obtained:

```
> rss(t, 10);
```

$$1, 31, 631, 10631, 1510631, 211510631, 28211510631,$$

$$3628211510631, 453628211510631, 55453628211510631,$$

We define the Smarandche Mirror Sequence (sms) as follows:

$$sms_1 = u_1,$$

$$sms_n = conc(conc(u_n, sms_{n-1}), u_n)$$

```
> sms_n :=(u, n) − > if n = 1 then
    >                    u(1)
>                    else
>                    conc(conc(u(n), sms_n(u, n−1)), u(n))
> sms :=(u, n) − >seq(sms_n(u, i), i=1 · · · n):
```

The first 10 terms of the Smarandache mirror sequence introduced by Smarandache are:

```
> sms(n − >, 10);
```

$$1, 212, 32123, 4321234, 543212345, 65432123456, 7654321234567,$$

$$876543212345678, 98765432123456789, 109876543212345678910$$

We are interested in the Smarandache mirror sequence of triangular numbers. The first 10 terms are:

```
> sms(t, 10);
```

$$1, 313, 63136, 106313610, 1510631361015, 21151063136101521,$$

$$2821151063136101521 28, 3628211510631361015212836,$$

$$453628211510631361015212836 45, 5545362821151063136101521283645 55,$$

Finally, we define the Smarandache Symmetric Sequence (sss). For that we introduce the function "But Last Digit" (bld):

```
> bld :=n − > iquo(n,10):
> bld(123);
```

$$12$$

If the integer number is a one-digit number, then function bld returns zero:

```
> bld(3);
```

$$0$$

This is important: with our conc function, the concatenation of zero with a positive integer n gives n.

```
> conc(bld(1), 3);
```

$$3$$

The Smarandache Symmetric Sequence (sss) is now easily defined, appealing to the Smarandache consecutive, and reversed Smarandache sequences:

$$sss_{2n-1} = conc(bld(scs_{2n-1}), rss_{2n-1}),$$

$$sss_{2n} = conc(scs_{2n}, rss_{2n}),$$

$n \in N$. In Maple, we give the following definitions:

```
> sss_n :=(u, n) − > if type(n, odd) then
>                    conc(bld(scs_n(u, (n+1)/2), rss_(u, (n+1)/2))
>                    else
>                    conc(scs_n(u, n/2), rss_n(u, n/2))
>                    fi:
> sss :=(u, n) − > seq(sss_n(u, i), i=1···n):
```

The first terms of Smarandache's symmetric sequence are

```
> sss(n − > n, 10);
```

$$1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321$$

while the first 10 terms of the Smarandache symmetric sequence of triangular numbers are

```
> sss(t, 10);
```

$$1, 11, 131, 1331, 13631, 136631, 136110631, 1361010631, 1361011510631, 13610151510631,$$

One interesting question is to find prime numbers in the above defined Smarandache sequences of triangular numbers. We will restrict our search to the first 1000 terms of each sequence. All computations were done with Maple 9 runing on a 2.00Ghz Pentium 4 with 256Mb RAM. We begin by collecting four lists with the first 1000 terms of the consecutive, reversed, mirror, and symmetric Smarandache sequences of triangular numbers:

```
> st:=time(): Lscs1000:=[scs(t, 1000)]: print("%a seconds", round(time()-st));
```

20 seconds

```
> st:=time(): Lrss1000:=[rss(t, 1000)]: print("%a seconds", round(time()-st));
```

75 seconds

```
> st:=time(): Lsms1000:=[sms(t, 1000)]: print("%a seconds", round(time()-st));
```

212 seconds

```
> st:=time(): Lsss1000:=[sss(t, 1000)]: print("%a seconds", round(time()-st));
```

26 seconds

We note that scs_{1000} and rss_{1000} are positive integer numbers with 5354 digits;

```
> length(Lscs1000[1000]), length(Lrss1000[1000]);
```

$$5354, 5354$$

while sms_{1000} and sss_{1000} have, respectively, 10707 and 4708 digits.

```
> length(Lsms1000[1000]), length(Lsss1000[1000]);
```

$$10707, 4708$$

There exist two primes (13 and 136101521) among the first 1000 terms of the Smarandache consecutive sequence of triangular numbers;

```
> st := time()
> select(isprime, Lscs1000);
> printf("%a minutes", round((time()-st)/60));
```

$$[13, 136101521]$$

9 minutes

six primes among the first 1000 terms of the reversed Smarandache sequence of triangular numbers;

```
> st := time()
> select(isprime, Lrss1000);
> printf("%a minutes", round((time()-st)/60));
```

$$[31, 631, 10631, 55453628211510631, 786655453628211510631, 10591786655453628211510631]$$

31 minutes

only one prime (313) among the first 600 terms of the Smarandache mirror sequence of triangular numbers;

```
> length(Lsms1000[600]); # sms_{600} is a number with 5907 digits
```

$$5907$$

```
> st := time()
> select(isprime, Lsms1000[1...600]);
> printf("%a minutes", round((time()-st)/60));
```

$$[313]$$

3 minutes

and five primes among the first 1000 terms of the Smarandache symmetric sequence of triangular numbers (the fifth prime is an integer with 336 digits).

```
> st := time()
> select(isprime, Lsss1000);
> printf("%a minutes", round((time()-st)/60));
```

[11,131,136110631,13610152128364555667891105120120105917866554536282115106 31,

13610152128364555667891105120136153171190210231253276300325351378406435465496528561595630666703741780820861903946990103510811128117612251275132613781431148515401596165316531596154014851431137813261275122511761128108110359909469038618207807417036666306595561528496465435406378351325300276253231210190171153136120105917866554536282115106 31]

19 minutes

```
> length(%[5]);
```

$$336$$

How many primes are there in the above defined Smarandache sequences of triangular numbers? This seems to be an open question. Another interesting question is to find triangular

numbers in the Smarandache sequences of triangular numbers. We begin by defining in Maple the boolean function istriangular.

> istriangular := n − > evalb(nops(select(i− >evalb(whattype(i)=integer),)[solve(t(k)=n)])> 0)

There exist two triangular numbers (1 and 136) among the first 1000 terms of the Smarandache consecutive sequence of triangular numbers;

> st := time()
> select(istriangular, Lscs1000);
> printf("%a seconds", round(time()-st));

$$[1, 136]$$

6 seconds

while the other Smarandache sequences of triangular numbers only show, among the first 1000 terms, the trival triangular number 1:

> st := time()
> select(istriangular, Lrss1000);
> printf("%a seconds", round(time()-st));

$$[1]$$

6 seconds

> st := time()
> select(istriangular, Lsms1000);
> printf("%a seconds", round(time()-st));

$$[1]$$

10 seconds

> st := time()
> select(istriangular, Lsss1000);
> printf("%a seconds", round(time()-st));

$$[1]$$

6 seconds

Does exist more triangular numbers in the Smarandache sequences of triangular numbers? This is, to the best of our knowledge, an open question needing further investigations. Since checking if a number is triangular is much faster than to check if a number if prime, we invite the readers to continue our research of triangular numbers for besides the 1000th term of the Smarandache sequences of triangular numbers. We look forward to readers discoveries.

References

[1] Shyam Sunder Gupta, Smarandache Sequence of Triangular Numbers, Smarandache Notions Journal, **14**(2004), 366-368 .

Scientia Magna
Vol. 1 (2005), No. 2, 46-48

On the square-free number sequence

Ren Dongmei

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is to study the number of the square-free number sequence, and give two interesting asymptotic formulas for it. At last, give another asymptotic formula and a corollary.

Keywords Square-free number sequence; Asymptotic formula.

§1. Introduction

A number is called a square-free number if its digits don't contain the numbers: 0, 1, 4, 9. Let $\mathcal{A}$ denote the set of all square-free numbers. In reference [1], Professor F. Smarandache asked us to study the properties of the square-free number sequence. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary method to study the number of the square-free number sequence, and obtain two interesting asymptotic formulas for it. That is, let $S(x) = \sum_{n\leq x, n\in\mathcal{A}} 1$, we shall prove the followings:

Theorem 1. For any real number $x \geq 1$, we have the asymptotic formula

$$\ln S(x) = \frac{\ln 6}{\ln 10} \times \ln x + O(1).$$

Theorem 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x, n\in\mathcal{B}} 1 = x + O\left(x^{\frac{2\ln 2}{\ln 10}}\right),$$

where $\mathcal{B}$ denote the complementary set of those numbers whose all digits are square numbers.

Let $\mathcal{B}'$ denote the set of those numbers whose all digits are square numbers. Then we have the following:

Theorem 3. For any real number $x \geq 1$,we have the asymptotic formula

$$\sum_{n\leq x, n\in\mathcal{B}} \frac{1}{n} = \ln x + \gamma - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right),$$

where C is a computable constant, γ denotes the Euler's constant.

Let $\mathcal{A}'$ denote the complementary set of $\mathcal{A}$, we have following:

Corollary. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\le x,n\in\mathcal{A}'}\frac{1}{n}=\ln x+\gamma-D+O\left(x^{-\frac{\ln\frac{5}{3}}{\ln 10}}\right),$$

where D is a computable constant.

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. First we need the following one simple lemma.

Lemma. For any real number $x\ge 1$, we have the asymptotic formula

$$\sum_{n\le x,n\in\mathcal{B}'}\frac{1}{n}=C+O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).$$

Proof. In the interval $[10^{r-1},10^r),(r\ge 2)$, there are $3\times 4^{r-1}$ numbers belong to $\mathcal{B}'$, and every number's reciprocal isn't greater than $\frac{1}{10^{r-1}}$; when $r=1$, there are 4 numbers belong to $\mathcal{B}'$ and their reciprocals aren't greater than 1. Then we have

$$\sum_{n\in\mathcal{B}'}\frac{1}{n}<3+\sum_{r=1}^{\infty}3\times\frac{4^r}{10^r},$$

then $\sum_{n\in\mathcal{B}'}1$ is convergent to a constant C. So

$$\sum_{n\le x,n\in\mathcal{B}'}\frac{1}{n}=\sum_{n\in\mathcal{B}'}\frac{1}{n}-\sum_{n>x,n\in\mathcal{B}'}\frac{1}{n}=C+O\left(\sum_{r=k}^{\infty}\frac{3\times 4^r}{10^r}\right)=C+O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).$$

Now we come to prove Theorem 1. First for any real number $x\ge 1$, there exists a non-negative integer k, such that $10^k\le x<10^{k+1}(k\ge 1)$ therefore $k\le\log x<k+1$. If a number belongs to $\mathcal{A}$, then its digits only contain these six numbers: 2, 3, 5, 6, 7, 8.

So in the interval $[10^{r-1},10^r)(r\ge 1)$, there are 6^r numbers belong to $\mathcal{A}$. Then we have

$$\sum_{n\le x,n\in\mathcal{A}}1\le\sum_{r=1}^{k+1}6^r=\frac{6}{5}\times(6^{k+1}-1)<\frac{6^{k+2}}{5}<\frac{6^2}{5}\times x^{\frac{\ln 6}{\ln 10}},$$

and

$$\sum_{n\le x,n\in\mathcal{A}}1\ge\sum_{r=1}^{k}6^r=\frac{6}{5}\times(6^k-1)\ge 6^k>\frac{1}{6}\times x^{\frac{\ln 6}{\ln 10}}.$$

So we have

$$\frac{1}{6}\times x^{\frac{\ln 6}{\ln 10}}<\sum_{n\le x,n\in\mathcal{A}}1<\frac{6^2}{5}\times x^{\frac{\ln 6}{\ln 10}}.$$

Taking the logarithm computation on both sides of the above, we get

$$\ln(x^{\frac{\ln 6}{\ln 10}}) + (-\ln 6) < \sum_{n\le x, n\in\mathcal{A}} 1 < \ln\left(x^{\frac{\ln 6}{\ln 10}}\right) + (2\times\ln 6 - \ln 5).$$

So

$$\ln S(x) = \ln\left(\sum_{n\le x, n\in\mathcal{A}} 1\right) = \ln\left(x^{\frac{\ln 6}{\ln 10}}\right) + O(1) = \frac{\ln 6}{\ln 10}\times\ln x + O(1).$$

This proves the Theorem 1.

Now we prove Theorem 2. It is clear that if a number doesn't belong to $\mathcal{B}$, then all of its digits are square numbers. So in the interval $[10^{r-1}, 10^r), (r\ge 2)$, there are $3\times 4^{r-1}$ numbers don't belong to $\mathcal{B}$; when $r=1$, there are 4 numbers don't belong to $\mathcal{B}$. Then we have

$$\begin{aligned}
&\sum_{n\le x, n\in\mathcal{B}} 1 = \sum_{n\le x} 1 - \sum_{n\le x, n\in\mathcal{B}'} 1\\
&= x + O\left(4 + 3\times 4 + 3\times 4^2 + \cdots + 3\times 4^k\right)\\
&= x + O\left(4^{k+1}\right) = x + O\left(x^{\frac{2\times\ln 2}{\ln 10}}\right).
\end{aligned}$$

This completes the proof of the Theorem 2. Now we prove the Theorem 3. In reference [2], we know the asymptotic formula:

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is the Euler's constant.

Then from this asymptotic formula and the above Lemma, we have

$$\begin{aligned}
&\sum_{n\le x, n\in\mathcal{B}}\frac{1}{n} = \sum_{n\le x}\frac{1}{n} - \sum_{n\le x, n\in\mathcal{B}'}\frac{1}{n}\\
&= \ln x + \gamma + O\left(\frac{1}{x}\right) - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right)\\
&= \ln x + \gamma - C + O\left(x^{-\frac{\ln\frac{5}{2}}{\ln 10}}\right).
\end{aligned}$$

This completes the proof of the Theorem 3. Now the Corollary immediately follows from the Lemma and Theorem 3.

Reference

[1] F.Smarandache, Only problems, Not Solutions, Xiquan Publ. House, Chicago, 1993.

[2] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

Scientia Magna
Vol. 1 (2005), No. 2, 49-51

On finite Smarandache near-rings

T.Ramaraj [†] and N.Kannappa [‡]

Lecture in Selection Grade Department of Mathematics T.B.M.L.College
Porayar-609307 Nagappattinam-(Dt), Tamil Nadu, India
Reader in Mathematics A.V.V.M.Pushpam College (Autonomous)
Poondi-613 503 Tanjore-(Dt), Tamil Nadu, India

Abstract In this paper we study the Finite Smarandache-2-algebraic structure of Finite-near-ring, namely, Finite-Smarandache-near-ring, written as Finite-S-near-ring. We define Finite Smarandache near-ring with examples. We introduce some equivalent conditions for Finite S-near-ring and obtain some of its properties.
Keywords Finite-S-near-ring; Finite-Smarandache-near-ring.

§1. Introduction

In this paper, we studied Finite-Smarandache 2-algebraic structure of Finite-near-rings, namely, Finite-Smarandache-near-ring, written as Finite-S-near-ring. A Finite-Smarandache 2-algebraic structure on a Finite-set N means a weak algebraic structure A_0 on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure A_1, stronger algebraic structure means satisfying more axioms, by proper subset means a subset different from the empty set, from the unit element if any, from the whole set [5]. By a Finite-near-ring N, we mean a zero-symmetric Finite- right-near-ring. For basic concept of near-ring we refer to Gunter Pilz [2].

Definition 1. A Finite-near-ring N is said to be Finite-Smarandache-near-ring. If a proper subset M of N is a Finite-near-field under the same induced operations in N.

Example 1 [2]. Let $N = \{0, n_1, n_2, n_3\}$ be the Finite-near-ring defined by:

Let $M = \{0, n_1\} \subset N$ be a Finite-near-field. Defined by

Now $(N, +, .)$ is a Finite-S-near-ring .

Example 2 [4]. Let $N = \{0, 6, 12, 18, 24, 30, 36, 42, 48, 54\}$ (mod 60) be the Finite-near-ring since every ring is a near-ring. Now N is a Finite-near-ring, Whose proper subset $M = \{0, 12, 24, 36, 48\}$ (mod 60) is a Finite-field. Since every field is a near-field, then M is a Finite-near-field. Therefore N is a Finite-S-near-ring.

Theorem 1. Let N be a Finite-near-ring. N is a Finite-S-near-ring if and only if there exist a proper subset M of N , either $M \cong M_c(z_2)$ or Z_p, integers modulo p, a prime number.

Proof. **Part-I**: We assume that N is a Finite-S-near-ring. By definition, there exist a proper subset M of N is a Finite-near-field. By Gunter Pilz Theorem (8.1)[2], either $M \cong$

$M_c(z_2)$ or zero-symmetric. Since Z_p^S is zero-symmetric and Finite-fields implies Z_p, S are zero-symmetric and Finite-near-fields because every field is a near-field. Therefore in particular M is Z_p.

Part-II: We assume that a proper subset M of N, either $M \cong M_c(z_2)$ or Z_p. Since $M_c(z_2)$ and Z_p are Finite-near-fields. Then M is a Finite-near-field. By definition, N is a Finite-S-near-ring.

Theorem. Let N be a Finite-near-ring. N is a Finite-S-near-ring if and only if there exist a proper subset M of N such that every element in M satisfying the polynomial $x^{pm} - x$.

Proof. **Part-I**: We assume that N is a Finite-S-near-ring. By definition, there exist a proper subset M of N is a Finite-near-field. By Gunter Pilz, Theorem (8.13)[2]. If M is a Finite-near-field, then there exist $p \in P, \exists m \in M$ such that $\mid M \mid = p^m$. According to I.N.Herstein[3]. If the Finite-near-field M has p^m element, then every $a \in M$ satisfies $a^{pm} = a$, since every field is a near-field. Now M is a Finite-near-field having p^m element, every element a in M satisfies $a^{pm} = a$. Therefore every element in M satisfying the polynomial $x^{pm} - x$.

Part-II: We assume that there exist a proper subset M of N such that every element in M satisfying the polynomial $x^{pm} - x$, which implies M has p^m element. According to I.N.Herstein[3], For every prime number p and every positive integer m, there is a unique field having p^m element. Hence M is a Finite-field implies M is a Finite-near-field. By definition, N is a Finite-S-near-ring.

Theorem 3. Let N be a Finite-near-ring. N is a Finite-S-near-ring if and only if M has no proper left ideals and $M_0 \neq M$. Where M is a proper sub near-ring of N, in which idempotent commute and for each $x \in M$, there exist $y \in M$ such that $yx \neq 0$.

Proof. **Part-I** :We assume that N is a Finite-S-near-ring. By definition A proper subset M of N is a Finite-near-field. In [1] Theorem (4),it is zero-symmetric and hence every left-ideal is a M-subgroup. Let $M_1 \neq 0$ be a M-subgroup and $m_1 \neq 0 \in M_1$. Then $m_1^{-1}m_1 = 1 \in M_1$. therefore $M = M_1$. Hence M has no proper M-subgroup, which implies M has no proper left ideal.

Part-II: We assume that a proper sub-near-ring M of N has no proper left ideals and $M_0 \neq M$, in which idempotent commute and for each $x \in M$ there exist $y \in M$ such that $yx \neq 0$. Let $x \neq 0$ in M. Let $F(x) = \{m \in M \mid mx = 0\}$. Clearly $F(x)$ is a left ideal. Since there exist $y \in M$ such that $yx \neq 0$. Then $y \notin F(x)$. Hence $F(x) = 0$. Let $\phi : (M, +) \longrightarrow (Mx, +)$ given by $\phi(m) = mx$. Then ϕ is an isomorphism. Since M is finite then $Mx = M$. Now by a theroem(2) in [1], M is a Finite-near-field. Therefore, by definition N is a Finite-S-near-ring.

We summarize what has been studied in

Theorem 4. Let N be a Finite-near-ring. Then the following conditions are equivalent.

1. A proper subset M of N, either $M \cong M_c(z_2)$ or Z_p, integers modulo p, a prime number.
2. A proper subset M of N such that every element in M satisfying the polynomial $x^{pm} - x$.
3. M has no proper left ideals and $M_0 \neq M$. Where M is a proper sub near-ring of N, in which idempotent commute and for each $x \in M$, there exist $y \in M$ such that $yx \neq 0$.

Theorem 5. Let N be a Finite-near-ring. If a proper subset M, sub near-ring of N, in which M has left identity and M is 0-primitive on M^M. Then N is a Finite-S-near-ring.

Proof. By Theorem(8.3)[2], the following conditions are equivalent:

(1) M is a Finite-near-field;

(2) M has left identity and M is 0-primitive on M^M.

Now Theorem is immediate.

Theorem 6. Let N be a Finite-near-ring. If a proper subset M, sub near-ring of N, in which M has left identity and M is simple. Then N is a Finite-S-near-ring.

Proof. By Theorem(8.3)[2], the following conditions are equivalent:

(1) M is a Finite-near-field;

(2) M has left identity and M is simple. Now the Theorem is immediate.

Theorem 7. Let N be a Finite-near-ring. If a proper subset M, sub near-ring of N is a Finite-near-domain, then N is a Finite-S-near-ring.

Proof. By Theorem(8.43)[2], a Finite-near- domain is a Finite-near-field. Therefore M is a Finite-near-field. By definition N is a Finite-S-near-ring.

Theorem 8. Let N be a Finite-near-ring. If a proper subset M of N is a Finite-Integer-domain. Then N is a Finite-S-near-ring.

Proof. By I.N.Herstein[3], every Finite-Integer-domain is a field, since every field is a near-field. Now M is a Finite-near-field. By definition N is a Finite-S-near-ring.

Theorem 9. Let N be a Finite-near-ring. If a proper subset M of N is a Finite-division-ring. Then N is a Finite-S-near-ring.

Proof. By Wedderburn's Theorem(7.2.1)[3], a Finite-division-ring is a necessarily commutative field, which gives M is a field, implies M is a Finite-near-field. By definition N is a Finite-S-near-ring.

References

[1] P.Dheena, On near-field, J. Pure. Appl. Math., **17(3)** (1986), 332-326.

[2] G.Pilz, Near-ring, North Holland, Amsterdam, 1997.

[3] I.N.herstein, Topics in algebra, Wiley Eastern Limited, New Delhi, 1993

[4] R.Padilla, Smarandache algebraic structures,presented to the Universidade do Minho, Baraga, Portugal, 18-23, June, 1999.

[5] PlanetMath, Smarandache n-structure.

Scientia Magna
Vol. 1 (2005), No. 2, 52-54

Some interesting properties of the Smarandache function

Kang Xiaoyu

Editorial Board of Journal of Northwest University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary method to study the property of the Smarandache function, and give an interesting result.
Keywords Smarandache function; Additive property; Greatest prime divisor.

§1. Introduction and results

Let n be an positive integer, the famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

About this function and many other Smarandache type function, many scholars have studied its properties, see [1], [2], [3] and [4]. Let $p(n)$ denotes the greatest prime divisor of n, it is clear that $S(n) \geq p(n)$. In fact, $S(n) = p(n)$ for almost all n, as noted by Erdös [5]. This means that the number of $n \leq x$ for which $S(n) \neq p(n)$, denoted by $N(x)$, is $o(x)$. It is easily to show that $S(p) = p$ and $S(n) < n$ except for the case $n = 4, n = p$. So there have a closely relationship between $S(n)$ and $\pi(x)$:

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n}\right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x. For two integer m and n, can you say $S(mn) = S(m) + S(n)$ is true or false? It is difficult to say. For some m an n, it is true, but for some other numbers it is false.

About this problem, J.Sandor [7] proved an very important conclusion. That is, for any positive integer k and any positive integers $m_1, m_2, \cdots, m_k$, we have the inequality

$$S\left(\prod_{i=1}^{k} m_i\right) \leq \sum_{i=1}^{k} S(m_i).$$

This paper as a note of [7], we shall prove the following two conclusions:

Theorem 1. *For any integer $k \geq 2$ and positive integers $m_1, m_2, \cdots, m_k$, we have the inequality*

$$S\left(\prod_{i=1}^{k} m_i\right) \leq \prod_{i=1}^{k} S(m_i).$$

Theorem 2. *For any integer $k \geq 2$, we can find infinite group numbers $m_1, m_2, \cdots, m_k$ such that:*

$$S\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} S(m_i).$$

§2. Proof of the theorems

In this section, we will complete the proof of the Theorems. First we prove a special case of Theorem 1. That is, for any positive integers m and n, we have

$$S(m)S(n) \geq S(mn).$$

If $m = 1$ (or $n = 1$), then it is clear that $S(m)S(n) \geq S(mn)$. Now we suppose $m \geq 2$ and $n \geq 2$, so that $S(m) \geq 2$, $S(n) \geq 2$, $mn \geq m + n$ and $S(m)S(n) \geq S(m) + S(n)$. Note that $m|S(m)!$, $n|S(n)!$, we have $mn|S(m)!S(n)!|((S(m) + S(n))!$. Because $S(m)S(n) \geq S(m) + S(n)$, we have $(S(m) + S(n))!|(S(m)S(n))!$. That is, $mn|S(m)!S(n)!|(S(m) + S(n))!|(S(m)S(n))!$. From the definition of $S(n)$ we may immediately deduce that

$$S(mn) \leq S(m)S(n).$$

Now the theorem 1 follows from $S(mn) \leq S(m)S(n)$ and the mathematical induction.

Proof of Theorem 2. For any integer n and prime p, if $p^{\alpha} \| n!$, then we have

$$\alpha = \sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right].$$

Let n_i are positive integers such that $n_i \neq n_j$, if $i \neq j$, where $1 \leq i, j \leq k$, $k \geq 2$ is any positive integer. Since

$$\sum_{r=1}^{\infty} \left[\frac{p^{n_i}}{p^r}\right] = p^{n_i - 1} + p^{n_i - 2} + \cdots + 1 = \frac{p^{n_i} - 1}{p - 1}.$$

For convenient, we let $u_i = \frac{p^{n_i} - 1}{p-1}$. So we have

$$S(p^{u_i}) = p^{n_i}, \qquad i = 1, 2, \cdots, k. \tag{1}$$

In general, we also have

$$\sum_{r=1}^{\infty} \left[\frac{\sum_{i=1}^{k} p^{n_i}}{p^r}\right] = \sum_{i=1}^{k} \frac{p^{n_i} - 1}{p - 1} = \sum_{i=1}^{k} u_i.$$

So

$$S\left(p^{u_1 + u_2 + \cdots + u_k}\right) = \sum_{i=1}^{k} p^{n_i}. \tag{2}$$

Combining (1) and (2) we may immediately obtain

$$S\left(\prod_{i=1}^{k} p^{u_i}\right) = \sum_{i=1}^{k} S(p^{u_i}).$$

Let $m_i = p^{u_i}$, noting that there are infinity primes p and n_i, we can easily get Theorem 2.

This completes the proof of the theorems.

References

[1] C.Ashbacher, Some Properties of the Smarandache-Kurepa and Smarandache-Wagstaff Functions. Mathematics and Informatics Quarterly, **7**(1997), 114-116.

[2] A.Begay, Smarandache Ceil Functions Bulletin of Pure and Applied Sciences, **16**(1997), 227-229.

[3] Mark Farris and Patrick Mitchell, Bounding the Smarandache function Smarandache Notions Journal,**13**(2002), 37-42.

[4] Kevin Ford, The normal behavior of the Smarandache function, Smarandache Notions Journal, **10**(1999), 81-86.

[5] P.Erdös, Problem 6674 Amer. Math. Monthly, **98**(1991), 965.

[6] Pan Chengdong and Pan Chengbiao, Element of the analytic number theory, Science Press, Beijing, (1991).

[7] J.Sandor, On a inequality for the Smarandache function, Smarandache Notions Journal, **10**(1999), 125-127.

Scientia Magna
Vol. 1 (2005), No. 2, 55-73

On automorphism groups of maps, surfaces and Smarandache geometries [1]

Linfan Mao
Institute of Systems, Academy of Mathematics and System Sciences,
Chinese Academy of Sciences, Beijing, P.R.China

Abstract A combinatorial map is a connected topological graph cellularly embedded in a surface. This report concentrates on the automorphism group of a map, which is related to the automorphism groups of a Klein surface and a Smarandache manifold, also applied to the enumeration of unrooted maps on orientable and non-orientable surfaces. A number of results for the automorphism groups of maps, Klein surfaces and Smarandache manifolds and the enumeration of unrooted maps underlying a graph on orientable and non-orientable surfaces are discovered. An classification for the closed s-manifolds by maps is found. Open problems related the combinatorial maps with the differential geometry, Riemann geometry and Smarandache geometries are also presented in this report for the further applications of the combinatorial maps to the classical mathematics.

Keywords Automorphism group; Surface; Map; Smarandache geometries; Map geometries; Classification.

Part I. Terminology and Notations

§1.1 Klein Surfaces

A *Klein surface* is a Hausdorff, connected, topological space S with a family $\sum = \{(U_i, \phi_i)|i \in I\}$ such that the chart $\{U_i|i \in I\}$ is an open covering of S, each map $\phi_i : U_i \longrightarrow A_i$ is a homeomorphism onto an open subset A_i of $\mathcal{C}$ or $\mathcal{C}^+ = \{z \in \mathcal{C} : Imz \geq 0\}$ and the transition functions

$$\phi_{ij} = \phi_i\phi_j^- : \phi_j(U_i \bigcap U_j) \longrightarrow \phi_i(U_i \bigcap U_j).$$

are dianalytic, where a mapping $f : A \longrightarrow \mathcal{C}$ is said dianalytic if $\frac{\partial f}{\partial \bar{z}} = 0$ (*Cauchy-Riemann equation*) or $\frac{\partial f}{\partial z} = 0$.

§1.2 {Riemann Surfaces}$\subset$ {Klein surfaces}

§1.3 Embedding and Combinatorial Maps

Embedding of a graph:

For any connected graph $\Gamma = (V(\Gamma), E(\Gamma))$ and a surface S, an *embedding* of the graph Γ in the surface S is geometrical defined to be a continuous $1-1$ mapping $\tau : \Gamma \to S$. The image

[1] Reported at the Academy of Mathematics and Systems of Chinese Academy of Sciences.

$\tau(\Gamma)$ is contained in the 1-skeleton of a triangulation of the surface S. If each component in $S - \tau(\Gamma)$ homeomorphic to an open disk, then the embedding is an embedding.

Map:

A combinatorial map is a connected topological graph cellularly embedded in a surface.

The Algebraic Definition of Maps:

A *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of quadricells Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the Klein group, satisfying the following two conditions:

(Ci) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(Cii) the group $\Psi_J =< \alpha, \beta, \mathcal{P} >$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

§1.4 Orientation

If the group $\Psi_I =< \alpha\beta, \mathcal{P} >$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

§1.5 An Example of Maps K_4 on the torus.

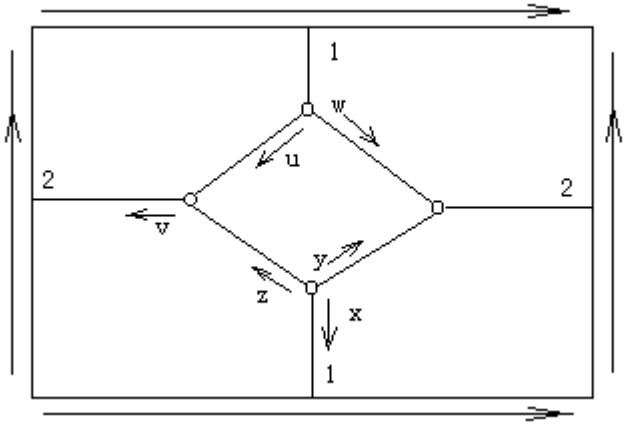


Fig.1

$(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$:

$$\begin{aligned}\mathcal{X}_{\alpha,\beta} &= \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z, \alpha u, \alpha v,\\ &\quad \alpha w, \beta x, \beta y, \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y,\\ &\quad \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\}\end{aligned}$$

$$\begin{aligned}\mathcal{P} &= (x,y,z)(\alpha\beta x,u,w)(\alpha\beta z,\alpha\beta u,v)\\ &\times (\alpha\beta y,\alpha\beta v,\alpha\beta w)(\alpha x,\alpha z,\alpha y)(\beta x,\alpha w,\alpha u)\\ &\times (\beta z,\alpha v,\beta u)(\beta y,\beta w,\beta v)\end{aligned}$$

Vertices:

$v_1 = \{(x,y,z),(\alpha x,\alpha z,\alpha y)\}$
$v_2 = \{(\alpha\beta x,u,w),(\beta x,\alpha w,\alpha u)\}$
$v_3 = \{(\alpha\beta z,\alpha\beta u,v),(\beta z,\alpha v,\beta u)\}$
$v_4 = \{(\alpha\beta y,\alpha\beta v,\alpha\beta w),(\beta y,\beta w,\beta v)\}$

Edges:

$\{e,\alpha e,\beta e,\alpha\beta e\},\ e\in\{x,y,z,u,v,w\}$

Faces:

$f_1 = \{(x,u,v,\alpha\beta w,\alpha\beta x,y,\alpha\beta v,\alpha\beta z),(\beta x,\alpha z,\alpha v,\beta y,\alpha x,\alpha w,\beta v,\beta u)\}$
$f_2 = \{(z,\alpha\beta u,w,\alpha\beta y),(\beta z,\alpha y,\beta w,\alpha u)\}$

§1.6 Isomorphism of Maps

Two maps $M_1 = (\mathcal{X}^1_{\alpha,\beta},\mathcal{P}_1)$ and $M_2 = (\mathcal{X}^2_{\alpha,\beta},\mathcal{P}_2)$ are said to be *isomorphic* if there exists a bijection ξ

$$\xi : \mathcal{X}^1_{\alpha,\beta} \longrightarrow \mathcal{X}^2_{\alpha,\beta}$$

such that for $\forall x\in\mathcal{X}^1_{\alpha,\beta}$,

$$\xi\alpha(x) = \alpha\xi(x), \xi\beta(x) = \beta\xi(x), \xi\mathcal{P}_1(x) = \mathcal{P}_2\xi(x).$$

§1.7 Equivalence

Two maps M_1, M_2 underlying graph Γ are equivalent if there exists an isomorphism ζ between them induced by an element $\xi, \xi\in \mathrm{Aut}\Gamma$. Call ζ an equivalence between M_1, M_2. Certainly, on an orientable surface, an equivalence preserve the orientation on this surface.

Theorem 1.1. Let $M = (\mathcal{X}_{\alpha,\beta},\mathcal{P})$ be a map with an underlying graph Γ, $\forall g\in Aut\Gamma$. Then the extend action of g on $\mathcal{X}_{\alpha,\beta}$ with $X = E(\Gamma)$ is an automorphism of map M iff $\forall v\in V(M)$, g preserves the cyclic order of v.

§1.8 Covering of Maps

For two maps $\widetilde{M} = (\widetilde{\mathcal{X}_{\alpha,\beta}},\widetilde{\mathcal{P}})$ and $M = (\mathcal{X}_{\alpha,\beta},\mathcal{P})$, call the map $\widetilde{M}$ covering the map M if there is a mapping $\pi : \widetilde{\mathcal{X}_{\alpha,\beta}} \to \mathcal{X}_{\alpha,\beta}$ such that $\forall x\in\widetilde{\mathcal{X}_{\alpha,\beta}}$,

$$\alpha\pi(x) = \pi\alpha(x),\ \beta\pi(x) = \pi\beta(x), \pi\widetilde{\mathcal{P}}(x) = \mathcal{P}\pi(x).$$

Theorem 1.2. Let $\pi : \widetilde{\mathcal{X}_{\alpha,\beta}} \to \mathcal{X}_{\alpha,\beta}$ be a covering mapping. Then π is an isomorphism iff π is an $1-1$ mapping.

§1.9 Voltage Map

Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a map and G a finite group. Call a pair (M, ϑ) a voltage map with group G if $\vartheta : \mathcal{X}_{\alpha,\beta} \to G$, satisfying the following condition:

(i) $\forall x \in \mathcal{X}_{\alpha,\beta}, \vartheta(\alpha x) = \vartheta(x), \vartheta(\alpha\beta x) = \vartheta(\beta x) = \vartheta^{-1}(x)$;

(ii) $\forall F = (x, y, \cdots, z)(\beta z, \cdots, \beta y, \beta x) \in F(M)$, the face set of M, $\vartheta(F) = \vartheta(x)\vartheta(y)\cdots\vartheta(z)$ and $< \vartheta(F)|F \in F(u), u \in V(M) >= G$, where, $F(u)$ denotes all the faces incident with the vertex u.

§1.10 Lifting of a Voltage Map

For a voltage map (M, ϑ) with group G, the map $M^\vartheta = (\mathcal{X}^\vartheta_{\alpha,\beta}, \mathcal{P}^\vartheta)$ is called its lifting map.

Theorem 1.3. An finite group G is a fixed-free automorphism group of a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ on $V(M)$ iff there is a voltage map $(M/G, G)$ with an assignment $\vartheta : \mathcal{X}_{\alpha,\beta}/G \to G$ such that $M \cong (M/G)^\vartheta$.

(A permutation group G action on Ω is called *fixed-free* if $G_x = \mathbf{1}_G$ for $\forall x \in \Omega$.)

§1.11 Semi-Arcs of a Graph

An edge $e = uv \in E(\Gamma)$ can be divided into two semi-arcs e_u, e_v.

$X_{\frac{1}{2}}(\Gamma)$— the set of semi-arcs.

Incidence of Semi-Arcs:

Call u the root vertex in the semi-arc e_u. Two semi-arc e_u, f_v are said v-incident or e-incident if $u = v$ or $e = f$.

§1.12 A Semi-Arc Automorphism

An $1-1$ mapping ξ on $X_{\frac{1}{2}}(\Gamma)$ such that $\forall e_u, f_v \in X_{\frac{1}{2}}(\Gamma)$, $\xi(e_u)$ and $\xi(f_v)$ are v-incident or e-incident if e_u and f_v are v-incident or e-incident, is called a semi-arc automorphism of the graph Γ.

$\mathrm{Aut}_{\frac{1}{2}}\Gamma$— the semi-arc automorphism group of Γ

For $\forall g \in \mathrm{Aut}\Gamma$, there is also an induced action $g|^{\frac{1}{2}}$ on $X_{\frac{1}{2}}(\Gamma)$, $g : X_{\frac{1}{2}}(\Gamma) \to X_{\frac{1}{2}}(\Gamma)$, as follows:

$$\forall e_u \in X_{\frac{1}{2}}(\Gamma), g(e_u) = (g(e)_{g(u)}.$$

All induced action of the elements in $\mathrm{Aut}\Gamma$ on $X_{\frac{1}{2}}(\Gamma)$ is denoted by $\mathrm{Aut}\Gamma|^{\frac{1}{2}}$. Notice that

$$\mathrm{Aut}\Gamma \cong \mathrm{Aut}\Gamma|^{\frac{1}{2}}.$$

Theorem 1.4. For a graph Γ without loops,

$$\mathrm{Aut}_{\frac{1}{2}}\Gamma = \mathrm{Aut}\Gamma|^{\frac{1}{2}}.$$

Theorem 1.5. For two maps $M_1 = (\mathcal{X}_{\alpha,\beta}, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_{\alpha,\beta}, \mathcal{P}_2)$ underlying a graph Γ, then

(i) M_1, M_2 are equivalent iff M_1, M_2 are in one orbits of $\mathrm{Aut}_{\frac{1}{2}}\Gamma$ action on $X_{\frac{1}{2}}(\Gamma)$;

(ii) M_1, M_2 are isomorphic iff M_1, M_2 are in one orbits of $\mathrm{Aut}_{\frac{1}{2}}\Gamma \times < \alpha >$ action on $\mathcal{X}_{\alpha,\beta}$.

Part II Automorphisms of Maps and Klein Surfaces

§2.1 Relation of Maps with Klein Surfaces

Angles incident with a Quadricell:

For a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, $x \in \mathcal{X}_{\alpha,\beta}$, the permutation pair $\{(x, \mathcal{P}x), (\alpha x, \mathcal{P}^{-1}\alpha x)\}$ is called an *angle* incident with x.

Theorem 2.1. Any automorphism of a map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is conformal.

Theorem 2.2. If M is a locally orientable map of genus q, then $\mathrm{Aut}M$ is isomorphic to a group of comformal transformations of a compact Klein surface of genus q.

(For Riemann surfaces, the same result gotten by Jones and Singerman in 1978.)

§2.2 The Euler Characteristic of Lifting Map

Theorem 2.3. The Euler characteristic $\chi(M^\vartheta)$ of the lifting map M^ϑ of the voltage map (M, G) is

$$\chi(M^\vartheta) = |G|(\chi(M) + \sum_{m\in\mathcal{O}(F(M))}(-1 + \frac{1}{m})),$$

where $\mathcal{O}(F(M))$ denotes the order $o(F)$ set of the faces in M.

§2.3 A Group Being That of a Map

Theorem 2.4 If a group $G, G \preceq \mathrm{Aut}M$, is fixed-free on $V(M)$, then

$$|G|(\chi(M/G) + \sum_{m\in\mathcal{O}(F(M/G))}(-1 + \frac{1}{m})) = \chi(M).$$

Corollary 2.1. If M is an orientable map of genus p, $G \preceq \mathrm{Aut}M$ is fixed-free on $V(M)$ and the quotient map M/G with genus γ, then

$$|G| = \frac{2p-2}{2\gamma - 2 + \sum\limits_{m\in\mathcal{O}(F(M/G))}(1 - \frac{1}{m}))}.$$

Particularly, if M/G is planar, then

$$|G| = \frac{2p-2}{-2 + \sum\limits_{m\in\mathcal{O}(F(M/G))}(1 - \frac{1}{m}))}.$$

Corollary 2.2. If M is a non-orientable map of genus q, $G \preceq \mathrm{Aut}M$ is fixed-free on $V(M)$ and the quotient map M/G with genus δ, then

$$|G| = \frac{q-2}{\delta - 2 + \sum\limits_{m\in\mathcal{O}(F(M/G))}(1 - \frac{1}{m}))}.$$

Particularly, if M/G is projective planar, then

$$|G| = \frac{q-2}{-1 + \sum\limits_{m\in\mathcal{O}(F(M/G))}(1 - \frac{1}{m}))}.$$

Theorem 2.5. If a group $G, G \preceq \mathrm{Aut}M$, then

$$\chi(M) + \sum_{g\in G, g\neq \mathbf{1}_G} (|\Phi_v(g)| + |\Phi_f(g)|) = |G|\chi(M/G),$$

where, $\Phi_v(g) = \{v|v \in V(M), v^g = v\}$ and $\Phi_f(g) = \{f|f \in F(M), f^g = f\}$, and if G is fixed-free on $V(M)$, then

$$\chi(M) + \sum_{g\in G, g\neq \mathbf{1}_G} |\Phi_f(g)| = |G|\chi(M/G).$$

Corollary 2.3. If a finite group $G, G \preceq \mathrm{Aut}M$ is fixed-free on $V(M)$ and transitive on $F(M)$, for example, M is regular and $G = \mathrm{Aut}M$, then M/G is an one face map and

$$\chi(M) = |G|(\chi(M/G) - 1) + \phi(M)$$

Corollary 2.4. For an one face map M, if G, $G \preceq \mathrm{Aut}M$ is fixed-free on $V(M)$, then

$$\chi(M) - 1 = |G|(\chi(M/G) - 1),$$

and $|G|$, especially, $|\mathrm{Aut}M|$ is an integer factor of $\chi(M) - 1$.

