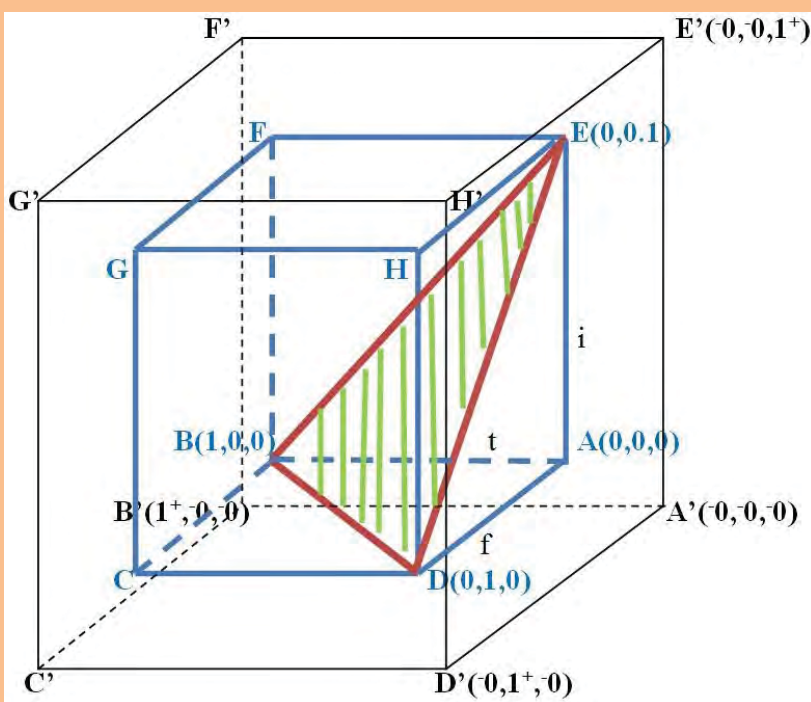


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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea $\langle A \rangle$ together with its opposite or negation $\langle \text{anti}A \rangle$ and with their spectrum of neutralities $\langle \text{neut}A \rangle$ in between them (i.e. notions or ideas supporting neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$). The $\langle \text{neut}A \rangle$ and $\langle \text{anti}A \rangle$ ideas together are referred to as $\langle \text{non}A \rangle$.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on $\langle A \rangle$ and $\langle \text{anti}A \rangle$ only).

According to this theory every idea $\langle A \rangle$ tends to be neutralized and balanced by $\langle \text{anti}A \rangle$ and $\langle \text{non}A \rangle$ ideas - as a state of equilibrium.

In a classical way $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ (and $\langle \text{non}A \rangle$ of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]0, 1+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the $\langle \text{neut}A \rangle$, which means neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$.

$\langle \text{neut}A \rangle$, which of course depends on $\langle A \rangle$, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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BMBJ-neutrosophic ideals in BCK/BCI -algebras

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Abstract: The concepts of a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal are introduced, and several properties are investigated. Conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI -algebras are provided. Characterizations of BMBJ-neutrosophic ideal are discussed. Relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal are considered.

Keywords: MBJ-neutrosophic set; BMBJ-neutrosophic subalgebra; BMBJ-neutrosophic ideal; BMBJ-neutrosophic $\circ$ -subalgebra.

1 Introduction

Smarandache introduced the notion of neutrosophic set which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set (see [11, 12]). Neutrosophic set theory is applied to various part which is referred to the site

<http://fs.gallup.unm.edu/neutrosophy.htm>.

Jun and his colleagues applied the notion of neutrosophic set theory to BCK/BCI -algebras (see [4, 5, 6, 7, 10, 13, 14]). Borzooei et al. [2] studied commutative generalized neutrosophic ideals in BCK -algebras. Mohseni et al. [9] introduced the notion of MBJ-neutrosophic sets which is another generalization of neutrosophic set. They introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI -algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI -algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. Bordbar et al. [1] introduced the notion of BMBJ-neutrosophic subalgebras, and investigated related properties.

In this paper, we apply the notion of MBJ-neutrosophic sets to ideals of BCK/BI -algebras. We introduce the concepts of a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal, and investigate several properties. We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI -algebras, and discuss characterizations of BMBJ-neutrosophic ideal. We consider relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal.

2 Preliminaries

By a *BCI-algebra*, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

$$(I) ((x * y) * (x * z)) * (z * y) = 0,$$

$$(II) (x * (x * y)) * y = 0,$$

$$(III) x * x = 0,$$

$$(IV) x * y = 0, y * x = 0 \Rightarrow x = y$$

for all $x, y, z \in X$. If a *BCI-algebra* X satisfies the following identity:

$$(V) (\forall x \in X) (0 * x = 0),$$

then X is called a *BCK-algebra*.

By a *weakly BCK-algebra* (see [3]), we mean a *BCI-algebra* X satisfying $0 * x \leq x$ for all $x \in X$.

Every *BCK/BCI-algebra* X satisfies the following conditions:

$$(\forall x \in X) (x * 0 = x), \quad (2.1)$$

$$(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x), \quad (2.2)$$

$$(\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \quad (2.3)$$

$$(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y) \quad (2.4)$$

where $x \leq y$ if and only if $x * y = 0$. Any *BCI-algebra* X satisfies the following conditions (see [3]):

$$(\forall x, y \in X) (x * (x * (x * y)) = x * y), \quad (2.5)$$

$$(\forall x, y \in X) (0 * (x * y) = (0 * x) * (0 * y)). \quad (2.6)$$

A *BCI-algebra* X is said to be *p-semisimple* (see [3]) if

$$(\forall x \in X) (0 * (0 * x) = x). \quad (2.7)$$

In a *p-semisimple BCI-algebra* X , the following holds:

$$(\forall x, y \in X) (0 * (x * y) = y * x, x * (x * y) = y). \quad (2.8)$$

A *BCI-algebra* X is said to be *associative* (see [3]) if

$$(\forall x, y, z \in X) ((x * y) * z = x * (y * z)). \quad (2.9)$$

By an *(S)-BCK-algebra*, we mean a *BCK-algebra* X such that, for any $x, y \in X$, the set

$$\{z \in X \mid z * x \leq y\}$$

has the greatest element, written by $x \circ y$ (see [8]).

A nonempty subset S of a BCK/BCI -algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI -algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \quad (2.10)$$

$$(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I). \quad (2.11)$$

A subset I of a BCI -algebra X is called a *closed ideal* of X (see [3]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in I \Rightarrow 0 * x \in I). \quad (2.12)$$

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I , where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers.

Let X be a nonempty set. A function $A : X \rightarrow [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X . Let $[I]^X$ stand for the set of all IVF sets in X . For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A , where $A^- : X \rightarrow I$ and $A^+ : X \rightarrow I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X , respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [11]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X\}$$

where $A_T : X \rightarrow [0, 1]$ is a truth membership function, $A_I : X \rightarrow [0, 1]$ is an indeterminate membership function, and $A_F : X \rightarrow [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X\}.$$

We refer the reader to the books [3, 8] for further information regarding BCK/BCI -algebras, and to the site “<http://fs.gallup.unm.edu/neutrosophy.htm>” for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X (see [9]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X , which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X\}.$$

Let X be a BCK/BCI -algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a *BMBJ-neutrosophic subalgebra* of X (see [1]) if it satisfies:

$$(\forall x \in X)(M_A(x) + B_A^-(x) \leq 1, B_A^+(x) + J_A(x) \leq 1) \quad (2.13)$$

and

$$(\forall x, y \in X) \begin{pmatrix} M_A(x * y) \geq \min\{M_A(x), M_A(y)\} \\ B_A^-(x * y) \leq \max\{B_A^-(x), B_A^-(y)\} \\ B_A^+(x * y) \geq \min\{B_A^+(x), B_A^+(y)\} \\ J_A(x * y) \leq \max\{J_A(x), J_A(y)\} \end{pmatrix}. \quad (2.14)$$

3 BMBJ-neutrosophic ideals

Definition 3.1. Let X be a BCK/BCI -algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a *BMBJ-neutrosophic ideal* of X if it satisfies (2.13) and

$$(\forall x \in X) \begin{pmatrix} M_A(0) \geq M_A(x) \\ B_A^-(0) \leq B_A^-(x) \\ B_A^+(0) \geq B_A^+(x) \\ J_A(0) \leq J_A(x) \end{pmatrix}, \quad (3.1)$$

$$(\forall x, y \in X) \begin{pmatrix} M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} \\ B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} \\ J_A(x) \leq \max\{J_A(x * y), J_A(y)\} \end{pmatrix}. \quad (3.2)$$

A BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of a BCI -algebra X is said to be *closed* if

$$(\forall x \in X) \begin{pmatrix} M_A(0 * x) \geq M_A(x) \\ B_A^-(0 * x) \leq B_A^-(x) \\ B_A^+(0 * x) \geq B_A^+(x) \\ J_A(0 * x) \leq J_A(x) \end{pmatrix}. \quad (3.3)$$

Example 3.2. Consider a set $X = \{0, 1, 2, a\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	a
0	0	0	0	a
1	1	0	0	a
2	2	2	0	a
a	a	a	a	0

$(X; *, 0)$ is a BCI -algebra (see [3]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed MBJ-neutrosophic ideal of X .

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	[0.02, 0.08]	0.2
1	0.5	[0.02, 0.06]	0.2
2	0.4	[0.02, 0.06]	0.7
a	0.3	[0.02, 0.06]	0.7

Proposition 3.3. Let X be a BCK/BCI-algebra. Then every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.

$$x * y \leq z \Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ B_A^-(x) \leq \max\{B_A^-(y), B_A^-(z)\}, \\ B_A^+(x) \geq \min\{B_A^+(y), B_A^+(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases} \quad (3.4)$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$M_A(x * y) \geq \min\{M_A((x * y) * z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$B_A^-(x * y) \leq \max\{B_A^-((x * y) * z), B_A^-(z)\} = \max\{B_A^-(0), B_A^-(z)\} = B_A^-(z),$$

$$B_A^+(x * y) \geq \min\{B_A^+((x * y) * z), B_A^+(z)\} = \min\{B_A^+(0), B_A^+(z)\} = B_A^+(z),$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * z), J_A(z)\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(y), M_A(z)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} = \max\{B_A^-(y), B_A^-(z)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \min\{B_A^+(y), B_A^+(z)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = \max\{J_A(y), J_A(z)\}.$$

This completes the proof. $\square$

We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI -algebras.

Theorem 3.4. *Every MBJ-neutrosophic set in a BCK/BCI -algebra X satisfying (3.1) and (3.4) is a BMBJ-neutrosophic ideal of X .*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X satisfying (3.1) and (3.4). Note that $x * (x * y) \leq y$ for all $x, y \in X$. It follows from (3.4) that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X . $\square$

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a BCK/BCI -algebra X , we consider the following sets.

$$\begin{aligned} U(M_A; t) &:= \{x \in X \mid M_A(x) \geq t\}, \\ L(B_A^-; \alpha^-) &:= \{x \in X \mid B_A^-(x) \leq \alpha^-\}, \\ U(B_A^+; \alpha^+) &:= \{x \in X \mid B_A^+(x) \geq \alpha^+\}, \\ L(J_A; s) &:= \{x \in X \mid J_A(x) \leq s\} \end{aligned}$$

where $t, s, \alpha^-, \alpha^+ \in [0, 1]$.

Theorem 3.5. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a BCK/BCI -algebra X is an MBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-, \alpha^+ \in [0, 1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X . Let $t, s, \alpha^-, \alpha^+ \in [0, 1]$ be such that $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are non-empty. Obviously, $0 \in U(M_A; t) \cap L(B_A^-; \alpha^-) \cap U(B_A^+; \alpha^+) \cap L(J_A; s)$. For any $x, y, a, b, p, q, u, v \in X$, if $x * y \in U(M_A; t)$, $y \in U(M_A; t)$, $a * b \in L(B_A^-; \alpha^-)$, $b \in L(B_A^-; \alpha^-)$, $p * q \in U(B_A^+; \alpha^+)$, $q \in U(B_A^+; \alpha^+)$, $u * v \in L(J_A; s)$ and $v \in L(J_A; s)$, then

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * y), M_A(y)\} \geq \min\{t, t\} = t, \\ B_A^-(a) &\leq \max\{B_A^-(a * b), B_A^-(b)\} \leq \max\{\alpha^-, \alpha^-\} = \alpha^-, \\ B_A^+(p) &\geq \min\{B_A^+(p * q), B_A^+(q)\} \geq \min\{\alpha^+, \alpha^+\} = \alpha^+, \\ J_A(u) &\leq \max\{J_A(u * v), J_A(v)\} \leq \min\{s, s\} = s, \end{aligned}$$

and so $x \in U(M_A; t)$, $a \in L(B_A^-; \alpha^-)$, $p \in U(B_A^+; \alpha^+)$ and $u \in L(J_A; s)$. Therefore $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X .

Conversely, assume that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-, \alpha^+ \in [0, 1]$. Assume that $M_A(0) < M_A(a)$, $B_A^-(0) > B_A^-(a)$, $B_A^+(0) < B_A^+(a)$ and $J_A(0) > J_A(a)$ for some $a \in X$. Then $0 \notin U(M_A; M_A(a)) \cap L(B_A^-; B_A^-(a)) \cap U(B_A^+; B_A^+(a)) \cap L(J_A; J_A(a))$, which is a contradiction. Hence $M_A(0) \geq M_A(x)$, $B_A^-(0) \leq B_A^-(x)$, $B_A^+(0) \geq B_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. If $M_A(a_0) < \min\{M_A(a_0 * b_0), M_A(b_0)\}$ for some $a_0, b_0 \in X$, then $a_0 * b_0 \in U(M_A; t_0)$ and $b_0 \in U(M_A; t_0)$ but $a_0 \notin U(M_A; t_0)$ for $t_0 := \min\{M_A(a_0 * b_0), M_A(b_0)\}$. This is a contradiction, and thus $M_A(a) \geq \min\{M_A(a * b), M_A(b)\}$ for all $a, b \in X$. Similarly, we can show that $J_A(a) \leq \max\{J_A(a * b), J_A(b)\}$ for all $a, b \in X$. Suppose that $B_A^-(a_0) > \max\{B_A^-(a_0 * b_0), B_A^-(b_0)\}$ for some $a_0, b_0 \in X$. Taking $\alpha^- = \max\{B_A^-(a_0 * b_0), B_A^-(b_0)\}$ implies that $a_0 * b_0 \in L(B_A^-; \alpha^-)$ and $b_0 \in L(B_A^-; \alpha^-)$ but $a_0 \notin L(B_A^-; \alpha^-)$. This is a contradiction. Thus $B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\}$ for all $x, y \in X$. Similarly, we obtain $B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\}$ for all $x, y \in X$. Consequently $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X . $\square$

Theorem 3.6. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in a BCK/BCI-algebra X is a BMBJ-neutrosophic ideal of X if and only if (M_A, B_A^-) and (B_A^+, J_A) are intuitionistic fuzzy ideals of X .

Proof. Straightforward. $\square$

Theorem 3.7. Given an ideal I of a BCK/BCI-algebra X , let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by

$$\begin{aligned} M_A(x) &= \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} & B_A^-(x) &= \begin{cases} \alpha^- & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases} \\ B_A^+(x) &= \begin{cases} \alpha^+ & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} & J_A(x) &= \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where $t, \alpha^+ \in (0, 1]$, $s, \alpha^- \in [0, 1)$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X such that $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$.

Proof. It is clear that $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$. Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$ and so

$$\begin{aligned} M_A(x) &= t = \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) &= \alpha^- = \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &= \alpha^+ = \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &= s = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If any one of $x * y$ and y is contained in I , say $x * y \in I$, then $M_A(x * y) = t$, $B_A^-(x * y) = \alpha^-$, $J_A(x * y) = s$, $M_A(y) = 0$, $B_A^-(y) = 1$, $B_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned} M_A(x) &\geq 0 = \min\{t, 0\} = \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) &\leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &\geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &\leq 1 = \max\{s, 1\} = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If $x * y, y \notin I$, then $M_A(x * y) = 0 = M_A(y)$, $B_A^-(x * y) = 1 = B_A^-(y)$, $B_A^+(x * y) = 0 = B_A^+(y)$ and $J_A(x * y) = 1 = J_A(y)$. It follows that

$$\begin{aligned} M_A(x) &\geq 0 = \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) &\leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &\geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &\leq 1 = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

It is obvious that $M_A(0) \geq M_A(x)$, $B_A^-(0) \leq B_A^-(x)$, $B_A^+(0) \geq B_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X . $\square$

Theorem 3.8. For any non-empty subset I of X , let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.7. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X , then I is an ideal of X .

Proof. Obviously, $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $M_A(x * y) = t = M_A(y)$, $B_A^-(x * y) = \alpha^- = B_A^-(y)$, $B_A^+(x * y) = \alpha^+ = B_A^+(y)$ and $J_A(x * y) = s = J_A(y)$. Thus

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * y), M_A(y)\} = t, \\ B_A^-(x) &\leq \max\{B_A^-(x * y), B_A^-(y)\} = \alpha^-, \\ B_A^+(x) &\geq \min\{B_A^+(x * y), B_A^+(y)\} = \alpha^+, \\ J_A(x) &\leq \max\{J_A(x * y), J_A(y)\} = s, \end{aligned}$$

and hence $x \in I$. Therefore I is an ideal of X . $\square$

Theorem 3.9. In a BCK -algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of a BCK -algebra X . Since $(x * y) * x \leq y$ for all $x, y \in X$, it follows from Proposition 3.3 that

$$\begin{aligned} M_A(x * y) &\geq \min\{M_A(x), M_A(y)\}, \\ B_A^-(x * y) &\leq \max\{B_A^-(x), B_A^-(y)\}, \\ B_A^+(x * y) &\geq \min\{B_A^+(x), B_A^+(y)\}, \\ J_A(x * y) &\leq \max\{J_A(x), J_A(y)\} \end{aligned}$$

for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of a BCK -algebra X . $\square$

The converse of Theorem 3.9 may not be true as seen in the following example.

Example 3.10. Consider a BCK -algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X , but it is not a BMBJ-neutrosophic ideal of X since

$$B_A^+(1) \not\geq \min\{B_A^+(1 * 2), B_A^+(2)\}.$$

We provide a condition for a BMBJ-neutrosophic subalgebra to be a BMBJ-neutrosophic ideal in a BCK -algebra.

Table 3: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 4: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	[0.03, 0.08]	0.2
1	0.4	[0.02, 0.06]	0.3
2	0.4	[0.03, 0.08]	0.4
3	0.6	[0.02, 0.06]	0.5

Theorem 3.11. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic subalgebra of a BCK-algebra X satisfying the condition (3.4). Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X .

Proof. For any $x \in X$, we get

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$B_A^-(0) = B_A^-(x * x) \leq \max\{B_A^-(x), B_A^-(x)\} = B_A^-(x),$$

$$B_A^+(0) = B_A^+(x * x) \geq \min\{B_A^+(x), B_A^+(x)\} = B_A^+(x),$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * y), M_A(y)\}, \\ B_A^-(x) &\leq \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &\geq \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &\leq \max\{J_A(x * y), J_A(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X . □

Theorem 3.9 is not true in a BCI -algebra as seen in the following example.

Example 3.12. Let $(Y, *, 0)$ be a BCI -algebra and let $(\mathbb{Z}, -, 0)$ be an adjoint BCI -algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then $X = Y \times \mathbb{Z}$ is a BCI -algebra and $I = Y \times \mathbb{N}$ is an ideal of X where $\mathbb{N}$ is the set of all non-negative integers (see [3]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.7. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.7. But it is not a BMBJ-neutrosophic subalgebra of X since

$$M_A((0, 0) * (0, 1)) = M_A((0, -1)) = 0 < t = \min\{M_A((0, 0)), M_A(0, 1)\},$$

$$B_A^-((0, 0) * (0, 2)) = B_A^-((0, -2)) = 1 > \alpha^- = \max\{B_A^-((0, 0)), B_A^-(0, 2)\},$$

$$B_A^+((0, 0) * (0, 2)) = B_A^+((0, -2)) = 0 < \alpha^+ = \min\{B_A^+((0, 0)), B_A^+(0, 2)\},$$

and/or

$$J_A((0, 0) * (0, 3)) = J_A((0, -3)) = 1 > s = \max\{J_A((0, 0)), J_A(0, 3)\}.$$

Definition 3.13. A BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of a BCI -algebra X is said to be *closed* if

$$(\forall x \in X)(M_A(0 * x) \geq M_A(x), B_A^-(0 * x) \leq B_A^-(x), B_A^+(0 * x) \geq B_A^+(x), J_A(0 * x) \leq J_A(x)). \quad (3.5)$$

Theorem 3.14. In a BCI -algebra, every closed BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a closed BMBJ-neutrosophic ideal of a BCI -algebra X . Using (3.2), (2.3), (III) and (3.3), we have

$$M_A(x * y) \geq \min\{M_A((x * y) * x), M_A(x)\} = \min\{M_A(0 * y), M_A(x)\} \geq \min\{M_A(y), M_A(x)\},$$

$$B_A^-(x * y) \leq \max\{B_A^-((x * y) * x), B_A^-(x)\} = \max\{B_A^-(0 * y), B_A^-(x)\} \leq \max\{B_A^-(y), B_A^-(x)\},$$

$$B_A^+(x * y) \geq \min\{B_A^+((x * y) * x), B_A^+(x)\} = \min\{B_A^+(0 * y), B_A^+(x)\} \geq \min\{B_A^+(y), B_A^+(x)\},$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * x), J_A(x)\} = \max\{J_A(0 * y), J_A(x)\} \leq \max\{J_A(y), J_A(x)\}$$

for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X . □

Theorem 3.15. In a weakly BCK -algebra, every BMBJ-neutrosophic ideal is closed.

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of a weakly BCK -algebra X . For any $x \in X$, we obtain

$$M_A(0 * x) \geq \min\{M_A((0 * x) * x), M_A(x)\} = \min\{M_A(0), M_A(x)\} = M_A(x),$$

$$B_A^-(0 * x) \leq \max\{B_A^-((0 * x) * x), B_A^-(x)\} = \max\{B_A^-(0), B_A^-(x)\} = B_A^-(x),$$

$$B_A^+(0 * x) \geq \min\{B_A^+((0 * x) * x), B_A^+(x)\} = \min\{B_A^+(0), B_A^+(x)\} = B_A^+(x),$$

and

$$J_A(0 * x) \leq \max\{J_A((0 * x) * x), J_A(x)\} = \max\{J_A(0), J_A(x)\} = J_A(x).$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed BMBJ-neutrosophic ideal of X . $\square$

Corollary 3.16. *In a weakly BCK-algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.*

The following example shows that any BMBJ-neutrosophic subalgebra is not a BMBJ-neutrosophic ideal in a BCI-algebra.

Example 3.17. Consider a BCI-algebra $X = \{0, a, b, c, d, e\}$ with the $*$ -operation in Table 5.

Table 5: Cayley table for the binary operation “ $*$ ”

$*$	0	a	b	c	d	e
0	0	0	c	b	c	c
a	a	0	c	b	c	c
b	b	b	0	c	0	0
c	c	c	b	0	b	b
d	d	b	a	c	0	a
e	e	b	a	c	a	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 6.

Table 6: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	[0.14, 0.19]	0.3
a	0.4	[0.04, 0.45]	0.6
b	0.7	[0.14, 0.19]	0.3
c	0.7	[0.14, 0.19]	0.3
d	0.4	[0.04, 0.45]	0.6
e	0.4	[0.04, 0.45]	0.6

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X . But it is not a BMBJ-neutrosophic ideal of X since

$$M_A(d) < \min\{M_A(d * c), M_A(c)\},$$

$$B_A^-(d) > \max\{B_A^-(d * c), B_A^-(c)\},$$

$$B_A^+(d) < \min\{B_A^+(d * c), B_A^+(c)\},$$

and/or

$$J_A(d) > \max\{J_A(d * c), J_A(c)\}.$$

Theorem 3.18. *In a p -semisimple BCI-algebra X , the following are equivalent.*

- (1) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed BMBJ-neutrosophic ideal of X .
- (2) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X .

Proof. (1) $\Rightarrow$ (2). See Theorem 3.14.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X . For any $x \in X$, we get

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$B_A^-(0) = B_A^-(x * x) \leq \max\{B_A^-(x), B_A^-(x)\} = B_A^-(x),$$

$$B_A^+(0) = B_A^+(x * x) \geq \min\{B_A^+(x), B_A^+(x)\} = B_A^+(x),$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Hence $M_A(0 * x) \geq \min\{M_A(0), M_A(x)\} = M_A(x)$, $B_A^-(0 * x) \leq \max\{B_A^-(0), B_A^-(x)\} = B_A^-(x)$, $B_A^+(0 * x) \geq \min\{B_A^+(0), B_A^+(x)\} = B_A^+(x)$ and $J_A(0 * x) \leq \max\{J_A(0), J_A(x)\} = J_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} M_A(x) &= M_A(y * (y * x)) \geq \min\{M_A(y), M_A(y * x)\} \\ &= \min\{M_A(y), M_A(0 * (x * y))\} \\ &\geq \min\{M_A(x * y), M_A(y)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x) &= B_A^-(y * (y * x)) \leq \max\{B_A^-(y), B_A^-(y * x)\} \\ &= \max\{B_A^-(y), B_A^-(0 * (x * y))\} \\ &\leq \max\{B_A^-(x * y), B_A^-(y)\} \end{aligned}$$

$$\begin{aligned}
B_A^+(x) &= B_A^+(y * (y * x)) \geq \min\{B_A^+(y), B_A^+(y * x)\} \\
&= \min\{B_A^+(y), B_A^+(0 * (x * y))\} \\
&\geq \min\{B_A^+(x * y), B_A^+(y)\}
\end{aligned}$$

and

$$\begin{aligned}
J_A(x) &= J_A(y * (y * x)) \leq \max\{J_A(y), J_A(y * x)\} \\
&= \max\{J_A(y), J_A(0 * (x * y))\} \\
&\leq \max\{J_A(x * y), J_A(y)\}.
\end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed BMBJ-neutrosophic ideal of X . □

Since every associative BCI -algebra is p -semisimple, we have the following corollary.

Corollary 3.19. *In an associative BCI -algebra X , the following are equivalent.*

- (1) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a closed BMBJ-neutrosophic ideal of X .
- (2) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of X .

Definition 3.20. Let X be an (S) - BCK -algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a BMBJ-neutrosophic $\circ$ -subalgebra of X if the following assertions are valid.

$$\begin{aligned}
M_A(x \circ y) &\geq \min\{M_A(x), M_A(y)\}, \\
B_A^-(x \circ y) &\leq \max\{B_A^-(x), B_A^-(y)\}, \\
B_A^+(x \circ y) &\geq \min\{B_A^+(x), B_A^+(y)\}, \\
J_A(x \circ y) &\leq \max\{J_A(x), J_A(y)\}
\end{aligned} \tag{3.6}$$

for all $x, y \in X$.

Lemma 3.21. *Every BMBJ-neutrosophic ideal of a BCK/BCI -algebra X satisfies the following assertion.*

$$(\forall x, y \in X) (x \leq y \Rightarrow M_A(x) \geq M_A(y), B_A^-(x) \leq B_A^-(y), B_A^+(x) \geq B_A^+(y), J_A(x) \leq J_A(y)). \tag{3.7}$$

Proof. Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y),$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} = \max\{B_A^-(0), B_A^-(y)\} = B_A^-(y),$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \min\{B_A^+(0), B_A^+(y)\} = B_A^+(y),$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y).$$

This completes the proof. □

Theorem 3.22. *In an (S) -BCK-algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic $\circ$ -subalgebra.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of an (S) -BCK-algebra X . Note that $(x \circ y) * x \leq y$ for all $x, y \in X$. Using Lemma 3.21 and (3.2) implies that

$$M_A(x \circ y) \geq \min\{M_A((x \circ y) * x), M_A(x)\} \geq \min\{M_A(y), M_A(x)\},$$

$$B_A^-(x \circ y) \leq \max\{B_A^-((x \circ y) * x), B_A^-(x)\} \leq \max\{B_A^-(y), B_A^-(x)\},$$

$$B_A^+(x \circ y) \geq \min\{B_A^+((x \circ y) * x), B_A^+(x)\} \geq \min\{B_A^+(y), B_A^+(x)\},$$

and

$$J_A(x \circ y) \leq \max\{J_A((x \circ y) * x), J_A(x)\} \leq \max\{J_A(y), J_A(x)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic $\circ$ -subalgebra of X . □

We provide a characterization of a BMBJ-neutrosophic ideal in an (S) -BCK-algebra.

Theorem 3.23. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in an (S) -BCK-algebra X . Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X if and only if the following assertions are valid.*

$$\begin{aligned} M_A(x) &\geq \min\{M_A(y), M_A(z)\}, B_A^-(x) \leq \max\{B_A^-(y), B_A^-(z)\}, \\ B_A^+(x) &\geq \min\{B_A^+(y), B_A^+(z)\}, J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{aligned} \quad (3.8)$$

for all $x, y, z \in X$ with $x \leq y \circ z$.

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X and let $x, y, z \in X$ be such that $x \leq y \circ z$. Using (3.1), (3.2) and Theorem 3.22, we have

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * (y \circ z)), M_A(y \circ z)\} \\ &= \min\{M_A(0), M_A(y \circ z)\} \\ &= M_A(y \circ z) \geq \min\{M_A(y), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x) &\leq \max\{B_A^-(x * (y \circ z)), B_A^-(y \circ z)\} \\ &= \max\{B_A^-(0), B_A^-(y \circ z)\} \\ &= B_A^-(y \circ z) \leq \max\{B_A^-(y), B_A^-(z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x) &\geq \min\{B_A^+(x * (y \circ z)), B_A^+(y \circ z)\} \\ &= \min\{B_A^+(0), B_A^+(y \circ z)\} \\ &= B_A^+(y \circ z) \geq \min\{B_A^+(y), B_A^+(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x) &\leq \max\{J_A(x * (y \circ z)), J_A(y \circ z)\} \\ &= \max\{J_A(0), J_A(y \circ z)\} \\ &= J_A(y \circ z) \leq \max\{J_A(y), J_A(z)\}. \end{aligned}$$

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in an (S) -BCK-algebra X satisfying the condition (3.8) for all $x, y, z \in X$ with $x \leq y \circ z$. Since $0 \leq x \circ x$ for all $x \in X$, it follows from (3.8) that

$$M_A(0) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$B_A^-(0) \leq \max\{B_A^-(x), B_A^-(x)\} = B_A^-(x),$$

$$B_A^+(0) \geq \min\{B_A^+(x), B_A^+(x)\} = B_A^+(x),$$

and

$$J_A(0) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Note that $x \leq (x * y) \circ y$ for all $x, y \in X$. Hence we have

$$\begin{aligned} M_A(x) &\geq \min\{M_A(x * y), M_A(y)\}, B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &\geq \min\{B_A^+(x * y), B_A^+(y)\} \text{ and } J_A(x) \leq \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X . □

4 Conclusions

As a generalization of neutrosophic set, Mohseni et al. [9] have introduced the notion of MBJ-neutrosophic sets, and have applied it to BCK/BCI-algebras. BMBJ-neutrosophic set has been introduced in [1] with an application in BCK/BCI-algebras. In this article, we have applied the notion of MBJ-neutrosophic sets to ideals of BCK/BCI-algebras. We have introduced the concepts of a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal, and have investigated several properties. We have provided conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI-algebras, and have discussed characterizations of BMBJ-neutrosophic ideal. We have considered relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic $\circ$ -subalgebra and a (closed) BMBJ-neutrosophic ideal. Using the results and ideas in this paper, our future work will focus on the study of several algebraic structures and substructures.

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Neutrosophic General Finite Automata

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Abstract: The constructions of finite switchboard state automata is known to be an extension of finite automata in the view of commutative and switching automata. In this research, the idea of a neutrosophic is incorporated in the general fuzzy finite automata and general fuzzy finite switchboard automata to introduce neutrosophic general finite automata and neutrosophic general finite switchboard automata. Moreover, we define the notion of the neutrosophic subsystem and strong neutrosophic subsystem for both structures. We also establish the relationship between the neutrosophic subsystem and neutrosophic strong subsystem.

Keywords: Neutrosophic set, General fuzzy automata; switchboard; subsystems.

1 Introduction

It is well-known that the simplest and most important type of automata is finite automata. After the introduction of fuzzy set theory by [47] Zadeh in 1965, the first mathematical formulation of fuzzy automata was proposed by [46] Wee in 1967, considered as a generalization of fuzzy automata theory. Consequently, numerous works have been contributed towards the generalization of finite automata by many authors such as Cao and Ezawac [9], Jin et al [18], Jun [20], Li and Qiu [27], Qiu [34], Sato and Kuroki [36], Srivastava and Tiwari [41], Santos [35], Jun and Kavikumar [21], Kavikumar et al, [22, 23, 24] especially the simplest one by Mordeson and Malik [29]. In 2005, the theory of general fuzzy automata was firstly proposed by Doostfateme and Kermer [11] which is used to resolve the problem of assigning membership values to active states of the fuzzy automaton and its multi-membership. Subsequently, as a generalization, the concept of intuitionistic general fuzzy automata has been introduced and studied by Shamsizadeh and Zahedi [37], while Abolpour and Zahedi [6] proposed general fuzzy automata theory based on the complete residuated lattice-valued. As a further

extension, Kavikumar et al [25] studied the notions of general fuzzy switchboard automata. For more details see the recent literature as [5, 12, 13, 14, 15, 16, 17].

The notions of neutrosophic sets was proposed by Smarandache [38, 39], generalizing the existing ordinary fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy set in which each element of the universe has the degrees of truth, indeterminacy and falsity and the membership values are lies in $]0^-, 1^+[$, the nonstandard unit interval [40] it is an extension from standard interval $[0,1]$. It has been shown that fuzzy sets provides limited platform for computational complexity but neutrosophic sets is suitable for it. The neutrosophic sets is an appropriate mechanism for interpreting real-life philosophical problems but not for scientific problems since it is difficult to consolidate. In neutrosophic sets, the degree of indeterminacy can be defined independently since it is quantified explicitly which led to different from intuitionistic fuzzy sets. Single-valued neutrosophic set and interval neutrosophic set are the subclasses of the neutrosophic sets which was introduced by Wang et al. [44, 45] in order to examine kind of real-life and scientific problems. The applications of fuzzy sets have been found very useful in the domain of mathematics and elsewhere. A number of authors have been applied the concept of the neutrosophic set to many other structures especially in algebra [19, 28], decision-making [1, 2, 10, 30], medical [3, 4, 8], water quality management [33] and traffic control management [31, 32].

1.1 Motivation

In view of exploiting neutrosophic sets, Tahir et al. [43] introduced and studied the concept of single valued Neutrosophic finite state machine and switchboard state machine. Moreover, the fuzzy finite switchboard state machine is introduced into the context of the interval neutrosophic set in [42]. However, the realm of general structure of fuzzy automata in the neutrosophic environment has not been studied yet in the literature so far. Hence, it is still open to many possibilities for innovative research work especially in the context of neutrosophic general automata and its switchboard automata. The fundamental advantage of incorporating neutrosophic sets into general fuzzy automata is the ability to bring indeterminacy membership and nonmembership in each transitions and active states which help us to overcome the uncertain situation at the time of predicting next active state. Motivated by the work of [11], [36] and [38] the concept of neutrosophic general automata and neutrosophic general switchboard automata are introduced in this paper.

1.2 Main Contribution

The purpose of this paper is to introduce the primary algebraic structure of neutrosophic general finite automata and neutrosophic switchboard finite automata. The subsystem and strong subsystem of neutrosophic general finite automata and neutrosophic general finite switchboard f automata are exhibited. The relationship between these subsystems have been discussed and the characterizations of switching and commutative are discussed in the neutrosophic backdrop. We prove that the implication of a strong subsystem is a subsystem of neutrosophic general finite automata. The remainder of this paper is organised as follows. Section 2 provides the results and definitions concerning the general fuzzy automata. Section 3 describes the algebraic properties of the neutrosophic general finite automata. Finally, in section 4, the notion of the neutrosophic general finite switchboard automata is introduced. The paper concludes with Section 5.

2 Preliminaries

”For a nonempty set X , $\tilde{P}(X)$ denotes the set of all fuzzy sets on X .

Definition 2.1. [11] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ where

- (a) Q is a finite set of states, $Q = \{q_1, q_2, \dots, q_n\}$,
- (b) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \dots, a_m\}$,
- (c) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,
- (d) Z is a finite set of output symbols, $Z = \{b_1, b_2, \dots, b_k\}$,
- (e) $\omega : Q \rightarrow Z$ is the non-fuzzy output function,
- (f) $F_1 : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is the membership assignment function,
- (g) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \xrightarrow{F_1(\mu, \delta)} [0, 1]$ is the augmented transition function,
- (h) $F_2 : [0, 1]^* \rightarrow [0, 1]$ is a multi-membership resolution function.

Noted that the function $F_1(\mu, \delta)$ has two parameters μ and δ , where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition. The usual options for the function $F(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$,

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in \text{Domain}(Q_{act}(t_i))$ and $T \subset \text{Domain}(Q_{act}(t_i))$. Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$. The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 2.2. [11] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t + 1$, the following algorithm will assign a unified membership value to it:

1. Each transition weight $\tilde{\delta}(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1 , and will produce a membership value. Call this v_i ,

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

2. These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2 .

3. The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j ,

$$\mu^{t+1}(q_j) = F_{2i=1}^n[v_i] = F_{2i=1}^n[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))],$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t + 1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k .
- $\mu^t(q_i)$ is the membership value of q_i at time t .
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t + 1$.

Definition 2.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 2.1. The max-min general fuzzy automata is defined of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2),$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), \dots\}$ and for every $i, i \geq 0$:

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & \text{otherwise} \end{cases}$$

and for every $i, i \geq 1$: $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$$

and recursively

$$\begin{aligned} \tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) &= \bigvee \{ \tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots \wedge \\ &\tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1}) \}, \end{aligned}$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n - 1$.

Definition 2.4. [13] Let $\tilde{F}^*$ be a max-min GFA, $p \in Q, q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Then p is called a successor of q with threshold α if there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q, \mu^{t_j}(q)), x, p) > \alpha$.

Definition 2.5. [13] Let $\tilde{F}^*$ be a max-min GFA, $q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α . If $T \subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T) = \bigcup \{S_\alpha(q) : q \in T\}$.

Definition 2.6. [38] Let X be an universe of discourse. The neutrosophic set is an object having the form $A = \{\langle x, \mu_1(x), \mu_2(x), \mu_3(x) \rangle | \forall x \in X\}$ where the functions can be defined by $\mu_1, \mu_2, \mu_3 : X \rightarrow]0, 1[$ and μ_1 is the degree of membership or truth, μ_2 is the degree of indeterminacy and μ_3 is the degree of non-membership or false of the element $x \in X$ to the set A with the condition $0 \leq \mu_1(x) + \mu_2(x) + \mu_3(x) \leq 3$.

3 Neutrosophic General Finite Automata

Definition 3.1. An eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is called neutrosophic general finite automata (NGFA for short), where

1. Q is a finite set of states, $Q = \{q_1, q_2, \dots, q_n\}$,
2. Σ is a finite set of input symbols, $\Sigma = \{u_1, u_2, \dots, u_m\}$,
3. $\tilde{R} = \{(q, \mu_1^{t_0}(q), \mu_2^{t_0}(q), \mu_3^{t_0}(q)) | q \in R\}$ is the set of fuzzy start states, $R \subseteq \tilde{P}(Q)$,
4. Z is a finite set of output symbols, $Z = \{b_1, b_2, \dots, b_k\}$,
5. $\tilde{\delta} : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma \times Q \xrightarrow{F_1(\mu, \delta)} [0, 1] \times [0, 1] \times [0, 1]$ is the neutrosophic augmented transition function,
6. $\omega : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \rightarrow Z$ is the non-fuzzy output function,
7. $F_1 = (F_1^\wedge, F_1^{\wedge\vee}, F_1^\vee)$, where $F_1^\wedge : [0, 1] \times [0, 1] \rightarrow [0, 1]$, $F_2^{\wedge\vee} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $F_3^\vee : [0, 1] \times [0, 1] \rightarrow [0, 1]$ are the truth, indeterminacy and false membership assignment functions, respectively. $F_1^\wedge(\mu_1, \tilde{\delta}_1)$, $F_2^{\wedge\vee}(\mu_2, \tilde{\delta}_2)$ and $F_3^\vee(\mu_3, \tilde{\delta}_3)$ are motivated by two parameters μ_1, μ_2, μ_3 and $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$ where μ_1, μ_2 and μ_3 are the truth, indeterminacy and false membership value of a predecessor and $\tilde{\delta}_1, \tilde{\delta}_2$ and $\tilde{\delta}_3$ are the truth, indeterminacy and false membership value of a transition,
8. $F_2 = (F_2^\wedge, F_2^{\wedge\vee}, F_2^\vee)$, where $F_2^\wedge : [0, 1]^* \rightarrow [0, 1]$, $F_2^{\wedge\vee} : [0, 1]^* \rightarrow [0, 1]$ and $F_2^\vee : [0, 1]^* \rightarrow [0, 1]$ are the truth, indeterminacy and false multi-membership resolution function.

Remark 3.2. In Definition 3.1, the process that takes place upon the transition from the state q_i to q_j on an input u_k is represented by

$$\begin{aligned} \mu_1^{t_{k+1}}(q_j) &= \tilde{\delta}_1((q_i, \mu_1^{t_k}(q_i)), u_k, q_j) = F_1^\wedge(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)) = \bigwedge(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)), \\ \mu_2^{t_{k+1}}(q_j) &= \tilde{\delta}_2((q_i, \mu_2^{t_k}(q_i)), u_k, q_j) = F_1^{\wedge\vee}(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) = \begin{cases} \bigvee(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k < t_{k+1} \\ \bigwedge(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k \geq t_{k+1} \end{cases}, \\ \mu_3^{t_{k+1}}(q_j) &= \tilde{\delta}_3((q_i, \mu_3^{t_k}(q_i)), u_k, q_j) = F_1^\vee(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)) = \bigvee(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)), \end{aligned}$$

where

$$\begin{aligned} \tilde{\delta}((q_i, \mu^t(q_i)), u_k, q_j) &= (\tilde{\delta}_1((q_i, \mu_1^t(q_i)), u_k, q_j), \tilde{\delta}_2((q_i, \mu_2^t(q_i)), u_k, q_j), \tilde{\delta}_3((q_i, \mu_3^t(q_i)), u_k, q_j)) \text{ and} \\ \delta(q_i, u_k, q_j) &= (\delta_1(q_i, u_k, q_j), \delta_2(q_i, u_k, q_j), \delta_3(q_i, u_k, q_j)). \end{aligned}$$

Remark 3.3. The algorithm for truth, indeterminacy and false multi-membership resolution for transition function is same as Algorithm 2.2 but the computation depends (see Remark 3.2) on the truth, indeterminacy and false membership assignment function.

Definition 3.4. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a NGFA. We define the max-min neutrosophic general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$, where $\tilde{\delta}^* : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ and define a neutrosophic set $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ and for every $i, i \geq 0$:

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu^{t_i}(q)), \Lambda, p) &= \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases}, \\ \tilde{\delta}_2^*((q, \mu^{t_i}(q)), \Lambda, p) &= \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases}, \\ \tilde{\delta}_3^*((q, \mu^{t_i}(q)), \Lambda, p) &= \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases},\end{aligned}$$

and for every $i, i \geq 1$:

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), u_i, p) &= \tilde{\delta}_1((q, \mu^{t_{i-1}}(q)), u_i, p), \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_2((q, \mu^{t_{i-1}}(q)), u_i, p) \\ \tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), u_i, p) &= \tilde{\delta}_3((q, \mu^{t_{i-1}}(q)), u_i, p)\end{aligned}$$

and recursively,

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) &= \bigvee \{ \tilde{\delta}_1((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots \wedge \\ &\quad \tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \}, \\ \tilde{\delta}_2^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) &= \bigwedge \{ \tilde{\delta}_2((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee \\ &\quad \tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \}, \\ \tilde{\delta}_3^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) &= \bigwedge \{ \tilde{\delta}_3((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_3((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee \\ &\quad \tilde{\delta}_3((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \},\end{aligned}$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Example 3.5. Consider the NGFA in Figure 1 with several transition overlaps. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$ be a set of states,
- $\Sigma = \{a, b\}$ be a set of input symbols,
- $\tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$, set of initial states,
- the operation of $F_1^\wedge, F_1^{\wedge\vee}$ and $F_1^\vee$ are according to Remark 3.2,
- $Z = \emptyset$ and ω are not applicable (output mapping is not of our interest in this paper),
- $\tilde{\delta} : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma \times Q \xrightarrow{F_1(\mu, \delta)} [0, 1] \times [0, 1] \times [0, 1]$, the neutrosophic augmented transition function.

Assuming that $\tilde{F}$ starts operating at time t_0 and the next three inputs are a, b, b respectively (one at a time), active states and their membership values at each time step are as follows:

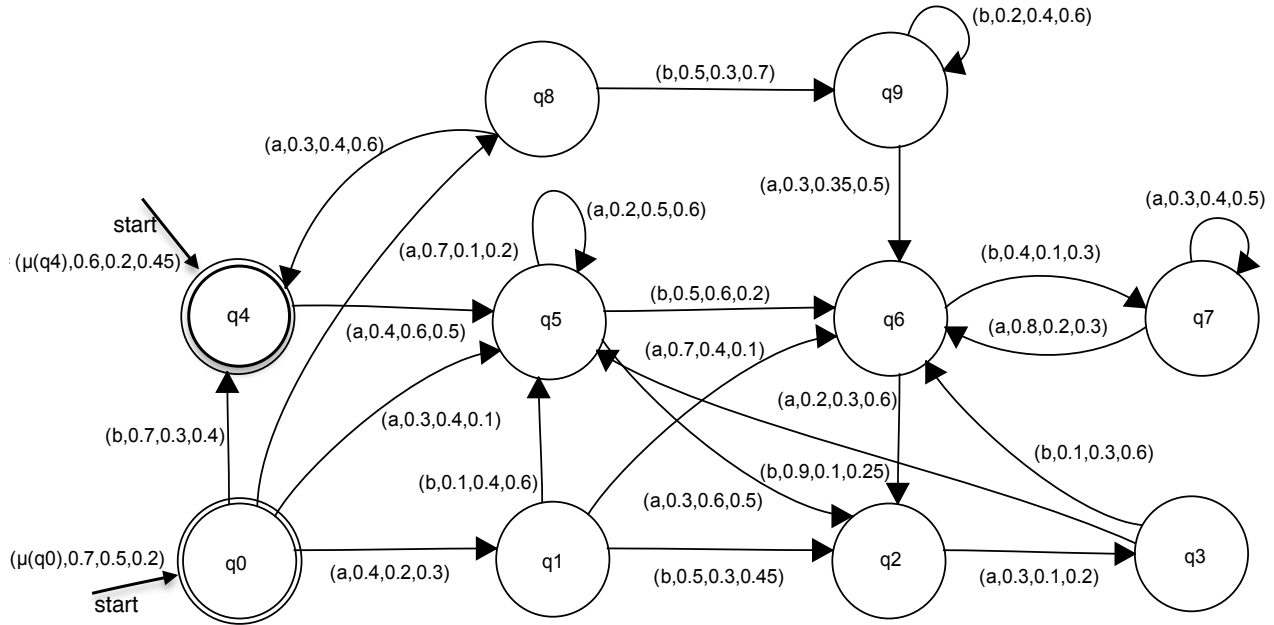


Figure 1: The NGFA of Example 3.5

- At time t_0 : $Q_{act}(t_0) = \tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$
- At time t_1 , input is a . Thus q_1, q_5 and q_8 get activated. Then:

$$\begin{aligned}\mu^{t_1}(q_1) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_1) \\ &= [F_1^\wedge(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_1)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_1)), F_1^\vee(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_1))] \\ &= [F_1^\wedge(0.7, 0.4), F_1^{\wedge\vee}(0.5, 0.2), F_1^\vee(0.2, 0.3)] = (0.4, 0.2, 0.3),\end{aligned}$$

$$\begin{aligned}\mu^{t_1}(q_8) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_8) \\ &= [F_1^\wedge(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_8)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_8)), F_1^\vee(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_8))] \\ &= [F_1^\wedge(0.7, 0.7), F_1^{\wedge\vee}(0.5, 0.1), F_1^\vee(0.2, 0.2)] = (0.7, 0.1, 0.2),\end{aligned}$$

but q_5 is multi-membership at t_1 . Then

$$\begin{aligned}\mu^{t_1}(q_5) &= F_2 \left[F_1[\mu^{t_0}(q_i), \delta(q_i, a, q_5)] \right]_{i=0 \& 4} \\ &= F_2 [F_1[\mu^{t_0}(q_0), \delta(q_0, a, q_5)], F_1[\mu^{t_0}(q_4), \delta(q_4, a, q_5)]] \\ &= F_2 [F_1[(0.7, 0.5, 0.2), (0.3, 0.4, 0.1)], F_1[(0.6, 0.2, 0.45), (0.4, 0.6, 0.5)]] \\ &= (F_2^\wedge[F_1^\wedge(0.7, 0.3), F_1^\wedge(0.6, 0.4)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.5, 0.4), F_1^{\wedge\vee}(0.2, 0.6)], \\ &\quad F_2^\vee[F_1^\vee(0.2, 0.1), F_1^\vee(0.45, 0.5)]) \\ &= (F_2^\wedge(0.3, 0.4), F_2^{\wedge\vee}(0.4, 0.2), F_2^\vee(0.2, 0.5)) = (0.3, 0.2, 0.5).\end{aligned}$$

Then we have:

$$\begin{aligned} Q_{act}(t_1) &= \{(q_1, \mu^{t_1}(q_1)), (q_5, \mu^{t_1}(q_5)), (q_8, \mu^{t_1}(q_8))\} \\ &= \{(q_1, 0.4, 0.2, 0.3), (q_5, 0.3, 0.2, 0.5), (q_8, 0.7, 0.1, 0.2)\}. \end{aligned}$$

- At t_2 input is b . q_2, q_5, q_6 and q_9 get activated. Then

$$\begin{aligned} \mu^{t_2}(q_5) &= \tilde{\delta}((q_1, \mu^{t_1}(q_1), \mu^{t_1}(q_1), \mu^{t_1}(q_1)), b, q_5) \\ &= [F_1^\wedge(\mu^{t_1}(q_1), \delta_1(q_1, b, q_5)), F_1^{\wedge\vee}(\mu^{t_1}(q_1), \delta_2(q_1, b, q_5)), F_1^\vee(\mu^{t_1}(q_1), \delta_3(q_1, b, q_5))] \\ &= [F_1^\wedge(0.4, 0.1), F_1^{\wedge\vee}(0.2, 0.4), F_1^\vee(0.3, 0.6)] = (0.1, 0.2, 0.6), \end{aligned}$$

$$\begin{aligned} \mu^{t_2}(q_6) &= \tilde{\delta}((q_5, \mu^{t_1}(q_5), \mu^{t_1}(q_5), \mu^{t_1}(q_5)), b, q_6) \\ &= [F_1^\wedge(\mu^{t_1}(q_5), \delta_1(q_5, b, q_6)), F_1^{\wedge\vee}(\mu^{t_1}(q_5), \delta_2(q_5, b, q_6)), F_1^\vee(\mu^{t_1}(q_5), \delta_3(q_5, b, q_6))] \\ &= [F_1^\wedge(0.3, 0.5), F_1^{\wedge\vee}(0.2, 0.6), F_1^\vee(0.5, 0.2)] = (0.3, 0.2, 0.5), \end{aligned}$$

$$\begin{aligned} \mu^{t_2}(q_9) &= \tilde{\delta}((q_8, \mu^{t_1}(q_8), \mu^{t_1}(q_8), \mu^{t_1}(q_8)), b, q_9) \\ &= [F_1^\wedge(\mu^{t_1}(q_8), \delta_1(q_8, b, q_9)), F_1^{\wedge\vee}(\mu^{t_1}(q_8), \delta_2(q_8, b, q_9)), F_1^\vee(\mu^{t_1}(q_8), \delta_3(q_8, b, q_9))] \\ &= [F_1^\wedge(0.7, 0.5), F_1^{\wedge\vee}(0.1, 0.3), F_1^\vee(0.2, 0.7)] = (0.5, 0.1, 0.7), \end{aligned}$$

but q_2 is multi-membership at t_2 . Then:

$$\begin{aligned} \mu^{t_2}(q_2) &= F_2 \big[F_1[\mu^{t_1}(q_i), \delta(q_i, b, q_2)] \big]_{i=1\&5} \\ &= F_2 [F_1[\mu^{t_1}(q_1), \delta(q_1, b, q_2)], F_1[\mu^{t_1}(q_5), \delta(q_5, b, q_2)]] \\ &= F_2 [F_1[(0.4, 0.2, 0.3), (0.5, 0.3, 0.45)], F_1[(0.3, 0.2, 0.5), (0.1, 0.4, 0.6)]] \\ &= (F_2^\wedge[F_1^\wedge(0.4, 0.5), F_1^\wedge(0.3, 0.1)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.2, 0.3), F_1^{\wedge\vee}(0.2, 0.4)], \\ &\quad F_2^\vee[F_1^\vee(0.3, 0.45), F_1^\vee(0.5, 0.6)]) \\ &= (F_2^\wedge(0.4, 0.1), F_2^{\wedge\vee}(0.2, 0.2), F_2^\vee(0.3, 0.5)) = (0.1, 0.2, 0.5). \end{aligned}$$

Then we have:

$$\begin{aligned} Q_{act}(t_2) &= \{(q_2, \mu^{t_2}(q_2)), (q_5, \mu^{t_2}(q_5)), (q_6, \mu^{t_2}(q_6)), (q_9, \mu^{t_2}(q_9))\} \\ &= \{(q_2, 0.1, 0.2, 0.5), (q_5, 0.1, 0.2, 0.6), (q_6, 0.3, 0.2, 0.5), (q_9, 0.5, 0.1, 0.7)\}. \end{aligned}$$

- At t_3 input is b . q_2, q_6, q_7 and q_9 get activated and none of them is multi-membership. It is easy to verify that:

$$\begin{aligned} Q_{act}(t_3) &= \{(q_2, \mu^{t_3}(q_2)), (q_6, \mu^{t_3}(q_6)), (q_7, \mu^{t_3}(q_7)), (q_9, \mu^{t_3}(q_9))\} \\ &= \{(q_2, 0.1, 0.1, 0.6), (q_6, 0.1, 0.2, 0.6), (q_7, 0.3, 0.1, 0.5), (q_9, 0.3, 0.1, 0.5)\}. \end{aligned}$$

Proposition 3.6. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a max-min NGFA, then for every $i \geq 1$,*

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \bigvee_{r \in Q_{act}(t_i)} \left[\tilde{\delta}_1^*((p, \mu^{t_{i-1}}(p)), x, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_{i-1}}(r)), y, q) \right], \\ \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \bigwedge_{r \in Q_{act}(t_i)} \left[\tilde{\delta}_2^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_2^*((r, \mu^{t_{i-1}}(r)), y, q) \right], \\ \tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \bigwedge_{r \in Q_{act}(t_i)} \left[\tilde{\delta}_3^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_3^*((r, \mu^{t_{i-1}}(r)), y, q) \right],\end{aligned}$$

for all $p, q \in Q$ and $x, y \in \Sigma^*$.

Proof. Since $p, q \in Q$ and $x, y \in \Sigma^*$, we prove the result by induction on $|y| = n$. First, we assume that $n = 0$, then $y = \Lambda$ and so $xy = x\Lambda = x$. Thus, for all $r \in Q_{act}(t_i)$

$$\begin{aligned}\bigvee \left[\tilde{\delta}_1^*((p, \mu^{t_{i-1}}(p)), x, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_{i-1}}(r)), y, q) \right] &= \bigvee \left[\tilde{\delta}_1^*((p, \mu^{t_{i-1}}(p)), x, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_{i-1}}(r)), \Lambda, q) \right] \\ &= \tilde{\delta}_1^*((p, \mu^{t_{i-1}}(p)), x, r) = \tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), xy, p), \\ \bigwedge \left[\tilde{\delta}_2^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_2^*((r, \mu^{t_{i-1}}(r)), y, q) \right] &= \bigwedge \left[\tilde{\delta}_2^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_2^*((r, \mu^{t_{i-1}}(r)), \Lambda, q) \right] \\ &= \tilde{\delta}_2^*((p, \mu^{t_{i-1}}(p)), x, r) = \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), xy, p), \\ \bigwedge \left[\tilde{\delta}_3^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_3^*((r, \mu^{t_{i-1}}(r)), y, q) \right] &= \bigwedge \left[\tilde{\delta}_3^*((p, \mu^{t_{i-1}}(p)), x, r) \vee \tilde{\delta}_3^*((r, \mu^{t_{i-1}}(r)), \Lambda, q) \right] \\ &= \tilde{\delta}_3^*((p, \mu^{t_{i-1}}(p)), x, r) = \tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), xy, p).\end{aligned}$$

The result holds for $n = 0$. Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n - 1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$ and $u \in \Sigma^*$. Then

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), xua, p) = \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), xu, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_i}(r)), a, p) \right) \\ &= \bigvee_{r \in Q_{act}(t_i)} \left(\bigvee_{s \in Q_{act}(t_i)} (\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), x, s) \wedge \tilde{\delta}_1^*((s, \mu^{t_{i-1}}(s)), u, r)) \wedge \tilde{\delta}_1^*((r, \mu^{t_i}(r)), a, p) \right) \\ &= \bigvee_{r, s \in Q_{act}(t_i)} (\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), x, s) \wedge \tilde{\delta}_1^*((s, \mu^{t_{i-1}}(s)), u, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_i}(r)), a, p)) \\ &= \bigvee_{s \in Q_{act}(t_i)} (\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), x, s) \wedge \left(\bigvee_{r \in Q_{act}(t_i)} (\tilde{\delta}_1^*((s, \mu^{t_{i-1}}(s)), u, r) \wedge \tilde{\delta}_1^*((r, \mu^{t_i}(r)), a, p)) \right)) \\ &= \bigvee_{s \in Q_{act}(t_i)} (\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), x, s) \wedge \tilde{\delta}_1^*((s, \mu^{t_i}(r)), ua, p)) = \bigvee_{s \in Q_{act}(t_i)} (\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), x, s) \wedge \tilde{\delta}_1^*((s, \mu^{t_i}(r)), y, p)), \\ \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_2^*((r, \mu^{t_i}(r)), a, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_2^*((s, \mu^{t_{i-1}}(s)), u, r)) \vee \tilde{\delta}_2^*((r, \mu^{t_i}(r)), a, p) \right) \\ &= \bigwedge_{r, s \in Q_{act}(t_i)} (\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_2^*((s, \mu^{t_{i-1}}(s)), u, r) \vee \tilde{\delta}_2^*((r, \mu^{t_i}(r)), a, p)) \\ &= \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \left(\bigwedge_{r \in Q_{act}(t_i)} (\tilde{\delta}_2^*((s, \mu^{t_{i-1}}(s)), u, r) \vee \tilde{\delta}_2^*((r, \mu^{t_i}(r)), a, p)) \right)) \\ &= \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_2^*((s, \mu^{t_i}(r)), ua, p)) = \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_2^*((s, \mu^{t_i}(r)), y, p)),\end{aligned}$$

$$\begin{aligned}
\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_3^*((r, \mu^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_3^*((s, \mu^{t_{i-1}}(s)), u, r)) \vee \tilde{\delta}_3^*((r, \mu^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r, s \in Q_{act}(t_i)} (\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_3^*((s, \mu^{t_{i-1}}(s)), u, r) \vee \tilde{\delta}_3^*((r, \mu^{t_i}(r)), a, p)) \\
&= \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \left(\bigwedge_{r \in Q_{act}(t_i)} (\tilde{\delta}_3^*((s, \mu^{t_{i-1}}(s)), u, r) \vee \tilde{\delta}_3^*((r, \mu^{t_i}(r)), a, p)) \right)) \\
&= \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_3^*((s, \mu^{t_i}(r)), ua, p)) = \bigwedge_{s \in Q_{act}(t_i)} (\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), x, s) \vee \tilde{\delta}_3^*((s, \mu^{t_i}(r)), y, p)).
\end{aligned}$$

Hence the result is valid for $|y| = n$. This completes the proof. $\square$

Definition 3.7. Let $\tilde{F}^*$ be a max-min NGFA, $p \in Q, q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Then p is called a successor of q with threshold α if there exists $x \in \Sigma^*$ such that $\tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p) > \alpha, \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p) < \alpha$ and $\tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p) < \alpha$.

Definition 3.8. Let $\tilde{F}^*$ be a max-min NGFA, $q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α . If $T \subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T) = \bigcup \{S_\alpha(q) : q \in T\}$.

Definition 3.9. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a neutrosophic subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$ if for every $j, 1 \leq j \leq k$ such that $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p), \mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p), \mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p). \forall q, p \in Q$ and $x \in \Sigma^*$.

Example 3.10. Let $Q = \{p, q\}, \Sigma = \{a\}$. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q such that $\mu_1^{t_j}(p) = 0.8, \mu_2^{t_j}(p) = 0.7, \mu_3^{t_j}(p) = 0.5, \mu_1^{t_j}(q) = 0.5, \mu_2^{t_j}(q) = 0.6, \mu_3^{t_j}(q) = 0.8, \delta_1(q, x, p) = 0.7, \delta_2(q, x, p) = 0.9$ and $\delta_3(q, x, p) = 0.7$. Then

$$\begin{aligned}
\tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p) &= F_1^\wedge(\mu_1^{t_j}(q), \delta_1(q, x, p)) = \min\{0.5, 0.7\} = 0.5 \leq \mu_1^{t_j}(p), \\
\tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p) &= F_2^{\wedge\vee}(\mu_2^{t_j}(q), \delta_2(q, x, p)) = \max\{0.6, 0.9\} = 0.9 \geq \mu_2^{t_j}(p), \text{ (since } t < t_j) \\
\tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p) &= F_3^\vee(\mu_3^{t_j}(q), \delta_3(q, x, p)) = \max\{0.8, 0.7\} = 0.8 \geq \mu_3^{t_j}(p).
\end{aligned}$$

Hence μ is a neutrosophic subsystem of $\tilde{F}^*$.

Theorem 3.11. Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a neutrosophic subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p), \mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p), \mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p),$ for all $q \in Q_{act}(t_j), p \in Q$ and $x \in \Sigma^*$.

Proof. Suppose that μ is a neutrosophic subsystem of $\tilde{F}^*$. Let $q \in Q_{act}(t_j), p \in Q$ and $x \in \Sigma^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), \Lambda, p) = F_1^\wedge(\mu_1^{t_i}(p), \tilde{\delta}_1(p, \Lambda, p)) = \mu_1^{t_i}(p), \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), \Lambda, p) = F_1^{\wedge\vee}(\mu_2^{t_i}(p), \tilde{\delta}_2(p, \Lambda, p)) = \mu_2^{t_i}(p), \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), \Lambda, p) = F_1^\vee(\mu_3^{t_i}(p), \tilde{\delta}_3(p, \Lambda, p)) = \mu_3^{t_i}(p).$

If $q \neq p$, then $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = F_1^\wedge(\mu_1^{t_i}(q), \tilde{\delta}_1(q, \Lambda, p)) = 0 \leq \mu_1^{t_j}(p), \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = F_1^{\wedge\vee}(\mu_2^{t_i}(q), \tilde{\delta}_2(q, \Lambda, p)) = 1 \geq \mu_2^{t_j}(p), \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = F_1^\vee(\mu_3^{t_i}(q), \tilde{\delta}_3(q, \Lambda, p)) = 1 \geq \mu_3^{t_j}(p).$

Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $y \in \Sigma^*$ with $|y| = n-1$, $n > 0$. For the y above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma, i = 1, 2, \dots, n$. Then

$$\begin{aligned} \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) = \bigvee \left(\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, r_1) \wedge \cdots \wedge \tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p) \right) \\ &\leq \bigvee \left(\tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigvee \mu_1^{t_j}(p) = \mu_1^{t_j}(p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p) \right) \\ &\leq \bigwedge \left(\tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigwedge \mu_2^{t_j}(p) = \mu_2^{t_j}(p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p) \right) \\ &\leq \bigwedge \left(\tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigwedge \mu_3^{t_j}(p) = \mu_3^{t_j}(p), \end{aligned}$$

where $r_1 \in Q_{(act)}(t_{i+1}) \cdots r_{n-1} \in Q_{(act)}(t_{i+n})$. Hence $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. The converse is trivial. This proof is completed. $\square$

Definition 3.12. Let $\tilde{F}^*$ be a NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a neutrosophic strong subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every $i, 1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 3.13. Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a strong neutrosophic subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.

Proof. Suppose that μ is a strong neutrosophic subsystem of $\tilde{F}^*$. Let $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\delta_1^*((p, \mu_1^{t_i}(p)), \Lambda, p) = 1$, $\delta_2^*((p, \mu_2^{t_i}(p)), \Lambda, p) = 0$, $\delta_3^*((p, \mu_3^{t_i}(p)), \Lambda, p) = 0$ and $\mu_1^{t_j}(p) = \mu_1^{t_j}(p)$, $\mu_2^{t_j}(p) = \mu_2^{t_j}(p)$, $\mu_3^{t_j}(p) = \mu_3^{t_j}(p)$. If $q \neq p$, then $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = F_1^\wedge(\mu_1^{t_i}(q), \tilde{\delta}_1(q, \Lambda, p)) = c \leq \mu_1^{t_j}(p)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = F_1^{\wedge\vee}(\mu_2^{t_i}(q), \tilde{\delta}_2(q, \Lambda, p)) = d \geq \mu_2^{t_j}(p)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = F_1^\vee(\mu_3^{t_i}(q), \tilde{\delta}_3(q, \Lambda, p)) = e \geq \mu_3^{t_j}(p)$. Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $u \in \Sigma^*$ with $|u| = n-1$, $n > 0$. For the u above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma^*, i = 1, 2, \dots, n$. Suppose that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > c$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < d$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < e$. Then

$$\begin{aligned} \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) &= \bigvee \left\{ \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, p_1) \wedge \cdots \wedge \tilde{\delta}_1^*((p_{n-1}, \mu_1^{t_{i+n}}(p_{n-1})), u_n, p) \right\} > c, \\ \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) &= \bigwedge \left\{ \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_2^*((p_{n-1}, \mu_2^{t_{i+n}}(p_{n-1})), u_n, p) \right\} < d, \\ \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) &= \bigwedge \left\{ \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_3^*((p_{n-1}, \mu_3^{t_{i+n}}(p_{n-1})), u_n, p) \right\} < e, \end{aligned}$$

where $p_1 \in Q_{(act)}(t_i), \dots, p_{n-1} \in Q_{(act)}(t_{i+n})$.

This implies that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, p_1) > c, \dots, \tilde{\delta}_1^*((p_{n-1}, \mu_1^{t_{i+n}}(p_{n-1})), u_n, p) > c, \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, p_1) < d, \dots, \tilde{\delta}_2^*((p_{n-1}, \mu_2^{t_{i+n}}(p_{n-1})), u_n, p) < d, \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, p_1) < e, \dots, \tilde{\delta}_3^*((p_{n-1}, \mu_3^{t_{i+n}}(p_{n-1})), u_n, p) < e$. Hence $\mu_1^{t_j}(p) \geq \mu_1^{t_{i+n}}(p_{n-1}), \mu_1^{t_{i+n}}(p) \geq \mu_1^{t_{i+n-1}}(p_{n-2}), \dots, \mu_1^{t_i}(p_1) \geq \mu_1^{t_j}(q), \mu_2^{t_j}(p) \leq \mu_2^{t_{i+n}}(p_{n-1}), \mu_2^{t_{i+n}}(p) \leq \mu_2^{t_{i+n-1}}(p_{n-2}), \dots, \mu_2^{t_i}(p_1) \leq \mu_2^{t_j}(q), \mu_3^{t_j}(p) \leq \mu_3^{t_{i+n}}(p_{n-1}), \mu_3^{t_{i+n}}(p) \leq \mu_3^{t_{i+n-1}}(p_{n-2}), \dots, \mu_3^{t_i}(p_1) \leq \mu_3^{t_j}(q)$. Thus $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q), \mu_2^{t_j}(p) \leq \mu_2^{t_j}(q), \mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$. The converse is trivial. The proof is completed. $\square$

4 Neutrosophic General Finite Switchboard Automata

Definition 4.1. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ be a neutrosophic set in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma \times Q$ in Q . Then

1. $\tilde{F}^*$ is switching, if it satisfies $\forall p, q \in Q, a \in \Sigma$ and for every $i, i \geq 0$,
 $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), a, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), a, q), \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), a, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), a, q),$
 $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), a, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), a, q).$
2. $\tilde{F}^*$ is commutative, if it satisfies $\forall p, q \in Q, a, b \in \Sigma$ and for every $i, i \geq 1$, $\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ab, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ba, p), \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ab, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ba, p),$
 $\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ab, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ba, p).$
3. $\tilde{F}^*$ is Neutrosophic General Finite Switchboard Automata (NGFSA, for short), if $\tilde{F}^*$ satisfies both switching and commutative.

Proposition 4.2. Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a commutative NGFSA, then for every $i \geq 1$,

$$\begin{aligned} \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ax, p), \\ \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ax, p), \\ \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ax, p), \end{aligned}$$

for all $q \in Q_{act}(t_{i-1}), p \in S_c(q), a \in \Sigma$ and $x \in \Sigma^*$.

Proof. Since $p \in S_c(q)$ then $q \in Q_{act}(t_{i-1})$ and $|x| = n$. If $n = 0$, then $x = \Lambda$. Thus

$$\begin{aligned} \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), \Lambda a, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), a, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), a\Lambda, p) \\ &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ax, p), \\ \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), \Lambda a, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), a, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), a\Lambda, p) \\ &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ax, p), \\ \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), \Lambda a, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), a, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), a\Lambda, p) \\ &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ax, p). \end{aligned}$$

Suppose the result is true for all $u \in \Sigma^*$ with $|u| = n - 1$, where $n > 0$. Let $x = ub$, where $b \in \Sigma$. Then

$$\begin{aligned} \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uba, p) = \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ba, p) \right) \\ &= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ab, p) \right) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uab, p) \\ &= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), ua, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p) \right) \\ &= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), au, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p) \right) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ax, p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ba, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ab, p) \right) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uab, p) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p) \right) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ax, p), \end{aligned}$$

$$\begin{aligned} \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xa, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ba, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ab, p) \right) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uab, p) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p) \right) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ax, p). \end{aligned}$$

This completes the proof. $\square$

Proposition 4.3. Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a switching NGFSFA, then for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$, for all $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$.

Proof. Since $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $x = \Lambda$, whenever $n = 0$. Then we have $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), \Lambda, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), \Lambda, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), \Lambda, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$. Thus, the theorem holds for $x = \Lambda$. Now, we assume that the results holds for all $u \in \Sigma^*$ such that $|u| = n - 1$ and $n > 0$. Let

$a \in \Sigma$ and $x \in \Sigma^*$ be such that $x = ua$. Then

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), ua, p) = \bigvee_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_1((q, \mu_1^{t_i}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_{i+1}}(r)), a, p) \right) \\ &= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((r, \mu_1^{t_i}(r)), u, q) \wedge \tilde{\delta}_1((p, \mu_1^{t_{i+1}}(p)), a, r) \right) = \bigvee_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_1((p, \mu_1^{t_i}(p)), a, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_{i+1}}(r)), u, q) \right) \\ &= \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), au, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), ua, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q),\end{aligned}$$

$$\begin{aligned}\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), ua, p) = \bigwedge_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_2((q, \mu_2^{t_i}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_{i+1}}(r)), a, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((r, \mu_2^{t_i}(r)), u, q) \vee \tilde{\delta}_2((p, \mu_2^{t_{i+1}}(p)), a, r) \right) = \bigwedge_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_2((p, \mu_2^{t_i}(p)), a, r) \vee \tilde{\delta}_2((r, \mu_2^{t_{i+1}}(r)), u, q) \right) \\ &= \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), au, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), ua, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q),\end{aligned}$$

$$\begin{aligned}\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), ua, p) = \bigwedge_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_3((q, \mu_3^{t_i}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_{i+1}}(r)), a, p) \right) \\ &= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((r, \mu_3^{t_i}(r)), u, q) \vee \tilde{\delta}_3((p, \mu_3^{t_{i+1}}(p)), a, r) \right) = \bigwedge_{r \in Q_{act}(t_{i+1})} \left(\tilde{\delta}_3((p, \mu_3^{t_i}(p)), a, r) \vee \tilde{\delta}_3((r, \mu_3^{t_{i+1}}(r)), u, q) \right) \\ &= \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), au, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), ua, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q).\end{aligned}$$

Hence, the result is true for $|u| = n$. This completes the proof. $\square$

Proposition 4.4. Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a NGFSA, then for every $i \geq 1$, $\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), yx, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), yx, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), yx, q)$ for all $p, q \in Q$ and $x, y \in \Sigma^*$.

Proof. Since $p, q \in Q$ and $x, y \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $n = 0$, then $x = \Lambda$. Thus

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), yx, p), \\ \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), yx, p), \\ \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), yx, p).\end{aligned}$$

Suppose that

$$\begin{aligned}\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xu, p) &= \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), ux, q), \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xu, p) = \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), ux, q), \\ \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xu, p) &= \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), ux, q), \text{ for every } u \in \Sigma^*.\end{aligned}$$

Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n - 1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$

and $u \in \Sigma^*$. Then

$$\begin{aligned}
\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xua, p) = \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), xu, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), a, p) \right) \\
&= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), ux, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), a, p) \right) \\
&= \bigvee_{r \in Q_{act}(t_{i-1})} \left(\tilde{\delta}_1((r, \mu_1^{t_{i-1}}(r)), ux, q) \wedge \tilde{\delta}_1((p, \mu_1^{t_i}(p)), a, r) \right) \\
&= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)), a, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ux, q) \right) \\
&= \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), aux, q) = \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)), au, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), x, q) \right) \\
&= \bigvee_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)), ua, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), x, q) \right) \\
&= \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), yx, p), \\
\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), ux, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r \in Q_{act}(t_{i-1})} \left(\tilde{\delta}_2((r, \mu_2^{t_{i-1}}(r)), ux, q) \vee \tilde{\delta}_2((p, \mu_2^{t_i}(p)), a, r) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)), a, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ux, q) \right) \\
&= \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), aux, q) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)), au, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), x, q) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)), ua, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), x, q) \right) \\
&= \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), yx, p), \\
\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) &= \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), ux, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), a, p) \right) \\
&= \bigwedge_{r \in Q_{act}(t_{i-1})} \left(\tilde{\delta}_3((r, \mu_3^{t_{i-1}}(r)), ux, q) \vee \tilde{\delta}_3((p, \mu_3^{t_i}(p)), a, r) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)), a, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ux, q) \right) \\
&= \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), aux, q) = \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)), au, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), x, q) \right) \\
&= \bigwedge_{r \in Q_{act}(t_i)} \left(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)), ua, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), x, q) \right) \\
&= \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), yx, p).
\end{aligned}$$

This completes the proof. $\square$

Definition 4.5. Let $\tilde{F}^*$ be a GNFSFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1]) \times \Sigma^* \times Q$ be

a neutrosophic set in Q . Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every j , $1 \leq j \leq k$ such that $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. $\forall q, p \in Q$ and $x \in \Sigma$.

Theorem 4.6. Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$.

Proof. The proof of the theorem is similar to Theorem 3.11 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. This proof is completed. $\square$

Definition 4.7. Let $\tilde{F}^*$ be a NGFSA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a neutrosophic strong switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i , $1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 4.8. Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q . Then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.

Proof. The proof of the theorem is similar to Theorem 3.13 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. The proof is completed. $\square$

Theorem 4.9. Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q . If μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$.

Proof. Assume that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > 0$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < 1$ and $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < 1$, for all $x \in \Sigma$. Since μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, we have

$$\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p), \quad \mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p), \quad \mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p).$$

for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma$. As μ is switching, then we have

$$\begin{aligned} \mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q) = \mu_1^{t_j}(q), \\ \mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q) = \mu_2^{t_j}(q), \\ \mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q) = \mu_3^{t_j}(q). \end{aligned}$$

As μ is commutative, then $x = uv$, we have

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), uv, p) \\
&= \bigvee \left\{ \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u, r) \wedge \tilde{\delta}_1^*((r, \mu_1^{t_{i+1}}(r)), v, p) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigvee \left\{ \tilde{\delta}_1^*((r, \mu_1^{t_i}(r)), u, q) \wedge \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), v, r) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigvee \left\{ \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), v, r) \wedge \tilde{\delta}_1^*((r, \mu_1^{t_i}(r)), u, q) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), x, q) \geq \mu_1^{t_j}(q), \\
\mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), uv, p) \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u, r) \vee \tilde{\delta}_2^*((r, \mu_2^{t_{i+1}}(r)), v, p) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((r, \mu_2^{t_i}(r)), u, q) \vee \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), v, r) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), v, r) \vee \tilde{\delta}_2^*((r, \mu_2^{t_i}(r)), u, q) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), x, q) \leq \mu_2^{t_j}(q), \\
\mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), uv, p) \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u, r) \vee \tilde{\delta}_3^*((r, \mu_3^{t_{i+1}}(r)), v, p) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((r, \mu_3^{t_i}(r)), u, q) \vee \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), v, r) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), v, r) \vee \tilde{\delta}_3^*((r, \mu_3^{t_i}(r)), u, q) \mid r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), x, q) \leq \mu_3^{t_j}(q).
\end{aligned}$$

Hence μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$. □

Theorem 4.10. Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q . If μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$, then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$.

Proof. Let $q, p \in Q$. Since μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ and μ is switching, we have for all $x \in \Sigma$, since $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > 0$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < 1$ and $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < 1$, $\forall x \in \Sigma$,

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \mu_1^{t_j}(q) \geq \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q) \geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p), \\
\mu_2^{t_j}(p) &\leq \mu_2^{t_j}(q) \leq \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q) \leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p), \\
\mu_3^{t_j}(p) &\leq \mu_3^{t_j}(q) \leq \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q) \leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p).
\end{aligned}$$

It is clear that μ is commutative. Thus μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$. □

5 Conclusions

This paper attempt to develop and present a new general definition for neutrosophic finite automata. The general definition for (strong) subsystem also examined and discussed their properties. A comprehensive analysis and an appropriate methodology to manage the essential issues of output mapping in general fuzzy

automata were studied by Doostfatemen and Kremer [11]. Their approach is consistent with the output which is either associated with the states (Moore model) or with the transitions (Mealy model). Interval-valued fuzzy subsets have many applications in several areas. The concept of interval-valued fuzzy sets have been studied in various algebraic structures, see [7, 26]. On the basis [11] and [7], the future work will focus on general interval-valued neutrosophic finite automata with output respond to input strings.

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Generalized Neutrosophic Exponential map

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Abstract: The concept of $g\aleph$ compact open topology is introduced. Some characterization of this topology are discussed.

Keywords: $g\aleph$ locally Compact Hausdorff space; $g\aleph$ product topology; $g\aleph$ compact open topology; $g\aleph$ homeomorphism; $g\aleph$ evaluation map; $g\aleph$ Exponential map.

1 Introduction

Ever since the introduction of fuzzy sets by Zadeh [12] and fuzzy topological space by Chang [5], several authors have tried successfully to generalize numerous pivot concepts of general topology to the fuzzy setting. The concept of intuitionistic fuzzy set was introduced are studied by Atanassov [1] and many works by the same author and his colleagues appeared in the literature [[2],[3],[4]]. The concepts of generalized intuitionistic fuzzy closed set was introduced by Dhavaseelan et al[6]. The concepts of Intuitionistic Fuzzy Exponential Map Via Generalized Open Set by Dhavaseelan et al[8]. After the introduction of the neutrosophic set concept [[10], [11]]. The concepts of Neutrosophic Set and Neutrosophic Topological Spaces was introduced by A.A.Salama and S.A.Alblowi[9].

In this paper the concept of $g\aleph$ compact open topology are introduced. Some interesting properties are discussed. In this paper the concepts of $g\aleph$ local compactness and generalized $\aleph$ - product topology are developed. We have Throughout this paper neutrosophic topological spaces (briefly NTS) $(S_1, \xi_1), (S_2, \xi_2)$ and (S_3, ξ_3) will be replaced by S_1, S_2 and S_3 , respectively.

2 Preliminaries

Definition 2.1. [10, 11] Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
 $sup_I = i_{sup}, inf_I = i_{inf}$
 $sup_F = f_{sup}, inf_F = f_{inf}$
 $n - sup = t_{sup} + i_{sup} + f_{sup}$
 $n - inf = t_{inf} + i_{inf} + f_{inf}$. T, I, F are $\aleph$ - components.

Definition 2.2. [10, 11] Let S_1 be a non-empty fixed set. A $\aleph$ -set (briefly N -set) Λ is an object such that $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$ where $\mu_\Lambda(x)$, $\sigma_\Lambda(x)$ and $\gamma_\Lambda(x)$ which represents the degree of membership function (namely $\mu_\Lambda(x)$), the degree of indeterminacy (namely $\sigma_\Lambda(x)$) and the degree of non-membership (namely $\gamma_\Lambda(x)$) respectively of each element $x \in S_1$ to the set Λ .

Remark 2.1. [10, 11]

- (1) An N -set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$ can be identified to an ordered triple $\langle \mu_\Lambda, \sigma_\Lambda, \gamma_\Lambda \rangle$ in $]0^-, 1^+[$ on S_1 .
- (2) In this paper, we use the symbol $\Lambda = \langle \mu_\Lambda, \sigma_\Lambda, \gamma_\Lambda \rangle$ for the N -set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$.

Definition 2.3. [7] Let $S_1 \neq \emptyset$ and the N -sets Λ and Γ be defined as

$\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$, $\Gamma = \{\langle x, \mu_\Gamma(x), \sigma_\Gamma(x), \gamma_\Gamma(x) \rangle : x \in S_1\}$. Then

- (a) $\Lambda \subseteq \Gamma$ iff $\mu_\Lambda(x) \leq \mu_\Gamma(x)$, $\sigma_\Lambda(x) \leq \sigma_\Gamma(x)$ and $\gamma_\Lambda(x) \geq \gamma_\Gamma(x)$ for all $x \in S_1$;
- (b) $\Lambda = \Gamma$ iff $\Lambda \subseteq \Gamma$ and $\Gamma \subseteq \Lambda$;
- (c) $\bar{\Lambda} = \{\langle x, \gamma_\Lambda(x), \sigma_\Lambda(x), \mu_\Lambda(x) \rangle : x \in S_1\}$; [Complement of Λ]
- (d) $\Lambda \cap \Gamma = \{\langle x, \mu_\Lambda(x) \wedge \mu_\Gamma(x), \sigma_\Lambda(x) \wedge \sigma_\Gamma(x), \gamma_\Lambda(x) \vee \gamma_\Gamma(x) \rangle : x \in S_1\}$;
- (e) $\Lambda \cup \Gamma = \{\langle x, \mu_\Lambda(x) \vee \mu_\Gamma(x), \sigma_\Lambda(x) \vee \sigma_\Gamma(x), \gamma_\Lambda(x) \wedge \gamma_\Gamma(x) \rangle : x \in S_1\}$;
- (f) $[\]\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), 1 - \mu_\Lambda(x) \rangle : x \in S_1\}$;
- (g) $\langle \rangle \Lambda = \{\langle x, 1 - \gamma_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$.

Definition 2.4. [7] Let $\{\Lambda_i : i \in J\}$ be an arbitrary family of N -sets in S_1 . Then

- (a) $\bigcap \Lambda_i = \{\langle x, \wedge \mu_{\Lambda_i}(x), \wedge \sigma_{\Lambda_i}(x), \vee \gamma_{\Lambda_i}(x) \rangle : x \in S_1\}$;
- (b) $\bigcup \Lambda_i = \{\langle x, \vee \mu_{\Lambda_i}(x), \vee \sigma_{\Lambda_i}(x), \wedge \gamma_{\Lambda_i}(x) \rangle : x \in S_1\}$.

Since our main purpose is to construct the tools for developing NTS, we must introduce the $\aleph$ -sets 0_N and 1_N in X as follows:

Definition 2.5. [7] $0_N = \{\langle x, 0, 0, 1 \rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0 \rangle : x \in X\}$.

Definition 2.6. [7] A $\aleph$ -topology (briefly N -topology) on $S_1 \neq \emptyset$ is a family ξ_1 of N -sets in S_1 satisfying the following axioms:

- (i) $0_N, 1_N \in \xi_1$,
- (ii) $G_1 \cap G_2 \in \xi_1$ for any $G_1, G_2 \in \xi_1$,
- (iii) $\bigcup G_i \in \xi_1$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \xi_1$.

In this case the ordered pair (S_1, ξ_1) or simply S_1 is called an NTS and each N -set in ξ_1 is called a $\aleph$ -open set (briefly N -open set). The complement $\bar{\Lambda}$ of an N -open set Λ in S_1 is called a $\aleph$ -closed set (briefly N -closed set) in S_1 .

Definition 2.7. [7] Let Λ be an N -set in an NTS S_1 . Then

$Nint(\Lambda) = \bigcup \{G \mid G \text{ is an } N\text{-open set in } S_1 \text{ and } G \subseteq \Lambda\}$ is called the $\aleph$ -interior (briefly N -interior) of Λ ;
 $Ncl(\Lambda) = \bigcap \{G \mid G \text{ is an } N\text{-closed set in } S_1 \text{ and } G \supseteq \Lambda\}$ is called the $\aleph$ -closure (briefly N -cl) of Λ .

Definition 2.8. [7] Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$ then the $\aleph$ -set $x_{r,t,s}$ is called a $\aleph$ -point (in short NP) in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$. where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Definition 2.9. [7] Let (S_1, ξ_1) be a NTS . A $\aleph$ -set Λ in (S_1, ξ_1) is said to be a $g\aleph$ closed set if $Ncl(\Lambda) \subseteq \Gamma$ whenever $\Lambda \subseteq \Gamma$ and Γ is a $\aleph$ -open set. The complement of a $g\aleph$ closed set is called a $g\aleph$ open set.

Definition 2.10. [7] Let (X, T) be a $\aleph$ -topological space and Λ be a $\aleph$ -set in X . Then the $\aleph$ -generalized closure and $\aleph$ -generalized interior of Λ are defined by,

- (i) $NGcl(\Lambda) = \bigcap \{G : G \text{ is a generalized } \aleph\text{-closed set in } S_1 \text{ and } \Lambda \subseteq G\}.$
- (ii) $NGint(\Lambda) = \bigcup \{G : G \text{ is a generalized } \aleph\text{-open set in } S_1 \text{ and } \Lambda \supseteq G\}.$

3 Neutrosophic Compact Open Topology

Definition 3.1. Let S_1 and S_2 be any two NTS . A mapping $f : S_1 \rightarrow S_2$ is generalized neutrosophic [briefly $g\aleph$] continuous iff for every $g\aleph$ open set V in S_2 , there exists a $g\aleph$ open set U in S_1 such that $f(U) \subseteq V$.

Definition 3.2. A mapping $f : S_1 \rightarrow S_2$ is said to be $g\aleph$ homeomorphism if f is bijective, $g\aleph$ continuous and $g\aleph$ open.

Definition 3.3. Let S_1 be a NTS . S_1 is said to be $g\aleph$ Hausdorff space or T_2 space if for any two $\aleph$ -sets A and B with $A \cap B = 0_\sim$, there exist $g\aleph$ open sets U and V , such that $A \subseteq U, B \subseteq V$ and $U \cap V = 0_\sim$.

Definition 3.4. A NTS S_1 is said to be $g\aleph$ locally compact iff for any $\aleph$ -set A , there exists a $g\aleph$ open set G , such that $A \subseteq G$ and G is $g\aleph$ compact. That is each $g\aleph$ open cover of G has a finite subcover.

Remark 3.1. Let S_1 and S_2 be two NTS with S_2 $\aleph$ -compact. Let $x_{r,t,s}$ be any $\aleph$ -point in S_1 . The $\aleph$ -product space $S_1 \times S_2$ containing $\{x_{r,t,s}\} \times S_2$. It is clear that $\{x_{r,t,s}\} \times S_2$ is $\aleph$ -homeomorphic to S_2 .

Remark 3.2. Let S_1 and S_2 be two NTS with S_2 $\aleph$ -compact. Let $x_{r,t,s}$ be any $\aleph$ -point in S_1 . The $\aleph$ -product space $S_1 \times S_2$ containing $\{x_{r,t,s}\} \times S_2$. $\{x_{r,t,s}\} \times S_2$ is $\aleph$ -compact.

Remark 3.3. A $\aleph$ -compact subspace of a $\aleph$ -Hausdorff space is $\aleph$ -closed.

Proposition 3.1. A $g\aleph$ Hausdorff topological space S_1 , the following conditions are equivalent.

- (a) S_1 is $g\aleph$ locally compact

(b) for each $\aleph$ set A , there exists a $\mathfrak{g}\aleph$ open set G in S_1 such that $A \subseteq G$ and $NGcl(G)$ is $\mathfrak{g}\aleph$ compact

Proof. (a) $\Rightarrow$ (b) By hypothesis for each $\aleph$ - set A in S_1 , there exists a $\mathfrak{g}\aleph$ open set G , such that $A \subseteq G$ and G is $\mathfrak{g}\aleph$ compact. Since S_1 is $\mathfrak{g}\aleph$ Hausdorff, by Remark 3.3 ($\mathfrak{g}\aleph$ compact subspace of $\mathfrak{g}\aleph$ Hausdorff space is $\mathfrak{g}\aleph$ closed), G is $\mathfrak{g}\aleph$ closed, thus $G = NGcl(G)$. Hence $A \subseteq G = NGcl(G)$ and $NGcl(G)$ is $\mathfrak{g}\aleph$ compact.

(b) $\Rightarrow$ (a) Proof is simple.

Proposition 3.2. Let S_1 be a $\mathfrak{g}\aleph$ Hausdorff topological space. Then S_1 is $\mathfrak{g}\aleph$ locally compact on an $\aleph$ - set A in S_1 iff for every $\mathfrak{g}\aleph$ open set G containing A , there exists a $\mathfrak{g}\aleph$ open set V , such that $A \subseteq V$, $NGcl(V)$ is $\mathfrak{g}\aleph$ compact and $NGcl(V) \subseteq G$.

Proof. Suppose that S_1 is $\mathfrak{g}\aleph$ locally compact on an $\aleph$ - set A . By Definition 3.4, there exists a $\mathfrak{g}\aleph$ open set G , such that $A \subseteq G$ and G is $\mathfrak{g}\aleph$ compact. Since S_1 is $\mathfrak{g}\aleph$ Hausdorff space, by Remark 3.3 ($\mathfrak{g}\aleph$ compact subspace of $\mathfrak{g}\aleph$ Hausdorff space is $\mathfrak{g}\aleph$ closed), G is $\mathfrak{g}\aleph$ closed, thus $G = NGcl(G)$. Consider an $\aleph$ - set $A \subseteq G$. Since S_1 is $\mathfrak{g}\aleph$ Hausdorff space, by Definition 3.3, for any two $\aleph$ - sets A and B with $A \cap B = 0_\sim$, there exist $\mathfrak{g}\aleph$ open sets C and D , such that $A \subseteq C$, $B \subseteq D$ and $C \cap D = 0_\sim$. Let $V = C \cap G$. Hence $V \subseteq G$ implies $NGcl(V) \subseteq NGcl(G) = G$. Since $NGcl(V)$ is $\mathfrak{g}\aleph$ closed and G is $\mathfrak{g}\aleph$ compact, by Remark 3.3 (every $\mathfrak{g}\aleph$ closed subset of a $\mathfrak{g}\aleph$ compact space is $\mathfrak{g}\aleph$ compact) it follows that $NGcl(V)$ is $\aleph$ - compact. Thus $A \subseteq NGcl(V) \subseteq G$ and $NGcl(G)$ is $\mathfrak{g}\aleph$ compact.

The converse follows from Proposition 3.1(b).

Definition 3.5. Let S_1 and S_2 be two NTS. The function $T : S_1 \times S_2 \rightarrow S_2 \times S_1$ defined by $T(x, y) = (y, x)$ for each $(x, y) \in S_1 \times S_2$ is called a $\aleph$ - switching map.

