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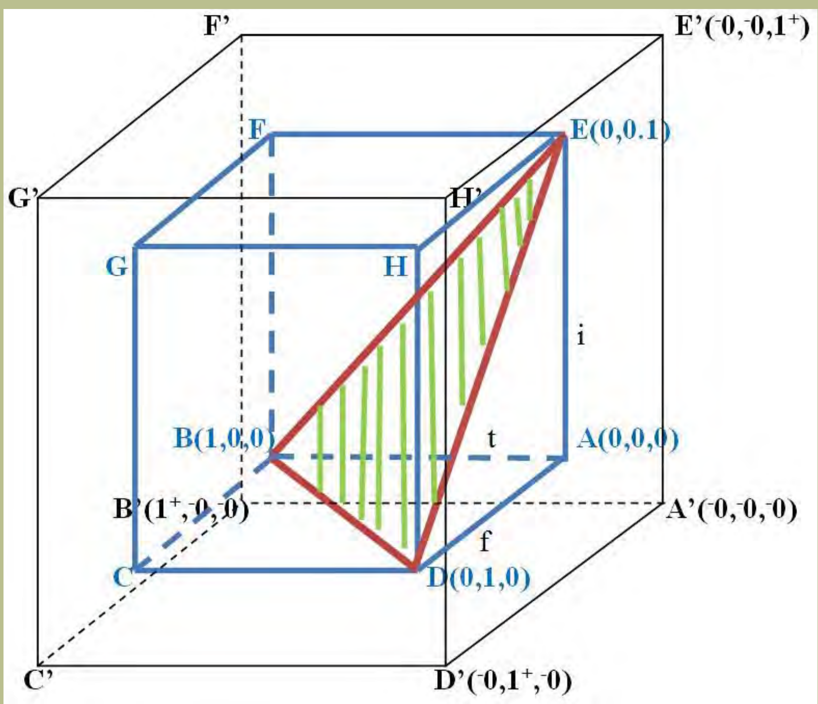

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Behaviour of ring ideal in neutrosophic and soft sense

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Abstract . This article enriches the idea of neutrosophic soft ideal (NSI). The notion of neutrosophic soft prime ideal (NSPI) is also introduced here. The characteristics of both NSI and NSPI are investigated. Their relations are drawn with the concept of ideal and prime ideal in crisp sense. Any neutrosophic soft set (Nss) can be made into NSI or NSPI using the respective cut set under a situation. The homomorphic characters of ideal and prime ideal in this new class are also drawn critically.

Keywords : Neutrosophic soft ideal (NSI); Neutrosophic soft prime ideal (NSPI); Homomorphic image.

1 Introduction

In today's world, the most of our routine activities are full of uncertainty and ambiguity. Whenever solving any problem arisen in decision making, political affairs, medicine, management, industrial and many other different real worlds, analysts suffer from a major confusion instead of directly moving towards a positive decision. The situation can be nicely conducted by practice of Neutrosophic set (N_S) theory introduced by Smarandache [7,8]. This theory represents an object by an additional value namely indeterministic function beside another two characters seen in Attanasov's theory [16]. So, Attanasov's theory can not be a proper choice in uncertain situation. Hence, the N_S theory is more reliable to an analyst, since an object is estimated here by three independent characters namely true value, indeterminate value and false value. The analysis of uncertain fact is possible in a more convenient way on the availability of adequate parameters. The soft set theory innovated by Molodtsov [5] brought that opportunity to practice the different theories in uncertain atmosphere.

Researchers are trying to extend the various mathematical structures over fuzzy set, intuitionistic fuzzy set, soft set from the very beginning. Some attempts [1,2,3,4,6,11,12,21,32,33,45] allied to group and ring theory are pointed out. Maji [22] took a successful effort to combine the neutrosophic logic with soft set theory and thus the Nss theory was brought forth. Later, modifying the different operations of Nss theory using t-norm and s-norm, Deli and Broumi [13] gave this Nss theory a new look. Doing the habit of this modified formation, Bera and Mahapatra [36] began to study the notion of NSI. From initiation, the authors are making attempt to unite with the neutrosophic logic in different mathematical areas and in many real sectors. These [9,10,14,15, 17-20, 23-31, 34-44] are some accomplishments.

The present study investigates the characteristics of NSI. Section 2 states some necessary definitions to carry on the main result. In Section 3, the structural characteristics of NSIs are investigated. Section 4 introduces and develops the concept of NSPI. Section 5 describes the nature of homomorphic image of NSI and the conclusion is given in Section 6.

2 Preliminaries

We shall remember some definitions here to make out the main thought.

2.1 Definition [38]

1. A continuous t-norm $\triangle$ maps $[0,1] \times [0,1] \rightarrow [0,1]$ and satisfies the followings.
(i) $\triangle$ is continuous and associative.
(ii) $m \triangle q = q \triangle m,\ \forall m, q \in [0,1]$.
(iii) $m \triangle 1 = 1 \triangle m = m,\ \forall m \in [0,1]$.
(iv) $m \triangle q \leq n \triangle s$ if $m \leq n,\ q \leq s$ with $m, q, n, s \in [0,1]$.
$m \triangle q = mq, m \triangle q = \min\{m, q\}, m \triangle q = \max\{m + q - 1, 0\}$ are some necessary continuous t-norms.
2. A continuous t - conorm (s - norm) $\bigtriangledown$ maps $[0,1] \times [0,1] \rightarrow [0,1]$ and obeys the followings.
(i) $\bigtriangledown$ is continuous and associative.
(ii) $w \bigtriangledown p = p \bigtriangledown w,\ \forall w, p \in [0,1]$.
(iii) $w \bigtriangledown 0 = 0 \bigtriangledown w = w,\ \forall w \in [0,1]$.
(iv) $w \bigtriangledown p \leq v \bigtriangledown q$ if $w \leq v,\ p \leq q$ with $w, v, p, q \in [0,1]$.
$w \bigtriangledown p = w + p - wp, w \bigtriangledown p = \max\{w, p\}, w \bigtriangledown p = \min\{w + p, 1\}$ are some useful continuous s-norms.

2.2 Definition [7]

An element u of a universal set X is described under an $N_S H$ by three characters $viz.$ truth-membership T_H, indeterminacy-membership I_H and falsity-membership F_H such that $T_H(u), I_H(u), F_H(u) \in]^-0, 1^+[$ and $^-0 \leq \sup T_H(u) + \sup I_H(u) + \sup F_H(u) \leq 3^+$. For $1^+ = 1 + \epsilon$, 1 is the standard part and ϵ is the non-standard part and so on for $^-0$ also. The non-standard subsets of $]^-0, 1^+[$ is practiced in philosophical ground but in real atmosphere, only the standard subsets of $]^-0, 1^+[$ i.e., $[0,1]$ is used. Thus the $N_S H$ is put as : $\{< u, (T_H(u), I_H(u), F_H(u)) >: u \in X\}$.

2.3 Definition [5]

Suppose X be the universe of discourse and E be a parametric set. Then for $B \subseteq E$ and $\wp(X)$ being the set of all subsets of X, a soft set is narrated by a pair (G, B) when G maps $B \rightarrow \wp(X)$.

2.4 Definition [22]

Suppose X be the universe of discourse and E be a parametric set. Then for $B \subseteq E$ and $N_S(X)$ being the set of all N_Ss over X, an Nss is narrated by a pair (G, B) when G maps $B \rightarrow N_S(X)$.
The Nss theory appeared in a new look by Deli and Broumi [13] as follows.

2.5 Definition [13]

Suppose X be the universe of discourse and E being a parametric set describes the elements of X. An Nss D over (X, E) is put as : $\{(b, h_D(b)) : b \in E\}$ where h_D maps $E \rightarrow N_S(X)$ given by $h_D(b) = \{< u, (T_{h_D(b)}(u), I_{h_D(b)}(u), F_{h_D(b)}(u)) >: u \in X\}$. $T_{h_D(b)}, I_{h_D(b)}, F_{h_D(b)} \in [0,1]$ are three characters of $h_D(b)$ as mentioned in Definition [7] and they are connected by the relation $0 \leq T_{h_D(b)}(u) + I_{h_D(b)}(u) + F_{h_D(b)}(u) \leq 3$.

2.5.1 Definition [13]

Over (X, E), suppose P, Q be two Nss. $\forall b \in E$ and $\forall u \in X$, if $T_{h_P(b)}(u) \leq T_{h_Q(b)}(u),\ \ I_{h_P(b)}(u) \geq I_{h_Q(b)}(u)$, $F_{h_P(b)}(u) \geq F_{h_Q(b)}(u)$, then P is called a neutrosophic soft subset of Q (denoted as $P \subseteq Q$)

2.6 Proposition [34]

A neutrosophic soft group (NSG) D is an Nss on (V, o), a classical group, obeying the inequalities mentioned below with respect to $m \bigtriangleup q = \min\{m, q\}$ and $p \bigtriangledown n = \max\{p, n\}$.

$T_{h_D(b)}(uov^{-1}) \geq T_{h_D(b)}(u) \bigtriangleup T_{h_D(b)}(v),\ I_{h_D(b)}(uov^{-1}) \leq I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$ and
$F_{h_D(b)}(uov^{-1}) \leq F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v),\ \forall u, v \in V, \forall b \in E.$

2.7 Definition [36]

1. For a neutrosophic soft ring (NSR) D on a ring $(S, +, \cdot)$ in crisp sense if each $h_D(b)$ is a neutrosophic left ideal for $b \in E$, then D is called a neutrosophic soft left ideal (NSLI) i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_D(b)}(x.y) \geq T_{h_D(b)}(y),\ I_{h_D(b)}(x.y) \leq I_{h_D(b)}(y),\ F_{h_D(b)}(x.y) \leq F_{h_D(b)}(y);$ for $x, y \in S$.

2. For an NSR D on $(S, +, \cdot)$ if each $h_D(b)$ is a neutrosophic right ideal for $b \in E$, then D is called a neutrosophic soft right ideal (NSRI) i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_D(b)}(x.y) \geq T_{h_D(b)}(x),\ I_{h_D(b)}(u.v) \leq I_{h_D(b)}(x),\ F_{h_D(b)}(x.y) \leq F_{h_D(b)}(x);$ for $x, y \in S$.

3. For an NSR D on $(S, +, \cdot)$ if each $h_D(b)$ is an NSLI as well as NSRI for $b \in E$, then D is called an NSI i.e.,

(i) $h_D(b)$ is a neutrosophic subgroup of $(S, +)$ for every $b \in E$ and

(ii) $T_{h_D(b)}(x.y) \geq \max\{T_{h_D(b)}(x), T_{h_D(b)}(y)\},\ I_{h_D(b)}(x.y) \leq \min\{I_{h_D(b)}(x), I_{h_D(b)}(y)\}$ and
$F_{h_D(b)}(x.y) \leq \min\{F_{h_D(b)}(x), F_{h_D(b)}(y)\};$ for $x, y \in S$.

2.8 Definition [35]

1. Let M be an N_S on the universe of discourse X. Then $M_{(\sigma,\eta,\delta)}$ is called (σ, η, δ)-cut of M and is described as a set $\{u \in X : T_M(u) \geq \sigma, I_M(u) \leq \eta, F_M(u) \leq \delta\}$ where $\sigma, \eta, \delta \in [0, 1]$ and $0 \leq \sigma + \eta + \delta \leq 3$. This $M_{(\sigma,\eta,\delta)}$ is called (σ, η, δ)-level set or (σ, η, δ)-cut set of the N_S M and clearly, $M_{(\sigma,\eta,\delta)} \subset X$.

2. Let D be an Nss on (X, E). Then the soft set $D_{(\sigma,\eta,\delta)} = \{(b, [h_D(b)]_{(\sigma,\eta,\delta)}) : b \in E\}$ is called (σ, η, δ)-level soft set or (σ, η, δ)-cut soft set for $\sigma, \eta, \delta \in [0, 1]$ with $0 \leq \sigma + \eta + \delta \leq 3$. Here each $[h_D(b)]_{(\sigma,\eta,\delta)}$ is an (σ, η, δ)-level set of the N_S $h_D(b)$ over X.

In the main results, we shall restrict ourselves by the t-norm as $m \bigtriangleup q = \min\{m, q\}$ and s-norm as $p \bigtriangledown n = \max\{p, n\}$ and shall take $b \in E$, a parametric set, as an arbitrary parameter.

3 Neutrosophic soft ideal

Some features of NSI are studied by developing a number of theorems here.

3.1 Proposition

Let K be an NSLI (NSRI) on (S, E). If 0_S is the additive identity of the ring S, then
(i) $T_{h_K(b)}(u) \leq T_{h_K(b)}(0_S)$, $I_{h_K(b)}(u) \geq I_{h_K(b)}(0_S)$, $F_{h_K(b)}(u) \geq F_{h_K(b)}(0_S)$, $\forall u \in R$ and $\forall b \in E$.
(ii) $K_{(\sigma,\eta,\delta)}$ is a left (right) ideal for $0 \leq \sigma \leq T_{h_K(b)}(0_S)$, $I_{h_K(b)}(0_S) \leq \eta \leq 1$, $F_{h_K(b)}(0_S) \leq \delta \leq 1$.

Proof. (i) Here, for every $b \in E$, $h_K(b)$ is a neutrosophic subgroup of $(S, +)$. Then $\forall u \in S$ and $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(0_S) &= T_{h_K(b)}(u-u) \geq T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(u) = T_{h_K(b)}(u), \\ I_{h_K(b)}(0_S) &= I_{h_K(b)}(u-u) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(u) = I_{h_K(b)}(u), \\ F_{h_K(b)}(0_S) &= F_{h_K(b)}(u-u) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(u) = F_{h_K(b)}(u); \end{aligned}$$

(ii) Let $u, v \in K_{(\sigma,\eta,\delta)}$ and $r \in S$. Then,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(v) \geq \sigma \bigtriangleup \sigma = \sigma, \\ I_{h_K(b)}(u-v) &\leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \leq \eta \bigtriangledown \eta = \eta, \\ F_{h_K(b)}(u-v) &\leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \leq \delta \bigtriangledown \delta = \delta; \end{aligned}$$

and $T_{h_K(b)}(ru) \geq T_{h_K(b)}(u) \geq \sigma$, $I_{h_K(b)}(ru) \leq I_{h_K(b)}(u) \leq \eta$, $F_{h_K(b)}(ru) \leq F_{h_K(b)}(u) \leq \delta$.
Hence $u - v$, $ru \in K_{(\sigma,\eta,\delta)}$ and so $K_{(\sigma,\eta,\delta)}$ is a left ideal of S. Similarly, one right ideal of S is $K_{(\sigma,\eta,\delta)}$ also.

3.2 Theorem

(i) Q be a non-empty ideal of crisp ring S if and only if $\exists$ an NSI K on (S, E) where $h_K : E \longrightarrow N_S(S)$ is given as, $\forall b \in E$,

$$T_{h_K(b)}(u) = \begin{cases} p_1 & \text{if } u \in Q \\ s_1 \ (< p_1) & \text{if } u \notin Q. \end{cases} \quad I_{h_K(b)}(u) = \begin{cases} p_2 & \text{if } u \in Q \\ s_2 \ (> p_2) & \text{if } u \notin Q. \end{cases} \quad F_{h_K(b)}(u) = \begin{cases} p_3 & \text{if } u \in Q \\ s_3 \ (> p_3) & \text{if } u \notin Q. \end{cases}$$

Briefly stated $$h_K(b)(u) = \begin{cases} (p_1, p_2, p_3) & \text{when } u \in Q \\ (s_1, s_2, s_3) & \text{when } u \notin Q. \end{cases}$$

where $s_1 < p_1, s_2 > p_2, s_3 > p_3$ and $p_i, s_i \in [0, 1]$ for all $i = 1, 2, 3$.
(ii) Specifically, Q is a non empty ideal of a crisp ring S iff it's characteristic function λ_Q is an NSI on (S, E) where $\lambda_Q : E \longrightarrow N_S(S)$ is given as, $\forall b \in E$,

$$T_{\lambda_Q(b)}(u) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q. \end{cases} \quad I_{\lambda_Q(b)}(u) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q. \end{cases} \quad F_{\lambda_Q(b)}(u) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q. \end{cases}$$

*Proof.***(i)** First let Q be a non empty ideal of S in crisp sense and consider an Nss K on (S, E). We now take the following cases.
Case 1 : When $u, v \in Q$, then $u - v \in Q$, an ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= p_1 = p_1 \bigtriangleup p_1 = T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= p_2 = p_2 \bigtriangledown p_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= p_3 = p_3 \bigtriangledown p_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 2 : If $u \in Q$ but $v \notin Q$, then $u - v \notin Q$. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= s_1 = p_1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= s_2 = p_2 \triangledown s_2 = I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= s_3 = p_3 \triangledown s_3 = F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v) \end{aligned}$$

Case 3 : If $u, v \notin Q$, then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq s_1 = s_1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &\leq s_2 = s_2 \triangledown s_2 = I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &\leq s_3 = s_3 \triangledown s_3 = F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v) \end{aligned}$$

Thus in any case $\forall u, v \in R$ and $\forall b \in E$,
$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v)$, $I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v)$ and $F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v)$.

We shall now test the 2nd condition of the Definition [2.7].

Case 1 : When $u \in Q$ then $uv,\ vu \in Q$, an ideal on S, for $v \in S$. So, $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(uv) &= T_{h_K(b)}(vu) = p_1 = T_{h_K(b)}(u), \\ I_{h_K(b)}(uv) &= I_{h_K(b)}(vu) = p_2 = I_{h_K(b)}(u), \\ F_{h_K(b)}(uv) &= F_{h_K(b)}(vu) = p_3 = F_{h_K(b)}(u); \end{aligned}$$

Case 2 : If $u \notin Q$ then either $uv \in Q$ or $uv \notin Q$ and so, $\forall b \in E$,

$$\begin{aligned} &T_{h_K(b)}(uv) \geq s_1 = T_{h_K(b)}(u),\ T_{h_K(b)}(vu) \geq s_1 = T_{h_K(b)}(u), \\ &I_{h_K(b)}(uv) \leq s_2 = I_{h_K(b)}(u),\ I_{h_K(b)}(vu) \leq s_2 = I_{h_K(b)}(u), \\ &F_{h_K(b)}(uv) \leq s_3 = F_{h_K(b)}(u),\ F_{h_K(b)}(vu) \leq s_3 = F_{h_K(b)}(u); \end{aligned}$$

This shows that K is NSLI and also NSRI on (S, E). Thus K is an NSI on (S, E).

Reversely, suppose K be an NSI on (S, E) in the specified form. We are to show $Q(\neq \phi)$ is a crisp ideal of S. Let $u, v \in Q$ and $a \in S$. Then $T_{h_K(b)}(u) = T_{h_K(b)}(v) = p_1$, $I_{h_K(b)}(u) = I_{h_K(b)}(v) = p_2$, $F_{h_K(b)}(u) = F_{h_K(b)}(v) = p_3$. Now,
$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) = p_1$, $I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \triangledown I_{h_K(b)}(v) = p_2$ and $F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \triangledown F_{h_K(b)}(v) = p_3$.

Further, as K is an NSI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,
$T_{h_K(b)}(u-v) \leq T_{h_K(b)}(0_S) \leq p_1$, $I_{h_K(b)}(u-v) \geq I_{h_K(b)}(0_S) \geq p_2$, $F_{h_K(b)}(u-v) \geq F_{h_K(b)}(0_S) \geq p_3$.

This implies $T_{h_K(b)}(u-v) = p_1$, $I_{h_K(b)}(u-v) = p_2$, $F_{h_K(b)}(u-v) = p_3$ and so by construction of K, $u-v \in Q$.

Next, K is an NSLI over (S, E) and so,
$T_{h_K(b)}(au) \geq T_{h_K(b)}(u) = p_1$, $I_{h_K(b)}(au) \leq I_{h_K(b)}(u) = p_2$, $F_{h_K(b)}(au) \leq F_{h_K(b)}(u) = p_3$.

Again K is an NSLI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,
$T_{h_K(b)}(au) \leq T_{h_K(b)}(0_S) \leq p_1$, $I_{h_K(b)}(au) \geq I_{h_K(b)}(0_S) \geq p_2$, $F_{h_K(b)}(au) \geq F_{h_K(b)}(0_S) \geq p_3$.

This shows $T_{h_K(b)}(au) = p_1, I_{h_K(b)}(au) = p_2, F_{h_K(b)}(au) = p_3$. So, $au \in Q$ by structure of K. In a same corner, $ua \in Q$. Therefore, Q is a crisp ideal of S.

(ii) First suppose Q be a non empty crisp ideal of S and on (S, E), λ_Q be an Nss. Following cases are needed to discuss.

Case 1 : When $u, v \in Q$, then $u - v \in Q$, an ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &= 1 = 1 \triangle 1 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &= 0 = 0 \bigtriangledown 0 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &= 0 = 0 \bigtriangledown 0 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Case 2 : If $u \in Q$ but $v \notin Q$, then $u - v \notin Q$. Then $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &= 0 = 1 \triangle 0 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &= 1 = 0 \bigtriangledown 1 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &= 1 = 0 \bigtriangledown 1 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Case 3 : If $u, v \notin Q$, then $\forall b \in E$,

$$\begin{aligned} T_{\lambda_{Q(b)}}(u-v) &\geq 0 = 0 \triangle 0 = T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) \\ I_{\lambda_{Q(b)}}(u-v) &\leq 1 = 1 \bigtriangledown 1 = I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) \\ F_{\lambda_{Q(b)}}(u-v) &\leq 1 = 1 \bigtriangledown 1 = F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) \end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{\lambda_{Q(b)}}(u-v) \geq T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v)$, $I_{\lambda_{Q(b)}}(u-v) \leq I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v)$ and $F_{\lambda_{Q(b)}}(u-v) \leq F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v)$.

We shall now test the 2nd condition of Definition [2.7].

Case 1 : When $u \in Q$ then $uv,\ vu \in Q$, an ideal of S, for $v \in S$. So, $\forall b \in E$,

$T_{\lambda_{Q(b)}}(uv) = T_{\lambda_{Q(b)}}(vu) = 1 = T_{\lambda_{Q(b)}}(u)$, $I_{\lambda_{Q(b)}}(uv) = I_{\lambda_{Q(b)}}(vu) = 0 = I_{\lambda_{Q(b)}}(u)$ and $F_{\lambda_{Q(b)}}(uv) = F_{\lambda_{Q(b)}}(vu) = 0 = F_{\lambda_{Q(b)}}(u)$.

Case 2 : If $u \notin Q$ then either $uv \in Q$ or $uv \notin Q$ and so $\forall b \in E$,

$$\begin{aligned} &T_{\lambda_{Q(b)}}(uv) \geq 0 = T_{\lambda_{Q(b)}}(u),\ T_{\lambda_{Q(b)}}(vu) \geq 0 = T_{\lambda_{Q(b)}}(u), \\ &I_{\lambda_{Q(b)}}(uv) \leq 1 = I_{\lambda_{Q(b)}}(u),\ \ I_{\lambda_{Q(b)}}(vu) \leq 1 = I_{\lambda_{Q(b)}}(u), \\ &F_{\lambda_{Q(b)}}(uv) \leq 1 = F_{\lambda_{Q(b)}}(u),\ F_{\lambda_{Q(b)}}(vu) \leq 1 = F_{\lambda_{Q(b)}}(u); \end{aligned}$$

This shows that λ_Q is NSLI and NSRI on (S, E). Thus λ_Q is NSI on (S, E).

Reversely, let λ_Q be an NSI over (S, E) in the prescribed form. We shall have to show $Q(\neq \phi)$ is a crisp ideal of S. Let $u, v \in Q$ and $a \in S$. Then $T_{\lambda_{Q(b)}}(u) = T_{\lambda_{Q(b)}}(v) = 1$, $I_{\lambda_{Q(b)}}(u) = I_{\lambda_{Q(b)}}(v) = 0$, $F_{\lambda_{Q(b)}}(u) = F_{\lambda_{Q(b)}}(v) = 0$. Now,

$T_{\lambda_{Q(b)}}(u-v) \geq T_{\lambda_{Q(b)}}(u) \triangle T_{\lambda_{Q(b)}}(v) = 1$, $I_{\lambda_{Q(b)}}(u-v) \leq I_{\lambda_{Q(b)}}(u) \bigtriangledown I_{\lambda_{Q(b)}}(v) = 0$ and $F_{\lambda_{Q(b)}}(u-v) \leq F_{\lambda_{Q(b)}}(u) \bigtriangledown F_{\lambda_{Q(b)}}(v) = 0$.

Further, as λ_Q is an NSI over (S, E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{\lambda_{Q(b)}}(u-v) \leq T_{\lambda_{Q(b)}}(0_S) \leq 1$, $I_{\lambda_{Q(b)}}(u-v) \geq I_{\lambda_{Q(b)}}(0_S) \geq 0$, $F_{\lambda_{Q(b)}}(u-v) \geq F_{\lambda_{Q(b)}}(0_S) \geq 0$.

This implies $T_{\lambda_{Q(b)}}(u-v) = 1$, $I_{\lambda_{Q(b)}}(u-v) = 0$, $F_{\lambda_{Q(b)}}(u-v) = 0$ and so by construction of λ_Q, $u - v \in Q$.

Next, λ_Q is an NSLI over (S, E) and so,

$T_{\lambda_{Q(b)}}(au) \geq T_{\lambda_{Q(b)}}(u) = 1$, $I_{\lambda_{Q(b)}}(au) \leq I_{\lambda_{Q(b)}}(u) = 0$, $F_{\lambda_{Q(b)}}(au) \leq F_{\lambda_{Q(b)}}(u) = 0$.

Again λ_Q is an NSLI over (S,E) and as either $0_S \in Q$ or $0_S \notin Q$,

$T_{\lambda_Q(b)}(au) \leq T_{\lambda_Q(b)}(0_S) \leq 1,\ I_{\lambda_Q(b)}(au) \geq I_{\lambda_Q(b)}(0_S) \geq 0,\ F_{\lambda_Q(b)}(au) \geq F_{\lambda_Q(b)}(0_S) \geq 0.$

This shows $T_{\lambda_Q(b)}(au) = 1, I_{\lambda_Q(b)}(au) = 0, F_{\lambda_Q(b)}(au) = 0$. So, $au \in Q$ by structure of λ_Q. By same logic, $ua \in Q$. Thus, Q is a crisp ideal of S.

3.3 Theorem

Consider an NSLI (NSRI) Q over (S,E). Then, $Q_0 = \{u \in S : T_{h_Q(b)}(u) = T_{h_Q(b)}(0_S),\ I_{h_Q(b)}(u) = I_{h_Q(b)}(0_S), F_{h_Q(b)}(u) = F_{h_Q(b)}(0_S)\}$ is a crisp left (right) ideal of S for $b \in E$.

Proof. Following the reverse part of Theorem [3.2], it will be as usual.

3.4 Theorem

Q, an Nss on (S,E), is an NSLI (NSRI) iff $\widehat{Q} = \{u \in S : T_{h_Q(b)}(u) = 1,\ I_{h_Q(b)}(u) = 0,\ F_{h_Q(b)}(u) = 0\}$ with $0_S \in \widehat{Q}$ is a crisp left (right) ideal of S.

Proof. We can put Q, an Nss on (S,E), as given below, $\forall b \in E$,

$$h_Q(b)(u) = \begin{cases} (1,0,0) & \text{when } u \in \widehat{Q} \\ (s_1,s_2,s_3) & \text{when } u \notin \widehat{Q}. \end{cases}$$

where $0 \leq s_1 < 1,\ 0 < s_2 \leq 1,\ 0 < s_3 \leq 1$. Assume $\widehat{Q}$ be a crisp left ideal of S for Q being an Nss on (S,E). We shall now take the cases stated below.

Case 1 : When $u, v \in \widehat{Q}$, then $u - v \in \widehat{Q}$, a crisp left ideal. So, $\forall b \in E$,

$$\begin{aligned} T_{h_Q(b)}(u-v) &= 1 = 1 \bigtriangleup 1 = T_{h_Q(b)}(u) \bigtriangleup T_{h_Q(b)}(v) \\ I_{h_Q(b)}(u-v) &= 0 = 0 \bigtriangledown 0 = I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v) \\ F_{h_Q(b)}(u-v) &= 0 = 0 \bigtriangledown 0 = F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v) \end{aligned}$$

Case 2 : If $u \in \widehat{Q}$ but $v \notin \widehat{Q}$, then $u - v \notin \widehat{Q}$. Then $\forall b \in E$,

$$\begin{aligned} T_{h_Q(b)}(u-v) &= s_1 = 1 \bigtriangleup s_1 = T_{h_Q(b)}(u) \bigtriangleup T_{h_Q(b)}(v) \\ I_{h_Q(b)}(u-v) &= s_2 = 0 \bigtriangledown s_2 = I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v) \\ F_{h_Q(b)}(u-v) &= s_3 = 0 \bigtriangledown s_3 = F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v) \end{aligned}$$

Case 3 : If $u, v \notin \widehat{Q}$, then $\forall b \in E$,

$$\begin{aligned} T_{h_Q(b)}(u-v) &\geq s_1 = s_1 \bigtriangleup s_1 = T_{h_Q(b)}(u) \bigtriangleup T_{h_Q(b)}(v) \\ I_{h_Q(b)}(u-v) &\leq s_2 = s_2 \bigtriangledown s_2 = I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v) \\ F_{h_Q(b)}(u-v) &\leq s_3 = s_3 \bigtriangledown s_3 = F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v) \end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_Q(b)}(u-v) \geq T_{h_Q(b)}(u) \bigtriangleup T_{h_Q(b)}(v),\quad I_{h_Q(b)}(u-v) \leq I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v)$ and $F_{h_Q(b)}(u-v) \leq F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v)$.

We are to test now the 2nd condition of Definition [2.7].

Case 1 : If $u \in \widehat{Q}$ then $vu \in \widehat{Q}$, a crisp left ideal on S, for $v \in S$. So, $\forall b \in E$,

$$T_{h_Q(b)}(vu) = 1 = T_{h_Q(b)}(u),\ I_{h_Q(b)}(vu) = 0 = I_{h_Q(b)}(u),\ F_{h_Q(b)}(vu) = 0 = F_{h_Q(b)}(u).$$

Case 2 : If $u \notin \widehat{Q}$ then either $vu \in \widehat{Q}$ or $vu \notin \widehat{Q}$ for $v \in R$ and so $\forall b \in E$,

$$T_{h_Q(b)}(vu) \geq s_1 = T_{h_Q(b)}(u),\ I_{h_Q(b)}(vu) \leq s_2 = I_{h_Q(b)}(u),\ F_{h_Q(b)}(vu) \leq s_3 = F_{h_Q(b)}(u).$$

This shows that Q is an NSLI over (S, E).

Conversely, let Q be an NSLI on (S, E) in the assumed structure. Let $u, v \in \widehat{Q}$ and $a \in S$. Then $T_{h_Q(b)}(u) = T_{h_Q(b)}(v) = 1,\ I_{h_Q(b)}(u) = I_{h_Q(b)}(v) = 0,\ F_{h_Q(b)}(u) = F_{h_Q(b)}(v) = 0$. Now,

$$T_{h_Q(b)}(u - v) \geq T_{h_Q(b)}(u) \bigtriangleup T_{h_Q(b)}(v) = 1,\ \ I_{h_Q(b)}(u - v) \leq I_{h_Q(b)}(u) \bigtriangledown I_{h_Q(b)}(v) = 0 \quad \text{and}$$
$$F_{h_Q(b)}(u - v) \leq F_{h_Q(b)}(u) \bigtriangledown F_{h_Q(b)}(v) = 0.$$

Further, as Q is an NSLI over (R, E) and as either $0_S \in \widehat{Q}$ or $0_S \notin \widehat{Q}$,

$$T_{h_Q(b)}(u - v) \leq T_{h_Q(b)}(0_S) \leq 1,\ I_{h_Q(b)}(u - v) \geq I_{h_Q(b)}(0_S) \geq 0,\ F_{h_Q(b)}(u - v) \geq F_{h_Q(b)}(0_S) \geq 0.$$

This implies $T_{h_Q(b)}(u - v) = 1,\ I_{h_Q(b)}(u - v) = 0, F_{h_Q(b)}(u - v) = 0$ and so by construction of Q, $u - v \in \widehat{Q}$. Next, Q is an NSLI over (R, E) and so,

$$T_{h_Q(b)}(au) \geq T_{h_Q(b)}(u) = 1,\ I_{h_Q(b)}(au) \leq I_{h_Q(b)}(u) = 0,\ F_{h_Q(b)}(au) \leq F_{h_Q(b)}(u) = 0.$$

Again Q is an NSLI over (R, E) and as either $0_R \in \widehat{Q}$ or $0_R \notin \widehat{Q}$,

$$T_{h_Q(b)}(au) \leq T_{h_Q(b)}(0_R) \leq 1,\ I_{h_Q(b)}(au) \geq I_{h_Q(b)}(0_R) \geq 0,\ F_{h_Q(b)}(au) \geq F_{h_Q(b)}(0_R) \geq 0.$$

This shows $T_{h_Q(b)}(au) = 1, I_{h_Q(b)}(au) = 0, F_{h_Q(b)}(au) = 0$ i.e., $au \in \widehat{Q}$. Therefore, $\widehat{Q}$ is a crisp left ideal of S and so is $\widehat{Q}$ over S similarly.

3.5 Theorem

Let K be an Nss over (S, E). Then K is an NSLI (NSRI) iff each nonempty cut set $[h_K(b)]_{(\delta,\eta,\sigma)}$ of the N_S $h_K(b)$ is a crisp left (right) ideal of S for $\delta \in Im\, T_{h_K(b)},\ \eta \in Im\, I_{h_K(b)},\ \sigma \in Im\, F_{h_K(b)}$.

Proof. Let K be an NSLI (NSRI) over (S, E) and $u, v \in [h_K(b)]_{(\delta,\eta,\sigma)}, r \in S$. Then,

$$T_{h_K(b)}(u - v) \geq T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(v) \geq \delta \bigtriangleup \delta = \delta$$
$$I_{h_K(b)}(u - v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \leq \eta \bigtriangledown \eta = \eta$$
$$F_{h_K(b)}(u - v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \leq \sigma \bigtriangledown \sigma = \sigma \ \text{ and}$$

$$T_{h_K(b)}(ru) \geq T_{h_K(b)}(u) \geq \delta,\ I_{h_K(b)}(ru) \leq I_{h_K(b)}(u) \leq \eta,\ \ F_{h_K(b)}(ru) \leq F_{h_K(b)}(u) \leq \sigma.$$

Hence $u - v,\ ru \in [h_K(b)]_{(\delta,\eta,\sigma)}$ and so $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp left ideal of S. By same way, $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a right ideal of S.

Reversely, assume $[h_K(b)]_{(\delta,\eta,\sigma)}$ be a crisp left (right) ideal of S and $u, v \in S$. If possible, let

$$T_{h_K(b)}(u - v) < T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(v),\ I_{h_K(b)}(u - v) > I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \text{ and}$$
$$F_{h_K(b)}(u - v) > F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v).$$

If $T_{h_K(b)}(u) \bigtriangleup T_{h_K(b)}(v) = s$ (say), then $T_{h_K(b)}(u) \geq s$ and $T_{h_K(b)}(v) \geq s$. As cut set is a crisp left ideal, so $T_{h_K(b)}(u - v) \geq s$ is natural. It shows a contradiction for $T_{h_K(b)}(u - v) < s$. Hence $T_{h_K(b)}(u - v) \geq T_{h_K(b)}(u) \bigtriangleup T_{h_K(x)}(v)$. Other two can be shown as usual.

For $r \in S$, let, $T_{h_K(b)}(ru) < T_{h_K(b)}(u),\ I_{h_K(b)}(ru) > I_{h_K(b)}(u)$ and $F_{h_K(b)}(ru) > F_{h_K(b)}(u)$.

If $T_{h_K(b)}(u) = t$, then $T_{h_K(b)}(ru) < t$. As cut set is a crisp left ideal, then $T_{h_K(x)}(ru) \geq t$ is obvious. It is against our assumption. So, $T_{h_K(x)}(ru) \geq T_{h_K(x)}(u)$. Other two can be set naturally. Thus K is an NSLI on

(S, E). K can also be shown an NSRI over (S, E) by same path and thus the theorem is ended.

4 Neutrosophic soft prime ideal

This section defines and illustrates NSPI along with the development of some theorems.

4.1 Definition

A constant Nss K on (S, E) is one whose $h_K(b)$ is constant $\forall b \in E$. It means, for every $b \in E$, the triplet $(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u))$ always gives same value $\forall u \in S$.
If for every $b \in E$, the triplet $(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u))$ is at least of two different kinds $\forall u \in S$, then K is called a nonconstant Nss.

4.2 Definition

Let C, D be two Nss on (S, E). Then CoD $(= P$, say) is also an Nss on (S, E). $\forall b \in E$ and $\forall u \in S$, it is defined as :

$$T_{h_P(b)}(u) = \begin{cases} \max_{u=vz}[T_{h_C(x)}(v) \triangle T_{h_D(x)}(z)] \\ 0 \quad \text{if } u \text{ is not put as } u = vz. \end{cases}$$

$$I_{h_P(b)}(u) = \begin{cases} \min_{u=vz}[I_{h_C(x)}(v) \bigtriangledown I_{h_D(x)}(z)] \\ 1 \quad \text{if } u \text{ is not put as } u = vz. \end{cases}$$

$$F_{h_P(b)}(u) = \begin{cases} \min_{u=vz}[F_{h_C(x)}(v) \bigtriangledown F_{h_D(x)}(z)] \\ 1 \quad \text{if } x \text{ is not put as } u = vz. \end{cases}$$

4.3 Definition

An NSI K over (S, E) is called an NSPI when (i) K is not constant NSI, (ii) for any two NSIs C, D over (S, E), $CoD \subseteq K$ implies either $C \subseteq K$ or $D \subseteq K$.

4.3.1 *Example*

Consider the integer set Z and the parametric set $E = \{b_1, b_2, b_3\}$. Take a division Z into $3Z$ and $Z - 3Z$. Consider an Nss K on (Z, E) given below.

Table 1 : Tabular form of Nss K

	$h_K(b_1)$	$h_K(b_2)$	$h_K(b_3)$
$3Z$	$(0.9, 0.4, 0.1)$	$(0.4, 0.3, 0.4)$	$(0.8, 0.7, 0.3)$
$Z - 3Z$	$(0.6, 0.7, 0.5)$	$(0.1, 0.6, 0.5)$	$(0.2, 0.9, 0.4)$

Now the following several cases are taken into consideration.
Case 1 : If $u, v \in 3Z$ then $u - v, uv \in 3Z$.
Case 2 : If $u, v \in Z - 3Z$ then $u - v \in 3Z$ or $Z - 3Z, uv \in Z - 3Z$.
Case 3 : If $u \in 3Z, v \in Z - 3Z$ then $u - v \in Z - 3Z$ and $uv \in 3Z$.

Obviously, K is an NSI on (Z, E). To make out that, consider Case 3 with respect to the parameter b_1. Other two are as usual.

$$\begin{cases} T_{h_K(b_1)}(u-v) = 0.6 = \min\{0.9, 0.6\} = T_{h_K(b_1)}(u) \triangle T_{h_K(b_1)}(v) \\ I_{h_K(b_1)}(u-v) = 0.7 = \max\{0.4, 0.7\} = I_{h_K(b_1)}(u) \bigtriangledown I_{h_K(b_1)}(v) \\ F_{h_K(b_1)}(u-v) = 0.5 = \max\{0.1, 0.5\} = F_{h_K(b_1)}(u) \bigtriangledown F_{h_K(b_1)}(v). \end{cases}$$

$$\begin{cases} T_{h_K(b_1)}(uv) = 0.9 = \max\{0.9, 0.6\} = \max\{T_{h_K(b_1)}(u), T_{h_K(b_1)}(v)\} \\ I_{h_K(b_1)}(uv) = 0.4 = \min\{0.4, 0.7\} = \min\{I_{h_K(b_1)}(u), I_{h_K(b_1)}(v)\} \\ F_{h_K(b_1)}(uv) = 0.1 = \min\{0.1, 0.5\} = \min\{F_{h_K(b_1)}(u), F_{h_K(b_1)}(v)\}. \end{cases}$$

To prove K as NSPI, we now let another two NSIs C (by Table 2) and D (by Table 3) on (Z, E). Table 4 refers the operation CoD.

Table 2 : Table for NSI C

	$h_C(b_1)$	$h_C(b_2)$	$h_C(b_3)$
$3Z$	$(0.3, 0.4, 0.6)$	$(0.7, 0.2, 0.5)$	$(0.6, 0.5, 0.1)$
$Z-3Z$	$(0.1, 0.5, 0.8)$	$(0.1, 0.6, 0.7)$	$(0.3, 0.8, 0.2)$

Table 3 : Table for NSI D

	$h_D(b_1)$	$h_D(b_2)$	$h_D(b_3)$
$3Z$	$(0.6, 0.4, 0.5)$	$(0.3, 0.5, 0.6)$	$(0.4, 0.8, 0.4)$
$Z-3Z$	$(0.2, 0.8, 0.9)$	$(0.1, 0.7, 0.8)$	$(0.1, 1.0, 0.5)$

Table 4 : Table for $CoD = Q$(say)

	$h_Q(b_1)$	$h_Q(b_2)$	$h_Q(b_3)$
$3Z$	$(0.3, 0.4, 0.6)$	$(0.3, 0.5, 0.6)$	$(0.4, 0.8, 0.4)$
$Z-3Z$	$(0.1, 0.8, 0.9)$	$(0.1, 0.7, 0.8)$	$(0.1, 1.0, 0.5)$

The discussion of $h_Q(b_1)$ is provided to convince the Table 4.
When $uv \in 3Z$, then either $u, v \in 3Z$ or $u \in 3Z, v \in Z-3Z$ or $u \in Z-3Z, v \in 3Z$.
When $uv \in Z-3Z$, then $u, v \in Z-3Z$ only. Now for $w = uv \in 3Z$,

$$T_{h_Q(b_1)}(w) = \max_w\{T_{h_C(b_1)}(u) \triangle T_{h_D(b_1)}(v)\} = \max\{0.3 \triangle 0.6, 0.3 \triangle 0.2, 0.1 \triangle 0.6\} = 0.3$$
$$I_{h_Q(b_1)}(w) = \min_w\{I_{h_C(b_1)}(u) \bigtriangledown I_{h_D(b_1)}(v)\} = \min\{0.4 \bigtriangledown 0.4, 0.4 \bigtriangledown 0.8, 0.5 \bigtriangledown 0.4\} = 0.4$$
$$F_{h_Q(b_1)}(w) = \min_w\{F_{h_C(b_1)}(u) \bigtriangledown F_{h_D(b_1)}(v)\} = \min\{0.6 \bigtriangledown 0.5, 0.6 \bigtriangledown 0.9, 0.8 \bigtriangledown 0.5\} = 0.6$$

Next for $u = uv \in Z-3Z$,

$$T_{h_Q(b_1)}(u) = \max_u\{T_{h_C(b_1)}(u) \triangle T_{h_D(b_1)}(v)\} = \max\{0.1 \triangle 0.2\} = 0.1$$
$$I_{h_Q(b_1)}(u) = \min_u\{I_{h_C(b_1)}(u) \bigtriangledown I_{h_D(b_1)}(v)\} = \min\{0.5 \bigtriangledown 0.8\} = 0.8$$
$$F_{h_Q(b_1)}(u) = \min_u\{F_{h_C(b_1)}(u) \bigtriangledown F_{h_D(b_1)}(v)\} = \min\{0.8 \bigtriangledown 0.9\} = 0.9$$

Table 1, Table 3, Table 4 execute that $D \subset K$ and $CoD \subset K$. Therefore, K is an NSPI on (Z, E).

4.4 Theorem

Consider an NSPI K on (S, E). Then $\forall b \in E$, $h_K(b)$ exactly attains two distinct values on S i.e., $|h_K(b)| = 2$.

Proof. As K is non-constant, hence $|h_K(b)| \geq 2$, $\forall b \in E$. Let $|h_K(b)| > 2$. Take $x = glb\{T_{h_K(b)}(u)\}$, $y = lub\{I_{h_K(b)}(u)\}, z = lub\{F_{h_K(b)}(u)\}$. Then $\exists\, s_1, p_1, s_2, p_2, s_3, p_3$ such that $x \leq s_1 < p_1 < T_{h_K(b)}(0_S)$, $y \geq s_2 > p_2 > I_{h_K(b)}(0_S)$, $z \geq s_3 > p_3 > F_{h_K(b)}(0_S)$. Define two Nss C, D on (S, E) as :

$T_{h_C(b)}(u) = \frac{1}{2}(s_1 + p_1),\ I_{h_C(b)}(u) = \frac{1}{2}(s_2 + p_2),\ \ F_{h_C(b)}(u) = \frac{1}{2}(s_3 + p_3),\ \forall u \in S$ and

$T_{h_D(b)}(u) = x,\ I_{h_D(b)}(u) = y,\ F_{h_D(b)}(u) = z$ if $u \notin K_{(p_1,p_2,p_3)}$,

$T_{h_D(b)}(u) = T_{h_K(b)}(0_S),\ I_{h_D(b)}(u) = I_{h_K(b)}(0_S),\ F_{h_D(b)}(u) = F_{h_K(b)}(0_S)$ if $u \in K_{(p_1,p_2,p_3)}$.

Clearly, C is an NSI on (S, E). We are to prove that D is an NSI over (S, E). Since K is an NSI on (S, E) then $K_{(p_1,p_2,p_3)}$ is a crisp ideal of S. Let $u, v \in S$. Following facts are considered.

Case 1 : When $u, v \in K_{(p_1,p_2,p_3)}$ then $u - v \in K_{(p_1,p_2,p_3)}$. So,

$$\begin{aligned}
T_{h_D(b)}(u-v) &= T_{h_K(b)}(0_S) = T_{h_K(b)}(0_S) \triangle T_{h_K(b)}(0_S) = T_{h_D(b)}(u) \triangle T_{h_D(b)}(v)\\
I_{h_D(b)}(u-v) &= I_{h_K(b)}(0_S) = I_{h_K(b)}(0_S) \bigtriangledown I_{h_K(b)}(0_S) = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)\\
F_{h_D(b)}(u-v) &= F_{h_K(b)}(0_S) = F_{h_K(b)}(0_S) \bigtriangledown F_{h_K(b)}(0_S) = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)
\end{aligned}$$

Case 2 : When $u \in K_{(p_1,p_2,p_3)}$, $v \notin K_{(p_1,p_2,p_3)}$ then $u - v \notin K_{(p_1,p_2,p_3)}$ and so,

$$\begin{aligned}
T_{h_D(b)}(u-y) &= x = T_{h_K(b)}(0_S) \triangle x = T_{h_D(b)}(u) \triangle T_{h_D(b)}(v)\\
I_{h_D(b)}(u-v) &= y = I_{h_K(b)}(0_S) \bigtriangledown y = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)\\
F_{h_D(b)}(u-v) &= z = F_{h_K(b)}(0_S) \bigtriangledown z = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)
\end{aligned}$$

Case 3 : When $u, v \notin K_{(p_1,p_2,p_3)}$ then,

$$\begin{aligned}
T_{h_D(b)}(u-v) &\geq x = x \triangle x = T_{h_D(b)}(u) \triangle T_{h_D(b)}(v)\\
I_{h_D(b)}(u-v) &\leq y = y \bigtriangledown y = I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)\\
F_{h_D(b)}(u-v) &\leq z = z \bigtriangledown z = F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)
\end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_D(b)}(u-v) \geq T_{h_D(b)}(u) \triangle T_{h_D(b)}(v)$, $\ I_{h_D(b)}(u-v) \leq I_{h_D(b)}(u) \bigtriangledown I_{h_D(b)}(v)$ and $F_{h_D(b)}(u-v) \leq F_{h_D(b)}(u) \bigtriangledown F_{h_D(b)}(v)$.

We are to test the 2nd condition of Definition [2.7].

Case 1 : When $u \in K_{(p_1,p_2,p_3)}$ then $uv,\ vu \in K_{(p_1,p_2,p_3)}$, a crisp ideal of S, for $u, v \in S$. So,

$$\begin{aligned}
T_{h_D(b)}(uv) &= T_{h_D(b)}(vu) = T_{h_K(b)}(0_S) = T_{h_D(b)}(u)\\
I_{h_D(b)}(uv) &= I_{h_D(b)}(vu) = I_{h_K(b)}(0_S) = I_{h_D(b)}(u)\\
F_{h_D(b)}(uv) &= F_{h_D(b)}(vu) = F_{h_K(b)}(0_S) = F_{h_D(b)}(u)
\end{aligned}$$

Case 2 : If $u \notin K_{(p_1,p_2,p_3)}$ then,

$$\begin{aligned}
&T_{h_D(b)}(uv) \geq x = T_{h_D(b)}(u),\ \ T_{h_D(b)}(vu) \geq x = T_{h_D(b)}(u)\\
&I_{h_D(b)}(uv) \leq y = I_{h_D(b)}(u),\ \ I_{h_D(b)}(vu) \leq y = I_{h_D(b)}(u)\\
&F_{h_D(b)}(uv) \leq z = F_{h_D(b)}(u),\ \ F_{h_D(b)}(vu) \leq z = F_{h_D(b)}(u)
\end{aligned}$$

This shows that D is both NSLI and NSRI over (S, E). So, D is an NSI on (S, E). We claim $CoD \subseteq K$. We require following cases to analyse.

Case 1 : Tell $P = CoD$. For $u = 0_S$,

$$\begin{aligned}
T_{h_P(b)}(u) &= \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] \leq \frac{1}{2}(s_1 + p_1) \triangle T_{h_K(b)}(0_S) \\
&< T_{h_K(b)}(0_S) \triangle T_{h_K(b)}(0_S) \text{ [as } s_1 < p_1 < T_{h_K(b)}(0_S)] = T_{h_K(b)}(0_S) \\
I_{h_P(b)}(u) &= \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] \geq \frac{1}{2}(s_2 + p_2) \bigtriangledown I_{h_K(b)}(0_S) \\
&> I_{h_K(b)}(0_S) \bigtriangledown I_{h_K(b)}(0_S) \text{ [as } s_2 > p_2 > I_{h_K(b)}(0_S)] = I_{h_K(b)}(0_S) \\
F_{h_P(b)}(u) &= \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] \geq \frac{1}{2}(s_3 + p_3) \bigtriangledown F_{h_K(b)}(0_S) \\
&> F_{h_K(b)}(0_S) \bigtriangledown F_{h_K(b)}(0_S) \text{ [as } s_3 > p_3 > F_{h_K(b)}(0_S)] = F_{h_K(b)}(0_S)
\end{aligned}$$

Case 2 : For $u \neq 0_S$ but $u \in K_{(p_1,p_2,p_3)}$,

$$\begin{aligned}
T_{h_P(b)}(u) &= \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] \leq \frac{1}{2}(s_1 + p_1) \triangle T_{h_K(b)}(0_S) \\
&= \frac{1}{2}(s_1 + p_1) \quad \text{[as } s_1 < p_1 < T_{h_K(b)}(0_S)] \\
&< p_1 \quad \text{[as } s_1 < p_1] \leq T_{h_K(b)}(u) \\
I_{h_P(b)}(u) &= \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] \geq \frac{1}{2}(s_2 + p_2) \bigtriangledown I_{h_K(b)}(0_S) \\
&= \frac{1}{2}(s_2 + p_2) \quad \text{[as } s_2 > p_2 > I_{h_K(b)}(0_S)] \\
&> p_2 \quad \text{[as } t_2 > m_2] \geq I_{h_K(b)}(u) \\
F_{h_P(b)}(u) &= \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] \geq \frac{1}{2}(s_3 + p_3) \bigtriangledown F_{h_K(b)}(0_S) \\
&= \frac{1}{2}(s_3 + p_3) \quad \text{[as } s_3 > p_3 > F_{h_K(b)}(0_S)] \\
&> p_3 \quad \text{[as } s_3 > p_3] \geq F_{h_K(b)}(u)
\end{aligned}$$

Case 3 : When $0_S \neq u \notin K_{(p_1,p_2,p_3)}$, for $v, w \in S$ such that $u = vw$, $v \notin K_{(p_1,p_2,p_3)}$ and $w \notin K_{(p_1,p_2,p_3)}$,

$$\begin{aligned}
T_{h_P(b)}(u) &= \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \frac{1}{2}(s_1 + p_1) \triangle x = x \quad \text{[as } x \leq s_1 < p_1] \leq T_{h_K(b)}(u) \\
I_{h_P(b)}(u) &= \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(s_2 + p_2) \bigtriangledown y = y \quad \text{[as } y \geq s_2 > p_2] \geq I_{h_K(b)}(u) \\
F_{h_P(b)}(u) &= \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(s_3 + p_3) \bigtriangledown z = z \quad \text{[as } z \geq s_3 > p_3] \geq F_{h_K(b)}(u)
\end{aligned}$$

Therefore, $CoD \subseteq K$. Lastly, let $v \in S$ such that $T_{h_K(b)}(v) = s_1$, $I_{h_K(b)}(v) = s_2$, $F_{h_K(b)}(v) = s_3$. Then, $T_{h_C(b)}(v) = \frac{1}{2}(s_1 + p_1) > T_{h_K(b)}(v)$. Then $C \nsubseteq K$. Again assume $w \in S$ for which $T_{h_K(b)}(w) = p_1$, $I_{h_K(b)}(w) = p_2$, $F_{h_K(b)}(w) = p_3$ i.e., $w \in K_{(p_1,p_2,p_3)}$. Then $T_{h_D(b)}(w) = T_{h_K(b)}(0_S) > p_1 = T_{h_K(b)}(w)$ imply $D \nsubseteq K$. Hence, neither $C \nsubseteq K$ nor $D \nsubseteq K$ if $CoD \subseteq K$. Therefore, K is not an NSPI on (S, E) and it is against the hypothesis. So, $h_K(b)$ exactly attains two distinct values on S for $b \in E$ i.e., $|h_K(b)| = 2$.

4.5 Theorem

If K is an NSPI on (S, E), then $T_{h_K(b)}(0_S) = 1,\ I_{h_K(b)}(0_S) = 0,\ F_{h_K(b)}(0_S) = 0,\ \forall b \in E$.

Proof. For K being an NSPI on (S, E), $|h_K(b)| = 2, \forall b \in E$. Assume $T_{h_K(b)}(0_S) < 1,\ I_{h_K(b)}(0_S) > 0,\ F_{h_K(b)}(0_S) > 0$. For K being nonconstant, $\exists\, u \in S$ for which $T_{h_K(b)}(u) < T_{h_K(b)}(0_S),\ I_{h_K(b)}(u) > I_{h_K(b)}(0_S),\ F_{h_K(b)}(u) > F_{h_K(b)}(0_S)$. Let $T_{h_K(b)}(u) = p_1, T_{h_K(b)}(0_S) = m_1, I_{h_K(b)}(u) = p_2, I_{h_K(b)}(0_S) = m_2, F_{h_K(b)}(u) = p_3, F_{h_K(b)}(0_S) = m_3$. Take s_1, s_2, s_3 for that $p_1 < m_1 < s_1 \leq 1,\ p_2 > m_2 > s_2 \geq 0,\ p_3 > m_3 > s_3 \geq 0$. We assume two Nss C, D on (S, E) so that,

$T_{h_C(b)}(u) = \frac{1}{2}(p_1 + m_1),\ I_{h_C(b)}(u) = \frac{1}{2}(p_2 + m_2),\ F_{h_C(b)}(u) = \frac{1}{2}(p_3 + m_3),\ \forall u \in S$ and

$T_{h_D(b)}(u) = p_1,\ I_{h_D(b)}(u) = p_2,\ F_{h_D(b)}(u) = p_3$ for $u \notin K_0$,

$T_{h_D(b)}(u) = s_1,\ I_{h_D(b)}(u) = s_2,\ F_{h_D(b)}(u) = s_3$ if $u \in K_0$

where $K_0 = \{u \in S : T_{h_K(b)}(u) = T_{h_K(b)}(0_S), I_{h_K(b)}(u) = I_{h_K(b)}(0_S), F_{h_K(b)}(u) = F_{h_K(b)}(0_S)\}$.
Clearly, C is an NSI on (S, E). D is an NSI on (S, E) for K_0 being an ideal of S. We are now to show that $CoD \subseteq K$. Following facts are needed to consider.