Remark 2.1. For an one face planar map, i.e., the plane tree, the only fixed-free automorphism group on its vertices is the trivial group by the Corollary 2.4.

§2.4 The Non-Euclid Area of a Map

For a given voltage map (M, G), its *non-Euclid area* $\mu(M, G)$ is

$$\mu(M, G) = 2\pi(-\chi(M) + \sum_{m\in\mathcal{O}(F(M))} (-1 + \frac{1}{m})).$$

Particularly, since any map M can be viewed as a voltage map $(M, \mathbf{1}_G)$, we get the non-Euclid area of a map M

$$\mu(M) = \mu(M, \mathbf{1}_G) = -2\pi\chi(M).$$

Theorem 2.6. (Riemann-Hurwitz formula) If $G \preceq \mathrm{Aut}M$ is fixed-free on $V(M)$, then

$$|G| = \frac{\mu(M)}{\mu(M/G, \vartheta)}.$$

Theorem 2.7. The non-Euclid area $\mu(\Delta)$ of a triangle Δ on a surface $\mathcal{S}$ with internal angles η, θ, σ is

$$\mu(\Delta) = \eta + \theta + \sigma - \pi.$$

§2.5 A Combinatorial Refinement of Huriwtz Theorem

Graphical property P:

Define a *graphical property* P to be a kind of subgraphs of a graph Γ, such as, regular subgraphs, circuits, trees, stars, wheels, $\cdots$.

Call a subset A of $\mathcal{X}_{\alpha,\beta}$ of $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ has the graphical property P if the underlying graph of A has property P.

$\mathcal{A}(P, M)$— the set of all the A subset with property P in the map M.

Theorem 2.8. Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a map. Then for $\forall G \preceq \mathrm{Aut}M$,

$$[|v^G||v \in V(M)] \mid |G|$$

$$|G| \mid |A||\mathcal{A}(P, M)|,$$

where $[a, b, \cdots]$ denotes least common multiple of $a, b, \cdots$.

Corollary 2.5. Let $\mathcal{T}r_2$ be the set of tours with each edge appearing 2 times. Then for $\forall G \preceq \mathrm{Aut}M$,

$$|G| \mid (l|\mathcal{T}r_2|,\ l = |T| = \frac{|T|}{2} \geq 1,\ T \in \mathcal{T}r_2,).$$

Let $\mathcal{T}r_1$ be the set of tours without repeat edges. Then

$$|G| \mid (2l|\mathcal{T}r_1|, l = |T| = \frac{|T|}{2} \geq 1,\ T \in \mathcal{T}r_1,).$$

Particularly, denote by $\phi(i, j)$ the number of faces in M with facial length i and singular edges j, then

$$|G| \mid ((2i - j)\phi(i, j), i, j \geq 1),$$

where, $(a, b, \cdots)$ denotes the greatest common divisor of $a, b, \cdots$.

Corollary 2.6. Let $\mathcal{T}$ be the set of trees in the map M. Then for $\forall G \preceq \mathrm{Aut}M$,

$$|G| \mid (2lt_l, l \geq 1),$$

where t_l denotes the number of trees with l edges.

Corollary 2.7. Let v_i be the number of vertices with valence i. Then for $\forall G \preceq \mathrm{Aut}M$,

$$|G| \mid (2iv_i, i \geq 1).$$

Theorem 2.9. Let M be an orientable map of genus $g(M) \geq 2$. Then for $\forall G \preceq \mathrm{Aut}^+M$,

$$|G| \ \leq \ 84(g(M) - 1)$$

and for $\forall G \preceq \mathrm{Aut}M$,

$$|G| \leq 168(g(M) - 1).$$

Corollary 2.8. For any Riemann surface $\mathcal{S}$ of genus $g \geq 2$,

$$4g(\mathcal{S}) + 2 \leq |\mathrm{Aut}^+\mathcal{S}| \leq 84(g(\mathcal{S}) - 1)$$

$$8g(\mathcal{S}) + 4 \leq |\mathrm{Aut}\mathcal{S}| \leq 168(g(\mathcal{S}) - 1).$$

Theorem 2.10. Let M be a non-orientable map of genus $g'(M) \geq 3$. Then for $\forall G \preceq \mathrm{Aut}^+M$,

$$|G| \ \leq \ 42(g'(M) - 2)$$

and for $\forall G \preceq \mathrm{Aut}M$,

$$|G| \leq 84(g'(M) - 2),$$

with the equality hold iff M is a regular map with vertex valence 3 and face valence 7 or vice via.

Corollary 2.9. For any Klein surface $\mathcal{K}$ underlying a non-orientable surface of genus $q \geq 3$,

$$|\mathrm{Aut}^+\mathcal{K}| \leq 42(q-2) \quad and \quad |\mathrm{Aut}\mathcal{K}| \leq 84(q-2).$$

§2.6 The Cyclic Group of a Klein Surface

Theorem 2.11. Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a map and $N = p_1^{r_1} \cdots p_k^{r_k}, p_1 < p_2 < \cdots < p_k$, be the arithmetic decomposition of the integer N. Then for any voltage assignment $\vartheta : \mathcal{X}_{\alpha,\beta} \to Z_N$,

(i) if M is orientable, the minimum genus g_{min} of the lifting map M^ϑ which admits an automorphism of order N, fixed-free on $V(M^\vartheta)$, is

$$g_{min} = 1 + N\{g(M) - 1 + (1 - \sum_{m \in \mathcal{O}(F(M))} \frac{1}{p_1})\lfloor \frac{\phi(M)}{2} \rfloor\}.$$

(ii) if M is non-orientable, the minimum genus g'_{min} of the lifting map M^ϑ which admits an automorphism of order N, fixed-free on $V(M^\vartheta)$, is

$$g'_{min} = 2 + N\{g(M) - 2 + 2(1 - \frac{1}{p_1})\lfloor \frac{\phi(M)}{2} \rfloor\}. \qquad \natural$$

Theorem 2.12. The maximum order N_{max} of an automorphism g of an orientable map M of genus≥ 2 is

$$N_{max} \leq 2g(M) + 1$$

and the maximum order N'_{max} of anautomorphism g of a non-orientable map of genus≥ 3 is

$$N'_{max} \leq g(M) + 1,$$

where $g(M)$ is the genus of the map M.

Corollary 2.10. The maximum order of an automorphism of a Riemann surface of genus≥ 2 is $2g(M) + 1$, and the maximum order of an automorphism of a non-orientable Klein surface of genus≥ 3 without boundary is $g(M) + 1$.

§2.7 The Subgroup of a Graph Being That of Maps

Theorem 2.13. Let Γ be a connected graph. If $G \preceq \mathrm{Aut}\Gamma$, then G is an automorphism group of a map underlying the graph Γ iff for $\forall v \in V(\Gamma)$, the stabler $G_v \preceq < v > \times < \alpha >$.

Theorem 2.14. Let Γ be a connected graph. If $G \preceq \mathrm{Aut}\Gamma$, then G is an orientation-preserving automorphism group of a map underlying the graph Γ iff for $\forall v \in V(\Gamma)$, the stabler $G_v \preceq < v >$ is a cyclic group.

Theorem 2.15. Let M be a map underlying the graph G and $o_{max}(M, g), o_{max}(G, g)$ be the maximum order of orientation-preserving automorphism in $\mathrm{Aut}M$ and in $\mathrm{Aut}_{\frac{1}{2}}G$. Then

$$o_{max}(M, g) \leq o_{max}(G, g),$$

and the equality hold for at least one map underlying the graph G.

Corollary 2.11. For any positive integer n, there exists a vertex transitive map M underlying a circultant such that Z_n is an orientation-preserving automorphism group of the map M.

Corollary 2.12. The maximum order of an orientation - preserving automorphism of a complete map $\mathcal{K}_n, n \geq 3$, is at most n.

Part III The representation of Automorphisms of a Map

§3.1 Complete Maps

A map underlying a complete graph K_n is called a complete map. Let K_n be a complete graph of order n. Label its vertices by integers $1, 2, ..., n$. Then its edge set is $\{ij|1 \leq i, j \leq n, i \neq j\, ij = ji\}$, and

$$\begin{aligned} \mathcal{X}_{\alpha,\beta}(K_n) &= \{i^{j+} : 1 \leq i, j \leq n, i \neq j\} \\ &\bigcup \{i^{j-} : 1 \leq i, j \leq n, i \neq j\} \end{aligned}$$

$$\alpha = \prod_{1 \leq i,j \leq n, i \neq j} (i^{j+}, i^{j-}),$$

$$\beta = \prod_{1 \leq i,j \leq n, i \neq j} (i^{j+}, i^{j+})(i^{j-}, i^{j-}).$$

Theorem 3.1. All orientation-preserving automorphisms of non-orientable complete maps of order≥ 4 are extended actions of elements in

$$\mathcal{E}_{[s^{\frac{n}{s}}]}, \quad \mathcal{E}_{[1,s^{\frac{n-1}{s}}]},$$

and all orientation-reversing automorphisms of non-orientable complete maps of order≥ 4 are extended actions of elements in

$$\alpha\mathcal{E}_{[(2s)^{\frac{n}{2s}}]}, \quad \alpha\mathcal{E}_{[(2s)^{\frac{4}{2s}}]}, \quad \alpha\mathcal{E}_{[1,1,2]},$$

where, $\mathcal{E}_\theta$ denotes the conjugatcy class containing element θ in the symmetry group S_n.

Theorem 3.2. All orientation-preserving automorphisms of orientable complete maps of order≥ 4 are extended actions of elements in

$$\mathcal{E}_{[s^{\frac{n}{s}}]}, \quad \mathcal{E}_{[1,s^{\frac{n-1}{s}}]}$$

and all orientation-reversing automorphisms of orientable complete maps of order≥ 4 are extended actions of elements in

$$\alpha\mathcal{E}_{[(2s)^{\frac{n}{2s}}]}, \quad \alpha\mathcal{E}_{[(2s)^{\frac{4}{2s}}]}, \quad \alpha\mathcal{E}_{[1,1,2]},$$

where,$\mathcal{E}_\theta$ denotes the conjugatcy class containing θ in S_n.

§3.2 Semi-Regular Maps

A graph is *semi-regular* if it is simple and its automorphism group action on its ordered pair of adjacent vertices is fixed-free and a map is semi-regular if it underlying a semi-regular graph.

Theorem 3.3. Let Γ be a semi-regular graph. Then all the automorphisms of orientable maps underlying the graph Γ are

$$g|^{\mathcal{X}_{\alpha,\beta}} \ and \ \alpha h|^{\mathcal{X}_{\alpha,\beta}}, g, h \in \mathrm{Aut}\Gamma \ with \ h \equiv 0(mod 2).$$

and all the automorphisms of non-orientable maps underlying the graph Γ are also

$$g|^{\mathcal{X}_{\alpha,\beta}} \ and \ \alpha h|^{\mathcal{X}_{\alpha,\beta}}, g, h \in \mathrm{Aut}\Gamma \ with \ h \equiv 0(mod 2).$$

§3.3 One Face Maps

Theorem 3.4. Let B_n be a bouquet with n edges $1, 2, \cdots, n$. Then the automorphisms $(g; h_1, h_2, \cdots, h_n)$ of orientable maps underlying a $B_n, n \geq 1$, are respective

$(O1)$ $g \in \mathcal{E}_{[k^{\frac{n}{k}}]}$, $h_i = 1, i = 1, 2, \cdots, n$;

$(O2)$ $g \in \mathcal{E}_{[k^{\frac{n}{k}}]}$ and if

$$g = \prod_{i=1}^{n/k} (i_1, i_2, \cdots i_k),$$

where $i_j \in \{1, 2, \cdots, n\}, n/k \equiv 0(mod 2)$, then $h_{i_1} = (1, \alpha\beta), i = 1, 2, \cdots, \frac{n}{k}$ and $h_{i_j} = 1 \ for \ j \geq 2$;

$(O3)$ $g \in \mathcal{E}_{[k^{2s}, (2k)^{\frac{n-2ks}{2k}}]}$ and if

$$g = \prod_{i=1}^{2s} (i_1, i_2, \cdots i_k) \prod_{j=1}^{(n-2ks)/2k} (e_{j_1}, e_{j_2}, \cdots, e_{j_{2k}}),$$

where $i_j, e_{j_t} \in \{1, 2, \cdots, n\}$, then $h_{i_1} = (1, \alpha\beta), i = 1, 2, \cdots, s, h_{i_l} = 1$ for $l \geq 2$ and $h_{j_t} = 1$ for $t = 1, 2, \cdots, 2k$ and the automorphisms $(g; h_1, h_2, \cdots, h_n)$ of non-orientable maps underlying a $B_n, n \geq 1$, are respective

$(N1)$ $g \in \mathcal{E}_{[k^{\frac{n}{k}}]}$, $h_i = 1, i = 1, 2, \cdots, n$;

$(N2)$ $g \in \mathcal{E}_{[k^{\frac{n}{k}}]}$ and if

$$g = \prod_{i=1}^{n/k} (i_1, i_2, \cdots i_k),$$

where $i_j \in \{1, 2, \cdots, n\}, n/k \equiv 0(mod 2)$, then $h_{i_1} = (1, \alpha\beta), (1, \beta)$ with at least one $h_{i_{01}}(1, \beta), i = 1, 2, \cdots, \frac{n}{k}$ and $h_{i_j} = 1 \ for \ j \geq 2$;

$(N3)$ $g \in \mathcal{E}_{[k^{2s}, (2k)^{\frac{n-2ks}{2k}}]}$ and if

$$g = \prod_{i=1}^{2s} (i_1, i_2, \cdots i_k) \prod_{j=1}^{(n-2ks)/2k} (e_{j_1}, e_{j_2}, \cdots, e_{j_{2k}}),$$

where $i_j, e_{j_t} \in \{1, 2, \cdots, n\}$, then $h_{i_1} = (1, \alpha\beta), (1, \beta)$ with at least one $h_{i_{01}} = (1, \beta), i = 1, 2, \cdots, s, h_{i_l} = 1$ for $l \geq 2$ and $h_{j_t} = 1$ for $t = 1, 2, \cdots, 2k$.

Part IV The Enumeration of Unrooted Maps

§4.1 A Scheme for Enumeration

Theorem 4.1. For a given graph Γ, let $\mathcal{E} \subset \mathcal{E}^L(\Gamma)$, then the numbers $n(\mathcal{E}, \Gamma)$ and $\eta(\mathcal{E}, \Gamma)$ of non-isomorphic unrooted maps and non-equivalent embeddings in $\mathcal{E}$ are respective

$$n(\mathcal{E}, \Gamma) = \frac{1}{2|\mathrm{Aut}_{\frac{1}{2}}\Gamma|} \sum_{g \in \mathrm{Aut}_{\frac{1}{2}}\Gamma} |\Phi_1(g)|,$$

where, $\Phi_1(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}$ and $\mathcal{P}^g = \mathcal{P}$ or $\mathcal{P}^{g\alpha} = \mathcal{P}\}$ and

$$\eta(\mathcal{E}, \Gamma) = \frac{1}{|\mathrm{Aut}_{\frac{1}{2}}\Gamma|} \sum_{g \in \mathrm{Aut}_{\frac{1}{2}}\Gamma} |\Phi_2(g)|,$$

where, $\Phi_2(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}$ and $\mathcal{P}^g = \mathcal{P}\}$.

Corollary 4.1. The numbers $n^O(\Gamma), n^N(\Gamma)$ and $n^L(\Gamma)$ of non-isomorphic unrooted orientable maps ,non-orientable maps and locally orientable maps underlying a graph Γ are respective

$$n^O(\Gamma) = \frac{1}{2|\mathrm{Aut}_{\frac{1}{2}}\Gamma|} \sum_{g \in \mathrm{Aut}_{\frac{1}{2}}\Gamma} |\Phi_1^O(g)|;$$

$$n^N(\Gamma) = \frac{1}{2|\mathrm{Aut}_{\frac{1}{2}}\Gamma|} \sum_{g \in \mathrm{Aut}_{\frac{1}{2}}\Gamma} |\Phi_1^N(g)|;$$

$$n^L(\Gamma) = \frac{1}{2|\mathrm{Aut}_{\frac{1}{2}}\Gamma|} \sum_{g \in \mathrm{Aut}_{\frac{1}{2}}\Gamma} |\Phi_1^L(g)|,$$

where, $\Phi_1^O(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^O(\Gamma)$ and $\mathcal{P}^g = \mathcal{P}$ or $\mathcal{P}^{g\alpha} = \mathcal{P}\}$, $\Phi_1^N(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^N(\Gamma)$ and $\mathcal{P}^g = \mathcal{P}$ or $\mathcal{P}^{g\alpha} = \mathcal{P}\}$, $\Phi_1^L(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^L(\Gamma)$ and $\mathcal{P}^g = \mathcal{P}$ or $\mathcal{P}^{g\alpha} = \mathcal{P}\}$.

§4.2 The Number of Complete Maps

Theorem 4.2. The number $n^L(K_n)$ of complete maps of order $n \geq 5$ on surfaces is

$$n^L(K_n) = \frac{1}{2}\Big(\sum_{k|n} + \sum_{k|n, k\equiv 0(mod 2)}\Big) \frac{2^{\alpha(n,k)}(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} + \sum_{k|(n-1), k\neq 1} \frac{\phi(k) 2^{\beta(n,k)} (n-2)!^{\frac{n-1}{k}}}{n-1},$$

where,

$$\alpha(n,k) = \begin{cases} \frac{n(n-3)}{2k}, & \text{if} \quad k \equiv 1(mod 2); \\ \frac{n(n-2)}{2k}, & \text{if} \quad k \equiv 0(mod 2), \end{cases}$$

and

$$\beta(n,k) = \begin{cases} \frac{(n-1)(n-2)}{2k}, & \text{if} \quad k \equiv 1(mod2); \\ \frac{(n-1)(n-3)}{2k}, & \text{if} \quad k \equiv 0(mod2). \end{cases}$$

and $n^L(K_4) = 11$.

Theorem 4.3. The number $n^O((K_n)$ of complete maps of order $n \geq 5$ on orientable surfaces is

$$n^O(K_n) \quad = \quad \frac{1}{2}(\sum_{k|n} + \sum_{k|n, k\equiv 0(mod2)}) \frac{(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} + \sum_{k|(n-1), k\neq 1} \frac{\phi(k)(n-2)!^{\frac{n-1}{k}}}{n-1}.$$

and $n(K_4) = 3$. For K_4 on the surfaces, see the Fig.2

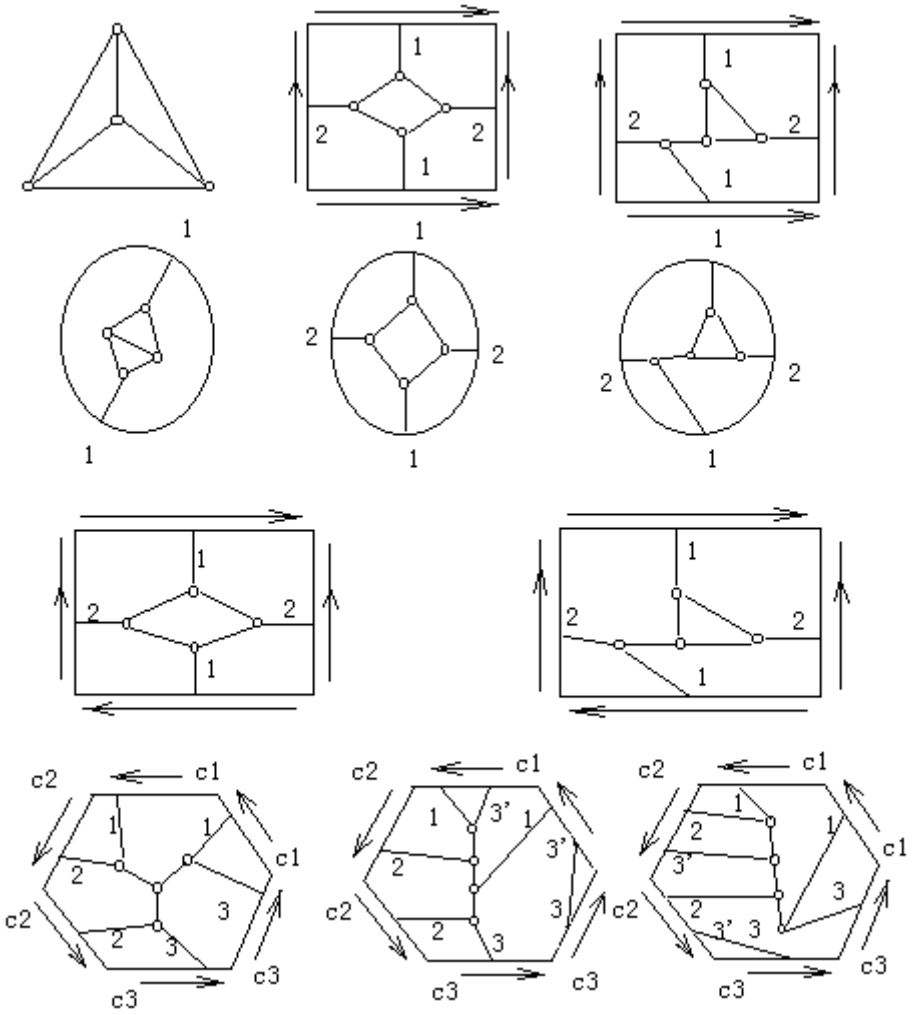


Fig.2

§4.3 The Number of Semi-Regular Maps

Theorem 4.4. Let Γ be a semi-regular graph. Then the numbers of unrooted maps on orientable and non-orientable surfaces underlying the graph Γ are

$$n^O(\Gamma) = \frac{1}{|\mathrm{Aut}\Gamma|}(\sum_{\xi\in \mathrm{Aut}\Gamma}\lambda(\xi)\prod_{x\in T_\xi^V}(\frac{d(x)}{o(\xi|_{N_\Gamma(x)})}-1)!$$

and

$$n^N(\Gamma) = \frac{1}{|\mathrm{Aut}\Gamma|} \times \sum_{\xi\in \mathrm{Aut}\Gamma}(2^{|T_\xi^E|-|T_\xi^V|}-1)\lambda(\xi)\prod_{x\in T_\xi^V}(\frac{d(x)}{o(\xi|_{N_\Gamma(x)})}-1)!,$$

where $\lambda(\xi)=1$ if $o(\xi)\equiv 0(mod 2)$ and $\frac{1}{2}$, otherwise.

Corollary 4.2. Let $\Gamma = \mathrm{Cay}(Z_p : S)$ be connected graph of prime order p with $(p-1,|S|) = 2$. Then

$$n^O(\Gamma,\mathcal{M}) = \frac{(|S|-1)!^p + 2p(|S|-1)!^{\frac{p+1}{2}}}{4p} + \frac{(p-1)(|S|-1)!}{4p}$$

and

$$\begin{aligned} n^N(\Gamma,\mathcal{M}) &= \frac{(2^{\frac{p|S|}{2}-p}-1)(|S|-1)!^p}{2p} + \frac{2(2^{\frac{p|S|-2p-2)}{4}}-1)p(|S|-1)!^{\frac{p+1}{2}}}{2p} \\ &+ \frac{(2^{\frac{|S|-2}{2}}-1)(p-1)(|S|-1)!}{4p}. \end{aligned}$$

§4.4 The Number of One Vertex Maps

Theorem 4.5. The number $n^O(B_n)$ of non -isomorphic maps on orientable surfaces underlying a graph B_n is

$$\begin{aligned} n^O(B_n) &= \sum_{k|2n,k\neq 2n} k^{\frac{2n}{k}-1}(\frac{2n}{k}-1)!\frac{1}{(\frac{2n}{k})!}\frac{\partial^{\frac{2n}{k}}(Z(S_n[S_2]))}{\partial s_k^{\frac{2n}{k}}}|_{s_k=0} \\ &+ \phi(2n)\frac{\partial(Z(S_n[S_2]))}{\partial s_{2n}}|_{s_{2n}=0} \end{aligned}$$

Theorem 4.6. he number $n^N(B_n)$ of non -isomorphic maps on the non-orientable surfaces with an underlying graph B_n is

$$\begin{aligned} n^N(B_n) &= \frac{(2n-1)!}{n!} + \sum_{k|2n,3\leq k<2n}(2k)^{\frac{2n}{k}-1}(\frac{2n}{k}-1)!\frac{\partial^{\frac{2n}{k}}(Z(S_n[S_2]))}{\partial s_k^{\frac{2n}{k}}}|_{s_k=0} \\ &+ \frac{1}{2^n n!}(\sum_{s\geq 1}\frac{n!}{(n-2s)!s!} + 4^n(n-1)!(\frac{\partial^n(Z(S_n[S_2]))}{\partial s_2^n}|_{s_2=0} - \lfloor\frac{n}{2}\rfloor)). \end{aligned}$$

For B_2 on the surfaces, see the Fig.3.

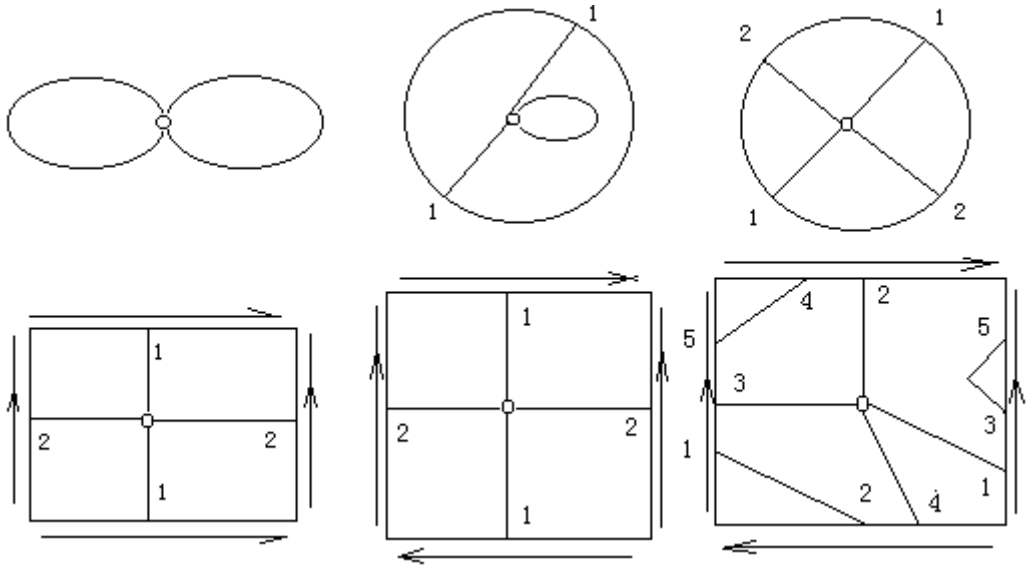


Fig.3

Part V Map Geometry

§5.1 What are the Contribution of Maps to Mathematics

Klein Erlanger Program:

Any geometry is finding invariant properties under the transformation group of this geometry (This is essentially the group action idea.)

The following problems are applications of the Klein Erlanger Program in maps:

(*i*)*determine isomorphism maps or rooted maps;*

(*ii*)*determine equivalent embeddings of a graph;*

(*iii*)*determine an embedding whether exists;*

(*iv*)*enumerate maps or rooted maps on a surface;*

(*v*)*enumerate embeddings of a graph on a surface;*

(*vi*) $\cdots$, etc.

What are their importance to classical mathematics?

What are their contributions to science?

§5.2 Smarandache Geometries

Classical geometries:

The axiom system of *Euclid geometry* is the following:

(A1)*there is a straight line between any two points.*

(A2)*a finite straight line can produce a infinite straight line continuously.*

(A3)*any point and a distance can describe a circle.*

(A4)*all right angles are equal to one another.*

(A5)*if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

The axiom (A5) can be also replaced by:

(A5')*given a line and a point exterior this line, there is one line parallel to this line.*

The *Lobachevshy-Bolyai-Gauss geometry*, also called hyperbolic geometry, is a geometry with axioms $(A1) - (A4)$ and the following axiom $(L5)$:

(L5) *there are infinitely many line parallels to a given line passing through an exterior point.*

The *Riemann geometry* is a geometry with axioms $(A1) - (A4)$ and the following axiom $(R5)$:

there is no parallel to a given line passing through an exterior point.

Smarandache introduced the *paradoxist geometry*, *non-geometry*, *counter-proje -ctive geometry* and *anti-geometry* by contradicts the axioms $(A1) - (A5)$ in Euclid geometry, generalize the classical geometries. For example, the axioms of a Paradoxist geometry are $(A1) - (A4)$ and with one of the following as the axiom $(P5)$:

(i)*there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(ii)*there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(iii)*there are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $l_1, l_2, \cdots, l_k, k \geq 2$ *pass through the point and do not intersect the initial line.*

(iv)*there are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(v)*there are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

F. Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

The Smarandache geometries are defined as follows.

Definition 5.1. An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969).

A nice model for the Smarandache geometries, called s-manifolds, is found by Isier, which is defined by Mao using maps as follows:

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \leq i \leq n$ satisfying the following conditions:

(i) *Each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i, j \leq n$ and $i \neq j$;*

(ii) *Each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.*

H.Iseri, *Smarandache manifolds*, American Research Press, Rehoboth, NM,2002.

L.F.Mao, *Automorphism groups of maps, surfaces and Smarandache geometries*, American Research Press, Rehoboth, NM,2005.

§5.3 A Classification of Smarandache Manifolds

Classical Type:

(1) $\Delta_1 = \{5 - regular\ triangular\ maps\}$ (*elliptic*);

(2) $\Delta_2 = \{6 - regular\ triangular\ maps\}$(*euclidean*);

(3) $\Delta_3 = \{7 - regular\ triangular\ maps\}$(*hyperbolic*).

Smarandache Type:

(4) $\Delta_4 = \{triangular\ maps\ with\ vertex\ valency\ 5\ \text{and}\ 6\}$ (*euclid-elliptic*);

(5) $\Delta_5 = \{triangular\ maps\ with\ vertex\ valency\ 5\ \text{and}\ 7\}$ (*elliptic-hyperbolic*);

(6) $\Delta_6 = \{triangular\ maps\ with\ vertex\ valency\ 6\ \text{and}\ 7\}$ (*euclid-hyperbolic*);

(7) $\Delta_7 = \{triangular\ maps\ with\ vertex\ valency\ 5, 6\ \text{and}\ 7\}$ (*mixed*).

Theorem 5.1. $|\Delta_1| = 2$, $|\Delta_5| \geq 2$ and $|\Delta_i|, i = 2, 3, 4, 6, 7$ are infinite.

Iseri proposed a question: *Do the other closed 2-manifolds correspond to s-manifolds with only hyperbolic vertices?*. Since $|\Delta_3|$ is infinite, the answer is affirmative for this question.

§5.4 Map Geometry

Definition 5.2. For a combinatorial map M with each vertex valency≥ 3, associates a real number $\mu(u), 0 < \mu(u) < \pi$, to each vertex $u, u \in V(M)$. Call (M, μ) the fundamental map space, $\mu(u)$ the angle factor of the vertex u and to be orientable or non-orientable if the map M is orientable or not.

Definition 5.3. A point u in a map space (M, μ) is called elliptic, euclidean or hyperbolic if $\rho(u)\mu(u) < 2\pi$, $\rho(u)\mu(u) = 2\pi$ or $\rho(u)\mu(u) > 2\pi$.

Definition 5.4. Let (M, μ) be a map space. An m-line in (M, μ) is a curve with a constant curvature. Points in (M, μ) are called m-points.

We have the following result for map geometries.

Theorem 5.2. For any planar map M with order≥ 3 and vertex valency≥ 3, there is an angle factor μ such that (M, μ) is a Smarandache geometry by denial the axiom (A5) with the axioms (A5), (L5) and (R5).

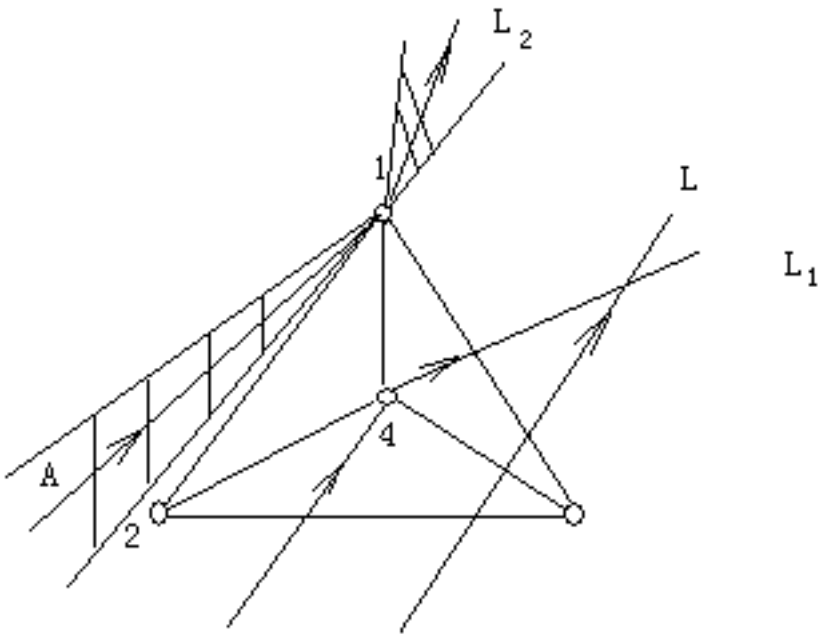


Fig.4

Theorem 5.3. For any map M on an orientable surface with order≥ 3 and vertex valency≥ 3, there is an angle factor μ such that (M, μ) is a Smarandache geometry by denial the axiom (A5) with the axioms (A5),(L5) and (R5).

Theorem 5.4. Let P be a k-polygon in a map space with each line segment passes through at most one elliptic or hyperbolic point. If H is the set of elliptic points and hyperbolic points on the line segment of P, then the sum of the internal angles in P is

$$(k + |H| - 2)\pi - \frac{1}{2}\sum_{u\in H}\rho(u)\mu(u).$$

Corollary 5.1. Let $\triangle$ be a triangle in a map space. Then

(i) if $\triangle$ is euclidean, then then the sum of its internal angles is equal to π;

(ii) if $\triangle$ is elliptic, then the sum of its internal angles is less than π;

(iii) if $\triangle$ is hyperbolic, then the sum of its internal angles is more than π.

Theorem 5.5. The number $n^O(\Gamma, g)$ of non -equivalent orientable map geometries underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) is

$$\frac{3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|},$$

where $\rho(v)$ is the valency of the vertex v in the graph G.

Part VI Open Problems for Combinatorial Maps

§6.1 The Uniformization Theorem for Simple Connected Riemann Surfaces

The *uniformization theorem* for simple connected Riemann surfaces is one of those beautiful results in the Riemann surface theory, which is stated as follows:

If $\mathcal{S}$ is a simple connected Riemann surface, then $\mathcal{S}$ is conformally equivalent to one and only one of the following three:

(a) $\mathcal{C}\bigcup\infty$;

(b) $\mathcal{C}$;

(c) $\triangle = \{z \in \mathcal{C} || z| < 1\}$.

How can we define the conformal equivalence for maps enabling us to get the uniformization theorem of maps?

What is the correspondence class maps with the three type $(a)-(c)$ Riemann surfaces?

§6.2 Combinatorial Construction of an Algebraic Curve of Genus

A complex plane algebraic curve $\mathcal{C}_l$ is a homogeneous equation $f(x,y,z)=0$ in $P_2\mathcal{C} = (C^2 \setminus (0,0,0))/\sim$, where $f(x,y,z)$ is a polynomial in x,y and z with coefficients in $\mathcal{C}$. The degree of $f(x,y,z)$ is said the *degree of the curve* $\mathcal{C}_l$. For a Riemann surface S, a well-known result is ([2]) there is a holomorphic mapping $\varphi : S \to P_2\mathcal{C}$ such that $\varphi(S)$ is a complex plane algebraic curve and

$$g(S) = \frac{(d(\varphi(S))-1)(d(\varphi(S))-2)}{2}.$$

By map theory, we know a combinatorial map also is on a surface with genus. Then

whether we can get an algebraic curve by all edges in a map or by make operations on the vertices or edges of the map to get plane algebraic curve with given k-multiple points?

how do we find the equation $f(x,y,z)=0$?

§6.3 Classification of s-Manifolds by Maps

We present an elementary classification for the closed s-manifolds in the Part V. For the general s-manifolds, their correspondence combinatorial model is the maps on surfaces with boundary, founded by Bryant and Singerman in 1985 (R.P.Bryant and D.Singerman, Foundations of the theory of maps on surfaces with boundary,*Quart.J.Math.Oxford*(2),36(1985), 17-41.). The later is also related to the modular groups of spaces and need to investigate further itself. The questions are

(i) **how can we combinatorially classify the general s-manifolds by maps with boundary?**

(ii) **how can we find the automorphism group of an s-manifold?**

(iii) **how can we know the numbers of non-isomorphic s-manifolds, with or without root?**

(iv) **find rulers for drawing an s-manifold on a surface, such as, the torus, the projective plane or Klein bottle, not the plane.**

§6.4 Map Geometries

(i) **For a given graph, determine properties of the map geometries underlying this graph.**

(ii) **For a given locally orientable surface, determine the properties of map geometries on this surface.**

(iii) **Classify map geometries on a locally orientable surface.**

(iv) **Enumerate non-equivalent map geometries underlying a graph or on a locally orientable surface.**

(v) **Establish the surface geometry by map geometries.**

(vi) **Applying map geometries to classical mathematics or other sciences.**

§6.5 Gauss Mapping Among Surfaces

In the classical differential geometry, a *Gauss mapping* among surfaces is defined as follows:

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. The mapping $N : \mathcal{S} \to R^3$ takes its value in the unit sphere

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. The map $N : \mathcal{S} \to S^2$, thus defined, is called the Gauss mapping.

we know that for a point $P \in \mathcal{S}$ such that the Gaussian curvature $K(P) \neq 0$ and V a connected neighborhood of P with K does not change sign,

$$K(P) = \lim_{A \to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$. The questions are

(i) **what is its combinatorial meaning of the Gauss mapping? How to realizes it by maps?**

(ii) **how we can define various curvatures for maps and rebuilt the results in the classical differential geometry?**

§6.6 The Gauss-Bonnet Theorem

Let $\mathcal{S}$ be a compact orientable surface. Then

$$\int \int_{\mathcal{S}} K d\sigma = 2\pi\chi(\mathcal{S}),$$

where K is Gaussian curvature on $\mathcal{S}$.

This is the famous Gauss-Bonnet theorem for compact surface. The questions are

(i) **what is its combinatorial mean of the Gauss curvature?**

(ii) **how can we define the angle, area, volume, curvature, $\cdots$, of a map?**

(iii)**can we rebuilt the Gauss-Bonnet theorem by maps? or can we get a generalization of the classical Gauss-Bonnet theorem by maps?**

Scientia Magna
Vol. 1 (2005), No. 2, 74-77

On the mean value of Smarandache ceil function[1]

Ding Liping

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China, 710069

Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N \mid n \mid x^k\} \, (\forall n \in N^*) \, .$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it.

Keywords Smarandache ceil function; Mean value; Asymptotic formula.

§1. Introduction

For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N \mid n \mid x^k\} \, (\forall n \in N^*) \, .$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots$. This was introduced by Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function. For example, Ibstedt [2] and [3] studied this function both theoretically and computationally, and got the following conclusions:

$$(\forall a, b \in N^*)(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b),$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} . \cdots . p_r^{\alpha_r}) = S_(p_1^{\alpha_1}) . \cdots . S_(p_r^{\alpha_r}).$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 2$, for any fixed positive integer k, we have the asymptotic formula

$$\sum_{n \leq x} S_k(n) = \frac{x^2}{2} \zeta(2k-1) \prod_p \left[1 - \frac{1}{p(p+1)} \left(1 + \frac{1}{p^{2k-3}} \right) \right] + O\left(x^{\frac{3}{2}+\epsilon} \right) .$$

where $\zeta(s)$ is the Riemann zeta function, $\prod_p$ denotes the product over all prime p, and ϵ be any fixed positive number.

[1]This work is supported by the N.S.F(60472068) and P.N.S.F of P.R.China

This solved a conjecture of [4].

From this theorem we may immediately deduce the following:

Corollary 1. For any real number $x \geq 2$, we have the asymptotic formula:

$$\sum_{n \leq x} S_2(n) = \frac{3x^2}{\pi^2}\zeta(3) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Corollary 2. Let $x \geq 1$ and $Sc(n)$ denotes the smallest cube greater than or equal to n, then we have

$$\sum_{n \leq x} (Sc(n) - n) = \frac{9}{10}x^{\frac{5}{3}} + O\left(x^{\frac{4}{3}}\right).$$

§2. A Lemma

To complete the proof of the theorem, we need the following famous Perron formula [5]:

Lemma. Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, with $s = \sigma + it$ is convergent absolutely for $\sigma > \beta$, and that there exist a positive λ and a positive increasing function $A(s)$ such that

$$\sum_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \ll (\sigma - \beta)^{-1}, \sigma \to \beta + 0$$

and

$$a(n) \ll A(n), n = 1, 2, \cdots.$$

Then for any $b > 0$, $b + \sigma > \beta$, and x not to be an integer, we have

$$\begin{aligned}\sum_{n \leq x} a(n)n^{-s_0} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s_0+\omega)\frac{x^{\omega}}{\omega}\mathrm{d}\omega + O\left(\frac{x^b}{T(b+\sigma-\beta)^{\lambda}}\right) \\ &\quad + O\left(\frac{A(2x)x^{1-\sigma}\log x}{T \,||\, x \,||}\right),\end{aligned}$$

where $|| \, x \, ||$ is the nearest integer to x.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{S_k(n)}{n^s},$$

where $\mathrm{Re}(s) > 3$.