Proposition 3.3. The $\aleph$ - switching map $T : S_1 \times S_2 \rightarrow S_2 \times S_1$ defined as above is $\mathfrak{g}\aleph$ continuous.

We now introduce the concept of $\mathfrak{g}\aleph$ compact open topology in the set of all $\mathfrak{g}\aleph$ continuous functions from a NTS S_1 to a NTS S_2 .

Definition 3.6. Let S_1 and S_2 be two NTS and let $S_2^{S_1} = \{f : S_1 \rightarrow S_2 \text{ such that } f \text{ is } \mathfrak{g}\aleph \text{ continuous}\}$. We give this class $S_2^{S_1}$ a topology called the $\mathfrak{g}\aleph$ compact open topology as follows: Let $\mathcal{K} = \{K \in I_1^S : K \text{ is } \mathfrak{g}\aleph \text{ compact } S_1\}$ and $\mathcal{V} = \{V \in I_2^S : V \text{ is } \mathfrak{g}\aleph \text{ open in } S_2\}$. For any $K \in \mathcal{K}$ and $V \in \mathcal{V}$, let $S_{K,V} = \{f \in S_2^{S_1} : f(K) \subseteq V\}$.

The collection of all such $\{S_{K,V} : K \in \mathcal{K}, V \in \mathcal{V}\}$ generates an $\aleph$ - structure on the class $S_2^{S_1}$.

4 Generalized Neutrosophic Evaluation Map and Generalized Neutrosophic Exponential Map

We now consider the $\mathfrak{g}\aleph$ product topological space $S_2^{S_1} \times S_1$ and define a $\mathfrak{g}\aleph$ continuous map from $S_2^{S_1} \times S_1$ into S_2 .

Definition 4.1. The mapping $e : S_2^{S_1} \times S_1 \rightarrow S_2$ defined by $e(f, A) = f(A)$ for each $\aleph$ - set A in S_1 and $f \in S_2^{S_1}$ is called the $\mathfrak{g}\aleph$ evaluation map.

Definition 4.2. Let S_1, S_2 and S_3 be three NTS and $f : S_3 \times S_1 \rightarrow S_2$ be any function. Then the induced map $\hat{f} : S_1 \rightarrow S_2^{S_3}$ is defined by $(\hat{f}(A_1))(A_2) = f(A_2, A_1)$ for $\aleph$ - sets A_1 of S_1 and A_2 of S_3 .

Conversely, given a function $\hat{f} : S_1 \rightarrow S_2^{S_3}$, a corresponding function f can be also be defined by the same rule.

Proposition 4.1. Let S_1 be a $g\aleph$ locally compact Hausdorff space. Then the $g\aleph$ evaluation map $e : S_2^{S_1} \times S_1 \rightarrow S_2$ is $g\aleph$ continuous.

Proof. Consider $(f, A_1) \in S_2^{S_1} \times S_1$, where $f \in S_2^{S_1}$ and $\aleph$ -set A_1 of S_1 . Let V be a $g\aleph$ open set containing $f(A_1) = e(f, A_1)$ in S_2 . Since S_1 is $g\aleph$ locally compact and f is $g\aleph$ continuous, by Proposition 3.2, there exists a $g\aleph$ open set U in S_1 , such that $A_1 \subseteq NGcl(U)$ and $NGcl(U)$ is $g\aleph$ compact and $f(NGcl(U)) \subseteq V$.

Consider the $g\aleph$ open set $S_{NGcl(U),V} \times U$ in $S_2^{S_1} \times S_1$. (f, A_1) is such that $f \in S_{NGcl(U),V}$ and $A_1 \subseteq U$. Let (g, A_2) be such that $g \in S_{NGcl(U),V}$ and $A_2 \subseteq U$ be arbitrary, thus $g(NGcl(U)) \subseteq V$. Since $A_2 \subseteq U$, we have $g(A_2) \subseteq V$ and $e(g, A_2) = g(A_2) \subseteq V$. Thus $e(S_{NGcl(U),V} \times U) \subseteq V$. Hence e is $g\aleph$ continuous.

Proposition 4.2. Let S_1 and S_2 be two NTS with S_2 is $g\aleph$ compact. Let A_1 be any $\aleph$ -set in S_1 and N be a $g\aleph$ open set in the $g\aleph$ product space $S_1 \times S_2$ containing $\{A_1\} \times S_2$. Then there exists some $g\aleph$ open W with $A_1 \subseteq W$ in S_1 , such that $\{A_1\} \times S_2 \subseteq W \times S_2 \subseteq N$.

Proof. It is clear that by Remark 3.1, $\{A_1\} \times S_2$ is $g\aleph$ homeomorphism to S_2 and hence by Remark 3.2, $\{A_1\} \times S_2$ is $g\aleph$ compact. We cover $\{A_1\} \times S_2$ by the basis elements $\{U \times V\}$ (for the $g\aleph$ product topology) lying in N . Since $\{A_1\} \times S_2$ is $g\aleph$ compact, $\{U \times V\}$ has a finite subcover, say a finite number of basis elements $U_1 \times V_1, \dots, U_n \times V_n$. Without loss of generality we assume that $\{A_1\} \subseteq U_i$ for each $i = 1, 2, \dots, n$. Since otherwise the basis elements would be superfluous.

Let $W = \bigcap_{i=1}^n U_i$. Clearly W is $g\aleph$ open and $A_1 \subseteq W$. We show that $W \times S_2 \subseteq \bigcup_{i=1}^n (U_i \times V_i)$. Let (A_1, B) be an $\aleph$ -set in $W \times S_2$. Now $(A_1, B) \subseteq U_i \times V_i$ for some i , thus $B \subseteq V_i$. But $A_1 \subseteq U_i$ for every $i = 1, 2, \dots, n$ (because $A_1 \subseteq W$). Therefore, $(A_1, B) \subseteq U_i \times V_i$ as desired. But $U_i \times V_i \subseteq N$ for all $i = 1, 2, \dots, n$ and $W \times S_2 \subseteq \bigcup_{i=1}^n (U_i \times V_i)$, therefore $W \times S_2 \subseteq N$.

Proposition 4.3. Let S_3 be a $g\aleph$ locally compact Hausdorff space and S_1, S_2 be arbitrary NTS. Then a map $f : S_3 \times S_1 \rightarrow S_2$ is $g\aleph$ continuous iff $\hat{f} : S_1 \rightarrow S_2^{S_3}$ is $g\aleph$ continuous, where $\hat{f}$ is defined by the rule $(\hat{f}(A_1))(A_2) = f(A_2, A_1)$.

Proof. Suppose that $\hat{f}$ is $g\aleph$ continuous. Consider the functions $S_3 \times S_1 \xrightarrow{i_Z} \times \hat{f} S_3 \times S_2^{S_3} \xrightarrow{t} S_2^{S_3} \times S_3 \xrightarrow{e} S_2$, where i_Z denote the $\aleph$ -identity function on Z , t denote the $\aleph$ -switching map and e denote the $g\aleph$ evaluation map. Since $et(i_Z \times \hat{f})(A_2, A_1) = et(A_2, \hat{f}(A_1)) = e(\hat{f}(A_1), A_2) = (\hat{f}(A_1))(A_2) = f(A_2, A_1)$ it follows that $f = et(i_Z \times \hat{f})$ and f being the composition of $g\aleph$ continuous functions is itself $g\aleph$.

Conversely, suppose that f is $g\aleph$ continuous, let A_1 be any arbitrary $\aleph$ -set in S_1 . We have $\hat{f}(A_1) \in S_2^{S_3}$. Consider $S_{K,U} = \{g \in S_2^{S_3} : g(K) \subseteq U, K \in I^{S_3} \text{ is } g\aleph \text{ compact and } U \in I^{S_2} \text{ is } g\aleph \text{ open}\}$, containing $\hat{f}(A_1)$. We need to find a $g\aleph$ open W with $A_1 \subseteq W$, such that $\hat{f}(A_1) \subseteq S_{K,U}$; this will suffice to prove $\hat{f}$ to be a $g\aleph$ continuous map.

For any $\aleph$ -set A_2 in K , we have $(\hat{f}(A_1))(A_2) = f(A_2, A_1) \in U$ thus $f(K \times \{A_1\}) \subseteq U$, that is $K \times \{A_1\} \subseteq f^{-1}(U)$. Since f is $g\aleph$ continuous, $f^{-1}(U)$ is a $g\aleph$ open set in $S_3 \times S_1$. Thus $f^{-1}(U)$ is a $g\aleph$ open set $S_3 \times S_1$ containing $K \times \{A_1\}$. Hence by Proposition 4.2, there exists a $g\aleph$ open W with $A_1 \subseteq W$ in S_1 , such that $K \times \{A_1\} \subseteq K \times W \subseteq f^{-1}(U)$. Therefore $f(K \times W) \subseteq U$. Now for any $A_1 \subseteq W$ and $A_2 \subseteq K$, $f(A_2, A_1) = (\hat{f}(A_1))(A_2) \subseteq U$. Therefore $\hat{f}(A_1)(K) \subseteq U$ for all $A_1 \subseteq W$. That is $\hat{f}(A_1) \in S_{K,U}$ for all $A_1 \subseteq W$. Hence $\hat{f}(W) \subseteq S_{K,U}$ as desired.

Proposition 4.4. Let S_1 and S_3 be two $g\aleph$ locally compact Hausdorff spaces. Then for any NTS S_2 , the function $E : S_2^{S_3 \times S_1} \rightarrow (S_2^{S_3})^{S_1}$ defined by $E(f) = \hat{f}$ (that is $E(f)(A_1)(A_2) = f(A_2, A_1) = (\hat{f}(A_1))(A_2)$) for all $f : S_3 \times S_1 \rightarrow S_2$ is a $g\aleph$ homeomorphism.

Proof.

(a) Clearly E is onto.

- (b) For E to be injective. Let $E(f) = E(g)$ for $f, g : S_3 \times S_1 \rightarrow S_2$. Thus $\hat{f} = \hat{g}$, where $\hat{f}$ and $\hat{g}$ are the induced maps of f and g respectively. Now for any $\aleph$ -set A_1 in S_1 and any $\aleph$ -set A_2 in S_3 , $f(A_2, A_1) = (\hat{f}(A_1))(A_2) = (\hat{g}(A_1))(A_2) = g(A_2, A_1)$; thus $f = g$.
- (c) For proving the $\mathfrak{g}\aleph$ continuity of E , consider any $\mathfrak{g}\aleph$ subbasis neighbourhood V of $\hat{f}$ in $(S_2^{S_3})^{S_1}$, that is V is of the form $S_{K,W}$ where K is a $\mathfrak{g}\aleph$ compact subset of S_1 and W is $\mathfrak{g}\aleph$ open in $S_2^{S_3}$. Without loss of generality we may assume that $W = S_{L,U}$, where L is a $\mathfrak{g}\aleph$ compact subset of S_3 and U is a $\mathfrak{g}\aleph$ open set in S_2 . Then $\hat{f}(K) \subseteq S_{L,U} = W$ and this implies that $\hat{f}(K)(L) \subseteq U$. Thus for any $\aleph$ -set $A_1 \subseteq K$ and for all $\aleph$ -sets $A_2 \subseteq L$. We have $(\hat{f}(A_1))(A_2) \subseteq U$, that is $f(A_2, A_1) \subseteq U$ and therefore $f(L \times K) \subseteq U$. Now since L is $\mathfrak{g}\aleph$ compact in S_3 and K is $\mathfrak{g}\aleph$ compact in S_1 , $L \times K$ is also $\mathfrak{g}\aleph$ compact in $S_3 \times S_1$ [6] and since U is a $\mathfrak{g}\aleph$ open set in S_2 , we conclude that $f \in S_{L \times K, U} \subseteq S_2^{S_3 \times S_1}$. We assert that $E(S_{L \times K, U}) \subseteq S_{K,W}$. Let $g \in S_{L \times K, U}$ be arbitrary. Thus $g(L \times K) \subseteq U$, that is $g(A_2, A_1) = (\hat{g}(A_1))(A_2) \subseteq U$ for all $\aleph$ -sets $A_2 \subseteq L$ in S_3 and for all $\aleph$ -sets $A_1 \subseteq K$ in S_1 . So $(\hat{g}(A_1))(L) \subseteq U$ for all $\aleph$ -sets $A_1 \subseteq K$ in S_1 , that is $\hat{g}(A_1) \subseteq S_{L,U} = W$ for all $\aleph$ -sets $A_1 \subseteq K$ in U . Hence we have $\hat{g}(K) \subseteq W$, that is $\hat{g} = E(g) \in S_{K,W}$ for any $g \in S_{L \times K, U}$. Thus $E(S_{L \times K, U}) \subseteq S_{K,W}$. This proves that E is $\mathfrak{g}\aleph$ continuous.
- (d) For proving the $\mathfrak{g}\aleph$ continuity of E^{-1} , we consider the following $\mathfrak{g}\aleph$ evaluation maps: $e_1 : (S_2^{S_3})^{S_1} \times S_1 \rightarrow S_2^{S_3}$ defined by $e_1(\hat{f}, A_1) = \hat{f}(A_1)$ where $\hat{f} \in (S_2^{S_3})^{S_1}$ and A_1 is an $\aleph$ -set in S_1 and $e_2 : S_2^{S_3} \times S_3 \rightarrow S_2$ defined by $e_2(g, A_2) = g(A_2)$ where $g \in S_2^{S_3}$ and A_2 is a $\aleph$ -set in S_3 . Let ψ denote the composition of the following $\mathfrak{g}\aleph$ continuous functions $\psi : (S_3 \times S_1) \times (S_2^{S_3})^{S_1} \xrightarrow{T} (S_2^{S_3})^{S_1} \times (S_3 \times S_1) \xrightarrow{i \times t} (S_2^{S_3})^{S_1} \times (S_1 \times S_3) \xrightarrow{\cong} ((S_2^{S_3})^{S_1} \times S_1) \times S_3 \xrightarrow{e_1 \times i_Z} (S_2^{S_3}) \times S_3 \xrightarrow{e_2} S_2$, where i, i_Z denote the $\aleph$ -identity maps on $(S_2^{S_3})^{S_1}$ and S_3 respectively and T, t denote the $\aleph$ -switching maps. Thus $\psi : (S_3 \times S_1) \times (S_2^{S_3})^{S_1} \rightarrow S_2$ that is $\psi \in S_2^{(S_3 \times S_1) \times (S_2^{S_3})^{S_1}}$. We consider the map $\tilde{E} : S_2^{(S_3 \times S_1) \times (S_2^{S_3})^{S_1}} \rightarrow (S_2^{(S_3 \times S_1)})^{(S_2^{S_3})^{S_1}}$ (as defined in the statement of the proposition in fact it is E). So $\tilde{E}(\psi) : (S_2^{S_3})^{S_1} \rightarrow S_2^{(S_3 \times S_1)}$. Now for any $\aleph$ -sets A_2 in S_3, A_1 in S_1 and $f \in S_2^{(S_3 \times S_1)}$, again to check that $(\tilde{E}(\psi) \circ E)(f)(A_2, A_1) = f(A_2, A_1)$; hence $\tilde{E}(\psi) \circ E = \text{identity}$. Similarly for any $\hat{g} \in (S_2^{S_3})^{S_1}$ and $\aleph$ -sets A_1 in S_1, A_2 in S_3 , again to check that $(E \circ \tilde{E}(\psi))(\hat{g})(A_1, A_2) = (\hat{g}(A_1))(A_2)$; hence $E \circ \tilde{E}(\psi) = \text{identity}$. Thus E is a $\mathfrak{g}\aleph$ homeomorphism.

Definition 4.3. The map E in Proposition 4.4 is called the $\mathfrak{g}\aleph$ exponential map.

As easy consequence of Proposition 4.4 is as follows.

Proposition 4.5. Let S_1, S_2 and S_3 be three $\mathfrak{g}\aleph$ locally compact Hausdorff spaces. Then the map $N : S_2^{S_1} \times S_3^{S_2} \rightarrow S_3^{S_1}$ defined by $N(f, g) = g \circ f$ is $\mathfrak{g}\aleph$ continuous.

Proof. Consider the following compositions: $S_1 \times S_2^{S_1} \times S_3^{S_2} \xrightarrow{T} S_2^{S_1} \times S_3^{S_2} \times S_1 \xrightarrow{t \times i_X} S_3^{S_2} \times S_2^{S_1} \times S_1 \xrightarrow{\cong} S_3^{S_2} \times (S_2^{S_1} \times S_1) \xrightarrow{i \times e_2} S_3^{S_2} \times S_2 \xrightarrow{e_2} S_3$ where T, t denote the $\aleph$ -switching maps, i_X, i denote the $\aleph$ -identity functions on S_1 and $S_3^{S_2}$ respectively and e_2 denote the $\mathfrak{g}\aleph$ evaluation maps. Let $\varphi = e_2 \circ (i \times e_2) \circ (t \times i_X) \circ T$. By proposition 4.4, we have an exponential map. $E : S_3^{S_1 \times S_2^{S_1} \times S_3^{S_2}} \rightarrow (S_3^{S_1})^{S_2^{S_1} \times S_3^{S_2}}$. Since $\varphi \in S_3^{S_1 \times S_2^{S_1} \times S_3^{S_2}}$, $E(\varphi) \in (S_3^{S_1})^{S_2^{S_1} \times S_3^{S_2}}$. Let $N = E(\varphi)$, that is $N : S_2^{S_1} \times S_3^{S_2} \rightarrow S_3^{S_1}$ is an $\mathfrak{g}\aleph$ continuous. For $f \in S_2^{S_1}, g \in S_3^{S_2}$ and for any $\aleph$ -set A_1 in S_1 , it is easy to see that $N(f, g)(A_1) = g(f(A_1))$.

5 Conclusions

In this paper, we introduced the concept of gN compact open topology and Some characterization of this topology are discussed.

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Implementation of Neutrosophic Function Memberships Using MATLAB Program

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Abstract. Membership function (MF) plays a key role for getting an output of a system and hence it influences system's performance directly. Therefore choosing a MF is an essential task in fuzzy logic and neutrosophic logic as well. Uncertainty is usually represented by MFs. In this paper, a novel Matlab code is derived for trapezoidal neutrosophic function and the validity of the proposed code is proved with illustrative graphical representation

Keywords: Membership function, Matlab code, Trapezoidal neutrosophic function, Graphical representation

1 Introduction

The membership function (MF) designs a structure of practical relationship to relational structure numerically where the elements lies between 0 and 1. By determining the MFs one can model the relationship between the cognitive and stimuli portrayal in fuzzy set theory [1]. The computed MF will provide a solution to the problem and the complete process can be observed as a training and acceptable approximation to the function from the behavior of the objects [2]. This kind of MFs can be utilized for the fuzzy implication appeared in the given rules to examine more examples [3].

The MFs of fuzzy logic is nothing but a stochastic representation and are used to determine a probability space and its value may be explained as probabilities. The stochastic representation will to know the reasoning and capability of fuzzy control [4]. MFs which are characterized in a single domain where the functions are in terms of single variable are playing a vital role in fuzzy logic system. FMFs determine the degree of membership (M/S) which is a crisp value. Generally MFs are considered as either triangular or trapezoidal as they are adequate, can be design easily and flexible [5].

MFs can be carried out using hardware [6]. MFs are taking part in most of the works done under fuzzy environment without checking their existence for sure and also in the connection between a studied characteristic for sure and its reference set won't be problematic as it is a direct measurement [7]. It is adorable to have continuously differentiable MFs with less parameters [8]. MFs plays an important role in fuzzy classifier (FC). In traditional FC, the domain of every input variable is separated into various intervals. All these intervals is assumed to be a FS and a correlated MF is determined. Hence the input space is separated again into various sub regions which are all parallel in to input axes and a fuzzy rule is defined for all these sub regions if the input belongs to the sub region then it is also belongs to the associated class with the sub region.

Further the degrees of M/S of an unidentified input for all the FSs are evaluated and the input is restricted into the class with maximum degree of M/S. Thus the MFs are directly control the performance of the fuzzy classifier [10]. If the position of the MF is changed then the direct methods maximize the understanding rate of the training data by calculating the total increase directly [11]. Estimation of the MF is usually based on the level of information gained with the experiment transferred by the numerical data [12]. Due to the important role of MFs, concepts of fuzzy logic have been applied in many of the control systems for controlling the robot, nuclear reactor, climate, speed of the car, power systems, memory device under fuzzy logic, aircraft flight, mobile robots and focus of a camcorder.

There has been a habit of restrain the MFs into a well-known formats like triangular, trapezoidal and standard Gaussian or sigmoid types [13]. In information systems the incomplete information can be designed by rough sets [20]. Neutrosophy has established the base for the entire family of novel mathematical theories which generalizes the counterparts of the conventional and fuzzy sets [21]. The success of an approach depends on the MFs and hence designing MFs is an important task for the process and the system. Theory of FSs contributes the way of handling impreciseness, uncertainty and vagueness in the software metrics. The uncertainty of the problem can be solved by considering MFs in an expert system under fuzzy setting. Triangular and trapezoidal MFs are flexible representation of domain expert knowledge and where the computational complexity is less. Hence the derivation of the MF is need to be clarified.

The MFs are continuous and maps from any closed interval to $[0,1]$. Also which are all either monotonically decreasing or increasing or both [22]. A connectively flexible aggregation of crisp and imprecise knowledge is possible with the horizontal MFs which are capable of introducing uncertainty directly [23]. There are effective methods for calculating MFs of FSs connected with few multi criteria decision making problem [25]. Due to the possibility of having some degree of hesitation, one could not define the non-membership degree by subtracting membership degree from 1 [26]. The degree of the fuzzy sets will be determined by FMFs. [30] Crisp value is converted into fuzzy during fuzzification process. If uncertainty exists on the variable then becomes fuzzy and could be characterized by MFs. The degree of MF is determined by fuzzification.

In the real world problems satisfaction of the decision maker is not possible at most of the time due to impreciseness and incompleteness of the information of the data. Fuzziness exist in the FS is identified by the MF [27]. The uncertainty measure is the possible MF of the FS and is interpreted individually. This is the advantage of MFs especially one needs to aggregate the data and human expert knowledge. Designing MFs vary according to the ambition of their use. Membership functions influence a quality of inference [31].

Neutrosophy is the connecting idea with its opposite idea also with non-committal idea to get the common parts with unknown things [36]. Artificial network, fuzzy clustering, genetic algorithm are some methods to determine the MFs and all these consume time with complexities. The MFs plays a vital role in getting the output. The methods are uncertain due to noisy data and difference of opinion of the people. The most suitable shape and widely used MFs in fuzzy systems are triangular and trapezoidal [37]. Properties and relations of multi FSs and its extension are depending on the order relations of the MFs [38]. FS is the class of elements with a continuum of grades of M/S [39].

The logic of neutrosophic concept is an explicit frame trying to calculate the truth, indeterminacy and falsity. Smarandache observes the dissimilarity of intuitionistic fuzzy logic (IFL) and neutrosophic logic (NL). NL could differentiate absolute truth (AT) and relative truth (RT) by assigning 1^+ for AT and 1 for RT and is also applied in the field of philosophy. Hence the standard interval $[0,1]$ used in IFS is extended to non-standard $]0,1^+[$ in NL. There is not condition on truth, indeterminacy and falsity which are all the subsets of non-standard unitary interval. This is the reason of considering $0^- \leq \inf T \leq \inf I \leq \inf F \leq \sup T \leq \sup I \leq \sup F \leq 3^+$ and which is useful to characterize para consistent and incomplete information [40]. The generalized form of trapezoidal FNs, trapezoidal IFNs, triangular FN and TIFNs are the trapezoidal and triangular neutrosophic fuzzy number [48].

2 Review of Literature

The authors of, [Zysno 1] presented a methodology to determine the MFs analytically. [Sebag and Schoenauer 2] Established algorithms to determine functions from examples. [Bergadano and Cutello 3] proposed an effective technique to learn MFs for fuzzy predicates. [Hansson 4] introduce a stochastic perception of the MFs based on fuzzy logic. [Kelly and Painter 5] proposed a methodology to define N-dimensional fuzzy MFs (FMFs) which is a generalized form of one dimensional MF generally used in fuzzy systems. [Peterson et al. 6] presented a hardware implementation of MF. [Royo and Verdegay 7] examined about the characterization of the different cases where the endurance of the MF is assured.

[Grauel and L. A. Ludwig 8] proposed a class of MFs for symmetrically and asymmetrically in exponential order and constructed a more adaptive MFs. [Straszecka 9] presented preliminaries and methodology to define the MFs of FSs and discussed about application of FS with its universe, certainty of MFs and format. [Abe 10] examined the influence of the MFs in fuzzy classifier. [Abe 11] proved that by adjusting the slopes and positions the performance of the fuzzy rule classification can be improved [Pedrycz and G. Vukovich 12] imposed on an influential issue of determining MF. [J. M. Garibaldi and R. I. John 13] focused more MFs which considered as the alternatives in fuzzy systems [T. J. Ross 14] established the methodology of MFs.

[Brennan, E. Martin 15] proposed MFs for dimensional proximity. [Hachani et al. 16] Proposed a new incremental method to represent the MFs for linguistic terms. [Gasparovica et al. 17] examined about the suitable MF for data analysis in bioinformatics. [Zade and Ismayilova 18] investigated a class of MFs which

conclude the familiar types of MFs for FSs. [Bilgic 19] proposed a method of measuring MFs. [Broumi et al. 20] established rough neutrosophic sets and their properties. [Salama et al. 21] proposed a technique for constructing. [Yadava and Yadav 22] proposed an approach for constructing the MFs of software metrics. [Piegat and M. Landowski 23] proposed horizontal MFs to determine the FS instead of usual vertical MFs. [Mani 24] reviewed the relation between different meta theoretical concepts of probability and rough MFs critically.

[Sularia 25] showed their interest of multi-criteria decision analysis under fuzzy environment. [Ali and F. Smarandache 26] Introduced complex NS. [Goyal et al. 27] proposed a circuit model for Gaussian MF. [Can and Ozguven 28] proposed fuzzy logic controller with neutrosophic MFs. [Ali et al. 29] introduced δ -equalities and their properties of NSs. [Radhika and Parvathi 30] introduced different fuzzification methods for intuitionistic fuzzy environment. [Porebski and Straszecka 31] examined diagnosing rules for driving data which can be described by human experts. [Hong et al. 32] accumulated the concepts of fuzzy MFs using fuzzy c-means clustering method.

[Kundu 33] proposed an improved method of approximation of piecewise linear MFs with the support of approximation of cut function obtained by sigmoid function. [Wang 34] proposed the operational laws of fuzzy ellipsoid numbers and straight connection between the MFs which are located on the junctions and edges. [Mani 35] studied the contemplation of theory of probability over rough MFs. [Christianto and Smarandache 36] offered a new perception at Liquid church and neutrosophic MF. [Asanka and A. S. Perera 37] introduced a new approach of using box plot to determine fuzzy Function with some conditions. [Sebastian and F. Smarandache 38] generalized the concepts of NSs and its extension method. [Reddy 39] proposed a FS with two MFs such as Belief and Disbelief. [Lupianeza 40] determined NSs and Topology.

[Zhang et al. 41] derived FMFs analytically. [Wang 42] framed a framework theoretically to construct MFs in a hierarchical order. [Germashev et al. 43] proposed convergence of series of FNs along with Unimodal membership. [Marlen and Dorzhigulov 44] implemented FMF with Memristor. [Ahmad et al. 45] introduced MFs and fuzzy rules for Harumanis examinations [Buhentala et al. 46] explained about the procedure and process of the Takagi-Sugeno fuzzy model. [Broumi et al. 47-55] proposed few concepts of NSs, triangular and trapezoidal NNs.

From this literature study, to the best our knowledge there is no contribution of work on deriving membership function using Matlab under neutrosophic environment and hence it's a motivation of the present work.

3 Preliminaries

Definition: A trapezoidal neutrosophic number $a = \langle (a, b, c, d); w_a, u_a, y_a \rangle$ is a special neutrosophic set on the real number set $\mathbb{R}$, whose truth-membership, indeterminacy- membership and falsity-membership functions are defined as follows:

$$\mu_a(x) = \begin{cases} \frac{(x-a)}{(b-a)} w_a & , a \leq x \leq b \\ w_a & , b \leq x \leq c \\ \frac{(d-x)}{(d-c)} w_a & , c \leq x \leq d \\ 0 & , \text{otherwise} \end{cases} \quad \nu_a(x) = \begin{cases} \frac{(b-x)+u_a(x-a)}{(b-a)} & , a \leq x \leq b \\ u_a & , b \leq x \leq c \\ \frac{(x-c)+u_a(d-x)}{(d-c)} & , c \leq x \leq d \\ 1 & , \text{otherwise} \end{cases}$$

$$\lambda_a(x) = \begin{cases} \frac{(b-x)+y_a(x-a)}{(b-a)} & , a \leq x \leq b \\ y_a & , b \leq x \leq c \\ \frac{(x-c)+y_a(d-x)}{(d-c)} & , c \leq x \leq d \\ 1 & , \text{otherwise} \end{cases}$$

4. Proposed Matlab code to find Trapezoidal Neutrosophic Function

In this section, trapezoidal neutrosophic function has been proposed using Matlab program and for the differentnt membership values, pictorical representation is given and the Matlab code is designed as follows.

```
Trapezoidal neutrosophic Function (trin)
%x=45:70;
%[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.6)%
```

```
U truth membership
V indterminacy membership
W :falsemembership
```

```
function [y,z,t]=trin(x,a,b,c,d,u,v,w)
y=zeros(1,length(x));
z=zeros(1,length(x));
t=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
    y(j)=0;
    z(j)=1;
    t(j)=1;
elseif(x(j)>=a)&&(x(j)<=b)
    y(j)=u*(((x(j)-a)/(b-a)));
    z(j)=(((b-x(j))+v*(x(j)-a))/(b-a));
    t(j)=(((b-x(j))+w*(x(j)-a))/(b-a));
elseif(x(j)>=b)&&(x(j)<=c)
    y(j)=u;
    z(j)=v;
    t(j)=w;
elseif(x(j)>=c)&&(x(j)<=d)
    y(j)=u*(((d-x(j))/(d-c)));
    z(j)=(((x(j)-c)+v*(d-x(j)))/(d-c));
    t(j)=(((x(j)-c)+w*(d-x(j)))/(d-c));
elseif(x(j)>=d)
    y(j)=0;
    z(j)=1;
    t(j)=1;
end
end
plot(x,y,x,z,x,t)
legend('Membership function','indeterminate function','Non-membership function')
end
```

4.1 Example

The figure 1 portrayed the pictorical representation of the trapezoidal neutrosophic function $a = \langle (0.3, 0.5, 0.6, 0.7); 0.4, 0.2, 0.3 \rangle$

The line command to show this function in Matlab is written below:

```
x=0:0.01:1;
[y,z,t]=trin(x,0.3,0.5,0.6,0.7, 0.4, 0.2,0.3)
```

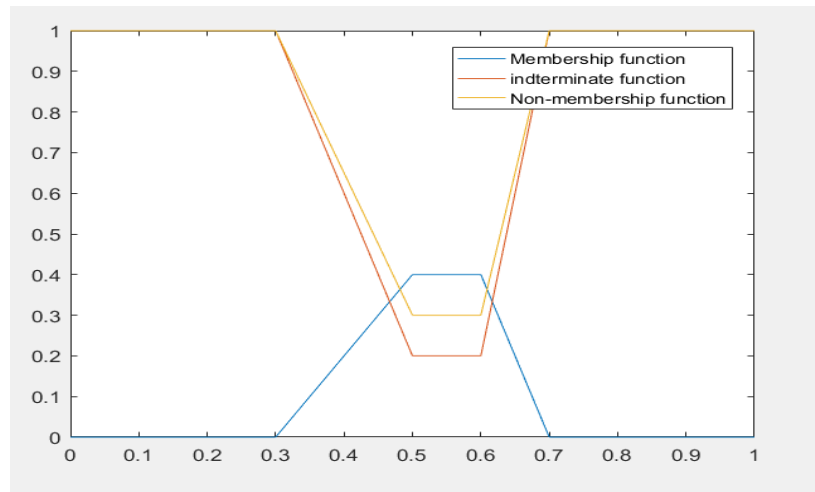


Figure 1: Trapezoidal neutrosophic function for example 4.1

4.2 Example

The figure 2 portrayed the trapezoidal neutrosophic function of $a = \langle (50, 55, 60, 65); 0.6, 0.4, 0.3 \rangle$

The line command to show this function in Matlab is written below:

```
>> x=45:70;
```

```
[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.3)
```

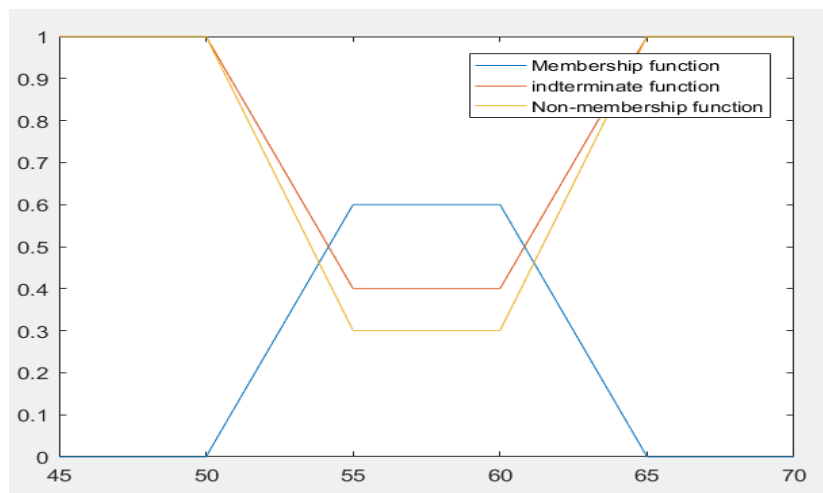


Figure 2: Trapezoidal neutrosophic function for example 4.2

4.3 Example

The figure 3 portrayed the triangular neutrosophic function of $a = \langle (0.3, 0.5, 0.5, 0.7); 0.4, 0.2, 0.3 \rangle$

The line command to show this function in Matlab is written below:

```
x= 0:0.01:1;
```

```
[y,z,t]=trin(x,0.3, 0.5,0.5,0.7, 0.4, 0.2,0.3)
```

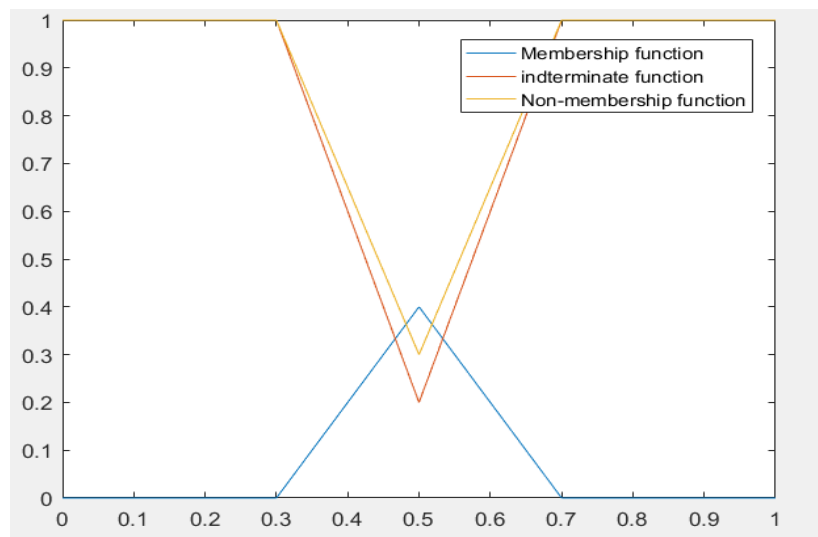


Figure 3: Triangular neutrosophic function for example 3

Remark: if $b = c$, the trapezoidal neutrosophic function degenerate to triangular neutrosophic function as portrayed in figure 3.

5. Qualitative analysis of different types of graphs

The following analysis helps to know the importance of the neutrosophic graph where the limitations are possible as mentioned in the table for fuzzy and intuitionistic fuzzy graphs.

Types of graphs	Advantages	Limitations
Graphs	- Models of relations - describing information involving relationship between objects - Objects are represented by vertices and relations by edges - Vertex and edge sets are crisp	Unable to handle fuzzy relation (FR)
Fuzzy graphs (FGs)	- Symmetric binary fuzzy relation on a fuzzy subset - Uncertainty exist in the description of the objects or in the relationships or in both - Able to handle FR with membership value - FGs models are more useful and practical in nature	Not able to deal interval data
Interval valued FGs	Edge set of a graphs is a collection of intervals	Unable to deal the case of non membership
Intuitionistic fuzzy graphs (IntFGs)	- Gives more certainty into the problems - Minimize the cost of operation and enhance efficiency - Contributes a adjustable model to define uncertainty and vagueness exists in decision making - Able to deal non membership of a relation	Unable to handle interval data

Interval valued IntFGs	• Capable of dealing interval data	• Unable to deal indeterminacy
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6. Conclusion

Choosing a MF is an essential task of all the fuzzy and neutrosophic system (Control system or decision making process). Due to the simplicity (less computational complexity) and flexibility triangular and trapezoidal membership functions are widely used in many real world applications. In this paper, trapezoidal neutrosophic membership function is derived using Matlab with illustrative example. In future, this work may be extended to interval valued trapezoidal and triangular neutrosophic membership functions.

Notes

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neutrosophic pre-continuous multifunctions and almost pre-continuous multifunctions

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Abstract: In this paper, we introduce neutrosophic upper and neutrosophic lower almost pre-continuous-multifunctions as a generalization of neutrosophic multifunctions. Some characterizations and several properties concerning neutrosophic upper and neutrosophic lower almost pre-continuous- multifunctions are obtained. further characterizations and several properties concerning neutrosophic upper (lower) pre-continuous continuous multifunctions are obtained. The relationship between these multifunctions and their graphs are investigated.

Keywords: neutrosophic topology, neutrosophic pre-continuous multifunctions, neutrosophic pre open set, neutrosophic continuous multifunctions, neutrosophic upper (lower) pre-continuous.

1 Introduction

The fundamental concept of the fuzzy sets was first introduced by Zadeh in his classical paper [12] of 1965. The idea of "intuitionistic fuzzy sets" was first published by Atanassov [7] and many works by the same author and his colleagues appeared in the literature [15, 16]. The theory of fuzzy topological spaces was introduced and developed by Chang [6] and since then various notions in classical topology have been extended to fuzzy topological spaces. In 1997, Coker [5] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space. F. Smarandache defined the notion of neutrosophic topology on the non-standard interval [13, 14, 18, 19, 20]. Also in various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). Also, (Zhang, Smarandache, and Wang, 2005) introduced the notion of interval neutrosophic set which is an instance of neutrosophic set and studied various properties. Recently, Wadei Al-Omeri and Smarandache [9, 10, 14, 21, 22] introduce and study a number of the definitions of neutrosophic continuity, neutrosophic open sets, and obtain several preservation properties and some characterizations concerning neutrosophic functions and neutrosophic connectedness. the theory of multifunctions plays an important role in functional analysis and fixed point theory. It also has a wide range of applications in artificial intelligence, economic theory, decision theory, non-cooperative games.

The concepts of the upper and lower pre-continuous multifunctions was introduced in [17]. In this paper we introduce and study the neutrosophic version of upper and lower pre-continuous multifunctions. Inspired

by the research works of Smarandache [13, 2], we introduce and study the notions of neutrosophic upper pre-continuous and neutrosophic upper pre-continuous multifunctions in this paper. Further, we present some characterizations and properties.

This paper is arranged as follows. In Section 2, we will recall some notions which will be used throughout this paper. In Section 3, neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) are introduced and investigate its basic properties. In Section 4, we study upper almost neutrosophic pre-continuous (lower almost neutrosophic pre-continuous) and study some of their properties. Finally, the applications are vast and the researchers in the field are exploring these realms of research and proved.

2 Preliminaries

Definition 2.1. [4] Let $\mathcal{R}$ be a non-empty fixed set. A neutrosophic set (NS for short) $\tilde{S}$ is an object having the form $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$, where $\mu_{\tilde{S}}(r)$, $\sigma_{\tilde{S}}(r)$, and $\gamma_{\tilde{S}}(r)$ are represent the degree of member ship function, the degree of indeterminacy, and the degree of non-membership, respectively, of each element $r \in \mathcal{R}$ to the set $\tilde{S}$.

Neutrosophic sets in $\mathcal{S}$ will be denoted by $\tilde{S}, \lambda, \psi, \mathcal{W}, B, G$, etc., and although subsets of $\mathcal{R}$ will be denoted by $\tilde{R}, \tilde{B}, T, B, p_0, r$, etc.

A neutrosophic set $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$ can be identified to an ordered triple $\langle \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r) \rangle$ in $]0^-, 1^+]$ on $\mathcal{R}$.

Remark 2.2. [4] A neutrosophic set $\tilde{S}$ is an object having the form $\tilde{S} = \{r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$ for the NS $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$.

Definition 2.3. [1] Let $\tilde{S} = \langle \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r) \rangle$ be an NS on $\mathcal{R}$. Maybe the complement of the set $\tilde{S}(C(\tilde{S}))$, for short) definitionned as follows.

- (i) $C(\tilde{S}) = \{\langle r, 1 - \mu_{\tilde{S}}(r), 1 - \gamma_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$,
- (ii) $C(\tilde{S}) = \{\langle r, \gamma_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \mu_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$
- (iii) $C(\tilde{S}) = \{\langle r, \gamma_{\tilde{S}}(r), 1 - \sigma_{\tilde{S}}(r), \mu_{\tilde{S}}(r) \rangle : r \in \mathcal{R}\}$

Definition 2.4. [4] Let r be a non-empty set, and $GNSs \tilde{S}$ and B be in the form $\tilde{S} = \{r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$, $B = \{r, \mu_B(r), \sigma_B(r), \gamma_B(r)\}$. Then $(\tilde{S} \subseteq B)$ definitionned as follows.

- (i) Type 1: $\tilde{S} \subseteq B \Leftrightarrow \mu_{\tilde{S}}(r) \leq \mu_B(r), \sigma_{\tilde{S}}(r) \geq \sigma_B(r)$, and $\gamma_{\tilde{S}}(r) \leq \gamma_B(r)$ or
- (ii) Type 2: $\tilde{S} \subseteq B \Leftrightarrow \mu_{\tilde{S}}(r) \leq \mu_B(r), \sigma_{\tilde{S}}(r) \geq \sigma_B(r)$, and $\gamma_{\tilde{S}}(r) \geq \gamma_B(r)$.

Definition 2.5. [4] Let $\{\tilde{S}_j : j \in J\}$ be an arbitrary family of an NSs in $\mathcal{R}$. Then

- (i) $\cap \tilde{S}_j$ definitionned as:
 - Type 1: $\cap \tilde{S}_j = \langle r, \bigwedge_{j \in J} \mu_{\tilde{S}_j}(r), \bigwedge_{j \in J} \sigma_{\tilde{S}_j}(r), \bigvee_{j \in J} \gamma_{\tilde{S}_j}(r) \rangle$
 - Type 2: $\cap \tilde{S}_j = \langle r, \bigwedge_{j \in J} \mu_{\tilde{S}_j}(r), \bigvee_{j \in J} \sigma_{\tilde{S}_j}(r), \bigvee_{j \in J} \gamma_{\tilde{S}_j}(r) \rangle$.

(ii) $\cup \tilde{S}_j$ definitionned as:

-Type 1: $\cup \tilde{S}_j = \langle r, \bigvee_{j \in J} \mu_{\tilde{S}_j}(r), \bigvee_{j \in J} \sigma_{\tilde{S}_j}(r), \bigwedge_{j \in J} \gamma_{\tilde{S}_j}(r) \rangle$

-Type 2: $\cup \tilde{S}_j = \langle r, \bigvee_{j \in J} \mu_{\tilde{S}_j}(r), \bigwedge_{j \in J} \sigma_{\tilde{S}_j}(r), \bigwedge_{j \in J} \gamma_{\tilde{S}_j}(r) \rangle$

Definition 2.6. [2] A neutrosophic topology (*NT* for short) and a non empty set $\mathcal{R}$ is a family $\mathcal{T}$ of neutrosophic subsets of $\mathcal{R}$ satisfying the following axioms

(i) $0_N, 1_N \in \mathcal{T}$

(ii) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$

(iii) $\cup G_i \in \mathcal{T}, \forall \{G_i | i \in J\} \subseteq \mathcal{T}$.

The pair $(\mathcal{R}, \mathcal{T})$ is called a neutrosophic topological space (*NTS* for short).

Definition 2.7. [4] Let $\tilde{S}$ be an *NS* and $(\mathcal{R}, \mathcal{T})$ an *NT* where $\tilde{S} = \{r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$. Then,

(i) $NCL(\tilde{S}) = \cap \{K : K \text{ is an NCS in } \mathcal{R} \text{ and } \tilde{S} \subseteq K\}$

(ii) $NInt(\tilde{S}) = \cup \{G : G \text{ is an NOS in } \mathcal{R} \text{ and } G \subseteq \tilde{S}\}$

It can be also shown that $NCL(\tilde{S})$ is an *NCS* and $NInt(\tilde{S})$ is an *NOS* in $\mathcal{R}$. We have

(i) $\tilde{S}$ is in $\mathcal{R}$ iff $NCL(\tilde{S})$.

(ii) $\tilde{S}$ is an *NCS* in $\mathcal{R}$ iff $NInt(\tilde{S}) = \tilde{S}$.

Definition 2.8. [4] Let $\tilde{S} = \{\mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$ be a neutrosophic open sets and $B = \{\mu_B(r), \sigma_B(r), \gamma_B(r)\}$ a neutrosophic set on a neutrosophic topological space $(\mathcal{R}, \mathcal{T})$. Then

(i) $\tilde{S}$ is called neutrosophic regular open iff $\tilde{S} = NInt(NCl(\tilde{S}))$.

(ii) The complement of neutrosophic regular open is neutrosophic regular closed.

Definition 2.9. [9] Let $\tilde{S}$ be an *NS* and $(\mathcal{R}, \mathcal{T})$ an *NT*. Then

(i) Neutrosophic semiopen set (*NSOS*) if $\tilde{S} \subseteq NCl(NInt(\tilde{S}))$,

(ii) Neutrosophic preopen set (*NPOS*) if $\tilde{S} \subseteq NInt(NCl(\tilde{S}))$,

(iii) Neutrosophic α -open set (*N α OS*) if $\tilde{S} \subseteq NInt(NCl(NInt(\tilde{S})))$

(iv) Neutrosophic β -open set (*N β OS*) if $\tilde{S} \subseteq NCl(NInt(NCl(\tilde{S})))$

Definition 2.10. [11] Let $(\mathcal{R}, \mathcal{T})$ be a topological space in the classical sense and $(\mathcal{S}, \mathcal{T}_1)$ be a neutrosophic topological space. $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is called a neutrosophic multifunction if and only if for each $r \in \mathcal{R}$, $F(r)$ is a neutrosophic set in $\mathcal{S}$.

Definition 2.11. [11] For a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, the upper inverse $F^+(\lambda)$ and lower inverse $F^-(\lambda)$ of a neutrosophic set λ in $\mathcal{S}$ are dened as follows:

$$F^+(\lambda) = \{r \in \mathcal{R} | F(r) \leq \lambda\} \quad (2.1)$$

and

$$F^-(\lambda) = \{r \in \mathcal{R} | F(r) q \lambda\} \quad (2.2)$$

Lemma 2.12. [11] In a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, we have $F^-(1 - \lambda) = \mathcal{R} - F^+(\lambda)$, for any neutrosophic set λ in $\mathcal{S}$.

A neutrosophic set $\mathcal{S}$ in $\mathcal{S}$ is said to be q -coincident with a neutrosophic set ψ , denoted by $Sq\psi$, if and only if there exists $p \in \mathcal{S}$ such that $S(p) + \psi(p) > 1$. A neutrosophic set $\mathcal{S}$ of $\mathcal{S}$ is called a neutrosophic neighbourhood of a fuzzy point p_ϵ in $\mathcal{S}$ if there exists a neutrosophic open set ψ in $\mathcal{S}$ such that $p_\epsilon \in \psi \leq S$.

3 Neutrosophic Pre-continuous multifunctions

Definition 3.1. In a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is said to be

- (i) neutrosophic lower pre-continuous at a point $p_0 \in \mathcal{R}$, if for any neutrosophic open set $\mathcal{W} \leq \mathcal{S}$ such that $F(p_0)q\mathcal{W}$ there exists $\tilde{R} \in NPO(\mathcal{R})$ containing p_0 such that $F(\tilde{R})q\mathcal{W}$, $\forall r \in \tilde{R}$.
- (ii) neutrosophic upper pre-continuous at a point $p_0 \in \mathcal{R}$, if for any neutrosophic open set $\mathcal{W} \leq \mathcal{S}$ such that $F(p_0) \leq \mathcal{W}$ there exists $\tilde{R} \in NPO(\mathcal{R})$ containing p_0 such that $F(\tilde{R}) \leq \mathcal{W}$.
- (iii) neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) if it is neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) at every point of $\mathcal{R}$.

A subset $\tilde{R}$ of a neutrosophic topological space $(\mathcal{R}, \mathcal{T})$ is said to be neutrosophic neighbourhood (resp. neutrosophic-preneighbourhood) of a point $r \in \mathcal{R}$ if there exists a neutrosophic-open (resp. neutrosophic-preopen) set $\tilde{S}$ such that $r \in \tilde{S} \subseteq \tilde{R}$, neutrosophic neighbourhood (resp. neutrosophic pre-neighbourhood) write briefly neutrosophic nbh (resp. neutrosophic pre-nbh).

Theorem 3.2. A neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, the the following statements are equivalent:

- (i) F is neutrosophic upper pre-continuous at p_0 ;
- (ii) $F^+(\tilde{S}) \in NPO(X)$ for any neutrosophic open set $\tilde{S}$ of $\mathcal{S}$,
- (iii) $F^-(T)$ is neutrosophic pre-closed in $\mathcal{R}$ for any neutrosophic closed set T of $\mathcal{S}$,
- (iv) $pNCl(F^-(\mathcal{W})) \subseteq F^-(NCl(\mathcal{W}))$ for each neutrosophic set $\mathcal{W}$ of $\mathcal{S}$.
- (v) for each point $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, $F^+(\tilde{S})$ is a neutrosophic pre-nbh of p_0 ,
- (vi) for each point $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, there exists a neutrosophic pre-nbh of p_0 such that $F(\tilde{R}) \leq \tilde{S}$,

(vii) $F^+(NInt(\mathcal{W})) \subseteq pNInt(F^+(\mathcal{W}))$ for every neutrosophic subset $\mathcal{W}$ of $I^\mathcal{S}$,

(viii) $F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S})))$ for every neutrosophic open subset $\tilde{S}$ of $I^\mathcal{S}$,

(ix) for each point $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, $Cl(F^+(\tilde{S}))$ is a neighbourhood of r .

Proof. (i) $\Rightarrow$ (ii): Let $\tilde{S}$ be any arbitrary NOS of $\mathcal{S}$ and $p_0 \in F^+(\tilde{S})$. Then $F(p_0) \in \mathcal{S}$. There exists an NPO set $\tilde{R}$ of $\mathcal{R}$ containing p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Since

$$p_0 \in \tilde{R} \subseteq NInt(NCl(\tilde{R})) \subseteq NInt(NCl(F^+(\tilde{S}))) \quad (3.1)$$

and so we have

$$F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S}))). \quad (3.2)$$

Hence $F^+(\tilde{S})$ is an NPO in $\mathcal{R}$.

(ii) $\Rightarrow$ (iii): It follows from the fact that $F^+(\mathcal{S} - B) = \mathcal{R} - F^-(\mathcal{W})$ for any subset $\mathcal{W}$ of $\mathcal{S}$.

(iii) $\Rightarrow$ (iv): For any subset $\mathcal{W}$ of $\mathcal{S}$, $NCl(\mathcal{W})$ is an NCS in $\mathcal{S}$ and then $F^-(NCl(\mathcal{W}))$ is neutrosophic pre-closed in $\mathcal{R}$. Hence,

$$pNCl(F^-(\mathcal{W})) \subseteq pNCl(F^-(NCl(\tilde{R}))) \subseteq F^-(NCl(\tilde{R})). \quad (3.3)$$

(iv) $\Rightarrow$ (iii): Let β be any arbitrary NCS of $\mathcal{S}$. Then

$$pNCl(F^-(M)) \subseteq F^-(NCl(\beta)) = F^-(\beta), \quad (3.4)$$

and hence $F^-(\beta)$ is NPC in $\mathcal{R}$.

(ii) $\Rightarrow$ (v): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be a nbh of $F(p_0)$. There exists an NOS $\tilde{B}$ of $\mathcal{S}$ such that

$$F(p_0) \subseteq \tilde{B} \subseteq \tilde{S}. \quad (3.5)$$

Then we have $p_0 \in F^+(\tilde{B}) \subseteq F^+(\mathcal{S})$ and since $F^+(\mathcal{S})$ is neutrosophic pre-open in $\mathcal{R}$, $F^+(\tilde{S})$ is a neutrosophic pre-nbh of p_0 .

(v) $\Rightarrow$ (vi): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(p_0)$. Put $\tilde{R} = F^+(\tilde{S})$. By (v) $\tilde{R}$ is a neutrosophic pre-nbh of p_0 and $F(\tilde{R}) \subseteq \tilde{S}$.

(vi) $\Rightarrow$ (i): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be any neutrosophic open set of $\mathcal{S}$ such that $F(p_0) \subseteq \tilde{S}$. Then $\tilde{S}$ is a neutrosophic nbh of $F(p_0)$ and there exists a neutrosophic pre-nbh $\tilde{R}$ of p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Therefore, there exists an NPO $\tilde{B}$ in $\mathcal{R}$ such that $p_0 \in \tilde{B} \subseteq \tilde{R}$ and so $F(\tilde{B}) \subseteq \tilde{S}$.

(ii) $\Rightarrow$ (vii): Let $\mathcal{W}$ be an NOS set of $\mathcal{S}$, $NInt(\mathcal{W})$ is an NO in $\mathcal{S}$ and then $F^+(NInt(\mathcal{W}))$ is NPO in $\mathcal{R}$. Hence,

$$F^+(NInt(\mathcal{W})) \subseteq pNInt(F^+(\mathcal{W})). \quad (3.6)$$

(vii) $\Rightarrow$ (ii): Let $\tilde{S}$ be any neutrosophic open set of $\mathcal{S}$. By (vii) $F^+(\tilde{S}) = F^+(Int(\tilde{S})) \subseteq pNInt(F^+(\tilde{S}))$ and hence $F^+(\tilde{S})$ is an NPO in $\mathcal{R}$.

(viii) $\Rightarrow$ (ix): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(r)$. Then

$$p_0 \in F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S}))) \subseteq NCl(F^+(\tilde{S})), \quad (3.7)$$

and hence $NCl(F^+(\tilde{S}))$ is a neutrosophic nbh of p_0 .

(viii) $\Rightarrow$ (ix): Let $\tilde{S}$ be any open set of $\mathcal{S}$ and

$$p_0 \in F^+(\tilde{S}). \quad (3.8)$$

Then

$$NCl(F^+(\tilde{S})) \quad (3.9)$$

is a neutrosophic nbh of p_0 and thus

$$NInt(NCl(F^+(\tilde{S}))). \quad (3.10)$$

Hence,

$$F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S}))). \quad (3.11)$$

□

Theorem 3.3. For a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, the following statements are equivalent:

- (i) F is neutrosophic lower pre-continuous at p_0 ;
- (ii) $F^+(\tilde{S}) \in NPO(X)$ for any NOS $\tilde{S}$ of $\mathcal{S}$,
- (iii) $F^+(T) \in NPC(X)$ for any neutrosophic closed set T of $\mathcal{S}$,
- (iv) for each $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ which intersects $F(r)$, $F^-(\tilde{S})$ is a neutrosophic pre-nbh of p_0 ,
- (v) for each $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ which intersects $F(r)$, there exists a neutrosophic pre-neighbourhood $\tilde{R}$ of p_0 such that $F(u) \cap \tilde{S} \neq \phi$ or any $u \in \tilde{R}$,
- (vi) $pNCl(F^+(\mathcal{W})) \subseteq F^+(NCl(\mathcal{W}))$ for any neutrosophic set $\mathcal{W}$ of $\mathcal{S}$.
- (vii) $F^-(NInt(\mathcal{W})) \subseteq pNInt(F^-(\mathcal{W}))$ for every neutrosophic subset $\mathcal{W}$ of $I^{\mathcal{S}}$,
- (viii) $F^-(\tilde{S}) \subseteq NInt(NCl(F^-(\tilde{S})))$ for any NOS subset $\tilde{S}$ of $I^{\mathcal{S}}$,
- (ix) for each point $p_0 \in \mathcal{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, $Cl(F^-(\tilde{S}))$ is a neighbourhood of r .

Proof. (i) $\Rightarrow$ (ii): Let $\tilde{S}$ be any arbitrary NOS of $\mathcal{S}$ and $p_0 \in F^+(\tilde{S})$. Then by (a), there exists an NPO set $\tilde{R}$ of $\mathcal{R}$ containing p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Since

$$p_0 \in \tilde{R} \subseteq NInt(NCl(\tilde{R})) \subseteq NInt(NCl(F^-(\tilde{S}))) \quad (3.12)$$

and so we have

$$F^-(\tilde{S}) \subseteq NInt(NCl(F^-(\tilde{S}))), \quad (3.13)$$

and hence

$$F^-(\tilde{S}) \in NPO(\mathcal{R}). \quad (3.14)$$

(ii) $\Rightarrow$ (iii): It follows from the fact that

$$F^+(\mathcal{S} \setminus B) = \mathcal{R} \setminus F^-(\mathcal{W}) \quad (3.15)$$

for any $\mathcal{W} \in \mathcal{S}$.

(iii) $\Rightarrow$ (vi): Let $\mathcal{W}$ in $\mathcal{S}$, $NCl(\mathcal{W})$ is an NCS in $\mathcal{S}$. By (iii) $F^+(NCl(\mathcal{W}))$ is neutrosophic pre-closed in $\mathcal{R}$. Hence,

$$pNCl(F^+(\mathcal{W})) \subseteq pNCl(F^+(NCl(\tilde{R}))) \subseteq F^+(NCl(\tilde{R})). \quad (3.16)$$

(iv) $\Rightarrow$ (iii): Let β be NCS of $\mathcal{S}$. Then

$$pNCl(F^+(M)) \subseteq pNCl(F^+(F^+(NCl(\beta)))) \subseteq F^+(NCl(\beta)) = F^+(\beta) \Rightarrow F^-(\beta) \quad (3.17)$$

is NPC in $\mathcal{R}$.

(ii) $\Rightarrow$ (v): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(p_0)$. There exists an NOS $\tilde{B}$ of $\mathcal{S}$ such that

$$F(p_0) \subseteq \tilde{B} \subseteq \tilde{S}. \quad (3.18)$$

Then we have

$$p_0 \in F^-(\tilde{B}) \subseteq F^-(\mathcal{S}), \quad (3.19)$$

and since $F^-(V)$ is neutrosophic pre-open in $\mathcal{R}$, by (ii) $F^-(\tilde{S})$ is a neutrosophic pre-nbh of p_0 .

(v) $\Rightarrow$ (vi): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(p_0)$. Put $\tilde{R} = F^-(\tilde{S})$. By (v) $\tilde{R}$ is a neutrosophic pre-nbh of p_0 and $F(\tilde{R}) \subseteq \tilde{S}$.

(vi) $\Rightarrow$ (i): Let $p_0 \in \mathcal{R}$ and $\tilde{S}$ be any NOS of $\mathcal{S}$ such that $F(p_0) \subseteq \tilde{S}$. Then $\tilde{S}$ is a neutrosophic nbh of $F(p_0)$ by (vi) there exists a neutrosophic pre-nbh $\tilde{R}$ of p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Therefore, there exists an NPO $\tilde{B}$ in $\mathcal{R}$ such that

$$p_0 \in \tilde{B} \subseteq \tilde{R} \quad (3.20)$$

and so

$$\tilde{S} \subseteq F^-(\tilde{B}). \quad (3.21)$$

(ii) $\Rightarrow$ (vii): Let $\mathcal{W}$ be an NOS set of $\mathcal{S}$, $NInt(\mathcal{W})$ is an NO in $\mathcal{S}$ and then $F^+(NInt(\mathcal{W}))$ is NPO in $\mathcal{R}$. Hence, $F^+(NInt(\mathcal{W})) \subseteq pNInt(F^+(\mathcal{W}))$.

(vii) $\Rightarrow$ (ii): Let $\tilde{S}$ be any NOS of $\mathcal{S}$. By (vii)

$$F^+(\tilde{S}) = F^-(Int(\tilde{S})) \subseteq pNInt(F^-(\tilde{S})), \quad (3.22)$$

and hence $F^-(\tilde{S})$ is an NPO in $\mathcal{R}$.

(vi) $\Rightarrow$ (vii): Let $\mathcal{W}$ be any neutrosophic open set of $\mathcal{S}$, then

$$[F^-(NInt(\mathcal{W}))]^c = F^+(NCl(\mathcal{W}^c)) \supset pNCl(F^+(NInt(NCl(\mathcal{W}^c)))) \quad (3.23)$$

$$= pNCl(F^+(NCl(NInt(\mathcal{W})))^c) = pNCl(F^-(NCl(NInt(\mathcal{W}))))^c \quad (3.24)$$

$$= [pNInt(F^-(NCl(NInt(\mathcal{W}))))]^c. \quad (3.25)$$

Thus we obtained

$$F^-(NInt(\mathcal{W})) \supset pNInt(F^-(NCl(NInt(\mathcal{W}))). \quad (3.26)$$

(vi) $\Rightarrow$ (vii): Obvious. □

We now show by means of the following examples that
lower neutrosophic pre-continuous $\nRightarrow$ upper neutrosophic pre-continuous.

Example 3.1. Let $\mathcal{R} = \{u, v, w\}$ and $\mathcal{S} = [0, 1]$. Let $\mathcal{T}$ and $\mathcal{T}_1$, be respectively the topology on $\mathcal{R}$ and neutrosophic topology on $\mathcal{S}$, given by $\mathcal{T} = \{\mathcal{R}_N, \phi_N, \{u, w\}\}$, $\mathcal{T}_1 = \{C_o, C, \mu_{\tilde{S}}, \sigma_{\tilde{S}}, \gamma_{\tilde{S}}, (\mu_{\tilde{S}} \cup \sigma_{\tilde{S}}), (\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})\}$. Where $\mu_{\tilde{S}}(r) = r$, $\sigma_{\tilde{S}}(r) = 1 - r$, for $r \in \mathcal{S}$, and

$$\mu(r) = \begin{cases} r, & \text{if } 0 \leq r \leq \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \quad (3.27)$$

We define a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ by letting $F(u) = (\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})$, $F(v) = \sigma_{\tilde{S}}$ and $F(w) = \gamma_{\tilde{S}}$.

$\{u, w\}$ is neutrosophic open set in $\mathcal{R}$ and therefore $\{u, w\}$ is neutrosophic pre-open set. The other neutrosophic pre-open set in $(\mathcal{R}, \mathcal{T})$ are $\{u, v\}$, $\{v, w\}$, $\{u\}$ and $\{w\}$. Then $\{u\}$ is not neutrosophic pre-open set in $(\mathcal{R}, \mathcal{T})$. From definition of $\mu_{\tilde{S}}$ and $\sigma_{\tilde{S}}$ we find that,

$$(\mu_{\tilde{S}} \cup \sigma_{\tilde{S}})(r) = \begin{cases} 1 - r, & \text{if } 0 \leq r \leq \frac{1}{2}, \\ r, & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \quad (3.28)$$

$$(\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})(r) = \begin{cases} r, & \text{if } 0 \leq r \leq \frac{1}{2}, \\ 1 - r, & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \quad (3.29)$$

Now $\sigma_{\tilde{S}} \in \mathcal{T}_1$ but $F^+(\sigma_{\tilde{S}}) = \{v\}$ which is not neutrosophic pre-open set in $(\mathcal{R}, \mathcal{T})$. Hence F is not upper neutrosophic pre-continuous. Then $F^-(\sigma_{\tilde{S}}) = \{v\}$ which is not neutrosophic pre-open set in $(\mathcal{R}, \mathcal{T})$. Therefore F is not lower neutrosophic pre-continuous.

Remark 3.4. [11] A subset μ of a topological space $(\mathcal{R}, \mathcal{T})$ can be considered as a neutrosophic set with characteristic function defined by

$$\mu(r) = \begin{cases} 1, & \text{if } u \in \mu, \\ 0, & \text{if } v \notin \mu, \end{cases} \quad (3.30)$$

Let $(\mathcal{S}, \mathcal{T}_1)$ be a neutrosophic topological space. The neutrosophic sets of the form $\mu \times \nu$ with $\mu \in \mathcal{T}$ and $\nu \in \mathcal{T}_1$ make a basis for the product neutrosophic topology $\mathcal{T} \times \mathcal{T}_1$ on $\mathcal{R} \times \mathcal{S}$, where for any $(u, v) \in \mathcal{R} \times \mathcal{S}$, $(\mu \times \nu)(u, v) = \min\{\mu(u), \nu(v)\}$.

Definition 3.5. [11] For a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, the neutrosophic graph multifunction $F_G : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{S}$ of F is defined by $F_G(r) = r_1 \times F(r)$ for every $r \in \mathcal{R}$.

Lemma 3.6. In a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$, the following hold: a) $F_G^+(\tilde{R} \times \tilde{S}) = \tilde{R} \cap F^+(\tilde{S})$
b) $F_G^-(\tilde{R} \times \tilde{S}) = \tilde{R} \cap F^-(\tilde{S})$ for all subsets $\tilde{R} \in \mathcal{R}$ and $\tilde{S} \in \mathcal{S}$.

Theorem 3.7. If the neutrosophic graph multifunction F_G of a neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is neutrosophic lower precontinuous, then F is neutrosophic lower precontinuous.

Proof. Suppose that F_G is neutrosophic lower precontinuous and $s \in \mathcal{R}$. Let B be an $NOs \in \mathcal{S}$ such that $F(r)qB$. Then there exists $r \in \mathcal{S}$ such that $(F(r))(r) + A(r) > 1$. Then

$$(F_G(r))(r, r) + (\mathcal{R} \times B)(r, r) = (F(r))(r) + B(r) > 1. \quad (3.31)$$

Hence, $F_G(r)q(\mathcal{R} \times B)$. Since F_G is neutrosophic lower precontinuous, there exists an open set $A \in \mathcal{R}$ such that $r \in A$ and $F_G(b)q(\mathcal{R} \times B) \forall a \in A$. Let there exists $a_0 \in A$ such that $F(a_0)qB$. Then $\forall r \in \mathcal{S}$, $(F(a_0))(r) + B(r) < 1$. For any $(b, c) \in \mathcal{R} \times \mathcal{S}$, we have

$$(F_G(a_0))(b, c) \subseteq (F(a_0))(c), \quad (3.32)$$

and

$$(\mathcal{R} \times B)(b, c) \subseteq B(c). \quad (3.33)$$

Since $\forall r \in \mathcal{S}$, $(F(a_0))(r) + B(r) < 1$,

$$(F_G(a_0))(b, c) + (\mathcal{R} \times B)(b, c) < 1. \quad (3.34)$$

Thus, $F_G(a_0)q(\mathcal{R} \times B)$, where $a_0 \in A$. This is a contradiction since

$$F_G(a)q(\mathcal{R} \times B), \forall a \in A, \quad (3.35)$$

Therefore, F is neutrosophic lower precontinuous. $\square$

Definition 3.8. A neutrosophic space $(\mathcal{R}, \mathcal{T})$ is said to be neutrosophic pre-regular (NP-regular) if for every $NCs F$ and a point $u \in F$, there exist disjoint neutrosophic-preopen sets $\tilde{R}$ and $\tilde{S}$ such that $F \subseteq \tilde{R}$ and $u \in \tilde{S}$.

Theorem 3.9. Let $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ be a neutrosophic multifunction and $F_G : \mathcal{R} \longrightarrow \mathcal{R} \times \mathcal{S}$ the graph multifunction of F . If F_G is neutrosophic upper pre-continuous (neutrosophic lower pre-continuous, then F is neutrosophic upper pre-continuous. (neutrosophic lower pre-continuous.) and $\mathcal{R}$ is NP-regular.

Proof. Let F_G be a neutrosophic upper pre-continuous multifunction and $\tilde{S}$ be a neutrosophic open set containing $F(r)$ such that $r \in F^+(\tilde{S})$. Then $\mathcal{R} \times \tilde{S}$ is a neutrosophic open set of $\mathcal{R} \times \mathcal{S}$ containing $F_G(r)$. Since F_G is neutrosophic upper pre-continuous, there exists an $NPOs \tilde{R}$ of $\mathcal{R}$ containing r such that $\tilde{R}_p^- \subseteq F_G^+(\mathcal{R} \times \tilde{S})$. Therefore we obtain

$$\tilde{R}_p^- \subseteq F^+(\tilde{S}). \quad (3.36)$$

Now we show that $\mathcal{R}$ is NP-regular. Let $\tilde{R}$ be any $NPOs$ of $\mathcal{R}$ containing r . Since

$$F_G(r) \in \tilde{R} \times \mathcal{S}, \quad (3.37)$$

and $\tilde{R} \times \mathcal{S}$ is neutrosophic open in $\mathcal{R} \times \mathcal{S}$, there exists an $NPOs$ set U of $\mathcal{R}$ such that

$$U_p^- \subseteq F_G^+(\tilde{R} \times \mathcal{S}). \quad (3.38)$$

Therefore we have

$$r \in U \subseteq U_p^- \subseteq \tilde{R}. \quad (3.39)$$

This shows that $\mathcal{R}$ is NP-regular.

The proof for neutrosophic upper lower-continuous is similar. $\square$

Theorem 3.10. *Let $\mathcal{R}$ is NP-regular. A neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is neutrosophic lower pre-continuous iff $F_G(r)$ is neutrosophic lower pre-continuous.*

Proof. $\implies$ Let $r \in \mathcal{R}$ and A be any NPOs of $\mathcal{R} \times \mathcal{S}$ such that $r \in F_G(A)$. Since

$$A \cap (\{r\} \times F(r)) \neq \phi, \quad (3.40)$$

there exists $s \in F(r)$ such that $(r, s) \in A$. Hence

$$(r, s) \in \tilde{R} \times \tilde{S} \subseteq A \quad (3.41)$$

for some NOs $\tilde{R} \subseteq \mathcal{R}$ and $\tilde{S} \subseteq \mathcal{S}$. Since $\mathcal{R}$ is NP-regular, there exists $B \in NPO(\mathcal{R}, r)$ such that

$$r \in B \subseteq B_p^- \subseteq \tilde{R}. \quad (3.42)$$

Since F is neutrosophic lower pre-continuous, there exists $W \in NPO(\mathcal{R}, r)$ such that

$$W_p^- \subseteq F^-(\tilde{S}). \quad (3.43)$$

By Lemma 3.6, we have

$$W_p^- \cap B_p^- \subseteq \tilde{R} \cap F^-(\tilde{S}) = F_G^-(\tilde{R} \times \tilde{S}) \subseteq F_G^-(B). \quad (3.44)$$

Moreover, we have $B \cap W \in NPO(\mathcal{R}, r)$ and hence $F_G(r)$ is neutrosophic lower pre-continuous.

$\Leftarrow$ Let $r \in \mathcal{R}$ and $\tilde{S}$ be any NOs $\in \mathcal{S}$ such that $r \in F^-(\tilde{S})$. Then $\mathcal{R} \times \tilde{S}$ is

$$NO \in \mathcal{R} \times \mathcal{S}. \quad (3.45)$$

Since F_G is neutrosophic lower pre-continuous and lemma,

$$F_G^-(\mathcal{R} \times \tilde{S}) = \mathcal{R} \cap F^-(\tilde{S}) = F^-(\tilde{S}) \quad (3.46)$$

is NPOs $\in \mathcal{R}$. This shows that F is neutrosophic lower pre-continuous. $\square$

Definition 3.11. [11] A neutrosophic set Δ of a neutrosophic topological space $\mathcal{S}$ is said to be neutrosophic compact relative to $\mathcal{S}$ if every cover $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ of Δ by neutrosophic open sets of $\mathcal{S}$ has a finite subcover $\{\Delta_i\}_{i=1}^n$ of Δ .

Theorem 3.12. *Let $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ be a neutrosophic multifunction such that $F(r)$ is compact for each $r \in \mathcal{R}$. And $\mathcal{R}$ is a NP-regular space. If F is neutrosophic upper pre-continuous then F_G is neutrosophic upper pre-continuous.*

Proof. Let $r \in \mathcal{R}$ and A be any NOs $\in \mathcal{R} \times \mathcal{S}$ containing $F_G(r)$. For each $s \in F(r)$, there exist open sets $\tilde{R}(s) \subseteq \mathcal{R}$ and $\tilde{S}(s) \subseteq \mathcal{S}$ such that

$$(r, s) \in \tilde{R}(s) \times \tilde{S}(s) \subseteq A. \quad (3.47)$$

The family $\{\tilde{S}(s) : s \in F(r)\}$ is a neutrosophic open cover of $F(r)$. Since $F(r)$ is compact, there exists a finite number of points $\{s_j\}_{j=1}^n$ in $F(r)$ such that

$$F(r) \subseteq \cup\{\tilde{S}(s_j) : j = 1, \dots, n\}. \quad (3.48)$$

Use $\tilde{R} = \cap \{\tilde{R}(s_j) : j = 1, \dots, n\}$ and $\tilde{S} = \{\tilde{S}(s_j) : j = 1, \dots, n\}$. Then $\tilde{R}$ and $\tilde{S}$ are $NOs \in \mathcal{R}$ and $\mathcal{S}$, respectively and

$$\{r\} \times F(r) \subseteq \tilde{R} \times \tilde{S} \subseteq A. \quad (3.49)$$

Since F is neutrosophic upper pre-continuous, there exists $S \in NPO(\mathcal{R}, r)$ such that

$$S_p^- \subseteq F^+(\tilde{S}). \quad (3.50)$$

Since $\mathcal{R}$ is NP-regular, there exists $G \in NPO(\mathcal{R}, r)$ such that

$$r \in G \subseteq G_p^- \subseteq \tilde{R}. \quad (3.51)$$

Hence, we have

$$\{r\} \times F(r) \subseteq G_p^- \times \tilde{S} \subseteq \tilde{R} \times \tilde{S} \subseteq A. \quad (3.52)$$

Then we have

$$(S \cap G)_p^- \subseteq SG_p^- \cap G_p^- \subseteq F^+(\tilde{S}) \cap G_p^- = F_G^+(G_p^- \times \tilde{S}) \subseteq F_G^+(A). \quad (3.53)$$

Moreover, we obtain $S \cap G \in NPO(\mathcal{R}, r)$ and hence F_G is neutrosophic upper pre-continuous. $\square$

Proposition 3.2. Let B and $\mathcal{R}_o$ be subsets of neutrosophic topological space $(\mathcal{R}, \mathcal{T})$.

- (i) If $B \in NPO(\mathcal{R})$ and $\mathcal{R}_o$ is NSO in $\mathcal{R}$, then $B \cap \mathcal{R}_o \in NPO(\mathcal{R}_o)$.
- (ii) If $B \in NPO(\mathcal{R}_o)$ and $\mathcal{R}_o \in NPO(\mathcal{R})$, then $B \in NPO(\mathcal{R})$.

Proposition 3.3. Let B and $\mathcal{R}$ be subsets of neutrosophic topological space $(\mathcal{R}, \mathcal{T})$, $B \subseteq \mathcal{R} \subseteq \mathcal{R}$. Let the neutrosophic pre-closure $(B_p^-)_{\mathcal{R}}$ of B in the neutrosophic subspace $\mathcal{R}_o$:

- (i) If $\mathcal{R}$ is NSO in $\mathcal{R}$, then $(B_p^-)_{\mathcal{R}_o} \subseteq (B_p^-)_{\mathcal{R}}$.
- (ii) If B in $NPO(\mathcal{R}_o)$ and $\mathcal{R}_o$ in $NPO(\mathcal{R})$, then $B_p^- \subseteq (B_p^-)_{\mathcal{R}_o}$.

Theorem 3.13. A neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is upper almost neutrosophic pre-continuous. (lower almost neutrosophic pre-continuous) if $\forall r \in \mathcal{R}$ there exists an $NPOs \mathcal{R}_o$ containing r such that $F|_{\mathcal{R}_o} : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is upper almost neutrosophic pre-continuous. (lower almost neutrosophic pre-continuous).

Proof. Let $r \in \mathcal{R}$ and $\tilde{S}$ be a neutrosophic open set of $\mathcal{S}$ containing $F(r)$ such that $r \in F^+(\tilde{S})$ and there exists $\mathcal{R}_o \in NPO(\mathcal{R}, r)$ such that

$$F|_{\mathcal{R}_o} : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1), \quad (3.54)$$

is upper almost neutrosophic pre-continuous. Therefore, there exists $\tilde{R}$ in $NPO(\mathcal{R}_o, r)$ such that

$$(\tilde{R}_p^-)_{\mathcal{R}_o} \subseteq (F|_{\mathcal{R}_o})^+(\tilde{S}). \quad (3.55)$$

By Proposition 3.2 and Proposition 3.3, $\tilde{R}$ in $NPO(X, r)$ and

$$\tilde{R}_p^- \subseteq (\tilde{R}_p^-)_{\mathcal{R}_o}. \quad (3.56)$$

Therefore

$$F(\tilde{R}_p^-) = (F|_{\mathcal{R}_o})(\tilde{R}_p^-) \subseteq (F|_{\mathcal{R}_o})((\tilde{R}_p^-)_{\mathcal{R}_o}) \subseteq \tilde{S}. \quad (3.57)$$

This shows that F is upper almost neutrosophic pre-continuous. □

4 Almost Neutrosophic pre-continuous multifunctions

Definition 4.1. Let $(\mathcal{R}, \mathcal{T})$ be a neutrosophic topological space and $(\mathcal{S}, \mathcal{T}_1)$ a topological space. A neutrosophic multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ is said to be:

- (i) upper almost neutrosophic pre-continuous at a point $r \in \mathcal{R}$ if for each open set $\tilde{S}$ of $I^{\mathcal{S}}$ such that $r \in F^+(\tilde{S})$, there exists a neutrosophic pre-open set $\tilde{R}$ of $\mathcal{R}$ containing r such that $\tilde{R} \subseteq F^+(NInt(NCl(\tilde{S})))$;
- (ii) lower almost neutrosophic pre-continuous at a point $r \in \mathcal{R}$ if for each neutrosophic open set $\tilde{S} \in \mathcal{S}$ such that $r \in F(\tilde{S})$, there exists a neutrosophic pre-open $\tilde{R}$ of $\mathcal{R}$ containing r such that $\tilde{R} \subseteq F^-(NInt(NCl(\tilde{S})))$;
- (iii) upper (resp. lower) almost neutrosophic pre-continuous if F has this property at each point of $\mathcal{R}$.

Theorem 4.2. Let $(\mathcal{R}, \mathcal{T})$ be a neutrosophic topological space and $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ a neutrosophic multifunction from a neutrosophic topological space $(\mathcal{R}, \mathcal{T})$ to a topological space $(\mathcal{S}, \mathcal{T}_1)$. Then the following properties are equivalent:

- (i) F is upper almost neutrosophic-pre-continuous;
- (ii) for any $r \in \mathcal{R}$ and for all NOs $\tilde{S}$ of $\mathcal{S}$ such that $F(r) \subseteq \tilde{S}$, there exists a neutrosophic pre-open $\tilde{R}$ of $\mathcal{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \subseteq NInt(NCl(\tilde{S}))$;
- (iii) for any $r \in \mathcal{R}$ and for all NROs G of $\mathcal{S}$ such that $F(r) \subseteq G$, there exists a neutrosophic pre-open $\tilde{R}$ of $\mathcal{R}$ containing r such that $F(\tilde{R}) \subseteq G$;
- (iv) for any $r \in \mathcal{R}$ and for all closed set $F^+(\mathcal{S} - M)$, there exists a neutrosophic pre-closed N of $\mathcal{R}$ such that $r \in \mathcal{R} - N$ and $F^-(NInt(NCl(M))) \subseteq N$;
- (v) $F^+(NInt(NCl(\tilde{S})))$ is neutrosophic pre-open in $\mathcal{R}$ for any NOs $\tilde{S}$ of $\mathcal{S}$;
- (vi) $F^-(NCl(NInt(F)))$ is neutrosophic pre-closed in $\mathcal{R}$ for each closed set F of $\mathcal{S}$;
- (vii) $F^+(G)$ is neutrosophic pre-open in $\mathcal{R}$ for each regular open G of $\mathcal{S}$;
- (viii) $F^+(H)$ is neutrosophic pre-closed in $\mathcal{R}$ for each regular closed H of $\mathcal{S}$;
- (ix) for any point $r \in \mathcal{R}$ and each neutrosophic-nbh $\tilde{S}$ of $F(r)$, $F^+(NInt(NCl(\tilde{S})))$ is a neutrosophic pre-nbh of r ;
- (x) for any point $r \in \mathcal{R}$ and each neutrosophic-nbh $\tilde{S}$ of $F(r)$, there exists a neutrosophic pre-nbh $\tilde{R}$ of r such that $F(\tilde{R}) \subseteq NInt(NCl(\tilde{S}))$;
- (xi) $pNCl(F^-(NCl(NInt(A)))) \subseteq F^-(NCl(NInt(NCl(A))))$ for any subset A of $\mathcal{S}$;
- (xii) $F^+(NInt(NCl(NInt(A)))) \subseteq pNInt(F^+(NInt(NCl(A))))$ for any subset A of $\mathcal{S}$.

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let $r \in \mathcal{R}$ and G be a regular open set of $\mathcal{S}$ such that $F(r) \subseteq G$. By (ii), there exists an $NPOs$ $\tilde{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \subseteq NInt(NCl(G)) = G$. We obtain $F(\tilde{R}) \subseteq G$.

(iii) $\Rightarrow$ (ii): Let $r \in \mathcal{R}$ and $\tilde{S}$ be an NOs set of $\mathcal{S}$ such that $F(r) \subseteq \tilde{S}$. Then, $NInt(NCl(\tilde{S}))$ is $NROs$ in $\mathcal{S}$. By (iii), there exists an $NPOS$ of $\mathcal{R}$ containing r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \quad (4.1)$$

(ii) $\Rightarrow$ (iv): Let $r \in \mathcal{R}$ and M be an NCs of $\mathcal{S}$ such that $r \in F^+(\mathcal{S} - M)$. By (ii), there exists an $NPOs$ $\tilde{R}$ of $\mathcal{R}$ containing r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\mathcal{S}M)). \quad (4.2)$$

We have

$$NInt(NCl(\mathcal{S} - M)) = \mathcal{S} - NCl(NInt(M)) \quad (4.3)$$

and

$$\tilde{R} \subseteq F^+(\mathcal{S} - NCl(NInt(M))) = \mathcal{R} - F^-(NCl(NInt(M))). \quad (4.4)$$

We get

$$F^-(NCl(NInt(M))) \subseteq \mathcal{R} - \tilde{R}. \quad (4.5)$$

Let $N = \mathcal{R} - \tilde{R}$. Then, $r \in \mathcal{R} - N$ and N is an $NPCs$.

(iv) $\Rightarrow$ (ii): The proof is similar to (ii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (v): Let $\tilde{S}$ be any neutrosophic open set of $\mathcal{S}$ and $r \in F^+(NInt(NCl(\tilde{S})))$. By (i), there exists an $NPOs$ $\tilde{R}_r$ of $\mathcal{R}$ containing r such that

$$\tilde{R}_r \subseteq F^+(NInt(NCl(\tilde{S}))). \quad (4.6)$$

Hence, we obtain

$$F^+(NInt(NCl(\tilde{S}))) = \cup_r \in F^+(NInt(NCl(\tilde{S})))\tilde{R}_r. \quad (4.7)$$

Therefore, $F^+(NInt(NCl(\tilde{S})))$ is an $NPOs$ of $\mathcal{R}$.

(v) $\Rightarrow$ (i): Let $\tilde{S}$ be any neutrosophic open set of $\mathcal{S}$ and $r \in F^+(\tilde{S})$. By (v), $F^+(NInt(NCl(\tilde{S})))$ is $NPOs$ in $\mathcal{R}$. Let $\tilde{R} = F^+(NInt(NCl(\tilde{S})))$. Then,

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \quad (4.8)$$

Therefore, F is upper neutrosophic pre-continuous.

(v) $\Rightarrow$ (vi): Let F be any neutrosophic closed set of $\mathcal{S}$. Then, $\mathcal{S} - F$ is an NOs of $\mathcal{S}$. By (v), $F^+(NInt(NCl(\mathcal{S} - F)))$ is $NPOs$ in $\mathcal{R}$. Since $NInt(NCl(\mathcal{S} - F)) = \mathcal{S} - NCl(NInt(F))$, it follows that $F^+(NInt(NCl(\mathcal{S} - F))) = F^+(\mathcal{S} - NCl(NInt(F))) = \mathcal{R} - F^-(NCl(NInt(F)))$. We obtain that $F^-(NCl(NInt(F)))$ is $NPCs$ in $\mathcal{R}$.

(vi) $\Rightarrow$ (v): The proof is similar to (v) $\Rightarrow$ (vi).

(v) $\Rightarrow$ (vii): Let G be any $NROs$ of $\mathcal{S}$. By (v),

$$F^+(NInt(NCl(G))) = F^+(G) \quad (4.9)$$

is $NPOs \in \mathcal{R}$.

(vii) $\Rightarrow$ (v): Let $\tilde{S}$ be any neutrosophic-open set of $\mathcal{S}$. Then, $NInt(NCl(\tilde{S}))$ is $NROsin\mathcal{S}$. By (vii),

$$F^+(NInt(NCl(\tilde{S}))) \quad (4.10)$$

is $NPOs \in \mathcal{R}$.

(vi) $\Rightarrow$ (viii): The proof is similar to (v) $\Rightarrow$ (vii).

(viii) $\Rightarrow$ (vi): The proof is similar to (vii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (ix): Let $r \in \mathcal{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(r)$. Then, there exists an open set G of $\mathcal{S}$ such that

$$F(r) \subseteq G \subseteq \tilde{S}. \quad (4.11)$$

Hence, we obtain $r \in F^+(G) \subseteq F^+(\tilde{S})$. Since $F^+(NInt(NCl(G)))$ is $NPOs \in \mathcal{R}$, $F^+(NInt(NCl(\tilde{S})))$ is a neutrosophic pre-nbh of r .

(ix) $\Rightarrow$ (x): Let $r \in \mathcal{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(r)$. By (ix), $F^+(NInt(NCl(\tilde{S})))$ is a neutrosophic-nbh of r . Let $\tilde{R} = F^+(NInt(NCl(\tilde{S})))$. Then,

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \quad (4.12)$$

(x) $\Rightarrow$ (i): Let $r \in \mathcal{R}$ and $\tilde{S}$ be any NOs of $\mathcal{S}$ such that $F(r) \subseteq \tilde{S}$. Then, $\tilde{S}$ is a neutrosophic-nbh of $F(r)$. By (r), there exists a neutrosophic pre-nbh $\tilde{R}$ of r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \quad (4.13)$$

Hence, there exists an $NPOs$ G of $\mathcal{R}$ such that

$$r \in G \subseteq \tilde{R}, \quad (4.14)$$

and hence

$$F(G) \subseteq F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \quad (4.15)$$

We obtain that F is upper almost neutrosophic pre-continuous.

(vi) $\Rightarrow$ (xi): For every subset A of $\mathcal{S}$, $NCl(A)$ is NCs in $\mathcal{S}$. By (vi), $F^-(NCl(NInt(NCl(A))))$ is $NPCs \in \mathcal{R}$. Hence, we obtain

$$pNCl(F^-(NCl(NInt(A)))) \subseteq F^-(NCl(NInt(NCl(A)))). \quad (4.16)$$

(xi) $\Rightarrow$ (vi): For any NCs F of $\mathcal{S}$. Then we have

$$pNCl(F^-(F))) \subseteq F^-(NCl(NInt(NCl(F)))) = F^-(NCl(NInt(F))). \quad (4.17)$$

Thus, $F^-(NCl(NInt(F)))$ is $NPCs \in \mathcal{R}$.

(v) $\Rightarrow$ (xii): For every subset A of $\mathcal{S}$, $NInt(A)$ is $NO \in \mathcal{S}$. By (v),

$$F^+(NInt(NCl(NInt(A)))) \quad (4.18)$$

is $NPOs$ in $\mathcal{R}$. Therefore, we obtain

$$F^+(NInt(NCl(NInt(A)))) \subseteq pNInt(F^+(NInt(NCl(A)))). \quad (4.19)$$

(xii) $\Rightarrow$ (v): Let $\tilde{S}$ be any subset of $\mathcal{S}$. Then

$$F^+(NInt(NCl(\tilde{S}))) \subseteq pNInt(F^+(NInt(NCl(\tilde{S})))). \quad (4.20)$$

Therefore, $F^+(NInt(NCl(\tilde{S})))$ is $NPOs \in \mathcal{R}$. $\square$

Remark 4.3. If $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ are neutrosophic upper pre-continuous multifunctions, then F is a neutrosophic upper almost pre-continuous multifunction.

The implication is not reversible.

Example 4.1. Let $\mathcal{R} = \{\mu, \nu, \omega\}$ and $\mathcal{S} = \{u, v, w, t, h\}$. Let $(\mathcal{R}, \mathcal{T})$ be a neutrosophic topology on $\mathcal{R}$ and $\sigma_{\tilde{S}}$ a topology on $\mathcal{S}$ given by $\mathcal{T} = \{\phi_N, \{v\}, \{w\}, \{v, w\}, \mathcal{R}_N\}$ and $\sigma_{\tilde{S}} = \{\phi_N, \{u, v, w, t\}, \mathcal{S}_N\}$. Define the multifunction $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ by $F(\mu) = \{w\}$, $F(\nu) = \{v, t\}$ and $F(\omega) = \{u, h\}$. Then F is upper almost neutrosophic precontinuous but not upper neutrosophic precontinuous, since $\{u, v, w, t, h\} \in \sigma_{\tilde{S}}$ and $F^+(\{u, v, w, t, h\}) = \{\mu, \nu\}$ is not neutrosophic pre-open in $\mathcal{R}$.

Theorem 4.4. Let $F : (\mathcal{R}, \mathcal{T}) \longrightarrow (\mathcal{S}, \mathcal{T}_1)$ be a multifunction from a neutrosophic topological space $(\mathcal{R}, \mathcal{T})$ to a topological space $(\mathcal{S}, \mathcal{T}_1)$. Then the following properties are equivalent:

- (i) F is lower almost neutrosophic-precontinuous;
- (ii) for each $r \in \mathcal{R}$ and for each open set $\tilde{S}$ of $\mathcal{S}$ such that $F(r) \cap \tilde{S} \neq \phi$, there exists a neutrosophic-preopen $\tilde{R}$ of $\mathcal{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \cap NInt(NCl(\tilde{S})) \neq \phi$;
- (iii) for each $r \in \mathcal{R}$ and for each regular open set G of $\mathcal{S}$ such that $F(r) \cap G \neq \phi$, there exists a neutrosophic-peropen $\tilde{R}$ of $\mathcal{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \cap G \neq \phi$;
- (iv) for each $r \in \mathcal{R}$ and for each closed set M of $\mathcal{S}$ such that $r \in F^+(\mathcal{S} - M)$, there exists a neutrosophic-perclosed N of $\mathcal{R}$ such that $r \in \mathcal{R} - N$ and $F^+(NCl(NInt(M))) \subseteq N$;
- (v) $F^-(NInt(NCl(\tilde{S})))$ is neutrosophic-pre-open in $\mathcal{R}$ for any $NOs \tilde{S}$ of $\mathcal{S}$;
- (vi) $F^+(NCl(NInt(F)))$ is neutrosophic-pre-closed in $\mathcal{R}$ for any $NCs F$ of $\mathcal{S}$;
- (vii) $F^-(G)$ is neutrosophic-per-open in $\mathcal{R}$ for any $NROs G$ of $\mathcal{S}$;
- (viii) $F^+(H)$ is neutrosophic-perclosed in $\mathcal{R}$ for any $NRCs H$ of $\mathcal{S}$;
- (ix) $pNCl(F^+(NCl(NInt(B)))) \subseteq F^+(NCl(NInt(NCl(B))))$ for every subset B of $\mathcal{S}$;
- (x) $F^-(NInt(NCl(NInt(B)))) \subseteq pNInt(F^-(NInt(NCl(B))))$ for every subset B of $\mathcal{S}$.