Case 1 : Take $Q = CoD$. For $u = 0_S$,

$$\begin{aligned} T_{h_Q(b)}(u) &= \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \max[\frac{1}{2}(p_1 + m_1) \triangle p_1, \frac{1}{2}(p_1 + m_1) \triangle s_1] \\ &= \max[p_1, \frac{1}{2}(p_1 + m_1)] = \frac{1}{2}(p_1 + m_1) < m_1 = T_{h_K(b)}(0_S) \\ I_{h_Q(b)}(u) &= \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) > m_2 = I_{h_K(b)}(0_S) \\ F_{h_Q(b)}(u) &= \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) > m_3 = F_{h_K(b)}(0_S) \end{aligned}$$

Case 2 : When $0_S \neq u = vw \in K_0$ for $v, w \in K_0 \subset S$,

$$T_{h_Q(b)}(u) = \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \frac{1}{2}(p_1 + m_1) \triangle s_1 = \frac{1}{2}(p_1 + m_1) < m_1 = T_{h_K(b)}(0_S) = T_{h_K(b)}(u)$$
$$I_{h_Q(b)}(u) = \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) \triangle s_2 = \frac{1}{2}(p_2 + m_2) > m_2 = I_{h_K(b)}(0_S) = I_{h_K(b)}(u)$$
$$F_{h_Q(b)}(u) = \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) \triangle s_3 = \frac{1}{2}(p_3 + m_3) > m_3 = F_{h_K(b)}(0_S) = F_{h_K(b)}(u)$$

Case 3 : When $0_S \neq u = vw \notin K_0$ for $v, w \in S - K_0$,

$$T_{h_Q(b)}(u) = \max_{u=vw}[T_{h_C(b)}(v) \triangle T_{h_D(b)}(w)] = \frac{1}{2}(p_1 + m_1) \triangle p_1 = p_1 = T_{h_K(b)}(u)$$
$$I_{h_Q(b)}(u) = \min_{u=vw}[I_{h_C(b)}(v) \bigtriangledown I_{h_D(b)}(w)] = \frac{1}{2}(p_2 + m_2) \bigtriangledown p_2 = p_2 = I_{h_K(b)}(u)$$
$$F_{h_Q(b)}(u) = \min_{u=vw}[F_{h_C(b)}(v) \bigtriangledown F_{h_D(b)}(w)] = \frac{1}{2}(p_3 + m_3) \bigtriangledown p_3 = p_3 = F_{h_K(b)}(u)$$

So including all, $CoD \subseteq K$. As $T_{h_K(b)}(0_S) = m_1 < s_1 = T_{h_D(b)}(0_S)$, so $D \nsubseteq K$. Further $\exists\, u \in S$ so that $T_{h_K(b)}(u) = p_1 < \frac{1}{2}(p_1 + m_1) = T_{h_C(b)}(u)$ impliy $C \nsubseteq K$. This means that K is not an NSPI which is against the hypothesis. Therefore $T_{h_K(b)}(0_S) = 1,\ I_{h_K(b)}(0_S) = 0,\ F_{h_K(b)}(0_S) = 0,\ \forall b \in E$.

4.6 Theorem

For an Nss K on (S, E), let $|h_K(b)| = 2$ and $T_{h_K(b)}(0_S) = 1$, $I_{h_K(b)}(0_S) = 0$, $F_{h_K(b)}(0_S) = 0, \forall b \in E$. If $K_0 = \{u \in S : T_{h_K(b)}(u) = T_{h_K(b)}(0_S), I_{h_K(b)}(u) = I_{h_K(b)}(0_S), F_{h_K(b)}(u) = F_{h_K(b)}(0_S)\}$ is a prime ideal on S, then K is an NSPI on (S, E).

Proof. By hypothesis, $\exists$ one $u \in S$ with $s_1 = T_{h_K(b)}(u) < 1, s_2 = I_{h_K(b)}(u) > 0, s_3 = F_{h_K(b)}(u) > 0$. The facts stated below are taken.

Case 1 : When $u, v \in K_0$, then $u - v \in K_0$, an ideal. So $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= T_{h_K(b)}(0) = 1 = 1 \triangle 1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= I_{h_K(b)}(0) = 0 = 0 \bigtriangledown 0 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= I_{h_K(b)}(0) = 0 = 0 \bigtriangledown 0 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 2 : If $u \in K_0$ but $v \notin K_0$, then $u - v \notin K_0$. Then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &= s_1 = 1 \triangle s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &= s_2 = 0 \bigtriangledown s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &= s_3 = 0 \bigtriangledown s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Case 3 : If $u, v \notin K_0$, then $\forall b \in E$,

$$\begin{aligned} T_{h_K(b)}(u-v) &\geq s_1 = T_{h_K(b)}(u) \triangle T_{h_K(b)}(v) \\ I_{h_K(b)}(u-v) &\leq s_2 = I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v) \\ F_{h_K(b)}(u-v) &\leq s_3 = F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v) \end{aligned}$$

Thus in any case $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_K(b)}(u-v) \geq T_{h_K(b)}(u) \triangle T_{h_K(b)}(v),\ I_{h_K(b)}(u-v) \leq I_{h_K(b)}(u) \bigtriangledown I_{h_K(b)}(v)$ and
$F_{h_K(b)}(u-v) \leq F_{h_K(b)}(u) \bigtriangledown F_{h_K(b)}(v)$.

To verify the final item, we consider the following cases.

Case 1 : When $u \in K_0$ then $uv,\ vu \in K_0$, an ideal over S, for $v \in s$. So $\forall b \in E$,

$T_{h_K(b)}(uv) = T_{h_K(b)}(vu) = 1 = T_{h_K(b)}(u),\ \ I_{h_K(b)}(uv) = I_{h_K(b)}(vu) = 0 = I_{h_K(b)}(u),$
$F_{h_K(b)}(uv) = F_{h_K(b)}(vu) = 0 = F_{h_K(b)}(u)$.

Case 2 : If $u \notin K_0$ then,

$$\begin{aligned} &T_{h_K(b)}(uv) \geq s_1 = T_{h_K(b)}(u),\ T_{h_K(b)}(vu) \geq s_1 = T_{h_K(b)}(u) \\ &I_{h_K(b)}(uv) \leq s_2 = I_{h_K(b)}(u),\ I_{h_K(b)}(vu) \leq s_2 = I_{h_K(b)}(u) \\ &F_{h_K(b)}(uv) \leq s_3 = F_{h_K(b)}(u),\ F_{h_K(b)}(vu) \leq s_3 = F_{h_K(b)}(u) \end{aligned}$$

This shows that K is NSI over (S, E). Let $CoD \subseteq K$ but $C \nsubseteq K$, $D \nsubseteq K$ for C, D being two NSIs on (S, E). So, $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_C(b)}(u) > T_{h_K(b)}(u),\ I_{h_C(b)}(u) < I_{h_K(b)}(u),\ F_{h_C(b)}(u) < F_{h_K(b)}(u)$ and
$T_{h_D(b)}(v) > T_{h_K(b)}(v),\ I_{h_D(b)}(v) < I_{h_K(b)}(v),\ F_{h_D(b)}(v) < F_{h_K(b)}(v)$.

Clearly, these $u, v \notin K_0$ otherwise $T_{h_C(b)}(u) > T_{h_K(b)}(u) = T_{h_K(b)}(0_S) = 1$ and $T_{h_D(b)}(u) > T_{h_K(b)}(u) = T_{h_K(b)}(0_S) = 1$ which are impossible. Then $rv,\ urv \notin K_0$, a prime ideal of S, for $r \in S$. Thus,

$T_{h_{K(b)}}(urv) = s_1 = T_{h_{K(b)}}(u) = T_{h_{K(b)}}(v),\ I_{h_{K(b)}}(urv) = s_2 = I_{h_{K(b)}}(u) = I_{h_{K(b)}}(v)$ and
$F_{h_{K(b)}}(urv) = s_3 = F_{h_{K(b)}}(u) = F_{h_{K(b)}}(v)$.

Now, if $Q = CoD$ then $\forall b \in E$ and $\forall w \in S$,

$$\begin{aligned} T_{h_{Q(b)}}(w) &= \max_{w=yz}[T_{h_{C(b)}}(y) \triangle T_{h_{D(b)}}(z)] \geq T_{h_{C(b)}}(u) \triangle T_{h_{D(b)}}(rv) \geq T_{h_{C(b)}}(u) \triangle T_{h_{D(b)}}(v) \\ &> T_{h_{K(b)}}(u) \triangle T_{h_{K(b)}}(v) = s_1 \triangle s_1 = T_{h_{K(b)}}(w) \end{aligned}$$

Hence $CoD \nsubseteq K$. Then either $C \subseteq K$ or $D \subseteq K$ implies K is an NSPI on (S, E).

4.7 Theorem

For an NSPI K on (S, E), $K_0 = \{u \in R : T_{h_{K(b)}}(u) = T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) = I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) = F_{h_{K(b)}}(0_S)\}$ is a crisp prime ideal of S.

Proof. Here, K_0 is a crisp ideal of S by Theorem [3.3]. To prove K_0 being prime, let A, B be two crisp ideals of K_0 with $AB \subseteq K_0$. Assume C, D as two Nss on (S, E) as given below, $\forall b \in E$,

$$h_C(b) = \begin{cases} (T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(0_S)) & \text{if } u \in A \\ (0, 1, 1) & \text{if } u \notin A. \end{cases}$$
$$h_D(b) = \begin{cases} (T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(0_S)) & \text{if } u \in B \\ (0, 1, 1) & \text{if } u \notin B. \end{cases}$$

Clearly C, D are two NSIs on (R, E) by Theorem [3.2]. We are to prove $CoD \subseteq K$. Consider the following facts.

Case 1 : If $Q = CoD$ and $u \in K_0$,

$$T_{h_{Q(b)}}(u) = \max_{u=vz}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(z)] \leq T_{h_{K(b)}}(0_S) \triangle T_{h_{K(b)}}(0_S) = T_{h_{K(b)}}(0_S) = T_{h_{K(b)}}(u)$$
$$I_{h_{Q(b)}}(u) = \min_{u=vz}[I_{h_{C(b)}}(v) \triangledown I_{h_{D(b)}}(z)] \geq I_{h_{K(b)}}(0_S) \triangledown I_{h_{K(b)}}(0_S) = I_{h_{K(b)}}(0_S) = I_{h_{K(b)}}(u)$$
$$F_{h_{Q(b)}}(u) = \min_{u=vz}[F_{h_{C(b)}}(v) \triangledown F_{h_{D(b)}}(z)] \geq F_{h_{K(b)}}(0_S) \triangledown F_{h_{K(b)}}(0_S) = F_{h_{K(b)}}(0_S) = F_{h_{K(b)}}(u)$$

Case 2 : If $u \notin K_0$ then for $v, z \in R$ such that $u = vz,\ v \notin K_0$ and $z \notin K_0$. Now,

$$T_{h_{Q(b)}}(u) = \max_{u=vz}[T_{h_{C(b)}}(v) \triangle T_{h_{D(b)}}(z)] = 0 \leq T_{h_{K(b)}}(u)$$
$$I_{h_{Q(b)}}(u) = \min_{u=vz}[I_{h_{C(b)}}(v) \triangledown I_{h_{D(b)}}(z)] = 1 \geq I_{h_{K(b)}}(u)$$
$$F_{h_{Q(b)}}(u) = \min_{u=vz}[F_{h_{C(b)}}(v) \triangledown F_{h_{D(b)}}(z)] = 1 \geq F_{h_{K(b)}}(u)$$

Thus in either case $CoD \subseteq K$. Then either $C \subseteq K$ or $D \subseteq K$, an NSPI over (S, E). Suppose $C \subseteq K$ but $A \nsubseteq K_0$. Then $\exists u \in A$ such that $u \notin K_0$ i.e., $T_{h_{K(b)}}(u) \neq T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) \neq I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) \neq F_{h_{K(b)}}(0_S)$, $\forall x \in E$. This implies $T_{h_{K(b)}}(u) < T_{h_{K(b)}}(0_S), I_{h_{K(b)}}(u) > I_{h_{K(b)}}(0_S), F_{h_{K(b)}}(u) > F_{h_{K(b)}}(0_S)$ by Proposition [3.1](i). Thus $T_{h_{C(b)}}(u) = T_{h_{K(b)}}(0_S) > T_{h_{K(b)}}(u)$, $I_{h_{C(b)}}(u) = I_{h_{K(b)}}(0_S) < I_{h_{K(b)}}(u)$, $F_{h_{C(b)}}(u) = F_{h_{K(b)}}(0_S) < F_{h_{K(b)}}(u)$ which is against the assumption $C \subseteq K$. So, $A \subseteq K_0$. Identically, $D \subseteq K \Rightarrow B \subseteq K_0$. Hence $AB \subseteq K_0 \Rightarrow$ either $A \subseteq K_0$ or $B \subseteq K_0$ implies K_0 is a prime ideal.

4.8 Theorem

(i) Q is a non empty crisp prime ideal of S if and only if $\exists$ an NSPI M on (S, E) where $h_M : E \longrightarrow N_S(S)$ is put as, $\forall b \in E$,

$$h_M(b) = \begin{cases} (1,0,0) & \text{when } u \in Q \\ (p_1, p_2, p_3) & \text{when } u \notin Q. \end{cases}$$

with $0 \leq p_1, p_2, p_3 \leq 1$.

(ii) Particularly, Q is a non empty crisp prime ideal of S if and only if it's characteristic function λ_Q is an NSPI on (S, E) when $\lambda_Q : E \longrightarrow N_S(S)$ is put as, $\forall b \in E$,

$$\lambda_Q(b)(u) = \begin{cases} (1,0,0) & \text{when } u \in Q \\ (0,1,1) & \text{when } u \notin Q. \end{cases}$$

Proof. **(i)** If Q be a crisp prime ideal, then M is an NSI on (S, E) by Theorem [3.2]. Consider two NSIs C, D on (S, E) with $CoD \subseteq M$ but $C \nsubseteq M$ and $D \nsubseteq M$. For $u, v \in S$ and $b \in E$,

$T_{h_{C(b)}}(u) > T_{h_{M(b)}}(u),\ I_{h_{C(b)}}(u) < I_{h_{M(b)}}(u),\ F_{h_{C(b)}}(u) < F_{h_{M(b)}}(u)$ and

$T_{h_{D(b)}}(v) > T_{h_{M(b)}}(v),\ I_{h_{D(b)}}(v) < I_{h_{M(b)}}(v),\ F_{h_{D(b)}}(v) < F_{h_{M(b)}}(v)$.

Obviously $u, v \notin Q$ otherwise $T_{h_{C(b)}}(u) > 1, I_{h_{C(b)}}(u) < 0,\ F_{h_{C(b)}}(u) < 0$ and $T_{h_{D(b)}}(v) > 1, I_{h_{D(b)}}(v) < 0, F_{h_{D(b)}}(v) < 0$ which are impossible. Then $z = uv \notin Q$ i.e., $T_{h_{M(b)}}(z) = p_1,\ I_{h_{M(b)}}(z) = p_2,\ F_{h_{M(b)}}(z) = p_3$. Now since $CoD \subseteq M$, then

$$p_1 = T_{h_{M(b)}}(z) \geq T_{h_{CoD(b)}}(z) = \max_{z=uv}[T_{h_{C(b)}}(u) \triangle T_{h_{D(b)}}(v)] > T_{h_{M(b)}}(u) \triangle T_{h_{M(b)}}(v) = p_1 \triangle p_1 = p_1$$

So $p_1 > p_1$ makes a contradiction and thus $C \nsubseteq M$ and $D \nsubseteq M$ are false. Hence $CoD \subseteq M$ implies either $C \subseteq M$ or $D \subseteq M$ i.e., M is an NSPI on (S, E).

The 'only if' part can be drawn from Theorem [4.7] by taking $T_{h_{M(b)}}(0_S) = 1,\ I_{h_{M(b)}}(0_S) = 0,\ F_{h_{M(b)}}(0_S) = 0$.

(ii) Following the sense of 1st part, it can be easily proved.

4.9 Theorem

An Nss K on (S, E) with $|h_K(b)| = 2,\ \forall b \in E$ is an NSPI over (S, E) if and only if $\widehat{K} = \{u \in S : T_{h_{K(b)}}(u) = 1,\ I_{h_{K(b)}}(u) = 0,\ F_{h_{K(b)}}(u) = 0,\ \forall b \in E\}$ with $0_S \in \widehat{K}$ is a crisp prime ideal of S.

Proof. Combining Theorem [4.7] and Theorem [4.8], it can be proved.

4.10 Theorem

An Nss K on (S, E) is an NSPI iff each nonempty cut set $[h_K(b)]_{(\delta,\eta,\sigma)}$ of $h_K(b)$, an N_S, is a crisp prime ideal of S when $\delta \in Im\, T_{h_{K(b)}},\ \eta \in Im\, I_{h_{K(b)}},\ \sigma \in Im\, F_{h_{K(b)}},\ \forall b \in E$.

Proof. Let K be an NSPI over (S, E). Then, by Theorem [3.5], $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp ideal of S. Consider another two crisp ideals A, B of S so as $AB \subseteq [h_K(b)]_{(\delta,\eta,\sigma)}$. On (S, E), define two Nss C, D as :

$$h_C(b) = \begin{cases} (\delta,0,0) & \text{if } u \in A \\ (0,\eta,\sigma) & \text{otherwise}. \end{cases} \quad h_D(b) = \begin{cases} (\delta,0,0) & \text{if } u \in B \\ (0,\eta,\sigma) & \text{otherwise}. \end{cases}$$

Then C, D are two NSIs over (R, E) and $CoD \subseteq K$. Since K is an NSPI over (R, E) then either $C \subseteq K$ or $D \subseteq K$. Now if possible, suppose $A \nsubseteq [h_K(b)]_{(\delta,\eta,\sigma)}$. Then $\exists\, u \in A$ such that $u \notin [h_K(b)]_{(\delta,\eta,\sigma)}$ i.e.,

$T_{h_K(b)}(u) < \delta,\ I_{h_K(b)}(u) > \eta,\ F_{h_K(b)}(u) > \sigma$. Now for $u \in A$,

$T_{h_C(b)}(u) = \delta > T_{h_K(b)}(u),\ I_{h_C(b)}(u) = 0 \leq \eta < I_{h_K(b)}(u),\ \ F_{h_C(b)}(u) = 0 \leq \sigma < F_{h_K(b)}(u)$.

This shows $C \nsubseteq K$. Also $D \nsubseteq K$ similarly. These are against the situation. Therefore $A \subseteq [h_K(b)]_{(\delta,\eta,\sigma)}$ means $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp prime ideal of S.

Reversely, we need to clear that K is an NSPI over (S, E) if $[h_K(b)]_{(\delta,\eta,\sigma)}$ is a crisp prime ideal of S. Take two NSIs C, D on (S, E) so as $CoD \subseteq K$. Let $C \nsubseteq K,\ D \nsubseteq K$. Then $\forall u, v \in S$ and $\forall b \in E$,

$T_{h_C(b)}(u) > T_{h_K(b)}(u),\ I_{h_C(b)}(u) < I_{h_K(b)}(u),\ F_{h_C(b)}(u) < F_{h_K(b)}(u)$ and
$T_{h_D(b)}(v) > T_{h_K(b)}(v),\ I_{h_D(b)}(v) < I_{h_K(b)}(v),\ F_{h_D(b)}(v) < F_{h_K(b)}(v)$.

Clearly $T_{h_K(b)}(u) \neq 1,\ I_{h_K(b)}(u) \neq 0,\ F_{h_K(b)}(u) \neq 0$ and $T_{h_K(b)}(v) \neq 1,\ I_{h_K(b)}(v) \neq 0,\ F_{h_K(b)}(v) \neq 0$. Let $T_{h_K(b)}(u) = T_{h_K(b)}(v) = p,\ I_{h_K(b)}(u) = I_{h_K(b)}(v) = q,\ F_{h_K(b)}(u) = F_{h_K(b)}(v) = r$. Then $T_{h_C(b)}(u) > p,\ I_{h_C(b)}(u) < q,\ F_{h_C(b)}(u) < r$ and $T_{h_D(b)}(v) > p,\ I_{h_D(b)}(v) < q,\ F_{h_D(b)}(v) < r$ i.e., $u \in [h_C(b)]_{(p,q,r)}$ and $v \in [h_D(b)]_{(p,q,r)}$. Now since $CoD \subseteq K$,

$$T_{h_K(b)}(z) \geq \max_{z=uv}[T_{h_C(b)}(u) \triangle T_{h_D(b)}(v)] > T_{h_C(b)}(u) \triangle T_{h_D(b)}(v) > p$$
$$I_{h_K(b)}(z) \leq \min_{z=uv}[I_{h_C(b)}(u) \bigtriangledown I_{h_D(b)}(v)] < I_{h_C(b)}(u) \bigtriangledown I_{h_D(b)}(v) < q$$
$$F_{h_K(b)}(z) \leq \min_{z=uv}[F_{h_C(b)}(u) \bigtriangledown F_{h_D(b)}(v)] < F_{h_C(b)}(u) \bigtriangledown F_{h_D(b)}(v) < r$$

Thus $z = uv \in [h_K(b)]_{(p,q,r)}$ i.e., $[h_C(b)]_{(p,q,r)}[h_D(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$, a crisp prime ideal of S. Then either $[h_C(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$ or $[h_D(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$. If $[h_C(b)]_{(p,q,r)} \subseteq [h_K(b)]_{(p,q,r)}$, then $u \in [h_C(b)]_{(p,q,r)}$ implies $u \in [h_K(b)]_{(p,q,r)}$. This means $T_{h_C(b)}(u) \geq p \Rightarrow T_{h_K(b)}(u) \geq p,\ I_{h_C(b)}(u) \leq q \Rightarrow I_{h_K(b)}(u) \leq q,\ F_{h_C(b)}(u) \leq r \Rightarrow F_{h_K(b)}(u) \leq r$ i.e., $T_{h_K(b)}(u) \geq T_{h_C(b)}(u),\ I_{h_K(b)}(u) \leq I_{h_C(b)}(u),\ F_{h_K(b)}(u) \leq F_{h_C(b)}(u)$. It is against the assumption. Therefore, $C \subseteq K$ or $D \subseteq K$ and the proof is reached.

5 Homomorphic image of NSI and NSPI

The homomorphic image of NSI and NSPI are analysed here. We let R_1, R_2 as two crisp rings and $\pi : R_1 \longrightarrow R_2$ being a ring homomorphism throughout this section.

5.1 Definition

If C, D be two Nss on $(R_1, E), (R_2, E)$ respectively, then $\pi(C), \pi^{-1}(D)$ are also Nss over $(R_2, E), (R_1, E)$ respectively and these are described as :

(i) $\pi(C)(v) = \{(T_{h_{\pi(C)}(b)}(v), I_{h_{\pi(C)}(b)}(v), F_{h_{\pi(C)}(b)}(v)) : b \in E\}, \forall v \in R_2$ where

$$T_{h_{\pi(C)}(b)}(v) = \begin{cases} \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 0 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$
$$I_{h_{\pi(C)}(b)}(v) = \begin{cases} \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 1 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$
$$F_{h_{\pi(C)}(b)}(v) = \begin{cases} \min\{F_{h_C(b)}(u) : u \in \pi^{-1}(v)\}, & \text{if } \pi^{-1}(v) \neq \phi \\ 1 & \text{if } \pi^{-1}(v) = \phi. \end{cases}$$

(ii) $\pi^{-1}(D)(u) = \{(T_{h_{\pi^{-1}(D)}(b)}(u), I_{h_{\pi^{-1}(D)}(b)}(u), F_{h_{\pi^{-1}(D)}(b)}(u)) : b \in E\}, \forall u \in R_1$ where

$T_{h_{\pi^{-1}(D)}(b)}(u) = T_{h_D(b)}[\pi(u)],\ \ I_{h_{\pi^{-1}(D)}(b)}(u) = I_{h_D(b)}[\pi(u)]$ and $F_{h_{\pi^{-1}(D)}(b)}(u) = F_{h_D(b)}[\pi(u)]$.

5.2 Proposition

Let C and D be two NSLIs (NSRIs) on (R_1, E) and (R_2, E) respectively. Then,
(i) $\pi(C)$ is an NSLIs (NSRIs) over (R_2, E) if π is epimorphism.
(ii) $\pi^{-1}(D)$ is an NSLIs (NSRIs) over (R_1, E).

Proof. (i) Let $v_1, v_2, s \in R_2$. If $\pi^{-1}(v_1) = \phi$ or $\pi^{-1}(v_2) = \phi$, the proof is usual. So, let $\exists\, u_1, u_2, r \in R_1$ so as $\pi(u_1) = v_1, \pi(u_2) = v_2, \pi(r) = s$. Now,

$$T_{h_{\pi(C)}(b)}(v_1 - v_2) = \max_{\pi(u)=v_1-v_2} \{T_{h_C(b)}(b)\} \geq T_{h_C(b)}(u_1 - u_2) \geq T_{h_C(b)}(u_1) \triangle T_{h_C(b)}(u_2),$$
$$T_{h_{\pi(C)}(b)}(sv_1) = \max_{\pi(u)=sv_1} \{T_{h_C(b)}(u)\} \geq T_{h_C(b)}(ru_1) \geq T_{h_C(b)}(u_1)$$

As all the inequalities are carried $\forall u_1, u_2, r \in R_1$ obeying $\pi(u_1) = v_1, \pi(u_2) = v_2, \pi(r) = s$ hence,

$$T_{h_{\pi(C)}(b)}(v_1 - v_2) \geq (\max_{\pi(u_1)=v_1} \{T_{h_C(b)}(u_1)\}) \triangle (\max_{\pi(u_2)=v_2} \{T_{h_C(b)}(u_2)\}) = T_{h_{\pi(C)}(b)}(v_1) \triangle T_{h_{\pi(C)}(b)}(v_2),$$
$$T_{h_{\pi(C)}(b)}(sv_1) \geq \max_{\pi(u_1)=v_1} \{T_{h_C(b)}(u_1)\} = T_{h_{\pi(C)}(b)}(v_1). \text{ Next,}$$
$$I_{h_{\pi(C)}(b)}(v_1 - v_2) = \min_{\pi(u)=v_1-v_2} \{I_{h_C(b)}(u)\} \leq I_{h_C(b)}(u_1 - u_2) \leq I_{h_C(b)}(u_1) \bigtriangledown I_{h_C(b)}(u_2),$$
$$I_{h_{\pi(C)}(b)}(sv_1) = \min_{\pi(u)=sv_1} \{I_{h_C(b)}(u)\} \leq I_{h_C(b)}(ru_1) \leq I_{h_C(b)}(u_1).$$

As all the inequalities are carried $\forall u_1, u_2, r \in R_1$ obeying $\pi(u_1) = y_1, \pi(u_2) = v_2, \pi(r) = s$ hence,

$$I_{h_{\pi(C)}(b)}(v_1 - v_2) \leq (\min_{\pi(u_1)=v_1} \{I_{h_C(b)}(u_1)\}) \bigtriangledown (\min_{\pi(u_2)=v_2} \{I_{h_C(b)}(u_2)\}) = I_{h_{\pi(C)}(b)}(v_1) \bigtriangledown I_{h_{\pi(C)}(b)}(v_2),$$
$$I_{h_{\pi(C)}(b)}(sv_1) \leq \min_{\pi(u_1)=v_1} \{I_{h_C(b)}(u_1)\} = I_{h_{\pi(C)}(b)}(v_1).$$

Similarly, we can show that
$F_{h_{\pi(C)}(b)}(v_1 - v_2) \leq F_{h_{\pi(C)}(b)}(v_1) \bigtriangledown F_{h_{\pi(C)}(b)}(v_2), \quad F_{h_{\pi(C)}(b)}(sv_1) \leq F_{h_{\pi(C)}(b)}(v_1).$
This brings the 1st result.
(ii) For $u_1, u_2 \in R_1$, we have,

$$\begin{aligned}
T_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) &= T_{h_D(b)}[\pi(u_1 - u_2)] = T_{h_D(b)}[\pi(u_1) - \pi(u_2)] \\
&\geq T_{h_D(b)}[\pi(u_1)] \triangle T_{h_D(b)}[\pi(u_2)] = T_{h_{\pi^{-1}(D)}(b)}(u_1) \triangle T_{h_{\pi^{-1}(D)}(b)}(u_2), \\
T_{h_{\pi^{-1}(D)}(b)}(ru_1) &= T_{h_D(b)}[\pi(ru_1)] = T_{h_D(b)}[\pi(r)\pi(u_1)] = T_{h_D(b)}[s\pi(u_1)] \\
&\geq T_{h_D(b)}[\pi(u_1)] = T_{h_{\pi^{-1}(D)}(b)}(u_1), \\
I_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) &= I_{h_D(b)}[\pi(u_1 - u_2)] = I_{h_D(b)}[\pi(u_1) - \pi(u_2)] \\
&\leq I_{h_D(b)}[\pi(u_1)] \bigtriangledown I_{h_D(b)}[\pi(u_2)] = I_{h_{\pi^{-1}(D)}(b)}(u_1) \bigtriangledown I_{h_{\pi^{-1}(D)}(b)}(u_2), \\
I_{h_{\pi^{-1}(D)}(b)}(ru_1) &= I_{h_D(b)}[\pi(ru_1)] = I_{h_D(b)}[\pi(r)\pi(u_1)] = I_{h_D(b)}[s\pi(u_1)] \\
&\leq I_{h_D(b)}[\pi(u_1)] = I_{h_{\pi^{-1}(D)}(b)}(u_1).
\end{aligned}$$

In a similar fashion,
$F_{h_{\pi^{-1}(D)}(b)}(u_1 - u_2) \leq F_{h_{\pi^{-1}(D)}(b)}(u_1) \bigtriangledown F_{h_{\pi^{-1}(D)}(b)}(u_2), \quad F_{h_{\pi^{-1}(D)}(b)}(ru_1) \leq F_{h_{\pi^{-1}(D)}(b)}(u_1).$
This brings the 2nd result.

5.3 Proposition

Take two NSLIs (NSRIs) C, D over (R_1, E) and (R_2, E), respectively. If $0_1, 0_2$ are the additive identities of R_1, R_2 respectively, then (i) $\pi(C)(0_2) = C(0_1)$ (ii) $\pi^{-1}(D)(0_1) = D(0_2)$

Proof. (i) Here $\pi(C)(0_2) = \{(T_{h_{\pi(C)}(b)}(0_2), I_{h_{\pi(C)}(b)}(0_2), F_{h_{\pi(C)}(b)}(0_2)) : b \in E\}$ and
$C(0_1) = \{(T_{h_C(b)}(0_1), I_{h_C(b)}(0_1), F_{h_C(b)}(0_1)) : b \in E\}$; Now,

$T_{h_{\pi(C)}(b)}(0_2) = \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \geq T_{h_C(b)}(0_1)$ [as $\pi(0_1) = 0_2$]

Since C is an NSLIs over (R_1, E), so $\forall u \in R$ and $\forall b \in E$,

$T_{h_C(b)}(u) \leq T_{h_C(b)}(0_1) \Rightarrow \max\{T_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \leq T_{h_C(b)}(0_1) \Rightarrow T_{h_{\pi(C)}(b)}(0_2) \leq T_{h_C(b)}(0_1)$

Thus $T_{h_{\pi(C)}(b)}(0_2) = T_{h_C(b)}(0_1)$. Next,

$I_{h_{\pi(C)}(b)}(0_2) = \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \leq I_{h_C(b)}(0_1)$ [as $\pi(0_1) = 0_2$]

Since C is an NSLIs over (R_1, E), so $\forall u \in R$ and $\forall b \in E$,

$I_{h_C(b)}(u) \geq I_{h_C(b)}(0_1) \Rightarrow \min\{I_{h_C(b)}(u) : u \in \pi^{-1}(0_2)\} \geq I_{h_C(b)}(0_1) \Rightarrow I_{h_{\pi(C)}(b)}(0_2) \geq I_{h_C(b)}(0_1)$.

Thus $I_{h_{\pi(C)}(b)}(0_2) = I_{h_C(b)}(0_1)$. Similarly, $F_{h_{\pi(C)}(b)}(0_2) = F_{h_C(b)}(0_1)$ and this follows the 1st result.

(ii) Here, we have

$T_{h_{\pi^{-1}(D)}(b)}(0_1) = T_{h_D(b)}[\pi(0_1)] = T_{h_D(b)}(0_2)$, $I_{h_{\pi^{-1}(D)}(b)}(0_1) = I_{h_D(b)}[\pi(0_1)] = I_{h_D(b)}(0_2)$ and
$F_{h_{\pi^{-1}(D)}(b)}(0_1) = F_{h_D(b)}[\pi(0_1)] = F_{h_D(b)}(0_2)$. This follows the 2nd result.

5.4 Definition

Consider two nonempty sets X, E and a lattice $[0, 1]$. Then $K = \{(T_{h_K(b)}, I_{h_K(b)}, F_{h_K(b)})|b \in E\} : X \longrightarrow [0, 1] \times [0, 1] \times [0, 1]$ attains the sup property when $T_{h_K(b)}(X) = \{T_{h_K(b)}(x) : x \in X\}$ (the image of $T_{h_K(b)}$) admits a maximal element and each of $I_{h_K(b)}(X) = \{I_{h_K(b)}(x) : x \in X\}$, $F_{h_K(b)}(X) = \{F_{h_K(b)}(x) : x \in X\}$ (the image of $I_{h_K(b)}, F_{h_K(b)}$ respectively) admits a minimal element $\forall b \in E$.

5.5 Proposition

For two NSLIs (NSRIs) K, L on (R_1, E) and (R_2, E), respectively, followings hold.
(i) $\pi(K_0) \subseteq (\pi(K))_0$ (Theorem [3.3] describes K_0).
(ii) $\pi(K_0) = (\pi(K))_0$ when K attains sup property.
(iii) $\pi^{-1}(L_0) = (\pi^{-1}(L))_0$.

Proof. (i) If $v \in \pi(K_0)$ signifies $v = \pi(u)$ for $u \in K_0 \subset R_1$ so as $T_{h_K(b)}(u) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(u) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(u) = F_{h_K(b)}(0_1)$. Now,

$$\begin{aligned} T_{h_{\pi(K)}(b)}(v) &= \max\{T_{h_K(b)}(u) : u \in \pi^{-1}(v)\} = \max\{T_{h_K(b)}(0_1)\} = T_{h_K(b)}(0_1) = T_{h_{\pi(K)}(b)}(0_2) \\ I_{h_{\pi(K)}(b)}(v) &= \min\{I_{h_K(b)}(u) : u \in \pi^{-1}(v)\} = \min\{I_{h_K(b)}(0_1)\} = I_{h_K(b)}(0_1) = I_{h_{\pi(K)}(b)}(0_2) \end{aligned}$$

Similarly, $F_{h_{\pi(K)}(b)}(v) = F_{h_{\pi(K)}(b)}(0_2)$. It signifies $v \in (\pi(K))_0$ when $v \in \pi(K_0)$ i.e., $\pi(K_0) \subseteq (\pi(K))_0$.

(ii)Take $u \in R_1$ so as $v = \pi(u) \in (\pi(K))_0 \subset R_2$. Then $\forall b \in E$,

$T_{h_{\pi(K)}(b)}(0_2) = T_{h_{\pi(K)}(b)}(v) \Rightarrow T_{h_K(b)}(0_1) = \max\{T_{h_K(b)}(t) : t \in \pi^{-1}(v)\} = T_{h_K(b)}(t)$

for $t \in R_1$ so as $t \in \pi^{-1}(v)$. Further,

$I_{h_{\pi(K)}(b)}(0_2) = I_{h_{\pi(K)}(b)}(v) \Rightarrow I_{h_K(b)}(0_1) = \min\{I_{h_K(b)}(t) : t \in \pi^{-1}(v)\} = I_{h_K(b)}(t)$

for $t \in R_1$ so as $t \in \pi^{-1}(v)$.

Identical picture is drawn for F and thus $t \in K_0$ i.e., $\pi(t) \in \pi(K_0) \Rightarrow v = \pi(u) \in \pi(K_0)$. Therefore $(\pi(K))_0 \subseteq \pi(K_0)$. Then $\pi(K_0) = (\pi(K))_0$ using (i).

(iii) $u \in \pi^{-1}(L_0) \subset R_1$

$\Leftrightarrow$ $T_{h_L(b)}[\pi(u)] = T_{h_L(b)}(0_2) = T_{h_L(b)}[\pi(0_1)], I_{h_L(b)}[\pi(u)] = I_{h_L(b)}(0_2) = I_{h_L(b)}[\pi(0_1)]$ and $F_{h_L(b)}[\pi(u)] = F_{h_L(b)}(0_2) = F_{h_L(b)}[\pi(0_1)]$;

$\Leftrightarrow$ $T_{h_{\pi^{-1}(L)}(b)}(u) = T_{h_{\pi^{-1}(L)}(b)}(0_1), I_{h_{\pi^{-1}(L)}(b)}(u) = I_{h_{\pi^{-1}(L)}(b)}(0_1), F_{h_{\pi^{-1}(L)}(b)}(u) = F_{h_{\pi^{-1}(L)}(b)}(0_1)$;

$\Leftrightarrow$ $u \in (\pi^{-1}(L))_0$

Therefore, $\pi^{-1}(L_0) = (\pi^{-1}(L))_0$.

5.6 Definition

Take a classical function $\pi : R_1 \longrightarrow R_2$ and an Nss $K(u) = \{(T_{h_K(b)}(u), I_{h_K(b)}(u), F_{h_K(b)}(u)) : b \in E\}$, $u \in R_1$. Then K is said to be π- invariant if $\pi(u) = \pi(v) \Rightarrow K(u) = K(v)$ for $u, v \in R_1$. $K(u) = K(v)$ hold if $T_{h_K(b)}(u) = T_{h_K(b)}(v), I_{h_K(b)}(u) = I_{h_K(b)}(v), F_{h_K(b)}(u) = F_{h_K(b)}(v)$, $\forall b \in E$.

5.7 Theorem

Let $\pi : R_1 \longrightarrow R_2$ be an epimorphism and K be a π- invariant NSI on (R_1, E). Then the followings hold.
(i) If K attains sup property, then $(\pi(K))_0$ is a crisp prime ideal of R_2 when K_0 is a prime ideal of R_1.
(ii) If $K(R_1)$ is finite and K_0 is prime ideal of R_1, then $\pi(K_0)$ is so of R_2 and $\pi(K_0) = (\pi(K))_0$.
(iii) If K is an NSPI over (R_1, E), then $\pi(K)$ is also an NSPI over (R_2, E).

Proof. (i) By Theorem [5.5], $\pi(K_0) = (\pi(K))_0$ obviously. Let $y, z \in R_2$ such that $yz \in \pi(K_0) = (\pi(K))_0$. Then there exists $u, v \in R_1$ so as $\pi(u) = y, \pi(v) = z$ and $\pi(uv) = \pi(u)\pi(v) = yz \in (\pi(K))_0$. Then $\forall b \in E$,

$$T_{h_{\pi(K)}(b)}[\pi(uv)] = T_{h_{\pi(K)}(b)}(0_2) \Rightarrow \max\{T_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = T_{h_K(b)}(0_1),$$
$$I_{h_{\pi(K)}(b)}[\pi(uv)] = I_{h_{\pi(K)}(b)}(0_2) \Rightarrow \min\{I_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = I_{h_K(b)}(0_1),$$
$$F_{h_{\pi(K)}(b)}[\pi(uv)] = F_{h_{\pi(K)}(b)}(0_2) \Rightarrow \min\{F_{h_K(b)}(t) : t \in \pi^{-1}(yz)\} = F_{h_K(b)}(0_1).$$

For $w \in \pi^{-1}(yz)$ i.e., for $\pi(w) = yz = \pi(uv)$, sup property tells,
$T_{h_K(b)}(w) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(w) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(w) = F_{h_K(b)}(0_1)$.
But as K is π-invariant, so $K(w) = K(uv)$. Then $\forall b \in E$,
$T_{h_K(b)}(uv) = T_{h_K(b)}(0_1)$, $I_{h_K(b)}(uv) = I_{h_K(b)}(0_1)$, $F_{h_K(b)}(uv) = F_{h_K(b)}(0_1)$.
Therefore, $uv \in K_0$. As K_0 is a crisp prime ideal of R_1, so $u \in K_0$ or $v \in K_0$. It refers $\pi(u) \in \pi(K_0)$ or $\pi(v) \in \pi(K_0)$. This furnishes the proof.

(ii) Combining the 1st part and Theorem [5.5], the proof is onward.

(iii) By Proposition [5.2](i), $\pi(K)$ is an NSI over (R_2, E). Since K is an NSPI over (R, E), then $|h_K(b)| = 2$, $[h_K(b)](0_1) = (1, 0, 0)$, $\forall b \in E$ and using Theorems [4.4, 4.5, 4.7], K_0 is a prime ideal. But $[h_{\pi(K)}(b)](0_2) = [h_K(b)](0_1) = (1, 0, 0)$, $\forall b \in E$ and by 1st part, $(\pi(K))_0$ is a prime ideal of R_2. As $|h_K(b)| = 2$, $\exists\, u \in R_1$ so as $[h_K(b)](u) = (p_1, p_2, p_3)$ for $b \in E$. Then,

$$\begin{aligned} T_{h_{\pi(K)}(b)}(\pi(u)) &= \max\{T_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_1 \\ I_{h_{\pi(K)}(b)}(\pi(u)) &= \min\{I_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_2 \\ F_{h_{\pi(K)}(b)}(\pi(u)) &= \min\{F_{h_K(b)}(u) : u \in \pi^{-1}(\pi(u))\} = p_3 \end{aligned}$$

So, $[h_{\pi(K)}(b)](\pi(u)) = (p_1, p_2, p_3) = [h_K(b)](u)$ for $b \in E$. Then $[h_K(b)](R_1) = [h_{\pi(K)}(b)](R_2)$ as π is epimorphism and u is arbitrary. Now consider two NSIs L, M over (R_2, E) such that $LoM \subseteq \pi(K)$ but $L \nsubseteq \pi(K)$ and $M \nsubseteq \pi(K)$. Then for all $y, z \in R_2$,

$T_{h_L(b)}(y) > T_{h_{\pi(K)}(b)}(y),\ I_{h_L(b)}(y) < I_{h_{\pi(K)}(b)}(y),\ F_{h_L(b)}(y) < F_{h_{\pi(K)}(b)}(y)$ and
$T_{h_M(b)}(z) > T_{h_{\pi(K)}(b)}(z),\ I_{h_M(b)}(z) < I_{h_{\pi(K)}(b)}(z),\ F_{h_M(b)}(z) < F_{h_{\pi(K)}(b)}(z)$.

For $y, z \in R_2 - (\pi(K))_0$, consider $T_{h_{\pi(K)}(b)}(y) = T_{h_{\pi(K)}(b)}(z) = p_1$, $I_{h_{\pi(K)}(b)}(y) = I_{h_{\pi(K)}(b)}(z) = p_2$ and $F_{h_{\pi(K)}(b)}(y) = F_{h_{\pi(K)}(b)}(z) = p_3$. Then,

$T_{h_L(b)}(y) > p_1,\ I_{h_L(b)}(y) < p_2,\ F_{h_L(b)}(y) < p_3$ and $T_{h_M(b)}(z) > p_1,\ I_{h_M(b)}(z) < p_2,\ F_{h_M(b)}(z) < p_3$.

Clearly, $yz \notin (\pi(K))_0$ as $y, z \notin (\pi(K))_0$, a prime ideal of R_2.

Then, $T_{h_{\pi(K)}(b)}(yz) = p_1,\ I_{h_{\pi(K)}(b)}(yz) = p_2,\ F_{h_{\pi(K)}(b)}(yz) = p_3$.

Now, $p_1 = T_{h_{\pi(K)}(b)}(yz) \geq T_{h_{LoM}(b)}(yz) = T_{h_L(b)}(y) \triangle T_{h_M(b)}(z) > p_1 \triangle p_1 = p_1$

The opposition $p_1 > p_1$ ensures $L \subseteq \pi(K), M \subseteq \pi(K)$ and this furnishes the 1st part.

5.8 Theorem

Let Q be an NSI over (R_2, E) and π is onto homomorphism. Then,
(i) $(\pi^{-1}(Q))_0$ is a crisp prime ideal on R_1 when Q_0 is so over R_2.
(ii) $\pi^{-1}(Q)$ is NSPI on (R_1, E) when Q is an NSPI over (R_2, E).

Proof. (i) We have by Theorem [5.5], $\pi^{-1}(Q_0) = (\pi^{-1}(Q))_0$. Let $u, v \in R_1$ so as $uv \in \pi^{-1}(Q_0)$. Then $\pi(uv) = \pi(u)\pi(v) \in Q_0$. Again $\pi(u) \in Q_0$ or $\pi(v) \in Q_0$ as Q_0 is a prime ideal.

$\pi(u) \in Q_0 \Rightarrow T_{h_Q(b)}[\pi(u)] = T_{h_Q(b)}(0_2) \Rightarrow T_{h_{\pi^{-1}(Q)}(b)}(u) = T_{h_{\pi^{-1}(Q)}(b)}(0_1) \Rightarrow u \in (\pi^{-1}(Q))_0$.

Identically, $v \in (\pi^{-1}(Q))_0$ when $\pi(v) \in Q_0$. Therefore, $uv \in (\pi^{-1}(Q))_0$ refers $u \in (\pi^{-1}(Q))_0$ or $v \in (\pi^{-1}(Q))_0$. Hence, the 1st part follows.

(ii) By Theorem [5.2], $\pi^{-1}(Q)$ is an NSI over (R_1, E) and by Theorem [5.3], $\pi^{-1}(Q)(0_1) = Q(0_2)$. Also since Q is an NSPI over (R_2, E), then $|h_Q(b)| = 2$, $[h_Q(b)](0_2) = (1, 0, 0)$ and Q_0 is a crisp prime ideal of R_2 respectively by Theorem [4.4], Theorem [4.5] and Theorem [4.7]. Then, by 1st result, $(\pi^{-1}(Q))_0$ is a crisp prime ideal of R_1 and $[h_{\pi^{-1}(Q)}(b)](0_1) = (1, 0, 0)$. Construct $[h_Q(b)](R_2) = \{(1, 0, 0) \cup (q_1, q_2, q_3)\}$ for a fixed $b \in E$ with $(1, 0, 0) \neq (q_1, q_2, q_3)$. Let $[h_Q(b)](v) = (q_1, q_2, q_3)$ for $v \in R_2$. Then $\exists\, u \in R_1$ for which $\pi(u) = v$ and $[h_{\pi^{-1}(Q)}(b)](u) = [h_Q(b)](v) = (q_1, q_2, q_3)$. Therefore, $[\pi^{-1}(Q)](R_1) = Q(R_2)$ as $b \in E$ is arbitrary and π is epimorphism.

For two NSIs A, B on (R_1, E), let $AoB \subseteq \pi^{-1}(Q)$ with $A \nsubseteq \pi^{-1}(Q)$ and $B \nsubseteq \pi^{-1}(Q)$. Then $\forall\, u, v \in R_1$,

$T_{h_A(b)}(u) > T_{h_{\pi^{-1}(Q)}(b)}(u),\ I_{h_A(b)}(u) < I_{h_{\pi^{-1}(Q)}(b)}(u),\ F_{h_A(b)}(u) < F_{h_{\pi^{-1}(Q)}(b)}(u)$ and
$T_{h_B(b)}(v) > T_{h_{\pi^{-1}(Q)}(b)}(v),\ I_{h_B(b)}(v) < I_{h_{\pi^{-1}(Q)}(b)}(v),\ F_{h_B(b)}(v) < F_{h_{\pi^{-1}(Q)}(b)}(v)$.

For $u, v \in R_1 - (\pi^{-1}(Q))_0$, let $T_{h_{\pi^{-1}(Q)}(b)}(u) = T_{h_{\pi^{-1}(Q)}(b)}(v) = q_1$, $I_{h_{\pi^{-1}(Q)}(b)}(u) = I_{h_{\pi^{-1}(Q)}(b)}(v) = q_2$ and $F_{h_{\pi^{-1}(Q)}(b)}(u) = F_{h_{\pi^{-1}(Q)}(b)}(v) = q_3$. Then,

$T_{h_A(b)}(u) > q_1,\ I_{h_A(b)}(u) < q_2,\ F_{h_A(b)}(u) < q_3$ and $T_{h_B(b)}(v) > q_1,\ I_{h_B(b)}(v) < q_2,\ F_{h_B(b)}(v) < q_3$.

It indicates $uv \notin (\pi^{-1}(Q))_0$ as $u, v \notin (\pi^{-1}(Q))_0$, a prime ideal of R_1.
Then, $T_{h_{\pi^{-1}(Q)}(b)}(uv) = q_1,\ I_{h_{\pi^{-1}(Q)}(b)}(uv) = q_2,\ F_{h_{\pi^{-1}(Q)}(b)}(uv) = q_3$ and
so, $q_1 = T_{h_{\pi^{-1}(Q)}(b)}(uv) \geq T_{h_{AoB}(b)}(uv) = T_{h_A(b)}(u) \triangle T_{h_B(b)}(v) > q_1 \triangle q_1 = q_1$
The opposition $q_1 > q_1$ ensures $A \subseteq \pi^{-1}(Q), B \subseteq \pi^{-1}(Q)$ and this leads the 2nd part.

6 Conclusion

This effort is made to extend the notion of ideal and prime ideal of a classical ring in the parlance of N_S theory and soft set theory. Their structural behaviours are innovated by developing a number of properties and theorems. Using neutrosophic cut set, it is shown how an Nss will be an NSI or NSPI. The nature of homomorphic image of NSI and NSPI are also studied in different aspect. This theoretical attempt will help to cultivate the N_S theory in several mode in future, we think.

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Separation Axioms in Neutrosophic Crisp Topological Spaces

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Abstract.The main idea of this research is to define a new neutrosophic crisp points in neutrosophic crisp topological space namely [NCP_N] ,the concept of neutrosophic crisp limit point was defind using [NCP_N],with some of its properties, the separation axioms [N-$\mathcal{T}_i$-space,i= 0,1,2] were constructed in neutrosophic crisp topological space using [NCP_N] and examine the relationship between them in details.

Keywords: Neutrosophic crisp topological spaces, neutrosophic crisp limit point, separation axioms.

Introduction

Smarandache [1,2,3] introduced the notions of neutrosophic theory and introduced the neutrosophic. components (T,I,F) which represent the membership , indeterminacy , and non membership values respectively, where $]-0,1^+[$ is a non standard unit interval. In [4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19 ,20] many scientists presented the concepts of the neutrosophic set theory in their works. Salama et al. [21,22] provided natural foundations to put mathematical treatments for the neutrosophic pervasively phenomena in our real world and for building new branches of neutrosophic mathematics.

Salama et al [23,24] put some basic concepts of the neutrosophic crisp set and their operations, and because of their wide applications and their grate flexibility to solve the problem, we used these concepts to define new types of neutrosophic points, that we called neutrosophic crisp points [NCP_N].

Fainally ,we used these points [NCP_N] to define the concept of neutrosophic crisp limit point, with some of its properties and constructe the separation axioms[N-$\mathcal{T}_i$-space,i=0,1,2] in neutrosophic crisp topological and examine the relationship between them in details.

Throughout this paper,(NCTS) means a neutrosophic crisp topological space. Also, simply we denote neighborhood by (nhd).

1 Basic Concepts

1.1 Definition [25]

Let $\mathcal{X}$ be a non-empty fixed set . A neutrosophic crisp set [NCS for short] B is an object having the form $B = <B_1, B_2, B_3>$ where B_1, B_2 and B_3 are subsets of $\mathcal{X}$.

1.2 Definition [25]

The object having the form $B=<B_1, B_2, B_3>$ is called :

1. A neutrosophic crisp set of Type1 [NCS/Type1] if satisfying
$B_1 \cap B_2 = \emptyset, B_1 \cap B_3 = \emptyset$ and $B_2 \cap B_3 = \emptyset$.
2. A neutrosophic crisp set of Type2 [NCS/Type2] if satisfying
$B_1 \cap B_2 = \emptyset, B_1 \cap B_3 = \emptyset$ and $B_2 \cap B_3 = \emptyset$, $B_1 \cup B_2 \cup B_3 = \mathcal{X}$.
3. A neutrosophic crisp set of Type3 [NCS/Type3] if satisfying
$B_1 \cap B_2 \cap B_3 = \emptyset$, $B_1 \cup B_2 \cup B_3 = \mathcal{X}$

1.3 Definition [25]

Types of NCSs $\emptyset_N$ & $\mathcal{X}_N$ in $\mathcal{X}$ as follows :

1 . $\emptyset_N$ may be defined in many ways as a NCS as follows :
1. Type1 : $\emptyset_N = >\varphi , \varphi , \mathcal{X} <$
2. Type2 : $\emptyset_N = >\varphi , \mathcal{X} , \mathcal{X} <$
3. Type3 : $\emptyset_N = >\varphi , \mathcal{X}, \varphi <$
4. Type4 : $\emptyset_N = >\varphi , \varphi , \varphi <$

2 . $\mathcal{X}_N$ may be defined in many ways as a NCS as follows :
1. Type1: $\mathcal{X}_N = > \mathcal{X}, \varphi , \varphi <$

2. Type2: $\mathcal{X}_N = > \mathcal{X}, \mathcal{X}, \varphi <$
3. Type3: $\mathcal{X}_N = > \mathcal{X}, \varphi, \mathcal{X} <$
4. Type4: $\mathcal{X}_N = > \mathcal{X}, \mathcal{X}, \mathcal{X} <$

1.4 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1, C_2, C_3>$, $D = <D_1, D_2, D_3>$ then we may consider two possible definitions for subsets $C \subseteq D$, may be defined in two ways :

1. $C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $C_2 \subseteq D_2$ and $D_3 \subseteq C_3$
2. $C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $D_2 \subseteq C_2$ and $D_3 \subseteq C_3$

1.5 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1, C_2, C_3>$, $D = <D_1, D_2, D_3>$ then :

1. $C \cap D$ may be defined in two ways as a NCS as follows :
   - $C \cap D = [C_1 \cap D_1], [C_2 \cup D_2], [C_3 \cup D_3]$
   - $C \cap D = [C_1 \cap D_1], [C_2 \cap D_2], [C_3 \cup D_3]$
2. $C \cup D$ may be defined in two ways as a NCS as follows :
   - $C \cup D = [C_1 \cup D_1], [C_2 \cup D_2], [C_3 \cap D_3]$
   - $C \cup D = [C_1 \cup D_1], [C_2 \cap D_2], [C_3 \cap D_3]$

1.6 Definition [25]

A neutrosophic crisp topology (NCT) on a non-empty set $\mathcal{X}$ is a family $\mathcal{T}$ of neutrosophic crisp subsets in $\mathcal{X}$ satisfying the following axioms :

1. $\emptyset_N, \mathcal{X}_N \in \mathcal{T}$
2. $C \cap D \in \mathcal{T}$, for any $C, D \in \mathcal{T}$
3. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$

The pair $(\mathcal{X}, \mathcal{T})$ is said to be a neutrosophic crisp topological space (NCTS) in $\mathcal{X}$. Moreover The elements in $\mathcal{T}$ are said to be neutrosophic crisp open sets (NCOS), a neutrosophic crisp set F is closed (NCCS) iff its complement F^C is an open neutrosophic crisp set.

1.7 Definition [25]

Let $\mathcal{X}$ be a non-empty set and the NCS D in the form $D = <D_1, D_2, D_3>$ .Then D^c may be defined in three ways as a NCS as follows :

$D^c = < D_1^c, D_2^c, D_3^c>$, $D^c = < D_3, D_2, D_1>$ or $D^c = < D_3, D_2^c, D_1>$

1.8 Definition [25]

Let ($\mathcal{X}, \mathcal{T}$) be neutrosophic crisp topological space (NCTS). A be neutrosophic crisp set then: The intersection of any neutrosophic crisp closed sets contained, A is called neutrosophic crisp closure of A (NC-Cl(A) for short) .

2 Neutrosophic crisp limit point :

In this section, we will introduce the neutrosophic crisp limit points with some of its properties. This work contains an adjustment for the above-mentioned definitions 1.4 & 1.5, this was necessary to homogeneous suitable results for the upgrade of this research.

2.1 Definition

Let $\mathcal{X}$ be a non-empty set and the NCSs C & D in the form $C = <C_1, C_2, C_3>$, $D = <D_1, D_2, D_3>$ then the additional new ways for the intersection , union and inclusion between C & D are

$C \cap D = [C_1 \cap D_1], [C_2 \cap D_2], [C_3 \cap D_3]$

$C \cup D = [C_1 \cup D_1], [C_2 \cup D_2], [C_3 \cup D_3]$

$C \subseteq D \Leftrightarrow C_1 \subseteq D_1$, $C_2 \subseteq D_2$ and $C_3 \subseteq D_3$

2.2 Definition

For all x, y, z belonging to a non-empty set $\mathcal{X}$. Then the neutrosophic crisp points related to x, y, z are defined as follows :

- $x_{N_1} = <\{x\},\emptyset,\emptyset>$, is called a neutrosophic crisp point (NCP_{N_1}) in $\mathcal{X}$.
- $y_{N_2} = <\emptyset,\{y\},\emptyset>$, is called a neutrosophic crisp point (NCP_{N_2}) in $\mathcal{X}$.
- $z_{N_3} = <\emptyset,\emptyset,\{z\}>$, is called a neutrosophic crisp point (NCP_{N_3}) in $\mathcal{X}$.

The set of all neutrosophic crisp points ($NCP_{N_1}, NCP_{N_2}, NCP_{N_3}$) is denoted by NCP_N.

2.3 Definition

Let $\mathcal{X}$ be to a non-empty set and x ,y, z $\in\mathcal{X}$. Then the neutrosophic crisp point:

- x_{N_1} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $x_{N_1} \in B$, if $x \in B_1$,wherein x_{N_1} does not belong to the neutrosophic crisp set B denoted by $x_{N_1} \notin B$, if $x \notin B_1$.
- y_{N_2} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $y_{N_2} \in B$, if $y \in B_2$. In contrast y_{N_2} does not belong to the neutrosophic crisp set B, denoted by $y_{N_2} \notin B$, if $y \notin B_2$.
- z_{N_3} is belonging to the neutrosophic crisp set B=<B_1,B_2,B_3>, denoted by $z_{N_3} \in B$, if $z \in B_3$. In contrast z_{N_3} does not belong to the neutrosophic crisp set B ,denoted by $z_{N_3} \notin B$, if $z \notin B_3$.