By Euler product formula [6], we have

$$\begin{aligned} f(s) &= \prod_p \left(1+\frac{S_k(p)}{p^s}+\frac{S_k(p^2)}{p^{2s}}+\cdots+\frac{S_k(p^k)}{p^{ks}}+\cdots\right) \\ &= \prod_p \left(1+\frac{p}{p^s}+\frac{p}{p^{2s}}+\cdots+\frac{p}{p^{ks}}+\frac{p^2}{p^{(k+1)s}}+\cdots+\frac{p^2}{p^{2ks}}+\cdots\right) \\ &= \prod_p \left(1+\frac{1}{p^{s-1}}\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}+\frac{p^2}{p^{(k+1)s}}\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}+\cdots\right) \\ &= \prod_p \left(1+\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\frac{\frac{1}{p^{s-1}}}{1-\frac{1}{p^{ks-1}}}\right) \\ &= \frac{\zeta(s)\zeta(s-1)\zeta(ks-1)}{\zeta(2s-2)}\prod_p\left(1-\frac{1}{1+\frac{1}{p^{s-1}}}\left(\frac{1}{p^{ks-1}}+\frac{1}{p^s}\right)\right) \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function.

Taking $s_0=0$, $b=3$, $T=x^{\frac{5}{2}}$ in the Lemma, we have

$$\sum_{n\le x} S_k(n) = \frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s)\zeta(s-1)\zeta(ks-1)}{\zeta(2s-2)}R(s)\frac{x^s}{s}ds+O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s)=\prod_p\left(1-\frac{1}{1+\frac{1}{p^{s-1}}}\left(\frac{1}{p^{ks-1}}+\frac{1}{p^s}\right)\right).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s)\zeta(s-1)\zeta(ks-1)}{\zeta(2s-2)}R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s=3\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function

$$f(s)=\frac{\zeta(s)\zeta(s-1)\zeta(ks-1)x^s}{\zeta(2s-2)s}R(s)$$

has a simple pole point at $s=2$ with residue $\frac{x^2}{2}\zeta(2k-1)R(2)$. So we have

$$\begin{aligned} &\frac{1}{2i\pi}\left(\int_{3-iT}^{3+iT}+\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(s)\zeta(s-1)\zeta(ks-1)x^s}{\zeta(2s-2)s}R(s)ds \\ =\ &\frac{x^2}{2}\zeta(2k-1)\prod_p\left[1-\frac{1}{p(p+1)}\left(1+\frac{1}{p^{2k-3}}\right)\right]. \end{aligned}$$

Note that

$$\frac{1}{2i\pi}\left(\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(s)\zeta(s-1)\zeta(ks-1)x^s}{\zeta(2s-2)s}R(s)ds\ll x^{\frac{3}{2}+\epsilon}$$

From the above, we may immediately get the asymptotic formula:

$$\sum_{n\le x}S_k(n)=\frac{x^2}{2}\zeta(2k-1)\prod_p\left[1-\frac{1}{p(p+1)}\left(1+\frac{1}{p^{2k-3}}\right)\right]+O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only problems, Not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Ibstedt, Surfing on the Ocean of Numbers-A Few Smarandache Notions and Similar Topics, Erhus University press, New Mexico,USA. 1997

[3] Ibstedt, Computational Aspects of Number Sequences, American Research Press, Lupton USA, 1999

[4] S.Tabirca and T. Tabirca, Smarandache Notions Journal, 13, 2002, 30-36

[5] Pan Chengdong and Pan Chengbiao, Goldbach conjecture, Science Press, Beijing, 1992, 145

[6] Tom M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976

Scientia Magna
Vol. 1 (2005), No. 2, 78-80

An equation concerning the Smarandache function[1]

Maohua Le

Department of Mathematics, Zhanjiang Normal College
29 Cunjin Road, Chikan Zhanjiang, Guangdong, P. R. China

Abstract In this paper, we solve an open question concerning the Smarandache function.
Keywords Smarandache function; Euler totient function; Diophantine equation.

For any positive integer n, let $S(n)$ and $\varphi(n)$ denote the Smarandache function and the Euler totient function respectively. In [1], Bencze asked that solve the equation

$$S\left(\sum_{k=1}^{n} n^k\right) = \varphi(n) \prod_{k=1}^{n} S(k) \tag{1}$$

in positive integers n. In this paper we solve this question as follows.

Theorem. The equation (1) has only positive integer solution $n = 1$.

The proof of our Theorem depends on the following lemmas.

Lemma 1([3]). If a is a positive integer with $a > 1$, then $S(a) > 1$.

Lemma 2([3]). If a and b are coprime positive integers, then we have $S(ab) = \max(S(a), S(b))$.

Lemma 3([2]). If p is a prime and α is a positive integer, then we have $S(p^\alpha) \leq p\alpha$ and $p|S(p^\alpha)$.

Proof of Theorem. It is easy to see that (1) has only solution $n = 1$ with $n \leq 5$. We now suppose that n is a positive integer solution of (1) with $n > 5$. Since $\gcd\left(n, 1 + n + \cdots + n^{n-1}\right)^{\delta} = 1$, by Lemma 2, we get

$$\begin{aligned} S\left(\sum_{k=1}^{n} n^k\right) &= S\left(n\left(1 + n + \cdots + n^{n-1}\right)\right) \\ &= \max\left(S(n), S(1 + n + \cdots + n^{n-1})\right). \end{aligned} \tag{2}$$

If $S(n) \geq S(1 + n + \cdots + n^{n-1})$, then from (1) and (2) we obtain

$$1 = \varphi(n) \prod_{k=1}^{n-1} S(k). \tag{3}$$

Since $n \geq 5$, by Lemma 1, we get $S(n-1) > 1$ and (3) is impossible. So we have $S(n) < S(1 + n + \cdots + n^{n-1})$.

This work is supported by N.S.F. of P. R. China(No.10271104), the Guangdong Provincial Natural Science [1]Foundation(No.011781) and the Natural Science Foundation of the Education Department of Guangdong Province(No.0161).

Then, by (1) and (2), we get

$$S(1+n+\cdots+n^{n-1}) = \varphi(n)\prod_{k=1}^{n} S(k). \tag{4}$$

Let

$$1+n+\cdots+n^{n-1} = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \tag{5}$$

be the factorization of $1+n+\cdots+n^{n-1}$. By Lemma 2, we have

$$S(1+n+\cdots+n^{n-1}) = \max(S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), \cdots, S(p_r^{\alpha_r})). \tag{6}$$

It implies that

$$S(1+n+\cdots+n^{n-1}) = S(p^{\alpha}), \tag{7}$$

where

$$p^{\alpha} = p_j^{\alpha_j}, \qquad 1 \le j \le r. \tag{8}$$

Hence, by (1) and (7), we get

$$S(p^{\alpha}) = \varphi(n)\prod_{k=1}^{n} S(n). \tag{9}$$

Since p is a prime, we find from (9) that $p|S(p^{\alpha})$ and

$$p|\varphi(n) \qquad \text{or} \qquad p|S(k), \qquad 1 \le k \le n. \tag{10}$$

On the other hand, by Lemma 3, we have $S(p^{\alpha}) \le \alpha p$. Therefore, we get from (9) that

$$\alpha p \ge \varphi(n)\prod_{k=1}^{n} S(k). \tag{11}$$

Since $n \ge 5$, we have $\varphi(n) > 1$ and $S(k) > 1$ for $k = 2, \cdots, n$. hence, by (10) and (11), we get

$$\alpha \ge \frac{1}{p}\varphi(n)\prod_{k=1}^{n} S(k) > 2^{n-1}. \tag{12}$$

However, since $1+n+\cdots+n^{n-1}$ is odd, we see from (5) and (8) that $p^{\alpha} < n^n$ and

$$\alpha < \frac{n\log n}{\log p} < \frac{n\log n}{\log 3} < n\log n. \tag{13}$$

The combination of (12) and (13) yields

$$n\log n > 2^{n-1}, \qquad n > 5, \tag{14}$$

a contradiction. Thus, (1) has only solution $n = 1$. The theorem is proved.

References

[1] M.Bencze, Open questions for the Smarandache function, Smarandache Notions Journal. **12**(2001), 201-203.

[2] M.Farris and P.Mitchell, Bounding the Smarandache function, Smarandache Notions Journal. **13**(2002), 37-42.

[3] F.Smarandache, A function in number theory, An. Univ. Timisoara, **18**(1980).

Scientia Magna
Vol. 1 (2005), No. 2, 81-88

An extension of ABC-theorem[1]

Morteza Bayat , Hossein Teimoori and Mehdi Hassani
Institute for Advanced Studies in Basic Sciences
P.O. Box 45195-159 Zanjan, Iran.
email: bayat, teimoori, mhassani@iasbs.ac.ir

Abstract In this paper we generalize the abc-theorem for n-polynomials over $\mathbb{F}[x]$ in which $\mathbb{F}$ is an algebraically closed field of characteristic zero. This generalization is obtained by considering the Wronskian of functions over $\mathbb{F}[x]$. We also show that the Diophantine equation (The generalized Fermat-Catalan equation)

$$a_1^{m_1} + a_2^{m_2} + \cdots + a_{n-1}^{m_{n-1}} = a_n^{m_n},$$

where $a_1, a_2, \cdots, a_n \in \mathbb{F}[x]$ such that at most one of a_i's is constant, and $m_1, m_2, \cdots, m_n \in \mathbb{N}$, has no solution for which $a_i (i = 1, \cdots, n)$ are relatively prime by pairs provided that $n(n-2) \leq \min_{1 \leq i \leq n} \{m_i\}$.

Keywords abc-theorem; abc-conjecture; algebraically closed field; Wronskian; Diophantine equations.

§1. Introduction

Although the arithmetic abc-conjecture is a great mystery, its algebraic counterpart is a rather easy theorem (abc-theorem). It looks like it was first noticed by W.W. Stothers [1]. Later on it was generalized and rediscovered independently by several people, including R.C. Mason [2] and J.H. Silverman [3].

Discovering the abc-theorem, opened a new way for investigating the Fermat's last theorem over the polynomials with coefficients in an algebraically closed field of characteristic zero. This theorem presented a very elementary proof of the Fermat's last theorem for polynomials. This led mathematician to give a variant of this theorem over the ring of integer numbers. Of course, this result has been stated as a conjecture and this conjecture has not been proved yet. Today this conjecture is known as the abc-conjecture. Let us state the original abc-theorem [1-4,8,9]. To do this, we need to introduce some notations. We denote the set of all polynomials of one variable x over $\mathbb{F}$ by $\mathbb{F}[x]$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero. We also consider the non-zero elements of $\mathbb{F}[x]$, as follows

$$f(x) = c \prod_{i=1}^{r} (x - \alpha_i)^{m_i},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the distinct roots of f, $c \neq 0$ is a constant, and the positive integers m_i $(i = 1, 2, \cdots, r)$ are the multiplicities of the roots. The degree of the polynomial f is

$$\deg f = m_1 + m_2 + \cdots + m_r.$$

[1]The first and second authors are supported in part by the Institute for Advanced Studies in Basic Sciences Zanjan, IRAN

The number of distinct roots of f will be denoted by $n_0(f)$. Thus, we have $n_0(f) = r$. If f, g are two nonzero polynomials, then in general

$$n_0(fg) \leq n_0(f) + n_0(g),$$

and the equality holds whenever f, g are relatively prime. Now, we state the abc-theorem.

The abc-Theorem (Stothers, Mason, Silverman). *Let $a, b, c \in \mathbb{F}[x]$ be non-constant relatively prime polynomials satisfying $a + b = c$. Then*

$$\max\{\deg a, \deg b, \deg c\} \leq n_0(abc) - 1.$$

The similar result for the ring of integers is well-known as the abc-conjecture. This conjecture has been stated by Oesterle and Masser [5,6] in 1986.

The abc-Conjecture (Oesterle, Masser). *Given $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that for all $a, b, c \in \mathbb{Z}$ with $a + b = c$, we have the inequality*

$$\max\{|a|, |b|, |c|\} \leq C(\varepsilon)(N_0(abc))^{1+\varepsilon},$$

in which $N_0(abc)$ denotes the radical of abc. By radical function we mean

$$N_0(n) = \prod_{p|n} p \qquad (p \text{ is prime and } n \in \mathbb{N}).$$

Note that Stewart and Tijdeman gave some lower bounds for $C(\varepsilon)$ (cf [7]).

§2. Generalizing ABC-Theorem

Now, we generalized the abc-theorem for n-functions. To do this, we need the following lemmas:

Lemma 1. *Suppose f is a nonzero polynomial in $\mathbb{F}[x]$. Then, we have*

$$\deg f - m.n_0(f) \leq \deg(f, f', \cdots, f^{(m)}), \tag{1}$$

where $(f, f', \cdots, f^{(m)})$ is the greatest common divisor of $f, f', \cdots, f^{(m)}$.

Needless to say that the derivative is considered as a purely algebraic operator over the elements of $\mathbb{F}[x]$. However, all known rules for derivatives in calculus text book can be easily proved by means of simple algebraic tools.

Proof of Lemma 1. Suppose $f(x) = c\prod_{i=1}^{r}(x - \alpha_i)^{m_i}$, in which $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the distinct roots of f with multiplicities $m_1, m_2, \cdots, m_r$ respectively.

Case I. Suppose for any $i(1 \leq i \leq r)$ we have $m_i \leq m$. Then we get

$$\deg f = \sum_{i=1}^{r} m_i \leq mr = m.n_0(f) \leq m.n_0(f) + \deg(f, f', \cdots, f^{(m)}).$$

Case II. Now, we suppose that there exists an i such that $m_i > m$. Therefore, we have

$$(x - \alpha_i)^{m_i - m} \Big| f^{(j)} \qquad (j = 0, 1, \cdots, m),$$

and consequently,

$$(x-\alpha_i)^{m_i-m} \big| (f, f', \cdots, f^{(m)}).$$

It is clear to see that,

$$\prod_{\substack{0<m_i-m \\ 1\leq i\leq r}} (x-\alpha_i)^{m_i-m} \big| (f, f', \cdots, f^{(m)}).$$

Considering the degrees of the both sides of the above result, we obtain

$$\sum_{\substack{0<m_i-m \\ 1\leq i\leq r}} (m_i-m) \leq \deg(f, f', \cdots, f^{(m)}).$$

Since

$$\sum_{i=1}^{r}(m_i-m) \leq \sum_{\substack{0<m_i-m \\ 1\leq i\leq r}} (m_i-m),$$

we get

$$\sum_{i=1}^{r} (m_i-m) \leq \deg(f, f', \cdots, f^{(m)}),$$

or equivalently

$$\deg f - m.n_0(f) \leq \deg(f, f', \cdots, f^{(m)}),$$

and this completes our proof.

Remark 1. If $\mathrm{char}(\mathbb{F}) = 0$, then we conclude that

$$\deg f - m.n_0(f) \leq \deg(f, f^{(m)}) = \sum_{\substack{0<m_i-m \\ 1\leq i\leq r}} (m_i-m).$$

Definition 1. *Let $f_1, f_2, \cdots, f_n$ be functions over the ring $\mathbb{F}[x]$. The Wronskian of these functions is defined by,*

$$W[f_1, f_2, \cdots, f_n] = \det \left| f_j^{(i-1)} \right|_{1\leq i,j\leq n}.$$

Lemma 2. *If $\mathrm{char}(\mathbb{F}) = 0$ and $f_1, f_2, \cdots, f_n$ be linearly independent functions over $\mathbb{F}$ in $\mathbb{F}[x]$, then there exists an element x in $\mathbb{F}$, such that $W[f_1, f_2, \cdots, f_n](x) \neq 0$ (i.e. $W[f_1, f_2, \cdots, f_n](x)$ is a nonzero polynomial).*

Proof. Suppose for every $x \in \mathbb{F}$, we have

$$W[f_1, f_2, \cdots, f_n](x) = 0.$$

Therefore, there are constant numbers $c_i (i = 1, 2, \cdots, n)$ in $\mathbb{F}$, such that at least one of these c_i is nonzero and

$$c_1 \begin{pmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(n-1)}(x) \end{pmatrix} + \cdots + c_n \begin{pmatrix} f_n(x) \\ f_n'(x) \\ \vdots \\ f_n^{(n-1)}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

or

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0,$$

which is a contradiction with the linearly independence of $f_1, f_2, \cdots, f_n$.

Lemma 3. *Suppose* $\mathrm{char}(\mathbb{F}) = 0$ *and* $f_1, f_2, \cdots, f_n$ *are nonzero functions in* $\mathbb{F}[x]$. *Then, for* $W[f_1, f_2, \cdots, f_n] \neq 0$, *we have*

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2}. \tag{2}$$

Whenever $\deg f_1 = \cdots = \deg f_n$, *we get*

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2} - 1. \tag{3}$$

Proof. We proceed it by mathematical induction on n. The initialization step $n = 1$, is clear. Suppose it holds for $n-1$ nonzero functions. By expanding the Wronskian determinant $W[f_1, f_2, \cdots, f_n]$ with respect to the first row, we obtain

$$W[f_1, f_2, \cdots, f_n] = \sum_{i=1}^{n} (-1)^{i+1} f_i . W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']. \tag{4}$$

We have the following inequality for degrees

$$\deg W[f_1, f_2, \cdots, f_n] \leq \max_{1 \leq i \leq n} \{\deg f_i + \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']\},$$

and since $W[f_1, f_2, \cdots, f_n] \neq 0$, there exists an i such that the right-hand side has the greatest degree, namely

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg f_i + \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n']. \tag{5}$$

Now, considering the induction hypothesis for the set of $(n-1)$-functions

$$f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n',$$

we get

$$\begin{aligned} \deg W[f_1', \cdots, f_{i-1}', f_{i+1}', \cdots, f_n'] &\leq \deg(f_1' \cdots f_{i-1}' f_{i+1}' \cdots f_n') - \frac{(n-1)(n-2)}{2} \\ &\leq \deg(f_1 \cdots f_{i-1} f_{i+1} \cdots f_n) - \frac{n(n-1)}{2}. \end{aligned} \tag{6}$$

Finally, by (5) and (6), we have

$$\deg W[f_1, f_2, \cdots, f_n] \leq \deg(f_1 f_2 \cdots f_n) - \frac{n(n-1)}{2}.$$

For proving (3), it is necessary to show that after expanding the determinant of $W[f_1, f_2, \cdots, f_n]$, the term with the highest degree is vanished. We prove this by induction on n, with $n \geq 2$. First we investigate the case $n = 2$. Since $\deg f_1 = \deg f_2$, we have $f_1(x) = a_k x^k + P(x)$ and

$f_2(x) = b_k x^k + Q(x)$, where $P(x)$ and $Q(x)$ are two polynomials of degree at most $(k-1)$. So, we have

$$\begin{aligned} W[f_1, f_2] &= \begin{vmatrix} a_k x^k + P(x) & b_k x^k + Q(x) \\ k a_k x^{k-1} + P'(x) & k b_k x^{k-1} + Q'(x) \end{vmatrix} \\ &= a_k x^k Q'(x) + k b_k x^{k-1} P(x) - b_k x^k P'(x) - k a_k x^{k-1} Q(x). \end{aligned}$$

Now, assume its validity for any arbitrary $(n-1)$-functions. Then the proof is straight forward considering the relation (4). Now, we are ready to state our main result

Theorem 1. *Let $f_n = f_1 + f_2 + \cdots + f_{n-1}$, in which f_i's are relatively prime by pairs in $\mathbb{F}[x]$ with $\mathrm{char}(\mathbb{F}) = 0$ and at most one of them is constant. Then, we have*

$$\max_{1 \le i \le n} \deg f_i \le (n-2) n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}, \tag{7}$$

and also

$$\min_{1 \le i \le n} \deg f_i \le (n-2) n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2} - 1. \tag{8}$$

Proof. For proving the first inequality, we distinguish between two cases. The proof of Case I, is analogous with [9, Theorem 1.2].

Case I. Let $f_1, f_2, \cdots, f_{n-1}$ be linearly dependent over $\mathbb{F}$. Now, the proof proceeds by induction on n. For $n = 3$, it is true; considering the results in [1-4]. Assume that the theorem is true for all cases $n', 3 \le n' < n$, and consider n polynomials. In equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, assume that $f_i (i = 1, 2, \cdots, n-1)$, are linearly dependent over $\mathbb{F}$. Note that, at most one of the $f_i (i = 1, 2, \cdots, n-1)$, is constant. Let $\{f_{i_1}, \cdots, f_{i_q}\}$, $q < n-1$, be a maximal linearly independent subset of the $f_i (i = 1, 2, \cdots, n-1)$. Since $n - 1 \ge 2$, and f_j's are relatively prime by pairs, it follows that $q \ge 2$. So each $f_j, 1 \le j \le n-1$; j not one of the i_k, is a linear combination of the f_{i_k}, of the form

$$f_j = \lambda_1 f_{i_1} + \cdots + \lambda_q f_{i_q}, \tag{9}$$

where the $\lambda_k \in \mathbb{F}$, and at least two of these λ_k are not zero. Using our inductive hypothesis we apply the theorem to (9). This yields that if $\lambda_k \neq 0$, then

$$\deg f_{i_k} \le (q-1) n_0(f_j \prod_{k=1}^{q} f_{i_k}) - \frac{q(q-1)}{2},$$

and so that

$$\deg f_{i_k} \le (q-1) n_0(\prod_{i=1}^{n} f_i) - \frac{q(q-1)}{2}. \tag{10}$$

Now, since at most one of f_i is a constant, i.e. $n - 1 \le n_0(\prod_{i=1}^{n} f_i)$, we yield that

$$(q-1) n_0(\prod_{i=1}^{n} f_i) - \frac{q(q-1)}{2} \le (n-2) n_0(\prod_{i=1}^{n} f_i) - \frac{(n-1)(n-2)}{2}. \tag{11}$$

Now, using (10) and (11), we have

$$\deg f_{i_k} \leq (n-2)n_0(\prod_{i=1}^{n} f_i) - \frac{(n-1)(n-2)}{2}. \tag{12}$$

From (9) the same estimate as in (12) follows for $\deg f_j$. Thus the theorem is proved for such f_j and f_{i_k}. Inserting all the relations of the from (9) into the right side of equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, yields an equation of the form

$$f_r = \kappa_1 f_{i_1} + \cdots + \kappa_q f_{i_q}, \tag{13}$$

where the $\kappa_j \in \mathbb{F}$. Moreover, if one of these $\kappa_\nu = 0$, then the corresponding f_{i_ν} must be appeared in one of the equations (9) with a nonzero λ_ν. Hence, (12) is established for this f_{i_ν}. Finally, for those $\kappa_\nu \neq 0$, we treat (13) exactly as we did (9), (note that $q+1 < n$), and obtain the estimate (12) for $\deg f_{i_\nu}$, and $\deg f_n$. This completes the induction in this case.

Case II. $f_1, f_2, \cdots, f_{n-1}$ are linearly independent over $\mathbb{F}$. By using Lemma 2, we have $W[f_1, f_2, \cdots, f_{n-1}] \neq 0$. Without lost of generality, we suppose that f_n has the greatest degree, and therefore it is necessary to prove that

$$\deg f_n \leq (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}.$$

Considering the equality $f_n = f_1 + f_2 + \cdots + f_{n-1}$, we have

$$W[f_1, \cdots, f_{n-2}, f_{n-1}] = W[f_1, \cdots, f_{n-2}, f_n].$$

It can be easily seen for any i $(i = 1, \cdots, n)$,

$$(f_i, f_i', \cdots, f_i^{(n-2)}) \Big| W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Since f_i's are relatively prime by pairs, we conclude that $(f_i, f_i' \cdots, f_i^{(n-2)})$'s are relatively prime. So, we get

$$\prod_{i=1}^{n} (f_i, f_i', \cdots, f_i^{(n-2)}) \Big| W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Now since $W[f_1, \cdots, f_{n-2}, f_{n-1}] \neq 0$, we conclude that

$$\sum_{i=1}^{n} \deg(f_i, f_i', \cdots, f_i^{(n-2)}) \leq \deg W[f_1, \cdots, f_{n-2}, f_{n-1}].$$

Using the relations (1) and (2), we obtain

$$\sum_{i=1}^{n} (\deg f_i - (n-2)n_0(f_i)) \leq \deg(f_1 f_2 \cdots f_{n-1}) - \frac{(n-1)(n-2)}{2}$$

or equivalently,

$$\deg f_n \leq (n-2)n_0(f_1 f_2 \cdots f_n) - \frac{(n-1)(n-2)}{2}.$$

For proving (8), it is necessary to consider the case $\deg f_1 = \cdots = \deg f_n$. Now the proof is clear using the relation (3).

Remark 2. In the case where the number of constant polynomials are more than one, the inequality (7) is not valid in general case. For example if $f_1 = \cdots = f_5 = 1, f_6 = x$ and $f_7 = x + 5$, then it is not true. Indeed, finding similar inequality for the case that constant polynomials are more than one is an open question yet.

As an immediate result of the relation (7), we have:

Corollary 1. *With the assumption of the Theorem 1, we have*

$$\deg(f_1 f_2 \cdots f_n) \leq n(n-2)n_0(f_1 f_2 \cdots f_n) - \frac{n(n-1)(n-2)}{2}.$$

Corollary 2. *For $n \geq 3$, suppose $f_1, f_2, \cdots, f_n$ are non-constant and relatively prime by pairs. Then we obtain*

$$\frac{1}{n-2} < \frac{n_0(f_1)}{\deg f_1} + \frac{n_0(f_2)}{\deg f_2} + \cdots + \frac{n_0(f_n)}{\deg f_n}.$$

Proof. Without loss of generality, we suppose that $\deg f_1 \leq \cdots \leq \deg f_n$. Applying Theorem 1, yields

$$\deg f_n < (n-2)(n_0(f_1) + \cdots + n_0(f_n)).$$

Dividing the both sides of the above inequality by $(n-2)\deg f_n$, completes the proof.

§3. Application to the generalized Fermat-Catalan Equation

Now, we deal with the generalized Fermat-Catalan equation [8].

Theorem 2. *Consider the generalized Fermat-Catalan equation as follows*

$$a_1^{m_1} + a_2^{m_2} + \cdots + a_{n-1}^{m_{n-1}} = a_n^{m_n}, \tag{14}$$

in which $a_1, a_2, \cdots, a_n$ are elements of $\mathbb{F}[x]$ with $\mathrm{char}(\mathbb{F}) = 0$, such that they are relatively prime by pairs and at most one of a_i's is constant. Then the equation (14) with condition $n(n-2) \leq m = \min\limits_{1 \leq i \leq n} \{m_i\}$ has no solution in $\mathbb{F}[x]$.

Proof. Suppose $f_1 = a_1^{m_1}, f_2 = a_2^{m_2}, \cdots, f_n = a_n^{m_n}$. These functions satisfy the conditions of Theorem 1. Thus we have

$$\deg(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}) \leq n(n-2)n_0(a_1^{m_1} \cdots a_n^{m_n}) - \frac{n(n-1)(n-2)}{2}. \tag{15}$$

We also have,

$$m \deg(a_1 a_2 \cdots a_n) \leq \deg(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}), \tag{16}$$

and

$$n_0(a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}) = n_0(a_1 a_2 \cdots a_n) \leq \deg(a_1 a_2 \cdots a_n). \tag{17}$$

Now considering the both relations (15)-(17), we get

$$m \deg(a_1 a_2 \cdots a_n) \leq n(n-2)\deg(a_1 a_2 \cdots a_n) - \frac{n(n-1)(n-2)}{2}, \tag{18}$$

or equivalently,

$$(m - n(n-2))\deg(a_1 a_2 \cdots a_n) \leq -\frac{n(n-1)(n-2)}{2}. \tag{19}$$

The last inequality result in $m - n(n-2) < 0$, which is in contradiction with our theorem's hypothesis. Therefore, we conclude that the Diophantine equation (14) has no solution in $\mathbb{F}[x]$.

Of course, there is in [10] a natural extension of the above result for several variables using the generalized Wronskian.

Acknowledgment

The authors thanks J. Browkin for his interesting suggestions about the abc-conjecture.

References

[1] W.Stothers, Polynomial identities and hauptmoduln, Quart. Math.Oxford, **32**(1981), 349-370.

[2] R.C.Mason, Diophantine equations over function fields, Londen Math. Soc. Lecture note series, Vol. 96, Cambridge University Press, 1984.

[3] J.H. Silverman, The S-unit equation over function fields, Math. Proc. Cambridge PhiLos. Soc. **95**(1984), No.1, 3-4.

[4] S.Lang, Math Talks for Undergraduates, Spinger-Verlag 1999.

[5] J.Oesterle, Nouvelles approches du "theorem" de Fermat. (New approches to Fermat's last theorem) Semin. Bourbaki, 40eme Annee, Vol. 1987/88, Exp. No. 694 Asterisque 161/162, 165-186(1988).

[6] D.W.Masser, Note on a Conjecture of Szpiro. Les pinceaux de courbes elliptiques, semin., Paris/Fr. 1988, Asterisque 183, 19-23(1990).

[7] C.L.Stewart and R.Tijdeman, On the Oesterle-Masser Conjecture, Monatshefte Math. **102**(1986), 251-257.

[8] L.N.Vaserstein, Quantum (abc)-Theorems, Journal of Number Theory, **81**(2000), 351-358.

[9] H.N.Shapiro and G.H.Sparer, Extention of a Theorem of Mason, Comm. Pure and Appl. Math., **47**(1994), 711-718.

[10] M.Bayat and H.Teimoori, A new bound for an extension of Mason Theorem for functions of several variables, Archiv der Mathematik, **82** (2004), 230-239.

Scientia Magna
Vol. 1 (2005), No. 2, 89-90

An equation involving the Smarandache function

Ma Jinping
Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China,710069

Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function, and $\phi(n)$ is the Euler function. The main purpose of this paper is using the elementary method to study the solutions of the equation $S(n) = \phi(n)$, and give all solutions for it.
Keywords Smrandache function; Equation; Solutions.

§1. Introduction

For any positive integer n, the Smarandache function $S(n)$ is defined as the smallest integer m such that $n|m!$. From the definition and the properties of $S(n)$, one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$S(n) = \max_{1 \le i \le k} \{S(p_i^{\alpha_i})\}.$$

About the arithmetical properities of $S(n)$, many people had studied it before, see references [3], [4] and [5].

If $n \ge 1$, the Euler function $\phi(n)$ is defined to be the number of all positive integers not exceeding n, which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function.

In this paper, we shall use the elementary method to study the solutions of the equation $S(n) = \phi(n)$, and give all solutions for it. That is, we shall prove the following:

Theorem. The equation $S(n) = \phi(n)$ have only 4 solutions, namely,

$$n = 1, 8, 9, 12.$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, and let

$$S(n) = \max_{1 \le i \le k} \{S({p_i}^{\alpha_i})\} = S(p^{\alpha}).$$

Then from the definitions of $S(n)$ and $\phi(n)$ we have

$$\begin{aligned} \phi(n) &= p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1) \\ &= \phi(p^{\alpha})\phi(n_1) = p^{\alpha - 1}(p - 1)\phi(n_1) = S(p^{\alpha}). \end{aligned}$$

It is clear that $n = 1$ is a solution of the equation $S(n) = \phi(n)$. If $n > 1$, then we will discuss the problem in three cases:

(I) If $\alpha = 1$ and $n = p$, then $S(n) = p \neq p - 1 = \phi(n)$. That is, there is no any prime satisfied the equation. If $\alpha = 1$ and $n = n_1 p$, then $S(n) = p \neq (p-1)\phi(n_1) = \phi(n_1 p)$. So the equation has also no solution.

(II) If $\alpha = 2$, then $S(p^2) = 2p$ and $\phi(p^2 n_1) = p(p-1)\phi(n_1)$. So in this case $S(n) = \phi(n)$ if and only if

$$(p-1)\phi(n_1) = 2.$$

This time, there are two cases: $p - 1 = 1$, $\phi(n_1) = 2$; $p - 1 = 2$, $\phi(n_1) = 1$. That is, $p = 2$, $n_1 = 3$; $p = 3$, $n_1 = 1$. So in this case, the equation has two solutions: $n = 12$, 9.

(III) If $\alpha = 3$, it is clear that $S(2^3) = \phi(2^3) = 4$, so $n = 8$ satisfied the equation.

If $\alpha \geq 3$ and $p > 2$, noting that

$$p^{\alpha-2} > 2^{\alpha-2} = (1+1)^{\alpha-2} = 1 + \alpha - 2 + \cdots + 1 > \alpha.$$

That is,

$$p^{\alpha-1} > \alpha p \Rightarrow p^{\alpha-1}(p-1)\phi(n_1) > \alpha p,$$

but

$$S(p^\alpha) \leq \alpha p.$$

So this time, the equation has no solution.

Now combining the above three cases, we may immediately get all 4 solutions of equation $S(n) = \phi(n)$, namely

$$n = 1, 8, 9, 12.$$

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[3] Wang Yongxing, On the Smarandache function, Research on Smarandache problems in number theory, Hexis, 2005, pp. 103-106.

[4]Ma Jinping, The Smaranache Multiplicative Function, Scientia Magna,**1**(2005), 125-128.

[5] Li Hailong and Zhao Xiaopeng, On the Smarandache function and the K-th roots of a positive integer, Research on Smarandache problems in number theory, Hexis, 2004, pp. 119-122.

Scientia Magna
Vol. 1 (2005), No. 2, 91-95

Inequalities for the polygamma functions with application[1]

Chaoping Chen
Department of Applied Mathmatics, Hennan Polytechnic University
Jiaozuo, Hennan, P. R. China

Abstract We present some inequalities for the polygamma funtions. As an application, we give the upper and lower bounds for the expression $\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma$, where $\gamma=0.57721\cdots$ is the Euler's constant.

Keywords Inequality; Polygamma function; Harmonic sequence; Euler's constant.

§1. Inequalities for the Polygamma Function

The gamma function is usually defined for $Rez>0$ by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function and the polygamma functions can be expressed as

$$\psi(z)=\frac{\Gamma^{'}(z)}{\Gamma(z)}=-\gamma+\sum_{k=0}^{\infty}\left(\frac{1}{1+k}-\frac{1}{z+k}\right),$$

$$\psi^n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}$$

for $Rez>0$ and $n=1,2,\cdots$, where $\gamma=0.57721\cdots$ is the Euler's constant.

M. Merkle [2] established the inequality

$$\frac{1}{x}+\frac{1}{2x^2}+\sum_{k=1}^{2N}\frac{B_{2k}}{x^{2k+1}}<\sum_{k=0}^{\infty}\frac{1}{(x+k)^2}<\frac{1}{x}+\sum_{k=1}^{2N+1}\frac{B_{2k}}{x^{2k+1}}$$

for all real $x>0$ and all integers $N\geq 1$, where B_k denotes Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}t^j.$$

The first five Bernoulli numbers with even indices are

$$B_2=\frac{1}{6},B_4=-\frac{1}{30},B_6=\frac{1}{42},B_8=-\frac{1}{30},B_{10}=\frac{5}{66}.$$

[1]This work is supported in part by SF of Henan Innovation Talents at University of P. R. China

The following theorem 1 establishes a more general result.

Theorem 1. Let $m \geq 0$ and $n \geq 1$ be integers, then we have for $x > 0$,

$$\ln x - \frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} \tag{1}$$

and

$$\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}$$

$$< (-1)^{n+1}\psi^{(n)}(x) < \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}. \tag{2}$$

Proof. From Binet's formula [6, p. 103]

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t^2} dt,$$

we conclude that

$$\psi(x) = \ln x - \frac{1}{2x} - \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t} dt \tag{3}$$

and therefore

$$(-1)^{n+1}\psi^{(n)}(n) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) t^{n-1} e^{-xt} dt. \tag{4}$$

It follows from Problem 154 in Part I, Chapter 4, of [3] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} t^{2j} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} t^{2j} \tag{5}$$

for all integers $m > 0$. The inequality (5) can be also found in [4].

From (3) and (5) we conclude (1), and we obtain (2) from (4) and (5). This completes the proof of the theorem 1.

Note that $\psi(x+1) = \psi(x) + \frac{1}{x}$ (see [1, p. 258]), (1) can be written as

$$\frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x+1) - \ln x < \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} \tag{6}$$

and (2) can be written as

$$\frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}$$

$$< (-1)^{n+1}\psi^{(n)}(x) < \frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}. \tag{7}$$

In particular, taking in (6) $m = 0$ we obtain for $x > 0$,

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x} \tag{8}$$

and taking in (7) $m = 1$ and $n = 1$, we obtain for $x > 0$

$$\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \tag{9}$$

The inequalities (8) and (9) play an important role in the proof of the theorem 2 in Section 2.

§2. Inequalities for Euler's Constant

Euler's constant $\gamma = 0.57721\cdots$ is defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right).$$

It is of interest to investigate the bounds for the expression $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln n - \gamma$. The inequality

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n}$$

is called in literature Franel's inequality [3, Ex. 18].

It is given in [1, p. 258] that $\psi(n) = \sum\limits_{k=1}^{n-1} \frac{1}{k} - \gamma$, and then we have get

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma = \psi(n+1) - \ln n. \tag{10}$$

Taking in (6) $x = n$ we obtain that

$$\frac{1}{2n} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j}\frac{1}{n^{2j}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j}\frac{1}{n^{2j}}. \tag{11}$$

The inequality (11) provides closer bounds for $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln n - \gamma$.

L.Tóth [5, p. 264] proposed the following problems:

(i) Prove that for every positive integer n we have

$$\frac{1}{2n + \frac{2}{5}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.$$

(ii) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ can not be replaced by a slightly larger number.

The following Theorem 2 answers the problem due to L.Tóth.

Theorem 2. For every positive integer n,

$$\frac{1}{2n + a} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + b}, \tag{12}$$

with the best possible constants

$$a = \frac{1}{1-\gamma} - 2 \qquad \text{and} \qquad b = \frac{1}{3}$$

Proof. By (10), the inequality (12) can be rearranged as

$$b < \frac{1}{\psi(n+1) - \ln n} - 2n \le a.$$

Define for $x > 0$

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x.$$

Differentiating ϕ and utilizing (8) and (9) reveals that for $x > \frac{12}{5}$

$$(\psi(x+1) - \ln x)^2 \phi^{'}(x) = \frac{1}{x} - \psi^{'}(x+1) - 2(\psi(x+1) - \ln x)^2$$
$$< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 = \frac{12 - 5x}{360x^5} < 0,$$

and then the function ϕ strictly decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155\cdots,$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752\cdots,$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115\cdots.$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \qquad n \in N$$

is strictly decreasing. This leads to

$$\lim_{n\to\infty} \phi(n) < \phi(n) \le \phi(1) = \frac{1}{1-\gamma} - 2.$$

Making use of asymptotic formula of ψ (see [1, p. 259])

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \qquad (x \to \infty),$$

we conclude that

$$\lim_{n\to\infty} \phi(n) = \lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})} = \frac{1}{3}.$$

This completes the proof of the theorem 2.

References

[1] M. Abramowitz and I.Stegun (Eds), Handbook of Mathematical Function with Formulas, Graphs, and Mathematical Tables, 4th printing, with corrections, Applied Mathematics Series 55, National Bureau of Standards, Washington, 1965.

[2] M. Merkle, Logarithmic convexity inequalities for the gemma function, J. Math. Anal. Appl, **203**(1996), 369–380.

[3] G. Pólya and G. Szegö, Problems and Theorems in analysis, Vol.I and II, Springer-Verlag, Berlin, Heidelberg, 1972.

[4] Z. Sasvári, Inequalities for binomial coefficients, J. Math. Anal. Appl. **236**(1999), 223-226.

[5] L. Tóth, E 3432, Amer. Math. Monthly **98**(1991), 264; **99**(1992), 684-685.

[6] Zh-x.Wang and D.-R.Guo, Introduction to Special Function, the Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000(Chinese).

Scientia Magna
Vol. 1 (2005), No. 2, 96-108

On the number of Smarandache zero-divisors and Smarandache weak zero-divisors in loop rings

W.B.Vasantha and Moon K.Chetry
Department of Mathematics
I.I.T.Madras,Chennai

Abstract In this paper we find the number of smarandache zero divisors (S-zero divisors) and smarandache weak zero divisors (S-weak zero divisors) for the loop rings $Z_2L_n(m)$ of the loops $L_n(m)$ over Z_2. We obtain the exact number of S-zero divisors and S-weak zero divisors when $n = p^2$ or p^3 or pq where p, q are odd primes. We also prove $ZL_n(m)$ has infinitely many S-zero divisors and S-weak zero divisors, where Z is the ring of integers. For any loop L we give conditions on L so that the loop ring Z_2L has S-zero divisors and S-weak zero divisors.

§0 . Introduction

This paper has four sections. In the first section, we just recall the definitions of S-zero divisors and S-weak zero divisors and some of the properties of the new class of loops $L_n(m)$. In section two, we obtain the number of S-zero divisors of the loop rings $Z_2L_n(m)$ and show when $n = p^2$, where p is an odd prime, $Z_2L_n(m)$ has $p(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Also when $n = p^3$, p an odd prime, $Z_2L_n(m)$ has $p(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r) + p^2(1 + \sum_{r=2,reven}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Again when $n = pq$, where p, q are odd primes, $Z_2L_n(m)$ has $p+q+p(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r)+q(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Further we prove $ZL_n(m)$ has infinitely many S-zero divisors. In section three, we find the number of S-weak zero divisors for the loop ring $Z_2L_n(m)$ and prove that when $n = p^2$, where p is an odd prime, $Z_2L_n(m)$ has $2p(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Also when $n = p^3$, where p is an odd prime, $Z_2L_n(m)$ has $2p(\sum_{r=2,reven}^{p^2-1} {}^{p^2+1}C_r) + 2p^2(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Again when $n = pq$, where p, q are odd primes, $Z_2L_n(m)$ has $2[p(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)]$ S-weak zero divisors. We prove $ZL_n(m)$ has infinitely many S-weak zero divisors. The final section gives some unsolved problems and some conclusions based on our study.