Proof. It is similar to that of Theorem 4.2. $\square$

5 Conclusions and/or Discussions

Topology on lattice is a type of theory developed on lattice which involves many problems on ordered structure. For instance, complete distributivity of lattices is a pure algebraic problem that establishes a connection between algebra and analysis. neutrosophic topology is a generalization of fuzzy topology in classical mathematics, but it also has its own marked characteristics. Some scholars used tools for examining neutrosophic topological spaces and establishing new types from existing ones. Attention has been paid to define and characterize new weak forms of continuity.

We have introduced neutrosophic upper and neutrosophic lower almost pre-continuous-multifunctions as a generalization of neutrosophic multifunctions over neutrosophic topology space. Many results have been established to show how far topological structures are preserved by these neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous). We also have provided examples where such properties fail to be preserved. In this paper we have introduced the concept of upper and lower pre-continuous multifunction and study some properties of these functions together with the graph of upper and lower pre-continuous as well as upper and lower weakly pre-continuous multifunction.

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A Novel on $\mathcal{NSS}$ Contra Strong Precontinuity

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Abstract: In this paper, the concept of $\mathcal{NSS}$ contra continuous function is introduced. Several types of contra continuous functions in $\mathcal{NSS}$ spaces are discussed. Some interesting properties of $\mathcal{NSS}$ contra strongly precontinuous function is established.

Keywords: $\mathcal{NR}$, $\mathcal{NSS} - \mathcal{CCF}$, $\mathcal{NSS} - \mathcal{CACF}$, $\mathcal{NSS} - \mathcal{CpreCF}$ and $\mathcal{NSS} - \mathcal{CStrpreCF}$.

1 Introduction

L. A. Zadeh introduced the idea of fuzzy sets in 1965[16] and later Atanassov [1] generalized it and offered the concept of intuitionistic fuzzy sets. Intuitionistic fuzzy set theory has applications in many fields like medical diagnosis, information technology, nanorobotics, etc. The idea of intuitionistic L-fuzzy subring was introduced by K. Meena and V. Thomas [9]. R. Narmada Devi et al. [10, 11, 12] introduced the concept of contra strong precontinuity with respect to the intuitionistic fuzzy structure ring spaces and B. Krteska and E. Ekici [5, 7, 8] introduced the idea of intuitionistic fuzzy contra continuity. The concept of α continuity in intuitionistic fuzzy topological spaces was introduced by J. K. Jeon et al. [6]. F. Smarandache introduced the important and useful concepts of neutrosophy and neutrosophic set [[14], [15]]. A. A. Salama and S. A. Alblowi were established the concepts of neutrosophic crisp set and neutrosophic crisp topological space[13]. In this paper, the concept of $\mathcal{NSS}$ contra continuous function is introduced. Several types of contra-continuous functions in $\mathcal{NSS}$ spaces are discussed. Some interesting properties of $\mathcal{NSS}$ contra strongly precontinuous function is established.

2 Preliminaries

Definition 2.1. [14, 15] Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$, with

$$(i) \sup_T = t_{sup}, \inf_T = t_{inf}$$

- (ii) $sup_I = i_{sup}, inf_I = i_{inf}$
- (iii) $sup_F = f_{sup}, inf_F = f_{inf}$
- (iv) $n - sup = t_{sup} + i_{sup} + f_{sup}$
- (v) $n - inf = t_{inf} + i_{inf} + f_{inf}$.

Observe that T, I, F are neutrosophic components.

Definition 2.2. [14, 15] Let S_1 be a non-empty fixed set. A neutrosophic set (briefly N -set) Λ is an object such that $\Lambda = \{\langle u, \mu_\Lambda(u), \sigma_\Lambda(u), \gamma_\Lambda(u) \rangle : u \in S_1\}$ where $\mu_\Lambda(u), \sigma_\Lambda(u)$ and $\gamma_\Lambda(u)$ which represents the degree of membership function (namely $\mu_\Lambda(u)$), the degree of indeterminacy (namely $\sigma_\Lambda(u)$) and the degree of non-membership (namely $\gamma_\Lambda(u)$) respectively of each element $u \in S_1$ to the set Λ .

Definition 2.3. [13] Let $S_1 \neq \emptyset$ and the N -sets Λ and Γ be defined as

$\Lambda = \{\langle u, \mu_\Lambda(u), \sigma_\Lambda(u), \Gamma_\Lambda(u) \rangle : u \in S_1\}, \Gamma = \{\langle u, \mu_\Gamma(u), \sigma_\Gamma(u), \Gamma_\Gamma(u) \rangle : u \in S_1\}$. Then

- (a) $\Lambda \subseteq \Gamma$ iff $\mu_\Lambda(u) \leq \mu_\Gamma(u), \sigma_\Lambda(u) \leq \sigma_\Gamma(u)$ and $\Gamma_\Lambda(u) \geq \Gamma_\Gamma(u)$ for all $u \in S_1$;
- (b) $\Lambda = \Gamma$ iff $\Lambda \subseteq \Gamma$ and $\Gamma \subseteq \Lambda$;
- (c) $\bar{\Lambda} = \{\langle u, \Gamma_\Lambda(u), \sigma_\Lambda(u), \mu_\Lambda(u) \rangle : u \in S_1\}$; [Complement of Λ]
- (d) $\Lambda \cap \Gamma = \{\langle u, \mu_\Lambda(u) \wedge \mu_\Gamma(u), \sigma_\Lambda(u) \wedge \sigma_\Gamma(u), \Gamma_\Lambda(u) \vee \Gamma_\Gamma(u) \rangle : u \in S_1\}$;
- (e) $\Lambda \cup \Gamma = \{\langle u, \mu_\Lambda(u) \vee \mu_\Gamma(u), \sigma_\Lambda(u) \vee \sigma_\Gamma(u), \Gamma_\Lambda(u) \wedge \Gamma_\Gamma(u) \rangle : u \in S_1\}$;
- (f) $[\]\Lambda = \{\langle u, \mu_\Lambda(u), \sigma_\Lambda(u), 1 - \mu_\Lambda(u) \rangle : u \in S_1\}$;
- (g) $\langle \rangle \Lambda = \{\langle u, 1 - \Gamma_\Lambda(u), \sigma_\Lambda(u), \Gamma_\Lambda(u) \rangle : u \in S_1\}$.

Definition 2.4. [13] Let $\{\Lambda_i : i \in J\}$ be an arbitrary family of N -sets in S_1 . Then

- (a) $\bigcap \Lambda_i = \{\langle u, \wedge \mu_{\Lambda_i}(u), \wedge \sigma_{\Lambda_i}(u), \vee \Gamma_{\Lambda_i}(u) \rangle : u \in S_1\}$;
- (b) $\bigcup \Lambda_i = \{\langle u, \vee \mu_{\Lambda_i}(u), \vee \sigma_{\Lambda_i}(u), \wedge \Gamma_{\Lambda_i}(u) \rangle : u \in S_1\}$.

Definition 2.5. [13] $0_N = \{\langle u, 0, 0, 1 \rangle : u \in S\}$ and $1_N = \{\langle u, 1, 1, 0 \rangle : u \in S\}$.

Definition 2.6. [4] A neutrosophic topology (briefly N -topology) on $S_1 \neq \emptyset$ is a family ξ_1 of N -sets in S_1 satisfying the following axioms:

- (i) $0_N, 1_N \in \xi_1$,
- (ii) $H_1 \cap H_2 \in \xi_1$ for any $H_1, H_2 \in \xi_1$,
- (iii) $\bigcup H_i \in \xi_1$ for arbitrary family $\{H_i \mid i \in \Lambda\} \subseteq \xi_1$.

In this case the ordered pair (S_1, ξ_1) or simply S_1 is called an NTS and each N -set in ξ_1 is called a neutrosophic open set (briefly N -open set). The complement $\bar{\Lambda}$ of an N -open set Λ in S_1 is called a neutrosophic closed set (briefly N -closed set) in S_1 .

Definition 2.7. [13] Let D be any neutrosophic set in an neutrosophic topological space S . Then the neutrosophic interior and neutrosophic closure of D are defined and denoted by

- (i) $Nint(D) = \bigcup \{H \mid H \text{ is an } NS \text{ open set in } S \text{ and } H \subseteq D\}.$
- (ii) $Ncl(D) = \bigcap \{H \mid H \text{ is a neutrosophic closed set in } S \text{ and } H \supseteq D\}.$

Proposition 2.1. [13] For any neutrosophic set D in (S, τ) we have $Ncl(C(D)) = C(Nint(D))$ and $Nint(C(D)) = C(Ncl(D))$.

Corollary 2.1. [4] Let $D, D_i (i \in J)$ and $U, U_j (j \in K)$ IFSs in S_1 and S_2 and $\phi : S_1 \rightarrow S_2$ a function. Then

- (i) $D \subseteq \phi^{-1}(\phi(D))$ (If ϕ is injective, then $D = \phi^{-1}(\phi(D))$),
- (ii) $\phi(\phi^{-1}(U)) \subseteq U$ (If ϕ is surjective, then $\phi(\phi^{-1}(U)) = U$),
- (iii) $\phi^{-1}(\bigcup U_j) = \bigcup \phi^{-1}(U_j)$ and $\phi^{-1}(\bigcap U_j) = \bigcap \phi^{-1}(U_j)$,
- (iv) $\phi^{-1}(1_{\sim}) = 1_{\sim}$ and $\phi^{-1}(0_{\sim}) = 0_{\sim}$,
- (v) $\phi^{-1}(\overline{U}) = \overline{\phi^{-1}(U)}$.

Definition 2.8. [5]

An IFS D of an IFTS is called an intuitionistic fuzzy α -open set (IF α OS) if $D \subseteq int(cl(int(D)))$. The complement of an IF α OS is called an intuitionistic fuzzy α -closed set (IF α CS).

Definition 2.9. [3] A $\phi : X \rightarrow Y$ be a function.

- (i) If $B = \{\langle v, \mu_B(v), \gamma_B(v) \rangle : v \in Y\}$ is an IFS in Y , then the preimage of B under ϕ (denoted by $\phi^{-1}(B)$) is defined by $\phi^{-1}(B) = \{\langle u, \phi^{-1}(\mu_B)(u), \phi^{-1}(\gamma_B)(u) \rangle : u \in X\}$.
- (ii) If $A = \{\langle u, \lambda_A(u), \vartheta_A(u) \rangle : u \in X\}$ is an IFS in X , then the image of A under ϕ (denoted by $\phi(A)$) is defined by $\phi(A) = \{\langle v, \phi(\lambda_A(v)), (1 - \phi(1 - \vartheta_A))(v) \rangle : v \in Y\}$.

Definition 2.10. [3] Let (X, τ) and (Y, σ) be two IFTSs and $\phi : X \rightarrow Y$ be a function. Then ϕ is said to be intuitionistic fuzzy continuous if the preimage of each IFS in σ is an IFS in τ .

Definition 2.11. [8,9] Let R be a ring. An intuitionistic fuzzy set $A = \langle u, \mu_A, \gamma_A \rangle$ in R is called an intuitionistic fuzzy ring on R if it satisfies the following conditions:

- (i) $\mu_A(u + v) \geq \mu_A(u) \wedge \mu_A(v)$ and $\mu_A(uv) \geq \mu_A(u) \wedge \mu_A(v)$.
- (ii) $\gamma_A(u + v) \leq \gamma_A(u) \vee \gamma_A(v)$ and $\gamma_A(uv) \leq \mu_A(u) \vee \gamma_A(v)$.

for all $u, v \in R$.

3 Neutrosophic structure ring contra strong precontinuous function

In this section, the concepts of neutrosophic ring, neutrosophic structure ring space are introduced. Also some interesting properties of neutrosophic structure ring contra strong precontinuous function and their characterizations are studied.

Definition 3.1. Let $\mathfrak{R}$ be a ring. A neutrosophic set $\Lambda = \{\langle u, \mu_{\Lambda}(u), \sigma_{\Lambda}(u), \gamma_{\Lambda}(u) \rangle : u \in R\}$ in $\mathfrak{R}$ is called a neutrosophic ring[briefly $\mathfrak{NR}$] on $\mathfrak{R}$ if it satisfies the following conditions:

- (i) $\mu_{\Lambda(u+v)} \geq \mu_{\Lambda(u)} \wedge \mu_{\Lambda(v)}$ and $\mu_{\Lambda(uv)} \geq \mu_{\Lambda(u)} \wedge \mu_{\Lambda(v)}$.
- (ii) $\sigma_{\Lambda(u+v)} \geq \sigma_{\Lambda(u)} \wedge \sigma_{\Lambda(v)}$ and $\sigma_{\Lambda(uv)} \geq \sigma_{\Lambda(u)} \wedge \sigma_{\Lambda(v)}$.
- (iii) $\gamma_{\Lambda(u+v)} \leq \gamma_{\Lambda(u)} \vee \gamma_{\Lambda(v)}$ and $\gamma_{\Lambda(uv)} \leq \gamma_{\Lambda(x)} \vee \gamma_{\Lambda(y)}$.

for all $u, v \in \mathfrak{R}$.

Definition 3.2. Let $\mathfrak{R}$ be a ring. A family $\mathcal{S}$ of a $\mathfrak{NR}$'s in $\mathfrak{R}$ is said to be neutrosophic structure ring on $\mathfrak{R}$ if it satisfies the following axioms:

- (i) $0_N, 1_N \in \mathcal{S}$.
- (ii) $H_1 \cap H_2 \in \mathcal{S}$ for any $H_1, H_2 \in \mathcal{S}$.
- (iii) $\cup H_k \in \mathcal{S}$ for arbitrary family $\{H_k \mid k \in J\} \subseteq \mathcal{S}$.

The ordered pair $(\mathfrak{R}, \mathcal{S})$ is called a neutrosophic structure ring($\mathfrak{NSR}$) space. Every member of $\mathcal{S}$ is called a $\mathfrak{N}$ open ring (briefly $\mathfrak{NOR}$) in $(\mathfrak{R}, \mathcal{S})$. The complement of a $\mathfrak{NOR}$ in $(\mathfrak{R}, \mathcal{S})$ is a $\mathfrak{N}$ closed ring ($\mathfrak{NCR}$) in $(\mathfrak{R}, \mathcal{S})$.

Definition 3.3. Let D be a $\mathfrak{N}$ ring in $\mathfrak{NSR}$ space $(\mathfrak{R}, \mathcal{S})$. Then $\mathfrak{NSR}$ interior and $\mathfrak{NSR}$ closure of D are defined and denoted by

- (i) $\mathfrak{Nint}_{\mathfrak{R}}(D) = \bigcup \{H \mid H \text{ is a } \mathfrak{NOR} \text{ in } \mathfrak{R} \text{ and } H \subseteq D\}$.
- (ii) $\mathfrak{Ncl}_{\mathfrak{R}}(D) = \bigcap \{H \mid H \text{ is a } \mathfrak{NCR} \text{ in } \mathfrak{R} \text{ and } H \supseteq D\}$.

Proposition 3.1. For any $\mathfrak{NR}$ D in $(\mathfrak{R}, \mathcal{S})$ we have

- (i) $\mathfrak{Ncl}_{\mathfrak{R}}(C(D)) = C(\mathfrak{Nint}_{\mathfrak{R}}(D))$
- (ii) $\mathfrak{Nint}_{\mathfrak{R}}(C(D)) = C(\mathfrak{Ncl}_{\mathfrak{R}}(D))$

Definition 3.4. A $\mathfrak{NR}$ D of a $\mathfrak{NSR}$ space $(\mathfrak{R}, \mathcal{S})$ is said to be a

- (i) $\mathfrak{N}$ regular open structure ring ($\mathfrak{NRegOSSR}$), if $D = \mathfrak{Nint}_{\mathfrak{R}}(\mathfrak{Ncl}_{\mathfrak{R}}(D))$
- (ii) $\mathfrak{N}\alpha$ -open structure ring ($\mathfrak{N}\alpha\text{OSSR}$), if $D \subseteq \mathfrak{Nint}_{\mathfrak{R}}(\mathfrak{Ncl}_{\mathfrak{R}}(\mathfrak{Nint}_{\mathfrak{R}}(D)))$
- (iii) $\mathfrak{N}$ semiopen structure ring ($\mathfrak{NSemiOSSR}$), if $D \subseteq \mathfrak{Ncl}_{\mathfrak{R}}(\mathfrak{Nint}_{\mathfrak{R}}(D))$
- (iv) $\mathfrak{N}$ preopen structure ring ($\mathfrak{NPreOSSR}$), if $D \subseteq \mathfrak{Nint}_{\mathfrak{R}}(\mathfrak{Ncl}_{\mathfrak{R}}(D))$

(v) $\aleph\beta$ -open structure ring ($\aleph\beta\text{OSR}$), if $D \subseteq \aleph cl_{\aleph}(\aleph int_{\aleph}(\aleph cl_{\aleph}(D)))$

Note 3.1. Let $(\aleph, \mathcal{S})$ be a $\aleph\text{SR}$ space. Then the complement of a $\aleph\text{RegOSR}$ (resp. $\aleph\alpha\text{OSR}$, $\aleph\text{SemiOSR}$, $\aleph\text{PreOSR}$ and $\aleph\beta\text{OSR}$) is a $\aleph$ regular closed structure ring ($\aleph\text{RegCSR}$) (resp. $\aleph\alpha$ -closed structure ring ($\aleph\alpha\text{CSR}$), $\aleph$ semiclosed structure ring ($\aleph\text{SemiCSR}$), $\aleph$ preclosed structure ring ($\aleph\text{PreCSR}$), $\aleph\beta$ -closed structure ring ($\aleph\beta\text{CSR}$)).

Definition 3.5. The $\aleph\text{SR}$ preinterior and $\aleph\text{SR}$ preclosure of $\aleph\aleph D$ of a $\aleph\text{SR}$ space are defined and denoted by

(i) $\aleph pint_{\aleph}(D) = \bigcup \{H : H \text{ is a } \aleph\text{PreOSR} \text{ in } (R, \mathcal{S}) \text{ and } H \subseteq D\}.$

(ii) $\aleph pcl_{\aleph}(D) = \bigcap \{H : H \text{ is a } \aleph\text{PreCSR} \text{ in } (R, \mathcal{S}) \text{ and } D \subseteq H\}.$

Remark 3.1. For any $\aleph\aleph D$ of a $\aleph\text{SR}$ space $(\aleph, \mathcal{S})$, then

(i) $\aleph pint_{\aleph}(D) = D$ if and only if D is a $\aleph\text{PreOSR}$.

(ii) $\aleph pcl_{\aleph}(D) = D$ if and only if D is a $\aleph\text{PreCSR}$.

(iii) $\aleph int_{\aleph}(D) \subseteq \aleph pint_{\aleph}(D) \subseteq D \subseteq \aleph pcl_{\aleph}(D) \subseteq \aleph cl_{\aleph}(D)$

Definition 3.6. A $\aleph\aleph D$ of a $\aleph\text{SR}$ space $(\aleph, \mathcal{S})$ is called a $\aleph$ strongly preopen structure ring ($\aleph\text{stronglyPreOSR}$), if $D \subseteq \aleph int_{\aleph}(\aleph pcl_{\aleph}(D))$. The complement of a $\aleph\text{stronglyPreOSR}$ is a $\aleph$ strongly preclosed structure ring (briefly $\aleph\text{stronglyPreCSR}$).

Definition 3.7. The $\aleph\text{SR}$ strongly preinterior and $\aleph\text{SR}$ strongly preclosure of $\aleph\aleph D$ of a $\aleph\text{SR}$ space are defined and denoted by

(i) $\aleph spint_{\aleph}(D) = \bigcup \{H : H \text{ is a } \aleph\text{stronglyPreOSR} \text{ in } (R, \mathcal{S}) \text{ and } H \subseteq D\}.$

(ii) $\aleph spcl_{\aleph}(D) = \bigcap \{H : H \text{ is a } \aleph\text{stronglyPreCSR} \text{ in } (R, \mathcal{S}) \text{ and } D \subseteq H\}.$

Remark 3.2. For any $\aleph\aleph D$ of a $\aleph\text{SR}$ space $(\aleph, \mathcal{S})$, then

(i) $\aleph spint_{\aleph}(D) = D$ if and only if D is a $\aleph\text{stronglyPreOSR}$.

(ii) $\aleph spcl_{\aleph}(D) = D$ if and only if D is a $\aleph\text{stronglyPreCSR}$.

(iii) $\aleph int_{\aleph}(D) \subseteq \aleph spint_{\aleph}(D) \subseteq D \subseteq \aleph spcl_{\aleph}(D) \subseteq \aleph cl_{\aleph}(D)$

Proposition 3.2. A $\aleph\aleph$ of a $\aleph\text{SR}$ space $(\aleph, \mathcal{S})$ is a $\aleph\alpha\text{OSR}$ if and only if it is both $\aleph\text{SemiOSR}$ and $\aleph\text{stronglyPreOSR}$.

Definition 3.8. Let $(\aleph_1, \mathcal{S}_1)$ and $(\aleph_2, \mathcal{S}_2)$ be any two $\aleph\text{SR}$ spaces. A function $\phi : (\aleph_1, \mathcal{S}_1) \rightarrow (\aleph_2, \mathcal{S}_2)$ is called a $\aleph\text{SR}$

(i) contra continuous function ($\aleph\text{SR} - \mathcal{CCF}$) if $\phi^{-1}(U)$ is a $\aleph\text{OR}$ in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph\text{CR}$ U in $(\aleph_2, \mathcal{S}_2)$.

(ii) contra α -continuous function ($\aleph\text{SR} - \mathcal{CACF}$) if $\phi^{-1}(U)$ is a $\aleph\alpha\text{OR}$ in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph\text{CR}$ U in $(\aleph_2, \mathcal{S}_2)$.

(iii) contra precontinuous function ($\aleph\text{SR} - \mathcal{CpreCF}$) if $\phi^{-1}(U)$ is a $\aleph\text{PreOSR}$ in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph\text{CR}$ U in $(\aleph_2, \mathcal{S}_2)$.

- (iv) contra strongly precontinuous function ($\text{NSSR} - \mathcal{CStrpreCF}$) if $\phi^{-1}(U)$ is a NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$.

Proposition 3.3. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two NSSR spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. If ϕ is a $\text{NSSR} - \mathcal{CCF}$, then ϕ is a $\text{NSSR} - \mathcal{C}\alpha\mathcal{CF}$.

Proof:

Let U be a NCR in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since ϕ is $\text{NSSR} - \mathcal{CCF}$, $\phi^{-1}(U)$ is a NOR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. By Remark 3.1(iii), $\phi^{-1}(U) \subseteq \text{Ncl}_{\mathfrak{R}_1}(\phi^{-1}(U))$.

Since $\phi^{-1}(U)$ is a NOR in $(\mathfrak{R}_1, \mathcal{S}_1)$, $\phi^{-1}(U) \subseteq \text{Ncl}_{\mathfrak{R}_1}(\text{Nint}_{\mathfrak{R}_1}(\phi^{-1}(U)))$. Hence, $\phi^{-1}(U)$ is a NSemiOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$. By Remark 3.1 (iii), $\phi^{-1}(U) \subseteq \text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U))$. Taking interior on both sides, $\text{Nint}_{\mathfrak{R}_1}(\phi^{-1}(U)) \subseteq \text{Nint}_{\mathfrak{R}_1}(\text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)))$. Since $\phi^{-1}(U)$ is a NOR in $(\mathfrak{R}_1, \mathcal{S}_1)$, $\phi^{-1}(U) \subseteq \text{Nint}_{\mathfrak{R}_1}(\text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)))$. Hence, $\phi^{-1}(U)$ is a NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$. Therefore, $\phi^{-1}(U)$ is both NSemiOSR and NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$. By Proposition 3.1, $\phi^{-1}(\phi)$ is a $\text{N}\alpha\text{OSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. Hence, ϕ is a $\text{NSSR} - \mathcal{C}\alpha\mathcal{CF}$.

Proposition 3.4. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two NSSR spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. If ϕ is a $\text{NSSR} - \mathcal{C}\alpha\mathcal{CF}$, then ϕ is a $\text{NSSR} - \mathcal{CStrpreCF}$.

Proof:

Let U be any NCR in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since ϕ is a $\text{NSSR} - \mathcal{C}\alpha\mathcal{CF}$, $\phi^{-1}(U)$ is a $\text{N}\alpha\text{OSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. By Proposition 3.1, $\phi^{-1}(U)$ is both NSemiOSR and NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. Therefore, $\phi^{-1}(U)$ is a NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. Hence ϕ is a $\text{NSSR} - \mathcal{CStrpreCF}$.

Proposition 3.5. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two NSSR spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. If ϕ is a $\text{NSSR} - \mathcal{CStrpreCF}$, then ϕ is a $\text{NSSR} - \mathcal{CpreCF}$.

Proof:

Let U be any NCR in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since ϕ is a $\text{NSSR} - \mathcal{CStrpreCF}$, $\phi^{-1}(U)$ is a NstronglyPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$, that is,

$$\phi^{-1}(U) \subseteq \text{Nint}_{\mathfrak{R}_1}(\text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U))) \quad (3.1)$$

By Remark 3.1(iii),

$$\text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)) \subseteq \text{Ncl}_{\mathfrak{R}_1}(\phi^{-1}(U)) \quad (3.2)$$

Substitute (3.2) in (3.1), we get $\phi^{-1}(U) \subseteq \text{Nint}_{\mathfrak{R}_1}(\text{Ncl}_{\mathfrak{R}_1}(\phi^{-1}(U)))$. Therefore, $\phi^{-1}(U)$ is a NPreOSR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$. Hence, ϕ is a $\text{NSSR} - \mathcal{CpreCF}$.

Proposition 3.6. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two NSSR spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. If ϕ is a $\text{NSSR} - \mathcal{CCF}$, then ϕ is a $\text{NSSR} - \mathcal{CStrpreCF}$.

Proof:

Let U be any NCR in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since ϕ is a $\text{NSSR} - \mathcal{CCF}$, $\phi^{-1}(U)$ is a NOR in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each NCR U in $(\mathfrak{R}_2, \mathcal{S}_2)$, that is, $\phi^{-1}(U) = \text{Nint}_{\mathfrak{R}_1}(\phi^{-1}(U))$. By Remark 3.1(iii),

$$\phi^{-1}(U) \subseteq \text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)) \quad (3.3)$$

Taking interior on both sides in (3.3),

$$\phi^{-1}(U) = \text{Nint}_{\mathfrak{R}_1}(\phi^{-1}(U)) \subseteq \text{Nint}_{\mathfrak{R}_1}(\text{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U))).$$

Hence, $\phi^{-1}(U)$ is a $\aleph$ stronglyPreOS $\aleph$ in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph$ CR U in $(\aleph_2, \mathcal{S}_2)$. Thus, ϕ is a $\aleph$ SR – $\mathcal{CStrpreCF}$.

Proposition 3.7. Let $(\aleph_1, \mathcal{S}_1)$ and $(\aleph_2, \mathcal{S}_2)$ be any two $\aleph$ SR spaces. Let $\phi : (\aleph_1, \mathcal{S}_1) \rightarrow (\aleph_2, \mathcal{S}_2)$ be a function. If ϕ is a $\aleph$ SR – $\mathcal{CCF}$, then ϕ is a $\aleph$ SR – $\mathcal{CpreCF}$.

Proof:

Let U be any $\aleph$ CR in $(\aleph_2, \mathcal{S}_2)$. Since ϕ is a $\aleph$ SR – $\mathcal{CCF}$, $\phi^{-1}(U)$ is a $\aleph$ OR in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph$ CR U in $(\aleph_2, \mathcal{S}_2)$, that is, $\phi^{-1}(U) = \aleph\text{int}_{\aleph_1}(\phi^{-1}(U))$. By Remark 3.1(iii),

$$\phi^{-1}(B) \subseteq \aleph\text{cl}_{\aleph_1}(\phi^{-1}(U)) \quad (3.4)$$

Taking interior on both sides in (3.4),

$$\phi^{-1}(U) = \aleph\text{int}_{\aleph_1}(\phi^{-1}(U)) \subseteq \aleph\text{int}_{\aleph_1}(\aleph\text{cl}_{\aleph_1}(\phi^{-1}(U))).$$

Hence, $\phi^{-1}(U)$ is a $\aleph$ PreOS $\aleph$ in $(\aleph_1, \mathcal{S}_1)$, for each $\aleph$ CR U in $(\aleph_2, \mathcal{S}_2)$. Thus, ϕ is a $\aleph$ SR – $\mathcal{CpreCF}$.

Remark 3.3. The converses of the Proposition 3.2, Proposition 3.3, Proposition 3.4, Proposition 3.5 and Proposition 3.6 need not be true as it is shown in the following example.

Example 3.1. Let $\aleph = \{a, b, c\}$ be a nonempty set with two binary operations as follows:

+	u	v	w
u	u	v	w
v	v	w	u
w	w	u	v

and

*	u	v	w
u	u	u	u
v	u	v	w
w	u	w	v

Then $(\aleph, +, *)$ is a ring. Define $\aleph$'s L , M and P as follows:

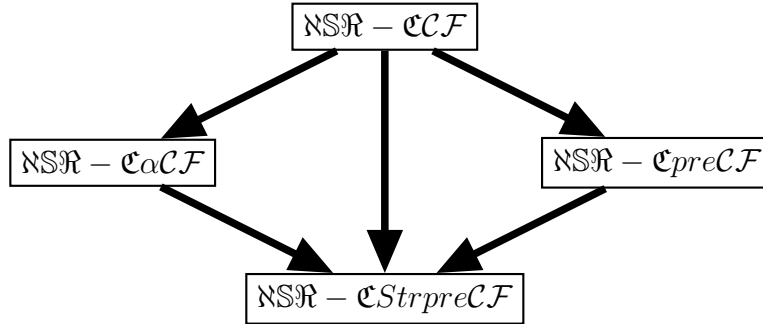
$$\begin{aligned} \mu_L(u) &= 0.4, \mu_L(v) = 0.8, \mu_L(w) = 0.2; \\ \mu_M(u) &= 0.7, \mu_M(v) = 0.9, \mu_M(w) = 0.4; \\ \mu_P(u) &= 0.5, \mu_P(v) = 0.7, \mu_P(w) = 0.3; \\ \sigma_L(u) &= 0.4, \sigma_L(v) = 0.8, \sigma_L(w) = 0.2; \\ \sigma_M(u) &= 0.7, \sigma_M(v) = 0.9, \sigma_M(w) = 0.4; \\ \sigma_P(u) &= 0.5, \sigma_P(v) = 0.7, \sigma_P(w) = 0.3; \\ \gamma_L(u) &= 0.1, \gamma_L(v) = 0.1, \gamma_L(w) = 0.1; \\ \gamma_M(u) &= 0.1, \gamma_M(v) = 0.1, \gamma_M(w) = 0.1; \text{ and} \\ \gamma_P(u) &= 0.1, \gamma_P(v) = 0.1, \gamma_P(w) = 0.1. \end{aligned}$$

Then $\mathcal{S}_1 = \{0_N, 1_N, L, M\}$, $\mathcal{S}_2 = \{0_N, 1_N, P\}$, $\mathcal{S}_3 = \{0_N, 1_N, C(L)\}$ and $\mathcal{S}_4 = \{0_N, 1_N, P\}$ are the $\aleph$ SR's on $\aleph$.

Then the identity function $\phi : (\aleph, \mathcal{S}_2) \rightarrow (\aleph, \mathcal{S}_3)$ is a $\aleph$ SR – $\mathcal{CpreCF}$, but ϕ is neither $\aleph$ SR – $\mathcal{CCF}$ nor $\aleph$ SR – $\mathcal{CStrpreCF}$.

Similarly the identity function $\phi : (\mathfrak{R}, \mathcal{S}_1) \rightarrow (\mathfrak{R}, \mathcal{S}_4)$ is a $\mathfrak{NSR} - \mathfrak{CStrpreCF}$ but ϕ is neither $\mathfrak{NSR} - \mathfrak{CCF}$ nor $\mathfrak{NSR} - \mathfrak{C}\alpha\mathfrak{CF}$

Remark 3.4. Clearly the following diagram holds.



Proposition 3.8. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two $\mathfrak{NSR}$ spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. Then the following are equivalent.

- (i) ϕ is $\mathfrak{NSR} - \mathfrak{CStrpreCF}$.
- (ii) $\phi^{-1}(U)$ is a $\mathfrak{NstronglyPreCSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{NO}\mathfrak{R}$ U in $(\mathfrak{R}_2, \mathcal{S}_2)$.

Proof:

(i) $\Rightarrow$ (ii)

Let U be any $\mathfrak{NO}\mathfrak{R}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$, then $C(U)$ is a $\mathfrak{NCR}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since ϕ is a $\mathfrak{NSR} - \mathfrak{CStrpreCF}$, $\phi^{-1}(C(U))$ is a $\mathfrak{NstronglyPreOSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{NCR}$ $C(U)$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. By Remark 3.2(i), $\phi^{-1}(C(U)) = C(\phi^{-1}(U)) \subseteq \mathfrak{Nint}_{\mathfrak{R}_1}(\mathfrak{Npcl}_{\mathfrak{R}_1}(C(\phi^{-1}(U))))$. Therefore, $\phi^{-1}(U)$ is a $\mathfrak{NstronglyPreCSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{NO}\mathfrak{R}$ U in $(\mathfrak{R}_2, \mathcal{S}_2)$.

(ii) $\Rightarrow$ (i)

Let $C(U)$ be any $\mathfrak{NO}\mathfrak{R}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Then U is a $\mathfrak{NCR}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Since $\phi^{-1}(C(U))$ is a $\mathfrak{NstronglyPreCSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{NO}\mathfrak{R}$ $C(U)$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. We have, $\phi^{-1}(U)$ is a $\mathfrak{NstronglyPreOSR}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{NCR}$ U in $(\mathfrak{R}_2, \mathcal{S}_2)$. Hence, ϕ is a $\mathfrak{NSR} - \mathfrak{CStrpreCF}$.

Proposition 3.9. Let $(\mathfrak{R}_1, \mathcal{S}_1)$ and $(\mathfrak{R}_2, \mathcal{S}_2)$ be any two $\mathfrak{NSR}$ spaces. Let $\phi : (\mathfrak{R}_1, \mathcal{S}_1) \rightarrow (\mathfrak{R}_2, \mathcal{S}_2)$ be a function. Suppose if one of the following statement hold.

- (i) $\phi^{-1}(\mathfrak{Ncl}_{\mathfrak{R}_2}(V)) \subseteq \mathfrak{Nint}_{\mathfrak{R}_1}(\mathfrak{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(V)))$, for each $\mathfrak{NR}$ V in $(\mathfrak{R}_2, \mathcal{S}_2)$.
- (ii) $\mathfrak{Ncl}_{\mathfrak{R}_1}(\mathfrak{Npint}_{\mathfrak{R}_1}(\phi^{-1}(V))) \subseteq \phi^{-1}(\mathfrak{Nint}_{\mathfrak{R}_2}(V))$, for each $\mathfrak{NR}$ V in $(\mathfrak{R}_2, \mathcal{S}_2)$.
- (iii) $\phi(\mathfrak{Ncl}_{\mathfrak{R}_1}(\mathfrak{Npint}_{\mathfrak{R}_1}(V))) \subseteq \mathfrak{Nint}_{\mathfrak{R}_2}(\phi(V))$, for each $\mathfrak{NR}$ V in $(\mathfrak{R}_1, \mathcal{S}_1)$.
- (iv) $\phi(\mathfrak{Ncl}_{\mathfrak{R}_1}(V)) \subseteq \mathfrak{Nint}_{\mathfrak{R}_2}(\phi(V))$, for each $\mathfrak{NPreOSR}$ V in $(\mathfrak{R}_1, \mathcal{S}_1)$.

Then, ϕ is a $\mathfrak{NSR} - \mathfrak{CStrpreCF}$.

Proof:

(i) $\Rightarrow$ (ii)

Let U be any $\mathfrak{NR}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Then, $\phi^{-1}(\mathfrak{Ncl}_{\mathfrak{R}_2}(U)) \subseteq \mathfrak{Nint}_{\mathfrak{R}_1}(\mathfrak{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)))$ By taking complement on both sides,

$$C(\mathfrak{Nint}_{\mathfrak{R}_1}(\mathfrak{Npcl}_{\mathfrak{R}_1}(\phi^{-1}(U)))) \subseteq C(\phi^{-1}(\mathfrak{Ncl}_{\mathfrak{R}_2}(U)))$$

$$\begin{aligned}\mathfrak{N}cl_{\mathfrak{R}_1}(C(\mathfrak{N}pcl_{\mathfrak{R}_1}(\phi^{-1}(U)))) &\subseteq \phi^{-1}(C(\mathfrak{N}cl_{\mathfrak{R}_2}(U))) \\ \mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(C(\phi^{-1}(U)))) &\subseteq \phi^{-1}(\mathfrak{N}int_{\mathfrak{R}_2}(C(U)))\end{aligned}$$

Therefore, $\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(V))) \subseteq \phi^{-1}(\mathfrak{N}int_{\mathfrak{R}_2}(V))$, for each $\mathfrak{N}\mathfrak{R} V = C(U)$ in $(\mathfrak{R}_2, \mathcal{S}_2)$.

(ii) $\Rightarrow$ (iii)

Let U be any $\mathfrak{N}\mathfrak{R}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Let V be any $\mathfrak{N}\mathfrak{R}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$ such that $U = \phi(V)$. Then $V \subseteq \phi^{-1}(U)$. By (ii), $\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U))) \subseteq \phi^{-1}(\mathfrak{N}int_{\mathfrak{R}_2}(U))$. We have

$$\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(V)) \subseteq \mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U))) \subseteq \phi^{-1}(\mathfrak{N}int_{\mathfrak{R}_2}(\phi(V)))$$

Therefore, $\phi(\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(V))) \subseteq \mathfrak{N}int_{\mathfrak{R}_2}(\phi(V))$, for each $\mathfrak{N}\mathfrak{R} V$ in $(\mathfrak{R}_1, \mathcal{S}_1)$.

(iii) $\Rightarrow$ (iv)

Let V be any $\mathfrak{N}Pre\mathfrak{OS}\mathfrak{R}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$. Then $\mathfrak{N}pint_{\mathfrak{R}_1}(V) = V$. By (iii),

$$\phi(\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(V))) = \phi(\mathfrak{N}cl_{\mathfrak{R}_1}(V)) \subseteq \mathfrak{N}int_{\mathfrak{R}_2}(\phi(V)).$$

Therefore, $\phi(\mathfrak{N}cl_{\mathfrak{R}_1}(V)) \subseteq \mathfrak{N}int_{\mathfrak{R}_2}(\phi(V))$, for each $\mathfrak{N}Pre\mathfrak{OS}\mathfrak{R} V$ in $(\mathfrak{R}_1, \mathcal{S}_1)$.

Suppose that (iv) holds. Let U be any $\mathfrak{N}\mathfrak{OS}\mathfrak{R}$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Then $\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U))$ is a $\mathfrak{N}Pre\mathfrak{OS}\mathfrak{R}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$. By (iv),

$$\begin{aligned}\phi(\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U)))) &\subseteq \mathfrak{N}int_{\mathfrak{R}_2}(\phi(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U)))) \\ &\subseteq \mathfrak{N}int_{\mathfrak{R}_2}(\phi(\phi^{-1}(U))) \subseteq \mathfrak{N}int_{\mathfrak{R}_2}(U) = U.\end{aligned}$$

We have, $\phi^{-1}(\phi(\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U)))) \subseteq \phi^{-1}(U)$.

Then $\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U))) \subseteq \phi^{-1}(U)$. This implies that $\phi^{-1}(U)$ is a $\mathfrak{N}Pre\mathfrak{CS}\mathfrak{R}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$. Taking complement on both sides, $C(\phi^{-1}(U)) \subseteq C(\mathfrak{N}cl_{\mathfrak{R}_1}(\mathfrak{N}pint_{\mathfrak{R}_1}(\phi^{-1}(U))))$. This implies that $\phi^{-1}(C(U)) \subseteq \mathfrak{N}int_{\mathfrak{R}_1}(\mathfrak{N}pcl_{\mathfrak{R}_1}(\phi^{-1}(C(U))))$. Therefore $\phi^{-1}(C(U))$ is a $\mathfrak{N}stronglyPre\mathfrak{OS}\mathfrak{R}$ in $(\mathfrak{R}_1, \mathcal{S}_1)$, for each $\mathfrak{N}\mathfrak{CR} C(U)$ in $(\mathfrak{R}_2, \mathcal{S}_2)$. Hence, ϕ is a $\mathfrak{NS}\mathfrak{R} - \mathfrak{C}Strpre\mathfrak{CF}$.

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Neutrosophic Supra Topological Applications in Data Mining Process

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Abstract: The primary aim of this paper is to introduce the neutrosophic supra topological spaces. Neutrosophic subspaces and neutrosophic mappings are presented by which some contradicting examples of the statements of Abd-Monsef and Ramadan^[9] in fuzzy supra topological spaces are derived. Finally, a new method is proposed to solve medical diagnosis problems by using single valued neutrosophic score function.

Keywords: Fuzzy topology, Intuitionistic topology, Neutrosophic topology, Neutrosophic subspaces, Neutrosophic supra topology.

1 Introduction

The concept of fuzzy set was introduced by A. Zadeh [1] in 1965 which is a generalization of crisp set to analyse imprecise mathematical information. Adlassnig [2] applied fuzzy set theory to formalize medical relationships and fuzzy logic to computerized diagnosis system. This theory [3, 4, 5] has been used in the fields of artificial intelligence, probability, biology, control systems and economics. C.L Chang [6] introduced the fuzzy topological spaces and further the properties of fuzzy topological spaces are studied by R. Lowen [7]. By relaxing one topological axiom, Mashhour et al. [8] introduced supra topological space in 1983 and discussed its properties. Abd-Monsef and Ramadan [9] introduced fuzzy supra topological spaces and its continuous mappings. K. Atanassov [10] considered the degree of non-membership of an element along with the degree of membership and introduced intuitionistic fuzzy sets. Dogan Coker [11] introduced intuitionistic fuzzy topology. Saadati [12] further studied the basic concept of intuitionistic fuzzy point. S.K.De et al. [13] was the first one to develop the applications of intuitionistic fuzzy sets in medical diagnosis. Several researchers [14, 15, 16] further studied intuitionistic fuzzy sets in medical diagnosis. Hung and Tuan [17] noted that the approach in [13] has some questionable results on false diagnosis of patients' symptoms. Generally it is recognized that the available information about the patient and medical relationships is inherently uncertain. There may be indeterminacy components in real life problems for data mining and neutrosophic logic can be used in this regard. Neutrosophic logic is a generalization of fuzzy, intuitionistic, boolean, paraconsistent logics etc. Compared to all other logics, neutrosophic logic introduces a percentage of "indeterminacy" and this logic allows

each component t true, i indeterminate, f false to "boil over" 100 or "freeze" under 0. Here no restriction on T, I, F , or the sum $n = t + i + f$, where t, i, f are real values from the ranges T, I, F . For instance, in some tautologies $t > 100$, called "overtrue". As a generalization of Zadeh's fuzzy set and Atanassov's intuitionistic fuzzy set, Florentin Smarandache [18] introduced neutrosophic set. Neutrosophic set A consists of three independent objects called truth-membership $\mu_A(x)$, indeterminacy-membership $\sigma_A(x)$ and falsity-membership $\gamma_A(x)$ whose values are real standard or non-standard subset of unit interval $]^{-0}, 1^{+}[$. In data analysis, many methods have been introduced [19, 20, 21] to measure the similarity degree between fuzzy sets. But these are not suitable for the similarity measures of neutrosophic sets. The single-valued neutrosophic set is a neutrosophic set which can be used in real life engineering and scientific applications. The single valued neutrosophic set was first initiated by Smarandache [22] in 1998 and further studied by Wang et al. [23]. Majumdar and Samanta [24] defined some similarity measures of single valued neutrosophic sets in decision making problems. Recently many researchers [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] introduced several similarity measures and single-valued neutrosophic sets in medical diagnosis. The notion of neutrosophic crisp sets and topological spaces were introduced by A. A. Salama and S. A. Alblawi [46,47].

In section 2 of this paper, we present some basic preliminaries of fuzzy, intuitionistic, neutrosophic sets and topological spaces. The section 3 introduces the neutrosophic subspaces with its properties. In section 4, we define the concept of neutrosophic supra topological spaces. In section 5, we introduce neutrosophic supra continuity, S^* -neutrosophic continuity and give some contradicting examples in fuzzy supra topological spaces^[9]. As a real life application, a common method for data analysis under neutrosophic supra topological environment is presented in section 6. In section 7, we solve numerical examples of above proposed method and the last section states the conclusion and future work of this paper.

2 Preliminary

This section studies some of the basic definitions of fuzzy, intuitionistic, neutrosophic sets and respective topological spaces which are used for further study.

Definition 2.1. [1] Let X be a non empty set, then $A = \{(x, \mu_A(x)) : x \in X\}$ is called a fuzzy set on X , where $\mu_A(x) \in [0, 1]$ is the degree of membership function of each $x \in X$ to the set A . For X , I^X denotes the collection of all fuzzy sets of X .

Definition 2.2. [10] Let X be a non empty set, then $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ is called an intuitionistic set on X , where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$, $\mu_A(x), \gamma_A(x) \in [0, 1]$ are the degree of membership and non membership functions of each $x \in X$ to the set A respectively. The set of all intuitionistic sets of X is denoted by $I(X)$.

Definition 2.3. [23] Let X be a non empty set, then $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$ is called a neutrosophic set on X , where $^{-0} \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^{+}$ for all $x \in X$, $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in]^{-0}, 1^{+}[$ are the degree of membership (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) of each $x \in X$ to the set A respectively. For X , $N(X)$ denotes the collection of all neutrosophic sets of X .

Definition 2.4. [18] The following statements are true for neutrosophic sets A and B on X :

- (i) $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$ if and only if $A \subseteq B$.
- (ii) $A \subseteq B$ and $B \subseteq A$ if and only if $A = B$.
- (iii) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\sigma_A(x), \sigma_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.
- (iv) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \max\{\sigma_A(x), \sigma_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.

More generally, the intersection and the union of a collection of neutrosophic sets $\{A_i\}_{i \in \Lambda}$, are defined by $\bigcap_{i \in \Lambda} A_i = \{(x, \inf_{i \in \Lambda} \{\mu_{A_i}(x)\}, \inf_{i \in \Lambda} \{\sigma_{A_i}(x)\}, \sup_{i \in \Lambda} \{\gamma_{A_i}(x)\}) : x \in X\}$ and $\bigcup_{i \in \Lambda} A_i = \{(x, \sup_{i \in \Lambda} \{\mu_{A_i}(x)\}, \sup_{i \in \Lambda} \{\sigma_{A_i}(x)\}, \inf_{i \in \Lambda} \{\gamma_{A_i}(x)\}) : x \in X\}$.

Notation 2.5. Let X be a non empty set. We consider the fuzzy, intuitionistic, neutrosophic empty set as $\emptyset = \{(x, 0) : x \in X\}$, $\emptyset = \{(x, 0, 1) : x \in X\}$, $\emptyset = \{(x, 0, 0, 1) : x \in X\}$ respectively and the fuzzy, intuitionistic, neutrosophic whole set as $X = \{(x, 1) : x \in X\}$, $X = \{(x, 1, 0) : x \in X\}$, $X = \{(x, 1, 1, 0) : x \in X\}$ respectively.

Definition 2.6. [24] A neutrosophic set $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$ is called a single valued neutrosophic set on a non empty set X , if $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in [0, 1]$ and $0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3$ for all $x \in X$ to the set A . For each attribute, the single valued neutrosophic score function (shortly SVNSF) is defined as $SVNSF = \frac{1}{3m} [\sum_{i=1}^m [2 + \mu_i - \sigma_i - \gamma_i]]$.

Definition 2.7. [6] Let X be a non empty set. A subcollection τ_f of I^X is said to be fuzzy topology on X if the sets X and $\emptyset$ belong to τ_f , τ_f is closed under arbitrary union and τ_f is closed under finite intersection. Then (X, τ_f) is called fuzzy topological space (shortly fts), members of τ_f are known as fuzzy open sets and their complements are fuzzy closed sets.

Definition 2.8. [11] Let X be a non empty set and a subfamily τ_i of $I(X)$ is called intuitionistic fuzzy topology on X if X and $\emptyset \in \tau_i$, τ_i is closed under arbitrary union and τ_i is closed under finite intersection. Then (X, τ_i) is called intuitionistic fuzzy topological space (shortly ifts), elements of τ_i are called intuitionistic fuzzy open sets and their complements are intuitionistic fuzzy closed sets.

Definition 2.9. [46, 47] Let X be a non empty set. A neutrosophic topology on X is a subfamily τ_n of $N(X)$ such that X and $\emptyset$ belong to τ_n , τ_n is closed under arbitrary union and τ_n is closed under finite intersection. Then (X, τ_n) is called neutrosophic topological space (shortly nts), members of τ_n are known as neutrosophic open sets and their complements are neutrosophic closed sets. For a neutrosophic set A of X , the interior and closure of A are respectively defined as: $int_n(A) = \bigcup \{G : G \subseteq A, G \in \tau_n\}$ and $cl_n(A) = \bigcap \{F : A \subseteq F, F^c \in \tau_n\}$.

Corollary 2.10. [18] The following statements are true for the neutrosophic sets A, B, C and D on X :

- (i) $A \cap C \subseteq B \cap D$ and $A \cup C \subseteq B \cup D$, if $A \subseteq B$ and $C \subseteq D$.
- (ii) $A \subseteq B \cap C$, if $A \subseteq B$ and $A \subseteq C$. $A \cup B \subseteq C$, if $A \subseteq C$ and $B \subseteq C$.
- (iii) $A \subseteq C$, if $A \subseteq B$ and $B \subseteq C$.

Definition 2.11. [48] Let $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$, $B = \{(y, \mu_B(y), \sigma_B(y), \gamma_B(y)) : y \in Y\}$ be two neutrosophic sets and $f : X \rightarrow Y$ be a function.

- (i) $f^{-1}(B) = \{(x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\gamma_B)(x)) : x \in X\}$ is a neutrosophic set on X called the pre-image of B under f .
- (ii) $f(A) = \{(y, f(\mu_A)(y), f(\sigma_A)(y), (1 - f(1 - \gamma_A))(y)) : y \in Y\}$ is a neutrosophic set on Y called the image of A under f , where

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f(\sigma_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \sigma_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$(1 - f(1 - \gamma_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_-(\gamma_A)$ for $(1 - f(1 - \gamma_A))$.

3 Neutrosophic Subspaces

This section introduce differences of two fuzzy, intuitionistic and neutrosophic sets on X . We also introduce neutrosophic subspaces with its proprieties.

Definition 3.1. The difference of neutrosophic sets A and B on X is a neutrosophic set on X , defined as $A \setminus B = \{(x, |\mu_A(x) - \mu_B(x)|, |\sigma_A(x) - \sigma_B(x)|, 1 - |\gamma_A(x) - \gamma_B(x)|) : x \in X\}$. Clearly $X^c = X \setminus X = (x, 0, 0, 1) = \emptyset$ and $\emptyset^c = X \setminus \emptyset = (x, 1, 1, 0) = X$.

Definition 3.2. Let A, B be two intuitionistic fuzzy sets of X , then the difference of A and B is a intuitionistic fuzzy set on X , defined as $A \setminus B = \{(x, |\mu_A(x) - \mu_B(x)|, 1 - |\gamma_A(x) - \gamma_B(x)|) : x \in X\}$. Clearly $X^c = X \setminus X = (x, 0, 1) = \emptyset$ and $\emptyset^c = X \setminus \emptyset = (x, 1, 0) = X$.

Definition 3.3. Let A, B be two fuzzy sets of X , then the difference of A and B is a fuzzy set on X , defined as $A \setminus B = \{(x, |\mu_A(x) - \mu_B(x)|) : x \in X\}$. Clearly $X^c = X \setminus X = (x, 0) = \emptyset$ and $\emptyset^c = X \setminus \emptyset = (x, 1) = X$.

Corollary 3.4. The following statements are true for the neutrosophic sets $\{A\}_{i=1}^\infty, A, B$ on X :

- (i) $(\bigcap_{i \in \Lambda} A_i)^c = \bigcup_{i \in \Lambda} A_i^c, (\bigcup_{i \in \Lambda} A_i)^c = \bigcap_{i \in \Lambda} A_i^c$.
- (ii) $(A^c)^c = A. B^c \subseteq A^c$, if $A \subseteq B$.

Proof. : Part(i): $(\bigcap_{i \in \Lambda} A_i)^c = \{(x, |1 - \inf_{i \in \Lambda} \{\mu_{A_i}(x)\}|, |1 - \inf_{i \in \Lambda} \{\sigma_{A_i}(x)\}|, 1 - |0 - \sup_{i \in \Lambda} \{\gamma_{A_i}(x)\}|) : x \in X\} = \{(x, \sup_{i \in \Lambda} (|1 - \mu_{A_i}(x)|), \sup_{i \in \Lambda} (|1 - \sigma_{A_i}(x)|), \inf_{i \in \Lambda} (|1 - \gamma_{A_i}(x)|) : x \in X\} = \bigcup_{i \in \Lambda} A_i^c$. Similarly we can prove $(\bigcup_{i \in \Lambda} A_i)^c = \bigcap_{i \in \Lambda} A_i^c$ and part(ii). $\square$

Generally, in the sense of Chang^[6] every fuzzy topology is intuitionistic fuzzy topology as well as neutrosophic topology. The following lemmas show that every intuitionistic fuzzy topology τ_i induce two fuzzy topologies on X and every neutrosophic topology τ_n induce three fuzzy topologies on X .

Lemma 3.5. In an intuitionistic fuzzy topological space (X, τ_i) , each of the following collections form fuzzy topologies on X :

- (i) $\tau_{f_1} = \{A = (x, \mu_A(x)) : (x, \mu_A(x), \gamma_A(x)) \in \tau_i\}$.
- (ii) $\tau_{f_2} = \{A = (x, 1 - \gamma_A(x)) : (x, \mu_A(x), \gamma_A(x)) \in \tau_i\}$.

Proof. : Here we shall prove part (ii) only and similarly we can prove part (i). Clearly $\emptyset = (x, 0)$ and $X = (x, 1)$ are belong to τ_{f_2} . If $\{A_j\}_{j \in \Lambda} \in \tau_{f_2}$, then $\{(x, \mu_{A_j}(x), \gamma_{A_j}(x))\}_{j \in \Lambda} \in \tau_i$ and $(x, \sup_{j \in \Lambda} \{\mu_{A_j}(x)\}, \inf_{j \in \Lambda} \{\gamma_{A_j}(x)\}) \in \tau_i$. Therefore $(x, \sup_{j \in \Lambda} \{1 - \gamma_{A_j}(x)\}) = (x, 1 - \inf_{j \in \Lambda} \{\gamma_{A_j}(x)\}) \in \tau_{f_2}$ and so $\cup_{j \in \Lambda} A_j \in \tau_{f_2}$. If $\{A_j\}_{j=1}^m \in \tau_{f_2}$, then $\{(x, \mu_{A_j}(x), \gamma_{A_j}(x))\}_{j=1}^m \in \tau_i$ and $(x, \inf_j \{\mu_{A_j}(x)\}, \sup_j \{\gamma_{A_j}(x)\}) \in \tau_i$. Therefore $(x, \inf_j \{1 - \gamma_{A_j}(x)\}) = (x, 1 - \sup_j \{\gamma_{A_j}(x)\}) \in \tau_{f_2}$ and so $\cap_{j=1}^m A_j \in \tau_{f_2}$. $\square$

Lemma 3.6. In a neutrosophic topological space (X, τ_n) , each of the following collections form fuzzy topologies on X :

- (i) $\tau_{f_1} = \{A = (x, \mu_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n\}$.
- (ii) $\tau_{f_2} = \{A = (x, \sigma_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n\}$.
- (iii) $\tau_{f_3} = \{A = (x, 1 - \gamma_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n\}$.

Proof. : **Part (i):** Clearly $\emptyset = (x, 0)$ and $X = (x, 1)$ are belong to τ_{f_1} . If $\{A_j\}_{j \in \Lambda} \in \tau_{f_1}$, then $\{(x, \mu_{A_j}(x), \sigma_{A_j}(x), \gamma_{A_j}(x))\}_{j \in \Lambda} \in \tau_n$ and $(x, \sup_{j \in \Lambda} \{\mu_{A_j}(x)\}, \sup_{j \in \Lambda} \{\sigma_{A_j}(x)\}, \inf_{j \in \Lambda} \{\gamma_{A_j}(x)\}) \in \tau_n$. Therefore $(x, \sup_{j \in \Lambda} \{\mu_{A_j}(x)\}) \in \tau_{f_1}$ and so $\cup_{j \in \Lambda} A_j \in \tau_{f_1}$. If $\{A_j\}_{j=1}^m \in \tau_{f_1}$, then $\{(x, \mu_{A_j}(x), \sigma_{A_j}(x), \gamma_{A_j}(x))\}_{j=1}^m \in \tau_n$ and $(x, \inf_j \{\mu_{A_j}(x)\}, \inf_j \{\sigma_{A_j}(x)\}, \sup_j \{\gamma_{A_j}(x)\}) \in \tau_n$. Therefore $(x, \inf_j \{\mu_{A_j}(x)\}) \in \tau_{f_1}$ and so $\cap_{j=1}^m A_j \in \tau_{f_1}$. In similar manner we can prove part (ii) and (iii). $\square$

Corollary 3.7. Let A be a neutrosophic set of (X, τ_n) , then the collection $(\tau_n)_A = \{A \cap O : O \in \tau_n\}$ is a neutrosophic topology on A , called the induced neutrosophic topology on A and the pair $(A, (\tau_n)_A)$ is called neutrosophic subspace of nts (X, τ_n) . The elements of $(\tau_n)_A$ are called $(\tau_n)_A$ -open sets and their complements are called $(\tau_n)_A$ -closed sets.

Proof. : Obviously $\emptyset = (x, \min(\mu_A(x), 0), \min(\sigma_A(x), 0), \max(\gamma_A(x), 1)) = A \cap \emptyset \in (\tau_n)_A$ and $A = (x, \min(\mu_A(x), 1), \min(\sigma_A(x), 1), \max(\gamma_A(x), 0)) = A \cap X \in (\tau_n)_A$. Take $\{A_j\}_{j \in \Lambda} \in (\tau_n)_A$, there exist $O_j \in \tau_n, j \in \Lambda$, such that $A_j = A \cap O_j$ for each $j \in \Lambda$. Then $A' = \cup_{j \in \Lambda} A_j = \{(x, \sup_{j \in \Lambda} \{\mu_{A_j}(x)\}, \sup_{j \in \Lambda} \{\sigma_{A_j}(x)\}, \inf_{j \in \Lambda} \{\gamma_{A_j}(x)\})\} = \{(x, \sup_{j \in \Lambda} (\min\{\mu_A(x), \mu_{O_j}(x)\}), \sup_{j \in \Lambda} (\min\{\sigma_A(x), \sigma_{O_j}(x)\}), \inf_{j \in \Lambda} (\max\{\gamma_A(x), \gamma_{O_j}(x)\}))\} = \{(x, \min(\sup_{j \in \Lambda} \{\mu_A(x), \mu_{O_j}(x)\}), \min(\sup_{j \in \Lambda} \{\sigma_A(x), \sigma_{O_j}(x)\}), \max(\inf_{j \in \Lambda} \{\gamma_A(x), \gamma_{O_j}(x)\}))\} = \{(x, \mu_{A \cap (\cup_{j \in \Lambda} O_j)}(x), \sigma_{A \cap (\cup_{j \in \Lambda} O_j)}(x), \gamma_{A \cap (\cup_{j \in \Lambda} O_j)}(x))\} \in (\tau_n)_A$. Therefore $(\tau_n)_A$ is closed under arbitrary union on A . If we take $\{A_j\}_{j=1}^m \in (\tau_n)_A$, there exist $O_j \in \tau_n, j = 1, 2, \dots, m$, such that $A_j = A \cap O_j$ for each $j \in \Lambda$. Then $A' = \cap_{j=1}^m A_j = \{(x, \inf_j \{\mu_{A_j}(x)\}, \inf_j \{\sigma_{A_j}(x)\}, \sup_j \{\gamma_{A_j}(x)\})\} = \{(x, \inf_j (\min\{\mu_A(x), \mu_{O_j}(x)\}), \inf_j (\min\{\sigma_A(x), \sigma_{O_j}(x)\}), \sup_j (\max\{\gamma_A(x), \gamma_{O_j}(x)\}))\} = \{(x, \min(\inf_j \{\mu_A(x), \mu_{O_j}(x)\}), \min(\inf_j \{\sigma_A(x), \sigma_{O_j}(x)\}), \max(\sup_j \{\gamma_A(x), \gamma_{O_j}(x)\}))\} = \{(x, \mu_{A \cap (\cap_{j \in \Lambda} O_j)}(x), \sigma_{A \cap (\cap_{j \in \Lambda} O_j)}(x), \gamma_{A \cap (\cap_{j \in \Lambda} O_j)}(x))\} \in (\tau_n)_A$. Therefore $(\tau_n)_A$ is a neutrosophic topology on A . $\square$

Corollary 3.8. Let A be a fuzzy set (resp. intuitionistic fuzzy set) of fts (X, τ_f) (resp. ifts (X, τ_i)), then the collection $(\tau_f)_A = \{A \cap O : O \in \tau_f\}$ (resp. $(\tau_i)_A = \{A \cap O : O \in \tau_i\}$) is a fuzzy topology (resp. intuitionistic fuzzy topology) on A , called the induced fuzzy topology (resp. induced intuitionistic fuzzy topology) on A and the pair $(A, (\tau_f)_A)$ (resp. $(A, (\tau_i)_A)$) is called fuzzy subspace (resp. intuitionistic fuzzy subspace).

Proof. : Proof follows from the above corollary. $\square$

Lemma 3.9. Let $(A, (\tau_n)_A)$ be a neutrosophic subspace of nts (X, τ_n) and $B \subseteq A$. If B is $(\tau_n)_A$ -open in $(A, (\tau_n)_A)$ and A is neutrosophic open in nts (X, τ_n) , then B is neutrosophic open in (X, τ_n) .

Proof. : Since B is $(\tau_n)_A$ -open in $(A, (\tau_n)_A)$, $B = A \cap O$ for some neutrosophic open set O in (X, τ_n) and so B is neutrosophic open in (X, τ_n) . $\square$

Lemma 3.10. Let $(A, (\tau_f)_A)$ (resp. $(A, (\tau_i)_A)$) be a fuzzy subspace (resp. intuitionistic fuzzy subspace) of fts (X, τ_f) (resp. of ifts (X, τ_i)) and $B \subseteq A$. If B is $(\tau_f)_A$ -open (resp. $(\tau_i)_A$ -open) in $(A, (\tau_f)_A)$ (resp. $(A, (\tau_i)_A)$) and A is fuzzy open (resp. intuitionistic fuzzy open) in fts (X, τ_f) (resp. ifts (X, τ_i)), then B is fuzzy open (resp. intuitionistic fuzzy open) in (X, τ_f) (resp. ifts (X, τ_i)).

Proof. : Proof is similar as above lemma. $\square$

Remark 3.11. In classical topology, we know that if (A, τ_A) is a subspace of (X, τ) and $B \subseteq A$, then

- (i) $B = A \cap F$, where F is closed in X if and only if B is closed in A .
- (ii) B is closed in X , if B is closed in A and A is closed in X .

The following examples illustrate that these are not true in fuzzy, intuitionistic fuzzy and neutrosophic topological spaces.

Example 3.12. Let $X = \{a, b, c\}$ with $\tau_n = \{\emptyset, X, ((1, 1, 1), (0, 0, 0), (0.7, 0.7, 0.7)), ((0.6, 0.6, 0.6), (0, 0, 0), (0, 0, 0)), ((1, 1, 1), (0, 0, 0), (0, 0, 0)), ((0.6, 0.6, 0.6), (0, 0, 0), (0.7, 0.7, 0.7))\}$. Then $(\tau_n)^c = \{X, \emptyset, ((0, 0, 0), (1, 1, 1), (0.3, 0.3, 0.3)), ((0.4, 0.4, 0.4), (1, 1, 1), (1, 1, 1)), ((0, 0, 0), (1, 1, 1), (1, 1, 1)), ((0.4, 0.4, 0.4), (1, 1, 1), (0.3, 0.3, 0.3))\}$. Let $A = ((0.6, 0.6, 0.2), (1, 0, 1), (0.8, 0.7, 0.6))$, then $(\tau_n)_A = \{\emptyset, A, ((0.6, 0.6, 0.2), (0, 0, 0), (0.8, 0.7, 0.7)), ((0.6, 0.6, 0.2), (0, 0, 0), (0.8, 0.7, 0.6))\}$ and $((\tau_n)_A)^c = \{A, \emptyset, ((0, 0, 0), (1, 0, 1), (1, 1, 0.9)), ((0, 0, 0), (1, 0, 1), (1, 1, 1))\}$. Clearly $B = ((0, 0, 0), (1, 0, 1), (1, 1, 0.9))$ is $(\tau_n)_A$ -closed in $(A, (\tau_n)_A)$ and $B \neq A \cap F$ for every neutrosophic closed set F in (X, τ_n) . Since $A = ((0, 0, 0), (1, 1, 1), (0.3, 0.3, 0.3))$ is neutrosophic closed in (X, τ_n) , then $(\tau_n)_A = \{\emptyset, A, ((0, 0, 0), (0, 0, 0), (0.7, 0.7, 0.7)), ((0, 0, 0), (0, 0, 0), (0.3, 0.3, 0.3))\}$ and $((\tau_n)_A)^c = \{A, \emptyset, ((0, 0, 0), (1, 1, 1), (0.6, 0.6, 0.6)), ((0, 0, 0), (1, 1, 1), (1, 1, 1))\}$. Clearly $B = ((0, 0, 0), (1, 1, 1), (0.6, 0.6, 0.6))$ is $(\tau_n)_A$ -closed in $(A, (\tau_n)_A)$, but it is not neutrosophic closed in (X, τ_n) .

Example 3.13. Let $X = \{a, b, c\}$ with $\tau_i = \{\emptyset, X, ((0.4, 0.4, 0.3), (0.6, 0.6, 0.7)), ((0.3, 0.8, 0.1), (0.7, 0.2, 0.9)), ((0.3, 0.4, 0.1), (0.7, 0.6, 0.9)), ((0.4, 0.8, 0.3), (0.6, 0.2, 0.7))\}$. Then $(\tau_i)^c = \{X, \emptyset, ((0.6, 0.6, 0.7), (0.4, 0.4, 0.3)), ((0.7, 0.2, 0.9), (0.3, 0.8, 0.1)), ((0.7, 0.6, 0.9), (0.3, 0.4, 0.1)), ((0.6, 0.2, 0.7), (0.4, 0.8, 0.3))\}$. Since $A = ((0.7, 0.2, 0.9), (0.3, 0.8, 0.1))$ is intuitionistic fuzzy closed in (X, τ_i) , then $(\tau_i)_A = \{\emptyset, A, ((0.4, 0.2, 0.3), (0.6, 0.8, 0.7)), ((0.3, 0.2, 0.1), (0.7, 0.8, 0.9))\}$ and $((\tau_i)_A)^c = \{A, \emptyset, ((0.3, 0, 0.6), (0.7, 1, 0.4)), ((0.4, 0, 0.8), (0.6, 1, 0.2))\}$. Clearly $B = ((0.3, 0, 0.6), (0.7, 1, 0.4))$ is $(\tau_i)_A$ -closed in $(A, (\tau_i)_A)$, but $B \neq A \cap F$ for every intuitionistic fuzzy closed set F in (X, τ_i) and B is not intuitionistic fuzzy closed in (X, τ_i) .

Example 3.14. Let $X = \{a, b, c\}$ with $\tau_f = \{\emptyset, X, (0.2, 0.3, 0.1), (0.7, 0.1, 0.8), (0.2, 0.1, 0.1), (0.7, 0.3, 0.8)\}$. Then $(\tau_f)^c = \{X, \emptyset, (0.8, 0.7, 0.9), (0.3, 0.9, 0.2), (0.8, 0.9, 0.9), (0.3, 0.7, 0.2)\}$. Since $A = (0.8, 0.7, 0.9)$ is fuzzy closed in (X, τ_f) , then $(\tau_f)_A = \{\emptyset, A, (0.2, 0.3, 0.1), (0.7, 0.1, 0.8), (0.2, 0.1, 0.1), (0.7, 0.3, 0.8)\}$ and $((\tau_f)_A)^c = \{A, \emptyset, (0.6, 0.4, 0.8), (0.1, 0.6, 0.1), (0.6, 0.6, 0.8), (0.1, 0.4, 0.1)\}$. Clearly $B = (0.6, 0.6, 0.8)$ is $(\tau_f)_A$ -closed in $(A, (\tau_f)_A)$, but $B \neq A \cap F$ for every fuzzy closed set F in (X, τ_f) and B is not fuzzy closed in (X, τ_f) .

4 Neutrosophic Supra Topological Spaces

In this section, we introduce neutrosophic supra topological spaces and also establish its properties.

Definition 4.1. A subcollection τ_n^* of neutrosophic sets on a non empty set X is said to be a neutrosophic supra topology on X if the sets $\emptyset, X \in \tau_n^*$ and $\bigcup_{i=1}^{\infty} A_i \in \tau_n^*$, for $\{A_i\}_{i=1}^{\infty} \in \tau_n^*$. Then (X, τ_n^*) is called neutrosophic supra topological space on X (for short nsts). The members of τ_n^* are known as neutrosophic supra open sets and its complement is called neutrosophic supra closed. A neutrosophic supra topology τ_n^* on X is said to be an associated neutrosophic supra topology with neutrosophic topology τ_n if $\tau_n \subseteq \tau_n^*$. Every neutrosophic topology on X is neutrosophic supra topology on X .

Remark 4.2. The following table illustrates the comparision of fuzzy supra topological spaces, intuitionistic supra topological spaces, neutrosophic supra topological spaces.

Comparison Table

S.No	Fuzzy supra topological spaces	Intuitionistic supra topological spaces	Neutrosophic supra topological spaces
1	It deals with fuzzy sets	It deals with intuitionistic sets	It deals with neutrosophic sets
2	A subcollection τ_f^* of fuzzy sets on a non empty set X is said to be a fuzzy supra topology on X if the sets $\emptyset, X \in \tau_f^*$ and $\bigcup_{i=1}^{\infty} A_i \in \tau_f^*$, for $\{A_i\}_{i=1}^{\infty} \in \tau_f^*$.	A subcollection τ_i^* of intuitionistic sets on a non empty set X is said to be a intuitionistic supra topology on X if the sets $\emptyset, X \in \tau_i^*$ and $\bigcup_{i=1}^{\infty} A_i \in \tau_i^*$, for $\{A_i\}_{i=1}^{\infty} \in \tau_i^*$.	A subcollection τ_n^* of neutrosophic sets on a non empty set X is said to be a neutrosophic supra topology on X if the sets $\emptyset, X \in \tau_n^*$ and $\bigcup_{i=1}^{\infty} A_i \in \tau_n^*$, for $\{A_i\}_{i=1}^{\infty} \in \tau_n^*$.
3	A non empty set X together with the collection τ_f^* is called fuzzy supra topological space on X (for short fst) denoted by the ordered pair (X, τ_f^*) .	A non empty set X together with the collection τ_i^* is called intuitionistic supra topological space on X (for short ist) denoted by the ordered pair (X, τ_i^*) .	A non empty set X together with the collection τ_n^* is called neutrosophic supra topological space on X (for short nst) denoted by the ordered pair (X, τ_n^*) .
4	The members of τ_f^* are known as fuzzy supra open sets.	The members of τ_i^* are known as intuitionistic supra open sets.	The members of τ_n^* are known as neutrosophic supra open sets.
5	It is a generalization of classical supra topological spaces.	It is a generalization of fuzzy supra topological spaces.	It is a generalization of intuitionistic supra topological spaces.
6	Every fuzzy topology is fuzzy supra topology.	Every intuitionistic topology is intuitionistic supra topology.	Every neutrosophic topology is neutrosophic supra topology.

Proposition 4.3. The collection $(\tau_n^*)^c$ of all neutrosophic supra closed sets in (X, τ_n^*) satisfies: $\emptyset, X \in (\tau_n^*)^c$ and $(\tau_n^*)^c$ is closed under arbitrary intersection.

Proof. : The proof is obvious. □

Lemma 4.4. As Proposition 3.4, every neutrosophic supra topology τ_n^* induce three fuzzy supra topologies $\tau_{f_1}^* = \{A = (x, \mu_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n^*\}$, $\tau_{f_2}^* = \{A = (x, \sigma_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n^*\}$ and $\tau_{f_3}^* = \{A = (x, 1 - \gamma_A(x)) : (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) \in \tau_n^*\}$ on X .

Definition 4.5. The neutrosophic supra topological interior $int_{\tau_n^*}(A)$ and closure $cl_{\tau_n^*}(A)$ operators of a neutrosophic set A are respectively defined as: $int_{\tau_n^*}(A) = \cup\{G : G \subseteq A \text{ and } G \in \tau_n^*\}$ and $cl_{\tau_n^*}(A) = \cap\{F : A \subseteq F \text{ and } F^c \in \tau_n^*\}$.

Theorem 4.6. The following are true for neutrosophic sets A and B of nsts (X, τ_n^*) :

- (i) $A = cl_{\tau_n^*}(A)$ if and only if A is neutrosophic supra closed.
- (ii) $A = int_{\tau_n^*}(A)$ if and only if A is neutrosophic supra open.
- (iii) $cl_{\tau_n^*}(A) \subseteq cl_{\tau_n^*}(B)$, if $A \subseteq B$.
- (iv) $int_{\tau_n^*}(A) \subseteq int_{\tau_n^*}(B)$, if $A \subseteq B$.
- (v) $cl_{\tau_n^*}(A) \cup cl_{\tau_n^*}(B) \subseteq cl_{\tau_n^*}(A \cup B)$.
- (vi) $int_{\tau_n^*}(A) \cup int_{\tau_n^*}(B) \subseteq int_{\tau_n^*}(A \cup B)$.
- (vii) $cl_{\tau_n^*}(A) \cap cl_{\tau_n^*}(B) \supseteq cl_{\tau_n^*}(A \cap B)$.
- (viii) $int_{\tau_n^*}(A) \cap int_{\tau_n^*}(B) \supseteq int_{\tau_n^*}(A \cap B)$.
- (ix) $int_{\tau_n^*}(A^c) = (cl_{\tau_n^*}(A))^c$.

Proof. : Here we shall prove parts (iii), (v) and (ix) only. The remaining parts similarly follows. Part (iii): $cl_{\tau_n^*}(B) = \cap\{G : G^c \in \tau_n^*, B \subseteq G\} \supseteq \cap\{G : G^c \in \tau_n^*, A \subseteq G\} = cl_{\tau_n^*}(A)$. Thus, $cl_{\tau_n^*}(A) \subseteq cl_{\tau_n^*}(B)$. Part (v): Since $A \cup B \supseteq A, B$, then $cl_{\tau_n^*}(A) \cup cl_{\tau_n^*}(B) \subseteq cl_{\tau_n^*}(A \cup B)$. Part (ix): $cl_{\tau_n^*}(A) = \cap\{G : G^c \in \tau_n^*, G \supseteq A\}$, $(cl_{\tau_n^*}(A))^c = \cup\{G^c : G^c \text{ is a neutrosophic supra open in } X \text{ and } G^c \subseteq A^c\} = int_{\tau_n^*}(A^c)$. Thus, $(cl_{\tau_n^*}(A))^c = int_{\tau_n^*}(A^c)$. □

Remark 4.7. In neutrosophic topological space, we have $cl_{\tau_n}(A \cup B) = cl_{\tau_n}(A) \cup cl_{\tau_n}(B)$ and $int_{\tau_n}(A \cap B) = int_{\tau_n}(A) \cap int_{\tau_n}(B)$. These equalities are not true in neutrosophic supra topological spaces as shown in the following examples.

Example 4.8. Let $X = \{a, b, c\}$ with neutrosophic topology $\tau_n^* = \{\emptyset, X, ((0.5, 1, 0), (0.5, 1, 0), (0.5, 0, 1)), ((0.25, 0, 1), (0.25, 0, 1), (0.75, 1, 0)), ((0.5, 1, 1), (0.5, 1, 1), (0.5, 0, 0))\}$. Then $(\tau_n^*)^c = \{X, \emptyset, ((0.5, 0, 1), (0.5, 0, 1), (0.5, 1, 0)), ((0.75, 1, 0), (0.75, 1, 0), (0.25, 0, 1)), ((0.5, 0, 0), (0.5, 0, 0), (0.5, 1, 1))\}$. Let $C = ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1))$ and $D = ((0.5, 0, 0.5), (0.5, 0, 0.5), (0.5, 1, 0.5))$, then $cl_{\tau_n^*}(C) = ((0.75, 1, 0), (0.75, 1, 0), (0.25, 0, 1))$ and $cl_{\tau_n^*}(D) = ((0.5, 0, 1), (0.5, 0, 1), (0.5, 1, 0))$, so $cl_{\tau_n^*}(C) \cup cl_{\tau_n^*}(D) = ((0.75, 1, 1), (0.75, 1, 1), (0.25, 0, 0))$. But $C \cup D = ((0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5))$ and $cl_{\tau_n^*}(C \cup D) = ((1, 1, 1), (1, 1, 1), (0, 0, 0)) = X$. Therefore $cl_{\tau_n^*}(C \cup D) \neq cl_{\tau_n^*}(C) \cup cl_{\tau_n^*}(D)$.

Let $E = ((0.5, 1, 0.25), (0.5, 1, 0.25), (0.5, 0, 0.75))$ and $F = ((0.5, 0.5, 1), (0.5, 0.5, 1), (0.5, 0.5, 0))$. Then $int_{\tau_n^*}(E) = ((0.5, 1, 0), (0.5, 1, 0), (0.5, 0, 1))$ and $int_{\tau_n^*}(F) = ((0.25, 0, 1), (0.25, 0, 1), (0.75, 1, 0))$, so $int_{\tau_n^*}(E) \cap int_{\tau_n^*}(F) = ((0.25, 0, 0), (0.25, 0, 0), (0.75, 1, 1))$. But $E \cap F = ((0.5, 0.5, 0.25), (0.5, 0.5, 0.25), (0.5, 0.5, 0.75))$ and $int_{\tau_n^*}(E \cap F) = ((0, 0, 0), (0, 0, 0), (1, 1, 1)) = \emptyset$. Therefore $int_{\tau_n^*}(E \cap F) \neq int_{\tau_n^*}(E) \cap int_{\tau_n^*}(F)$.

5 Mappings of Neutrosophic Spaces

In this section, we define and establish the properties of some mappings in neutrosophic supra topological spaces and neutrosophic subspaces.

Definition 5.1. Let τ_n^* and σ_n^* be associated neutrosophic supra topologies with respect to τ_n and σ_n . A mapping f from a nts (X, τ_n) into nts (Y, σ_n) is said to be S^* -neutrosophic open if the image of every neutrosophic open set in (X, τ_n) is neutrosophic supra open in (Y, σ_n^*) and $f : X \rightarrow Y$ is said to be S^* -neutrosophic continuous if the inverse image of every neutrosophic open set in (Y, σ_n) is neutrosophic supra open in (X, τ_n^*) .

Definition 5.2. Let τ_n^* and σ_n^* be associated neutrosophic supra topologies with respect to nts's τ_n and σ_n . A mapping f from a nts (X, τ_n) into a nts (Y, σ_n) is said to be supra neutrosophic open if the image of every neutrosophic supra open set in (X, τ_n^*) is a neutrosophic supra open in (Y, σ_n^*) and $f : X \rightarrow Y$ is said to be supra neutrosophic continuous if the inverse image of every neutrosophic supra open set in (Y, σ_n^*) is neutrosophic supra open in (X, τ_n^*) .

A mapping f of nts (X, τ_n) into nts (Y, σ_n) is said to be a mapping of neutrosophic subspace $(A, (\tau_n)_A)$ into neutrosophic subspace $(B, (\sigma_n)_B)$ if $f(A) \subset B$.

Definition 5.3. A mapping f of neutrosophic subspace $(A, (\tau_n)_A)$ of nts (X, τ_n) into neutrosophic subspace $(B, (\sigma_n)_B)$ of nts (Y, σ_n) is said to be relatively neutrosophic continuous if $f^{-1}(O) \cap A \in (\tau_n)_A$ for every $O \in (\sigma_n)_B$. If $f(O') \in (\sigma_n)_B$ for every $O' \in (\tau_n)_A$, then f is said to be relatively neutrosophic open.

Theorem 5.4. If a mapping f is neutrosophic continuous from nts (X, τ_n) into nts (Y, σ_n) and $f(A) \subset B$. Then f is relatively neutrosophic continuous from neutrosophic subspace $(A, (\tau_n)_A)$ of nts (X, τ_n) into neutrosophic subspace $(B, (\sigma_n)_B)$ of nts (Y, σ_n) .

Proof. : Let $O \in (\sigma_n)_B$, then there exists $G \in \sigma_n$ such that $O = B \cap G$ and $f^{-1}(G) \in \tau_n$. Therefore $f^{-1}(O) \cap A = f^{-1}(B) \cap f^{-1}(G) \cap A = f^{-1}(G) \cap A \in (\tau_n)_A$. $\square$

Remark 5.5. (i) Every neutrosophic continuous (resp. neutrosophic open) mapping is S^* -neutrosophic continuous (resp. S^* -neutrosophic open), but converse need not be true.

(ii) Every supra neutrosophic continuous (resp. supra neutrosophic open) mapping is S^* -neutrosophic continuous (resp. S^* -neutrosophic open), but converse need not be true.

(iii) Supra neutrosophic continuous and neutrosophic continuous mappings are independent each other.

(iv) Supra neutrosophic open and neutrosophic open mappings are independent each other.

Proof. : The proof follows from the definition, the converse and independence are shown in the following example. $\square$

Example 5.6. Let $Y = \{x, y, z\}$, $X = \{a, b, c\}$ with neutrosophic topologies $\tau_n = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1))\}$ and $\sigma_n = \{\emptyset, Y, ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))\}$. Let $\tau_n^* = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1)), ((0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5)), ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))\}$ and $\sigma_n^* = \sigma_n$ be associated neutrosophic supra topologies with respect to τ_n and σ_n . Define a mapping $f : X \rightarrow Y$ by $f(c) = z, f(b) = y, f(a) = x$, then $f^{-1}(((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))) = ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5)) \in \tau_n^*$. Clearly f is supra neutrosophic continuous and S^* -neutrosophic continuous but not neutrosophic continuous.

Let $Y = \{x, y, z\}$, $X = \{a, b, c\}$ with neutrosophic topologies $\tau_n = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1)), ((0.5, 0.25, 0), (0.5, 0.25, 0), (0.5, 0.75, 1))\}$ and $\sigma_n = \{\emptyset, Y, ((0.5, 0.25, 0), (0.5, 0.25, 0), (0.5, 0.75, 1))\}$. Let $\tau_n^* = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1)), ((0.5, 0.25, 0), (0.5, 0.25, 0), (0.5, 0.75, 1)), ((1, 0.5, 0), (1, 0.5, 0), (0, 0.5, 1))\}$ and $\sigma_n^* = \{\emptyset, Y, ((0.5, 0.25, 0), (0.5, 0.25, 0), (0.5, 0.75, 1)), ((0.3, 0.25, 0.5), (0.3, 0.25, 0.5), (0.7, 0.75, 0.5)), ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))\}$ be associated neutrosophic supra topologies with respect to τ_n and σ_n . Consider a mapping $f : X \rightarrow Y$ by $f(c) = z, f(b) = y, f(a) = x$, then $f^{-1}(((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))) = ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5)) \notin \tau_n^*$. Therefore f is neutrosophic continuous and S^* -neutrosophic continuous but not supra neutrosophic continuous. If we consider a mapping $g : Y \rightarrow X$ by $g(z) = c, g(y) = b, g(x) = a$, then g is neutrosophic open and S^* -neutrosophic open but not supra neutrosophic open.

Let $Y = \{x, y, z\}$, $X = \{a, b, c\}$ with neutrosophic topologies $\tau_n = \{\emptyset, X, ((1, 0.5, 0.3), (1, 0.5, 0.3), (0, 0.5, 0.7))\}$ and $\sigma_n = \{\emptyset, Y, ((1, 0.3, 0.5), (1, 0.3, 0.5), (0, 0.7, 0.5))\}$. Let $\sigma_n^* = \{\emptyset, Y, ((1, 0.3, 0.5), (1, 0.3, 0.5), (0, 0.7, 0.5)), ((1, 0.5, 0.3), (1, 0.5, 0.3), (0, 0.5, 0.7)), ((1, 0.5, 0.5), (1, 0.5, 0.5), (0, 0.5, 0.5))\}$ and $\tau_n^* = \tau_n$ be associated neutrosophic supra topologies with respect to σ_n and τ_n . Then $f : X \rightarrow Y$ defined by $f(c) = z, f(b) = y, f(a) = x$ is S^* -neutrosophic open and supra neutrosophic open but not neutrosophic open.

Observation 5.7. The following are the examples of contradicting the statements of Abd-Monsef and Ramadan^[9]. In fuzzy supra topological space, consider $Y = \{x, y, z\}$, $X = \{a, b, c\}$ with fuzzy topologies $\tau_f = \{\emptyset, X, (0.5, 0.5, 0), (0.5, 0.25, 0)\}$ and $\sigma_f = \{\emptyset, Y, (0.5, 0.25, 0)\}$. Let $\tau_f^* = \{\emptyset, X, (0.5, 0.5, 0), (0.5, 0.25, 0), (1, 0.5, 0)\}$ and $\sigma_f^* = \{\emptyset, Y, (0.5, 0.25, 0), (0.3, 0.25, 0.5), (0.5, 0.25, 0.5)\}$ be associated fuzzy supra topologies with respect to τ_f and σ_f . Consider a mapping $h : X \rightarrow Y$ by $h(c) = z, h(b) = y, h(a) = x$, then $h^{-1}((0.5, 0.25, 0.5)) = (0.5, 0.25, 0.5) \notin \tau_f^*$. Then h is fuzzy continuous but not supra fuzzy continuous. If we define a mapping $g : Y \rightarrow X$ by $g(z) = c, g(y) = b, g(x) = a$, then g is fuzzy open but not supra fuzzy open.

Theorem 5.8. The following statements are equivalent for the mapping f of nts (X, τ_n) into nts (Y, σ_n) :

- (i) The mapping $f : X \rightarrow Y$ is S^* -neutrosophic continuous.
- (ii) The inverse image of every neutrosophic closed set in (Y, σ_n) is neutrosophic supra closed in (X, τ_n^*) .
- (iii) For each neutrosophic set A in Y , $cl_{\tau_n^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{\sigma_n}(A))$.
- (iv) For each neutrosophic set B in X , $f(cl_{\tau_n^*}(B)) \subseteq cl_{\sigma_n}(f(B))$.

(v) For each neutrosophic set A in Y , $int_{\tau_n^*}(f^{-1}(A)) \supseteq f^{-1}(int_{\sigma_n}(A))$.

Proof. : (i) $\Rightarrow$ (ii): Let f be a S^* -neutrosophic continuous and A be a neutrosophic closed set in (Y, σ_n) , $f^{-1}(Y - A) = X - f^{-1}(A)$ is neutrosophic supra open in (X, τ_n^*) and so $f^{-1}(A)$ is neutrosophic supra closed in (X, τ_n^*) .

(ii) $\Rightarrow$ (iii): $cl_{\sigma_n}(A)$ is neutrosophic closed in (Y, σ_n) , for each neutrosophic set A in Y , then $f^{-1}(cl_{\sigma_n}(A))$ is neutrosophic supra closed in (X, τ_n^*) . Thus $f^{-1}(cl_{\sigma_n}(A)) = cl_{\tau_n^*}(f^{-1}(cl_{\sigma_n}(A))) \supseteq cl_{\tau_n^*}(f^{-1}(A))$.

(iii) $\Rightarrow$ (iv): $f^{-1}(cl_{\sigma_n}(f(B))) \supseteq cl_{\tau_n^*}(f^{-1}(f(B))) \supseteq cl_{\tau_n^*}(B)$, for each neutrosophic set B in X and so $f(cl_{\tau_n^*}(B)) \subseteq cl_{\sigma_n}(f(B))$.

(iv) $\Rightarrow$ (ii): Let $B = f^{-1}(A)$, for each neutrosophic closed set A in Y , then $f(cl_{\tau_n^*}(B)) \subseteq cl_{\sigma_n}(f(B)) \subseteq cl_{\sigma_n}(A) = A$ and $cl_{\tau_n^*}(B) \subseteq f^{-1}(f(cl_{\tau_n^*}(B))) \subseteq f^{-1}(A) = B$. Therefore $B = f^{-1}(A)$ is neutrosophic supra closed in X .

(ii) $\Rightarrow$ (i): Let A be a neutrosophic open set in Y , then $X - f^{-1}(A) = f^{-1}(Y - A)$ is neutrosophic supra closed in X , since $Y - A$ is neutrosophic closed in Y . Therefore $f^{-1}(A)$ is neutrosophic supra open in X .

(i) $\Rightarrow$ (v): $f^{-1}(int_{\sigma_n}(A))$ is neutrosophic supra open in X , for each neutrosophic set A in Y and $int_{\tau_n^*}(f^{-1}(A)) \supseteq int_{\tau_n^*}(f^{-1}(int_{\sigma_n}(A))) = f^{-1}(int_{\sigma_n}(A))$.

(v) $\Rightarrow$ (i): $f^{-1}(A) = f^{-1}(int_{\sigma_n}(A)) \subseteq int_{\tau_n^*}(f^{-1}(A))$, for each neutrosophic open set A in Y and so $f^{-1}(A)$ is neutrosophic supra open in X . $\square$

Theorem 5.9. The following statements are equivalent for the mapping f of nts (X, τ_n) into nts (Y, σ_n) :

(i) A mapping $f : (X, \tau_n^*) \rightarrow (Y, \sigma_n^*)$ is neutrosophic supra continuous.

(ii) The inverse image of every neutrosophic supra closed set in (Y, σ_n^*) is neutrosophic supra closed in (X, τ_n^*) .

(iii) For each neutrosophic set A in Y , $cl_{\tau_n^*}(f^{-1}(A)) \subseteq f^{-1}(cl_{\sigma_n^*}(A)) \subseteq f^{-1}(cl_{\sigma_n}(A))$.

(iv) For each neutrosophic set B in X , $f(cl_{\tau_n^*}(B)) \subseteq cl_{\sigma_n^*}(f(B)) \subseteq cl_{\sigma_n}(f(B))$.

(v) For each neutrosophic set A in Y , $int_{\tau_n^*}(f^{-1}(A)) \supseteq f^{-1}(int_{\sigma_n^*}(A)) \supseteq f^{-1}(int_{\sigma_n}(A))$.

Proof. : The proof is straightforward from theorem 5.8. $\square$

Theorem 5.10. If $f : X \rightarrow Y$ is S^* -neutrosophic continuous and $g : Y \rightarrow Z$ is neutrosophic continuous, then $g \circ f : X \rightarrow Z$ is S^* -neutrosophic continuous.

Proof. : The proof follows directly from the definition. $\square$

Theorem 5.11. If $f : X \rightarrow Y$ is supra neutrosophic continuous and $g : Y \rightarrow Z$ is S^* -neutrosophic continuous (or neutrosophic continuous), then $g \circ f : X \rightarrow Z$ is S^* -neutrosophic continuous.

Proof. : It follows from the definition. $\square$

Theorem 5.12. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are supra neutrosophic continuous (resp. supra neutrosophic open) mappings, then $g \circ f : X \rightarrow Z$ is supra neutrosophic continuous (resp. supra neutrosophic open).