2.4 Remark

If $B = <B_1,B_2,B_3>$ is a NCS in a non-empty set $\mathcal{X}$ then :

$B\backslash x_{N_1} = < B_1\backslash\{x\}, B_2, B_3 >$. $B\backslash x_{N_1}$ means that the component B doesn't contain x_{N_1}.

$B\backslash y_{N_2} = < B_1, B_2\backslash\{y\}, B_3 >$. $B\backslash y_{N_2}$ means that the component B doesn't contain y_{N_2}.

$B\backslash z_{N_3} = < B_1, B_2, B_3\backslash\{z\} >$. $B\backslash z_{N_3}$ means that the component B doesn't contain z_{N_3}.

2.5 Example

If $B = <\{a, b\},\{c, b\},\{c, a\}>$ is an NCS in $\mathcal{X} = \{a, b, c\}$, then:

$B\backslash a_{N_1} = <\{ b \}, \{c, b\}, \{c, a\}>$

$B\backslash b_{N_2} = <\{a, b\} , \{c\}, \{c, a\}>$

$B\backslash c_{N_3} = <\{a, b\} , \{c, b\}, \{ b \}>$

2.6 Remark

If $B = <B_1,B_2,B_3>$ is a NCS in a non-empty set $\mathcal{X}$ then :

$B = (\cup\{ x_{N_1}: x_{N_1} \in B \}) \cup (\cup\{ y_{N_2}: y_{N_2} \in B \}) \cup (\cap\{ z_{N_3}: z_{N_3} \in B \})$

$= (\cup\{<\{x\}, \emptyset,\emptyset> : x \in \mathcal{X}\}) \cup (\cup\{<\emptyset, \{y\}, \emptyset> : y \in \mathcal{X}\}) \cup (\cap\{<\emptyset, \emptyset, \{z\}> : z \in \mathcal{X}\})$

or $B = (\cup\{ x_{N_1}: x_{N_1} \in B \}) \cup (\cup\{ y_{N_2}: y_{N_2} \in B \}) \cup (\cup \{ z_{N_3}: z_{N_3} \in B \})$

$= (\cup\{<\{x\}, \emptyset,\emptyset> : x \in \mathcal{X}\}) \cup (\cup\{<\emptyset, \{y\}, \emptyset> : y \in \mathcal{X} \}) \cup (\cup\{<\emptyset, \emptyset,\{z\}> : z \in \mathcal{X}\})$.

2.7 Definition

Let $(\mathcal{X},\mathcal{T})$ be NCTS , $P \in NCP_N$ in $\mathcal{X}$, a neutrosophic crisp set $B = <B_1,B_2,B_3> \in \mathcal{T}$ is called neutrosophic crisp open nhd of P in $(\mathcal{X},\mathcal{T})$ if $P \in B$.

2.8 Definition

Let $(\mathcal{X},\mathcal{T})$ be NCTS , $P \in NCP_N$ in $\mathcal{X}$, a neutrosophic crisp set $B = < B_1 , B_2 , B_3 > \in \mathcal{T}$ is called neutrosophic crisp nhd of P in $(\mathcal{X},\mathcal{T})$, if there is neutrosophic crisp open set $A =< A_1, A_2, A_3 >$ containing P such that $A \subseteq B$.

2.9 Note

Every neutrosophic crisp open nhd of any point $P \in NCP_N$ in $\mathcal{X}$ is neutrosophic crisp nhd of P, but in general the inverse is not true , the following example illustrates this fact.

2.10 Example

If $\mathcal{X} = \{x,y,z\}$, $\mathcal{T} = \{\mathcal{X}_N ,\emptyset_N , A, B, C\}$,

$A = < \{x\}, \emptyset, \emptyset>$, $B = <\{y\}, \emptyset, \emptyset>$, $G = <\{x, y\}, \emptyset, \emptyset>$

If we take $U = <\{x, y\},\{z\}, \emptyset>$.

Then $G = <\{x ,y\}, \emptyset, \emptyset>$ is an open set containing $P = x_{N_1} = < \{x\}, \emptyset, \emptyset>$ and $G \subseteq U$. That is U is a neutrosophic crisp nhd of P in $(\mathcal{X},\mathcal{T})$, while it is not a neutrosophic crisp open nhd of P .

2.11 Definition

Let $(\mathcal{X},\mathcal{T})$ be NCTS and $B = < B_1 , B_2 , B_3 >$ be NCS of $\mathcal{X}$. A neutrosophic crisp point $P \in NCP_N$ in $\mathcal{X}$ is called a neutrosophic crisp limit point of $B = < B_1 ,B_2 ,B_3 >$ iff every neutrosophic crisp open set containing P must contains at least one neutrosophic crisp point of B different from P . It is easy to say that the point P is not neutrosophic crisp limit point of B if there is a neutrosophic crisp open set G of P and $B \cap (G\backslash P) = \emptyset_N$.

2.12 Definition

The set of all neutrosophic crisp limit points of a neutrosophic crisp set B is called neutrosophic crisp derived set of B , denoted by $NCD(B)$.

2.13 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N , \emptyset_N , A, B, C\}$, $A = < \{x\}, \emptyset, \emptyset >$, $B = <\{y\}, \emptyset, \emptyset >$, $G = <\{x, y\}, \emptyset, \emptyset >$. If we take $D =<\{x, y\},\emptyset,\emptyset>$, Then $P= Z_{N_1}=<\{Z\},\emptyset,\emptyset>$ is the only neutrosophic crisp limit point of D. i.e. $NCD(D) =\{Z_{N_1}\}$

2.14 Remarks

- Let B be any neutrosophic crisp set of $\mathcal{X}$, If $P= <\{x\},\emptyset,\emptyset> \in \mathcal{T}$ in any NCT space $(\mathcal{X},\mathcal{T})$, then $P \in NCD(B)$.
- Let B be any neutrosophic crisp set of $\mathcal{X}$, the following facts is true:
  $NCD(B) \not\subset B$, $B \not\subset NCD(B)$, and sometimes $NCD(B) \cap B = \emptyset_N$ or $NCD(B) \cap B \neq \emptyset_N$.
- In any NCT space $(\mathcal{X},\mathcal{T})$, we have $NCD(\emptyset) = \emptyset_N$.

2.15 Theorem

Let $(\mathcal{X},\mathcal{T})$ be NCTS and $B = < B_1 ,B_2 ,B_3 >$ be a neutrosophic crisp set of $\mathcal{X}$,then B is neutrosophic crisp closed set (NCCS for short) iff $NCD(B) \subseteq B$

Proof

Let B be NCCS, then $(\mathcal{X}\backslash B)$ is neutrosophic crisp open set (NCOS for short) this implies that for each neutrosophic crisp point $P \in NCP_N$ in $(\mathcal{X}\backslash B)$, $P \notin B$, there is a neutrosophic crisp open set G of P and $G \subseteq (\mathcal{X}\backslash B)$.

Since $B \cap (\mathcal{X}\backslash B) = \emptyset_N$, then P is not neutrosophic crisp limit point of B, thus $G \cap B = \emptyset_N$,which implies that $P \notin NCD(B)$.Hence $NCD(B) \subseteq B$

Conversely , assume that $P \notin NCD(B)$, implis that P is not neutrosophic crisp limit point of B, hence, there is a neutrosophic crisp open set G of P and $G \cap B = \emptyset_N$ which means that $G \subseteq (\mathcal{X}\backslash B)$ and since $(\mathcal{X}\backslash B)$ is a neutrosophic crisp open set . Hence B is neutrosophic crisp closed set. .

2.16 Theorem

Let $(\mathcal{X},\mathcal{T})$ be NCTS , B, G be a neutrosophic crisp sets of $\mathcal{X}$, then the following properties hold :

(1) $NCD(\emptyset_N) = \emptyset_N$

(2) If $B \subseteq G$, then $NCD(B) \subseteq NCD(G)$

(3) $NCD(B \cap G) \subseteq NCD(B) \cap NCD(G)$

(4) $NCD(B \cup G) = NCD(B) \cup NCD(G)$

Proof (1) the proof is, directly.

Proof (2)

Assume that NCD(B) be a neutrosophic crisp set containing a neutrosophic crisp point $P \in NCP_N$, then by definition 2.11, for each neutrosophic crisp open set V of P , we have $B \cap V\backslash P \neq \emptyset_N$,but $B \subseteq G$, hence $G \cap V\backslash P \neq \emptyset_N$, this means that $P \in NCD(G)$. Hence , $NCD(B) \subseteq NCD(G)$

Proof (3)

Since $B \cap G \subseteq B$, then by (2) $NCD(B \cap G) \subseteq NCD(B)$ (1)

$B \cap G \subseteq G$, implies $NCD(B \cap G) \subseteq NCD(G)$ (2)

From (1) & (2) $NCD(B \cap G) \subseteq NCD(B) \cap NCD(G)$

Proof (4)

Let $P \in NCP_N$ such that $P \notin NCD(B) \cup NCD(G)$, then either $P \notin NCD(B)$ and $P \notin NCD(G)$,then there is a neutrosophic crisp open set K of P and $B \cap K\backslash P = \emptyset_N$ and $G \cap K\backslash P = \emptyset_N$, this implies that $(B \cup G) \cap K\backslash P = \emptyset_N$, i.e $P \notin NCD(B \cup G)$, hence $NCD(B \cup G) \subseteq NCD(B) \cup NCD(G)$ (3)

Conversely, since $B \subseteq B \cup G$,$G \subseteq B \cup G$,then by property (2) $NCD(B) \subseteq NCD(B \cup G)$ and $NCD(G) \subseteq NCD(B \cup G)$, thus $NCD(B \cup G) \supseteq NCD(B) \cup NCD(G)$ (4)
from (3) and (4) we have $NCD(B \cup G) = NCD(B) \cup NCD(G)$.

2.17 Remark

In general, the inverse of property 2 & 3 in Th.(2.16) is not true. The following examples act as an evidence to this claim.

2.18 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N , \emptyset_N, B\}$, B =<∅,{$x$},∅> . If we take A = < ∅,{ x }, ∅ >, G = < ∅,{ y },∅ > Notes that; $NCD(A) = <\emptyset, \{y, z\}, \emptyset>$, $NCD(G) = <\emptyset, \{y, z\}, \emptyset>$ and $NCD(A) \subseteq NCD(G)$, but $A \not\subset G$.

2.19 Example

If $\mathcal{X} = \{x, y, z\}$, $\mathcal{T} = \{\mathcal{X}_N , \emptyset_N, B\}$, B =<∅,{$x$},∅> . If we take A =<∅, {x},∅>, G =<∅, {y},∅> .
Notes that; $NCD(B \cap G) \not\supset NCD(B) \cap NCD(G)$.

2.20 Theorem

For any neutrosophic crisp set B over the universe $\mathcal{X}$, then NC-Cl(B) = $B \cup NCD(B)$

Proof

Let us first prove that $B \cup NCD(B)$ is a neutrosophic crisp closed set that is
$\mathcal{X}_N \backslash (B \cup NCD(B)) = (\mathcal{X}_N \backslash B) \cap (\mathcal{X}_N \backslash NCD(B))$ is a neutrosophic crisp open set .
Now for a neutrosophic crisp point $P \in (\mathcal{X}_N \backslash (B)) \cap (\mathcal{X}_N \backslash NCD(B))$, then $P \in (\mathcal{X}_N \backslash (B))$ and $P \in \mathcal{X}_N \backslash NCD(B)$, thus $P \notin B$ and $P \notin NCD(B)$. So by definition 2.12, there is a neutrosophic crisp set R of P S.t $R \cap B = \emptyset_N$, hence $R \subseteq \mathcal{X}_N \backslash B$.
Now for each $P_1 \in R$, then $P_1 \notin NCD(B)$, then $R \cap NCD(B) = \emptyset_N$, this implies that $R \subseteq \mathcal{X}_N \backslash NCD(B)$ [i.e $R \subseteq (\mathcal{X}_N \backslash B) \cap (\mathcal{X}_N \backslash NCD(B))$] .Thus $(\mathcal{X}_N \backslash B) \cap (\mathcal{X}_N \backslash NCD(B))$ is a neutrosophic crisp nhd of all its elements and hence $(\mathcal{X}_N \backslash B) \cap (\mathcal{X}_N \backslash NCD(B))$ is a neutrosophic crisp open set and thus $B \cup NCD(B)$ is a neutrosophic crisp closed set containing B, therefore NC-Cl(B) $\subseteq B \cup NCD(B)$.
S ince NC-Cl(B) is a neutrosophic crisp closed set (see definition 2.12) and NC-Cl(B) contains all its neutrosophic crisp limits points .Thus $NCD(B) \subseteq$ NC-Cl(B) and $B \subseteq$ NC-Cl(B), hence NC-Cl(B) = $B \cup NCD(B)$.

3 Separation Axioms In a neutrosophic Crisp Topological Space

3.1 Definition

A neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$ is called:

- N_1-$\mathcal{T}_o$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other.
- N_2-$\mathcal{T}_o$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other .
- N_3-$\mathcal{T}_o$-space if $\forall\, x_{N_3} \neq y_{N_3} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open set G in $\mathcal{X}$ containing one of them but not the other .
- N_1-$\mathcal{T}_1$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_1} \in G_1$, $y_{N_1} \notin G_1$ and $x_{N_1} \notin G_2$, $y_{N_1} \in G_2$
- N_2-$\mathcal{T}_1$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$
- N_3-$\mathcal{T}_1$-space if $\forall\, x_{N_3} \neq y_{N_3} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_3} \in G_1$, $y_{N_3} \notin G_1$ and $x_{N_3} \notin G_2$, $y_{N_3} \in G_2$
- N_1-$\mathcal{T}_2$-space if $\forall\, x_{N_1} \neq y_{N_1} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_1} \in G_1$, $y_{N_1} \notin G_1$ and $x_{N_1} \notin G_2$, $y_{N_1} \in G_2$ with $G_1 \cap G_2 = \emptyset$.
- N_2-$\mathcal{T}_2$-space if $\forall\, x_{N_2} \neq y_{N_2} \in \mathcal{X}$ $\exists$ a neutrosophic crisp open sets G_1 , G_2 in $\mathcal{X}$ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$ with $G_1 \cap G_2 = \emptyset$.

- N_3-$\mathcal{T}_2$-space if $\forall\ x_{N_3} \neq y_{N_3} \in \mathcal{X}\ \exists$ a neutrosophic crisp open sets G_1, G_2 in $\mathcal{X}$ such that $x_{N_3} \in G_1$, $y_{N_3} \notin G_1$ and $x_{N_3} \notin G_2$, $y_{N_3} \in G_2$ with $G_1 \cap G_2 = \emptyset$.

3.2 Definition

A neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$ is called:

- N-$\mathcal{T}_o$-space if $(\mathcal{X}, \mathcal{T})$ is N_1-$\mathcal{T}_o$-space, N_2-$\mathcal{T}_o$-space and N_3-$\mathcal{T}_o$-space
- N-$\mathcal{T}_1$-space if $(\mathcal{X}, \mathcal{T})$ is N_1-$\mathcal{T}_1$-space, N_2-$\mathcal{T}_1$-space and N_3-$\mathcal{T}_1$-space
- N-$\mathcal{T}_2$-space if $(\mathcal{X}, \mathcal{T})$ is N_1-$\mathcal{T}_2$-space, N_2-$\mathcal{T}_2$-space and N_3-$\mathcal{T}_2$-space

3.3 Remark

For a neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$

- Every N-$\mathcal{T}_o$-space is N_1-$\mathcal{T}_o$-space
- Every N-$\mathcal{T}_0$- space is N_2-$\mathcal{T}_0$-space
- Every N-$\mathcal{T}_0$- space is N_3-$\mathcal{T}_0$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark 3.3 is not true, the following example explain this state.

3.4 Example

If $\mathcal{X} = \{x, y\}$, $\mathcal{T}_1 = \{\mathcal{X}_N, \emptyset_N, A\}$, $\mathcal{T}_2 = \{\mathcal{X}_N, \emptyset_N, B\}$, $\mathcal{T}_3 = \{\mathcal{X}_N, \emptyset_N, G\}$, $A = <\{x\}, \emptyset, \emptyset>$, $B = <\emptyset, \{y\}, \emptyset>$, $G = <\emptyset, \emptyset, \{x\}>$, Then $(\mathcal{X}, \mathcal{T}_1)$ is N_1-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space, $(\mathcal{X}, \mathcal{T}_2)$ is N_2-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space, $(\mathcal{X}, \mathcal{T}_3)$ is N_3-$\mathcal{T}_o$-space but it is not N-$\mathcal{T}_o$-space.

3.5 Remark

For a neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$

- Every N-$\mathcal{T}_1$-space is N_1-$\mathcal{T}_1$-space
- Every N-$\mathcal{T}_1$- space is N_2-$\mathcal{T}_1$-space
- Every N-$\mathcal{T}_1$- space is N_3-$\mathcal{T}_1$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark (3.5) is not true as it is shown in the following example,

3.6 Example

If $\mathcal{X} = \{x, y\}$, $\mathcal{T}_1 = \{\mathcal{X}_N, \emptyset_N, A, B\}$, $\mathcal{T}_2 = \{\mathcal{X}_N, \emptyset_N, G, F\}$, $A = <\{x\}, \{y\}, \emptyset>$, $B = <\{y\}, \{x\}, \emptyset>$, $G = <\emptyset, \emptyset, \{x\}>$, $F = <\emptyset, \emptyset, \{y\}>$, Then $(\mathcal{X}, \mathcal{T}_1)$ is N_1-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space. $(\mathcal{X}, \mathcal{T}_1)$ is N_2-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space. $(\mathcal{X}, \mathcal{T}_2)$ is N_3-$\mathcal{T}_1$-space but it is not N-$\mathcal{T}_1$-space

3.7 Remark

For a neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$

- Every N-$\mathcal{T}_2$-space is N_1-$\mathcal{T}_2$-space
- Every N-$\mathcal{T}_2$-space is N_2-$\mathcal{T}_2$-space
- Every N-$\mathcal{T}_2$-space is N_3-$\mathcal{T}_2$-space

Proof the proof is directly from definition 3.2 .

The inverse of remark (3.7) is not true as it is shown in the example (3.6).

3.8 Remark

For a neutrosophic crisp topological space $(\mathcal{X}, \mathcal{T})$

- Every N-$\mathcal{T}_1$-space is N-$\mathcal{T}_0$-space
- Every N-$\mathcal{T}_2$-space is N-$\mathcal{T}_1$-space

Proof the proof is directly.

The inverse of remark (3.8) is not true as it is shown in the following example :

3.9 Example

If $= \{x, y\}$, $\mathcal{T} = \{\mathcal{X}_N, \emptyset_N, A, B, G\}$

$A = <\{x\}, \emptyset, \emptyset>$, $B = <\emptyset, \{y\}, \emptyset>$, $G = <\emptyset, \emptyset, \{x\}>$,

Then $(\mathcal{X}, \mathcal{T})$ is N-$\mathcal{T}_o$-space but not N-$\mathcal{T}_1$-space

Conclusion

- We defined a new neutrosophic crisp points in neutrosophic crisp topological space
- We introduced the concept of neutrosophic crisp limit point, with some of its properties
- We constructed the separation axioms [N-$\mathcal{T}_i$-space , i= 0,1,2] in neutrosophic crisp topological and examine the relationship between them in details.

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Some operators with IVGSVTrN-numbers and their applications to multiple criteria group decision making

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Abstract: Interval valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN-number), which permits the membership degrees of an element to a set expressed with intervals rather than exact numbers, is considered to be very useful to describe uncertain information for analyzing multiple criteria decision making (MCDM) problems. In this paper, we firstly introduced the concept of IVGSVTrN-number with some operations based on neutrosophic number. Then, we presented some aggregation and geometric operators. Finally, we developed a approaches for multiple criteria group decision making problems based on the proposed operators and we applied the method to a numerical example to illustrate proposed approach.

Keywords: Neutrosophic set, interval eutrosophic set, neutrosophic numbers, IVGSVTrN-numbers, aggregation and geometric operators, multiple criteria group decision making.

1 Introduction

Since the nature of real world and limited knowledge and perception capability of human beings, their real life contain different styles of vagueness, inexact and imprecise information. To handle and analyze various kinds of vagueness, inexact and imprecise information a number of methods and theories have been developed. For example; in 1965, fuzzy set theory [53] has gradually become the mainstream in the field of representing and handling vagueness, inexact and imprecise information in decision-making, pattern recognition, game theory and so on. After fuzzy set theory, various classes of extensions have been defined and extended successively such as; intuitionistic fuzzy sets introduced Atanassov[3], neutrosophic sets by proposed by Smarandache[43], interval neutrosophic sets by developed by Wang et al. [44]. Recently, some studies on the sets have been researched by many authors (e.g. [7, 8, 9, 10, 16, 21, 22, 36, 41, 42, 50, 54]).

In recent years, many researchers have realized the need for a set that has the ability to accurately model and represent intuitionistic information in [48]. As a theory to model different styles of uncertainty, intuitionistic fuzzy set is usually employed to analyze uncertain MCDM problems through intuitionistic fuzzy number. As an important representation of fuzzy numbers, intuitionistic trapezoidal fuzzy numbers in [33]. Some of the recent research done on the MCDM of intuitionistic fuzzy number were presented in [28, 37].

In some real problems, a information can be modelling with intervals rather than exact numbers. Therefore, Wan [46] presented interval-valued intuitionistic trapezoidal fuzzy numbers which is its membership function and non-membership function are intervals rather than exact numbers. After Wan [46] some authors studied on the interval-valued intuitionistic trapezoidal fuzzy numbers in [4, 12, 27, 34, 38, 39]. Then, Wei [47] introduced some aggregating oprators and gave an illustrative example.

To modelling an ill-known quantity some decision making problems Deli and Şubaş [23, 24] defined single valued neutrosophic numbers. Some of the recent researchs done on the MCDM of neutrosophic numbers such as; on triangular neutrosophic numbers [1, 5, 18, 35] and on trapezoidal neutrosophic numbers [6, 13, 15, 17, 19, 26, 30, 31, 32, 40, 49, 51, 52]. Although single valued neutrosophic numbers can characterize possible membership degrees of x into the set A in a exact number way, it may lose some original information. For this, interval valued single valued neutrosophic trapezoidal numbers studied in [2, 11, 14, 25, 29]. This paper is organized as follows; in section 2, we presented a literature review that presents papers about fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, single valued neutrosophic sets and single valued neutrosophic numbers. In section 3, we gave the concept of interval valued generalized single valued neutrosophic trapezoidal number(IVGSVTrN-number) which is a generalization of fuzzy number, intuitionistic fuzzy number, neutrosophic number, and so on. In section 4, we presented some aggregation is called IVGSVTrN ordered weighted aggregation operator, IVGSVTrN ordered hybrid weighted aggregation operator. In section 5 proposed some geometric operators is called IVGSVTrN ordered weighted geometric operator, IVGSVTrN ordered hybrid weighted geometric operator. In section 6, we developed a approaches for multiple criteria decision making problems based on the operator and we applied the method to a numerical example to illustrate the practicality and effectiveness of the proposed approach. In section 7, we concluded the research and determines the future directions of the work.

2 Preliminary

In this section, we recall some of the necessary notions related to fuzzy sets, neutrosophic sets, single valued neutrosophic sets and single valued neutrosophic numbers.

From now on we use $I_n = \{1, 2, ..., n\}$ and $I_m = \{1, 2, ..., m\}$ as an index set for $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively.

Definition 2.1. [53] Let E be a universe. Then a fuzzy set X over E is defined by

$$X = \{(\mu_X(x)/x) : x \in E\}$$

where μ_X is called membership function of X and defined by $\mu_X : E \to [0.1]$. For each $x \in E$, the value $\mu_X(x)$ represents the degree of x belonging to the fuzzy set X.

Definition 2.2. [54] t-norm a function such that $t : [0, 1] \times [0, 1] \to [0, 1]$

1. $t(0, 0) = 0$ and $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x),\ x \in E$
2. If $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}(x), \mu_{X_4}(x))$
3. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$
4. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

Definition 2.3. [54] s-norm a function such that $s : [0, 1] \times [0, 1] \to [0, 1]$ with the following conditions:

1. $s(1, 1) = 1 \ and \ s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$

2. if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$
3. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$
4. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

t-norm and t-conorm are related in a sense of lojical duality as;

$$t(\mu_{X_1}(x), \mu_{X_2}(x)) = 1 - s(1 - \mu_{X_3}(x), 1 - \mu_{X_4}(x))$$

Some t-norm and t-conorm are given as;

1. Drastic product:

$$t_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} min\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & max\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 1 \\ 0, & otherwise \end{cases}$$

2. Drastic sum:

$$s_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} max\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & min\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 0 \\ 1, & otherwise \end{cases}$$

3. Bounded product:

$$t_1(\mu_{X_1}(x), \mu_{X_2}(x)) = max\{0, \mu_{X_1}(x) + \mu_{X_2}(x) - 1\}$$

4. Bounded sum:

$$s_1(\mu_{X_1}(x), \mu_{X_2}(x)) = min\{1, \mu_{X_1}(x) + \mu_{X_2}(x)\}$$

5. Einstein product:

$$t_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{2 - [\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)]}$$

6. Einstein sum:

$$s_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x)}{1 + \mu_{X_1}(x).\mu_{X_2}(x)}$$

7. Algebraic product:

$$t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x).\mu_{X_2}(x)$$

8. Algebraic sum:

$$s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)$$

9. Hamacher product:

$$t_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)}$$

10. Hamacher sum:

$$s_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x) - 2.\mu_{X_1}(x).\mu_{X_2}(x)}{1 - \mu_{X_1}(x).\mu_{X_2}(x)}$$

11. Minumum:

$$t_3(\mu_{X_1}(x), \mu_{X_2}(x)) = min\{\mu_{X_1}(x), \mu_{X_2}(x)\}$$

12. Maximum:

$$s_3(\mu_{X_1}(x), \mu_{X_2}(x)) = max\{\mu_{X_1}(x), \mu_{X_2}(x)\}$$

Definition 2.4. [45] Let E be a universe. An single valued neutrosophic set (SVN-set) over E defined by

$$T_A : E \to [0, 1], \quad I_A : E \to [0, 1], \quad F_A : E \to [0, 1]$$

such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.5. [44] Let U be a universe. Then, an interval value neutrosophic set (IVN-sets) A in U is given as;

$$A = \{\langle T_A(u), I_A(u), F_A(u)\rangle/u : u \in U\}$$

In here, $(T_A(u), I_A(u), F_A(u)) = ([infT_A(u), supT_A(u)], [infI_A(u), supI_A(u)]], [infF_A(u), supF_A(u)])$ is called interval value neutrosophic number for all $u \in U$ and all interval value neutrosophic numbers over U will be denoted by $IVN(U)$.

%begindefinition[24]

3 Interval valued generalized SVTrN -numbers

In this section, we give definitions of interval valued generalized SVTrN-numbers with operations. Some of it is quoted from application in [2, 11, 23, 24, 25].

Definition 3.1. [2, 11, 25, 29] A interval valued generalized single valued trapezoidal neutrosophic number (IVGSVTrN-number)

$$\tilde{a} = \langle (a_1, b_1, c_1, d_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}]\rangle$$

is a special neutrosophic set on the set of real numbers $\mathbb{R}$, whose truth-membership, indeterminacy-membership and falsity-membership functions are respectively defined by

$$T^-_{\tilde{a}}(x) = \begin{cases} (x - a_1)T^-_{\tilde{a}}/(b_1 - a_1) & (a_1 \leq x < b_1) \\ T^-_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ (d_1 - x)T^-_{\tilde{a}}/(d_1 - c_1) & (c_1 < x \leq d_1) \\ 0 & otherwise, \end{cases}$$

$$T^+_{\tilde{a}}(x) = \begin{cases} (x - a_1)T^+_{\tilde{a}}/(b_1 - a_1) & (a_1 \leq x < b_1) \\ T^+_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ (d_1 - x)T^+_{\tilde{a}}/(d_1 - c_1) & (c_1 < x \leq d_1) \\ 0 & otherwise, \end{cases}$$

$$I^-_{\tilde{a}}(x) = \begin{cases} (b_1 - x + I^-_{\tilde{a}}(x - a_1))/(b_1 - a_1) & (a_1 \le x < b_1) \\ I^-_{\tilde{a}} & (b_1 \le x \le c_1) \\ (x - c_1 + I^-_{\tilde{a}}(d_1 - x))/(d_1 - c_1) & (c_1 < x \le d_1) \\ 1 & otherwise \end{cases}$$

$$I^+_{\tilde{a}}(x) = \begin{cases} (b_1 - x + I^+_{\tilde{a}}(x - a_1))/(b_1 - a_1) & (a_1 \le x < b_1) \\ I^+_{\tilde{a}} & (b_1 \le x \le c_1) \\ (x - c_1 + I^+_{\tilde{a}}(d_1 - x))/(d_1 - c_1) & (c_1 < x \le d_1) \\ 1 & otherwise \end{cases}$$

$$F^-_{\tilde{a}}(x) = \begin{cases} (b_1 - x + F^-_{\tilde{a}}(x - a_1))/(b_1 - a_1) & (a_1 \le x < b_1) \\ F^-_{\tilde{a}} & (b_1 \le x \le c_1) \\ (x - c_1 + F^-_{\tilde{a}}(d_1 - x))/(d_1 - c_1) & (c_1 < x \le d_1) \\ 1 & otherwise \end{cases}$$

and

$$F^+_{\tilde{a}}(x) = \begin{cases} (b_1 - x + F^+_{\tilde{a}}(x - a_1))/(b_1 - a_1) & (a_1 \le x < b_1) \\ F^+_{\tilde{a}} & (b_1 \le x \le c_1) \\ (x - c_1 + F^+_{\tilde{a}}(d_1 - x))/(d_1 - c_1) & (c_1 < x \le d_1) \\ 1 & otherwise \end{cases}$$

If $a_1 \ge 0$ and at least $d_1 > 0$, then $\tilde{a} = \langle (a_1, b_1, c_1, d_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle$, is called a positive IVGSVTrN, denoted by $\tilde{a} > 0$. Likewise, if $d_1 \le 0$ and at least $a_1 < 0$, then $\tilde{a} = \langle (a_1, b_1, c_1, d_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle$, is called a negative IVGSVTrN, denoted by $\tilde{a} < 0$.

Note that the set of all IVGSVTrN-number on $\mathbb{R}$ will be denoted by Ω.

[2, 11, 25, 29] give some operations based algebraic sum-product norms on interval valued generalized SVTrN -numbers . We now give alternative operations based maximum-minimum norms on interval valued generalized SVTrN -numbers as;

Definition 3.2. Let $\tilde{a} = \langle (a_1, b_1, c_1, d_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle$, $\tilde{b} = \langle (a_2, b_2, c_2, d_2); [T^-_{\tilde{b}}, T^+_{\tilde{b}}], [I^-_{\tilde{b}}, I^+_{\tilde{b}}], [F^-_{\tilde{b}}, F^+_{\tilde{b}}] \rangle \in \Omega$ and $\Omega \neq 0$ be any real number. Then,

1. sum of $\tilde{a}$ and $\tilde{b}$, denoted by $\tilde{a} + \tilde{b}$, defined as;

$$\begin{aligned} \tilde{a} + \tilde{b} = \ & \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \\ & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle \end{aligned} \tag{3.1}$$

2.

$$\begin{aligned} \tilde{a} - \tilde{b} = \ & \langle (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2); \\ & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle \end{aligned} \tag{3.2}$$

3.

$$\tilde{a}\tilde{b} = \begin{cases} \langle (a_1a_2, b_1b_2, c_1c_2, d_1d_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 > 0, d_2 > 0) \\ \langle (a_1d_2, b_1c_2, c_1b_2, d_1a_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 < 0, d_2 > 0) \\ \langle (d_1d_2, c_1c_2, b_1b_2, a_1a_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 < 0, d_2 < 0) \end{cases} \quad (3.3)$$

4.

$$\tilde{a}/\tilde{b} = \begin{cases} \langle (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 > 0, d_2 > 0) \\ \langle (d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 < 0, d_2 > 0) \\ \langle (d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2); & [min\{T^-_{\tilde{a}}, T^-_{\tilde{b}}\}, min\{T^+_{\tilde{a}}, T^+_{\tilde{b}}\}], [max\{I^-_{\tilde{a}} \vee I^-_{\tilde{b}}\}, max\{I^+_{\tilde{a}}, I^+_{\tilde{b}}\}], \\ & [max\{F^-_{\tilde{a}}, F^-_{\tilde{b}}\}, max\{F^+_{\tilde{a}}, F^+_{\tilde{b}}\}] \rangle (d_1 < 0, d_2 < 0) \end{cases} \quad (3.4)$$

5.

$$\tilde{a} = \begin{cases} \langle (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle & (\gamma > 0) \\ \langle (\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle & (\gamma < 0) \end{cases} \quad (3.5)$$

6.

$$\tilde{a}^{\gamma} = \begin{cases} \langle (a_1^{\gamma}, b_1^{\gamma}, c_1^{\gamma}, d_1^{\gamma}); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle & (\gamma > 0) \\ \langle (d_1^{\gamma}, c_1^{\gamma}, b_1^{\gamma}, a_1^{\gamma}); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle & (\gamma < 0) \end{cases} \quad (3.6)$$

7.

$$\tilde{a}^{-1} = \langle (1/d_1, 1/c_1, 1/b_1, 1/a_1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle \ (\tilde{a} \neq 0). \quad (3.7)$$

Definition 3.3. Let $\tilde{a} = \langle (a, b, c, d); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle \in \Omega$. Then, we defined a method to normalize $\tilde{a}$ as;

$$\langle (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}, 1); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle$$

such that $d \neq 0$.

Definition 3.4. Let $\tilde{a} = \langle (a, b, c, d); [T^-_{\tilde{a}}, T^+_{\tilde{a}}], [I^-_{\tilde{a}}, I^+_{\tilde{a}}], [F^-_{\tilde{a}}, F^+_{\tilde{a}}] \rangle \in \Omega$, then

$$S(\tilde{a}) = \frac{1}{16}[a + b + c + d] \times [4 + (T^-_{\tilde{a}} - I^-_{\tilde{a}} - F^-_{\tilde{a}}) + (T^+_{\tilde{a}} - I^+_{\tilde{a}} - F^+_{\tilde{a}})] \quad (3.8)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a + b + c + d] \times [4 + (T^-_{\tilde{a}} - I^-_{\tilde{a}} + F^-_{\tilde{a}}) + (T^+_{\tilde{a}} - I^+_{\tilde{a}} + F^+_{\tilde{a}})] \quad (3.9)$$

is called the score and accuracy degrees of $\tilde{a}$, respectively.

Example 3.5. Let $\tilde{a} = \langle(0.3, 0.4, 0.8, 0.9); [0.5, 0.7], [0.4, 0.6], [0.3, 0.7]\rangle$ be a IVGSVTrN-number then, based on Equation 3.8 and 3.9, $S(\tilde{a})$ and $A(\tilde{a})$ is computed as;

$$S(\tilde{a}) = \frac{1}{16}[0.3 + 0.4 + 0.8 + 0.9] \times [4 + (0.5 - 0.4 - 0.3) + (0.7 - 0.6 - 0.7)] = 0.533$$

$$A(\tilde{a}) = \frac{1}{16}[0.3 + 0.4 + 0.8 + 0.9] \times [4 + (0.5 - 0.4 + 0.3) + (0.7 - 0.6 + 0.7)] = 0.866$$

Definition 3.6. Let $\tilde{a}_1, \tilde{a}_2 \in \Omega$. Then,

1. If $S(\tilde{a}_1) < S(\tilde{a}_2) \Rightarrow \tilde{a}_1 < \tilde{a}_2$
2. If $S(\tilde{a}_1) > S(\tilde{a}_2) \Rightarrow \tilde{a}_1 > \tilde{a}_2$
3. If $S(\tilde{a}_1) = S(\tilde{a}_2)$;
   (a) If $A(\tilde{a}_1) < A(\tilde{a}_2) \Rightarrow \tilde{a}_1 < \tilde{a}_2$
   (b) If $A(\tilde{a}_1) > A(\tilde{a}_2) \Rightarrow \tilde{a}_1 > \tilde{a}_2$
   (c) If $A(\tilde{a}_1) = A(\tilde{a}_2) \Rightarrow \tilde{a}_1 = \tilde{a}_2$

4 Aggregation operators on IVGSVTrN-numbers

In this section, three IVGSVTrN weighted aggregation operator of IVGSVTrN-numbers is given. Some of it is quoted from application in [2, 11, 23, 24, 25].

Definition 4.1. Let $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); [T^-_{\tilde{a}_j}, T^+_{\tilde{a}_j}], [I^-_{\tilde{a}_j}, I^+_{\tilde{a}_j}], [F^-_{\tilde{a}_j}, F^+_{\tilde{a}_j}]\rangle \in \Omega$ $(j \in I_n)$. Then IVGSVTrN weighted aggregation operator, denoted by K_{ao}, is defined as;

$$K_{ao} : \Omega^n \to \Omega, \quad K_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{i=1}^{n} \omega_i \tilde{a}_i \tag{4.1}$$

where, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is a weight vector associated with the K_{ao} operator, for every $j \in I_n$ such that, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 4.2. *Let* $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); [T^-_{\tilde{a}_j}, T^+_{\tilde{a}_j}], [I^-_{\tilde{a}_j}, I^+_{\tilde{a}_j}], [F^-_{\tilde{a}_j}, F^+_{\tilde{a}_j}]\rangle \in \Omega$ $(j \in I)$, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *be a weight vector of* $\tilde{a}_j$, *for every* $j \in I_n$ *such that* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$. *Then, their aggregated value by using* K_{ao} *operator is also a IVGSVTrN-number and*

$$K_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \Big\langle \Big(\sum_{j=1}^{n} \omega_j a_j, \sum_{j=1}^{n} \omega_j b_j, \sum_{j=1}^{n} \omega_j c_j, \sum_{j=1}^{n} \omega_j d_j \Big); [\min_{1\le j\le n}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le n}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le n}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{I^+_{\tilde{a}_j}\}], [\max_{1\le j\le n}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{F^+_{\tilde{a}_j}\}] \Big\rangle \tag{4.2}$$

Proof: The proof can be made by using mathematical induction on n as; Assume that,

$$\tilde{a}_1 = \langle(a_1, b_1, c_1, d_1); [T^-_{\tilde{a}_1}, T^+_{\tilde{a}_1}], [I^-_{\tilde{a}_1}, I^+_{\tilde{a}_1}], [F^-_{\tilde{a}_1}, F^+_{\tilde{a}_1}]\rangle$$

and

$$\tilde{a}_2 = \langle (a_2, b_2, c_2, d_2); [T^-_{\tilde{a}_2}, T^+_{\tilde{a}_2}], [I^-_{\tilde{a}_2}, I^+_{\tilde{a}_2}], [F^-_{\tilde{a}_2}, F^+_{\tilde{a}_2}] \rangle$$

be two IVGSVTrN-numbers then, for $n = 2$, we have

$$\omega_1\tilde{a}_1 + \omega_2\tilde{a}_2 = \Bigg\langle \Bigg(\sum_{j=1}^{2}\omega_j a_j, \sum_{j=1}^{2}\omega_j b_j, \sum_{j=1}^{2}\omega_j c_j, \sum_{j=1}^{2}\omega_j d_j \Bigg); [\min_{1\le j\le 2}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le 2}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le 2}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le 2}\{I^+_{\tilde{a}_j}\}], [\max_{1\le j\le 2}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le 2}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle \quad (4.3)$$

If holds for $n = k$, that is

$$\omega_1\tilde{a}_1 + \omega_2\tilde{a}_2 + \cdots + \omega_k\tilde{a}_k = \Bigg\langle \Bigg(\sum_{j=1}^{k}\omega_j a_j, \sum_{j=1}^{k}\omega_j b_j, \sum_{j=1}^{k}\omega_j c_j, \sum_{j=1}^{k}\omega_j d_j \Bigg); [\min_{1\le j\le k}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le k}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le k}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le k}\{I^+_{\tilde{a}_j}\}], [\max_{1\le j\le k}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le k}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle \quad (4.4)$$

then, when $n = k + 1$, by the operational laws in Definition 3.2, I have

$$\begin{aligned} \omega_1\tilde{a}_1 + \omega_2\tilde{a}_2 + \cdots + \omega_{k+1}\tilde{a}_{k+1} = & \Bigg\langle \Bigg(\sum_{j=1}^{k}\omega_j a_j, \sum_{j=1}^{k}\omega_j b_j, \sum_{j=1}^{k}\omega_j c_j, \sum_{j=1}^{k}\omega_j d_j \Bigg); \\ & [\min_{1\le j\le k}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le k}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le k}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le k}\{I^+_{\tilde{a}_j}\}], \\ & [\max_{1\le j\le k}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le k}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle + \\ & \Bigg\langle \Bigg(\omega_{k+1}\, a_{k+1}, \omega_{k+1}\, b_{k+1}, \omega_{k+1} c_{k+1}, \omega_{k+1} d_{k+1} \Bigg); \\ & [T^-_{\tilde{a}_{k+1}}, T^+_{\tilde{a}_{k+1}}], [I^-_{\tilde{a}_{k+1}}, I^+_{\tilde{a}_{k+1}}], [F^-_{\tilde{a}_{k+1}}, F^+_{\tilde{a}_{k+1}}] \Bigg\rangle \\ = & \Bigg\langle \Bigg(\sum_{j=1}^{k+1}\omega_j a_j, \sum_{j=1}^{k+1}\omega_j b_j, \sum_{j=1}^{k+1}\omega_j c_j, \sum_{j=1}^{k+1}\omega_j d_j \Bigg); \\ & [\min_{1\le j\le k+1}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le k+1}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le k+1} I^-_{\tilde{a}_j}, \max_{1\le j\le k+1}\{I^+_{\tilde{a}_j}\}], \\ & [\max_{1\le j\le k+1}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le k+1}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle \end{aligned} \quad (4.5)$$

Finally, based on Equation 4.3, 4.4 and 4.5, the proof is valid.

Example 4.3. Let

$$\tilde{a}_1 = \big\langle (0.125, 0.439, 0.754, 0.847); [0.5, 0.6], [0.4, 0.7], [0.6, 0.9] \big\rangle,$$

$$\tilde{a}_2 = \big\langle (0.326, 0.427, 0.648, 0.726); [0.8, 0.9], [0.2, 0.5], [0.4, 0.8] \big\rangle,$$

$$\tilde{a}_3 = \big\langle (0.427, 0.524, 0.578, 0.683); [0.4, 0.6], [0.3, 0, 8], [0, 5, 0.7] \big\rangle$$

be three IVGSVTrN-numbers, and $\omega = (0.4, 0.3, 0.3)^T$ be the weight vector of $\tilde{a}_j (j = 1, 2, 3)$. Then, based on

Equation 4.2,

$$K_{ao}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\big\langle(0.276,0.461,0.669,0.762);[0.4,0.6],[0.4,0.8],[0.6,0.9]\big\rangle$$

and, based on Equation 3.8, their score is 0.312.

Definition 4.4. Let $\tilde{a}_j=\langle(a_j,b_j,c_j,d_j);[T^-_{\tilde{a}_j},T^+_{\tilde{a}_j}],[I^-_{\tilde{a}_j},I^+_{\tilde{a}_j}],[F^-_{\tilde{a}_j},F^+_{\tilde{a}_j}]\rangle\in\Omega\,(j\in I_n)$.Then IVGSVTrN ordered weighted aggregation operator(K_{oao}) is defined as;

$$K_{oao}:\Omega^n\to\Omega,\quad K_{oao}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\sum_{k=1}^{n}\omega_k\tilde{b}_k \tag{4.6}$$

where $\tilde{b}_k=\langle(a_k,b_k,c_k,d_k);[T^-_{\tilde{a}_k},T^+_{\tilde{a}_k}],[I^-_{\tilde{a}_k},I^+_{\tilde{a}_k}],[F^-_{\tilde{a}_k},F^+_{\tilde{a}_k}]\rangle$ is the k-th largest of the n IVGSVTrN-numbers $\tilde{a}_j\,(j\in I_n)$ based on Equation 3.6.

Their aggregated value by using K_{oao} operator is also a IVGSVTrN-number and computed as;

$$\begin{aligned}K_{oao}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\;&\Big\langle\Big(\textstyle\sum_{k=1}^n\omega_ka_k,\sum_{k=1}^n\omega_kb_k,\sum_{k=1}^n\omega_kc_k\sum_{k=1}^n\omega_kd_k\Big);\\&[\min_{1\le j\le n}T^-_{\tilde{a}_j},\min_{1\le j\le n}T^+_{\tilde{a}_j}],[\max_{1\le j\le n}I^-_{\tilde{a}_j},\max_{1\le j\le n}I^+_{\tilde{a}_j}],\\&[\max_{1\le j\le n}F^-_{\tilde{a}_j},\max_{1\le j\le n}F^+_{\tilde{a}_j}]\Big\rangle\end{aligned} \tag{4.7}$$

Definition 4.5. Let $\tilde{a}_j=\langle(a_j,b_j,c_j,d_j);[T^-_{\tilde{a}_j},T^+_{\tilde{a}_j}],[I^-_{\tilde{a}_j},I^+_{\tilde{a}_j}],[F^-_{\tilde{a}_j},F^+_{\tilde{a}_j}]\rangle\in\Omega\;(j\in I_n)$. Then, IVGSVTrN ordered hybrid weighted averaging operator denoted by K_{hao} is defined as;denoted K_{hao}

$$K_{hao}:\Omega^n\to\Omega,\quad K_{hao}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\sum_{k=1}^{n}\omega_k\widehat{b}_k \tag{4.8}$$

where $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is a weight vector associated with the mapping K_{hao} such that $\omega_k\in[0,1]$ and $\sum_{k=1}^n\omega_k=1$, $\tilde{a}_j\in\Omega$ weighted with $n\varpi_j(j\in I_n)$ is denoted by $\tilde{A}_j$, i.e., $\tilde{A}_j=n\varpi_j\tilde{a}_j$, here n is regarded as a balance factor; $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ is a weight vector of the $\tilde{a}_j\in\Omega\,(j\in I_n)$n such that $\varpi_j\in[0,1]$ and $\sum_{j=1}^n\varpi_j=1$; $\widehat{b}_k$ is the k-th largest of the n IVGSVTrN-number $\tilde{A}_j\in\Omega\,(j\in I_n)$ based on Equation 3.6.

Their aggregated value by using K_{hao} operator is also a IVGSVTrN-number and computed as

$$\begin{aligned}K_{hao}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\;&\Big\langle\Big(\textstyle\sum_{k=1}^n\omega_ka_k,\sum_{k=1}^n\omega_kb_k,\sum_{k=1}^n\omega_kc_k,\sum_{k=1}^n\omega_kd_k\Big);\\&[\min_{1\le j\le n}T^-_{\tilde{a}_j},\min_{1\le j\le n}T^+_{\tilde{a}_j}],[\max_{1\le j\le n}I^-_{\tilde{a}_j},\max_{1\le j\le n}I^+_{\tilde{a}_j}],\\&[\max_{1\le j\le n}F^-_{\tilde{a}_j},\max_{1\le j\le n}F^+_{\tilde{a}_j}]\Big\rangle\end{aligned} \tag{4.9}$$

Example 4.6. Let

$$\tilde{a}_1=\big\langle(0.123,0.278,0.347,0.426);[0.7,0.8],[0.4,0.7],[0.1,0.6]\big\rangle,$$

$$\tilde{a}_2=\big\langle(0.133,0.268,0.357,0.416);[0.1,0.6],[0.7,0.8],[0.4,0.7]\big\rangle$$

and

$$\tilde{a}_3 = \big\langle (0.143, 0.258, 0.367, 0.406); [0.4, 0.7], [0.1, 0.6], [0.7, 0.8] \big\rangle$$

be three IVGSVTrN-numbers. Assume that $\varpi = (0.2, 0.3, 0.5)^T$ be a weight vector and $\omega = (0.5, 0.3, 0.2)^T$ be a position weight vector.Then evaluation of the three numbers by using the Equation 4.9 is given as;

Solving

$$\tilde{A}_1 = 3 \times 0.2 \times \tilde{a}_1 = \big\langle (0.074, 0.167, 0.208, 0.256); [0.7, 0.8], [0.4, 0.7], [0.1, 0.6] \big\rangle$$

Likewise, we obtain:

$$\tilde{A}_2 = 3 \times 0.3 \times \tilde{a}_2 = \big\langle (0.160, 0.322, 0.428, 0.499); [0.1, 0.6], [0.7, 0.8], [0.4, 0.7] \big\rangle$$

$$\tilde{A}_3 = 3 \times 0.5 \times \tilde{a}_3 = \big\langle (0.286, 0.516, 0.734, 0.812); [0.4, 0.7], [0.1, 0.6], [0.7, 0.8] \big\rangle$$

we obtain the scores of the IVGSVTrN-numbers $\tilde{A}_j$(j=1,2,3), based on Equation 3.8, as follows:

$$S(\tilde{A}_1) = \frac{1}{16}[0.074 + 0.167 + 0.208 + 0.256] \times (4 + (0.7 - 0.4 - 0.1) + (0.8 - 0.7 - 0.6)) = 0.163$$

$$S(\tilde{A}_2) = \frac{1}{16}[0.160 + 0.322 + 0.428 + 0.499] \times (4 + (0.1 - 0.7 - 0.4) + (0.6 - 0.8 - 0.7)) = 0.185$$

$$S(\tilde{A}_3) = \frac{1}{16}[0.286 + 0.516 + 0.734 + 0.812] \times (4 + (0.4 - 0.1 - 0.7) + (0.7 - 0.6 - 0.8)) = 0.426$$

respectively. Obviously, $S(\tilde{A}_3) > S(\tilde{A}_2) > S(\tilde{A}_1)$. Thereby, according to the Equation 3.6, we have

$$\hat{b}_1 = \tilde{A}_3 = \big\langle (0.143, 0.258, 0.367, 0.406); [0.4, 0.7], [0.1, 0.6], [0.7, 0.8] \big\rangle$$

$$\hat{b}_2 = \tilde{A}_2 = \big\langle (0.133, 0.268, 0.357, 0.416); [0.1, 0.6], [0.7, 0.8], [0.4, 0.7] \big\rangle$$

$$\hat{b}_3 = \tilde{A}_1 = \big\langle (0.123, 0.278, 0.347, 0.426); [0.7, 0.8], [0.4, 0.7], [0.1, 0.6] \big\rangle$$

It follows from Equation 4.9 that

$$\begin{aligned} K_{hao}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \;& \big\langle \big(0.143 \times 0.5 + 0.133 \times 0.3 + 0.123 \times 0.2, \\ & 0.258 \times 0.5 + 0.268 \times 0.3 + 0.278 \times 0.2, \\ & 0.367 \times 0.5 + 0.357 \times 0.3 + 0.347 \times 0.2, \\ & 0.406 \times 0.5 + 0.416 \times 0.3 + 0.326 \times 0.2\big); [0.1, 0.6], [0.7, 0.8], [0.7, 0.8] \big\rangle \\ = \;& \big\langle \big(0.1360, 0.2650, 0.3600, 0.4130\big); [0.1, 0.6], [0.7, 0.8], [0.7, 0.8] \big\rangle \end{aligned}$$

5 Geometric operators of the IVGSVTrN-number

In this section, we give three IVGSVTrN weighted geometric operator of IVGSVTrN-numbers. Some of it is quoted from application in [2, 11, 24, 33].

Definition 5.1. Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); [T^-_{\tilde{a}_j}, T^+_{\tilde{a}_j}], [I^-_{\tilde{a}_j}, I^+_{\tilde{a}_j}], [F^-_{\tilde{a}_j}, F^+_{\tilde{a}_j}] \rangle \in \Omega \; (j \in I_n)$. Then IVGSVTrN weighted geometric operator, denoted by L_{go}, is defined as;

$$L_{go} : \Omega^n \to \Omega, \quad L_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \prod_{i=1}^{n} \tilde{a}_i^{\omega_i} \tag{5.1}$$

where, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is a weight vector associated with the L_{go} operator, for every $j \in I_n$ such that, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Their aggregated value by using L_{go} operator is also a IVGSVTrN-number and computed as;

$$\begin{aligned} L_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \ & \Bigg\langle \Bigg(\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j}, \prod_{j=1}^{n} c_j^{\omega_j} \prod_{j=1}^{n} d_j^{\omega_j} \Bigg); \\ & [\min_{1\le j\le n}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le n}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le n}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{I^+_{\tilde{a}_j}\}], \\ & [\max_{1\le j\le n}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle \end{aligned} \tag{5.2}$$

Example 5.2. Let

$$\tilde{a}_1 = \big\langle (0.125, 0.439, 0.754, 0.847); [0.5, 0.6], [0.4, 0.7], [0.6, 0.9] \big\rangle,$$

$$\tilde{a}_2 = \big\langle (0.326, 0.427, 0.648, 0.726); [0.8, 0.9], [0.2, 0.5], [0.4, 0.8] \big\rangle,$$

$$\tilde{a}_3 = \big\langle (0.427, 0.524, 0.578, 0.683); [0.4, 0.6], [0.3, 0, 8], [0, 5, 0.7] \big\rangle$$

be four IVGSVTrN-numbers, and $w = (0.4, 0.3, 0.3)^T$ be the weight vector of $\tilde{a}_j (j = 1, 2, 3)$. Then, based on Equation 5.2,

$$L_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \big\langle (0.241, 0.459, 0.665, 0.758); [0.4, 0.6], [0.4, 0.8], [0.6, 0.9] \big\rangle$$

and, based on Equation 3.8, their score is 0.305.

Definition 5.3. Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); [T^-_{\tilde{a}_j}, T^+_{\tilde{a}_j}], [I^-_{\tilde{a}_j}, I^+_{\tilde{a}_j}], [F^-_{\tilde{a}_j}, F^+_{\tilde{a}_j}] \rangle \in \Omega\, (j \in I_n)$. Then IVGSVTrN ordered weighted geometric operator denoted by L_{ogo}, is defined as;

$$L_{ogo} : \Omega^n \to \Omega, \quad L_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \prod_{k=1}^{n} \tilde{b}_k^{w_k} \tag{5.3}$$

where $\omega_k \in [0, 1]$, $\sum_{k=1}^{n} \omega_k = 1$; $\tilde{b}_k = \langle (a_k, b_k, c_k, d_k); [T^-_{\tilde{a}_k}, T^+_{\tilde{a}_k}], [I^-_{\tilde{a}_k}, I^+_{\tilde{a}_k}], [F^-_{\tilde{a}_k}, F^+_{\tilde{a}_k}] \rangle$ is the k-th largest of the n neutrosophic sets $\tilde{a}_j\ (j \in I_n)$ based on Equation 3.6.

Their aggregated value by using L_{ogo} operator is also a IVGSVTrN-number and computed as;

$$\begin{aligned} L_{ogo} : \Omega^n \to \Omega, \quad & L_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \\ & \Bigg\langle \Bigg(\prod_{k=1}^{n} a_k^{w_k}, \prod_{k=1}^{n} b_k^{w_k}, \prod_{k=1}^{n} c_k^{w_k} \prod_{k=1}^{n} d_k^{w_k} \Bigg); \\ & [\min_{1\le j\le n}\{T^-_{\tilde{a}_j}\}, \min_{1\le j\le n}\{T^+_{\tilde{a}_j}\}], [\max_{1\le j\le n}\{I^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{I^+_{\tilde{a}_j}\}], \\ & [\max_{1\le j\le n}\{F^-_{\tilde{a}_j}\}, \max_{1\le j\le n}\{F^+_{\tilde{a}_j}\}] \Bigg\rangle \end{aligned} \tag{5.4}$$

Definition 5.4. Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); [T^-_{\tilde{a}_j}, T^+_{\tilde{a}_j}], [I^-_{\tilde{a}_j}, I^+_{\tilde{a}_j}], [F^-_{\tilde{a}_j}, F^+_{\tilde{a}_j}] \rangle \in \Omega\, (j \in I_n)$. Then IVGSVTrN ordered hybrid weighted geometric operator denoted by L_{hgo}, is defined as;

$$L_{hgo} : \Omega^n \to \Omega, \quad L_{hgo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \prod_{k=1}^{n} \hat{b}_k^{\omega_k} \tag{5.5}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$. $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ is a weight vector associated with the mapping L_{hgo}, $a_j \in \Omega$ a weight with $n\varpi(j \in I_n)$ is denoted by $\tilde{A}_j$ i.e., $\tilde{A}_j = n\varpi\tilde{a}_j$, here n is regarded as a balance factor $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is a weight vector of the $a_j \in \Omega\, (j \in I_n)$; $\hat{b}_k$ is the k-th largest of the n IVGSVTrN-numbers $\tilde{A}_j \in \Omega\, (j \in I_n)$ based on Equation 3.6.