§1. Basic Results

Here we just recollect some basic results to make this paper a self contained one.

Definition 1.1[4]. Let R be a ring. An element $a \in R\backslash\{0\}$ is said to be a S-zero divisor if $a.b = 0$ for some $b \neq 0$ in R and there exists $x, y \in R\backslash\{0, a, b\}$ such that

$$i. \quad a.x = 0 \quad or \quad x.a = 0$$

$$ii \quad b.y = 0 \quad or \quad y.b = 0$$

$$iii. \quad x.y \neq 0 \quad or \quad y.x \neq 0$$

Definition 1.2[4]. Let R be a ring. An element $a \in R\backslash\{0\}$ is a S-weak zero divisor if there exists $b \in R\backslash\{0, a\}$ such that $a, b = 0$ satisfying the following conditions: There exists $x, y \in R\backslash\{0, a, b\}$ such that

$$i. \quad a.x = 0 \quad or \quad x.a = 0$$

$$ii. \quad b.y = 0 \quad or \quad y.b = 0$$

$$iii. \quad x.y = 0 \quad or \quad y.x = 0$$

Definition 1.3[3]. Let $L_n(m) = \{e, 1, 2, 3 \cdots, n\}$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as follows:

$$i. \quad e.i = i.e \quad for \quad all \quad i \in L_n(m)\backslash\{e\}$$

$$ii. \quad i^2. = e \quad for \quad all \quad i \in L_n(m)$$

$$iii. \quad i.j = t, \quad where \quad t \equiv (mj - (m-1)i)(mod\, n) \quad for \quad all \quad i, j \in L_n(m), \quad i \neq e \quad and \quad j \neq e.$$

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by L_n.

Example 1.1[3]. Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

.	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

Theorem 1.1[3]. Let $L_n(m) \in L_n$. For every $t|n$ there exists t subloops of order $k + 1$, where $k = n/t$.

Theorem 1.2[3]. Let $L_n(m) \in L_n$. If H is a subloop of $L_n(m)$ of order $t+1$, then $t|n$.

Remark 1.2[3]. Lagrange's theorem is not satisfied by all subloops of the loop $L_n(m)$,i.e there always exists a subloop H of $L_n(m)$ which does not satisfy the Lagrange's theorem, i.e $o(H) \dagger o(L_n(m))$.

§2. Definition of the number of S-zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$

In this section, we give the number of S-zero divisors in $Z_2L_n(m)$. We prove $ZL_n(m)$ (where $n = p^2$ or pq, p and q are odd primes), has infinitely many S-zero divisors. Further we show any loop L of odd (or even) order if it has a proper subloop of even (or odd) order then the loop ring $Z_2L_n(m)$ over the field Z_2 has S-zero divisors. We first show if L is a loop of odd order and L has a proper subloop of even order, then $Z_2L_n(m)$ has S-zero divisors.

Theorem 2.1. Let L be a finite loop of odd order. $Z_2 = \{0, 1\}$, the prime field of characteristic 2. Suppose H is a subloop of L of even order, then Z_2L has S-zero divisors.

Proof. Let $|L| = n$; where n is odd. Z_2L be the loop ring of L over Z_2. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = \sum_{i=1}^{m} h_i$, then

$$X.Y = 0.$$

Now

$$(1 + g_t)X = 0, \quad g_t \in l \backslash H.$$

also

$$(1 + h_i + h_j + h_k)Y = 0, \quad h_i, h_j, h_k \in H.$$

so that

$$(1 + g_t)(1 + h_i + h_j + h_k) \neq 0.$$

Hence the claim.

Corollary 2.1. If L is a finite loop of even order n and H is a subloop of odd order m, then the loop ring Z_2L has S-zero divisors.

It is important here to mention that Z_2L may have other types of S-zero divisors. This theorem only gives one of the basic conditions for Z_2L to have S-zero divisors.

Example 2.1. Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m, 25) = 1$ and $(m-1, 25) = 1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6. Let H be one of the proper subloops of $L_{25}(m)$.

Now take

$$X = \sum_{i=1}^{26} g_i, \quad Y = \sum_{i=1}^{6} h_i, \quad g_i \in L_{25}(m), \quad h_i \in H,$$

then

$$(1 + g_i)X = 0, \quad g_i \in L_{25}(m) \backslash H$$

$$(1+h_i)Y = 0, \quad h_i \in H$$

but

$$(1+g_i)(1+h_i) \neq 0.$$

so X and Y are S-zero divisors in $Z_2L_{25}(m)$.

Theorem 2.2. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number,$n > 3$) with $n = p^2$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors.

Proof. Given $L_n(m)$ is a loop of order $n+1$, where $n = p^2$ (p an odd prime). Let $Z_2L_n(m)$ be the loop ring of the loop $L_n(m)$ over Z_2. Now clearly the loop $L_n(m)$ has exactly p subloops of order $p+1$. The number of S-zero divisors in $Z_2L_n(m)$ for $n = p^2$ can be enumerated in the following way: Let

$$X = \sum_{i=1}^{n+1} g_i \quad and \quad Y = \sum_{i=1}^{p+1} h_i$$

where $g_i \in L_n(m)$ and $h_i \in H_j$. For this

$$X.Y = 0$$

choose

$$a = (1+g), \quad g \in L_n(m) \backslash H_j$$

$$b = (h_i + h_j), \quad h_i, h_j \in H_j$$

then

$$a.X = 0 \quad and \quad b.Y = 0$$

but

$$a.b \neq 0.$$

So X and Y are S-zero divisors. There are p such S-zero divisors, as we have p subloops H_j $(j = 1, 2, \cdots, p)$ of $L_n(m)$.

Next consider, S-zero divisors of the form

$$(h_1 + h_2)\sum_{i=1}^{n+1} g_i = 0, \quad where \quad h_1, h_2 \in H_j, \quad g_i \in L_n(m)$$

put

$$X = (h_1 + h_2), \quad Y = \sum_{i=1}^{n+1} g_i$$

we have ${}^{p+1}C_2$ such S-zero divisors. This is true for each of the subloops. Hence there exists ${}^{p+1}C_2 \times p$ such S-zero divisors. Taking four elements h_1, h_2, h_3, h_4 from H_j at a time, we get

$$(h_1 + h_2 + h_3 + h_4)\sum_{i=1}^{n+1} g_i = 0$$

so we get ${}^{p+1}C_4 \times p$ such S-zero divisors. Continue in this way, we get

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n+1} g_i = 0, \quad where \quad h_1, h_2, \cdots, h_{p-1} \in H_j$$

So we get ${}^{p+1}C_{p-1} \times p$ such S-zero divisors. Adding all these S-zero divisors, we get

$$p\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

number of S-zero divisors in the loop ring $Z_2L_n(m)$. Hence the claim.

Example 2.2. Let $Z_2L_{49}(m)$ be the loop ring of the loop $L_{49}(m)$ over Z_2, where $(m, 49) = 1$ and $(m-1, 49) = 1$. Here $p = 7$, so from Theorem 2.2, $Z_2L_{49}(m)$ has

$$7\left(1 + \sum_{r=2,\ r\ even}^{6} {}^{7+1}C_r\right)$$

S-zero divisors i.e $7(1 + \sum_{r=2,\ r\ even}^{6} {}^{8}C_r) = 889$ S-zero divisors.

Theorem 2.3. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p\left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right) + p^2\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors.

Proof. We enumerate all the S-zero divisors of $Z_2L_n(m)$ in the following way:

Case I: As $p|p^3, L_n(m)$ has p proper subloops H_j each of order p^2+1. In this case I, we have $p^2 - 1$ types of S-zero divisors. We just index them by type I_1, type I_2, $\cdots$, type I_{p^2-1}.

Type I_1: Here

$$\sum_{i=1}^{n+1} g_i \sum_{i=1}^{p^2+1} h_i = 0, \quad g_i \in L_n(m), \quad h_i \in H_j, (j = 1, 2, \cdots, p)$$

So we will get p S-zero divisors of this type.

Type I_2:

$$(h_1 + h_2) \sum_{i=1}^{n+1} g_i = 0, \quad h_1, h_2 \in H_j (j = 1, 2, \cdots, p).$$

As in the Theorem 2.2, we will get ${}^{p^2+1}C_2 \times p$ S-zero divisors of this type.

Type I_3:

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{n+1} g_i = 0, \quad h_1, h_2, h_3, h_4 \in H_j (j = 1, 2, \cdots, p).$$

We will get ${}^{p^2+1}C_4 \times p$ S-zero divisors of this type.

Continue this way,

Type I_{p^2-1}:

$$(h_1 + h_2 + \cdots + h_{p^2-1}) \sum_{i=1}^{n+1} g_i = 0, \quad h_i \in H_j$$

We will get ${}^{p^2+1}C_{p^2-1} \times p$ S-zero divisors of this type. Hence adding all this types of S-zero divisors we will get

$$p\left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right)$$

S-zero divisors for case I.

Case II: Again $p^2|p^3$, so there are p^2 subloops H_j each of order $p+1$. Now we can enumerate all the S-zero divisors in this case exactly as in case I above. So there are

$$p^2(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors. Hence the total number of S-zero divisors in $Z_2L_n(m)$ is

$$p\left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right) + p^2\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

Hence the claim.

Example 2.3. Let $Z_2L_{27}(m)$ be the loop ring of the loop $L_{27}(m)$ over Z_2, where $(m, 27) = 1$ and $(m-1, 27) = 1$. Here $p = 3$, so from Theorem 2.3, $Z_2L_{27}(m)$ has

$$3(1 + \sum_{r=2,\ r\ even}^{8} {}^{3^2+1}C_r) + 3^2(1 + \sum_{r=2,\ r\ even}^{2} {}^{4}C_r)$$

S-zero divisors i.e $3\left(1 + \sum_{r=2,\ r\ even}^{8} {}^{10}C_r\right) + 9\left(1 + \sum_{r=2,\ r\ even}^{2} {}^{4}C_r\right) = 1533$ S-zero divisors.

Theorem 2.4. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, where p, q are odd primes. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p + q + p\left(1 + \sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right) + q\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors.

Proof. We will enumerate all the S-zero divisors in the following way:

Case I: As $p|pq$, $L_n(m)$ has p subloops H_j each of order $q+1$. Proceeding exactly in the same way as in the Theorem 2.3, we will get $p + p\left(1 + \sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)$ S-zero divisors for case I.

Case II: Again $q|pq$, so $L_n(m)$ has q subloops H_j each of order $p+1$. Now as above we will get $q + q\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$ S-zero divisors for case II. Hence adding all the S-zero

divisors in case I and case II, we get

$$p+q+p\left(1+\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)+q\left(1+\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors in $Z_2L_n(m)$.

Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ when $n = p^2$ or p^3 or pq, where p, q are odd primes, $ZL_n(m)$ has infinitely many S-zero divisors.

Theorem 2.5. Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n = p^2$ or p^3 or pq (p, q are odd primes), then $ZL_n(m)$ has infinitely many S-zero divisors.

Proof. Let $L_n(m)$ be a loop ring such that $n = p^2$. $L_n(M)$ has p subloops (say H_j) each of order $p + 1$.

Now the loop ring $ZL_n(m)$ has the following types of S-zero divisors:

$$X = a - bh_1 + bh_2 - ah_3 \quad and \quad Y = \sum_{i=1}^{n+1} g_i$$

where $a, b \in Z$ and $h_i \in H_i$, $g_i \in L_n(m)$ such that

$$(a - bh_1 + bh_2 - ah_3)\sum_{i=1}^{n+1} g_i = 0$$

Again

$$(1 - g_k)Y = 0, \quad g_k \in L_n(m)\backslash H_j$$

also

$$(a - bh_1 + bh_2 - ah_3)\sum h_i = 0, \quad h_i \in H_j$$

clearly

$$(1 - g_k)\left(\sum_{h_i \in H_j} h_i\right) \neq 0.$$

So X, Y are S-zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-zero divisors of this type for a and b can take infinite number of values in Z. For $n = p^2$ or p^3 or pq we can prove the results in a similar way. Hence the claim.

§3. Determination of the number of S-weak zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$

In this section, we give the number of S-weak zero divisors in the loop ring $Z_2L_n(m)$ when n is of the form p^2, p^3 or pq where p and q are odd primes. Before that we prove the existence of S-weak zero divisors in the loop ring Z_2L whenever L has a proper subloop.

Theorem 3.1. Let n be a finite loop of odd order. Suppose H is a subloop of L of even order, then Z_2L has S-weak zero divisors.

Proof. Let $|L| = n$; n odd. Z_2L be the loop ring. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = 1 + h_t, g_i \in L, h_t \in H$, then

$$X.Y = 0$$

Now

$$Y.\sum_{i=1}^{m} h_i = 0, \quad h_i \in H$$

also

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

so that

$$(1 + g_t)\sum_{i=1}^{m} h_i = 0.$$

Hence the claim.

Example 3.1. Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m, 25) = 1$ and $(m - 1, 25) = 1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6.

Take

$$X = \sum_{i=1}^{26} g_i, \quad Y = 1 + h_t, \quad g_i \in L_{25}(m), \quad h_t \in H$$

then

$$X.Y = 0$$

again

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

$$Y\sum_{i=1}^{6} h_i = 0, \quad h_i \in H$$

also

$$(1 + g_t)\sum_{i=1}^{6} h_i = 0,$$

So X and Y are S-weak zero divisors in $Z_2L_{25}(m)$.

Example 3.2. Let $Z_2L_{21}(m)$ be the loop ring of the loop $L_{21}(m)$ over Z_2, where where $(m, 21) = 1$ and $(m - 1, 21) = 1$. As $3|21$, so $L_{21}(m)$ has 3 proper subloops each of order 8.

Take

$$X = \sum_{i=1}^{8} h_i, \quad Y = 1 + h_t, \quad h_i, h_t \in H$$

then

$$X.Y = 0$$

again

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

$$Y\sum_{i=1}^{22} g_i = 0, \quad g_i \in L_{21}(m)$$

also

$$(1+g_t)\sum_{i=1}^{22} g_i = 0,$$

So X and Y are S-weak zero divisors in $Z_2L_{21}(m)$.

Theorem 3.2. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n>3$) with $n=p^2$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors.

Proof. Clearly the loop $L_n(m)$ has p subloops H_j each of order $p+1$. As in case of Theorem 2.3, we index the $p-1$ types of S-weak zero divisors by $I_1, I_2, \cdots, I_{p-1}$. Now the number of S-weak zero divisors in $Z_2L_n(m)$ for $n=p^2$ can be enumerated in the following way:

Type I_1. Let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{n+1} g_i$$

where $h_1, h_2 \in H_j$ and $g_i \in L_n(m)$ then

$$XY = 0$$

take

$$a = \sum_{i=1}^{p+1} h_i, \quad and \quad b = h_3 + h_4 \quad where \quad h_i \in H_j, \quad (j = 1, 2, \cdots, p)$$

then

$$aX = 0, \quad bY = 0$$

also

$$ab = 0$$

So for each proper subloop we will get ${}^{p+1}C_2$ S-weak zero divisors and as there are p proper subloops we will get ${}^{p+1}C_2 \times p$ such S-weak zero divisors.

Type I_2. Again let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{p+1} h_i, \quad h_i \in H_j$$

then

$$XY = 0$$

take

$$a = \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad b = h_1 + h_2, \quad h_1, h_2 \in H_j,$$

then

$$aX = 0, \quad bY = 0$$

also

$$ab = 0$$

Here also we will get ${}^{p+1}C_2 \times p$ such S-weak zero divisors of this type.

Type I_3.

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

As above we can say there are ${}^{p+1}C_4 \times p$ such S-weak zero divisors.

Type I_4.

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{p+1} h_i, \quad h_i \in H_j.$$

There are ${}^{p+1}C_4 \times p$ such S-weak zero divisors.

Continue this way,

Type I_{p-2}.

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

there are ${}^{p+1}C_{p-1} \times p$ such S-weak zero divisors.

Type I_{p-1}.

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n} h_i, \quad h_i \in H_j.$$

Again there are ${}^{p+1}C_{p-1} \times p$ such S-weak zero divisors of this type. Adding all these S-weak zero divisors we will get the total number of S-weak zero divisors in $Z_2L_n(m)$ as

$$2p \left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

Hence the claim.

Theorem 3.3. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p \left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r \right) + 2p^2 \left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

S-weak zero divisors.

Proof. We enumerate all the S-weak zero divisors of $Z_2L_n(m)$ in the following way:

Case I: As $p|p^3$, $L_n(m)$ has p proper subloops H_j each of order $p^2 + 1$. Now as in the Theorem 3.2.

Type I_1:

$$(h_1 + h_2) \sum_{i=1}^{n+1} g_i = 0, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_1.

Type I_2:

$$(h_1 + h_2)\sum_{i=1}^{p^2+1} h_i = 0, \quad h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_2.

Continue in this way

Type I_{p^2-2}:

$$(h_1 + h_2 + \cdots + h_{p^2-1})\sum_{i=1}^{n+1} g_i = 0,$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of this type.

Type I_{p^2-1}:

$$(h_1 + h_2 + \cdots + h_{p^2-1})\sum_{i=1}^{p^2+1} h_i = 0,$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of type I_{p^2-1}.

Adding all this S-weak zero divisors, we will get the total number of S-weak zero divisors (in case I) in $Z_2L_n(m)$ as $2p\left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right)$.

Case II: Again $p^2|p^3$, so there are p^2 proper subloops H_j each of order $p+1$. Now we can enumerate all the S-weak zero divisors in this case exactly as in case I above. So there are

$$2p^2\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors in case II.

Hence the total number of S-weak zero divisors in $Z_2L_n(m)$ is

$$2p\left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right) + 2p^2\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

Hence the claim.

Theorem 3.4. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, p, q are odd primes. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2\left[p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right) + q\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)\right]$$

S-weak zero divisors.

Proof. We will enumerate all the S-weak zero divisors in the following way:

Case I: As $p|pq$, $L_n(m)$ has p proper subloops H_j each of order $q+1$. Proceeding exactly same way as in Theorem 3.3, we will get

$$2p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)$$

S-weak zero divisors in case I.

Case II: Again as $q|pq, L_n(m)$ has q proper subloops H_j each of order $p+1$. So as above we will get

$$2q\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors in case II.

Hence adding all the S-weak zero divisors in case I and case II, we get

$$2\left[p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)+q\left(\sum_{r=2,\ r4\ even}^{p-1} {}^{p+1}C_r\right)\right]$$

S-weak zero divisors in $Z_2L_n(m)$.

Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ where $n=p^2$ or p^3 or pq, (p,q are odd primes), $ZL_n(m)$ has infinitely many S-weak zero divisors.

Theorem 3.5. Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n=p^2$ or p^3 or pq (p,q are odd primes), then $ZL_n(m)$ has infinitely many S-weak zero divisors.

Proof. Let $L_n(m)$ be a loop ring such that $n=p^2$. $L_n(M)$ has p subloops (say H_j) each of order $p+1$. Now the loop ring $ZL_n(m)$ has the following types of S-weak zero divisors:

$$X=a-bh_1+bh_2-ah_3 \quad and \quad Y=\sum_{i=1}^{n+1} g_i$$

where $a,b\in Z, g_i\in L_n(m)$ and $h_1,h_2,h_3\in H_j$ are such that

$$XY=0.$$

Again

$$X\sum_{i=1}^{p+1} h_i=0, \quad h_i\in H_j$$

also

$$(1-g_t)Y=0, \quad g_t(\neq h_t)\in H_j$$

clearly

$$(1-g_t)\left(\sum_{i=1}^{p+1} h_i\right)=0.$$

So X,Y are S-weak zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-weak zero divisors of this type for a and b can take infinite number of values in Z.

For $n=p^2$ or p^3 or pq we can prove the results in a similar way.

Hence the claim.

§4. Conclusions:

In this paper we find the exact number of S-zero divisors and S-weak zero divisors for the loop rings $Z_2L_n(m)$ in case of the special type of loops $L_n(m) \in L_n$ over Z_2, when $n = p^2$ or p^3 or pq (p, q are odd primes). We also prove for the loop ring $ZL_n(m)$ has infinite number of S-zero divisors and S-weak zero divisors. We obtain conditions for any loop L to have S-zero divisors and S-weak zero divisors. We suggest it would be possible to enumerate in the similar way the number of S-zero divisors and S-weak zero divisors for the loop ring $Z_2L_n(m)$ when $n = p^s, s > 3$; p a prime or when $p = p_1p_2 \cdots p_t$ where $p_1, p_2, \cdots, p_t$ are odd primes. However we find it difficult when we take Z_p instead of Z_2, where p can be odd prime or a composite number such that $(p, n+1 = 1)$ or $(p, n+1 = p)$ and n is of the form $n = p_1^{t_1}p_2^{t_2} \cdots p_r^{t_r}, t_i > 1, n$ is odd and $p_1, p_2, \cdots p_r$ are odd primes.

References

[1] R.H. Bruck, A survey of binary system, Spinger Verlag (1958).

[2] D.S.Passman, The algebraic structure of group rings, Wiley interscience, (1977).

[3] S.V.Singh, On a new class of loops and loop rings, PhD thesis, IIT Madras, (1994).

[4] Vasantha Kandasamy,W.B, Smarandache Zero divisors, (2001).
http://www.gallup.unm.edu/smarandache/Zero-divisor.pdf

Scientia Magna
Vol. 1 (2005), No. 2, 109-110

The function equation $S(n) = Z(n)$ [1]

Maohua Le

Department of Mathematics, Zhanjiang Normal College
29 Cunjin Road, Chikan Zhanjiang, Guangdong, P. R. China

Abstract For any positive integer n, let $S(n)$ and $Z(n)$ denote the Smarandache function and the pseudo Smarandache function respectively. In this paper we prove that the equation $S(n) = Z(n)$ has infinitely many positive integer solutions n.

Keywords Smarandache function; Pseudo Smarandache function; Diophantine equation.

For any positive integers n, let $S(n)$ and $Z(n)$ denote the Smarandache function and pseudo Smarandache function respectively. In [1], Ashbacher proposed two problems concerning the equation

$$S(n) = Z(n) \tag{1}$$

as follows.

Problem 1. Prove that if n is an even perfect number, then n satisfies (1).

Problem 2. Prove that (1) has infinitely many positive integer solutions n.

In this paper we completely solve these problems as follows.

Theorem 1. If n is an even perfect number, then (1) holds.

Theorem 2. (1) has infinitely many positive integer solutions n.

Proof of Theorem 1. By [2, Theorem 277], if n is an even perfect number, then

$$n = 2^{p-1}(2^p - 1), \tag{2}$$

where p is a prime. By [3] and [4], we have

$$S(n) = 2^p - 1. \tag{3}$$

On the other hand, since

$$\frac{1}{2}(2^p - 1)((2^p - 1) + 1) = n, \tag{4}$$

by (2), we get

$$Z(n) = 2^p - 1 \tag{5}$$

immediately. The combination of (3) and (5) yields (1). Thus, the theorem is proved.

[1] This work is supported by N.S.F. of P. R. China(10271104), the Guangdong Provincial Natural Science Foundation(011781) and the Natural Science Foundation of the Education Department of Guangdong Province(0161).

Proof of Theorem 2. Let p be an odd prime with $p \equiv 3(\mod 4)$. Since $S(2) = 2$ and $S(p) = p$, we have

$$S(2p) = \max(S(2), S(p)) = \max(2, p) = p. \tag{6}$$

Let $t = Z(2p)$, By the define of $Z(n)$, we have

$$\frac{1}{2}t(t+1) \equiv 0(\mathrm{mod}2p). \tag{7}$$

It implies that either $t \equiv 0(\mathrm{mod}p)$ or $t+1 \equiv 0(\mathrm{mod}p)$. Hence, we get $t \geq p-1$. If $t = p-1$, then from (7) we obtain

$$\frac{1}{2}(p-1)p \equiv 0(\mathrm{mod}2p). \tag{8}$$

whence we get

$$\frac{1}{2}(p-1)p \equiv 0(\mathrm{mod}2). \tag{9}$$

But, since $p \equiv 3(\mathrm{mod}4)$, (9) is impossible. So we have

$$t \geq p. \tag{10}$$

Since $p+1 \equiv 0(\mathrm{mod}4)$, we get

$$\frac{1}{2}p(p+1) \equiv 0(\mathrm{mod}2p) \tag{11}$$

and $t = p$ by (10). Therefore, by (6), $n = 2p$ is a solution of (1). Notice that there exist infinitely many primes p with $p \equiv 3(\mathrm{mod}4)$. It implies that (1) has infinitely many positive integer solutions n. The theorem is proved.

References

[1] C.Ashbacher, Problems, Smrandache Notions J. **9**(1998), 141-151.

[2] G.H.Hardy and E.M.Wright, An introduction to the theory of numbers, Oxford University Press, Oxford, 1938.

Scientia Magna
Vol. 1 (2005), No. 2, 111-113

On the Smarandache Pseudo-number Sequences

Li Zhanhu[†,‡]

† Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China
‡ Department of Mathematics, Xianyang Teacher's College
Xianyang, Shannxi, P.R.China

Abstract The main purpose of this paper is using elementary method to study the main value of the m-th power mean of the sum of all digits in the Smarandache pseudo-number sequence, and give some interesting asymptotic formulae for them.

Keywords Smarandache Pseudo-multiple of 5, pseudo-even, pseudo-odd sequence number; Sum of digits; Asymptotic formulae.

§1. Introdution

A number is called Smarandache pseudo-multiple of 5 if some permutation of the digits is a multiple of 5, including the identity permutation. For example: $51, 52, 53, 54, 56, 57, 58, 59, 101, 102, 103, 104, 106\cdots$ are Smarandache pseudo-multiple of 5 numbers. Similarly we can define the Smarandache pseudo-even numbers and the Smarandache pseudo-odd numbers. In reference [1], Professor F.Smarandache asked us to study the properties of the pseudo-multiple of 5, pseudo-even, pseudo-odd sequence. Let A denote the set of all Smarandache Pseudo-multiple of 5 numbers; Let B denote the set of all Smarandache Pseudo-even numbers and Let C denote the set of all Smarandache Pseudo-odd numbers. For convenience, denoted by $A(n)$, the sum of all the digits of the base 10 digits of n. That is

$$A(n) = \sum_{i=0}^{k} a_i$$

if $n = a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 10 + a_0$. In this paper, we shall use the element method to study the mean value of the m-power of the sum of all digits in the pseudo-number sequence, and give some interesting formulae for them. That is, we shall prove the following results:

Theorem 1. For any integer number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\in A\\ n\leq x}} A^m(n) = x\left(\frac{9}{2}\log x\right)^m + O\left(x(\log x)^{m-1}\right).$$

Theorem 2 For any integer number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\in B\\ n\leq x}} A^m(n) = x\left(\frac{9}{2}\log x\right)^m + O(x(\log x)^{m-1}).$$

Theorem 3 For any integer number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\in C\\ n\leq x}} A^m(n) = x\left(\frac{9}{2}\log x\right)^m + O(x(\log x)^{m-1}).$$

§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. For any integer number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} A^m(n) = x\left(\frac{9}{2}\log x\right)^m + O\left(x(\log x)^{m-1}\right).$$

Proof. See reference [1].

Lemma 2. For any integer number $x \geq 1$. Let D denotes the complementary set of A, then we have the asymptotic formula

$$\sum_{\substack{n\in D\\ n\leq x}} A^m(n) = O\left(x\frac{(\log x)^m}{(\frac{5}{4})^{\log x}}\right).$$

Proof. From the definition of the set D, we know that the base 10 digits of the numbers in D are $1, 2, 3, 4, 6, 7, 8, 9$, not including $0, 5$. So, there are 8^m m-digit number in D. Hence, for any integer n, there is a k such that $10^{k-1} \leq x < 10^k$. Then we have

$$\sum_{\substack{n\in D\\ n\leq x}} A^m(n) \leq \sum_{t=1}^{k} \sum_{\substack{10^{t-1}\leq n<10^t\\ n\in D}} A^m(n)$$

Noting that

$$\sum_{\substack{10^{t-1}\leq n<10^t\\ n\in D}} A^m(n) < (9t)^m \times 8^t,$$

we can write

$$\sum_{t=1}^{k} \sum_{\substack{10^{t-1}\leq n<10^t\\ n\in D}} A^m(n) < \sum_{t=1}^{k} (9t)^m \times 8^t < 9^m \times k^m \times 8^{k+1}.$$

Since $k \leq (\log x) + 1 < k + 1$, we have

$$\sum_{\substack{n\in D\\ n\leq x}} A^m(n) = O\left((\log x)^m \times 8^{\log x}\right) = O\left(x\frac{(\log x)^m}{(\frac{5}{4})^{\log x}}\right).$$

This proves Lemma 2.

Lemma 3. For any integer number $x \geq 1$. Let E denote the complementary set of B, then we have the asymptotic formula

$$\sum_{\substack{n\in E\\ n\leq x}} A^m(n) = O\left(x\frac{(\log x)^m}{2^{\log x}}\right).$$

Proof. By use the same method of proving Lemma 2, we can also get this Lemma.

§3. Proof of the theorems

Now we complete the proof of the theorems. First we prove Theorem 1. From the definition of Smarandache pseudo-multiple of 5 numbers, Lemma 1 and Lemma 2, we can get

$$\begin{aligned}\sum_{\substack{n\in A\\ n\le x}} A^m(n) &= \sum_{n\le x} A^m(n) - \sum_{\substack{n\in D\\ n\le x}} A^m(n)\\ &= x\left(\frac{9}{2}\log x\right)^m + O(x(\log x)^{m-1}) - O\left(x\frac{(\log x)^m}{(\frac{5}{4})^{\log x}}\right)\\ &= x\left(\frac{9}{2}\log x\right)^m + O(x(\log x)^{m-1}).\end{aligned}$$

This completes the proof of Theorem 1. Using the same method of proving Theorem 1, we can also deduce the other Theorems.

References

[1] F. Smarandache, Only problem, Not Solution, Chicago, Xiquan Publ. House, 1993.

[2] Harald Riede, Asymptotic estimation of a sum of digits, Fibonacci Quarterly, **36(1)**(1998), 72-75.

Scientia Magna
Vol. 1 (2005), No. 2, 114-115

The classical Smarandache function and a formula for twin primes

Dhananjay P.Mehendale
Department of Electronics, S.P.College, Pune, India

Abstract This short paper presents an exact formula for counting twin prime pairs less than or equal to x in terms of the classical Smarandache Function. An extension of the formula to count prime pairs $(p, p+2n), n > 1$ and a positive interger is also given.

§1. Introduction

The most known Smarandache function which has become a classical Smarandache function in number theory is defined as follows:

Definition. The classical Smarandache function, S, is a function

$S : N \to N, N$, the set of natural numbers such that $S(1) = 1$, and $S(n)$=The smallest integer such that n/S(n)!.

This function has been extensively studied and many interesting properties of it have been discovered [1]. Subsequently many Smarandache type functions have been defined and their interesting properties have been achieved. Ruiz and Perez have discussed some properties of several Smarandache type funcitons that are involved in many proposed, solved and unsolved problems [2].

An exact formula for counting primes less than or equal to given x in terms of classical Smarandache function has been discovered by L.Seagull[3]. Ruiz and Perez have quoted this result along with a proof while discussing some properties of the classical Smrandache function (Property 2.4) [2].

§2. A formula for twin prime pairs

We now proceed to obtain an exact formula for counting twin prime pairs less than or equal to given x in terms of the classical Smarandache function.

We denote by $T_2(x)$ the exact number of twin prime pairs less than or equal to x. Also $[m]$ denotes the integral part of m.

Theorem.

$$T_2(x) = -1 + \sum_{1 \le j \le x-2} \left[\frac{S(j) \cdot S(j+2)}{(j) \cdot (j+2)} \right],$$

where $S(k)$ denotes the value of classical Smarandache function evaluated at k.

Proof. It is well known that (1) $S(p) = p$ iff p is prime > 4, (2) $S(p) < p$ when p is not prime and $p \neq 4$, (3) $S(4) = 4$.

In the light of the above properties,

(1) $\left[\frac{S(2)\cdot S(4)}{(2)\cdot(4)}\right] = 1$,

therefore $(2, 4)$ will be counted as a twin prime pair in the sum given in the above formula. The term " -1" is added in the formula to eliminate this additional count. Also,

(2) $\left[\frac{S(j)\cdot S(j+2)}{(j)\cdot(j+2)}\right] = 1$

only when $(j, j+2)$ will be a twin prime pair and in all other cases

$\left[\frac{S(j)\cdot S(j+2)}{(j)\cdot(j+2)}\right] = 0$.

Hence the theorem is obvious.

Let us denote by $T_{2n}(x)$ the exact number of prime pairs $(p, p+2n)$, n is a positive integer and $n > 1$.

Corollary.

$$T_{2n}(x) = \sum_{1\leq j\leq x-2n} \left[\frac{S(j)\cdot S(j+2n)}{(j)\cdot(j+2n)}\right].$$

Proof. Since $n > 1$, the illegal appearance of the pair $(2, 4)$ as a prime pair is automatically prohibited, and the proof follows by proceeding on the similar lines.

§3. Conclusion

Like formula for counting primes up to given x, [3], one can obtain a similar formula for counting twin prime pairs as well as prime pairs in which the primes are separated by $2n$ in terms of the Classical Smarandache Functions by proceeding along the same lines.

Acknowledgements

The author is thankful to Dr. M.R.Modak and Dr. S.A.Katre, Bhaskaracharya Pratishthana, Pune, for their keen interest.

References

[1] e-books at http://www.gallup.unm.edu/ smarandache/math.htm.

[2] Sebastian Martin Ruiz and M.Perez, Properties and Problems Ralated to the Smarandache Type Functions, Mathematics Magazine, **2**(2004).

[3] L.Seagull, The Smarandache Function and the Number of Primes up to X, Mathematical Spectrum, University of Shielfield, **3**(1995), 53.

Scientia Magna
Vol. 1 (2005), No. 2, 116-119

On completely f-perfect numbers

József Sándor
Babeş-Bolyai University of Cluj, Romania

1. Let $f : \mathbf{N}^* \to \mathbf{N}$ be a given arithmetic function. Recently, J.L.Pe[3] has called a number n to be f-**perfect**, if

$$\sum_{i|n, i<n} f(i) = n, \tag{1}$$

where the sum is taken for all proper divisors i of n (i.e. $i|n, i < n$). Clearly for $f = I$ (where $I(n) = n$ for all $n \geq 1$)(1) given $\sigma(n) = 2n$, i.e. one reobtains the classical perfect numbers.

Let S, Z be the **Smarandache**, resp. **Pseudo-Smarandache functions**, defined by

$$S(n) = \min\{k \epsilon \mathbf{N} : n|n!\}, Z(n) = \min\{k \epsilon \mathbf{N} : n|\frac{k(k+1)}{2}\} \tag{2}$$

Since $0! = 1$, we may assume $S(1) = 0$. With this assumption, recently Ch. Ashbacher [1] showed that for $n \leq 10^6$ the only S-perfect number is $n = 12$, while the Z-perfect numbers in this range are $n = 4, 6, 471544$.

2. In what follows, we shall call a number n **completely** f-**perfect**, if

$$\sum_{i|n} f(i) = n, \tag{3}$$

where the sum is over **all** divisors of n. We note that this notion generalizes again the classical notion of a perfect number, since for $f = I - \varphi$ (where φ is Euler's totient), clearly $f(n) = n - \varphi \geq 0$ for all n, and by Gauss' relation $\sum_{i|n} \varphi(i) = n$, (3) implies $\sigma(n) = 2n$. Thus, the completely $I - \varphi$-perfect numbers are the perfect numbers.

3. By assuming $S(1) = 0$, P.Gronas [2] has shown that for $f = S$, all solutions of equation (3) are the following: $n = p$ (prime), and $n = 9, 16, 24$. Thus:

Theorem 1. All completely S-perfect numbers are the primes, and the numbers 9,16,24.

Remark. It is important to note, that if one defines $S(n)$ by $S(n) = \min\{k \epsilon \mathbf{N}^* : n|k!\}$, then clearly $S(1) = 1$, and Theorem 1 above, as well as Aschbacher's result, are no more valid. Indeed, when $S(1) = 0$, then for $f = S$, (1) has the form

$$\sum_{i|n, 1<i<n} S(i) = n, \tag{4}$$

while if $S(1) = 1$, then (1) becomes

$$\sum_{i|n,1<i<n} f(i) = n - 1. \tag{5}$$

Thus we have two distinct equations, namely (4) at one part, and (5) at another part. On the other hand, from (3) we can deduce the two distinct equations (the first one solved by Theorem 1):

$$\sum_{i|n,1<i<n} f(i) = n - S(n), \tag{6}$$

and

$$\sum_{i|n,1<i<n} f(i) = n - S(n) - 1. \tag{7}$$

Then, since $S(2) = 2$, $S(3) = 3$ and 2, 3 are the only proper divisors of 6, $n = 6$ is a solution to (5), but not (4). Therefore one can have **two distinct notions** of "S-perfect" (as well as "completely S-perfect") numbers. Let us call n to be S-perfect in the case 1, if (4) holds, and S-perfect in the sense 2, if (5) holds. The following little result is true:

Theorem 2. Let p,q be distinct primes. Then the only S-perfect number n of the form $n = pq$ in the sense 2 is $n = 6$. There are no S-perfect numbers of this form in sense 1. The only S-perfect numbers of this form in sense 2.

Proof. Let $n = pq$ in (5), and assume $p < q$. Then since $S(p) = p$, $S(q) = q$, one obtains the equation $p + q = pq - 1$ i.e. $(p - 1)(q - 1) = 2$, giving $p - 1 = 1$,$q - 1 = 2$, i.e. $p = 2$, $q = 3$, implying $n = 6$. The equation (4) gives $p + q = pq$, which cannot have a solution. Let now $n = p^2q$. The proper divisors are p, q, p^2, pq, and since $S(p^2) = 2p$, $S(pq) = q$, (4) implies the equation

$$3p + 2q = p^2q.$$

Since $p|2q$, clearly $p|2$, so $p = 2$. This implies $q = 3$, so $n = 2^2 \cdot 3 = 12$. The equation

$$3p + 2q = p^2q - 1$$

can not have solution, since for $p = 2$ this gives $7 = 2q$ (impossible); while for p, q odd, $p^2q-1 =$ even, $3p + 2q =$ odd.

In the similar way, one can prove:

Thoerem 3. There are no completely S-perfect numbers of the form $n = pq$ in both sense. There are no completely S-perfect numbers of the form $n = p^2q$ in sense 1. The only completely S-perfect number of this form in sense 2, is $n = 28$.

Proof. Let $n = pq$ ($p > q$ primes) in (6), resp. (7). Then one gets $p + q = pq - q$, resp. $p + q = pq - q - 1$. The first equation, i.e. $p + 2q = pq$ forces $q|p$, impossible; while the second one, i.e. $p + 2q + 1 = pq$ for $p = 2$ gives $3 = 0$, while for p,$q \geq 3$ left side = even, right side n odd.

Now let $n = p^2q$. Since $S(p^2) = 2p$, $S(pq) = q$ and $S(p^2q) = \max\{S(p^2), S(q)\} = \max\{2p, q\}$, one can deduce the following equations:

i) $3p + 2q = p^2q - max\{2p, q\}$;

ii)$3p + 2q = p^2q - max\{2p, q\} - 1$.

i) a) $2p > q \Rightarrow 5p + 2q = p^2q$. Since $p|2q$, this gives $p = 2$, when $2q = 10$, impossible.

b) $2p < q \Rightarrow 3p + 3q = p^2q$, giving $p|3q$, so $p = 3$ and $9 = 6p$, impossible again.

ii)a) $\Rightarrow 5p + 2q = p^2q - 1$. For $p = 2$ one has $q = -5$, impossible, while for $p, q \geq 3$ left side = odd, right side = even.

b) $\Rightarrow 3p + 3q = p^2q - 1$. Remark that $p = 2$,$q = 7$ is a solution of this equation and this satisfies condition $2p < q$ since $4 < 7$. Now, for $p, q \geq 3$ write the equation in the form

$$q(pq - 3) = 3p + 1$$

and remark that by $q > p \geq 3$ one has $q \geq 5$ so $q(pq - 3) \geq 5(5p - 3) > 3p + 1$, i.e. $22p > 16$, which is true. Thus, there are no other solutions.

4. The solutions $n = 6$ of (5) and $n = 28$ of (7) are ordinary prefect numbers. Having in view to determine all these solutions, we first prove the following result:

Theorem 4. Let $n = 2^kp$, where p is an odd prime, $k \geq 1$ and $p \geq 2k$. Then n cannot be a solution to equation (4) or (6). The number n is a solution of (5) iff $n = 6$. The only solution of this type of equation (7) is $n = 28$.

Proof. We first calculate $S = \sum_{i|n, 1<i<n} S(n)$. Since the proper divisors of $n = 2^kp$ are $2, 2^2, \cdots, 2^k, p, 2p, 2^2p, \cdots, 2^{k-1}p$, one has

$$S(n) = S(2) + S(2^2) + \cdots + S(2^k) + S(1 \cdot p) + S(2 \cdot p) + \cdots + S(2^{k-1} \cdot p)$$

Now

$$S(2^lp) = \max\{S(2^l), \quad S(p)\} = \max\{S(2^l), p\}$$

and since it is well-known that $S(2^l) \leq 2l$, by $2l \leq 2(k - 1) < 2k < p$ we get

$$S \leq 2 + 2 \cdot 2 + \cdots + 2 \cdot k + kp = \frac{2(k + 1)k}{2} + kp = k(k + 1) + kp,$$

so

$$S \leq k(k + 1) + kp \tag{8}$$

Therefore, by (4), (5), (6), (7) we have to solve the equations

$$S(2) + S(2^2) + \cdots + S(2^k) + kp = \begin{cases} 2^kp & (4') \\ 2^kp - 1 & (5') \\ 2^kp - p & (6') \\ 2^kp - p - 1 & (7') \end{cases}$$

a) For (4') remark that by (8) we must have $2^kp \leq kp + k(k + 1)$, so $p(2^k - k) \leq k(k + 1)$. Since $p > 2k$, on the other hand we have $p(2^k - k) > 2k(2^k - 2) \geq k(k + 1)$ by the inequality $2(2^k - k) \geq k + 1$, i.e.

$$2^{k+1} \geq 3k + 1, k \geq 1 \tag{9}$$

It is easy to verify by induction that (9) holds true for all $k \geq 1$. Therefore, equation (4') is impossible.