Proof. : The proof follows obviously from the definition. $\square$

Remark 5.13. Abd-Monsef and Ramadan^[9] stated that if $g : X \rightarrow Y$ is supra fuzzy continuous and $h : Y \rightarrow Z$ is fuzzy continuous, then $h \circ g : X \rightarrow Z$ is supra fuzzy continuous. But in general this is not true, for example consider $Z = \{p, q, r\}$, $Y = \{x, y, z\}$, and $X = \{a, b, c\}$ with fuzzy topologies $\tau_f = \{\emptyset, X, (1, 0.5, 0), (0.3, 0.3, 0)\}$, $\sigma_f = \{\emptyset, Y, (0.5, 0.5, 0), (0.5, 0.25, 0)\}$ and $\eta_f = \{\emptyset, Z, (0.5, 0.25, 0)\}$ on X, Y and Z respectively. Let $\tau_f^* = \{\emptyset, X, (0.5, 0.5, 0), (0.5, 0.25, 0), (1, 0.5, 0), (0.3, 0.3, 0)\}$, $\sigma_f^* = \{\emptyset, Y, (0.5, 0.5, 0), (0.5, 0.25, 0), (1, 0.5, 0)\}$ and $\eta_f^* = \{\emptyset, Z, (0.5, 0.25, 0), (0.3, 0.25, 0.5), (0.5, 0.25, 0.5)\}$ be associated fuzzy supra topologies with respect to τ_f, σ_f and η_f . Then the mapping $g : X \rightarrow Y$ defined by $g(c) = z, g(b) = y, g(a) = x$ is supra fuzzy continuous and the mapping $h : Y \rightarrow Z$ by $h(z) = r, h(y) = q, h(x) = p$ is fuzzy continuous. But $h \circ g : X \rightarrow Z$ is not supra fuzzy continuous, since $(g \circ h)^{-1}((0.3, 0.25, 0.5)) = (0.3, 0.25, 0.5) \notin \tau_f^*$.

Remark 5.14. In general the composition of two supra neutrosophic continuous mappings is again supra neutrosophic continuous, but the composition of two S^* -neutrosophic continuous mappings may not be S^* -neutrosophic continuous. Let $Z = \{p, q, r\}$, $Y = \{x, y, z\}$, and $X = \{a, b, c\}$ with neutrosophic topologies $\tau_n = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1))\}$, $\sigma_n = \{\emptyset, Y, ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))\}$ and $\eta_n = \{\emptyset, Z, ((0.3, 0.7, 0.5), (0.3, 0.7, 0.5), (0.7, 0.3, 0.5))\}$ on X, Y and Z respectively. Let $\tau_n^* = \{\emptyset, X, ((0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1)), ((0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5)), ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5))\}$ and $\sigma_n^* = \{\emptyset, Y, ((0.5, 0.25, 0.5), (0.5, 0.25, 0.5), (0.5, 0.75, 0.5)), ((0.3, 0.7, 0.5), (0.3, 0.7, 0.5), (0.7, 0.3, 0.5)), ((0.5, 0.7, 0.5), (0.5, 0.7, 0.5), (0.5, 0.3, 0.5))\}$ be associated neutrosophic supra topologies with respect to τ_n and σ_n . Then the mappings $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are defined respectively by $f(c) = z, f(b) = y, f(a) = x$ and $g(z) = r, g(y) = q, g(x) = p$ are S^* -neutrosophic continuous. But $g \circ f : X \rightarrow Z$ is not S^* -neutrosophic continuous, since $(g \circ f)^{-1}(((0.3, 0.7, 0.5), (0.3, 0.7, 0.5), (0.7, 0.3, 0.5)))) = ((0.3, 0.7, 0.5), (0.3, 0.7, 0.5), (0.7, 0.3, 0.5)) \notin \tau_n^*$.

Theorem 5.15. If mappings $f : (A, (\tau_n)_A) \rightarrow (B, (\sigma_n)_B)$ from neutrosophic subspace $(A, (\tau_n)_A)$ of nts (X, τ_n) into neutrosophic subspace $(B, (\sigma_n)_B)$ of nts (Y, σ_n) and $g : (B, (\sigma_n)_B) \rightarrow (C, (\eta_n)_C)$ from neutrosophic subspace $(B, (\sigma_n)_B)$ of nts (Y, σ_n) into neutrosophic subspace $(C, (\eta_n)_C)$ of nts (Z, η_n) are relatively neutrosophic continuous (resp. relatively neutrosophic open) mappings, then $g \circ f : (A, (\tau_n)_A) \rightarrow (C, (\eta_n)_C)$ is relatively neutrosophic continuous (resp. relatively neutrosophic open) from neutrosophic subspace $(A, (\tau_n)_A)$ of nts (X, τ_n) into neutrosophic subspace $(C, (\eta_n)_C)$ of nts (Z, η_n) .

Proof. : Let $O \in (\eta_n)_C$, then $g^{-1}(O) \cap B \in (\sigma_n)_B$ and $f^{-1}(g^{-1}(O) \cap B) \cap A \in (\tau_n)_A$. Since $B \supset f(A)$, then $(g \circ f)^{-1}(O) \cap A = f^{-1}(g^{-1}(O) \cap B) \cap A$. Therefore $g \circ f$ is relatively neutrosophic continuous. Let $U \in (\tau_n)_A$, then $f(U) \in (\sigma_n)_B$ and $g(f(U)) = (g \circ f)(U) \in (\eta_n)_C$. Therefore $g \circ f$ is relatively neutrosophic open. $\square$

6 Neutrosophic Supra Topology in Data Mining

In this section, we present a methodical approach for decision-making problem with single valued neutrosophic information. The following necessary steps are proposed the methodical approach to select the proper attributes and alternatives in the decision-making situation.

Step 1: Problem field selection:

Consider multi-attribute decision making problems with m attributes $A_1, A_2, \dots, A_m$ and n alternatives $C_1, C_2, \dots, C_n$ and p attributes $D_1, D_2, \dots, D_p$, ($n \leq p$).

	C_1	C_2	.	.	.	C_n
A_1	(a_{11})	(a_{12})	.	.	.	(a_{1n})
A_2	(a_{21})	(a_{22})	.	.	.	(a_{2n})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
A_m	(a_{m1})	(a_{m2})	.	.	.	(a_{mn})

	A_1	A_2	.	.	.	A_m
D_1	(d_{11})	(d_{12})	.	.	.	(d_{1m})
D_2	(d_{21})	(d_{22})	.	.	.	(d_{2m})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
D_p	(d_{p1})	(d_{p2})	.	.	.	(d_{pm})

Here all the attributes a_{ij} and d_{ki} ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, p$) are single valued neutrosophic numbers.

Step 2: Form neutrosophic supra topologies for (C_j) and (D_k) :

- (i) $\tau_j^* = A \cup B$, where $A = \{1_N, 0_N, a_{1j}, a_{2j}, \dots, a_{mj}\}$ and $B = \{a_{1j} \cup a_{2j}, a_{1j} \cup a_{3j}, \dots, a_{m-1j} \cup a_{mj}\}$.
- (ii) $\nu_k^* = C \cup D$, where $C = \{1_N, 0_N, d_{k1}, d_{k2}, \dots, d_{km}\}$ and $D = \{d_{k1} \cup d_{k2}, d_{k1} \cup d_{k3}, \dots, d_{km-1} \cup d_{km}\}$.

Step 3: Find Single valued neutrosophic score functions:

Single valued neutrosophic score functions (shortly SVNSF) of A , B , C , D , C_j and D_k are defined as follows.

- (i) $SVNSF(A) = \frac{1}{3(m+2)} [\sum_{i=1}^{m+2} [2 + \mu_i - \sigma_i - \gamma_i]]$, and $SVNSF(B) = \frac{1}{3q} [\sum_{i=1}^q [2 + \mu_i - \sigma_i - \gamma_i]]$, where q is the number of elements of B . For $j = 1, 2, \dots, n$,
- $$SVNSF(C_j) = \begin{cases} SVNSF(A) & \text{if } SVNSF(B) = 0 \\ \frac{1}{2} [SVNSF(A) + SVNSF(B)] & \text{otherwise} \end{cases}.$$
- (ii) $SVNSF(C) = \frac{1}{3(m+2)} [\sum_{i=1}^{m+2} [2 + \mu_i - \sigma_i - \gamma_i]]$ and $SVNSF(D) = \frac{1}{3r} [\sum_{i=1}^r [2 + \mu_i - \sigma_i - \gamma_i]]$, where r is the number of elements of D . For $k = 1, 2, \dots, p$,
- $$SVNSF(D_k) = \begin{cases} SVNSF(C) & \text{if } SVNSF(D) = 0 \\ \frac{1}{2} [SVNSF(C) + SVNSF(D)] & \text{otherwise} \end{cases}.$$

Step 4: Final Decision

Arrange single valued neutrosophic score values for the alternatives $C_1 \leq C_2 \leq \dots \leq C_n$ and the attributes $D_1 \leq D_2 \leq \dots \leq D_p$. Choose the attribute D_p for the alternative C_1 and D_{p-1} for the alternative C_2 etc. If $n < p$, then ignore D_k , where $k = 1, 2, \dots, n - p$.

7 Numerical Example

Medical diagnosis has increased volume of information available to physicians from new medical technologies and comprises of uncertainties. In medical diagnosis, very difficult task is the process of classifying different set of symptoms under a single name of a disease. In this section, we exemplify a medical diagnosis problem for effectiveness and applicability of above proposed approach.

Step 1: Problem field selection:

Consider the following tables giving informations when consulted physicians about four patients P_1, P_2, P_3, P_4 and symptoms are Temperature, Cough, Blood Plates, Joint Pain, Insulin. We need to find the patient and to find the disease such as Tuberculosis, Diabetes, Chikungunya, Swine Flu, Dengue of the patient. The data in Table 1 are explained by the membership, the indeterminacy and the non-membership functions. From Table 2, we can observe that for tuberculosis, cough is high ($\mu = 0.9, \sigma = 0.1, \gamma = 0.1$), but for chikungunya, cough is low ($\mu = 0, \sigma = 0.1, \gamma = 0.9$).

Table 1.

Patients \ Symptoms	P ₁	P ₂	P ₃	P ₄
Temperature	(0.8,0,0.2)	(0.1,0,0.7)	(0.9,0.1,0)	(0,0.1,0.9)
Cough	(0.1,0.2,0.7)	(0.1,0.1,0.8)	(0,0.3,0.7)	(0.8,0.1,0.2)
Blood Plates	(0.8,0,0.2)	(0.2,0.1,0.6)	(0.3,0.1,0.6)	(0.3,0.1,0.6)
Joint Pain	(0.4,0.2,0.5)	(0.4,0.2,0.5)	(0.9,0,0.1)	(0.2,0.2,0.7)
Insulin	(0.3,0.2,0.5)	(0.9,0,0.1)	(0.2,0.1,0.7)	(0.4,0.3,0.2)

Table 2.

Symptoms \ Diagnosis	Temperature	Cough	Blood Plates	Joint Pain	Insulin
Tuberculosis	(0.6,0.3,0.1)	(0.9,0.1,0.1)	(0,0.2,0.8)	(0,0.1,0.8)	(0,0.1,0.9)
Diabetes	(0.1,0.1,0.8)	(0.1,0.1,0.8)	(0.2,0.2,0.1)	(0.1,0.4,0.6)	(0.9,0,0.1)
Chikungunya	(0.9,0,0.1)	(0,0.1,0.9)	(0.7,0.2,0.1)	(0.9,0.1,0.1)	(0.2,0,0.8)
Swine Flu	(0.2,0.5,0.3)	(0.1,0.4,0.3)	(0.2,0.4,0.1)	(0.1,0.3,0.5)	(0.2,0.4,0.1)
Dengue	(0.9,0,0.1)	(0.2,0.6,0.4)	(0.2,0.6,0.4)	(0.3,0.1,0.6)	(0.2,0.1,0.7)

Step 2: Form neutrosophic supra topologies for (C_j) and (D_k) :

- (i) $\tau_1^* = A \cup B$, where $A = \{(1, 1, 0), (0, 0, 1), (0.8, 0, 0.2), (0.1, 0.2, 0.7), (0.4, 0.2, 0.5), (0.3, 0.2, 0.5)\}$ and $B = \{(0.8, 0.2, 0.2)\}$.
- (ii) $\tau_2^* = A \cup B$, where $A = \{(1, 1, 0), (0, 0, 1), (0.1, 0, 0.7), (0.1, 0.1, 0.8), (0.2, 0.1, 0.6), (0.4, 0.2, 0.5), (0.9, 0, 0.1)\}$ and $B = \{(0.1, 0.1, 0.7), (0.9, 0.1, 0.1), (0.9, 0.2, 0.1)\}$.
- (iii) $\tau_3^* = A \cup B$, where $A = \{(1, 1, 0), (0, 0, 1), (0.9, 0.1, 0), (0, 0.3, 0.7), (0.3, 0.1, 0.6), (0.9, 0, 0.1), (0.2, 0.1, 0.7)\}$ and $B = \{(0.9, 0.3, 0), (0.3, 0.3, 0.6), (0.9, 0.3, 0.1), (0.2, 0.3, 0.7), (0.9, 0.1, 0.1)\}$.
- (iv) $\tau_4^* = A \cup B$, where $A = \{(1, 1, 0), (0, 0, 1), (0, 0.1, 0.9), (0.8, 0.1, 0.2), (0.3, 0.1, 0.6), (0.2, 0.2, 0.7), (0.4, 0.3, 0.2)\}$ and $B = \{(0.8, 0.2, 0.2), (0.8, 0.3, 0.2), (0.3, 0.2, 0.6)\}$.
- (i) $\nu_1^* = C \cup D$, where $C = \{(1, 1, 0), (0, 0, 1), (0.6, 0.3, 0.1), (0.9, 0.1, 0.1), (0, 0.2, 0.8), (0, 0.1, 0.8), (0, 0.1, 0.9)\}$ and $D = \{(0.9, 0.3, 0.1), (0.9, 0.2, 0.1)\}$.

- (ii) $\nu_2^* = C \cup D$, where $C = \{(1, 1, 0), (0, 0, 1), (0.1, 0.1, 0.8), (0.2, 0.2, 0.1), (0.1, 0.4, 0.6), (0.9, 0, 0.1)\}$ and $D = \{(0.9, 0.1, 0.1), (0.2, 0.4, 0.1), (0.9, 0.2, 0.1), (0.9, 0.4, 0.1)\}$.
- (iii) $\nu_3^* = C \cup D$, where $C = \{(1, 1, 0), (0, 0, 1), (0.9, 0, 0.1), (0, 0.1, 0.9), (0.7, 0.2, 0.1), (0.9, 0.1, 0.1), (0.2, 0, 0.8)\}$ and $D = \{(0.9, 0.2, 0.1), (0.2, 0.1, 0.8)\}$.
- (iv) $\nu_4^* = C \cup D$, where $C = \{(1, 1, 0), (0, 0, 1), (0.2, 0.5, 0.3), (0.1, 0.4, 0.3), (0.2, 0.4, 0.3), (0.2, 0.4, 0.1), (0.1, 0.3, 0.5)\}$ and $D = \{(0.2, 0.5, 0.1)\}$.
- (v) $\nu_k^* = C \cup D$, where $C = \{(1, 1, 0), (0, 0, 1), (0.9, 0, 0.1), (0.2, 0.6, 0.4), (0.3, 0.1, 0.6), (0.2, 0.1, 0.7)\}$ and $D = \{(0.9, 0.6, 0.1), (0.9, 0.1, 0.1), (0.3, 0.6, 0.4)\}$.

Step 3: Find Single valued neutrosophic score functions:

- (i) $\text{SVNSF}(A) = 0.5611$ and $\text{SVNSF}(B) = 0.8$, where $q = 1$. $\text{SVNSF}(C_1) = 0.6801$.
- (ii) $\text{SVNSF}(A) = 0.5524$ and $\text{SVNSF}(B) = 0.7333$, where $q = 3$. $\text{SVNSF}(C_2) = 0.6428$.
- (iii) $\text{SVNSF}(A) = 0.6$ and $\text{SVNSF}(B) = 0.6933$, where $q = 5$. $\text{SVNSF}(C_3) = 0.6466$.
- (iv) $\text{SVNSF}(A) = 0.5381$ and $\text{SVNSF}(B) = 0.6888$, where $q = 3$. $\text{SVNSF}(C_4) = 0.6135$.
- (i) $\text{SVNSF}(C) = 0.5238$ and $\text{SVNSF}(D) = 0.85$, where $r = 2$. $\text{SVNSF}(D_1) = 0.6869$.
- (ii) $\text{SVNSF}(C) = 0.5555$ and $\text{SVNSF}(D) = 0.7833$, where $r = 4$. $\text{SVNSF}(D_2) = 0.6694$.
- (iii) $\text{SVNSF}(C) = 0.6333$ and $\text{SVNSF}(B) = 0.65$, where $r = 2$. $\text{SVNSF}(D_3) = 0.6416$.
- (iv) $\text{SVNSF}(C) = 0.4888$ and $\text{SVNSF}(B) = 0.5333$, where $r = 1$. $\text{SVNSF}(D_4) = 0.5111$.
- (v) $\text{SVNSF}(C) = 0.5555$ and $\text{SVNSF}(B) = 0.6888$, where $r = 3$. $\text{SVNSF}(D_5) = 0.6222$.

Step 4: Final Decision:

Arrange single valued neutrosophic score values for the alternatives C_1, C_2, C_3, C_4 and the attributes D_1, D_2, D_3, D_4, D_5 in ascending order. We get the following sequences $C_4 \leq C_2 \leq C_3 \leq C_1$ and $D_4 \leq D_5 \leq D_3 \leq D_2 \leq D_1$. Thus the patient P_4 suffers from tuberculosis, the patient P_2 suffers from diabetes, the patient P_3 suffers from chikungunya and the patient P_1 suffers from dengue.

8 Conclusion and Future Work

Neutrosophic topological space is one of the research areas in general fuzzy topological spaces to deal the concept of vagueness. This paper introduced neutrosophic supra topological spaces and its real life application. Moreover we have discussed some mappings in neutrosophic supra topological spaces and derived some contradicting examples in fuzzy supra topological spaces. This theory can be develop and implement to other research areas of general topology such as rough topology, digital topology and so on.

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Slope Stability Assessment Method Using the Arctangent and Tangent Similarity Measure of Neutrosophic Numbers

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Abstract. Slope is a typical disaster in open-pit mining. Then it is a kind of non-continuous and uncertain natural geological body. In this case, common assessment approaches cannot assess the slope stability problems with both certain and uncertain information. Then, a neutrosophic number (NN) can easily represent the certain and uncertain information. Unfortunately NNs have not been applied in slope stability analysis so far. Therefore, this paper proposes the arctangent and tangent similarity measures of NNs and a slope stability assessment method using the arctangent and tangent similarity measures of NNs. By the similarity measure between the classification grades of slope stability and a slope sample with NNs, we can determine the assessment grade of the slope sample/case. Then, the assessment results of ten slope samples/cases demonstrate the same as actual grades of the ten slope cases, which indicate the effectiveness and feasibility of the proposed slope stability assessment method. The advantages are that the slope stability assessment method based on the arctangent and tangent similarity measures of NNs is simple and suitable and can assess slope stability problems in NN setting.

Keywords: Neutrosophic number, Arctangent similarity measure, Tangent similarity measure, Slope stability, Classification grade

1 Introduction

Geological disasters caused by open-pit mining have been gradually increased, while one of them is the slope instability of open-pit mining. Therefore, one is convinced of the importance of slope stability evaluation. The majority of slope stability evaluation performed in practical engineering still use deterministic analytical methods, such as traditional limit equilibrium methods [1-4] and finite element methods [5,6]. However, the slope is a kind of non-continuous and uncertain natural geological body along with the effect of external various factors, such as geological environment, topography, hydrology, and climate, which will affect the analysis of slope stability. However, the slope data usually imply imprecise and indeterminate information because of uncertain natural geological body and various external factors. In real situations, the slope data usually contain the indeterminate and determinate information. To represent it, a neutrosophic number (NN) was proposed firstly by Smarandache [7-9] and denoted by $N = c + eI$, which is composed of its certain part c and its uncertain part eI . Especially, NN is reduced to its certain part $N = c$ if $eI = 0$ (the best case) and its uncertain part $N = eI$ if $c = 0$ (the worst case). Clearly, NNs can show the advantage of representing the certain and uncertain information. Hence, the misfire fault diagnosis method of gasoline engines was presented by the cosine similarity measure of NNs [10]. Then, the fault diagnosis method of steam turbine was introduced by the exponential similarity measure of NNs [11]. A NN optimization model was used for the optimal design of truss structures [12]. Further, the multiple attribute decision making approach of clay-brick selection was provided by the projection model of NNs [13]. Recently, NNs have been also used for linear and nonlinear programming problems [14-17], neutrosophic traffic flow problems [18], as well as a neutrosophic neural network for NN function approximations [19]. Furthermore, NN functions and neutrosophic statistic methods were utilized for scale effect and anisotropy analyses of rock joint roughness coefficient in rock mechanics [20-22].

Slope is a typical disaster in open-pit mining. Then it is a kind of non-continuous and uncertain natural geological body. In this case, it is difficult to assess the slope stability with certain and uncertain information. However, existing slope stability assessment approaches [1-6] cannot handle the assessment problems of slope stability in certain and uncertain setting. As mentioned above, NNs can easily represent the advantage of certain and uncertain information. Unfortunately, NNs have not been applied in slope stability analysis so far. Therefore, we

need to develop a new slope stability assessment method for assessing slope stability problems with NNs in the geotechnical engineering field. To do so, this paper presents the arctangent and tangent similarity measures of NNs and their assessment method in slope stability problems with NN information.

Then, this study indicates the contribution of two new arctangent and tangent similarity measures of NNs and their slope stability assessment method for assessing slope stability problems in NN setting. Then, the main advantages are that the developed assessment approach is simpler and more suitable than existing common ones [1-6] and can assess slope stability problems with NN information, which traditional limit equilibrium methods [1-4] and finite element methods [5,6] cannot do.

The structure of this paper is given as the following framework. Some concepts of NNs are described in Section 2. Section 3 presents the arctangent and tangent similarity measures of NNs. A slope stability assessment method is established by the arctangent and tangent similarity measures, and then the effectiveness and feasibility are indicated by ten slope samples/cases in Section 4. Lastly, Section 5 presents conclusions and future work.

2 Some NN concepts

The NN presented by Smarandache [1] includes the certain part c and uncertain part eI , which is represented by $N = c + eI$ for $c, e \in R$ (all real numbers) and the indeterminacy $I \in [\inf I, \sup I]$. For instance, a NN is $N = 5 + 3I$, and then it is equivalent to $N \in [8, 11]$ for $I \in [1, 2]$ and $N \in [11, 14]$ for $I \in [2, 3]$. Generally, NN implies a changeable interval number regarding $I \in [\inf I, \sup I]$.

Let $N_1 = c_1 + e_1I$ and $N_2 = c_2 + e_2I$ be two NNs. Smarandache [7-9] introduced the operational relations of NNs:

- (1) $N_1 + N_2 = c_1 + c_2 + (e_1 + e_2)I$;
- (2) $N_1 - N_2 = c_1 - c_2 + (e_1 - e_2)I$;
- (3) $N_1 \times N_2 = c_1c_2 + (c_1e_2 + e_1c_2 + e_1e_2)I$;
- (4) $N_1^2 = (c_1 + e_1I)^2 = c_1^2 + (2c_1e_1 + e_1^2)I$;
- (5) $\frac{N_1}{N_2} = \frac{c_1 + e_1I}{c_2 + e_2I} = \frac{c_1}{c_2} + \frac{e_1c_2 - c_1e_2}{c_2(c_2 + e_2)} \cdot I$ for $c_2 \neq 0$ and $e_2 \neq -c_2$;

$$(6) \sqrt{N_1} = \sqrt{c_1 + e_1I} = \begin{cases} \sqrt{c_1} - (\sqrt{c_1} - \sqrt{c_1 + e_1})I \\ \sqrt{c_1} - (\sqrt{c_1} + \sqrt{c_1 + e_1})I \\ -\sqrt{c_1} + (\sqrt{c_1} + \sqrt{c_1 + e_1})I \\ -\sqrt{c_1} + (\sqrt{c_1} - \sqrt{c_1 + e_1})I \end{cases}.$$

3 Arctangent and tangent similarity measures between NNs

This section presents similarity measures of NNs based on arctangent and tangent functions.

It is well known that arctangent and tangent functions, $\arctan(y)$ for $y \in [0, 1]$ and $\tan(y)$ for $y \in [0, \pi/4]$ are two increasing functions. If their function values are defined within $[0, 1]$, we can present the arctangent and tangent similarity measures between NNs.

Definition 1. Set $P = \{N_{P1}, N_{P2}, \dots, N_{Pn}\}$ and $Q = \{N_{Q1}, N_{Q2}, \dots, N_{Qn}\}$ as two sets of NNs, where $N_{Pj} = c_{Pj} + e_{Pj}I$ and $N_{Qj} = c_{Qj} + e_{Qj}I$ ($j = 1, 2, \dots, n$) for $c_{Pj}, e_{Pj}, c_{Qj}, e_{Qj} \geq 0, I \in [\inf I, \sup I]$, and $N_{Pj}, N_{Qj} \subseteq [0, 1]$. Then, the arctangent and tangent similarity measures between P and Q are defined by

$$AT(P, Q) = 1 - \frac{4}{n\pi} \sum_{j=1}^n \arctan \left[\frac{1}{2} \left(\frac{|c_{Pj} + e_{Pj} \inf I - c_{Qj} - e_{Qj} \inf I|}{|c_{Pj} + e_{Pj} \sup I - c_{Qj} - e_{Qj} \sup I|} \right) \right], \quad (1)$$

$$T(P, Q) = 1 - \frac{1}{n} \sum_{j=1}^n \left\{ \tan \left[\frac{\pi}{8} \left(\frac{|c_{Pj} + e_{Pj} \inf I - c_{Qj} - e_{Qj} \inf I|}{|c_{Pj} + e_{Pj} \sup I - c_{Qj} - e_{Qj} \sup I|} \right) \right] \right\}. \quad (2)$$

Obviously, the arctangent and tangent similarity measures should satisfy the following properties [23]:

- (1) $0 \leq AT(P, Q) \leq 1$ and $0 \leq T(P, Q) \leq 1$;
- (2) $AT(P, Q)=1$ and $T(P, Q)=1$ if and only if $P = Q$;
- (3) $AT(P, Q) = AT(Q, P)$ and $T(P, Q) = T(Q, P)$;
- (4) If $S = \{N_{S1}, N_{S2}, \dots, N_{Sn}\}$ is a set of NNs, $AT(P, S) \leq AT(P, Q)$, $AT(P, S) \leq AT(Q, S)$, $T(P, S) \leq T(P, Q)$, and $T(P, S) \leq T(Q, S)$.

In practical applications, the importance of each element is considered in the sets of NNs. If we assume that the weight of elements N_{Pj} and N_{Qj} is w_j for $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, then the weighted arctangent and tangent similarity measures between P and Q can be introduced below:

$$AT_w(P, Q) = 1 - \frac{4}{\pi} \sum_{j=1}^n w_j \arctan \left[\frac{1}{2} \left(\frac{|c_{Pj} + e_{Pj} \inf I - c_{Qj} - e_{Qj} \inf I|}{|c_{Pj} + e_{Pj} \sup I - c_{Qj} - e_{Qj} \sup I|} \right) \right], \quad (3)$$

$$T_w(P, Q) = 1 - \sum_{j=1}^n \left\{ w_j \tan \left[\frac{\pi}{8} \left(\frac{|c_{Pj} + \inf(e_{Pj} I) - c_{Qj} - \inf(e_{Qj} I)|}{|c_{Pj} + \sup(e_{Pj} I) - c_{Qj} - \sup(e_{Qj} I)|} \right) \right] \right\}. \quad (4)$$

Clearly, the weighted arctangent and tangent similarity measures should satisfy the above properties (1)-(4).

4 Slope stability assessment method using the similarity measures

Because of the complexity of practical engineering, the assessment information of slope stability often is incomplete and indeterminate. By using the evaluation method of slope stability regarding determinate information, however, it is difficult to reasonably evaluate whether the slope is unstable/stable due to missing indeterminate information in certain and uncertain setting.

Then, how can we give a proper evaluation method of slope stability with indeterminate and determinate information? A slope stability assessment method is presented to help engineers' proper evaluation of slope stability problems in NN setting.

4.1 Classification grades of slope stability

First, the stability of slopes is divided into four classification grades: stability (I), basic stability (II), relative instability (III), and instability (IV). Among them, the stability (I) means the slope is in a safe state, while the basic stability (II) means the slope may imply a possible safe state, and other two grades imply unsafe states. Then, we choose some slope samples as actual cases from the south central area of Zhejiang province in China. Through the investigation and statistics of the slopes in this area, a set of main impact factors is selected for the slope stability assessment. In this study, we choose eight impact factors, including the structural surface occurrence (h_1), the degree of weathering (h_2), the integrity of rock mass (h_3), the slope angle (h_4), the slope height (h_5), the degree of vegetation coverage (h_6), the annual rainfall (h_7), and the degree of human activities (h_8). However, these actual values obtained from slope samples/cases need to be normalized and shown in Table 1.

h_j	I	II	III	IV
h_1	[0,0.3]	[0.3,0.5]	[0.5,0.7]	[0.7,1]
h_2	[0,0.2]	[0.2,0.6]	[0.6,0.8]	[0.8,1]
h_3	[0,0.25]	[0.25,0.45]	[0.45,0.65]	[0.65,1]
h_4	[0,0.33]	[0.33,0.5]	[0.5,0.67]	[0.67,1]
h_5	[0,0.33]	[0.33,0.5]	[0.5,0.67]	[0.67,1]
h_6	[0,0.3]	[0.3,0.6]	[0.6,0.8]	[0.8,1]
h_7	[0,0.25]	[0.25,0.5]	[0.5,0.75]	[0.75,1]
h_8	[0,0.3]	[0.3,0.6]	[0.6,0.8]	[0.8,1]

Table 1: Data of slope stability between the eight impact factors and the four classification grades.

Since NNs imply the changable interval values depending on ranges of the indeterminacy $I \in [\inf I, \sup I]$, they can represent indeterminate data effectively and reasonably in indeterminate setting. Hence, the interval values in Table 1 can be transformed into NNs for $I \in [0, 0.25]$, as shown in Table 2.

h_j	I	II	III	IV
h_1	$0+1.2I$	$0.3+0.8I$	$0.5+0.8I$	$0.7+1.2I$
h_2	$0+0.8I$	$0.2+0.16I$	$0.6+0.8I$	$0.8+0.8I$
h_3	$0+I$	$0.25+0.8I$	$0.45+0.8I$	$0.65+1.4I$
h_4	$0+1.32I$	$0.33+0.68I$	$0.5+0.68I$	$0.67+1.32I$
h_5	$0+1.32I$	$0.33+0.68I$	$0.5+0.68I$	$0.67+1.32I$
h_6	$0+1.2I$	$0.3+1.2I$	$0.6+0.8I$	$0.8+0.8I$
h_7	$0+I$	$0.25+I$	$0.5+I$	$0.75+I$
h_8	$0+1.2I$	$0.3+1.2I$	$0.6+0.8I$	$0.8+0.8I$

Table 2: NNs of slope stability between the eight impact factors and the four classification grades.

4.2 Slope samples/cases with NNs

The actually measured data of the eight impact factors obtained by the slope samples are all the forms of single values, which can be also considered as special cases of NNs without the uncertain part. For instance, suppose there is a number 0.4, then it can be considered as NN $0.4 + 0I$ or $[0.4, 0.4]$. Thus, we choose ten slope samples/cases from the south central area of Zhejiang province in China, where the data are shown in Tables 3 and 4. The actual grades in Tables 3 and 4 are given by using the limit equilibrium method for convenient comparison.

h_j	T_1	T_2	T_3	T_4	T_5
h_1	0.35	0.75	0.85	0.45	0.15
h_2	0.4	0.7	0.6	0.4	0.2
h_3	0.4	0.8	0.85	0.4	0.3
h_4	0.76	0.78	0.81	0.62	0.56
h_5	0.81	0.79	1	0.59	0.3
h_6	0.5	0.4	0.25	0.35	0.4
h_7	0.71	0.71	0.86	0.86	0.86
h_8	0.4	0.2	0.2	0.4	0.3
Actual grade	II	IV	IV	II	I

Table 3: Data of slope samples/cases.

h_j	T_6	T_7	T_8	T_9	T_{10}
h_1	0.7	0.1	0.43	0.6	0.2
h_2	0.6	0.6	0.8	0.6	0.1
h_3	0.55	0.15	0.38	0.5	0.15
h_4	0.7	0.67	0.72	0.83	0.61
h_5	0.33	0.61	0.23	0.21	0.43
h_6	0.75	0.7	0.8	0.4	0.15
h_7	0.71	0.71	0.7	0.7	0.86
h_8	0.3	0.2	0.4	0.4	0.1
Actual grade	III	I	II	III	I

Table 4: Data of slope samples/cases.

4.3 Stability assessment of slope samples/cases based on the arctangent and tangent similarity measures

Set $G = \{G_1, G_2, G_3, G_4\} = \{I, II, III, IV\}$ as a set of four classification grades for slope stability assessment and $T = \{T_1, T_2, \dots, T_{10}\}$ as a set of ten slope samples/cases. If we consider $w = (0.35, 0.17, 0.39, 0.0115, 0.027, 0.0215, 0.005, 0.025)$ as the weight vector of the eight impact factors, the slope stability assessment method can be applied to the slope stability assessment problems with NNs.

Thus, we calculate the similarity measure between the slope sample T_s ($s = 1, 2, \dots, 10$) and the classification grade G_k ($k = 1, 2, 3, 4$) for the eight impact factors h_j ($j = 1, 2, \dots, 8$) by the following formula:

$$AT_w(T_s, G_k) = 1 - \frac{4}{\pi} \sum_{j=1}^n w_j \arctan \left[\frac{1}{2} \left(\frac{|c_{sj} + e_{sj} \inf I - c_{kj} - e_{kj} \inf I|}{|c_{sj} + e_{sj} \sup I - c_{kj} - e_{kj} \sup I|} \right) \right], \quad (5)$$

$$\text{or } T_w(T_s, G_k) = 1 - \sum_{j=1}^n \left\{ w_j \tan \left[\frac{\pi}{8} \left(\frac{|c_{sj} + e_{sj} \inf I - c_{kj} - e_{kj} \inf I|}{|c_{sj} + e_{sj} \sup I - c_{kj} - e_{kj} \sup I|} \right) \right] \right\}. \quad (6)$$

Then, the measure values of Eq. (5) or Eq. (6) are given in Tables 5 and 6. The maximum measure value indicates the corresponding assessment grade of slope stability. For the results in Tables 5 and 6, the arctangent and tangent similarity measures can carry out all the classification recognitions because all the obtained assessment grades are the same as the actual grades of the ten slope cases.

s	$AT_w(T_s, G_1)$	$AT_w(T_s, G_2)$	$AT_w(T_s, G_3)$	$AT_w(T_s, G_4)$	Assessment grade	Actual grade
1	0.6678	0.8347	0.7282	0.4592	II	II
2	0.3053	0.5455	0.7571	0.7637	IV	IV
3	0.2708	0.4988	0.6800	0.7405	IV	IV
4	0.6365	0.8428	0.7725	0.4898	II	II
5	0.7994	0.7809	0.5572	0.3041	I	I
6	0.4403	0.7076	0.8586	0.6965	III	III
7	0.7284	0.7025	0.5604	0.3194	I	I
8	0.5822	0.7972	0.7933	0.5536	II	II

Table 5: Results of the arctangent measures and the slope stability grades

s	$T_w(T_s, G_1)$	$T_w(T_s, G_2)$	$T_w(T_s, G_3)$	$T_w(T_s, G_4)$	Assessment grade	Actual grade
1	0.9963	0.9982	0.9970	0.9938	II	II
2	0.9916	0.9949	0.9973	0.9974	IV	IV
3	0.9910	0.9942	0.9965	0.9971	IV	IV
4	0.9960	0.9983	0.9975	0.9942	II	II
5	0.9978	0.9976	0.9950	0.9916	I	I
6	0.9935	0.9968	0.9985	0.9966	III	III
7	0.9969	0.9967	0.9950	0.9917	I	I
8	0.9952	0.9978	0.9977	0.9949	II	II

Table 6: Results of the tangent measures and the slope stability grades

Hence, the arctangent similarity measure and the tangent similarity measure can be suitable for handling slope stability assessment in these slope samples/cases. They demonstrate that the assessment results of slope stability corresponding to the arctangent and tangent similarity measures are the effectiveness and feasibility of the slope stability assessment method proposed in this paper.

Conclusion

Since existing assessment approaches [1-6] cannot cope with the evaluation problems of slope stability with NNs in uncertain setting, this paper put forward the arctangent and tangent similarity measures between NNs and their slope stability assessment method in NN setting. Further, Ten slope samples/cases with NN information were given to indicate the applicability of the slope stability assessment approach. It is obvious that the assessment results corresponding to the arctangent and tangent similarity measures demonstrated the same as actual grades of the ten slope cases, which indicated the effectiveness and feasibility of the developed slope stability assessment approach in NN setting. Then, the main advantages in this study are that the developed assessment approach is simpler and more effective than existing ones and can assess slope stability problems with NN information, which traditional limit equilibrium methods [1-4] and finite element methods [5,6] cannot do. For the future work, the proposed similarity measures of NNs will be extended to other applications, such as slope clustering analysis and slope failure recognition.

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A Neutrosophic Similarity Approach to Selection of Department for Student Transiting from JSS3 to SSS1 Class in Nigerian Education System

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Abstract. Single-valued neutrosophic set has been of valued importance in multi-criteria decision making problems using similarity measure. Department selection for students moving from JSS to SSS class in Nigerian Education System is such an area where decision taking has been critical as the future career of a student depends of the choice of Department in SSS class. Neutrosophic similarity measure is proposed for this department selection.

Keywords: Similarity measure, BECE, Neutrosophy, UBE, Humanities, Grade legend.

1 Introduction

In 2006, Nigeria adopted the 9-3-4 system education, Universal Basic Education (UBE), which replaces the 6-3-3-4 system of education. This implies a first nine years of Basic and compulsory education i.e. from Primary (6 years) to Junior Secondary School (JSS) (3 years), three (3) years in Senior Secondary School (SSS) and four (7) years in tertiary institution. This new arrangement organized the over-crowded nature of subjects done at basic education level [21][10]. Moving from one class to another on the first 9-year is almost automatic but transiting to next stage requires an assessment to determine who and who qualifies for the next stage in the educational system. Basic Education Certificate Examination (BECE) is written by JSS3 students at the end of this section to determine what the student does next. There are two options; which are either the SSS class or Technical school where the next three (3) years in the education system will be spent. In most of the time, the school admits the academically good students to SSS class while the remaining are referred to Technical school where they school and learn a skill.

The SSS section has four Departments viz Sciences, Humanities, Business Studies and Technology. The basic requirement for any of these departments is majorly based on the students' performance in some key subjects of BECE though the interest of the student would be additional. This problem presents a multi-factor decision which must be handled by appropriate tool for a fair decision making process. Choice making is a delicate exercise that must be carefully managed as it could involve the processes of experimentation, trial and error, decision-making and finally the decision [18]. Single-valued neutrosophic decision making model has been experimented in the choice school for children as determined by their parents. This model is based on hybridization of grey system theory and single valued neutrosophic set considering a real life scenario of five criteria in the choice of school. This model has been proved to be helpful in solving a real life problem in taking correct and appropriate decision [13].

When decision making involves selecting among various contending attributes it is known as Multi-Attributes Decision Making (MADM) and in solving problems like this there is need to involve the processes of sorting and ranking [16]. Recently in research, multi-criteria decision making (MCDM) has been gaining attention especially when it is important to select the best alternatives from list of varying list of alternatives available in relation to some predefined attributes as presented for a particular problem at hand. Decision making becomes much more difficult when alternatives are not precisely stated. This may be due to the fact that information about the attributes/criteria are vague, uncertain or indeterminate. In his work, Smarandache introduced a new philoso-

phy called ‘neutrosophy’. This new concept is able to encompass all the past theories in expressing uncertainty [20]. Neutrosophic logic has been noted to be applied in several areas of human endeavours which include but not limited to Science, Engineering, Information Technology, game theory etc [4].

Neutrosophic logic is described as more suitable to be used in decision making as compared to fuzzy and intuitionistic fuzzy logic for the fact that vagueness and impreciseness information are always needed to be put into consideration in solving uncertainty problems. These are as identified in voting for election, football games, rule of penalty etc. The concept of Smarandache about neutrosophy is nearer to human reasoning as it has a better representation of the third component which is indeterminate (neither true nor false) for uncertainty element [6]. A multi-attribute decision method based on Sine, Cosine and Cotangent similarity measures under interval rough neutrosophic environment has been developed and these new methods have been proved to be useful with some appropriate examples in decision making considering interval rough neutrosophic environment [17].

A similarity measure based on single valued neutrosophic sets has been developed. This had been demonstrated to be applicable in single valued neutrosophic multi-criteria decision making. The results as compared to existing decision making methods were better-off. This could be attributed to the fact that this new approach could automatically take into account the indeterminate information provided by decision makers [7]. In the like manner, a tangent similarity measure based multi-attribute decision making of single valued neutrosophic set has been proposed and applied to solve problem in selection of educational stream and medical diagnosis. This concept has also been suggested to be useful in other multi-valued attribute decision making problems especially of neutrosophic nature [12]. Zou and Deng have also proposed a distance function to measure similarity between two single valued neutrosophic sets. In their work, an additional achievement was a new method developed to transform the single valued neutrosophic set into probability assignment. The efficiency of the new method has been proved by applying it to a multi-criteria decision making problem [24].

Some distance and similarity measures between two interval neutrosophic sets have been defined. The measures were applied to solve multi-criteria decision making problems (MCDM). The results of these compared to the results of the existing ones especially Ye’s work was reported to be more precise and specific [23]. These proposed similarity measures are useful in real life applications of Science and Engineering such as medical diagnosis, pattern recognition, education etc [8]. The sugar selection device for diabetes patients has been analyzed using Neutrosophic TOPSIS on a MCDM problem where the method (TOPSIS) produced more realistic and reliable results than other existing MCDM techniques as evaluation was based on Spearman’s coefficient [1]. Two new algorithms for medical diagnosis have been developed using distance formulas and similarity measures. Examples have been evaluated numerically and the results thus compared with other existing methods based on normalized Hamming and normalized Euclidean distances [19]. A new framework has also been proposed with four phases for solving the problem of selection process in MCDM. This framework integrated two techniques of Analytical Network Process (ANP) and VIKOR (ViseKriterijumska Optimizacija I Kompromisno Resenje) in neutrosophic environment by using triangular neutrosophic number to present linguistic variable [2]. Neutrosophic TOPSIS method of the type 2 neutrosophic number has also been applied to the solving selection problems as the regards getting the best suppliers for importing cars [3]. Choosing the appropriate personnel for specific job is another area TOPSIS had been applied. This would go a long way in optimizing production cost and assist in meeting corporate goals [15]. While considering classical facts, Oddgram, sumSquare and set-based tri-gram similarity measures were proposed by Akinwale and Niewiadomski for evaluation of electronic text at subjective examination, retrieval of text matching from the medical database and word list. Their experiment revealed that the proposed methods as compared to existing classical methods of generalized n-gram, bi-gram and tri-gram assigned high values of similarity and performance to price with low running time. They concluded that their proposed methods are very useful in the application areas of the experiment [5].

In this paper, a new neutrosophic similarity measure is proposed in multi-criteria decision making and applied in educational sector where it concerns students’ choice of department as they transit from Junior Secondary School (JSS) to Senior Secondary School (SSS) based on their performance in BECE and interests in the various Departments available.

The rest of this paper is arranged as follows, section 2 discusses preliminaries where neutrosophic set, single-valued neutrosophic set and axioms of neutrosophic similarity measures are presented. Section 3 presents the proposed neutrosophic similarity measure for multi-criteria decision making problems and the decision model. The proposed method of application to Department selection for transition from JSS3 to SSS1 is discussed in section 4 with the associated data set. Results, discussion and evaluation of the experiment are discussed in section 5 while the conclusion is finally presented in section 6.

2 Preliminaries

In this section, some definitions of Neutrosophic set, Single-Valued Neutrosophic set (SVNS) and axioms of Neutrosophic Similarity measure are presented.

Definition 1 : Neutrosophic Set

A neutrosophic set A on the universe of discourse X is defined as:

$$A = \{ \langle x: T_A(x), I_A(x), F_A(x) \rangle, x \in X \} \quad (1)$$

where the functions $T_A, I_A, F_A : X \rightarrow]^{-}0, 1^{+}[$ and

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}. \quad (2)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. Therefore, instead of $]^{-}0, 1^{+}[$ the interval $[0, 1]$ is taken for technical applications, because $]^{-}0, 1^{+}[$ will be difficult to apply in the real applications such as in scientific and engineering problems. For example, the fact that a person could win an election could be 0.7 true, 0.2 false and 0.1 indeterminate. This presents neutrosophy in voting election result.

Definition 2: Single-Valued Neutrosophic Set (SVNS)

A single valued neutrosophic set A is denoted by $A_{SVNS} = (T_A(x), I_A(x), F_A(x))$ for any x in X [11]. For two Single-Valued Neutrosophic sets A and B , let $A_{SVNS} = \{ \langle x: T_A(x), I_A(x), F_A(x) \rangle \mid x \in X \}$ and $B_{SVNS} = \{ \langle x: T_B(x), I_B(x), F_B(x) \rangle \mid x \in X \}$ then two relations are defined in [22] as follows:

- i. $A_{SVNS} \subseteq B_{SVNS}$ if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$ (3)
- ii. $A_{SVNS} = B_{SVNS}$ if and only if $T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$ (4)

Other properties as presented in [11] are:

$$A_{SVNS} \cup B_{SVNS} = (\max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x))) \quad (5) \text{ and}$$

$$A_{SVNS} \cap B_{SVNS} = (\min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x))) \quad (6)$$

Definition 3: Axioms of Neutrosophic Similarity measure

A mapping $S(A, B): NS(x) \times NS(x) \rightarrow [0, 1]$, where $NS(x)$ denotes the set of all neutrosophic sets in $x = \{x_1, \dots, x_n\}$, is said to be the degree of similarity between A and B in [23][14][9] if it satisfies the following conditions:

$$1) 0 \leq S(A, B) \leq 1 \quad (7)$$

$$2) S(A, A) = 1 \text{ (Reflexive)} \quad (8)$$

$$2) S(A, B) = 1 \text{ if and only if } A = B, \forall A, B \in NS \text{ (Local-Reflexive)} \quad (9)$$

$$3) S(A, B) = S(B, A) \text{ (Symmetric)} \quad (10)$$

$$4) S(A, C) \leq S(A, B) \text{ and } S(A, C) \leq S(B, C) \text{ if } A \leq B \leq C \text{ for a SVNS } C. \quad (11)$$

where all x is in X . (Transitive)

3 Methodology: Neutrosophic Similarity Measure (N-Sim)

Let $A_{SVNS} = \langle x: T_A(x), I_A(x), F_A(x) \rangle$ and $B_{SVNS} = \langle x: T_B(x), I_B(x), F_B(x) \rangle$ be two single valued neutrosophic numbers, presented here (eq. 12) is the proposed neutrosophic similarity measure which decides the measure of closeness between any two entities A and B be presented as follows:

$$N-Sim(A, B) = \frac{\sum_{i=1}^n \sum_{j=1}^m \min(T_A(x_i), T_B(x_j)) + \min(I_A(x_i), I_B(x_j)) + \min(F_A(x_i), F_B(x_j))}{\sum_{i=1}^n \sum_{j=1}^m \max(T_A(x_i), T_B(x_j)) + \max(I_A(x_i), I_B(x_j)) + \max(F_A(x_i), F_B(x_j))} \quad (12)$$

Proposition 1

Suppose the proposed neutrosophic similarity **N-Sim(A,B)** satisfies the similarity measure axioms as stated in eq.(12) then::

1. $0 \leq N-Sim(A, B) \leq 1$
2. $N-Sim(A, B) = 1$ iff $A = B$

3. $N\text{-Sim}(A,B) = N\text{-Sim}(B, A)$
4. If C is a SVN in X and $A \subset B \subset C$ then $N\text{-Sim}(A,C) \leq N\text{-Sim}(A,B)$ and $N\text{-Sim}(A,C) \leq N\text{-Sim}(B,C)$

Proofs:

1. Since $T_A(x), I_A(x), F_A(x) \in [0, 1]$, then $N\text{-Sim}(A,B) \leq [0, 1]$
2. For any two SNVS A and B if $A = B$, this implies that $T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$, then

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^m \min(T_A(x_i), T_B(x_j)) + \min(I_A(x_i), I_B(x_j)) + \min(F_A(x_i), F_B(x_j)) \\ &= \sum_{i=1}^n \sum_{j=1}^m \max(T_A(x_i), T_B(x_j)) + \max(I_A(x_i), I_B(x_j)) + \max(F_A(x_i), F_B(x_j)) \end{aligned} \quad (13)$$

Thus $N\text{-Sim}(A,B) = 1$, conversely $N\text{-Sim}(B,A) = 1$

3. The proof is clear as stated in (2)
4. If $A \subset B \subset C$ then $T_A(x) \leq T_B(x) \leq T_C(x), I_A(x) \geq I_B(x) \geq I_C(x), F_A(x) \geq F_B(x) \geq F_C(x)$, then the following inequalities hold:

$$\begin{aligned} & \frac{\sum_{i=1}^n \sum_{j=1}^m \min(T_A(x_i), T_C(x_j)) + \min(I_A(x_i), I_C(x_j)) + \min(F_A(x_i), F_C(x_j))}{\sum_{i=1}^n \sum_{j=1}^m \max(T_A(x_i), T_C(x_j)) + \max(I_A(x_i), I_C(x_j)) + \max(F_A(x_i), F_C(x_j))} \leq \\ & \frac{\sum_{i=1}^n \sum_{j=1}^m \min(T_A(x_i), T_B(x_j)) + \min(I_A(x_i), I_B(x_j)) + \min(F_A(x_i), F_B(x_j))}{\sum_{i=1}^n \sum_{j=1}^m \max(T_A(x_i), T_B(x_j)) + \max(I_A(x_i), I_B(x_j)) + \max(F_A(x_i), F_B(x_j))} \end{aligned} \quad (14)$$

Thus ; $N\text{-Sim}(A,C) \leq N\text{-Sim}(A,B)$

Also,

$$\begin{aligned} & \frac{\sum_{i=1}^n \sum_{j=1}^m \min(T_A(x_i), T_C(x_j)) + \min(I_A(x_i), I_C(x_j)) + \min(F_A(x_i), F_C(x_j))}{\sum_{i=1}^n \sum_{j=1}^m \max(T_A(x_i), T_C(x_j)) + \max(I_A(x_i), I_C(x_j)) + \max(F_A(x_i), F_C(x_j))} \leq \\ & \frac{\sum_{i=1}^n \sum_{j=1}^m \min(T_B(x_i), T_C(x_j)) + \min(I_B(x_i), I_C(x_j)) + \min(F_B(x_i), F_C(x_j))}{\sum_{i=1}^n \sum_{j=1}^m \max(T_B(x_i), T_C(x_j)) + \max(I_B(x_i), I_C(x_j)) + \max(F_B(x_i), F_C(x_j))} \end{aligned} \quad (15)$$

Thus; $N\text{-Sim}(A,C) \leq N\text{-Sim}(B,C)$

3.1 Single-Valued Neutrosophic Multi-Criteria Decision Making (MCDM) model

Let $S_1, S_2, \dots, S_m$ be a discrete set of students, $C_1, C_2, \dots, C_n$ be the set of selection factors (criteria) of each student and $D_1, D_2, \dots, D_k$ are the available departments (alternatives) for each student. The ranking alternatives would be deduced by the decision-maker as it affects each student's situation. The ranking thus presents the performances of student $S(i = 1, 2, \dots, m)$ against the criteria $C_j (j = 1, 2, \dots, n)$ [12]. The different values associated with the alternatives for MCDM problem for this study are detailed out in section 4.

4 Experiment: Application of Neutrosophic Similarity Measure in Department Selection From JSS to SSS Class

BECE marks the end of compulsory basic education and the student's performance in this examination determines the actual department to put the student in the SSS section which is a basic foundation for the student's chosen career of higher institution [10]. A student's performance in BECE is graded; Distinction (A), Credit (C), Pass (P) or Fail (F). Out of maximum of ten (10) subjects offered by students in JSS, five (5) subjects are taken to be factors in consideration for the approved Departments in SSS section and the least grade expected for these subjects is Credit (C) for consideration into any desired Department. As a major factor is also the interest of concerned student in the chosen Department. The approved Departments are Science (D1), Humanities (D2), Business Studies (D3) and Technology (D4). The required JSS subject(s) to be passed at credit level of each Department is shown in Table 1.

S/NO	Department	Core Subject(s)
1	Science	(1)Mathematics and (2) Basic Science & Technology (BST)
2	Humanities	(1) English language and (2) Religion & Value Education (RVE)
3	Business Studies	(1) Business Studies (BUS)
4	Technology	(1)Mathematics and (2) Basic Science & Technology (BST)

Table 1: SSS Departments and JSS core subjects

For the purpose of this study, five (5) selection factors for any Department will be considered. These are performance in Maths and BST (C1), performance in English and RVE (C2), performance in BUS (C3), performance in English and Maths (C4) and student's interest in the Departments (C5). Figure 1 presents the proposed algorithm to determine the student's department in SSS considering these various factors in a neutrosophic environment.

Algorithm: Neutrosophic-Based Department Selection for student transiting From JSS to SSS class
<i>Inputs: (1): The grade legends for BECE result in some selected subjects (2): The neutrosophic values denoting interest of students in various departments</i>
<i>Output: Similarity values between the students and various departments</i>
<i>Procedure:</i>
<i>1. Determine of the relation between departments and selection factors</i>
<i>2. Evaluate the neutrosophic values of each student's grade in the subject</i>
<i>3. Evaluate the relation between the student and the selection factor</i>
<i>4. Determine of the similarity value between the student and the Departments using $N-Sim(A, B)$</i>
<i>5. Choose the best Department for the student as the highest value from 4</i>
<i>6. End</i>

Figure 1: Algorithm on Neutrosophic-Based Department Selection from JSS to SSS

4.1 Neutrosophication of grades legends

S/NO	GRADE LEGEND	Interpretation	Neutrosophic values
1	A	Excellent	(1.0, 0, 0)
2	C	Credit	(0.6, 0.2, 0.1)
3	P	Pass	(0.4, 0.4, 0.4)
4	F	Fail	(0, 0, 1.0)

Table 2: Neutrosophic values of grade legends of BECE result

This could be determined using the extent of goodness of these grades as mostly desired, this is as presented in table 2. The neutrosophic values for the grades were deduced with the assistance of Senoir teachers in Secondary schools. These values were based on the desired expectation to achieve success in the education sector in accordance with the set goals and objectives of education as designed by Nigerian government.

For Departments where two (2) subjects are factors, the average neutrosophic value is determined (see illustration 1). Table 3 presents the proposed Neutrosophic relation between the Departments and the selection factors as determined by Senior Teachers in Secondary schools with the guidance of 9-3-4 Nigeria Basic Education curriculum.

	C1	C2	C3	C4	C5
D1	(0.8, 0.2, 0.2)	(0.6, 0.2, 0.4)	(0.2, 0.2, 0.7)	(0.7, 0.1, 0.1)	(0.8, 0.1, 0.1)
D2	(0.4, 0.3, 0.4)	(0.8, 0.2, 0.1)	(0.4, 0.3, 0.6)	(0.7, 0.1, 0.1)	(0.7, 0.1, 0.2)
D3	(0.5, 0.3, 0.2)	(0.6, 0.3, 0.2)	(0.8, 0.2, 0.1)	(0.6, 0.2, 0.2)	(0.6, 0.2, 0.4)
D4	(0.7, 0.2, 0.2)	(0.5, 0.3, 0.3)	(0.2, 0.2, 0.7)	(0.7, 0.1, 0.1)	(0.7, 0.2, 0.3)

Table 3: Relation between Departments and Selection factors

In evaluating a scenario, there is the need to get a relationship between student and the selection factors. Table 4 depicts a relation for student and selection factors based on required subject performances (i.e C1, C2, C3 and C4). Also, for the selection factor which is based on student's interest in the department (C5), there is another table, as the student's interest for each department varies, thus the need to get a relationship between each student's interest and the various departments available as depicted in Table 5.

	C1	C2		Cn
S1	(T ₁₁ , I ₁₁ , F ₁₁)	(T ₁₂ , I ₁₂ , F ₁₂)		(T _{1n} , I _{1n} , F _{1n})

Table 4: Student and selection factors relation based on required subject performance

	D1	D2	D3	D4
C5	(T ₁₁ , I ₁₁ , F ₁₁)	(T ₁₂ , I ₁₂ , F ₁₂)	(T ₁₃ , I ₁₃ , F ₁₃)	(T ₁₄ , I ₁₄ , F ₁₄)

Table 5: Student's interest and Department relation

4.2 Illustration 1

Given the grades of a student as English (C), Maths (A), BST (C), RVE(C) and BUS (C). Also, the interests (C5) in various Departments as expressed by a particular student be given as; D1(0.7, 0.3, 0.2), D2(0.5, 0.2, 0.1), D3(0.2, 0.4, 0.2) and D4(0.1, 0.3, 0.6) as depicted in Table 5. Using Table 2, the neutrosophic representation of each required subject is thus computed based on the associated grades as; English(0.6, 0.2, 0.1), Maths(1, 0, 0), BST(0.6, 0.2, 0.1), RVE(0.4, 0.4, 0.4) and BUS(0, 0, 1). The relationship between student and selection factors would be as presented in Table 6a.

	C1	C2	C3	C4
S1	(0.8, 0.1, 0.05)	(0.6, 0.2, 0.1)	(0.6, 0.2, 0.1)	(0.8, 0.1, 0.05)

Table 6a: Student and selection factor for illustration 1

For selection factor C1, the required subjects are Maths and BST, this student's subject grade neutrosophic representation will be computed thus:

$$C1 = (\text{average}(1+0.6), \text{average}(0+0.2), \text{average}(0+0.1)) = (0.8, 0.1, 0.05).$$

C2 through C4 are also computed in the same way.

The neutrosophic values of student's interest in each Department, C5, are as expressed by each student when asked. Suppose Table 6b represents the interest of the student whose result is presented in this scenario.

	D1	D2	D3	D4
C5	(0.7, 0.3, 0.2)	(0.5, 0.2, 0.1)	(0.2, 0.4, 0.2)	(0.1, 0.3, 0.6)

Table 6b: Student and Department relation for Illustration 1

Using eq(12); thus we evaluate the similarity between this student's result with the interest and the available Departments;

$$\begin{aligned}
 N - \text{Sim}(S1, D1) &= \frac{(0.8 + 0.1 + 0.05) + (0.6 + 0.2 + 0.1) + (0.2 + 0.2 + 0.1) + (0.7 + 0.1 + 0.05) + (0.7 + 0.1 + 0.1)}{(0.8 + 0.2 + 0.2) + (0.6 + 0.2 + 0.4) + (0.6 + 0.2 + 0.7) + (0.8 + 0.1 + 0.1) + (0.8 + 0.3 + 0.2)} \\
 &= \frac{4.1}{6.2} = 0.6613
 \end{aligned}$$

Similarly, $N - \text{Sim}(S1, D2) = 0.5968$, $N - \text{Sim}(S1, D3) = 0.6333$ and $N - \text{Sim}(S1, D4) = 0.5539$

This illustration presents Science Dept (D1) as the best option as it has the highest similarity value, then Business Studies, Humanities and lastly Technology. These results use the combinations of student's BECE result grades and student's interest in the available Departments.

4.3 Data Set

The data set for this study comprises of 20 students' BECE results for consideration into SSS class. This data spanned some selected Secondary Schools in Abeokuta North Local Government of Ogun State, Nigeria. These schools comprised of both public and private schools. In order to deduce appropriate conclusion from this study, the result of this experiment showing the students' grade legends, the neutrosophic values of students' interests in all departments and the ranking of Department selection method (presented in this study) which shows the extent of recommendation were presented to 50 seasoned teachers not below the rank of Level 12 to rate on a Likert scale of five i.e. Strongly Agree, Agree, Indifferent, Disagree and Strongly Disagree. Then, percentage of each of the acceptance criteria is calculated to know the effectiveness of the proposed neutrosophic recommendation system as compared with experts' judgment. A sample of expert's judgment is shown in figure2.

5 Result, Discussion and Evaluation

BECE results and neutrosophic values of students' interest in each Department were taken as input to get the ranking of Department selection for such student using the proposed neutrosophic Similarity Measure.

S/No	Eng	Maths	Basic Sci & Tech	RVE	Bus Stud	Interest in Science (D1)	Interest in Humanities (D2)	Interest in Business Stud (D3)	Interest in Technology (D4)
1	C	A	C	C	C	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.8, 0.4, 0.2	0.1, 0.3, 0.6
2	C	A	C	C	C	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.2, 0.4, 0.2	0.1, 0.3, 0.6
3	A	P	F	A	C	0.7, 0.3, 0.2	0.6, 0.2, 0.1	0.4, 0.4, 0.2	0.1, 0.3, 0.6

Table 7: Sample of Students' results and neutrosophic interest in each Department

The neutrosophic input for table 7 is presented in table 8 with the result which reveals the order of proposed Department from most preferred to the least recommended.

Students	C1	C2	C3	C4	Interest in Science (D1)	Interest in Humanities (D2)	Interest in Business Stud (D3)	Interest in Technology (D4)	Rank of Department Selection
1	0.80, 0.10, 0.05	0.60, 0.20, 0.10	0.60, 0.20, 0.10	0.80, 0.10, 0.05	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.8, 0.4, 0.2	0.1, 0.3, 0.6	D3, D1, D2, D4
2	0.80, 0.10, 0.05	0.60, 0.20, 0.10	0.60, 0.20, 0.10	0.80, 0.10, 0.05	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.2, 0.4, 0.1	0.1, 0.3, 0.6	D1, D3, D2, D4
3	0.20, 0.20, 0.70	1.00, 0.00, 0.00	0.60, 0.20, 0.10	0.70, 0.20, 0.20	0.7, 0.3, 0.2	0.6, 0.2, 0.1	0.4, 0.4, 0.2	0.8, 0.3, 0.2	D2, D3, D1, D4

Table 8: Sample of Neutrosophic inputs and the Department rank selection output

- Evaluation

A sample of evaluation sheet is thus presented in figure 2 for this experiment.

S/No	Eng	math	Basic Sci & Tech	RVE	Bus Stud	Interest in Science (D1)	Interest in Humanities (D2)	Interest in Business (D3)	Interest in Technology (D4)	Order of dept decision rank	Strongly agree	Agree	Indifferent	Disagree	Strongly disagree
1	C	A	C	C	C	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.8, 0.4, 0.2	0.1, 0.3, 0.6	D3, D1, D2, D4	✓				
2	C	A	C	C	C	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.2, 0.4, 0.1	0.1, 0.3, 0.6	D1, D3, D2, D4		✓			
3	A	D	F	A	C	0.7, 0.3, 0.2	0.6, 0.2, 0.1	0.4, 0.4, 0.2	0.1, 0.3, 0.6	D2, D3, D1, D4	✓				
4	A	D	F	A	C	0.7, 0.3, 0.2	0.6, 0.2, 0.1	0.4, 0.4, 0.2	0.8, 0.3, 0.2	D2, D3, D1, D4	✓				
5	C	A	C	C	F	0.7, 0.3, 0.2	0.5, 0.2, 0.1	0.8, 0.4, 0.2	0.1, 0.3, 0.6	D1, D3, D2, D4		✓			
6	C	A	C	C	F	0.4, 0.3, 0.2	0.5, 0.2, 0.1	0.8, 0.4, 0.2	0.1, 0.3, 0.6	D3, D1, D2, D4				✓	
7	F	A	F	C	C	0.5, 0.3, 0.2	0.8, 0.2, 0.1	0.6, 0.2, 0.2	0.6, 0.3, 0.5	D3, D4, D2, D1	✓				
8	F	A	F	C	C	0.5, 0.3, 0.2	0.8, 0.2, 0.1	0.4, 0.2, 0.2	0.4, 0.3, 0.6	D3, D2, D4, D1	✓				
9	C	A	A	C	P	0.7, 0.2, 0.2	0.2, 0.2, 0.1	0.3, 0.2, 0.2	0.8, 0.1, 0.1	D1, D4, D2, D3	✓				
10	P	A	F	P	C	0.5, 0.3, 0.2	0.8, 0.2, 0.1	0.6, 0.2, 0.2	0.6, 0.3, 0.5	D3, D4, D2, D1		✓			

The total number and percentage of responses is presented in Table 9 from 992 responses. Considering the desirable results i.e. Strongly Agree and Agree responses a total percentage of 82.86 is obtained which presents that the proposed method is highly rated by experts to be used as a selection tool in deciding the student's Department of choice for SSS class.

S/No	Choice of response	Number or responses	%
1	Strongly Agree	385	38.81
2.	Agree	437	44.05
3.	Indifferent	73	7.36
4.	Disagree	76	7.66
5.	Strongly Disagree	21	2.12
Total		992	

Table 9: Number and percentage of responses

The new neutrosophic similarity method has been implemented using JAVA programming language embedded in NetBean IDE 8.0.1. This was analyzed on HP laptop with an Intel Pentium 2.20GHz dual core CPU and 2.00GB memory running a 64-bit Windows 10 operating system. An application was developed where grades of students in the required subjects and their interest rating were taken as inputs and the output produces the similarity value for each student with the available departments as shown in figure 3. The application also selects the best option for the Department based on analysis made and it could also save in a specified file for future reference.

Department Selection For Student in Secondary School

Enter Student Name:

BECE Grade		Rate Student's Interest in Departments		
		T	I	F
English	C			
Mathematics	A	0.7	0.3	0.2
Basic Science & Tech	C	0.5	0.2	0.1
Religious & National Value	C	0.2	0.4	0.2
Business Studies	C	0.1	0.3	0.6

Science: 0.6613 Humanities: 0.5968 Bus. Stud: 0.6333 Technology: 0.5538

Student will be suitable for Science Department

Figure 3: Sample of Automated Department Selection Process

Conclusion

In this paper, a neutrosophic similarity measure has been proposed to assist in taking decision with multi-criteria with single valued neutrosophic value set. An application on selection of Departments for students transiting from Junior Secondary school to Senior Secondary Class has been done with a high percentage of acceptance of 82.86 for the proposed method from teachers who are mostly involved in this kind of exercise. An application also developed to enhance the usage of the new method.

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Characterizations of Group Theory under Q-Neutrosophic Soft Environment

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Abstract: Neutrosophic set theory was initiated as a method to handle indeterminate uncertain data. It is identified via three independent memberships represent truth T , indeterminate I and falsity F membership degrees of an element. As a generalization of neutrosophic set theory, Q-neutrosophic set theory was established as a new hybrid model that keeps the features of Q-fuzzy soft sets which handle two-dimensional information and the features of neutrosophic soft sets in dealing with uncertainty. Different extensions of fuzzy sets have been already implemented to several algebraic structures, such as groups, symmetric groups, rings and lie algebras. Group theory is one of the most essential algebraic structures in the field of algebra. The inspiration of the current work is to broaden the idea of Q-neutrosophic soft set to group theory. In this paper the concept of Q-neutrosophic soft groups is presented. Numerous properties and basic attributes are examined. We characterize the thought of Q-level soft sets of a Q-neutrosophic soft set, which is a bridge between Q-neutrosophic soft groups and soft groups. The concept of Q-neutrosophic soft homomorphism is defined and homomorphic image and preimage of a Q-neutrosophic soft groups are investigated. Furthermore, the cartesian product of Q-neutrosophic soft groups is proposed and some relevant properties are explored.

Keywords: Group, Neutrosophic set, Neutrosophic group, Neutrosophic soft group, Q-neutrosophic set, Q-neutrosophic soft set, Soft group.

1 Introduction

Neutrosophic sets (NSs), one of the fundamental models that deal with uncertainty, first appeared in mathematics in 1998 by Smarandache [1, 2] as an extension of the concepts of the classical sets, fuzzy sets [3] and intuitionistic fuzzy sets [4]. A NS is identified via three independent membership degrees which are standard or non-standard subsets of the interval $] -0, 1+[$ where $-0 = 0 - \delta$, $1+ = 1 + \delta$; δ is an infinitesimal number. These memberships represent the degrees of truth (T), indeterminacy (I), and falsity (F). This structure makes the NS an effective common framework and empowers it to deal with indeterminate information which were not considered by fuzzy and intuitionistic fuzzy sets. Molodtsov [5] raised the notion of soft sets, based on the theory of adequate parametrization, as another approach to handle uncertain data. Since its initiation, a plenty of hybrid models of soft sets have been produced, for example, soft multi set theory [6], soft expert sets [7], fuzzy soft sets [8] and neutrosophic soft sets (NSS) [9]. Recently, NSs and NSSs were studied deeply by different researchers [10]–[19].

However, none of the above models can deal with two-dimensional indeterminate, uncertain and incompatible data. This propelled researchers to amplify them to have the capacity to deal with such circumstances, for example, Q-fuzzy soft sets [20, 21], Q-neutrosophic soft sets (Q-NSSs) [22] and Q-linguistic neutrosophic variable sets [23]. A Q-NSS is an expanded model of NSSs characterized via three two-dimensional independent membership degrees to tackle two-dimensional indeterminate issues that show up in real world. It gave an appropriate parametrization notion to handle imprecise, indeterminate and inconsistent two-dimensional information. Hence, it fits the indeterminacy and two-dimensionality simultaneously. Thus, Q-NSSs were further explored by Abu Qamar and Hassan by discussing their basic operations [24], relations [22], measures of distance, similarity and entropy [25] and also extended it further to the concept of generalized Q-neutrosophic soft expert set [26].

Hybrid models of fuzzy sets and soft sets were extensively applied in different fields of mathematics, in particular they were extremely applied in classical algebraic structures. This was started by Rosenfeld in 1971 [27] when he established the idea of fuzzy subgroup, by applying fuzzy sets to the theory of groups. Since then, the theories and approaches of fuzzy soft sets on different algebraic structures developed rapidly. Mukherjee and Bhattacharya [28] studied fuzzy groups, Sharma [29] discussed intuitionistic fuzzy groups, Aktas and Cagman [30] defined soft groups and Aygunoglu and Aygun presented the concept of fuzzy soft groups [31]. Recently, many researchers have applied different hybrid models of fuzzy sets to several algebraic structures such as groups, semigroups, rings, fields and BCK/BCI-algebras [32]-[38]. NSs and NSSs have received more attention in studying the algebraic structures dealing with uncertainty. Cetkin and Aygun [39] established the concept of neutrosophic subgroups. Bera and Mahapatra introduced neutrosophic soft groups [40], neutrosophic soft rings [41], (α, β, γ) -cut of neutrosophic soft sets and its application to neutrosophic soft groups [42] and neutrosophic normal soft groups [43]. Neutrosophic triplet groups, rings and fields and many other structures were discussed in [44, 45, 46]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. Solairaju and Nagarajan [47] introduced the notion of Q-fuzzy groups. Thiruvani and Solairaju defined the concept of neutrosophic Q-fuzzy subgroups [48], while Rasuli [49] established Q-fuzzy and anti Q-fuzzy subrings.

Inspired by the above discussion, in the present work we combine the idea of Q-NSS and group theory to conceptualize the notion of Q-neutrosophic soft groups (Q-NSGs) as a generalization of neutrosophic soft groups and soft groups; it is a new algebraic structure that deals with two-dimensional universal set under uncertain and indeterminate data. Some properties and basic characteristics are explored. Additionally, we define the Q-level soft set of a Q-NSS, which is a bridge between Q-NSGs and soft groups. The concept of Q-neutrosophic soft homomorphism (Q-NS hom) is defined and homomorphic image and preimage of a Q-NSG are investigated. Furthermore, the cartesian product of Q-NSGs is defined and some pertinent properties are examined. To clarify the novelty and originality of the proposed model a few contributions of numerous authors toward Q-NSGs are appeared in Table 1.

Table 1: Contributions toward Q-NSG.

Authors	Year	Contributions
Rosenfeld [27]	1971	Introduction of fuzzy subgroup.
Aktas and Cagman [30]	2007	Introduction of soft group.
Aygunoglu and Aygun [31]	2009	Introduction to fuzzy soft groups.
Cetkin and Aygun [39]	2015	Introduction of neutrosophic subgroup.
Bera and Mahapatra [40]	2016	Introduction of neutrosophic soft group.
Solairaju and Nagarajan [47]	2009	Introduction of Q-fuzzy group.
Thiruvani and Solairaju [48]	2018	Introduction of neutrosophic Q-fuzzy subgroup.
Abu Qamar and Hassan	This paper	Introduction of Q-NSG.

2 Preliminaries

We recall the elementary aspects of soft set, Q-NS and Q-NSS relevant to this study.

Definition 2.1. [5] A pair (f, E) is a soft set over X if f is a mapping given by $f : E \rightarrow \mathcal{P}(X)$. That is, the soft set is a parametrized family of subsets of X .

Definition 2.2. [30] A soft set (f, E) over a group G is called a soft group over G if $f(a)$ is a subgroup of G , $\forall a \in E$.

Definition 2.3. [31] A fuzzy soft set (F, E) over a group G is called a fuzzy soft group over G if $\forall a \in E$, $F(a)$ is a fuzzy subgroup of G in Rosenfeld's sense.

Abu Qamar and Hassan [22] proposed the notion of Q-neutrosophic set (Q-NS) in the following way.

Definition 2.4. [22] A Q-NS Γ_Q in X is an object of the form

$$\Gamma_Q = \left\{ \langle (s, p), T_{\Gamma_Q}(s, p), I_{\Gamma_Q}(s, p), F_{\Gamma_Q}(s, p) \rangle : s \in X, p \in Q \right\},$$

where $Q \neq \phi$ and $T_{\Gamma_Q}, I_{\Gamma_Q}, F_{\Gamma_Q} : X \times Q \rightarrow]-0, 1+[$ are the true, indeterminacy and false membership functions, respectively with $-0 \leq T_{\Gamma_Q} + I_{\Gamma_Q} + F_{\Gamma_Q} \leq 3^+$.

Definition 2.5. [22] Let X be a universal set, Q be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on X with dimension $l = 1$. A pair (Γ_Q, A) is called a Q-NSS over X , where $\Gamma_Q : A \rightarrow \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

A Q-NSS can be presented as

$$(\Gamma_Q, A) = \{(e, \Gamma_Q(e)) : e \in A, \Gamma_Q \in \mu^l QNS(X)\}.$$

Definition 2.6 ([24]). Let $(\Gamma_Q, A), (\Psi_Q, B) \in Q - NSS(X)$. Then, (Γ_Q, A) is a Q-neutrosophic soft subset of (Ψ_Q, B) , denoted by $(\Gamma_Q, A) \subseteq (\Psi_Q, B)$, if $A \subseteq B$ and $\Gamma_Q(e) \subseteq \Psi_Q(e)$ for all $e \in A$, that is $T_{\Gamma_Q(e)}(s, p) \leq T_{\Psi_Q(e)}(s, p)$, $I_{\Gamma_Q(e)}(s, p) \geq I_{\Psi_Q(e)}(s, p)$, $F_{\Gamma_Q(e)}(s, p) \geq F_{\Psi_Q(e)}(s, p)$, for all $(s, p) \in X \times Q$.

Definition 2.7. [24] The union of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Λ_Q, C) written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and $\forall c \in C, (s, p) \in X \times Q$, the membership degrees of (Λ_Q, C) are:

$$T_{\Lambda_Q(c)}(s, p) = \begin{cases} T_{\Gamma_Q(c)}(s, p) & \text{if } c \in A - B, \\ T_{\Psi_Q(c)}(s, p) & \text{if } c \in B - A, \\ \max\{T_{\Gamma_Q(c)}(s, p), T_{\Psi_Q(c)}(s, p)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_Q(c)}(s, p) = \begin{cases} I_{\Gamma_Q(c)}(s, p) & \text{if } c \in A - B, \\ I_{\Psi_Q(c)}(s, p) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_Q(c)}(s, p), I_{\Psi_Q(c)}(s, p)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_Q(c)}(s, p) = \begin{cases} F_{\Gamma_Q(c)}(s, p) & \text{if } c \in A - B, \\ F_{\Psi_Q(c)}(s, p) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_Q(c)}(s, p), F_{\Psi_Q(c)}(s, p)\} & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.8. [24] The intersection of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Ξ_Q, C) written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Xi_Q, C)$, where $C = A \cap B$ and $\forall c \in C$ and $(s, p) \in X \times Q$, the membership degrees of (Ξ_Q, C) are:

$$\begin{aligned} T_{\Xi_Q(c)}(s, p) &= \min\{T_{\Gamma_Q(c)}(s, p), T_{\Psi_Q(c)}(s, p)\}, \\ I_{\Xi_Q(c)}(s, p) &= \max\{I_{\Gamma_Q(c)}(s, p), I_{\Psi_Q(c)}(s, p)\}, \\ F_{\Xi_Q(c)}(s, p) &= \max\{F_{\Gamma_Q(c)}(s, p), F_{\Psi_Q(c)}(s, p)\}. \end{aligned}$$

Definition 2.9. [24] If (Γ_Q, A) and (Ψ_Q, B) are two Q-NSSs on X , then (Γ_Q, A) AND (Ψ_Q, B) is the Q-NSS denoted by $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ and introduced by $(\Gamma_Q, A) \wedge (\Psi_Q, B) = (\Theta_Q, A \times B)$, where $\Theta_Q(a, b) = \Gamma_Q(a) \cap \Psi_Q(b) \forall (a, b) \in A \times B$ and $(s, p) \in X \times Q$, the membership degrees of $(\Theta_Q, A \times B)$ are:

$$\begin{aligned} T_{\Theta_Q(a,b)}(s, p) &= \min\{T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p)\}, \\ I_{\Theta_Q(a,b)}(s, p) &= \max\{I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p)\}, \\ F_{\Theta_Q(a,b)}(s, p) &= \max\{F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p)\}. \end{aligned}$$

Definition 2.10. [24] If (Γ_Q, A) and (Ψ_Q, B) are two Q-NSSs on X , then (Γ_Q, A) OR (Ψ_Q, B) is the Q-NSS denoted by $(\Gamma_Q, A) \vee (\Psi_Q, B)$ and introduced by $(\Gamma_Q, A) \vee (\Psi_Q, B) = (\Upsilon_Q, A \times B)$, where $\Upsilon_Q(a, b) = \Gamma_Q(a) \cup \Psi_Q(b) \forall (a, b) \in A \times B$ and $(s, p) \in X \times Q$, the membership degrees of $(\Upsilon_Q, A \times B)$ are:

$$\begin{aligned} T_{\Upsilon_Q(a,b)}(s, p) &= \max\{T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p)\}, \\ I_{\Upsilon_Q(a,b)}(s, p) &= \min\{I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p)\}, \\ F_{\Upsilon_Q(a,b)}(s, p) &= \min\{F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p)\}. \end{aligned}$$

Definition 2.11. [24] If (Γ_Q, A) is a Q-NSS on X , then the necessity $\Box(\Gamma_Q, A)$ and the possibility $\Diamond(\Gamma_Q, A)$ operations of (Γ_Q, A) are defined as: for all $e \in A$

$$\Box(\Gamma_Q, A) = \left\{ \left\langle e, [(s, p), T_{\Gamma_Q}(s, p), I_{\Gamma_Q}(s, p), 1 - T_{\Gamma_Q}(s, p)] \right\rangle : (s, p) \in X \times Q \right\}$$

and

$$\Diamond(\Gamma_Q, A) = \left\{ \left\langle e, [(s, p), 1 - F_{\Gamma_Q}(s, p), I_{\Gamma_Q}(s, p), F_{\Gamma_Q}(s, p)] \right\rangle : (s, p) \in X \times Q \right\}.$$

3 Q-Neutrosophic soft groups

In the current section, we propose the notion of Q-NSG and investigate some related properties. In this paper G will denote a classical group.

Definition 3.1. Let (Γ_Q, A) be a Q-NSS over G . Then, (Γ_Q, A) is said to be a Q-NSG over G if for all $e \in A$, $\Gamma_Q(e)$ is a Q-neutrosophic subgroup of G , where $\Gamma_Q(e)$ is a mapping given by $\Gamma_Q(e) : G \times Q \rightarrow [0, 1]^3$.

Definition 3.2. Let (Γ_Q, A) be a Q-NSS over G . Then, (Γ_Q, A) is said to be a Q-NSG over G if for all $s, t \in G, p \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_Q(e)}(st, p) \geq \min\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\}$, $I_{\Gamma_Q(e)}(st, p) \leq \max\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\}$ and $F_{\Gamma_Q(e)}(st, p) \leq \max\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\}$.

2. $T_{\Gamma_Q(e)}(s^{-1}, p) \geq T_{\Gamma_Q(e)}(s, p)$, $I_{\Gamma_Q(e)}(s^{-1}, p) \leq I_{\Gamma_Q(e)}(s, p)$ and $F_{\Gamma_Q(e)}(s^{-1}, p) \leq F_{\Gamma_Q(e)}(s, p)$.

Example 3.3. Let $G = (\mathbb{Z}, +)$ be a group and $A = 3\mathbb{Z}$ be the parametric set. Define a Q-NSS (Γ_Q, A) as follows
for $p \in Q$ and $s, m \in \mathbb{Z}$

$$T_{\Gamma_Q(3m)}(s, p) = \begin{cases} 0.50 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{\Gamma_Q(3m)}(s, p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.20 & \text{otherwise,} \end{cases}$$

$$F_{\Gamma_Q(3m)}(s, p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.25 & \text{otherwise.} \end{cases}$$

It is clear that $(\Gamma_Q, 3\mathbb{Z})$ is a Q-NSG over G .

Theorem 3.4. Let (Γ_Q, A) be a Q-NSG over G . Then, for all $s \in G$ and $p \in Q$ the following valid:

1. $T_{\Gamma_Q(e)}(s^{-1}, p) = T_{\Gamma_Q(e)}(s, p)$, $I_{\Gamma_Q(e)}(s^{-1}, p) = I_{\Gamma_Q(e)}(s, p)$ and $F_{\Gamma_Q(e)}(s^{-1}, p) = F_{\Gamma_Q(e)}(s, p)$.
2. $T_{\Gamma_Q(e)}(\acute{e}, p) \geq T_{\Gamma_Q(e)}(s, p)$, $I_{\Gamma_Q(e)}(\acute{e}, p) \leq I_{\Gamma_Q(e)}(s, p)$ and $F_{\Gamma_Q(e)}(\acute{e}, p) \leq F_{\Gamma_Q(e)}(s, p)$.

Proof. 1. $T_{\Gamma_Q(e)}(s, p) = T_{\Gamma_Q(e)}((s^{-1})^{-1}, p) \geq T_{\Gamma_Q(e)}(s^{-1}, p)$, $I_{\Gamma_Q(e)}(s, p) = I_{\Gamma_Q(e)}((s^{-1})^{-1}, p) \leq I_{\Gamma_Q(e)}(s^{-1}, p)$, and $F_{\Gamma_Q(e)}(s, p) = F_{\Gamma_Q(e)}((s^{-1})^{-1}, p) \leq F_{\Gamma_Q(e)}(s^{-1}, p)$. Now, from Definition 3.2 the result follows.

2. For the identity element $\acute{e}$ in G

$$\begin{aligned} T_{\Gamma_Q(e)}(\acute{e}, p) &= T_{\Gamma_Q(e)}(ss^{-1}, p) \\ &\geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(s, p)\} \\ &= T_{\Gamma_Q(e)}(s, p), \\ I_{\Gamma_Q(e)}(\acute{e}, p) &= I_{\Gamma_Q(e)}(ss^{-1}, p) \\ &\leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(s, p)\} \\ &= I_{\Gamma_Q(e)}(s, p) \end{aligned}$$

and

$$\begin{aligned} F_{\Gamma_Q(e)}(\acute{e}, p) &= F_{\Gamma_Q(e)}(ss^{-1}, p) \\ &\leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(s, p)\} \\ &= F_{\Gamma_Q(e)}(s, p). \end{aligned}$$

Therefore, the result is proved. □

Theorem 3.5. A Q -NSS (Γ_Q, A) over G is a Q -NSG if and only if for all $s, t \in G, p \in Q$ and $e \in A$

1. $T_{\Gamma_Q(e)}(st^{-1}, p) \geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\},$
2. $I_{\Gamma_Q(e)}(st^{-1}, p) \leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\}$ and
3. $F_{\Gamma_Q(e)}(st^{-1}, p) \leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\}.$

Proof. Suppose that (Γ_Q, A) is a Q -NSG over G . By Definition 3.2 we have

$$\begin{aligned} T_{\Gamma_Q(e)}(st^{-1}, p) &\geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t^{-1}, p)\} \geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\}, \\ I_{\Gamma_Q(e)}(st^{-1}, p) &\leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t^{-1}, p)\} \leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\}, \\ F_{\Gamma_Q(e)}(st^{-1}, p) &\leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t^{-1}, p)\} \leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\}. \end{aligned}$$

Thus, conditions 1,2 and 3 hold.

Conversely, suppose conditions 1,2 and 3 are satisfied. We show that for each $e \in A$ (Γ_Q, A) is a Q -neutrosophic subgroup of G . From Theorem 3.4 we have $T_{\Gamma_Q(e)}(s^{-1}, p) \geq T_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(s^{-1}, p) \leq I_{\Gamma_Q(e)}(s, p)$ and $F_{\Gamma_Q(e)}(s^{-1}, p) \leq F_{\Gamma_Q(e)}(s, p)$. Next,

$$\begin{aligned} T_{\Gamma_Q(e)}(st, p) &= T_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t^{-1}, p)\} \\ &\geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\}, \\ I_{\Gamma_Q(e)}(st, p) &= I_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t^{-1}, p)\} \\ &\leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\} \end{aligned}$$

and

$$\begin{aligned} F_{\Gamma_Q(e)}(st, p) &= F_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t^{-1}, p)\} \\ &\leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.6. Let (Γ_Q, A) and (Ψ_Q, B) be two Q -NSGs over G . Then, $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ and $(\Gamma_Q, A) \cap (\Psi_Q, B)$ are also Q -NSGs over G .

Proof. We know that $(\Gamma_Q, A) \wedge (\Psi_Q, B) = (\Theta_Q, A \times B)$, where for all $(a, b) \in A \times B$ and $(s, p) \in X \times Q$

$$\begin{aligned} T_{\Theta_Q(a,b)}(s, p) &= \min \{T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p)\}, \\ I_{\Theta_Q(a,b)}(s, p) &= \max \{I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p)\}, \\ F_{\Theta_Q(a,b)}(s, p) &= \max \{F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p)\}. \end{aligned}$$

Now, since (Γ_Q, A) and (Ψ_Q, B) are Q-NSGs over G , $\forall s, t \in G, p \in Q$ and $(a, b) \in A \times B$, we get

$$\begin{aligned} T_{\Theta_Q(a,b)}(st, p) &= \min \left\{ T_{\Gamma_Q(a)}(st, p), T_{\Psi_Q(b)}(st, p) \right\} \\ &\geq \min \left\{ \min \left\{ T_{\Gamma_Q(a)}(s, p), T_{\Gamma_Q(a)}(t, p) \right\}, \min \left\{ T_{\Psi_Q(b)}(s, p), T_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \min \left\{ \min \left\{ T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p) \right\}, \min \left\{ T_{\Gamma_Q(a)}(t, p), T_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \min \left\{ T_{\Theta_Q(a,b)}(s, p), T_{\Theta_Q(a,b)}(t, p) \right\}, \end{aligned}$$

$$\begin{aligned} I_{\Theta_Q(a,b)}(st, p) &= \max \left\{ I_{\Gamma_Q(a)}(st, p), I_{\Psi_Q(b)}(st, p) \right\} \\ &\leq \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(s, p), I_{\Gamma_Q(a)}(t, p) \right\}, \max \left\{ I_{\Psi_Q(b)}(s, p), I_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p) \right\}, \max \left\{ I_{\Gamma_Q(a)}(t, p), I_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \max \left\{ I_{\Theta_Q(a,b)}(s, p), I_{\Theta_Q(a,b)}(t, p) \right\} \end{aligned}$$

and

$$\begin{aligned} F_{\Theta_Q(a,b)}(st, p) &= \max \left\{ F_{\Gamma_Q(a)}(st, p), F_{\Psi_Q(b)}(st, p) \right\} \\ &\leq \max \left\{ \max \left\{ F_{\Gamma_Q(a)}(s, p), F_{\Gamma_Q(a)}(t, p) \right\}, \max \left\{ F_{\Psi_Q(b)}(s, p), F_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \max \left\{ \max \left\{ F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p) \right\}, \max \left\{ F_{\Gamma_Q(a)}(t, p), F_{\Psi_Q(b)}(t, p) \right\} \right\} \\ &= \max \left\{ F_{\Theta_Q(a,b)}(s, p), F_{\Theta_Q(a,b)}(t, p) \right\}. \end{aligned}$$

Also,

$$\begin{aligned} T_{\Theta_Q(a,b)}(s^{-1}, p) &= \min \left\{ T_{\Gamma_Q(a)}(s^{-1}, p), T_{\Psi_Q(b)}(s^{-1}, p) \right\} \\ &\geq \min \left\{ T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p) \right\} \\ &= T_{\Theta_Q(a,b)}(s, p), \end{aligned}$$

$$\begin{aligned} I_{\Theta_Q(a,b)}(s^{-1}, p) &= \max \left\{ I_{\Gamma_Q(a)}(s^{-1}, p), I_{\Psi_Q(b)}(s^{-1}, p) \right\} \\ &\leq \max \left\{ I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p) \right\} \\ &= I_{\Theta_Q(a,b)}(s, p), \end{aligned}$$

$$\begin{aligned}
F_{\Theta_Q(a,b)}(s^{-1}, p) &= \max \left\{ F_{\Gamma_Q(a)}(s^{-1}, p), F_{\Psi_Q(b)}(s^{-1}, p) \right\} \\
&\leq \max \left\{ F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p) \right\} \\
&= F_{\Theta_Q(a,b)}(s, p).
\end{aligned}$$

This shows that $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ is a Q-NSG. The proof of $(\Gamma_Q, A) \cap (\Psi_Q, B)$ is similar to the proof of $(\Gamma_Q, A) \wedge (\Psi_Q, B)$. $\square$

Remark 3.7. For two Q-NSGs (Γ_Q, A) and (Ψ_Q, B) over G , $(\Gamma_Q, A) \cup (\Psi_Q, B)$ is not generally a Q-NSG over G .

For example, let $G = (\mathbb{Z}, +)$ and $E = 2\mathbb{Z}$. Define the two Q-NSGs (Γ_Q, E) and (Ψ_Q, E) over G as the following for $s, m \in \mathbb{Z}, p \in Q$

$$\begin{aligned}
T_{\Gamma_Q(2m)}(s, p) &= \begin{cases} 0.50 & \text{if } x = 4rm, \exists r \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \\
I_{\Gamma_Q(2m)}(s, p) &= \begin{cases} 0 & \text{if } x = 4rm, \exists r \in \mathbb{Z}, \\ 0.25 & \text{otherwise,} \end{cases}
\end{aligned}$$

$$F_{\Gamma_Q(2m)}(s, p) = \begin{cases} 0 & \text{if } x = 4rm, \exists r \in \mathbb{Z}, \\ 0.10 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned}
T_{\Psi_Q(2m)}(s, p) &= \begin{cases} 0.67 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \\
I_{\Psi_Q(3m)}(s, p) &= \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.20 & \text{otherwise,} \end{cases} \\
F_{\Psi_Q(3m)}(s, p) &= \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.17 & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, E)$. For $m = 3, s = 12, t = 18$ we have

$$T_{\Lambda_Q(6)}(12.18^{-1}, p) = T_{\Lambda_Q(6)}(-6, p) = \max \left\{ T_{\Gamma_Q(6)}(-6, p), T_{\Psi_Q(6)}(-6, p) \right\} = \max \{0, 0\} = 0$$

and

$$\begin{aligned}
&\min \left\{ T_{\Lambda_Q(6)}(12, p), T_{\Lambda_Q(6)}(18, p) \right\} \\
&= \min \left\{ \max \left\{ T_{\Gamma_Q(6)}(12, p), T_{\Psi_Q(6)}(12, p) \right\}, \max \left\{ T_{\Gamma_Q(6)}(18, p), T_{\Psi_Q(6)}(18, p) \right\} \right\} \\
&= \min \left\{ \max \{0.50, 0.67\}, \max \{0, 0.67\} \right\} \\
&= \min \{0.67, 0.67\} = 0.67.
\end{aligned}$$

Hence, $T_{\Lambda_Q(6)}(12.18^{-1}, p) = 0 < \min \{T_{\Lambda_Q(6)}(12, p), T_{\Lambda_Q(6)}(18, p)\} = 0.67$; i.e. $(\Lambda_Q, E) = (\Gamma_Q, A) \cup (\Psi_Q, B)$ is not a Q-NSG.