Their aggregated value by using L_{hgo} operator is also a IVGSVTrN-number and computed as

$$\begin{aligned} L_{hgo} : \Omega^n \to \Omega, \quad & L_{hgo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \\ & \Big\langle \Big(\prod_{k=1}^{n} a_k^{w_k}, \prod_{k=1}^{n} b_k^{w_k}, \prod_{k=1}^{n} c_k^{w_k} \prod_{k=1}^{n} d_k^{w_k} \Big); \\ & [\min_{1\le j\le n} T^-_{\tilde{a}_j}, \min_{1\le j\le n} T^+_{\tilde{a}_j}], [\max_{1\le j\le n} I^-_{\tilde{a}_j}, \max_{1\le j\le n} I^+_{\tilde{a}_j}], \\ & [\max_{1\le j\le n} F^-_{\tilde{a}_j}, \max_{1\le j\le n} F^+_{\tilde{a}_j}] \Big\rangle \end{aligned} \tag{5.6}$$

6 IVGSVTrN-multi-criteria decision-making method

In this section, we define a multi-criteria decision making method as follows. Some of it is quoted from application in [2, 11, 23, 24, 25].

Definition 6.1. Let $X = (x_1, x_2, ..., x_m)$ be a set of alternatives, $U = (u_1, u_2, ..., u_n)$ be the set of attributes. If $\tilde{a}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); [T^-_{\tilde{a}_{ij}}, T^+_{\tilde{a}_{ij}}], [I^-_{\tilde{a}_{ij}}, I^+_{\tilde{a}_{ij}}], [F^-_{\tilde{a}_{ij}}, F^+_{\tilde{a}_{ij}}] \rangle \in \Omega$, then

$$[\tilde{a}_{ij}]_{m\times n} = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{pmatrix} \end{array} \tag{6.1}$$

is called an IVGSVTrN-multi-criteria decision-making matrix of the decision maker.

Now, we can give an algorithm of the IVGSVTrN-multi-criteria decision-making method as follows;
Algorithm:

Step 1. Construct the decision-making matrix $[\tilde{a}_{ij}]_{m\times n}$ for decision based on Equation 6.1;

Step 2. Compute the IVGSVTrN-numbers $\tilde{A}_{ij} = n\varpi_i\tilde{a}_{ij}\ (i \in I_m;\ j \in I_n)$ and write the decision-making matrix $[\tilde{A}_{ij}]_{m\times n}$;

Step 3. Obtain the scores of the IVGSVTrN-numbers $\tilde{A}_{ij}\ (i \in I_m;\ j \in I_n)$ based on Equation 3.8;

Step 4. Rank all IVGSVTrN-numbers $\tilde{A}_{ij}(i \in I_m;\ j \in I_n)$ by using the ranking method of IVGSVTrN-numbers and determine the IVGSVTrN-numbers $[b_i]_{1\times n} = \tilde{b}_{ik}(i \in I_m;\ k \in I_n)$ where $\tilde{b}_{ik}$ is k-th largest of $\tilde{A}_{ij}$ for $j \in I_n$ based on Equation 3.6 ;

Step 5. Give the decision matrix $[b_i]_{1\times n}$ for $i = 1, 2, 3, 4$;

Step 6. Compute $K_{hao}(\tilde{b}_{i1}, \tilde{b}_{i2}, ..., \tilde{b}_{in})$ for $i \in I_m$ based on Equation 4.9;

Step 7. Compute $L_{hgo}(\tilde{b}_{i1}, \tilde{b}_{i2}, ..., \tilde{b}_{in})$ for $i \in I_m$ based on Equation 5.6;

Step 8. Rank all alternatives x_i by using the Equation 3.6 and determine the best alternative.

Example 6.2. Let us consider the decision-making problem adapted from [24, 52]. There is an investment company, which wants to invest a sum of money in the best option. There is a panel with the set of the four alternatives is denoted by $X = \{x_1$= car company, x_2=food company, x_3=computer company, x_4=arms company$\}$ to invest the money. The investment company must take a decision according to the set of the four attributes is denoted by $U = \{u_1 = risk, u_2 = growth, u_3 = environmental\ impact, u_4 = performance\}$. Then, the weight vector of the attributes is $\varpi = (0.2, 0.3, 0.2, 0.3)^T$ and the position weight vector is $\omega = (0.3, 0.2, 0.3, 0.2)^T$ by using the weight determination based on the normal distribution. For the evaluation of an alternative x_i $(i = 1, 2, 3, 4)$ with respect to a criterion u_j $(j = 1, 2, 3, 4)$, it is obtained from the questionnaire of a domain expert. Then, the four possible alternatives are to be evaluated under the above three criteria by corresponding to linguistic values of IVGSVTrN-numbers for linguistic terms (adapted from [24]), as shown in Table 1.

Linguistic terms	Linguistic values of IVGSVTrN-numbers
Absolutely low	$\langle (0.1, 0.2, 0.3, 0.4); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9] \rangle$
Low	$\langle (0.1, 0.3, 0.4, 0.7); [0.2, 0.4], [0.7, 0.8], [0.6, 0.8] \rangle$
Fairly low	$\langle (0.1, 0.4, 0.5, 0.7); [0.3, 0.5], [0.6, 0.7], [0.5, 0.7] \rangle$
Medium	$\langle (0.2, 0.4, 0.5, 0.8); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5] \rangle$
Fairly high	$\langle (0.4, 0.5, 0.6, 0.8); [0.6, 0.7], [0.4, 0.5], [0.3, 0.5] \rangle$
High	$\langle (0.5, 0.6, 0.7, 0.9); [0.7, 0.8], [0.3, 0.4], [0.2, 0.3] \rangle$
Absolutely high	$\langle (0.6, 0.7, 0.8, 0.9); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2] \rangle$

Table 1: IVGSVTrN-numbers for linguistic terms

Step 1. The decision maker construct the decision matrix $[\tilde{a}_{ij}]_{4x4}$ based on Equation 6.1 as follows:

$$\left(\begin{array}{cc} \langle (0.2, 0.4, 0.5, 0.8); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5] \rangle & \langle (0.1, 0.4, 0.5, 0.7); [0.3, 0.5], [0.6, 0.7], [0.5, 0.7] \rangle \\ \langle (0.1, 0.3, 0.4, 0.7); [0.2, 0.4], [0.7, 0.8], [0.6, 0.8] \rangle & \langle (0.1, 0.2, 0.3, 0.4); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9] \rangle \\ \langle (0.6, 0.7, 0.8, 0.9); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2] \rangle & \langle (0.4, 0.5, 0.6, 0.8); [0.6, 0.7], [0.4, 0.5], [0.3, 0.5] \rangle \\ \langle (0.5, 0.6, 0.7, 0.9); [0.7, 0.8], [0.3, 0.4], [0.2, 0.3] \rangle & \langle (0.2, 0.4, 0.5, 0.8); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5] \rangle \end{array} \right.$$

$$\left. \begin{array}{cc} \langle (0.1, 0.4, 0.5, 0.7); [0.3, 0.5], [0.6, 0.7], [0.5, 0.7] \rangle & \langle (0.1, 0.3, 0.4, 0.7); [0.2, 0.4], [0.7, 0.8], [0.6, 0.8] \rangle \\ \langle (0.2, 0.4, 0.5, 0.8); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5] \rangle & \langle (0.6, 0.7, 0.8, 0.9); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2] \rangle \\ \langle (0.6, 0.7, 0.8, 0.9); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2] \rangle & \langle (0.1, 0.2, 0.3, 0.4); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9] \rangle \\ \langle (0.1, 0.2, 0.3, 0.4); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9] \rangle & \langle (0.5, 0.6, 0.7, 0.9); [0.7, 0.8], [0.3, 0.4], [0.2, 0.3] \rangle \end{array} \right)$$

Step 2. Compute $\tilde{A}_{ij} = n\varpi_i\tilde{a}_{ij}\,(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ as follows:

$$\begin{aligned}\tilde{A}_{11} = &\ 4\times 0.2\times\big\langle\big(0.2, 0.4, 0.5, 0.8\big); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\big\rangle\\ = &\ \big\langle\big(0.16, 0.32, 0.40, 0.64\big); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\big\rangle\end{aligned}$$

Likewise, we can obtain other IVGSVTrN-numbers $\tilde{A}_{ij} = n\varpi_i\tilde{a}_{ij}\ (i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ which are given by the IVGSVTrN-decision matrix $[\tilde{A}_{ij}]_{4\times 4}$ as follows:

$$\left(\begin{array}{cccc}\langle(0.16, 0.32, 0.40, 0.64); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\rangle & \langle(0.12, 0.48, 0.60, 0.84); [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]\rangle & \langle(0.08, 0.32, 0.40, 0.56); [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]\rangle & \langle(0.12, 0.36, 0.48, 0.84); [0.2, 0.4], [0.7, 0.8], [0.6, 0.8]\rangle\\ \langle(0.08, 0.24, 0.32, 0.56); [0.2, 0.4], [0.7, 0.8], [0.6, 0.8]\rangle & \langle(0.12, 0.24, 0.36, 0.48); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\rangle & \langle(0.16, 0.32, 0.40, 0.64); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\rangle & \langle(0.72, 0.84, 0.96, 1.08); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2]\rangle\\ \langle(0.48, 0.56, 0.64, 0.72); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2]\rangle & \langle(0.48, 0.60, 0.72, 0.96); [0.6, 0.7], [0.4, 0.5], [0.3, 0.5]\rangle & \langle(0.08, 0.16, 0.24, 0.32); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\rangle & \langle(0.12, 0.24, 0.36, 0.48); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\rangle\\ \langle(0.40, 0.48, 0.56, 0.72); [0.7, 0.8], [0.3, 0.4], [0.2, 0.3]\rangle & \langle(0.24, 0.48, 0.60, 0.96); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\rangle & \langle(0.48, 0.56, 0.64, 0.72); [0.8, 0.9], [0.1, 0.2], [0.1, 0.2]\rangle & \langle(0.60, 0.72, 0.84, 1.08); [0.7, 0.8], [0.3, 0.4], [0.2, 0.3]\rangle\end{array}\right)$$

Step 3. We can obtain the scores of the IVGSVTrN-numbers $\tilde{A}_{ij}$ of the alternatives $x_j\ (j = 1, 2, 3, 4)$ on the four attributes $u_i\ (i = 1, 2, 3, 4)$ based on Equation 3.8 as follows:

$$\begin{array}{llll}S(\tilde{A}_{11}) = 0.295 & S(\tilde{A}_{12}) = 0.293 & S(\tilde{A}_{13}) = 0.196 & S(\tilde{A}_{14}) = 0.191\\ S(\tilde{A}_{21}) = 0.128 & S(\tilde{A}_{22}) = 0.068 & S(\tilde{A}_{23}) = 0.295 & S(\tilde{A}_{24}) = 1.148\\ S(\tilde{A}_{31}) = 0.765 & S(\tilde{A}_{32}) = 0.621 & S(\tilde{A}_{33}) = 0.045 & S(\tilde{A}_{34}) = 0.068\\ S(\tilde{A}_{41}) = 0.581 & S(\tilde{A}_{42}) = 0.442 & S(\tilde{A}_{43}) = 0.765 & S(\tilde{A}_{44}) = 0.871\end{array}$$

respectively.

Step 4. The ranking order of all IVGSVTrN-numbers $\tilde{A}_{ij}(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ based on Equation 3.6 as follows;

$$\begin{array}{c}\tilde{A}_{11} > \tilde{A}_{12} > \tilde{A}_{13} > \tilde{A}_{14}\\ \tilde{A}_{24} > \tilde{A}_{23} > \tilde{A}_{21} > \tilde{A}_{22}\\ \tilde{A}_{31} > \tilde{A}_{32} > \tilde{A}_{34} > \tilde{A}_{33}\\ \tilde{A}_{44} > \tilde{A}_{43} > \tilde{A}_{41} > \tilde{A}_{42}\end{array}$$

Thus, we have:

$$\begin{array}{c}\tilde{b}_{11} = \tilde{A}_{11},\ \tilde{b}_{12} = \tilde{A}_{12},\ \tilde{b}_{13} = \tilde{A}_{13},\ \tilde{b}_{14} = \tilde{A}_{14}\\ \tilde{b}_{21} = \tilde{A}_{24},\ \tilde{b}_{22} = \tilde{A}_{23},\ \tilde{b}_{23} = \tilde{A}_{21},\ \tilde{b}_{24} = \tilde{A}_{22}\\ \tilde{b}_{31} = \tilde{A}_{31},\ \tilde{b}_{32} = \tilde{A}_{32},\ \tilde{b}_{33} = \tilde{A}_{34},\ \tilde{b}_{34} = \tilde{A}_{33}\\ \tilde{b}_{41} = \tilde{A}_{44},\ \tilde{b}_{42} = \tilde{A}_{43},\ \tilde{b}_{43} = \tilde{A}_{41},\ \tilde{b}_{44} = \tilde{A}_{42}\end{array}$$

Step 5. The decision matrix $[b_i]_{1\times n}$ for $i = 1, 2, 3, 4$ are given by;

$$\tilde{b}_1 = \Big(\langle(0.16,0.32,0.40,0.64);[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle,\langle(0.12,0.48,0.60,0.84);[0.3,0.5],[0.6,0.7],[0.5,0.7]\rangle,$$
$$\langle(0.08,0.32,0.40,0.56);[0.3,0.5],[0.6,0.7],[0.5,0.7]\rangle,\langle(0.12,0.36,0.48,0.84);[0.2,0.4],[0.7,0.8],[0.6,0.8]\rangle\Big)$$

$$\tilde{b}_2 = \Big(\langle(0.72,0.84,0.96,1.08);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle,\langle(0.16,0.32,0.40,0.64);[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle,$$
$$\langle(0.08,0.24,0.32,0.56);[0.2,0.4],[0.7,0.8],[0.6,0.8]\rangle,\langle(0.12,0.24,0.36,0.48);[0.1,0.2],[0.8,0.9],[0.8,0.9]\rangle\Big)$$

$$\tilde{b}_3 = \Big(\langle(0.48,0.56,0.64,0.72);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle,\langle(0.48,0.60,0.72,0.96);[0.6,0.7],[0.4,0.5],[0.3,0.5]\rangle,$$
$$\langle(0.12,0.24,0.36,0.48);[0.1,0.2],[0.8,0.9],[0.8,0.9]\rangle,\langle(0.08,0.16,0.24,0.32);[0.1,0.2],[0.8,0.9],[0.8,0.9]\rangle\Big)$$

$$\tilde{b}_4 = \Big(\langle(0.60,0.72,0.84,1.08);[0.7,0.8],[0.3,0.4],[0.2,0.3]\rangle,\langle(0.48,0.56,0.64,0.72);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle,$$
$$\langle(0.40,0.48,0.56,0.72);[0.7,0.8],[0.3,0.4],[0.2,0.3]\rangle,\langle(0.24,0.48,0.60,0.96);[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle\Big)$$

Step 6. We can calculate the IVGSVTrN-numbers based on Equation 4.9 $K_{hao}(b_i) = K_{hao}(\tilde{b}_{i1},\tilde{b}_{i2},\tilde{b}_{i3},\tilde{b}_{i4})$ for $i = 1,2,3,4$ as follows:

$$\begin{aligned} K_{hao}(b_1) = & K_{hao}(\tilde{b}_{11},\tilde{b}_{12},\tilde{b}_{13},\tilde{b}_{14}) \\ = & \big\langle\big(0.16\times0.3+0.72\times0.2+0.48\times0.3+0.60\times0.2, \\ & 0.32\times0.3+0.84\times0.2+0.56\times0.3+0.72\times0.2, \\ & 0.40\times0.3+0.96\times0.2+0.64\times0.3+0.84\times0.2, \\ & 0.64\times0.3+1.08\times0.2+0.72\times0.3+1.08\times0.2\big);[0.5,0.6],[0.5,0.6],[0.4,0.5]\big\rangle \\ = & \big\langle\big(0.456,0.576,0.672,0.840\big);[0.5,0.6],[0.5,0.6],[0.4,0.5]\big\rangle \end{aligned}$$

$$\begin{aligned} K_{hao}(b_2) = & K_{hao}(\tilde{b}_{21},\tilde{b}_{22},\tilde{b}_{23},\tilde{b}_{24}) \\ = & \big\langle\big(0.12\times0.3+0.16\times0.2+0.48\times0.3+0.48\times0.2, \\ & 0.48\times0.3+0.32\times0.2+0.60\times0.3+0.56\times0.2, \\ & 0.60\times0.3+0.40\times0.2+0.72\times0.3+0.64\times0.2, \\ & 0.84\times0.3+0.64\times0.2+0.96\times0.3+0.72\times0.2\big);[0.3,0.5],[0.6,0.7],[0.7,0.8]\big\rangle \\ = & \big\langle\big(0.308,0.500,0.604,0.812\big);[0.3,0.5],[0.6,0.7],[0.7,0.8]\big\rangle \end{aligned}$$

$$\begin{aligned} K_{hao}(b_3) = & K_{hao}(\tilde{b}_{31},\tilde{b}_{32},\tilde{b}_{33},\tilde{b}_{34}) \\ = & \big\langle\big(0.08\times0.3+0.08\times0.2+0.12\times0.3+0.40\times0.2, \\ & 0.32\times0.3+0.24\times0.2+0.24\times0.3+0.48\times0.2, \\ & 0.40\times0.3+0.32\times0.2+0.36\times0.3+0.56\times0.2, \\ & 0.56\times0.3+0.56\times0.2+0.48\times0.3+0.72\times0.2\big);[0.1,0.2],[0.8,0.9],[0.8,0.9]\big\rangle \\ = & \big\langle\big(0.156,0.312,0.404,0.568\big);[0.1,0.2],[0.8,0.9],[0.8,0.9]\big\rangle \end{aligned}$$

and

$$\begin{aligned} K_{hao}(b_4) = & K_{hao}(\tilde{b}_{41}, \tilde{b}_{42}, \tilde{b}_{43}, \tilde{b}_{44}) \\ = & \big\langle \big(0.12 \times 0.3 + 0.12 \times 0.2 + 0.08 \times 0.3 + 0.24 \times 0.2, \\ & 0.36 \times 0.3 + 0.24 \times 0.2 + 0.16 \times 0.3 + 0.48 \times 0.2, \\ & 0.48 \times 0.3 + 0.36 \times 0.2 + 0.24 \times 0.3 + 0.60 \times 0.2, \\ & 0.84 \times 0.3 + 0.48 \times 0.2 + 0.32 \times 0.3 + 0.96 \times 0.2\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \\ = & \big\langle \big(0.132, 0.300, 0.408, 0.636\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \end{aligned}$$

Step 7. We can calculate the IVGSVTrN-numbers $L_{hgo}(b_i) = L_{hgo}(\tilde{b}_{i1}, \tilde{b}_{i2}, \tilde{b}_{i3}, \tilde{b}_{i4})$ for $i = 1, 2, 3, 4$ based on Equation 5.6 as follows:

$$\begin{aligned} L_{hao}(b_1) = & L_{hao}(\tilde{b}_{11}, \tilde{b}_{12}, \tilde{b}_{13}, \tilde{b}_{14}) \\ = & \big\langle \big(0.16^{0.3} + 0.72^{0.2} + 0.48^{0.3} + 0.60^{0.2}, \\ & 0.32^{0.3} + 0.84^{0.2} + 0.56^{0.3} + 0.72^{0.2}, \\ & 0.40^{0.3} + 0.96^{0.2} + 0.64^{0.3} + 0.84^{0.2}, \\ & 0.64^{0.3} + 1.08^{0.2} + 0.72^{0.3} + 1.08^{0.2}\big); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\big\rangle \\ = & \big\langle \big(0.391, 0.540, 0.636, 0.817\big); [0.5, 0.6], [0.5, 0.6], [0.4, 0.5]\big\rangle \end{aligned}$$

$$\begin{aligned} L_{hao}(b_2) = & L_{hao}(\tilde{b}_{21}, \tilde{b}_{22}, \tilde{b}_{23}, \tilde{b}_{24}) \\ = & \big\langle \big(0.12^{0.3} + 0.16^{0.2} + 0.48^{0.3} + 0.48^{0.2}, \\ & 0.48^{0.3} + 0.32^{0.2} + 0.60^{0.3} + 0.56^{0.2}, \\ & 0.60^{0.3} + 0.40^{0.2} + 0.72^{0.3} + 0.64^{0.2}, \\ & 0.84^{0.3} + 0.64^{0.2} + 0.96^{0.3} + 0.72^{0.2}\big); [0.3, 0.5], [0.6, 0.7], [0.7, 0.8]\big\rangle \\ = & \big\langle \big(0.254, 0.488, 0.592, 0.803\big); [0.3, 0.5], [0.6, 0.7], [0.7, 0.8]\big\rangle \end{aligned}$$

$$\begin{aligned} L_{hao}(b_3) = & L_{hao}(\tilde{b}_{31}, \tilde{b}_{32}, \tilde{b}_{33}, \tilde{b}_{34}) \\ = & \big\langle \big(0.08^{0.3} + 0.08^{0.2} + 0.12^{0.3} + 0.40^{0.2}, \\ & 0.32^{0.3} + 0.24^{0.2} + 0.24^{0.3} + 0.48^{0.2}, \\ & 0.40^{0.3} + 0.32^{0.2} + 0.36^{0.3} + 0.56^{0.2}, \\ & 0.56^{0.3} + 0.56^{0.2} + 0.48^{0.3} + 0.72^{0.2}\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \\ = & \big\langle \big(0.125, 0.301, 0.396, 0.562\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \end{aligned}$$

and

$$\begin{aligned} L_{hao}(b_4) = & L_{hao}(\tilde{b}_{41}, \tilde{b}_{42}, \tilde{b}_{43}, \tilde{b}_{44}) \\ = & \big\langle \big(0.12^{0.3} + 0.12^{0.2} + 0.08^{0.3} + 0.24^{0.2}, \\ & 0.36^{0.3} + 0.24^{0.2} + 0.16^{0.3} + 0.48^{0.2}, \\ & 0.48^{0.3} + 0.36^{0.2} + 0.24^{0.3} + 0.60^{0.2}, \\ & 0.84^{0.3} + 0.48^{0.2} + 0.32^{0.3} + 0.96^{0.2}\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \\ = & \big\langle \big(0.122, 0.276, 0.385, 0.577\big); [0.1, 0.2], [0.8, 0.9], [0.8, 0.9]\big\rangle \end{aligned}$$

Step 8. The scores of $K_{hao}(\tilde{b}_i)$ for $i = 1, 2, 3, 4$ can be obtained based on Equation 3.8 as follows:

$$S(K_{hao}(b_1)) = 0.493$$

$$S(K_{hao}(b_2)) = 0.320$$

$$S(K_{hao}(b_3)) = 0.081$$

$$S(K_{hao}(b_4)) = 0.083$$

respectively. It is obvious based on Equation 3.6 that

$$K_{hao}(b_1) > K_{hao}(b_2) > K_{hao}(b_4) > K_{hao}(b_3)$$

Therefore, the ranking order of the alternatives x_j $(j = 1, 2, 3, 4)$ is generated as follows:

$$x_1 \succ x_2 \succ x_4 \succ x_3$$

The best supplier for the enterprise is x_1.

Similarly, the scores of $L_{hgo}(\tilde{b}_i)$ for $i = 1, 2, 3, 4$ can be obtained based on Equation 3.8 as follows:

$$S(L_{hgo}(b_1)) = 0.462$$

$$S(L_{hgo}(b_2)) = 0.307$$

$$S(L_{hgo}(b_3)) = 0.078$$

$$S(L_{hgo}(b_4)) = 0.077$$

respectively. It is obvious that

$$L_{hgo}(b_1) > L_{hgo}(b_2) > L_{hgo}(b_3) > L_{hgo}(b_4)$$

Therefore, the ranking order of the alternatives x_j $(j = 1, 2, 3, 4)$ is generated based on Equation 3.6 as follows:

$$x_1 \succ x_2 \succ x_3 \succ x_4$$

The best supplier for the enterprise is x_1.

7 Conclusion

The paper gave the concept of interval valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN-number) which is a generalization of fuzzy number, intuitionistic fuzzy number, neutrosophic number, and so on. An IVGSVTrN-number is a special interval neutrosophic set on the set of real numbers $\mathbb{R}$. To aggregating the information with IVGSVTrN-numbers, we give some operations on IVGSVTrN-numbers.

Also, we presented some aggregation and geometric operators is called IVGSVTrN weighted aggregation operator, IVGSVTrN ordered weighted aggregation operator, IVGSVTrN ordered hybrid weighted aggregation operator, IVGSVTrN weighted geometric operator, IVGSVTrN ordered weighted geometric operator, IVGSVTrN ordered hybrid weighted geometric operator. Furtermore, for these operators, we examined some desirable properties and special cases. Finally, we developed a approach for multiple criteria decision making problems based on the operator and we applied the method to a numerical example to demonstrate its practicality and effectiveness. In the future, we shall focus on the multiple criteria group decision making problems problems with IVGSVTrNs in which the information of attributes weights is partially unknown in advance.

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SINGLE VALUED (2N+1) SIDED POLYGONAL NEUTROSOPHIC NUMBERS AND SINGLE VALUED (2N) SIDED POLYGONAL NEUTROSOPHIC NUMBERS

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Abstract. This paper introduces a single valued (2n as well as 2n+1) sided polygonal neutrosophic numbers in continuation with other defined single valued neutrosophic numbers. The paper provides basic algebra like addition, subtraction and multiplication of a single valued (2n as well as 2n+1) sided polygonal neutrosophic numbers with examples. In addition, the paper introduces matrix for single valued (2n as well as 2n+1) sided polygonal neutrosophic matrix and its properties.

Keywords: Fuzzy numbers, Intuitionistic fuzzy numbers, Single valued trapezoidal neutrosophic numbers, Single valued triangular neutrosophic numbers, Neutrosophic matrix.

1 Introduction

In the real world problems, uncertainty occurs in many situations which cannot be handled precisely via crisp set theory. To approximate those uncertainties exists in the given linguistics words the fuzzy set theory is introduced by Zadeh [10]. After that, Dubois and Prade [2] defined the fuzzy number as a generalization of real number. In continuation, many authors [5-8, 11-23] introduced various types of fuzzy numbers such as triangular, trapezoidal, pentagonal, hexagonal fuzzy numbers etc. with their membership functions. Atanassov [1] introduced the concept of intuitionistic fuzzy sets that provides precise solutions to the problems in uncertain situations than fuzzy sets with membership and non-membership functions. After developing intuitionistic fuzzy sets, authors in [4, 6, 10, 19] defined various types of intuitionistic fuzzy numbers and different types of operations on intuitionistic fuzzy sets are also established by suitable examples. Smarandache [9] introduced the generalization of both fuzzy and intuitionistic fuzzy sets and named it as neutrosophic set. The Single valued neutrosophic number and its applications are described in [3]. The results of the problems using neutrosophic sets are more accurate than the results given by fuzzy and intuitionistic fuzzy sets [11-20]. Due to which it is applied in various fields for multi-decision tasks [20-32]. The applications of n-valued neutrosophic set [24-26] in data analytics research fields given a thrust to study the neutrosophic numbers. This paper focuses on introducing mathematical operation of 2n and 2n+1 sided polygonal neutrosophic numbers and its matrices with examples.

The rest of the paper is organized as follows: The section 2 contains preliminaries. Section 3 explains single valued 2n+1 polygonal neutrosophic numbers whereas the Section 4 demonstrates Single valued 2n side polygonal neutrosophic numbers. Section 5 provides conclusions followed by acknowledgements and references.

2. Preliminaries

Definition 1 (Fuzzy Number)[4]: A fuzzy number is nothing but an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each of the possible value has its own weight between 0 and 1. This weight is called the membership function. The complex fuzzy set for a given fuzzy number $\widetilde{A}$ can be defined as $\mu_{\widetilde{A}}(x)$ is non-decreasing for $x \le x_0$ and non-increasing for $\ge x_0$. Similarly other properties can be defined.

Definition 2 (Triangular fuzzy number [4]): A fuzzy number $\widetilde{A}$= {a, b, c } is said to be a triangular fuzzy number if its membership function is given by, where $a \le b \le c$

$$\mu_{\widetilde{A}}(x)=\begin{cases} \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ \frac{(c-x)}{(c-b)} & for\ b \le x \le c \\ 0 & otherwise \end{cases}$$

Definition 3 (Trapezoidal fuzzy number [4])

A Trapezoidal fuzzy number (TrFN) denoted by $\widetilde{A}_P$ is defined as (a, b, c, d), where the membership function

$$\mu_{\widetilde{A}_P}(x)=\begin{cases} 0 & for\ x \le a \\ \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ 1 & for\ b \le x \le c \\ \frac{(d-x)}{(d-c)} & for\ c \le x \le d \\ 0 & for\ x \ge d \end{cases}$$

Or, $\mu_{\widetilde{A}_P}(x)$= **max (min (**$\frac{(x-a)}{(b-a)}$**, 1,** $\frac{(d-x)}{(d-c)}$ **),0)**

Definition 4 (Generalized Trapezoidal Fuzzy Number) (GTrFNs)

A Generalized Fuzzy Number (a, b, c, d, w), is called a Generalized Trapezoidal Fuzzy Number "x" if its membership function is given by

$$(x)=\begin{cases} 0 & for\ x \le a \\ \frac{(x-a)}{(b-a)}w & for\ a \le x \le b \\ w & for\ b \le x \le c \\ \frac{(d-x)}{(d-c)}w & for\ c \le x \le d \\ 0 & for\ x \ge d \end{cases}$$

Or, $\mu_{\widetilde{A}_P}(x)$= **max (min (w**$\frac{(x-a)}{(b-a)}$**, w,**$w\frac{(d-x)}{(d-c)}$**) ,0)**

Definition 5 (Pentagonal fuzzy number [4])

A pentagonal fuzzy number (PFN) of a fuzzy set $\widetilde{A}_P$= {a, b, c, d, e} and its membership function is given by,

$$\mu_{\widetilde{A}_P}(x)=\begin{cases} 0 & for\ x < a \\ \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ \frac{(x-b)}{(c-b)} & for\ b \le x \le c \\ 1 & x = c \\ \frac{(d-x)}{(d-c)} & for\ c \le x \le d \\ \frac{(e-x)}{(e-d)} & for\ d \le x \le e \\ 0 & for\ x > d \end{cases}$$

Definition 6 (Hexagonal fuzzy number [4])

A Hexagonal fuzzy number (HFN) of a fuzzy set $\widetilde{\boldsymbol{A}}_P$= {a, b, c, d, e, f} and its membership function is given by,

$$\mu_{\tilde{A}_P}(x)=\begin{cases} 0 & for\ x < a \\ \frac{1}{2}(\frac{x-a}{b-a}) & for\ a \le x \le b \\ \frac{1}{2}+\frac{1}{2}(\frac{x-b}{c-b}) & for\ b \le x \le c \\ 1 & c \le x \le d \\ 1-\frac{1}{2}(\frac{x-d}{e-d}) & for\ c \le x \le d \\ \frac{1}{2}(\frac{f-x}{f-e}) & for\ d \le x \le e \\ 0 & for\ x > d \end{cases}$$

Definition 7 (Octagonal fuzzy number [4])

A Octagonal fuzzy number (OFN) of a fuzzy set $\widetilde{\boldsymbol{A}}_P$= $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ and its membership function is given by,

$$\mu_{\tilde{A}_P} = \begin{cases} k\dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ k, & a_2 \le x \le a_3 \\ k+(1-k)\dfrac{x-a_3}{a_4-a_3}, & a_3 \le x \le a_4 \\ 1, & a_4 \le x \le a_5 \\ k+(1-k)\dfrac{a_6-x}{a_6-a_5}, & a_5 \le x \le a_6 \\ k, & a_6 \le x \le a_7 \\ k\dfrac{a_8-x}{a_8-a_7}, & a_7 \le x \le a_8 \\ 0, & \text{Otherwise} \end{cases}$$

Where k= max$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$

Definition 8 (A triangular intuitionistic fuzzy number)[4]

A triangular intuitionistic fuzzy number $\tilde{a}$ is denoted as $\tilde{a} = ((a,b,c),(a',b',c'))$, where $a' \le a \le b \le b' \le c \le c'$ with the following membership function $\mu_{\tilde{a}}(x)$ and non-membership function $\nu_{\tilde{a}}(x)$

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le b \\ \dfrac{c-x}{c-b}, & b \le c \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \dfrac{b-x}{b-a'}, & a' \le b \\ \dfrac{x-b}{c'-b}, & b \le c' \\ 1, & \text{otherwise} \end{cases}$$

Definition 9 (Trapezoidal Intuitionistic fuzzy number)

$$\boldsymbol{\mu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} 0 & x \leq 0 \\ \frac{(x-a)}{(b-a)} & \text{for } a < x < b \\ w & \text{for } b \leq x \leq c \\ \frac{(d-x)}{(d-c)} & \text{for } c < x < d \\ 0 & \text{otherwise} \end{cases}, \quad \boldsymbol{\nu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} 1 & x \leq 0 \\ \frac{(b-x+u_{\tilde{a}}(x-a))}{(b-a)} & \text{for } a < x < b \\ u_{\tilde{a}} & \text{for } b \leq x \leq c \\ \frac{(x-c+u_{\tilde{a}}(d-x))}{(d-c)} & \text{for } c < x < d \\ 1 & \text{otherwise} \end{cases}$$

Definition 10 (Single valued triangular neutrosophic number [3]):
A triangular neutrosophic number $\tilde{a}$=< (a, b, c) ;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$\boldsymbol{\mu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(x-a)}{(b-a)} w_{\tilde{a}} & \text{for } a \leq x \leq b \\ w_{\tilde{a}} & \text{for } x = b \\ \frac{(c-x)}{(c-b)} w_{\tilde{a}} & \text{for } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}, \quad \boldsymbol{\nu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(b-x+u_{\tilde{a}}(x-a))}{(b-a)} & \text{for } a \leq x \leq b \\ u_{\tilde{a}} & \text{for } x = b \\ \frac{(x-b+u_{\tilde{a}}(c-x))}{(c-b)} & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

$$\boldsymbol{\lambda}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(b-x+y_{\tilde{a}}(x-a))}{(b-a)} & \text{for } a \leq x \leq b \\ y_{\tilde{a}} & \text{for } x = b \\ \frac{(x-b+y_{\tilde{a}}(c-x))}{(c-b)} & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

A triangular neutrosophic number $\tilde{a}$ =< (a, b, c) ;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$> may express an ill-known quantity about b which is approximately equal to b.

Definition 11 (Single valued trapezoidal neutrosophic number [3]):
A triangular neutrosophic number $\tilde{a}$=< $(a, b, c,$ d) ;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership function are defined as follows:

$$\boldsymbol{\mu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(x-a)}{(b-a)} w_{\tilde{a}} & \text{for } a \leq x \leq b \\ w_{\tilde{a}} & \text{for } b \leq x \leq c \\ \frac{(d-x)}{(d-c)} w_{\tilde{a}} & \text{for } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}, \quad \boldsymbol{\nu}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(b-x+u_{\tilde{a}}(x-a))}{(b-a)} & \text{for } a \leq x \leq b \\ u_{\tilde{a}} & \text{for } b \leq x \leq c \\ \frac{(x-c+u_{\tilde{a}}(d-x))}{(d-c)} & \text{for } c \leq x \leq d \\ 1 & \text{otherwise} \end{cases}$$

$$\boldsymbol{\lambda}_{\tilde{a}}(\boldsymbol{x})=\begin{cases} \frac{(b-x+y_{\tilde{a}}(x-a))}{(b-a)} & \text{for } a \leq x \leq b \\ y_{\tilde{a}} & \text{for } b \leq x \leq c \\ \frac{(x-c+y_{\tilde{a}}(d-x))}{(d-c)} & \text{for } c \leq x \leq d \\ 1 & \text{otherwise} \end{cases}$$

The single valued trapezoidal neutrosophic numbers are a generalization of the intuitionistic trapezoidal fuzzy numbers, Thus, the neutrosophic number may express more uncertainty than the intuitionstic fuzzy number.

3. Single valued 2n+1 polygonal neutrosophic numbers

Definition 12 (Single valued 2n+1 polygonal neutrosophic number**):**
A single valued 2n+1 sided polygonal neutrosophic number $\tilde{a}$=< $(a_1, a_2, \ldots, a_n, \ldots, a_{2n}, a_{2n+1})$;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$T_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} w_{\tilde{a}}, & a_1 \le a_2 \\ \dfrac{x-a_2}{a_3-a_2} w_{\tilde{a}}, & a_2 \le a_3 \\ \dots \\ \dots \\ \dots \\ \dfrac{x-a_n}{a_{n+1}-a_n} w_{\tilde{a}}, & a_n \le a_{n+1} \\ w_{\tilde{a}} & x = a_{n+1} \\ \dfrac{a_{n+2}-x}{a_{n+2}-a_{n+1}} w_{\tilde{a}}, & a_{n+1} \le a_{n+2} \\ \dfrac{a_{n+3}-x}{a_{n+3}-a_{n+2}} w_{\tilde{a}}, & a_{n+2} \le a_{n+3} \\ \dots \\ \dots \\ \dots \\ \dfrac{a_{2n+1}-x}{a_{2n+1}-a_{2n}} w_{\tilde{a}}, & a_{2n} \le a_{2n+1} \\ 0, & \text{Otherwise} \end{cases}$$

$$I_A(x) = \begin{cases} \dfrac{a_2-x+u_{\tilde{a}}(x-a_1)}{a_2-a_1}, & a_1 \le a_2 \\ \dfrac{a_3-x+u_{\tilde{a}}(x-a_2)}{a_3-a_2}, & a_2 \le a_3 \\ \dots \\ \dots \\ \dots \\ \dfrac{a_{n+1}-x+u_{\tilde{a}}(x-a_n)}{a_{n+1}-a_n}, & a_n \le a_{n+1} \\ u_{\tilde{a}}, & x = a_{n+1} \\ \dfrac{x-a_{n+1}+u_{\tilde{a}}(a_{n+2}-x)}{a_{n+2}-a_{n+1}}, & a_{n+1} \le a_{n+2} \\ \dfrac{x-a_{n+2}+u_{\tilde{a}}(a_{n+3}-x)}{a_{n+3}-a_{n+2}}, & a_{n+2} \le a_{n+3} \\ \dots \\ \dots \\ \dots \\ \dfrac{x-a_{2n}+u_{\tilde{a}}(a_{2n+1}-x)}{a_{2n+1}-a_{2n}}, & a_{2n} \le a_{2n+1} \\ 1, & \text{Otherwise} \end{cases}$$

$$F_A(x) = \begin{cases} \dfrac{a_2-x+y_{\tilde{a}}(x-a_1)}{a_2-a_1}, & a_1 \le a_2 \\ \dfrac{a_3-x+y_{\tilde{a}}(x-a_2)}{a_3-a_2}, & a_2 \le a_3 \\ \dots \\ \dots \\ \dots \\ \dfrac{a_{n+1}-x+y_{\tilde{a}}(x-a_n)}{a_{n+1}-a_n}, & a_n \le a_{n+1} \\ y_{\tilde{a}}, & x = a_{n+1} \\ \dfrac{x-a_{n+1}+y_{\tilde{a}}(a_{n+2}-x)}{a_{n+2}-a_{n+1}}, & a_{n+1} \le a_{n+2} \\ \dfrac{x-a_{n+2}+y_{\tilde{a}}(a_{n+3}-x)}{a_{n+3}-a_{n+2}}, & a_{n+2} \le a_{n+3} \\ \dots \\ \dots \\ \dfrac{x-a_{2n}+y_{\tilde{a}}(a_{2n+1}-x)}{a_{2n+1}-a_{2n}}, & a_{2n} \le a_{2n+1} \\ 1, & \text{Otherwise} \end{cases}$$

Example:1 If $w_{\tilde{a}}$ = 0.2 ,$u_{\tilde{a}}$ = 0.4 $y_{\tilde{a}}$ = 0.3 and n= 4 , then we have an nanogonal neutrosophic number $\tilde{a}$ and it is taken as $\tilde{a} =< (3,6,8,10,11,21,43,44,56) >$. Figure 1 demonstrates the Example 1.

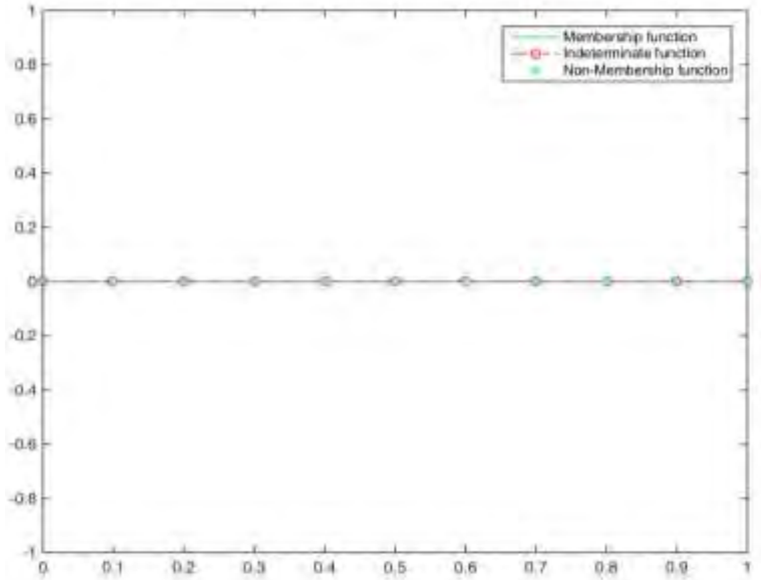

Figure: 1

Example: 2

If $w_{\tilde{a}}$ = 0.2 ,$u_{\tilde{a}}$ = 0.4 $y_{\tilde{a}}$ = 0.3 and n= 4 , then we have an nanogonal neutrosophic number $\tilde{a}$ and it is taken as $\tilde{a} =< (3,6,8,10,1,2,4,7,5) >$. Figure 2 demonstrates the Example 2 and its neutrosophic membership.

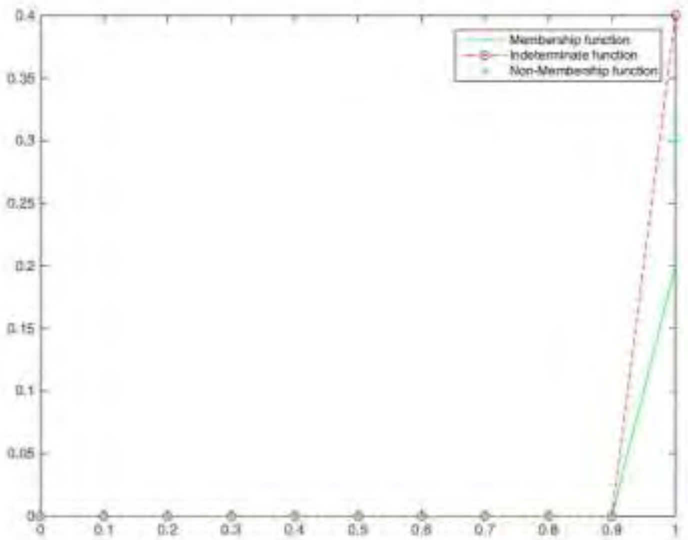

Figure: 2

Note

The single valued triangular neutrosophic number can be generalized to a single valued 2n+1 polygonal neutrosophic number, where n=1,2,3,…,n

$\tilde{a}$=< (a_1,a_2 ,….,a_n,…,a_{2n},a_{2n+1}) ;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$>, where$\tilde{a}$ may express an ill –known quantity about a_n which is gradually equal to a_n.

We mean that $a_2$approximatesa_n, $a_3$approximatesa_n a littel better thana_2,………………. a_{n-1}approximatesa_n a litte better than all previous a_1, a_2,…a_n,

Remark
If $0 \leq w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \leq 1$, $0 \leq w_{\tilde{a}} + u_{\tilde{a}} + y_{\tilde{a}} \leq 1$, $y_{\tilde{a}} = 0$ and the single valued 2n+1 sided polygonal neutrosophic number reduced to the case single valued 2n+1 sided polygonal fuzzy number.

3.1. Operations of single valued 2n+1 sided polygonal neutrosophic numbers
Following are the three operations that can be performed on single valued 2n+1 polygonal neutrosophic numbers suppose A_{PNN}=< $(a_1,a_2,....,a_n,...,a_{2n},a_{2n+1})$; $w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}$> and B_{PNN}=< $(b_1,b_2,....,b_n,...,b_{2n},b_{2n+1})$;$w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}$ >are two single valued 2n+1 polygonal neutrosophic numbers then

(i) Addition:
$A_{PNN} + B_{PNN}$= < $(a_1+b_1, a_2+b_2, \ldots, a_n+b_n, \ldots, a_{2n}+b_{2n}, a_{2n+1}+b_{2n+1})$; $w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}} \cdot w_{\tilde{b}}$, $u_{\tilde{b}} \cdot u_{\tilde{b}}$,$y_{\tilde{a}} \cdot y_{\tilde{b}}$>

(ii) Subtraction:
$A_{PNN} - B_{PNN}$ = < $(a_1-b_1, a_2-b_2, \ldots, a_n-b_n, \ldots, a_{2n}-b_{2n}, a_{2n+1}-b_{2n+1})$; $w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}} \cdot w_{\tilde{b}}$, $u_{\tilde{b}} \cdot u_{\tilde{b}}$,$y_{\tilde{a}} \cdot y_{\tilde{b}}$ >
Multiplication:
$A_{PNN} * B_{PNN}$ = < $(a_1 \cdot b_1, a_2 \cdot b_2, ..., a_n \cdot b_n, ..., a_{2n} \cdot b_{2n}, a_{2n+1} \cdot b_{2n+1})$;$w_{\tilde{a}} \cdot w_{\tilde{b}}$,$u_{\tilde{a}} + u_{\tilde{b}} - u_{\tilde{a}} \cdot u_{\tilde{b}}$,$y_{\tilde{a}} + y_{\tilde{b}} - y_{\tilde{a}} \cdot y_{\tilde{b}}$>

Remark
If $w_{\tilde{a}} = 1$,$u_{\tilde{a}} = 0$ $y_{\tilde{a}} = 0$ then **single valued 2n+1 sided polygonal neutrosophic number** A_{PNN}=< $(a_1,a_2,....,a_n,...,a_{2n},a_{2n+1})$;$w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}$> **reduced to the case of single valued 2n+1 sided polygonal fuzzy number**A_{PFN}=< $(a_1,a_2,....,a_n,...,a_{2n},a_{2n+1})$>, n=1,2,3,…,n.

Remark
If $0 \leq w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \leq 1$, $0 \leq w_{\tilde{a}} + u_{\tilde{a}} + y_{\tilde{a}} \leq 3$, and n=1, the single valued 2n+1 -sided polygonal neutrosophic number reduced to the case of the single valued triangular neutrosophic numberA_{PNN}=< (a_1,a_2,a_3);$w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}$>**[3].**

Example 3: Let $w_{\tilde{a}} = 1$, $u_{\tilde{a}} = 0$, $y_{\tilde{a}} = 0$ and n= 1

If $w_{\tilde{a}} = 1$, $u_{\tilde{a}} = 0$, $y_{\tilde{a}} = 0$ and n= 2, then we have an Pentagonal fuzzy number [5]:

Let A=(1, 2, 3 , 4, 5) and B=(2, 3,4,5,6) be two Pentagonal fuzzy numbers, then

i. A + B = (3, 5, 7, 9,11)
ii. A – B = (-1,-1, -1,-1,-1)
iii. 2A = (2, 4, 6, 8, 10)
iv. A.B = (2, 6, 12, 20, 30)

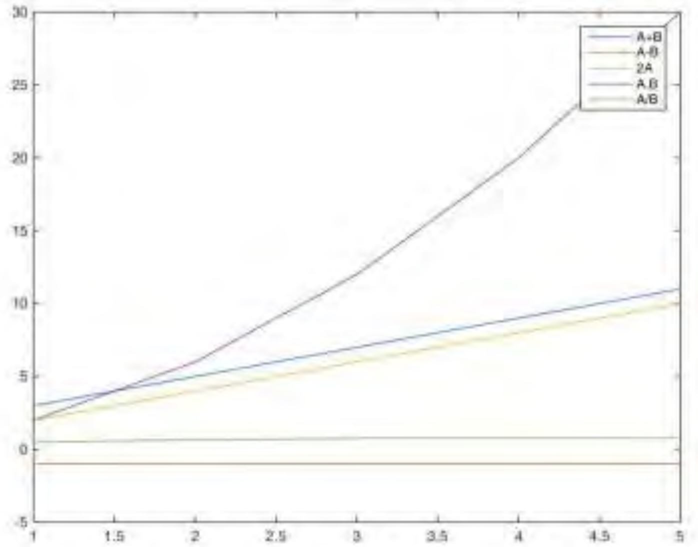

Figure: 3

Figure 3 demonstrates operation given in Example 3. The single valued 2n+1 polygonal neutrosophic number are generalization of the Pentagonal fuzzy number numbers [5] , and single valued triangular neutrosophic number [3]

4. Single valued 2n-sided polygonal neutrosophic numbers

Definition 13: The single valued trapezoidal neutrosophic number can be extended to a single valued 2n sided polygonal neutrosophic number $\tilde{a}=<(a_1,a_2,\ldots,a_n,a_{n+1},\ \ldots,a_{2n-1},a_{2n});\ \boldsymbol{w_{\tilde{a}}},\ \boldsymbol{u_{\tilde{a}}},\boldsymbol{y_{\tilde{a}}}>$ where n=1,2,3,…,n, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$T_A(x) = \begin{cases} k\dfrac{x-a_1}{a_2-a_1}w_{\tilde{a}}, & a_1 \le x \le a_2 \\ k+(1-k)\dfrac{x-a_2}{a_3-a_2}w_{\tilde{a}}, & a_2 \le x \le a_3 \\ \ldots \\ \ldots \\ \ldots \\ k+(1-mk)\dfrac{x-a_{n-1}}{a_n-a_{n-1}}w_{\tilde{a}}, & a_{n-1} \le x \le a_n \\ w_{\tilde{a}}, & a_n \le x \le a_{n+1} \\ k+(1-mk)\dfrac{a_{n+2}-x}{a_{n+2}-a_{n+1}}w_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\ \ldots \\ \ldots \\ \ldots \\ k+(1-k)\dfrac{a_{2n-1}-x}{a_{2n-1}-a_{2n-2}}w_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\ k\dfrac{a_{2n}-x}{a_{2n}-a_{2n-1}}w_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\ 0, & \text{Otherwise} \end{cases}$$

$$I_A(x) = \begin{cases} k+(1-mk)\dfrac{a_2-x}{a_2-a_1}u_{\tilde{a}}, & a_1 \le x \le a_2 \\ k+(1-(m-1)k)\dfrac{a_3-x}{a_3-a_2}u_{\tilde{a}}, & a_2 \le x \le a_3 \\ \dots \\ \dots \\ \dots \\ k+(1-k)\dfrac{a_{n-1}-x}{a_{n-1}-a_{n-2}}u_{\tilde{a}}, & a_{n-2} \le x \le a_{n-1} \\ k\dfrac{a_n-x}{a_n-a_{n-1}}u_{\tilde{a}}, & a_{n-1} \le x \le a_n \\ 0, & a_n \le x \le a_{n+1} \\ k\dfrac{x-a_{n+1}}{a_{n+2}-a_{n+1}}u_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\ k+(1-k)\dfrac{x-a_{n+2}}{a_{n+3}-a_{n+2}}u_{\tilde{a}}, & a_{n+2} \le x \le a_{n+3} \\ \dots \\ \dots \\ \dots \\ k+(1-(m-1)k)\dfrac{x-a_{2n-2}}{a_{2n-1}-a_{2n-2}}u_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\ k+(1-mk)\dfrac{x-a_{2n-1}}{a_{2n}-a_{2n-1}}u_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\ 1, & \text{Otherwise} \end{cases}$$

$$F_A(x) = \begin{cases} k+(1-mk)\dfrac{a_2-x}{a_2-a_1}y_{\tilde{a}}, & a_1 \le x \le a_2 \\ k+(1-(m-1)k)\dfrac{a_3-x}{a_3-a_2}y_{\tilde{a}}, & a_2 \le x \le a_3 \\ \dots \\ \dots \\ \dots \\ k+(1-k)\dfrac{a_{n-1}-x}{a_{n-1}-a_{n-2}}y_{\tilde{a}}, & a_{n-2} \le x \le a_{n-1} \\ k\dfrac{a_n-x}{a_n-a_{n-1}}y_{\tilde{a}}, & a_{n-1} \le x \le a_n \\ 0, & a_n \le x \le a_{n+1} \\ k\dfrac{x-a_{n+1}}{a_{n+2}-a_{n+1}}y_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\ k+(1-k)\dfrac{x-a_{n+2}}{a_{n+3}-a_{n+2}}y_{\tilde{a}}, & a_{n+2} \le x \le a_{n+3} \\ \dots \\ \dots \\ \dots \\ k+(1-(m-1)k)\dfrac{x-a_{2n-2}}{a_{2n-1}-a_{2n-2}}y_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\ k+(1-mk)\dfrac{x-a_{2n-1}}{a_{2n}-a_{2n-1}}y_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\ 1, & \text{Otherwise} \end{cases}$$

where $\tilde{a}$ may represent an ill–known quantity of range, which is gradually approximately equal to the interval $[a_n, a_{n+1}]$.
We mean that (a_2, a_{2n-1}) approximates $[a_n, a_{n+1}]$,
(a, a_{2n-2}) approximates $[a_n, a_{n+1}]$ a little better than (a_2, a_{2n-1}), …………………(a_n, a_{n+1}) approximates $[a_n, a_{n+1}]$ a little better than all previous intervals.

Remark

If $0 \le w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \le 1$, $0 \le w_{\tilde{a}} + u_{\tilde{a}} + y_{\tilde{a}} \le 1$, $y_{\tilde{a}} = 0$ and the single valued 2n -sided polygonal neutrosophic number reduced to the case of single valued 2n-sided polygonal fuzzy number.

4.1 Single valued 2n-sided polygonal neutrosophic number

Following are the three operations that can be performed on single valued 2n-sided polygonal neutrosophic numbers suppose $\boldsymbol{A_{PNN}}$=< $(a_1,a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$;$\boldsymbol{w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}}$> and $\boldsymbol{B_{PNN}}$=<$(b_1,b_2, \dots, b_n, b_{n+1}, \dots, b_{2n-1}, b_{2n})$;$\boldsymbol{w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}}$>are two **2n-sided polygonal neutrosophic number.**

(i) **Addition:** $\boldsymbol{A_{PNN}}$+$\boldsymbol{B_{PNN}}$=$(a_1+b_1, a_2+b_2, \dots, a_n+b_n, a_{n+1}+b_{n+1}, \dots, a_{2n-1}+b_{2n-1}, a_{2n}+b_{2n})$;$\boldsymbol{w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{b}}\cdot u_{\tilde{b}}, y_{\tilde{a}}\cdot y_{\tilde{b}}}$>

(ii) **Subtraction:**$\boldsymbol{A_{PNN}}$-$\boldsymbol{B_{PNN}}$=<$(a_1-b_{2n}, a_2-b_{2n-1}, \dots, a_n-b_n, a_{n+1}-b_{n-1}, \dots, a_{2n-1}-b_2, a_{2n}-b_1)$;$\boldsymbol{w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{b}}\cdot u_{\tilde{b}}, y_{\tilde{a}}\cdot y_{\tilde{b}}}$>

(iii) **Multiplication:**$\boldsymbol{A_{PNN}}$*$\boldsymbol{B_{PNN}}$ =<$(a_1\cdot b_1, a_2\cdot b_2, \dots, a_n\cdot b_n, a_{n+1}\cdot b_{n+1}, \dots, a_{2n-1}\cdot b_{2n-1}, a_{2n}\cdot b_{2n})$;$\boldsymbol{w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{a}}+u_{\tilde{b}}-u_{\tilde{a}}\cdot u_{\tilde{b}}, y_{\tilde{a}}+y_{\tilde{b}}-y_{\tilde{a}}\cdot y_{\tilde{b}}}$>

Remark

If $w_{\tilde{a}} = 1$,$u_{\tilde{a}} = 0$ $y_{\tilde{a}} = 0$ then **single valued 2n**sided**polygonal neutrosophic number** $\boldsymbol{A_{PNN}}$=<$(a_1,a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$;$\boldsymbol{w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}}$> **reduced to the case of single valued 2n-** sided **polygonal fuzzy number**$\boldsymbol{A_{PFN}}$=<$(a_1,a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$ **for** n=1,2,3,…,n.

Remark
If 0≤ $w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$ ≤1 , 0≤ $w_{\tilde{a}}$+ $u_{\tilde{a}}$+$y_{\tilde{a}}$ ≤3, and n=2, the single valued 2n-sided polygonal neutrosophic number reduced to the case of single valued trapezoidal neutrosophic number $\boldsymbol{A_{PNN}}$=< $(\boldsymbol{a_1},\boldsymbol{a_2},\boldsymbol{a_3}, \boldsymbol{a_4})$;$\boldsymbol{w_{\tilde{a}}}$, $\boldsymbol{u_{\tilde{a}}}$,$\boldsymbol{y_{\tilde{a}}}$>[x].