Remark. The solution $n = 12 = 2^2 \cdot 3$ with $p = 3$, $k = 2$ doesn't satisfy $p > 2k$.

b) Similarly, for (5'), by (8) we should have satisfied the inequality $2^k p - 1 \leq kp + k(k+1)$. Now, by $p > 2k$ we get

$$p(2^k - k) < 2k(2^k - k) > k(k+1) + 1 \Leftrightarrow k(2^{k+1} - 3k - 1) \geq 2.$$

Now, the inequality

$$2^{k+1} \geq 3k + 2, k \geq 2 \tag{10}$$

holds true. Thus for $k \geq 2$ we cannot have a solution. For $k = 1$, however, by Theorem 2 we get the solution $n = 2^1 \cdot 3$ when $p = 3 > 2 \cdot 1 = 2$.

c) For (6') remark, that similarly we must have $2^k p - p \leq k(k+1) + kp$, or $p(2^k - k - 1) \leq k(k+1)$. Now, by $p > 2k$, and the inequality

$$2^{k+1} > 3k + 3, k \geq 3 \tag{11}$$

it follows that $p(2^k - k - 1) > 2k(2^k - k - 1) > k(k+1)$. Thus we could have eventually $k = 1$ or $k = 2$. By Theorem 3 we cannot have solutions.

d) The equation (7'), by (8) implies $2^k p - p - 1 \leq k(k+1) + kp$ so $p(2^k - k - 1) - 1 \leq k(k+1)$. Now, by $p > 2k$, and $2k(2^k - k - 1) > k(k+1) + 1 \Leftrightarrow k(2^{k+1} - 3k - 3) > 1$, this is true by

$$2^{k+1} \geq 3k + 4, k \geq 3, \tag{12}$$

so we could have eventually $k = 1$ or $k = 2$, i.e. $n = 2p$ or $n = 2^2 p$. By Theorem 3 this is possible only when $p = 7$, when $p > 2k$, i.e. $7 > 4$ is satisfied.

Corollary. There are no ordinary even perfect numbers which are S-perfect or completely S-perfect in sense 1. The only even perfect number which is S-perfect in sense 2 is $n = 6$. The only even perfect number which is completely S-perfect in sense 2 is $n = 28$.

Proof. Let n be an even perfect number. Then, by Euclid-Euler's theorem, n can be written as $n = 2^k p$, where p is a prime of the form $p = 2^{k+1} - 1$. Now, $p > 2k$ is true, since $2^{k+1} > 2k + 1$, $k \geq 1$. This follows e.g. by induction, and we omit the details. Theorem 4 implies the corollary.

4. Finally, note that in paper [4] we have proved that the only completely d-perfect numbers are $n = 1, 3, 18$ and 36 (here $d(n)$ is the number of distinct divisors of n).

References

[1] Ch.Ashbacher, On numbers that are pseudo-Smarandache and Smarandache perfect, Smarandache Notions Journal, **41**(2004), 40-42.

[2] P.Gronas, The solution of the diophantine equation $\sigma(n) = n$, Smarandanche Notions Journal, **4-5**(1994), 14-16.

[3] J.L.Pe, On a generalization of perfect numbers, J.Recr. Math. (to appear).

[4] J.Sándor, On completely d-perfect numbers(to appear).

Scientia Magna
Vol. 1 (2005), No. 2, 120-133

Parallel bundles in planar map geometries

Linfan Mao

Institute of Systems, Academy of Mathematics and System Sciences,
Chinese Academy of Sciences, Beijing, P.R.China
E-mail: maolinfan@163.com

Abstract Parallel lines are very important objects in Euclid plane geometry and its behaviors can be gotten by one's intuition. But in a planar map geometry, a kind of the Smarandache geometries, the situation is complex since it may contains elliptic or hyperbolic points. This paper concentrates on the behaviors of parallel bundles in planar map geometries, a generalization of parallel lines in plane geometry and obtains characteristics for parallel bundles.
Keywords Parallel bundle; Planar map; Smarandache geometry; Map geometry; Classification.

§1. Introduction

A *map* is a connected topological graph cellularly embedded in a surface. On the past century, many works are concentrated on to find the combinatorial properties of maps, such as to determine whether exists a particularly embedding on a surface ([7], [11]) or to enumerate a family of maps ([6]). All these works are on the side of algebra, not the object itself, i.e., geometry. For the later, more attentions are given to its element's behaviors, such as, the line, angle, area, curvature, $\cdots$, see also [12] and [14]. For returning to its original face, the conception of map geometries is introduced in [10]. It is proved in [10] that the map geometries are nice model of the Smarandache geometries. They are also a new kind of intrinsic geometry of surfaces ([1]). The main purpose of this paper is to determine the behaviors of parallel bundles in planar geometries, a generalization of parallel lines in the Euclid plane geometry.

An axiom is said *Smarandachely denied* if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A *Smarandache geometry* is a geometry which has at least one Smarandachely denied axiom(1969)([5], [13]).

In [3] and [4], Iseri presented a nice model of the Smarandache geometries, called s-manifolds by using equilateral triangles, which is defined as follows([3], [5] and [9]):

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \leq i \leq n$ satisfying the following conditions:

(i) Each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i, j \leq n$ and $i \neq j$;

(ii) Each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.

The conception of map geometries without boundary is defined as follows ([10]).

Definition 1.1 *For a given combinatorial map M, associates a real number $\mu(u), 0 < \mu(u) < \pi$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry without boundary, $\mu(u)$ the angle factor of the vertex u and to be orientablle or non-orientable if M is orientable or not.*

In [10], it has proved that map geometries are the Smarandache geometries. The realization of each vertex $u, u \in V(M)$ in R^3 space is shown in the Fig.1 for each case of $\rho(u)\mu(u) > 2\pi$, $= 2\pi$ or $< 2\pi$, call *elliptic point, euclidean point* and *hyperbolic point*, respectively.

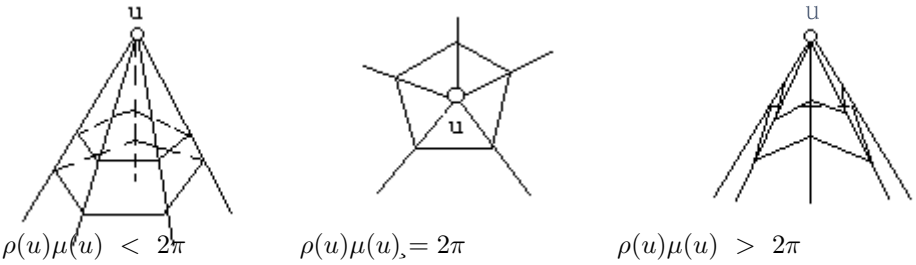


Fig.1

Therefore, a line passes through an elliptic vertex, an euclidean vertex or a hyperbolic vertex u has angle $\frac{\rho(u)\mu(u)}{2}$ at the vertex u. It is not 180° if the vertex u is elliptic or hyperbolic. Then what is the angle of a line passes through a point on an edge of a map? It is 180°? Since we wish the change of angles on an edge is smooth, the answer is not. For the Smarandache geometries, the parallel lines in them are need to be given more attention. We have the following definition.

Definition 1.2 *A family $\mathcal{L}$ of infinite lines not intersecting each other in a planar geometry is called a parallel bundle.*

In the Fig.2, we present all cases of parallel bundles passing through an edge in planar geometries, where, (a) is the case of points u, v are same type with $\rho(u)\mu(u) = \rho(v)\mu(v)$, (b) and (c) the cases of same types with $\rho(u)\mu(u) > \rho(v)\mu(v)$ and (d) the case of u is elliptic and v hyperbolic.

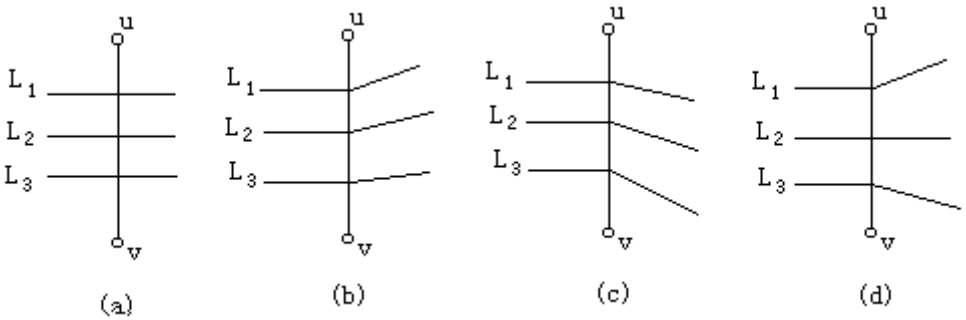


Fig.2

Here, we assume the angle at the intersection point is in clockwise, that is, a line passing

through an elliptic point will bend up and a hyperbolic point will bend down, such as the cases (b),(c) in the Fig.2. For a vector $\overrightarrow{O}$ on the Euclid plane, call it an *orientation*. We classify parallel bundles in planar map geometries along an orientation $\overrightarrow{O}$.

§2. A condition for parallel bundles

We investigate the behaviors of parallel bundles in the planar map geometries. For this object, we define a function $f(x)$ of angles on an edge of a planar map as follows.

Definition 2.1 *Denote by $f(x)$ the angle function of a line L passing through an edge uv at the point of distance x to u on the edge uv.*

Then we get the following result.

Proposition 2.1 *A family $\mathcal{L}$ of parallel lines passing through an edge uv is a parallel bundle iff*

$$\left.\frac{df}{dx}\right|_{+} \geq 0.$$

Proof. If $\mathcal{L}$ is a parallel bundle, then any two lines L_1, L_2 will not intersect after them passing through the edge uv. Therefore, if θ_1, θ_2 are the angles of L_1, L_2 at the intersect points of L_1, L_2 with uv and L_2 is far from u than L_2, then we know that $\theta_2 \geq \theta_1$. Whence, for any point with x distance from u and $\Delta x > 0$, we have that

$$f(x + \Delta x) - f(x) \geq 0.$$

Therefore, we get that

$$\left.\frac{df}{dx}\right|_{+} = \lim_{\Delta x \to +0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

As the cases in the Fig.1.

Now if $\left.\frac{df}{dx}\right|_{+} \geq 0$, then $f(y) \geq f(x)$ if $y \geq x$. Since $\mathcal{L}$ is a family of parallel lines before meeting uv, whence, any two lines in $\mathcal{L}$ will not intersect each other after them passing through uv. Therefore, $\mathcal{L}$ is a parallel bundle. ♮

A general condition for a family of parallel lines passing through a cut of a planar map being a parallel bundle is the following.

Proposition 2.2 *Let (M, μ) be a planar map geometry, $C = \{u_1v_1, u_2v_2, \cdots, u_lv_l\}$ a cut of the map M with order $u_1v_1, u_2v_2, \cdots, u_lv_l$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively, also see the Fig.3.*

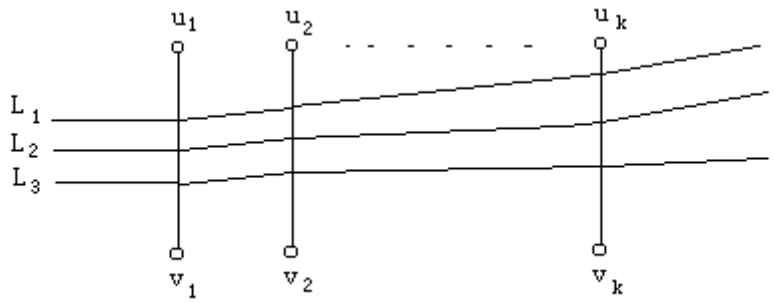


Fig.3

Then a family $\mathcal{L}$ of parallel lines passing through C is a parallel bundle iff for any $x, x \geq 0$,

$$\begin{aligned} f_1'(x) &\geq 0 \\ f_{1+}'(x) + f_{2+}'(x) &\geq 0 \\ f_{1+}'(x) + f_{2+}'(x) + f_{3+}'(x) &\geq 0 \\ &\cdots\cdots\cdots\cdots \\ f_{1+}'(x) + f_{2+}'(x) + \cdots + f_{l+}'(x) &\geq 0. \end{aligned}$$

Proof. According to the Proposition 2.1, see the following Fig.4,

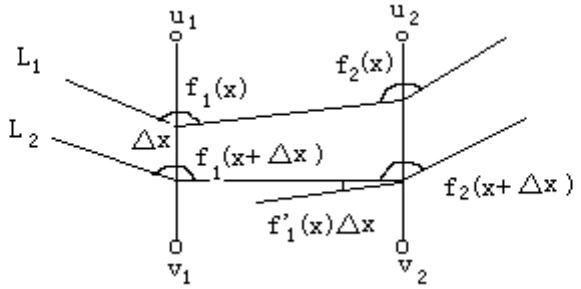


Fig.4

we know that any lines will not intersect after them passing through u_1v_1 and u_2v_2 iff for $\forall \Delta x > 0$ and $x \geq 0$,

$$f_2(x + \Delta x) + f_{1+}'(x)\Delta x \geq f_2(x).$$

That is,

$$f_{1+}'(x) + f_{2+}'(x) \geq 0.$$

Similarly, any lines will not intersect after them passing through u_1v_1, u_2v_2 and u_3v_3 iff for $\forall \Delta x > 0$ and $x \geq 0$,

$$f_3(x + \Delta x) + f_{2+}'(x)\Delta x + f_{1+}'(x)\Delta x \geq f_3(x).$$

That is,

$$f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) \geq 0.$$

Generally, any lines will not intersect after them passing through $u_1v_1, u_2v_2, \cdots, u_{l-1}v_{l-1}$ and u_lv_l iff for $\forall \Delta x > 0$ and $x \geq 0$,

$$f_l(x + \Delta x) + f'_{l-1+}(x)\Delta x + \cdots + f'_{1+}(x)\Delta x \geq f_l(x).$$

Whence, we get that

$$f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l+}(x) \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel lines passing through C is a parallel bundle iff for any $x, x \geq 0$, we have that

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ &\cdots\cdots\cdots\cdots \\ f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l+}(x) &\geq 0. \end{aligned}$$

This completes the proof. ♮.

Corollary 2.1 *Let (M, μ) be a planar map geometry, $C = \{u_1v_1, u_2v_2, \cdots, u_lv_l\}$ a cut of the map M with order $u_1v_1, u_2v_2, \cdots, u_lv_l$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$. Then a family $\mathcal{L}$ of parallel lines passing through C is still parallel lines after them leaving C iff for any $x, x \geq 0$,*

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ &\cdots\cdots\cdots\cdots \\ f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l-1+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l+}(x) &= 0. \end{aligned}$$

Proof. According to the Proposition 2.2, we know the condition is a necessary and sufficient condition for $\mathcal{L}$ being a parallel bundle. Now since lines in $\mathcal{L}$ are parallel lines after them leaving C iff for any $x \geq 0$ and $\Delta x \geq 0$, there must be that

$$f_l(x + \Delta x) + f'_{l-1+}(x)\Delta x + \cdots + f'_{1+}(x)\Delta x = f_l(x).$$

Therefore, we get that

$$f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l+}(x) = 0 \qquad \natural$$

When do the parallel lines parallel the initial parallel lines after them passing through a cut C in a planar map geometry? The answer is in the following result.

Proposition 2.3 *Let (M, μ) be a planar map geometry, $C = \{u_1v_1, u_2v_2, \cdots, u_lv_l\}$ a cut of the map M with order $u_1v_1, u_2v_2, \cdots, u_lv_l$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$. Then the parallel lines parallel the initial parallel lines after them passing through C iff for $\forall x \geq 0$,*

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ &\cdots\cdots\cdots\cdots \\ f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l-1+}(x) &\geq 0 \end{aligned}$$

and

$$f_1(x) + f_2(x) + \cdots + f_l(x) = l\pi.$$

Proof. According to the Proposition 2.2 and Corollary 2.1, we know the parallel lines passing through C is a parallel bundle.

We calculate the angle $\alpha(i, x)$ of a line L passing through an edge $u_iv_i, 1 \leq i \leq l$ with the line before it meeting C at the intersection of L with the edge u_iv_i, where x is the distance of the intersection point to u_1 on u_1v_1, see also the Fig.4. By the definition, we know the angle $\alpha(1, x) = f(x)$ and $\alpha(2, x) = f_2(x) - (\pi - f_1(x)) = f_1(x) + f_2(x) - \pi$.

Now if $\alpha(i, x) = f_1(x) + f_2(x) + \cdots + f_i(x) - (i-1)\pi$, then similar to the case $i = 2$, we know that $\alpha(i+1, x) = f_{i+1}(x) - (\pi - \alpha(i, x)) = f_{i+1}(x) + \alpha(i, x) - \pi$. Whence, we get that

$$\alpha(i+1, x) = f_1(x) + f_2(x) + \cdots + f_{i+1}(x) - i\pi.$$

Notice that a line L parallel the initial parallel line after it passing through C iff $\alpha(l, x) = \pi$, i.e.,

$$f_1(x) + f_2(x) + \cdots + f_l(x) = l\pi.$$

This completes the proof. $\natural$

§3. Linear condition and combinatorial realization for parallel bundles

For the simplicity, we can assume the function $f(x)$ is linear and denoted it by $f_l(x)$. We can calculate $f_l(x)$ as follows.

Proposition 3.1 *The angle function $f_l(x)$ of a line L passing through an edge uv at the point with distance x to u is*

$$f_l(x) = (1 - \frac{x}{d(uv)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(uv)}\frac{\rho(v)\mu(v)}{2},$$

where, $d(uv)$ is the length of the edge uv.

Proof. Since $f_l(x)$ is linear, we know that $f_l(x)$ satisfies the following equation.

$$\frac{f_l(x) - \frac{\rho(u)\mu(u)}{2}}{\frac{\rho(v)\mu(v)}{2} - \frac{\rho(u)\mu(u)}{2}} = \frac{x}{d(uv)},$$

Calculation shows that

$$f_l(x) = (1 - \frac{x}{d(uv)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(uv)}\frac{\rho(v)\mu(v)}{2}. \quad \natural$$

Corollary 3.1 *Under the linear assumption, a family $\mathcal{L}$ of parallel lines passing through an edge uv is a parallel bundle iff*

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}.$$

Proof. According to the Proposition 2.1, a family of parallel lines passing through an edge uv is a parallel bundle iff for $\forall x, x \geq 0$, $f'(x) \geq 0$, i.e.,

$$\frac{\rho(v)\mu(v)}{2d(uv)} - \frac{\rho(u)\mu(u)}{2d(uv)} \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel lines passing through an edge uv is a parallel bundle iff

$$\rho(v)\mu(v) \geq \rho(u)\mu(u).$$

Whence,

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}. \quad \natural$$

For a family of parallel lines pass through a cut, we have the following condition for it being a parallel bundle.

Proposition 3.2 *Let (M, μ) be a planar map geometry, $C = \{u_1v_1, u_2v_2, \cdots, u_lv_l\}$ a cut of the map M with order $u_1v_1, u_2v_2, \cdots, u_lv_l$ from the left to the right, $l \geq 1$. Then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle iff the angle factor μ satisfies the following linear inequality system*

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2v_2)}$$

. . . . . .

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} + \cdots + \frac{\rho(v_l)\mu(v_l)}{d(u_lv_l)} \\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)} + \cdots + \frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

Proof. Under the linear assumption, for any integer $i, i \geq 1$, we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_iv_i)}$$

by the Proposition 3.1. Whence, according to the Proposition 2.2, we get that a family L of parallel lines passing through C is a parallel bundle iff the angle factor μ satisfies the following linear inequality system

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2v_2)} + \cdots + \frac{\rho(v_l)\mu(v_l)}{d(u_lv_l)} \\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)} + \cdots + \frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

This completes the proof. ♮

For planar maps underlying a regular graph, we have the following interesting results for parallel bundles.

Corollary 3.2 *Let (M,μ) be a planar map geometry with M underlying a regular graph, $C = \{u_1v_1, u_2v_2, \cdots, u_lv_l\}$ a cut of the map M with order $u_1v_1, u_2v_2, \cdots, u_lv_l$ from the left to the right, $l \geq 1$. Then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle iff the angle factor μ satisfies the following linear inequality system*

$$\mu(v_1) \geq \mu(u_1)$$

$$\frac{\mu(v_1)}{d(u_1v_1)} + \frac{\mu(v_2)}{d(u_2v_2)} \geq \frac{\mu(u_1)}{d(u_1v_1)} + \frac{\mu(u_2)}{d(u_2v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\mu(v_1)}{d(u_1v_1)} + \frac{\mu(v_2)}{d(u_2v_2)} + \cdots + \frac{\mu(v_l)}{d(u_lv_l)} \geq \frac{\mu(u_1)}{d(u_1v_1)} + \frac{\mu(u_2)}{d(u_2v_2)} + \cdots + \frac{\mu(u_l)}{d(u_lv_l)}$$

and particularly, if assume that all the lengths of edges in C are the same, then

$$\begin{array}{rcl}\mu(v_1) & \geq & \mu(u_1)\\ \mu(v_1)+\mu(v_2) & \geq & \mu(u_1)+\mu(u_2)\\ \cdots\cdots & \cdots & \cdots\cdots\\ \mu(v_1)+\mu(v_2)+\cdots+\mu(v_l) & \geq & \mu(u_1)+\mu(u_2)+\cdots+\mu(u_l).\end{array}$$

Certainly, by choosing different angle factors, we can also get combinatorial conditions for existing parallel bundles under the linear assumption.

Proposition 3.3 *Let (M,μ) be a planar map geometry, $C=\{u_1v_1,u_2v_2,\cdots,u_lv_l\}$ a cut of the map M with order $u_1v_1,u_2v_2,\cdots,u_lv_l$ from the left to the right, $l\geq 1$. If for any integer $i, i\geq 1$,*

$$\frac{\rho(u_i)}{\rho(v_i)}\leq\frac{\mu(v_i)}{\mu(u_i)},$$

then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle.

Proof. Notice that under the linear assumption, for any integer $i, i\geq 1$, we know that

$$f'_{i+}(x)=\frac{\rho(v_i)\mu(v_i)-\rho(u_i)\mu(u_i)}{2d(u_iv_i)}$$

by the Proposition 3.1. Whence, $f'_{i+}(x)\geq 0$ for $i=1,2,\cdots,l$. Therefore, we get that

$$\begin{array}{r}f'_1(x)\geq 0\\ f'_{1+}(x)+f'_{2+}(x)\geq 0\\ f'_{1+}(x)+f'_{2+}(x)+f'_{3+}(x)\geq 0\\ \cdots\cdots\cdots\\ f'_{1+}(x)+f'_{2+}(x)+\cdots+f'_{l+}(x)\geq 0.\end{array}$$

By the Proposition 2.2, we know that a family L of parallel lines passing through C is a parallel bundle. ♮

§4. Classification of parallel bundles

For a cut C in a planar map geometry and $e\in C$, denote by $f_e(x)$ the angle function on the edge e, $f(C,x)=\sum\limits_{e\in C}f_e(x)$. If $f(C,x)$ is independent on x, then we abbreviate it to $f(C)$. According to the results in the Section 2 and 3, we can classify the parallel bundles with a given orientation $\overrightarrow{O}$ in planar map geometries into the following 15 classes, where, each class is labelled by a 4-tuple $0,1$ code.

Classification of parallel bundles

(1) $\mathcal{C}_{1000}$: *for any cut C along $\overrightarrow{O}$, $f(C)=|C|\pi$;*

(2) $\mathcal{C}_{0100}$: *for any cut* C *along* $\overrightarrow{O}$, $f(C) < |C|\pi$;

(3) $\mathcal{C}_{0010}$: *for any cut* C *along* $\overrightarrow{O}$, $f(C) > |C|\pi$;

(4) $\mathcal{C}_{0001}$: *for any cut* C *along* $\overrightarrow{O}$, $f'_+(C,x) > 0$ *for* $\forall x, x \geq 0$;

(5) $\mathcal{C}_{1100}$: *There exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$ *and* $f(C_2) = c < |C_2|\pi$;

(6) $\mathcal{C}_{1010}$: *there exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$ *and* $f(C_2) > |C_2|\pi$;

(7) $\mathcal{C}_{1001}$: *there exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$ *and* $f'_+(C_2,x) > 0$ *for* $\forall x, x \geq 0$;

(8) $\mathcal{C}_{0110}$: *there exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) < |C_1|\pi$ *and* $f(C_2) > |C_2|\pi$;

(9) $\mathcal{C}_{0101}$: *there exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) < |C_1|\pi$ *and* $f'_+(C_2,x) > 0$ *for* $\forall x, x \geq 0$;

(10) $\mathcal{C}_{0011}$: *there exist cuts* C_1, C_2 *along* $\overrightarrow{O}$, *such that* $f(C_1) > |C_1|\pi$ *and* $f'_+(C_2,x) > 0$ *for* $\forall x, x \geq 0$;

(11) $\mathcal{C}_{1110}$: *there exist cuts* C_1, C_2 *and* C_3 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$, $f(C_2) < |C_2|\pi$ *and* $f(C_3) > |C_3|\pi$;

(12) $\mathcal{C}_{1101}$: *there exist cuts* C_1, C_2 *and* C_3 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$, $f(C_2) < |C_2|\pi$ *and* $f'_+(C_3,x) > 0$ *for* $\forall x, x \geq 0$;

(13) $\mathcal{C}_{1011}$: *there exist cuts* C_1, C_2 *and* C_3 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$, $f(C_2) > |C_2|\pi$ *and* $f'_+(C_1,x) > 0$ *for* $\forall x, x \geq 0$;

(14) $\mathcal{C}_{0111}$: *there exist cuts* C_1, C_2 *and* C_3 *along* $\overrightarrow{O}$, *such that* $f(C_1) < |C_1|\pi$, $f(C_2) > |C_2|\pi$ *and* $f'_+(C_1,x) > 0$ *for* $\forall x, x \geq 0$;

(15) $\mathcal{C}_{1111}$: *there exist cuts* C_1, C_2, C_3 *and* C_4 *along* $\overrightarrow{O}$, *such that* $f(C_1) = |C_1|\pi$, $f(C_2) < |C_2|\pi$, $f(C_3) > |C_3|\pi$ *and* $f'_+(C_4,x) > 0$ *for* $\forall x, x \geq 0$.

Notice that only the first three classes may be parallel lines after them passing through the cut C. All of the other classes are only parallel bundles, not parallel lines in the usual meaning.

Proposition 4.1 *For an orientation* $\overrightarrow{O}$, *the* 15 *classes* $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$ *are all the parallel bundles in planar map geometries.*

Proof. Not loss of generality, we assume $C_1, C_2, \cdots, C_m, m \geq 1$, are all the cuts along $\overrightarrow{O}$ in a planar map geometry (M, μ) from the upon side of $\overrightarrow{O}$ to its down side. We find their structural characters for each case in the following discussion.

$\mathcal{C}_{1000}$: By the Proposition 2.3, a family $\mathcal{L}$ of parallel lines parallel their initial lines before meeting M after the passing through M.

$\mathcal{C}_{0100}$: By the definition, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if

$$f(C_1) \leq f(C_2) \leq \cdots \leq f(C_m) < \pi.$$

Otherwise, some lines in $\mathcal{L}$ will intersect. According to the Corollary 2.1, they parallel each other after they passing through M only if

$$f(C_1) = f(C_2) = \cdots = f(C_m) < \pi.$$

$\mathcal{C}_{0010}$: Similar to the case $\mathcal{C}_{0100}$, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if

$$\pi \ < \ f(C_1) \leq f(C_2) \leq \cdots \leq f(C_m)$$

and parallel each other after they passing through M only if

$$\pi \ < \ f(C_1) = f(C_2) = \cdots = f(C_m).$$

$\mathcal{C}_{0001}$: Notice that by the proof of the Proposition 2.3, a line has angle $f(C,x) - (|C|-1)\pi$ after it passing through C with the initial line before meeting C. In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if for $\forall x_i, x_i \geq 0, 1 \leq i \leq m$,

$$f(C_1,x_1) \leq f(C_2,x_2) \leq \cdots \leq f(C_m,x_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{1100}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, 2 \leq k \leq m$, such that

$$f(C_1) \leq f(C_2) \leq \cdots \leq f(C_{k-1}) \ < \ f(C_k) = f(C_{k+1}) = \cdots = f(C_m) = \pi.$$

Otherwise, they will intersect.

$\mathcal{C}_{1010}$: Similar to the case $\mathcal{C}_{1100}$, in this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, 2 \leq k \leq m$, such that

$$\pi = f(C_1) = f(C_2) = \cdots = f(C_k) \ < \ f(C_{k+1}) \leq \cdots \leq f(C_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{1001}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, l, 1 \leq k \ < \ l \leq m$, such that for $\forall x_i, x_i \geq 0, 1 \leq i \leq k$ or $l \leq i \leq m$,

$$\begin{aligned} f(C_1,x_1) \ &\leq \ f(C_2,x_2) \leq \cdots \leq f(C_k,x_k) \ < \ f(C_{k+1}) \\ &= \ f(C_{k+2}) = \cdots = f(C_{l-1}) = \pi \ < \ f(C_l,x_l) \leq \cdots \leq f(C_m,x_m). \end{aligned}$$

Otherwise, they will intersect.

$\mathcal{C}_{0110}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is integers $k, 1 \leq k \ < \ m$, such that

$$f(C_1) \leq f(C_2) \leq \cdots \leq f(C_k) \ < \ \pi \ < \ f(C_{k+1}) \leq \cdots \leq f(C_m).$$

Otherwise, they will intersect.

$\mathcal{C}_{0101}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is integers $k, 1 \leq k \leq m$, such that for $\forall x_i, x_i \geq 0, 1 \leq i \leq m$,

$$f(C_1,x_1) \leq f(C_2,x_2) \leq \cdots \leq f(C_k,x_k) \ < \pi \ \leq f(C_{k+1},x_{k+1}) \leq \cdots \leq f(C_m,x_m),$$

and there must be a constant in $f(C_1,x_1), f(C_2,x_2), \cdots, f(C_k,x_k)$.

$\mathcal{C}_{0011}$: In this case, the situation is similar to the case $\mathcal{C}_{0101}$ and there must be a constant in $f(C_{k+1}, x_{k+1}), f(C_{k+2}, x_{k+2}), \cdots, f(C_m, x_m)$.

$\mathcal{C}_{1110}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, l, 1 \leq k \ < \ l \leq m$, such that

$$\begin{aligned} f(C_1) \ &\leq \ f(C_2) \leq \cdots \leq f(C_k) \ < \ f(C_{k+1}) \\ &= \ \cdots = f(C_{l-1}) = \pi \ < \ f(C_l) \leq \cdots \leq f(C_m). \end{aligned}$$

Otherwise, they will intersect.

$\mathcal{C}_{1101}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, l, 1 \leq k \ < \ l \leq m$, such that for $\forall x_i, x_i \geq 0, 1 \leq i \leq k$ or $l \leq i \leq m$,

$$\begin{aligned} f(C_1, x_1) \ &\leq \ f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) \ < \ f(C_{k+1}) \\ &= \ \cdots = f(C_{l-1}) = \pi \ < \ f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in $f(C_1, x_1), f(C_2, x_2), \cdots, f(C_k, x_k)$. Otherwise, they will intersect.

$\mathcal{C}_{1011}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, l, 1 \leq k \ < \ l \leq m$, such that for $\forall x_i, x_i \geq 0, 1 \leq i \leq k$ or $l \leq i \leq m$,

$$\begin{aligned} f(C_1, x_1) \ &\leq \ f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) \ < \ f(C_{k+1}) \\ &= \ \cdots = f(C_{l-1}) = \pi \ < \ f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \cdots, f(C_m, x_m)$. Otherwise, they will intersect.

$\mathcal{C}_{0111}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, 1 \leq k \leq m$, such that for $\forall x_i, x_i \geq 0$,

$$f(C_1, x_1) \leq f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) \ < \ \pi \ < \ f(C_l, x_l) \leq \cdots \leq f(C_m, x_m)$$

and there must be a constant in $f(C_1, x_1), f(C_2, x_2), \cdots, f(C_k, x_k)$ and a constant in $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \cdots, f(C_m, x_m)$. Otherwise, they will intersect.

$\mathcal{C}_{1111}$: In this case, a family $\mathcal{L}$ of parallel lines is a parallel bundle along $\overrightarrow{O}$ only if there is an integer $k, l, 1 \leq k \ < \ l \leq m$, such that for $\forall x_i, x_i \geq 0, 1 \leq i \leq k$ or $l \leq i \leq m$,

$$\begin{aligned} f(C_1, x_1) \ &\leq \ f(C_2, x_2) \leq \cdots \leq f(C_k, x_k) \ < \ f(C_{k+1}) \\ &= \ \cdots = f(C_{l-1}) = \pi \ < \ f(C_l, x_l) \leq \cdots \leq f(C_m, x_m) \end{aligned}$$

and there must be a constant in $f(C_1, x_1), f(C_2, x_2), \cdots, f(C_k, x_k)$ and a constant in $f(C_l, x_l), f(C_{l+1}, x_{l+1}), \cdots, f(C_m, x_m)$. Otherwise, they will intersect.

Following the structural characters of the classes $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$, by the Proposition 2.2, 2.3 and Proposition 3.1, we know that any parallel bundle is in one of the classes $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$ and each class in $\mathcal{C}_{1000} \sim \mathcal{C}_{1111}$ is non-empty. This completes the proof. $\natural$

A example of parallel bundle in a planar map geometry is shown in the Fig.5, in where the number on a vertex u denotes the number $\rho(u)\mu(u)$.

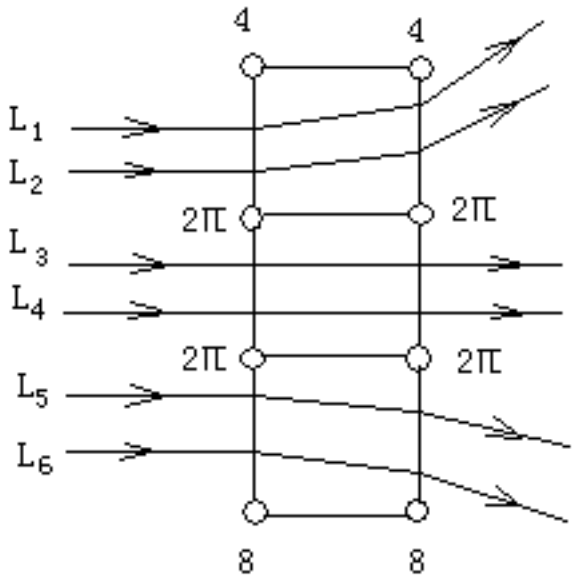


Fig.5

§5. Generalization

All the planar map geometries considered in this paper are without boundary. For planar map geometries with boundary, i.e., some faces are deleted ([10]), which are correspondence with the maps with boundary ([2]). We know that they are the Smarandache *non-geometries*, satisfying one or more of the following conditions:

$(A1^-)$*It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

$(A2^-)$*It is not always possible to extend by continuity a finite line to an infinite line.*

$(A3^-)$*It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

$(A4^-)$*Not all the right angles are congruent.*

$(A5^-)$*If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

Notice that for an one face planar map geometry $(M, \mu)^{-1}$ with boundary, if we choose all points being euclidean, then $(M, \mu)^{-1}$ is just the Poincaré's model for the hyperbolic geometry.

Using the neutrosophic logic idea, we can also define the conception of *neutrosophic surface* as follow, comparing also with the surfaces in [8] and [14].

Definition 5.1 *A neutrosophic surface is a Hausdorff, connected, topological space S such that every point v is elleptic, euclidean, or hyperbolic.*

For this kind of surface, we present the following problem for the further researching.

Problem 5.1 *To determine the behaviors of elements, such as, the line, angle, area, $\cdots$, in neutrosophic surfaces.*

Notice that results in this paper are just the behaviors of line bundles in a neutrosophic plane.

References

[1]A.D.Aleksandrov and V.A.Zalgaller, Intrinsic geometry of surfaces, American Mathematical Society, 1967.

[2]R.P.Bryant and D.Singerman, Foundations of the theory of maps on surfaces with boundary, Quart.J.Math.Oxford, **36(2)**(1985), 17-41.

[3]H.Iseri, Smarandache manifolds, American Research Press, Rehoboth, NM, 2002.

[4]H.Iseri, Partially Paradoxist Smarandache Geometries,

http://www.gallup.unm. edu/šmarandache/Howard-Iseri-paper.htm.

[5]L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, Mathematics Magazine, Aurora, Canada,**12**(2003)

[6]Y.P.Liu, Enumerative Theory of Maps, Kluwer Academic Publisher, Dordrecht, Boston, London, (1999).

[7]Y.P.Liu, Embeddability in Graphs, Kluwer Academic Publisher, Dordrecht, Boston, London, (1995).

[8]Mantredo P.de Carmao, Differential Geometry of Curves and Surfaces, Pearson Education asia Ltd, (2004).

[9]L.F.Mao, Automorphism groups of maps, surfaces and Smarandache geometries, American Research Press, Rehoboth, NM, 2005.

[10]L.F.Mao, A new view of combinatorial maps by Smarandache's notion, arXiv, Math.GM/0506232, will also appear in Smarandache Notions Journal.

[11]B.Mohar and C.Thomassen, Graphs on Surfaces, The Johns Hopkins University Press, London, 2001.

[12]V.V.Nikulin and I.R.Shafarevlch, Geometries and Groups, Springer-Verlag Berlin Heidelberg, (1987).

[13]F. Smarandache, Mixed noneuclidean geometries, eprint arXiv, math/0010119, 10/2000.

[14]J.Stillwell, Classical topology and combinatorial group theory, Springer-Verlag New York Inc., (1980).

Scientia Magna
Vol. 1 (2005), No. 2, 134-138

On the asymptotic properties of odd sieve sequence

Yang Hai† and Fu Ruiqin‡

† Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China
‡ School of Science, Department of Mathematics, xi'an Shiyou University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using elementary method and an analytic method to study the asymptotic properties of the odd sieve sequence, and give two interesting asymptotic formulae.
Keywords Odd sieve sequence; Mean value; Asymptotic formula.

§1. Introduction And Results

According to reference [1], the definition of the odd sieve is: subtract 2 from all prime numbers and obtain a temporary sequence, and choose all odd number that do not belong to the temporary one. For example: 7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 79, 83, 85, 91, 93, 97,... are all belong to odd sieve sequence. Let A denote the set of all odd sieve. In reference [1], Professor F.Smarandache asked us to study the properties of the odd sieve sequence. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we use elementary method and analytic method to study the asymptotic properties of odd sieve sequence, and obtain two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{\substack{a\in A\\ a\leq x}} a = x^2 - \frac{x^2}{2\log x} - \frac{x^2}{4\log^2 x} + O\left(x\right).$$

Theorem 2. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{\substack{a\in A\\ a\leq x}} d(a) = \frac{1}{2}x\log x + Bx + O\left(x^{\frac{1}{2}}\log^2 x\right),$$

where $B = 4C - \frac{1}{2} + \frac{3}{2}\log 2 - \frac{315\zeta(3)}{6\pi^4}$, C is Euler constant, $d(n)$ be the Dirichlet divisor function.

§2. Several Lemmas

To complete the proof of the theorems, we need the following several simple lemmas. Firstly, we have

Lemma 1. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O\left(x^{\alpha}\right). \qquad (\alpha \geq 1)$$

Proof. (See reference [2]).

Taking $\alpha = 0, 1$ in the lemma 1, we may immediately obtain the following:

Corollary. *For any real number $x \geq 1$, then we have*

$$\sum_{n \leq x} 1 = x + O\left(1\right)$$

and

$$\sum_{n \leq x} n = \frac{1}{2}x^2 + O\left(x\right).$$

Lemma 2. *For any fixed real number x, let $\pi(x)$ denote the number of all primes not exceeding x, then we have*

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right).$$

Proof. (See reference [3]).

Lemma 3. *For any real number $x \geq 3$, let p be a prime, then we have*

$$\sum_{p \leq x} p = \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + O\left(\frac{x^2}{\log^3 x}\right).$$

Proof. From Lemma 2 and by Abel's identity, we can easily deduce that

$$\begin{aligned}
\sum_{p \leq x} p &= \pi(x)x - \int_1^x \pi(x)dt \\
&= \frac{x^2}{\log x} + \frac{x^2}{\log^2 x} + O\left(\frac{x^2}{\log^3 x}\right) \\
&- \int_2^x \frac{t}{\log t}dt - \int_2^x \frac{t}{\log^2 t}dt + O\left(\int_2^x \frac{t}{\log^3 t}dt\right) \\
&= \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + O\left(\frac{x^2}{\log^3 x}\right).
\end{aligned}$$

This complets the proof of Lemma 3.

Lemma 4. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(n) = x\log x + (2C-1)x + O\left(\sqrt{x}\right),$$

where C is Euler constant, and $d(n)$ be the Dirichlet divisor function.

Proof. This result may be immediately got from [2].

Lemma 5. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(2n) = \frac{3}{2}x\log x + (\frac{\log 2}{2} - \frac{3}{2})x + O\left(x^{\frac{1}{2}}\log^2 x\right).$$

Proof. Let $s=\sigma+it$ be a complex number and $h(s)=\sum\limits_{n=1}^{\infty}\frac{d(2n)}{n^s}$. Note that $d(2n)\ll n^{\epsilon}$, so it is clear that $h(s)$ is a Dirichlet series absolutely convergent for Re$(s)>1$, by the Euler Product formula [2] and the definition of $d(n)$ we get

$$\begin{aligned}
h(s) &= \prod_p \sum_{m=0}^{\infty}\frac{d(2p^m)}{p^{ms}} \\
&= \sum_{m=0}^{\infty}\frac{d(2^{m+1})}{2^{ms}}\cdot\prod_{p>2}\sum_{m=0}^{\infty}\frac{d(2p^m)}{p^{ms}} \\
&= 2\zeta^2(s)\cdot\frac{(\prod\limits_{p>2}\sum\limits_{m=0}^{\infty}\frac{d(p^m)}{p^{ms}})\cdot(\sum\limits_{m=0}^{\infty}\frac{d(2^{m+1})}{2^{ms}})}{\prod\limits_p\sum\limits_{m=0}^{\infty}\frac{d(p^m)}{p^{ms}}} \\
&= 2\zeta^2(s)\cdot\frac{\sum\limits_{m=0}^{\infty}\frac{d(2^{m+1})}{2^{ms}}}{\sum\limits_{m=0}^{\infty}\frac{d(2^m)}{2^{ms}}} \\
&= \zeta^2(s)(2-\frac{1}{2^s}), \qquad (1)
\end{aligned}$$

where $\zeta(s)$ is the Riemann Zeta-function and $\prod\limits_p$ denotes the product over all primes.