Theorem 3.8. *If (Γ_Q, A) is a Q-NSG over G , then $\square(\Gamma_Q, A)$ and $\diamond(\Gamma_Q, A)$ are Q-NSGs over G .*

Proof. Let (Γ_Q, A) be a Q-NSG over G . Then, for each $e \in A, s, t \in G$ and $p \in Q$ we have

$$\begin{aligned} F_{\square_{\Gamma_Q(e)}}(st^{-1}, p) &= 1 - T_{\Gamma_Q(e)}(st^{-1}, p) \\ &\leq 1 - \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\} \\ &= \max \{1 - T_{\Gamma_Q(e)}(s, p), 1 - T_{\Gamma_Q(e)}(t, p)\} \\ &= \max \{F_{\square_{\Gamma_Q(e)}}(s, p), F_{\square_{\Gamma_Q(e)}}(t, p)\}. \end{aligned}$$

Hence, $\square(\Gamma_Q, A)$ is a Q-NSG. Similarly, we can prove the second part. $\square$

Definition 3.9. Let (Γ_Q, A) be a Q-NSG over G . Let $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$. Then $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)}$ is a Q-level soft set of (Γ_Q, A) defined by

$$(\Gamma_Q, A)_{(\alpha, \beta, \gamma)} = \left\{ s \in G, p \in Q : T_{\Gamma_Q(e)}(s, p) \geq \alpha, I_{\Gamma_Q(e)}(s, p) \leq \beta, F_{\Gamma_Q(e)}(s, p) \leq \gamma \right\}$$

for all $e \in A$.

The next theorem provides a bridge between Q-NSG and soft group.

Theorem 3.10. *Let (Γ_Q, A) be a Q-NSS over G . Then, (Γ_Q, A) is a Q-NSG over G if and only if for all $\alpha, \beta, \gamma \in [0, 1]$ the Q-level soft set $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)} \neq \phi$ is a soft group over G .*

Proof. Let (Γ_Q, A) be a Q-NSG over G , $s, t \in (\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$ and $p \in Q$, for arbitrary $\alpha, \beta, \gamma \in [0, 1]$ and $e \in A$.

Then we have $T_{\Gamma_Q(e)}(s, p) \geq \alpha, I_{\Gamma_Q(e)}(s, p) \leq \beta, F_{\Gamma_Q(e)}(s, p) \leq \gamma$. Since (Γ_Q, A) is a Q-NSG over G , then we have

$$\begin{aligned} T_{\Gamma_Q(e)}(st, p) &\geq \min \{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\} \geq \{\alpha, \alpha\} = \alpha, \\ I_{\Gamma_Q(e)}(st, p) &\leq \max \{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\} \leq \{\beta, \beta\} = \beta, \\ F_{\Gamma_Q(e)}(st, p) &\leq \max \{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\} \leq \{\gamma, \gamma\} = \gamma. \end{aligned}$$

Therefore, $st \in (\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$. Furthermore $T_{\Gamma_Q(e)}(s^{-1}, p) \geq \alpha, I_{\Gamma_Q(e)}(s^{-1}, p) \leq \beta, F_{\Gamma_Q(e)}(s^{-1}, p) \leq \gamma$. So, $s^{-1} \in (\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$. Hence $(\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$ is a subgroup over $G, \forall e \in A$.

Conversely, suppose (Γ_Q, A) is not a Q-NSG over G . Then, there exists $e \in A$ such that $\Gamma_Q(e)$ is not a Q-neutrosophic subgroup of G . Then, there exist $s_1, t_1 \in G$ and $p \in Q$ such that

$$\begin{aligned} T_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) &< \min \{T_{\Gamma_Q(e)}(s_1, p), T_{\Gamma_Q(e)}(t_1, p)\}, \\ I_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) &> \max \{I_{\Gamma_Q(e)}(s_1, p), I_{\Gamma_Q(e)}(t_1, p)\} \end{aligned}$$

and

$$F_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) > \max \left\{ F_{\Gamma_Q(e)}(s_1, p), F_{\Gamma_Q(e)}(t_1, p) \right\}.$$

Let us assume that, $T_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) < \min \left\{ T_{\Gamma_Q(e)}(s_1, p), T_{\Gamma_Q(e)}(t_1, p) \right\}$. Let $T_{\Gamma_Q(e)}(s_1, p) = \alpha_1$, $T_{\Gamma_Q(e)}(t_1, p) = \alpha_2$ and $T_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) = \alpha_3$. If we take $\alpha = \min \{ \alpha_1, \alpha_2 \}$, then $s_1 t_1^{-1} \notin (\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$. But, since

$$T_{\Gamma_Q(e)}(s_1, p) = \alpha_1 \geq \min \{ \alpha_1, \alpha_2 \} = \alpha$$

and

$$T_{\Gamma_Q(e)}(t_1, p) = \alpha_2 \geq \min \{ \alpha_1, \alpha_2 \} = \alpha.$$

For $I_{\Gamma_Q(e)}(s_1, p) \leq \beta$, $I_{\Gamma_Q(e)}(t_1, p) \leq \beta$, $F_{\Gamma_Q(e)}(s_1, p) \leq \gamma$, $F_{\Gamma_Q(e)}(t_1, p) \leq \gamma$, we have $s_1, t_1 \in (\Gamma_Q(e))_{(\alpha, \beta, \gamma)}$. This contradicts with the fact that $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)}$ is a soft group over G .

Similarly, we can show that $I_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) > \max \left\{ I_{\Gamma_Q(e)}(s_1, p), I_{\Gamma_Q(e)}(t_1, p) \right\}$ and $F_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) > \max \left\{ F_{\Gamma_Q(e)}(s_1, p), F_{\Gamma_Q(e)}(t_1, p) \right\}$. $\square$

4 Homomorphism of Q-neutrosophic soft groups

In the following, we define the Q-neutrosophic soft function (Q-NS fn), and then define the image and preimage of a Q-NSS under Q-NS fn. Moreover, we define the Q-neutrosophic soft homomorphism (Q-NS hom) and prove that the homomorphic image and pre-image of a Q-NSG are also Q-NSGs.

Definition 4.1. Let $g : X \times Q \rightarrow Y \times Q$ and $h : A \rightarrow B$ be two functions where A and B are parameter sets for the sets $X \times Q$ and $Y \times Q$, respectively. Then, the pair (g, h) is called a Q-NS fn from $X \times Q$ to $Y \times Q$.

Definition 4.2. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NSSs defined over $X \times Q$ and $Y \times Q$, respectively, and (g, h) be a Q-NS fn from $X \times Q$ to $Y \times Q$. Then,

1. The image of (Γ_Q, A) under (g, h) , denoted by $(g, h)(\Gamma_Q, A)$, is a Q-NSS over $Y \times Q$ and is defined by:

$$(g, h)(\Gamma_Q, A) = \left(g(\Gamma_Q), h(A) \right) = \left\{ \left\langle b, g(\Gamma_Q)(b) : b \in h(A) \right\rangle \right\},$$

where for all $b \in h(A)$, $t \in Y$, $p \in Q$,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t, p) &= \begin{cases} \max_{g(s, p) = (t, p)} \max_{h(a) = b} [T_{\Gamma_Q(a)}(s, p)] & \text{if } (s, p) \in g^{-1}(t, p), \\ 0 & \text{otherwise,} \end{cases} \\ I_{g(\Gamma_Q)(b)}(t, p) &= \begin{cases} \min_{g(s, p) = (t, p)} \min_{h(a) = b} [I_{\Gamma_Q(a)}(s, p)] & \text{if } (s, p) \in g^{-1}(t, p), \\ 1 & \text{otherwise,} \end{cases} \\ F_{g(\Gamma_Q)(b)}(t, p) &= \begin{cases} \min_{g(s, p) = (t, p)} \min_{h(a) = b} [F_{\Gamma_Q(a)}(s, p)] & \text{if } (s, p) \in g^{-1}(t, p), \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

2. The preimage of (Ψ_Q, B) under (g, h) , denoted by $(g, h)^{-1}(\Psi_Q, B)$, is a Q-NSS over $X \times Q$ and is defined by:

$$(g, h)^{-1}(\Psi_Q, B) = \left(g^{-1}(\Psi_Q), h^{-1}(B) \right) = \left\{ \left\langle a, g^{-1}(\Psi_Q)(a) : a \in h^{-1}(B) \right\rangle \right\},$$

where, for all $a \in h^{-1}(B), s \in X, p \in Q$,

$$T_{g^{-1}(\Psi_Q)(a)}(s, p) = T_{\Psi_Q[h(a)]}(g(s, p)),$$

$$I_{g^{-1}(\Psi_Q)(a)}(s, p) = I_{\Psi_Q[h(a)]}(g(s, p)),$$

$$F_{g^{-1}(\Psi_Q)(a)}(s, p) = F_{\Psi_Q[h(a)]}(g(s, p)).$$

If g and h are injective (surjective), then (g, h) is injective (surjective).

Definition 4.3. Let (g, h) be a Q-NS fn from $X \times Q$ to $Y \times Q$. If g is a homomorphism from $X \times Q$ to $Y \times Q$, then (g, h) is said to be a Q-NS hom. If g is an isomorphism from $X \times Q$ to $Y \times Q$ and h is a one-to-one mapping from A to B , then (g, h) is said to be a Q-neutrosophic soft isomorphism.

Theorem 4.4. Let (Γ_Q, A) be a Q-NSG over a group G_1 and (g, h) be a Q-NS hom from $G_1 \times Q$ to $G_2 \times Q$. Then, $(g, h)(\Gamma_Q, A)$ is a Q-NSG over G_2 .

Proof. Let $b \in h(E), t_1, t_2 \in G_2$ and $p \in Q$. For $g^{-1}(t_1, p) = \phi$ or $g^{-1}(t_2, p) = \phi$, the proof is clear.

So, suppose there exist $s_1, s_2 \in G_1$ and $p \in Q$ such that $g(s_1, p) = (t_1, p)$ and $g(s_2, p) = (t_2, p)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1 t_2, p) &= \max_{g(s, p) = (t_1 t_2, p)} \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s, p) \right] \\ &\geq \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_1 s_2, p) \right] \\ &\geq \max_{h(a) = b} \left[\min \left\{ T_{\Gamma_Q(a)}(s_1, p), T_{\Gamma_Q(a)}(s_2, p) \right\} \right] \\ &= \min \left\{ \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_1, p) \right], \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_2, p) \right] \right\} \end{aligned}$$

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1^{-1}, p) &\geq \max_{g(s, p) = (t_1^{-1}, p)} \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s, p) \right] \\ &\geq \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_1^{-1}, p) \right] \\ &\geq \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_1, p) \right]. \end{aligned}$$

Since, the inequality is hold for each $s_1, s_2 \in G_1$ and $p \in Q$, which satisfy $g(s_1, p) = (t_1, p)$ and $g(s_2, p) = (t_2, p)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1 t_2, p) &\geq \min \left\{ \max_{g(s_1, p) = (t_1, p)} \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_1, p) \right], \max_{g(s_2, p) = (t_2, p)} \max_{h(a) = b} \left[T_{\Gamma_Q(a)}(s_2, p) \right] \right\} \\ &= \min \left\{ T_{g(\Gamma_Q)(b)}(t_1, p), T_{g(\Gamma_Q)(b)}(t_2, p) \right\}. \end{aligned}$$

Also,

$$T_{g(\Gamma_Q)(b)}(t_1^{-1}, p) \geq \max_{g(s_1, p) = (t_1, p)} \max_{h(a) = b} [T_{\Gamma_Q(a)}(s_1, p)] = T_{g(\Gamma_Q)(b)}(t_1, p).$$

Similarly, we can obtain

$$\begin{aligned} I_{g(\Gamma_Q)(b)}(t_1 t_2, p) &\leq \max \left\{ I_{g(\Gamma_Q)(b)}(t_1, p), I_{g(\Gamma_Q)(b)}(t_2, p) \right\}, I_{g(\Gamma_Q)(b)}(t_1^{-1}, p) \leq I_{g(\Gamma_Q)(b)}(t_1, p), \\ F_{g(\Gamma_Q)(b)}(t_1 t_2, p) &\leq \max \left\{ F_{g(\Gamma_Q)(b)}(t_1, p), F_{g(\Gamma_Q)(b)}(t_2, p) \right\}, F_{g(\Gamma_Q)(b)}(t_1^{-1}, p) \leq F_{g(\Gamma_Q)(b)}(t_1, p). \end{aligned}$$

This completes the proof. $\square$

Theorem 4.5. Let (Ψ_Q, B) be a Q -NSG over a group G_2 and (g, h) be a Q -NS hom from $G_1 \times Q$ to $G_2 \times Q$. Then, $(g, h)^{-1}(\Psi_Q, B)$ is a Q -NSG over G_1 .

Proof. For $a \in h^{-1}(B)$, $s_1, s_2 \in G_1$ and $p \in Q$, we have

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(s_1 s_2, p) &= T_{\Psi_Q[h(a)]}(g(s_1 s_2, p)) \\ &= T_{\Psi_Q[h(a)]}(g(s_1, p)g(s_2, p)) \\ &\geq \min \left\{ T_{\Psi_Q[h(a)]}(g(s_1, p)), T_{\Psi_Q[h(a)]}(g(s_2, p)) \right\} \\ &= \min \left\{ T_{g^{-1}(\Psi_Q)(a)}(s_1, p), T_{g^{-1}(\Psi_Q)(a)}(s_2, p) \right\} \end{aligned}$$

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(s_1^{-1}, p) &= T_{\Psi_Q[h(a)]}(g(s_1^{-1}, p)) \\ &= T_{\Psi_Q[h(a)]}(g(s_1, p)^{-1}) \\ &\geq T_{\Psi_Q[h(a)]}(g(s_1, p)) \\ &= T_{g^{-1}(\Psi_Q)(a)}(s_1, p). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} I_{g^{-1}(\Psi_Q)(a)}(s_1 s_2, p) &\leq \min \left\{ I_{g^{-1}(\Psi_Q)(a)}(s_1, p), I_{g^{-1}(\Psi_Q)(a)}(s_2, p) \right\}, \\ I_{g^{-1}(\Psi_Q)(a)}(s_1^{-1}, p) &= I_{g^{-1}(\Psi_Q)(a)}(s_1, p), \\ F_{g^{-1}(\Psi_Q)(a)}(s_1 s_2, p) &\leq \min \left\{ F_{g^{-1}(\Psi_Q)(a)}(s_1, p), F_{g^{-1}(\Psi_Q)(a)}(s_2, p) \right\}, \\ F_{g^{-1}(\Psi_Q)(a)}(s_1^{-1}, p) &= F_{g^{-1}(\Psi_Q)(a)}(s_1, p). \end{aligned}$$

Thus, the theorem is proved. $\square$

5 Cartesian product of Q-neutrosophic soft groups

In this section, we introduce the cartesian product of Q-NSGs and discuss some of its properties.

Definition 5.1. Let (Γ_Q, A) and (Ψ_Q, B) be two Q -NSGs over the groups G_1 and G_2 , respectively. Then their cartesian product is $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ where $\Omega_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a, b) \in A \times B$. Analytically,

$$\Omega_Q(a, b) = \left\{ \left\langle ((s, t), p), T_{\Omega_Q(a, b)}((s, t), p), I_{\Omega_Q(a, b)}((s, t), p), F_{\Omega_Q(a, b)}((s, t), p) \right\rangle : s \in G_1, t \in G_2, p \in Q \right\}$$

where,

$$\begin{aligned} T_{\Omega_Q(a, b)}((s, t), p) &= \min \left\{ T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(t, p) \right\}, \\ I_{\Omega_Q(a, b)}((s, t), p) &= \max \left\{ I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(t, p) \right\}, \\ F_{\Omega_Q(a, b)}((s, t), p) &= \max \left\{ F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(t, p) \right\}. \end{aligned}$$

Theorem 5.2. Let (Γ_Q, A) and (Ψ_Q, B) be two Q -NSGs over the groups G_1 and G_2 . Then their cartesian product $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ is also a Q -NSG over $G_1 \times G_2$.

Proof. Let $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ where $\Omega_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a, b) \in A \times B$. Then for $((s_1, t_1), p), ((s_2, t_2), p) \in (G_1 \times G_2) \times Q$

$$\begin{aligned} T_{\Omega_Q(a, b)}((s_1, t_1)(s_2, t_2), p) &= T_{\Omega_Q(a, b)}((s_1 s_2, t_1 t_2), p) \\ &= \min \left\{ T_{\Gamma_Q(a)}(s_1 s_2, p), T_{\Psi_Q(b)}(t_1 t_2, p) \right\} \\ &\geq \min \left\{ \min \left\{ T_{\Gamma_Q(a)}(s_1, p), T_{\Gamma_Q(a)}(s_2, p) \right\}, \min \left\{ T_{\Psi_Q(b)}(t_1, p), T_{\Psi_Q(b)}(t_2, p) \right\} \right\} \\ &= \min \left\{ \min \left\{ T_{\Gamma_Q(a)}(s_1, p), T_{\Psi_Q(b)}(t_1, p) \right\}, \min \left\{ T_{\Gamma_Q(a)}(s_2, p), T_{\Psi_Q(b)}(t_2, p) \right\} \right\} \\ &= \min \left\{ T_{\Omega_Q(a, b)}((s_1, t_1), p), T_{\Omega_Q(a, b)}((s_2, t_2), p) \right\}, \end{aligned}$$

also

$$\begin{aligned} I_{\Omega_Q(a, b)}((s_1, t_1)(s_2, t_2), p) &= I_{\Omega_Q(a, b)}((s_1 s_2, t_1 t_2), p) \\ &= \max \left\{ I_{\Gamma_Q(a)}(s_1 s_2, p), I_{\Psi_Q(b)}(t_1 t_2, p) \right\} \\ &\leq \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(s_1, p), I_{\Gamma_Q(a)}(s_2, p) \right\}, \max \left\{ I_{\Psi_Q(b)}(t_1, p), I_{\Psi_Q(b)}(t_2, p) \right\} \right\} \\ &= \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(s_1, p), I_{\Psi_Q(b)}(t_1, p) \right\}, \max \left\{ I_{\Gamma_Q(a)}(s_2, p), I_{\Psi_Q(b)}(t_2, p) \right\} \right\} \\ &= \max \left\{ I_{\Omega_Q(a, b)}((s_1, t_1), p), I_{\Omega_Q(a, b)}((s_2, t_2), p) \right\}, \end{aligned}$$

similarly, $F_{\Omega_Q(a,b)}((s_1, t_1)(s_2, t_2), p) \leq \max \left\{ F_{\Omega_Q(a,b)}((s_1, t_1), p), F_{\Omega_Q(a,b)}((s_2, t_2), p) \right\}$.

Next,

$$\begin{aligned} T_{\Omega_Q(a,b)}((s_1, t_1)^{-1}, p) &= T_{\Omega_Q(a,b)}((s_1^{-1}, t_1^{-1}), p) \\ &\geq \min \left\{ T_{\Gamma_Q(a)}(s_1^{-1}, p), T_{\Psi_Q(b)}(t_1^{-1}, p) \right\} \\ &\geq \min \left\{ T_{\Gamma_Q(a)}(s_1, p), T_{\Psi_Q(b)}(t_1, p) \right\} \\ &= T_{\Omega_Q(a,b)}((s_1, t_1), p), \end{aligned}$$

also

$$\begin{aligned} I_{\Omega_Q(a,b)}((s_1, t_1)^{-1}, p) &= I_{\Omega_Q(a,b)}((s_1^{-1}, t_1^{-1}), p) \\ &\leq \max \left\{ I_{\Gamma_Q(a)}(s_1^{-1}, p), I_{\Psi_Q(b)}(t_1^{-1}, p) \right\} \\ &\leq \max \left\{ I_{\Gamma_Q(a)}(s_1, p), I_{\Psi_Q(b)}(t_1, p) \right\} \\ &= I_{\Omega_Q(a,b)}((s_1, t_1), p), \end{aligned}$$

similarly, $F_{\Omega_Q(a,b)}((s_1, t_1)^{-1}, p) \leq F_{\Omega_Q(a,b)}((s_1, t_1), p)$. Hence, this proves that $(\Gamma_Q, A) \times (\Psi_Q, B)$ is a Q-NSG over $G_1 \times G_2$. $\square$

6 Conclusions

A Q-NSS is a NSS over two-dimensional universal set. Thus, a Q-NSS is a set with three components that can handle two-dimensional and indeterminate data simultaneously. The main goal of the current work is to utilize Q-NSSs to group theory. This study conceptualizes the notion of Q-NSGs as a new algebraic structure that deals with two-dimensional universal set. Some relevant properties and basic characteristics are explored. We define the Q-level soft set of a Q-NSS, which acts as a bridge between Q-neutrosophic soft groups and soft groups. Also, the concepts of image and preimage of a Q-NSG are investigated. Moreover, the cartesian product of Q-NSGs is discussed. The defined notion serves as the base for applying Q-NSSs to different algebraic structures such as semigroups, rings, hemirings, fields, lie subalgebras, BCK/BCI-algebras and in hyperstructure theory such as hypergroups and hyperrings following the discussion in [50, 51, 52, 53]. Moreover, these topics may be discussed using t -norm and s -norm.

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Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application

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Abstract. The purpose of MCDM is to determine the best option amongst all the probable options. Due to linguistic assessments, the traditional crisp techniques are not good to solve MCDM problems. This paper deals with the generalization of TOPSIS for neutrosophic hypersoft set primarily based issues explained in section 3. In section 4, the proposed technique is implemented. The proposed technique is easy to implement, and precise and sensible for fixing the MCDM problem with multiple-valued neutrosophic data. In the end, the applicability of the developed method, the problem of parking on which decision maker has normally vague and imprecise knowledge is used. It seems that the outcomes of these examinations are terrific.

Keywords: Uncertainties, Decision making, FNSS, FNHSS, Linguistic variable, Accuracy Function AF, TOPSIS

1 Introduction

To describe the characteristics people generally use apt values when they come across the decision-making problems. On the other hand, it is observed that in an environment of real decision making we face various complex and alterable factors and for these fuzzy expressions, the decision makers take help from the linguistic evaluations. For instance, the evaluation values are represented with the use of expressions like excellent, v. good, and good by decision makers. Zadeh [15-16] proposed a linguistic variable set to express the evaluation values. The idea of vague linguistic variables and the operational rules were devised by Xu [12]. The level of a linguistic variable just depicts the values of linguistic evaluation of a decision maker, but these can not aptly describe the vague level of decision maker particularly in the environment of linguistic evaluation. This flaw can be taken into account by adjoining the linguistic variables as well as by putting forward its other sets. For example, Ye [13] put forward an interval Neutrosophic linguistic set (INLS) and an interval Neutrosophic linguistic number (INLN); Ye [14] also found a single and multiple valued Neutrosophic linguistic set (SVNLS & MVNLS).

At the primary, soft set theory was planned by a Russian scientist [7] that was used as a standard mathematical mean to come back across the difficulty of hesitant and uncertainty. He additionally argues that however, the same theory of sentimental set is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, and applied mathematics. Neutrosophic set could be a terribly powerful tool to agitate incomplete and indeterminate data planned by F. Smarandache [10] and has attracted the eye of the many students [1], which might offer the credibleness of the given linguistic analysis worth and linguistic set can offer qualitative analysis values. Florentin [11] generalized soft set to hypersoft set by remodeling the function into a multi-attribute function, NHSS (Neutrosophic Hyper Soft Set) is additionally planned in his pioneer work.

[8] applies neutrosophic TOPSIS and AHP to reinforce the normal strategies of personal choice to realize the perfect solutions. To investigate and verify the factors influencing the choice of SCM suppliers, [2] used the neutrosophic set for deciding and analysis technique (DEMATEL). [3] offers a unique approach for estimating the sensible medical devices (SMDs) choice method in an exceedingly cluster deciding (GDM) in an exceedingly obscure call atmosphere. Neutrosophic with TOPSIS approach is applied within the decision-making method to handle the unclearness, incomplete knowledge and therefore the uncertainty, considering the selections criteria within the knowledge collected by the choice manufacturers (DMs) [3]. [4] projected a technique of the ANP method and therefore the VIKOR underneath the neutrosophic atmosphere for managing incomplete info and high order inexactitude. [9] used a neutrosophic soft set to predict FIFA 2018.

The sturdy ranking technique with neutrosophic set [5] to handle practices and performances in green supply chain management (GSCM). [6] projected T2NN, Type 2 neutrosophic number, which might accurately describe real psychological feature info.

In this paper, the generalization of TOPSIS for the neutrosophic hypersoft set is proposed. In the proposed method Fuzzy Neutrosophic Numbers FNNs are converted into crisp by using accuracy function $N(A)$.

2 Preliminaries

Linguistic Set [9]: In a crisp set, an element Y in the universe $\mathfrak{X}$ is either a member of some crisp set $\tilde{A}$ or not. It can be represented mathematically with indicator function: $\mu_{\tilde{A}}(Y) = \{1, \text{ if } Y \text{ belongs to } \tilde{A} \text{ and } 0, \text{ if } Y \text{ doesn't belong to } \tilde{A}\}$.

Fuzzy Set [10]: Fuzzy set μ in a universe $\mathfrak{X}$ is a mapping $\mu: \mathfrak{X} \rightarrow [0,1]$ which assigns a degree of membership to each element with symbol $\mu_{\tilde{A}}(y)$ such that $\mu_{\tilde{A}}(y) \in [0, 1]$.

Fuzzy Neutrosophic set: A Fuzzy Neutrosophic set FNs $\mathcal{A}$ over the universe of discourse $\mathfrak{X}$ is defined as

$$\mathcal{A} = \langle x, T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x) \rangle, \quad x \in \mathfrak{X} \text{ where } T, F, I: \mathfrak{X} \rightarrow [0, 1] \text{ \&}$$

$$0 \leq T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3.$$

Fuzzy Neutrosophic soft set: Let $\mathfrak{X}$ be the initial universal set and $\bar{E}$ be a set of parameters. Consider a non-empty set $\mathcal{A}, \mathcal{A} \subset \bar{E}$. Let $\mathcal{P}(\mathfrak{X})$ denote the set of all FNs of $\mathfrak{X}$.

Throughout this paper Fuzzy Neutrosophic soft set is denoted by FNS set / FNSS.

3 Algorithm

Let the function be

$$F: P_j \times P_k \times P_l \times \dots \times P_m \rightarrow P(\mathfrak{X}), \text{ such that } P_q = P_j, P_k, P_l, \dots, P_m$$

Where

$$P_j = p_1, p_2, p_3, \dots, p_n \quad 1 \leq j \leq n$$

$$P_k = p_1, p_2, p_3, \dots, p_n \quad 1 \leq k \leq n$$

$$P_l = p_1, p_2, p_3, \dots, p_n \quad 1 \leq l \leq n$$

⋮

$$P_m = p_1, p_2, p_3, \dots, p_n \quad 1 \leq m \leq n$$

are multiple valued neutrosophic attributes and $\mathfrak{X}$ is a universe of discourse.

Step 1: Construct a matrix of multiple-valued P_q of attributes of order $m \times n$.

$$A = [p_{qr}]_{m \times n}, \quad 1 \leq q \leq m, \quad 1 \leq r \leq n$$

Step 2: Fill the column values with zeros if multiple valued attributes are less than equal to n to form a matrix of order $m \times n$ as defined in the below example.

Step 3: Decision makers will assign fuzzy neutrosophic numbers (FNNs) to each multiple valued linguistic variables.

Step 4: Selection of the subset of NHSS.

Step 5: Conversion of fuzzy neutrosophic values of step: 4 into crisp numbers by using accuracy function $A(N)$.

$$A(N) = \left[\frac{P_{ij}}{3} \right]$$

Step 6: Calculate the relative closeness by using the TOPSIS technique of MCDM.

Step 7: Determine the rank of relative closeness by arranging in ascending order.

Remark 1: In step 2, if all values of each tuple of complete row or complete column are null, then eliminate that respective row or column.

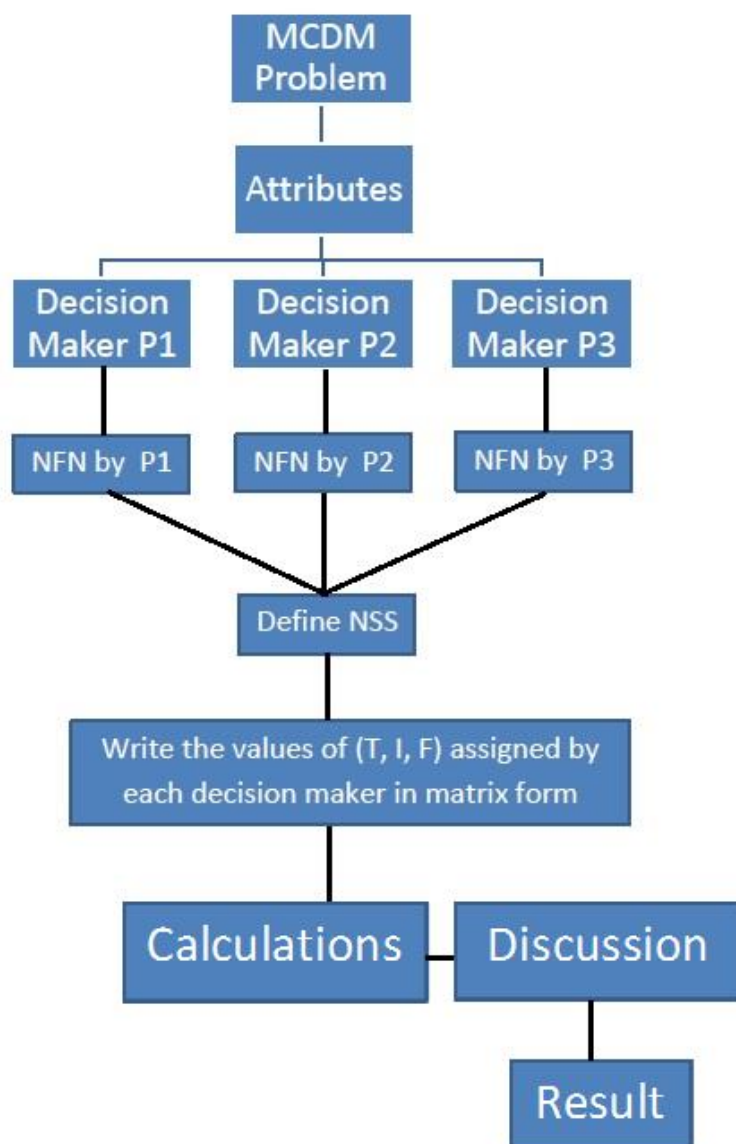


Figure 1: Algorithm design for the proposed technique

We apply the neutrosophic set theory to handle vague data, imprecise knowledge, incomplete information, and linguistic imprecision. The efficiency of the proposed method is evaluated by considering the parking problem as stated below.

The environment of decision making is a multi-criteria decision making surrounded by inconsistency and uncertainty. This paper contributes to supporting the parking problem by integrating a neutrosophic soft set with the technique for order preference by similarity to an ideal solution (TOPSIS) to illustrate an ideal solution amongst different alternatives.

4 Problem Statement

Environmental pollution strongly affects life in cities. The major issue of blockage is due to an excessive number of vehicles in the cities. This causes a major problem in finding a proper place for parking. Therefore, various techniques are implemented to cover this problem. Among them, an application of NHSS (Neutrosophic Hypersoft Set) is used.

In figure 2: there is an elaboration of the trip of a vehicle driver, to his final point. Now he has three numbers of choices to park his vehicle at different distances. So, by Using the NHSS algorithm he will be able to find the nearby spot to stand his vehicle. The driver is intended to go there in the minimum time. This work helped in the Following ways:

- Four Linguistic inputs and an output.
- During his trip how many traffic signals are sensed by the sensor?
- The measure of motor threshold on the way up to final spot is shown by PCU (Parking Car Unit) and
- The Separation between the parking slot and the final point.

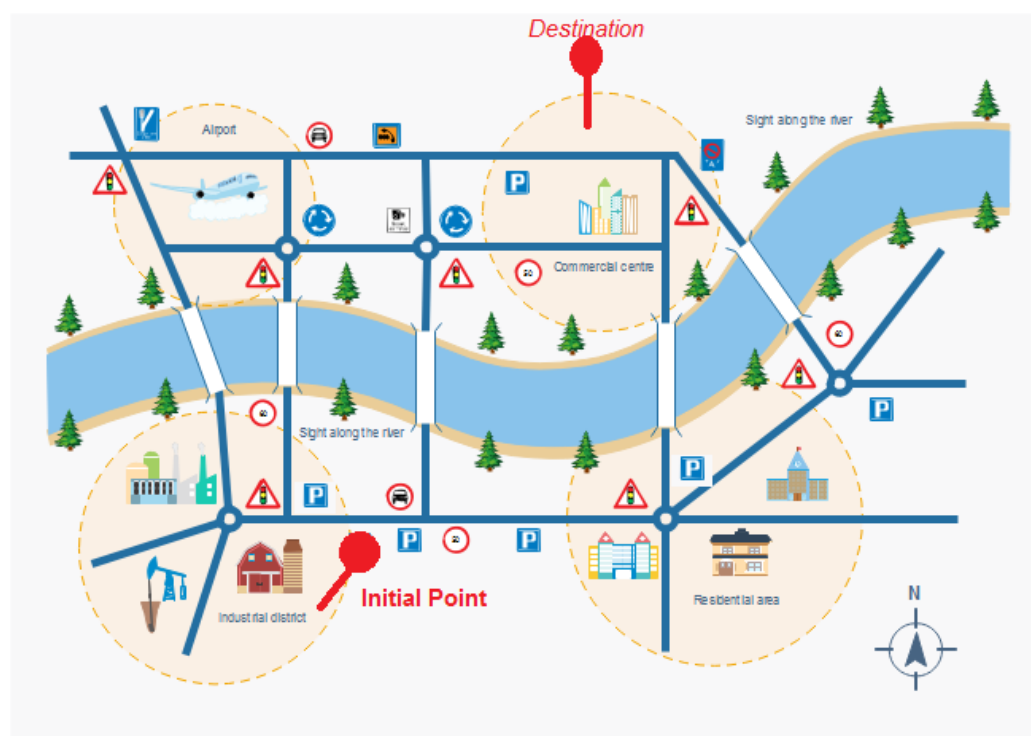


Figure 2: Initial Problem Model

5 Modelling problem into NHSS form

Due to fractional knowledge about the attributes as well as lack of information, mostly the decision makers are observed to be using certain linguistic variables instead of exact values for evaluating characteristics. In such a situation, preference information of alternatives provided by the decision makers may be vague, imprecise, or incomplete.

Sr. #	Linguistic Variable	Code	NFN 1	NFN 2	NFN 3	NFN 4
1	Normal	α	(0.4,0.1,0.0)	(0.3,0.3,0.2)	(0.7,0.2,0.3)	(1.0,1.0,1.0)
2	High	β	(0.3,0.5,0.2)	(0.1,0.1,0.1)	(0.5,0.5,0.3)	(0.5,0.3,0.5)
3	Medium	γ	(0.6,0.6,0.2)	(0.2,0.1,0.1)	(0.6,0.3,0.3)	(0.6,0.4,0.4)
4	Distance i.e., Near	N	(0.2,0.0,0.2)	(0.1,0.2,0.1)	(0.6,0.6,0.1)	(0.4,0.5,0.4)
5	Distance i.e., Far	$\bar{N}$	(0.4,0.4,0.1)	(0.1,0.3,0.4)	(0.1,0.4,0.1)	(0.4,0.2,0.1)
6	No. of Traffic Signal i.e., one	--	(0.5,0.2,0.4)	(0.5,0.4,0.3)	(0.3,0.3,0.3)	(0.6,0.6,0.2)

7	No. of Traffic Signal i.e., two	=	(0.5,0.2,0.1)	(0.5,0.2,0.1)	(0.6,0.5,0.5)	(1.0,1.0,1.0)
8	No. of Traffic Signal i.e., three	≡	(0.4,0.4,0.2)	(0.2,0.1,0.2)	(0.3,0.3,0.1)	(0.4,0.4,0.6)
9	Parking Space i.e., medium	<i>M</i>	(0.3,0.6,0.2)	(0.3,0.6,0.2)	(0.3,0.6,0.2)	(1.0,1.0,1.0)
10	Parking Space i.e., high	<i>H</i>	(0.3,0.1,0.4)	(0.3,0.3,0.3)	(0.1,0.2,0.5)	(0.6,0.6,0.2)

Table 1: Neutrosophic fuzzy number and corresponding linguistic variable.

6 Numerical calculations of problem

Let $F: P_1 \times P_2 \times P_3 \times P_4 \rightarrow \mathcal{P}(\mathcal{X})$, where $\mathcal{X}$ is the universe of discourse, such that

$$P_1 = \text{Traffic Threshold} = \{\alpha, \beta, \gamma\}$$

$$P_2 = \text{Distance of destination from initial point} = \{N, \mathfrak{N}\}$$

$$P_3 = \text{No. of traffic lights} = \{T_1, T_2, T_3\}$$

$$P_4 = \text{Distance from parking area to destination point} = \{M, h\}$$

Sr. #	PCU	Distance	No. of traffic signals	Parking space
1	A	<i>N</i>	--	<i>M</i>
2	B	$\mathfrak{N}$	=	<i>h</i>
3	γ		≡	

Table 2: Linguistic variables used in parking problem.

Consider a multiple valued neutrosophic hyper soft set $A = \{P_1, P_3, P_4\}$ such that

$$F(A) = F(\alpha, T_2, M) = \{\alpha(0.4,0.1,0.0), T_2(0.5,0.2,0.1), M(0.3,0.6,0.2), \alpha(0.7,0.3,0.2), T_2(0.6,0.5,0.5), M(0.1,0.5,0.4) \\ \alpha(1.0,1.0,1.0), T_2(1.0,1.0,1.0), M(1.0,1.0,1.0)\}$$

Step 1: Construct a matrix of multiple valued Pq of attributes of order $m \times n$.

$$\begin{bmatrix} \alpha & \beta & \gamma \\ N & \mathfrak{N} & \\ T_1 & T_2 & T_3 \\ M & h & \end{bmatrix}$$

Step 2: Fill the column values with zeros if multiple valued attributes are less than equal to n to form a matrix of order $m \times n$ as defined:

$$\begin{bmatrix} \alpha & \beta & \gamma \\ N & \mathfrak{N} & 0 \\ T_1 & T_2 & T_3 \\ M & h & 0 \end{bmatrix}$$

Step 3: The decision makers gives the values to the selected subset i.e. $F(\alpha, T_2, M)$.

$$\begin{bmatrix} (0.4,0.1,0.0) & (0.5,0.2,0.1) & (0.3,0.6,0.2) \\ (0.7,0.2,0.3) & (0.6,0.5,0.5) & (0.3,0.6,0.2) \\ (1.0,1.0,1.0) & (1.0,1.0,1.0) & (1.0,1.0,1.0) \end{bmatrix}$$

Step 4: Conversion of fuzzy neutrosophic values of step 4 into crisp numbers by using accuracy function $A(N)$.

$$\begin{bmatrix} (0.4 + 0.1 + 0.0)/3 & (0.5 + 0.2 + 0.1)/3 & (0.3 + 0.6 + 0.2)/3 \\ (0.7 + 0.2 + 0.3)/3 & (0.6 + 0.5 + 0.5)/3 & (0.3 + 0.6 + 0.2)/3 \\ (1.0 + 1.0 + 1.0)/3 & (1.0 + 1.0 + 1.0)/3 & (1.0 + 1.0 + 1.0)/3 \end{bmatrix}$$

$$\begin{bmatrix} 0.17 & 0.27 & 0.37 \\ 0.4 & 0.53 & 0.37 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 5: Now we will apply the TOPSIS on the resulting matrix.

	A	T_2	M
P_1	0.17	0.40	1
P_3	0.27	0.53	1
P_4	0.37	0.37	1

Table 3: Decision matrix of the parking problem.

Applying the technique of TOPSIS on the above-mentioned matrix obtained in step 5, the following are the results.

Si+	Si-	ci	Rank
0.177459059	0.015787	0.081693	3
0.081871695	0.117439	0.589226	2
0.084195951	0.163743	0.660417	1

Table 4: Results of calculations done by applying TOPSIS technique of MCDM

Graphical representation of the results obtained by applying the TOPSIS technique of MCDM is shown below in figure 3.

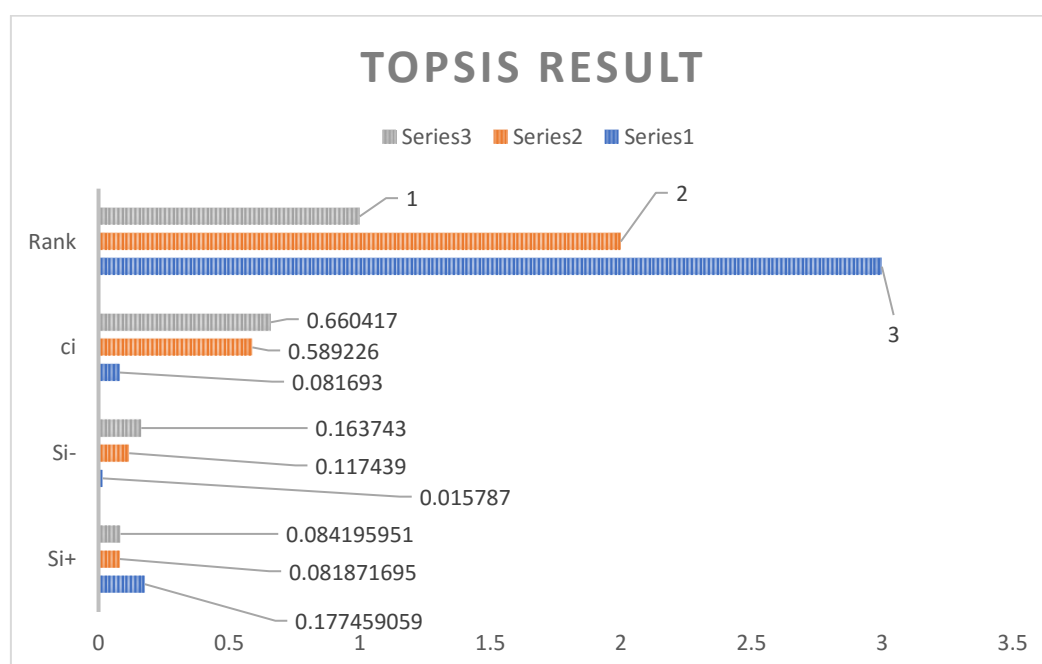


Figure 3: Graphical representation of results done by applying TOPSIS technique of MCDM

In figure 3, $P_1 = \text{series 1}$, $P_3 = \text{series 2}$ and $P_4 = \text{series 3}$ and the result shows that P_4 is the best alternative for the shortest time to reach the destination for the problem discussed above.

Conclusion

This paper introduces the Generalized Fuzzy TOPSIS by using an accuracy function for NHSS given in [4]. The proposed technique is used to solve a parking problem. Results show that the technique can be implemented to solve the MCDM problem with multiple-valued neutrosophic data in a vague and imprecise environment. In the future, the stability of the proposed technique is to be investigated and the proposed algorithm can be used in neutrosophic set (NS) theory to handle vague data, imprecise knowledge, incomplete information, and linguistic imprecision.

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Refined neutrosophic quadruple (po-)hypergroups and their fundamental group

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Abstract: After introducing the notion of hyperstructures about 80 years ago by F. Marty, a number of researches on its theory, generalization, and its applications have been done. On the other hand, the theory of Neutrosophy, the study of neutralities, was developed in 1995 by F. Smarandache as an extension of dialectics. This paper aims at finding a connection between refined neutrosophy of sets and hypergroups. In this regard, we define refined neutrosophic quadruple hypergroups, study their properties, and find their fundamental refined neutrosophic quadruple groups. Moreover, some results related to refined neutrosophic quadruple po-hypergroups are obtained.

Keywords: H_v -group, po-hypergroup, refined neutrosophic quadruple number, refined neutrosophic quadruple hypergroup, fundamental group.

1 Introduction

In 1934, Marty [19] introduced the concept of hypergroups by considering the quotient of a group by its subgroup. And this was the birth of an interesting new branch of Mathematics known as “Algebraic hyperstructures” which is considered as a generalization of algebraic structures. In algebraic structure, the composition of two elements is an element whereas in algebraic hyperstructure, the composition of two elements is a non-empty set. Since then, many different kinds of hyperstructures (hyperrings, hypermodules, hypervector spaces, ...) were introduced. And many studies were done on the theory of algebraic hyperstructures as well on their applications to various subjects of Sciences (see [12, 13, 30]). Later, in 1991, Vougioklis [28] generalized hyperstructures by introducing a larger class known as weak hyperstructures or H_v -structures. For more details about H_v -structures, see [28, 29, 30, 31].

In 1965, Zadeh [32] extended the classical notion of sets by introducing the notion of Fuzzy sets whose elements have degrees of membership. The theory of fuzzy sets is mainly concerned with the measurement of the degree of membership and non-membership of a given abstract situation. Despite its wide range of real life applications, fuzzy set theory can not be applied to models or problems that contain indeterminacy. This is the reason that arose the importance of introducing a new logic known as neutrosophic logic that contains the concept of indeterminacy. It was introduced by F. Smarandache in 1995, studied and developed by him

and by other authors. For more details about neutrosophic theory, we refer to [17, 22, 23, 25]. Recently, many authors are working on the applications of this important concept. For example in [2], Abdel-Baset et al. offered a novel approach for estimating the smart medical devices selection process in a group decision making in a vague decision environment and used neutrosophics in their methodology. Moreover, in [7], R. Alhabib et al. worked on some neutrosophic probability distribution. Other interesting applications of it are found in [1, 3, 4, 15, 20].

In 2015, Smarandache [22] introduced the concept of neutrosophic quadruple numbers and presented some basic operations on the set of neutrosophic quadruple numbers such as, addition, subtraction, multiplication, and scalar multiplication. After that, a connection between neutrosophy and algebraic structures was established where Agboola et al. [5] considered the set of neutrosophic quadruple numbers and used the defined operations on it to discuss neutrosophic quadruple algebraic structures. More results about neutrosophic algebraic structures are found in [11, 26]. A generalization of the latter work was done in 2016 where Akinleye et al. [6] considered the set of neutrosophic quadruple numbers and defined some hyperoperations on it and discussed neutrosophic quadruple hyperstructures. More specifically, the latter papers introduced the notions of neutrosophic groups, neutrosophic rings, neutrosophic hypergroups and neutrosophic hyperrings on a set of real numbers and studied their basic properties.

The authors in [9] discussed neutrosophic quadruple H_v -groups and studied their properties. Then in [10], they found the fundamental group of neutrosophic quadruple H_v -groups and proved that it is a neutrosophic quadruple group. This paper is an extension to the above mentioned results. In Section 2, some definitions related to weak hyperstructures have been presented while section 3 involves the refined neutrosophic quadruple hypergroup and the studying of its properties. As for section 4, an order on refined neutrosophic quadruple hypergroups is defined and some examples on refined neutrosophic quadruple po-hypergroups are presented. Finally, in section 5, the fundamental refined neutrosophic quadruple group of refined neutrosophic quadruple hypergroups with some important theorems, corollaries and propositions have been submitted.

2 Preliminaries

In this section, some definitions and theorems related to both: hyperstructure theory and neutrosophic theory are presented. (See [12, 13, 30].)

2.1 Basic notions of hypergroups

Definition 2.1. Let H be a non-empty set. Then, a mapping $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H , where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H . The couple $(H, \circ)$ is called a *hypergroupoid*.

In this definition, if A and B are two non-empty subsets of H and $x \in H$, then:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.2. A hypergroupoid $(H, \circ)$ is called a:

1. *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;

2. *quasi-hypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (The latter condition is called the reproduction axiom);
3. *hypergroup* if it is a semihypergroup and a quasi-hypergroup.

Definition 2.3. [13] Let $(H, \star)$ and $(K, \star')$ be two hypergroups. Then $f : H \rightarrow K$ is said to be *hypergroup homomorphism* if $f(x \star y) = f(x) \star' f(y)$ for all $x, y \in H$. $(H, \star)$ and $(K, \star')$ are called *isomorphic H_v -groups*, and written as $H \cong K$, if there exists a bijective function $f : R \rightarrow S$ that is also a homomorphism. The set of all isomorphism of $(H, \star)$ is denoted as $Aut(H)$.

T. Vougiouklis, the pioneer of H_v -structures, generalized the concept of algebraic hyperstructures to weak algebraic hyperstructures [28]. The latter concept is known as “weak” since the equality sign in the definitions of H_v -structures is more likely to be replaced by non-empty intersection. The concepts in H_v -structures are mostly used in representation theory [29].

A hypergroupoid $(H, \circ)$ is called an *H_v -semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$. An element $0 \in H$ is called an *identity* if $x \in (0 \circ x \cap x \circ 0)$ for all $x \in H$ and it is called a *scalar identity* if $x = 0 \circ x = x \circ 0$ for all $x \in H$. If the scalar identity exists then it is unique. A hypergroupoid $(H, \circ)$ is called an *H_v -group* if it is a quasi-hypergroup and an H_v -semigroup. A non empty subset S of an H_v -group $(H, \circ)$ is called *H_v -subgroup of H* if $(S, \circ)$ is an H_v -group.

Definition 2.4. [27] A hypergroup is called *cyclic* if there exist $h \in H$ such that $H = h \cup h^2 \cup \dots \cup h^i \cup \dots$ with $i \in \mathbb{N}$. If there exists $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \dots \cup h^s$ then H is a cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^s = \underbrace{h \star h \star \dots \star h}_{s \text{ times}}$.

Definition 2.5. [27] A hypergroup is called a *single power cyclic hypergroup* if there exist $h \in H$ and $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \dots \cup h^s \cup \dots$ and $h \cup h^2 \cup \dots \cup h^{m-1} \subset h^m$ for every $m \geq 1$. In this case, h is called a *generator of H* .

2.2 Refined neutrosophic quadruple hypergroups

Let T, I, F , represent the neutrosophic components truth, indeterminacy, and falsehood respectively. Symbolic (or Literal) Neutrosophic theory is referring to the use of these symbols in neutrosophics. In 2013, F. Smarandache [24] introduced the refined neutrosophic components. Where the neutrosophic literal components T, I, F can be split into respectively the following neutrosophic literal subcomponents:

$$T_1, \dots, T_p; I_1, \dots, I_r; F_1, \dots, F_s,$$

where p, r, s are positive integers with $\max\{p, r, s\} \geq 2$.

Definition 2.6. [25] Let X be a nonempty set and $p, r, s \in \mathbb{N}$ with $(p, r, s) \neq (1, 1, 1)$. A refined neutrosophic quadruple X -number is a number having the following form:

$$a + \sum_{i=1}^p b_i T_i + \sum_{j=1}^r c_j I_j + \sum_{k=1}^s b_k F_k,$$

where $a, b_i, c_j, d_k \in X$ and T, I, F have their usual neutrosophic logic meanings, and T_i, I_j, F_k are refinements of T, I, F respectively.

The set of all refined neutrosophic quadruple X -numbers is denoted by $RNQ(X)$, that is,

$$RNQ(X) = \{a + \sum_{i=1}^p b_i T_i + \sum_{j=1}^r c_j I_j + \sum_{k=1}^s d_k F_k : a, b_i, c_j, d_k \in X\}.$$

For simplicity, we write $a + \sum_{i=1}^p b_i T_i + \sum_{j=1}^r c_j I_j + \sum_{k=1}^s d_k F_k$ as

$$(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k).$$

In what follows, T_i, I_j, F_k are refinements of T, I, F respectively with $1 \leq i \leq p, 1 \leq j \leq r$ and $1 \leq k \leq s$. Let $(H, +)$ be a hypergroupoid with identity “0” and $0 + 0 = 0$ and define “ $\oplus$ ” on $RNQ(H)$ as follows:

$$\begin{aligned} & (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \oplus (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) \\ &= \{(x, \sum_{i=1}^p y_i T_i, \sum_{j=1}^r z_j I_j, \sum_{k=1}^s w_k F_k) : x \in a + a', y_i \in b_i + b'_i, z_j \in c_j + c'_j, w_k \in d_k + d'_k\}. \end{aligned}$$

3 New properties of refined neutrosophic quadruple hypergroups

In this section, refined neutrosophic quadruple hypergroups are defined and their properties are studied.

Proposition 3.1. *Let $(H, +)$ be a hypergroupoid with $0 \in H$ and T_i, I_j, F_k are refinements of T, I, F respectively. Then $(RNQ(H), \oplus)$ is a quasi-hypergroup with identity $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is a quasi-hypergroup with identity 0.*

Proof. Let $(H, +)$ be a quasi-hypergroup. We prove now that $(RNQ(H), \oplus)$ satisfies the reproduction axiom. That is, $\bar{x} \oplus RNQ(H) = RNQ(H) \oplus \bar{x} = RNQ(H)$ for all $\bar{x} = (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \in RNQ(H)$. We prove $\bar{x} \oplus RNQ(H) = RNQ(H)$ and the proof of $RNQ(H) \oplus \bar{x} = RNQ(H)$ is done in a similar manner. Let $\bar{y} = (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) \in RNQ(H)$, we have $\bar{x} \oplus \bar{y} = (a + a', \sum_{i=1}^p (b_i + b'_i) T_i, \sum_{j=1}^r (c_j + c'_j) I_j, \sum_{k=1}^s (d_k + d'_k) F_k) \subseteq RNQ(H)$ as $(a + a') \cup (b_i + b'_i) \cup (c_j + c'_j) \cup (d_k + d'_k) \subseteq H$. Thus $\bar{x} \oplus RNQ(H) \subseteq RNQ(H)$. Let $\bar{y} = (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) \in RNQ(H)$. Since $(H, +)$ satisfies the reproduction axiom and $a', b'_i, c'_j, d'_k \in H$, it follows that $a' \in a + H, b'_i \in b_i + H, c'_j \in c_j + H$ and $d'_k \in d_k + H$. The latter implies that there exist $a^*, b_i^*, c_j^*, d_k^* \in H$ such that $a' \in a + a^*, b'_i \in b_i + b_i^*, c'_j \in c_j + c_j^*$ and $d'_k \in d_k + d_k^*$. It is clear that $\bar{y} \in \bar{x} \oplus \bar{z}$ where $\bar{z} = (a^*, \sum_{i=1}^p b_i^* T_i, \sum_{j=1}^r c_j^* I_j, \sum_{k=1}^s d_k^* F_k) \in RNQ(H)$. Thus, $(RNQ(H), \oplus)$ satisfies the reproduction axiom.

Conversely, let $(RNQ(H), \oplus)$ be a quasi-hypergroup and $a \in H$. Since $0 \in H$, it follows that $\bar{a} = (a, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \in RNQ(H)$. Having $(RNQ(H), \oplus)$ a quasi-hypergroup implies that $\bar{a} \oplus RNQ(H) =$

$RNQ(H) \oplus \bar{a} = RNQ(H)$. The latter implies that $a + H = H + a = H$. $\square$

Proposition 3.2. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is a semi-hypergroup with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is a semi-hypergroup with identity element 0.*

Proof. Let $(H, +)$ be a semi-hypergroup and $\bar{x}, \bar{y}, \bar{z} \in RNQ(H)$ with

$$\bar{x} = (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k), \bar{y} = (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) \text{ and}$$

$$\bar{z} = (a'', \sum_{i=1}^p b''_i T_i, \sum_{j=1}^r c''_j I_j, \sum_{k=1}^s d''_k F_k). \text{ Having } a + (a' + a'') = (a + a') + a'', b_i + (b'_i + b''_i) = (b_i + b'_i) + b''_i, \\ c_j + (c'_j + c''_j) = (c_j + c'_j) + c''_j \text{ and } d_k + (d'_k + d''_k) = (d_k + d'_k) + d''_k \text{ implies that } \bar{x} \oplus (\bar{y} \oplus \bar{z}) = (\bar{x} \oplus \bar{y}) \oplus \bar{z}.$$

Let $(RNQ(H), \oplus)$ be a semi-hypergroup and $a, b, c \in H$. Then $\bar{a}, \bar{b}, \bar{c} \in RNQ(H)$ with $\bar{a} = (a, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$,

$$\bar{b} = (b, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \text{ and}$$

$$\bar{c} = (c, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k). \text{ Having } \bar{a} \oplus (\bar{b} \oplus \bar{c}) = (\bar{a} \oplus \bar{b}) \oplus \bar{c} \text{ implies that } a + (b + c) = (a + b) + c. \quad \square$$

Proposition 3.3. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is an H_v -semigroup with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is an H_v -semigroup with identity element 0.*

Proof. The proof is similar to that of Proposition 3.2 but instead of equality we have non-empty intersection. $\square$

Theorem 3.4. *Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus)$ is a hypergroup with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is a hypergroup with identity element 0.*

Proof. The proof is direct from Propositions 3.1 and 3.2. $\square$

Theorem 3.5. *Let $(H, +)$ be a hypergroupoid with $0 \in H$. Then $(RNQ(H), \oplus)$ is an H_v -group with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is an H_v -group with an identity element 0.*

Proof. The proof follows from Propositions 3.1 and 3.3. $\square$

Theorem 3.6. *Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus)$ is a commutative hypergroup (H_v -group) with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +)$ is a commutative hypergroup (H_v -group) with an identity 0.*

Proof. The proof is straightforward. $\square$

NOTATION 1. *Let $(H, +)$ be a hypergroup (H_v -group) with identity “0” satisfying $0+0 = 0$. Then $(RNQ(H), \oplus)$ is called a refined neutrosophic quadruple hypergroup (refined neutrosophic quadruple H_v -group).*

Corollary 3.7. Let $(H, +)$ be a hypergroup (H_v -group) containing an identity element 0 with the property that $0 + 0 = 0$. Then there are infinite number of refined neutrosophic quadruple hypergroups (H_v -groups).

Proof. Let $(H, +)$ be a hypergroup (H_v -group). Theorem 3.4 and Theorem 3.5 implies that $(RNQ(H), \oplus)$ is a neutrosophic quadruple hypergroup (H_v -group) with identity $\bar{0}$ and $\bar{0} \oplus \bar{0} = \bar{0}$. Applying Theorem 3.4 and Theorem 3.5 on $(RNQ(H), \oplus)$, we get $RNQ(RNQ(H))$ is a neutrosophic quadruple hypergroup (H_v -group). Continuing on this pattern, we get $RNQ(RNQ(\dots(RNQ(H))\dots))$ is a neutrosophic quadruple hypergroup (H_v -group). $\square$

Proposition 3.8. Let X be any set with a hyperoperation “+”. Then $RNQ(X)$ is a cyclic refined neutrosophic quadruple hypergroup if and only if X is a cyclic hypergroup with an identity element “ $0 \in X$ ” and $0 + 0 = 0$.

Proof. Let X be a cyclic hypergroup with identity “ $0 \in X$ ” and $0 + 0 = 0$. Then there exist $a \in X$ such that a is a generator of X . It is clear that $\bar{a}$ is a generator of $RNQ(X)$ where $\bar{a} = (a, \sum_{i=1}^p aT_i, \sum_{j=1}^r aI_j, \sum_{k=1}^s aF_k) \in RNQ(X)$.

Let $RNQ(X)$ be a cyclic quadruple hypergroup. Then there exist $\bar{x} \in RNQ(X)$ such that $\bar{x} = (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)$ is a generator of $RNQ(X)$. It is clear that a is a generator of X . $\square$

Example 3.9. Let T_1, T_2 be refinements of T , I_1, F_1 be refinements of I, F respectively, $H_1 = \{0, 1\}$ and define $(H_1, +_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	H_1

Since $(H_1, \oplus)$ is a commutative hypergroup with an identity 0 , it follows by Theorem 3.6 that $(RNQ(H_1), \oplus)$ is a commutative refined neutrosophic quadruple hypergroup with 32 elements and identity $\bar{0} = (0, 0T_1 + 0T_2, 0I_1, 0F_1)$. Moreover, having $H_1 = 1 +_1 1$ implies that 1 is a generator of $(H_1, +)$ and $(H_1, +)$ is a single-power cyclic hypergroup of period 2. Theorem 3.8 asserts that $(RNQ(H_1), \oplus)$ is a single power cyclic hypergroup of period 2 and the generator element is $(1, 1T_1 + 1T_2, 1I_1, 1F_1)$.

It is clear that $(1, 0T_1 + 0T_2, 1I_1, 1F_1) \oplus (1, 0T_1 + 1T_2, 0I_1, 1F_1) = \{(1, 0T_1 + 1T_2, 1I_1, 0F_1), (1, 0T_1 + 1T_2, 1I_1, 1F_1)\}$.

Definition 3.10. Let $(H, +)$ be a hypergroup (H_v -group). A subset X of $RNQ(H)$ with the property that $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \in X$ is called a *refined neutrosophic subhypergroup* (H_v -subgroup) of $RNQ(H)$ if there exists $S \subseteq H$ such that $X = RNQ(S)$ and $(X, \oplus)$ is a refined neutrosophic quadruple hypergroup (H_v -group).

Proposition 3.11. Let $(H, +)$ be a hypergroup (H_v -group) and $S \subseteq H$. A subset $X = RNQ(S) \subseteq RNQ(H)$ is a refined neutrosophic subhypergroup (H_v -subgroup) of $RNQ(H)$ if the following conditions are satisfied:

1. $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \in X$;
2. $\bar{x} \oplus X = X \oplus \bar{x} = X$ for all $\bar{x} \in X$.

Proof. The proof is straightforward. □

Theorem 3.12. Let $(H, +)$ be a hypergroup (H_v -group) with identity “0”, $S \subseteq H$ and $0 \in S$. Then $(RNQ(S), \oplus)$ is a refined neutrosophic quadruple subhypergroup (H_v -subgroup) of $(RNQ(H), \oplus)$ if and only if $(S, +)$ is a subhypergroup (H_v -subgroup) of $(H, +)$.

Proof. The proof is straightforward by applying Proposition 3.11. □

Example 3.13. Since $(H_1, +_1)$ in Example 3.9 has only two subhypergroups ($\{0\}$ and H_1), it follows by applying Theorem 3.12 that $(RNQ(H_1), \oplus)$ has only two refined neutrosophic quadruple subhypergroups: $(\{0\}, \oplus) = (RNQ(\{0\}), \oplus)$ and $(RNQ(H_1), \oplus)$.

Example 3.14. Let $H_2 = \{0, 1, 2, 3\}$ and define $(H_2, +_2)$ as follows:

$+_2$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	$\{0, 2\}$	1
3	3	0	1	2

It is clear that $(H_2, +_2)$ is a commutative H_v -group that has exactly three non-isomorphic H_v -subgroups containing 0: $\{0\}$, $\{0, 2\}$ and H_2 . We can deduce by Theorem 3.12 that $(RNQ(H_2), \oplus)$ is a commutative refined neutrosophic quadruple H_v -group and has three non-isomorphic refined neutrosophic quadruple H_v -subgroups: $RNQ(\{0\}) = \{0\}$, $RNQ(\{0, 2\})$ and $RNQ(H_2)$.

Proposition 3.15. Let $(H, +)$ be a hypergroup and $(S, +)$ be a subhypergroup of $(H, +)$ containing 0. Then $RNQ(S) \oplus RNQ(S) = RNQ(S)$.

Proof. The proof is straightforward. □

Definition 3.16. Let $(RNQ(H), \oplus_1)$ and $(RNQ(J), \oplus_2)$ be refined neutrosophic quadruple hypergroups with $0_H \in H$ and $0_J \in J$. A function $\phi : RNQ(H) \rightarrow RNQ(J)$ is called *refined neutrosophic homomorphism* if the following conditions are satisfied:

1. $\phi(0_H, \sum_{i=1}^p 0_H T_i, \sum_{j=1}^r 0_H I_j, \sum_{k=1}^s 0_H F_k) = (0_J, \sum_{i=1}^p 0_J T_i, \sum_{j=1}^r 0_J I_j, \sum_{k=1}^s 0_J F_k)$;
2. $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$ for all $x, y \in RNQ(H)$.

If ϕ is a refined neutrosophic bijective homomorphism then it is called *refined neutrosophic isomorphism* and we write $RNQ(H) \cong RNQ(J)$.

Example 3.17. Let $(H, +)$ be a hypergroup. Then the function $f : RNQ(H) \rightarrow RNQ(H)$ is an isomorphism, where $f(x) = x$ for all $x \in RNQ(H)$.

Example 3.18. Let $(H, +)$ be a hypergroup and $0 \in H$ and $f : RNQ(H) \rightarrow RNQ(H)$ be defined as follows:

$$f((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)) = (a, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k).$$

Then f is a refined neutrosophic homomorphism.

Proposition 3.19. Let $(H, +_1)$ and $(J, +_2)$ be hypergroups with $0_H \in H, 0_J \in J$. If there exist a homomorphism $f : H \rightarrow J$ with $f(0_H) = 0_J$ then there exist a refined neutrosophic homomorphism from $(RNQ(H), \oplus_1)$ to $(RNQ(J), \oplus_2)$.

Proof. Let $\phi : RNQ(H) \rightarrow RNQ(J)$ be defined as follows:

$$\phi((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)) = (f(a), \sum_{i=1}^p f(b_i)T_i, \sum_{j=1}^r f(c_j)I_j, \sum_{k=1}^s f(d_k)F_k).$$

It is clear that ϕ is a refined neutrosophic homomorphism. □

Corollary 3.20. Let $(H, +_1)$ and $(J, +_2)$ be isomorphic hypergroups with $0_H \in H, 0_J \in J$. Then $(RNQ(H), \oplus_1)$ and $(RNQ(J), \oplus_2)$ are isomorphic refined neutrosophic quadruple hypergroups.

Proof. The proof is straightforward by using Proposition 3.19. □

Definition 3.21. Let $(H, +)$ be a commutative hypergroup with an identity element “0” and $S \subseteq R$ be a subhypergroup of H . Then $(H/S, +')$ is a hypergroup with: S as an identity element and $S +' S = S$. Here “+’” is defined as follows: For all $x, y \in H$,

$$(x + S) +' (y + S) = (x + y) + S.$$

Proposition 3.22. Let $(S, +)$ be a subhypergroup of a commutative hypergroup $(H, +)$. Then $(RNQ(H/S), \oplus)$ is a hypergroup.

Proof. Since $(H, +)$ is commutative, it follows that “+’” is well defined. The proof follows from having $(H/S, +')$ a hypergroup with S as an identity, $S +' S = S$ and from Theorem 3.4. □

Proposition 3.23. Let $(S, +)$ be a subhypergroup of a commutative hypergroup $(H, +)$. Then $(RNQ(H/S), \oplus) \cong (RNQ(H)/RNQ(S), \oplus')$.

Proof. Let $g : RNQ(H)/RNQ(S) \rightarrow RNQ(H/S)$ be defined as follows:

$$\begin{aligned} g((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \oplus RNQ(S)) \\ = (a + S, \sum_{i=1}^p (b_i + S)T_i, \sum_{j=1}^r (c_j + S)I_j, \sum_{k=1}^s (d_k + S)F_k). \end{aligned}$$

Then g is a hypergroup isomorphism. This can be proved easily by applying a similar proof to that of Proposition 3.27 that was done by the authors in [9]. □

Example 3.24. Let $H_3 = \{0, 1, 2, 3, 4\}$ and define “+” on H_3 as follows: $x + y = \{x, y\}$ for all $x, y \in H_3$. It is clear that $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$ and H_3 are the only non-isomorphic subhypergroups of H_3 . By applying Proposition 3.23, we get $RNQ(H_3/\{0, 1\}) \cong RNQ(H_3)/RNQ(\{0, 1\})$, $RNQ(H_3/\{0, 1, 2\}) \cong RNQ(H_3)/RNQ(\{0, 1, 2\})$ and $RNQ(H_3/\{0, 1, 2, 3\}) \cong RNQ(H_3)/RNQ(\{0, 1, 2, 3\})$.

4 Ordered refined neutrosophic quadruple hypergroups

In this section, an order on refined neutrosophic quadruple hypergroups is defined and some examples and results on refined neutrosophic quadruple partially ordered hypergroups (po-hypergroups) are presented.

A partial order relation on a set X (Poset) is a binary relation “ $\leq$ ” on X which satisfies conditions reflexivity, antisymmetry and transitivity.

Let $(H, \leq)$ be a partial ordered set and define $(RNQ(H), \trianglelefteq)$ as follows:

$$(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \trianglelefteq (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)$$

if and only if $a \leq a'$, $b_i \leq b'_i$, $c_j \leq c'_j$ and $d_k \leq d'_k$. It is clear that $(RNQ(H), \trianglelefteq)$ is a partial ordered set.

Definition 4.1. [16] An algebraic hyperstructure $(H, \circ, \leq)$ is called a *partially ordered hypergroup* or *po-hypergroup*, if $(H, \circ)$ is a hypergroup and $\leq$ is a partial order relation on H such that the monotone condition holds as follows:

$$x \leq y \Rightarrow a \circ x \leq a \circ y \text{ for all } a, x, y \in H.$$

Let A, B be non-empty subsets of $(H, \leq)$. The inequality $A \leq B$ means that for any $a \in A$, there exist $b \in B$ such that $a \leq b$.

Theorem 4.2. Let $(H, +)$ be a hypergroupoid. Then $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(H, +, \leq)$ is a po-hypergroup with identity element 0.

Proof. Let $(H, +, \leq)$ be a po-hypergroup, $\bar{e} = (e, \sum_{i=1}^p f_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s h_k F_k) \in RNQ(H)$ and

$(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \trianglelefteq (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)$. We need to show that:

$$\bar{e} \oplus (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \trianglelefteq \bar{e} \oplus (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k).$$

Having $a \leq a'$, $b_i \leq b'_i$, $c_j \leq c'_j$, $d_k \leq d'_k$ and $(H, +, \leq)$ a po-hypergroup implies that $e + a \leq e + a'$, $f_i + b_i \leq f_i + b'_i$, $g_j + c_j \leq g_j + c'_j$ and $h_k + d_k \leq h_k + d'_k$. Let $\bar{a}^* = (a^*, \sum_{i=1}^p b_i^* T_i, \sum_{j=1}^r c_j^* I_j, \sum_{k=1}^s d_k^* F_k) \in \bar{e} \oplus (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)$. Then $a^* \in e + a$, $b_i^* \in f_i + b_i$, $c_j^* \in g_j + c_j$ and $d_k^* \in h_k + d_k$. Having $e + a \leq e + a'$, $f_i + b_i \leq f_i + b'_i$, $g_j + c_j \leq g_j + c'_j$ and $h_k + d_k \leq h_k + d'_k$ implies that there exist $a^{*'} \in e + a'$, $b_i^{*'} \in f_i + b'_i$, $c_j^{*'} \in g_j + c'_j$ and $d_k^{*'} \in h_k + d'_k$ such that $a^* \leq a^{*'}$, $b_i^* \leq b_i^{*'}$, $c_j^* \leq c_j^{*'}$ and $d_k^* \leq d_k^{*'}$. We get now that $\bar{a}^* \trianglelefteq \bar{a}^{*'}$ where $\bar{a}^{*'}$ is $(a^{*'}, \sum_{i=1}^p b_i^{*'} T_i, \sum_{j=1}^r c_j^{*'} I_j, \sum_{k=1}^s d_k^{*'} F_k)$ and $\bar{a}^{*'}$ is $\bar{e} \oplus (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)$.

Let $a, b, e \in H$ and $a \leq b$. Having $0 \leq 0$ implies that

$$(a, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \trianglelefteq (b, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k).$$

Since $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup, it follows that for

$$\bar{e} = (e, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k),$$

$$\bar{e} \oplus (a, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k) \trianglelefteq \bar{e} \oplus (b, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k).$$

It is clear that $e + a \leq e + b$. □

Corollary 4.3. Let $(H, +, \leq)$ be a po-hypergroup containing an identity element 0 with the property that $0 + 0 = 0$. Then there is infinite number of refined neutrosophic quadruple po-hypergroups.

Proof. The proof is straightforward using Theorem 4.2. □

Example 4.4. Let $H_1 = \{0, 1\}$ and define $(H_1, +_1)$ as in Example 3.9. It is clear that $(H_1, +_1, \leq)$ is a po-hypergroup. Here, the partial order relation “ $\leq$ ” is directed to the set $\{(0, 0), (1, 1)\}$. By using Theorem 4.2, we get $(RNQ(H_1), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.

Example 4.5. Let $(H, \leq)$ be any poset and define $(H, +)$ as the biset hypergroup, i.e. $x + y = \{x, y\}$ for all $x, y \in H$. Then $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.

Theorem 4.6. [16] Let $(H, \circ)$ be a hypergroup such that there exists an element $0 \in H$ and the following conditions hold:

1. $0 \circ 0 = 0$;
2. $\{0, x\} \subseteq 0 \circ x$ for all $x \in H$;
3. If $x \circ 0 = y \circ 0$ then $x = y$ for all $x, y \in H$.

Then there exist a relation “ $\leq$ ” on H such that $(H, \circ, \leq)$ is a po-hypergroup.

Heidari et al. [16], in their proof of Theorem 4.6, defined the binary relation “ $\leq$ ” on H as follows:
 $x \leq y \iff x \in y \circ 0$, for all $x, y \in H$.

Corollary 4.7. Let $(H, +)$ be a hypergroup satisfying conditions of Theorem 4.6. Then there exist a relation “ $\trianglelefteq$ ” on $RNQ(H)$ such that $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup.

Proof. The proof follows from Theorems 4.2 and 4.6. □

Example 4.8. Let $H = \{0, x, y\}$ and define “ $+$ ” by the following table:

+	0	x	y
0	0	$\{0, x\}$	H
x	$\{0, x\}$	H	H
y	H	H	H

Then $(RNQ(H), \oplus, \trianglelefteq)$ is a refined neutrosophic quadruple po-hypergroup. Here the partial order relation “ $\leq$ ” is directed to the set $\{(0, 0), (x, x), (y, y), (x, y), (0, x), (0, y)\}$ and $\trianglelefteq$ is defined in the usual way on $RNQ(H)$.

Definition 4.9. Let $(RNQ(H), \oplus_1, \trianglelefteq_1)$ and $(RNQ(J), \oplus_2, \trianglelefteq_2)$ be refined neutrosophic quadruple po-hypergroups. A function $\phi : RNQ(H) \rightarrow RNQ(J)$ is called an *ordered refined neutrosophic homomorphism* if the following conditions hold:

1. $\phi(0_H, \sum_{i=1}^p 0_H T_i, \sum_{j=1}^r p 0_H I_j, \sum_{k=1}^s 0_H F_k) = (0_J, \sum_{i=1}^p 0_J T_i, \sum_{j=1}^r 0_J I_j, \sum_{k=1}^s 0_J F_k);$
2. $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$ for all $x, y \in RNQ(H);$
3. if $x \trianglelefteq_1 y$ then $\phi(x) \trianglelefteq_2 \phi(y)$ for all $x, y \in RNQ(H).$

If ϕ is an ordered refined neutrosophic homomorphism and is bijective then it is called an *ordered refined neutrosophic isomorphism* and we say $RNQ(H)$ and $RNQ(J)$ are isomorphic refined neutrosophic quadruple po-hypergroups.

Example 4.10. Let $(H, +, \leq)$ be a po-hypergroup. Then $f : RNQ(H) \rightarrow RNQ(H)$ is an ordered refined neutrosophic isomorphism, where $f(x) = x$ for all $x \in RNQ(H).$

Example 4.11. Let $(H, +, \leq)$ be a po-hypergroup, $0 \in H$ and $f : RNQ(H) \rightarrow RNQ(H)$ be defined as follows:

$$f((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)) = (a, \sum_{i=1}^p 0 T_i, \sum_{j=1}^r 0 I_j, \sum_{k=1}^s 0 F_k).$$

Then f is an ordered refined neutrosophic homomorphism.

Example 4.12. Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be po-hypergroups, $0_H \in H, 0_J \in J$ and $g : H \rightarrow J$ be an ordered homomorphism. Then $f : RNQ(H) \rightarrow RNQ(J)$ is an ordered refined neutrosophic homomorphism. Here, f is defined as follows:

$$f((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)) = (g(a), \sum_{i=1}^p 0_J T_i, \sum_{j=1}^r 0_J I_j, \sum_{k=1}^s 0_J F_k).$$

Proposition 4.13. Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be po-hypergroups with $0_H \in H, 0_J \in J$. If there exist an ordered homomorphism $f : H \rightarrow J$ with $f(0_H) = 0_J$ then there exist an ordered refined neutrosophic homomorphism from $(RNQ(H), \oplus_1, \trianglelefteq_1)$ to $(RNQ(J), \oplus_2, \trianglelefteq_2)$.

Proof. Let $\phi : RNQ(H) \rightarrow RNQ(J)$ be defined as follows:

$$\phi((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k)) = (f(a), \sum_{i=1}^p f(b_i) T_i, \sum_{j=1}^r f(c_j) I_j, \sum_{k=1}^s f(d_k) F_k).$$

It is clear that ϕ is an ordered refined neutrosophic homomorphism. □

Corollary 4.14. Let $(H, +_1, \leq_1)$ and $(J, +_2, \leq_2)$ be isomorphic po-hypergroup with $0_H \in H, 0_J \in J$. Then $(RNQ(H), \oplus_1, \trianglelefteq_1)$ and $(RNQ(J), \oplus_2, \trianglelefteq_2)$ are isomorphic refined neutrosophic quadruple po-hypergroups.

Proof. The proof is straightforward by using Proposition 4.13. □

5 Fundamental group of refined neutrosophic quadruple hypergroups

This section presents the study of fundamental relation on refined neutrosophic quadruple hypergroups and finds their fundamental refined quadruple neutrosophic groups.

Theorem 5.1. *Let $(G, +)$ be a groupoid. Then $(RNQ(G), \oplus)$ is a group with identity element $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ if and only if $(G, +)$ is a group with identity element 0.*

Proof. It is clear that $\bar{0} = (0, \sum_{i=1}^p 0T_i, \sum_{j=1}^r 0I_j, \sum_{k=1}^s 0F_k)$ is the identity of $(RNQ(G), \oplus)$ if and only if 0 is the identity of G . Let $\bar{x} = (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \in RNQ(G)$. Then the inverse $-\bar{x} = (a, \sum_{i=1}^p (-b_i) T_i, \sum_{j=1}^r (-c_j) I_j, \sum_{k=1}^s (-d_k) F_k)$ of $\bar{x}$ exists if and only if the inverse $-y$ of y exists in G for all $y \in G$. The proof of $(RNQ(G), \oplus)$ is binary closed if and only if $(G, +)$ is binary closed is similar to that of Proposition 3.1. And the proof of $(RNQ(G), \oplus)$ is associative if and only if $(G, +)$ is associative is similar to that of Proposition 3.2. $\square$

NOTATION 2. *Let $(G, +)$ be a group with identity element “0”. Then $(RNQ(G), \oplus)$ is called refined neutrosophic quadruple group.*

Proposition 5.2. *Let G, G' be isomorphic groups. Then $RNQ(G)$ and $RNQ(G')$ are isomorphic neutrosophic quadruple groups.*

Definition 5.3. [14] For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \dots, a_n \text{ in } H \text{ such that } \{x, y\} \subseteq \prod_{i=1}^n a_i$$

Here, $\prod_{i=1}^n a_i = a_1 \circ a_2 \dots \circ a_n$. And we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H .

Koskas [18] introduced this relation as an important tool to connect hypergroups with groups. And due to its importance in connecting algebraic hyperstructures with algebraic structures, different researchers studied it on various hypergroups and some extended this definition to cover other types of hyperstructures. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β . Then β^* is called the *fundamental equivalence relation* on H and it is the smallest strongly regular relation on H . If H is a hypergroup then $\beta = \beta^*$ and H/β^* is called the *fundamental group*.

Throughout this section, β and β^* are the relation on H and β_N and β_N^* are the relations on $RNQ(H)$.

Theorem 5.4. *Let $(H, +)$ be a hypergroup with identity element “0” and $0 + 0 = 0$ and let $a, a', b_i, b'_i, c_j, c'_j, d_k, d'_k \in H$. Then*

$$(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \beta_N (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) \text{ if and only if } a\beta a', b_i\beta b'_i, c_j\beta c'_j \text{ and } d_k\beta d'_k.$$

Proof. Let $(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \beta_N (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)$. Then there exist $(a_t, \sum_{i=1}^p b_{it} T_i, \sum_{j=1}^r c_{jt} I_j, \sum_{k=1}^s d_{kt} F_k)$ with $t = 1, \dots, n$ such that

$$\{(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k), (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)\}$$

is a subset of

$$(a_1, \sum_{i=1}^p b_{i1} T_i, \sum_{j=1}^r c_{j1} I_j, \sum_{k=1}^s d_{k1} F_k) \oplus \dots \oplus (a_n, \sum_{i=1}^p b_{in} T_i, \sum_{j=1}^r c_{jn} I_j, \sum_{k=1}^s d_{kn} F_k).$$

The latter implies that $a, a' \in a_1 + \dots + a_n$, $b_i, b'_i \in b_{i1} + \dots + b_{in}$, $c_j, c'_j \in c_{j1} + \dots + c_{jn}$ and $d_k, d'_k \in d_{k1} + \dots + d_{kn}$. Thus, $a\beta a'$, $b_i\beta b'_i$, $c_j\beta c'_j$ and $d_k\beta d'_k$.

Conversely, let $a\beta a'$, $b_i\beta b'_i$, $c_j\beta c'_j$ and $d_k\beta d'_k$. Then there exist $t_1, t_2, t_3, t_4 \in \mathbb{N}$ and $x_1, \dots, x_{t_1}, y_{i1}, \dots, y_{it_2}, z_{j1}, \dots, z_{jt_3}, w_{k1}, \dots, w_{kt_4} \in H$ such that $a, a' \in x_1 + \dots + x_{t_1}$, $b_i, b'_i \in y_{i1} + \dots + y_{it_2}$, $c_j, c'_j \in z_{j1} + \dots + z_{jt_3}$ and $d_k, d'_k \in w_{k1} + \dots + w_{kt_4}$. By setting $t = \max\{t_1, t_2, t_3, t_4\}$ and $x_m = 0$ for $t_1 < m \leq t$, $y_{im} = 0$ for $t_2 < m \leq t$, $z_{jm} = 0$ for $t_3 < m \leq t$ and $w_{km} = 0$ for $t_4 < m \leq t$ and using the fact that $e \in 0 + e \cap e + 0$ for all $e \in H$, we get $a, a' \in x_1 + \dots + x_{t_1}$, $b_i, b'_i \in y_{i1} + \dots + y_{it}$, $c_j, c'_j \in z_{j1} + \dots + z_{jt}$ and $d_k, d'_k \in w_{k1} + \dots + w_{kt}$. The latter implies that $\{(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k), (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k)\}$ is a subset of $(x_1, \sum_{i=1}^p y_{i1} T_i, \sum_{j=1}^r z_{j1} I_j, \sum_{k=1}^s w_{k1} F_k) \oplus \dots \oplus (x_t, \sum_{i=1}^p y_{it} T_i, \sum_{j=1}^r z_{jt} I_j, \sum_{k=1}^s w_{kt} F_k)$. Thus,

$$(a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \beta_N (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k).$$

□

Theorem 5.5. Let $(H, +)$ be a hypergroup with identity “0” and $0 + 0 = 0$. Then $RNQ(H)/\beta_N \cong RNQ(H/\beta)$.

Proof. Let $\phi : RNQ(H)/\beta_N \rightarrow RNQ(H/\beta)$ be defined as

$$\phi(\beta_N((a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k))) = (\beta(a), \sum_{i=1}^p \beta(b_i) T_i, \sum_{j=1}^r \beta(c_j) I_j, \sum_{k=1}^s \beta(d_k) F_k).$$

Theorem 5.4 asserts that ϕ is well-defined and one-to-one. Also, it is clear that ϕ is onto. We need to show that ϕ is a group homomorphism. Let

$$\bar{a} = (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \text{ and } \bar{a'} = (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k). \text{ Since } \beta_N(\bar{a}) \boxplus \beta_N(\bar{a'}) = \beta_N(\bar{x})$$

where $\bar{x} = (x, \sum_{i=1}^p y_i T_i, \sum_{j=1}^r z_j I_j, \sum_{k=1}^s w_k F_k) \in (a, \sum_{i=1}^p b_i T_i, \sum_{j=1}^r c_j I_j, \sum_{k=1}^s d_k F_k) \oplus (a', \sum_{i=1}^p b'_i T_i, \sum_{j=1}^r c'_j I_j, \sum_{k=1}^s d'_k F_k) =$

$(a + a', \sum_{i=1}^p (b_i + b'_i)T_i, \sum_{j=1}^r (c_j + c'_j)I_j, \sum_{k=1}^s (d_k + d'_k)F_k)$, it follows that

$$\phi(\beta_N(\bar{a}) \boxplus \beta_N(\bar{a}')) = \phi(\bar{x}) = (\beta(x), \sum_{i=1}^p \beta(y_i)T_i, \sum_{j=1}^r \beta(z_j)I_j, \sum_{k=1}^s \beta(w_k)F_k).$$

Having $\beta(x) = \beta(a) \oplus \beta(a')$, $\beta(y_i) = \beta(b_i) \oplus \beta(b'_i)$, $\beta(z_j) = \beta(c_j) \oplus \beta(c'_j)$ and $\beta(w_k) = \beta(d_k) \boxplus \beta(d'_k)$ imply that $\phi(\beta_N(\bar{a}) \boxplus \beta_N(\bar{a}')) = \phi(\beta_N(\bar{a})) \oplus' \phi(\beta_N(\bar{a}'))$. $\square$

Corollary 5.6. *Let $(H, +)$ be a hypergroup with identity element “0” and $0 + 0 = 0$. If G is the fundamental group of H (up to isomorphism) then $RNQ(G)$ is the fundamental group of $RNQ(H)$ (up to isomorphism).*

Proof. The proof follows from Proposition 5.2 and Theorem 5.5. $\square$

Corollary 5.7. *Let $(H, +)$ be a hypergroup with identity element “0” and $0 + 0 = 0$. If H has a trivial fundamental group then $RNQ(H)$ has a trivial fundamental group.*

Proof. The proof is straightforward by applying Corollary 5.6. $\square$

Theorem 5.8. [8] *Every single power cyclic hypergroup has a trivial fundamental group.*

Corollary 5.9. *Let $(H, +)$ be a single power cyclic hypergroup with $0 \in H$ and $0 + 0 = 0$. Then $RNQ(H)$ has a trivial fundamental group.*

Proof. The proof follows from Corollary 5.7 and Theorem 5.8. $\square$

6 Conclusion

This paper contributed to the study of neutrosophic hyperstructures by introducing refined neutrosophic quadruple hypergroups (po-hypergroups) and determining their fundamental refined neutrosophic quadruple groups. Several interesting results related to these new hypergroups were obtained. For future work, it will be interesting to study new properties of other types of refined neutrosophic quadruple hyperstructures.

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Neutrosophic Triplet Metric Topology

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Abstract: Topology is a branch of mathematic that deals with the specific definitions given for spatial structure concepts, compares different definitions and explores the connections between the structures described on the sets. Also, neutrosophic triplet metric and neutrosophic triplet topology are a new concept in neutrosophy and they are completely different from classical structures. In this paper, we firstly study neutrosophic triplet metric topology. Furthermore, we give some new definitions and properties for neutrosophic triplet metric space, neutrosophic triplet topology and neutrosophic triplet metric topology. Thus, we obtain neutrosophic triplet metric topology using the neutrosophic triplet metric and neutrosophic triplet topology. Also, we show relationship between neutrosophic triplet metric space and neutrosophic triplet topology.

Keywords: topology, metric space, neutrosophic triplet metric space, neutrosophic triplet topology, neutrosophic triplet metric topology

1 Introduction

In 1980 Smarandache first introduced the concept of neutrosophy. In neutrosophy, there are neutrosophic set, neutrosophic logic and neutrosophic probability. Neutrosophic sets are, in fact, the generalized state of the previously described fuzzy sets [2] and intuitionistic fuzzy sets [3]. Because, unlike fuzzy sets and intuitionistic fuzzy sets, in neutrosophic sets, truth value (t), falsity value (f) and indeterminacy value (i) are completely independent of each other. Therefore, neutrosophic sets are more useful in coping with uncertainties. For this reason, many researchers have done many studies on neutrosophic structures [3 - 15].

Smarandache and Ali defined neutrosophic triplet (NT) sets and neutrosophic triplet (NT) groups [16] in 2016 and gave the properties of these structures. In order for a set to be a NT set, for each “n” element in this set must have a neutral element and an anti element. The neutral element does not have to be just one for all elements as in the classical group and must be different from the classical identity element. So there may be more than one neutral element in NT sets. Also, a “n” element of a NT set is shown in the form of $\langle n, \text{neut}(n), \text{anti}(n) \rangle$. Therefore, NT structures are different from classical structures. Also, many researchers have introduced NT structures.

Recently, Ali and Smarandache studied neutrosophic triplet ring and neutrosophic triplet field [17]; Şahin and Kargın obtained neutrosophic triplet normed space [18]; Şahin and Kargın introduced neutrosophic triplet inner product space [19]; Smarandache, Şahin and Kargın studied neutrosophic Triplet G- Module [20]; Bal, Shalla and Olgun obtained neutrosophic triplet cosets and quotient groups [21]; Şahin, Kargın and Çoban introduced fixed point theorem for neutrosophic triplet partial metric space [22]; Şahin and Kargın neutrosophic triplet v – generalized metric space [23]; Çelik, Shalla and Olgun studied fundamental homomorphism theorems for neutrosophic extended triplet groups [24].

Topology is a branch of mathematic that deals with the specific definitions given for spatial structure concepts, compares different definitions and explores the connections between the structures described on the sets. In mathematics it is a large area of study with many more specific subfields. Subfields of topology include algebraic topology, geometric topology, differential topology, manifold topology.

Topology has many different application areas in mathematic. For example, a curve, a surface, a family of curves, a set of functions or a metric space can be a topological space. Also, the topology has been studied on neutrosophic set, fuzzy set, intuitionistic fuzzy set and soft set. Many researchers have introduced the topology in [25-32]. Furthermore, Şahin, Kargın and Smarandache obtained NT topology [33].

In this paper, we obtain NT metric topology. In Section 2; we give definitions of NT set [16], NT metric space [18] and NT topology [33]. In Section 3, we obtain some properties of NT topology. We define base for NT topology. In Section 4, we obtain some properties of NT metric space. We define NT open balls for NT metric

space and isometric NT metric spaces. We define NT metric topological space. Also, we show relationship between NT metric spaces and NT topology. In Section 5, we give conclusions.

2 Preliminaries

In this section, we give definition of NT sets [16], NT metric space [18], NT topology [33] and NT open sets [33]. We give new properties and definitions for NT topology and NT metric space using these definitions. Also, we firstly obtain NT metric topology using the NT metric space and NT topology.

In this paper, we show neutrosophic triplet briefly with NT.

Definition 2.1: [16]: Let $\#$ be a binary operation. A NT set $(X, \#)$ is a set such that for $x \in X$,

- i) There is a neutral of “ x ” = $\text{neut}(x)$ such that $x\#\text{neut}(x) = \text{neut}(x)\#x = x$, for every $x \in X$;
- ii) There is an anti of “ x ” = $\text{anti}(x)$ such that $x\#\text{anti}(x) = \text{anti}(x)\#x = \text{neut}(x)$, for every $x \in X$;

Also, an element “ x ” is showed with $(x, \text{neut}(x), \text{anti}(x))$.

Furthermore, $\text{neut}(x)$ must different from classical identity element.

Definition 2.2: [18] A NT metric on a NT set $(N, *)$ is a function $d: N \times N \rightarrow \mathbb{R}$ such for every $n, m, s \in N$,

- i) $n * m \in N$
- ii) $d(n, m) \geq 0$
- iii) If $n = m$, then $d(n, m) = 0$
- iv) $d(n, m) = d(m, n)$
- v) If there is at least an element $s \in N$ for each $n, m \in N$ pair such that $d(n, m) \leq d(n, m * \text{neut}(s))$, then $d(n, m * \text{neut}(s)) \leq d(n, s) + d(s, m)$.