Example 4: if $w_{\tilde{a}}$ = 1 ,$u_{\tilde{a}}$ = 0 $y_{\tilde{a}}$ = 0 and n= 3 then we have an Hexagonal fuzzy number [7-8]:
Let A=(1, 2, 3 ,5 ,6) and B=(2, 4,6,8,10,12) be two Hexagonal fuzzy numbers then
A+ B= (3, 6,9, 13,16,19)

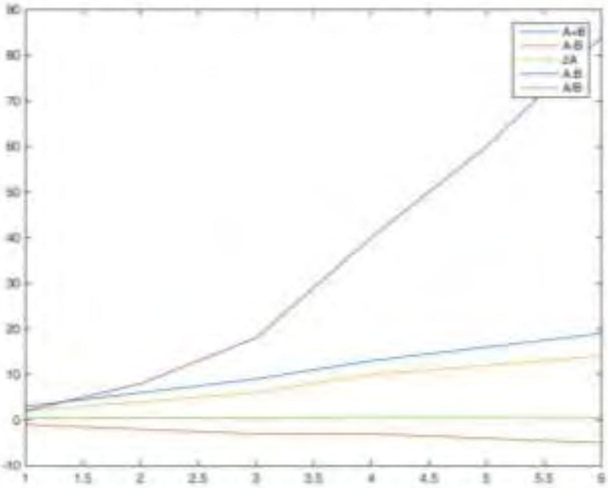

Figure: 4

Figure 4 demonstrates operation given in Example 4.
The single valued 2n-sided polygonal neutrosophic number are generalization of the hexagonal fuzzy numbers [8] ,intuitionistic trapezoidal fuzzy numbers[x] and single valued trapezoidal neutrosophic number [3] with its application [12-23] for multi-decision process [24-26].

5. Conclusion:
This paper introduces single valued (2n and 2n+1) sided polygonal neutrosophic numbers its addition, subtraction, multiplication as well as polygonal neutrosophic matrix with an illustrative example. In near future our focus will be on applications of single-valued 2n sided polygonal neutrosophic numbers and its other mathematical algebra.

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Interval Neutrosophic Vague Sets

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Abstract.
Recently, vague sets and neutrosophic sets have received great attention among the scholars and have been applied in many applications. But, the actual theoretical impacts of the combination of these two sets in dealing uncertainties are still not fully explored until now. In this paper, a new generalized mathematical model called interval neutrosophic vague sets is proposed, which is a combination of vague sets and interval neutrosophic sets and a generalization of interval neutrosophic vague sets. Some definitions of interval neutrosophic vague set such as union, complement and intersection are presented. Furthermore, the basic operations, the derivation of its properties and related example are included.
Keywords: neutrosophic set, vague set, interval neutrsophic vague set.

1 Introduction

Many attempts that used classical mathematics to model uncertain data may not be successful. This is due to the concept of uncertainty which is too complicated and not clearly defined. Therefore, many different theories were developed to solve uncertainty and vagueness including the fuzzy set theory [1], intuitionistic fuzzy set [2], rough sets theory [3], soft set [4], vague sets [5], soft expert set [6] and some other mathematical tools. There are many real applications were solved using these theories related to the uncertainty of these applications [7], [8], [9], [10] . However, these theories cannot deals with indeterminacy and consistent information. Furthermore, all these theories have their inherent difficulties and weakness. Therefore, neutrosophic set is developed by Smarandache in 1998 which is generalization of probability set, fuzzy set and intuitionistic fuzzy set [11]. The neutrosophic set contains three independent membership functions. Unlike fuzzy and intuitionistic fuzzy sets, the memberships in neutrosophic sets are truth, indeterminacy and falsity. The neutrosophic set has received more and more attention since its appearance. Hybrid neutrosophic set were introduced by many researchers [12], [13], [14], [15], [16]. In line with these developments, these extensions have been used in multi criteria decision making problem such as ANP, VIKOR, TOPSIS and DEMATEL with different application [17]–[20].

Vague sets have been introduced by Gau and Buehrar in 1993 as an extension of fuzzy set theory [5]. Vague sets is considered as an effective tool to deal with uncertainty since it provides more information as compared to fuzzy sets [21]. Several studies have revealed that, many researchers have combined vague sets with others theories. Xu et al. proposed vague soft sets and examined its properties [22]. Later, Hassan [23] have combined vague set with soft expert set and its operations were introduced. In addition, others hybrid theories such as complex vague soft set [24], interval valued vague soft set [25], generalized interval valued vague set [26] and possibility vague soft set [27] were presented to solve uncertainty problem in decision making. Recently, Al- Quran and Hassan [28] proposed new hybrid of neutrosophic vague such as [29], [30], [31] and [32].

Until now, there has been no study on interval neutrosophic vague set (INVS) and its combination particularly with vague sets. Therefore, the objective of this paper is to develop a mathematical tool to solve uncertainty problem, namely INVS which is a combination of vague sets and interval neutrosophic set and as a generalization of interval neutrosophic vague set. This set theory provides an interval-based membership structure to handle the neutrosophic vague data. This feature allows users to record their hesitancy in assigning membership values which in turn better capture the vagueness and uncertainties of these data.

This paper is structured in the following manner. Section 2 presents some basic mathematical concepts to enhance the understanding of INVS. Section 3 describes definitions IVNS and its properties together with example. Finally, conclusion of INVS is stated in section 4.

2 Preliminaries

Some basic concepts associated to neutrosophic sets and interval neutrosophic set are presented in this section.

2.1 Vague Set

Definition 2.1 [5]

Let e be a vague value, $e=\left[\bar{t}_e, 1-\bar{f}_e\right]$ where $\bar{t}_e \in[0,1]$, $\bar{f}_e \in[0,1]$ and $0 \leq \bar{t}_e \leq 1-\bar{f}_e \leq 1$. If $\bar{t}_e=1$ and $\bar{f}_e=0$ then e is named a unit vague value so as $e=[1,1]$. Meanwhile if $t_e=0$ and $f_e=1$, hence e is named a zero vague value such that $e=[0,0]$.

Definition 2.1.1 [5]

Let e and f be two vague values, where $e=\left[\bar{t}_e, 1-\bar{f}_e\right]$ and $f=\left[\bar{t}_f, 1-\bar{f}_f\right]$. If $\bar{t}_e=\bar{t}_f$ and $\bar{f}_e=\bar{f}_f$, then vague values e and f are named equal $\left(\text{i.e.} \left[\bar{t}_e, 1-\bar{f}_e\right]=\left[\bar{t}_f, 1-\bar{f}_f\right]\right)$.

Definition 2.1.2 [5]

Let p be a vague set of the universe E . If $\forall e_n \in E$, $t_p(e_n)=1$ and $f_p(e_n)=0$, then p is named a unit vague set where $1 \leq n \leq m$. If $t_p(e_n)=0$ and $t_p(e_n)=1$ hence p named a zero vague set where $1 \leq n \leq m$.

2.2 Neutrosophic Set

Definition 2.2 [11]

A neutrosophic set e in E is described by three functions: truth membership function $V_p(e)$, indeterminacy-membership function $W_p(e)$ and falsity-membership function $X_p(e)$ as $p=\{<e: V_p(e), W_p(e), X_p(e), e \in E\}$ where $V, W, X: E \rightarrow \ ]^{-}0,1^{+}[$ and ${}^{-}0 \leq \sup V_p(e)+\sup W_p(e)+\sup X_p(e) \leq 3^{+}$

2.3 Interval Neutrosophic Set

Definition 2.3 [12]

Let E be a universe. An interval neutrosophic set denoted as (INS) can be defined as follows:

$p=\left\{\left(<\left[V_p^L(e), V_p^U(e)\right],\left[W_p^L(e), W_p^U(e)\right], X_p^L(e), X_p^U(e)\right)>\mid e \in E\right\}$ where for each point $e \in E$, we have $V_p(e) \in[0,1]$, $W_p(e) \in[0,1]$, $X_p(e) \in[0,1]$ and ${}^{-}0 \leq \sup V_p(e)+\sup W_p(e)+\sup X_p(e) \leq 3^{+}$.

2.4 Neutrosophic Vague Set

Definition 2.4 [28]

A neutrosophic vague set p in E denoted (NVS) as an object of the form

$p_{NV}=\left\{<e: V_{p_{NV}}(e), W_{p_{NV}}(e), X_{p_{NV}}(e)>\mid e \in E\right\}$ and

$V_{p_{NV}}(e)=\left[V^{-}, V^{+}\right], \quad W_{p_{NV}}(e)=\left[W^{-}, W^{+}\right], \quad X_{p_{NV}}(e)=\left[X^{-}, X^{+}\right]$

where

$V^{+}=1-X^{-}, X^{+}=1-V^{-}$ and ${}^{-}0 \leq V^{-}+W^{-}+X^{-} \leq 2^{+}$.

Definition 2.4.1 [28]

Let α be a NVS in E . Then α is called a unit NVS where $1 \leq n \leq m$

$V_{\alpha_{NV}}(e)=[1,1],\ W_{\alpha_{NV}}(e)=[0,0],\ X_{\alpha_{NV}}(e)=[0,0]$

.

Definition 2.4.2 [28]

Let β be a NVS in E . Then β is called a zero NVS where $1 \leq n \leq m$.

$V_{\beta_{NV}}(e)=[0,0],\ W_{\beta_{NV}}(e)=[1,1],\ X_{\beta_{NV}}(e)=[1,1]$

For two NVS

$p_{NV} = \{< e : V_{p_{NV}}(e), W_{p_{NV}}(e), X_{p_{NV}}(e) > | e \in E\}$ and

$q_{NV} = \{< e : V_{q_{NV}}(e), W_{q_{NV}}(e), X_{q_{NV}}(e) > | e \in E\}$

The relations of NVS is presented as follows:

(i) $p_{NV} = q_{NV}$ if and only if $V_{p_{NV}}(e) = V_{q_{NV}}(e)$, $W_{p_{NV}}(e) = W_{q_{NV}}(e)$ and $X_{p_{NV}}(e) = X_{q_{NV}}(e)$.

(ii) $p_{NV} \subseteq q_{NV}$ if and only if $V_{p_{NV}}(e) \le V_{q_{NV}}(e)$, $W_{p_{NV}}(e) \ge \bar{I}_{q_{NV}}(e)$ and $X_{p_{NV}}(e) \ge X_{q_{NV}}(e)$.

(iii) The union of p and q is denoted by $R_{NV} = p_{NV} \cup q_{NV}$ is defined by;

$$V_{R_{NV}}(e) = \left[\max\left(V^-_{pNVe}, V^-_{qNVe}\right), \max\left(V^+_{pNVe}, V^+_{qNVe}\right)\right]$$

$$W_{R_{NV}}(e) = \left[\min\left(W^-_{pNVe}, W^-_{qNVe}\right), \min\left(W^+_{pNVe}, W^+_{qNVe}\right)\right],$$

$$X_{R_{NV}}(e) = \left[\min\left(X^-_{pNVe}, X^-_{qNVe}\right), \min\left(X^+_{pNVe}, X^+_{qNVe}\right)\right]$$

(iv) The intersection of p and q is denoted by $S_{NV} = p_{NV} \cap q_{NV}$ is defined by;

$$V_{S_{NV}}(e) = \left[\min\left(V^-_{pNVe}, V^-_{qNVe}\right), \min\left(V^+_{pNVe}, V^+_{qNVe}\right)\right]$$

$$W_{S_{NV}}(e) = \left[\max\left(W^-_{pNVe}, W^-_{qNVe}\right), \max\left(W^+_{pNVe}, W^+_{qNVe}\right)\right],$$

$$X_{S_{NV}}(e) = \left[\max\left(X^-_{pNVe}, X^-_{qNVe}\right), \max\left(X^+_{pNVe}, X^+_{qNVe}\right)\right]$$

(v) The complement of a NVS p_{NV} is denoted by p^c and is defined by

$$V^c_{p_{NV}}(e) = \left[1 - V^+, 1 - V^-\right]$$

$$W^c_{p_{NV}}(e) = \left[1 - W^+, 1 - W^-\right]$$

$$X^c_{p_{NV}}(e) = \left[1 - X^+, 1 - X^-\right]$$

3 Interval Neutrosophic Vague Sets

The formal definition of an INVS and its basic operations of complement, union and intersection are introduced. Related properties and suitable examples are presented in this section.

Definition 3.1

An interval valued neutrosophic vague set A_{INV} also known as INVS in the universe of discourse E. An INVS is characterized by truth membership, indeterminacy membership and fasilty-membership functions is defined as:

$$A_{INV} = \{e, [\tilde{V}^L_A(e), \tilde{V}^U_A(e)], [\tilde{W}^L_A(e), \tilde{W}^U_A(e)], [\tilde{X}^L_A(e), \tilde{X}^U_A(e)] > | e \in E\}$$

$$\tilde{V}^L_A(e) = \left[V^{L^-}, V^{L^+}\right], \tilde{V}^U_A(e) = \left[T^{U^-}, T^{U^+}\right],$$

$$\tilde{W}^L_A(e) = \left[W^{L^-}, W^{L^+}\right], \tilde{W}^U_A(e) = \left[W^{U^-}, W^{U^+}\right],$$

$$\tilde{X}^L_A(e) = \left[X^{L^-}, X^{L^+}\right], \tilde{X}^U_A(e) = \left[X^{U^-}, X^{U^+}\right]$$

where

$V^{L^+} = 1 - X^{L^-}, X^{L^+} = 1 - V^{L^-}, V^{U^+} = 1 - X^{U^-}, X^{U^+} = 1 - V^{U^-}$ and

$^{-}0 \le V^{L^-} + V^{U^-} + W^{L^-} + W^{U^-} + X^{L^-} + X^{U^-} \le 4^+$

$^{-}0 \le V^{L^+} + V^{U^+} + W^{L^+} + W^{U^+} + X^{L^+} + X^{U^+} \le 4^+$

An INVS A_{INV} when E is continuous is presented as follows:

$$A_{INV} = \int_E \frac{\left\langle \left[\tilde{V}_A^L(e), \tilde{V}_A^U(e)\right], \left[\tilde{W}_A^L(e), \tilde{W}_A^U(e)\right], \left[\tilde{X}_A^L(e), \tilde{X}_A^U(e)\right] \right\rangle}{e} : e \in E$$

and when E is discrete an INVS A_{INV} can be presented as follows:

$$A_{INV} = \sum_{i=1}^{n} \frac{\left\langle \left[\tilde{V}_A^L(e), \tilde{V}_A^U(e)\right], \left[\tilde{W}_A^L(e), \tilde{W}_A^U(e)\right], \left[\tilde{X}_A^L(e), \tilde{X}_A^U(e)\right] \right\rangle}{e_i} : e \in E$$

$0 \le \sup \tilde{V}_A(e) + \sup \tilde{W}_A(e) + \sup \tilde{X}_A(e) \le 3$

Example 3.1

Let $E = \{e_1, e_2, e_3\}$. Then

$$A_{INV} = \left\{ \begin{array}{l} \dfrac{e_1}{\langle \{[0.2,0.5],[0.2,0.3]\}, \{[0.1,0.6],[0.3,0.6]\}, \{[0.5,0.8],[0.7,0.8]\} \rangle}, \\ \dfrac{e_2}{\langle \{[0.4,0.5],[0.1,0.7]\}, \{[0.5,0.5],[0.1,0.3]\}, \{[0.5,0.6],[0.3,0.9]\} \rangle} \\ \dfrac{e_3}{\langle \{[0.6,0.9],[0.2,0.5]\}, \{[0.3,0.7],[0.4,0.6]\}, \{[0.1,0.4],[0.5,0.8]\} \rangle} \end{array} \right\}$$

is an INVS subset of E.

Consider Example 3.1 .Then we check the INVS for e_1 by Definition 3.1 as follows:

$V^{L^+} = 1 - X^{L^-} = 0.5 + 0.5 = 1$, $X^{L^+} = 1 - V^{L^-} = 0.8 + 0.2 = 1$

$V^{U^+} = 1 - X^{U^-} = 0.3 + 0.7 = 1$, $X^{U^+} = 1 - V^{U^-} = 0.8 + 0.2 = 1$

Using condition $^{-}0 \le V^{L^-} + V^{U^-} + W^{L^-} + W^{U^-} + X^{L^-} + X^{U^-} \le 4^+$,

therefore we have $0.2 + 0.2 + 0.1 + 0.3 + 0.5 + 0.7 = 2$ and

$^{-}0 \le V^{L^+} + V^{U^+} + W^{L^+} + W^{U^+} + X^{L^+} + X^{U^+} \le 4^+$, therefore we have

$0.5 + 0.3 + 0.6 + 0.6 + 0.8 + 0.8 = 3.6$

The calculations for INVS in Example 3.2, Example 3.3 are calculated similarly.

Definition 3.2

Consider Φ_{INV} be an INVS of the universe E where $\forall e_n \in E$,

$\tilde{V}^L_{\Phi_{INV}}(e) = [1,1], \tilde{V}^U_{\Phi_{INV}}(e) = [1,1]$,

$\tilde{W}^L_{\Phi_{INV}}(e) = [0,0], \tilde{W}^U_{\Phi_{INV}}(e) = [0,0]$,

$\tilde{X}^L_{\Phi_{INV}}(e) = [0,0], \tilde{X}^U_{\Phi_{INV}}(e) = [0,0]$

Therefore, Φ_{INV} is defined a unit INVS where $1 \le n \le m$

Consider δ_{INV} be a INVS of the universe E where $\forall e_n \in E$

$\tilde{V}^L_{\delta_{INV}}(e) = [0,0], \tilde{V}^U_{\delta_{INV}}(e) = [0,0]$,

$\tilde{W}^L_{\delta_{INV}}(e) = [1,1], \tilde{W}^U_{\delta_{INV}}(e) = [1,1]$,

$\tilde{X}^L_{\delta_{INV}}(e) = [1,1], \tilde{X}^U_{\delta_{INV}}(e) = [1,1]$

Therefore, δ_{INV} is defined a zero INVS where $1 \le n \le m$

Definition 3.3

Let A^c_{INV} is defined as complemnet of INVS

$$\left(\tilde{V}^L_A\right)^c(e) = \left[1 - V^{L^+}, 1 - V^{L^-}\right],\ \left(\tilde{V}^U_A\right)^c(e) = \left[1 - V^{U^+}, 1 - V^{U^-}\right],$$

$$\left(\tilde{W}^L_A\right)^c(e) = \left[1 - W^{L^+} 1-, W^{L^-}\right],\ \left(\tilde{W}^U_A\right)^c(e) = \left[1 - W^{U^+}, 1 - W^{U^-}\right],$$

$$\left(\tilde{X}^L_A\right)^c(e) = \left[1 - X^{L^+}, 1 - X^{L^-}\right],\ \left(\tilde{X}^U_A\right)^c(e) = \left[1 - X^{U^+}, 1 - X^{U^-}\right]$$

Example 3.2

Considering **Example 3.1**, by using Definition 3.3, we have

$$A^c_{INV} = \left\{ \begin{array}{l} \dfrac{e_1}{\langle \{[0.5,0.8],[0.7,0.8]\}, \{[0.4,0.9],[0.4,0.7]\}, \{[0.2,0.5],[0.2,0.3]\} \rangle}, \\ \dfrac{e_2}{\langle \{[0.5,0.6],[0.3,0.9]\}, \{[0.5,0.5],[0.7,0.9]\}, \{[0.4,0.5],[0.1,0.7]\} \rangle} \\ \dfrac{e_3}{\langle \{[0.1,0.4],[0.3,0.8]\}, \{[0.3,0.7],[0.4,0.6]\}, \{[0.6,0.9],[0.2,0.7]\} \rangle} \end{array} \right\}$$

Definition 3.5

Let A_{INV} and B_{INV} be two INVS of the universe. If $\forall e_n \in E$,

$\tilde{V}^L_A(e_n) = \tilde{V}^L_B(e_n)$, $\tilde{V}^U_A(e_n) = \tilde{V}^U_B(e_n)$, $\tilde{W}^L_A(e_n) = \tilde{W}^L_B(e_n)$, $\tilde{W}^U_A(e_n) = \tilde{W}^U_B(e_n)$, $\tilde{X}^L_A(e_n) = \tilde{X}^L_B(e_n)$ and $\tilde{X}^U_A(e_n) = \tilde{X}^U_B(e_n)$

Then the INVS A_{INV} and B_{INV} are equal, where $1 \le n \le m$

Definition 3.6

Let A_{INV} and B_{INV} be two INVS of the universe. If $\forall e_n \in E$,

$\tilde{V}^L_A(e_n) \le \tilde{V}^L_B(e_n)$ and $\tilde{V}^U_A(e_n) \le \tilde{V}^U_B(e_n)$,

$\tilde{W}^L_A(e_n) \ge \tilde{W}^L_B(e_n)$ and $\tilde{W}^U_A(e_n) \ge \tilde{W}^U_B(e_n)$,

$\tilde{X}^L_A(e_n) \ge \tilde{X}^L_B(e_n)$ and $\tilde{X}^U_A(e_n) \ge \tilde{X}^U_B(e_n)$

Then the INVS A_{INV} are included by B_{INV} denoted by $A_{INV} \subseteq B_{INV}$, where $1 \le n \le m$.

Definition 3.7

The union of two INVS A_{INV} and B_{INV} is a INVS C_{INV}, written as $C_{INV} = A_{INV} \cup B_{INV}$ is defined as follows:

$$\tilde{V}^L_A(e) = \left[\max\left(V^{L^-}_A, V^{L^-}_B\right), \max\left(V^{L^+}_A, V^{L^+}_B\right)\right] \text{ and } \tilde{V}^U_A(e) = \left[\max\left(V^{U^-}_A, V^{U^-}_B\right), \max\left(V^{U^+}_A, V^{U^+}_B\right)\right],$$

$$\tilde{W}^L_A(e) = \left[\min\left(W^{L^-}_A, W^{L^-}_B\right), \min\left(W^{L^+}_A, W^{L^+}_B\right),\right] \text{and } \tilde{W}^U_A(e) = \left[\min\left(W^{U^-}_A, W^{U^-}_B\right), \min\left(W^{U^+}_A, W^{U^+}_B\right),\right]$$

$$\tilde{X}^L_A(e) = \left[\min\left(X^{L^-}_A, X^{L^-}_B\right), \min\left(X^{L^+}_A, X^{L^+}_B\right)\right] \text{ and } \tilde{X}^U_A(e) = \left[\min\left(X^{U^-}_A, X^{U^-}_B\right), \min\left(X^{U^+}_A, X^{LU^+}_B\right)\right]$$

Definition 3.8

The intersection of two INVS A_{INV} and B_{INV} is a INVS C_{INV}, written as $D_{INV} = A_{INV} \cap B_{INV}$, is defined as below:

$$\widetilde{V}_A^L(e)=\left[\min\left(V_A^{L^-},V_B^{L^-}\right),\min\left(V_A^{L^+},V_B^{L^+}\right)\right] \text{ and } \widetilde{V}_A^U(e)=\left[\min\left(V_A^{U^-},V_B^{U^-}\right),\min\left(V_A^{U^+},V_B^{U^+}\right)\right],$$

$$\widetilde{W}_A^L(e)=\left[\max\left(W_A^{L^-},W_B^{L^-}\right),\max\left(W_A^{L^+},W_B^{L^+}\right),\right] \text{and } \widetilde{W}_A^U(e)=\left[\max\left(W_A^{U^-},W_B^{U^-}\right),\max\left(W_A^{U^+},W_B^{U^+}\right)\right],$$

$$\widetilde{X}_A^L(e)=\left[\max\left(X_A^{L^-},X_B^{L^-}\right),\max\left(X_A^{L^+},X_B^{L^+}\right)\right] \text{ and } \widetilde{X}_A^U(e)=\left[\max\left(X_A^{U^-},X_B^{U^-}\right),\max\left(X_A^{U^+},X_B^{LU+}\right)\right]$$

Example 3.3

Consider that there are two INVS A_{INV} and B_{INV} consist of $E=\{e_1,e_2,e_3\}$ defined as follows:

$$A_{INV}=\left\{\begin{array}{l}\dfrac{e_1}{\langle\{[0.2,0.5],[0.2,0.3]\},\{[0.1,0.6],[0.3,0.6]\},\{[0.5,0.8],[0.7,0.8]\}\rangle},\\ \dfrac{e_2}{\langle\{[0.4,0.5],[0.1,0.7]\},\{[0.5,0.5],[0.1,0.3]\},\{[0.5,0.6],[0.3,0.9]\}\rangle}\\ \dfrac{e_3}{\langle\{[0.6,0.9],[0.2,0.5]\},\{[0.3,0.7],[0.4,0.6]\},\{[0.1,0.4],[0.5,0.8]\}\rangle}\end{array}\right\}$$

$$B_{INV}=\left\{\begin{array}{l}\dfrac{e_1}{\langle\{[0.2,0.6],[0.4,0.9]\},\{[0.5,0.5],[0.3,0.6]\},\{[0.4,0.8],[0.1,0.6]\}\rangle},\\ \dfrac{e_2}{\langle\{[0.2,0.6],[0.2,0.3]\},\{[0.2,0.6],[0.5,0.5]\},\{[0.4,0.0.8],[0.7,0.8]\}\rangle}\\ \dfrac{e_3}{\langle\{[0.4,0.9],[0.2,0.5]\},\{[0.1,0.8],[0.2,0.6]\},\{[0.1,0.6],[0.5,0.8]\}\rangle}\end{array}\right\}$$

By using Definition 3.7, then we obtain INV union, $C_{INV} = A_{INV} \cup B_{INV}$ presented as follows:

$$C_{INV}=\left\{\begin{array}{l}\dfrac{e_1}{\langle\{[0.2,0.6],[0.4,0.9]\},\{[0.1,0.5],[0.3,0.6]\},\{[0.4,0.8],[0.1,0.6]\}\rangle},\\ \dfrac{e_2}{\langle\{[0.4,0.6],[0.2,0.7]\},\{[0.2,0.5],[0.1,0.3]\},\{[0.5,0.6],[0.3,0.8]\}\rangle}\\ \dfrac{e_3}{\langle\{[0.6,0.9],[0.2,0.5]\},\{[0.1,0.7],[0.2,0.6]\},\{[0.1,0.4],[0.5,0.8]\}\rangle}\end{array}\right\}$$

Moreover, by using Definition 3.8, we obtained INV intersection, $D_{INV} = A_{INV} \cap B_{INV}$ as follows:

$$D_{INV}=\left\{\begin{array}{l}\dfrac{e_1}{\langle\{[0.2,0.5],[0.2,0.3]\},\{[0.5,0.6],[0.3,0.6]\},\{[0.5,0.8],[0.7,0.8]\}\rangle},\\ \dfrac{e_2}{\langle\{[0.2,0.5],[0.1,0.3]\},\{[0.5,0.6],[0.5,0.5]\},\{[0.5,0.8],[0.7,0.9]\}\rangle}\\ \dfrac{e_3}{\langle\{[0.4,0.9],[0.2,0.5]\},\{[0.3,0.8],[0.4,0.6]\},\{[0.1,0.6],[0.5,0.8]\}\rangle}\end{array}\right\}$$

Proposition 3.1 Let A_{INV} and B_{INV} be two INVS in X. Then

(i) $A_{INV} \cup A_{INV} = A_{INV}$

(ii) $A_{INV} \cap A_{INV} = A_{INV}$

(iii) $A_{INV} \cup B_{INV} = B_{INV} \cup A_{INV}$

(iv) $A_{INV} \cap B_{INV} = B_{INV} \cap A_{INV}$

Proof (i):
If x is any arbitrary element in $A_{INV} \cup A_{INV} = A_{INV}$ by definition of union, we have $x \in A_{INV}$ or $x \in A_{INV}$. Hence $x \in A_{INV}$. Therefore $A_{INV} \cup A_{INV} \subseteq A_{INV}$. Conversely, If x is any arbitrary element in $A_{INV} \cup A_{INV} = A_{INV}$, then $x \in A_{INV}$ and $x \in A_{INV}$. Therefore,
$A_{INV} \cup A_{INV} \subseteq A_{INV} \therefore A_{INV} \cup A_{INV} = A_{INV}$

Proof (ii): similar to the proof of (i).

Proof (iii):
Let x is any arbitrary element in $A_{INV} \cup B_{INV} = B_{INV} \cup A_{INV}$, then by definition of union, $x \in A_{INV}$ and $x \in B_{INV}$. But, if x is in A_{INV} and B_{INV}, then it is in B_{INV} or A_{INV}, and by definition of union, this means $x \in B_{INV} \cup A_{INV}$. Therefore, $A_{INV} \cup B_{INV} \subseteq B_{INV} \cup A_{INV}$.
The other inclusion is identical: if x is any element $B_{INV} \cup A_{INV}$. Therefore, then we know $x \in B_{INV}$ or $x \in A_{INV}$. But, $x \in B_{INV}$ or $x \in A_{INV}$ implies that x is in A_{INV} or B_{INV} ; hence, $x \in A_{INV} \cup B_{INV}$. Therefore, $B_{INV} \cup A_{INV} \subseteq A_{INV} \cup B_{INV}$. Hence $A_{INV} \cup B_{INV} = B_{INV} \cup A_{INV}$.

Proof (iv): same to the proof (iii)

Proposition 3.2 Let A_{INV}, B_{INV} and C_{INV} be three INVS over the common universe X. Then,

(i) $A_{INV} \cup (B_{INV} \cup C_{INV}) = (A_{INV} \cup B_{INV}) \cup C_{INV}$

(ii) $A_{INV} \cap (B_{INV} \cap C_{INV}) = (A_{INV} \cap B_{INV}) \cap C_{INV}$

(iii) $A_{INV} \cup (B_{INV} \cap C_{INV}) = (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$

(iv) $A_{INV} \cap (B_{INV} \cup C_{INV}) = (A_{INV} \cap B_{INV}) \cup (A_{INV} \cap C_{INV})$

Proof (i):
First, let x be any element in $A_{INV} \cup (B_{INV} \cup C_{INV})$. This means that $x \in A_{INV}$ or $x \in (B_{INV} \cup C_{INV})$. If $x \in A_{INV}$ then $x \in (B_{INV} \cup C_{INV})$; hence, $x \in A_{INV} \cup (B_{INV} \cup C_{INV})$. On the other side, if $x \in A_{INV}$, then $x \in (B_{INV} \cup C_{INV})$. This means $x \in B_{INV}$ or $x \in C_{INV}$. If, $x \in B_{INV}$ then
$x \in A_{INV} \cup B_{INV} \Rightarrow x \in (A_{INV} \cup B_{INV}) \cup C_{INV}$. If $x \in C_{INV}$, Then, $x \in (A_{INV} \cup B_{INV}) \cup C_{INV}$ Hence,
$A_{INV} \cup (B_{INV} \cup C_{INV}) \subseteq (A_{INV} \cup B_{INV}) \cup C_{INV}$.
For the reverse inclusion, let x be any element of $(A_{INV} \cup B_{INV}) \cup C_{INV}$. Then, $x \in (A_{INV} \cup B_{INV})$ or $x \in C_{INV}$. If $x \in (A_{INV} \cup B_{INV})$, we know $x \in A_{INV}$ or $x \in B_{INV}$. If $x \in A_{INV}$, then $x \in A_{INV} \cup (B_{INV} \cup C_{INV})$. If $x \in B_{INV}$,then $x \in (B_{INV} \cup C_{INV})$. Hence $x \in A_{INV} \cup (B_{INV} \cup C_{INV})$. On the other side, if $x \in C_{INV}$, hence $x \in (B_{INV} \cup C_{INV})$, and so, $A_{INV} \cup (B_{INV} \cup C_{INV})$.
Therefore, $A_{INV} \cup (B_{INV} \cup C_{INV}) \subseteq (A_{INV} \cup B_{INV}) \cup C_{INV}$.
Thus, $A_{INV} \cup (B_{INV} \cup C_{INV}) = (A_{INV} \cup B_{INV}) \cup C_{INV}$.
Proof (ii): associativity of intersections is similar to the proof (i)

Proof (iii): Distributive Laws are satisfied for INVS
Let $x \in A_{INV} \cup (B_{INV} \cap C_{INV})$. If $x \in A_{INV} \cup (B_{INV} \cap C_{INV})$ then x is either in A_{INV} or in $(B_{INV} \cap C_{INV})$.
This means that $x \in A_{INV}$ or $x \in (B_{INV} \cap C_{INV})$.
If $x \in A_{INV}$ or $\{x \in B_{INV}$ and $x \in C_{INV}\}$.
Then, $\{x \in A_{INV}$ or $x \in B_{INV}\}$ and $\{x \in A_{INV}$ or $x \in C_{INV}\}$.
So we have,
$x \in A_{INV}$ or B_{INV} and $x \in A_{INV}$ or C_{INV}
$x \in (A_{INV} \cup B_{INV})$ and $x \in (A_{INV} \cup B_{INV})$
$x \in (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup B_{INV})$
Hence, $A_{INV} \cup (B_{INV} \cap C_{INV}) \Rightarrow (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$
Therefore,
$A_{INV} \cup (B_{INV} \cap C_{INV}) \subseteq (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$
Let $x \in (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$. If $x \in (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$ then x is in $(A_{INV}$ or $B_{INV})$ and x $(A_{INV}$ or $C_{INV})$.
So we have,
$x \in (A_{INV}$ or $B_{INV})$ and $x \in (A_{INV}$ or $C_{INV})$
$\{x \in A_{INV}$ or $x \in B_{INV}\}$ and $\{x \in A_{INV}$ or $x \in C_{INV}\}$
$x \in A_{INV}$ or $\{x \in B_{INV}$ and $x \in C_{INV}\}$
$x \in A_{INV} \cup \{x \in (B_{INV}$ and $C_{INV})\}$
$x \in A_{INV} \cup \{x \in (B_{INV} \cap C_{INV})\}$
$x \in A_{INV} \cup (B_{INV} \cap C_{INV})$
$x \in (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV}) \Rightarrow x \in A_{INV} \cup (B_{INV} \cap C_{INV})$
Therefore,
$(A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV}) \subseteq A_{INV} \cup (B_{INV} \cap C_{INV})$
$\therefore A_{INV} \cup (B_{INV} \cap C_{INV}) = (A_{INV} \cup B_{INV}) \cap (A_{INV} \cup C_{INV})$
Proof (iv): similar to the prove of (iii)

Proposition 3.3:
(i) $(A_{INV} \cup B_{INV})^c = A^c_{INV} \cap B^c_{INV}$
(ii) $(A_{INV} \cap B_{INV})^c = A^c_{INV} \cup B^c_{INV}$
(iii) $(A^c_{INV} \cup B^c_{INV} \cup C^c_{INV}) = (A_{INV} \cap B_{INV} \cap C_{INV})^c$
(iv) $(A^c_{INV} \cap B^c_{INV} \cap C^c_{INV}) = (A_{INV} \cup B_{INV} \cup C_{INV})^c$

Proof (i):
Let $x \in (A_{INV} \cup B_{INV})^c$
$\Rightarrow x \notin A_{INV} \cup B_{INV}$
$\Rightarrow x \notin A_{INV}$ and $x \notin B_{INV}$
$\Rightarrow x \in A^c_{INV}$ and $x \in B^c_{INV}$
Since for all $x \in (A_{INV} \cup B_{INV})^c$ such that $x \in A^c_{INV} \cap B^c_{INV}$
Therefore, $(A_{INV} \cup B_{INV})^c \subseteq A^c_{INV} \cap B^c_{INV}$

Proof (ii): similar to the prove of (i)

Proof (ii):

Let $x \in \left(A_{INV}^{c} \cup B_{INV}^{c} \cup C_{INV}^{c}\right)$

$\Rightarrow x \in A_{INV}^{c} \cup x \in B_{INV}^{c} \cup x \in C_{INV}^{c}$

$\Rightarrow x \notin A_{INV} \cup x \notin B_{INV} \cup x \notin C_{INV}$

$\Rightarrow x \notin \left(A_{INV} \cap B_{INV}\right) \cap C_{INV}$

$\Rightarrow x \notin \left(A_{INV} \cap B_{INV} \cap C_{INV}\right)$

$\Rightarrow x \in \left(A_{INV} \cap B_{INV} \cap C_{INV}\right)^{c}$

Since for all $x \in \left(A_{INV}^{c} \cup B_{INV}^{c} \cup C_{INV}^{c}\right)$ such that $x \in \left(A_{INV} \cap B_{INV} \cap C_{INV}\right)^{c}$

Therefore, $\left(A_{INV}^{c} \cup B_{INV}^{c} \cup C_{INV}^{c}\right) \subseteq \left(A_{INV} \cap B_{INV} \cap C_{INV}\right)^{c}$

Proof (iv): similar to the prove of (iv)

4 Conclusion

In this paper, the concept of interval neutrosophic vague was successfully established. The idea of this concept was taken from the theory of vague sets and interval neutrosophic. Neutrosophic set theory is mainly concerned with indeterminate and inconsistent information. However, interval nutrosophic vague sets were developed to improvise results in decision making problem. Meanwhile, vague set capturing vagueness of data. It is clear that, interval neutrosophic vague sets, can be utilize in solving decision making problems that inherited uncertainties. The basic operations involving union, complement, intersection for interval neutrosophic vague set was well defined. Subsequently, the basic properties of these operations related to interval neutrosophic vague set were given and mathematically proven. Finally, some examples are presented. In future, this new extension will broaden the knowledge of existing set theories and subsequently, can be used in practical decision making problem.

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A Note On Neutrosophic Chaotic Continuous Functions

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Abstract. Many real time problems are based on uncertainity and chaotic environment. To demonstrate this ambiguous suituation more precisely we intend to amalgamate the ideas of chaos theory and neutrosophy. Neutrosophy is a flourishing arena which conceptualizes the notions of true, falsity and indeterminacy attributes of an event. Chaos theory is another branch which brings out the concepts of periodic point, orbit and sensitive of a set. Hence in this paper we focus on the introducing the idea of chaotic periodic points, orbit sets, sensitive functions under neutrosophic settings. We start with defining a neutrosophic chaotic space and enlist its properties, As a futher extension we coin neutrosophic chaotic continuous functions and discuss its charaterizations and their interrelationships. We have also illustrated the above said concepts with suitable examples.

Keywords: Neutrosophic periodic points, neutrosophic orbit sets, neutrosophic chaotic sets, neutrosophic sensitive functions, neutrosophic orbit extremally disconnected spaces.

1 Introduction

The introduction of the idea of fuzzy set was introduced in the year 1965 by Zadeh[16]. He proposed that each element in a fuzzy set has a degree of membership. Following this concept K.Atanassov[1,2,3] in 1983 introduced the idea of intuitionistic fuzzy set on a universe X as a generalization of fuzzy set. Here besides the degree of membership a degree of non-membership for each element is also defined. Smarandache[11,12] originally gave the definition of a neutrosophic set and neutrosophic logic. The neutrosophic logic is a formal frame trying to measure the truth, indeterminacy and falsehood. The significance of neutrosophy is that it finds and indispensible place in decision making. Several authors[7, 8, 9, 10] have done remarkable achievements in this area. One of the prime discoveries of the 20^{th} century which has been widely investigated with significant progress and achievements is the theory of Chaos and fractals.It has become an exciting emerging interdisciplinary area in which a broad spectrum of technologies and methodologies have emerged to deal with large-scale, complex and dynamical systems and problems. In 1989, R.L. Deveney[4] defined chaotic function in general metric space. A breakthrough in the conventional general topology was intiated by T. Thrivikraman and P.B. Vinod Kumar[15] by defining Chaos and fractals in general topological spaces. M. Kousalyaparasakthi, E. Roja, M.K. Uma[6] introduced the above said idea to intuitionistic chaotic continuous functions. Tethering around this concept we introduce neutrosophic periodic points, neutrosophic orbit sets, neutrosophic sensitive functions, neutrosophic clopen chaotic sets and neutrossophic chaos spaces. The concepts of neutrosophic chaotic continuous functions, neutrosophic chaotic* continuous functions, neutrosophic chaotic** continuous functions, neutrosophic chaotic*** continuous functions are introduced and studied. Some interrelation are discussed with suitable examples. Also the concept of neutrosophic orbit extremally disconnected spaces, neutrosophic chaotic extremally disconnected spaces, neutrosophic orbit irresolute function are discussed.

2 Preliminaries

2.1 Definition [12]

Let X be a non empty set. A neutrosophic set (NS for short) V is an object having the form V = $<x, V^1, V^2, V^3>$ where V^1, V^2, V^3 represent the degree of membership, the degree of indeterminacy and the degree of non-membership respectively of each element x ∈ X to the set V.

2.2 Definition [12]

Let X be a non empty set, U = $<x, U^1, U^2, U^3>$ and V = $<x, V^1, V^2, V^3>$ be neutrosophic sets on X, and let $\{V_i: i \in J\}$ be an arbitrary family of neutrosophic sets in X, where $V^i = <x, V^1, V^2, V^3>$
(i) $U \subseteq V \Leftrightarrow U^1 \subseteq V^1, U^2 \supseteq V^2$ and $U^3 \supseteq V^3$

(ii) U = V $\Leftrightarrow$ $U \subseteq V$ and $V \subseteq U$.
(iii) $\overline{V}$ = <x, V^3, V^2, V^1>
(iv)$U \cap V$=<x, $U^1 \cap V^1$, $U^2 \cup V^2$, $U^3 \cup V^3$>
(v) $U \cup V$=<x, $U^1 \cup V^1$, $U^2 \cap V^2$, $U^3 \cap V^3$>
(vi) $\cup V_i$ = <x, $\cup V_i^1$, $\cap V_i^2$, $\cap V_i^3$>
(vii) $\cap V_i$ = <x, $\cap V_i^1$, $\cup V_i^2$, $\cup V_i^3$>
(viii)$U - V = U \cap \overline{V}$.
(ix) φ_N = <x, φ, X, X>; X_N = <x, X, φ, φ >.

2.3 Definition [14]

A neutrosophic topology (NT for short) on a nonempty set X is a family τ of neutrosophic set in X satisfying the following axioms:
(i) $\varphi_N, X_N \in \tau$.
(ii) $T_1 \cap T_2 \in \tau$ for any $T_1, T_2 \in \tau$.
(iii) $\cup T_i \in \tau$ for any arbitrary family $\{T_i : i \in J\} \subseteq \tau$.
In this case the pair (X, τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is called a neutrosophic open set (NOS for short) in X. The complement V of a neutrosophic open set V is called a neutrosophic closed set (NCS for short) in X.

2.4 Definition [14]

Let (X, τ) be a neutrosophic topological space and V = <X, V_1, V_2, V_3> be a set in X. Then the closure and interior of V are defined by
$Ncl(V) = \cap\{M : M \text{ is a neutrosophic closed set in X and } V \subseteq M\}$,
$Nint(V) = \cup\{N : N \text{ is a neutrosophic open set in X and } N \subseteq V\}$.
It can be also shown that Ncl(V) is a neutrosophic closed set and Nint(V) is a neutrosophic open set in X, and V is a neutrosophic closed set in X iff Ncl(V) = V; and V is a neutrosophic open set in X iff Nint(V) = V.
Where Ncl - neutrosophic closure and Nint – neutrosophic interior

2.5 Definition [5]

(a) If V = <y,V^1,V^2,V^3> is a neutrosophic set in Y , then the preimage of V under f, denoted by $f^{-1}(V)$, is the neutrosophic set in X defined by $f^{-1}(V)$ = <x,$f^{-1}(V^1)$,$f^{-1}(V^2)$,$f^{-1}(V^3)$>.
(b) If U = <x,U^1,U^2,U^3> is a neutrosophic set in X, then the image of U under f, denoted by f(U), is the neutrosophic set in Y defined by f(U) = <y,f(U^1),f(U^2),Y-f(X-U^3)> where

$$f(U^1)=\begin{cases}\sup_{x\in f^{-1}(y)} U^1 & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

$$f(U^2)=\begin{cases}\sup_{x\in f^{-1}(y)} U^2 & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

$$Y-f(X-U^3)=\begin{cases}\inf_{x\in f^{-1}(y)} U^3 & \text{if } f^{-1}(y)\neq\phi\\ 1 & \text{otherwise}\end{cases}$$

2.6 Definition [13]

Let (X, τ) and (Y,σ) be any two neutrosophic topological spaces and let f : X → Y be a function. Then f is said to be continuous if and only if the preimage of each neutrosophic set in σ is a neutrosophic set in τ.

2.7 Definition [13]

Let (X, τ) and (Y,σ) be two neutrosophic topological spaces and let f : (X, τ) → (Y,σ) be a function. Then f is said to be open iff the image of each neutrosophic set in τ is a neutrosophic set in σ.

2.8 Definition [4]

Orbit of a point x in X under the mapping f is $O_f(x)=\{x, f(x), f^2(x),...\}$

2.9 Definition [4]

x in X is called a periodic point of f if $f^n(x) = x$, for some $n \in Z^+$. Smallest of these n is called period of x.

2.10 Definition [4]

f is sensitive if for each $\delta > 0$ $\exists$ (a) $\varepsilon > 0$ (b) $y \in X$ and (c) $n \in Z_+ \ni d(x,y) < \delta$ and $d(f^n(x),f^n(y)) > \varepsilon$.

2.11 Definition [4]

f is chaotic on (X,d) if (i) Periodic points of f are dense in X (ii) Orbit of x is dense in X for some x in X and (iii) f is sensitive.

2.12 Definition [15]

Let (X, τ) be a topological space and f : (X, τ) → (X, τ) be continuous map. Then f is sensitive at x ∈ X if given any open set U containing x $\exists$ (i) y ∈ U (ii) $n \in Z^+$ and (iii) an open set V $\ni$ $f^n(x) \in V$, $f^n(y) \notin cl(V)$. We say that f is sensitive on a F if f|F is sensitive at every point of F.

2.13 Definition [15]

Let (X, τ) be a topological space and F ∈ K(X). Let f : F → F be a continuous. Then f is chaotic on F if

(i) $cl(O_f(x)) = F$ for some x ∈ F.
(ii) periodic points of f are dense in F.
(iii) f ∈ S(F).

2.14 Definition [15]

(i) C(F) = {f : F → F | f is chaotic on F} and (ii) CH(X) = {F ∈ NK(X) | C(F) ≠ φ}.

2.15 Definition [15]

A topological space (X, τ) is called a chaos space if CH(X) ≠ φ. The members of CH(X) are called chaotic sets.

3 Characterizations of neutrosophic chaotic continuous functions

3.1 Definition

Let (X, τ) be a neutrosophic topological space and $V = \langle X,V^1,V^2,V^3 \rangle$ be a neutrosophic set of X.

(i) Ncl(V) denotes neutrosophic closure of V.
(ii) Nint(V) denotes neutrosophic interior of V.
(iii) NK(X) denotes the collection of all non empty neutrosophic compact sets of X.
(iv) clopen denotes closed and open

3.2 Definition

Let (X, τ) be a neutrosophic topological space. An orbit of a point x in X under the function f : (X, τ) → (X, τ) is denoted and defined as $O_f(x) = \{x,f^1(x),f^2(x),...f^n(x)\}$ for x ∈ X and $n \in Z^+$.

3.3 Example

Let X = {p,q,r}. Let f : X → X be a function defined by f(p) = q, f(q) = r, and f(r) = p. If n = 1, then the orbit points $O_f(p) = \{p,q\}$, $O_f(q) = \{q,r\}$ and $O_f(r) = \{p,r\}$. If n = 2, then the orbit points $O_f(p) = X$, $O_f(q) = X$ and $O_f(r) = X$.

3.4 Definition

Let (X, τ) be a neutrosophic topological space. A neutrosophic orbit set in X under the function f : (X, τ) → (X, τ) is denoted and defined as $NO_f(x) = \langle x,O_{fT}(x),O_{fI}(x),O_{fF}(x) \rangle$ for x ∈ X.

3.5 Example

Let X = {p,q,r,s}. Let f : X → X be a function defined by f(p) = <q,s,q>, f(q) = <s,p.r>, f(r) = <p,q,s> and f(s) = <r,r,p>. If n = 1, then the neutrosophic orbit sets $NO_f(p) = \langle x,\{p,q\},\{p,s\},\{p,q\} \rangle$, $NO_f(q) = \langle x,\{q,s\},\{q,p\},\{q,r\} \rangle$, $NO_f(r) = \langle x,\{p,r\},\{q,r\},\{r,s\} \rangle$ and $NO_f(s) = \langle x,\{r,s\},\{r,s\},\{p,s\} \rangle$. If n = 2, then the neutrosophic orbit sets $NO_f(p) = \langle x,\{p,q,s\},\{p,r,s\},\{p,q,r\} \rangle$, $NO_f(q) = \langle x,\{q,r,s\},\{p,q,s\},\{q,r,s\} \rangle$, $NO_f(r) = \langle x,\{p,q,r\},\{p,q,r\},\{p,r,s\} \rangle$ and $NO_f(s) = \langle x,\{p,r,s\},\{q,r,s\},\{p,q,s\} \rangle$. If n = 3, then the neutrosophic orbit sets $NO_f(a) = \langle x,X,X,X \rangle$, $NO_f(b) = \langle x,X,X,X \rangle$, $NO_f(c) = \langle x,X,X,X \rangle$ and $NO_f(d) = \langle x,X,X,X \rangle$.

3.6 Definition

Let (X, τ) be a neutrosophic topological space and f : (X, τ) → (X, τ) be a neutrosophic continuous function. Then f is said to be neutrosophic sensitive at x ∈ X if given any neutrosophic open set $U = \langle x,U^1,U^2,U^3 \rangle$ containing x $\exists$ a neutrosophic open set $V = \langle x,V^1,V^2,V^3 \rangle$ $\ni$ $f^n(x) \in V$, $f^n(y) \notin Ncl(V)$ and y ∈ U, $n \in Z^+$. We say that f is neutrosophic sensitive on a neutrosophic compact set $F = \langle x,F^1,F^2.F^3 \rangle$ if f|F is neutrosophic sensitive at every point of F.

3.7 Example

Let X = {p,q,r,s}. Then the neutrosophic sets P, Q, R and S are defined by P = <x,{p,r,s},{p,q,r},{p,r,s}>, Q = <x,{r,s},{p,r},{p,s}>, R = <x,{r,s},{p,q,r},{p,r,s}> and S = <x,{p,r,s},{p,r},{p,s}>. Then the family $\tau = \{X_N,\varphi_N,P,Q,R,S\}$ is neutrosophic topology on X. Clearly, (X, τ) is an neutrosophic topological space. Let f : (X, τ) → (X, τ) be a function defined by f(p) = <r,q,s> f(q) = <s,s,r>, f(r) =< q,p,p> and f(s) = <p,r,q>. Let x = p and y = r. If n = 1,3,5, then the neutrosophic open set P = <x,{p,r,s},{p,q,r},{p,r,s}> containing x there exists an neutrosophic open set R = <x,{r,s},{p,q,r},{p,r,s}> such that $f^n(x) \in R, f^n(y) \notin Ncl(R)$ and y ∈ P. Hence the function f is called neutrosophic sensitive.

3.8 Notation

Let (X, τ) be a neutrosophic topological space. Let $F = <x,F^1,F^2,F^3> \subseteq X_N$ then $S(F) = <x,S(F)^1,S(F)^2,S(F)^3>$ where $S(F)^1$ = {f | f is neutrosophic sensitive on F}, $S(F)^2$ = {f | f is indeterminacy neutrosophic sensitive on F} and $S(F)^3$ = {f | f is not neutrosophic sensitive on F}.

3.9 Definition

Let (X, τ) be a two neutrosophic topological space. Let f : (X, τ) → (X, τ) be a function. A neutrosophic periodic set is denoted and defined as $NP_f(x) = <x,\{x \in X \mid f^n{}_T(x) = x\},\{x \in X \mid f^n{}_I(x) = x\},\{x \in X \mid f^n{}_F(x) = x\}>$

3.10 Example

Let X = {p,q,r}. Let f : X → X be a function defined by f(p) = <p,q,r>, f(q) = <r,p,q> and f(r) = <q,r,p>. If n = 1, then the neutrosophic periodic set $NP_f(p)$ = <x,{p},{q},{r}>, $NP_f(q)$ = <x,{r},{p},{q}>and $NP_f(q)$ = <x,{q},{r},{p}>. If n = 2, then the neutrosophic periodic sets $NP_f(p)$ = <x,{p},{p},{p}>, $NP_f(q)$ = <x,{q},{q},{q}> and $NP_f(r)$ = <x,{r},{r},{r}>.

3.11 Definition

Let (X, τ) be a neutrosophic topological space. A neutrosophic set $V = <X,V^1,V^2,V^3>$ of X is said to be a neutrosophic dense in X, if Ncl(V) = X.

3.12 Definition

Let (X, τ) be a neutrosophic topological space and $F = <x,F^1,F^2,F^3> \in NK(X)$. Let f : F → F be a neutrosophic continuous function. Then f is said to be neutrosophic chaotic on F if

(i) $Ncl(NO_f(x)) = F$ for some x ∈ F.
(ii) neutrosophic periodic points of f are neutrosophic dense in F. That is, $Ncl(NP_f(x)) = F$.
(iii) f ∈ S(F).

3.13 Notation

Let (X, τ) be a neutrosophic topological space then $C(F) = <x,C(F)^1,C(F)^2,C(F)^3>$ where $C(F)^1$ = {f : F → F | f is neutrosophic chaotic on F}, $C(F)^2$ = {f : F → F | f is indeterminacy neutrosophic chaotic on F},and $C(F)^3$ = {f : F → F | f is not neutrosophic chaotic on F}.

3.14 Notation

Let (X, τ) be a neutrosophic topological space then $CH(X) = \{F = <x,F^1,F^2,F^3> \in NK(X) \mid C(F) \neq \varphi\}$.

3.15 Definition

A neutrosophic topological space (X, τ) is called a neutrosophic chaos space if CH(X) ≠ φ. The members of CH(X) are called neutrosophic chaotic sets.

3.16 Definition

Let (X, τ) be a neutrosophic topological space. A neutrosophic set $V = <x,V^1,V^2,V^3>$ is neutrosophic clopen if it is both neutrosophic open and neutrosophic closed.

3.17 Definition

Let (X, τ) be a neutrosophic topological space.

(i) A neutrosophic open orbit set is a neutrosophic set which is both neutrosophic open and neutrosophic orbit.
(ii) A neutrosophic closed orbit set is a neutrosophic set which is both neutrosophic closed and neutrosophic orbit.
(iii) A neutrosophic clopen orbit set is a neutrosophic set which is both neutrosophic clopen and neutrosophic orbit.

3.18 Definition

Let (X, τ) be a neutrosophic topological space.

(i) A neutrosophic open chaotic set is a neutrosophic set which is both neutrosophic open and neutrosophic chaotic.
(ii) A neutrosophic closed chaotic set is a neutrosophic set which is both neutrosophic closed and neutrosophic chaotic.
(iii) A neutrosophic clopen chaotic set is a neutrosophic set which is both neutrosophic clopen and neutrosophic chaotic.

3.19 Definition

Let (X, τ) and (X,σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X,σ) is said to be neutrosophic chaotic continuous if for each periodic point x ∈ X and each neutrosophic clopen chaotic set $F = <x,F^1,F^2,F^3>$ of f(x) ∃ a neutrosophic open orbit set $NO_f(x)$ of the periodic point x ∍ $f(NO_f(x)) \subseteq F$.

3.20 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q and R are defined by M=<x,{q,r},{r},{p,r}>, N=<x,{p},{p,q},{p,s}>, O=<x,{p,q,r},φ,{p}>, P=<x,φ,{p,q,r},{p,r,s}>, Q=<x,{p,q,r},{r},{p}>, R=<x,{p},{r},{p,q,r}>. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <p,q,s> f(q) = <r,s,r>, f(r) =< q,r,p> and f(s) = <s,p,q>. Now the function f is called neutrosophic chaotic continuous.

3.21 Theorem

Let (X, τ) and (X,σ) be any two neutrosophic chaos spaces. Let f : (X, τ) → (X,σ) be a function. Then the following statements are equivalent:

(i) f is neutrosophic chaotic continuous.

(ii) Inverse image of every neutrosophic clopen chaotic set of (X,σ) is a neutrosophic open orbit set of (X, τ).

(iii) Inverse image of every neutrosophic clopen chaotic set of (X,σ) is a neutrosophic clopen orbit set of (X, τ).

Proof

(i)⇒ (ii) Let $F = <x,F^1,F^2,F^3>$ be a neutrosophic clopen chaotic set of (X, σ) and the periodic point $x \in f^{-1}(F)$. Then f(x) ∈ F. Since f is neutrosophic chaotic continuous, ∃ a neutrosophic open orbit set $NO_f(x)$ of (X, τ) ∍ $x \in NO_f(x)$, $f(NO_f(x)) \subseteq F$. That is, $x \in NO_f(x) \subseteq f^{-1}(F)$. Now, $f^{-1}(F) = \cup\{NO_f(x) : x \in f^{-1}(F)\}$. Since $f^{-1}(F)$ is union of neutrosophic open orbit sets. Therefore, $f^{-1}(F)$ is an neutrosophic open orbit set.

(ii) ⇒ (iii) Let F be a neutrosophic clopen chaotic set of (X, σ). Then X − F is also a neutrosophic clopen chaotic set, By (ii) $f^{-1}(X - F)$ is neutrosophic open orbit in (X, τ). So $X - f^{-1}(F)$ is a neutrosophic open orbit set in (X, τ). Hence, $f^{-1}(F)$ is neutrosophic closed orbit in (X, τ). By (ii), $f^{-1}(F)$ is a neutrosophic open orbit set of (X, τ). Therefore, $f^{-1}(F)$ is both neutrosophic open orbit and neutrosophic closed orbit in (X, τ). Hence, $f^{-1}(F)$ is a neutrosophic clopen orbit set of (X, τ).

(iii) ⇒ (i) Let x be a periodic point, x ∈ X and F be a neutrosophic clopen chaotic set containing f(x) then $f^{-1}(F)$ is a neutrosophic open orbit set of (X, τ) containing x and $f(f^{-1}(F)) \subseteq F$. Hence, f is neutrosophic chaotic continuous.