From (1) and Perron's formula [4], for $b=1+\epsilon, T\geq 1$ and $x\geq 1$ we have

$$\sum_{n\leq x}d(2n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}h(s)\frac{x^s}{s}ds+O\left|\frac{x^b}{T}\right|+O\left(\frac{xH(2x)\log x}{T}\right). \qquad (2)$$

Taking $a=\frac{1}{2}+\epsilon$, we move the integral line in (2). Then

$$\begin{aligned}
\sum_{n\leq x}d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} \\
&+ \frac{1}{2\pi i}\left|\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds \\
&+ O\left|\frac{x^b}{T}\right|+O\left|\frac{xH(2x)\log x}{T}\right|,
\end{aligned}$$

where

$$\left|\int_{b-iT}^{a-iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds\ll\frac{x}{T}$$

$$\int_{a-iT}^{a+iT}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds\ll x^{\frac{1}{2}}\log^2 T.$$

Hence, we have

$$\begin{aligned}\sum_{n\le x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}+O\left|\frac{x}{T}\right| \\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right)+O\left|\frac{x^b}{T}\right|+O\left|xH(2x)\frac{\log x}{T}\right| \\ &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}+O\left|\frac{x}{T}\right| \\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right)+O\left|x^{1+\varepsilon}\frac{\log x}{T}\right|. \end{aligned} \tag{3}$$

Taking $T = x^{\frac{1}{2}+\epsilon}$ in (3), then

$$\begin{aligned}\sum_{n\le x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}+O\left(x^{\frac{1}{2}-\varepsilon}\right)+O\left(x^{\frac{1}{2}}\log^2 x\right) \\ &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}+O\left(x^{\frac{1}{2}}\log^2 x\right). \end{aligned} \tag{4}$$

Now we can easily get the residue of the function $\zeta^2(s)(2-\frac{1}{2^s})\cdot\frac{x^s}{s}$ at second order pole point $s=1$

$$\operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} = \frac{3}{2}x\log x+(\frac{\log 2}{2}-\frac{3}{2})x. \tag{5}$$

Combining (4) and (5), we immediately get

$$\sum_{n\le x} d(2n) = \frac{3}{2}x\log x+(\frac{\log 2}{2}-\frac{3}{2})x+O\left(x^{\frac{1}{2}}\log^2 x\right).$$

This completes the proof of Lemma 5.

Lemma 6. *For any $x>0$, let a be any fixed positive integer, then we have*

$$\sum_{0<p-a\le x} d(p-a) = \frac{315\zeta(3)}{2\pi^4}\prod_{p|a}\frac{(p-1)^2}{p^2-p+1}x+O\left(x(\log x)^{-1+\varepsilon}\right),$$

where ϵ is any positive integer.

Proof. This result may be immediately got from [5].

§3. Proof of the Theorem

In this section, we will complete the proof of Theorem. First, we have

$$\begin{aligned}\sum_{\substack{a\in A\\ a\le x}} a &= \sum_{n\le x}(2n-1)-\sum_{p\le x}(p-2) \\ &= 2\sum_{n\le x} n-\sum_{n\le x}1-\sum_{p\le x}p+2\pi(x).\end{aligned}$$

From Lemma 1, Lemma 2 and Lemma 3, we have

$$\begin{aligned}\sum_{\substack{a\in A\\ a\le x}} a &= 2\left(\frac{x^2}{2}+O(x)\right)-(x+O(1))\\ &- (\frac{x^2}{2\log x}+\frac{x^2}{4\log^2 x}+O\left(\frac{x^2}{\log^3 x}\right))\\ &+ 2(\frac{x}{\log x}+\frac{x}{\log^2 x}+O\left(\frac{x}{\log^3 x}\right))\\ &= x^2-\frac{x^2}{2\log x}-\frac{x^2}{4\log^2 x}+O(x).\end{aligned}$$

This completes the proof of Theorem 1.

Now we will give the proof of Theorem 2.

From Lemma 4, Lemma 5 and Lemma 6, we can easily obtain

$$\begin{aligned}\sum_{\substack{a\in A\\ a\le x}} d(a) &= \sum_{n\le x} d(2n-1)-\sum_{p\le x} d(p-2)\\ &= \sum_{n\le 2x} d(n)-\sum_{n\le x} d(2n)-\sum_{p\le x} d(p-2)\\ &= 2x\log x+2(2C-1+\log 2)x+O\left(\sqrt{x}\right)\\ &- \frac{3}{2}x\log x+(\frac{\log 2}{2}-\frac{3}{2})x+O\left(x^{\frac{1}{2}}\log^2 x\right)\\ &- \frac{315\zeta(3)}{6\pi^4}x+O\left(x(\log x)^{-1+\varepsilon}\right)\\ &= \frac{1}{2}x\log x+Bx+O\left(x^{\frac{1}{2}}\log^2 x\right),\end{aligned}$$

where $B=4C-\frac{1}{2}+\frac{3}{2}\log 2-\frac{315\zeta(3)}{6\pi^4}$, C is Euler constant.

This completes the proof of Theorem 2.

References

[1] F. Smarandache, Only problems, Not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Tom M.Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[3] M.Ram Murty, Problems in Analytic Number Theory, New York, 2001, 35-36.

[4] Pan Chengdong and Pan Chengbiao, Foundation of Analytic number Theory, Beijing, Science Press, 1997.

[5] B.M.Bredihin, Binary Additive Problems of Indeterminate Type I(Russian), Izv. Akad. Nauk SSSR Ser. Mat., **27**(1963), 439-462.

Scientia Magna
Vol. 1 (2005), No. 2, 139-144

Smarandache quasigroup rings

Arun S. Muktibodh
Mohota Science College
Nagpur, India

Abstract In this paper, we have introduced Smarandache quasigroups which are Smarandache non-associative structures. W.B.Kandasamy [2] has studied groupoid ring and loop ring. We have defined Smarandache quasigroup rings which are again non-associative structures having two binary operations. Substructures of quasigroup rings are also studied.

Keywords Non-associative rings; Smarandache non-associative rings; Quasigroups; Smarandache quasigroups; Smarandache quasigroup rings.

§1. Introduction

In the paper [2] W.B.Kandasamy has introduced a new concept of groupoid rings. This structure provides number of examples of SNA-rings (Smarandache non-associative rings). SNA-rings are non-associative structure on which are defined two binary operations one associative and other being non-associative and addition distributes over multiplication both from right and left. We are introducing a new concept of quasigroup rings. These are non associative structures. In our view groupoid rings and quasigroup rings are the rich source of non-associative SNA-rings without unit since all other rings happen to be either associative or non-associative rings with unit. To make this paper self contained we recollect some definitions and results which we will use subsequently.

§2. Preliminaries

Definition 2.1. A groupoid S such that for all $a, b \in S$ there exist unique $x, y \in S$ such that $ax = b$ and $ya = b$ is called a quasigroup.

Thus a quasigroup does not have an identity element and it is also non-associative.
Here is a quasigroup that is not a loop.

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

We note that the definition of quasigroup Q forces it to have a property that every element of Q appears exactly once in every row and column of its operation table. Such a table is called a LATIN SQUARE. Thus, quasigroup is precisely a groupoid whose multiplication table is a LATIN SQUARE.

Definition 2.2. If a quasigroup $(Q, *)$ contains a group $(G, *)$ properly then the quasigroup is said to be Smarandache quasigroup.

Example 2.1. Let Q be a quasigroup defined by the following table:

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is a group w.r.t. $*$ which is a proper subset of Q. Therefore Q is a Smarandache quasigroup.

Definition 2.3. A quasigroup Q is idempotent if every element x in Q satisfies $x * x = x$.

Definition 2.4. A ring $(R, +, *)$ is said to be a non-associative ring if $(R, +)$ is an additive abelian group, $(R, *)$ is a non-associative semigroup (i.e. binary operation $*$ is non-associative) such that the distributive laws
$a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all a, b, c in R.

Definition 2.5. Let R be a commutative ring with one. G be any group (S any semigroup with unit) RG (RS the semigroup ring of the semigroup S over the ring R) the group ring of the group G over the ring R consists of finite formal sums of the form $\sum_{i=1}^{n} \alpha_i g_i, (n < \infty)$ i.e. i runs over a finite number where $\alpha_i \in R$ and $g_i \in G \quad (g_i \in S)$ satisfying the following conditions:

1. $\sum_{i=1}^{n} \alpha_i m_i = \sum_{i=1}^{n} \beta_i m_i \Leftrightarrow \alpha_i = \beta_i$, for $i = 1, 2, \cdots, n$
2. $\sum_{i=1}^{n} \alpha_i m_i + \sum_{i=1}^{n} \beta_i m_i \Leftrightarrow \sum_{i=1}^{n} (\alpha_i + \beta i) m_i$
3. $(\sum_{i=1}^{n} \alpha_i m_i)(\sum_{i=1}^{n} \beta_i m_i) = \sum_{i=1}^{n} \gamma_k m_k, m_k = m_i m_j$, where $\gamma_k = \sum \alpha_i \beta_i$
4. $r_i m_i = m_i r_i$ for all $r_i \in R$ and $m_i \in G (m_i \in S)$.
5. $r \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r r_i m_i$ for all $r \in R$ and $\sum_{i=1}^{n} r_i m_i \in RG$. RG is an associative ring with $0 \in R$ acts as its additive identity. Since $I \in R$ we have $G = IG \subseteq RG$ and $R.e = R \subseteq RG$ where e is the identity element of G.

If we replace the group G in the above definition by a quasigroup Q we get RQ the quasigroup ring which will satisfy all the five conditions 1 to 5 given in the definition. But RQ

will only be a non-associative ring without identy. As $I \in R$ we have $Q \subseteq RQ$. Thus we define quasigroup rings as follows:

Definition 2.6. For any quasigroup Q the quasigroup ring RQ is the quasigroup Q over the ring R consisting of all finite formal sums of the form $\sum_{i=1}^{n} r_i q_i, \quad (n < \infty)$ i.e. i runs over a finite number where $r_i \in R$ and $q_i \in Q$ satisfying conditions 1 to 5 given in the definition of group rings above.

Note that only when Q is a quasigroup with identity (i.e. then Q is a Loop) that the quasigroup ring RQ will be a non-associative ring with unit. Here we give examples of non-associative quasigroup rings.

Example 2.2. Let Z be the ring of integers and $(Q, *)$ be the quasigroup given by the following table:

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Clearly $(Q, *)$ is a quasigroup and does not posses an identity element. The quasigroup ring ZQ is a non-associative ring without unit element.

Example 2.3. Let R be the ring of reals and $(Q, *)$ be the quasigroup defined by the following table:

$*$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$(Q, *)$ is an idempotent quasigroup. Again RQ is a non-associative quasigroup ring without unit. Note that $R\langle 1\rangle$, $R\langle 2\rangle$, $R\langle 3\rangle$, $R\langle 4\rangle$ are the subrings of RQ which are associative.

Result: All quasigroup rings RQ of a quasigroup Q over the ring R are non-associative rings without unit.

The smallest non-associative ring without unit is quasigroup ring given by the following example. This example was quoted by W.B.Kandasamy [2] as a groupoid ring.

Example 2.4. Let $Z_2 = \{0, 1\}$ be the prime field of characteristic 2. $(Q, *)$ be a quasigroup of order 3 given by the following table:

$*$	q_1	q_2	q_3
q_1	q_1	q_2	q_3
q_2	q_3	q_1	q_2
q_3	q_2	q_3	q_1

Z_2Q is a quasigroup ring having only eight elements given by $\{0, q_1, q_2, q_3, q_1 + q_2, q_2 + q_3, q_1 + q_3, q_1 + q_2 + q_3\}$. Clearly, Z_2Q is a non-associative ring without unit. This happens to be the smallest non-associative ring without unit known to us.

§3. SNA-Quasigroup rings

We introduce Smarandache non-associative quasigroup rings. It is true that quasigroup rings are always non-associative. We write "Smarandache non-associative quasigroup ring" only to emphasize the fact that they are non-associative.

Definition 3.1. Let S be a quasigroup ring. S is said to be SNA-quasigroup ring (Smarandache non-associative quasigroup ring) if S contains a proper subset P such that P is an associative ring under the operations of S.

Example 3.1. Let Z be the ring of integers and Q be a quasigroup defined by the following table;

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is group and $ZQ \supset ZA$. Thus the quasigroup ring ZQ contains an associative ring properly. Hence ZQ is an SNA-quasigroup ring. Note that Q is a Smarandache quasigroup.

Example 3.2. Let R be the reals, $(Q, *)$ be the quasigroup defined by the following table;

$*$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	3	2	1	0
3	2	3	0	1

Then clearly RQ is an SNA-quasigroup ring as $RQ \supset R\langle 0,1\rangle$ and $R\langle 0,1\rangle$ is an associative ring.

Theorem 3.1. *Let Q be a quasigroup and R be any ring. Then the quasigroup ring RQ is not always an SNA-quasigroup ring.*

Proof. Since Q does not have an identity element, there is no guarantee that R is contained in RQ .

Example 3.3. Let R be an arbitrary ring and Q be a quasigroup defined by the table;

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Then clearly, RQ is not an SNA-quasigroup ring as the quasigroup ring RQ does not contain an associative ring.

Theorem 3.2. *If Q is a quasigroup with identity, then quasigroup ring RQ is SNA-quasigroup ring.*

Proof. Quasigroup with identity is a Loop. So, $RI \subseteq RQ$ and R serves as the associative ring in RQ. Thus RQ is an SNA-quasigroup ring.

Theorem 3.3. *Let R be a ring. If Q is a Smarandache quasigroup, then quasigroup ring RQ is an SNA-quasigroup ring. .*

Proof. Obviously RQ is a non-associative ring. As Q is a Smarandache quasigroup Q contains a group G properly. So $RQ \supset RG$ and RG is an associative ring contained in RQ. Therefore RQ is an SNA-quasigroup ring.

§4. Substructure of SNA-quasigroup rings

Definition 4.1. Let R be a SNA-quasigroup ring. Let S be a non-empty subset of R. Then S is said to be S-quasigroup subring of R if S itself is a ring and contains a proper subset P such that P is an associative ring under the operation of R.

Example 4.1. Let Z be the ring of integers. Let Q be the quasigroup defined by the following table:

$*$	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Clearly the quasigroup ring ZQ is a non-associative ring. Consider the subset $S = \{1, 2, 3, 4\}$ then S is a group and hence ZS is a group ring and hence also a quasigroup ring. Let $P = \{1, 2\}$. Note that ZS also contains ZP where $P = \{1, 2\}$. So, ZS is an S-quasigroup subring of SNA-quasigroup ring ZQ.

We have not yet been able to find a Smarandache non associative quasigroup subring for a given quasigroup ring. We think that it is not possible to obtain a subquasigroup for any quasigroup because for a quasigroup its composition table is a LATIN SQUARE.

Theorem 4.1. *Let R be a quasigroup ring, if R has a SNA-quasigroup subring S, then R itself is SNA-quasigroup ring.*

Proof. As S is an SNA-quasigroup surbring S contains an associative ring. As a result R contains an associtive ring. Thus R is an SNA-quasigroup ring.

References

[1] R.H.Bruck, A survey of binary systems, Springer Verlag, 1958.

[2] W.B.Kandasamy, Smarandache non-associative (SNA) rings, Smarandache Notions (book series), American Research Press, **14**(2004), 281-293.

[3] D.S. Passman, The algebraic structure of group rings, Wiley- interscience, 1977.

[4] J.S.Robinson Derek, A course in the theory of Groups, Springer Verlag, 1996.

Scientia Magna
Vol. 1 (2005), No. 2, 145-148

Two asymptotic formulae on the $k+1$-power free numbers

Shen Hong

Xian Yang vocational and technical college
Xian Yang, Shaanxi, P.R.China

Abstract The main purpose of this paper is to study the distributive properties of $k+1$-power free numbers, and give two interesting asymptotic formulae.
Keywords $k+1$-power free numbers; Asymptotic formula.

§1. Introduction

A natural number n is called a $k+1$-power free number if it can not be divided by any p^{k+1}, where p is a prime number. One can obtain all $k+1$-power free numbers by the following method:

From the set of natural numbers (except 0 and 1)

-take off all multiples of 2^{k+1}(i.e. 2^{k+1}, 2^{k+2}, $\cdots$).

-take off all multiples of 3^{k+1}.

-take off all multiples of 5^{k+1}.

$\cdots$ and so on (take off all multiples of all $k+1$-power primes).

In reference [1], Professor F. Smarandache asked us to study the properties of the $k+1$-power free numbers sequence. Yet we still know very little about it.

Now we define two new number-theoretic functions $U(n)$ and $V(n)$ as following,

$$U(1)=1, \quad U(n)=\prod_{p|n} p,$$

$$V(1)=1, \quad V(n)=V\left(p_1^{\alpha_1}\right)\cdots U\left(p_r^{\alpha_r}\right)=\left(p^{\alpha_1}-1\right)\cdots\left(p^{\alpha_r}-1\right),$$

where n is any natural number with the form $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$. Obviously they are both multiplicative functions. In this paper, we shall use the analytic method to study the distribution properties of this sequence, and obtain two interesting asymptotic formulae. That is, we have the following two theorems:

Theorem 1. *Let $\mathcal{A}$ denote the set of all $k+1$-power free numbers, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} U(n)=\frac{3x^2}{\pi^2}\prod_p\left(1+\frac{p^{2k-2}-1}{p^{2k+1}+p^{2k}-p^{2k-1}-p^{2k-2}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number and $\prod_p$ denotes the product of all the prime numbers.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} V(n) = \frac{x^2}{2}\prod_p\left(1-\frac{1}{p^{k+1}}-\frac{p^{2k+1}+p^{2k}-p-1}{p^{2k+3}+p^{2k+1}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

§2. Proof of Theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1, let

$$f(s) = 1+\sum_{\substack{n\in\mathcal{A}\\ n\leq x}}\frac{U(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $U(n)$, we may have

$$\begin{aligned} f(s) &= \prod_p\left(1+\frac{U(p)}{p^s}+\frac{U(p^2)}{p^{2s}}+\cdots+\frac{U(p^k)}{p^{ks}}\right)\\ &= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2s-1}}+\cdots+\frac{1}{p^{ks-1}}\right)\\ &= \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{p^{(k-1)s}-1}{p^{2s-1}(p^{(k-1)s}-p^{(k-2)s})}\right)\\ &= \frac{\zeta(s-1)}{\zeta(2(s-1))}\prod_p\left(1+\frac{p^{(k-1)s}-1}{(p^{2s-1}+p^s)(p^{(k-1)s}-p^{(k-2)s})}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. Obviously, we have the following two inequalities

$$|U(n)|\leq n, \qquad \left|\sum_{n=1}^{\infty}\frac{U(n)}{n^\sigma}\right| < \frac{1}{\sigma-2},$$

where $\sigma > 2$ is the real part of s. So by Perron formula [3]

$$\begin{aligned} \sum_{n\leq x}\frac{U(n)}{n^{s_0}} &= \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds+O\left(\frac{x^bB(b+\sigma_0)}{T}\right)\\ &\quad +O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right)+O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right), \end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x-N|$. Taking $s_0 = 0$, $b = 3$, $T = x^{\frac{3}{2}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma-2}$, we have

$$\sum_{n\leq x}U(n) = \frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s-1)}{\zeta(2(s-1))}R(s)\frac{x^s}{s}ds+O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s) = \prod_p\left(1+\frac{p^{2k-2}-1}{p^{2k+1}+p^{2k}-p^{2k-1}-p^{2k-2}}\right).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s-1)}{\zeta(2(s-1))}R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s=3\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function

$$f(s)=\frac{\zeta(s-1)x^s}{\zeta(2(s-1))s}R(s)$$

has a simple pole point at $s=2$ with residue $\frac{x^2}{2\zeta(2)}R(2)$. So we have

$$\begin{aligned}&\frac{1}{2i\pi}\left(\int_{3-iT}^{3+iT}+\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(s-1)x^s}{\zeta(2(s-1))s}R(s)ds\\&=\frac{x^2}{2\zeta(2)}\prod_p\left(1+\frac{p^{2k-2}-1}{p^{2k+1}+p^{2k}-p^{2k-1}-p^{2k-2}}\right).\end{aligned}$$

We can easily get the estimates

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(s-1)x^s}{\zeta(2(s-1))s}R(s)ds\right|\\&\ll\int_{\frac{3}{2}}^{3}\left|\frac{\zeta(\sigma-1+iT)}{\zeta(2(\sigma-1+iT))}R(s)\frac{x^3}{T}\right|d\sigma\ll\frac{x^3}{T}=x^{\frac{3}{2}}\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\frac{\zeta(s-1)x^s}{\zeta(2(s-2))s}R(s)ds\right|\ll\int_0^T\left|\frac{\zeta(1/2+it)}{\zeta(1+2it)}\frac{x^{\frac{3}{2}}}{t}\right|dt\ll x^{\frac{3}{2}+\varepsilon}.$$

Note the fact that $\zeta(2)=\frac{\pi^2}{6}$, then from the above we can obtain

$$\sum_{\substack{n\in\mathcal{A}\\n\le x}}U(n)=\frac{3x^2}{\pi^2}\prod_p\left(1+\frac{p^{2k-2}-1}{p^{2k+1}+p^{2k}-p^{2k-1}-p^{2k-2}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let

$$g(s)=1+\sum_{\substack{n\in\mathcal{A}\\n\le x}}\frac{V(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $V(n)$, we also have

$$\begin{aligned}g(s)&=\prod_p\left(1+\frac{V(p)}{p^s}+\frac{V(p^2)}{p^{2s}}+\cdots+\frac{V(p^k)}{p^{ks}}\right)\\&=\prod_p\left(1+\frac{p-1}{p^s}+\frac{p^2-1}{p^{2s}}+\cdots+\frac{p^k-1}{p^{ks}}\right)\\&=\prod_p\left(\frac{1-\frac{1}{p^{(k+1)(s-1)}}}{1-\frac{1}{p^{s-1}}}-\frac{1-\frac{1}{p^{ks}}}{p^s-1}\right)\\&=\zeta(s-1)\prod_p\left(1-\frac{1}{p^{(k+1)(s-1)}}-\frac{(p^{ks}-1)(p^{s-1}+1)}{(p^{ks}-p^{(k-1)s})p^{2s-1}}\right).\end{aligned}$$

Now applying Perron formula [3], and the method of proving Theorem 1, we can also obtain the result of Theorem 2.

This completes the proof of Theorems.

References

[1]F.Smarandache, Only problems, Not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Tom M.Apostol, Introduction to analytic number theory, New York, Springer-Verlag, 1976.

[3] Pan Chengdong and Pan Chengbiao, Foundation of analytic number theory, Beijing, Science Press, 1997.

Scientia Magna
Vol. 1 (2005), No. 2, 149-151

An equation involving the Smarandache ceil function

Ji Yongqiang

Danfeng Teacher's School, Shangluo, Shaanxi, P.R.China

Abstract In this paper, we use the elementary methods to study the properties of the infinity series $\sum_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}$, where $\mathcal{A}$ denotes the set of all positive integers n such that the equation $S_k(n)=a_k(n)$, and give an interesting identity for it.

Keywords Smarandache ceil function; Infinity series; Identity.

§1. Introduction

For any fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n)=\min\{m\in N: n|m^k\}.$$

This was introduced by Professor F.Smarandache. About this function, many scholar had studied its properties, see [1] and [2]. In [1], Ibstedt presented the following properties:

$$(\forall a,b\in N)\ (a,b)=1\Rightarrow S_k(a\cdot b)=S_k(a)\cdot S_k(b),$$

and $S_k(p^\alpha)=p^{\lceil\frac{\alpha}{k}\rceil}$, where p is a prime and $\lceil x\rceil$ denotes the smallest integer greater than x. That is, $S_k(n)$ is a multiplicative function. Therefore, if $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ is the prime power decomposition of n, then the following identity is obviously:

$$S_k(n)=S_k(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r})=p_1^{\lceil\frac{\alpha_1}{k}\rceil}p_2^{\lceil\frac{\alpha_2}{k}\rceil}\cdots p_r^{\lceil\frac{\alpha_r}{k}\rceil}.$$

For any positive integer n, the Smarandache $k-$th power complements $a_k(n)$ is the smallest positive integer such that $na_k(n)$ is a complete $k-$th power. That is,

$$a_k(n)=\min\{l\mid n\cdot l=m^k, l\geq 0, m\in N^+\}.$$

If $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denotes the factorization n into prime powers, then from the definition of $a_k(n)$, we know that it is also a multiplicative function, therefore

$$a_k(n)=a_k(p_1^{\alpha_1})a_k(p_2^{\alpha_2})\cdots a_k(p_r^{\alpha_r}).$$

Let $\mathcal{A}$ denotes the set of the positive integers n such that the equation $S_k(n)=a_k(n)$. That is, $\mathcal{A}=\{n\in N, S_k(n)=a_k(n)\}$. In this paper, we use the elementary methods to study

the properties of the infinity series $\sum\limits_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}$, and give an interesting identity for it. That is, we shall prove the following conclusion:

Theorem. *Let $k \geq 2$ be a given positive integer. Then for any real number $s > 1$, we have the identity:*

$$\sum_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}=\frac{\zeta\left((k-1)s\right)}{\zeta\left((k^2-1)s\right)},$$

where $\zeta(s)$ is the Riemann zeta-function.

Corollary 1. *Taking $k = 2$ and $s = 2$ in the above theorem, then we have the identities:*

$$\sum_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n^2}=\frac{315}{2\pi^4}.$$

Corollary 2. *Taking $k = 3$ and $s = 1$ in the above theorem, then we have the identities:*

$$\sum_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n}=\frac{1575}{\pi^6}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First, we define the arithmetical function $b(n)$ as follows:

$$b(n)=\begin{cases}1, & if \quad n\in\mathcal{A},\\ 0, & otherwise.\end{cases}$$

Now let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization n into prime powers. Then from the introduction of this paper we know that $S_k(n)$ and $a_k(n)$ both are multiplicative functions, so we only discuss the case of $n = p^i$.

If $i = lk + n, (l \geq 0, 0 \leq n < k)$, then from the definition of $S_k(n)$ and $a_k(n)$, we can easily get:

$$S_k(p^i)=p^{\lceil\frac{i}{k}\rceil}=\begin{cases}p^l, & if \quad n=0,\\ p^{l+1}, & if \quad 0<n<k.\end{cases}$$

$$a_k(p^i)=\begin{cases}1, & if \quad n=0,\\ p^{k-n}, & if \quad 0<n<k.\end{cases}$$

So $S_k(p^i) = a_k(p^i)$, if and only if $l+1 = k-n$, i.e. $n = k-l-1$, or $i = lk+n = lk+k-l-1 = (k-1)(l+1)$. Hence, $p^i = p^{(k-1)(l+1)}$, where $0 \leq l \leq k-1$.

For any real number $s > 1$, it is clear that

$$\sum_{\substack{n=1\\n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}<\sum_{n=1}^{\infty}\frac{1}{n^s},$$

and $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is convergent if $s>1$. Thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}$ is also convergent if $s>1$, so from the Euler product formula (see [3]), we have

$$\begin{aligned}\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\frac{b(n)}{n^s} &= \prod_p\left(1+\frac{b(p)}{p^s}+\frac{b(p^2)}{p^{2s}}+\cdots\right)\\ &= \prod_p\left(1+\sum_{l=0}^{k-1}\frac{b(p^{(k-1)(l+1)})}{p^{(k-1)(l+1)s}}\right)\\ &= \prod_p\left(1+\sum_{l=0}^{k-1}\frac{1}{p^{(k-1)(l+1)s}}\right)\\ &= \prod_p\left(1+\frac{1}{p^{(k-1)s}}\sum_{l=0}^{k-1}\frac{1}{p^{(k-1)ls}}\right)\\ &= \prod_p\left(1+\frac{1-\frac{1}{p^{k(k-1)s}}}{p^{(k-1)s}-1}\right)\\ &= \frac{\zeta\left((k-1)s\right)}{\zeta\left((k^2-1)s\right)},\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

This completes the proof of Theorem.

Note that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(6)=\frac{\pi^6}{945}$ and $\zeta(8)=\frac{\pi^8}{9450}$, we may immediately deduce the corollaries.

References

[1] Ibstedt Surfinig, On the ocean of number-A few Smarandache notions and similar topics, Erthus University Press, New Mexico, USA.

[2] Sabin Tabirca and Tatiana Tabirca, Some new results concerning the Smarandache ceil function, Smarandache notions journal, **13**(2002), 30-36.

[3] Tom M.Apstol, Introduction to analytic number theory, New York, Springer-Verlag, 1976.

Scientia Magna
Vol. 1 (2005), No. 2, 152-158

Smarandache Bcc-algebras

Young Bar Jun

Department of Mathematics Education, Gyeongsang National University, Chinju 660 − 701, Korea
E-mail address: ybjun@gsnu.ac.kr
jamjana@korea.com

Abstract The notion of Smarandache BCC-algebras and Smarandache BCC-ideas is introduced. Conditions for a (special) subset to be a Smarandache BCC-ideal are given.
Keywords Smarandache BCC-algebra; Smarandache BCC-ideal.

§1. Introduction

Generally, in any human field, a Smarandache Structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [10], W.B.Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoide, semi-normal subgroupoides, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results of congruences, and it was studied by R.Padilla [9]. In this paper, we discuss the Smarandache structure on BCC-algebras, and introduce the notion of Smarandache ideas, and investigate its properties. We give conditions for a (special) subset to be a Smarandache BCC-ideal.

§2. Preliminaries

BCC-algebras were introduced by Komori [7] in a connection with some problems on BCK-algebras in[11], and Dudek[1, 2] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCC-algebra if it satisfies the following conditions.

(a1)$(\forall x, y, z \in X)$ $(((x * y) * (z * y)) * (x * z))$,
(a1)$(\forall x \in X)$ $(0 * x = 0)$,
(a1)$(\forall x \in X)$ $(x * 0 = 0)$,
(a1)$(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Longrightarrow x = y)$.

Note that every BCK-algebra is BCC-algebra, but the converse is not true. A BCC-algebra which is not a BCK-algebra is called a proper BCC-algebra. The smallest proper BCC-algebra has four elements and for every $n \geq 4$ there exists at least one proper BCC-algebra [2].

A nonempty subset I of a BCC-algebra X is called a BCC-ideal of X if it satisfies the following assertions:

(a5) $0 \in I$,

(a6) $(\forall x, z \in X)\ (\forall y \in I)\ ((x * y) * z \in I \Longrightarrow x * z \in I)$.

Note that every BCC-algebra X satisfies the following assertions.

(b1) $(\forall x \in X)\ (x * x = 0)$,

(b2) $(\forall x, y \in X)\ (x * y \leq x)$,

(b3) $(\forall x, y, z \in X)\ (x * y \Longrightarrow x \leq z \leq y * z, z * y \leq z * x)$,
where $x \leq y$ if and only if $x \leq y = 0$.

§3. Smarandache BCC-algebra

We know that every proper BCC-algebra has at least four elements (see [2]), and that if X is a BCC-algebra then $\{0, a\}, a \in X$, is a BCC-algebra with respect to the same operation on X. Now let us consider a proper BCC-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table :

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	1	2
3	3	3	1	0	3
4	4	0	0	0	0

Table 3.1

Then $\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}$, {0,1,2 } and $\{0, 1, 3\}$ are BCC-algebras with respect to the operation $*$ on X, and note that X does not contain BCC-algebras of order 4. Based on this result, we give the following definition.

Definition 3.1. A Smarandache BCC-algebra is defined to be a BCC-algebra algebra X in which there exists a proper subset Q of X such that

(¡) $0 \in Q$, and $|Q| \geq 4$,

(¡¡) Q is a BCC-algebra with respect to the operation on X.

Note that any proper BCC-algebra X with four elements can be Smarandache. Hence if X is a Smarandache BCC-algebra, then $|x| \geq 5$. Notice that the BCC-algebra $X = \{0, 1, 2, 3, 4\}$ with Table 3.1 is not a Smarandache BCC-algebra.

Example 3.2. (1)Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Table 3.2

Then$(X; *, 0)$ is a Smarandache BCC-algebra. Note that $Q = \{0, a, b, c\}$ is a BCK-algebra which is contained in X.

(2)Let $\{X; *, 0\}$ be a finite BCK-chain containing at least four elements and let c be its maximal element. Let $Y = X \cup \{d\}$, where $d \notin X$, and define a binary operation $\odot$ on Y as follows:

$$x \odot y = \begin{cases} x * y, & \text{if } x, y \in X; \\ 0, & \text{if } x \in X, y = d; \\ d, & \text{if } x = d, y = 0; \\ c, & \text{if } x = d, y \in X. \end{cases}$$

Then $(Y; \odot, 0)$ is a Smarandache BCC-algebra.

(3)Let $\{X; *, 0\}$ be a BCC-algebra containing at least four elements in which a is the small atom. Let $Y = X \cup \{\omega\}$, where $\omega \notin X$,and define a binary operation $\odot$ on Y as follows:

$$x \odot y = \begin{cases} x * y, & \text{if } x, y \in X; \\ 0, & \text{if } y \in X, x = \omega; \\ d, & \text{if } x = \omega = y; \\ c, & \text{if } x \in X/\{0\}, y = \omega. \end{cases}$$

Then $(Y; \odot, 0)$ is a Smarandache BCC-algebra.

In what follows , let X and Q denote a Smarandache BCC-algebra and non-trivial BCC-algebra which is properly contained in X , respectively, unless otherwise specified.

Definition 3.3. A nonempty subset I of X is called a Smarandache BCC-ideal of X related to Q if it satisfies:

(c1) $0 \in I$,

(c2) $(\forall x, z \in Q)\ (\forall y \in I)\ ((x * y) * z \in I \Longrightarrow x * z \in I)$.

If I is a Smarandache BCC-ideal of X related to every non-trivial BCC-algebra contained in X, we simply say that I is a Smarandache BCC-ideal of X.

Example 3.4. (1)Let $X = \{0, a, b, c, d, e\}$ be the Smarandache BCC-algebra described in Example 3.2(1). Then $I = \{0, a\}$ and $J = \{0, a, b, c, d\}$ are Smarandache BCC-ideals of X related to $Q = \{0, a, b, c\}$.

Proposition 3.5. Every Smarandache BCC-ideals of X related to Q satisfies:

(c3) $(\forall x \in Q)\ (\forall a \in I)\ (x * a \in I \Longrightarrow x \in I)$,

(c4) $(\forall x \in Q)\ (\forall a \in I)\ (a * x \in I)$,

(c5) $(\forall x \in Q)\ (\forall a, b \in I)\ (x * ((x * a) * b \in I))$.

Proof. (c3) Taking $z = 0$ and $y = a$ in $(c2)$ and using $(a3)$ induces the desired implication.

(c4) For every $x \in Q$ and $a \in I$, we have $(a * a) * x = 0 * x = 0 \in I$ and so $a * x \in I$ by $(c2)$.

(c5) Let $x \in Q$ and $a, b \in I$. Then $(x * a) * (a * x) = 0 \in I$, and so $x * (x * a) \in I$ by $(c2)$. Since

$$((x * b) * ((x * a) * b)) * (x * (x * a)) = 0 \in I,$$

it follows from $(c3)$ that $(x * b) * ((x * a) * b) \in I$ so from $(c2)$ that $x * ((x * a) * b) \in I$.

Corollary 3.6. For every Smarandache BCC-ideal I of X related to Q, the following implication is valid:

$$(\forall x \in Q)(\forall a \in I)(x \leq a \Longrightarrow x \in I).$$

Corollary 3.7. Let I be a Smarandache BCC-ideal I of X related to Q . Then

$$(\forall x \in Q)(\forall a, b \in I)(x * a \leq b \Longrightarrow x \in I).$$

Theorem 3.8. Let Q_1 and Q_2 be non-trivial BCK-algebras which are properly contained in X such that $Q_1 \subset Q_2$. Then every Smarandache BCC-ideal of X related to Q_2 is a Smarandache BCC-ideal of X related to Q_1.

Proof. Straightforward.

Corollary 3.9. If Q is the largest BCK-algebra which is properly contained in X, then every Smarandache BCC-ideal of X related to Q is a Smarandache BCC-ideal of X.

The converse of Theorem 3.8 is not true in general as seen in the following example.

Example 3.10. Consider a Smarandache BCC-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	1	0	0	0	1
3	3	1	1	0	1	1
4	4	1	1	1	0	1
5	5	5	5	5	5	0

Table 3.3

Note that $Q_1 := \{0, 1, 2, 3\}$ and $Q_2 := \{0, 1, 2, 3, 4\}$ are BCK-algebras. Then the set Q_1 is a Smarandache BCC-ideal of X related to Q_1, but not Q_2. In fact, we know that $(4*2)*0 = 1 \in Q_1$ and $4 * 0 = 4Q_1$.

Remark 3.11. Note that every BCC-ideal of X is a Smarandache BCC-ideal of X, but the converse is not valid. Example 3.10 shows that there exists a BCC-algebra Q of order

$n \geq 4$ which is properly contained in a Smarandache BCC-algebra X such that a Smarandache BCC-ideal of X related to Q is not a BCC-ideal of X.

We provide convide conditions for a subset to be a Smarandache BCC-ideal.

Theorem 3.12. If I is a subset of Q that satisfies conditions $(c1)$ and $(c3)$, then I is a Smarandache BCC-ideal of X related to Q.

Proof. Let $x, y \in Q$ and $a \in I$ be such that $(x * a) * y \in I$. Since $a \in I \subseteq Q$ and Q is a BCK-algebra, it follows that $(x * y) * a = (x * a) * y \in I$. So from $(c3)$ that $x * y \in I$. Hence I is a Smarandache BCC-ideal of X related to Q.

Theorem 3.13. If a nonempty subset I of X satisfies conditions $(c1)$ and $(c5)$, then I is a Smarandache BCC-ideal of X related to Q.

Proof. Let $x, y \in Q$ and $a \in I$ be such that $(x * a) * y \in I$. Taking $b = 0$ in $(c5)$ and using $(a3)$, we have $(x * x) * a \in I$. It follows from $(a3), (a1)$ and $(c5)$ that

$$x * y = (x * y) * 0 = (x * y) * (((x * y) * ((x * a) * y)) * (x * (x * a))) \in I.$$

Thus I is a Smarandache BCC-ideal of X related to Q.

Theorem 3.14. Let H be a BCC-subalgebra of X. Then H is a Smarandache BCC-ideal of X related to Q if and only if it satisfies:

$(\forall x \in H)(\forall y, z \in Q)((y * x) * z \in H \Longrightarrow y * z \in H)$.

Proof. Straightforward.

Given an element $\omega \in X \backslash \{0\}$, consider the set

$$[0, \omega] := \{x \in X \| x \leq \omega\}.$$

which is called the initial segment of ω[5]. Obviously, $0 \in [0, \omega]$ for all $x \in \omega$. Since $x \leq \omega$ is equivalent to $x\omega = 0$, the initial segment of ω is defacto the left annihilator of ω. In general, $[0, \omega]$ is not a Smarandache BCC-ideal of X, but it is a subalgebra. For example, let X be the Smarandache BCC-algebra in example 3.2(1). Then $[0, e] = \{0, e\}$ is not a Smarandache BCC-ideal of X related to $Q = \{0, a, b, c\}$ since $(b * e) * d = 0 \in [0, e]$, but $b * d = a[0, e]$.

Theorem 3.15. For every $c \in X \backslash \{0\}$, if the inequality

$$(\forall x \in Q)(x * ((x * c) * c) \leq c$$

holds, then $[0, c]$ is a Smarandache BCC-ideal of X related to Q.

Proof. Let $x \in Q$. If $b \in [0, c]$, then $b \leq c$ and hence $(x * c) * c \leq (x * c) * b$ by $(b3)$. It follows from $(b3)$ and assumption that

$$x * ((x * c) * b) \leq x * ((x * c) * c) \leq c.$$

Now if $a \in [0, c]$, then $x * c \leq x * a$, and so

$$x * ((x * a) * b) \leq x * ((x * c) * b) \leq c.$$

This shows that $x * ((x * a) * b) \in [0, c]$. Applying Theorem 3.13, we conclude that $[0, c]$ is a Smarandache BCC-ideal of X related to Q.

Theorem 3.16. The initial segment $[0, c]$, where $c \in X/\{0\}$, is a Smarandache BCC-ideal of X related to Q if and only if the implication

$$(\forall x, y \in Q)((x * c) * y \leq c \Longrightarrow x * z \in I)$$

is valid.

Proof. Let $\forall x, y \in Q$ and $a \in [0, c]$ be such that $(x * a) * y \in [0, c]$. Then $a \leq c$ and $(x * a) * y \leq c$. The inequality $z \leq c$ implies that $(x * c) * y \leq (x * a) * y \leq c$ so from hypothesis that $x * y \leq c$, that is, $x * y \in [0, c]$ is a Smarandache BCC-ideal of X related to Q. Conversely assume that $[0, c], c \in X\backslash\{0\}$, is a Smarandache BCC-ideal of X related to Q and let $\forall x, y \in Q$ be such that $(x * c) * y \leq c$. Then $(x * c) * y \in [0, c]$. Since $[0, c]$ is a Smarandache BCC-ideal of X related to Q and $c \in [0, c]$, it follows from $(c2)$ that $x * y \in [0, c]$ so that $x * y \leq c$. This completes the proof.

Corollary 3.17. If $[0, c], c \in X/\{0\}$, is a Smarandache BCC-ideal of X related to Q, then

$$(\forall x, y \in Q)((x * c) \leq c \Longrightarrow x \leq c).$$

Theorem 3.18. For every $c \in X/\{0\}$, if the equality

$$(\forall x, y \in Q)(((x * c) * y) * c = (x * y) * c)$$

is valid, then $[0, c]$ is a Smarandache BCC-ideal of X related to Q.