Definition 2.3: [33] Let $(X, *)$ be a NT set, $P(X)$ be set family of each subset of X and $\mathcal{T}$ be a subset family of $P(X)$. If $\mathcal{T}$ and X are satisfied the following conditions, then $\mathcal{T}$ is called a NT topology on X .

- i) $A * B \in \mathcal{T}$, for every $A, B \in \mathcal{T}$
 - ii) $\emptyset, X \in \mathcal{T}$
 - iii) For $\forall i \in K$, If $A_i \in \mathcal{T}$, then $\bigcup_{i \in K} A_i \in \mathcal{T}$
 - iv) For $\forall i \in K$ (K is finite), If $A_i \in \mathcal{T}$, then $\bigcap_{i \in K} A_i \in \mathcal{T}$
- Also, $((X, *), \mathcal{T})$ is called NT topological space.

Definition 2.4: [33] Let $((X, *), \mathcal{T})$ be a NT topology. For every $A \in \mathcal{T}$, A is called a NT open set.

3 Some Properties for Neutrosophic Triplet Topology

Definition 3.1: Let $((X, *), \mathcal{T})$ be a NT topological space and $\mathcal{B} \subset P(X)$ be a set family. If $\mathcal{B}^\# = \mathcal{T}$ such that $\mathcal{B}^\# = \{A \subset X: A = \bigcup C, C \subset \mathcal{B}\}$, then it is said that $\mathcal{B}$ is a base of $\mathcal{T}$.

Theorem 3.2: Let $(X, *)$ be a NT set and $\mathcal{B} \subset P(X)$ be a base of NT topology. If the following conditions are satisfied, then $((X, *), \mathcal{B}^\#)$ is a NT topological space such that $\mathcal{B}^\# = \{A \subset X: A = \bigcup C, C \subset \mathcal{B}\}$.

- c_1) $x * y \in X$, for every $x, y \in X$
- c_2) $X = \bigcup C$ ($C \subset \mathcal{B}$)
- c_3) For every $C_1, C_2 \in \mathcal{B}$ and $x \in C_1 \cap C_2$; there is at least a set $C_3 \in \mathcal{B}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Proof: We suppose that conditions c_1, c_2 and c_3 are satisfied. We show that $((X, *), \mathcal{B}^\#)$ is a NT topological space.

In Definiton 2.3,

i) is equal to condition c_1 .

ii) It is clear that $\emptyset \in \mathcal{B}^\#$ since $\mathcal{B}^\# = \{A \subset X: A = \bigcup C, C \subset \mathcal{B}\}$. Also, we obtain $X \in \mathcal{B}^\#$ since condition c_2 ,

iii) We take $A_i \in \mathfrak{B}^\#$ ($i \in I$). $\cup A_i = \cup \cup C_i$ ($C_i \subset \mathfrak{B}$). Thus, we obtain $\cup A_i \in \mathfrak{B}^\#$.

iv) We take $A_1, A_2 \in \mathfrak{B}^\#$. If $x \in A_1 \cap A_2$, then $x \in A_1$ and $x \in A_2$. Thus, there is at least a pair element $C_1, C_2 \in \mathfrak{B}$ such that $x \in C_1 \subset A_1$ and $x \in C_2 \subset A_2$ since $\mathfrak{B}^\# = \{A \subset X: A = \cup C, C \subset \mathfrak{B}\}$. Then from c_3 , there are an element $C_3 \in \mathfrak{B}$ such that $x \in C_3 \subset C_1 \cap C_2 \subset A_1 \cap A_2$. Thus, we obtain $A_1 \cap A_2 \in \mathfrak{B}^\#$ since $\mathfrak{B}^\# = \{A \subset X: A = \cup C, C \subset \mathfrak{B}\}$.

4 Neutrosophic Triplet Metric Topology

Definition 4.1:

a) Let $((N, *), d)$ be a NT metric space and $a \in N$. The set $B(a, r) = \{x \in N: d(a, x) < r\}$ is called open ball centered at a with radius r ($r > 0$).

b) Let $((N, *), d)$ be a NT metric space and $a \in N$. The set $B[a, r] = \{x \in N: d(a, x) \leq r\}$ is called closed ball centered at a with radius r ($r > 0$).

b) Let $((N, *), d)$ be a NT metric space and $a \in N$. The set $S(a, r) = \{x \in N: d(a, x) = r\}$ is called sphere centered at a with radius r ($r > 0$).

Definition 4.2: Let $((N, *), d)$ be a NT metric space and $x \in N$. Then an open ball neighbourhood of x is a $B(x, \varepsilon)$, for some $\varepsilon > 0$.

Theorem 4.3: Let $((N, *), d)$ be a NT metric space and $B(a, \varepsilon)$ be an open ball in this space. If there is at least an element $x \in N$ for each $a, y \in N$ pair such that $d(a, y) \leq d(a, y * \text{neut}(x))$, then there is an open ball such that $B(x, r) \subset B(a, \varepsilon)$ for all $x \in B(a, \varepsilon)$.

Proof: We suppose that there is at least an element $x \in N$ for each $a, y \in N$ pair such that

$$d(a, y) \leq d(a, y * \text{neut}(x)). \quad (1)$$

Then, we take an element $x \in B(a, \varepsilon)$. Thus, from Definition 4.1 we obtain $d(a, x) < \varepsilon$. Also, we take a real number r such that

$$0 < r < \varepsilon - d(a, x). \quad (2)$$

Now, we take an element $y \in B(x, r)$. Thus, from Definition 4.1 we obtain $d(y, x) < r$. From (1) and Definition 2.2, we obtain

$$d(a, y) \leq d(a, x) + d(x, y). \quad (3)$$

From (2) and (3), we can write $d(a, y) < d(a, x) + r < \varepsilon$. Thus, we obtained $y \in B(a, \varepsilon)$ and $B(x, r) \subset B(a, \varepsilon)$.

Theorem 4.4: Let $((N, *), d)$ be a NT metric space and $\mathfrak{B}$ be set of all open ball of $((N, *), d)$. Then $\mathfrak{B}$ is a base of NT topology on $(N, *)$.

Proof: We show that $\mathfrak{B} = \{B(a, \varepsilon): a \in N, \varepsilon > 0\}$ is a base of a NT topology such that $\mathfrak{B}^\# = \{A \subset N: A = \cup C, C \subset \mathfrak{B}\}$. Thus, we show that $\mathfrak{B}$ satisfies the conditions in Theorem 3.2.

c_1) It is clear that since $((N, *), d)$ is a NT metric space.

c_2) For every $\varepsilon > 0$ and $a \in N$, we can write $N = \cup B(a, \varepsilon)$ since $a \in B(a, \varepsilon) \subset N$.

c_3) Let $C_1, C_2 \in \mathfrak{B}$ and $x \in C_1 \cap C_2$. From Theorem 4.3,

if $x \in C_1$, then there exists at least a $B(x, r_1)$ open ball such that $B(x, r_1) \subset C_1$ and $r_1 > 0$. Similarly,

if $x \in C_2$, then there exists at least a $B(x, r_2)$ ball such that $B(x, r_2) \subset \mathcal{B}$ and $r_2 > 0$. If we take $r = \min\{r_1, r_2\}$, then $x \in B(x, r) \subset C_1 \cap C_2$.

Thus, $\mathcal{B}$ is a base of NT topology on $(N, *)$ such that $\mathcal{B}^\# = \{A \subset N: A = \cup C, C \subset \mathcal{B}\}$.

Corollary 4.5: Let $((N, *), d)$ be a NT metric space and $\mathcal{B}$ be set of all open ball of $((N, *), d)$. From Theorem 4.4 and Definition 3.1, $\mathcal{B}^\# = \{A \subset N: A = \cup C, C \subset \mathcal{B}\}$ is a NT topology on $(N, *)$.

Corollary 4.6: Let $((N, *), d)$ be a NT metric space and $B(x, r)$ be an open ball in this space. From Corollary 4.5 and Definition 2.4, $B(x, r)$ is an open set.

Definition 4.7: Let $((N, *), d)$ be a NT metric space and $\mathcal{B}$ be set of all open ball of $((N, *), d)$. Then, $((N, *), \mathcal{T}_d = \mathcal{B}^\#)$ is called NT metric topological space such that $\mathcal{B}^\# = \{A \subset N: A = \cup C, C \subset \mathcal{B}\}$.

Example 4.8: Let $N = \{0, 2, 5, 6\}$ be a set. $(N, .)$ is a NT set under multiplication module 10 in $(\mathbb{Z}_{10}, .)$. Also, NT are $(0, 0, 0)$, $(2, 6, 2)$, $(5, 5, 5)$ and $(6, 6, 6)$.

Then we take that $d: N \times N \rightarrow N$ is a function such that $d(k, m) = (\lfloor 2^k - 2^m \rfloor)/8$.

Now we show that d is a NT metric.

i) It is clear that $k, m \in N$, for every $k, m \in N$.

ii) If $k = m$, then $d(k, m) = (\lfloor 2^k - 2^m \rfloor)/8 = (\lfloor 2^k - 2^k \rfloor)/8 = 0$. Also, $d(k, m) = (\lfloor 2^k - 2^m \rfloor)/8 \geq 0$.

iii) $d(k, m) = (\lfloor 2^k - 2^m \rfloor)/8 = (\lfloor 2^m - 2^k \rfloor)/8 = d(m, k)$.

iv)

It is clear that $d(0, 0) \leq d(0, 0.2) = d(0, 0)$. Also, $d(0, 0) = 0$ and $d(0, 2) = 3/8$. Thus, we obtain $d(0, 0) \leq d(0, 2) + d(2, 0)$.

It is clear that $d(0, 2) \leq d(0, 2.8) = d(0, 6)$. Also, $d(0, 2) = 3/8$ and $d(0, 6) = 63/8$. Thus, we obtain $d(0, 6) \leq d(0, 8) + d(8, 2)$.

It is clear that $d(0, 4) \leq d(0, 4.6) = d(0, 4)$. Also, $d(0, 4) = 15/8$, $d(0, 6) = 63/8$ and $d(6, 4) = 48/8 = 6$. Thus, we obtain

$d(0, 4) \leq d(0, 6) + d(6, 4)$.

It is clear that $d(0, 5) \leq d(0, 5.5) = d(0, 5)$. Also, $d(0, 5) = 31/8$ and $d(5, 5) = 0$. Thus, we obtain $d(0, 5) \leq d(0, 5) + d(5, 5)$.

It is clear that $d(0, 6) \leq d(0, 6.8) = d(0, 8)$. Also, $d(0, 6) = 63/8$, $d(8, 6) = 192/8 = 24$ and $d(0, 8) = 255/8$. Thus, we obtain

$d(0, 8) \leq d(0, 8) + d(8, 6)$.

It is clear that $d(0, 8) \leq d(0, 6.8) = d(0, 8)$. Also, $d(0, 6) = 63/8$, $d(8, 6) = 192/8 = 24$ and $d(0, 8) = 255/8$. Thus, we obtain

$d(0, 8) \leq d(0, 6) + d(6, 8)$.

It is clear that $d(2, 2) \leq d(2, 2.5) = d(2, 0)$. Also, $d(2, 0) = 3/8$ and $d(5, 2) = 28/8 = 7/2$. Thus, we obtain $d(2, 0) \leq d(2, 5) + d(5, 2)$.

It is clear that $d(2, 4) \leq d(2, 4.6) = d(2, 4)$. Also, $d(2, 6) = 60/8 = 15/2$ and $d(4, 6) = 48/8 = 6$.

Thus, we obtain

$d(2, 4) \leq d(2, 6) + d(6, 4)$.

It is clear that $d(2, 5) \leq d(2, 5.5) = d(2, 5)$. Also, $d(2, 5) = 28/8 = 7/2$ and $d(5, 5) = 0$. Thus, we obtain $d(2, 5) \leq d(2, 5) + d(5, 5)$.

It is clear that $d(2, 6) \leq d(2, 6.8) = d(2, 8)$. Also, $d(2, 8) = 254/8 = 127/4$ and $d(6, 8) = 192/8 = 24$. Thus, we obtain

$d(2, 8) \leq d(2, 8) + d(8, 6)$.

It is clear that $d(2, 8) \leq d(2, 6.8) = d(2, 8)$. Also, $d(2, 8) = 254/8 = 127/4$ and $d(8, 8) = 0$. Thus, we obtain

$d(2, 8) \leq d(2, 8) + d(8, 8)$.

It is clear that $d(4, 4) \leq d(4, 4.5) = d(4, 0)$. Also, $d(4, 5) = 16/8 = 2$ and $d(4, 0) = 15/8$. Thus, we obtain $d(4, 0) \leq d(4, 5) + d(5, 4)$.

It is clear that $d(4, 5) \leq d(4, 5.4) = d(4, 0)$. Also, $d(4, 5) = 16/8 = 2$ and $d(4, 0) = 15/8$. Thus, we obtain $d(4, 0) \leq d(4, 4) + d(4, 5)$.

It is clear that $d(4, 6) \leq d(4, 6.8) = d(4, 8)$. Also, $d(4, 8) = 240/8 = 30$ and $d(8, 6) = 192/8 = 24$. Thus, we obtain

$d(4, 8) \leq d(4, 8) + d(8, 6)$.

It is clear that $d(4, 8) \leq d(4, 8.6) = d(4, 8)$. Also, $d(4, 8) = 240/8 = 30$ and $d(8, 8) = 0$. Thus, we obtain $d(4, 8) \leq d(4, 8) + d(8, 8)$.

It is clear that $d(5, 5) \leq d(5, 5.0) = d(5, 0)$. Also, $d(0, 5) = 31/8$ and $d(5, 5) = 0$. Thus, we obtain $d(5, 0) \leq d(5, 0) + d(0, 5)$.

It is clear that $d(5, 6) \leq d(5, 6.8) = d(5, 8)$. Also, $d(5, 8) = 224/8 = 28$ and $d(8, 6) = 192/8 = 24$. Thus, we obtain $d(5, 8) \leq d(5, 8) + d(8, 6)$.

It is clear that $d(5, 8) \leq d(5, 8.6) = d(5, 8)$. Also, $d(5, 6) = 32/8 = 4$ and $d(8, 6) = 192/8 = 24$. Thus, we obtain $d(5, 8) \leq d(5, 8) + d(8, 6)$.

It is clear that $d(6, 6) \leq d(6, 6.8) = d(6, 8)$. Also, $d(6, 8) = 192/8 = 24$. Thus, we obtain $d(6, 8) \leq d(6, 8) + d(6, 8)$.

It is clear that $d(6, 8) \leq d(6, 8.6) = d(6, 8)$. Also, $d(6, 8) = 192/8 = 24$ and $d(8, 8) = 0$. Thus, we obtain $d(6, 8) \leq d(6, 8) + d(8, 8)$.

It is clear that $d(8, 8) \leq d(8, 8.2) = d(8, 6)$. Also, $d(6, 8) = 192/8 = 24$. Thus, we obtain $d(8, 6) \leq d(8, 6) + d(6, 8)$.

Therefore, $((N, .), d)$ is A NT metric space.

Now, we show that open balls in $((N, .), d)$.

For $0 \in N$,

$$\begin{aligned} B(0, 3/8) &= \{0\} \\ B(0, 15/8) &= \{0, 2\} \\ B(0, 31/8) &= \{0, 2, 4\} \\ B(0, 63/8) &= \{0, 2, 4, 5\} \\ B(0, 255/8) &= \{0, 2, 4, 5, 6\} \\ B(0, 32) &= \{0, 2, 4, 5, 6, 8\} \end{aligned}$$

For $2 \in N$,

$$\begin{aligned} B(2, 3/8) &= \{2\} \\ B(2, 3/2) &= \{0, 2\} \\ B(2, 7) &= \{0, 2, 4\} \\ B(2, 15/2) &= \{0, 2, 4, 5\} \\ B(2, 63/2) &= \{0, 2, 4, 5, 6\} \\ B(2, 32) &= \{0, 2, 4, 5, 6, 8\} \end{aligned}$$

For $4 \in N$,

$$\begin{aligned} B(4, 15/8) &= \{4\} \\ B(4, 3/2) &= \{0, 4\} \\ B(4, 2) &= \{0, 2, 4\} \\ B(4, 6) &= \{0, 2, 4, 5\} \\ B(4, 30) &= \{0, 2, 4, 5, 6\} \\ B(4, 31) &= \{0, 2, 4, 5, 6, 8\} \end{aligned}$$

For $5 \in N$,

$$\begin{aligned} B(5, 2) &= \{5\} \\ B(5, 31/8) &= \{4, 5\} \\ B(5, 4) &= \{0, 4, 5\} \\ B(5, 7) &= \{0, 4, 5, 6\} \\ B(5, 31/2) &= \{0, 2, 4, 5, 6\} \\ B(5, 16) &= \{0, 2, 4, 5, 6, 8\} \end{aligned}$$

For $6 \in N$,

$$\begin{aligned} B(6, 4) &= \{6\} \\ B(6, 63/8) &= \{5, 6\} \\ B(6, 15/2) &= \{0, 5, 6\} \\ B(6, 8) &= \{0, 2, 5, 6\} \\ B(6, 24) &= \{0, 2, 4, 5, 6\} \\ B(6, 25) &= \{0, 2, 4, 5, 6, 8\} \end{aligned}$$

For $8 \in N$,

$$\begin{aligned} B(8, 31/2) &= \{8\} \\ B(8, 24) &= \{5, 8\} \\ B(8, 30) &= \{5, 6, 8\} \\ B(8, 63/2) &= \{4, 5, 6, 8\} \\ B(8, 255/8) &= \{2, 4, 5, 6, 8\} \end{aligned}$$

$$B(8, 32) = \{0, 2, 4, 5, 6, 8\}.$$

Thus,

in $((N, .), d)$, we obtain set of every open ball $\mathfrak{B} = \{\{0\}, \{2\}, \{4\}, \{5\}, \{6\}, \{8\}, \{0, 2\}, \{0, 4\}, \{4, 5\}, \{5, 6\}, \{5, 8\}, \{5, 6, 8\}, \{0, 5, 6\}, \{0, 4, 5\}, \{0, 2, 4\}, \{4, 5, 6, 8\}, \{0, 2, 5, 6\}, \{0, 2, 4, 5\}, \{0, 4, 5, 6\}, \{2, 4, 5, 6, 8\}, \{0, 2, 4, 5, 6\}, \{0, 2, 4, 5, 6, 8\}\}.$

Also, from Definition 3.1, it is clear that $\mathfrak{B}$ is a base of $\mathcal{T} = \mathfrak{B}^\# = \{A \subset N: A = \bigcup C, C \subset \mathfrak{B}\}$. Furthermore, B and $\mathfrak{B}^\#$ satisfy the conditions in Theorem 3.2. Therefore, $((N, .), \mathfrak{B}^\#)$ is a NT topological space. Also, every open ball $A \in \mathfrak{B}$ is satisfies the conditions in Theorem 4.3. Finally, from Definition 4.7, $((N, .), \mathcal{T}_d = \mathfrak{B}^\#)$ is a NT metric topological space.

Corollary 4.9: Let $((N, *), d)$ be a NT metric space and $((N, *), \mathcal{T}_d)$ be a NT metric topology. Then, there is a unique $((N, *), \mathcal{T}_d)$ NT topology.

Definition 4.10: Let $((N, *), d)$ be a NT metric space, $A \subset N$ and $\mathfrak{B}$ be set of all open ball of $((N, *), d)$. If $A = \bigcup C_i$ ($i \in I$ and $C_i \subset \mathfrak{B}$) is satisfied, then it is called that A is an open set in $((N, *), d)$.

Theorem 4.11: Let $((N, *), d)$ be a NT metric space. Then $((N, *), \mathcal{T}_d)$ is a NT metric topology such that

$$\mathcal{T}_d = \{C_i \subset N: \text{for every } i \in I \text{ and } a \in C_i, \text{ there exists at least a } \varepsilon > 0 \text{ such that } a \in B(a, \varepsilon) \subset C_i\}.$$

Proof: We show that $\mathcal{T}_d = \{C_i \subset N: \text{for every } i \in I \text{ and } a \in C_i, \text{ there exists at least a } \varepsilon > 0 \text{ such that } a \in B(a, \varepsilon) \subset C_i\}$ is a NT topology.

i) It is clear that since $((N, *), d)$ is a NT metric space.

ii) For every $x \in N$, it is clear that $B(x, \varepsilon) \subset N$. Thus, $N \in \mathcal{T}_d$. Also, if we choose the ε large enough, then we can write $\emptyset \in \mathcal{T}_d$.

iii) Let $C_i \in \mathcal{T}_d$, for every $i \in I$. We take an element $x \in \bigcup_{i \in I} C_i$. Thus, there is a $j \in I$ such that $x \in C_j \in \mathcal{T}_d$. Where, there is a $\varepsilon > 0$ such that $B(x, \varepsilon) \subset C_j$. Hence,

$$B(x, \varepsilon) \subset C_j \subset \bigcup_{i \in I} C_i \text{ and } \bigcup_{i \in I} C_i \in \mathcal{T}_d.$$

iv) We suppose that $C_1, C_2, \dots, C_n \in \mathcal{T}_d$. If $C_1 \cap C_2 \cap \dots \cap C_n = \emptyset$, then we obtain $C_1 \cap C_2 \cap \dots \cap C_n = \emptyset \in \mathcal{T}_d$ since $\emptyset \in \mathcal{T}_d$. We suppose that $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$. Where, there is an element $x \in C_1 \cap C_2 \cap \dots \cap C_n$. Then,

if $x \in C_1$, then there is at least $r_1 > 0$ such that $B(x, r_1) \subset C_1$.

if $x \in C_2$, then there is at least $r_2 > 0$ such that $B(x, r_2) \subset C_2$.

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if $x \in C_n$, then there is at least $r_n > 0$ such that $B(x, r_n) \subset C_n$.

If we take $r = \min\{r_1, r_2, \dots, r_n\}$, then

$$B(x, r) \subset B(x, r_1) \cap B(x, r_2) \cap \dots \cap B(x, r_n) \subset C_1 \cap C_2 \cap \dots \cap C_n.$$

Thus, we obtain $C_1 \cap C_2 \cap \dots \cap C_n \in \mathcal{T}_d$.

Definition 4.12: Let $((N_1, *), d_1)$ and $((N_2, *), d_2)$ be NT metric spaces, $\mathcal{T}_{d_1}$ and $\mathcal{T}_{d_2}$ be NT metric topologies. If $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$, then it is called that $((N, *), d_1)$ is equal to $((N, *), d_2)$.

Definition 4.13: Let $((N_1, *), d_1)$, $((N_2, *), d_2)$ be NT metric spaces and $f: N_1 \rightarrow N_2$ be a function. If $d_1(f(a), f(b)) = d_2(a, b)$, for every $a, b \in N_1$, then f is called a NT isometry. Also, if f is one to one and surjective, then it is called that $((N_1, *), d_1)$ and $((N_2, *), d_2)$ are NT isometric spaces.

Example 4.14: From Example 4.8,

$N_1 = \{0, 2, 5, 6\}$. $(N_1, .)$ is a NT set under multiplication module 10 in $(\mathbb{Z}_{10}, .)$. Also, NT are $(0, 0, 0)$, $(2, 6, 2)$, $(5, 5, 5)$ and $(6, 6, 6)$. Thus, $d_1(k, m) = (|2^k - 2^m|)/8$ is a NT metric such that $d_1: N_1 \times N_1 \rightarrow N_1$.

Now, if we take that

$N_2 = \{10, 12, 15, 16\}$. $(N_2, .)$ is a NT set under multiplication module 10 in $(\mathbb{Z}_{10}, .)$.

Actually, $10 \equiv 0$, $12 \equiv 2$, $15 \equiv 5$ and $16 \equiv 6$ in $\mathbb{Z}_{10}$. Thus, NTs are $(0, 0, 0)$, $(2, 6, 2)$, $(5, 5, 5)$ and $(6, 6, 6)$.

Therefore, it is clear that $d_2(k, m) = (|2^k - 2^m|)/2^{13}$ is a NT metric such that $d_2: N_2 \times N_2 \rightarrow N_2$.

Now, we define function $f: N_1 \rightarrow N_2$ such that $f(x) = x + 10$. We show that f is a NT isometry.

$$d_2(f(a), f(b)) = d_2(a + 10, b + 10) = (|2^{a+10} - 2^{b+10}|)/2^{13} = 2^{10} \cdot |2^a - 2^b|/2^{13} = |2^a - 2^b|/2^3 = |2^a - 2^b|/8 = d_1(a, b).$$

Thus, from Definition 4.13, f is a NT isometry. Also, it is clear that f is one to one and surjective. Therefore, $((N_1, .), d_1)$ and $((N_2, .), d_2)$ are NT isometric spaces.

Definition 4.15: Let $((N_1, *), d_1)$ and $((N_2, *), d_2)$ be NT metric spaces, $x_0 \in N_1$ and $f: N_1 \rightarrow N_2$ be a function. f is continuous at point x_0 if and only if for every $B(f(x_0), \varepsilon)$, there is a $B(x_0, \delta)$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Definition 4.16: Let $((N_1, *), d_1)$ and $((N_2, *), d_2)$ be NT metric spaces, $x_0 \in N_1$ and $f: N_1 \rightarrow N_2$ be a function. f is continuous at point x_0 if and only if

There is a $\delta > 0$ depending on x_0 and ε such that it is $d_1(x_0, x) < \delta \Rightarrow d_2(f(x_0), f(x)) < \varepsilon$.

Conclusion

Topology has many different application areas in classical mathematic. Also, NT structures are a new concept in neutrosophy. In this paper, we introduce NT metric topology. We give some properties and definitions for NT topology, NT metric and NT metric topology. Also, we obtain that in a NT metric space, all NT open sets are a NT based for a NT topology. Hence, we can obtain a NT topology using each NT metric topology. Thus, we add NT metric topology to NT structures which is a new concept.

Furthermore, by utilizing NT metric topology, researcher can obtain new structure and properties. For example, researcher can define NT quasi – metric topology, NT Hausdorff metric topology, NT partial metric topology, NT v-generalized metric topology, NT b-metric topology.

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Neutrosophic α gs Continuity And Neutrosophic α gs Irresolute Maps

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Abstract. Neutrosophic Continuity functions very first introduced by A.A.Salama et.al.Aim of this present paper is, we introduce and investigate new kind of Neutrosophic continuity is called Neutrosophic α gs Continuity maps in Neutrosophic topological spaces and also discussed about some properties and characterization of Neutrosophic α gs Irresolute Maps.

Keywords: Neutrosophic α -closed sets, Neutrosophic semi-closed sets, Neutrosophic α gs-closed sets Neutrosophic α gs Continuity maps, Neutrosophic α gs irresolute maps

1. Introduction

Neutrosophic set theory concepts first initiated by F.Smarandache[11] which is Based on K. Atanassov's intuitionistic[6]fuzzy sets & L.A.Zadeh's [20]fuzzy sets. Also it defined by three parameters truth(T), indeterminacy (I),and falsity(F)-membership function. Smarandache's neutrosophic concept have wide range of real time applications for the fields of [1,2,3,4&5] Information Systems, Computer Science, Artificial Intelligence, Applied Mathematics, decision making. Mechanics, Electrical & Electronic, Medicine and Management Science etc.,

A.A.Salama[16] introduced Neutrosophic topological spaces by using Smarandache's Neutrosophic sets. I.Arokiaarani.[7] et.al., introduced Neutrosophic α -closed sets.P. Ishwarya, [13]et.al., introduced and studied Neutrosophic semi-open sets in Neutrosophic topological spaces. Neutrosophic continuity functions introduced by A.A.Salama[15]. Neutrosophic α gs-closed set[8] introduced by V.Banu priya&S.Chandrasekar. Aim of this present paper is, we introduce and investigate new kind of Neutrosophic continuity is called Neutrosophic α gs Continuity maps in Neutrosophic topological spaces and also we discussed about properties and characterization Neutrosophic α gs Irresolute Maps

2. PRELIMINARIES

In this section, we introduce the basic definition for Neutrosophic sets and its operations.

Definition 2.1 [11]

Let E be a non-empty fixed set. A Neutrosophic set λ writing the format is

$$\lambda = \{ \langle e, \eta_{\lambda}(e), \sigma_{\lambda}(e), \gamma_{\lambda}(e) \rangle : e \in E \}$$

Where $\eta_{\lambda}(e)$, $\sigma_{\lambda}(e)$ and $\gamma_{\lambda}(e)$ which represents Neutrosophic topological spaces the degree of membership function, indeterminacy and non-membership function respectively of each element $e \in E$ to the set λ .

Remark 2.2 [11]

A Neutrosophic set $\lambda = \{ \langle e, \eta_{\lambda}(e), \sigma_{\lambda}(e), \gamma_{\lambda}(e) \rangle : e \in E \}$ can be identified to an ordered triple $\langle \eta_{\lambda}, \sigma_{\lambda}, \gamma_{\lambda} \rangle$ in $[-0,1+[-$ on E.

Remark 2.3[11]

Neutrosophic set $\lambda = \{ \langle e, \eta_{\lambda}(e), \sigma_{\lambda}(e), \gamma_{\lambda}(e) \rangle : e \in E \}$ our convenient we can write $\lambda = \langle \eta_{\lambda}, \sigma_{\lambda}, \gamma_{\lambda} \rangle$.

Example 2.4 [11]

we must introduce the Neutrosophic set 0_N and 1_N in E as follows:

0_N may be defined as:

$$(0_1) 0_N = \{ \langle e, 0, 0, 1 \rangle : e \in E \}$$

$$(0_2) 0_N = \{ \langle e, 0, 1, 1 \rangle : e \in E \}$$

$$(0_3) 0_N = \{ \langle e, 0, 1, 0 \rangle : e \in E \}$$

$$(0_4) 0_N = \{ \langle e, 0, 0, 0 \rangle : e \in E \}$$

1_N may be defined as:

$$(1_1) 1_N = \{ \langle e, 1, 0, 0 \rangle : e \in E \}$$

$$(1_2) 1_N = \{ \langle e, 1, 0, 1 \rangle : e \in E \}$$

$$(1_3) 1_N = \{ \langle e, 1, 1, 0 \rangle : e \in E \}$$

$$(1_4) 1_N = \{ \langle e, 1, 1, 1 \rangle : e \in E \}$$

Definition 2.5 [11]

Let $\lambda = \langle \eta_\lambda, \sigma_\lambda, \gamma_\lambda \rangle$ be a Neutrosophic set on E, then λ^c defined as $\lambda^c = \{ \langle e, \gamma_\lambda(e), 1 - \sigma_\lambda(e), \eta_\lambda(e) \rangle : e \in E \}$

Definition 2.6 [11]

Let E be a non-empty set, and Neutrosophic sets λ and μ in the form

$$\lambda = \{ \langle e, \eta_\lambda(e), \sigma_\lambda(e), \gamma_\lambda(e) \rangle : e \in E \} \text{ and}$$

$$\mu = \{ \langle e, \eta_\mu(e), \sigma_\mu(e), \gamma_\mu(e) \rangle : e \in E \}.$$

Then we consider definition for subsets ($\lambda \subseteq \mu$).

$\lambda \subseteq \mu$ defined as: $\lambda \subseteq \mu \Leftrightarrow \eta_\lambda(e) \leq \eta_\mu(e), \sigma_\lambda(e) \leq \sigma_\mu(e) \text{ and } \gamma_\lambda(e) \geq \gamma_\mu(e) \text{ for all } e \in E$

Proposition 2.7 [11]

For any Neutrosophic set λ , then the following condition are holds:

$$(i) 0_N \subseteq \lambda, 0_N \subseteq 0_N$$

$$(ii) \lambda \subseteq 1_N, 1_N \subseteq 1_N$$

Definition 2.8 [11]

Let E be a non-empty set, and $\lambda = \langle e, \eta_\lambda(e), \sigma_\lambda(e), \gamma_\lambda(e) \rangle$, $\mu = \langle e, \eta_\mu(e), \sigma_\mu(e), \gamma_\mu(e) \rangle$ be two Neutrosophic sets. Then

$$(i) \lambda \cap \mu \text{ defined as } : \lambda \cap \mu = \langle e, \eta_\lambda(e) \wedge \eta_\mu(e), \sigma_\lambda(e) \vee \sigma_\mu(e), \gamma_\lambda(e) \vee \gamma_\mu(e) \rangle$$

$$(ii) \lambda \cup \mu \text{ defined as } : \lambda \cup \mu = \langle e, \eta_\lambda(e) \vee \eta_\mu(e), \sigma_\lambda(e) \wedge \sigma_\mu(e), \gamma_\lambda(e) \wedge \gamma_\mu(e) \rangle$$

Proposition 2.9 [11]

For all λ and μ are two Neutrosophic sets then the following condition are true:

$$(i) (\lambda \cap \mu)^c = \lambda^c \cup \mu^c$$

$$(ii) (\lambda \cup \mu)^c = \lambda^c \cap \mu^c.$$

Definition 2.10 [16]

A Neutrosophic topology is a non-empty set E is a family τ_N of Neutrosophic subsets in E satisfying the following axioms:

$$(i) 0_N, 1_N \in \tau_N,$$

$$(ii) G_1 \cap G_2 \in \tau_N \text{ for any } G_1, G_2 \in \tau_N,$$

$$(iii) \bigcup G_i \in \tau_N \text{ for any family } \{G_i \mid i \in J\} \subseteq \tau_N.$$

the pair (E, τ_N) is called a Neutrosophic topological space.

The element Neutrosophic topological spaces of τ_N are called Neutrosophic open sets.

A Neutrosophic set λ is closed if and only if λ^c is Neutrosophic open.

Example 2.11[16]

Let $E = \{e\}$ and

$$A_1 = \{ \langle e, .6, .6, .5 \rangle : e \in E \}$$

$$A_2 = \{ \langle e, .5, .7, .9 \rangle : e \in E \}$$

$$A_3 = \{ \langle e, .6, .7, .5 \rangle : e \in E \}$$

$$A_4 = \{ \langle e, .5, .6, .9 \rangle : e \in E \}$$

Then the family $\tau_N = \{0_N, 1_N, A_1, A_2, A_3, A_4\}$ is called a Neutrosophic topological space on E.

Definition 2.12[16]

Let (E, τ_N) be Neutrosophic topological spaces and $\lambda = \{ \langle e, \eta_\lambda(e), \sigma_\lambda(e), \gamma_\lambda(e) \rangle : e \in E \}$ be a Neutrosophic set in E.

Then the Neutrosophic closure and Neutrosophic interior of λ are defined by

$$\text{Neu-cl}(\lambda) = \bigcap \{D : D \text{ is a Neutrosophic closed set in } E \text{ and } \lambda \subseteq D\}$$

$$\text{Neu-int}(\lambda) = \bigcup \{C : C \text{ is a Neutrosophic open set in } E \text{ and } C \subseteq \lambda\}.$$

Definition 2.13

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

$$(i) \text{ Neutrosophic regular Closed set [7] (Neu-RCS in short) if } \lambda = \text{Neu-Cl}(\text{Neu-Int}(\lambda)),$$

$$(ii) \text{ Neutrosophic } \alpha\text{-Closed set [7] (Neu-}\alpha\text{CS in short) if } \text{Neu-Cl}(\text{Neu-Int}(\text{Neu-Cl}(\lambda))) \subseteq \lambda,$$

$$(iii) \text{ Neutrosophic semi Closed set [13] (Neu-SCS in short) if } \text{Neu-Int}(\text{Neu-Cl}(\lambda)) \subseteq \lambda,$$

$$(iv) \text{ Neutrosophic pre Closed set [18] (Neu-PCS in short) if } \text{Neu-Cl}(\text{Neu-Int}(\lambda)) \subseteq \lambda,$$

Definition 2.14

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

$$(i). \text{ Neutrosophic regular open set [7] (Neu-ROS in short) if } \lambda = \text{Neu-Int}(\text{Neu-Cl}(\lambda)),$$

$$(ii). \text{ Neutrosophic } \alpha\text{-open set [7] (Neu-}\alpha\text{OS in short) if } \lambda \subseteq \text{Neu-Int}(\text{Neu-Cl}(\text{Neu-Int}(\lambda))),$$

$$(iii). \text{ Neutrosophic semi open set [13] (Neu-SOS in short) if } \lambda \subseteq \text{Neu-Cl}(\text{Neu-Int}(\lambda)),$$

$$(iv). \text{ Neutrosophic pre open set [18] (Neu-POS in short) if } \lambda \subseteq \text{Neu-Int}(\text{Neu-Cl}(\lambda)),$$

Definition 2.15

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called

$$(i). \text{ Neutrosophic generalized closed set [9] (Neu-GCS in short) if } \text{Neu-cl}(\lambda) \subseteq U \text{ whenever } \lambda \subseteq U \text{ and } U \text{ is a Neu-}$$

- OS in E ,
- (ii).Neutrosophic generalized semi closed set[17] (Neu-GSCS in short) if $\text{Neu-scl}(\lambda) \subseteq U$ Whenever $\lambda \subseteq U$ and U is a Neu-OS in E ,
- (iii).Neutrosophic α generalized closed set [14](Neu- α GCS in short) if $\text{Neu-}\alpha\text{cl}(\lambda) \subseteq U$ whenever $\lambda \subseteq U$ and U is a Neu-OS in E ,
- (iv).Neutrosophic generalized alpha closed set [10] (Neu-G α CS in short) if $\text{Neu-}\alpha\text{cl}(\lambda) \subseteq U$ whenever $\lambda \subseteq U$ and U is a Neu- α OS in E .

The complements of the above mentioned Neutrosophic closed sets are called their respective Neutrosophic open sets.

Definition 2.16 [8]

Let (E, τ_N) be a Neutrosophic topological space. Then λ is called Neutrosophic α generalized Semi closed set (Neu- α GSCS in short) if $\text{Neu-}\alpha\text{cl}(\lambda) \subseteq U$ whenever $\lambda \subseteq U$ and U is a Neu-SOS in E

The complements of Neutrosophic α GS closed sets is called Neutrosophic α GS open sets.

3. Neutrosophic ags-Continuity maps

In this section we Introduce Neutrosophic α -generalized semi continuity maps and study some of its properties.

Definition 3.1.

A maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ is called a Neutrosophic α -generalized semi continuity (Neu- α GS continuity in short) $f^{-1}(\mu)$ is a Neu- α GSCS in (E_1, τ_N) for every Neu-CS μ of (E_2, σ_N)

Example 3.2.

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.7, .5, .8), (.5, .5, .4) \rangle$ and $V = \langle e_2, (1, .5, .9), (.2, .5, .3) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively.

Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is a Neu- α GS continuity maps.

Theorem 3.3.

Every Neu-continuity maps is a Neu- α GS continuity maps.

Proof.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu-continuity maps. Let λ be a Neu-CS in E_2 . Since f is a Neu-continuity maps, $f^{-1}(\lambda)$ is a Neu-CS in E_1 . Since every Neu-CS is a Neu- α GSCS, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Hence f is a Neu- α GS continuity maps.

Example 3.4.

Neu- α GS continuity maps is not Neu-continuity maps

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.5, .5, .3), (.7, .5, .8) \rangle$ and $V = \langle e_2, (.4, .5, .3), (.8, .5, .9) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic sets on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle y, (.3, .5, .4), (.9, .5, .8) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is a Neu- α GSCS but not Neu-CS in E_1 . Therefore f is a Neu- α GS continuity maps but not a Neu-continuity maps.

Theorem 3.5.

Every Neu- α continuity maps is a Neu- α GS continuity maps.

Proof.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α continuity maps. Let λ be a Neu-CS in E_2 . Then by hypothesis $f^{-1}(\lambda)$ is a Neu- α CS in E_1 . Since every Neu- α CS is a Neu- α GSCS, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Hence f is a Neu- α GS continuity maps.

Example 3.6.

Neu- α GS continuity maps is not Neu- α continuity maps

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.5, .5, .6), (.7, .5, .6) \rangle$ and $V = \langle e_2, (.3, .5, .9), (.5, .5, .7) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle e_2, (.9, .5, .3), (.7, .5, .5) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is a Neu- α GSCS continuity maps.

Remark 3.7.

Neu-G continuity maps and Neu- α GS continuity maps are independent of each other.

Example 3.8.

Neu- α GS continuity maps is not Neu-G continuity maps.

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.5, .5, .6), (.8, .5, .4) \rangle$ and $V = \langle e_2, (.7, .5, .4), (.9, .5, .3) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is Neu- α GS continuity maps but not Neu-G continuity maps.

Since $\lambda = \langle e_1, (.4, .5, .7), (.3, .5, .9) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda) = \langle e_2, (.4, .5, .7), (.7, .5, .3) \rangle$ is not Neu-GCS in E_1 .

Example 3.9.

Neu-G continuity maps is not Neu- α GS continuity maps.

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.6, .5, .4), (.8, .5, .2) \rangle$ and $V = \langle e_2, (.3, .5, .7), (.1, .5, .9) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is Neu-G continuity maps but not a Neu- α GS continuity maps.

Since $\lambda = \langle e_2, (.7, .5, .3), (.9, .5, .1) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda) = \langle e_1, (.7, .5, .3), (.9, .5, .1) \rangle$ is not Neu- α GSCS in E_1 .

Theorem 3.10.

Every Neu- α GS continuity maps is a Neu-GS continuity maps.

Proof.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps. Let λ be a Neu-CS in E_2 . Then by hypothesis $f^{-1}(\lambda)$ Neu- α GSCS in E_1 . Since every Neu- α GSCS is a Neu-GSCS, $f^{-1}(\lambda)$ is a Neu-GSCS in E_1 . Hence f is a Neu-GS continuity maps.

Example 3.11.

Neu-GS continuity maps is not Neu- α GS continuity maps.

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.8, .5, .4), (.9, .5, .2) \rangle$ and $V = \langle e_2, (.3, .5, .9), (.0, .1, .5, .9) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle e_2, (.9, .5, .3), (.9, .5, .1) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is Neu-GSCS in E_1 but not Neu- α GSCS in E_1 . Therefore f is a Neu-GS continuity maps but not a Neu- α GS continuity maps.

Remark 3.12.

Neu-P continuity maps and Neu- α GS continuity maps are independent of each other.

Example 3.13.

Neu-P continuity maps is not Neu- α GS continuity maps Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.3, .5, .7), (.4, .5, .6) \rangle$ and $V = \langle e_2, (.8, .5, .3), (.9, .5, .2) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle e_2, (.3, .5, .8), (.2, .5, .9) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is Neu-PCS in E_1 but not Neu- α GSCS in E_1 . Therefore f is a Neu-P continuity maps but not Neu- α GS continuity maps.

Example 3.14.

Neu- α GS continuity maps is not Neu-P continuity maps

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.4, .5, .8), (.5, .5, .7) \rangle$ and $V = \langle e_1, (.5, .5, .7), (.6, .5, .6) \rangle$ and $W = \langle e_2, (.8, .5, .4), (.5, .5, .7) \rangle$. Then $\tau_N = \{0_N, U, V, 1_N\}$ and $\sigma_N = \{0_N, W, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle y, (.4, .5, .8), (.7, .5, .5) \rangle$ is Neu- α GSCS but not Neu-PCS in E_2 , $f^{-1}(\lambda)$ is Neu- α GSCS in E_1 but not Neu-PCS in E_1 . Therefore f is a Neu- α GS continuity maps but not Neu-P continuity maps.

Theorem 3.15.

Every Neu- α GS continuity maps is a Neu- α G continuity maps.

Proof.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps. Let λ be a Neu-CS in E_2 . Since f is Neu- α GS continuity maps, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Since every Neu- α GSCS is a Neu- α GCS, $f^{-1}(\lambda)$ is a Neu- α GCS in E_1 . Hence f is a Neu- α G continuity maps.

Example 3.16.

Neu- α G continuity maps is not Neu- α GS continuity maps

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.1, .5, .7), (.3, .5, .6) \rangle$ and $V = \langle e_2, (.7, .5, .4), (.6, .5, .5) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle e_2, (.4, .5, .7), (.5, .5, .6) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is Neu- α GCS in E_1 but not Neu- α GSCS in E_1 . Therefore f is a Neu- α G continuity maps but not a Neu- α GS continuity maps.

Theorem 3.17.

Every Neu- α GS continuity maps is a Neu-G α continuity maps.

Proof.

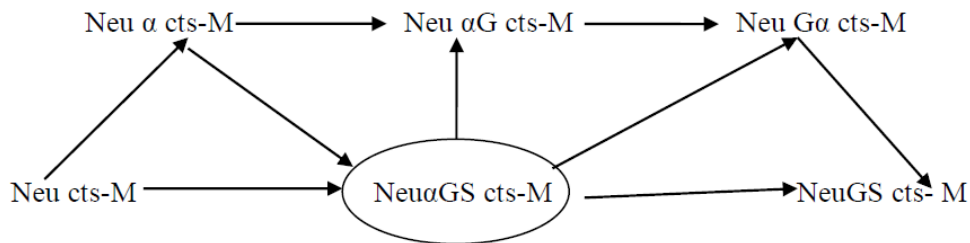
Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps. Let λ be a Neu-CS in E_2 . Since f is Neu- α GS continuity maps, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Since every Neu- α GSCS is a Neu-G α CS, $f^{-1}(\lambda)$ is a Neu-G α CS in E_1 . Hence f is a Neu-G α continuity maps.

Example 3.18.

Neu-G α continuity maps is not Neu- α GS continuity maps Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.5, .5, .7), (.3, .5, .9) \rangle$ and $V = \langle e_2, (.6, .5, .6), (.5, .5, .7) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Since the Neutrosophic set $\lambda = \langle y, (.6, .5, .6), (.7, .5, .5) \rangle$ is Neu-CS in E_2 , $f^{-1}(\lambda)$ is Neu-G α CS in E_1 but not Neu- α GSCS in E_1 . Therefore f is a Neu-G α continuity maps but not a Neu- α GS continuity maps.

Remark 3.19.

We obtain the following diagram from the results we discussed above.

**Theorem 3.20.**

A maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ is Neu- α GS continuity if and only if the inverse image of each Neutrosophic set in E_2 is a Neu- α GSOS in E_1 .

Proof.

first part Let λ be a Neutrosophic set in E_2 . This implies λ^C is Neu-CS in E_2 . Since f is Neu- α GS continuity, $f^{-1}(\lambda^C)$ is Neu- α GSCS in E_1 . Since $f^{-1}(\lambda^C) = (f^{-1}(\lambda))^C$, $f^{-1}(\lambda)$ is a Neu- α GSOS in E_1 .

Converse part Let λ be a Neu-CS in E_2 . Then λ^C is a Neutrosophic set in E_2 . By hypothesis $f^{-1}(\lambda^C)$ is Neu- α GSOS in E_1 . Since $f^{-1}(\lambda^C) = (f^{-1}(\lambda))^C$, $(f^{-1}(\lambda))^C$ is a Neu- α GSOS in E_1 . Therefore $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Hence f is Neu- α GS continuity.

Theorem 3.21.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a maps and $f^{-1}(\lambda)$ be a Neu-RCS in E_1 for every Neu-CS λ in E_2 . Then f is a Neu- α GS continuity maps.

Proof.

Let λ be a Neu-CS in E_2 and $f^{-1}(\lambda)$ be a Neu-RCS in E_1 . Since every Neu-RCS is a Neu- α GSCS, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Hence f is a Neu- α GS continuity maps.

Definition 3.22.

A Neutrosophic Topology (E, τ_N) is said to be an

- (i) Neu- $\alpha_{ga}U_{1/2}$ (in short Neu- $\alpha_{ga}U_{1/2}$) space, if every Neu- α GSCS in E is a Neu-CS in E ,
- (ii) Neu- $\alpha_{gb}U_{1/2}$ (in short Neu- $\alpha_{gb}U_{1/2}$) space, if every Neu- α GSCS in E is a Neu-GCS in E ,
- (iii) Neu- $\alpha_{gc}U_{1/2}$ (in short Neu- $\alpha_{gc}U_{1/2}$) space, if every Neu- α GSCS in E is a Neu-GSCS in E .

Theorem 3.23.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps, then f is a Neu-continuity maps if E_1 is a Neu- $\alpha_{ga}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2 . Then $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 , by hypothesis. Since E_1 is a Neu- $\alpha_{ga}U_{1/2}$, $f^{-1}(\lambda)$ is a Neu-CS in E_1 . Hence f is a Neu-continuity maps.

Theorem 3.24.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps, then f is a Neu-G continuity maps if E_1 is a Neu- $\alpha_{gb}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2 . Then $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 , by hypothesis. Since E_1 is a Neu- $\alpha_{gb}U_{1/2}$, $f^{-1}(\lambda)$ is a Neu-GCS in E_1 . Hence f is a Neu-G continuity maps.

Theorem 3.25.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps, then f is a Neu-GS continuity maps if E_1 is a Neu- $\alpha_{gc}U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2 . Then $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 , by hypothesis. Since E_1 is a Neu- $\alpha_{gc}U_{1/2}$, $f^{-1}(\lambda)$ is a Neu-GSCS in E_1 . Hence f is a Neu-GS continuity maps.

Theorem 3.26.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps and $g: (E_2, \sigma_N) \rightarrow (E_3, \rho_N)$ be an Neutrosophic continuity, then $g \circ f: (E_1, \tau_N) \rightarrow (E_3, \rho_N)$ is a Neu- α GS continuity.

Proof.

Let λ be a Neu-CS in E_3 . Then $g^{-1}(\lambda)$ is a Neu-CS in E_2 , by hypothesis. Since f is a Neu- α GS continuity maps, $f^{-1}(g^{-1}(\lambda))$ is a Neu- α GSCS in E_1 . Hence $g \circ f$ is a Neu- α GS continuity maps.

Theorem 3.27.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a maps from Neutrosophic Topology in E_1 into a Neutrosophic Topology E_2 . Then the following conditions set are equivalent if E_1 is a Neu- $\alpha_{ga}U_{1/2}$ space.

- (i) f is a Neu- α GS continuity maps.
- (ii) if μ is a Neutrosophic set in E_2 then $f^{-1}(\mu)$ is a Neu- α GSOS in E_1 .
- (iii) $f^{-1}(\text{Neu-int}(\mu)) \subseteq \text{Neu-int}(\text{Neu-Cl}(\text{Neu-int}(f^{-1}(\mu))))$ for every Neutrosophic set μ in E_2 .

Proof.

(i) $\rightarrow$ (ii): is obviously true.

(ii) $\rightarrow$ (iii): Let μ be any Neutrosophic set in E_2 . Then $\text{Neu-int}(\mu)$ is a Neutrosophic set in E_2 . Then $f^{-1}(\text{Neu-int}(\mu))$ is a Neu- α GSCS in E_1 . Since E_1 is a Neu- α ga $U_{1/2}$ space, $f^{-1}(\text{Neu-int}(\mu))$ is a Neutrosophic set in E_1 . Therefore $f^{-1}(\text{Neu-int}(\mu)) = \text{Neu-int}(f^{-1}(\text{Neu-int}(\mu))) \subseteq \text{Neu-int}(\text{Neu-Cl}(\text{Neu-int}(f^{-1}(\mu))))$.

(iii) $\rightarrow$ (i) Let μ be a Neu-CS in E_2 . Then its complement μ^c is a Neutrosophic set in E_2 . By Hypothesis $f^{-1}(\text{Neu-int}(\mu^c)) \subseteq \text{Neu-int}(\text{Neu-Cl}(\text{Neu-int}(f^{-1}(\text{Neu-int}(\mu^c)))))$. This implies that $f^{-1}(\mu^c) \subseteq \text{Neu-int}(\text{Neu-Cl}(\text{Neu-int}(f^{-1}(\text{Neu-int}(\mu^c)))))$. Hence $f^{-1}(\mu^c)$ is a Neu- α OS in E_1 . Since every Neu- α OS is a Neu- α GSCS, $f^{-1}(\mu^c)$ is a Neu- α GSCS in E_1 . Therefore $f^{-1}(\mu)$ is a Neu- α GSCS in E_1 . Hence f is a Neu- α GS continuity maps.

Theorem 3.28.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a maps. Then the following conditions set are equivalent if E_1 is a Neu- α ga $U_{1/2}$ space.

(i) f is a Neu- α GS continuity maps.

(ii) $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 for every Neu-CS λ in E_2 .

(iii) $\text{Neu-Cl}(\text{Neu-int}(\text{Neu-Cl}(f^{-1}(\lambda)))) \subseteq f^{-1}(\text{Neu-Cl}(\lambda))$ for every Neutrosophic set λ in E_2 .

Proof.

(i) $\rightarrow$ (ii): is obviously true.

(ii) $\rightarrow$ (iii): Let λ be a Neutrosophic set in E_2 . Then $\text{Neu-Cl}(\lambda)$ is a Neu-CS in E_2 . By hypothesis, $f^{-1}(\text{Neu-Cl}(\lambda))$ is a Neu- α GSCS in E_1 . Since E_1 is a Neu- α ga $U_{1/2}$ space, $f^{-1}(\text{Neu-Cl}(\lambda))$ is a Neu-CS in E_1 . Therefore $\text{Neu-Cl}(f^{-1}(\text{Neu-Cl}(\lambda))) = f^{-1}(\text{Neu-Cl}(\lambda))$. Now $\text{Neu-Cl}(\text{Neu-int}(\text{Neu-Cl}(f^{-1}(\lambda)))) \subseteq \text{Neu-Cl}(\text{Neu-int}(\text{Neu-Cl}(f^{-1}(\text{Neu-Cl}(\lambda))))) \subseteq f^{-1}(\text{Neu-Cl}(\lambda))$.

(iii) $\rightarrow$ (i): Let λ be a Neu-CS in E_2 . By hypothesis $\text{Neu-Cl}(\text{Neu-int}(\text{Neu-Cl}(f^{-1}(\lambda)))) \subseteq f^{-1}(\text{Neu-Cl}(\lambda)) = f^{-1}(\lambda)$. This implies $f^{-1}(\lambda)$ is a Neu- α CS in E_1 and hence it is a Neu- α GSCS in E_1 . Therefore f is a Neu- α GS continuity maps.

Definition 3.29.

Let (E, τ_N) be a Neutrosophic topology. The Neutrosophic alpha generalized semi closure ($\text{Neu-}\alpha\text{GSCl}(\lambda)$ in short) for any Neutrosophic set λ is Defined as follows. $\text{Neu-}\alpha\text{GSCl}(\lambda) = \cap \{ K \mid K \text{ is a Neu-}\alpha\text{GSCS in } E_1 \text{ and } \lambda \subseteq K \}$. If λ is Neu- α GSCS, then $\text{Neu-}\alpha\text{GSCl}(\lambda) = \lambda$.

Theorem 3.30.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS continuity maps. Then the following conditions set are hold.

(i) $f(\text{Neu-}\alpha\text{GSCl}(\lambda)) \subseteq \text{Neu-Cl}(f(\lambda))$, for every Neutrosophic set λ in E_1 .

(ii) $\text{Neu-}\alpha\text{GSCl}(f^{-1}(\mu)) \subseteq f^{-1}(\text{Neu-Cl}(\mu))$, for every Neutrosophic set μ in E_2 .

Proof.

(i) Since $\text{Neu-Cl}(f(\lambda))$ is a Neu-CS in E_2 and f is a Neu- α GS continuity maps, $f^{-1}(\text{Neu-Cl}(f(\lambda)))$ is Neu- α GSCS in E_1 . That is $\text{Neu-}\alpha\text{GSCl}(\lambda) \subseteq f^{-1}(\text{Neu-Cl}(f(\lambda)))$. Therefore $f(\text{Neu-}\alpha\text{GSCl}(\lambda)) \subseteq \text{Neu-Cl}(f(\lambda))$, for every Neutrosophic set λ in E_1 .

(ii) Replacing λ by $f^{-1}(\mu)$ in (i) we get $f(\text{Neu-}\alpha\text{GSCl}(f^{-1}(\mu))) \subseteq \text{Neu-Cl}(f(f^{-1}(\mu))) \subseteq \text{Neu-Cl}(\mu)$. Hence $\text{Neu-}\alpha\text{GSCl}(f^{-1}(\mu)) \subseteq f^{-1}(\text{Neu-Cl}(\mu))$, for every Neutrosophic set μ in E_2 .

4. Neutrosophic α -Generalized Semi Irresolute Maps

In this section we Introduce Neutrosophic α -generalized semi irresolute maps and study some of its characterizations.

Definition 4.1.

A maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ is called a Neutrosophic alpha-generalized semi irresolute (Neu- α GS irresolute) maps if $f^{-1}(\lambda)$ is a Neu- α GSCS in (E_1, τ_N) for every Neu- α GSCS λ of (E_2, σ_N)

Theorem 4.2.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS irresolute, then f is a Neu- α GS continuity maps.

Proof.

Let f be a Neu- α GS irresolute maps. Let λ be any Neu-CS in E_2 . Since every Neu-CS is a Neu- α GSCS, λ is a Neu- α GSCS in E_2 . By hypothesis $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Hence f is a Neu- α GS continuity maps.

Example 4.3.

Neu- α GS continuity maps is not Neu- α GS irresolute maps.

Let $E_1 = \{a_1, a_2\}$, $E_2 = \{b_1, b_2\}$, $U = \langle e_1, (.4, .5, .7), (.5, .5, .6) \rangle$ and $V = \langle e_2, (.8, .5, .3), (.4, .6, .7) \rangle$. Then $\tau_N = \{0_N, U, 1_N\}$ and $\sigma_N = \{0_N, V, 1_N\}$ are Neutrosophic Topologies on E_1 and E_2 respectively. Define a maps $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ by $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is a Neu- α GS continuity. We have $\mu = \langle e_2, (.2, .5, .9), (.6, .5, .5) \rangle$ is a Neu- α GSCS in E_2 but $f^{-1}(\mu)$ is not a Neu- α GSCS in E_1 . Therefore f is not a Neu- α GS irresolute maps.

Theorem 4.4.

Let $f: (E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS irresolute, then f is a Neutrosophic irresolute maps if E_1 is a Neu- α ga $U_{1/2}$ space.

Proof.

Let λ be a Neu-CS in E_2 . Then λ is a Neu- α GSCS in E_2 . Therefore $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 , by hypothesis. Since E_1 is a Neu- α ga $U_{1/2}$ space, $f^{-1}(\lambda)$ is a Neu-CS in E_1 . Hence f is a Neutrosophic irresolute maps.

Theorem 4.5.

Let $f:(E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ and $g:(E_2, \sigma_N) \rightarrow (E_3, \rho_N)$ be Neu- α GS irresolute maps, then $g \circ f:(E_1, \tau_N) \rightarrow (E_3, \rho_N)$ is a Neu- α GS irresolute maps.

Proof.

Let λ be a Neu- α GSCS in E_3 . Then $g^{-1}(\lambda)$ is a Neu- α GSCS in E_2 . Since f is a Neu- α GS irresolute maps, $f^{-1}((g^{-1}(\lambda)))$ is a Neu- α GSCS in E_1 . Hence $g \circ f$ is a Neu- α GS irresolute maps.

Theorem 4.6.

Let $f:(E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS irresolute and $g:(E_2, \sigma_N) \rightarrow (E_3, \rho_N)$ be Neu- α GS continuity maps, then $g \circ f:(E_1, \tau_N) \rightarrow (E_3, \rho_N)$ is a Neu- α GS continuity maps.

Proof.

Let λ be a Neu-CS in E_3 . Then $g^{-1}(\lambda)$ is a Neu- α GSCS in E_2 . Since f is a Neu- α GS irresolute, $f^{-1}((g^{-1}(\lambda)))$ is a Neu- α GSCS in E_1 . Hence $g \circ f$ is a Neu- α GS continuity maps.

Theorem 4.7.

Let $f:(E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a Neu- α GS irresolute, then f is a Neu-G irresolute maps if E_1 is a Neu- $\alpha_{gb}U_{1/2}$ space.

Proof.

Let λ be a Neu- α GSCS in E_2 . By hypothesis, $f^{-1}(\lambda)$ is a Neu- α GSCS in E_1 . Since E_1 is a Neu- $\alpha_{gb}U_{1/2}$ space, $f^{-1}(\lambda)$ is a Neu-GCS in E_1 . Hence f is a Neu-G irresolute maps.

Theorem 4.8.

Let $f:(E_1, \tau_N) \rightarrow (E_2, \sigma_N)$ be a maps from a Neutrosophic Topology E_1 Into a Neutrosophic Topology E_2 . Then the following conditions set are equivalent if E_1 and E_2 are Neu- $\alpha_{ga}U_{1/2}$ spaces.

- (i) f is a Neu- α GS irresolute maps.
- (ii) $f^{-1}(\mu)$ is a Neu- α GSOS in E_1 for each Neu- α GSOS μ in E_2 .
- (iii) $\text{Neu-Cl}(f^{-1}(\mu)) \subseteq f^{-1}(\text{Neu-Cl}(\mu))$ for each Neutrosophic set μ of E_2 .

Proof.

(i) $\rightarrow$ (ii) : Let μ be any Neu- α GSOS in E_2 . Then μ^c is a Neu- α GSCS in E_2 . Since f is Neu- α GS irresolute, $f^{-1}(\mu^c)$ is a Neu- α GSCS in E_1 . But $f^{-1}(\mu^c) = (f^{-1}(\mu))^c$. Therefore $f^{-1}(\mu)$ is a Neu- α GSOS in E_1 .

(ii) $\rightarrow$ (iii) : Let μ be any Neutrosophic set in E_2 and $\mu \subseteq \text{Neu-Cl}(\mu)$. Then $f^{-1}(\mu) \subseteq f^{-1}(\text{Neu-Cl}(\mu))$. Since $\text{Neu-Cl}(\mu)$ is a Neu-CS in E_2 , $\text{Neu-Cl}(\mu)$ is a Neu- α GSCS in E_2 . Therefore $(\text{Neu-Cl}(\mu))^c$ is a Neu- α GSOS in E_2 . By hypothesis, $f^{-1}((\text{Neu-Cl}(\mu))^c)$ is a Neu- α GSOS in E_1 . Since $f^{-1}((\text{Neu-Cl}(\mu))^c) = (f^{-1}(\text{Neu-Cl}(\mu)))^c$, $f^{-1}(\text{Neu-Cl}(\mu))$ is a Neu- α GSCS in E_1 . Since E_1 is Neu- $\alpha_{ga}U_{1/2}$ space, $f^{-1}(\text{Neu-Cl}(\mu))$ is a Neu-CS in E_1 . Hence $\text{Neu-Cl}(f^{-1}(\mu)) \subseteq \text{Neu-Cl}(f^{-1}(\text{Neu-Cl}(\mu))) = f^{-1}(\text{Neu-Cl}(\mu))$. That is $\text{Neu-Cl}(f^{-1}(\mu)) \subseteq f^{-1}(\text{Neu-Cl}(\mu))$.

(iii) $\rightarrow$ (i) : Let μ be any Neu- α GSCS in E_2 . Since E_2 is Neu- $\alpha_{ga}U_{1/2}$ space, μ is a Neu-CS in E_2 and $\text{Neu-Cl}(\mu) = \mu$. Hence $f^{-1}(\mu) = f^{-1}(\text{Neu-Cl}(\mu)) \supseteq \text{Neu-Cl}(f^{-1}(\mu))$. But clearly $f^{-1}(\mu) \subseteq \text{Neu-Cl}(f^{-1}(\mu))$. Therefore $\text{Neu-Cl}(f^{-1}(\mu)) = f^{-1}(\mu)$. This implies $f^{-1}(\mu)$ is a Neu-CS and hence it is a Neu- α GSCS in E_1 . Thus f is a Neu- α GS irresolute maps.

Conclusion

In this research paper using Neu- α GSCS (Neutrosophic α gs-closed sets) we are defined Neu- α GS continuity maps and analyzed its properties. After that we were compared already existing Neutrosophic continuity maps to Neu- α GSCS continuity maps. Furthermore we were extended to this maps to Neu- α GS irresolute maps. Finally This concepts can be extended to future Research for some mathematical applications.

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Neutrosophic α^m -continuity

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Abstract: In this paper, we introduce and study a new class of neutrosophic closed set called neutrosophic α^m -closed set. In this respect, we introduce the concepts of neutrosophic α^m -continuous, strongly neutrosophic α^m continuous, neutrosophic α^m -irresolute and present their basic properties.

Keywords: Neutrosophic α^m -closed set, neutrosophic α^m -continuous, strongly neutrosophic α^m -continuous, neutrosophic α^m -irresolute.

1 Introduction

In 1965, Zadeh [21] studied the idea of fuzzy sets and its logic. Later, Chang [8] introduced the concept of fuzzy topological spaces. Atanassov [1] discussed the concepts of intuitionistic fuzzy set[[2],[3],[4]]. The concepts of strongly fuzzy continuous and fuzzy gc-irresolute are introduced by G. Balasubramanian and P. Sundaram [6]. The idea of α^m -closed in topological spaces was introduced by M. Mathew and R. Parimelazhagan[16]. He also introduced and investigated, α^m -continuous maps in topological spaces together with S. Jafari[17]. The concept of fuzzy α^m -continuous function was introduced by R. Dhavaseelan[13]. After the introduction of the concept of neutrosophy and neutrosophic set by F. Smarandache [[19], [20]], the concepts of neutrosophic crisp set and neutrosophic crisp topological space were introduced by A. A. Salama and S. A. Alblowi[18]. In this paper, a new class of neutrosophic closed set called neutrosophic α^m closed set is studied. Furthermore, the concepts of neutrosophic α^m -continuous, strongly neutrosophic α^m -continuous, neutrosophic α^m -irresolute are introduced and obtain some interesting properties. Throughout this paper neutrosophic topological spaces (briefly *NTS*) $(S_1, \xi_1), (S_2, \xi_2)$ and (S_3, ξ_3) will be replaced by S_1, S_2 and S_3 , respectively.

2 Preliminaries

Definition 2.1. [19] Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
 $sup_I = i_{sup}, inf_I = i_{inf}$
 $sup_F = f_{sup}, inf_F = f_{inf}$
 $n - sup = t_{sup} + i_{sup} + f_{sup}$
 $n - inf = t_{inf} + i_{inf} + f_{inf}$. T, I, F are neutrosophic components.

Definition 2.2. [19] Let S_1 be a non-empty fixed set. A neutrosophic set (briefly N -set) Λ is an object such that $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$ where $\mu_\Lambda(x), \sigma_\Lambda(x)$ and $\gamma_\Lambda(x)$ which represents the degree of membership function (namely $\mu_\Lambda(x)$), the degree of indeterminacy (namely $\sigma_\Lambda(x)$) and the degree of non-membership (namely $\gamma_\Lambda(x)$) respectively of each element $x \in S_1$ to the set Λ .

Remark 2.3. [19]

- (1) An N -set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$ can be identified to an ordered triple $\langle \mu_\Lambda, \sigma_\Lambda, \gamma_\Lambda \rangle$ in $]0^-, 1^+[$ on S_1 .
- (2) In this paper, we use the symbol $\Lambda = \langle \mu_\Lambda, \sigma_\Lambda, \gamma_\Lambda \rangle$ for the N -set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$.

Definition 2.4. [18] Let $S_1 \neq \emptyset$ and the N -sets Λ and Γ be defined as $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$, $\Gamma = \{\langle x, \mu_\Gamma(x), \sigma_\Gamma(x), \gamma_\Gamma(x) \rangle : x \in S_1\}$. Then

- (a) $\Lambda \subseteq \Gamma$ iff $\mu_\Lambda(x) \leq \mu_\Gamma(x), \sigma_\Lambda(x) \leq \sigma_\Gamma(x)$ and $\gamma_\Lambda(x) \geq \gamma_\Gamma(x)$ for all $x \in S_1$;
- (b) $\Lambda = \Gamma$ iff $\Lambda \subseteq \Gamma$ and $\Gamma \subseteq \Lambda$;
- (c) $\bar{\Lambda} = \{\langle x, \gamma_\Lambda(x), \sigma_\Lambda(x), \mu_\Lambda(x) \rangle : x \in S_1\}$; [Complement of Λ]
- (d) $\Lambda \cap \Gamma = \{\langle x, \mu_\Lambda(x) \wedge \mu_\Gamma(x), \sigma_\Lambda(x) \wedge \sigma_\Gamma(x), \gamma_\Lambda(x) \vee \gamma_\Gamma(x) \rangle : x \in S_1\}$;
- (e) $\Lambda \cup \Gamma = \{\langle x, \mu_\Lambda(x) \vee \mu_\Gamma(x), \sigma_\Lambda(x) \vee \sigma_\Gamma(x), \gamma_\Lambda(x) \wedge \gamma_\Gamma(x) \rangle : x \in S_1\}$;
- (f) $[\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), 1 - \mu_\Lambda(x) \rangle : x \in S_1\}$;
- (g) $\langle \Lambda = \{\langle x, 1 - \gamma_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x) \rangle : x \in S_1\}$.

Definition 2.5. [10] Let $\{\Lambda_i : i \in J\}$ be an arbitrary family of N -sets in S_1 . Then

- (a) $\bigcap \Lambda_i = \{\langle x, \wedge \mu_{\Lambda_i}(x), \wedge \sigma_{\Lambda_i}(x), \vee \gamma_{\Lambda_i}(x) \rangle : x \in S_1\}$;
- (b) $\bigcup \Lambda_i = \{\langle x, \vee \mu_{\Lambda_i}(x), \vee \sigma_{\Lambda_i}(x), \wedge \gamma_{\Lambda_i}(x) \rangle : x \in S_1\}$.

In order to develop NTS we need to introduce the N -sets 0_N and 1_N in S_1 as follows:

Definition 2.6. [10] $0_N = \{\langle x, 0, 0, 1 \rangle : x \in S_1\}$ and $1_N = \{\langle x, 1, 1, 0 \rangle : x \in S_1\}$.

Definition 2.7. [10] A neutrosophic topology (briefly N -topology) on $S_1 \neq \emptyset$ is a family ξ_1 of N -sets in S_1 satisfying the following axioms:

- (i) $0_N, 1_N \in \xi_1$,
- (ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in \xi_1$,
- (iii) $\cup G_i \in \xi_1$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \xi_1$.

In this case the ordered pair (S_1, ξ_1) or simply S_1 is called an *NTS* and each N -set in ξ_1 is called a neutrosophic open set (briefly N -open set). The complement $\bar{\Lambda}$ of an N -open set Λ in S_1 is called a neutrosophic closed set (briefly N -closed set) in S_1 .

Definition 2.8. [10] Let Λ be an N -set in an *NTS* S_1 . Then

$Nint(\Lambda) = \bigcup \{G \mid G \text{ is an } N\text{-open set in } S_1 \text{ and } G \subseteq \Lambda\}$ is called the neutrosophic interior (briefly N -interior) of Λ ;

$Ncl(\Lambda) = \bigcap \{G \mid G \text{ is an } N\text{-closed set in } S_1 \text{ and } G \supseteq \Lambda\}$ is called the neutrosophic closure (briefly N -cl) of Λ .

Definition 2.9. Let $S_1 \neq \emptyset$. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$ then the N -set $x_{r,t,s}$ is called a neutrosophic point (briefly NP) in S_1 given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in S_1$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Now we shall define the image and preimage of N -sets. Let $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ and $\Omega : S_1 \rightarrow S_2$ be a map.

Definition 2.10. [10]

- (a) If $\Gamma = \{\langle y, \mu_\Gamma(y), \sigma_\Gamma(y), \Gamma_\Gamma(y) \rangle : y \in S_2\}$ is an N -set in S_2 , then the pre-image of Γ under Ω , denoted by $\Omega^{-1}(\Gamma)$, is the N -set in S_1 defined by

$$\Omega^{-1}(\Gamma) = \{\langle x, \Omega^{-1}(\mu_\Gamma)(x), \Omega^{-1}(\sigma_\Gamma)(x), \Omega^{-1}(\Gamma_\Gamma)(x) \rangle : x \in S_1\}.$$

- (b) If $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \Gamma_\Lambda(x) \rangle : x \in S_1\}$ is an N -set in S_1 , then the image of Λ under Ω , denoted by $\Omega(\Lambda)$, is the N -set in S_2 defined by

$$\Omega(\Lambda) = \{\langle y, \Omega(\mu_\Lambda)(y), \Omega(\sigma_\Lambda)(y), (1 - \Omega(1 - \Gamma_\Lambda))(y) \rangle : y \in S_2\}. \text{ where}$$

$$\begin{aligned} \Omega(\mu_\Lambda)(y) &= \begin{cases} \sup_{x \in \Omega^{-1}(y)} \mu_\Lambda(x), & \text{if } \Omega^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \\ \Omega(\sigma_\Lambda)(y) &= \begin{cases} \sup_{x \in \Omega^{-1}(y)} \sigma_\Lambda(x), & \text{if } \Omega^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \\ (1 - \Omega(1 - \Gamma_\Lambda))(y) &= \begin{cases} \inf_{x \in \Omega^{-1}(y)} \Gamma_\Lambda(x), & \text{if } \Omega^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases} \end{aligned}$$

In what follows, we use the symbol $\Omega_-(\Gamma_\Lambda)$ for $1 - \Omega(1 - \Gamma_\Lambda)$.

Corollary 2.11. [10] Let $\Lambda, \Lambda_i (i \in J)$ be N -sets in S_1 , $\Gamma, \Gamma_i (i \in K)$ be N -sets in S_1 and $\Omega : S_1 \rightarrow S_2$ a function. Then

- (a) $\Lambda_1 \subseteq \Lambda_2 \Rightarrow \Omega(\Lambda_1) \subseteq \Omega(\Lambda_2)$,
- (b) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow \Omega^{-1}(\Gamma_1) \subseteq \Omega^{-1}(\Gamma_2)$,
- (c) $\Lambda \subseteq \Omega^{-1}(\Omega(\Lambda))$ { If Ω is injective, then $\Lambda = \Omega^{-1}(\Omega(\Lambda))$ } ,
- (d) $\Omega(\Omega^{-1}(\Gamma)) \subseteq \Gamma$ { If Ω is surjective, then $\Omega(\Omega^{-1}(\Gamma)) = \Gamma$ } ,
- (e) $\Omega^{-1}(\bigcup \Gamma_j) = \bigcup \Omega^{-1}(\Gamma_j)$,
- (f) $\Omega^{-1}(\bigcap \Gamma_j) = \bigcap \Omega^{-1}(\Gamma_j)$,
- (g) $\Omega(\bigcup \Lambda_i) = \bigcup \Omega(\Lambda_i)$,
- (h) $\Omega(\bigcap \Lambda_i) \subseteq \bigcap \Omega(\Lambda_i)$ { If Ω is injective, then $\Omega(\bigcap \Lambda_i) = \bigcap \Omega(\Lambda_i)$ } ,
- (i) $\Omega^{-1}(1_N) = 1_N$,
- (j) $\Omega^{-1}(0_N) = 0_N$,
- (k) $\Omega(1_N) = 1_N$, if Ω is surjective
- (l) $\Omega(0_N) = 0_N$,
- (m) $\overline{\Omega(\Lambda)} \subseteq \Omega(\overline{\Lambda})$, if Ω is surjective,
- (n) $\Omega^{-1}(\overline{\Gamma}) = \overline{\Omega^{-1}(\Gamma)}$.

Definition 2.12. [11] An N -set Λ in an $NTS (S_1, \xi_1)$ is called

- 1) a neutrosophic semiopen set (briefly N -semiopen) if $\Lambda \subseteq Ncl(Nint(\Lambda))$.
- 2) a neutrosophic α open set (briefly $N\alpha$ -open set) if $\Lambda \subseteq Nint(Ncl(Nint(\Lambda)))$.
- 3) a neutrosophic preopen set (briefly N -preopen set) if $\Lambda \subseteq Nint(Ncl(\Lambda))$.
- 4) a neutrosophic regular open set (briefly N -regular open set) if $\Lambda = Nint(Ncl(\Lambda))$.
- 5) a neutrosophic semipre open or β open set (briefly $N\beta$ -open set) if $\Lambda \subseteq Ncl(Nint(Ncl(\Lambda)))$.

An N -set Λ is called a neutrosophic semiclosed set, neutrosophic α closed set, neutrosophic preclosed set, neutrosophic regular closed set and neutrosophic β closed set, respectively, if the complement of Λ is an N -semiopen set, $N\alpha$ -open set, N -preopen set, N -regular open set, and $N\beta$ -open set, respectively.

Definition 2.13. [10] Let (S_1, ξ_1) be an NTS . An N -set Λ in (S_1, ξ_1) is said to be a generalized neutrosophic closed set (briefly g - N -closed set) if $Ncl(\Lambda) \subseteq G$ whenever $\Lambda \subseteq G$ and G is an N -open set. The complement of a generalized neutrosophic closed set is called a generalized neutrosophic open set (briefly g - N -open set).

Definition 2.14. [10] Let (S_1, ξ_1) be an NTS and Λ be an N -set in S_1 . Then the neutrosophic generalized closure (briefly N - g - cl) and neutrosophic generalized interior (briefly N - g - Int) of Λ are defined by,

- (i) $NGcl(\Lambda) = \bigcap \{G: G \text{ is a } g\text{-}N\text{-closed set in } S_1 \text{ and } \Lambda \subseteq G\}$.
- (ii) $NGint(\Lambda) = \bigcup \{G: G \text{ is a } g\text{-}N\text{-open set in } S_1 \text{ and } \Lambda \supseteq G\}$.

3 Neutrosophic α^m continuous functions

Definition 3.1. An N -subset Λ of an $NTS (S_1, \xi_1)$ is called neutrosophic α^m -closed set (briefly $N\alpha^m$ -closed set) if $Nint(Ncl(\Lambda)) \subseteq U$ whenever $\Lambda \subseteq U$ and U is $N\alpha$ -open.

Definition 3.2. An N -subset λ of an $NTS (S_1, \xi_1)$ is called a neutrosophic αg -closed set (briefly $N\alpha g$ -closed set) if $\alpha Ncl(\lambda) \subseteq U$ whenever $\lambda \subseteq U$ and U is an $N\alpha$ -open set in S_1 .

Definition 3.3. An N -subset λ of an $NTS (S_1, \xi_1)$ is called a neutrosophic $g\alpha$ -closed set (briefly $Ng\alpha$ -closed set) if $\alpha Ncl(\Lambda) \subseteq U$ whenever $\Lambda \subseteq U$ and U is an N -open set in S_1 .

Remark 3.4. In an $NTS (S_1, \xi_1)$, the following statements are true:

- (i) Every N -closed set is an Ng -closed set.
- (ii) Every N -closed set is an $N\alpha$ -closed set.

Remark 3.5. In an $NTS (S_1, \xi_1)$, the following statements are true:

- (i) Every Ng -closed set is an $Ng\alpha$ -closed set.
- (ii) Every $N\alpha$ -closed set is an $N\alpha g$ -closed set.
- (iii) Every $N\alpha g$ -closed set is an $Ng\alpha$ -closed set.

Remark 3.6. In an $NTS (S_1, \xi_1)$, the following statements are true:

- (i) Every N -closed set is an $N\alpha^m$ -closed set.
- (ii) Every $N\alpha^m$ -closed set is an $N\alpha$ -closed set.
- (iii) Every $N\alpha^m$ -closed set is an $N\alpha g$ -closed set.
- (iv) Every $N\alpha^m$ -closed set is an $Ng\alpha$ -closed set.

Proof. (i) This follows directly from the definitions.

(ii) Let Λ be an $N\alpha^m$ -closed set in S_1 and U a N -open set such that $\Lambda \subseteq U$. Since every N -open set is an $N\alpha$ -open set and Λ is a $N\alpha^m$ -closed set, $Nint(Ncl(\Lambda)) \subseteq (Nint(Ncl(\Lambda))) \cup (Ncl(Nint(\Lambda))) \subseteq U$. Therefore, Λ is an $N\alpha$ -closed set in S_1 .

(iii) It is a consequence of (ii) and remark 3.5 (ii).

(iv) It is a consequence of (iii) and remark 3.5 (iii). □

Proposition 3.7. The intersection of an $N\alpha^m$ -closed set and an N -closed set is an $N\alpha^m$ -closed set.

Proof. Let Λ be an $N\alpha^m$ -closed set and Ψ an N -closed set. Since Λ is an $N\alpha^m$ -closed set, $Nint(Ncl(\Lambda)) \subseteq U$ whenever $\Lambda \subseteq U$, where U is an $N\alpha$ -open set. To show that $\Lambda \cap \Psi$ is an $N\alpha^m$ -closed set, it is enough to show that $Nint(Ncl(\Lambda \cap \Psi)) \subseteq U$ whenever $\Lambda \cap \Psi \subseteq U$, where U is an $N\alpha$ -open set. Let $M = S_1 - \Psi$. Then $\Lambda \subseteq U \cup M$. Since M is an N -open set, $U \cup M$ is an $N\alpha$ -open set and Λ is an $N\alpha^m$ -closed set, $Nint(Ncl(\Lambda)) \subseteq U \cup M$. Now, $Nint(Ncl(\Lambda \cap \Psi)) \subseteq Nint(Ncl(\Lambda)) \cap Nint(Ncl(\Psi)) \subseteq Nint(Ncl(\Lambda)) \cap \Psi \subseteq (U \cup M) \cap \Psi \subseteq (U \cap \Psi) \cup (M \cap \Psi) \subseteq (U \cap \Psi) \cup 0_N \subseteq U$. This implies that $\Lambda \cap \Psi$ is an $N\alpha^m$ -closed set. □

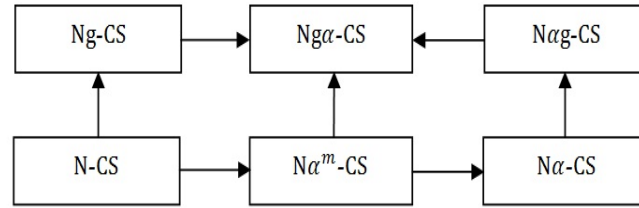


Figure 1: Implications of a neutrosophic α^m -closed set

Proposition 3.8. If Λ and Γ are two $N\alpha^m$ -closed sets in an $NTS (S_1, \xi_1)$, then $\Lambda \cap \Gamma$ is an $N\alpha^m$ -closed set in S_1 .

Proof. Let Λ and Γ be two $N\alpha^m$ -closed sets in an $NTS (S_1, \xi_1)$. Let U be a $N\alpha$ -open set in S_1 such that $\Lambda \cap \Gamma \subseteq U$. Now, $Nint(Ncl(\Lambda \cap \Gamma)) \subseteq Nint(Ncl(\Lambda)) \cap Nint(Ncl(\Gamma)) \subseteq U$. Hence $\Lambda \cap \Gamma$ is an $N\alpha^m$ -closed set. $\square$

Proposition 3.9. Every $N\alpha^m$ -closed set is $N\alpha$ -closed set.

The converse of the above Proposition 3.9 need not be true.

Example 3.10. Let $S_1 = \{a, b, c\}$. Define the N -subsets Λ and Γ as follows

$\Lambda = \{x, (\frac{a}{0.7}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.7}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.5})\}$, $\Gamma = \{x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.7}, \frac{b}{0.7}, \frac{c}{0.7})\}$. Then $\xi_1 = \{0_{S_1}, 1_{S_1}, \Lambda, \Gamma\}$ is an N -topology on S_1 . Clearly (S_1, ξ_1) is an NTS . Observe that the N -subset $\Sigma = \{x, (\frac{a}{0.3}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.3}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.7}, \frac{b}{0.4}, \frac{c}{0.5})\}$ is $N\alpha$ -closed but it is not $N\alpha^m$ -closed set.

Definition 3.11. Let (S_1, ξ_1) be an NTS and Λ an N -subset of S_1 . Then the neutrosophic α^m -interior (briefly $N\alpha^m-I$) and the neutrosophic $N\alpha^m$ -closure (briefly $N\alpha^m-cl$) of Λ are defined by,

$$\alpha^m Nint(\Lambda) = \cup \{U | U \text{ is } N\alpha^m\text{-open set in } S_1 \text{ and } \Lambda \supseteq U\}$$

$$\alpha^m Ncl(\Lambda) = \cap \{U | U \text{ is } N\alpha^m\text{-closed set in } S_1 \text{ and } \Lambda \subseteq U\}.$$

Proposition 3.12. If Λ is an $N\alpha^m$ -c-set and $\Lambda \subseteq \Gamma \subseteq Nint(Ncl(\Lambda))$, then Γ is $N\alpha^m$ -c-set.

Proof. Let Λ be an $N\alpha^m$ -c-set such that $\Lambda \subseteq \Gamma \subseteq Nint(Ncl(\Lambda))$. Let U be an $N\alpha$ -open set of S_1 such that $\Gamma \subseteq U$. Since Λ is $N\alpha^m$ -c-set, we have $Nint(Ncl(\Lambda)) \subseteq U$, whenever $\Lambda \subseteq U$. Since $\Lambda \subseteq \Gamma$ and $\Gamma \subseteq Nint(Ncl(\Lambda))$, then $Nint(Ncl(\Gamma)) \subseteq Nint(Ncl(Nint(Ncl(\Lambda)))) \subseteq Nint(Ncl(\Lambda)) \subseteq U$. Therefore $Nint(Ncl(\Gamma)) \subseteq U$. Hence Γ is an $N\alpha^m$ -c-set in S_1 . $\square$

Remark 3.13. The union of two $N\alpha^m$ -c-sets need not be an $N\alpha^m$ -c-set.

Remark 3.14. The following are the implications of an $N\alpha^m$ -c-set and the reverses are not true.

Definition 3.15. Let (S_1, ξ_1) and (S_2, ξ_2) be any two NTS .

- 1) A map $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is called neutrosophic α^m -continuous (briefly $N\alpha^m$ -cont) if the inverse image of every N -closed set in (S_2, ξ_2) is $N\alpha^m$ -c-set in (S_1, ξ_1) .
Equivalently if the inverse image of every N -open set in (S_2, ξ_2) is $N\alpha^m$ -open set in (S_1, ξ_1) .
- 2) A map $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is called neutrosophic α^m -irresolute (briefly $N\alpha^m$ -I) if the inverse image of every $N\alpha^m$ -c-set in (S_2, ξ_2) is $N\alpha^m$ -c-set in (S_1, ξ_1) .
Equivalently if the inverse image of every $N\alpha^m$ -open set in (S_2, ξ_2) is $N\alpha^m$ -open set in (S_1, ξ_1) .
- 3) A map $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is called strongly neutrosophic α^m -continuous (briefly $SN\alpha^m$ -cont) if the inverse image of every $N\alpha^m$ -c-set in (S_2, ξ_2) is N -closed set in (S_1, ξ_1) .
Equivalently if the inverse image of every $N\alpha^m$ -open set in (S_2, ξ_2) is N -open set in (S_1, ξ_1) .

Proposition 3.16. Let (S_1, ξ_1) and (S_2, ξ_2) be any two NTS . If $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is NC , then it is $N\alpha^m$ -cont.

Proof. Let Λ be any N -closed set in (S_2, ξ_2) . Since Ω is NC , $\Omega^{-1}(\Lambda)$ is N -closed in (S_1, ξ_1) . Since every N -closed set is $N\alpha^m$ -c-set, $\Omega^{-1}(\Lambda)$ is $N\alpha^m$ -c-set in (S_1, ξ_1) . Therefore Ω is $N\alpha^m$ -cont. $\square$

The converse of Proposition 3.16 need not be true as it is shown in the following example.

Example 3.17. Let $S_1 = \{a, b, c\}$ and $S_2 = \{a, b, c\}$. Define N -subsets E, F, G and D as follows
 $E = \{x, (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.7}, \frac{b}{0.5}, \frac{c}{0.7})\}$, $F = \{x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.6})\}$, $G = \{x, (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5})\}$, and $D = \{x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.5}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.5}), (\frac{a}{0.7}, \frac{b}{0.7}, \frac{c}{0.5})\}$. Then the family $\xi_1 = \{0_{S_1}, 1_{S_1}, E, F\}$ is an NT on S_1 and $\xi_2 = \{0_{S_2}, 1_{S_2}, G, D\}$ is an NT on S_2 . Thus (S_1, ξ_1) and (S_2, ξ_2) are NTS . Define $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ as $\Omega(a) = a, \Omega(b) = c, \Omega(c) = b$. Clearly Ω is $N\alpha^m$ -cont but Ω is not NC since $\Omega^{-1}(D) \notin \xi_1$ for $D \in \xi_2$.

Proposition 3.18. Let (S_1, ξ_1) and (S_2, ξ_2) be any two neutrosophic NTS . If $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is $N\alpha^m$ -I, then it is $N\alpha^m$ -cont.

Proof. Let Λ be an N -closed set in (S_2, ξ_2) . Since every N -closed set is $N\alpha^m$ -c-set, Λ is $N\alpha^m$ -c-set in S_2 . Since Ω is $N\alpha^m$ -I, $\Omega^{-1}(\Lambda)$ is $N\alpha^m$ -c-set in (S_1, ξ_1) . Therefore Ω is $N\alpha^m$ -cont. $\square$

The converse of Proposition 3.18 need not be true.

Example 3.19. Let $S_1 = \{a, b, c\}$. Define the N -subsets E, F and G as follows
 $E = \{x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.6})\}$, $F = \{x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5})\}$ and $G = \{x, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.7}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.7}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.3})\}$. Then $\xi_1 = \{0_{S_1}, 1_{S_1}, E, F\}$ and $\xi_2 = \{0_{S_1}, 1_{S_1}, G\}$ are N -topologies on S_1 . Define $\Omega : (S_1, \xi_1) \rightarrow (S_1, \xi_2)$ as follows $\Omega(a) = b, \Omega(b) = a, \Omega(c) = c$. Observe that Ω is $N\alpha^m$ -continuous. But Ω is not $N\alpha^m$ -I. Since $D = \{x, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.8})\}$ is $N\alpha^m$ -c-set in (S_1, ξ_2) , $\Omega^{-1}(D)$ is not $N\alpha$ -c-set in (S_1, ξ_1) .

Proposition 3.20. Let (S_1, ξ_1) and (S_2, ξ_2) be any two NTS . If $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ is $SN\alpha^m$ -I, then it is NC .

Proof. Let Λ be an N -closed set in (S_2, ξ_2) . Since every N -closed set is $N\alpha^m$ -c-set. Since Ω is $SN\alpha^m$ -cont, $\Omega^{-1}(\Lambda)$ is N -closed set in (S_1, ξ_1) . Therefore Ω is NC . $\square$

The converse of Proposition 3.20 need not be true.

Example 3.21. Let $S_1 = \{a, b, c\}$. Define the N -subsets E, F and G as follows

$E = \{x, (\frac{a}{0.1}, \frac{b}{0.1}, \frac{c}{0.1}), (\frac{a}{0.1}, \frac{b}{0.1}, \frac{c}{0.1}), (\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{0.9})\}$, $F = \{x, (\frac{a}{0.1}, \frac{b}{0.1}, \frac{c}{0}), (\frac{a}{0.1}, \frac{b}{0.1}, \frac{c}{0}), (\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{1})\}$ and $G = \{x, (\frac{a}{0.1}, \frac{b}{0}, \frac{c}{0.1}), (\frac{a}{0.1}, \frac{b}{0}, \frac{c}{0.1}), (\frac{a}{0.9}, \frac{b}{1}, \frac{c}{0.9})\}$. Then $\xi_1 = \{0_{S_1}, 1_{S_1}, E, F\}$ and $\xi_2 = \{0_{S_1}, 1_{S_1}, G\}$ are N -topologies on S_1 . Define $\Omega(S_1, \xi_1) \rightarrow (S_1, \xi_2)$ as follows $\Omega(a) = \Omega(b) = a$, $\Omega(c) = c$. Ω is NC but Ω is not $SN\alpha^m$ -cont. Since $D = \{x, (\frac{a}{0.05}, \frac{b}{0}, \frac{c}{0.1}), (\frac{a}{0.05}, \frac{b}{0}, \frac{c}{0.1}), (\frac{a}{0.95}, \frac{b}{1}, \frac{c}{0.9})\}$ is $N\alpha^m$ - c -set in (S_1, ξ_2) , $\Omega^{-1}(D)$ is not N -closed set in (S_1, ξ_1) .

Proposition 3.22. Let (S_1, ξ_1) , (S_2, ξ_2) and (S_3, ξ_3) be any three NTS . Suppose $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$, $\Xi : (S_2, \xi_2) \rightarrow (S_3, \xi_3)$ are maps. Assume Ω is $N\alpha^m$ - I and Ξ is $N\alpha^m$ -cont, then $\Xi \circ \Omega$ is $N\alpha^m$ -cont.