3.22 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be neutrosophic chaotic* continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F containing f(x), ∃ neutrosophic open orbit set $NO_f(x)$ containing x ∍ $f(Ncl(NO_f(x))) \subseteq F$.

3.23 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic* continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ containing x ∍ $f(NO_f(x)) \subseteq F$. Then $f^{-1}(F)$ is a neutrosophic clopen chaotic set of (X, σ). By (iii) of Theorem 3.21., $f^{-1}(F)$ is a neutrosophic clopen orbit set in (X, τ). Therefore, F is a neutrosophic closed chaotic set containing f(x) and $f^{-1}(F)$ is a neutrosophic open orbit set ∍ $f(f^{-1}(F)) \subseteq F$. Since $f^{-1}(F)$ is neutrosophic closed orbit set, $Ncl(f^{-1}(F)) = f^{-1}(F)$. This implies that, $f(Ncl(f^{-1}(F))) \subseteq F$. Hence, f is a neutrosophic chaotic* continuous function.

3.24 Remark

The converse of Theorem 3.23. need not be true as shown in Example 3.25.

3.25 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by M=<x,{p,r},{q,r},{r}>, N=<x,{r},{q},{p,q,r}>, O=<x,{r},{q,r},{p,q,r}>, P=<x,{p,r},{q},{r}>, Q=<x,{p,q,s},{q,s},{p,r}>, R=<x,{q,s},{p,q},{q,r}>, S=<x,{q,s},{p,q,s},{p,q,r}> and T=<x,{p,q,s},{q},{r}>. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R,S,T\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <q,p,s> f(q) = <s,r,p>, f(r) =< p,q,r> and f(s) = <r,s,q>. Now the function f is neutrosophic chaotic* continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic* continuous function need not be neutrosophic chaotic continuous function.

3.26 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be neutrosophic chaotic** continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F of f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ of the periodic point x ∍ $f(NO_f(x)) \subseteq Nint(F)$.

3.27 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic** continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ containing x ∋ $f(NO_f(x)) \subseteq F$. Since F is a neutrosophic open orbit set in (X, σ), F = Nint(F). This implies that, $f(NO_f(x)) \subseteq Nint(F)$. Hence, f is an neutrosophic chaotic** continuous function.

3.28 Remark

The converse of Theorem 3.27 need not be true as shown in the Example 3.29.

3.29 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by M=<x,{q,r},{r},{p,r}>, N=<x,{p,s},{p,q},{p,q}>,O=<x,φ,{p,q,r},{p,q,r}>,P=<x,X,φ,{p}>,Q=<x,{p,q,r},{r},{p,s}>,R=<x,{q},{q,r},{p,r}>, S=<x,{p,q,r},{r},{p}> and T=<x,{q},{r},{p,r,s}>. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R,S,T\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <p,q,s> f(q) = <r,s,r>, f(r) =< q,r,p> and f(s) = <s,p,q>. Now the function f is neutrosophic chaotic** continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic** continuous function need not be neutrosophic chaotic continuous function.

3.30 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be a neutrosophic chaotic*** continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F of f(x) ∃ a neutrosophic clopen orbit set $NO_f(x)$ of the periodic point x ∋ $f(Nint(NO_f(x))) \subseteq F$.

3.31 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic*** continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ containing x ∋ $f(NO_f(x)) \subseteq F$. This implies that, $NO_f(x) \subseteq f^{-1}(F)$. Then, $f^{-1}(F)$ is a neutrosophic clopen chaotic set of (X, σ). By (iii) of Theorem 3.21, $f^{-1}(F)$ is a neutrosophic clopen orbit set in (X, τ). Therefore, F is a neutrosophic closed chaotic set containing f(x) and $f^{-1}(F)$ is a neutrosophic open orbit set ∋ $f(f^{-1}(F)) \subseteq F$. Since $f^{-1}(F)$ is neutrosophic open orbit set, $Nint(f^{-1}(F)) = f^{-1}(F)$. This implies that, $f(Nint(f^{-1}(F))) \subseteq F$. Hence, f is a neutrosophic chaotic*** continuous function.

3.32 Remark

The converse of Theorem 3.31 need not be true as shown in the Example 3.33.

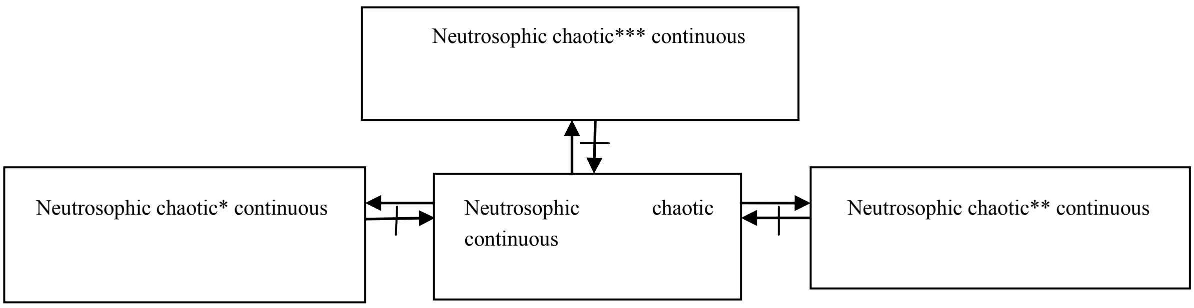
3.33 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by M=<x,{q,r},{r},{p,r}>, N=<x,{p,r},{r},{q,r}>, O=<x,{p,q,r},{r},{r}>, P=<x,{r},{r},{p,q,r}>, Q=<x,{p,q,r},{q,r},{p,s}>, R=<x,{q,r},{p,q},{r,s}>, S=<x, {p,q,r},{r},{p}> and T=<x,{q},{r},{p,r,s}>. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R,S,T\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <p,q,s> f(q) = <r,s,r>, f(r) =< q,r,p> and f(s) = <s,p,q>. Now the function f is neutrosophic chaotic*** continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic*** continuous function need not be neutrosophic chaotic continuous function.

3.34 Remark

The interrelation among the functions introduced are given clearly in the following diagram.

Figure 1:



4 Properties of neutrosophic chaotic continuous functions

4.1 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic orbit extremally disconnected space if the neutrosophic closure of every neutrosophic open orbit set is neutrosophic open orbit.

4.2 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function and (X, τ) is a neutrosophic orbit extremally disconnected space then f is a neutrosophic chaotic* continuous function.

Proof Let x be a periodic point and x ∈ X. Since f is neutrosophic chaotic continuous, F = <x,F^1,F^2,F^3> is a neutrosophic clopen chaotic set of (X, σ), ∃ a neutrosophic open orbit set $NO_f(x)$ of (X, τ) containing x ∋ f($NO_f(x)$) ⊆ F. Therefore, $NO_f(x)$ is a neutrosophic open orbit set $NO_f(x)$ of (X, τ). Since (X, τ) is neutrosophic orbit extremally disconnected, Ncl($NO_f(x)$) is a neutrosophic open orbit set. Therefore, F is a neutrosophic closed chaotic set containing f(x) ∃ a neutrosophic open orbit set Ncl($NO_f(x)$) ∋ f(Ncl($NO_f(x)$)) ⊆ F. Hence, f is neutrosophic chaotic* continuous.

4.3 Definition

A neutrosophic chaos space (X, τ) is said to be neutrosophic chaotic 0- dimensional if it has a neutrosophic base consisting of neutrosophic clopen chaotic sets.

4.4 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. Let f : (X, τ) → (X, σ) be a neutrosophic chaotic*** continuous function. If (X, σ) is neutrosophic chaotic 0-dimensional then f is a neutrosophic chaotic continuous function.

Proof Let the periodic point x ∈ X. Since (X, σ) is neutrosophic chaotic 0-dimensional, ∃ a neutrosophic clopen chaotic set F = <x,F^1,F^2,F^3> in (X, σ). Since f is a neutrosophic chaotic*** continuous function, ∃ a neutrosophic clopen orbit set $NO_f(x)$ ∋ f(Nint($NO_f(x)$)) ⊆ F. Since $NO_f(x)$ is a neutrosophic open orbit set, Nint($NO_f(x)$ = $NO_f(x)$. This implies that, f($NO_f(x)$) ⊆ F. Therefore, f is neutrosophic chaotic continuous.

4.5 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic orbit connected space if X_N cannot be expressed as the union of two neutrosophic open orbit sets $NO_f(x)$ and $NO_f(y)$, x,y ∈ X of (X, τ) with $NO_f(x) \cap NO_f(y) \neq \varphi_N$.

4.6 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic chaotic connected space if X_N cannot be expressed as the union of two neutrosophic open chaotic sets U = <x,U^1,U^2,U^3> and V = <x,V^1,V^2,V^3> of (X, τ) with $U \cap V \neq \varphi_N$.

4.7 Theorem

A neutrosophic chaotic continuous image of a neutrosophic orbit connected space is a neutrosophic chaotic connected space.

Proof Let (X, σ) be neutrosophic chaotic disconnected. Let F_1 = <x, F_1^1, F_1^2, F_1^3> and F_2 = <x, F_2^1, F_2^2, F_2^3> be a neutrosophic chaotic disconnected sets of (X, σ). Then $F_1 \neq \varphi_N$ and $F_2 \neq \varphi_N$ are neutrosophic clopen chaotic sets in (X, σ) and $Y_N = F_1 \cup F_2$ where $F_1 \cap F_2 = \varphi_N$. Now, $X_N = f^{-1}(Y_N) = f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2)$.Since f is neutrosophic chaotic continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are neutrosophic open orbit sets in (X, τ). Also $f^{-1}(F_1) \cap f^{-1}(F_2) = \varphi_N$. Therefore, (X, τ) is not neutrosophic orbit connected. Which is a contradiction. Hence, (X, σ) is neutrosophic chaotic connected.

4.8 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function and $NO_f(x)$ is neutrosophic open orbit set then the restriction f|$NO_f(x)$: $NO_f(x)$ → (X, σ) is neutrosophic chaotic continuous.

Proof Let F = <x,F^1,F^2,F^3> be a neutrosophic clopen chaotic set in (X, σ). Then, $(f|NO_f(x))^{-1}(F) = f^{-1}(F) \cap NO_f(x)$. Since f is neutrosophic chaotic continuous, $f^{-1}(F)$ is neutrosophic open orbit in (X, τ) and $NO_f(x)$ is a neutrosophic open orbit set. This implies that, $f^{-1}(F) \cap NO_f(x)$ is a neutrosophic open orbit set. Therefore, $(f|NO_f(x))^{-1}(F)$ is neutrosophic open orbit in (X, τ). Hence, f|$NO_f(x)$ is neutrosophic chaotic continuous.

4.9 Definition

Let (X, τ) be a neutrosophic chaos space. If a family {$NO_f(x_i)$: i ∈ J} of neutrosophic open orbit set in (X, τ) satisfies the condition $\cup NO_f(x_i) = X_N$, then it is called a neutrosophic open orbit cover of (X, τ).

4.10 Theorem

Let {$NO_f(x)_\gamma$: γ ∈ Γ} be any neutrosophic open orbit cover of a neutrosophic chaos space (X, τ). A function f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function if and only if the restriction f|$NO_f(x)_\gamma$: $NO_f(x)_\gamma$ → (X, σ) is neutrosophic chaotic continuous for each γ ∈ Γ.

Proof Let γ be an arbitrarily fixed index and $NO_f(x)_\gamma$ be a neutrosophic open orbit set of (X, τ). Let the periodic point x ∈ $NO_f(x)_\gamma$ and F = <x,F^1,F^2,F^3> is neutrosophic clopen chaotic set containing $(f|NO_f(x)_\gamma)(x)$ = f(x). Since f is neutrosophic chaotic continuous there exists a neutrosophic open orbit set $NO_f(x)$ containing x such that

$f(NO_f(x)) \subseteq F$. Since $(NO_f(x)_\gamma)$ is neutrosophic open orbit cover in (X, τ), $x \in NO_f(x) \cap NO_f(x)_\gamma$ and $(f|NO_f(x)_\gamma)(NO_f(x) \cap (NO_f(x)_\gamma) = f(NO_f(x) \cap (NO_f(x)_\gamma)) \subset f(NO_f(x) \subset F$. Hence $f|NO_f(x)_\gamma$ is a neutrosophic chaotic continuous function. Conversely, let the periodic point $x \in X$ and F be a neutrosophic chaotic set containing f(x). There exists an $\gamma \in \Gamma$ such that $x \in NO_f(x)_\gamma$. Since $(f|NO_f(x)_\gamma) : NO_f(x)_\gamma \to (X, \sigma)$ is neutrosophic chaotic continuous, there exists a $NO_f(x) \in NO_f(x)_\gamma$ containing x such that $(f|NO_f(x)_\gamma)(NO_f(x)) \subseteq F$. Since $NO_f(x)$ is neutrosophic open orbit in (X, τ), $f(NO_f(x)) \subseteq F$. Hence, f is neutrosophic chaotic continuous.

4.11 Theorem

If a function $f : (X, \tau) \to \Pi (X, \sigma)_\lambda$ is neutrosophic chaotic continuous then $P_\lambda \circ f : (X, \tau) \to (X, \sigma)_\lambda$ is neutrosophic chaotic continuous for each $\lambda \in \Lambda$, where P_λ is the projection of $\Pi (X, \sigma)_\lambda$ onto $(X, \sigma)_\lambda$.

Proof Let $F_\lambda = <x, F_\lambda^1, F_\lambda^2, F_\lambda^3>$ be any neutrosophic clopen chaotic set of $(X, \sigma)_\lambda$. Then $P_\lambda^{-1}(F_\lambda)$ is a neutrosophic clopen chaotic set in $\Pi (X, \sigma)_\lambda$ and hence $(P_\lambda \circ f)^{-1}(F_\lambda) = f^{-1}(P_\lambda^{-1}(F_\lambda))$ is a neutrosophic open orbit set in (X, τ). Therefore, $P_\lambda \circ f$ is neutrosophic chaotic continuous.

4.12 Theorem

If a function $f : \Pi (X, \tau)_\lambda \to \Pi (X, \sigma)_\lambda$ is neutrosophic chaotic continuous then $f_\lambda : (X, \tau)_\lambda \to (X, \sigma)_\lambda$ is a neutrosophic chaotic continuous function for each $\lambda \in \Lambda$.

Proof Let $F_\lambda =<x, F_\lambda^1, F_\lambda^2, F_\lambda^3>$ be any neutrosophic clopen chaotic set of $(X, \sigma)_\lambda$. Then $P_\lambda^{-1}(F_\lambda)$ is neutrosophic clopen chaotic in $\Pi (X, \sigma)_\lambda$ and $f^{-1}(P_\lambda^{-1}(F_\lambda)) = f_\lambda^{-1}(F_\lambda) \times \Pi \{(X, \tau)_\alpha : \alpha \in \Lambda - \{\lambda\}\}$. Since f is neutrosophic chaotic continuous, $f^{-1}(P_\lambda^{-1}(F_\lambda))$ is a neutrosophic open orbit set in $\Pi (X, \tau)_\lambda$. Since the projection P_λ of $\Pi (X, \tau)_\lambda$ onto $(X, \tau)_\lambda$ is a neutrosophic open function, $f_\lambda^{-1}(F_\lambda)$ is neutrosophic open orbit in $(X, \tau)_\lambda$. Hence, f_λ is neutrosophic chaotic continuous.

4.13 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function $f : (X, \tau) \to (X, \sigma)$ is said to be neutrosophic chaotic irresolute if for each neutrosophic clopen chaotic set $F = <x,F^1,F^2,F^3>$ in (X, σ), $f^{-1}(A)$ is a neutrosophic clopen chaotic set of (X, τ).

4.14 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If $f : (X, \tau) \to (X, \sigma)$ is a neutrosophic chaotic continuous function and $g : (X, \sigma) \to (X, \eta)$ is a neutrosophic chaotic irresolute function, then $g \circ f : (X, \tau) \to (X, \eta)$ is neutrosophic chaotic continuous.

Proof Let $F = <x,F^1,F^2,F^3>$ be a neutrosophic clopen set of (X, η). Since g is neutrosophic chaotic irresolute, $g^{-1}(F)$ is neutrosophic clopen chaotic set of (X, σ). Since f is neutrosophic chaotic continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is a neutrosophic open orbit set of (X, τ) such that $f^{-1}(g^{-1}(F)) \subseteq F$. Hence $g \circ f$ is neutrosophic chaotic continuous.

4.15 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function $f : (X, \tau) \to (X, \sigma)$ is said to be neutrosophic orbit irresolute if for each neutrosophic open orbit set $NO_f(x)$ in (X, σ), $f^{-1}(NO_f(x))$ is a neutrosophic open orbit set of (X, τ).

4.16 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. Let $f : (X, \tau) \to (X, \sigma)$ be a function. Then f is said to be a neutrosophic open orbit function if the image of every neutrosophic open orbit set in (X, τ) is neutrosophic open orbit in (X, σ).

4.17 Theorem

Let $f : (X, \tau) \to (X, \sigma)$ be neutrosophic orbit irresolute, surjective and neutrosophic open orbit function. Then $g \circ f : (X, \tau) \to (X, \eta)$ is neutrosophic chaotic continuous iff $g : (X, \sigma) \to (X, \eta)$ is neutrosophic chaotic continuous.

Proof Let $F_\lambda =<x, F_\lambda^1, F_\lambda^2, F_\lambda^3>$ be a neutrosophic clopen chaotic set of (X, η). Since g is neutrosophic chaotic continuous, $g^{-1}(F)$ is neutrosophic open orbit in (X, σ). Since f is neutrosophic orbit irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is neutrosophic open orbit in (X, τ). Hence $g \circ f$ is neutrosophic chaotic continuous. Conversely, let $g \circ f : (X, \tau) \to (X, \eta)$ be neutrosophic chaotic continuous function. Let F be a neutrosophic clopen chaotic set of (X, η), then $(g \circ f)^{-1}(F)$ is a neutrosophic open orbit set of (X, τ). Since f is neutrosophic open orbit and surjective, $f(f^{-1}(g^{-1}(F))$ is a neutrosophic open orbit set of (X, σ). Therefore, $g^{-1}(F)$ is a neutrosophic open orbit set in (X, σ). Hence, g is neutrosophic chaotic continuous.

Conclusion

In this paper, characterization of neutrosophic chaotic continuous functions are studied. Some interrelations are discussed with suitable examples. Also, neutrosophic orbit, extremally disconnected spaces and neutrosophic chaotic zero-dimensional spaces has been discussed with some interesting properties. This paper paves way in future to introduce and study the notions of neutrosophic orbit Co-kernal spaces, neutrosophic hardly open orbit spaces, neutrosophic orbit quasi regular spaces and neutrosophic orbit strongly complete spaces, neutrosophic orbit Co-kernal function, neutrosophic hardly open orbit function for which the above discussed set form the basis.

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Generalised single valued neutrosophic number and its application to neutrosophic linear programming

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Abstract . In this paper, the concept of single valued neutrosophic number (SVN-number) is presented in a generalized way. Using this notion, a crisp linear programming problem (LP-problem) is extended to a neutrosophic linear programming problem (NLP-problem). The coefficients of the objective function of a crisp LP-problem are considered as generalized single valued neutrosophic number (G_{SVN}-number). This modified form of LP-problem is here called an NLP-problem. An algorithm is developed to solve NLP-problem by simplex method. Finally, this simplex algorithm is applied to a real life problem. The problem is illustrated and solved numerically.

Keywords : Single valued neutrosophic number; Neutrosophic linear programming problem; Simplex method.

1 Introduction

Introduction of fuzzy set by Zadeh [10] and then intuitionistic fuzzy set by Atanassov [8] brought a golden opportunity to handle the uncertainty and vagueness in our daily life activities. The fuzzy sets are evaluated by the membership grade of an object only, whereas intuitionistic fuzzy set meets the membership and the non-membership grade of an object simultaneously. To deal with uncertainty more precisely, Smarandache [3,4] initiated the notion of neutrosophic set (NS), a generalised version of classical set, fuzzy set, intuitionistic fuzzy set etc. In the neutrosophic logic, each proposition is estimated by a triplet $viz.$, truth grade, indeterminacy grade and falsity grade. The indeterministic part of uncertain data, introduced in NS theory, plays an important role to make a proper decision which is not possible by intuitionistic fuzzy set theory. Since indeterminacy always appears in our routine activities, the NS theory can analyse the various situations smoothly. But it is too difficult to apply the NS theory in real life scenario for it's initial character as pointed out by Smarandache. So to apply in real spectrum, Wang et al. [6] brought the concept of single valued neutrosophic set (SVN-set). Ranking of fuzzy number and intuitionistic fuzzy number is an interesting subject needed in decision making, optimization, even in developing of various mathematical structures. From time to time, several ranking methods [2,5,9,13-15] have been adopted by researchers. Naturally, the ranking of neutrosophic number also was come into consideration from beginning of NS theory. Deli and Subas [7] considered a ranking way of neutrosophic numbers and have used it to a decision making problems. Abdel-Baset [11,12] solved group decision making problems based on TOPSIS technique by use of neutrosophic number. To estimate and solve the NLP-problem in different direction, some respective attempts [1,16] by researchers are seen.

This paper introduces the structure of SVN-number in a different way to opt the notion of generalized single valued trapezoidal neutrosophic number (G_{SVTN}-number), generalized single valued triangular neutrosophic number (G_{SVTrN}-number) and develops an algorithm to solve NLP-problem by simplex method. The proposed simplex algorithm is applied to a real life problem. The problem is illustrated and solved numerically.

The organisation of this paper is as follows. Section 2 deals some preliminary definitions. The concept of G_{SVN}-number, G_{SVTN}-number, G_{SVTrN}-number and their respective parametric form are presented in Section 3. The concept of NLP-problem and it's solution procedure are proposed in Section 4 and Section 5, respectively. In Section 6, the simplex method is illustrated by suitable examples. Finally, the present work is summarised in Section 7.

2 Preliminaries

Some basic definitions are provided to bring the main thought of this paper here.

2.1 Definition [18]

A continuous t- norm $*$ and t- conorm $\diamond$ are two continuous binary operations assigning $[0,1]\times[0,1]\rightarrow[0,1]$ and obey the under stated principles :
(i) $*$ and $\diamond$ are both commutative and associative.
(ii) $x*1=1*x=x$ and $x\diamond 0=0\diamond x=x,\ \forall x\in[0,1]$.
(iii) $x*y\leq p*q$ and $x\diamond y\leq p\diamond q$ if $x\leq p,\ y\leq q$ with $x,y,p,q\in[0,1]$.
$x*y=xy, x*y=\min\{x,y\}, x*y=\max\{x+y-1,0\}$ are most useful t-norms and
$x\diamond y=x+y-xy, x\diamond y=\max\{x,y\}, x\diamond y=\min\{x+y,1\}$ are most useful t-conorms.

2.2 Definition [3]

An NS Q on an initial universe X is presented by three characterisations namely true value T_Q, indeterminant value I_Q and false value F_Q so that $T_Q, I_Q, F_Q\ :\ X\ \rightarrow]^-0,1^+[$. Thus Q can be designed as : $\{<u,(T_Q(u),I_Q(u),F_Q(u))>: u\in X\}$ with $^-0\leq\sup T_Q(u)+\sup I_Q(u)+\sup F_Q(u)\leq 3^+$. Here $1^+=1+\delta$, where 1 is standard part and δ is non-standard part. Similarly $^-0=0-\delta$. The non-standard set $]^-0,1^+[$ is basically practiced in philosophical ground and because of the difficulty to adopt it in real field, the standard subset of $]^-0,1^+[$ i.e., [0,1] is applicable in real neutrosophic environment.

2.3 Definition [6]

An SVN-set Q over a universe X is a set $Q=\{<x,T_Q(x),I_Q(x),F_Q(x)>: x\in X$ and $T_Q(x),I_Q(x)$, $F_Q(x)\in[0,1]\}$ with $0\leq\sup T_Q(x)+\sup I_Q(x)+\sup F_Q(x)\leq 3$.

2.4 Definition [7]

Let $a_i,b_i,c_i,d_i\in\mathbf{R}$ (the set of all real numbers) with $a_i\leq b_i\leq c_i\leq d_i\ (i=1,2,3)$ and $w_{\tilde{p}},u_{\tilde{p}},y_{\tilde{p}}\in[0,1]\subset\mathbf{R}$. Then an SVN-number $\tilde{p}=\langle([a_1,b_1,c_1,d_1];w_{\tilde{p}}),([a_2,b_2,c_2,d_2];u_{\tilde{p}}),([a_3,b_3,c_3,d_3];y_{\tilde{p}})\rangle$ is a special SVN-set on $\mathbf{R}$ whose true value, indeterminant value, false value are respectively defined by the mappings $T_{\tilde{p}}:\mathbf{R}\rightarrow[0,w_{\tilde{p}}]$, $I_{\tilde{p}}:\mathbf{R}\rightarrow[u_{\tilde{p}},1]$, $F_{\tilde{p}}:\mathbf{R}\rightarrow[y_{\tilde{p}},1]$ and they are given as :

$$T_{\tilde{p}}(x)=\begin{cases} g_T^l(x), & a_1\leq x\leq b_1,\\ w_{\tilde{p}}, & b_1\leq x\leq c_1,\\ g_T^r(x), & c_1\leq x\leq d_1,\\ 0, & \text{otherwise.}\end{cases}\quad I_{\tilde{p}}(x)=\begin{cases} g_I^l(x), & a_2\leq x\leq b_2,\\ u_{\tilde{p}}, & b_2\leq x\leq c_2,\\ g_I^r(x), & c_2\leq x\leq d_2,\\ 1, & \text{otherwise.}\end{cases}\quad F_{\tilde{p}}(x)=\begin{cases} g_F^l(x), & a_3\leq x\leq b_3,\\ y_{\tilde{p}}, & b_3\leq x\leq c_3,\\ g_F^r(x), & c_3\leq x\leq d_3,\\ 1, & \text{otherwise.}\end{cases}$$

The functions $g_T^l:[a_1,b_1]\rightarrow[0,w_{\tilde{p}}], g_I^r:[c_2,d_2]\rightarrow[u_{\tilde{p}},1], g_F^r:[c_3,d_3]\rightarrow[y_{\tilde{p}},1]$ are continuous and non-decreasing functions satisfying : $g_T^l(a_1)=0, g_T^l(b_1)=w_{\tilde{p}}, g_I^r(c_2)=u_{\tilde{p}}, g_I^r(d_2)=1, g_F^r(c_3)=y_{\tilde{p}}, g_F^r(d_3)=1$.

The functions $g_T^r : [c_1, d_1] \to [0, w_{\tilde{p}}], g_I^l : [a_2, b_2] \to [u_{\tilde{p}}, 1], g_F^l : [a_3, b_3] \to [y_{\tilde{p}}, 1]$ are continuous and non-increasing functions satisfying : $g_T^r(c_1) = w_{\tilde{p}}, g_T^r(d_1) = 0, g_I^l(a_2) = 1, g_I^l(b_2) = u_{\tilde{p}}, g_F^l(a_3) = 1, g_F^l(b_3) = y_{\tilde{p}}$.

2.4.1 Definition [7]

If $[a_1, b_1, c_1, d_1] = [a_2, b_2, c_2, d_2] = [a_3, b_3, c_3, d_3]$, then the SVN-number $\tilde{p}$ is reduced to a single valued trapezoidal neutrosophic number as : $\tilde{p} = \langle([a_1, b_1, c_1, d_1]; w_{\tilde{p}}, u_{\tilde{p}}, y_{\tilde{p}})\rangle$.
$\tilde{p} = \langle([a_1, b_1, d_1]; w_{\tilde{p}}, u_{\tilde{p}}, y_{\tilde{p}})\rangle$ is called a single valued triangular neutrosophic number if $b_1 = c_1$.

2.5 Definition [17]

The (α, β, γ)-cut of an NS P is denoted by $P_{(\alpha,\beta,\gamma)}$ and is defined as : $P_{(\alpha,\beta,\gamma)} = \{x \in X : T_P(x) \geq \alpha, I_P(x) \leq \beta, F_P(x) \leq \gamma\}$ with $\alpha, \beta, \gamma \in [0, 1]$ and $0 \leq \alpha + \beta + \gamma \leq 3$. Clearly, it is a crisp subset X.

2.6 Definition [14]

In parametric form, a fuzzy number P is a pair (P_L, P_R) of functions $P_L(r), P_R(r), r \in [0, 1]$ satisfying the followings.
(i) Both are bounded functions.
(ii) P_L is monotone increasing left continuous and P_R is monotone decreasing right continuous function.
(iii) $P_L(r) \leq P_R(r), 0 \leq r \leq 1$.
A trapezoidal fuzzy number is put as $P = (x_0, y_0, \delta, \zeta)$ where $[x_0, y_0]$ is interval defuzzifier and $\delta(> 0), \zeta(> 0)$ are respectively called left fuzziness, right fuzziness. $(x_0 - \delta, y_0 + \zeta)$ is the support of P and it's membership function is :

$$P(x) = \begin{cases} \frac{1}{\delta}(x - x_0 + \delta), \; x_0 - \delta \leq x \leq x_0, \\ 1, \quad\quad x \in [x_0, y_0], \\ \frac{1}{\zeta}(y_0 - x + \zeta), \; y_0 \leq x \leq y_0 + \zeta, \\ 0, \quad\quad \text{otherwise.} \end{cases}$$

In parametric form $P_L(r) = x_0 - \delta + \delta r, \; P_R(r) = y_0 + \zeta - \zeta r$.
For arbitrary trapezoidal fuzzy numbers $P = (P_L, P_R), Q = (Q_L, Q_R)$ and scalar $k > 0$, the addition and scalar multiplication are $P + Q, kQ$ and they are defined by :
$(P + Q)_L(r) = P_L(r) + Q_L(r), \; (P + Q)_R(r) = P_R(r) + Q_R(r)$ and
$(kQ)_L(r) = kQ_L(r), \; (kQ)_R(r) = kQ_R(r)$.

3 Generalised single valued neutrosophic number

Here, the structure of G_{SVN}-number, G_{SVTN}-number and G_{SVTrN}-number have been presented.

3.1 Definition

• The support of three components of an SVN-set Q over X are given by a triplet $(S_{Q_T}, S_{Q_I}, S_{Q_F})$ where $S_{Q_T} = \{u \in X | T_Q(u) > 0\}, S_{Q_I} = \{u \in X | I_Q(u) < 1\}, S_{Q_F} = \{u \in X | F_Q(u) < 1\}$.
• The height of the components of Q are given by a triplet $(H_{Q_T}, H_{Q_I}, H_{Q_F})$ where $H_{Q_T} = \max\{T_Q(u) | u \in X\}, H_{Q_I} = \max\{I_Q(u) | u \in X\}, H_{Q_F} = \max\{F_Q(u) | u \in X\}$.

3.1.1 *Example*

Define an SVN-set Q on $\{0, 1, \cdots, 10\} \subset \mathbf{Z}$ (the set of integers) as : $\{< u, (\frac{u}{1+u}, 1-\frac{1}{2^u}, \frac{1}{1+u}) > | 0 \le u \le 10\}$. Then $S_{Q_T} = \{1, \cdots, 10\}$, $S_{Q_I} = \{0, \cdots, 10\}$, $S_{Q_F} = \{1, \cdots, 10\}$ and $H_{Q_T} = 0.909$ at $u = 10$, $H_{Q_I} = 0.999$ at $u = 10$, $H_{Q_F} = 1$ at $u = 0$.

3.2 Definition

A G_{SVN}-number $\tilde{p} = \langle([a_1, b_1, \sigma_1, \eta_1]; w_{\tilde{p}}), ([a_2, b_2, \sigma_2, \eta_2]; u_{\tilde{p}}), ([a_3, b_3, \sigma_3, \eta_3]; y_{\tilde{p}})\rangle$ is a special SVN-set on $\mathbf{R}$ where $\sigma_i(> 0), \eta_i(> 0)$ are respectively called left spreads, right spreads and $[a_i, b_i]$ are the modal intervals of truth, indeterminacy and falsity functions for $i = 1, 2, 3$ respectively in $\tilde{p}$ and $w_{\tilde{p}}, u_{\tilde{p}}, y_{\tilde{p}} \in [0, 1] \subset \mathbf{R}$. The truth, indeterminacy and falsity functions are defined as follows :

$$T_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_1} w_{\tilde{p}}(x - a_1 + \sigma_1), & a_1 - \sigma_1 \le x \le a_1, \\ w_{\tilde{p}}, & x \in [a_1, b_1], \\ \frac{1}{\eta_1} w_{\tilde{p}}(b_1 - x + \eta_1), & b_1 \le x \le b_1 + \eta_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_2}(a_2 - x + u_{\tilde{p}}(x - a_2 + \sigma_2)), & a_2 - \sigma_2 \le x \le a_2, \\ u_{\tilde{p}}, & x \in [a_2, b_2], \\ \frac{1}{\eta_2}(x - b_2 + u_{\tilde{p}}(b_2 - x + \eta_2)), & b_2 \le x \le b_2 + \eta_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$F_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_3}(a_3 - x + y_{\tilde{p}}(x - a_3 + \sigma_3)), & a_3 - \sigma_3 \le x \le a_3, \\ y_{\tilde{p}}, & x \in [a_3, b_3], \\ \frac{1}{\eta_3}(x - b_3 + y_{\tilde{p}}(b_3 - x + \eta_3)), & b_3 \le x \le b_3 + \eta_3, \\ 1, & \text{otherwise.} \end{cases}$$

In parametric form, a G_{SVN}-number $\tilde{p}$ consists of three pairs $(T^l_{\tilde{p}}, T^u_{\tilde{p}}), (I^l_{\tilde{p}}, I^u_{\tilde{p}}), (F^l_{\tilde{p}}, F^u_{\tilde{p}})$ of functions $T^l_{\tilde{p}}(r), T^u_{\tilde{p}}(r), I^l_{\tilde{p}}(r), I^u_{\tilde{p}}(r), F^l_{\tilde{p}}(r), F^u_{\tilde{p}}(r), r \in [0, 1]$ satisfying the followings.
(i) $T^l_{\tilde{p}}, I^u_{\tilde{p}}, F^u_{\tilde{a}}$ are bounded monotone increasing continuous function.
(ii) $T^u_{\tilde{p}}, I^l_{\tilde{p}}, F^l_{\tilde{a}}$ are bounded monotone decreasing continuous function.
(iii) $T^l_{\tilde{p}}(r) \le T^u_{\tilde{p}}(r), I^l_{\tilde{p}}(r) \ge I^u_{\tilde{p}}(r), F^l_{\tilde{p}}(r) \ge F^u_{\tilde{p}}(r), r \in [0, 1]$.

3.2.1 Definition

• The support of the components of a G_{SVN}-number $\tilde{p}$ are given by a triplet $(S_{P_T}, S_{P_I}, S_{P_F})$ where $S_{P_T} = \{x \in \mathbf{R} | T_{\tilde{p}}(x) > 0\}, S_{P_I} = \{x \in \mathbf{R} | I_{\tilde{p}}(x) < 1\}, S_{P_F} = \{x \in \mathbf{R} | F_{\tilde{p}}(x) < 1\}$.

• The height of the components of $\tilde{p}$ are given by a triplet $(H_{P_T}, H_{P_I}, H_{P_F})$ where $H\tilde{p}_T = w_{\tilde{p}}, H\tilde{p}_I = 1 - u_{\tilde{p}}, H\tilde{p}_F = 1 - y_{\tilde{p}}$.

• The boundaries of the truth function of $\tilde{p}$ are : $LB_{\tilde{p}_T} = (a_1 - \sigma_1, a_1)$ and $RB_{\tilde{p}_T} = (b_1, b_1 + \eta_1)$. $LB_{\tilde{p}_T}$ and $RB_{\tilde{p}_T}$ are respectively called left boundary and right boundary for truth function of $\tilde{p}$. Similarly, $LB_{\tilde{p}_I} = (a_2 - \sigma_2, a_2)$, $RB_{\tilde{p}_I} = (b_2, b_2 + \eta_2)$ and $LB_{\tilde{p}_F} = (a_3 - \sigma_3, a_3)$, $RB_{\tilde{p}_F} = (b_3, b_3 + \eta_3)$.

• The core for the truth function of $\tilde{p}$ is a set of points at which it's height is measured. Similarly, the core for other two components are defined.

3.2.2 *Example*

Consider a G_{SVN}-number $\tilde{p}$ on **R** whose three components are as follows :

$$T_{\tilde{p}}(x)=\begin{cases}\frac{0.6(x-11)}{4}, & x\in[11,15]\\ 0.6, & x\in[15,25]\\ \frac{0.6(36-x)}{11}, & x\in[25,36]\\ 0, & \text{otherwise.}\end{cases}\quad I_{\tilde{p}}(x)=\begin{cases}\frac{4.4-0.1x}{4}, & x\in[4,8]\\ 0.9, & x\in[8,13]\\ \frac{0.1x+5}{7}, & x\in[13,20]\\ 1, & \text{otherwise.}\end{cases}\quad F_{\tilde{p}}(x)=\begin{cases}\frac{26-x}{3}, & x\in[23,26]\\ 0, & x\in[26,30]\\ \frac{x-30}{8}, & x\in[30,38]\\ 1, & \text{otherwise.}\end{cases}$$

Then $S_{P_T}=(11,36)$, $S_{P_I}=(4,20)$ and $S_{P_F}=(23,38)$.
For that $\tilde{p}$, $H\tilde{p}_T=0.6$, $H\tilde{p}_I=0.1$, $H\tilde{p}_F=1$. Here,
$LB_{\tilde{p}_T}=(11,15), RB_{\tilde{p}_T}=(25,36)$; $LB_{\tilde{p}_I}=(4,8), RB_{\tilde{p}_I}=(13,20)$; $LB_{\tilde{p}_F}=(23,26), RB_{\tilde{p}_F}=(30,38)$.
The core of truth, indeterminacy and falsity function are $[15,25]$, $[8,13]$, $[26,30]$ respectively.

3.3 Definition

Let us assume two G_{SVN}-numbers $\tilde{p}$ and $\tilde{q}$ as follows :
$\tilde{p}=\langle([a_1,a_1',\sigma_1,\eta_1];w_{\tilde{p}}),([a_2,a_2',\sigma_2,\eta_2];u_{\tilde{p}}),([a_3,a_3',\sigma_3,\eta_3];y_{\tilde{p}})\rangle$,
$\tilde{q}=\langle([b_1,b_1',\xi_1,\delta_1];w_{\tilde{q}}),([b_2,b_2',\xi_2,\delta_2];u_{\tilde{q}}),([b_3,b_3',\xi_3,\delta_3];y_{\tilde{q}})\rangle$.
Then for any real number x,

(i) Image of $\tilde{p}$:
$-\tilde{p}=\langle([-a_1',-a_1,\eta_1,\sigma_1];w_{\tilde{p}}),([-a_2',-a_2,\eta_2,\sigma_2];u_{\tilde{p}}),([-a_3',-a_3,\eta_3,\sigma_3];y_{\tilde{p}})\rangle$.

(ii) Addition :
$\tilde{p}+\tilde{q}=\langle([a_1+b_1,a_1'+b_1',\sigma_1+\xi_1,\eta_1+\delta_1];w_{\tilde{p}}*w_{\tilde{q}}),([a_2+b_2,a_2'+b_2',\sigma_2+\xi_2,\eta_2+\delta_2];u_{\tilde{p}}\diamond u_{\tilde{q}}),$
$([a_3+b_3,a_3'+b_3',\sigma_3+\xi_3,\eta_3+\delta_3];y_{\tilde{p}}\diamond y_{\tilde{q}})\rangle$.

(iii) Scalar multiplication :
$x\tilde{p}=\langle([xa_1,xa_1',x\sigma_1,x\eta_1];w_{\tilde{p}}),([xa_2,xa_2',x\sigma_2,x\eta_2];u_{\tilde{p}}),([xa_3,xa_3',x\sigma_3,x\eta_3];y_{\tilde{p}})\rangle$
for $x>0$.
$x\tilde{p}=\langle([xa_1',xa_1,-x\eta_1,-x\sigma_1];w_{\tilde{p}}),([xa_2',xa_2,-x\eta_2,-x\sigma_2];u_{\tilde{p}}),([xa_3',xa_3,-x\eta_3,-x\sigma_3];y_{\tilde{p}})\rangle$
for $x<0$.

3.4 Corollary

Let $\tilde{p}=\langle([a_1,b_1,\sigma_1,\eta_1];w_{\tilde{p}}),([a_2,b_2,\sigma_2,\eta_2];u_{\tilde{p}}),([a_3,b_3,\sigma_3,\eta_3];y_{\tilde{p}})\rangle$ be an G_{SVN}-number.
1. Any α-cut set of the G_{SVN}-number $\tilde{p}$ for truth function is denoted by $\tilde{p}_\alpha$ and is given by a closed interval as :

$$\tilde{p}_\alpha=[L_{\tilde{p}}(\alpha),R_{\tilde{p}}(\alpha)]=[a_1-\sigma_1+\frac{\sigma_1\alpha}{w_{\tilde{p}}},b_1+\eta_1-\frac{\eta_1\alpha}{w_{\tilde{p}}}],\quad \text{for } \alpha\in[0,w_{\tilde{p}}].$$

The value of $\tilde{p}$ corresponding α-cut set is denoted by $V_T(\tilde{p})$ and is calculated as :

$$\begin{aligned}V_T(\tilde{p}) &= \int_0^{w_{\tilde{p}}}[(a_1-\sigma_1+\frac{\sigma_1\alpha}{w_{\tilde{p}}})+(b_1+\eta_1-\frac{\eta_1\alpha}{w_{\tilde{p}}})]\alpha\,d\alpha\\ &= \int_0^{w_{\tilde{p}}}[a_1+b_1+\eta_1-\sigma_1-\frac{(\eta_1-\sigma_1)\alpha}{w_{\tilde{p}}}]\alpha\,d\alpha\\ &= \frac{1}{6}(3a_1+3b_1-\sigma_1+\eta_1)w_{\tilde{p}}^2.\end{aligned}$$

2. Any β- cut set of the G_{SVN}-number $\tilde{p}$ for indeterminacy membership function is denoted by $\tilde{p}^{\beta}$ and is given by a closed interval as :

$$\begin{aligned}\tilde{p}^{\beta} &= [L'_{\tilde{p}}(\beta), R'_{\tilde{p}}(\beta)]\\ &= [\frac{(u_{\tilde{p}}-\beta)\sigma_2+(1-u_{\tilde{p}})a_2}{1-u_{\tilde{p}}}, \frac{(\beta-u_{\tilde{p}})\eta_2+(1-u_{\tilde{p}})b_2}{1-u_{\tilde{p}}}], \quad \text{for } \beta\in[u_{\tilde{p}},1].\end{aligned}$$

The value of $\tilde{p}$ corresponding β- cut set is denoted by $V_I(\tilde{p})$ and is calculated as :

$$\begin{aligned}V_I(\tilde{p}) &= \int_{u_{\tilde{p}}}^{1}[\frac{(u_{\tilde{p}}-\beta)\sigma_2+(1-u_{\tilde{p}})a_2}{1-u_{\tilde{p}}}+\frac{(\beta-u_{\tilde{p}})\eta_2+(1-u_{\tilde{p}})b_2}{1-u_{\tilde{p}}}](1-\beta)\,d\beta\\ &= \int_{u_{\tilde{p}}}^{1}[a_2+b_2-\sigma_2+\eta_2+\frac{(\sigma_2-\eta_2)(1-\beta)}{1-u_{\tilde{p}}}](1-\beta)\,d\beta\\ &= \frac{1}{6}(3a_2+3b_2-\sigma_2+\eta_2)(1-u_{\tilde{p}})^2.\end{aligned}$$

3. Any γ-cut set of the G_{SVN}-number $\tilde{p}$ for falsity membership function is denoted by ${}^{\gamma}\tilde{p}$ and is given by a closed interval as :

$$\begin{aligned}{}^{\gamma}\tilde{p} &= [L''_{\tilde{p}}(\gamma), R''_{\tilde{p}}(\gamma)]\\ &= [\frac{(u_{\tilde{p}}-\gamma)\sigma_3+(1-y_{\tilde{p}})a_3}{1-y_{\tilde{p}}}, \frac{(\gamma-y_{\tilde{p}})\eta_3+(1-y_{\tilde{p}})b_3}{1-y_{\tilde{p}}}], \quad \text{for } \gamma\in[y_{\tilde{p}},1].\end{aligned}$$

The value of $\tilde{p}$ corresponding γ-cut set is denoted by $V_F(\tilde{p})$ and is calculated as :

$$\begin{aligned}V_F(\tilde{p}) &= \int_{y_{\tilde{p}}}^{1}[\frac{(u_{\tilde{p}}-\gamma)\sigma_3+(1-y_{\tilde{p}})a_3}{1-y_{\tilde{p}}}+\frac{(\gamma-y_{\tilde{p}})\eta_3+(1-y_{\tilde{p}})b_3}{1-y_{\tilde{p}}}](1-\gamma)\,d\gamma\\ &= \int_{y_{\tilde{p}}}^{1}[a_3+b_3-\sigma_3+\eta_3+\frac{(\sigma_3-\eta_3)(1-\gamma)}{1-y_{\tilde{p}}}](1-\gamma)\,d\gamma\\ &= \frac{1}{6}(3a_3+3b_3-\sigma_3+\eta_3)(1-y_{\tilde{p}})^2.\end{aligned}$$

3.5 Definition

For $\kappa\in[0,1]$, the κ-weighted value of an G_{SVN}-number $\tilde{b}$ is denoted by $V_{\kappa}(\tilde{b})$ and is defined as :
$V_{\kappa}(\tilde{b})=\kappa^n V_T(\tilde{b})+(1-\kappa^n)V_I(\tilde{b})+(1-\kappa^n)V_F(\tilde{b}),\ n$ being any natural number.

Thus, the κ - weighted value for the G_{SVN}- number $\tilde{p}$ defined in Corollary 3.4 is :

$$\begin{aligned}V_{\kappa}(\tilde{p}) &= \frac{1}{6}[(3a_1+3b_1-\sigma_1+\eta_1)\kappa^n w_{\tilde{p}}^2+(3a_2+3b_2-\sigma_2+\eta_2)(1-\kappa^n)(1-u_{\tilde{p}})^2\\ &\quad +(3a_3+3b_3-\sigma_3+\eta_3)(1-\kappa^n)(1-y_{\tilde{p}})^2].\end{aligned}$$

3.5.1 Property of κ - weighted value function

The κ- weighted value $V_{\kappa}(\tilde{p})$ and $V_{\kappa}(\tilde{q})$ of two G_{SVN}-numbers $\tilde{p},\tilde{q}$ respectively obey the followings.
(i) $V_{\kappa}(\tilde{p}\pm\tilde{q})\leq V_{\kappa}(\tilde{p})+V_{\kappa}(\tilde{q}),\ V_{\kappa}(\tilde{p}+\tilde{q})\geq V_{\kappa}(\tilde{p})\sim V_{\kappa}(\tilde{q})$.
(ii) $V_{\kappa}(\tilde{p}-\tilde{p})=V_{\kappa}(\tilde{0}),\ V_{\kappa}(\mu\tilde{p})=\mu V_{\kappa}(\tilde{p}))$ for μ being any real number.

(iii) $V_\kappa(\tilde{p})$ is monotone increasing or decreasing or constant according as $V_T(\tilde{p}) > V_I(\tilde{p}) + V_F(\tilde{p})$ or $V_T(\tilde{p}) < V_I(\tilde{p}) + V_F(\tilde{p})$ or $V_T(\tilde{p}) = V_I(\tilde{p}) + V_F(\tilde{p})$ respectively.

Proof. We shall here prove (vi) only. Others can be easily verified by taking any two G_{SVN}-numbers. Here,

$$\begin{aligned} V_\kappa(\tilde{p}) &= \kappa^n V_T(\tilde{p}) + (1-\kappa^n)(V_I(\tilde{p}) + V_F(\tilde{p})) \\ \frac{dV_\kappa(\tilde{p})}{d\kappa} &= n\kappa^{n-1}[V_T(\tilde{p}) - (V_I(\tilde{p}) + V_F(\tilde{p}))] \end{aligned}$$

As $\kappa \in [0,1]$, so $\frac{dV_\kappa(\tilde{p})}{d\kappa} >, <, = 0$ for $[V_T(\tilde{p}) - (V_I(\tilde{p}) + V_F(\tilde{p}))] >, <, = 0$ respectively. This clears the fact.

3.6 Definition

Let $G_{SVN}(\mathbf{R})$ be the set of all G_{SVN}-numbers defined over $\mathbf{R}$. For $\kappa \in [0,1]$, a mapping $\Re_\kappa : G_{SVN}(\mathbf{R}) \longrightarrow \mathbf{R}$ is called a ranking function and it is defined as : $\Re_\kappa(\tilde{a}) = V_\kappa(\tilde{a})$ for $\tilde{a} \in G_{SVN}(\mathbf{R})$.

For $\tilde{a}, \tilde{b} \in G_{SVN}(\mathbf{R})$, their ranking is defined as :

$$\tilde{a} >_{\Re_\kappa} \tilde{b} \text{ iff } \Re_\kappa(\tilde{a}) > \Re_\kappa(\tilde{b}),\ \tilde{a} <_{\Re_\kappa} \tilde{b} \text{ iff } \Re_\kappa(\tilde{a}) < \Re_\kappa(\tilde{b}),\ \tilde{a} =_{\Re_\kappa} \tilde{b} \text{ iff } \Re_\kappa(\tilde{a}) = \Re_\kappa(\tilde{b}).$$

3.7 Definition

An G_{SVN}-number $\tilde{p}$ is called a G_{SVTN}-number if three modal intervals in $\tilde{p}$ are equal. Thus $\tilde{p} = \langle([a_0, b_0, \sigma_1, \eta_1]; w_{\tilde{p}}), ([a_0, b_0, \sigma_2, \eta_2]; u_{\tilde{p}}), ([a_0, b_0, \sigma_3, \eta_3]; y_{\tilde{p}})\rangle$ is an G_{SVTN}-number whose truth, indeterminacy and falsity functions are as follows :

$$T_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_1} w_{\tilde{p}}(x - a_0 + \sigma_1), & a_0 - \sigma_1 \le x \le a_0, \\ w_{\tilde{p}}, & x \in [a_0, b_0], \\ \frac{1}{\eta_1} w_{\tilde{p}}(b_0 - x + \eta_1), & b_0 \le x \le b_0 + \eta_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_2}(a_0 - x + u_{\tilde{p}}(x - a_0 + \sigma_2)), & a_0 - \sigma_2 \le x \le a_0, \\ u_{\tilde{p}}, & x \in [a_0, b_0], \\ \frac{1}{\eta_2}(x - b_0 + u_{\tilde{p}}(b_0 - x + \eta_2)), & b_0 \le x \le b_0 + \eta_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$F_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_3}(a_0 - x + y_{\tilde{p}}(x - a_0 + \sigma_3)), & a_0 - \sigma_3 \le x \le a_0, \\ y_{\tilde{p}}, & x \in [a_0, b_0], \\ \frac{1}{\eta_3}(x - b_0 + y_{\tilde{p}}(b_0 - x + \eta_3)), & b_0 \le x \le b_0 + \eta_3, \\ 1, & \text{otherwise.} \end{cases}$$

In parametric form for $r \in [0,1]$:

$$T^l_{\tilde{p}}(r) = a_0 - \sigma_1 + \frac{\sigma_1 r}{w_{\tilde{p}}},\ \ T^u_{\tilde{p}}(r) = b_0 + \eta_1 - \frac{\eta_1 r}{w_{\tilde{p}}};$$
$$I^l_{\tilde{p}}(r) = \frac{(1-u_{\tilde{p}})a_0 + (u_{\tilde{p}} - r)\sigma_2}{1 - u_{\tilde{p}}},\ I^u_{\tilde{p}}(r) = \frac{(1-u_{\tilde{p}})b_0 + (r - u_{\tilde{p}})\eta_2}{1 - u_{\tilde{p}}};$$
$$F^l_{\tilde{p}}(r) = \frac{(1-y_{\tilde{p}})a_0 + (y_{\tilde{p}} - r)\sigma_3}{1 - y_{\tilde{p}}},\ F^u_{\tilde{p}}(r) = \frac{(1-y_{\tilde{p}})b_0 + (r - y_{\tilde{p}})\eta_3}{1 - y_{\tilde{p}}}.$$

3.8 Definition

A G_{SVTN}-number $\tilde{p}$ is called a G_{SVTrN}-number if the modal interval in $\tilde{p}$ is reduced to a modal point. Thus $\tilde{p} = \langle([a_0, \sigma_1, \eta_1]; w_{\tilde{p}}), ([a_0, \sigma_2, \eta_2]; u_{\tilde{p}}), ([a_0, \sigma_3, \eta_3]; y_{\tilde{p}})\rangle$ is a G_{SVTrN}-number whose truth, indeterminacy and falsity functions are as follows :

$$T_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_1} w_{\tilde{p}}(x - a_0 + \sigma_1), & a_0 - \sigma_1 \le x \le a_0, \\ w_{\tilde{p}}, & x = a_0, \\ \frac{1}{\eta_1} w_{\tilde{p}}(a_0 - x + \eta_1), & a_0 \le x \le a_0 + \eta_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_2}(a_0 - x + u_{\tilde{p}}(x - a_0 + \sigma_2)), & a_0 - \sigma_2 \le x \le a_0, \\ u_{\tilde{p}}, & x = a_0, \\ \frac{1}{\eta_2}(x - a_0 + u_{\tilde{p}}(a_0 - x + \eta_2)), & a_0 \le x \le a_0 + \eta_2, \\ 1, & \text{otherwise.} \end{cases}$$

$$F_{\tilde{p}}(x) = \begin{cases} \frac{1}{\sigma_3}(a_0 - x + y_{\tilde{p}}(x - a_0 + \sigma_3)), & a_0 - \sigma_3 \le x \le a_0, \\ y_{\tilde{p}}, & x = a_0, \\ \frac{1}{\eta_3}(x - a_0 + y_{\tilde{p}}(a_0 - x + \eta_3)), & a_0 \le x \le a_0 + \eta_3, \\ 1, & \text{otherwise.} \end{cases}$$

3.8.1 Definition

Let $\tilde{a}$ and $\tilde{b}$ be two G_{SVTrN}-numbers as follows :

$\tilde{a} = \langle([a, \sigma_1, \eta_1]; w_{\tilde{a}}), ([a, \sigma_2, \eta_2]; u_{\tilde{a}}), ([a, \sigma_3, \eta_3]; y_{\tilde{a}})\rangle$,

$\tilde{b} = \langle([b, \xi_1, \delta_1]; w_{\tilde{b}}), ([b, \xi_2, \delta_2]; u_{\tilde{b}}), ([b, \xi_3, \delta_3]; y_{\tilde{b}})\rangle$.

Then for any real number x,

(i) Image of $\tilde{a}$:

$-\tilde{a} = \langle([-a, \eta_1, \sigma_1]; w_{\tilde{a}}), ([-a, \eta_2, \sigma_2]; u_{\tilde{a}}), ([-a, \eta_3, \sigma_3]; y_{\tilde{a}})\rangle$.

(ii) Addition :

$\tilde{a} + \tilde{b} = \langle([a + b, \sigma_1 + \xi_1, \eta_1 + \delta_1]; w_{\tilde{a}} * w_{\tilde{b}}), ([a + b, \sigma_2 + \xi_2, \eta_2 + \delta_2]; u_{\tilde{a}} \diamond u_{\tilde{b}}),$
$([a + b, \sigma_3 + \xi_3, \eta_3 + \delta_3]; y_{\tilde{a}} \diamond y_{\tilde{b}})\rangle$.

(iii) Scalar multiplication :

$x\tilde{a} = \langle([xa, x\sigma_1, x\eta_1]; w_{\tilde{a}}), ([xa, x\sigma_2, x\eta_2]; u_{\tilde{a}}), ([xa, x\sigma_3, x\eta_3]; y_{\tilde{a}})\rangle$ for $x > 0$.

$x\tilde{a} = \langle([xa, -x\eta_1, -x\sigma_1]; w_{\tilde{a}}), ([xa, -x\eta_2, -x\sigma_2]; u_{\tilde{a}}), ([xa, -x\eta_3, -x\sigma_3]; y_{\tilde{a}})\rangle$ for $x < 0$.