Proof. Let $\forall x, y \in Q$ and $a \in [0, c]$ be such that $(x * a) * y \in [0, c]$. Then $a \leq c$ and $(x * a) * y \leq c$. It follows that

$$(x * y) * c = ((x * c) * y) * c \leq ((x * a) * y) * c \leq c * c = 0,$$

so that $(x * y) * c = 0$, i.e., $x * y \leq c$. Hence $x * y \in [0, c]$ and therefore $[0, c]$ is a Smarandache BCC-ideal of X related to Q.

Acknowledgements

This paper was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).

References

[1] W.A.Dudek, The number of subalgebras of finite BCC-algebras, Bull.Inst.Math, Academia Sinica, **20**(1992), 129-136.

[2] W.A.Dudek, On proper BCC-algebras, Bull.Inst.Math, Academia Sinica **20**(1992), 137-150.

[3] W.A.Dudek, On constructions of BCC-algebras,selected Papers on BCK and BCI-algebras, Xi'an, China, Shaanxi Scientific and Technological Press, **1**(1992), 93-96.

[4] W.A.Dudek and X.H.Zhang, On ideals and congruences in BCC-algebras, Coravica, **4**(1998), 21-29.

[5] W.A.Dudek and X.H.Zhang, Initial segments in BCC-algebras, Math.Moreavica, **4**(2000), 27-34.

[6] J.Hao, Ideal theory of BCC-algebras, Sci.Math, **3**(1998), 378-381.

[7] Y.Komori, The class of CC-algebras is not a variety, Math. Japon, **29**(1984), 391-394.

[8] J.Meng and Y.B.Jun, BCC-algebras, Kyungmoonsa Ca.Seoul, Korea, (1994).

[9] R.Padilla, Smarandache algebraic structures, Bull Pure Appl.Sci.Delhi, **17E**(1998), 119-121; http://www.gallup.unm.edu/smarandache/alg-s-tx.txt.

[10] W.B.Vasantha Kandasamy, Smarandache groupoids,
http://www.gallup.unm.edu/smarandache/Groupoids.pdf.

[11] A.Wronski, BCK-algebras do not form a variety, Math. Japon, **28**(1983), 211-213.

Scientia Magna
Vol. 1 (2005), No. 2, 159-161

On the asymptotic property for Smarandache additive factorial complements

Yang Mingshun and Yang Qianli

Department of Mathematics, Weinan Teacher's College
Weinan, Shaanxi, P.R.China

Abstract The main purpose of this paper is to study the mean value properties of the Smarandache additive factorial complements, and give an interesting asymptotic formula for it.
Keywords Smarandache additive factorial complements; Asymptotic formula.

§1. Introduction

For any positive integer n, the Smarandache factorial complements quotients of n, denoted by $a(n)$, is defined as follows

$$c(n) = \min\{k|nk = m!, k \geq 0, m \in N^+\}.$$

In problem 45 of reference [1], professor F.Smarandache asked us to study the properties of the factorial complements. Analogously, we can define the Smarandache additive factorial complements $a(n)$:

$$a(n) = \min\{k|n + k = m!, k \geq 0, m \in N^+\}.$$

About arithmetical properties of the this sequence, it seems that none had studied it before. In this paper, we use the elementary method to study the mean value properties of the Smarandache additive factorial complements, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \frac{1}{a(n)+1} = \frac{\ln^2 x}{2 \ln \ln x} + O\left(\frac{\ln^2 x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. For any real number x, let positive integer m satisfy

$$M! \leq x < (M+1)!. \tag{1}$$

Taking the logistic computation on both sides of the inequality, we get

$$\sum_{i=1}^{M} \ln i \le \ln x < \sum_{i=1}^{M+1} \ln i.$$

And then using the Euler's summation formula we have

$$\sum_{i=1}^{M} \ln i = M \ln M - M + O(\ln M) \tag{2}$$

and

$$\sum_{i=1}^{M+1} \ln i = M \ln M - M + O(\ln M). \tag{3}$$

Combining (2) and (3), we can easily deduce that

$$\ln x = M \ln M - M + O(\ln M).$$

So we get

$$M = \frac{\ln x}{\ln M - 1} + O(1).$$

Similarly, taking logistic computation on both sides we have

$$\ln M = \ln \ln x + O(\ln \ln M) \tag{4}$$

and

$$M = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right). \tag{5}$$

According to the definition of $a(n)$ and (1), we can write

$$\begin{aligned}
\sum_{n\le x} \frac{1}{a(n)+1} &= \sum_{1\le m\le M-1} \left(\sum_{m!\le n<(m+1)!} \frac{1}{a(n)+1} \right) + \sum_{M!\le n\le x} \frac{1}{a(n)+n} \\
&= \sum_{1\le m\le M-1} \sum_{i=1}^{m\cdot m!} \frac{1}{i} + \sum_{n\le x-M!+1} \frac{1}{n} \\
&= \sum_{1\le m\le M-1} \left(\ln(m\cdot m!) + \gamma - 1 + O\left(\frac{1}{m\cdot m!}\right)\right) + O\left(\ln(M\cdot M!)\right),
\end{aligned}$$

where γ is the Euler's constant. Now combining the Stirling Formula (see reference [2]), we can get

$$\begin{aligned}
\sum_{n\le x} \frac{1}{a(n)+1} &= \sum_{1\le m\le M-1} \ln m! + \sum_{1\le m\le M-1} \ln m + O(M \ln M) \\
&= \sum_{1\le m\le M-1} m \ln(m+1) - \sum_{1\le m\le M-1} m + O(M \ln M).
\end{aligned}$$

Applying Abel's identity (see reference [3]), we have

$$\sum_{n\le x} \frac{1}{a(n)+1} = \frac{1}{2} M^2 \ln M - \frac{3}{4} M^2 + O(M \ln M).$$

So from (4) and (5), we obtain

$$\sum_{n\leq x}\frac{1}{a(n)+1}=\frac{\ln^2 x}{2\ln\ln x}+O\left(\frac{\ln^2 x\ln\ln\ln x}{(\ln\ln x)^2}\right).$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only problems, not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Pan Chengdong and Pan Chengbiao, Foundation of analytic number theory, Beijing, Science publishing house, 1999, pp49.

[3] Tom M.Apostol, Introduction to analytic number theory, New York, Springer-Verlag, 1976, pp77.

Scientia Magna
Vol. 1 (2005), No. 2, 162-166

The Smarandache minimum and maximum functions

József Sándor

Babes-Bolyai University of Cluj, Romania

Abstract This papers deals with the introduction and preliminary study of the Smarandache minimum and maximum functions.

Keywords Smarandache minimum and maximum functions; arithmetical properties.

1. Let f: $N^* \longrightarrow N$ be a given arithmetic function and $A \subset N$ a given set. The arithmetic function

$$F_f^A(n) = \min\{k \in A : n \mid f(k)\} \tag{1}$$

has been introduced in [4] and [5].

For $A = N, f(k) = k!$ one obtains the Smarandache function; For $A = N^*, A = p = \{2, 3, 5, \cdot\}$ = set of all primes, one obtains a function

$$P(n) = \min\{k \in P : n \mid k!\} \tag{2}$$

For the properties of this function, see [4] and [5]. The "dual" function of (1) has been defined by

$$G_g^A(n) = \max\{k \in A : g(k) \mid n\}, \tag{3}$$

where $g : N^* \longrightarrow N$ is a given function, and $A \in N$ is a given set. Particularly, for$A = N^*, g(k) = k!$, one obtains the dual of the Smarandache function,

$$S_*(n) = \max\{k \geq 1 : k! \mid n\} \tag{4}$$

For the properties of this function, see [4] and [5]. F.Luca [3], K.Atanassov [1] and L.le [2] have proved in the affirmative a conjecture of the author.

For $A = N^*$ and $f(k) = g(k) = \varphi(k)$ in (1), resp.(3) one obtains the Euler minimum, resp. maximum-function, defined by

$$E(n) = \min\{k \geq 1 : n \mid \varphi(k)\}, \tag{5}$$

$$E_*(n) = \max\{k \geq 1 : \varphi(k) \mid n\} \tag{6}$$

For the properties of these function, see [6]. When $A = N^*, f(k) = d(k)$ =number of divisors of k, one obtains the divisor minimum function (see [4], [5] and [7])

$$D(n) = \min\{k \geq 1 : n \mid d(k)\}. \tag{7}$$

It is interesting to note that the divisor maximum function (i.e., the " dual" of $D(n)$) given by

$$D_*(n) = \max\{k \geq 1 : d(k) \mid n\} \tag{8}$$

is not well defined! Indeed, for any prime p one has $d(p^{n-1}) = n \mid n$ and p^{n-1} is unbounded as $p \longrightarrow \infty$. For a finite set A, however $D_*^A(n)$ does exist. On one hand, it has been shown in [4] and [5] that

$$\sum(n) = \min\{k \geq 1 : n \mid \sigma(k)\} \tag{9}$$

(denoted there by $F_\sigma(n)$) is well defined. (Here $\sigma(k)$ denotes the sum of all divisors of k). The dual of the sum-of-divisors minimum function is

$$\sum{}_*(n) = \max(k \geq 1 : \sigma(k) \mid n\}) \tag{10}$$

Since $\sigma(1) = 1 \mid n$ and $\sigma(k) \geq k$, clearly $\sum_*(n) \leq n$, so this function is well defined (see [8]).

2. The Smarandache minimum function will be defined for $A = N^*, f(k) = S(k)$ in (1). Let us denote this function by $S_{\min}$:

$$S_{\min}(n) = \min\{k \geq 1 : n \mid S(k)\} \tag{11}$$

Let us assume that $S(1) = 1$, i. e., $S(n)$ is defined by (1) for $A = N^*, f(k) = k!$:

$$S(n) = \min\{k \geq 1 : n \mid k!\} \tag{12}$$

Otherwise (i.e.when $S(1) = 0$) by $n \mid 0$ for all n, by (11) for one gets the trivial function $S_{min}(n) = 0$. By this assumption, however, one obtains a very interesting (and difficult) function s_{min} given by (11). Since $n \mid S(n!) = n$, this function is correctly defined.

The Smarandache maximum function will be defined as the dual of S_{min} :

$$S_{\max}(n) = \max(k \geq 1 : S(k) \mid n\}. \tag{13}$$

We prove that this is well defined. Indeed, for a fixed n, there are a finite number of divisors of n, let $i \mid n$ be one of them. The equation

$$S(k) = i \tag{14}$$

is well-known to have a number of $d(i!) - d((i-1)!)$ solutions, i. e., in a finite number. This implies that for a given n there are at most finitely many k with $S(k) \mid k$, so the maximum in (13) is attained.

Clearly $S_{\min}(1) = 1, S_{\min}(2) = 2, S_{\min}(3) = 3, S_{\min}(4) = 4, S_{\min}(5) = 5, S_{\min}(6) = 9, S_{\min}(7) = 7, S_{\min}(8) = 32, S_{\min}(9) = 27, S_{\min}(10) = 25, S_{\min}(11) = 11$, etc, which can be determined from a table of Smarandache numbers:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
S(n)	1	2	3	4	5	3	7	4	6	5	11	4	13

n	14	15	16	17	18	19	20	21	22	23	24	25
S(n)	7	5	6	7	6	19	5	7	11	23	4	10

We first prove that:

Theorem 1. $S_{\min}(n) \geq n$ for all $n \geq 1$, with equality only for

$$n = 1, 4, p(p = \text{prime}) \tag{15}$$

Proof. Let $n \mid S(k)$. If we would have $k < n$, then since $S(k) \leq k < n$ we should get $S(k) < n$, in contradiction with $n \mid S(k)$. Thus $k \geq n$, and taking minimum, the inequality follows. There is equality for $n = 1$ and $n = 4$. Let now $n > 4$. If $n = p =$prime, then $p \mid S(p) = p$, but for $k < p, p \nmid S(k)$. Indeed, by $S(k) \leq k < p$ this is impossible. Reciprocally, if $min\{k \geq 1 : n \mid S(k)\} = n$, then $n \mid S(n)$, and by $S(n) \leq n$ this is possible only when $S(n) = n$, i. e., when $n = 1, 4, p(p = \text{prime})$.

Theorem 2. For all $n \geq 1$,

$$S_{\min}(n) \leq n! \leq S_{\max}(n) \tag{16}$$

Proof. Since S(n!)=n, definition (11) gives the left side of (16), while definition (13) gives the right side inequality.

Corollary. The series $\sum_{n\geq 1} \frac{1}{S_{\min}(n)}$ is divergent, while the series $\sum_{n\geq 1} \frac{1}{S_{\max}(n)}$ is convergent.

Proof. Since $\sum_{n\geq 1} \frac{1}{S_{\max}(n)} \leq \sum_{n\geq 1} \frac{1}{n!} = e - 1$ by (16), this series is convergent. On the other hand,

$$\sum_{n\geq 1} \frac{1}{S_{\min}(n)} \geq \sum_{p} \frac{1}{S_{min}(p)} = \sum_{p} \frac{1}{p} = +\infty,$$

so the first series is divergent.

Theorem 3. For all primes p one has

$$S_{\max}(p) = p! \tag{17}$$

Proof. Let $S(k) \mid p$. Then $S(k) = 1$ or $S(k) = p$. We prove that if $S(k) = p$, then $k \leq p!$. Indeed, this follows from the definition (12), since $S(k) = \min\{m \geq 1 : k \mid m!\} = p$ implies $k \mid p!$, so $k \leq p!$. Therefore the greatest value of k is $k = p!$, when $S(k) = p \mid p$. This proves relation (17).

Theorem 4. For all primes p,

$$S_{\min}(2p) \leq p^2 \leq S_{\max}(2p) \tag{18}$$

and more generally; for all $m \leq p$,

$$S_{\min}(mp) \leq p^m \leq S_{\max}(mp) \tag{19}$$

Proof. (19) follows by the known relation $S(p^m) = mp$ if $m \leq p$ and the definition (11), (13). Particularly, for $m = 2$, (19) reduces to (18). For $m = p$, (19) gives

$$S_{\min}(p^2) \leq p^p \leq S_{\max}(p^2) \tag{20}$$

This case when m is also an arbitrary prime is given in.

Theorem 5. For all odd primes p and $q, p < q$ one has

$$S_{\min}(pq) \leq q^p \leq p^q \leq S_{\max}(pq) \tag{21}$$

(21) holds also when $p = 2$ and $q \geq 5$.

Proof. Since $S(q^p) = pq$ and $S(p^q) = qp$ for primes p and q, the extreme inequalities of (21) follow from the definition (11) and (13). For the inequality $q^p < p^q$ remark that this is equivalent to $f(p) > f(q)$, where $f(x) = \frac{\ln x}{x} (x \geq 3)$.

Since $f'(x) = \frac{1-\ln x}{x^2} = 0 \Leftrightarrow x = e$ immediately follows that f is strictly decreasing for $x \geq e = 2.71\cdot$ From the graph of this function, since $\frac{\ln 2}{2} = \frac{\ln 4}{4}$ we get that

$$\frac{\ln 2}{2} < \frac{\ln 3}{3},$$

but

$$\frac{\ln 2}{2} > \frac{\ln q}{q}$$

for $q \geq 5$. Therefore (21) holds when $p = 2$ and $q \geq 5$. Indeed, $f(q) \leq f(5) < f(4) = f(2)$.

Remark. For all primes p, q

$$S_{\min}(pq) \leq \min\{p^q, q^p\} \tag{22}$$

and

$$S_{\max}(pq) \geq \max\{p^q, q^p\}. \tag{23}$$

For $p = q$ this implies relation (21).

Proof. Since $S(q^p) = S(p^q) = pq$, one has

$$S_{\min}(pq) \leq p^q, S_{\min}(pq) \leq q^p, S_{\max}(pq) \leq p^q, S_{\max}(pq) \leq q^p$$

.

References

[1] K.Atanassov, Remark on Jozsef Sandor and Florian Luca's theorem, C. R. Acad. Bulg. Sci., **55(10)**(2002), 9-14.

[2] M.Le, A conjecture concerning the Smarandache dual function, Smarandache Notions J., **14**(2004), 153-155.

[3] F.Luca, On a divisibility property involving factorials, C. R. Acad. Bulg. Sci., **53(6)**(2000), 35-38.

[4] J.Sandor, On certain generalizations of the Smarandache function, Notes Number Theorem Disci. Math., **5(2)**(1999),41-51.

[5] J.Sandor, On certain generalizations of the Smarandache function, Smarandache Notions Journal, **11**(2000), 202-212.

[6] J.Sandor, On the Euler minimum and maximum functions(to appear).

Scientia Magna
Vol. 1 (2005), No. 2, 167-172

Some properties of the Pseudo-Smarandache function

Richard Pinch

2 ELDON Road, Cheltenham, Glos GL52 6TU, U.K.
E-mail address: rgep@chalcedon.demon.co.uk

Abstract Charles Ashbacher [1] has posed a number of questions relating to the pseudo-smarandache function $Z(n)$. In this note we show that the ratio of consecutive values $Z(n+1)/Z(n)$ and $Z(n-1)/Z(n)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; that $n/Z(n)$ takes every integer value infinitely often; and that the $\sum_n 1/Z(n)^\alpha$ is convergent for any $\alpha > 1$.

§1. Introduction

We defined the m-th triangular number $T(m) = \frac{m(m+1)}{2}$. Kashihara [2] has defined the pseudo-Smarandache function $Z(n)$ by

$$Z(n) = min\{m : n|T(m)\}.$$

Charles Ashbacher [1] has posed a number of questions relating to pseudo-Smarandache function $Z(n)$. In this note, we show that the ratio of consecutive values $Z(n)/Z(n-1)$ and $Z(n)/Z(n+1)$ are unbounded; that $Z(2n)/Z(n)$ is unbounded; and that $n/Z(n)$ takes every integer value infinitely often. He notes that the series$\sum_n 1/Z(n)^\alpha$ is divergent for $\alpha = 1$ and asks whether it is convergent for $\alpha = 2$. He further suggests that the least value α for which the series converges " may never be known " . We resolve this problem by showing that the series converges for all $\alpha > 1$.

§2. Some Properties of t he Pseudo-Smarandache Function

We record some elementary properties of the funtion Z.

Lemma 1.(1) If $n \geq T(m)$, then $Z(n) \geq m, Z(T(m)) = m$.

(2)For all n we have $\sqrt{n} < Z(n)$.

(3)$Z(n) \leq 2n - 1$, and if n is odd, then $Z(n) \leq n - 1$.

(4)If p is an odd prime dividing n, then $Z(n) \geq p - 1$.

(5)$Z(2^k) = 2^{k+1} - 1$.

(6)If p is an odd prime, then $Z(p^k) = p^k - 1$ and $Z(2p^k) = p^k - 1$ or p^k according as $p^k \equiv 1$ or 3 mod 4.

We shall make use of Dirichlet's Theorem on primes in arithmetic progression in the following form.

Lemma 2. Let a, b be coprime integers. Then the arithmetic progression $a + bt$ is prime for infinitely many values of t.

§3. Successive Values of the Pseudo-Smarandache Function

Using the properties (3) and (5), Ashbacher observed that $|Z(2^k) - Z(2^k - 1)| > 2^k$ and so the difference between the consecutive of Z is unbounded. He asks about the ratio of consecutive values.

Theorem 1. For any given $L > 0$ there are infinitely many values of n such that $Z(n + 1)/Z(n) > L$, and there are infinitely many values of such that $Z(n - 1)/Z(n) > L$.

Proof. Choose $k \equiv 3 \bmod 4$, so that $T(k)$ is even and $(k+1)|(m+1)$. There are satisfied if $m \equiv k \bmod k(k + 1)$, that is , $m = k + k(k + 1)t$ for some t. We have $m(m + 1) = k(1 + (k + 1)t)(k + 1)(1 + kt)$, so that if $n = k(k + 1)(k + 1)(1 + kt)/2$, we have $n|T(m)$. Now consider $n + 1 = T(k) + 1 + kT(k)t$. We have $k|T(k)$, so $T(k) + 1$ is coprime to both k and $T(k)$. Thus the arithmetic progression $T(k) + 1 + kT(k)t$ has initial term coprime to its increment and by Dirichlet's Theorem contains infinitely many primes. We find that there are infinitely many values of t for which $n + 1$ is prime and so $Z(n) \leq m = k + k(k + 1)t$ and $Z(n + 1) = n = T(k)(1 + kt)$. Hence

$$\frac{Z(n+1)}{Z(n)} \geq \frac{n}{m} = \frac{T(k) + kT(k)t}{k + 2T(k)t} > \frac{k}{3}.$$

A similar argument holds if we consider the arithmetic progression $T(k) - 1 + kT(k)t$. We then find infinitely many values of t for which $n - 1$ is prime and

$$\frac{Z(n-1)}{Z(n)} \geq \frac{n-2}{m} = \frac{T(k) - 2 + kT(k)t}{k + 2T(k)t} > \frac{k}{4}.$$

The Theorem follows by taking $k > 4L$.

We note that this Theorem, combined with Lemma 1(2) , given another proof of the result that the differences of consecutive values is unbounded.

§4. Divisibility of the Pseudo-Smarandache Function

Theorem 2. For any integer $k \geq 2$, the equation $n/Z(n) = k$ has infinitely many solutions n.

Proof. Fix an integer $k \geq 2$. Let p be a prime $\equiv -1 \bmod 2k$ and put $p + 1 = 2kt$. Put $n = T(p)/t = p(p+1)/2t = pk$. Then $n|T(p)$ so that $Z(n) \leq p$. We have $p|n$, so $Z(n) \geq p - 1$; That is, $Z(n)$ must be either p or $p - 1$. Suppose, if possible, that it is the latter. In this case we have $2n|p(p+1)$ and $2n|(p-1)p$, so $2n$ divides $p(p+1) - (p-1) = 2p$; but this is impossible since $k > 1$ and so $n > p$. We conclude that $Z(n) = p$ and $n/Z(n) = k$ as required. Further, for any given value of k there are infinitely many prime values of p satisfying the congruence condition and infinitely many values of $n = Y(p)$ such that $Z/Z(n) = k$.

§5. Another Divisibility Question

Theorem 3. The ratio $Z(2n)/Z(n)$ is not bounded above.

Proof. Fix an integer k, let $p \equiv -1 \bmod 2^k$ be prime and put $n = T(p)$. Then $Z(n) = p$. Consider $Z(2n) = m$. We have $2^k p | p(p+1) = 2n$ and this divides $m(m+1)/2$. We have $m = \epsilon \bmod p$ and $m \equiv \delta \bmod 2^{k+1}$ where each of ϵ, δ can be either 0 or -1.

Let $m = pt + \epsilon$. Then $m \equiv \epsilon - t \equiv \delta \bmod 2^k$. This implies that either $t = 1$ or $t \geq 2^k - 1$. Now if $t = 1$ then $m \leq p$ and $T(m) \leq T(p) = n$, which is impossible since $2n \leq T(m)$. Hence $t \geq 2^k - 1$. Since $Z(2n)/Z(n) = m/p > t/2$, we see that the ratio $Z(2n)/Z(n)$ can be made as large as desired.

§6. Convergence of A Series

Ashbacher observes that the series $\sum_n 1/Z(n)^\alpha$ diverges for $\alpha = 1$ and asks whether it converges for $\alpha = 2$.

Lemma 3.

$$\log n \leq \sum_{m=1}^{n} 1/Z(n)^\alpha \leq 1 + \log n;$$

$$\frac{1}{2}(\log n)^2 - 0.257 \leq \sum_{m=1}^{n} \frac{\log m}{m} \leq \frac{1}{2}(\log n)^2 + 0.110,$$

for $n \geq 4$.

Proof. For the first part, we have $\frac{1}{m} \leq \frac{1}{t} \leq \frac{1}{m-1}$ for $t \in [m-1, m]$. Integrating,

$$\frac{1}{m} \leq \int_{m-1}^{m} \frac{1}{t} dt \leq \frac{1}{m-1}$$

Summing,

$$\sum_{2}^{n} \frac{1}{m} \leq \int_{1}^{n} \frac{1}{t} dt \leq \sum_{2}^{n} \frac{1}{m-1}$$

That is ,

$$\sum_{1}^{n} \frac{1}{m} \leq 1 + \log n$$

and

$$\log n \leq \sum_{1}^{n-1} \frac{1}{m}$$

The result follows.

For the second part, we similarly have $\log m/m \leq \log t/t \leq \log(m-1)/(m-1)$, for $t \in [m-1, m]$ when $m \geq 4$, since $\log x/x$ is monotonic decreasing for $x \geq e$.

Integrating,

$$\frac{\log m}{m} \leq \int_{m-1}^{m} \frac{\log t}{t} dt \leq \frac{m-1}{m}.$$

Summing,

$$\sum_4^n \frac{\log m}{m} \leq \int_3^n \frac{\log t}{t} dt \leq \sum_4^n \frac{m-1}{m}.$$

That is,

$$\sum_1^n \frac{\log m}{m} - \frac{\log 2}{2} - \frac{\log 3}{3}$$
$$\leq \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log 3)^2$$
$$\leq \sum_1^n \frac{\log m}{m} - \frac{\log n}{n} - \frac{\log 2}{2}.$$

We approximate the numerical values

$$\frac{\log 2}{2} + \frac{\log 3}{3} - \frac{1}{2}(\log 3)^2 < 0.110$$

and

$$\frac{\log 2}{2} - \frac{1}{2}(\log 3)^2 > -0.257$$

to obtain the result.

Lemma 4. Let $d(m)$ be the function which counts the divisors of m. For $n \geq 2$ we have

$$\sum_{m=1}^n d(m)/m < 7(\log n)^2.$$

Proof. We verify the assertion numerically for $n \leq 6$. Now assume that $n \geq 8 > e^2$, we have

$$\sum_{m=1}^n \frac{d(m)}{m} = \sum_{m=1}^n \sum_{de=m} \frac{1}{m} = \sum_{d\leq n} \sum_{de \leq n} \frac{1}{de}$$
$$= \sum_{d\leq n} \frac{1}{d} \sum_{e<n/d} \frac{1}{e} \leq \sum_{d\leq n} \frac{1}{d}(1 + \log(n/d))$$
$$\leq (1+\log n)^2 - \frac{1}{2}(\log n)^2 + 0.257$$
$$= 1.257 + 2\log n + \frac{1}{2}(\log n)^2$$
$$< \frac{4}{3}(\frac{\log n}{2})^2 + 2\log n(\frac{\log n}{2}) + \frac{1}{2}(\log n)^2$$
$$< 2(\log n)^2$$

Lemma 5. Fix an integer $t \geq 5$. Let $e^t > Y > e^{(t-1)/2}$. The number of integers n with $e^{t-1} > n > e^t$ such that $Z(n) \leq Y$ is at most $196Yt^2$.

Proof. Consider such an n with $m = Z(n) \leq Y$. Now $n|m(m+1)$, say $k_1 n_1 = m$ and $k_2 n_2 = m+1$, with $n = n_1 n_2$. Thus $k = k_1 k_2 = m(m+1)/n$ and $k_1 n_1 \leq Y$. The value

of k is bounded below by 2 and above by $m(m+1)/n \leq 2Y^2/e^{t-1} = K$, say. Given a pair (k_1, k_2), the possible values of n_1 are bounded above by Y/k_1 and must satisfy the congruence condition $k_1 n_1 + 1 \equiv 0$ modulo k_2: there are therefore at most $Y/k_1 k_2 + 1$ such values. Since $Y/k \geq Y/K = e^{t-1}/2Y > 1/2e$, we have $Y/k + 1 < (2e+1)Y/k < 7Y/k$. Given values for k_1, k_2 and n_1, the value of n_2 is fixed as $n_2 = (k_1 n_1 + 1)/k_2$. There are thus at most $\sum d(k)$ possible pairs (k_1, k_2) and hence at most $\sum 7Y d(k)/k$ possible quadruples (k_1, k_2, n_1, n_2). We have $K > 2$, so that the previous Lemma applies and we can deduce that the number of values of n satisfying the given conditions is most $49Y(logK)^2$. Now $K = 2Y^2/e^{t-1} < 2e^{t+1}$ so $\log K < t + 1 + \log 2 < 2t$. This establishes the claimed upper bound of $196Yt^2$.

Theorem 4. Fix $\frac{1}{2} < \beta < 1$ and integer $t \geq 5$. The number of integers n with $e^{t-1} < n < e^t$, such that $Z(n) < n^\beta$ is at most $196t^2 e^{\beta t}$.

Proof. We apply the previous result with $Y = e^{\beta t}$. The conditions of β ensure that the previous Lemma is applicable and the upper bound on the number of such n is $196t^2 e^{\beta t}$ as claimed.

Theorem 5. The series

$$\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}$$

is convergent for any $\alpha > \sqrt{2}$.

Proof. We note that if $\alpha > 2$ then $frac1Z(n)^\alpha < \frac{1}{n^\alpha}$ and the series is convergent . So we may assume$\sqrt{2} < \alpha < 2$. Fix β with $\frac{1}{\alpha} < \beta < \frac{\alpha}{2}$. We have $\frac{1}{2} < \beta < \sqrt{\frac{1}{2}} < \frac{\alpha}{2}$.

We split the positive integers $n > e^4$ into two classes A and B. We let class A be the union of the A_t where, for postive integer $t \geq 5$ we put into class A_t those integers n such that $e^{t-1} < n < e^t$ for integer t and $Z(n) \leq n^\beta$. All values of n with $Z(n) > n^\beta$ we put into class B. We consider the sum of $\frac{1}{Z(n)^\alpha}$ over each of the two classes. Since all terms are positive, it is sufficient to prove that each series separately is convergent.

Firstly we observe that for $n \in B$, we have $\frac{1}{Z(n)^\alpha} < \frac{1}{n^{\alpha\beta}}$ and since $\alpha\beta > 1$ the series summed over the class B is convergent.

Consider the elements n of A_t : so for such n we have $e^{t-1} < n < e^t$ and $Z(n) < n^\beta$. By the previous result, the number of values of n satisfying these conditions is at most $196t^2 e^{\beta t}$. For $n \in A_t$, we have $Z(n) > \sqrt{n}$, so $1/Z(n)^\alpha \leq 1/n^{\alpha/2} < 1/e^{\alpha(t-1)/2}$. Hence the sum of the subseries $\sum n \in A_t \frac{1}{Z(n)^\alpha}$ is at most $196t^2 e^{\alpha/2} e^{(\beta - \alpha/2)t}$. Since $\beta < \alpha/2$ for $\alpha > \sqrt{2}$, the sum over all t of these terms is finite.

We conclude that$\sum \frac{1}{Z(n)^\alpha}$ is convergent for any $\alpha > \sqrt{2}$.

Theorem 6. The series

$$\sum_{n=1}^{\infty} \frac{1}{Z(n)^\alpha}$$

is convergent for any $\alpha > 1$.

proof. We fix $\beta_0 = 1 > \beta_1 > \cdots > \beta_r = \frac{1}{2}$ with $\beta_j < \alpha\beta_{j+1}$ for $0 \leq j \leq r-1$. We defined a partition of the integers $e^{t-1} < n < e^t$ into classes B_t and $C_t(j)$, $1 \leq j \leq r-1$. Into B_t place those n with $Z(n) > n^{\beta_1}$. Into $C_t(j)$ place those n with $n^{\beta_{j+1}} < Z(n) < n^{\beta_j}$. Since $\beta_r = \frac{1}{2}$ we see that every n with $e^{t-1} < n < e^t$ is placed into one of the classes.

The number of elements in $C_t(j)$ is at most $196t^2e^{\beta_j t}$ and so

$$\sum_{n\in C_t(j)} \frac{1}{Z(n)^\alpha} < 196t^2e^{\beta_j t}e^{-\beta_j\alpha(t-1)} = 196t^2e^{\beta_{j+1}\alpha}e^{(\beta_j-\alpha\beta_{j+1})t}.$$

For each j we have $\beta_j < \alpha\beta_{j+1}$ so each sum over t converges.

The sum over the union of the B_t is bounded above by

$$\sum_n \frac{1}{n^{\alpha\beta_1}},$$

which is convergent since $\alpha\beta_1 > \beta_0 = 1$.

We conclude that $\sum_{n=1}^\infty \frac{1}{Z(n)^\alpha}$ is convergent.

References

[1] Charles Ashbacher, Pluckings from the tree of Smarandache sequences and functions, American Research Press, 1998.

http://www.gallup.unm.edu/ smarandache/ashbacher-pluckings.pdf.

[2] K.Kashihare, Comments and topics on Smarandache notions and problems, Erhus University Press,Vall, AZ, USA, 1996.

Scientia Magna
Vol. 1 (2005), No. 2, 173-175

An equation involving the Euler function and Smarandache function

Yi Yuan

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

Abstract For any positive integer n, let $\phi(n)$ be the Euler function, and $S(n)$ denotes the Smarandache function. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation $\phi(n) = S(n^k)$, where k is any fixed positive integer, and give all solutions for this equation.

Keywords Euler function; Smarandache function; Equation; Solutions.

§1. Introduction

For any positive integer n, let $S(n)$ denotes the Smarandache function, $S(n)$ is defined as the smallest positive integer m such that $n|m!$. From the definition one can easily deduce that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the factorization of n into prime powers, then $S(n) = \max\{S(p_i^{\alpha_i})\}$, where the maximum is taken over the i's from 1 to k. Let $\phi(n)$ denotes the Euler function. That is, $\phi(n)$ denotes the number of all positive integers not exceeding n which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function.

In this paper, we shall study the number of the solutions of the equation $\phi(n) = S(n^k)$, where k is any fixed positive integer. About this problem, it is easy to get that $n = 1$ be a solution of the equation, but now we don't know whether it have finite solutions or not. Here, we shall use the elementary methods to solve this problem, and give all solutions for this equation for any fixed positive integer k. That is, we shall prove the following conclusions:

Theorem 1. *The equation $\phi(n) = S(n^2)$ have three solutions, namely $n = 1$, 24, 50.*

Theorem 2. *The equation $\phi(n) = S(n^3)$ have three solutions, namely $n = 1$, 48, 98.*

Theorem 3. *The equation $\phi(n) = S(n^4)$ has one solution, namely $n = 1$.*

Note. *Using the similarly method, we can also deduce that the equation $\phi(n) = S(n^k)$ have the finite solutions, where k be any fixed positive integer.*

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First, we need one simple lemma which is necessary in the proof of Theorems.

Lemma *If p is prime, then $S(p^k) \le kp$. If $k < p$, then $S(p^k) = kp$, where k be any fixed positive integer.*

[1]This work is supported by the N.S.F(10271093) and P.N.S.F of P.R.China

Proof. (See reference [2]).

Now we shall complete the proofs of Theorems.

The proof of Theorem 1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then from the definitions of $S(n)$ and $\phi(n)$ we have

$$S(n^2) = max\{S(p_i^{2\alpha_i})\} = S(p^{2\alpha}),$$

where p be prime, and

$$\phi(n) = p^{\alpha-1}(p-1)\phi(n_1),$$

where $(n_1, p) = 1$. That is, the largest common factor n_1 and p is 1.

It is clear that $n = 1$ is a solution of the equation $\phi(n) = S(n^2)$. If $n > 1$ then we will discuss the problem in the following cases:

(i) Let $\alpha = 1$.

If $p = 2$, then $S(2^2) = 4$, $\phi(n) = (2-1)\phi(n_1)$, from $S(n^2) = S(2^2) = \phi(n) = \phi(n_1)$, we get $\phi(n_1) = 4$, so $n_1 = 5$, then $n = 2^2 \times 5$. But now $S(2^4 \cdot 5^2) = 10 \neq \phi(2^2 \times 5)$, hence the equation has no solution in this case.

If $p \geq 3$, then from Lemma we have $S(p^2) = 2p$, $\phi(n) = (p-1)\phi(n_1)$, note that $p \nmid (p-1)\phi(n_1)$, hence the equation has no solution in this case.

(ii) Let $\alpha = 2$.

If $p = 2$, then $S(2^4) = 6 = 2\phi(n_1)$, no solution.

If $p = 3$, then $S(3^4) = 9 = 3 \times 2\phi(n_1)$, no solution.

If $p = 5$, then $S(5^4) = 20 = 5 \times 4\phi(n_1)$, so $n_1 = 2$, hence $n = 5^2 \times 2$ is a solution.

If $p \geq 7$, then $S(p^4) = 4p = p(p-1)\phi(n_1)$, note that $p - 1 > 4$, hence no solution.

(iii) Let $\alpha = 3$.

If $p = 2$, then $S(2^6) = 8 = 4\phi(n_1)$, so $n_1 = 3$, hence $n = 2^3 \times 3$ is a solution.

If $p = 3$, then $S(3^6) = 15 = 3^2 \times 2\phi(n_1)$, no solution.

If $p = 5$, then $S(5^6) = 25 = 5^2 \times 4\phi(n_1)$, no solution.

If $p = 7$, then $S(7^6) = 42 = 7^2 \times 6\phi(n_1)$, no solution.

If $p > 7$, then $S(p^6) = 6p = p(p-1)\phi(n_1)$, note that $p - 1 > 6$, no solution.

(iv) Let $\alpha = 4$.

If $p = 2$, then $S(2^8) = 10 = 8\phi(n_1)$, no solution.

If $p \geq 3$, from Lemma we have $S(p^{2\alpha}) < 2p\alpha$, note that $\phi(n) = p^{\alpha-1}(p-1)\phi(n_1)$ and $p^{\alpha-1} > 2p\alpha$, no solution.

(v) Let $\alpha = 5$.

If $p = 2$, then $S(2^{10}) = 12 = 2^4\phi(n_1)$, no solution.

If $p \geq 3$, from Lemma we have $S(p^{2\alpha}) < 2p\alpha$, note that $\phi(n) = p^{\alpha-1}(p-1)\phi(n_1)$ and $p^{\alpha-1} > 2p\alpha$, no solution.

(vi) Let $\alpha \geq 6$.

If $p \geq 2$, from Lemma we have $S(p^{2\alpha}) < 2p\alpha$, note that $\phi(n) = p^{\alpha-1}(p-1)\phi(n_1)$ and $p^{\alpha-1} > 2p\alpha$, no solution.

Combining (i) to (vi), we may immediately get that the equation $\phi(n) = S(n^2)$ have three solutions, namely $n = 1$, 24, 50. This completes the proof of Theorem 1.

Similarly, using the same methods we can also deduce the results of Theorem 2 and Theorem 3. This complete the proofs of Theorems.

For the general positive integer k, from the methods of proving our Theorems we can get that the equation $\phi(n) = S(n^k)$ have finite solutions.

References

[1] F.Smarandache, Only problems, not solutions, Chicago, Xiquan Publ. House, 1993.

[2] Mark Farris and Patrick Mitchell, Bounding the Smarandache function, Smarandache notions journal, **13**(2002).

[3] Pan Chengddong and Pan Chengbiao, Elementary number theory. Beijing, Beijing University Press, 1992.

[4] Tom M.Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

Scientia Magna
Vol. 1 (2005), No. 2, 176-178

Recursive Palindromic Smarandache Values

Jason Earls

R.R. 1-43-05 Fritch, TX 79036
email: jason_earls@hotmail.com

Abstract In [1] Recursive Prime Numbers were studied and shown to be finite. This article deals with the same "recursive" topic, but applies the method to numbers whose Smarandache value, $S(n)$, gives a palindromic number. Here, $S(n)$ denotes the Smarandache function of least m such that n divides $m!$, and a palindrome is an integer that reads the same forwards and backwards (23432, for example). This sequence of recursive palindromic Smarandache values is shown to be finite with 1514384 being the last term.

Recursive palindromic Smarandache values (RPSV) are integers $n > 0$, such that $S(n)$ gives a palindromic value, and repeatedly deleting the rightmost digits of n and taking $S(n)$ at each step also gives a palindromic value until only a single digit remains. (Note that the numbers are not permitted to have zeroes.) Example:

n	S(n)
94649	1514384
373	151438
797	15143
1514	757
151	151
15	5
1	1

The same algorithm outlined in [1] was used to generate all RPSV sets beginning with each digit 1 through 9. To summarize the basic algorithm, two arrays are defined: A_1, contains only the initial digit, then A_2 is filled with any integers that give palindromic Smarandache values after multiplying the integers in A_1 by 10 and adding y, with $1 \leq y \leq 9$. A_1 is then updated with the A_2 values. This process is repeated until no solutions are found and thus A_2 is empty.

This is enough to prove that the sequence is finite. And using this algorithm, RPSVs were found to be finite with 1514384 being the last term.

As in [1], genetic trees can be constructed from each digit for visualization and comparison purposes. Below, only the genetic tree for the digit 5 is produced. It is left to readers so inclined to construct the other trees. (However, the full sequence of RPSV numbers is given at the end of this article.)

Tree of recursive palindromic Smarandache values with starting digit 5:

5

54 55 56

543 567

5436

54362 54365

543654

To show that the numbers in the genetic tree above are recursively palindromic when $S(n)$ is applied, let us demonstrate with 54365:

n	$S(n)$
54365	131
5436	151
543	181
54	9
5	5

Unsolved Questions: What is the sequence of RPSVs when the leftmost digits are repeatedly deleted? Is the sequence finite?

Full Sequence of RPSVs: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 21, 22, 24, 27, 28, 32, 33, 35, 36, 42, 44, 45, 48, 54, 55, 56, 63, 64, 66, 72, 77, 81, 84, 88, 96, 99, 112, 121, 126, 128, 144, 151, 154, 162, 165, 168, 181, 189, 216, 224, 242, 275, 288, 324, 336, 352, 353, 362, 363, 448, 453, 484, 543, 567, 648, 724, 726, 727, 847, 968, 1212, 1267, 1441, 1448, 1512, 1514, 1515, 1629, 1812, 1815, 1818, 2424, 2751, 2757, 2882, 3247, 3535, 3537, 3624, 3629, 3635,

3636, 4536, 4847, 4848, 4849, 5436, 7248, 7272, 7277, 8472, 12127, 12672, 15125, 15143, 18154, 18181, 24245, 27512, 27573, 27576, 32476, 35353, 36359, 36362, 48471, 54362, 54365, 72724, 72727, 72771, 126723, 151436, 151437, 151438, 181542, 181543, 275127, 275762, 363594, 363629, 484718, 543654, 1514384.