Proof. Let Λ be an N -closed set in (S_3, ξ_3) . Since Ξ is $N\alpha^m$ -cont, $\Xi^{-1}(\Lambda)$ is $N\alpha^m$ - c -set in (S_2, ξ_2) . Since Ω is $N\alpha^m$ - I , $\Omega^{-1}(\Xi^{-1}(\Lambda))$ is $N\alpha^m$ -closed in (S_1, ξ_1) . Thus $\Xi \circ \Omega$ is $N\alpha^m$ -cont. $\square$

Proposition 3.23. Let (S_1, ξ_1) , (S_2, ξ_2) and (S_3, ξ_3) be any three NTS . Let $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ and $\Xi : (S_2, \xi_2) \rightarrow (S_3, \xi_3)$ be maps such that Ω is $SN\alpha^m$ -cont and Ξ is $N\alpha^m$ -cont, then $\Xi \circ \Omega$ is NC .

Proof. Let Λ be an N - c -set in (S_3, ξ_3) . Since Ξ is $N\alpha^m$ -cont, $\Xi^{-1}(\Lambda)$ is $N\alpha^m$ - c -set in (S_2, ξ_2) . Moreover, since Ω is $SN\alpha^m$ -cont, $\Omega^{-1}(\Xi^{-1}(\Lambda))$ is N -closed in (S_1, ξ_1) . Thus $\Xi \circ \Omega$ is NC . $\square$

Proposition 3.24. Let (S_1, ξ_1) , (S_2, ξ_2) and (S_3, ξ_3) be any three NTS . Let $\Omega : (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ and $\Xi : (S_2, \xi_2) \rightarrow (S_3, \xi_3)$ be two maps. Assume Ω and Ξ are $N\alpha^m$ - I , then $\Xi \circ \Omega$ is $N\alpha^m$ - I .

Proof. Let Λ be an $N\alpha^m$ - c -set in (S_3, ξ_3) . Since Ξ is $N\alpha^m$ - I , $\Xi^{-1}(\Lambda)$ is $N\alpha^m$ - c -set in (S_2, ξ_2) . Since Ω is $N\alpha^m$ - I , $\Omega^{-1}(\Xi^{-1}(\Lambda))$ is an $N\alpha^m$ - c -set in (S_1, ξ_1) . Thus $\Xi \circ \Omega$ is $N\alpha^m$ - I . $\square$

4 Conclusions

In this paper, a new class of neutrosophic closed set called neutrosophic α^m closed set is introduced and studied. Furthermore, the basic properties of neutrosophic α^m -continuity are presented with some examples.

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Application of Bipolar Neutrosophic sets to Incidence Graphs

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Abstract: In this research paper, we apply the idea of bipolar neutrosophic sets to incidence graphs. We present some notions, including bipolar neutrosophic incidence graphs, bipolar neutrosophic incidence cycle and bipolar neutrosophic incidence tree. We define strong path, strength and incidence strength of strongest path in bipolar neutrosophic incidence graphs. We investigate some properties of bipolar neutrosophic incidence graphs. We also describe an application of bipolar neutrosophic incidence graphs.

Keywords: Bipolar neutrosophic incidence graphs; Bipolar neutrosophic incidence cycle; Bipolar neutrosophic incidence tree.

1 Introduction

Graph theory is a mathematical structure which is used to represent a relationship between objects. It has been very successful in engineering and natural sciences. Sometimes, in many cases, graph theoretical concepts may be imprecise. To handle such cases, in 1975, Rosenfeld [1] gave the idea of fuzzy graphs. He considered fuzzy relations and proposed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhutani and Rosenfeld [2] studied the strong edges in fuzzy graphs. By applying bipolar fuzzy sets [3] to graphs, Akram [4] introduced the notion of bipolar fuzzy graphs. He described the different methods to construct the bipolar fuzzy graphs and discussed the some of their properties. Broumi et al [5] introduced the single-valued neutrosophic graphs by applying the concept of single-valued neutrosophic sets to graphs. Later on, Akram and Sarwar [6] studied the novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. They developed the independent and dominating sets of bipolar

neutrosophic graphs. Ishfaq et al [13, 14] introduced the rough neutrosophic digraphs and their applications. Later Akram et al [15] introduced the decision making approach based on neutrosophic rough information. Dinesh [7, 8] studied the graph structures and introduced the fuzzy incidence graphs. Fuzzy incidence graphs not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. Fuzzy incidence graphs play an important role to interconnect the networks. Incidence relations have significant parts in human and natural made networks, including pipe, road, power and interconnection networks. Later Mathew and Mordeson [9] introduced the connectivity concepts in fuzzy incidence graphs and also introduced fuzzy influence graphs [10]. In this paper, we apply the idea of bipolar neutrosophic sets to incidence graphs and introduce a new concept, namely bipolar neutrosophic incidence graphs.

Some of essential preliminaries from [7] and [11] are given below:

Let V^* be a non-empty set. Then $G^* = (V^*, E^*, I^*)$ is an incidence graph, where E^* is a subset of $V^* \times V^*$ and I^* is a subset of $V^* \times E^*$.

A fuzzy incidence graph on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G' = (\mu', \lambda', \psi')$, where μ' is a fuzzy set on V^* , λ' is a fuzzy relation on V^* and ψ' is a fuzzy set on $V^* \times E^*$ such that

$$\psi'(y, yz) \leq \mu'(y) \wedge \lambda'(yz), \quad \forall y, z \in V^*.$$

A bipolar neutrosophic set on a non-empty set V^* is an object having the form

$$B = \{(b, T_Y^+(b), I_Y^+(b), F_Y^+(b), T_Y^-(b), I_Y^-(b), F_Y^-(b)) : b \in V^*\},$$

where, $T_b^+, I_b^+, F_b^+ : V^* \longrightarrow [0, 1]$ and $T_b^-, I_b^-, F_b^- : V^* \longrightarrow [-1, 0]$.

For other notations and applications, readers are referred to [15-21].

2 Bipolar Neutrosophic Incidence Graphs

Definition 2.1. A bipolar neutrosophic incidence graphs (BNIG) on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G = (X, Y, Z)$, where

- (1) X is a bipolar neutrosophic set on V^* .
- (2) Y is a bipolar neutrosophic relation on V^* .
- (3) Z is a bipolar neutrosophic set on $V^* \times E^*$ such that

$$\begin{aligned} T_Z^+(x, xy) &\leq \min\{T_X^+(x), T_Y^+(xy)\}, & T_Z^-(x, xy) &\geq \max\{T_X^-(x), T_Y^-(xy)\}, \\ I_Z^+(x, xy) &\leq \min\{I_X^+(x), I_Y^+(xy)\}, & I_Z^-(x, xy) &\geq \max\{I_X^-(x), I_Y^-(xy)\}, \\ F_Z^+(x, xy) &\geq \max\{F_X^+(x), F_Y^+(xy)\}, & F_Z^-(x, xy) &\leq \min\{F_X^-(x), F_Y^-(xy)\}, \forall x, y \in V^*. \end{aligned}$$

Example 2.2. Let $G^* = (V^*, E^*, I^*)$ be an incidence graph, as shown in Fig. 1, where $V^* = \{w, x, y, z\}$, $E^* = \{wx, xy, yz, zw\}$ and $I^* = \{(w, wx), (x, wx), (x, xy), (y, xy), (y, yz), (z, yz), (z, zw), (w, zw)\}$. Let X be a bipolar neutrosophic set on V^* given as

$$\begin{aligned} X = \{ & (w, 0.2, 0.4, 0.7, -0.1, -0.2, -0.4), (x, 0.3, 0.5, 0.9, -0.1, -0.6, -0.7), \\ & (y, 0.4, 0.6, 0.9, -0.1, -0.2, -0.8), (z, 0.5, 0.6, 0.8, -0.2, -0.8, -0.6) \}. \end{aligned}$$

Let Y be a bipolar neutrosophic relation on V^* given as

$$Y = \{(wx, 0.1, 0.2, 0.8, -0.1, -0.2, -0.9), (xy, 0.2, 0.4, 0.7, -0.2, -0.3, -0.9), \\ (yz, 0.1, 0.2, 0.8, -0.1, -0.2, -0.9), (zw, 0.2, 0.3, 0.6, -0.1, -0.2, -0.7)\}.$$

Let Z be a bipolar neutrosophic set on $V^* \times E^*$ given as

$$Z = \{((w, wx), 0.1, 0.1, 0.8, -0.2, -0.2, -0.9), ((x, wx), 0.1, 0.2, 0.8, -0.2, -0.3, -0.9), \\ ((x, xy), 0.2, 0.3, 0.8, -0.2, -0.4, -0.9), ((y, xy), 0.1, 0.1, 0.8, -0.2, -0.2, -0.9), \\ ((y, yz), 0.1, 0.2, 0.7, -0.2, -0.3, -0.9), ((z, yz), 0.1, 0.2, 0.7, -0.2, -0.3, -0.7), \\ ((z, zw), 0.1, 0.1, 0.8, -0.2, -0.2, -0.9), ((w, zw), 0.2, 0.3, 0.5, -0.3, -0.3, -0.8)\}.$$

Then $G = (X, Y, Z)$ is a BNIG as shown in Fig. 2.

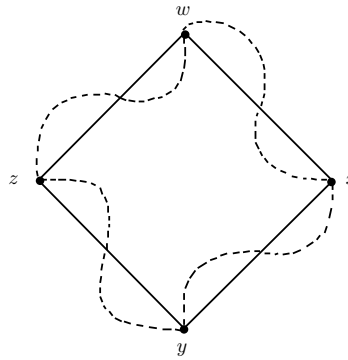


Figure 1: $G^* = (V^*, E^*, I^*)$

Definition 2.3. Let $G = (X, Y, Z)$ be a BNIG of G^* . Then *support* of $G = (X, Y, Z)$ is denoted by $\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$ such that

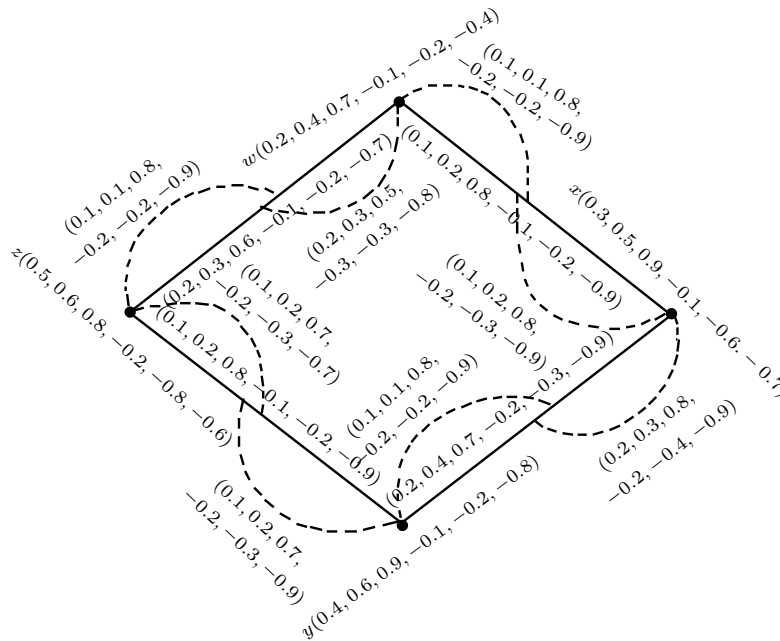
$$\text{supp}(X) = \{x \in X | T_X^+(x) > 0, I_X^+(x) > 0, F_X^+(x) > 0, \\ T_X^-(x) < 0, I_X^-(x) < 0, F_X^-(x) < 0\},$$

$$\text{supp}(Y) = \{xy \in Y | T_Y^+(xy) > 0, I_Y^+(xy) > 0, F_Y^+(xy) > 0, \\ T_Y^-(xy) < 0, I_Y^-(xy) < 0, F_Y^-(xy) < 0\},$$

$$\text{supp}(Z) = \{(x, xy) \in Z | T_Z^+(x, xy) > 0, I_Z^+(x, xy) > 0, F_Z^+(x, xy) > 0, \\ T_Z^-(x, xy) < 0, I_Z^-(x, xy) < 0, F_Z^-(x, xy) < 0\}.$$

Definition 2.4. A sequence

$x_0, (x_0, x_0x_1), x_0x_1, (x_1, x_0x_1), x_1, \dots, x_{n-1}, (x_{n-1}, x_{n-1}x_n), x_{n-1}x_n, (x_n, x_{n-1}x_n), x_n$
of vertices, edges and pairs in BNIG G is called *walk*.

Figure 2: BNIG $G = (X, Y, Z)$

If $x_0 = x_n$, it is a *close walk*.

If edges are distinct, it is a *trail*.

If pairs are distinct, it is an *incidence trail*.

If vertices are distinct, it is a *path*.

If pairs are distinct, it is an *incidence path*.

Example 2.5. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,

$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w, (w, wx), wx, (x, wx), x$ is a walk. It is not a path, trail and an incidence trail.

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z$$

is a path, trail and an incidence trail.

Definition 2.6. The BNIG $G = (X, Y, Z)$ is a *cycle* if and only if $\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$ is a cycle.

Example 2.7. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, consider a walk

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w.$$

which is a cycle. So $G = (X, Y, Z)$ is a cycle.

Definition 2.8. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic cycle* if and only if

$$\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$$

is a cycle and there exist at least two $xy \in \text{supp}(Y)$ such that

$$\begin{aligned} T_Y^+(xy) &= \min\{T_Y^+(uv) \mid uv \in \text{supp}(Y)\}, \\ I_Y^+(xy) &= \min\{I_Y^+(uv) \mid uv \in \text{supp}(Y)\}, \\ F_Y^+(xy) &= \max\{F_Y^+(uv) \mid uv \in \text{supp}(Y)\}, \\ T_Y^-(xy) &= \max\{T_Y^-(uv) \mid uv \in \text{supp}(Y)\}, \\ I_Y^-(xy) &= \max\{I_Y^-(uv) \mid uv \in \text{supp}(Y)\}, \\ F_Y^-(xy) &= \min\{F_Y^-(uv) \mid uv \in \text{supp}(Y)\}. \end{aligned}$$

Example 2.9. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, we have

$$\begin{aligned} T_Y^+(wx) &= 0.1 = \min\{T_Y^+(wx), T_Y^+(xy), T_Y^+(yz), T_Y^+(zw)\}, \\ I_Y^+(wx) &= 0.2 = \min\{I_Y^+(wx), I_Y^+(xy), I_Y^+(yz), I_Y^+(zw)\}, \\ F_Y^+(wx) &= 0.8 = \max\{F_Y^+(wx), F_Y^+(xy), F_Y^+(yz), F_Y^+(zw)\}, \\ T_Y^-(wx) &= -0.1 = \max\{T_Y^-(wx), T_Y^-(xy), T_Y^-(yz), T_Y^-(zw)\}, \\ I_Y^-(wx) &= -0.2 = \max\{I_Y^-(wx), I_Y^-(xy), I_Y^-(yz), I_Y^-(zw)\}, \\ F_Y^-(wx) &= -0.9 = \min\{F_Y^-(wx), F_Y^-(xy), F_Y^-(yz), F_Y^-(zw)\}. \end{aligned}$$

Also

$$\begin{aligned} T_Y^+(yz) &= 0.1 = \min\{T_Y^+(wx), T_Y^+(xy), T_Y^+(yz), T_Y^+(zw)\}, \\ I_Y^+(yz) &= 0.2 = \min\{I_Y^+(wx), I_Y^+(xy), I_Y^+(yz), I_Y^+(zw)\}, \\ F_Y^+(yz) &= 0.8 = \max\{F_Y^+(wx), F_Y^+(xy), F_Y^+(yz), F_Y^+(zw)\}, \\ T_Y^-(yz) &= -0.1 = \max\{T_Y^-(wx), T_Y^-(xy), T_Y^-(yz), T_Y^-(zw)\}, \\ I_Y^-(yz) &= -0.2 = \max\{I_Y^-(wx), I_Y^-(xy), I_Y^-(yz), I_Y^-(zw)\}, \\ F_Y^-(yz) &= -0.9 = \min\{F_Y^-(wx), F_Y^-(xy), F_Y^-(yz), F_Y^-(zw)\}. \end{aligned}$$

So $G = (X, Y, Z)$ is a bipolar neutrosophic cycle.

Definition 2.10. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic incidence cycle* if and only if it is a bipolar neutrosophic cycle and there exist at least two $(x, xy) \in \text{supp}(Z)$ such that

$$\begin{aligned} T_Z^+(x, xy) &= \min\{T_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ I_Z^+(x, xy) &= \min\{I_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ F_Z^+(x, xy) &= \max\{F_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ T_Z^-(x, xy) &= \max\{T_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ I_Z^-(x, xy) &= \max\{I_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ F_Z^-(x, xy) &= \min\{F_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}. \end{aligned}$$

Example 2.11. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,

we have

$$\begin{aligned}T_Z^+(w, wx) &= 0.1 = \min\{T_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\I_Z^+(w, wx) &= 0.1 = \min\{I_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\F_Z^+(w, wx) &= 0.8 = \max\{F_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\T_Z^-(w, wx) &= -0.2 = \max\{T_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\I_Z^-(w, wx) &= -0.2 = \max\{I_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\F_Z^-(w, wx) &= -0.9 = \min\{F_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}.\end{aligned}$$

and

$$\begin{aligned}T_Z^+(y, xy) &= 0.1 = \min\{T_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\I_Z^+(y, xy) &= 0.1 = \min\{I_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\F_Z^+(y, xy) &= 0.8 = \max\{F_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\T_Z^-(y, xy) &= -0.2 = \max\{T_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\I_Z^-(y, xy) &= -0.2 = \max\{I_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\F_Z^-(y, xy) &= -0.9 = \min\{F_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}.\end{aligned}$$

So $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle.

Definition 2.12. If $G = (X, Y, Z)$ is a BNIG, then $H = (X^*, Y^*, Z^*)$ is a *bipolar neutrosophic incidence subgraph* of G if

$$X^* \subseteq X, Y^* \subseteq Y, Z^* \subseteq Z.$$

$H = (X^*, Y^*, Z^*)$ is a *spanning subgraph* if $X = X^*$.

Definition 2.13. *Strength of the strongest path* from x to y in BNIG $G = (X, Y, Z)$ is defined as

$$\begin{aligned}T_{\rho^\infty}^+(x, y) &= \bigvee_{i=1}^k T_{\rho_i}^+(x, y), \quad I_{\rho^\infty}^+(x, y) = \bigvee_{i=1}^k I_{\rho_i}^+(x, y), \quad F_{\rho^\infty}^+(x, y) = \bigwedge_{i=1}^k F_{\rho_i}^+(x, y), \\T_{\rho^\infty}^-(x, y) &= \bigwedge_{i=1}^k T_{\rho_i}^-(x, y), \quad I_{\rho^\infty}^-(x, y) = \bigwedge_{i=1}^k I_{\rho_i}^-(x, y), \quad F_{\rho^\infty}^-(x, y) = \bigvee_{i=1}^k F_{\rho_i}^-(x, y).\end{aligned}$$

where $\rho(x, y)$ is the strength of path from x to y such that

$$\begin{aligned}T_\rho^+(x, y) &= \wedge \{T_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\I_\rho^+(x, y) &= \wedge \{I_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\F_\rho^+(x, y) &= \vee \{F_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\T_\rho^-(x, y) &= \vee \{T_Y^-(xy) \mid xy \in \text{supp}(Y)\}, \\I_\rho^-(x, y) &= \vee \{I_Y^-(xy) \mid xy \in \text{supp}(Y)\}, \\F_\rho^-(x, y) &= \wedge \{F_Y^-(xy) \mid xy \in \text{supp}(Y)\}.\end{aligned}$$

Definition 2.14. Incidence strength of the strongest path from x to wy in BNIG $G = (X, Y, Z)$ is defined as

$$\begin{aligned} T_{\psi\infty}^+(x, wy) &= \bigvee_{i=1}^k T_{\psi_i}^+(x, wy), & T_{\psi\infty}^-(x, wy) &= \bigwedge_{i=1}^k T_{\psi_i}^-(x, wy), \\ I_{\psi\infty}^+(x, wy) &= \bigvee_{i=1}^k I_{\psi_i}^+(x, wy), & I_{\psi\infty}^-(x, wy) &= \bigwedge_{i=1}^k I_{\psi_i}^-(x, wy), \\ F_{\psi\infty}^+(x, wy) &= \bigwedge_{i=1}^k F_{\psi_i}^+(x, wy), & F_{\psi\infty}^-(x, wy) &= \bigvee_{i=1}^k F_{\psi_i}^-(x, wy). \end{aligned}$$

where $\psi(x, wy)$ is the incidence strength of path from x to wy such that

$$\begin{aligned} T_{\psi}^+(x, wy) &= \bigwedge \{T_Z^+(x, wy) | (x, wy) \in \text{supp}(Z)\}, \\ I_{\psi}^+(x, wy) &= \bigwedge \{I_Z^+(x, wy) | (x, wy) \in \text{supp}(Z)\}, \\ F_{\psi}^+(x, wy) &= \bigvee \{F_Z^+(x, wy) | (x, wy) \in \text{supp}(Z)\}, \\ T_{\psi}^-(x, wy) &= \bigvee \{T_Z^-(x, wy) | (x, wy) \in \text{supp}(Z)\}, \\ I_{\psi}^-(x, wy) &= \bigvee \{I_Z^-(x, wy) | (x, wy) \in \text{supp}(Z)\}, \\ F_{\psi}^-(x, wy) &= \bigwedge \{F_Z^-(x, wy) | (x, wy) \in \text{supp}(Z)\}. \end{aligned}$$

Example 2.15. In a BNIG $G = (X, Y, Z)$ as shown in Fig.3

the strength of path $w, (w, wy), wy, (y, wy), y, (y, yz), yz, (z, yz), z$ is

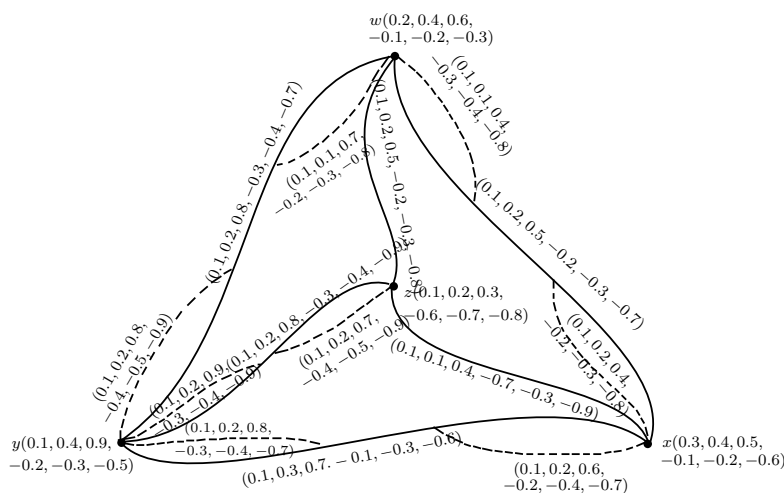


Figure 3: BNIG $G = (X, Y, Z)$

$$(0.1, 0.1, 0.8, -0.3, -0.4, -0.9),$$

the strength of path $w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z$ is

$$(0.1, 0.2, 0.8, -0.1, -0.3, -0.9),$$

the strength of the strongest path from w to z is

$$(0.1, 0.2, 0.8, -0.3, -0.4, -0.9).$$

In a BNIG $G = (X, Y, Z)$ as shown in Fig.3

the incidence strength of the path $w, (w, wy), wy, (y, wy), y, (y, yz), yz$ is

$$(0.1, 0.1, 0.9, -0.2, -0.3, -0.9),$$

the incidence strength of the path $w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz$ is

$$(0.1, 0.1, 0.8, -0.2, -0.3, -0.9),$$

the incidence strength of strongest path from w to yz is

$$(0.1, 0.1, 0.8, -0.2, -0.3, -0.9).$$

Definition 2.16. BNIG $G = (X, Y, Z)$ is called a *tree* if and only if $\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$ is a tree.

Definition 2.17. $G = (X, Y, Z)$ is a *bipolar single-valued neutrosophic tree* if and only if bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ is a tree such that

$$\begin{aligned} T_Y^+(xy) &< T_{\phi^\infty}^+(x, y), \quad I_Y^+(xy) < I_{\phi^\infty}^+(x, y), \quad F_Y^+(xy) > F_{\phi^\infty}^+(x, y), \\ T_Y^-(xy) &> T_{\phi^\infty}^-(x, y), \quad I_Y^-(xy) > I_{\phi^\infty}^-(x, y), \quad F_Y^-(xy) < F_{\phi^\infty}^-(x, y), \quad \forall xy \in \text{supp}(Y) \setminus \text{supp}(Y^*). \end{aligned}$$

where $\phi^\infty(x, y)$ is the strength of strongest path from x to y in $H = (X, Y^*, Z^*)$.

Definition 2.18. $G = (X, Y, Z)$ is a *bipolar neutrosophic incidence tree* if and only if bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ is a tree such that

$$\begin{aligned} T_Z^+(x, xy) &< T_{\tau^\infty}^+(x, xy), \quad I_Z^+(x, xy) < I_{\tau^\infty}^+(x, xy), \quad F_Z^+(x, xy) > F_{\tau^\infty}^+(x, xy), \\ T_Z^-(x, xy) &> T_{\tau^\infty}^-(x, xy), \quad I_Z^-(x, xy) > I_{\tau^\infty}^-(x, xy), \quad F_Z^-(x, xy) < F_{\tau^\infty}^-(x, xy), \quad \forall (x, xy) \in \text{supp}(Z) \setminus \text{supp}(Z^*). \end{aligned}$$

where $\tau^\infty(x, xy)$ is the strength of strongest path from x to xy in $H = (X, Y^*, Z^*)$.

Example 2.19. $G = (X, Y, Z)$ is a bipolar neutrosophic tree as shown in Fig.4 because a bipolar neutrosophic

incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ as shown in Fig.5 is a tree and

$$T_Y^+(wx) = 0.1 < 0.2 = T_{\phi\infty}^+(w, x),$$

$$I_Y^+(wx) = 0.1 < 0.2 = I_{\phi\infty}^+(w, x),$$

$$F_Y^+(wx) = 0.9 > 0.7 = F_{\phi\infty}^+(w, x),$$

$$T_Y^-(wx) = -0.1 > -0.2 = T_{\phi\infty}^-(w, x),$$

$$I_Y^-(wx) = -0.2 > -0.3 = I_{\phi\infty}^-(w, x),$$

$$F_Y^-(wx) = -0.9 < -0.8 = F_{\phi\infty}^-(w, x).$$

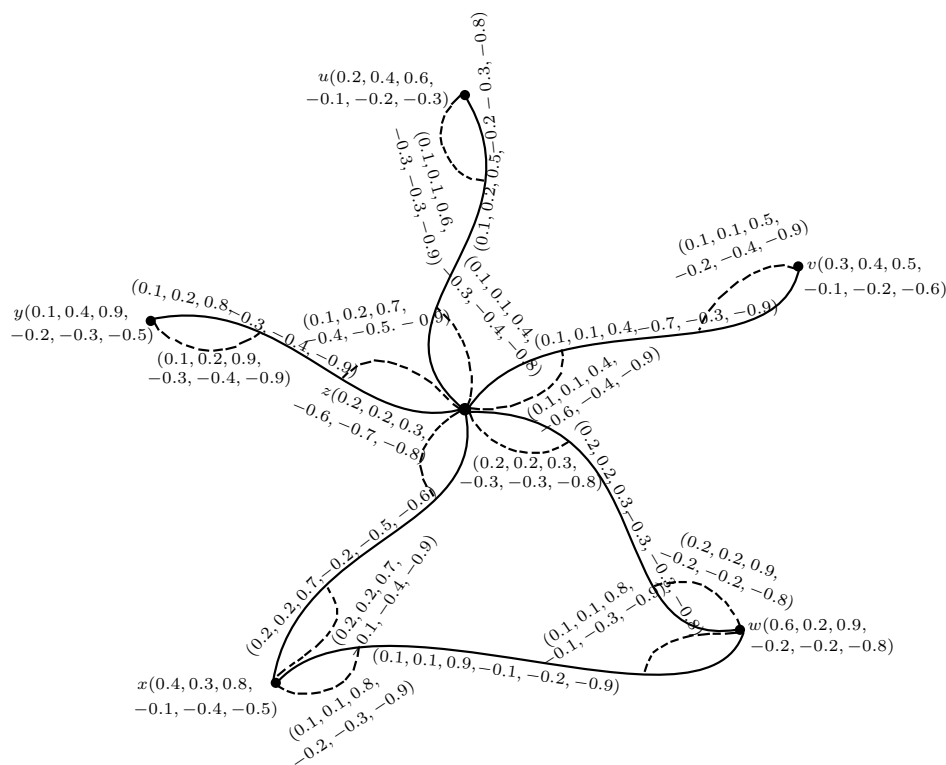
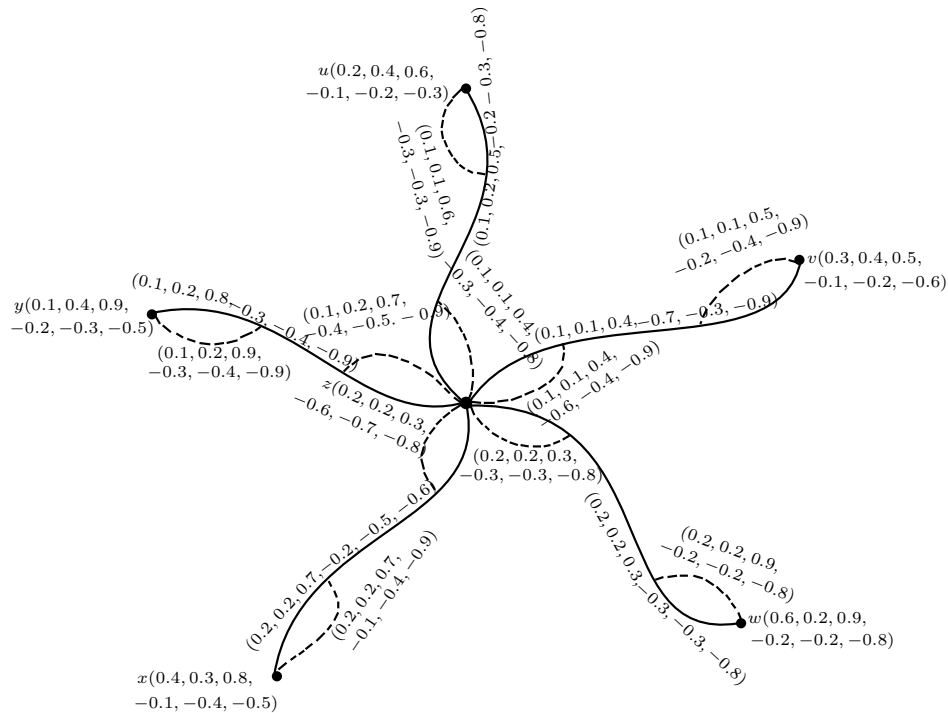


Figure 4: BNIG $G = (X, Y, Z)$

Theorem 2.20. Let $G = (X, Y, Z)$ be a cycle. Then $G = (X, Y, Z)$ is a bipolar neutrosophic cycle if and only if $G = (X, Y, Z)$ is not a bipolar neutrosophic tree.

Figure 5: $H = (X, Y^*, Z^*)$

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic cycle. So there exists $uv, xy \in \text{supp}(Y)$ such that

$$\begin{aligned} T_Y^+(uv) &= T_Y^+(xy) = \bigwedge \{T_Y^+(yz) | yz \in \text{supp}(Y)\}, \\ I_Y^+(uv) &= I_Y^+(xy) = \bigwedge \{I_Y^+(yz) | yz \in \text{supp}(Y)\}, \\ F_Y^+(uv) &= F_Y^+(xy) = \bigvee \{F_Y^+(yz) | yz \in \text{supp}(Y)\}, \\ T_Y^-(uv) &= T_Y^-(xy) = \bigvee \{T_Y^-(yz) | yz \in \text{supp}(Y)\}, \\ I_Y^-(uv) &= I_Y^-(xy) = \bigvee \{I_Y^-(yz) | yz \in \text{supp}(Y)\}, \\ F_Y^-(uv) &= F_Y^-(xy) = \bigwedge \{F_Y^-(yz) | yz \in \text{supp}(Y)\}. \end{aligned}$$

If $H = (X, Y^*, Z^*)$ is a spanning bipolar neutrosophic incidence tree of $G = (X, Y, Z)$, then $\text{supp}(Y) \setminus \text{supp}(Y^*) = \{yz\}$ for some $y, z \in V$ because $G = (X, Y, Z)$ is a cycle.

Hence there exists no path between y and z in $H = (X, Y^*, Z^*)$ such that

$$\begin{aligned} T_Y^+(yz) &< T_{\phi_\infty}^+(y, z), \quad I_Y^+(yz) < I_{\phi_\infty}^+(y, z), \quad F_Y^+(yz) > F_{\phi_\infty}^+(y, z), \\ T_Y^-(yz) &> T_{\phi_\infty}^-(y, z), \quad I_Y^-(yz) > I_{\phi_\infty}^-(y, z), \quad F_Y^-(yz) < F_{\phi_\infty}^-(y, z). \end{aligned}$$

Thus, $G = (X, Y, Z)$ is not a bipolar neutrosophic tree.

Conversely, let $G = (X, Y, Z)$ be not a bipolar neutrosophic tree. Because $G = (X, Y, Z)$ is a cycle, so for all

$yz \in \text{supp}(Y)$, $H = (X, Y^*, Z^*)$ is spanning bipolar neutrosophic incidence tree in $G = (X, Y, Z)$ such that

$$\begin{aligned} T_Y^+(yz) &\geq T_{\phi^\infty}^+(y, z), \quad I_Y^+(yz) \geq I_{\phi^\infty}^+(y, z), \quad F_Y^+(yz) \leq F_{\phi^\infty}^+(y, z), \\ T_Y^-(yz) &\leq T_{\phi^\infty}^-(y, z), \quad I_Y^-(yz) \leq I_{\phi^\infty}^-(y, z), \quad F_Y^-(yz) \geq F_{\phi^\infty}^-(y, z). \end{aligned}$$

where

$$\begin{aligned} T_{Y^*}^+(yz) &= 0, \quad I_{Y^*}^+(yz) = 0, \quad F_{Y^*}^+(yz) = 0, \\ T_{Y^*}^-(yz) &= 0, \quad I_{Y^*}^-(yz) = 0, \quad F_{Y^*}^-(yz) = 0. \end{aligned}$$

and

$$\begin{aligned} T_{Y^*}^+(uv) &= T_Y^+(uv), \quad I_{Y^*}^+(uv) = I_Y^+(uv), \quad F_{Y^*}^+(uv) = F_Y^+(uv), \\ T_{Y^*}^-(uv) &= T_Y^-(uv), \quad I_{Y^*}^-(uv) = I_Y^-(uv), \quad F_{Y^*}^-(uv) = F_Y^-(uv), \quad \forall uv \in \text{supp}(Y) \setminus \{yz\}. \end{aligned}$$

Hence, there exists more than one edge such that

$$\begin{aligned} T_Y^+(yz) &= \bigwedge \{T_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\ I_Y^+(yz) &= \bigwedge \{I_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\ F_Y^+(yz) &= \bigvee \{F_Y^+(xy) \mid xy \in \text{supp}(Y)\}, \\ T_Y^-(yz) &= \bigvee \{T_Y^-(xy) \mid xy \in \text{supp}(Y)\}, \\ I_Y^-(yz) &= \bigvee \{I_Y^-(xy) \mid xy \in \text{supp}(Y)\}, \\ F_Y^-(yz) &= \bigwedge \{F_Y^-(xy) \mid xy \in \text{supp}(Y)\}. \end{aligned}$$

Thus, $G = (X, Y, Z)$ is a bipolar neutrosophic cycle. □

Theorem 2.21. *If $G = (X, Y, Z)$ is a bipolar neutrosophic tree and $\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$ is not a tree, then there exists at least one edge $xy \in \text{supp}(Y)$ such that*

$$\begin{aligned} T_Y^+(xy) &< T_{\mu^\infty}^+(x, y), \quad I_Y^+(xy) < I_{\mu^\infty}^+(x, y), \quad F_Y^+(xy) > F_{\mu^\infty}^+(x, y), \\ T_Y^-(xy) &> T_{\mu^\infty}^-(x, y), \quad I_Y^-(xy) > I_{\mu^\infty}^-(x, y), \quad F_Y^-(xy) < F_{\mu^\infty}^-(x, y). \end{aligned}$$

where $\mu^\infty(x, y)$ is the strength of strongest path between u and v in $G = (X, Y, Z)$.

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic tree, then there exists a bipolar neutrosophic spanning subgraph $H = (X, Y^*, Z^*)$ that is tree and

$$\begin{aligned} T_Y^+(xy) &< T_{\rho^\infty}^+(x, y), \quad I_Y^+(xy) < I_{\rho^\infty}^+(x, y), \quad F_Y^+(xy) > F_{\rho^\infty}^+(x, y), \\ T_Y^-(xy) &> T_{\rho^\infty}^-(x, y), \quad I_Y^-(xy) > I_{\rho^\infty}^-(x, y), \quad F_Y^-(xy) < F_{\rho^\infty}^-(x, y), \quad \forall uv \in \text{supp}(Y) \setminus \text{supp}(Y^*). \end{aligned}$$

Also

$$\begin{aligned} T_{\rho^\infty}^+(x, y) &\leq T_{\mu^\infty}^+(x, y), \quad I_{\rho^\infty}^+(x, y) \leq I_{\mu^\infty}^+(x, y), \quad F_{\rho^\infty}^+(x, y) \geq F_{\mu^\infty}^+(x, y), \\ T_{\rho^\infty}^-(x, y) &\geq T_{\mu^\infty}^-(x, y), \quad I_{\rho^\infty}^-(x, y) \geq I_{\mu^\infty}^-(x, y), \quad F_{\rho^\infty}^-(x, y) \leq F_{\mu^\infty}^-(x, y). \end{aligned}$$

Thus,

$$T_Y^+(xy) < T_{\mu\infty}^+(x, y), \quad I_Y^+(xy) < I_{\mu\infty}^+(x, y), \quad F_Y^+(xy) > F_{\mu\infty}^+(x, y), \\ T_Y^-(xy) > T_{\mu\infty}^-(x, y), \quad I_Y^-(xy) > I_{\mu\infty}^-(x, y), \quad F_Y^-(xy) < F_{\mu\infty}^-(x, y), \quad \forall uv \in \text{supp}(Y) \setminus \text{supp}(Y^*)$$

and by hypothesis there exists at least one edge $xy \in \text{supp}(Y)$. □

Theorem 2.22. *Let $G = (X, Y, Z)$ be a cycle. Then $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle if and only if $G = (X, Y, Z)$ is not a bipolar neutrosophic incidence tree.*

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic incidence cycle. Then there exist at least two $(x, wy) \in \text{supp}(Z)$ such that

$$T_Z^+(x, yz) = \min\{T_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ I_Z^+(x, yz) = \min\{I_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ F_Z^+(x, yz) = \max\{F_Z^+(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ T_Z^-(x, yz) = \max\{T_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ I_Z^-(x, yz) = \max\{I_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}, \\ F_Z^-(x, yz) = \min\{F_Z^-(u, vw) \mid (u, vw) \in \text{supp}(Z)\}.$$

If $H = (X, Y^*, Z^*)$ is a spanning bipolar neutrosophic incidence tree of $G = (X, Y, Z)$, then $\text{supp}(Z) \setminus \text{supp}(Z^*) = \{(x, yz)\}$ for some $x \in V$ $yz \in \text{supp}(Y)$.

Hence there exists no path between x and yz in $H = (X, Y^*, Z^*)$ such that

$$T_Z^+(x, yz) < T_{\tau\infty}^+(x, yz), \quad I_Z^+(x, yz) < I_{\tau\infty}^+(x, yz), \quad F_Z^+(x, yz) > F_{\tau\infty}^+(x, yz), \\ T_Z^-(x, yz) > T_{\tau\infty}^-(x, yz), \quad I_Z^-(x, yz) > I_{\tau\infty}^-(x, yz), \quad F_Z^-(x, yz) < F_{\tau\infty}^-(x, yz).$$

Thus, $G = (X, Y, Z)$ is not a bipolar neutrosophic incidence tree.

Conversely, let $G = (X, Y, Z)$ be not a bipolar neutrosophic incidence tree. Then for all $(x, yz) \in \text{supp}(Z)$, $H = (X, Y^*, Z^*)$ is spanning bipolar neutrosophic incidence tree in $G = (X, Y, Z)$ such that

$$T_Z^+(x, yz) \geq T_{\tau\infty}^+(x, yz), \quad I_Z^+(x, yz) \geq I_{\tau\infty}^+(x, yz), \quad F_Z^+(x, yz) \leq F_{\tau\infty}^+(x, yz), \\ T_Z^-(x, yz) \leq T_{\tau\infty}^-(x, yz), \quad I_Z^-(x, yz) \leq I_{\tau\infty}^-(x, yz), \quad F_Z^-(x, yz) \geq F_{\tau\infty}^-(x, yz).$$

where

$$T_{Z^*}^+(x, yz) = 0, \quad I_{Z^*}^+(x, yz) = 0, \quad F_{Z^*}^+(x, yz) = 0, \\ T_{Z^*}^-(x, yz) = 0, \quad I_{Z^*}^-(x, yz) = 0, \quad F_{Z^*}^-(x, yz) = 0.$$

and

$$T_{Z^*}^+(u, vw) = T_Z^+(u, vw), \quad I_{Z^*}^+(u, vw) = I_Z^+(u, vw), \quad F_{Z^*}^+(u, vw) = F_Z^+(u, vw), \\ T_{Z^*}^-(u, vw) = T_Z^-(u, vw), \quad I_{Z^*}^-(u, vw) = I_Z^-(u, vw), \quad F_{Z^*}^-(u, vw) = F_Z^-(u, vw), \quad \forall (u, vw) \in \text{supp}(Z) \setminus \{(x, yz)\}.$$

Hence, there exists more than one pair such that

$$\begin{aligned}T_Z^+(u, vw) &= \wedge \{T_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\}, \\I_Z^+(u, vw) &= \wedge \{I_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\}, \\F_Z^+(u, vw) &= \vee \{F_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\}, \\T_Z^-(u, vw) &= \vee \{T_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\}, \\I_Z^-(u, vw) &= \vee \{I_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\}, \\F_Z^-(u, vw) &= \wedge \{F_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\}.\end{aligned}$$

Thus, $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle. $\square$

Definition 2.23. Let $G = (X, Y, Z)$ be a BNIG. An edge xy is called a *strong edge* if

$$\begin{aligned}T_Y^+(xy) &\geq T_{\epsilon^\infty}^+(x, y), \quad T_Y^-(xy) \leq T_{\epsilon^\infty}^-(x, y), \\I_Y^+(xy) &\geq I_{\epsilon^\infty}^+(x, y), \quad I_Y^-(xy) \leq I_{\epsilon^\infty}^-(x, y), \\F_Y^+(xy) &\leq F_{\epsilon^\infty}^+(x, y), \quad F_Y^-(xy) \geq F_{\epsilon^\infty}^-(x, y).\end{aligned}$$

An edge xy is called α -strong if

$$\begin{aligned}T_Y^+(xy) &> T_{\epsilon^\infty}^+(x, y), \quad T_Y^-(xy) < T_{\epsilon^\infty}^-(x, y), \\I_Y^+(xy) &> I_{\epsilon^\infty}^+(x, y), \quad I_Y^-(xy) < I_{\epsilon^\infty}^-(x, y), \\F_Y^+(xy) &< F_{\epsilon^\infty}^+(x, y), \quad F_Y^-(xy) > F_{\epsilon^\infty}^-(x, y).\end{aligned}$$

An edge xy is called β -strong if

$$\begin{aligned}T_Y^+(xy) &= T_{\epsilon^\infty}^+(x, y), \quad T_Y^-(xy) = T_{\epsilon^\infty}^-(x, y), \\I_Y^+(xy) &= I_{\epsilon^\infty}^+(x, y), \quad I_Y^-(xy) = I_{\epsilon^\infty}^-(x, y), \\F_Y^+(xy) &= F_{\epsilon^\infty}^+(x, y), \quad F_Y^-(xy) = F_{\epsilon^\infty}^-(x, y).\end{aligned}$$

where $\epsilon^\infty(x, y)$ is the strength of strongest path between x and y .

Definition 2.24. Let $G = (X, Y, Z)$ be a BNIG. An edge xy is called a δ -edge if

$$\begin{aligned}T_Y^+(xy) &< T_{\epsilon^\infty}^+(x, y), \quad T_Y^-(xy) > T_{\epsilon^\infty}^-(x, y), \\I_Y^+(xy) &< I_{\epsilon^\infty}^+(x, y), \quad I_Y^-(xy) > I_{\epsilon^\infty}^-(x, y), \\F_Y^+(xy) &> F_{\epsilon^\infty}^+(x, y), \quad F_Y^-(xy) < F_{\epsilon^\infty}^-(x, y).\end{aligned}$$

Definition 2.25. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is called a *strong pair* if

$$\begin{aligned}T_Z^+(w, xy) &\geq T_{\eta^\infty}^+(w, xy), \quad T_Z^-(w, xy) \leq T_{\eta^\infty}^-(w, xy), \\I_Z^+(w, xy) &\geq I_{\eta^\infty}^+(w, xy), \quad I_Z^-(w, xy) \leq I_{\eta^\infty}^-(w, xy), \\F_Z^+(w, xy) &\leq F_{\eta^\infty}^+(w, xy), \quad F_Z^-(w, xy) \geq F_{\eta^\infty}^-(w, xy).\end{aligned}$$

A pair (w, xy) is called α -strong if

$$\begin{aligned} T_Z^+(w, xy) &> T_{\eta^\infty}^+(w, xy), \quad T_Z^-(w, xy) < T_{\eta^\infty}^-(w, xy), \\ I_Z^+(w, xy) &> I_{\eta^\infty}^+(w, xy), \quad I_Z^-(w, xy) < I_{\eta^\infty}^-(w, xy), \\ F_Z^+(w, xy) &< F_{\eta^\infty}^+(w, xy), \quad F_Z^-(w, xy) > F_{\eta^\infty}^-(w, xy). \end{aligned}$$

A pair (w, xy) is called β -strong if

$$\begin{aligned} T_Z^+(w, xy) &= T_{\eta^\infty}^+(w, xy), \quad T_Z^-(w, xy) = T_{\eta^\infty}^-(w, xy), \\ I_Z^+(w, xy) &= I_{\eta^\infty}^+(w, xy), \quad I_Z^-(w, xy) = I_{\eta^\infty}^-(w, xy), \\ F_Z^+(w, xy) &= F_{\eta^\infty}^+(w, xy), \quad F_Z^-(w, xy) = F_{\eta^\infty}^-(w, xy). \end{aligned}$$

where $\eta^\infty(w, xy)$ is incidence strength of strongest path between w and xy .

Definition 2.26. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is called a δ -pair if

$$\begin{aligned} T_Z^+(w, xy) &< T_{\eta^\infty}^+(w, xy), \quad T_Z^-(w, xy) > T_{\eta^\infty}^-(w, xy), \\ I_Z^+(w, xy) &< I_{\eta^\infty}^+(w, xy), \quad I_Z^-(w, xy) > I_{\eta^\infty}^-(w, xy), \\ F_Z^+(w, xy) &> F_{\eta^\infty}^+(w, xy), \quad F_Z^-(w, xy) < F_{\eta^\infty}^-(w, xy). \end{aligned}$$

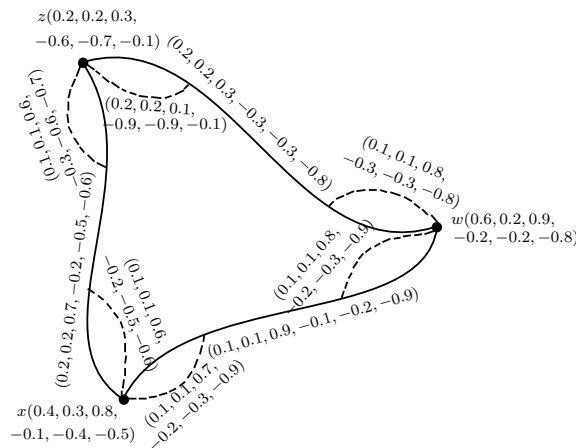
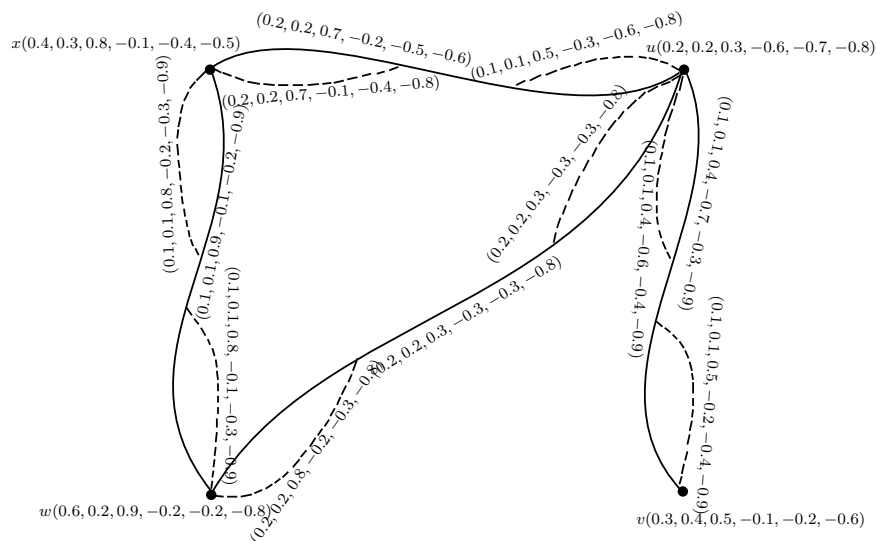


Figure 6: BNIG $G = (X, Y, Z)$

Example 2.27. In Fig.6 all edges except xw are strong. Indeed, wz and xz are α -strong edges. whereas, a pair (z, wz) is an α -strong pair and (w, xw) is a β -strong pair.

Definition 2.28. A path P in $G = (X, Y, Z)$ is called a *strong path* if all edges and pairs of P are strong. If strong path is closed, then it is called a *strong cycle*.

Example 2.29. In Fig.7 a path $x, (x, xu), xu, (u, xu), u, (u, uw), uw, (w, uw), w$ is strong path.

Figure 7: BNIG $G = (X, Y, Z)$

Theorem 2.30. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is strong if

$$\begin{aligned} T_Z^+(w, xy) &= \vee \{T_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ I_Z^+(w, xy) &= \vee \{I_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ F_Z^+(w, xy) &= \wedge \{F_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ T_Z^-(w, xy) &= \wedge \{T_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ I_Z^-(w, xy) &= \wedge \{I_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ F_Z^-(w, xy) &= \vee \{F_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}. \end{aligned}$$

Proof. Let $\psi^\infty(w, xy)$ be an incidence strength of strongest path between w and xy in $G = (X, Y, Z)$, then

$$\begin{aligned} T_{\psi^\infty}^+(w, xy) &\leq T_Z^+(w, xy), \quad T_{\psi^\infty}^-(w, xy) \geq T_Z^-(w, xy), \\ I_{\psi^\infty}^+(w, xy) &\leq I_Z^+(w, xy), \quad I_{\psi^\infty}^-(w, xy) \geq I_Z^-(w, xy), \\ F_{\psi^\infty}^+(w, xy) &\geq F_Z^+(w, xy), \quad F_{\psi^\infty}^-(w, xy) \leq F_Z^-(w, xy). \end{aligned}$$

If (w, xy) is only one pair such that

$$\begin{aligned} T_Z^+(w, xy) &= \vee \{T_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ I_Z^+(w, xy) &= \vee \{I_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ F_Z^+(w, xy) &= \wedge \{F_Z^+(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ T_Z^-(w, xy) &= \wedge \{T_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ I_Z^-(w, xy) &= \wedge \{I_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}, \\ F_Z^-(w, xy) &= \vee \{F_Z^-(u, vw) | (u, vw) \in \text{supp}(Z)\}. \end{aligned}$$

then for every path between u and vw , we have

$$\begin{aligned} T_{\psi\infty}^+(u, vw) &< T_Z^+(w, xy), \quad T_{\psi\infty}^-(u, vw) > T_Z^-(w, xy), \\ I_{\psi\infty}^+(u, vw) &< I_Z^+(w, xy), \quad I_{\psi\infty}^-(u, vw) > I_Z^-(w, xy), \\ F_{\psi\infty}^+(u, vw) &> F_Z^+(w, xy), \quad F_{\psi\infty}^-(u, vw) < F_Z^-(w, xy). \end{aligned}$$

hence

$$\begin{aligned} T_{\psi\infty}^+(w, xy) &< T_Z^+(w, xy), \quad T_{\psi\infty}^-(w, xy) > T_Z^-(w, xy), \\ I_{\psi\infty}^+(w, xy) &< I_Z^+(w, xy), \quad I_{\psi\infty}^-(w, xy) > I_Z^-(w, xy), \\ F_{\psi\infty}^+(w, xy) &> F_Z^+(w, xy), \quad F_{\psi\infty}^-(w, xy) < F_Z^-(w, xy). \end{aligned}$$

Thus, (w, xy) is an α -strong pair. If (w, xy) is not unique, then

$$\begin{aligned} T_Z^+(w, xy) &= T_{\psi\infty}^+(w, xy), \quad T_Z^-(w, xy) = T_{\psi\infty}^-(w, xy), \\ I_Z^+(w, xy) &= I_{\psi\infty}^+(w, xy), \quad I_Z^-(w, xy) = I_{\psi\infty}^-(w, xy), \\ F_Z^+(w, xy) &= F_{\psi\infty}^+(w, xy), \quad F_Z^-(w, xy) = F_{\psi\infty}^-(w, xy). \end{aligned}$$

Hence (w, xy) is β -strong pair. □

Theorem 2.31. *If $G = (X, Y, Z)$ is a bipolar neutrosophic incidence tree and P is a strong path between any two vertices x and y . Then P have maximum strength between x and y .*

Proof. Let P be only one strong path between x and y . Because P is strong, all edges and pairs of P are in the spanning bipolar neutrosophic incidence tree H of G . We prove that P is a path between x and y having maximum strength.

Suppose, on contrary that P is not a path having maximum strength from x to y and P' is such a path. Then P and P' are not equal, hence P and reversal of P' form a cycle. Since H^* is tree, so there exist no cycle in H . Hence any edge $x'y'$ of P' must not exist in H .

By definition of G , we have

$$\begin{aligned} T_Y^+(x'y') &< T_{\phi\infty}^+(x', y'), \quad I_Y^+(x'y') < I_{\phi\infty}^+(x', y'), \quad F_Y^+(x'y') > F_{\phi\infty}^+(x', y'), \\ T_Y^-(x'y') &> T_{\phi\infty}^-(x', y'), \quad I_Y^-(x'y') > I_{\phi\infty}^-(x', y'), \quad F_Y^-(x'y') < F_{\phi\infty}^-(x', y'). \end{aligned}$$

It means there exist a path between x' and y' in H and we can replace all edges $x'y'$ of P' which not exist in H by a path P^* from x to y in H . Hence P^* is at least as strong as P' . Hence P^* and P cannot be equal. So, P and reversal of P^* form a cycle in H , which is a contradiction to the fact that H^* is tree.

Hence our assumption P is not a path having maximum strength from x to y is wrong. □

3 Application to Illegal Migration

Suppose Mr.Kamran wants to travel from Bangladesh to India illegally. For this he use all borders line between Bangladesh and India. He have three ways, first one is a direct way, i.e. Bangladesh to India, second one is Bangladesh to Pakistan and Pakistan to India and the third one is Bangladesh to Bhutan, Bhutan to

Pakistan, Pakistan to Nepal and Nepal to India. Let $V = \{\text{Bangladesh}(BGD), \text{Bhutan}(BTN), \text{Pakistan}(PAK), \text{Nepal}(NPL), \text{India}(IND)\}$ be the set of countries and $E = \{(BGD, BTN), (BTN, PAK), (PAK, NPL), (NPL, IND), (BGD, PAK), (PAK, IND), (BGD, IND)\}$ a subset of $V \times V$.

Let X be the bipolar neutrosophic set on V , which is given as

$$X = \{(BGD, 0.3, 0.2, 0.6, -0.1, -0.2, -0.5), (BTN, 0.3, 0.6, 0.9, -0.2, -0.4, -0.6), \\ (PAK, 0.4, 0.5, 0.6, -0.1, -0.3, -0.4), (NPL, 0.9, 0.7, 0.8, -0.4, -0.3, -0.4), \\ (IND, 0.6, 0.9, 0.1, -0.1, -0.2, -0.3)\}.$$

Let Y be the bipolar neutrosophic relation on V , which is given as

$$Y = \{((BGD, BTN), 0.1, 0.2, 0.8, -0.2, -0.3, -0.7), ((BTN, PAK), 0.2, 0.5, 0.9, -0.3, -0.3, -0.7), \\ ((PAK, NPL), 0.3, 0.4, 0.7, -0.2, -0.4, -0.5), ((NPL, IND), 0.5, 0.6, 0.7, -0.2, -0.3, -0.5), \\ ((BGD, PAK), 0.3, 0.1, 0.6, -0.2, -0.2, -0.6), ((PAK, IND), 0.4, 0.4, 0.5, -0.1, -0.3, -0.5), \\ ((BGD, IND), 0.2, 0.1, 0.5, -0.1, -0.3, -0.6)\}.$$

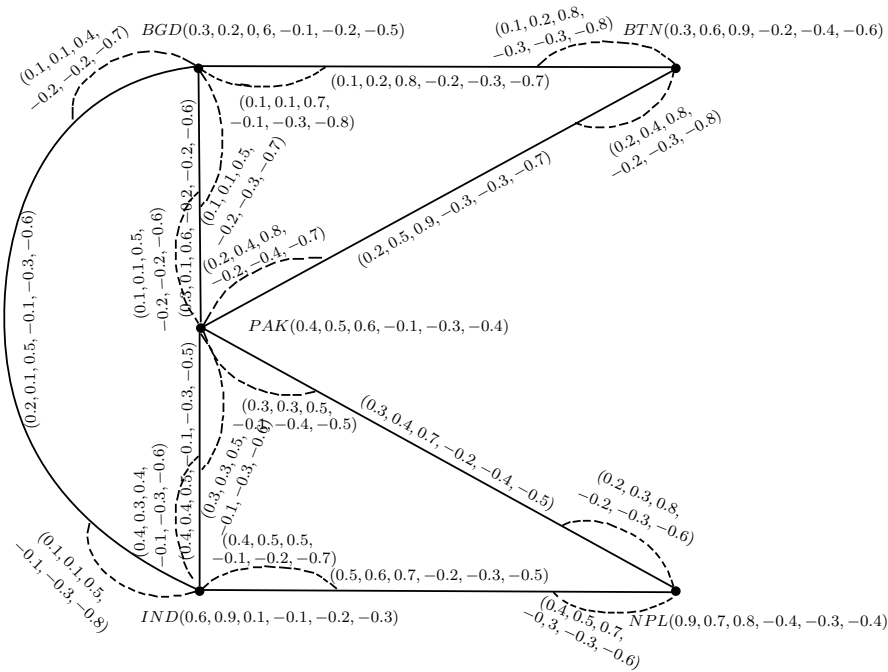
Let Z be the bipolar neutrosophic set on $V \times E$, which is given as

$$Z = \{((BGD, (BGD, BTN)), 0.1, 0.1, 0.7, -0.1, -0.3, -0.8), \\ ((BTN, (BGD, BTN)), 0.1, 0.2, 0.8, -0.3, -0.3, -0.8), \\ ((BTN, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.3, -0.8), \\ ((PAK, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.4, -0.7), \\ ((PAK, (PAK, NPL)), 0.3, 0.3, 0.5, -0.1, -0.4, -0.5), \\ ((NPL, (PAK, NPL)), 0.2, 0.3, 0.8, -0.2, -0.3, -0.6), \\ ((NPL, (NPL, IND)), 0.4, 0.5, 0.7, -0.3, -0.3, -0.6), \\ ((IND, (NPL, IND)), 0.4, 0.5, 0.5, -0.1, -0.2, -0.7), \\ ((BGD, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.3, -0.7), \\ ((PAK, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.2, -0.6), \\ ((PAK, (PAK, IND)), 0.3, 0.3, 0.5, -0.1, -0.3, -0.6), \\ ((IND, (PAK, IND)), 0.4, 0.3, 0.4, -0.1, -0.3, -0.6), \\ ((BGD, (BGD, IND)), 0.1, 0.1, 0.4, -0.2, -0.2, -0.7), \\ ((IND, (BGD, IND)), 0.1, 0.1, 0.5, -0.1, -0.3, -0.8)\}.$$

Thus, $G = (X, Y, Z)$ is a BNIG as shown in Fig.8.

Let $T_{\rho}^{+}(u, v)$ represent the degree of protection for an illegal immigrant to use u as origin and come to a destination v . There are three paths from BGD to IND

$$P_1 : BGD, (BGD, (BGD, IND)), (BGD, IND), (IND, (BGD, IND)), IND.$$

Figure 8: BNIG $G = (X, Y, Z)$

$$P_2 : BGD, (BGD, (BGD, PAK)), (BGD, PAK), (PAK, (BGD, PAK)), PAK, \\ (PAK, (PAK, IND)), (PAK, IND), (IND, (PAK, IND)), IND.$$

$$P_3 : BGD, (BGD, (BGD, BTN)), (BGD, BTN), (BTN, (BGD, BTN)), BTN, \\ (BTN, (BTN, PAK)), (BTN, PAK), (PAK, (BTN, PAK)), PAK, \\ (PAK, (PAK, NPL)), (PAK, NPL), (NPL, (PAK, NPL)), NPL, \\ (NPL, (NPL, IND)), (NPL, IND), (IND, (NPL, IND)), IND.$$

$\rho^\infty(BGD, IND)$ is the strength of strongest path between BGD and IND . This is the safest path between BGD and IND . To calculate the value of $\rho^\infty(BGD, IND)$, we need the strength of paths P_1, P_2 and P_3 , which is denoted by $\rho_{P_1}(BGD, IND)$, $\rho_{P_2}(BGD, IND)$ and $\rho_{P_3}(BGD, IND)$, respectively.

By calculation, we have

$$\rho_{P_1}(BGD, IND) = (0.2, 0.1, 0.5, -0.1, -0.3, -0.6),$$

$$\rho_{P_2}(BGD, IND) = (0.3, 0.1, 0.6, -0.1, -0.2, -0.6),$$

$$\rho_{P_3}(BGD, IND) = (0.1, 0.2, 0.9, -0.2, -0.3, -0.7).$$

$$\rho^\infty(BGD, IND) = (0.3, 0.2, 0.5, -0.2, -0.3, -0.6).$$

We see that

$$T_{\rho^\infty}^+(BGD, IND) = T_{\rho_{P_2}}^+(BGD, IND).$$

Hence P_2 is safest path for an illegal immigrant.

We present proposed method in the following Algorithm 3.1.

3.1 Algorithm

1. Input the vertex set V^* .
2. Input the edge set $E^* \subseteq V^* \times V^*$.
3. Input the bipolar neutrosophic set X on V^* .
4. Input the bipolar neutrosophic relation Y on V^* .
5. Input the bipolar neutrosophic set Z on $V^* \times E^*$.
6. Calculate the strength of path $\rho(x, y)$ from x to y such that

$$\begin{aligned} T_{\rho}^{+}(x, y) &= \wedge \{T_Y^{+}(xy) | xy \in \text{supp}(Y)\}, \\ I_{\rho}^{+}(x, y) &= \wedge \{I_Y^{+}(xy) | xy \in \text{supp}(Y)\}, \\ F_{\rho}^{+}(x, y) &= \vee \{F_Y^{+}(xy) | xy \in \text{supp}(Y)\}, \\ T_{\rho}^{-}(x, y) &= \vee \{T_Y^{-}(xy) | xy \in \text{supp}(Y)\}, \\ I_{\rho}^{-}(x, y) &= \vee \{I_Y^{-}(xy) | xy \in \text{supp}(Y)\}, \\ F_{\rho}^{-}(x, y) &= \wedge \{F_Y^{-}(xy) | xy \in \text{supp}(Y)\}. \end{aligned}$$

7. Calculate the incidence strength $\rho^{\infty}(x, y)$ of strongest path from x to y such that

$$\begin{aligned} T_{\rho^{\infty}}^{+}(x, y) &= \bigvee_{i=1}^k T_{\rho_i}^{+}(x, y), \quad I_{\rho^{\infty}}^{+}(x, y) = \bigvee_{i=1}^k I_{\rho_i}^{+}(x, y), \quad F_{\rho^{\infty}}^{+}(x, y) = \bigwedge_{i=1}^k F_{\rho_i}^{+}(x, y), \\ T_{\rho^{\infty}}^{-}(x, y) &= \bigwedge_{i=1}^k T_{\rho_i}^{-}(x, y), \quad I_{\rho^{\infty}}^{-}(x, y) = \bigwedge_{i=1}^k I_{\rho_i}^{-}(x, y), \quad F_{\rho^{\infty}}^{-}(x, y) = \bigvee_{i=1}^k F_{\rho_i}^{-}(x, y). \end{aligned}$$

8. The safest path is $S(v_k) = \bigvee_{i=1}^k T_{\rho_i}^{+}(x, y)$.
9. If v_k has more than one value then any path can be chosen.

4 Conclusion

Graph theory has become a branch of applied mathematics. Graph theory is considered as a mathematical tool for modeling and analyzing different mathematical structure, but it does not give the relationship between element and its relation pair. We have introduced BNIG which not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. We

have defined the bipolar neutrosophic incidence cycle and tree. An application to illegal migration is presented using strength of strongest path in BNIG.

Conflict of interest: The authors declare that they have no conflict of interest.

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Neutrosophic Hyper BCK-Ideals

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Abstract: In this paper we introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. Some relevant properties and their relations are indicated. Characterization of neutrosophic (weak) hyper BCK-ideal is considered. Conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal are discussed. Also, conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal are provided.

Keywords: Hyper BCK-algebra; hyper BCK-ideals; neutrosophic (strong, weak, s-weak) hyper BCK-ideal; reflexive neutrosophic hyper BCK-ideal.

1 Introduction

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty [17] when Marty defined hypergroups, began to analyze their properties, and applied them to groups and relational algebraic functions (See [17]). Since then, many papers and several books have been written on this topic. Hyperstructures have many applications to several sectors of both pure and applied sciences. (See [4, 5, 8, 11, 14, 19, 25]). In [16], Jun et al. applied the hyperstructures to *BCK*-algebras, and introduced the concept of a hyper *BCK*-algebra which is a generalization of a *BCK*-algebra. Since then, Jun et al. studied more notions and results in [12] and [15]. Also, several fuzzy versions of hyper *BCK*-algebras have been considered in [10] and [13]. The neutrosophic set, which is developed by Smarandache ([20], [21] and [22]), is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Borzooei et al. [6] studied neutrosophic deductive filters on

BL-algebras. Zhang et al. [26] applied the notion of neutrosophic set to pseudo-BCI algebras, and discussed neutrosophic regular filters and fuzzy regular filters. Neutrosophic set theory is applied to various parts and received attentions from many researches were proceed to develop, improve and expand the neutrosophic theory ([1], [2], [3], [7], [9], [18], [23] and [24]).

Our purpose is to introduce the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal, and reflexive neutrosophic hyper BCK-ideal. We consider their relations and related properties. We discuss characterizations of neutrosophic (weak) hyper BCK-ideal. We give conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal. We are interested in finding some provisions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal. We discuss conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal.

2 Preliminaries

In this section, we give the basic definitions of hyper BCK-ideals and neutrosophic set.

For a nonempty set H a function $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a hyper operation on H . If $A, B \subseteq H$, then $A \circ B = \cup \{a \circ b \mid a \in A, b \in B\}$.

A nonempty set H with a hyper operation “ $\circ$ ” and a constant 0 is called a hyper BCK-algebra (See [16]), if it satisfies the following conditions: for any $x, y, z \in H$,

$$(HBCK1) \quad (x \circ z) \circ (y \circ z) \ll x \circ y,$$

$$(HBCK2) \quad (x \circ y) \circ z = (x \circ z) \circ y,$$

$$(HBCK3) \quad x \circ H \ll \{x\},$$

$$(HBCK4) \quad x \ll y \text{ and } y \ll x \text{ imply } x = y,$$

where $x \ll y$ is defined by $0 \in x \circ y$. Also for any $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Lemma 2.1. ([16]) In a hyper BCK-algebra H , the condition (HBCK3) is equivalent to the following condition:

$$(\forall x, y \in H) (x \circ y \ll \{x\}). \quad (2.1)$$

Lemma 2.2. ([16]) Let H be a hyper BCK-algebra. Then

$$(i) \quad x \circ 0 \ll \{x\}, 0 \circ x \ll \{0\} \text{ and } 0 \circ 0 \ll \{0\}, \text{ for all } x \in H$$

$$(ii) \quad (A \circ B) \circ C = (A \circ C) \circ B, A \circ B \ll A \text{ and } 0 \circ A \ll \{0\}, \text{ for any nonempty subsets } A, B \text{ and } C \text{ of } H.$$

Lemma 2.3. ([16]) In any hyper BCK-algebra H , we have:

$$0 \circ 0 = \{0\}, 0 \ll x, x \ll x \text{ and } A \ll A, \quad (2.2)$$

$$A \subseteq B \text{ implies } A \ll B, \quad (2.3)$$

$$0 \circ x = \{0\} \text{ and } 0 \circ A = \{0\}, \quad (2.4)$$

$$A \ll \{0\} \text{ implies } A = \{0\}, \quad (2.5)$$

$$x \in x \circ 0, \quad (2.6)$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H .

Let $I \subseteq H$ be such that $0 \in I$. Then I is said to be (See [16] and [15])

- hyper BCK-ideal of H if

$$(\forall x, y \in H) (x \circ y \ll A, y \in A \Rightarrow x \in A). \quad (2.7)$$

- weak hyper BCK-ideal of H if

$$(\forall x, y \in H) (x \circ y \subseteq A, y \in A \Rightarrow x \in A). \quad (2.8)$$

- strong hyper BCK-ideal of H if

$$(\forall x, y \in H) ((x \circ y) \cap A \neq \emptyset, y \in A \Rightarrow x \in A). \quad (2.9)$$

A subset I of a hyper BCK-algebra H is said to be reflexive if $(x \circ x) \subseteq I$ for all $x \in H$.

Let H be a non-empty set. A neutrosophic set (NS) in H (See [21]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in H\}$$

where $A_T : H \rightarrow [0, 1]$ is a truth membership function, $A_I : H \rightarrow [0, 1]$ is an indeterminate membership function, and $A_F : H \rightarrow [0, 1]$ is a false membership function. For abbreviation, we continue to write $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in H\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a hyper BCK-algebra H and a subset S of H , by $*A_T$, $*A_T$, $*A_I$, $*A_I$, $*A_F$ and $*A_F$ we mean

$$\begin{aligned} *A_T(S) &= \inf_{a \in S} A_T(a) \text{ and } *A_T(S) = \sup_{a \in S} A_T(a), \\ *A_I(S) &= \inf_{a \in S} A_I(a) \text{ and } *A_I(S) = \sup_{a \in S} A_I(a), \\ *A_F(S) &= \inf_{a \in S} A_F(a) \text{ and } *A_F(S) = \sup_{a \in S} A_F(a), \end{aligned}$$

respectively.

Notation. From now on, in this paper, we assume that H is a hyper BCK-algebra.

3 Neutrosophic hyper BCK-ideals

In this section, we introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal, reflexive neutrosophic hyper BCK-ideal and discuss their properties.

Definition 3.1. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H . Then A is said to be a neutrosophic hyper BCK-ideal of H if it satisfies the following assertions for all $x, y \in H$,

$$\left(x \ll y \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases} \right), \quad (3.1)$$

$$\left(\begin{cases} A_T(x) \geq \min \{ {}_*A_T(x \circ y), A_T(y) \} \\ A_I(x) \geq \min \{ {}_*A_I(x \circ y), A_I(y) \} \\ A_F(x) \leq \max \{ {}_*A_F(x \circ y), A_F(y) \} \end{cases} \right). \quad (3.2)$$

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra. The hyper operation “ $\circ$ ” on H described by Table 1.

Table 1: Cayley table for the binary operation “ $\circ$ ”

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

We define a neutrosophic set $A = (A_T, A_I, A_F)$ on H by Table 2.

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.77	0.65	0.08
a	0.55	0.47	0.57
b	0.11	0.27	0.69

It is easy to check that $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H .

Proposition 3.3. For any neutrosophic hyper BCK-ideal $A = (A_T, A_I, A_F)$ of H , the following assertions are valid.

- (1) $A = (A_T, A_I, A_F)$ satisfies

$$(\forall x \in H) \left(\begin{cases} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{cases} \right). \quad (3.3)$$

(2) If $A = (A_T, A_I, A_F)$ satisfies

$$(\forall S \subseteq H)(\exists a, b, c \in S) \begin{pmatrix} A_T(a) = {}^*A_T(S) \\ A_I(b) = {}^*A_I(S) \\ A_F(c) = {}^*A_F(S) \end{pmatrix}, \quad (3.4)$$

then the following assertion is valid.

$$(\forall x, y \in H)(\exists a, b, c \in x \circ y) \begin{pmatrix} A_T(x) \geq \min\{A_T(a), A_T(y)\} \\ A_I(x) \geq \min\{A_I(b), A_I(y)\} \\ A_F(x) \leq \max\{A_F(c), A_F(y)\} \end{pmatrix}. \quad (3.5)$$

Proof. By (2.2) and (3.1) we have

$$A_T(0) \geq A_T(x), A_I(0) \geq A_I(x) \text{ and } A_F(0) \leq A_F(x).$$

Assume that $A = (A_T, A_I, A_F)$ satisfies the condition (3.4). For all $x, y \in H$, there exists $a_0, b_0, c_0 \in x \circ y$ such that

$$A_T(a_0) = {}^*A_T(x \circ y), A_I(b_0) = {}^*A_I(x \circ y) \text{ and } A_F(c_0) = {}^*A_F(x \circ y).$$

Now condition (3.2) implies that

$$\begin{aligned} A_T(x) &\geq \min\{{}^*A_T(x \circ y), A_T(y)\} = \min\{A_T(a_0), A_T(y)\} \\ A_I(x) &\geq \min\{{}^*A_I(x \circ y), A_I(y)\} = \min\{A_I(b_0), A_I(y)\} \\ A_F(x) &\leq \max\{{}^*A_F(x \circ y), A_F(y)\} = \max\{A_F(c_0), A_F(y)\}. \end{aligned}$$

This completes the proof. □

We define the following sets:

$$\begin{aligned} U(A_T, \varepsilon_T) &:= \{x \in H \mid A_T(x) \geq \varepsilon_T\}, \\ U(A_I, \varepsilon_I) &:= \{x \in H \mid A_I(x) \geq \varepsilon_I\}, \\ L(A_F, \varepsilon_F) &:= \{x \in H \mid A_F(x) \leq \varepsilon_F\}, \end{aligned}$$

where $A = (A_T, A_I, A_F)$ is a neutrosophic set in H and $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Lemma 3.4 ([12]). Let A be a subset of H . If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I .

Theorem 3.5. A neutrosophic set $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Proof. Assume that $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H and suppose that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$. It is easy to see that

$0 \in U(A_T, \varepsilon_T)$, $0 \in U(A_I, \varepsilon_I)$ and $0 \in L(A_F, \varepsilon_F)$. Let $x, y \in H$ be such that $x \circ y \ll U(A_T, \varepsilon_T)$ and $y \in U(A_T, \varepsilon_T)$. Then $A_T(y) \geq \varepsilon_T$ and for any $a \in x \circ y$ there exists $a_0 \in U(A_T, \varepsilon_T)$ such that $a \ll a_0$. We conclude from (3.1) that $A_T(a) \geq A_T(a_0) \geq \varepsilon_T$ for all $a \in x \circ y$. Hence $*A_T(x \circ y) \geq \varepsilon_T$, and so

$$A_T(x) \geq \min \{ *A_T(x \circ y), A_T(y) \} \geq \varepsilon_T,$$

that is, $x \in U(A_T, \varepsilon_T)$. Similarly, we show that if $x \circ y \ll U(A_I, \varepsilon_I)$ and $y \in U(A_I, \varepsilon_I)$, then $x \in U(A_I, \varepsilon_I)$. Hence $U(A_T, \varepsilon_T)$ and $U(A_I, \varepsilon_I)$ are hyper BCK-ideals of H . Let $x, y \in H$ be such that $x \circ y \ll L(A_F, \varepsilon_F)$ and $y \in L(A_F, \varepsilon_F)$. Then $A_F(y) \leq \varepsilon_F$. Let $b \in x \circ y$. Then there exists $b_0 \in L(A_F, \varepsilon_F)$ such that $b \ll b_0$, which implies from (3.1) that $A_F(b) \leq A_F(b_0) \leq \varepsilon_F$. Thus $*A_F(x \circ y) \leq \varepsilon_F$, and so

$$A_F(x) \leq \max \{ *A_F(x \circ y), A_F(y) \} \leq \varepsilon_F.$$

Hence $x \in L(A_F, \varepsilon_F)$, and therefore $L(A_F, \varepsilon_F)$ is a hyper BCK-ideal of H .

Conversely, suppose that the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$. Let $x, y \in H$ be such that $x \ll y$. Then

$$y \in U(A_T, A_T(y)) \cap U(A_I, A_I(y)) \cap L(A_F, A_F(y)),$$

and thus $x \ll U(A_T, A_T(y))$, $x \ll U(A_I, A_I(y))$ and $x \ll L(A_F, A_F(y))$. According to Lemma 3.4 we have $x \in U(A_T, A_T(y))$, $x \in U(A_I, A_I(y))$ and $x \in L(A_F, A_F(y))$ which imply that $A_T(x) \geq A_T(y)$, $A_I(x) \geq A_I(y)$ and $A_F(x) \leq A_F(y)$. For any $x, y \in H$, let $\varepsilon_T := \min \{ *A_T(x \circ y), A_T(y) \}$, $\varepsilon_I := \min \{ *A_I(x \circ y), A_I(y) \}$ and $\varepsilon_F := \max \{ *A_F(x \circ y), A_F(y) \}$. Then

$$y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F),$$

and for each $a_T, b_I, c_F \in x \circ y$ we have

$$A_T(a_T) \geq *A_T(x \circ y) \geq \min \{ *A_T(x \circ y), A_T(y) \} = \varepsilon_T,$$

$$A_I(b_I) \geq *A_I(x \circ y) \geq \min \{ *A_I(x \circ y), A_I(y) \} = \varepsilon_I$$

and

$$A_F(c_F) \leq *A_F(x \circ y) \leq \max \{ *A_F(x \circ y), A_F(y) \} = \varepsilon_F.$$

Hence $a_T \in U(A_T, \varepsilon_T)$, $b_I \in U(A_I, \varepsilon_I)$ and $c_F \in L(A_F, \varepsilon_F)$, and so $x \circ y \subseteq U(A_T, \varepsilon_T)$, $x \circ y \subseteq U(A_I, \varepsilon_I)$ and $x \circ y \subseteq L(A_F, \varepsilon_F)$. By (2.3), we have $x \circ y \ll U(A_T, \varepsilon_T)$, $x \circ y \ll U(A_I, \varepsilon_I)$ and $x \circ y \ll L(A_F, \varepsilon_F)$. It follows from (2.7) that

$$x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F).$$

Hence

$$A_T(x) \geq \varepsilon_T = \min \{ *A_T(x \circ y), A_T(y) \},$$

$$A_I(x) \geq \varepsilon_I = \min \{ {}_s A_I(x \circ y), A_I(y) \}$$

and

$$A_F(x) \leq \varepsilon_F = \max \{ {}_s A_F(x \circ y), A_F(y) \}.$$

Therefore $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H . $\square$

Theorem 3.6. If $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H , then the set

$$J := \{x \in H \mid A_T(x) = A_T(0), A_I(x) = A_I(0), A_F(x) = A_F(0)\} \quad (3.6)$$

is a hyper BCK-ideal of H .

Proof. It is easy to check that $0 \in J$. Let $x, y \in H$ be such that $x \circ y \ll J$ and $y \in J$. Then $A_T(y) = A_T(0)$, $A_I(y) = A_I(0)$ and $A_F(y) = A_F(0)$. Let $a \in x \circ y$. Then there exists $a_0 \in J$ such that $a \ll a_0$, and thus by (3.1), $A_T(a) \geq A_T(a_0) = A_T(0)$, $A_I(a) \geq A_I(a_0) = A_I(0)$ and $A_F(a) \leq A_F(a_0) = A_F(0)$. It follows from (3.2) that

$$A_T(x) \geq \min \{ {}_s A_T(x \circ y), A_T(y) \} \geq A_T(0),$$

$$A_I(x) \geq \min \{ {}_s A_I(x \circ y), A_I(y) \} \geq A_I(0)$$

and

$$A_F(x) \leq \max \{ {}_s A_F(x \circ y), A_F(y) \} \leq A_F(0).$$

Hence $A_T(x) = A_T(0)$, $A_I(x) = A_I(0)$ and $A_F(x) = A_F(0)$, that is, $x \in J$. Therefore J is a hyper BCK-ideal of H . $\square$

We provide conditions for a neutrosophic set $A = (A_T, A_I, A_F)$ to be a neutrosophic hyper BCK-ideal of H .

Theorem 3.7. Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$, and let $\{J_t \mid t \in \Lambda \subseteq [0, 0.5]\}$ be a collection of hyper BCK-ideals of H such that

$$H = \bigcup_{t \in \Lambda} J_t, \quad (3.7)$$

$$(\forall s, t \in \Lambda)(s > t \Leftrightarrow J_s \subset J_t). \quad (3.8)$$

Then a neutrosophic set $A = (A_T, A_I, A_F)$ in H defined by

$$\begin{aligned} A_T : H &\rightarrow [0, 1], \quad x \mapsto \sup \{t \in \Lambda \mid x \in J_t\}, \\ A_I : H &\rightarrow [0, 1], \quad x \mapsto \sup \{t \in \Lambda \mid x \in J_t\}, \\ A_F : H &\rightarrow [0, 1], \quad x \mapsto \inf \{t \in \Lambda \mid x \in J_t\} \end{aligned}$$

is a neutrosophic hyper BCK-ideal of H .

Proof. We first shows that

$$q \in [0, 1] \Rightarrow \bigcup_{p \in \Lambda, p \geq q} J_p \text{ is a hyper BCK-ideal of } H. \quad (3.9)$$

It is clear that $0 \in \bigcup_{p \in \Lambda, p \geq q} J_p$ for all $q \in [0, 1]$. Let $x, y \in H$ be such that $x \circ y = \{a_1, a_2, \dots, a_n\}$, $x \circ y \ll \bigcup_{p \in \Lambda, p \geq q} J_p$ and $y \in \bigcup_{p \in \Lambda, p \geq q} J_p$. Then $y \in J_r$ for some $r \in \Lambda$ with $q \leq r$, and for any $a_i \in x \circ y$ there exists $b_i \in \bigcup_{p \in \Lambda, p \geq q} J_p$, and so $b_i \in J_{t_i}$ for some $t_i \in \Lambda$ with $q \leq t_i$, such that $a_i \ll b_i$. If we let $t := \min\{t_i \mid i \in \{1, 2, \dots, n\}\}$, then $J_{t_i} \subset J_t$ for all $i \in \{1, 2, \dots, n\}$ and so $x \circ y \ll J_t$ with $q \leq t$. We may assume that $r > t$ without loss of generality, and so $J_r \subset J_t$. By (2.7), we have $x \in J_t \subset \bigcup_{p \in \Lambda, p \geq q} J_p$. Hence $\bigcup_{p \in \Lambda, p \geq q} J_p$ is a hyper BCK-ideal of H . Next, we consider the following two cases:

$$(i) \ t = \sup\{q \in \Lambda \mid q < t\}, \quad (ii) \ t \neq \sup\{q \in \Lambda \mid q < t\}. \quad (3.10)$$

If the first case is valid, then

$$x \in U(A_T, t) \Leftrightarrow x \in J_q \text{ for all } q < t \Leftrightarrow x \in \bigcap_{q < t} J_q,$$

and so $U(A_T, t) = \bigcap_{q < t} J_q$ which is a hyper BCK-ideal of H . Similarly, we know that $U(A_I, t)$ is a hyper BCK-ideal of H . For the second case, we will show that $U(A_T, t) = \bigcup_{q \geq t} J_q$. If $x \in \bigcup_{q \geq t} J_q$, then $x \in J_q$ for some $q \geq t$. Thus $A_T(x) \geq q \geq t$, and so $x \in U(A_T, t)$ which shows that $\bigcup_{q \geq t} J_q \subseteq U(A_T, t)$. Assume that $x \notin \bigcup_{q \geq t} J_q$. Then $x \notin J_q$ for all $q \geq t$, and so there exist $\delta > 0$ such that $(t - \delta, t) \cap \Lambda = \emptyset$. Thus $x \notin J_q$ for all $q > t - \delta$, that is, if $x \in J_q$ then $q \leq t - \delta < t$. Hence $x \notin U(A_T, t)$. This shows that $U(A_T, t) = \bigcup_{q \geq t} J_q$ which is a hyper BCK-ideal of H by (3.9). Similarly we can prove that $U(A_I, t)$ is a hyper BCK-ideal of H . Now we consider the following two cases:

$$s = \inf\{r \in \Lambda \mid s < r\} \text{ and } s \neq \inf\{r \in \Lambda \mid s < r\}. \quad (3.11)$$

The first case implies that

$$x \in L(A_F, s) \Leftrightarrow x \in J_r \text{ for all } s < r \Leftrightarrow x \in \bigcap_{s < r} J_r,$$

and so $L(A_F, s) = \bigcap_{s < r} J_r$ which is a hyper BCK-ideal of H . For the second case, there exists $\delta > 0$ such that $(s, s + \delta) \cap \Lambda = \emptyset$. If $x \in \bigcup_{s \geq r} J_r$, then $x \in J_r$ for some $s \geq r$. Thus $A_F(x) \leq r \leq s$, that is, $x \in L(A_F, s)$. Hence $\bigcup_{s \geq r} J_r \subseteq L(A_F, s)$. If $x \notin \bigcup_{s \geq r} J_r$, then $x \notin J_r$ for all $r \leq s$ and thus $x \notin J_r$ for all $r < s + \delta$. This shows that if $x \in J_r$ then $r \geq s + \delta$. Hence $A_F(x) \geq s + \delta > s$, i.e., $x \notin L(A_F, s)$.

Therefore $L(A_F, s) \subseteq \bigcup_{s \geq r} J_r$. Consequently, $L(A_F, s) = \bigcup_{s \geq r} J_r$ which is a hyper BCK-ideal of H by (3.9). It follows from Theorem 3.5 that $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H . $\square$

Definition 3.8. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic strong hyper BCK-ideal of H if it satisfies the following assertions.

$$\begin{aligned} {}^*A_T(x \circ x) &\geq A_T(x) \geq \min \left\{ \sup_{a_0 \in x \circ y} A_T(a_0), A_T(y) \right\}, \\ {}^*A_I(x \circ x) &\geq A_I(x) \geq \min \left\{ \sup_{b_0 \in x \circ y} A_I(b_0), A_I(y) \right\}, \\ {}^*A_F(x \circ x) &\leq A_F(x) \leq \max \left\{ \inf_{c_0 \in x \circ y} A_F(c_0), A_F(y) \right\} \end{aligned} \quad (3.12)$$

for all $x, y \in H$.

Example 3.9. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation “ $\circ$ ” which is given by Table 3.

Table 3: Cayley table for the binary operation “ $\circ$ ”

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H which is described in Table 4.

Table 4: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.86	0.75	0.09
a	0.65	0.57	0.17
b	0.31	0.37	0.29

It is routine to verify that $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H .

Theorem 3.10. For any neutrosophic strong hyper BCK-ideal $A = (A_T, A_I, A_F)$ of H , the following assertions are valid.

- (1) $A = (A_T, A_I, A_F)$ satisfies the conditions (3.1) and (3.3).

(2) $A = (A_T, A_I, A_F)$ satisfies

$$(\forall x, y \in H)(\forall a, b, c \in x \circ y) \left(\begin{array}{l} A_T(x) \geq \min\{A_T(a), A_T(y)\} \\ A_I(x) \geq \min\{A_I(b), A_I(y)\} \\ A_F(x) \leq \max\{A_F(c), A_F(y)\} \end{array} \right). \quad (3.13)$$

Proof. (1) Since $x \ll x$, i.e., $0 \in x \circ x$ for all $x \in H$, we get

$$\begin{aligned} A_T(0) &\geq {}^*A_T(x \circ x) \geq A_T(x), \\ A_I(0) &\geq {}^*A_I(x \circ x) \geq A_I(x), \\ A_F(0) &\leq {}^*A_F(x \circ x) \leq A_F(x), \end{aligned}$$

which shows that (3.3) is valid. Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$, and so

$${}^*A_T(x \circ y) \geq A_T(0), {}^*A_I(x \circ y) \geq A_I(0) \text{ and } {}^*A_F(x \circ y) \leq A_F(0).$$

It follows from (3.3) that

$$\begin{aligned} A_T(x) &\geq \min\{{}^*A_T(x \circ y), A_T(y)\} \geq \min\{A_T(0), A_T(y)\} = A_T(y), \\ A_I(x) &\geq \min\{{}^*A_I(x \circ y), A_I(y)\} \geq \min\{A_I(0), A_I(y)\} = A_I(y), \\ A_F(x) &\leq \max\{{}^*A_F(x \circ y), A_F(y)\} \leq \max\{A_F(0), A_F(y)\} = A_F(y). \end{aligned}$$

Hence $A = (A_T, A_I, A_F)$ satisfies the condition (3.1).

(2) Let $x, y, a, b, c \in H$ be such that $a, b, c \in x \circ y$. Then

$$\begin{aligned} A_T(x) &\geq \min\left\{\sup_{a_0 \in x \circ y} A_T(a_0), A_T(y)\right\} \geq \min\{A_T(a), A_T(y)\}, \\ A_I(x) &\geq \min\left\{\sup_{b_0 \in x \circ y} A_I(b_0), A_I(y)\right\} \geq \min\{A_I(b), A_I(y)\}, \\ A_F(x) &\leq \max\left\{\inf_{c_0 \in x \circ y} A_F(c_0), A_F(y)\right\} \leq \max\{A_F(c), A_F(y)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.11. If a neutrosophic set $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H , then the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic strong hyper BCK-ideal of H . Then $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H . Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$. Then there exist $a \in U(A_T, \varepsilon_T)$, $b \in U(A_I, \varepsilon_I)$ and $c \in L(A_F, \varepsilon_F)$, that is, $A_T(a) \geq \varepsilon_T$, $A_I(b) \geq \varepsilon_I$ and $A_F(c) \leq \varepsilon_F$. It follows from (3.3) that $A_T(0) \geq A_T(a) \geq \varepsilon_T$, $A_I(0) \geq A_I(b) \geq \varepsilon_I$ and $A_F(0) \leq A_F(c) \leq \varepsilon_F$. Hence

$$0 \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F).$$

Let $x, y, a, b, u, v \in H$ be such that $(x \circ y) \cap U(A_T, \varepsilon_T) \neq \emptyset$, $y \in U(A_T, \varepsilon_T)$, $(a \circ b) \cap U(A_I, \varepsilon_I) \neq \emptyset$, $b \in U(A_I, \varepsilon_I)$, $(u \circ v) \cap L(A_F, \varepsilon_F) \neq \emptyset$ and $v \in L(A_F, \varepsilon_F)$. Then there exist $x_0 \in (x \circ y) \cap U(A_T, \varepsilon_T)$, $a_0 \in (a \circ b) \cap U(A_I, \varepsilon_I)$ and $u_0 \in (u \circ v) \cap L(A_F, \varepsilon_F)$. It follows that

$$A_T(x) \geq \min \{ {}^*A_T(x \circ y), A_T(y) \} \geq \min \{ A_T(x_0), A_T(y) \} \geq \varepsilon_T,$$

$$A_I(a) \geq \min \left\{ \sup_{d \in a \circ b} A_I(d), A_I(b) \right\} \geq \min \{ A_I(a_0), A_I(b) \} \geq \varepsilon_I$$

and

$$A_F(u) \leq \max \left\{ \inf_{e \in u \circ v} A_F(e), A_F(v) \right\} \leq \max \{ A_F(u_0), A_F(v) \} \leq \varepsilon_F.$$

Hence $x \in U(A_T, \varepsilon_T)$, $a \in U(A_I, \varepsilon_I)$ and $u \in L(A_F, \varepsilon_F)$. Therefore $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H . $\square$

Theorem 3.12. For any neutrosophic set $A = (A_T, A_I, A_F)$ in H satisfying the condition

$$(\forall S \subseteq H)(\exists a, b, c \in S) \begin{pmatrix} A_T(a) = {}^*A_T(S) \\ A_I(b) = {}^*A_I(S) \\ A_F(c) = {}^*A_F(S) \end{pmatrix}, \quad (3.14)$$

if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$, then $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H .