(iv) The κ - weighted value $V_\kappa(\tilde{a})$ of $\tilde{a}$ is given as :

$$V_\kappa(\tilde{a}) = \frac{1}{6}[(6a - \sigma_1 + \eta_1)\kappa^n w_{\tilde{a}}^2 + \{(6a - \sigma_2 + \eta_2)(1 - u_{\tilde{a}})^2 + (6a - \sigma_3 + \eta_3)(1 - y_{\tilde{a}})^2\}(1 - \kappa^n)].$$

3.8.2 *Remark*

Definition 2.4.1 shows that the supports (i.e. the bases of trapeziums (triangles)) for truth, indeterminacy and falsity function are all same. Then the value of truth, indeterminacy and falsity function (i.e., the area of individual trapezium (triangle)) differs in respect to their corresponding height only. But by Definition 3.7, we consider different supports (i.e. bases of trapeziums (triangles) formed) for truth, indeterminacy and falsity functions. Thus we can allow the supports and heights together to differ the value of truth, indeterminacy and

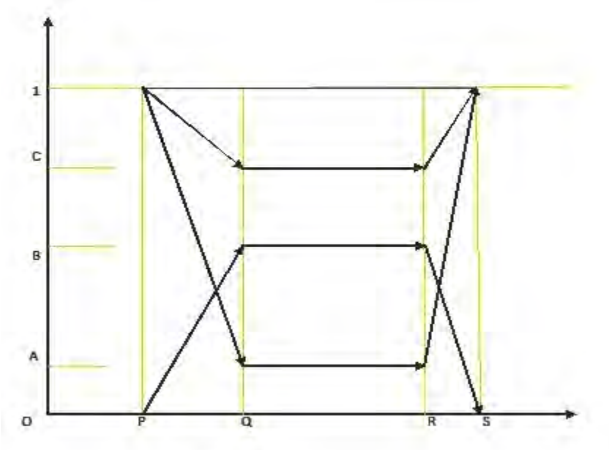

Figure – 1

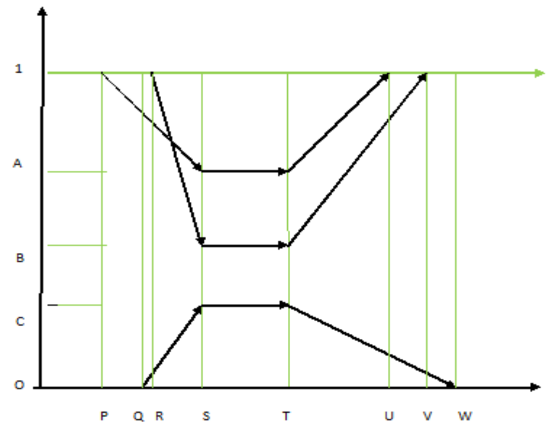

Figure - 2

falsity functions in the present study. Briefly, Definition 2.4.1 is a particular case of Definition 3.7. Hence decision maker has a scope of flexibility to choose and compare different G_{SVN}-numbers in their study. The facts are shown by the graphical Figure 1 and 2. Figure 1 and Figure 2 represent Definition 2.4.1 and Definition 3.7 respectively.

3.9 Definition

1. The zero G_{SVTN}-number is denoted by $\tilde{0}$ and is defined as :
$\tilde{0} = \langle([0,0,0,0];1),([0,0,0,0];0),([0,0,0,0];0)\rangle$.
2. The zero G_{SVTrN}-number is denoted by $\tilde{0}$ and is defined as :
$\tilde{0} = \langle([0,0,0];1),([0,0,0];0),([0,0,0];0)\rangle$.

4 Neutrosophic Linear Programming Problem

Before to discuss the main result, we shall remember the crisp concept of an LP-problem. The standard form of an LP-problem is :

$$\text{Max } z \;=\; cx \quad \text{such that } Ax = b,\; x \geq 0$$

where $c = (c_1, c_2, \cdots, c_n), b = (b_1, b_2, \cdots, b_n)^t$ and $A = [a_{ij}]_{m\times n}$.

In this problem, all the parameters are crisp. we shall now define NLP-problem.

4.1 Definition

An LP-problem having some parameters as G_{SVN}-number is called an NLP-problem. Considering the coefficient of the variables in the objective function in an LP-problem in term of G_{SVN}-numbers, an NLP-problem is designed as follows :

$$\begin{aligned} \text{Max } \tilde{z} &=_{\Re_\kappa} \tilde{c}x \\ \text{such that } \quad Mx &= b;\; x \geq 0 \end{aligned} \tag{4.1}$$

where $b \in \mathbf{R}^m, x \in \mathbf{R}^n, M \in \mathbf{R}^{m\times n}, \tilde{c}^t \in (G_{SVN}(\mathbf{R}))^n$ and $\Re_\kappa$ is a ranking function.

4.2 Definition

1. $x \in \mathbf{R}^n$ is a feasible solution to equation (4.1) if x satisfies the constraints of that.

2. A feasible solution x^* is an optimal solution if for all solutions x to (4.1), $\tilde{c}x^* \geq_{\Re_\kappa} \tilde{c}x$.

3. For the NLP-problem (4.1), suppose rank(M, b) = rank$(M) = m$. M is partitioned as $[B, N]$ where B is a non-singular $m \times m$ matrix i.e., rank$(B) = m$. A feasible solution $x = (x_B, x_N)^t$ to (4.1) obtained by setting $x_B = B^{-1}b, x_N = 0$ is called a neutrosophic basic feasible solution (N_{BFS}). Here B and N are respectively called basis and non basis matrix. x_B is called a basic variable and x_N is called a non-basic variable.

4. In an N_{BFS} if all components of $x_B > 0$, then x is non-degenerate N_{BFS} and if at least one component of $x_B = 0$, then x is degenerate N_{BFS}.

5 Simplex Method for NLP-problem

The NLP-problem (4.1) can be put as follows :

$$\text{Max } \tilde{z} =_{\Re_\kappa} \tilde{c}_B x_B + \tilde{c}_N x_N$$
$$\text{such that} \qquad Bx_B + Nx_N = b; \quad x_B, x_N \geq 0$$

where the characters B, N, x_B and x_N are already stated. Then we have,

$$\begin{aligned} & x_B + B^{-1}Nx_N = B^{-1}b & (5.1)\\ \Rightarrow\ & \tilde{c}_B x_B + \tilde{c}_B B^{-1}Nx_N =_{\Re_\kappa} \tilde{c}_B B^{-1}b \\ \Rightarrow\ & \tilde{z} - \tilde{c}_N x_N + \tilde{c}_B B^{-1}Nx_N =_{\Re_\kappa} \tilde{c}_B B^{-1}b \\ \Rightarrow\ & \tilde{z} + (\tilde{c}_B B^{-1}N - \tilde{c}_N)x_N =_{\Re_\kappa} \tilde{c}_B B^{-1}b. & (5.2)\end{aligned}$$

For an N_{BFS}, treating $x_N = 0$, we have $x_B = B^{-1}b$ and $\tilde{z} =_{\Re_\kappa} \tilde{c}_B B^{-1}b$ from (5.1) and (5.2), respectively. We can rewrite the NLP-problem as given in Table 1.

Table 1 : Tabular form of an NLP-problem.

	$\tilde{c}_j$	$\tilde{c}_B$	$\tilde{c}_N$	
	$\tilde{z}$	x_B	x_N	R.H.S
x_B	0	1	$B^{-1}N$	$B^{-1}b$
$\tilde{z}$	1	0	$\tilde{c}_B B^{-1}N - \tilde{c}_N$	$\tilde{c}_B B^{-1}b$

We can get all required initial information to proceed with the simplex method from Table 1. The neutrosophic cost row in the Table 1 is $\tilde{\lambda}_j =_{\Re_\kappa} (\tilde{c}_B B^{-1}a_j - c_j)_{a_j \notin B}$ giving $\tilde{\lambda}_j =_{\Re_\kappa} (\tilde{z}_j - \tilde{c}_j)$ for non-basic variables. The optimality arises if $\tilde{\lambda}_j \geq_{\Re_\kappa} \tilde{0}, \forall a_j \notin B$. If $\tilde{\lambda}_l <_{\Re_\kappa} \tilde{0}$ for any $a_l \notin B$, we need to replace x_{B_i} by x_l. We then compute $y_l = B^{-1}a_l$. If $y_l \leq 0$, then x_l can be increased indefinitely and so the problem admits unbounded optimal solution. But if y_l has at least one positive component, then one of the current basic variables blocks that increase, which drops to zero.

5.1 Theorem

In every column a_j of M, if $\tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0}$ holds for an N_{BFS} x_B of the NLP-problem (4.1) then it is an optimal solution to that.

Proof. Let $M = [a_{ij}]_{m\times n} = [a_1, a_2, \cdots, a_n]$ where each $a_l = (a_{1l}, a_{2l}, \cdots, a_{ml})^t$ is m component column vector. Suppose $B = [\eta_1, \eta_2, \cdots, \eta_m]$ is the basis matrix and $\tilde{z}_B =_{\Re_\kappa} \tilde{c}_B x_B =_{\Re_\kappa} \sum_{i=1}^{m} \tilde{c}_{B_i} x_{B_i}$, where $\tilde{c}_{B_i}$ is the price corresponding to the basic variable x_{B_i}. Then any column a_l of M may be put as a linear combination of the vectors $\eta_1, \eta_2, \cdots, \eta_m$ of B. Let

$$a_l = y_{1l}\eta_1 + y_{2l}\eta_2 + \cdots + y_{ml}\eta_m = \sum_{i=1}^{m} y_{il}\eta_i = By_l \qquad \Rightarrow \ y_l = B^{-1}a_l.$$

where $y_l = (y_{1l}, y_{2l}, \cdots, y_{ml})^t$ being m component scalars represents a_l, the l-th vector of M. Assume that $\tilde{z}_l =_{\Re_\kappa} \tilde{c}_B y_l =_{\Re_\kappa} \sum_{i=1}^{m} \tilde{c}_{B_i} y_{il}$.

Let $x = [x_1, x_2, \cdots, x_n]^t$ be any other feasible solution of the NLP-problem (4.1) and $\tilde{z}$ be the corresponding objective function. Then,

$$Bx_B = b = Mx \qquad \Rightarrow \ x_B = B^{-1}(Mx) = (B^{-1}M)x = yx$$

where $B^{-1}M = y = [y_{ij}]_{m\times n} = [y_1, y_2, \cdots, y_n]$ with y_l defined as above. Thus,

$$\begin{pmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Equating i-th component from both sides, we have $x_{B_i} = \sum_{j=1}^{n} y_{ij} x_j$. Now,

$$\begin{aligned}
& \tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0} \Rightarrow (\tilde{z}_j - \tilde{c}_j)x_j \geq_{\Re_\kappa} \tilde{0} \text{ [as } x_j > 0 \text{]} \Rightarrow \sum_{j=1}^{n} (\tilde{z}_j - \tilde{c}_j)x_j \geq_{\Re_\kappa} \tilde{0} \\
\Rightarrow \ & \sum_{j=1}^{n} \tilde{z}_j x_j - \sum_{j=1}^{n} \tilde{c}_j x_j \geq_{\Re_\kappa} \tilde{0} \Rightarrow \sum_{j=1}^{n} x_j(\tilde{c}_B y_j) - \tilde{z} \geq_{\Re_\kappa} \tilde{0} \\
\Rightarrow \ & \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} \tilde{c}_{B_i} y_{ij}) - \tilde{z} \geq_{\Re_\kappa} \tilde{0} \Rightarrow \sum_{i=1}^{m} \tilde{c}_{B_i}(\sum_{j=1}^{n} y_{ij} x_j) - \tilde{z} \geq_{\Re_\kappa} \tilde{0} \\
\Rightarrow \ & \sum_{i=1}^{m} \tilde{c}_{B_i} \tilde{x}_{B_i} - \tilde{z} \geq_{\Re_\kappa} \tilde{0} \Rightarrow \tilde{z}_B - \tilde{z} \geq_{\Re_\kappa} \tilde{0}.
\end{aligned}$$

Thus $\tilde{z}_B$ is the maximum value of the objective function. This optimality criterion holds for all non-basic vectors of M. If a_l be in the basis matrix B, say $a_l = \eta_l$, then

$a_l = \eta_l = 0.\eta_1 + 0.\eta_2 + \cdots + 0.\eta_{l-1} + 1.\eta_l + 0.\eta_{l+1} + \cdots + 0.\eta_m$

i.e., y_l is a unit vector e_l with l-th component unity.

Since $a_l = \eta_l$, we have $\tilde{c}_l = \tilde{c}_{B_l}$ and so

$\tilde{z}_l - \tilde{c}_l =_{\Re_\kappa} (\tilde{c}_B y_l - \tilde{c}_l) =_{\Re_\kappa} (\tilde{c}_B e_l - \tilde{c}_l) =_{\Re_\kappa} (\tilde{c}_{B_l} - \tilde{c}_{B_l}) =_{\Re_\kappa} \tilde{0}.$

Thus as a whole $\tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0}$ is the necessary condition for optimality.

5.2 Theorem

A non-degenerate N_{BFS} $x_B = B^{-1}b, x_N = 0$ is optimal to NLP-problem (4.1) iff $\tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0}, \forall 1 \leq j \leq n$.

Proof. Suppose $x^* = (x_B^t, x_N^t)^t$ be an N_{BFS} to (4.1) where $x_B = B^{-1}b, x_N = 0$. If $\tilde{z}^*$ be the objective function corresponding to x^*, then $\tilde{z}^* =_{\Re_\kappa} \tilde{c}_B x_B =_{\Re_\kappa} \tilde{c}_B B^{-1}b$. Let $x = [x_1, x_2, \cdots, x_n]^t$ be another feasible solution of NLP-problem (4.1) and $\tilde{z}$ be the corresponding objective function. Then,

$$\tilde{z} \;=_{\Re_\kappa}\; \tilde{c}_B x_B + \tilde{c}_N x_N =_{\Re_\kappa} \tilde{c}_B B^{-1}b - \sum_{a_j \notin B}(\tilde{c}_B B^{-1}a_j - \tilde{c}_j)x_j =_{\Re_\kappa} \tilde{z}^* - \sum_{a_j \notin B}(\tilde{z}_j - \tilde{c}_j)x_j$$

This shows that the solution is optimal iff $\tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0}$ for all $1 \leq j \leq n$.

5.3 Theorem

For any N_{BFS} to NLP-problem (4.1), if there is some column not in basis such that $\tilde{z}_l - \tilde{c}_l <_{\Re_\kappa} \tilde{0}$ and $y_{il} \leq 0$, $i = 1, 2, \cdots, m$, then (4.1) admits an unbounded solution.

Proof. Let x_B be a basic solution to the NLP-problem (4.1). Re-writing the constraints,

$$\begin{aligned}
& Bx_B + Nx_N = b \\
\Rightarrow\;& x_B + B^{-1}Nx_N = B^{-1}b \\
\Rightarrow\;& x_B + B^{-1}\sum_j (a_j x_j) = B^{-1}b,\; a_j s \text{ are the columns of } N \\
\Rightarrow\;& x_B + \sum_j (B^{-1}a_j x_j) = B^{-1}b \\
\Rightarrow\;& x_B + \sum_j (y_j x_j) = y_0,\; \text{where } a_j = By_j, a_j \notin B \\
\Rightarrow\;& x_{B_i} + \sum_j (y_{ij} x_j) = y_{i0}, 1 \leq i \leq m, 1 \leq j \leq n \\
\Rightarrow\;& x_{B_i} = y_{i0} - \sum_j (y_{ij} x_j), 1 \leq i \leq m, 1 \leq j \leq n.
\end{aligned}$$

If x_l enters into the basis, then $x_l > 0$ and $x_j = 0$ for $j \neq B_i \cup l$. Since $y_{il} \leq 0,\ 1 \leq i \leq m$ hence $y_{i0} - y_{il}x_l \geq 0$. So, the basic solution remains feasible and for that, the objective function is :

$$\begin{aligned}
\tilde{z}^* \;&=_{\Re_\kappa}\; \tilde{c}_B x_B + \tilde{c}_N x_N =_{\Re_\kappa} \sum_{i=1}^m \tilde{c}_{B_i}(y_{i0} - y_{il}x_l) + \tilde{c}_l x_l =_{\Re_\kappa} \sum_{i=1}^m \tilde{c}_{B_i} y_{i0} - (\sum_{i=1}^m \tilde{c}_{B_i} y_{il} - \tilde{c}_l)x_l \\
&=_{\Re_\kappa}\; \tilde{c}_B y_0 - (\tilde{c}_B y_l - \tilde{c}_l)x_l =_{\Re_\kappa} \tilde{z} - (\tilde{z}_l - \tilde{c}_l)x_l.
\end{aligned}$$

It shows that $\tilde{z}^* >_{\Re_\kappa} \tilde{z}$, as $\tilde{z}_l - \tilde{c}_l <_{\Re_\kappa} \tilde{0}$ and this completes the fact.

5.4 Simplex algorithm for solving NLP-problem

To solve any NLP-problem by simplex method, the existence of an initial basic feasible solution is always assumed. This solution will be optimised through some iterations. The required steps are as follows :

Step 1. Check whether the objective function of the given NLP-problem is to be maximized or minimized. If it is to be minimized, then it is converted into a maximization problem by using the result $Min(\tilde{z}) = -Max(-\tilde{z})$.

Step 2. Convert all the inequations of the constraints ($\leq$ type) into equations by introducing slack variables. Put the costs of the respective variables equal to $\tilde{0}$.

Step 3. Obtain an N_{BFS} to the problem in the form $x_B = B^{-1}b = y_0$ and $x_N = 0$. The corresponding objective function is $\tilde{z} =_{\Re_\kappa} \tilde{c}_B B^{-1}b =_{\Re_\kappa} \tilde{c}_B y_0$.
Step 4. For each basic variable, put $\tilde{\lambda}_B =_{\Re_\kappa} \tilde{z}_B - \tilde{c}_B =_{\Re_\kappa} \tilde{0}$. For each non-basic variable, calculate $\tilde{\lambda}_j =_{\Re_\kappa} \tilde{z}_j - \tilde{c}_j =_{\Re_\kappa} \tilde{c}_B B^{-1}a_j - \tilde{c}_j$ in the current iteration. If all $\tilde{z}_j - \tilde{c}_j \geq_{\Re_\kappa} \tilde{0}$, then the present solution is optimal.
Step 5. If for some non-basic variables, $\tilde{\lambda}_j =_{\Re_\kappa} \tilde{z}_j - \tilde{c}_j <_{\Re_\kappa} \tilde{0}$ then find out $\tilde{\lambda}_l = \min\{\tilde{\lambda}_j\}$. If $y_{il} < 0$ for all $i = 1, \cdots, m$, then the given problem will have unbounded solution and stop the iteration. Otherwise to determine the index of the variable x_{B_r} that is to be removed from the current basis, compute

$$\frac{y_{r0}}{y_{rl}} = \min\{\frac{y_{i0}}{y_{il}} : y_{il} > 0, 1 \leq i \leq m\}.$$

Step 6. Update y_{i0} by replacing $y_{i0} - \frac{y_{r0}}{y_{rl}} y_{il}$ for $i \neq r$ and y_{r0} by $\frac{y_{r0}}{y_{rl}}$.
Step 7. Construct new basis and repeat the Step 4, Step 5 until the optimality is reached.
Step 8. Find the optimal solution and hence the optimal value of objective function.

6 Numerical Example

The NLP-problems with both G_{SVTN}-number and G_{SVTrN}-number are solved by the use of proposed algorithm. For simplicity, we define the κ-weighted value function for $n = 1$ in rest of the paper.

6.1 *Example*

Two friends F_1 and F_2 wish to invest in a raising share market. They choose two particular shares S_1 and S_2 of two multinational companies. They also decide to purchase equal unit of two shares individually. The maximum investment of F_1 is Rs. 4000 and that of F_2 is Rs. 7000. The price per unit of S_1 and S_2 are Re. 1 and Rs. 3, respectively when F_1 purchases. These are Rs. 2 and Rs. 5 at the time of purchasing of share by F_2. The current value of share S_1 and S_2 per unit is Rs. $\tilde{c}_1$ and Rs. $\tilde{c}_2$ (given in G_{SVN}-numbers), respectively. Now if they sell their shares, formulate an NLP-problem to maximize their returns.

The problem can be summarised as follows :

Table 2

Friends $\Downarrow$	Shares : S_1	S_2	Purchasing capacity $\Downarrow$
F_1	Re. 1	Rs. 3	Rs. 4000
F_2	Rs. 2	Rs. 5	Rs. 7000
Price per unit $\Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	

Let they individually purchase x_1 units of share S_1 and x_2 units of share S_2. The problem is formulated as :

$$\text{Max } \tilde{z} =_{\Re_\kappa} \tilde{c}_1 x_1 + \tilde{c}_2 x_2$$
$$\text{such that} \quad x_1 + 3x_2 \leq 4000$$
$$2x_1 + 5x_2 \leq 7000; \quad x_1, x_2 \geq 0$$

It is an NLP-problem where $\tilde{c}_1 = \langle([5, 8, 1, 3]; 0.2), ([5, 8, 3, 4]; 0.3), ([5, 8, 2, 1]; 0.4)\rangle$ and $\tilde{c}_2 = \langle([3, 7, 2, 4]; 0.3), ([3, 7, 1, 3]; 0.5), ([3, 7, 2, 5]; 0.6)\rangle$ are two G_{SVTN}-numbers with a pre-assigned $\kappa = 0.45$.

Rewriting the given constraints by introducing slack variables :

$$x_1 + 3x_2 + x_3 = 4000$$
$$2x_1 + 5x_2 + x_4 = 7000$$
$$x_1, x_2, x_3, x_4 \geq 0$$

We take the t-norm and s-norm as $p * q = \min\{p, q\}$ and $p \diamond q = \max\{p, q\}$, respectively. The first feasible simplex table is as follows :

Table 3 : First iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_3	1	3	1	0	4000
x_4	2	5	0	1	7000$\rightarrow$
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(1)} \uparrow$	$\tilde{c}_2^{(1)}$	$\tilde{c}_3^{(1)}$	$\tilde{c}_4^{(1)}$	

Here $\tilde{c}_1^{(1)} = -\tilde{c}_1 = \langle([-8, -5, 3, 1]; 0.2), ([-8, -5, 4, 3]; 0.3), ([-8, -5, 1, 2]; 0.4)\rangle$,
$\tilde{c}_2^{(1)} = -\tilde{c}_2 = \langle([-7, -3, 4, 2]; 0.3), ([-7, -3, 3, 1]; 0.5), ([-7, -3, 5, 2]; 0.6)\rangle$
and $V_\kappa(\tilde{c}_3^{(1)}) = V_\kappa(\tilde{c}_4^{(1)}) = V_\kappa(\tilde{0})$.
Then $V_\kappa(\tilde{c}_1^{(1)}) = \frac{1}{6}(31.64\kappa - 33.28)$ and $V_\kappa(\tilde{c}_2^{(1)}) = \frac{1}{6}(10.4\kappa - 13.28)$ by Definition 3.5.
Clearly $V_\kappa(\tilde{c}_1^{(1)}) < 0,\ V_\kappa(\tilde{c}_2^{(1)}) < 0$ and $V_\kappa(\tilde{c}_1^{(1)}) - V_\kappa(\tilde{c}_2^{(1)}) < 0$ for $\kappa = 0.45$.
Then $\tilde{c}_1^{(1)} <_{\Re_\kappa} \tilde{c}_2^{(1)}$. So x_1 enters in the basis and as $\min\{4000/1, 7000/2\} = 3500$, the leaving variable is x_4. The revised table is :

Table 4 : Second iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_3	0	1/2	1	-1/2	500
x_1	1	5/2	0	1/2	3500
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(2)}$	$\tilde{c}_2^{(2)}$	$\tilde{c}_3^{(2)}$	$\tilde{c}_4^{(2)}$	$3500\tilde{c}_1$

where $V_\kappa(\tilde{c}_1^{(2)}) = V_\kappa(\tilde{c}_3^{(2)}) = V_\kappa(\tilde{0})$ and

$$\begin{aligned}
\tilde{c}_2^{(2)} &= \frac{5}{2}\tilde{c}_1 - \tilde{c}_2 \\
&= 2.5\langle([5, 8, 1, 3]; 0.2), ([5, 8, 3, 4]; 0.3), ([5, 8, 2, 1]; 0.4)\rangle \\
&\quad -\langle([3, 7, 2, 4]; 0.3), ([3, 7, 1, 3]; 0.5), ([3, 7, 2, 5]; 0.6)\rangle \\
&= \langle([5.5, 17, 6.5, 9.5]; 0.2), ([5.5, 17, 10.5, 11]; 0.5), ([5.5, 17, 10, 4.5]; 0.6)\rangle. \\
\tilde{c}_4^{(2)} &= \frac{1}{2}\tilde{c}_1 = \langle([2.5, 4, 0.5, 1.5]; 0.2), ([2.5, 4, 1.5, 2]; 0.3), ([2.5, 4, 1, 0.5]; 0.4)\rangle.
\end{aligned}$$

Then $V_\kappa(\tilde{c}_2^{(2)}) = \frac{1}{6}(26.92 - 24.1\kappa)$ and $V_\kappa(\tilde{c}_4^{(2)}) = \frac{1}{6}(16.64 - 15.82\kappa)$ by Definition 3.5.
Clearly $V_\kappa(\tilde{c}_2^{(2)}) > 0$ and $V_\kappa(\tilde{c}_4^{(2)}) > 0$ for $\kappa = 0.45$.

Hence the optimality arises and Max $\tilde{z} =_{\Re_\kappa} 3500\tilde{c}_1$, which, using κ - weighted function, becomes Rs. 11107 approximately. Then corresponding return of F_1 and F_2 becomes Rs. 7607 and of Rs. 4107 respectively.

6.1.1 *Example*

Consider the NLP-problem defined in Example 6.1 with a pre-assigned $\kappa = 0.96$.

The initial simplex table (Table 5) is same as Table 3.

Table 5 : First iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_3	1	3	1	0	4000$\rightarrow$
x_4	2	5	0	1	7000
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(1)}$	$\tilde{c}_2^{(1)} \uparrow$	$\tilde{c}_3^{(1)}$	$\tilde{c}_4^{(1)}$	

Here $V_\kappa(\tilde{c}_3^{(1)}) = V_\kappa(\tilde{c}_4^{(1)}) = V_\kappa(\tilde{0})$ and $V_\kappa(\tilde{c}_1^{(1)}) < 0$, $V_\kappa(\tilde{c}_2^{(1)}) < 0$ with $V_\kappa(\tilde{c}_1^{(1)}) - V_\kappa(\tilde{c}_2^{(1)}) > 0$ for $\kappa = 0.96$. Then $\tilde{c}_1^{(1)} >_{\Re_\kappa} \tilde{c}_2^{(1)}$. So x_2 enters in the basis and as $\min\{\frac{4000}{3}, \frac{7000}{5}\} = \frac{4000}{3}$, the leaving variable is x_3. The revised table is :

Table 6 : Second iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_2	1/3	1	1/3	0	4000/3
x_4	1/3	0	-5/3	1	1000/3 $\rightarrow$
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(2)} \uparrow$	$\tilde{c}_2^{(2)}$	$\tilde{c}_3^{(2)}$	$\tilde{c}_4^{(2)}$	$\frac{4000}{3}\tilde{c}_2$

where $V_\kappa(\tilde{c}_2^{(2)}) = V_\kappa(\tilde{c}_4^{(2)}) = V_\kappa(\tilde{0})$ and

$$\begin{aligned}\tilde{c}_1^{(2)} &= \frac{1}{3}\tilde{c}_2 - \tilde{c}_1 = \langle([-7,-8/3,11/3,7/3];0.2),([-7,-8/3,13/3,4];0.5),([-7,-8/3,5/3,11/3];0.6)\rangle,\\ \tilde{c}_3^{(2)} &= \frac{1}{3}\tilde{c}_2 = \langle([1,7/3,2/3,4/3];0.3),([1,7/3,1/3,1];0.5),([1,7/3,2/3,5/3];0.6)\rangle.\end{aligned}$$

Then $V_\kappa(\tilde{c}_1^{(2)}) = \frac{1}{18}(31.32\kappa - 34.96)$ and $V_\kappa(\tilde{c}_3^{(2)}) = \frac{1}{18}(13.28 - 10.4\kappa)$.

Clearly $V_\kappa(\tilde{c}_1^{(2)}) < 0$ and $V_\kappa(\tilde{c}_3^{(2)}) > 0$ for $\kappa = 0.96$. So x_1 enters in the basis and as $\min\{\frac{4000/3}{1/3}, \frac{1000/3}{1/3}\} = 1000$, the leaving variable is x_4. The revised table is :

Table 7 : Third iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_2	0	1	2	-1	1000 $\rightarrow$
x_1	1	0	-5	3	1000
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(3)}$	$\tilde{c}_2^{(3)}$	$\tilde{c}_3^{(3)} \uparrow$	$\tilde{c}_4^{(3)}$	$1000(\tilde{c}_1 + \tilde{c}_2)$

where $V_\kappa(\tilde{c}_1^{(3)}) = V_\kappa(\tilde{c}_2^{(3)}) = V_\kappa(\tilde{0})$ and

$$\tilde{c}_3^{(3)} = -5\tilde{c}_1 + 2\tilde{c}_2 = \langle([-34,-11,19,13];0.2),([-34,-11,22,21];0.5),([-34,-11,9,20];0.6)\rangle,$$
$$\tilde{c}_4^{(3)} = 3\tilde{c}_1 - \tilde{c}_2 = \langle([8,21,7,11];0.2),([8,21,12,13];0.5),([8,21,11,5];0.6)\rangle.$$

Then $V_\kappa(\tilde{c}_3^{(3)}) = \frac{1}{6}(48.2\kappa - 53.84) < 0$ and $V_\kappa(\tilde{c}_4^{(3)}) = \frac{1}{6}(34.96 - 31.32\kappa) > 0$ for $\kappa = 0.96$. So x_3 enters in the basis and the leaving variable is x_2. The revised table is :

Table 8 : Fourth iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_3	0	1/2	1	-1/2	500
x_1	1	5/2	0	1/2	3500
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(4)}$	$\tilde{c}_2^{(4)}$	$\tilde{c}_3^{(4)}$	$\tilde{c}_4^{(4)}$	$3500\tilde{c}_1$

where $V_\kappa(\tilde{c}_1^{(4)}) = V_\kappa(\tilde{c}_3^{(4)}) = V_\kappa(\tilde{0})$ and $\tilde{c}_2^{(4)} = \frac{5}{2}\tilde{c}_1 - \tilde{c}_2$ and $\tilde{c}_4^{(4)} = \frac{1}{2}\tilde{c}_1$. Then $V_\kappa(\tilde{c}_2^{(4)}) = \frac{1}{6}(26.92 - 24.1\kappa) > 0$ and $V_\kappa(\tilde{c}_4^{(4)}) = \frac{1}{6}(16.64 - 15.82\kappa) > 0$ for $\kappa = 0.96$.

Hence the optimality arises and the optimal solution is $x_1 = 3500, x_2 = 0$.

6.1.2 *Remark*

From Example 6.1 and Example 6.1.1, it is seen that the final simplex tables in both cases are same. So, if the optimality exists for an NLP-problem, the optimal solutions are always unique whatever the value of κ assigned. Depending upon the chosen κ, the number of iteration to reach at optimality stage may vary but it does not affect the optimal solutions. However, the character κ plays an important role to assign the optimal value of the objective function in a problem. The fact is shown in Table 9. So, the value of κ is an important factor in any such NLP-problem. Since the share market depends on so many factors, we claim κ as the degree of political turmoil of the country in the present problem.

6.1.3 Sensitivity analysis in post optimality stage

We shall analyse the results of the problem in Example 6.1 for different values of κ in post optimality stage, shown by the Table 9.

Table 9 : Sensitivity analysis

κ	0	0.1	0.2	0.3	0.4
x_1	3500	3500	3500	3500	3500
x_2	0	0	0	0	0
$V_\kappa(\tilde{z})$	19413.33	17567.67	15722	13876.33	12030.67

κ	0.5	0.6	0.7	0.8	0.9	1
x_1	3500	3500	3500	3500	3500	3500
x_2	0	0	0	0	0	0
$V_\kappa(\tilde{z})$	10185	8339.33	6493.67	4648	2802.33	956.67

6.2 *Example*

$$\begin{aligned} \text{Max } \tilde{z} \ &=_{\Re_\kappa} \ \tilde{c}_1 x_1 + \tilde{c}_2 x_2 \\ \text{s.t.} \ & \quad 2x_1 + 3x_2 \leq 4 \\ & \quad 5x_1 + 4x_2 \leq 15 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

is an NLP-problem where $\tilde{c}_1 = \langle([8,1,3];0.6),([8,3,4];0.2),([8,2,1];0.5)\rangle$ and $\tilde{c}_2 = \langle([6,2,6];0.7),([6,4,3];0.4),([6,3,5];0.3)\rangle$ are two G_{SVTrN}-numbers with a pre-assigned $\kappa = 0.9$.

Rewriting the given constraints by introducing slack variables :

$$\begin{aligned} & 2x_1 + 3x_2 + x_3 = 4 \\ & 5x_1 + 4x_2 + x_4 = 15 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The t-norm and s-norm are $p * q = \max\{p+q-1, 0\}$ and $p \diamond q = \min\{p+q, 1\}$, respectively. The first feasible simplex table is as follows :

Table 10 : First iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_3	2	3	1	0	4 $\rightarrow$
x_4	5	4	0	1	15
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(1)}$	$\tilde{c}_2^{(1)} \uparrow$	$\tilde{c}_3^{(1)}$	$\tilde{c}_4^{(1)}$	

Here $\tilde{c}_1^{(1)} = -\tilde{c}_1 = \langle([-8,3,1];0.6),([-8,4,3];0.2),([-8,1,2];0.5)\rangle$,
$\tilde{c}_2^{(1)} = -\tilde{c}_2 = \langle([-6,6,2];0.7),([-6,3,4];0.4),([-6,5,3];0.3)\rangle$
and $V_\kappa(\tilde{c}_3^{(1)}) = V_\kappa(\tilde{c}_4^{(1)}) = V_\kappa(\tilde{0})$.
Then $V_\kappa(\tilde{c}_1^{(1)}) = \frac{1}{6}(25.11\kappa - 43.11)$ and $V_\kappa(\tilde{c}_2^{(1)}) = \frac{1}{6}(11.62\kappa - 31.22)$ by Definition 3.8.1.
Clearly $V_\kappa(-\tilde{c}_1) < 0$, $V_\kappa(-\tilde{c}_2) < 0$ and $V_\kappa(-\tilde{c}_1) - V_\kappa(-\tilde{c}_2) > 0$ for $\kappa = 0.9$.

So x_2 enters in the basis and as $\min\{4/3, 15/4\} = 4/3$, the leaving variable is x_3. The revised table is as :

Table 11 : Second iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_2	2/3	1	1/3	0	4/3 $\rightarrow$
x_4	7/3	0	-4/3	1	29/3
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(2)} \uparrow$	$\tilde{c}_2^{(2)}$	$\tilde{c}_3^{(2)}$	$\tilde{c}_4^{(2)}$	$\frac{4}{3}\tilde{c}_2$

where $V_\kappa(\tilde{c}_2^{(2)}) = V_\kappa(\tilde{c}_4^{(2)}) = V_\kappa(\tilde{0})$ and

$$\tilde{c}_1^{(2)} = \frac{2}{3}\tilde{c}_2 - \tilde{c}_1 = \langle([-4, 13/3, 5]; 0.3), ([-4, 20/3, 5]; 0.6), ([-4, 3, 16/3]; 0.8)\rangle,$$
$$\tilde{c}_3^{(2)} = \frac{1}{3}\tilde{c}_2 = \langle([2, 2/3, 2]; 0.7), ([2, 4/3, 1]; 0.4), ([2, 1, 5/3]; 0.3)\rangle.$$

Then $V_\kappa(\tilde{c}_1^{(2)}) = \frac{1}{18}(8.62\kappa - 14.92)$ and $V_\kappa(\tilde{c}_3^{(2)}) = \frac{1}{18}(31.22 - 11.62\kappa)$ by Definition 3.8.1.
Clearly, $V_\kappa(\tilde{c}_1^{(2)}) < 0$ but $V_\kappa(\tilde{c}_3^{(2)}) > 0$ for $\kappa = 0.9$. So x_1 enters in the basis and as $\min\{\frac{4/3}{2/3}, \frac{29/3}{7/3}\} = 2$, the leaving variable is x_2. The revised table is :

Table 12 : Third iteration

$\tilde{c}_j \Rightarrow$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{0}$	$\tilde{0}$	
$x_B \Downarrow$	x_1	x_2	x_3	x_4	R.H.S
x_1	1	3/2	1/2	0	2
x_4	0	-7/2	-5/2	1	5
$\tilde{z} \Rightarrow$	$\tilde{c}_1^{(3)}$	$\tilde{c}_2^{(3)}$	$\tilde{c}_3^{(3)}$	$\tilde{c}_4^{(3)}$	$2\tilde{c}_1$

where $V_\kappa(\tilde{c}_1^{(3)}) = V_\kappa(\tilde{c}_4^{(3)}) = V_\kappa(\tilde{0})$ and

$$\tilde{c}_2^{(3)} = \frac{3}{2}\tilde{c}_1 - \tilde{c}_2 = \langle([6, 7.5, 6.5]; 0.3), ([6, 7.5, 10]; 0.6), ([6, 8, 4.5]; 0.8)\rangle,$$
$$\tilde{c}_3^{(3)} = \frac{1}{2}\tilde{c}_1 = \langle([4, 0.5, 1.5]; 0.6), ([4, 1.5, 2]; 0.2), ([4, 1, 0.5]; 0.5)\rangle.$$

Then $V_\kappa(\tilde{c}_2^{(3)}) = \frac{1}{6}(7.46 - 4.31\kappa)$ and $V_\kappa(\tilde{c}_3^{(3)}) = \frac{1}{6}(21.555 - 12.555\kappa)$ by Definition 3.8.1.
Obviously, $V_\kappa(\tilde{c}_2^{(3)}) > 0$ and $V_\kappa(\tilde{c}_3^{(3)}) > 0$ for $\kappa = 0.9$. Hence the optimality arises. The optimal solution is $x_1 = 2, x_2 = 0$ and so Max $\tilde{z} =_{\Re_\kappa} 2\tilde{c}_1$.

7 Conclusion

In this paper, the crisp LP-problem has been generalised by considering the coefficients of the objective function as G_{SVN}-numbers. This generalised form of crisp LP-problem is called NLP-problem. Then a simplex algorithm has been proposed to solve such NLP-problems. Finally, the newly developed simplex algorithm has been applied to a real life problem. The concept has been illustrated by suitable examples using both G_{SVTN}-numbers and G_{SVTrN}-numbers. In future, the concept of a linear programming problem may be extended in more generalised way by considering some or all of the parameters as G_{SVN}-numbers.

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Neutrosophic Soft Topological K-Algebras

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Abstract: In this paper, we propose the notion of single-valued neutrosophic soft topological K-algebras. We discuss certain concepts, including interior, closure, C_5-connected, super connected, Compactness and Hausdorff in single-valued neutrosophic soft topological K-algebras. We illustrate these concepts with examples and investigate some of their related properties. We also study image and pre-image of single-valued neutrosophic soft topological K-algebras.

Keywords: K-algebras, Single-valued neutrosophic soft sets, Compactness, C_5-connectedness, Super connectedness, Hausdorff.

1 Introduction

A K-algebra $(G, \cdot, \odot, e)$ is a new class of logical algebra, introduced by Dar and Akram [1] in 2003. A K-algebra is constructed on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G and attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general, non-commutative and non-associative with a right identity e. If the given group G is not an elementary abelian 2-group, then the K-algebra is proper . Therefore, a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ is abelian and non-abelian, proper and improper purely depends upon the base group G. In 2004, a K-algebra renamed as $K(G)$-algebra due to its structural basis G and characterized by left and right mappings when the group G is abelian and non-abelian by Dar and Akram in [2, 3] . In 2007, Dar and Akram [4] investigated the K-homomorphisms of K-algebras.
Non-classical logic leads to classical logic due to various aspects of uncertainty. It has become a conventional tool for computer science and engineering to deal with fuzzy information and indeterminate data and executions. In our daily life, the most frequently encountered uncertainty is incomparability. Zadeh's fuzzy set theory [5] revolutionized the systems, accomplished with vagueness and uncertainty. A number of researchers extended the conception of Zadeh and presented different theories regarding uncertainty which includes intuitionistic fuzzy set theory, interval-valued intuitionistic fuzzy set theory [6] and so on. In addition, Smarandache [7] generalized intuitionistic fuzzy set by introducing the concept of neutrosophic set in 1998. It is such a branch of philosophy which studies the origin, nature, and scope of neutralities as well as their interactions with different ideational spectra. To have real life applications of neutrosophic sets such as in engineering and science, Wang et al. [8] introduced the single-valued neutrosophic set in 2010. In 1999, Molodtsov [9] introduced another mathematical approach to deal with ambiguous data, called soft set theory. Soft set theory gives a parameterized outlook to uncertainty. Maji [10] defined the notion of neutrosophic soft set by unifying

the fundamental theories of neutrosophic set and soft set to deal with inconsistent data in a much-unified mode. A large number of theories regarding uncertainty with their respective topological structures have been introduced. In 1968, Chang [11] introduced the concept of fuzzy topology. Chattopadhyay and Samanta [12], Pu and Liu [13] and Lowan [14] defined some certain notions related to fuzzy topology. Recently, Tahan et al. [15] presented the notion of topological hypergroupoids. Onasanya and Hoskova-Mayerova [16] discussed some topological and algebraic properties of $\alpha-$level subsets of fuzzy subsets. Coker [17] considered the notion of an intuitionistic fuzzy topology. Salama and Alblowi [18] studied the notion of neutrosophic topological spaces. In 2017, Bera and Mahapatra [19] described neutrosophic soft topological spaces. Akram and Dar [20, 21] considered fuzzy topological K-algebras and intuitionistic topological K-algebras. Recently, Akram et al. [22, 23, 24, 25] presented some notions, including single-valued neutrosophic K-algebras, single-valued neutrosophic topological K-algebras and single-valued neutrosophic Lie algebras. In this research article, In this paper, we propose the notion of single-valued neutrosophic soft topological K-algebras. We discuss certain concepts, including interior, closure, C_5-connected, super connected, Compactness and Hausdorff in single-valued neutrosophic soft topological K-algebras. We illustrate these concepts with examples and investigate some of their related properties. We also study image and pre-image of single-valued neutrosophic soft topological K-algebras.
The rest of the paper is organized as follows: In Section 2, we review some elementary concepts related to K-algebras, single-valued neutrosophic soft sets and their topological structures. In Section 3, we define the concept of single-valued neutrosophic soft topological K-algebras and discuss certain concepts with some numerical examples. In Section 4, we present concluding remarks.

2 Preliminaries

This section consists of some basic definitions and concepts, which will be used in the next sections.

Definition 2.1. [1] A K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ is an algebra of the type (2, 2, 0) defined on the group $(G, \cdot, e)$ in which each non-identity element is not of order 2 with the following $\odot-$ axioms:

(K1) $(x \odot y) \odot (x \odot z) = (x \odot (z^{-1} \odot y^{-1})) \odot x = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y) = (x \odot y^{-1}) \odot x = (x \odot (e \odot y)) \odot x$,

(K3) $(x \odot x) = e$,

(K4) $(x \odot e) = x$,

(K5) $(e \odot x) = x^{-1}$

for all x, y, $z \in G$.

Definition 2.2. [1] A nonempty set $\mathcal{S}$ in a K-algebra $\mathcal{K}$ is called a *subalgebra* of $\mathcal{K}$ if for all $x, y \in \mathcal{S}$, $x \odot y \in \mathcal{S}$.

Definition 2.3. [1] Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras. A mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ is called a *homomorphism* if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}$.

Definition 2.4. [7] Let Z be a nonempty set of objects. A single-valued neutrosophic set H in Z is of the form $H = \{s \in Z : \mathcal{T}_H(s), \mathcal{I}_H(s), \mathcal{F}_H(s)\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} : Z \to [0, 1]$ for all $s \in Z$ with $0 \leq \mathcal{T}_H(s) + \mathcal{I}_H(s) + \mathcal{F}_H(s) \leq 3$.

Definition 2.5. [22] Let $H = (\mathcal{T}_H, \mathcal{I}_H, \mathcal{F}_H)$ be a single-valued neutrosophic set in $\mathcal{K}$, then H is said to be a *single-valued neutrosophic K-subalgebra* of $\mathcal{K}$ if it possess the following properties:

(a) $\mathcal{T}_H(s \odot t) \geq \min\{\mathcal{T}_H(s), \mathcal{T}_H(t)\}$,

(b) $\mathcal{I}_H(s \odot t) \geq \min\{\mathcal{I}_H(s), \mathcal{I}_H(t)\}$,

(c) $\mathcal{F}_H(s \odot t) \leq \max\{\mathcal{F}_H(s), \mathcal{F}_H(t)\}$ for all $s, t \in \mathcal{K}$.

A K-subalgebra also satisfies the following conditions:
$\mathcal{T}_H(e) \geq \mathcal{T}_H(s), \mathcal{I}_H(e) \geq \mathcal{I}_H(s), \mathcal{F}_H(e) \leq \mathcal{F}_H(s)$ for all $s \neq e \in \mathcal{K}$.

Definition 2.6. [26] A *t-norm* is a two-valued function defined by a binary operation $*$, where $* : [0, 1] \times [0, 1] \to [0, 1]$. A t-norm is an associative, monotonic and commutative function possess the following properties, for all $a, b, c, d \in [0, 1]$,

(i) $*$ is a commutative binary operation.

(ii) $*$ is an associative binary operation.

(iii) $*(0, 0) = 0$ and $*(a, 1) = *(1, a) = a$.

(iv) If $a \leq c$ and $b \leq d$, then $*(a, b) \leq *(c, d)$.

Definition 2.7. [26] A *t-conorm (s-norm)* is a two-valued function defined by a binary operation $\circ$ such that $\circ : [0, 1] \times [0, 1] \to [0, 1]$. A t-conorm is an associative, monotonic and commutative two-valued function, possess the following properties, for all $a, b, c, d \in [0, 1]$,

(i) $\circ$ is a commutative binary operation.

(ii) $\circ$ is an associative binary operation.

(iii) $\circ(1, 1) = 1$ and $\circ(a, 0) = \circ(0, a) = a$.

(iv) If $a \leq c$ and $b \leq d$, then $\circ(a, b) \leq \circ(c, d)$.

Definition 2.8. [23] Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic topology over $\mathcal{K}$. Let H be a single-valued neutrosophic K-algebra of $\mathcal{K}$ and χ_H be a single-valued neutrosophic topology on H. Then H is called a *single-valued neutrosophic topological K-algebra* over $\mathcal{K}$ if the self map $\rho_a : (H, \chi_H) \to (H, \chi_H)$ for all $a \in \mathcal{K}$, defined as $\rho_a(s) = s \odot a$, is relatively single-valued neutrosophic continuous.

Definition 2.9. [9] Let Z be a universe of discourse and E be a universe of parameters. Let $P(Z)$ denotes the set of all subsets of Z and $A \subseteq E$. Then a *soft set* F_A over Z is represented by a set-valued function ζ_A, where $\zeta_A : E \to P(Z)$ such that $\zeta_A(\theta) = \emptyset$ if $\theta \in E - A$. In other words, F_A can be represented in the form of a collection of parameterized subsets of Z such as $F_A = \{(\theta, \zeta_A(\theta)) : \theta \in E, \zeta_A(\theta) = \emptyset \text{ if } \theta \in E - A\}$.

Definition 2.10. [27] Let Z be a universe of discourse and E be a universe of parameters. A *single-valued neutrosophic soft set* H in Z is defined by a set-valued function ζ_H, where $\zeta_H : E \to P(Z)$ and $P(Z)$ denotes the power set set of Z. In other words, a single-valued neutrosophic soft set is a parameterized family of single-valued neutrosophic sets in Z and therefore can be written as:
$H = \{(\theta, \langle u, \mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{F}_{\zeta_H(\theta)}(u)\rangle : u \in Z) : \theta \in E\}$, where $\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}$ are called truth , indeterminacy and falsity membership functions of $\zeta_H(\theta)$, respectively.

Definition 2.11. [27] Let H be a single-valued neutrosophic soft set. The *compliment* of H, denoted by H^c, is defined as follows:

$$H^c = \{(\theta,\ \langle u, \mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{T}_{\zeta_H(\theta)}(u)\rangle\ : u \in Z) : \theta \in E\}.$$

Definition 2.12. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then H is called a *neutrosophic soft subset* of J, denoted by $H \subseteq J$, if the following conditions hold:

(i) $\mathcal{T}_{\zeta_H(\theta)}(u) \leq \mathcal{T}_{\eta_J(\theta)}(u)$,

(ii) $\mathcal{I}_{\zeta_H(\theta)}(u) \leq \mathcal{I}_{\eta_J(\theta)}(u)$,

(iii) $\mathcal{F}_{\zeta_H(\theta)}(u) \geq \mathcal{F}_{\eta_J(\theta)}(u)$ for all $\theta \in E, u \in Z$.

Throughout this article, we take the t-norm $(*)$ as $\min(a, b)$ and t-conorm $(\circ)$ as $\max(a, b)$ for intersection of two single-valued neutrosophic soft sets and $(*)$ as $\max(a, b)$ and t-conorm $(\circ)$ as $\min(a, b)$ for union of two single-valued neutrosophic soft sets. The union and the intersection for two single-valued neutrosophic soft sets are defined as follows.

Definition 2.13. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then the *union* of H and J is denoted by $H \cup J = L$ and defined as:

$$L = \Big\{\Big(\theta,\ \langle u, \mathcal{T}_{\vartheta_L(\theta)}(u), \mathcal{I}_{\vartheta_L(\theta)}(u), \mathcal{F}_{\vartheta_L(\theta)}(u)\rangle\ : u \in Z\Big) : \theta \in E\Big\},$$

where

$$\begin{aligned}
\mathcal{T}_{\vartheta_L(\theta)}(u) &= \{\mathcal{T}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{T}_{\eta_J(\theta)}(u)\},\\
\mathcal{I}_{\vartheta_L(\theta)}(u) &= \{\mathcal{I}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{I}_{\eta_J(\theta)}(u)\},\\
\mathcal{F}_{\vartheta_L(\theta)}(u) &= \{\mathcal{F}_{\zeta_H(\theta)}(u) \circ \mathcal{F}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{F}_{\eta_J(\theta)}(u)\}.
\end{aligned}$$

Definition 2.14. [27] Let H and J be two single-valued neutrosophic soft sets over (Z, E). Then their *intersection* is denoted by $H \cap J = L$ and defined as:

$$L = \Big\{\Big(\theta,\ \langle u, \mathcal{T}_{\vartheta_L(\theta)}(u), \mathcal{I}_{\vartheta_L(\theta)}(u), \mathcal{F}_{\vartheta_L(\theta)}(u)\rangle\ : u \in Z\Big) : \theta \in E\Big\},$$

where

$$\begin{aligned}
\mathcal{T}_{\vartheta_L(\theta)}(u) &= \{\mathcal{T}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{T}_{\zeta_H(\theta)}(u), \mathcal{T}_{\eta_J(\theta)}(u)\},\\
\mathcal{I}_{\vartheta_L(\theta)}(u) &= \{\mathcal{I}_{\zeta_H(\theta)}(u) * \mathcal{T}_{\eta_J(\theta)}(u)\} = \min\{\mathcal{I}_{\zeta_H(\theta)}(u), \mathcal{I}_{\eta_J(\theta)}(u)\},\\
\mathcal{F}_{\vartheta_L(\theta)}(u) &= \{\mathcal{F}_{\zeta_H(\theta)}(u) \circ \mathcal{F}_{\eta_J(\theta)}(u)\} = \max\{\mathcal{F}_{\zeta_H(\theta)}(u), \mathcal{F}_{\eta_J(\theta)}(u)\}.
\end{aligned}$$

Definition 2.15. [27] A single-valued neutrosophic soft set H over the universe Z is termed to be an *empty or null single-valued neutrosophic soft set* with respect to the parametric set E if $\mathcal{T}_{\zeta_H(\theta)}(u) = 0, \mathcal{I}_{\zeta_H(\theta)}(u) = 0$, $\mathcal{F}_{\zeta_H(\theta)}(u) = 1$, for all $u \in Z$, $\theta \in E$, denoted by $\emptyset_E$ and can be written as:

$$\emptyset_E(u) = \{u \in Z : \mathcal{T}_{\zeta_H(\theta)}(u) = 0, \mathcal{I}_{\zeta_H(\theta)}(u) = 0, \mathcal{F}_{\zeta_H(\theta)}(u) = 1 : \theta \in E\}.$$

Definition 2.16. [27] A single-valued neutrosophic soft set H over the universe Z is called an *absolute or a whole single-valued neutrosophic soft set* if $\mathcal{T}_{\zeta_{H(\theta)}}(u) = 1$, $\mathcal{I}_{\zeta_{H(\theta)}}(u) = 1$, $\mathcal{F}_{\zeta_{H(\theta)}}(u) = 0$, for all $u \in Z$, $\theta \in E$, denoted by 1_E and can be written as:

$$1_E(u) = \{u \in Z : \mathcal{T}_{\zeta_{H(\theta)}}(u) = 1, \mathcal{I}_{\zeta_{H(\theta)}}(u) = 1, \mathcal{F}_{\zeta_{H(\theta)}}(u) = 0 : \theta \in E\}.$$

Definition 2.17. [10] Let (Z_1, E) and (Z_2, E) be two initial universes. Then a pair (φ, ρ) is called a *single-valued neutrosophic soft function* from (Z_1, E) into (Z_2, E), where $\varphi : Z_1 \to Z_2$ and $\rho : E \to E$, and E is a parametric set of Z_1 and Z_2.

Definition 2.18. [10] Let (H, E) and (J, E) be two single-valued neutrosophic soft sets over G_1 and G_2, respectively. If (φ, ρ) is a single-valued neutrosophic soft function from (G_1, E) into (G_2, E), then under this single-valued neutrosophic soft function (φ, ρ), *image* of (H, E) is a single-valued neutrosophic soft set on $\mathcal{K}_2$, denoted by $(\varphi, \rho)(H, E)$ and defined as follows:
for all $m \in \rho(E)$ and $y \in G_2$, $(\varphi, \rho)(H, E) = (\varphi(H), \rho(E))$, where

$$\mathcal{T}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigvee_{\varphi(x)=y} \bigvee_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 1, & \text{otherwise}, \end{cases}$$
$$\mathcal{I}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigvee_{\varphi(x)=y} \bigvee_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 1, & \text{otherwise}, \end{cases}$$
$$\mathcal{F}_{\varphi(\zeta)_m}(y) = \begin{cases} \bigwedge_{\varphi(x)=y} \bigwedge_{\rho(a)=m} \zeta_a(x) & \text{if } x \in \rho^{-1}(y), \\ 0, & \text{otherwise}. \end{cases}$$

The *preimage* of (J, E), denoted by $(\varphi, \rho)^{-1}(J, E)$, is defined as $\forall\, l \in \rho^{-1}(E)$ and for all $x \in G_1$, $(\varphi, \rho)^{-1}(J, E) = (\varphi^{-1}(J), \rho^{-1}(E))$, where

$$\mathcal{T}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{T}_{\eta_{\rho(l)}}(\varphi(x)),$$
$$\mathcal{I}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{I}_{\eta_{\rho(l)}}(\varphi(x)),$$
$$\mathcal{F}_{\varphi^{-1}(\eta)_l}(x) = \mathcal{F}_{\eta_{\rho(l)}}(\varphi(x)).$$

Proposition 2.19. Let Z_1 and Z_2 be two initial universes with parametric set E_1 and E_2, respectively. Let H, $(H_i, i \in I)$ be a single-valued neutrosophic soft set in Z_1 and J be a single-valued neutrosophic soft set in Z_2. Let $f : Z_1 \to Z_2$ be a function. Then

(i) $f(1_{E_1}) = 1_{E_2}$, if f is a surjective function.

(ii) $f(\emptyset_{E_1}) = \emptyset_{E_2}$.

(iii) $f^{-1}(1_{E_2}) = 1_{E_1}$.

(iv) $f^{-1}(\emptyset_{E_2}) = \emptyset_{E_1}$.

(v) $f^{-1}(\bigcup_{i=1}^{n} H_i) = \bigcup_{i=1}^{n} f^{-1}(H_i)$.

Through out this article, Z is considered as initial universe, E is a parametric set and $\theta \in E$ an arbitrary parameter.

3 Single-Valued Neutrosophic Soft Topological K-Algebras

Definition 3.1. Let Z be a nonempty set and E be a universe of parameters. A collection χ of single-valued neutrosophic soft sets is called a *single-valued neutrosophic soft topology* if the following properties hold:

(1) $\emptyset_E, 1_E \in \chi$.

(2) The intersection of any two single-valued neutrosophic soft sets of χ belongs to χ.

(3) The union of any collection of single-valued neutrosophic soft sets of χ belongs to χ.

The triplet (Z, E, χ) is called a single-valued neutrosophic soft topological space over (Z, E). Each element of χ is called a single-valued neutrosophic soft open set and compliment of each single-valued neutrosophic soft open set is a single-valued neutrosophic soft closed set in χ . A single-valued neutrosophic soft topology which contains all single-valued neutrosophic soft subsets of Z is called a discrete single-valued neutrosophic soft topology and indiscrete single-valued neutrosophic soft topology if it consists of $\emptyset_E$ and 1_E.