References

[1] S. Tabirca and K. Reynolds, Recursive prime numbers, Smarandache Notions Journal **14**(2004), 133-138.

Scientia Magna
Vol. 1 (2005), No. 2, 179-187

Smarandache Idempotents in finite ring Z_n and in Group Ring Z_nG

W.B.Vasantha and Moon K.Chetry

Department of Mathematics
I.I.T. Madras, Chennai

Abstract In this paper we analyze and study the Smarandache idempotents (S-idempotents) in the ring Z_n and in the group ring Z_nG of a finite group G over the finite ring Z_n. We have shown the existance of Smarandache idempotents (S-idempotents) in the ring Z_n when $n = 2^m p$ (or $3p$), where p is a prime > 2 (or p a prime > 3). Also we have shown the existance of Smarandache idempotents (S-idempotents) in the group ring Z_2G and Z_2S_n where $n = 2^m p$ (p a prime of the form $2^m t + 1$).

§1. Introduction

This paper has 4 sections. In section 1, we just give the basic definition of S-idempotents in rings. In section 2, we prove the existence of S-idempotents in the ring Z_n where $n = 2^m p, m \in N$ and p is an odd prime. We also prove the existence of S-idempotents for the ring Z_n where n is of the form $n = 3p$, p is a prime greater than 3. In section 3, we prove the existence of S-idempotents in group rings Z_2G of cyclic group G over Z_2 where order of G is n, $n = 2^m p$ (p a prime of the form $2^m t + 1$). We also prove the existence of S-idempotents for the group ring Z_2S_n where $n = 2^m p$ (p a prime of the form $2^m t + 1$). In the final section, we propose some interesting number theoretic problems based on our study.

Here we just recollect the definition of Smarandache idempotents (S-idempotent) and some basic results to make this paper a self contained one.

Definition 1.1[5]. *Let R be a ring. An element $x \in R \setminus \{0\}$ is said to be a Smarandache idempotent (S-idempotent) of R if $x^2 = x$ and there exist $a \in R \setminus \{x, 0\}$ such that*

$$i. \quad a^2 = x$$

$$ii. \quad xa = x \quad or \quad ax = a.$$

Example 1.1. Let $Z_10 = \{0, 1, 2, \ldots, 9\}$ be the ring of integers modulo 10. Here

$$6^2 \equiv 6(\text{mod}10), \quad 4^2 \equiv 6(\text{mod}10)$$

and

$$6 \cdot 4 \equiv 4(\text{mod}10).$$

So 6 is a S-idempotent in Z_{10}.

Example 1.2. Take $Z_{12} = \{0, 1, 2, \ldots, 11\}$ the ring of integers modulo 12. Here

$$4^2 \equiv 4(\text{mod}12), \quad 8^2 \equiv 4(\text{mod}12)$$

and

$$4 \cdot 8 \equiv 8(\text{mod}12).$$

So 4 is a S-idempotent in Z_{12}.

Example 1.3. In $Z_{30} = \{0, 1, 2, \ldots, 29\}$ the ring of integers modulo 30, 25 is a S-idempotent. As

$$25^2 \equiv 25(\text{mod}30), \quad 5^2 \equiv 25(\text{mod}30)$$

and

$$25 \cdot 5 \equiv 5(\text{mod}30).$$

So 25 is a S-idempotent in Z_{30}.

Theorem 1.1 [5]. *Let R be a ring. If $x \in R$ is a S-idempotent then it is an idempotent in R.*

Proof. From the very definition of S-idempotents.

§2. S-idempotents in the finite ring Z_n

In this section, we find conditions for Z_n to have S-idempotents and prove that when n is of the form $2^m p$, p a prime ¿2 or $n = 3p$ (p a prime ¿3) has S-idempotents. We also explicitly find all the S-idempotents.

Theorem 2.1. *$Z_p = \{0, 1, 2, \ldots, p-1\}$, the prime field of characteristic p, where p is a prime has no non-trivial S-idempotents.*

Proof. Straightforward, as every S-idempotents are idempotents and Z_p has no non-trivial idempotents.

Theorem 2.2: *The ring Z_{2p}, where p is an odd prime has S-idempotents.*

Proof. Here p is an odd prime, so p must be of the form $2m + 1$ i.e $p = 2m + 1$. Take

$$x = p + 1 \qquad \textit{and} \qquad a = p - 1.$$

Here

$$\begin{aligned} p^2 = (2m+1)^2 &= 4m^2 + 4m + 1 \\ &= 2m(2m+1) + 2m + 1 \\ &= 2pm + p \\ &\equiv p(\text{mod}2p). \end{aligned}$$

So

$$p^2 \equiv p(\text{mod}2p).$$

Again

$$\begin{aligned} x^2 = (p+1)^2 &\equiv p^2 + 1(\mathrm{mod}2p) \\ &\equiv p + 1(\mathrm{mod}2p). \end{aligned}$$

Therefore

$$x^2 = x.$$

Also

$$a^2 = (p-1)^2 \equiv p + 1(\mathrm{mod}2p),$$

therefore

$$a^2 = x.$$

And

$$\begin{aligned} xa &= (p+1)(p-1) \\ &= p^2 - 1 \\ &\equiv p - 1(\mathrm{mod}2p) \end{aligned}$$

therefore

$$xa = a.$$

So $x = p + 1$ is a S-idempotent in Z_{2p}.

Example 2.1. Take $Z_6 = Z_{2\cdot 3} = \{0, 1, 2, 3, 4, 5\}$ the ring of integers modulo 6. Then $x = 3 + 1 = 4$ is a S-idempotent. As

$$x^2 = 4^2 \equiv 4(\mathrm{mod}6),$$

take $a = 2$, then $a^2 = 2^2 \equiv 4(\mathrm{mod}6)$.
Therefore

$$a^2 = x,$$

and

$$xa = 4 \cdot 2 \equiv 2(\mathrm{mod}6)$$

i.e

$$xa = a.$$

Theorem 2.3. *The ring Z_{2^2p}, p a prime > 2 and is of the form $4m + 1$ or $4m + 3$ has (at least) two S-idempotents.*

Proof. Here p is of the form $4m + 1$ or $4m + 3$.

If $p = 4m + 1$, then $p^2 \equiv p(\mathrm{mod}2^2p)$. As

$$\begin{aligned} p^2 &= (4m+1)^2 \\ &= 16m^2 + 8m + 1 \\ &= 4m(4m+1) + 4m + 1 \\ &= 4pm + p \\ &\equiv p(\mathrm{mod}2^2p), \end{aligned}$$

therefore

$$p^2 \equiv p(\mathrm{mod}2^2p).$$

Now, take $x = 3p + 1$ and $a = p - 1$ then

$$\begin{aligned} x^2 = (3p+1)^2 &= 9p^2 + 6p + 1 \\ &\equiv 9p + 6p + 1(\mathrm{mod}2^2p) \\ &\equiv 3p + 1(\mathrm{mod}2^2p) \end{aligned}$$

therefore

$$a^2 = x.$$

And

$$\begin{aligned} xa &= (3p+1)(p-1) \\ &= 3p^2 - 3p + p - 1 \\ &\equiv p - 1(\mathrm{mod}2^2p) \end{aligned}$$

therefore

$$xa = a.$$

So x is an S-idempotent.

Similarly, we can prove that $y = p$, (here take $a = 3p$) is another S-idempotent. These are the only two S-idempotents in Z_{2^2p} when $p = 4m + 1$. If $p = 4m + 3$, then $p^2 \equiv 3p(\mathrm{mod}2^2p)$.

As above, we can show that $x = p + 1, (a = 3p - 1)$ and $y = 3p, (a = p)$ are the two S-idempotents. So we are getting a nice pattern here for S-idempotents in Z_{2^2p}:

I. If $p = 4m + 1$, then $x = 3p + 1$, $(a = p - 1)$ and $y = p$, $(a = 3p)$ are the two S-idempotents.

II. If $p = 4m+3$, $x = p+1$, $(a = 3p-1)$ and $y = 3p$, $(a = p)$ are the two S-idempotents.

Example 2.2. Take $Z_{2^2.5} = \{0, 1, \ldots, 19\}$, here $5 = 4 \cdot 1 + 1$. So $x = 3 \cdot 5 + 1 = 16$, $(a = 5 - 1 = 4)$ is an S-idempotent. As $16^2 \equiv 16(\mathrm{mod}20), 4^2 \equiv 16(\mathrm{mod}20)$ and $16 \cdot 4 \equiv 4(\mathrm{mod}20)$. Also $y = 5, (a = 3 \cdot 5 = 15)$ is another S-idempotent. As $5^2 \equiv 5(\mathrm{mod}20), 15^2 \equiv 5(\mathrm{mod}20)$ and $5 \cdot 15 \equiv 15(\mathrm{mod}20)$.

Example 2.3. In the ring $Z_{2^2.7} = \{0, 1, \ldots, 27\}$, here $7 = 4 \cdot 1 + 3, x = 7 + 1 = 8, (a = 3 \cdot 7 - 1 = 20)$ is an S-idempotent. As $8^2 \equiv 8(\mathrm{mod}28), 20^2 \equiv 8(\mathrm{mod}28)$ and $8 \cdot 20 \equiv 20(\mathrm{mod}28)$. Also $y = 3 \cdot 7 = 21, (a = 7)$ is another S-idempotent. As $21^2 \equiv 21(\mathrm{mod}28), 7^2 \equiv 21(\mathrm{mod}28)$ and $21 \cdot 7 \equiv 7(\mathrm{mod}28)$.

Theorem 2.4. *The ring Z_{2^3p}, p a prime > 2 has (at least) two S-idempotents of $\phi(2^3)$ types (where $\phi(n)$ is the number of integer less than n and relatively prime to n).*

Proof. As p is prime > 2. So p is one of the $8m + 1, 8m + 3, 8m + 5, 8m + 7$. Now we will get the following two S-idempotents for each $\phi(2^3) = 4$ types of prime p.

I. If $p = 8m + 1$, then $x = 7p + 1, (a = p - 1)$ and $y = p$, $(a = 7p)$ are S-idempotents.

II. If $p = 8m + 3$, then $x = 5p + 1, (a = 3p - 1)$ and $y = 3p$, $(a = 5p)$ are S-idempotents.

III. If $p = 8m + 5$, then $x = 3p + 1, (a = 5p - 1)$ and $y = 5p$, $(a = 3p)$ are S-idempotents.

IV. If $p = 8m + 7$, then $x = p + 1, (a = 7p - 1)$ and $y = 7p$, $(a = p)$ are S-idempotents.

Example 2.4. In the ring $Z_{2^3.3} = \{0, 1, \ldots, 23\}$, here $3 = 8 \cdot 0 + 3$. So $x = 5 \cdot 3 + 1 = 16, (a = 3 \cdot 3 - 1 = 8)$ is an S-idempotent. As $16^2 \equiv 16(\text{mod}24), 8^2 \equiv 16(\text{mod}24)$ and $16 \cdot 8 \equiv 8(\text{mod}24)$. Also $y = 3 \cdot 3 = 9, (a = 5 \cdot 3 = 15)$ is another S-idempotent. As $9^2 \equiv 9(\text{mod}24), 15^2 \equiv 9(\text{mod}24)$ and $9 \cdot 15 \equiv 15(\text{mod}24)$.

Example 2.5. Take $Z_{2^3.13} = Z_{104} = \{0, 1, \ldots, 103\}$, here $13 = 8 \cdot 1 + 5$. So $x = 3 \cdot 13 + 1 = 40, (a = 5 \cdot 13 - 1 = 64)$ is an S-idempotent. As $40^2 \equiv 40(\text{mod}104), 64^2 \equiv 40(\text{mod}104)$ and $40 \cdot 64 \equiv 64(\text{mod}104)$. Also $y = 5 \cdot 13 = 65, (a = 3 \cdot 13 = 39)$ is another S-idempotent. As $65^2 \equiv 65(\text{mod}104), 39^2 \equiv 65(\text{mod}104)$ and $65 \cdot 39 \equiv 39(\text{mod}104)$.

Theorem 2.5. *The ring Z_{2^4p}, p a prime > 2 has (at least) two S-idempotents for each of $\phi(2^4)$ types of prime p.*

Proof. As above, we can list the S-idempotents for all $\phi(2^4) = 8$ types of prime p.

I. If $p = 16m + 1$, then $x = 15p + 1, (a = p - 1)$ and $y = p$, $(a = 15p)$ are S-idempotents.

II. If $p = 16m + 3$, then $x = 13p + 1, (a = 3p - 1)$ and $y = 3p$, $(a = 13p)$ are S-idempotents.

III. If $p = 16m + 5$, then $x = 11p + 1, (a = 5p - 1)$ and $y = 5p$, $(a = 11p)$ are S-idempotents.

IV. If $p = 16m + 7$, then $x = 9p + 1, (a = 7p - 1)$ and $y = 7p$, $(a = 9p)$ are S-idempotents.

V. If $p = 16m + 9$, then $x = 7p + 1, (a = 9p - 1)$ and $y = 9p$, $(a = 7p)$ are S-idempotents.

VI. If $p = 16m + 11$, then $x = 5p + 1, (a = 11p - 1)$ and $y = 11p$, $(a = 5p)$ are S-idempotents.

VII. If $p = 16m + 13$, then $x = 3p + 1, (a = 13p - 1)$ and $y = 13p$, $(a = 13p)$ are S-idempotents.

VIII. If $p = 16m + 15$, then $x = p + 1, (a = 15p - 1)$ and $y = 15p$, $(a = p)$ are S-idempotents.

Example 2.6. In the ring $Z_{2^4.17} = Z_{272} = \{0, 1, \ldots, 271\}$, here $17 = 16 \cdot 1 + 1$. So $x = 15 \cdot 17 + 1 = 256, (a = 17 - 1 = 16)$ is an S-idempotent. As $256^2 \equiv 256(\text{mod}272), 16^2 \equiv 256(\text{mod}272)$ and $256 \cdot 16 \equiv 16(\text{mod}272)$. Also $y = 17, (a = 15 \cdot 17 = 255)$ is another S-idempotent. As $17^2 \equiv 17(\text{mod}272), 255^2 \equiv 17(\text{mod}272)$ and $17 \cdot 255 \equiv 255(\text{mod}272)$.

We can generalize the above result as followings:

Theorem 2.6. *The ring Z_{2^np}, p a prime > 2 has (at least) two S-idempotents for each of $\phi(2^n)$ types of prime p.*

Proof. Here p is one of the $\phi(2^n)$ form:

$$2^n m_1 + 1, \quad 2^n m_2 + 3, \quad \ldots \quad 2^n m_{\phi(2^n)} + (2^n - 1).$$

We can find the two S-idempotents for each p as above. We are showing here for the prime $p = 2^n m_1 + 1$ only. If

$$p = 2^n m_1 + 1,$$

then

$$x = (2^n - 1)p + 1, \quad (a = p - 1)$$

and

$$y = p, \quad (a = (2^n - 1)p)$$

are S-idempotents.

Similarly we can find S-idempotents for each of the $\phi(2^n)$ form of prime p.

Theorem 2.7. *The ring Z_{3p}, p a prime > 3 has (at least) two S-idempotents of $\phi(3)$ types.*

Proof. Here p can be one of the form $3m+1$ or $3m+2$. We can apply the Theorem 2.6 for Z_{3p} also.

I. If $p = 3m+1$, then $x = 2p+1, (a = p-1)$ and $y = p$, $(a = 2p)$ are S-idempotents.

II. If $p = 3m+2$, then $x = p+1, (a = 2p-1)$ and $y = 2p$, $(a = p)$ are S-idempotents.

Example 2.7. In the ring $Z_{3\cdot 5} = Z_{15} = \{0, 1, \ldots, 14\}$, here $5 = 3 \cdot 1 + 2$. So $x = 5+1 = 6, (a = 2 \cdot 5 - 1 = 9)$ is an S-idempotent. As $6^2 \equiv 6(\text{mod}15), 9^2 \equiv 6(\text{mod}15)$ and $6 \cdot 9 \equiv 9(\text{mod}15)$. Also $y = 2 \cdot 5 = 10, (a = 5)$ is another S-idempotent. As $10^2 \equiv 10(\text{mod}15), 5^2 \equiv 10(\text{mod}15)$ and $10 \cdot 5 \equiv 5(\text{mod}15)$.

Remark: The above result is not true for the ring Z_{3^2p}, p prime > 3. As, for $p = 9m+5; x = 4p+1, (a = 5p-1)$ should be an S-idempotent from the above result. But we see it is not the case in general; for take the ring $Z_{3^2\cdot 23} = Z_{207} = \{0, 1, \ldots, 206\}$. Here $p = 9 \cdot 2 + 5$. Now take

$$x = 4 \cdot 23 + 1 = 93 \quad \textit{and} \quad a = 5 \cdot 23 - 1 = 114.$$

But

$$x^2 \not\equiv x(\text{mod}207).$$

So x is not even an idempotent. So $x = 4p+1$ is not an S-idempotent of Z_{3^2p}.

§3. S-idempotents in the group rings Z_2G

Here we prove the existance of Smarandache idempotents for the group rings Z_{3^2p} of the cyclic group G of order 2^np where p is a prime of the form 2^nt+1.

Example 3.2. Let $G = \{g/g^{52} = 1\}$ be the cyclic group of order $2^2 \cdot 13$. Consider the group ring Z_2G of the group G over Z_2. Take

$$x = 1 + g^4 + g^8 + g^{12} + \ldots + g^{44} + g^{48}$$

and

$$a = 1 + g^2 + g^4 + \ldots + g^{22} + g^{24}$$

then

$$x^2 = x, \quad \textit{and} \quad a^2 = x$$

also

$$x \cdot a = x.$$

So $x = 1 + g^4 + g^8 + g^{12} + \ldots + g^{44} + g^{48}$ is a S-idempotent in Z_2G.

Theorem 3.1. *Let Z_2G be the group ring of the finite cyclic group G of order 2^2p, where p is a prime of the form 2^2m+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^2p, where p of the form 2^2m+1.

Take

$$x = 1 + g^4 + g^8 + \ldots + g^{16m}$$

and

$$a = 1 + g^2 + g^4 + \ldots + g^{8m}$$

then

$$\begin{aligned} x^2 &= (1+g^4+g^8+\ldots+g^{16m})^2 \\ &= 1+g^4+g^8+\ldots+g^{16m} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1+g^2+g^4+\ldots+g^{8m})^2 \\ &= 1+(g^2)^2+(g^4)^2+\ldots+(g^{8m})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x\cdot a &= (1+g^4+g^8+\ldots+g^{16m})(1+g^2+g^4+\ldots+g^{8m}) \\ &= 1+g^4+g^8+\ldots+g^{16m} \\ &= x. \end{aligned}$$

So $x = 1+g^4+g^8+\ldots+g^{16m}$ is a S-idempotent in Z_2G.

Example 3.3. Let $G = \{g/g^{136} = 1\}$ be the cyclic group of order $2^3 \cdot 17$. Consider the group ring Z_2G of the group G over Z_2.

Take

$$x = 1+g^8+g^{16}+\ldots+g^{128}$$

and

$$a = 1+g^4+g^8+\ldots+g^{64}$$

then

$$\begin{aligned} x^2 &= (1+g^8+g^{16}+\ldots+g^{128})^2 \\ &= 1+g^8+g^{16}+\ldots+g^{128} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1+g^4+g^8+\ldots+g^{64})^2 \\ &= 1+(g^4)^2+(g^8)^2+\ldots+(g^{64})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x\cdot a &= (1+g^8+g^{16}+\ldots+g^{128})(1+g^4+g^8+\ldots+g^{64}) \\ &= 1+g^8+g^{64}+\ldots+g^{128} \\ &= x. \end{aligned}$$

So $x = 1+g^8+g^{16}+\ldots+g^{128}$ is a S-idempotent in Z_2G.

Theorem 3.2. *Let Z_2G be the group ring of a finite cyclic group G of order 2^3p, where p is a prime of the form 2^3m+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^3p, where p of the form 2^3m+1.

Take

$$x = 1 + g^8 + g^{16} + \ldots + g^{8(p-1)}$$

and

$$a = 1 + g^4 + g^8 + \ldots + g^{4(p-1)}$$

then

$$\begin{aligned} x^2 &= (1 + g^8 + g^{16} + \ldots + g^{8(p-1)})^2 \\ &= 1 + g^8 + g^{16} + \ldots + g^{8(p-1)} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1 + g^4 + g^8 + \ldots + g^{4(p-1)})^2 \\ &= 1 + (g^4)^2 + (g^8)^2 + \ldots + (g^{8(p-1)})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x \cdot a &= (1 + g^8 + g^{16} + \ldots + g^{8(p-1)})(1 + g^4 + g^8 + \ldots + g^{4(p-1)}) \\ &= 1 + g^8 + g^{16} + \ldots + g^{8(p-1)} \\ &= x. \end{aligned}$$

So $x = 1 + g^8 + g^{16} + \ldots + g^{8(p-1)}$ is a S-idempotent in Z_2G.

We can generalize the above two results as followings:

Theorem 3.3. *Let Z_2G be the group ring of a finite cyclic group G of order 2^np, where p is a prime of the form 2^nt+1, then the group ring Z_2G has non-trivial S-idempotents.*

Proof. Here G is a cyclic group of order 2^np, where p of the form 2^nt+1.

Take

$$x = 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)}$$

and

$$a = 1 + g^{2^{n-1}} + g^{2^{n-1} \cdot 2} + \ldots + g^{2^{n-1} \cdot (p-1)}$$

then

$$\begin{aligned} x^2 &= (1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)})^2 \\ &= 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n(p-1)} \\ &= x. \end{aligned}$$

And

$$\begin{aligned} a^2 &= (1 + g^{2^{n-1}} + g^{2^{n-1} \cdot 2} + \ldots + g^{2^{n-1} \cdot (p-1)})^2 \\ &= 1 + (g^{2^{n-1}})^2 + (g^{2^{n-1} \cdot 2})^2 + \ldots + (g^{2^{n-1} \cdot (p-1)})^2 \\ &= x. \end{aligned}$$

Also

$$\begin{aligned} x \cdot a &= (1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n (p-1)})(1 + g^{2^{n-1}} + g^{2^{n-1} \cdot 2} + \ldots + g^{2^{n-1} \cdot (p-1)}) \\ &= 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n (p-1)} \\ &= x. \end{aligned}$$

So $x = 1 + g^{2^n} + g^{2^n \cdot 2} + \ldots + g^{2^n (p-1)}$ is a S-idempotent in Z_2G.

Corollary 3.1. *Let Z_2S_n be the group ring of a symmetric group S_n where $n = 2^n p$, and p is a prime of the form $2^n t + 1$, then the group ring Z_2S_n has non-trivial S-idempotents.*

Proof. Here Z_2S_n is a group ring where $n = 2^n p$, and p of the form $2^n t + 1$. Clearly Z_2S_n contains a finite cyclic group of order $2^n p$. Then by the Theorem 3.3, Z_2S_n has a non-trivial S-idempotent.

§4. Conclusions

Here we have mainly proved the existance of S-idempotents in certain types of group rings. But it is interesting to enumerate the number of S-idempotents for the group rings Z_2G and Z_2S_n in the Theorem 3.3 and Corollary 3.1. We feel that Z_2G can have only one S-idempotent but we are not in a position to give a proof for it. Also, the problem of finding S-idempotents in Z_pS_n (and Z_pG) where $(p, n) = 1$ (and $(p, |G|) = 1$) or $(p, n) = d \neq 1$ (and $(p, |G|) = d \neq 1$) are still interesting number theoretic problems.

References

[1] I.G.Connel, On the group ring, Can.J.Math. **15**(1963), 650-685.

[2] Kim E.T., Idempotents in some group rings, Bull Korean math.soc., **2**(1987), 77-81.

[3] C.P.Milies and S.K.Sehgal, An Introduction to group rings, algebras and applications, Kluwer Academic Publishers, Dordrecht, Netherlands,(2002).

[4] D.S.Passman, The algebraic structure of group rings, Wiley interscience (1997).

[5] W.B.Vasantha Kandasamy, Smarandache rings, American research press, Rehoboth (2002).

Scientia Magna
Vol. 1 (2005), No. 2, 188-195

Smarandache Idempotents in Loop Rings $Z_tL_n(m)$ of the Loops $L_n(m)$

W.B.Vasantha and Moon K. Chetry

Department of Mathematics, I.I.T.Madras, Chennai

Abstract In this paper we establish the existence of S-idempotents in case of loop rings $Z_tL_n(m)$ for a special class of loops $L_n(m)$; over the ring of modulo integers Z_t for a specific value of t. These loops satisfy the conditions g_i^2 for every $g_i \in L_n(m)$. We prove $Z_tL_n(m)$ has an S-idempotent when t is a perfect number or when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$(p_1 and p_2 are distinct odd primes), It is important to note that we are able to prove only the existence of a single S-idempotent; however we leave it as an open problem whether such loop rings have more than one S-idempotent.

§1. Basic Results

This paper has three sections. In section one, we give the basic notions about the loops $L_n(m)$ and recall the definition of S-idempotents in rings. In section two, we establish the existence of S-idempotents in the loop ring $Z_tL_n(m)$. In the final section, we suggest some interesting problems based on our study.

Here we just give the basic notions about the loops $L_n(m)$ and the definition of S-idempotents in rings.

Definition 1.1 [4]. Let R be a ring. An element $x \in R \setminus \{0\}$ is said to be a Smarandache idempotents (S-idempotent) of R if $x^2 = x$ and there exist $a \in R \setminus \{x, 0\}$ such that

$$i. \quad a^2 = x$$
$$ii. \quad xa = x \text{ or } ax = a.$$

For more about S-idempotent please refer [4].

Definition 1.2 [2]. A positive integer n is said to be a perfect number if n is equal to the sum of all its positive divisors, excluding n itself. e.g. 6 is a perfect number. As $6 = 1 + 2 + 3$.

Definition 1.3 [1]. A non-empty set L is said to form a loop, if in L is defined a binary operation, called product and denoted by $'.'$ such that

1. For $a, b \in L$ we have $a.b \in L$. (closure property.)

2. There exists an element $e \in L$ such that $a.e = e.a = a$ for all $a \in L$. (e is called the identity element of L.)

3. For every ordered pair $(a, b) \in L \times L$ there exists a unique pair $(x, y) \in L \times L$ such that $ax = b$ and $ya = b$.

Definition 1.4 [3]. Let $L_n(m) = \{e, 1, 2, 3, \cdots, n\}$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as following:

$i.$ $e.i = i.e = i$ for all $i \in L_n(m) \setminus \{e\}$

$ii.$ $i^2. = e$ for all $i \in L_n(m)$

$iii.$ $i.j = t$, where $t \equiv (mj - (m-1)i)(\text{mod}n)$ for all $i, j \in L_n(m)$, $i \neq e$ and $j \neq e$.

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by

$$L_n = \{L_n(m) | n > 3, n \text{ is odd and } (m, n) = 1, (m - 1, n) = 1 \text{ with } m < n\}.$$

Example 1.1 [3]. Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

$\cdot$	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

§2. Existence of S-idempotents in the Loop Rings $Z_tL_n(m)$

In this section we will prove the existence of an S-idempotent in the loop ring $Z_tL_n(m)$ when t is an even perfect number. Also we will prove that the loop ring $Z_tL_n(m)$ has an S-idempotent when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$ (p_1 and p_2 are distinct odd primes).

Theorem 2.1. Let $Z_tL_n(m)$ be a loop ring, where t is an even perfect number of the form $t = 2^s(2^{s+1} - 1)$ for some $s > 1$, then $\alpha = 2^s + 2^sg_i \in Z_tL_n(m)$ is an S-idempotent.

Proof. As t is an even perfect number, t must be of the form

$$t = 2^s(2^{s+1} - 1), \qquad \text{for some } s > 1$$

where $2^{s+1} - 1$ is a prime.

Consider

$$\alpha = 2^s + 2^sg_i \in Z_tL_n(m).$$

Choose

$$\beta = (t - 2^s) + (t - 2^s)g_i \in Z_tL_n(m).$$

Clearly

$$\begin{aligned}\alpha^2 &= (2^s + 2^s g_i)^2 \\ &= 2.2^{2s}(1 + g_i) \\ &\equiv 2^s(1 + g_i) \quad [2^s.2^{s+1} \equiv 2^s (\text{mod } t)] \\ &= \alpha.\end{aligned}$$

Now

$$\begin{aligned}\beta^2 &= ((t - 2^s) + (t - 2^s)g_i)^2 \\ &= 2.(t - 2^s)^2(1 + g_i) \\ &\equiv 2^s(1 + g_i) \\ &= \alpha.\end{aligned}$$

Also

$$\begin{aligned}\alpha\beta &= [2^s + 2^s g_i][(t - 2^s) + (t - 2^s)g_i] \\ &= 2^s(1 + g_i)(t - 2^s)(1 + g_i) \\ &\equiv -2.2^s.2^s(1 + g_i) \\ &\equiv (t - 2^s)(1 + g_i) \\ &= \beta.\end{aligned}$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{and} \quad \alpha\beta = \beta.$$

Therefore $\alpha = 2^s + 2^s g_i$ is an S-idempotent.

Example 2.1. Take the loop ring $Z_6L_n(m)$. Here 6 is an even perfect number. As $6 = 2.(2^s - 1)$, so $\alpha = 2 + 2g_i$ is an S-idempotent. For

$$\begin{aligned}\alpha^2 &= (2 + 2g_i)^2 \\ &\equiv 2 + 2g_i \\ &= \alpha.\end{aligned}$$

Choose now

$$\beta = (6 - 2) + (6 - 2)g_i.$$

then

$$\begin{aligned}\beta^2 &= (4 + 4g_i)^2 \\ &\equiv (2 + 2g_i) \\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\alpha\beta &= (2 + 2g_i)(4 + 4g_i) \\ &= 8 + 8g_i + 8g_i + 8 \\ &\equiv 4 + 4g_i \\ &= \beta.\end{aligned}$$

So $\alpha = 2 + 2g_i$ is an S-idempotent.

Theorem 2.2. Let $Z_{2p}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq 1$, then $\alpha = 2^{t_0} + 2^{t_0}g_i \in Z_{2p}L_n(m)$ is an S-idempotent.

Proof. Suppose $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq 1$. Take $\alpha = 2^{t_0} + 2^{t_0}g_i \in Z_{2p}L_n(m)$ and $\beta = (2p - 2^{t_0}) + (2p - 2^{t_0})g_i \in Z_{2p}L_n(m)$.

Clearly

$$\begin{aligned} \alpha^2 &= (2^{t_0} + 2^{t_0}g_i)^2 \\ &= 2.2^{2t_0}(1 + g_i) \\ &= 2^{t_0+1}.2^{t_0}(1 + g_i) \\ &\equiv 2^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

As

$$2^{t_0}.2^{t_0+1} \equiv 2^{t_0}(mod\ 2p)$$

Since

$$2^{t_0+1} \equiv 1(mod\ p)$$

$$\Leftrightarrow 2^{t_0}.2^{t_0+1} \equiv 2^{t_0}(mod\ 2p) \quad \text{for} \quad gcd(2^{t_0}, 2p) = 2, \quad t_0 \geq 1.$$

Also

$$\begin{aligned} \beta^2 &= [(2p - 2^{t_0}) + (2p - 2^{t_0})g_i]^2 \\ &= 2(2p - 2^{t_0})^2(1 + g_i) \\ &\equiv 2.2^{2t_0}(1 + g_i) \\ &= 2^{t_0+1}.2^{t_0}(1 + g_i) \\ &\equiv 2^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \alpha\beta &= [2^{t_0} + 2^{t_0}g_i][(2p - 2^{t_0}) + (2p - 2^{t_0})g_i] \\ &\equiv -2^{t_0}(1 + g_i)2^{t_0}(1 + g_i) \\ &= -2.2^{t_0}(1 + g_i) \\ &\equiv (2p - 2^{t_0})(1 + g_i) \\ &= \beta. \end{aligned}$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \qquad \text{and} \quad \alpha\beta = \beta.$$

Hence $\alpha = 2^{t_0} + 2^{t_0}g_i$ is an S-idempotent.

Example 2.2. Consider the loop ring $Z_{10}L_n(m)$. Here $5 \mid 2^{3+1} - 1$, so $t_0 = 3$.

Take

$$\alpha = 2^3 + 2^3 g_i \text{ and } \beta = 2 + 2g_i.$$

Now

$$\begin{aligned} \alpha^2 &= (8 + 8g_i)^2 \\ &= 64 + 128g_i + 64 \\ &\equiv 8 + 8g_i \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \beta^2 &= (2 + 2g_i)^2 \\ &= 4 + 8g_i + 4 \\ &\equiv 8 + 8g_i \\ &= \alpha. \end{aligned}$$

Also

$$\begin{aligned} \alpha\beta &= (8 + 8g_i)(2 + 2g_i) \\ &= 16 + 16g_i + 16g_i + 16 \\ &\equiv 2 + 2g_i \\ &= \beta. \end{aligned}$$

So $\alpha = 8 + 8g_i$ is an S-idempotent.

Theorem 2.3. Let $Z_{2^i p}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq i$, then $\alpha = 2^{t_0} + 2^{t_0} g_i \in Z_{2^i p}L_n(m)$ is an S-idempotent.

Proof. Note that $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq i$.

Therefore

$$2^{t_0+1} \equiv 1 (mod\ p) \text{ for some } t_0 \geq i$$

$$\Leftrightarrow 2^{t_0}.2^{t_0+1} \equiv 2^{t_0} (mod\ 2^i p) \text{ as } gcd(2^{t_0}, 2^i p) = 2^i, \quad t_0 \geq 1.$$

Now take

$$\alpha = 2^{t_0} + 2^{t_0} g_i \in Z_{2^i p}L_n(m) \text{ and } \beta = (2^i p - 2^{t_0}) + (2^i p - 2^{t_0}) g_i \in Z_{2^i p}L_n(m).$$

Then it is easy to see that

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{and} \quad \alpha\beta = \beta.$$

Hence $\alpha = 2^{t_0} + 2^{t_0} g_i$ is an S-idempotent.

Example 2.3. Take the loop ring $Z_{2^3.7}L_n(m)$. Here $7 \mid 2^{5+1} - 1$, so $t_0 = 5$.

Take

$$\alpha = 2^5 + 2^5 g_i \text{ and } \beta = (2^3.7 - 2^5) + (2^3.7 - 2^5) g_i.$$

Now

$$\begin{aligned}\alpha^2 &= (32+32g_i)^2\\ &= 1024+2048g_i+1024\\ &\equiv 32+32g_i\\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\beta^2 &= (24+24g_i)^2\\ &= 576+1152g_i+576\\ &\equiv 24+24g_i\\ &= \alpha.\end{aligned}$$

Also

$$\begin{aligned}\alpha\beta &= (32+32g_i)(24+24g_i)\\ &\equiv 24+24g_i\\ &= \beta.\end{aligned}$$

So $\alpha = 32+32g_i$ is an S-idempotent.

Theorem 2.4. Let $Z_{3^i p}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2.3^{t_0}-1$ for some $t_0 \geq i$, then $\alpha = 3^{t_0}+3^{t_0}g_i \in Z_{3^i p}L_n(m)$ is an S-idempotent.

Proof. Suppose $p \mid 2.3^{t_0}-1$ for some $t_0 \geq i$.

Take

$$\alpha = 3^{t_0}+3^{t_0}g_i \in Z_{3^i p}L_n(m) \text{ and } \beta = (3^i p - 3^{t_0}) + (3^i p - 3^{t_0})g_i \in Z_{3^i p}L_n(m).$$

Then

$$\begin{aligned}\alpha^2 &= (3^{t_0}+3^{t_0}g_i)^2\\ &= 2.3^{2t_0}(1+g_i)\\ &= 2.3^{t_0}3^{t_0}(1+g_i)\\ &\equiv 3^{t_0}(1+g_i)\\ &= \alpha.\end{aligned}$$

As

$$2.3^{t_0} \equiv 1(mod\ p) \text{ for some } t_0 \geq i$$

$$\Leftrightarrow 2.3^{t_0}.3^{t_0} \equiv 3^{t_0}(mod\ 3^i p) \text{ as } gcd(3^{t_0}, 3^i p) = 3^i, \quad t_0 \geq 1.$$

Similarly

$$\beta^2 = \alpha \quad \text{and} \quad \alpha\beta = \beta.$$

So $\alpha = 3^{t_0}+3^{t_0}g_i$ is an S-idempotent.

Example 2.4. Take the loop ring $Z_{3^2.5}L_n(m)$. Here $5 \mid 2.3^5 - 1$, so $t_0 = 5$.

Take

$$\alpha = 3^5 + 3^5 g_i \text{ and } \beta = (3^2.5 - 3^5) + (3^2.5 - 3^5)g_i.$$

Now

$$\begin{aligned} \alpha^2 &= (18 + 18g_i)^2 \\ &\equiv 18 + 18g_i \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \beta^2 &= (27 + 27g_i)^2 \\ &\equiv 18 + 18g_i \\ &= \alpha. \end{aligned}$$

Also

$$\alpha\beta = \beta.$$

So $\alpha = 3^5 + 3^5 g_i$ is an S-idempotent.

We can generalize Theorem 2.3 and Theorem 2.4 as following:

Theorem 2.5. Let $Z_{p_1^i p_2}L_n(m)$ be a loop ring where p_1 and p_2 are distinct odd primes and $p_2 \mid 2.p_1^{t_0} - 1$ for some $t_0 \geq i$, then $\alpha = p_1^{t_0} + p_1^{t_0} g_i \in Z_{p_1^i p_2}L_n(m)$ is an S-idempotent.

Proof. Suppose $p_2 \mid 2.p_1^{t_0} - 1$ for some $t_0 \geq i$.

Take

$$\alpha = p_1^{t_0} + p_1^{t_0} g_i \in Z_{p_1^i p_2}L_n(m) \text{ and } \beta = (p_1^i p_2 - p_1^{t_0}) + (p_1^i p_2 - p_1^{t_0})g_i \in Z_{p_1^i p_2}L_n(m).$$

Then

$$\begin{aligned} \alpha^2 &= (p_1^{t_0} + p_1^{t_0} g_i)^2 \\ &= 2.p_1^{2t_0}(1 + g_i) \\ &= 2.p_1^{t_0} p_1^{t_0}(1 + g_i) \\ &\equiv p_1^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

As

$$2.p_1^{t_0} \equiv 1 (mod\ p_2) \text{ for some } t_0 \geq i$$

.

$$\Leftrightarrow 2.p_1^{t_0}.p_1^{t_0} \equiv p_1^{t_0} (mod\ p_1^i p_2) \text{ as } gcd(p_1^{t_0}, p_1^i p_2) = p_1^i, \quad t_0 \geq i.$$

Similarly

$$\beta^2 = \alpha \quad \text{and } \alpha\beta = \beta.$$

So $\alpha = p_1^{t_0} + p_1^{t_0} g_i$ is an S-idempotent.

§3. Conclusion

We see in all the 5 cases described in the Theorem 2.1 to 2.5 we are able to establish the existence of one non-trivial S-idempotent. however we are not able to prove the uniqueness of this S-idempotent. Hence we suggest the following problems:

• Does the loop rings described in the Theorems 2.1 to 2.5 can have more than one S-idempotent?

• Does the loop rings $Z_tL_n(m)$ have S-idempotent when t is of the form $t = p_1p_2 \dots p_s$ where $p_1p_2 \dots p_s$ are distinct odd primes?

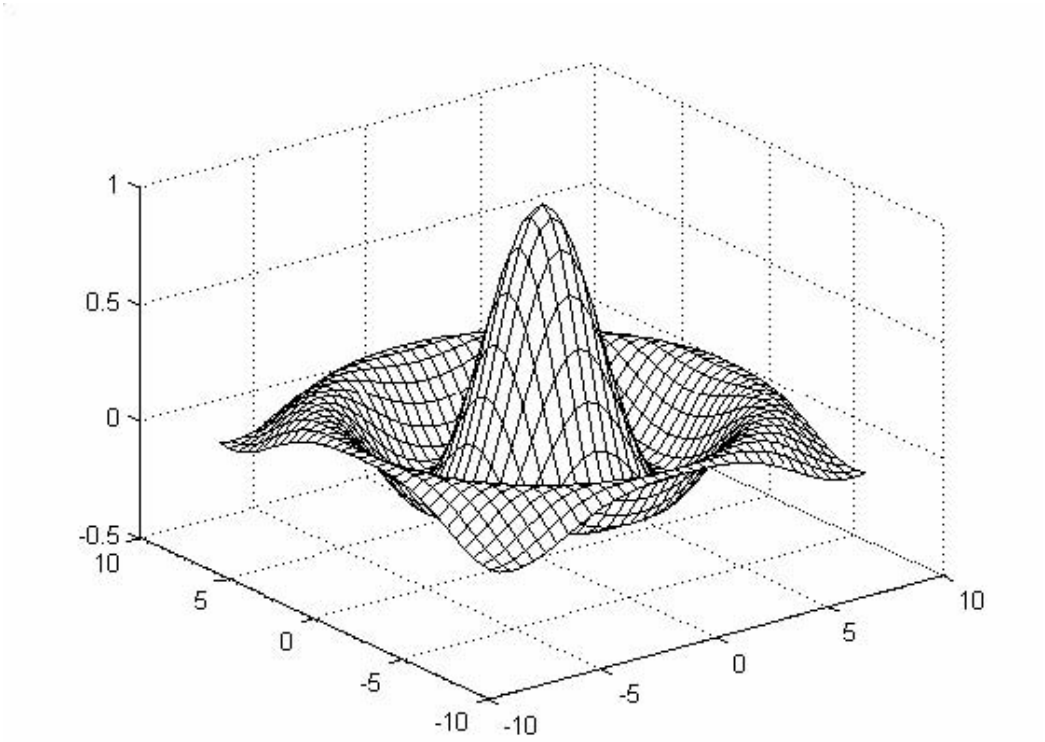
References

[1] Bruck R.H, A survey of binary system, Spring Verlag, 1958.

[2] Burton David, Elementary Number Theory, Universal Book Stall. New Delhi, 1998.

[3] Singh S.V., On a new class of loops and loop rings. PhD thesis, IIT Madras, 1994.

[4] Vasantha Kandasamy, W.B. Smarandache Rings. American Reseach Press, Rehoboth, 2002.



SCIENTIA MAGNA

an international book series

ISBN 1-59973-002-2

90000>

9 781599 730028