Proof. Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty and strong hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$. For any $x, y, z \in H$, such that $x \in U(A_T, \varepsilon_T)$, $y \in U(A_I, \varepsilon_I)$ and $z \in L(A_F, \varepsilon_F)$, since $x \circ x \ll x$, $y \circ y \ll y$ and $z \circ z \ll z$ by (2.1), we have $x \circ x \ll U(A_T, \varepsilon_T)$, $y \circ y \ll U(A_I, \varepsilon_I)$ and $z \circ z \ll L(A_F, \varepsilon_F)$. By Lemma 3.4, $x \circ x \subseteq U(A_T, \varepsilon_T)$, $y \circ y \subseteq U(A_I, \varepsilon_I)$ and $z \circ z \subseteq L(A_F, \varepsilon_F)$. Hence $a \in U(A_T, \varepsilon_T)$, $b \in U(A_I, \varepsilon_I)$ and $c \in L(A_F, \varepsilon_F)$ for all $a \in x \circ x$, $b \in y \circ y$ and $c \in z \circ z$. Therefore ${}^*A_T(x \circ x) \geq A_T(x)$, ${}^*A_I(y \circ y) \geq A_I(y)$ and ${}^*A_F(z \circ z) \leq A_F(z)$. Now, let $\varepsilon_T := \min \{ {}^*A_T(x \circ y), A_T(y) \}$, $\varepsilon_I := \min \{ {}^*A_I(x \circ y), A_I(y) \}$ and $\varepsilon_F := \max \{ {}^*A_F(x \circ y), A_F(y) \}$. By (3.14), we have

$$A_T(a_0) = {}^*A_T(x \circ y) \geq \min \{ {}^*A_T(x \circ y), A_T(y) \} = \varepsilon_T,$$

$$A_I(b_0) = {}^*A_I(x \circ y) \geq \min \{ {}^*A_I(x \circ y), A_I(y) \} = \varepsilon_I$$

and

$$A_F(c_0) = {}^*A_F(x \circ y) \leq \max \{ {}^*A_F(x \circ y), A_F(y) \} = \varepsilon_F$$

for some $a_0, b_0, c_0 \in x \circ y$. Hence $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$ which imply that

$$(x \circ y) \cap U(A_T, \varepsilon_T), (x \circ y) \cap U(A_I, \varepsilon_I) \text{ and } (x \circ y) \cap L(A_F, \varepsilon_F)$$

are nonempty. Since $y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$, it follows from (2.9) that $x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$. Thus

$$A_T(x) \geq \varepsilon_T = \min \{ {}^*A_T(x \circ y), A_T(y) \},$$

$$A_I(x) \geq \varepsilon_I = \min \{ {}^*A_I(x \circ y), A_I(y) \}$$

and

$$A_F(x) \leq \varepsilon_F = \max \{ {}_*A_F(x \circ y), A_F(y) \}.$$

Consequently, $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H . $\square$

Since any neutrosophic set $A = (A_T, A_I, A_F)$ satisfies the condition (3.14) in a finite hyper BCK-algebra, we have the following corollary.

Corollary 3.13. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a finite hyper BCK-algebra H . Then $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Definition 3.14. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic weak hyper BCK-ideal of H if it satisfies the following assertions.

$$\begin{aligned} A_T(0) &\geq A_T(x) \geq \min \{ {}_*A_T(x \circ y), A_T(y) \}, \\ A_I(0) &\geq A_I(x) \geq \min \{ {}_*A_I(x \circ y), A_I(y) \}, \\ A_F(0) &\leq A_F(x) \leq \max \{ {}^*A_F(x \circ y), A_F(y) \} \end{aligned} \quad (3.15)$$

for all $x, y \in H$.

Definition 3.15. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic s-weak hyper BCK-ideal of H if it satisfies the conditions (3.3) and (3.5).

Example 3.16. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation “ $\circ$ ” which is given by Table 5.

Table 5: Cayley table for the binary operation “ $\circ$ ”

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H which is described in Table 6.

It is routine to verify that $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H .

Table 6: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.98	0.85	0.02
a	0.81	0.69	0.19
b	0.56	0.43	0.32
c	0.34	0.21	0.44

Theorem 3.17. Every neutrosophic s-weak hyper BCK-ideal is a neutrosophic weak hyper BCK-ideal.

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic s-weak hyper BCK-ideal of H and let $x, y \in H$. Then there exist $a, b, c \in x \circ y$ such that

$$\begin{aligned} A_T(x) &\geq \min\{A_T(a), A_T(y)\} \geq \min\left\{\inf_{a_0 \in x \circ y} A_T(a_0), A_T(y)\right\}, \\ A_I(x) &\geq \min\{A_I(b), A_I(y)\} \geq \min\left\{\inf_{b_0 \in x \circ y} A_I(b_0), A_I(y)\right\}, \\ A_F(x) &\leq \max\{A_F(c), A_F(y)\} \leq \max\left\{\sup_{c_0 \in x \circ y} A_F(c_0), A_F(y)\right\}. \end{aligned}$$

Hence $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H . $\square$

We can conjecture that the converse of Theorem 3.17 is not true. But it is not easy to find an example of a neutrosophic weak hyper BCK-ideal which is not a neutrosophic s-weak hyper BCK-ideal.

Now we provide a condition for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal.

Theorem 3.18. If $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H which satisfies the condition (3.4), then $A = (A_T, A_I, A_F)$ is a neutrosophic s-weak hyper BCK-ideal of H .

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic weak hyper BCK-ideal of H in which the condition (3.4) is true. Then there exist $a_0, b_0, c_0 \in x \circ y$ such that $A_T(a_0) = {}_s A_T(x \circ y)$, $A_I(b_0) = {}_s A_I(x \circ y)$ and $A_F(c_0) = {}_s A_F(x \circ y)$. Hence

$$\begin{aligned} A_T(x) &\geq \min\{{}_s A_T(x \circ y), A_T(y)\} = \min\{A_T(a_0), A_T(y)\}, \\ A_I(x) &\geq \min\{{}_s A_I(x \circ y), A_I(y)\} = \min\{A_I(b_0), A_I(y)\}, \\ A_F(x) &\leq \max\{{}_s A_F(x \circ y), A_F(y)\} = \max\{A_F(c_0), A_F(y)\}. \end{aligned}$$

Therefore $A = (A_T, A_I, A_F)$ is a neutrosophic s-weak hyper BCK-ideal of H . $\square$

Remark 3.19. In a finite hyper BCK-algebra, every neutrosophic set satisfies the condition (3.4). Hence the concept of neutrosophic s-weak hyper BCK-ideal and neutrosophic weak hyper BCK-ideal coincide in a finite hyper BCK-algebra.

Theorem 3.20. A neutrosophic set $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are weak hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Proof. The proof is similar to the proof of Theorem 3.5. $\square$

Definition 3.21. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a reflexive neutrosophic hyper BCK-ideal of H if it satisfies

$$(\forall x, y \in H) \begin{pmatrix} {}^*A_T(x \circ x) \geq A_T(y) \\ {}^*A_I(x \circ x) \geq A_I(y) \\ {}^*A_F(x \circ x) \leq A_F(y) \end{pmatrix}, \quad (3.16)$$

and

$$(\forall x, y \in H) \begin{pmatrix} A_T(x) \geq \min \{ {}^*A_T(x \circ y), A_T(y) \} \\ A_I(x) \geq \min \{ {}^*A_I(x \circ y), A_I(y) \} \\ A_F(x) \leq \max \{ {}^*A_F(x \circ y), A_F(y) \} \end{pmatrix}. \quad (3.17)$$

Theorem 3.22. Every reflexive neutrosophic hyper BCK-ideal is a neutrosophic strong hyper BCK-ideal.

Proof. Straightforward. $\square$

Theorem 3.23. If $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H , then the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Proof. Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$. Let $a \in U(A_T, \varepsilon_T)$, $b \in U(A_I, \varepsilon_I)$ and $c \in L(A_F, \varepsilon_F)$. If $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H , then by Theorem 3.22, $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H , and so it is a neutrosophic hyper BCK-ideal of H . It follows from Theorem 3.5 that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H . For each $x \in H$, let $a_0, b_0, c_0 \in x \circ x$. Then

$$A_T(a_0) \geq \inf_{u \in x \circ x} A_T(u) \geq A_T(a) \geq \varepsilon_T,$$

$$A_I(b_0) \geq \inf_{v \in x \circ x} A_I(v) \geq A_I(b) \geq \varepsilon_I,$$

$$A_F(c_0) \leq \sup_{w \in x \circ x} A_F(w) \leq A_F(c) \leq \varepsilon_F,$$

and so $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$. Hence $x \circ x \subseteq U(A_T, \varepsilon_T)$, $x \circ x \subseteq U(A_I, \varepsilon_I)$ and $x \circ x \subseteq L(A_F, \varepsilon_F)$. Therefore $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H . $\square$

Lemma 3.24 ([15]). Every reflexive hyper BCK-ideal is a strong hyper BCK-ideal.

We consider the converse of Theorem 3.23 by adding a condition.

Theorem 3.25. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H satisfying the condition (3.14). If the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$, then $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H .

Proof. If the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H , then by Lemma 3.24 they are strong hyper BCK-ideals of H . By Theorem 3.12 that $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H . Hence the condition (3.17) is valid. Let $x, y \in H$. Then the sets $U(A_T, A_T(y))$, $U(A_I, A_I(y))$ and $L(A_F, A_F(y))$ are reflexive hyper BCK-ideals of H , and so $x \circ x \subseteq U(A_T, A_T(y))$, $x \circ x \subseteq U(A_I, A_I(y))$ and $x \circ x \subseteq L(A_F, A_F(y))$. Hence $A_T(a) \geq A_T(y)$, $A_I(b) \geq A_I(y)$ and $A_F(c) \leq A_F(y)$ for all $a, b, c \in x \circ x$ and so $*A_T(x \circ x) \geq A_T(y)$, $*A_I(x \circ x) \geq A_I(y)$ and $*A_F(x \circ x) \leq A_F(y)$. Therefore $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H . $\square$

We provide conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

Theorem 3.26. Let $A = (A_T, A_I, A_F)$ be a neutrosophic strong hyper BCK-ideal of H which satisfies the condition (3.14). Then $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H if and only if the following assertion is valid.

$$(\forall x \in H) \left(\begin{array}{l} *A_T(x \circ x) \geq A_T(0) \\ *A_I(x \circ x) \geq A_I(0) \\ *A_F(x \circ x) \leq A_F(0) \end{array} \right). \quad (3.18)$$

Proof. It is clear that if $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H , then the condition (3.18) is valid.

Conversely, assume that $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H which satisfies the conditions (3.14) and (3.18). Then $A_T(0) \geq A_T(y)$, $A_I(0) \geq A_I(y)$ and $A_F(0) \leq A_F(y)$ for all $y \in H$. Hence

$$*A_T(x \circ x) \geq A_T(y), *A_I(x \circ x) \geq A_I(y) \text{ and } *A_F(x \circ x) \leq A_F(y).$$

For any $x, y \in H$, let

$$\begin{aligned} \varepsilon_T &:= \min \{ *A_T(x \circ y), A_T(y) \}, \\ \varepsilon_I &:= \min \{ *A_I(x \circ y), A_I(y) \}, \\ \varepsilon_F &:= \max \{ *A_F(x \circ y), A_F(y) \}. \end{aligned}$$

Then $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H by Theorem 3.11. Since $A = (A_T, A_I, A_F)$ satisfies the condition (3.14), there exist $a_0, b_0, c_0 \in x \circ y$ such that

$$A_T(a_0) = *A_T(x \circ y), A_I(b_0) = *A_I(x \circ y), A_F(c_0) = *A_F(x \circ y).$$

Hence $A_T(a_0) \geq \varepsilon_T$, $A_I(b_0) \geq \varepsilon_I$ and $A_F(c_0) \leq \varepsilon_F$, that is, $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$. Hence $(x \circ y) \cap U(A_T, \varepsilon_T) \neq \emptyset$, $(x \circ y) \cap U(A_I, \varepsilon_I) \neq \emptyset$ and $(x \circ y) \cap L(A_F, \varepsilon_F) \neq \emptyset$. Since $y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$, by (2.9), $x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$. Thus

$$\begin{aligned} A_T(x) &\geq \varepsilon_T = \min \{ *A_T(x \circ y), A_T(y) \}, \\ A_I(x) &\geq \varepsilon_I = \min \{ *A_I(x \circ y), A_I(y) \}, \\ A_F(x) &\leq \varepsilon_F = \max \{ *A_F(x \circ y), A_F(y) \}. \end{aligned}$$

Therefore $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H . □

4 Conclusions

We have introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. We have considered their relations and related properties. We have discussed characterizations of neutrosophic (weak) hyper BCK-ideal, and have given conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal. We have provided conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and have provided conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

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PESTEL analysis with neutrosophic cognitive maps to determine the factors that affect rural sustainability. Case Study of the South-Eastern plain of the province of Pinar del Río.

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Abstract. Neutrosophic cognitive maps and their application in decision-making have become an important subject for researchers and practitioners. Especially, PESTEL analysis based on neutrosophic cognitive maps is a useful method, which permits to analyse specific topics statically. In the present paper strategies for the external factors that contribute to the identification of agricultural contexts in the South-Eastern plain of the province of Pinar del Río, Cuba are studied based on PESTEL analysis and neutrosophic cognitive maps. Here, PEST analysis incorporates Ecological and Legal factors and their characteristics. This study aims to determine which factors affect the agricultural sustainability of the South - Eastern plain of the province of Pinar del Río. The main contribution of the present paper is that it was identified quantitatively the factors that affect the agricultural sustainability, they are, the technological, political and economic ones.

Keywords: PESTEL analysis, Neutrosophy theory, cognitive maps, agricultural sustainability.

1 Introduction

The term sustainable, lasting or sustainable development applies to socio-economic development. It was formalized for the first time in the so-called document Brundtland Report in 1987, as the result of the work of the World Commission on Environment and Development of the United Nations, created in the United Nations Assembly in 1983. This definition assumes the Principle of 3rd Rio Declaration of 1992, according the needs of the present without compromising the ability of future generations to meet their own needs.

The scope of the development conceptually sustainable is divided into three parts, viz., environmental, economic and social. The social aspect is defined as the relationship that exists among social welfare, environment and the economic bonanza.

Based on the aforementioned conceptualization, it is noteworthy that agricultural sustainability in Cuba, as a small island and as a developing state, presents a high degree of vulnerability to the impacts of global environmental problems. In particular climate change, which they are intensely reflected through ecological, environmental, legal and industrial factors of the key sectors of the economy, such as agriculture, tourism, construction, transport and fishing, substantially affecting their objective of achieving real sustainable development, see [1].

The South-Eastern plain of the province of Pinar del Río in Cuba has a livestock production, which is one of the most important agricultural branches, for its role in human nutrition. It is referred in [1] that the flat relief is favorable for agricultural sustainability since the climate in this area of study is favorable for the forage plants growth, which guarantees the feeding of livestock.

Similarly, other papers of the aforementioned author refer to the agricultural production in the Southern-East plains of the province of Pinar del Río. Aridity and other adverse factors affect this territory, increasing the unfavourable effects on soils, and causing the accelerated loss of agro-productivity.

Some of these adverse effects are, the groundwater contamination because of saline wedges penetrations, the lower availability of water resources for irrigation due to the recurrence and extension of droughts, the loss of crops due to the appearance of pests and diseases, the decrease in yields due to the rise in temperature and the

losses due to the occurrence of extreme hydro-meteorological phenomena.

Moreover, let us note that Cuban agriculture is a source of significant direct and indirect revenues to the economy in terms of convertible currencies. These are products containing a recognized quality, which are exportable funds or essential attractions for the tourism and the domestic foreign exchange market.

Among the main products that contribute to the existence of sustainable agriculture in Cuba are, coffee, cocoa, citrus, sugar cane, rum and honey. All of them are affected by the impacts caused by climate change, in one way or another.

Based on the aforementioned elements and the analysis of the effects produced by the adverse factors that affect the sustainable agricultural development in the area of study, PESTEL analysis is applied.

The PESTEL analysis is a strategic analysis technique to determine the external environment that affects the following factors, namely, Political, Economic, Socio-cultural, Technological, Ecological and Legal. This analysis consists in determining the social forces that affect the microenvironment, i.e., to analyze all those general factors (nationals and internationals) that determine the framework in which the institutions of a given region act and that affect their specific environment: sector, market, customers, competition, suppliers, among others.

PESTEL analysis, as it is reported in [2], is a technique for analyzing business that permits and determines the context in which it moves, in turn, enables the design of strategies to defend themselves, take advantage or adapt to anything that affects the sector or market. The categories contemplated by this analysis are the following:

- Political factors
- Economic factors
- Socio-cultural factors
- Technological Factors
- Ecological factors
- Legislative factors

In this analysis, it is necessary to differentiate two levels of the environment; general and specific. In our case of study, the general environment refers to the external environment surrounding agricultural sustainability in the South-Eastern plain of the province of Pinar del Río, from a generic perspective. This analysis is realized in two ways. Firstly, the macroeconomic figures of the environment are detailed, as well as the evolution of the agricultural sector, to put it in a context that serves as a starting point, and secondly, the sector has been analyzed using the PESTEL model.

The analysis with the PESTEL model, according to [3], has gained ground in the literature in recent years. The aforementioned author reports that the term PESTEL was used for the first time by the authors Johnson and Scholes in their book "Exploring Corporate Strategy", in the sixth edition of the year 2002, without claiming the invention of the acronym PESTEL. For our case of study, the PESTEL model integrates the factors shown in Figure 1.

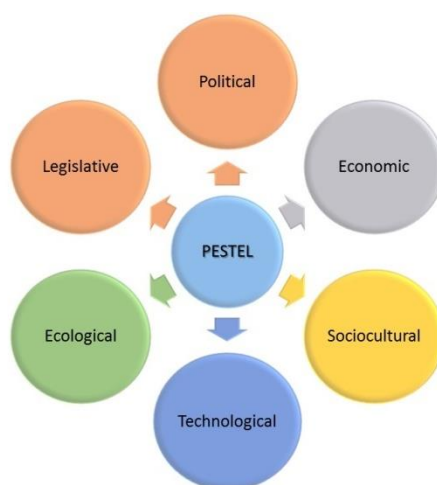


Figure 1. Factors that integrate PESTEL analysis and which affect the agricultural sustainability of the South-Eastern plain of the province of Pinar del Río.

A group of environmental factors, according to [4], using the PESTEL model and specified in Figure 1, are determined to identify the variables that have major incidence on the agricultural sustainability of the South-Eastern of the province of Pinar del Río.

Neutrosophy theory was proposed by Florentin Smarandache, for the treatment of neutralities. It generalizes crisp and fuzzy set theories, among others, introducing for the first time new concepts like neutrosophic sets and

neutrosophic logic [5].

Neutrosophic PESTEL analysis, which combines PESTEL analysis with Neutrosophic Cognitive Maps has been previously applied by other authors, see [6] where it was used to determine the factors that affect Food Industry. However, the problem we study in this research is particularly complex, because these six aspects are interrelated each other, in such a way that does not make sense to study only one of them independently of the other ones, and all the population is concerned.

Therefore, our main motivation is to propose strategies to solve this complex problem, which is a universal one. Additionally, we intend to demonstrate that neutrosophic PESTEL can be applied to solve problems of such magnitude. These are our aims to write this paper.

Neutrosophy is a useful theory that is increasing the number of its applications in many fields. In this case, the inclusion of this theory enriches the possibilities of PESTEL analysis, mainly because of two issues, firstly, the addition of the notion of indeterminacy and secondly the possibility to calculate using linguistic terms.

In the present study, PESTEL analysis using neutrosophic cognitive maps facilitates greater interpretability of the obtained results and contributes to the correlation between the characteristics of the factors of study. The analysis of the characteristics of each factor in PESTEL model eases to obtain the most important of them that influences in a greater agricultural sustainability of the South-Eastern plain of the province of Pinar del Río.

Furthermore, neutrosophy theory has significantly enhance crisp techniques, tools and methods. For instance, it has been successfully used in conjoint with methods like TOPSIS, VIKOR, ANP and DEMATEL, see [7-9].

The present paper is divided as follows, Section 2 of Materials and Methods summarizes the basic concepts necessities to achieve the solution of this problem. Section 3 of Results exposes the application of Neutrosophic PESTEL in the solution of the case of agricultural sustainability in the Province of Pinar del Río, Cuba. Finally, conclusions are drawn.

2 Materials and methods

In the present study, PESTEL analysis with neutrosophic cognitive maps are applied to determine the factors that affect the agricultural sustainability of the South-Eastern plain of the province of Pinar del Río. This is based on a descriptive methodology with a quantitative method. The result was achieved by using the descriptive methodology, which demonstrates that it is feasible to define the characteristics of the factors that intervene in the PESTEL model related to agricultural sustainability of the South-Eastern plain of the province of Pinar del Río, Cuba.

Firstly, let us formally expose the original definition of neutrosophic logic as it is shown in [10].

Definition 1. Let $N = \{(T, I, F): T, I, F \in [0,1]\}$ be a *neutrosophic set of evaluation*. $v: \mathbf{P} \rightarrow N$ is a mapping of a group of propositional formulas into N , i.e., each sentence $p \in \mathbf{P}$ is associated to a value in N , as it is exposed in the Equation 1, meaning that p is $T\%$ true, $I\%$ indeterminate and $F\%$ false.

$$v(p) = (T, I, F) \quad (1)$$

Hence, the neutrosophic logic is a generalization of fuzzy logic, based on the concept of neutrosophy according to [5, 11].

Definition 2. (See [12-13]) Let K be the ring of real numbers. The ring generated by $K \cup I$ is called a *neutrosophic ring* if it involves the indeterminacy factor in it, where I satisfies $I^2 = I$, $I+I = 2I$ and in general, $I+I+\dots+I = nI$, if $k \in K$, then $k.I = kI$, $0I = 0$. The neutrosophic ring is denoted by $K(I)$, which is generated by $K \cup I$, i.e., $K(I) = \langle K \cup I \rangle$, where $\langle K \cup I \rangle$ denotes the ring generated by K and I .

Definition 3. A *neutrosophic matrix* is a matrix $A = [a_{ij}]_{ij}$ $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$; $m, n \geq 1$, such that each $a_{ij} \in K(I)$, where $K(I)$ is a neutrosophic ring, see [14].

Let us observe that an element of the matrix can have the form $a+bI$, where a and b are real numbers, whereas I is the indeterminacy factor. The usual operations of neutrosophic matrices can be extended from the classical matrix operations.

$$\text{For example, } \begin{pmatrix} -1 & I & 5I \\ I & 4 & 7 \end{pmatrix} \begin{pmatrix} I & 9I & 6 \\ 0 & I & 0 \\ -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -21I & 27I & -6 + 25I \\ -28 + I & 49 + 13I & 35 + 6I \end{pmatrix}.$$

Additionally, a *neutrosophic graph* is a graph that has at least one indeterminate edge or one indeterminate node [15]. The *neutrosophic adjacency matrix* is an extension of the adjacency matrix in classical graph theory. $a_{ij} = 0$ means nodes i and j are not connected, $a_{ij} = 1$ means that these nodes are connected and $a_{ij} = I$ means the connection is indeterminate (unknown if it is or if not). Fuzzy set theory does not use such notions.

On the other hand, if the indetermination is introduced in a cognitive map as it is referred in [16], then this cognitive map is called a *neutrosophic cognitive map*, which is especially useful in the representation of causal knowledge [5, 17]. It is formally defined in Definition 4.

Definition 4. A *Neutrosophic Cognitive Map* (NCM) is a neutrosophic directed graph with concepts like policies, events, among others, as nodes and causalities or indeterminates as edges. It represents the causal relationship between concepts.

Neutrosophic Cognitive Maps are used in this paper, according to the proposed objective to include an indeterminate framework in the PESTEL analysis. The proposed framework is shown in Figure 2.

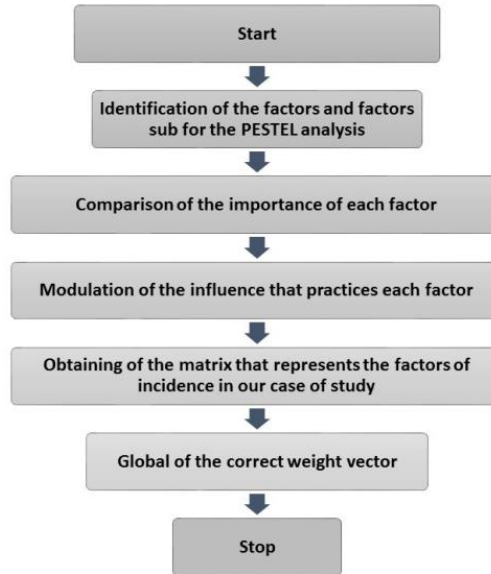


Figure 2. Framework to obtain the characteristics analyzed in each factor of the PESTEL model based on neutrosophic cognitive maps.

Neutrosophic cognitive maps are a generalization of fuzzy cognitive maps. Fuzzy cognitive maps are introduced by Axelrod, see [18], where nodes represent concepts or variables in a particular area of study and arcs indicate either positive or negative influences, and which are considered like causal relationships. They have been applied in various areas, especially in supporting decision making and in the analysis of complex systems as it is referred in [19]. Static analysis in a cognitive neutrosophic map focuses on the selection of the most important concepts, characteristics or factors in the modeled system [20].

The framework proposed in Figure 2 guides the process to obtain the characteristics of each analyzed factor, for agricultural sustainability in the South-Eastern plain of the province of Pinar del Río, with the PESTEL model. Integrated structure factors corresponding to an analysis of PESTEL and characteristics are modeled using neutrosophic cognitive maps, which contributes to obtain quantitative analysis of the characteristics that correspond to factor analysis.

The measures described below are used in the proposed model, they are based on the absolute values of the adjacency matrix [21]:

- Outdegree (v_i) is the sum of the row elements in the neutrosophic adjacency matrix. It reflects the strength of the outgoing relationships (c_{ij}) of the variable.

$$od(v_i) = \sum_{j=1}^n c_{ij} \quad (2)$$

- Indegree (v_i) is the sum of the column elements. It reflects the strength of relations (c_{ij}) outgoing from the variable.

$$id(v_i) = \sum_{j=1}^n c_{ji} \quad (3)$$

- Total centrality (total degree $td(v_i)$), is the sum of the indegree and the outdegree of the variable.

$$td(v_i) = od(v_i) + id(v_i) \quad (4)$$

The static analysis is applied using the adjacency matrix, taking into consideration the absolute value of the weights [20]. Static analysis in Neutrosophic Cognitive Maps (NCM), see [22], initially contains the neutrosophic number of the form $(a + bI)$, where I = indetermination [23]. It requires a process of de-neutrosophication as proposed in [21] by Salmeron and Smarandache, where $I \in [0, 1]$ and it is replaced by their values maximum and minimum.

Finally, we work with the average of the extreme values, which is calculated using Equation 5, which is useful to obtain a single value as it is referred in [24]. This value contributes to the identification of the characteristics to be attended, according to the factors obtained with the PESTEL model, for our case study.

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (5)$$

Then,

$$A > B \Leftrightarrow \frac{a_1 + a_2}{2} > \frac{b_1 + b_2}{2} \quad (6)$$

3 Results

Figure 3 shows the factors and characteristics of the PESTEL model obtained for the analysis of agricultural sustainability in the South-Eastern plain of the province of Pinar del Río.

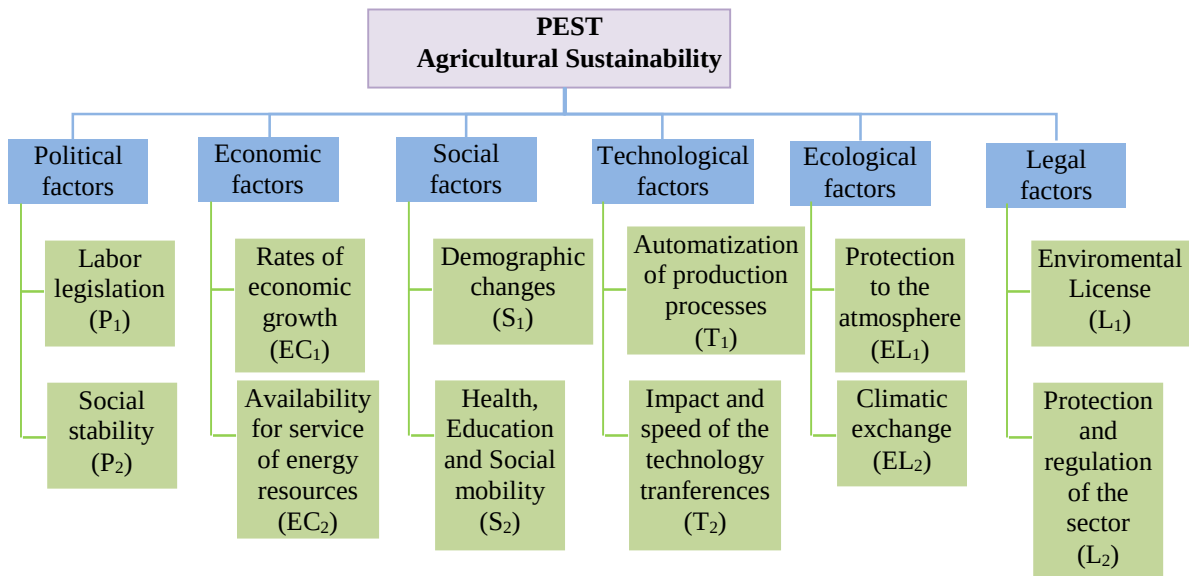


Figure 3. PESTEL hierarchical model for the analysis of Agricultural Sustainability in the South-Eastern Plain of Pinar del Río.

After obtaining the characteristics corresponding to the PESTEL model factors, we analyzed them keeping in mind that the PESTEL model is a strategic analysis technique to define the context of a determined area about the analysis of external factors, as it is referred in [25]. It is noteworthy that the PESTEL analysis incorporates the ecological and legal factors into the PEST analysis so that in the present investigation a PEST analysis was previously applied. Thus, the factors analyzed by using the PEST technique according to [26] are the following:

- **Political factors:** The first element of PEST Analysis is that we must make a study of political factors. In this sense, for our case of study, the political factors to evaluate are related to the impact of any political or legislative change that may affect agricultural sustainability in the South-Eastern plain of the province of Pinar del Río.
- **Economic factors:** Political factors do not operate independently, and public policy decisions have economic implications. All companies are affected by economic factors of national, international or global order. Behavioural, purchasing power is related to the stage of boom, recession, stagnation or recovery throughout which an economy growth. Economic factors affect the purchasing power of resources necessary for agricultural sustainability and the cost of capital for the company responsible for maintaining agricultural sustainability.
- **Social factors:** Agricultural sustainability focuses on the forces that act in the society and affect attitudes, interests and opinions of those who are influenced by decision-making. In reference [27] it is assured that social factors vary and include aspects as diverse as demography changes.
- **Technological factor:** it is another one of the factors to be taken into account, since technology is a driving force that contributes to an improvement in quality, it reduces entrance barriers.

After factors of the macro environment were defined with the aid of PEST technique, we defined the external factors that affect the agricultural sustainability in the South-Eastern plain, again by using the PESTEL model. Factors that are obtained with the purpose of defining this environment to measure the agricultural sustainability of this region and discussed according to [18], are the following:

1. **Ecological factors**
2. **Legal factors**

In the present study, the analyzed ecological factors correspond to the characteristics related to the protection of the environment and climate change. On the other hand, and with regard to legal factors, the characteristics

related to environmental licenses and the protection and regulation of the agricultural sector in the study area are analyzed.

In [28] it is pointed out that the obtained results with the PEST and PESTEL analysis, in particular, for each characteristic representing the factors under study, are presented in form of linguistic terms, therefore, in order to obtain a greater interpretability, a linguistic treatment is necessary to be able to quantify them. For this reason, in the present study, neutrosophic cognitive maps are used, as a tool for modeling the characteristics that are related to the factors that affect the agricultural development of this region.

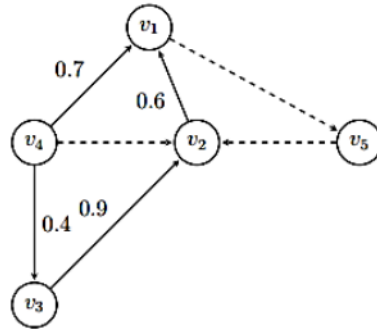


Figure4. Neutrosophic cognitive map. **Source:** [29]

Essentially, to perform static analysis on a NCM should follow the steps shown in Figure 5.

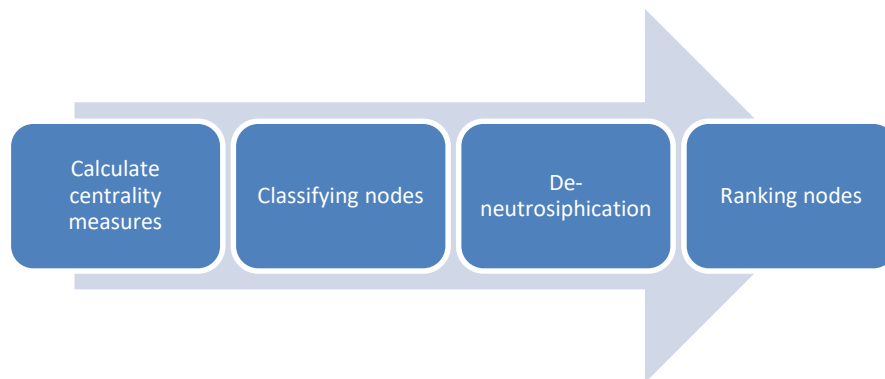


Figure 5. Steps to follow for static analysis in a neutrosophical cognitive map. **Source:** [29]

For the evaluation of the PESTEL factors with a neutrosophic cognitive map, the factors obtained with the PESTEL technique and the characteristics related to each factor that was represented in a hierarchical way in Figure 3 are taken into account. The MCN, for our case study, is developed through the capture of knowledge. The neutrosophic adjacency matrix generated is shown in Table 1.

	P ₁	P ₂	EC ₁	EC ₂	S ₁	S ₂	T ₁	T ₂	EL ₁	EL ₂	L ₁	L ₂
P ₁	0	0	0	-0.3	0	0	0	0	0	0	0	0
P ₂	0	0	0	0	0	0	0.25	0	0	0	0	0
EC ₁	0	0	0	0	0	0	0	0	0	0	0	0
EC ₂	0	0	0	0	0	0	0	0.3	0	0	0	0
S ₁	0.4	1	0	0	0	0	0	0	0	0	0	0
S ₂	0	0	0	0	0	0	0	0	0	0	0	0
T ₁	0	0	0	0	0	0	0	0	0	0	0	0
T ₂	0	0	0	0.35	0	0	0	0	0	0	0	0
EL ₁	0	0	0	0	0	0	0	0	0.25	0	0	0
EL ₂	0	0	0	0	0	0	0	0	0	0	0	0
L ₁	0	0	0	0	0	0	0	0	0	0.30	0	0
L ₂	0	0	0	0	0	0	0	0	0	0	0	0.20

Table 1. Neutrosophic adjacency matrix.

The measures of centrality are calculated using the Outdegree measures and Indegree, using Equations 2 and 3, the results are shown in Table 2.

Node	Id	Od
P ₁	0.4	0.3
P ₂	I	0.25
EC ₁	0	0.2
EC ₂	1.05	0.3
S ₁	I	0.7 + I
S ₂	0	I
T ₁	0.55	0.2
T ₂	0.3	0.35
EL ₁	0.25	0
EL ₂	0.30	0
L ₁	0	0.30
L ₂	0	0.20

Table 2. Measures of centrality, outdegree, indegree.

Once the measures of centrality were calculated, the nodes of the neutrosophic cognitive map were classified. This classification is shown in Table 3.

	Node transmitter	Receiving node	Ordinary
P ₁			X
P ₂			X
EC ₁			X
EC ₂	X		
S ₁			X
S ₂	X		
T ₁			X
T ₂			X
EL ₁			X
EL ₂			X
L ₁			X
L ₂			X

Table 3. Classification of the nodes.

According to the results shown in Table 3, E2 and S2 are receiving nodes. The rest of them are ordinary.

The total centrality (total degree $td(v_i)$), is calculated using Equation 4, the results for our case of study are shown in Table 4.

	td
P ₁	0.7
P ₂	0.25 + I
EC ₁	0.2
EC ₂	1.35
S ₁	0.7 + 2I
S ₂	I
T ₁	0.75
T ₂	0.65
EL ₁	0.25

EL ₂	0.30
L ₁	0.30
L ₂	0.20

Table 4. Total, centrality.

Next, the process of des-neutrosophication is applied as it is referred by Salmeron and Smarandache in [30]. $I \in [0,1]$ is replaced by values maximum and minimum. The Interval values are displayed in Table 5.

	td
P ₁	0.7
P ₂	[0.25, 1.25]
EC ₁	0.2
EC ₂	1.35
S ₁	[0.7, 2.7]
S ₂	[0, 1]
T ₁	0.75
T ₂	0.65
EL ₁	1.25
EL ₂	1.30
L ₁	1.30
L ₂	1.20

Table 5. Total, des - neutrosophication of the values of total centrality.

Based on Equation 5, the mean of the extreme values are obtained to analyze the characteristics to be attended according to the factors obtained with the PESTEL technique in the present study. The results are shown in Table 6.

We conclude the factors that address the sustainability of the agricultural sector in the province of Pinar del Rio are the technological, political and economic factors. The measurements of the central position of the obtained factors with the PESTEL technique and analyzed according to the use of the neutrosophic cognitive maps are shown in Figure 6.

Then, the priorities can be ordered as follows:

$$S_1 > EC_2 > EL_2 \sim L_1 > EL_1 > L_2 > P_2 \sim T_1 > P_1 > T_2 > S_2 > EC_1.$$

	Td
P ₁	0.7
P ₂	0.75
EC ₁	0.2
EC ₂	1.35
S ₁	1.7
S ₂	0.5
T ₁	0.75
T ₂	0.65
EL ₁	1.25
EL ₂	1.30
L ₁	1.30
L ₂	1.20

Table 6. Median of the extremes values.

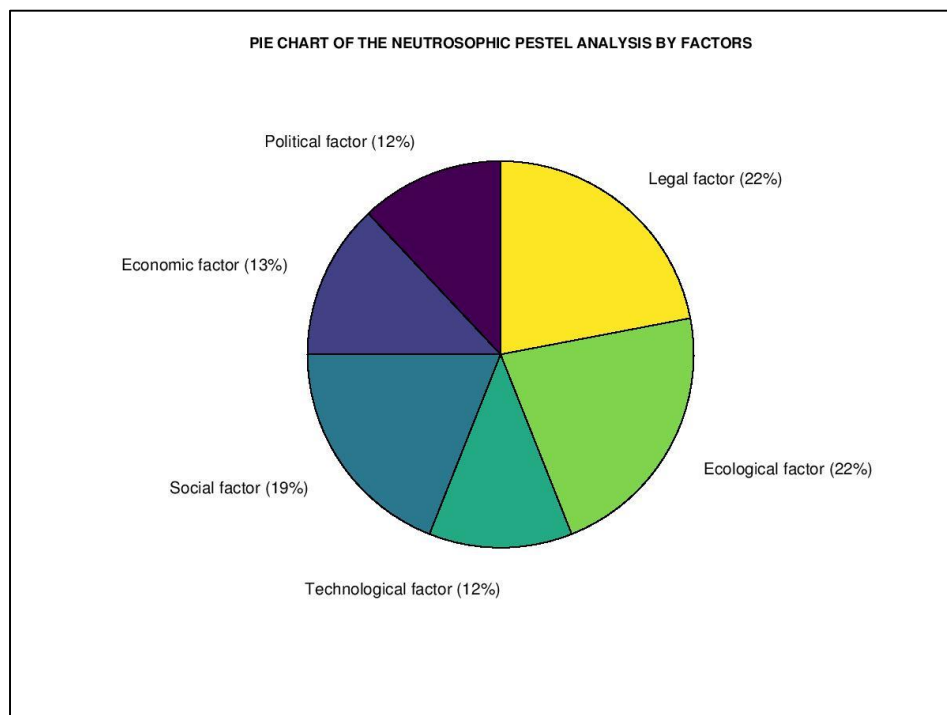


Figure 6. Values of central position for factors.

Conclusions

In the present study a characterization of the agricultural sustainability of the South - Eastern plain of the province of Pinar del Río, Cuba, is made. The PESTEL technique is used, contributing to environmental analysis, identifying key factors that have significant impact on the agricultural sector.

The most influential characteristics for the region and the agricultural sustainability of each identified factor are described.

The characteristics were modeled using neutrosophic cognitive maps, taking into account the interdependencies between the characteristics and the factors identified with the PESTEL technique, from which a quantitative analysis was applied, based on the static analysis provided by the use of neutrosophic cognitive maps.

It is shown that in order to achieve agricultural sustainability, technological, political and economic factors must be addressed. This result is the main contribution of this paper. Nevertheless, other contribution is that it was demonstrated that Neutrosophic PESTEL can be applied to the solution of complex problems like it is.

A future direction of this study is to assess the impact of the technological, political and economic measures that should be applied to reverse this negative situation.

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Extending PESTEL technique to neutrosophic environment for decisions making in business management

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Abstract Recently, Neutrosophy theory and its application in decision-making has become in a significant issue for the scientific community. In this paper, an extension of the PESTEL (Political, Economic, Social, Technological, Ecological and Legal) technique adapted to the neutrosophic environment is proposed for decision making in management. The proposal is specially tailored for the dynamic analysis of the different factors in an uncertain environment. The framework developed for the extension of the PESTEL technique to the neutrosophic environment is compound for six fundamental activities. A system of recommendation is designed for practical purposes. A case study is developed for the use of the neutrosophic PESTEL to show the applicability of the technique. The paper ends with conclusions and recommendations for future work.

Keywords: PESTEL, business management, neutrosophy theory, decision making.

1 Introduction

The start-up of a company or a new business unit requires detailed knowledge of the context in which it is going to develop. There exist numerous external factors that will condition its operation, hence the analysis of the environment is the key to know future trends and define in advance the business strategy to follow. A useful tool to achieve this goal is the PESTEL (Political, Economic, Social, Technological, Ecological and Legal) technique, which allows a detailed investigation of the issues that most influence in the development of business activity or project that it is needed to promote.

This tool allows forecasting future trends in the short and medium term, offering to the organization a wide range of action and improving its ability to adapt to the changes that are anticipated. It also provides objective criteria to define the strategic position and provides information to take advantage of the opportunities that arise in certain markets. This is achieved through the description of variables that provide arguments about the behavior of the environment in the future.

PESTEL technique is useful to perform a strategic analysis to define the external environment of a company, as it is referred in [1-4], due to the advantages that this method provides, since it constitutes a research guide of the context surrounding the company. Among the advantages that most stand out are the following:

- **It adapts to each case.** There exist factors that can be included within others. For example, the legislature can be easily integrated into the political and industry can be included in economics. The ecological factor can also be easily included in social one. That depends on the area in which the activity of the company takes place and the peculiarities of its sector.
- **It helps to make decisions.** This is because it permits to know about the market and the factors that produce no growth or decline, their potential and attractiveness, the simple identification and control of the present risks which in turn can be potentially determined, and finally, about the convenience or not to enter on it. Therefore, it is useful when it is applied in internationalization processes.
- **It has a proactive approach.** It allows anticipating changes and glimpsing future trends in such a way that the organization will go one-step ahead and shall not have to suddenly react to the new market characteristics. In addition, it facilitates planning and minimizes the impact of adverse scenarios.
- **It is of broad application.** Whether it is to make decisions about the foundation of a new company, the

opening of an office in a foreign country or region, the redefinition of the brand, a possible acquisition or the association with partners, in every of these situations PESTEL analysis allows knowing in detail the trends that will mark the future of the market.

The aforementioned author refers that PESTEL factors serve to know the tendencies and redesign the business strategy. The variables that integrate these factors are the following:

- **Political variables:** These are the governmental aspects that directly affect the company. This variable involves tax policies or business incentives in certain sectors, employment regulations, the promotion of foreign trade, government stability, the government system, international treaties or the existence of conflicts of any kind, like internal, current external or future ones. In addition, it is related to the way in which the different local, regional and national administrations are organized. The projects of the major parties on the company are also included in this section.
- **Economic variables:** This variable is useful for analyzing macroeconomic data, the evolution of Gross National Product (GNP), interest rates, inflation, the unemployment rate, the level of income, exchange rates, access to resources, the level of development and economic cycles. Current and future economic scenarios and economic policies should also be investigated.
- **Social variables:** In this variable, the relevant factors are demographic evolution, social mobility and changes in lifestyle. Additionally, the educational level and other cultural patterns, religion, beliefs, gender roles, tastes, fashions and consumption habits of society should be included. In short, it contains the social trends that may affect the business project.
- **Technological variables:** This variable is somewhat more complex to analyze because of the accelerated changes in this area. It is necessary to know the public investment in research and the promotion of technological development, the penetration of technology, the degree of obsolescence, the level of coverage, the digital division, funds destined to Research and Development (R&D), as well as the trends in the use of new technologies.
- **Ecological variables:** This variable analyses factors related to the conservation of the environment, environmental legislation, climate change and variations in temperatures, natural risks, levels of recycling, energy regulation and possible regulatory changes in this area.
- **Legal variables:** This variable concerns all the legislation that has a direct relationship with the project, the information about licenses, labor legislation, intellectual property, sanitary laws, the regulated sectors, among others.

The decision-making process has been a central issue in the study and the configurations of organizational structures. This structure determines the definition of the organizational units, their objectives, functions, charges, and associated tasks, as well as the levels of authority-subordination, and consequently, the system of formal relations, see [5].

According to [5], in order to support decision-making in business management, it is necessary to define variables, which identify the most important aspects that are included in the business environment and that affect the future business environment, as well as those variables that identify factors less decisive and irrelevant to the operation of a business, business unit or project. For this purpose, it is recommended to start the analysis by the most general factors and finish with those that are more specific or characteristic of the company.

Once the factors of greater and less importance are obtained, it is possible to carry out a comparative analysis, assigning a qualification to each factor. This assignment facilitates the study of several characteristics that contribute to the knowledge of the environment, in particular, to know which is the most favorable or suitable environment for the purposes of the company. The weight assigned to each variable depends on the type of business, environment, among others.

These studies are often carried out using the PESTEL technique, which is an accessible tool, easy to apply and widely used by companies in different sectors and of different sizes. According to [1], PESTEL serves to evaluate the main external factors that influence a project or business. This technique facilitates the support for making early decisions because it guides the direction of the company towards future scenarios in order to determine the development of the activity. The results obtained by using the PESTEL technique are expressed qualitatively, whose results require to be treated for the solution of indeterminate problems that are obtained when applying PESTEL.

To solve the aforementioned drawback, it is proposed to translate the technique of PESTEL to the neutrosophic field. Neutrosophy is a branch of science that brings significant results when there exist problems containing indeterminacy. One example of the presence of indeterminacy in real life situations can be encountered when analysing the factors to consider for business management.

In business management area, neutrosophy theory is useful to include for obtaining greater data interpretability. It is a tool for supporting decision making, taking advantage of opposing positions as well as the neutral or

ambiguous ones. Assuming that every idea $\langle A \rangle$ tends to be neutralized, diminished, balanced by ideas, in clear rupture with the binary doctrines for explaining or understanding the phenomena [6]. Neutrosophy theory has been successfully hybridized with other decision-making techniques and methods like, with DEMATEL method, see [7], VIKOR and ANP, [8], and TOPSIS technique, [9].

Based on the aforementioned ideas, in the present study we propose the combination of PESTEL with the neutrosophic recommendation models, to make this combination a neutrosophic PESTEL technique, capable of supporting decision-making, providing a set of options in order to satisfy the business environments expectations [10]. Essentially the proposal is related to a recommendation model based on the knowledge obtained by applying the PESTEL technique.

The proposed model includes Single-Valued Neutrosophic Numbers (SVNN), which facilitates the use of linguistic terms [11]. PESTEL analysis based on a recommendation model takes into account the analysis of the factors, in order to support decision making for obtaining efficient business management. The neutrosophic PESTEL technique, which is proposed in the present work, has the possibility of treating the interdependence between the analyzed factors, feedback and treats the uncertainty.

Summarizing, in this paper a neutrosophic PESTEL analysis is designed with the purpose to be applied in business management. This model is the basis of a proposed system of recommendation, where the PESTEL analysis provides the knowledge, which is stored in a database. The data in dataset are represented in form of linguistic terms, and the calculi are made by using SVNNs. To illustrate the applicability of the model we utilize an actual example.

2 Preliminary

In this section, a brief review of the PESTEL technique and the interdependence of its factors are provided. Then we summarize those neutrosophic foundations that are able to adapt PESTEL technique to neutrosophy.

2.1 PESTEL Analysis

PESTEL analysis is applied to aid and consider subjects like, namely, political, economic, social, technological, legal and environmental. It is a tool of interest to understand the increase or decline of some specific market and consequently, the position, potential and direction of a business. PESTEL works as a frame to analyze such situations, or either to revise the strategy. In others words, PESTEL measures market potentialities and actual situation. It allows us to understand, present, discuss and take decisions about external factors. The aspects to measure PESTEL are displayed in Figure 1.

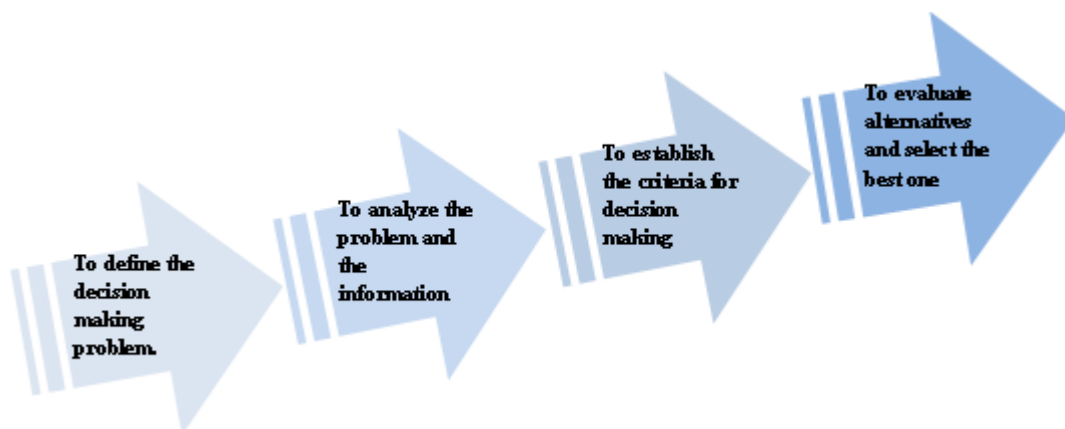


Figure 1. Pyramid PESTEL factors.

A recommendation model based on the knowledge acquired from the application of PESTEL analysis, has an integrated structure among the factors, it is modeled by a neutrosophic recommendation model and the quantitative analysis is developed from a static analysis that allows to classify and reduce the number of analysed factors.

2.2 Neutrosophic PESTEL and recommendation models based on knowledge, to support decision-making

Decision-making has been historically studied using multiple disciplines from classical ones such as philosophy, statistics, mathematics, and economics, to the most recent ones such as Artificial Intelligence [12, 13]. The developed theories and models point to rational support for making complex decisions [12]. The process for solving a decision-making problem according to [14] is shown in Figure 2.

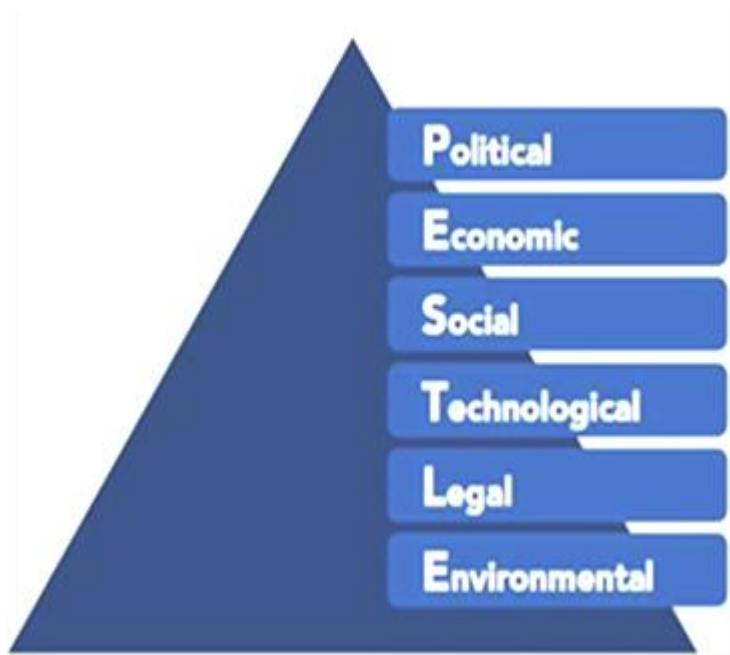


Figure 2. Phases for the solution of a decision-making problem ([14]).

Knowledge-based recommendation models provide suggestions by making inferences about the desired needs and preferences [15, 16]. The knowledge-based approach is distinguished because knowledge is used about how a particular object can satisfy the desired needs. Therefore, the knowledge-based recommendation models have the capacity to reason about the relationship between one need and the possible recommendation that is provided.

From the mathematical point of view, the proposed recommendation model is distinguished by X , which is called the universe of study. A set of unique *Single-Valued Neutrosophic Set* (SVNS) A on X , is an object that satisfies the formula given in Equation 1.

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\} \quad (1)$$

Where, $T_A: X \rightarrow [0,1]$, $I_A: X \rightarrow [0,1]$ and $F_A: X \rightarrow [0,1]$ satisfy $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, for every $x \in A$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership, indeterminate-membership and false-membership of x in A . A *Single-Valued Neutrosophic Number* (SVNN) is expressed as: $= (T, I, F)$, where $T, I, F \in [0,1]$, and $0 \leq T + I + F \leq 3$.

The models of recommendation based on sideways constructions are structures of knowledge which learn either by themselves or by processes of inference. Thus, they can be enriched with the use of natural language expressions [17, 18].

3 Proposed Framework

The proposed framework to support decision making in business management using the PESTEL technique, with a neutrosophic environment consists of four fundamental phases, they are graphically shown in Figure 3. This framework is based on [17, 19], for knowledge-based recommendation systems allowing the representation of linguistic terms and indetermination in form of SVNNs.

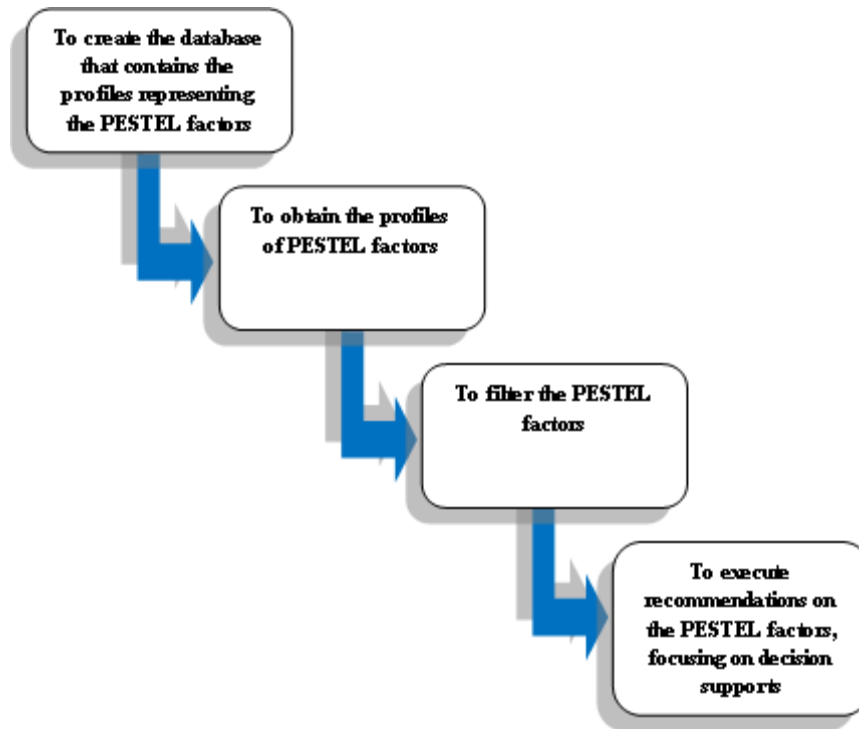


Figure 3. Proposed model for the PESTEL analysis.

3.1 Creation of the Database with the profiles that represent the factors of PESTEL

Each of the PESTEL factors are represented by a_i , which are described by a set of characteristics that will make up the profile of the factors. They are mathematically expressed by the set $C = \{c_1, \dots, c_k, \dots, c_l\}$.

In order to obtain the PESTEL factors database, the profile of each PESTEL factor is obtained by SVNNS [20, 21].

Let $A^* = \{A_1^*, A_2^*, \dots, A_n^*\}$ be a vector of SVNNS, such that $A_j^* = (a_j^*, b_j^*, c_j^*)$, for $j = 1, 2, \dots, n$. Additionally, let $B_i = (B_{i1}, B_{i2}, \dots, B_{im})$, $i = 1, 2, \dots, m$, be m vectors of n SVNNS, where $B_{ij} = (a_{ij}, b_{ij}, c_{ij})$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then, the Euclidean distance between A^* and B_i is defined in Equation 2, see [20].

$$d_i = \left(\frac{1}{3} \sum_{j=1}^n \left\{ (a_{ij} - a_j^*)^2 + (b_{ij} - b_j^*)^2 + (c_{ij} - c_j^*)^2 \right\} \right)^{\frac{1}{2}} \quad (2)$$

Where $i = 1, 2, \dots, m$.

The Euclidean distance calculated with Equation 2 defines a measure of similarity, according to it is referred in [21].

The measure of similarity varies in correspondence with the alternative A_i . Closer is A^* to the profile that represents the PESTEL factors, greater is the measure of similarity (s_i), favouring the establishment of an order between alternatives [23]. The profile that represents the PESTEL factors can be obtained directly from experts' criterion. The formula of s_i is given in Equation 3.

$$s_i = 1 - d_i \quad (3)$$

The evaluation of the PESTEL characteristics of the factors a_j , are expressed using a linguistic scale S ,

Where $S = \{s_1, \dots, s_g\}$ is the set of linguistic terms defined to assess the corresponding characteristics of each PESTEL factor, c_k , which is evaluated by using the SVNNS. For this end, the linguistic terms to employ are defined. Once the factors are described, they are included in the previously created database, like $A = \{a_1, \dots, a_j, \dots, a_n\}$.

The system outputs a recommendation about the best factors for either maintain or improve the current market situation. The recommendation is obtained from previous experiences and strategies of the company in this situation, which were stored in the database.

3.2 Obtaining profiles by PESTEL factors

In this phase we obtain the company information related to the PESTEL factors, these preferences are profiles

that are stored in the database, mathematically they are expressed as shown in Equation 4.

$$P_e = \{p_1^e, \dots, p_k^e, \dots, p_l^e\} \quad (4)$$

Where $P_k^e \in S$.

The profiles of the obtained PESTEL factors, which have been analyzed according to the preferences of the company are integrated by a set of attributes as shown in Equation 5.

$$C_e = \{c_1^e, \dots, c_k^e, \dots, c_l^e\} \quad (5)$$

Where $c_k^e \in S$.

3.3. Filtering of the PESTEL factors

In this phase, the PESTEL factors are filtered according to the profile of each obtained factor, in order to find out what PESTEL factors need to be addressed for supporting decision making in a company. For this purpose, the similarity between the profiles of each PESTEL factor, P_e and the characteristics corresponding to each PESTEL factor, a_j , previously registered in the Database, was calculated. Equation 6 is used to calculate the total similarity.

$$F_{a_j} = \{v_1^j, \dots, v_k^j, \dots, v_l^j\} \quad (6)$$

$j = 1, 2, \dots, n; v_k^e \in S$.

The function s_i calculates the similarity between the values of the user profile attributes and that of the products, a_j , according to [24].

3.4. Execute recommendations of the factors of PESTEL attend for the decision making support

The similarity between the profile of the PESTEL factors of the Database and each one of the characteristics corresponding to each PESTEL factor is calculated, see [25], these are ordered according to the obtained similarity, which is represented by the similarity vector denoted by $D = (d_1, d_2, \dots, d_n)$.

The best results are those that best meet the needs of the profile of the PESTEL factors determined in a company to support decision-making. The system outputs the factors with the best performance.

As a kind of discussion it is noteworthy to remark that our research contribution is the design of a system of recommendation based on PESTEL analysis for business management, as well as the extension of this technique to the framework of neutrosophy theory.

This approach has the advantage that decision makers can interact with the system using linguistic terms. Furthermore, the system can be coded to create a Decision Support System, which shall be useful for decision makers.

Nevertheless, we have to acknowledge that we could encounter difficulties that managers accept to apply these results in their companies, which is the main objective of our work and this is the way to validate the results. To overcome this difficult we should design strategies to introduce this techniques in real life.

4 Case study

In the present paper, we use a model of companies with specialized treatment in Cuba as the case study. These companies are not totally financed by the Cuban State, therefore, they aims to be self-financed in most of the economic aspects.

The aspects of PESTEL study are reflected in Figure 4

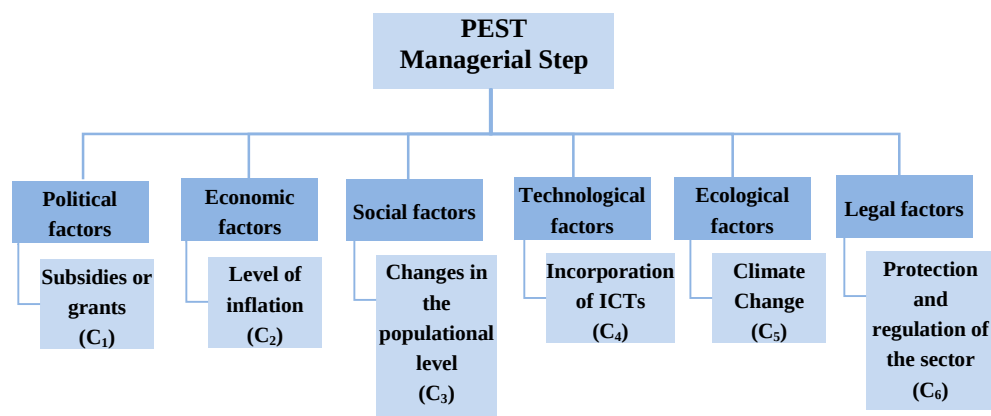


Figure 4. Feature to attend related to the PESTEL factors for business management.

According to Figure 4, specialists considered that one characteristic per factor are sufficient to determine the factors that are adequate and those that decision makers have to improve. These characteristics are the following:

C₁: Subsidies or grants, which are indicators of how the Cuban State politically supports these enterprises.

C₂: Level of inflation, which is an important economic indicator mainly for a company that sell products and the management success depends on fair prices.

C₃: Changes of the population level, which is the social factor that indicates the potential users constitute a good market for the products sold by these companies.

C₄: Incorporation of Information and Communication Technologies (ICTs), it is an important factor, because efficiency is consequence of the well use of ICTs.

C₅: Climate change, because ecology is a state policy of the Cuban government, taking into account that these are enterprises partially supported by the Cuban State. On the other side, Cuba is affected by natural disasters, mainly, hurricanes.

C₆: Protection and regulation of the sector, is a factor that guarantees that the rights of the enterprises shall be respected, and enterprises shall fulfil with their legal duties and obligations.

For this case study, we count on a database that contains all the factors profiles and their characteristics related to the analysis carried out with the PESTEL technique. These profiles are represented by a vector of shape $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$.

Where, a_1 corresponds to the Political Factor, a_2 to the Economic Factor, a_3 to the Social Factor, a_4 to the Technological Factor, a_5 to the Ecological Factor and a_6 to the Legal Factor.

The vector that describes the profiles of the PESTEL factors and their characteristics related to the realized analysis, in our case study are represented in form of neutrosophic attributes. Then, the characteristic corresponding to each factor of each PESTEL of neutrosophic attributes are measured using linguistic scales, which are shown in Table 1.

Linguistic term	SVNN
Extremely good (EG)	(1,0,0)
Very very good (VVG)	(0.9, 0.1, 0.1)
Very good (VG)	(0.8,0.15,0.20)
Good (G)	(0.70,0.25,0.30)
Moderately good (MDG)	(0.60,0.35,0.40)
Average (A)	(0.50,0.50,0.50)
Moderately bad (MDB)	(0.40,0.65,0.60)
Bad (B)	(0.30,0.75,0.70)
Very bad (VB)	(0.20,0.85,0.80)
Very very bad (VVB)	(0.10,0.90,0.90)
Extremely bad (EB)	(0,1,1)

Table 1: Linguistic terms associated to a SVNN, see [20].

We have $P_e = \{\text{MDG, VG, VVG, VG, VG, G}\}$, see Equation 4, as the results of the characteristic evaluation by the specialists, which their meanings can be read in Table 1.

In our case study, we concluded that the Political Factor has characteristics that make it “Fairly good” for the achievement of adequate business management, within the framework of the characteristics of the companies with specialized treatment in Cuba.

- It is obtained that the PESTEL factor related to the Economic Factor, is assessed as “Very good” according to the characteristic that identifies it.
- The Social Factor of PESTEL is evaluated as “Very very good” since there exists continuous changes in the population level.
- The Technological Factor of PESTEL, with the incorporation of Information and Communication Technologies (ICTs), helps companies to obtain “Very good” results.
- The Ecological Factor of PESTEL obtains “Good” results, however, in this factor it is necessary to specify what means the term climate change, to mitigate the deficiencies existing on the subject at the country and business levels.
- Concerning to the Legal Factor of PESTEL, for organizational management in companies with the previously mentioned characteristics, possesses “Very good” result, given by the protection and regulation that exists in the business sector in Cuba.

Having the PESTEL factors and the their characteristics, to support decision making in the interest of efficient business management, we calculated the similarity between the profile of the analyzed factors with PESTEL and the characteristics corresponding to each factor previously stored in the database. The result is shown in Table 2.

a1	a2	a3	a4	a5	a6
0.43	0.80	0.41	0.83	0.75	0.33

Table 2. Similarity between the PESTEL factors and the characteristics related to the factors.

During the recommendation phase, the system recommends the characteristic corresponding to each PESTEL factor which is the nearest to the factors profile. The obtained ranking of the PESTEL factors based on this comparison is the following: $a_4 > a_2 > a_5 > a_1 > a_3 > a_6$.

Whether we have to give a recommendation about the more similar factors according to the characteristics of the enterprises for achieving an appropriate managerial step, system recommends to maintain a_4 and a_2 , i.e., those that represent the Technological and the Economic Factors with the analysis of PESTEL, respectively.

Whereas, the system does not recommend a_1 , a_3 or a_6 , because the results had a small level of similarity.

Conclusions

In the present study, an extension of the PESTEL technique was proposed to the neutrosophic environment to support decision-making in business management, taking into account uncertainty. The integrated structure of PESTEL has factors that are modeled.

To illustrate the scope of application of the proposed model, companies with specialized treatment in Cuba was used as case study. These companies are affected by economic, political, social and technological factors.

On the other hand, a study of the recommendation models was carried out to address problems encountered in the measurement and evaluation process of PESTEL analysis. The integrated structure of PESTEL was modeled by a recommendation model. The proposed recommender system compares the stored knowledge in a database, extracted from PESTEL analysis, with the current evaluation of the company. This assessment is facilitated by using linguistic terms for calculation, which allows a better communication between the decision makers and the system.

Future work will focus on the development of a software tool and the use of aggregation operators to indicate interdependency among subfactors in the PESTEL analysis.

Future directions will consist in exploring the hybridization of the model that we exposed in this paper with other decision-making techniques, with the aims to improve the results in management decision, e.g., the SWOT method.

Additionally, a challenge consists in accelerating the search in the database when the number of items is big. Therefore, we also will explore to incorporate heuristics for optimizing the selection of the best option in the database, as well as to deepen the knowledge contained in the database incorporating an expert system model.

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An Integrated of Neutrosophic-ANP Technique for Supplier Selection

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Abstract. This study provides a novel integrated multi-criteria decision-making approach to supplier selection problems in neutrosophic environment. The main objective is to study the Analytic network process (ANP) technique in environment of neutrosophic and present a new method for formulation problem of Multi-Criteria Decision-Making (MCDM) in network structure out of neutrosophic and present a way of checking and calculating consistency consensus degree of decision makers. We have used neutrosophic set theory in ANP to overcome the problem that the decision makers might be have restricted knowledge or differences opinions of individuals participating in group decision making to specify deterministic valuation values to comparison judgments. We have formulated that each pairwise comparison judgment as a trapezoidal neutrosophic number. The decision makers specify the weight criteria of each criteria in the problem and compare between each criteria and effect of each criteria on other criteria Whenever number of alternatives increasing it's difficult to make a consistent judgments because the workload of giving judgments by each expert. We have introduced a real life example in the research of how to select personnel mobile according to opinion of decision makers. Through solution of a numerical example we present steps of how formulate problem in ANP by Neutrosophic.

Keywords: Triangular neutrosophic number; ANP method; supplier selection; Consistency; MCDM

1 Introduction

This The Analytic Network Process (ANP) is a new theory that extends the Analytic Hierarchy Process (AHP) to cases of dependence and feedback and generalizes on the supermatrix approach introduced in Saaty (1980) for the AHP [1]. This research focuses on ANP method, which is a generalization of AHP. Analytical Hierarchy Process (AHP) [2] is a multi-criteria decision making method that given the criteria and alternative solutions of a specific model, a graph structure is created and the decision maker is asked to pairwise compare the components, in order to determine their priorities. On the other hand, ANP supports feedback and interaction by having inner and outer dependencies among the models components [2]. We deal with the problem and analyze it and specify alternatives and the critical factors that change the decision. ANP consider one of the most technique that used for dealing with multi criteria decision making using network hierarchy.

The ANP is an expansion of AHP and it's a multi-criteria decision making technique. It's advanced by Saaty in 1996 for considering dependency and feedback between elements of decision making problem. The analytic network process models the decision making problems as a network not as hierarchies as with the analytic hierarchy process. In the analytic hierarchy process it's assumed that the alternatives depend on criteria and criteria depend on goal. So, in AHP the criteria don't depend on alternatives, criteria don't affect depend on each other and also alternatives don't depend on each other. But in the analytic network process the dependencies between decision making elements are allowed. The differences between ANP and AHP presented with the structural graph as in Fig.1. The upper side of Fig.1 shows the hierarchy of AHP in which elements from the lower level have influence on the higher level or in other words the upper level depends on the lower level. But in the lower side of Fig.1

which shows the network model of ANP, we have a cluster network and there exist some dependencies between them. The dependencies may be inner-dependencies when the cluster influence itself or may be outer-dependencies when cluster depend on another one. The complex decision making problem in real life may be contain dependencies between problem elements but AHP doesn't consider this, so it may lead to less optimal decisions and ANP is more appropriate.

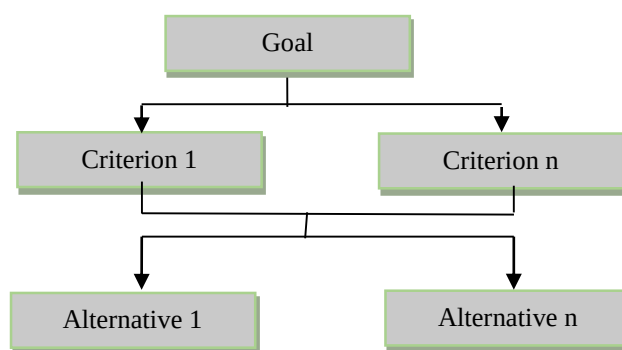
Neutrosophic is a generalization of the intuitionistic fuzzy set whilst fuzzy using true and false for express relationship, Neutrosophic using true membership, false membership and indeterminacy membership [3, 12]. ANP using network structure, dependence and feedback [4, 11]. (MCDM) is a formal and structured decision making methodology for dealing with complex problems. ANP fuzzy integrated with many researches as SWOT method. An overview of integrated ANP with intuitionistic fuzzy. Then, this research of proposed model ANP with neutrosophic represents ANP in neutrosophic environments.

The main achievements of this research are:

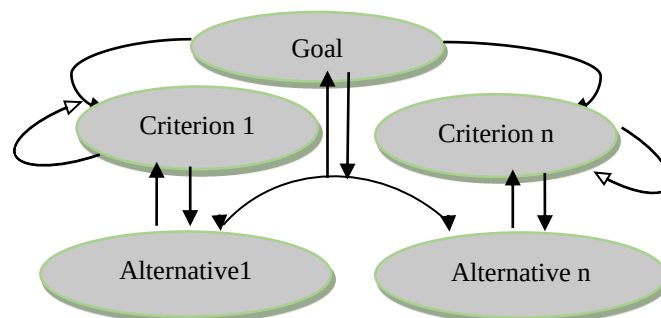
- Considering the significance of integrating of ANP method and VIKOR method under the environment of neutrosophic.
- Recognizing a comprehensive the most effective criteria for supplier's selection.

The research is organized as it is assumed up:

Section 2 gives an insight into some basic preliminaries about neutrosophic. Section 3 explains the proposed methodology of neutrosophic ANP group decision making model. Section 4 introduces numerical example. Lastly, presents conclusion.



(a) The AHP hierarchy.



(b) The ANP network.

Figure 1: The structural difference between hierarchy and network model.

2 Preliminaries

In this section, the essential definitions involving neutrosophic set, single valued neutrosophic sets, trapezoidal neutrosophic numbers and operations on trapezoidal neutrosophic numbers are defined.

Definition 1. [5, 6, 10] Let X be a space of points and $x \in X$. A neutrosophic set A in X is definite by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]0, 1+[$. That is $T_A(x):X \rightarrow]0, 1+[$, $I_A(x):X \rightarrow]0, 1+[$ and $F_A(x):X \rightarrow]0, 1+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0 \leq \sup(x) + \sup x + \sup x \leq 3+$.

Definition 2. [5, 6, 7] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object taking the form $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$, where $T_A(x):X \rightarrow [0, 1]$, $I_A(x):X \rightarrow [0, 1]$ and $F_A(x):X \rightarrow [0, 1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A , respectively. For convenience, a SVN number is represented by $A = (a, b, c)$, where $a, b, c \in [0, 1]$ and $a+b+c \leq 3$.

Definition 3. [8, 9] Suppose that $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [0, 1]$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set $\mathbb{R}$ whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x - a_3 + \beta_{\tilde{a}}(a_4 - x))}{(a_4 - a_3)} & (a_3 \leq x \leq a_4) \\ 1 & \text{otherwise} \end{cases}, \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \theta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x - a_3 + \theta_{\tilde{a}}(a_4 - x))}{(a_4 - a_3)} & (a_3 \leq x \leq a_4) \\ 1 & \text{otherwise} \end{cases}, \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x - a_3 + \beta_{\tilde{a}}(a_4 - x))}{(a_4 - a_3)} & (a_3 \leq x \leq a_4) \\ 1 & \text{otherwise} \end{cases}, \quad (3)$$

Where $\alpha_{\tilde{a}}, \theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ represent the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued trapezoidal neutrosophic number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ may express an ill-defined quantity of the range, which is approximately equal to the interval $[a_2, a_3]$.

Definition 4. [6, 8] Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3, b_4); \alpha_{\tilde{b}}, \theta_{\tilde{b}}, \beta_{\tilde{b}} \rangle$ be two single valued trapezoidal neutrosophic numbers and $\gamma \neq 0$ be any real number. Then,

1. Addition of two trapezoidal neutrosophic numbers
 $\tilde{a} + \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$
2. Subtraction of two trapezoidal neutrosophic numbers
 $\tilde{a} - \tilde{b} = \langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$

3. Inverse of trapezoidal neutrosophic number

$$\tilde{a}^{-1} = \left(\left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1} \right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \right) \quad \text{where } (\tilde{a} \neq 0)$$

4. Multiplication of trapezoidal neutrosophic number by constant value

$$Y\tilde{a} = \begin{cases} \langle (Ya_1, Ya_2, Ya_3, Ya_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & \text{if } (Y > 0) \\ \langle (Ya_4, Ya_3, Ya_2, Ya_1); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & \text{if } (Y < 0) \end{cases}$$

5. Division of two trapezoidal neutrosophic numbers

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \left(\left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right) & \text{if } (a_4 > 0, b_4 > 0) \\ \left(\left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1} \right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right) & \text{if } (a_4 < 0, b_4 > 0) \\ \left(\left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4} \right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right) & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

6. Multiplication of trapezoidal neutrosophic numbers

$$\tilde{a}\tilde{b} = \begin{cases} \langle (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \langle (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \langle (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

3 Methodology

In this study, we present the steps of proposed model we identify criteria, evaluating them and decision makers also evaluate their judgments using neutrosophic trapezoidal numbers. Since most previous researches using AHP to solve problems but AHP using hierarchy structure so not use in problems with feedback and interdependence so we presenting ANP with neutrosophic to deal with the complex problems. We present a new scale from 0 to 1 to avoid this drawbacks. We use $(n-1)$ judgments to obtain consistent trapezoidal neutrosophic preference relations instead of $\frac{n \times (n-1)}{2}$ to decrease the workload and not tired decision makers. ANP is used for ranking and selecting the alternatives. The model of ANP with neutrosophic quantifies four criteria to combine them for decision making into one global variable. To do this, we first present the concept of ANP and determine the weight of each criteria based on opinion of decision makers. Then each alternative is evaluated with other criteria and considering the effects of relationship among criteria. The ANP technique composed of four steps.

The steps of our model ANP-Neutrosophic can be introduced as:

Step1 Constructing model and problem structuring

1. Selection of decision makers (DMs).
2. Form the problem in a network
3. Preparing the consensus degree

Step2 Pairwise comparison matrices and determine weighting

1. Identify the alternatives of a problem $A = \{A_1, A_2, A_3 \dots A_m\}$.
2. Identify the criteria and sub criteria and the interdependence between it $C = \{C_1, C_2, C_3 \dots C_m\}$.
3. Determine the weighting matrix of criteria that is defined by DMs for each criteria W_1 .
4. Determine the relationship interdependence among the criteria and weight of effect of each criteria on another in range from 0 to 1.
5. Determine the interdependence matrix from multiplication of weighting matrix in step 3 and interdependence matrix in step 4.

6. Decision makers make pairwise comparisons matrix between Alternatives compared to each criterion.

$$\tilde{R} = \begin{bmatrix} (l_{11}, m_{11l}, m_{11u}, u_{11}) & (l_{11}, m_{11l}, m_{11u}, u_{11}) & \dots & (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) & (l_{22}, m_{22l}, m_{22u}, u_{22}) & \dots & (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \dots & \dots & \dots & \dots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) & (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) & \dots & (l_{nn}, m_{nnl}, m_{nnu}, u_{nn}) \end{bmatrix} \quad (4)$$

7. After making the matrix is consistent we transform neutrosophic matrix to pairwise comparisons deterministic matrix by adding (α, θ, β) and using the following equation to calculate the accuracy and score

$$S(\tilde{a}_{ij}) = \frac{1}{16} [a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (5)$$

Step3: formulation of supermatrix

1. Determine Scale and weighting data for the n alternatives against n criteria $w_{21}, w_{22}, w_{23}, w_{2n}$
2. Determine the interdependence weighting matrix of criteria comparing it to another criteria in range from 0 to 1 is defined as
3. We obtaining the weighting criteria $W_c = W_3 \times W_1$
4. Determine the interdependence matrix $\tilde{A}_{criteria}$ among the alternatives with respect to each criterion.

Step4 selection of the best alternatives

1. Determine the priorities matrix of the alternatives with respect to each of the n criteria W_{An} where n number of criteria.

$$\text{Then } W_{A1} = W_{\tilde{A}_{C1}} \times W_{21}$$

$$W_{A2} = W_{\tilde{A}_{C1}} \times W_{22}$$

$$W_{A3} = W_{\tilde{A}_{C1}} \times W_{23}$$

$$W_{An} = W_{\tilde{A}_{Cn}} \times W_{2n}$$

$$\text{Then } W_A = [W_{A1}, W_{A2}, W_{A3}, \dots, W_{An}]$$

2. In the last we ranking the priorities of criteria and obtaining the best alternatives by multiplication of the W_A matrix by the Weighting criteria matrix W_c .

$$= W_A \times W_c$$

4 Practical example

In this section, to illustrate the ANP Neutrosophic we present an example. This example is that the selecting the best personnel mobile from four alternative Samsung that is alternative A1, Huawei that is alternative A2, iPhone that is alternative A3, Infinix is alternative A4. With four criteria, the four criteria are as follows: C_1 for price, C_2 for processor, C_3 for color, C_4 for model. The criteria to be considered is the supplier selections are determined by the experts from a decision group.

Step 1: In order to compare the criteria, the decision makers assuming that there is no interdependence among criteria. The weighting matrix of criteria that is defined by decision makers is as $W_1 = (P, P, C \text{ and } M) = (0.33, 0.40, 0.22 \text{ and } 0.05)$

Step 2: Assuming that there is no interdependence among the four alternatives, (A_1, A_2, A_3, A_4) , they are compared against each criterion yielding. Decision makers determine the relationships between each criterion and Alternatives Determine the neutrosophic Decision matrix between four Alternatives (A_1, A_2, A_3, A_4) and four criteria (C_1, C_2, C_3, C_4)

$$R = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (0.3, 0.5, 0.2, 0.5; 0.3, 0.4, 0.6) & (0.6, 0.7, 0.9, 0.1; 0.4, 0.3, 0.5) & (0.7, 0.2, 0.4, 0.6; 0.8, 0.4, 0.2) & (0.3, 0.6, 0.4, 0.7; 0.4, 0.5, 0.6) \\ (0.6, 0.3, 0.4, 0.7; 0.2, 0.5, 0.8) & (0.2, 0.3, 0.6, 0.9; 0.6, 0.2, 0.5) & (0.6, 0.7, 0.8, 0.9; 0.2, 0.5, 0.7) & (0.3, 0.5, 0.2, 0.5; 0.5, 0.7, 0.8) \\ (0.3, 0.5, 0.2, 0.5; 0.4, 0.5, 0.7) & (0.3, 0.7, 0.4, 0.3; 0.2, 0.5, 0.9) & (0.8, 0.2, 0.4, 0.6; 0.4, 0.6, 0.5) & (0.2, 0.5, 0.6, 0.8; 0.4, 0.3, 0.8) \\ (0.4, 0.3, 0.1, 0.6; 0.2, 0.3, 0.5) & (0.1, 0.4, 0.2, 0.8; 0.7, 0.3, 0.6) & (0.5, 0.3, 0.2, 0.4; 0.3, 0.4, 0.7) & (0.6, 0.2, 0.3, 0.4; 0.6, 0.3, 0.4) \end{bmatrix} \end{matrix}$$

$$S(\tilde{a}_{ij}) = \frac{1}{16} [a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}})$$

The deterministic matrix can obtain by S ($\tilde{a}_{ij}$) equation in the following step:

$$R = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_3 \end{matrix} & \begin{bmatrix} 0.122 & 0.23 & 0.261 & 0.163 \\ 0.113 & 0.238 & 0.188 & 0.10 \\ 0.113 & 0.085 & 0.163 & 0.17 \\ 0.123 & 0.169 & 0.105 & 0.178 \end{bmatrix} \end{matrix}$$

Scale and weighting data for four alternatives against four criteria is derived by dividing each element by sum of each column. The comparison matrix of four alternatives and four criteria is the following: Scale and weighting data for four alternatives against four criteria:

$$\begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_3 \end{matrix} & \begin{bmatrix} 0.259 & 0.319 & 0.364 & 0.268 \\ 0.240 & 0.329 & 0.262 & 0.164 \\ 0.240 & 0.118 & 0.227 & 0.278 \\ 0.261 & 0.234 & 0.146 & 0.291 \end{bmatrix} \\ w_{21} & w_{22} & w_{23} & w_{24} \end{matrix}$$

Step 3: The interdependence among the criteria is next considered by decision makers. The interdependence weighting matrix of criteria is defined as:

$$w_3 = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix} & \begin{bmatrix} 1 & 0.8 & 0.4 & 0 \\ 0 & 0.2 & 0.5 & 0.6 \\ 0 & 0 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \end{matrix}$$

$$w_c = w_3 \times w_1 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0 \\ 0 & 0.2 & 0.5 & 0.6 \\ 0 & 0 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \times \begin{bmatrix} 0.33 \\ 0.40 \\ 0.22 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.738 \\ 0.220 \\ 0.037 \\ 0.005 \end{bmatrix}$$

Thus, it is derived that $w_c = (C_1, C_2, C_3 \text{ and } C_4) = (0.738, 0.220, 0.037, 0.005)$.

Step 4: Interdependence among the alternatives with respect to each criterion

a. First Criteria

$$\tilde{A}_{C1} = \begin{matrix} & A_1 & A_2 & A_3 & A_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (0.5, 0.5, 0.5, 0.5) & (0.3, 0.2, 0.4, 0.5; 0.7, 0.2, 0.5) & (0.1, 0.1, 0.3, 0.8; 0.5, 0.2, 0.1) & (0.1, 0.3, 0.2, 1.0; 0.5, 0.2, 0.1) \\ (0.5, 0.6, 0.8, 0.7; 0.7, 0.2, 0.5) & (0.5, 0.5, 0.5, 0.5) & (0.1, 0.2, 0.4, 0.8; 0.4, 0.5, 0.6) & (0.1, 0.2, 0.5, 1.0; 0.5, 0.1, 0.2) \\ (0.2, 0.7, 1.0, 1.0; 0.8, 0.2, 0.1) & (0.0, 0.4, 1.0, 1.0; 0.3, 0.1, 0.5) & (0.5, 0.5, 0.5, 0.5) & (0.2, 0.3, 0.4, 0.7; 0.7, 0.2, 0.5) \\ (1.0, 0.8, 1.0, 1.0; 0.6, 0.2, 0.3) & (0.2, 0.5, 1.0, 1.0; 0.6, 0.2, 0.3) & (0.3, 0.6, 0.7, 0.8; 0.9, 0.4, 0.6) & (0.5, 0.5, 0.5, 0.5) \end{bmatrix} \end{matrix}$$

Then, Sure that the matrix be deterministic or transform the previous matrix to be deterministic pairwise comparisons matrix and calculate the weight of each criteria using Eq.5. The deterministic matrix can obtain by S ($\tilde{a}_{ij}$) equation in the following step:

$$\tilde{A}_{C1} = \begin{bmatrix} 0.5 & 0.175 & 0.179 & 0.22 \\ 0.325 & 0.5 & 0.122 & 0.25 \\ 0.453 & 0.265 & 0.5 & 0.2 \\ 0.38 & 0.354 & 0.285 & 0.5 \end{bmatrix}$$

We present the weight of each alternatives according to each criteria from the deterministic matrix easily by dividing each entry by the sum of the column, we obtain the following matrix as:

$$\tilde{A}_{C1} = \begin{bmatrix} 0.30 & 0.135 & 0.165 & 0.188 \\ 0.196 & 0.386 & 0.112 & 0.214 \\ 0.273 & 0.198 & 0.460 & 0.171 \\ 0.229 & 0.274 & 0.262 & 0.427 \end{bmatrix}$$

b. Second Criteria

We present the weight of each alternatives according to each criteria from the deterministic matrix easily by dividing each entry by the sum of the column, we obtain the following matrix as:

$$\tilde{A}_{c2} = \begin{bmatrix} 0.50 & 0.215 & 0.244 & 0.192 \\ 0.216 & 0.503 & 0.161 & 0.175 \\ 0.273 & 0.182 & 0.495 & 0.197 \\ 0.229 & 0.356 & 0.259 & 0.436 \end{bmatrix}$$

c. Third Criteria

$$\tilde{A}_{c3} = \begin{bmatrix} 0.43 & 0.27 & 0.30 & 0.22 \\ 0.08 & 0.35 & 0.26 & 0.20 \\ 0.15 & 0.16 & 0.31 & 0.30 \\ 0.33 & 0.21 & 0.12 & 0.27 \end{bmatrix}$$

d. Four Criteria

$$\tilde{A}_{c4} = \begin{bmatrix} 0.40 & 0.16 & 0.16 & 0.15 \\ 0.19 & 0.43 & 0.14 & 0.19 \\ 0.23 & 0.23 & 0.5 & 0.23 \\ 0.18 & 0.18 & 0.18 & 0.42 \end{bmatrix}$$

Step 4: The overall priorities for the candidate alternatives are finally calculated by multiplying W_A and W_c and given by and presented in Fig.2.

$$= W_A \times W_c = \begin{bmatrix} 0.199 & 0.303 & 0.327 & 0.222 \\ 0.172 & 0.294 & 0.209 & 0.216 \\ 0.273 & 0.251 & 0.210 & 0.305 \\ 0.299 & 0.347 & 0.241 & 0.250 \end{bmatrix} \times \begin{bmatrix} 0.738 \\ 0.220 \\ 0.005 \\ 0.037 \end{bmatrix} = \begin{bmatrix} 0.426 \\ 0.400 \\ 0.507 \\ 0.365 \end{bmatrix}$$

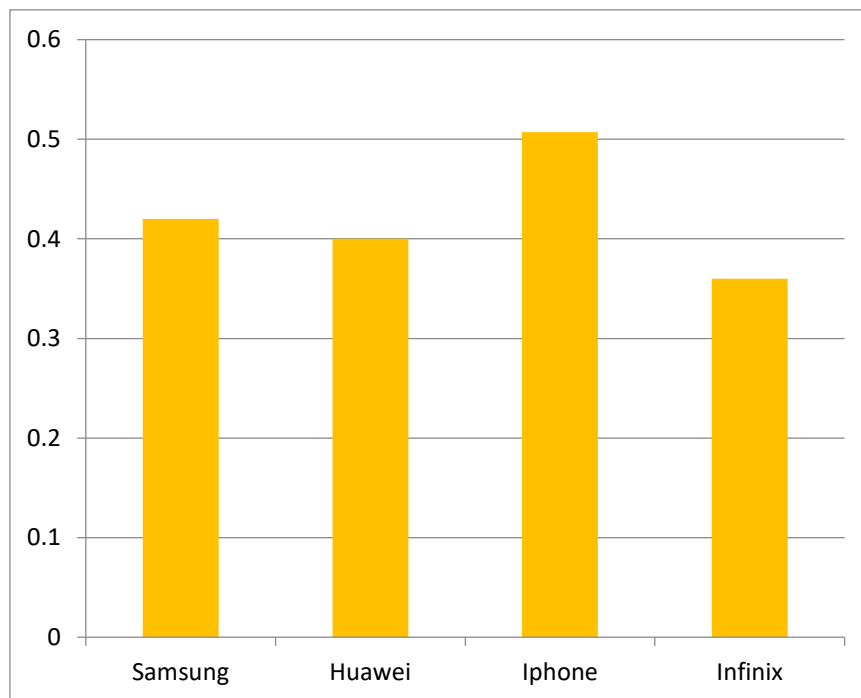


Figure 2: Ranking the alternatives using ANP under Neutrosophic.

5 Conclusion

This research presented the technique of ANP in the neutrosophic environments for solving complex problem with network structure not hierarchy and show the interdependence among criteria and feedback and relative weight of DMs. Firstly, we have presented ANP and how determine the weight for each criteria. Next, we show the interdependence among criteria and calculating effecting of each criteria on another and calculating the

weighting of each criteria to each alternatives. We have using a new scale from 0 to 1 instead of 1-9. In the future, we will apply ANP in environments of neutrosophic by integrating it by other technique such as TOPSIS and other technique. The case study we have presented is a real life example about selecting the best personnel mobile for using that the DMs specify the criteria and how select the best alternatives.

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Abstract

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