Definition 3.2. Let H be a single-valued neutrosophic soft set over a K-algebras $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft K-subalgebra* of $\mathcal{K}$ if the following conditions hold:

$$\begin{array}{l}
\text{(i)}\ \ \mathcal{T}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{T}_{\zeta_\theta}(s), \mathcal{T}_{\zeta_\theta}(t)\}, \\
\text{(ii)}\ \ \mathcal{I}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{I}_{\zeta_\theta}(s), \mathcal{I}_{\zeta_\theta}(t)\}, \\
\text{(iii)}\ \mathcal{F}_{\zeta_\theta}(s \odot t) \leq \max\{\mathcal{F}_{\zeta_\theta}(s), \mathcal{F}_{\zeta_\theta}(t)\} \text{ for all } s, t \in G \text{ and } \theta \in E.
\end{array}$$

Note that

$$\begin{array}{l}
\mathcal{T}_{\zeta_\theta}(e) \geq \mathcal{T}_{\zeta_\theta}(s), \\
\mathcal{I}_{\zeta_\theta}(e) \geq \mathcal{I}_{\zeta_\theta}(s), \\
\mathcal{F}_{\zeta_\theta}(e) \leq \mathcal{F}_{\zeta_\theta}(s), \text{ for all } s \neq e \in G.
\end{array}$$

Example 3.3. Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on a group $(G, \cdot)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ is the cyclic group of order 8 and $\odot$ is given by the following Cayley's table as:

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6	x^7
e	e	x^7	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^7	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^7	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^7	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^7	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^7	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e	x^7
x^7	x^7	x^6	x^5	x^4	x^3	x^2	x	e

Let E be a set of parameters defined as $E = \{l_1, l_2\}$. We define single-valued neutrosophic soft sets H, J and L in $\mathcal{K}$ as:

$$\begin{array}{l}
\zeta_H(l_1) = \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\}, \\
\zeta_H(l_2) = \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},
\end{array}$$

$$\zeta_J(l_1) = \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_J(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},$$

$$\zeta_L(l_1) = \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},$$
$$\zeta_L(l_2) = \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}$$

for all $h \neq e \in G$.
The collection $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, L\}$ is a single-valued neutrosophic soft topology on $\mathcal{K}$ and the triplet $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft topological space over $\mathcal{K}$. It is interesting to note that corresponding to each parameter $\theta \in E$, we get a single-valued neutrosophic topology over $\mathcal{K}$ which means that a single-valued neutrosophic soft topological space gives a parameterized family of single-valued neutrosophic topological space on $\mathcal{K}$. Now, we define a single-valued neutrosophic soft set Q in $\mathcal{K}$ as:

$$\zeta_Q(l_1) = \{(e, 0.8, 0.5, 0.1), (h, 0.6, 0.4, 0.3)\},$$
$$\zeta_Q(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\}.$$

Clearly, by Definition 3.2, Q is a single-valued neutrosophic soft K-subalgebra over $\mathcal{K}$.

Proposition 3.4. Let $(\mathcal{K}, E, \chi'_{\mathcal{K}})$ and $(\mathcal{K}, E, \chi''_{\mathcal{K}})$ be two single-valued neutrosophic topological spaces over $\mathcal{K}$. If $\chi'_{\mathcal{K}} \cap \chi''_{\mathcal{K}} = M'$, where M' is a single-valued neutrosophic soft set from the set of all single-valued neutrosophic soft sets in $\mathcal{K}$, then $\chi'_{\mathcal{K}} \cap \chi''_{\mathcal{K}}$ is also a single-valued neutrosophic soft topology on $\mathcal{K}$.

Remark 3.5. The union of two single-valued neutrosophic soft topologies over $\mathcal{K}$ may not be a single-valued neutrosophic soft topology over $\mathcal{K}$.

Example 3.6. Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ is the cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3. We take $E = \{l_1, l_2\}$ and two single-valued neutrosophic soft topological spaces $\chi'_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$, $\chi''_{\mathcal{K}} = \{\emptyset_E, 1_E, R, S\}$ on $\mathcal{K}$, where $R = H$ and single-valued neutrosophic soft set S is defined as:

$$\zeta_S(l_1) = \{(e, 0.7, 0.6, 0.2), (h, 0.5, 0.5, 0.6)\},$$
$$\zeta_S(l_2) = \{(e, 0.9, 0.8, 0.2), (h, 0.7, 0.7, 0.3)\}.$$

Suppose that $\chi'''_{\mathcal{K}} = \chi'_{\mathcal{K}} \cup \chi''_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, S\}$. We see that $\chi'''_{\mathcal{K}}$ is not a single-valued neutrosophic soft topology over $\mathcal{K}$ since $S \cap J \notin \chi'''_{\mathcal{K}}$.

Definition 3.7. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$, where $\chi_{\mathcal{K}}$ is a single-valued neutrosophic soft topology over $\mathcal{K}$. Let F be a single-valued neutrosophic soft set in $\mathcal{K}$, then $\chi_F = \{F \cap H : H \in \chi_{\mathcal{K}}\}$ is called a single-valued neutrosophic soft topology on F and (F, E, χ_F) is called a *single-valued neutrosophic soft subspace* of $(\mathcal{K}, E, \chi_{\mathcal{K}})$.

Definition 3.8. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are two K-algebras. Then, a mapping $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is called *single-valued neutrosophic soft continuous mapping* of single-valued neutrosophic soft topological spaces if it the following properties hold:

(i) For each single-valued neutrosophic soft set $H \in \chi_2$, $f^{-1}(H) \in \chi_1$.

(ii) For each single-valued neutrosophic soft K-subalgebra $H \in \chi_2$, $f^{-1}(H)$ is a single-valued neutrosophic soft K-subalgebra $\in \chi_1$.

Definition 3.9. Let H and J be two single-valued neutrosophic soft sets in a K-algebra $\mathcal{K}$ and $f : (H, E, \chi_H) \to (J, E, \chi_J)$. Then, f is called a *relatively single-valued neutrosophic soft open* function if for every single-valued neutrosophic soft open set V in χ_H, the image $f(V) \in \chi_J$.

Definition 3.10. If f is a mapping such that $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$. Then f is a mapping from (H, E, χ_H) into (J, E, χ_J) if $f(H) \subset J$, where (H, E, χ_H) and (J, E, χ_J) are two single-valued neutrosophic soft subspaces of $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$, respectively.

Definition 3.11. A mapping f such that $f : (H, E, \chi_H) \to (J, E, \chi_J)$ is called *relatively single-valued neutrosophic soft continuous* if for every single-valued neutrosophic soft open set $Y_J \in \chi_J$, $f^{-1}(y_J) \cap H \in \chi_H$.

Definition 3.12. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces. Then, a function $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is called a *single-valued neutrosophic soft homomorphism* if it satisfies the following properties:

(i) f is a bijective function.

(ii) Both f and f^{-1} are single-valued neutrosophic soft continuous functions.

Proposition 3.13. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft continues mapping and (H, E, χ_H) and (J, E, χ_J) two single-valued neutrosophic soft topological subspaces of $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$, respectively. If $f(H) \subseteq J$, then f is a relatively single-valued neutrosophic soft continuous mapping from (H, E, χ_H) into (J, E, χ_J).

Proposition 3.14. Let $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces, where χ_1 is a single-valued neutrosophic soft topology on $\mathcal{K}_1$ and χ_2 is an indiscrete single-valued neutrosophic soft topology on $\mathcal{K}_2$. Then for each $\theta \in E$, every function $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continues function.

Proof. Let χ_1 be a single-valued neutrosophic soft topology on $\mathcal{K}_1$ and χ_2 an indiscrete single-valued neutrosophic soft topology on $\mathcal{K}_2$ such that $\chi_2 = \{\emptyset_E, 1_E\}$. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be any function. Now, to prove that f is a single-valued neutrosophic soft continues function for each $\theta \in E$, we show that f satisfies both conditions of Definition 3.8. Clearly, every member of χ_2 is a single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_2$ for each $\theta \in E$. Now, there is only need to show that for all $H \in \chi_2$ and for each $\theta \in E$, $f^{-1}(H) \in \chi_1$. For this purpose, let us assume that $\emptyset_\theta \in \chi_2$, for any $u \in \mathcal{K}_1$ and $\theta \in E$, we have $f^{-1}(\emptyset_\theta)(u) = \emptyset_\theta(f(u)) = \emptyset_\theta(u) \Rightarrow \emptyset_\theta \in \chi_1$. Similarly, $f^{-1}(1_\theta)(u) = 1_\theta(f(u)) = 1_\theta(u) \Rightarrow 1_\theta \in \chi_1$. For an arbitrary choice of θ, result holds for each $\theta \in E$. This shows that f is a single-valued neutrosophic soft continues function. □

Proposition 3.15. Let χ_1 and χ_2 be any two discrete single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively and $(\mathcal{K}_1, E, \chi_1)$ and $(\mathcal{K}_2, E, \chi_2)$ two discrete single-valued neutrosophic soft topological spaces. Then for each $\theta \in E$, every homomorphism $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continuous function.

Proof. Let $H = \{(\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}) : \theta \in E\}$ be a single-valued neutrosophic soft set in $\mathcal{K}_2$ defined by a set-valued function ζ_H. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a homomorphism (not a usual inverse homomorphism). Since χ_1 and χ_2 be two discrete single-valued neutrosophic soft topologies, then for every $H \in \chi_2$, $f^{-1}(H) \in \chi_1$. Now, we show that for each $\theta \in E$, the mapping $f^{-1}(H)$ is a single-valued neutrosophic soft K-subalgebra of K-algebra $\mathcal{K}_1$. Then for any $s, t \in \mathcal{K}_1$ and $\theta \in E$, we have

$$\begin{aligned} f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{T}_{\zeta_H(\theta)}\big(f(s \odot t)\big) \\ &= \mathcal{T}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\ &\geq \min\{\mathcal{T}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{T}_{\zeta_H(\theta)}(f(t))\} \\ &= \min\{f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{T}_{\zeta_H(\theta)})(t)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{I}_{\zeta_H(\theta)}(f(s \odot t)) \\ &= \mathcal{I}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\ &\geq \min\{\mathcal{I}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{I}_{\zeta_H(\theta)}(f(t))\} \\ &= \min\{f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{I}_{\zeta_H(\theta)})(t)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(s \odot t) &= \mathcal{F}_{\zeta_H(\theta)}(f(s \odot t)) \\ &= \mathcal{F}_{\zeta_H(\theta)}(f(s) \odot f(t)) \\ &\geq \min\{\mathcal{F}_{\zeta_H(\theta)}(f(s)) \odot \mathcal{F}_{\zeta_H(\theta)}(f(t))\} \\ &= \min\{f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(s), f^{-1}(\mathcal{F}_{\zeta_H(\theta)})(t)\}, \end{aligned}$$

Therefore, $f^{-1}(H)$ is single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_1$. Hence $f^{-1}(H) \in \chi_2$ which shows that f is a single-valued neutrosophic soft continuous function from $(\mathcal{K}_1, E, \chi_1)$ into $(\mathcal{K}_2, E, \chi_2)$. □

Proposition 3.16. Let χ_1 and χ_2 be any two single-valued neutrosophic soft topological spaces on $\mathcal{K}$ and $(\mathcal{K}, E, \chi_1)$ and $(\mathcal{K}, E, \chi_2)$ be two single-valued neutrosophic soft topological spaces. Then for each $\theta \in E$, every homomorphism $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft continuous function.

Definition 3.17. Let χ be a single-valued neutrosophic soft topology on K-algebra $\mathcal{K}$. Let $H = (\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H})$ be a single-valued neutrosophic soft K-algebra (K-subalgebra) of $\mathcal{K}$ and χ_H a single-valued neutrosophic soft topology over H. Then H is called a *single-valued neutrosophic soft topological* K-algebra of $\mathcal{K}$ if the self mapping $\rho_a : (H, E, \chi_H) \to (H, E, \chi_H)$ defined as $\rho_a(u) = u \odot a$, $\forall\, a \in \mathcal{K}$, is a relatively single-valued neutrosophic soft continuous mapping.

Theorem 3.18. Let χ_1 and χ_2 be two single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a homomorphism of K-algebras such that $f^{-1}(\chi_2) = \chi_1$. If for each $\theta \in E$, $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$, then for each $\theta \in E$, $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$.

Proof. In order to prove that $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of K-algebra $\mathcal{K}_1$. Firstly, we show that $f^{-1}(H)$ is a single-valued neutrosophic soft K-algebra of $\mathcal{K}_1$. One can easily show

that for all $s \neq e \in G$ and $\theta \in E$, $\mathcal{T}_{\zeta_\theta}(e) \geq \mathcal{T}_{\zeta_\theta}(s), \mathcal{I}_{\zeta_\theta}(e) \geq \mathcal{I}_{\zeta_\theta}(s), \mathcal{F}_{\zeta_\theta}(e) \leq \mathcal{F}_{\zeta_\theta}(s)$.
Let for any $s, t \in \mathcal{K}_1$ and $\theta \in E$,

$$\begin{aligned}\mathcal{T}_{f^{-1}(H)}(s \odot t) &= \mathcal{T}_H(f(s \odot t))\\ &\geq \min\{\mathcal{T}_H(f(s)), \mathcal{T}_H(f(t))\}\\ &= \min\{\mathcal{T}_{f^{-1}(H)}(s), \mathcal{T}_{f^{-1}(H)}(t)\},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{f^{-1}(H)}(s \odot t) &= \mathcal{I}_H(f(s \odot t))\\ &\geq \min\{\mathcal{I}_H(f(s)), \mathcal{I}_H(f(t))\}\\ &= \min\{\mathcal{I}_{f^{-1}(H)}(s), \mathcal{I}_{f^{-1}(H)}(t)\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{f^{-1}(H)}(s \odot t) &= \mathcal{F}_H(f(s \odot t))\\ &\geq \min\{\mathcal{F}_H(f(s)), \mathcal{F}_H(f(t))\}\\ &= \min\{\mathcal{F}_{f^{-1}(H)}(s), \mathcal{F}_{f^{-1}(H)}(t)\}.\end{aligned}$$

This shows that $f^{-1}(H)$ is a single-valued neutrosophic soft K-algebra of $\mathcal{K}_1$.
Since f is a homomorphism and also a single-valued neutrosophic soft continuous mapping, then clearly, f is relatively single-valued neutrosophic soft continuous mapping from (H, E, χ_H) into $(f^{-1}(H), E, \chi_{f^{-1}(H)})$ such that for a single-valued neutrosophic soft set V in χ_H, and a single-valued neutrosophic soft set U in $\chi_{(f^{-1}(H)}$,

$$f^{-1}(V) = U. \tag{1}$$

Now, we prove that the self mapping $\rho_a : (f^{-1}(H), E, \chi_{f^{-1}(H)}) \to (f^{-1}(H), E, \chi_{f^{-1}(H)})$ is relatively single-valued neutrosophic soft continuous mapping. Now, for any $a \in \mathcal{K}_1$ and $\theta \in E$, we have

$$\begin{aligned}\mathcal{T}_{\rho_a^{-1}(U)}(s) &= \mathcal{T}_{(U)}(\rho_a(s)) = \mathcal{T}_{(U)}(s \odot a)\\ &= \mathcal{T}_{f^{-1}(V)}(s \odot a) = \mathcal{T}_{(V)}(f(s \odot a))\\ &= \mathcal{T}_{(V)}(f(s) \odot f(a)) = \mathcal{T}_{(V)}(\rho_{f(a)}(f(s)))\\ &= \mathcal{T}_{\rho^{-1}f(a)V}(f(s)) = \mathcal{T}_{f^{-1}}(\rho_{f(a)}^{-1}(V)(s)),\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{\rho_a^{-1}(U)}(s) &= \mathcal{I}_{(U)}(\rho_a(s)) = \mathcal{I}_{(U)}(s \odot a)\\ &= \mathcal{I}_{f^{-1}(V)}(s \odot a) = \mathcal{I}_{(V)}(f(s \odot a))\\ &= \mathcal{I}_{(V)}(f(s) \odot f(a)) = \mathcal{I}_{(V)}(\rho_{f(a)}(f(s)))\\ &= \mathcal{I}_{\rho^{-1}f(a)V}(f(s)) = \mathcal{I}_{f^{-1}}(\rho_{f(a)}^{-1}(V)(s)),\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\rho_a^{-1}(U)}(s) &= \mathcal{F}_{(U)}(\rho_a(s)) = \mathcal{F}_{(U)}(s \odot a)\\ &= \mathcal{F}_{f^{-1}(V)}(s \odot a) = \mathcal{F}_{(V)}(f(s \odot a))\\ &= \mathcal{F}_{(V)}(f(s) \odot f(a)) = \mathcal{F}_{(V)}(\rho_{f(a)}(f(s)))\\ &= \mathcal{F}_{\rho^{-1}f(a)V}(f(s)) = \mathcal{F}_{f^{-1}}(\rho_{f(a)}^{-1}(V)(s)).\end{aligned}$$

This implies that $\rho_a^{-1}(U) = f^{-1}(\rho_{f(a)}^{-1}(V))$. Thus, $\rho_a^{-1}(U) \cap f^{-1}(H) = f^{-1}(\rho_{f(a)}^{-1}(V)) \cap f^{-1}(H)$ is a single-valued neutrosophic soft set in $f^{-1}(H)$ and a single-valued neutrosophic soft set in $\chi_{f^{-1}(H)}$. Hence $f^{-1}(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$. This completes the proof. □

Theorem 3.19. Let χ_1 and χ_2 be two single-valued neutrosophic soft topologies on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively and $f : \mathcal{K}_1 \to \mathcal{K}_2$ an isomorphism of K-algebras such that $f(\chi_1) = \chi_2$. If for each $\theta \in E$, $H = \{(\mathcal{T}_{\zeta_H(\theta)}, \mathcal{I}_{\zeta_H(\theta)}, \mathcal{F}_{\zeta_H(\theta)}) : \theta \in E\}$ is a single-valued neutrosophic soft topological K-algebra of K-algebra $\mathcal{K}_1$, then for each $\theta \in E$, $f(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$.

Proof. Let H be a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_1$. For $u, v \in \mathcal{K}_2$.

Let $t_o \in f^{-1}(u)$, $s_o \in f^{-1}(v)$ such that

$$\mathcal{T}_H(t_o) = \sup_{t\in f^{-1}(u)} \mathcal{T}_H(t), \mathcal{T}_H(y_o) = \sup_{t\in f^{-1}(v)} \mathcal{T}_H(t).$$

We now have,

$$\begin{aligned}\mathcal{T}_{f(H)}(u \odot v) =& \sup_{t\in f^{-1}(u\odot v)} \mathcal{T}_H(t)\\ \geq&\ \mathcal{T}_H(t_o, s_o)\\ \geq&\ \min\{\mathcal{T}_H(t_o), \mathcal{T}_H(s_o)\}\\ =&\ \min\{\sup_{t\in f^{-1}(u)} \mathcal{T}_H(t), \sup_{a\in f^{-1}(v)} \mathcal{T}_H(t)\}\\ =&\ \min\{\mathcal{T}_{f(H)}(u), \mathcal{T}_{f(H)}(v)\},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{f(H)}(u \odot v) =& \sup_{t\in f^{-1}(u\odot v)} \mathcal{I}_H(t)\\ \geq&\ \mathcal{I}_H(t_o, s_o)\\ \geq&\ \min\{\mathcal{I}_H(t_o), \mathcal{I}_H(s_o)\}\\ =&\ \min\{\sup_{t\in f^{-1}(u)} \mathcal{I}_H(t), \sup_{t\in f^{-1}(v)} \mathcal{I}_H(t)\}\\ =&\ \min\{\mathcal{I}_{f(H)}(u), \mathcal{I}_{f(H)}(v)\},\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{f(H)}(u \odot v) = & \inf_{t \in f^{-1}(u \odot v)} \mathcal{F}_H(t) \\
\leq & \ \mathcal{F}_H(t_o, s_o) \\
\leq & \ \max\{\mathcal{F}_H(t_o), \mathcal{F}_H(s_o)\} \\
= & \ \max\{\inf_{t \in f^{-1}(u)} \mathcal{F}_H(t), \inf_{t \in f^{-1}(v)} \mathcal{F}_H(t)\} \\
= & \ \max\{\mathcal{F}_{f(H)}(u), \mathcal{F}_{f(H)}(v)\}.
\end{aligned}$$

Hence $f(H)$ is a single-valued neutrosophic soft K-subalgebra of $\mathcal{K}_2$. To show that $f(H)$ is a single-valued neutrosophic soft topological K-algebra of $\mathcal{K}_2$, i.e., the self map $\rho_b : (f(H), \chi_{f(H)}) \to (f(H), \chi_{f(H)})$, defined as $\rho_b(v) = v \odot b$, $\forall\, b \in \mathcal{K}_2$ is a relatively single-valued neutrosophic soft continuous mapping. Let Y_H be a single-valued neutrosophic soft set in χ_H, then there exists a single-valued neutrosophic soft set Y in χ_1 be such that $Y_H = Y \cap H$.

$$\rho^{-1}{}_b(Y_{f(H)}) \cap f(H) \in \chi_{f(H)}$$

Then $f(Y_H) = f(Y \cap H) = f(Y) \cap f(H)$ is a single-valued neutrosophic soft set in $\chi_{f(H)}$ since f is an injective function. Thus, f is relatively single-valued neutrosophic soft open. Since f is also an onto function, then for all $b \in \mathcal{K}_2$ and $a \in \mathcal{K}_1$, $a = f(b)$, we have

$$\begin{aligned}
\mathcal{T}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{T}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u) \\
&= \mathcal{T}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big) \\
&= \mathcal{T}_{(Y_{f(H)})}\big(\rho_{f(a)}\big(f(u)\big)\big) \\
&= \mathcal{T}_{(Y_{f(H)})}\big(f(u) \odot f(a)\big) \\
&= \mathcal{T}_{f^{-1}(Y_{f(H)})}(u \odot a) \\
&= \mathcal{T}_{f^{-1}(Y_{f(H)})}(\rho_a(u)) \\
&= \mathcal{T}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{I}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u) \\
&= \mathcal{I}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big) \\
&= \mathcal{I}_{(Y_{f(H)})}\big(\rho_{f(a)}\big(f(u)\big)\big) \\
&= \mathcal{I}_{(Y_{f(H)})}\big(f(u) \odot f(a)\big) \\
&= \mathcal{I}_{f^{-1}(Y_{f(H)})}(u \odot a) \\
&= \mathcal{I}_{f^{-1}(Y_{f(H)})}(\rho_a(u)) \\
&= \mathcal{I}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u),
\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{f^{-1}(\rho^{-1}{}_b(Y_{f(H)}))}(u) &= \mathcal{F}_{f^{-1}(\rho^{-1}{}_{f(a)}(Y_{f(H)}))}(u)\\ &= \mathcal{F}_{\rho^{-1}{}_{f(a)}(Y_{f(H)})}\big(f(u)\big)\\ &= \mathcal{F}_{(Y_{f(H)})}\big(\rho_{f(a)}\big(f(u)\big)\big)\\ &= \mathcal{F}_{(Y_{f(H)})}\big(f(u)\odot f(a)\big)\\ &= \mathcal{F}_{f^{-1}(Y_{f(H)})}(u\odot a)\\ &= \mathcal{F}_{f^{-1}(Y_{f(H)})}(\rho_a(u))\\ &= \mathcal{F}_{\rho^{-1}{}_{(a)}}(f^{-1}(Y_{f(H)}))(u).\end{aligned}$$

This shows that $f^{-1}(\rho_{(b)}^{-1}((Y_{f(H)}))) = \rho_{(a)}^{-1}(f^{-1}(Y_{(H)}))$. Since $\rho_a : (H,\chi_H) \to (H,\chi_H)$ is relatively single-valued neutrosophic soft continuous mapping and f is also relatively single-valued neutrosophic soft continues function. Therefore, $f^{-1}(\rho_{(b)}^{-1}((Y_{f(H)}))) \cap H = \rho_{(a)}^{-1}(f^{-1}(Y_{(H)})) \cap H$ is a single-valued neutrosophic soft set in χ_H. Thus, $f(f^{-1}(\rho_{(b)}((Y_{f(H)}))) \cap \mathcal{A}) = \rho_{(b)}^{-1}(Y_{f(\mathcal{A})}) \cap f(\mathcal{A})$ is a single-valued neutrosophic soft set in $\chi_{\mathcal{A}}$. □

Example 3.20. Consider a K-algebra $\mathcal{K}$ on a cyclic group of order 8 and Cayley's table for $\odot$ is given Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. Consider a set of parameters $E = \{l_1, l_2\}$ and single-valued neutrosophic soft sets H, J, L defined as:

$$\zeta_H(l_1) = \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\},$$
$$\zeta_H(l_2) = \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},$$

$$\zeta_J(l_1) = \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_J(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},$$

$$\zeta_L(l_1) = \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},$$
$$\zeta_L(l_2) = \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}$$

for all $h \neq e \in G$. Then the family $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, L\}$ is a single-valued neutrosophic soft topology on $\mathcal{K}$ and $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft topological space over $\mathcal{K}$. We define another single-valued neutrosophic soft set Q in $\mathcal{K}$ as:

$$\zeta_Q(l_1) = \{(e, 0.8, 0.5, 0.1), (h, 0.6, 0.4, 0.3)\},$$
$$\zeta_Q(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\}.$$

It is obvious that Q is a single-valued neutrosophic soft K-algebra of $\mathcal{K}$.
Now, we prove that the self map $\rho_a : (Q, E, \chi_Q) \to (Q, E, \chi_Q)$, defined as $\rho_a(s) = s \odot a$ for all $a \in \mathcal{K}$, is a relatively single-valued neutrosophic soft continuous mapping.
We get $Q \cap \emptyset_E = \emptyset_E, Q \cap 1_E = 1_E, Q \cap H = R_1, Q \cap J = R_2, Q \cap L = R_3$, where R_1, R_2, R_3 are as follows:

$$\zeta_{R_1}(l_1) = \{(e, 0.8, 0.5, 0.2), (h, 0.6, 0.4, 0.4)\},$$
$$\zeta_{R_1}(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.6)\},$$

$$\zeta_{R_2}(l_1) = \{(e, 0.7, 0.5, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_{R_2}(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.4, 0.7)\},$$

$$\zeta_{R_3}(l_1) = \{(e, 0.8, 0.5, 0.1), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_{R_3}(l_2) = \{(e, 0.5, 0.6, 0.5), (h, 0.3, 0.4, 0.7)\}.$$

Thus, $\chi_Q = \{\emptyset_E, 1_E, R_1, R_2, R_3\}$ is a relatively topology of Q and (Q, E, χ_Q) is a single-valued neutrosophic soft subspace of $(\mathcal{K}, E, \chi_{\mathcal{K}})$. Since ρ_a is a homomorphism, then for a single-valued neutrosophic soft set $R \in \chi_Q$, $\rho_a^{-1}(R) \cap Q \in \chi_Q$. Which shows that $\rho_a : (Q, E, \chi_Q) \to (Q, E, \chi_Q)$ is relatively single-valued neutrosophic soft continuous mapping. Therefore, Q is a single-valued neutrosophic soft topological K-algebra.

4 Single-Valued Neutrosophic Soft C_5-connected K-Algebras

In this section, we discuss single-valued neutrosophic soft C_5-connected K-algebras.

Definition 4.1. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. A *single-valued neutrosophic soft separation* of $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a pair of nonempty single-valued neutrosophic soft open sets H, J if the following conditions hold:

(i) $H \cup J = 1_E$.

(ii) $H \cap J = \emptyset_E$.

Definition 4.2. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. Then $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is called a *single-valued neutrosophic soft C_5-disconnected* if there exists a single-valued neutrosophic soft separation of $(\mathcal{K}, E, \chi_{\mathcal{K}})$, otherwise C_5-connected.

Definition 4.2 can be written as:

Definition 4.3. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. If there exists a single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set L such that $L \neq 1_E$ and $L \neq \emptyset_E$, then $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is called a *single-valued neutrosophic soft C_5-disconnected*, otherwise $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is called a single-valued neutrosophic soft C_5-connected.

Example 4.4. By considering Example 3.3, we consider a single-valued neutrosophic soft topological space $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J, L\}$. Since $H \cap J \neq \emptyset_E$, $H \cap L \neq \emptyset_E$, $J \cap L \neq \emptyset_E$ and $H \cup J \neq 1_E$, $H \cup L \neq 1_E$, $J \cup L \neq 1_E$. Thus, $\chi_{\mathcal{K}}$ is a single-valued neutrosophic soft C_5-connected.

Example 4.5. Every indiscrete single-valued neutrosophic soft space is C_5-connected since the only single-valued neutrosophic soft sets in single-valued neutrosophic soft indiscrete space that are both single-valued neutrosophic soft open and single-valued neutrosophic soft closed are $\emptyset_E$ and 1_E.

Theorem 4.6. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space on K-algebra $\mathcal{K}$. Then $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft C_5-connected if and only if $\chi_{\mathcal{K}}$ contains only $\emptyset_E$ and 1_E which are both single-valued neutrosophic soft open and single-valued neutrosophic soft closed.

Proof. Straightforward. □

Proposition 4.7. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras and $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$, $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ two single-valued neutrosophic soft topological spaces on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a single-valued neutrosophic soft continuous surjective function. If $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ is a single-valued neutrosophic soft C_5-connected space, then $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ is also single-valued neutrosophic soft C_5-connected.

Proof. Let $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ and $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ be two single-valued neutrosophic soft topological spaces and $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ be a single-valued neutrosophic soft C_5-connected space. We prove that $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ is also single-valued neutrosophic soft C_5-connected. Let us suppose on contrary that $(\mathcal{K}_2, \chi_2)$ be a single-valued neutrosophic soft C_5-disconnected space. According to Definition 4.3, we have both single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set L such that $L \neq 1_{SN}$ and $L \neq \emptyset_{SN}$. Then $f^{-1}(L) = 1_{SN}$ or $f^{-1}(L) = \emptyset_{SN}$ since f is a single-valued neutrosophic soft continuous surjective mapping , where $f^{-1}(L)$ is both single-valued neutrosophic soft open set and single-valued neutrosophic soft closed set. Therefore, $L = f(f^{-1}(L)) = f(1_{SN}) = 1_{SN}$ and $L = f(f^{-1}(L)) = f(\emptyset_{SN}) = \emptyset_{SN}$, a contradiction. Hence $(\mathcal{K}_2, E, \chi_2)$ is a single-valued neutrosophic soft C_5-connected space. □

5 Single-Valued Neutrosophic Soft Super Connected K-Algebras

Definition 5.1. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$ and $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ a single-valued neutrosophic soft set in $\mathcal{K}$. Then the *interior* and *closure* of H in a K-algebra $\mathcal{K}$ is defines as:

$$H^{Int} = \bigcup\{O : O \text{ is a single-valued neutrosophic soft open set in } \mathcal{K} \text{ and } O \subseteq H\},$$
$$H^{Clo} = \bigcap\{C : C \text{ is a single-valued neutrosophic soft closed set in } \mathcal{K} \text{ and } H \subseteq C\}.$$

It is interesting to note that H^{Int}, being union of single-valued neutrosophic soft open sets is single-valued neutrosophic soft open and H^{Clo}, being intersection of single-valued neutrosophic soft closed set is single-valued neutrosophic soft closed.

Theorem 5.2. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space on $\mathcal{K}$. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft set in $\chi_{\mathcal{K}}$. Then H^{Int} is the largest single-valued neutrosophic soft open set contained in H.

Proof. Obvious. □

Proposition 5.3. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. Then the following properties hold:

(i) $(1_E)^{Int} = 1_E$.

(ii) $(\emptyset_E)^{Clo} = \emptyset_E$.

(iii) $\overline{(H)}^{Int} = \overline{(H)^{Clo}}$.

(iv) $\overline{(H)}^{Clo} = \overline{(H)^{Int}}$.

Corollary 5.4. If H is a single-valued neutrosophic soft set in $\mathcal{K}$, then H is single-valued neutrosophic soft open if and only if $H^{Int} = H$ and H is a single-valued neutrosophic soft closed if and only if $H^{Clo} = H$.

Definition 5.5. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space on $\mathcal{K}$ and $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft open set in $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft regular open* if

$$H = (H^{Clo})^{Int}.$$

Remark 5.6. (1) Every single-valued neutrosophic soft regular is single-valued neutrosophic soft open.
(2) Every single-valued neutrosophic soft clopen set is single-valued neutrosophic soft regular open.

Definition 5.7. Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Then $\mathcal{K}$ is called a *single-valued neutrosophic soft super disconnected* if there exists a single-valued neutrosophic soft regular open set $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ such that $1_E \neq H$ and $\emptyset_E \neq H$. But if there does not exist such a single-valued neutrosophic soft regular open set H such that $1_E \neq H$ and $\emptyset_E \neq H$, then $\mathcal{K}$ is called *single-valued neutrosophic soft super connected.*

Example 5.8. Consider a K-algebra on a cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. We have a single-valued neutrosophic soft topology $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$, where H, J with a parametric set $E = \{l_1, l_2\}$ are given as:

$$\zeta_H(l_1) = \{(e, 0.8, 0.7, 0.2), (h, 0.6, 0.5, 0.4)\},$$
$$\zeta_H(l_2) = \{(e, 0.7, 0.7, 0.2), (h, 0.6, 0.6, 0.5)\},$$

$$\zeta_J(l_1) = \{(e, 0.7, 0.7, 0.2), (h, 0.4, 0.1, 0.5)\},$$
$$\zeta_J(l_2) = \{(e, 0.4, 0.6, 0.6), (h, 0.3, 0.5, 0.7)\},$$

for all $h \neq e \in G$.
Let L be a single-valued neutrosophic soft set in $\mathcal{K}$, defined by:

$$\zeta_L(l_1) = \{(e, 0.9, 0.8, 0.1), (h, 0.7, 0.6, 0.4)\},$$
$$\zeta_L(l_2) = \{(e, 0.9, 0.7, 0.1), (h, 0.7, 0.6, 0.4)\}.$$

Now, we have single-valued neutrosophic soft open sets : $\emptyset_E, 1_E, H, J$.
single-valued neutrosophic soft closed sets : $(\emptyset_E)^c = 1_E,\ (1_E)^c = \emptyset_E,\ (H)^c = H',\ (J)^c = J'$, where H', J' are obtained as:

$$\zeta_{H'}(l_1) = \{(e, 0.2, 0.7, 0.8), (h, 0.4, 0.5, 0.6)\},$$
$$\zeta_{H'}(l_2) = \{(e, 0.2, 0.7, 0.7), (h, 0.5, 0.6, 0.6)\},$$

$$\zeta_{J'}(l_1) = \{(e, 0.2, 0.7, 0.7), (h, 0.5, 0.1, 0.4)\},$$
$$\zeta_{J'}(l_2) = \{(e, 0.6, 0.6, 0.4), (h, 0.7, 0.5, 0.3)\},$$

for all $h \neq e \in G$. Then, interior and closure of a single-valued neutrosophic soft set L is obtained as:

$$L^{Int} = H,$$
$$L^{Clo} = 1_E.$$

For L to be a single-valued neutrosophic soft regular open, then $L = (L^{Clo})^{Int}$. But since $L = (1_E)^{Int} = 1_E \neq L$. This shows that $1_E \neq L \neq \emptyset_E$ is not a single-valued neutrosophic soft regular open set. By Definition 5.7, defined K-algebra is a single-valued neutrosophic soft super connected K-algebra.

6 Single-Valued Neutrosophic Soft Compactness K-Algebras

Definition 6.1. Let $\chi_{\mathcal{K}}$ be a single-valued neutrosophic soft topology on $\mathcal{K}$. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. A collection $\Omega = \{(\mathcal{T}_{\zeta_{H_i}}, \mathcal{I}_{\zeta_{H_i}}, \mathcal{F}_{\zeta_{H_i}}) : i \in I\}$ of single-valued neutrosophic soft sets in $\mathcal{K}$ is called a *single-valued neutrosophic soft open covering* of H if $H \subseteq \bigcup \Omega$. A finite sub-collection of Ω say $(\Omega^{'})$ is also a single-valued neutrosophic soft open covering of H, called a *finite subcovering* of H.

Definition 6.2. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space of $\mathcal{K}$. Let H be a single-valued neutrosophic soft set in $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft compact* if every single-valued neutrosophic soft open covering Ω of H has a finite sub-covering $(\Omega^{'})$.

Example 6.3. A single-valued neutrosophic soft topological space $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is single-valued neutrosophic soft compact if either $\mathcal{K}$ is finite or $\chi_{\mathcal{K}}$ is a finite single-valued neutrosophic soft topology on $\mathcal{K}$.

Proposition 6.4. Let $f : (\mathcal{K}_1, E, \chi_{\mathcal{K}_1}) \to (\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ be a single-valued neutrosophic soft continuous mapping, where $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$ and $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$ are two single-valued neutrosophic soft topological spaces of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. If H is a single-valued neutrosophic soft compact in $(\mathcal{K}_1, E, \chi_{\mathcal{K}_1})$, then $f(H)$ is single-valued neutrosophic soft compact in $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$.

Proof. Let f be a single-valued neutrosophic soft continuous map from $\mathcal{K}_1$ into $\mathcal{K}_2$. Let $\Omega = \{f^{-1}(H_i : i \in I)\}$ be a single-valued neutrosophic soft open covering of H and $\Delta = \{H_i : i \in I\}$ a single-valued neutrosophic soft open covering of $f(H)$. Then there exists a single-valued neutrosophic soft finite sub-covering $\bigcup\limits_{I=1}^{n} f^{-1}(H_i)$ such that

$$H \subseteq \bigcup_{i=1}^{n} f^{-1}\big(H_i\big).$$

Thus,

$$f(H) \subseteq \bigcup_{i=1}^{n} \big(H_i\big)$$

$$H \subseteq \bigcup_{i=1}^{n} f^{-1}(H_i)$$
$$f(H) \subseteq f\big(\bigcup_{i=1}^{n} f^{-1}(H_i)\big)$$
$$f(H) \subseteq \bigcup_{i=1}^{n} \big(f(f^{-1}(H_i))\big)$$
$$f(H) \subseteq \bigcup_{i=1}^{n} (H_i).$$

This shows that there exists a single-valued neutrosophic soft finite sub-covering of $f(H)$. Therefore, $f(H)$ is single-valued neutrosophic soft compact in $(\mathcal{K}_2, E, \chi_{\mathcal{K}_2})$. □

7 Single-Valued Neutrosophic Soft Hausdorff K-Algebras

Definition 7.1. Let $H = \{\mathcal{T}_{\zeta_H}, \mathcal{I}_{\zeta_H}, \mathcal{F}_{\zeta_H}\}$ be a single-valued neutrosophic soft set in a $\mathcal{K}$. Then H is called a *single-valued neutrosophic soft point* if, for $\theta \in E$

$$\zeta_H(\theta) \neq \emptyset_E,$$

and

$$\zeta_H(\theta') = \emptyset_E,$$

for all $\theta' \in E - \{\theta\}$. A single-valued neutrosophic soft point in H is denoted by θ_H.

Definition 7.2. A single-valued neutrosophic soft point θ_H is said to *belong* to a single-valued neutrosophic soft set J, i.e., $\theta_H \in J$ if, for $\theta \in E$

$$\zeta_H(\theta) \leq \zeta_J(\theta).$$

Definition 7.3. Let $(\mathcal{K}, E, \chi_{\mathcal{K}})$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$ and θ_L, θ_Q be two single-valued neutrosophic soft points in $\mathcal{K}$. If for these two single-valued neutrosophic soft points, there exist two disjoint single-valued neutrosophic soft open sets H, J such that $\theta_L \in H$ and $\theta_Q \in J$. Then $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is called a *single-valued neutrosophic soft Hausdorff topological space* over $\mathcal{K}$ and $\mathcal{K}$ is called a single-valued neutrosophic soft Hausdorff K-algebra.

Example 7.4. Consider a K-algebra $\mathcal{K}$ on a cyclic group of order 8 and Cayley's table for $\odot$ is given in Example 3.3, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. Let $E = \{l\}$ and $\chi_{\mathcal{K}} = \{\emptyset_E, 1_E, H, J\}$ be a single-valued neutrosophic soft topological space over $\mathcal{K}$. We define two single-valued neutrosophic soft points l_L, l_Q such that

$$l_L = \{(e, 1, 0, 1), (h, 0, 0, 1)\},$$
$$l_Q = \{(e, 0, 0, 1), (h, 0, 1, 0)\}.$$

Since for $l \in E, \zeta_L(l) \neq \emptyset_E, \zeta_Q(l) \neq \emptyset_E$, and $l_L \neq l_Q$, then clearly l_L and l_Q are two single-valued neutrosophic soft points. Now, consider two single-valued neutrosophic soft open sets H and J defined as:

$$\zeta_H(l) = \{(e,1,1,0),(h,0,0,1)\},$$
$$\zeta_J(l) = \{(e,0,0,1),(h,1,1,0)\},$$

for all $h \neq e \in G$. Since $\zeta_L(l) \leq \zeta_H(l)$ and $\zeta_Q(l) \leq \zeta_J(l)$, i.e., $l_L \in H$ and $l_Q \in J$ and $H \cap J = \emptyset_E$. Thus, $(\mathcal{K}, E, \chi_{\mathcal{K}})$ is a single-valued neutrosophic soft Hausdorff space and $\mathcal{K}$ is a single-valued neutrosophic soft Hausdorff K-algebra.

Theorem 7.5. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft homomorphism. Then $\mathcal{K}_1$ is a single-valued neutrosophic soft Hausdorff space if and only if $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra.

Proof. Let $f : (\mathcal{K}_1, E, \chi_1) \to (\mathcal{K}_2, E, \chi_2)$ be a single-valued neutrosophic soft homomorphism and χ_1, χ_2 be two single-valued neutrosophic soft topologies on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Suppose that $\mathcal{K}_1$ is a single-valued neutrosophic soft Hausdorff space. To prove that $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra, Let for $l \in E$, l_L and l_Q be two single-valued neutrosophic soft points in χ_2 such that $l_L \neq l_Q$ with $u, v \in \mathcal{K}_1$, $u \neq v$. Then for these two distinct single-valued neutrosophic soft points, there exist two single-valued neutrosophic soft open sets H and J such that $l_L \in H$, $l_Q \in J$ with $H \bigcap J = \emptyset_E$. For $x \in \mathcal{K}_1$, we consider

$$(f^{-1}(l_L))(x) = l_L(f^{-1}(x)) = \begin{cases} s \in (0,1] & if x = f^{-1}(u), \\ 0 & otherwise. \end{cases}$$
$$= ((f^{-1}(l))_L(x))$$

Therefore, $f^{-1}(l_L) = (f^{-1}(l))_L$. Likewise, $f^{-1}(l_Q) = (f^{-1}(l))_Q$. Since f is a single-valued neutrosophic soft continuous function from $\mathcal{K}_1$ into $\mathcal{K}_2$ and also f^{-1} is a single-valued neutrosophic soft continuous function from $\mathcal{K}_2$ into $\mathcal{K}_1$, then there exist two disjoint single-valued neutrosophic soft open sets $f(H)$ and $f(J)$ of single-valued neutrosophic soft points l_L and l_Q, respectively be such that $f(H) \bigcap f(J) = f(\emptyset_E) = \emptyset_E$. This shows that $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra. The proof of converse part is straightforward. □

Theorem 7.6. let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a bijective single-valued neutrosophic soft continuous function, where $\mathcal{K}_1$ is a single-valued neutrosophic soft compact K-algebra and $\mathcal{K}_2$ is a single-valued neutrosophic soft Hausdorff K-algebra. Then mapping f is a $\mathcal{K}_1$ is a single-valued neutrosophic soft homomorphism.

Proof. Let f be a bijective single-valued neutrosophic soft mapping from a single-valued neutrosophic soft compact K-algebra into a single-valued neutrosophic soft Hausdorff K-algebra. Then clearly, f is a single-valued neutrosophic soft homomorphism. We only prove that f is single-valued neutrosophic soft closed since f is a bijective mapping. Let a single-valued neutrosophic soft set $Q = \{\mathcal{T}_{\zeta_Q}, \mathcal{I}_{\zeta_Q}, \mathcal{F}_{\zeta_Q}\}$ be closed in K-algebra $\mathcal{K}_1$. Now if $Q = \emptyset_E$, then $f(Q) = \emptyset_E$ is single-valued neutrosophic soft closed in $\mathcal{K}_2$. But if $Q \neq \emptyset_E$, then being a subset of a single-valued neutrosophic soft compact K-algebra, Q is single-valued neutrosophic soft compact. Also $f(Q)$ is single-valued neutrosophic soft compact, being a single-valued neutrosophic soft continuous image of a single-valued neutrosophic soft compact K-algebra. Hence f is closed thus, f is a single-valued neutrosophic soft homomorphism. □

8 Conclusions

In 1998, Smarandache originally considered the concept of neutrosophic set from philosophical point of view. The notion of a single-valued neutrosophic set is a subclass of the neutrosophic set from a scientific and engineering point of view, and an extension of intuitionistic fuzzy sets [32]. In 1999, Molodtsov introduced the idea of soft set theory as another powerful mathematical tool to handle indeterminate and inconsistent data. This theory fixes the problem of establishing the membership function for each specific case by giving a parameterized outlook to indeterminacy. By using a hybrid model of these two mathematical techniques with a topological structure, we have developed the concept of single-valued neutrosophic soft topological K-algebras to analyze the element of indeterminacy in K-algebras. We have defined some certain concepts such as the interior, closure, C_5-connected, super connected, compactness and Hausdorff of single-valued neutrosophic soft topological K-algebras. In future, we aim to extend our notions to (1) Rough neutrosophic K-algebras, (2) Soft rough neutrosophic K-algebras, (3) Bipolar neutrosophic soft K-algebras, and (4) Rough neutrosophic K-algebras.

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On Multi-Criteria Decision Making problem via Bipolar Single-Valued Neutrosophic Settings

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Abstract. In this paper, the idea bipolar single-valued neutrosophic (BSVN) set was introduced. We also introduce bipolar single-valued neutrosophic topological space and some of its properties were characterized. Comparing Bipolar single-valued neutrosophic sets with score function, certainty function and accuracy function .Bipolar single-valued neutrosophic weighted average operator (A_ω) and bipolar single-valued neutrosophic weighted geometric operator (G_ω) were developed and based on Bipolar single-valued neutrosophic set, a multiple decision making problem were evaluated through an example to select the desirable one.

Keywords: Bipolar single-valued neutrosophic set, bipolar single-valued neutrosophic topological space, bipolar single-valued neutrosophic average operator, bipolar single-valued neutrosophic geometric operator, score, certainty and accuracy functions.

1. Introduction

Fuzzy Logic resembles the human decision making methodology.Zadeh [39] who was considered as the Father of Fuzzy Logic introduced the fuzzy sets in 1965 and it is a tool in learning logical subject. He put forth the concept of fuzzy sets to deal with contrasting types of uncertainties. Using single value $\mu_A(x)\in[0, 1]$, the degree of membership of the fuzzy set is in classic fuzzy, which is defined on a universal scale, they cannot grasp convinced cases where it is hard to define μ_A by one specific value.

Intuitionistic fuzzy sets which was proposed by Atanassov [2] is the extension of Zadeh's Fuzzy Sets to overthrown the lack of observation of non-membership degrees. Intuitionistic fuzzy sets generally tested in solving multi-criteria decision making problems. Intuitionistic fuzzy sets detailed into the membership degree, non-membership degree and simultaneously with degree of indeterminancy.

Neutrosophic is the base for the new mathematical theories derives both their classical and fuzzy depiction. Smarandache [4,5] introduced the neutrosophic set . Neutrosophic set has the capability to induce classical sets, fuzzy set, Intuitionistic fuzzy sets. Introduceing the components of the neutrosophic set are True(T), Indeterminacy(I), False(F) which represent the membership, indeterminacy, and non-membership values respectively.The notion of classical set, fuzzy set [17], interval-valued fuzzy set [39], Intuitionistic fuzzy [2], etc were generalized by the neutrosophic set. Majumdar & Samant [19] recommended the Single-valued neutrosophic sets (SVNSs), which is a variation of Neutrosophic Sets. Wang, et.al [38] describe an example of neutrosophic set and sgnify single valued Neutrosophic set (SVNs).They give many properties of Single-Valued Neutrosophic Set, which are associated to the operations and relations by Single-Valued Neutrosophic Sets.The correlation coefficient of SVNSs placed on the development of the correlation coefficient of Intuitionistic fuzzy sets and tested that the cosine similarity measure of SVNS is a special case of the correlation coefficient and correlated it to single valued neutrosophic multicriteria decision-making problems which was presented by Jun Ye [7]. For solving multi-criteria decision-making problems, he overworked similarity measure for interval valued neutrosophic set. Single valued neutrosophic sets (SVNSs) can handle the undetermined and uncertain information and also symbolize, which fuzzy sets and Intuitionistic fuzzy sets cannot define and finalize.

Turksen [37] proposed the Interval-valued fuzzy set is similar as Intuitionistic fuzzy set. The concept is to hook the anxiety of class of membership . Interval- valued fuzzy set need an interval value $[\mu_A{}^L(a), \mu_A{}^U(a)]$ with $0\leq\mu_A{}^L(a)\leq\mu_A{}^U(a)\leq1$ to represent the class of membership of a fuzzy set A. But it is not suffient to take only the membership function, but also to have the non-membership function .

Bipolar fuzzy relations was given by Bosc and Pivert [3] where a pair of satisfaction degrees is made with each tuple. In 1994, an development of fuzzy set termed bipolar fuzzy was given by Zhang [40].By the notion of fuzzy sets, Lee [16] illustrate bipolar fuzzy sets. Manemaran and Chellappa [20] provide some applications in groups are called the bipolar fuzzy groups, fuzzy d-ideals of groups under (T-S) norm. They also explore few properties of the groups and the relations. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras were researched by K. J. Lee[17]. Multiple attribute decision-making method situated on single-valued neutrosophic was granted by P. Liu and Y. Wang[18].

In bipolar neutrosophic environment, bipolar neutrosophic sets(BNS) was developed by Irfan Deli [6] and et.al. The application based on multi-criteria decision making problems were also given by them in bipolar neutrosophic set. To collect bipolar neutrosophic information, they defined score, accuracy, and certainty functions to compare BNS and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators. In the study, a Multi Criteria Decision Making approach were discussed on the basis of score, accuracy, and certainty functions, bipolar Neutrosophic Weighted Average and bipolar Neutrosophic Weighted Geometric operators were calculated. Fuzzy neutrosophic sets and its Topological spaces was introduced by I.Arockiarani and J.Martina Jency [1].

Positive and Negative effects count on Decision making . Multiple decision-making problems have gained very much attention in the area of systemic optimization, urban planning, operation research, management science and many other fields. Correlation Coefficient between Single Valued Neutrosophic Sets and its Multiple Attribute Decision Making Method given by Jun Ye [7]. A Neutrosophic Multi-Attribute Decision making with Unknown Weight data was investigated by Pranab Biswas, Surapati Pramanik, Bibhas C. Giri[30]. Neutrosophic Tangent Similarity Measure and its Application was given by Mondal, Surapati Pramanik [11]. Many of the authors[8-14,21,22,24-29,31,32,33,35,36] studied and examine different and variation of neutrosophic set theory in Decision making problems.

Here, we introduce bipolar single-valued neutrosophic set which is an expansion of the fuzzy sets, Intuitionistic fuzzy sets, neutrosophic sets and bipolar fuzzy sets. Bipolar single-valued neutrosophic topological spaces were also proposed. Bipolar single-valued neutrosophic topological spaces characterized a few of its properties and a numerical example were illustrated. Bipolar single-valued neutrosophic sets were compared with score function, certainty function and accuracy function. Then,the bipolar single-valued Neutrosophic weighted average operator (A_ω) and bipolar single-valued neutrosophic weighted geometric operator (G_ω) are developed to aggregate the data.To determine the application and the performance of this method to choose the best one, atlast a numerical example of the method was given.

2 Preliminaries

2.1 Definition [34]: Let X be a non-empty fixed set. A neutrosophic set B is an object having the form $B=\{<x,\mu_B(x),\sigma_B(x),\gamma_B(x)>x\in X\}$ Where $\mu_B(x),\sigma_B(x)$ and $\gamma_B(x)$ which represent the degree of membership function , the degree of indeterminacy and the degree of non-membership respectively of each element $x\in X$ to the set B .

2.2 Definition [38]: Let a universe X of discourse. Then $A_{NS}=\{<x,F_A(x),T_A(x)I_A(x)>x\in X\}$ defined as a single-valued neutrosophic set where truth-membership function T_A:X→[0,1],an indeterminacy-membership function I_A: X→[0,1] and a falsity-membership function F_A: X → [0,1].No restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0\leq \sup T_A(x) \leq \sup I_A(x) \leq \sup F_A(x) \leq 3$. $\widetilde{A}$ =<T, I, F> is denoted as a single-valued neutrosophic number.

2.3 Definition [23]: Let two single-valued neutrosophic number be $\widetilde{A}_1=<T_1, I_1, F_1>$ and $\widetilde{A}_2=<T_2, I_2, F_2>$. Then,the operations for NNs are defined as follows:

i. $\lambda \widetilde{A} =<1-(1-T_1)^\lambda,I_1^\lambda,F_1^\lambda>$
ii. $\widetilde{A}_1^\lambda =<(T_1^\lambda,1-(1-I_1)^\lambda,1-(1-F_1)^\lambda>$
iii. $\widetilde{A}_1+\widetilde{A}_2=<T_1+T_2-T_1T_2,I_1I_2,F_1F_2>$
iv. $\widetilde{A}_1.\widetilde{A}_2=<T_1T_2,I_1+I_2-I_1I_2,F_1+F_2-F_1F_2>$

2.4 Definition [15]: Let a single-valued neutrosophic number be $\widetilde{B}_1=<T_1, I_1, F_1>$. Then, SNN are defined as

i. score function $s(\widetilde{B}_1) = (T_1+1- I_1+ 1-F_1)/3$;
ii. accuracy function $a(\widetilde{B}_1) = T_1-F_1$;
iii. certainty function $c(\widetilde{B}_1) = T_1$.

2.5 Definition [23]: Let two single-valued neutrosophic number be $\widetilde{B}_1=<T_1, I_1, F_1>$ and $\widetilde{B}_2=<T_2, I_2, F_2>$. The comparison method defined as:

i. if $s(\widetilde{B}_1) > s(\widetilde{B}_2)$, then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by $\widetilde{B}_1>\widetilde{B}_2$.
ii. if $s(\widetilde{B}_1) = s(\widetilde{B}_2)$ and $a(\widetilde{B}_1)>a(\widetilde{B}_2)$, then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by

$\widetilde{B}_1 < \widetilde{B}_2$.

iii.if s($\widetilde{B}_1$) = s($\widetilde{B}_2$) and a($\widetilde{B}_1$)=a($\widetilde{B}_2$) and c($\widetilde{B}_1$) >c($\widetilde{B}_2$), then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by $\widetilde{B}_1 > \widetilde{B}_2$.

iv.if s($\widetilde{B}_1$) = s($\widetilde{B}_2$) and a($\widetilde{B}_1$)=a($\widetilde{B}_2$) and c($\widetilde{B}_1$)=c($\widetilde{B}_2$), then $\widetilde{B}_1$ is equal to $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is indifferent to $\widetilde{B}_2$, denoted by $\widetilde{B}_1 = \widetilde{B}_2$.

2.6 Definition [6]: In *X,* a bipolar neutrosophic set B is defined in the form

$$B=<x, (T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)):x\in X>$$

Where T^+, I^+, F^+: X→ [1, 0] and T^- , I^- , F^- : X [-1, 0].The positive membership degree denotes the truth membership $T^+(x)$, indeterminate membership I^+ (x) and false membership F^+ (x) of an element x∈X corresponding to the set *A* and the negative membership degree denotes the truth membership $T^-(x)$, indeterminate membership $I^-(x)$ and false membership $F^-(x)$ of an element x∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set .

2.7 Definition [39, 2]: Each element had a degree of membership (T) in the fuzzy set . The Intuitionistic fuzzy set on a universe, where the degree of membership $\mu_B(x) \in[0,1]$ of each element x∈X to a set B, there was a degree of non-membership $\nu_B(x)\in[0,1]$,such that $\forall$ x∈X, $\mu_B(x) + \nu_B(x) \leq 1$.

2.8 Definition [15, 20]: Let a non-empty set be X. Then, B_{BF}= {<x, $\mu^+_B(x)$, $\mu^-_B(x)$>: x∈X}is a bipolar-valued fuzzy set denoted by B_{BF} ,where μ^+_B: X → [0, 1] and μ^-_B: X → [0, 1]. The positive Membership degree $\mu^+_B(x)$ denotes the satisfaction degree of an element x to the property corresponding to B_{BF} and the negative membership degree $\mu_B^-(x)$ denotes the satisfaction degree of x to some implicit counter property of B_{BF} .

In this section, we give the concept bipolar single-valued neutrosophic set and its operations. We also developed the bipolar single-valued neutrosophic weighted (A_ω) average operator and geometric operator (G_ω). Some of it is quoted from [2, 5, 7, 10, and 14].

3. Bipolar single-valued Neutrosophic set(BSVN):

3.1 Definition : A Bipolar Single-Valued Neutrosophic set (BSVN) S in X is defined in the form of

$$\mathbf{BSVN\ (S) =<v,(T_{BSVN}^+,T_{BSVN}^-),(I_{BSVN}^+,I_{BSVN}^-),(F_{BSVN}^+,F_{BSVN}^-):v\in X>}$$

where $(T_{BSVN}^+,I_{BSVN}^+,F_{BSVN}^+)$:X→[0,1] and $(T_{BSVN}^-,I_{BSVN}^-,F_{BSVN}^-)$:X→ [-1,0] .In this definition, there T_{BSVN}^+ and T_{BSVN}^- are acceptable and unacceptable in past. Similarly I_{BSVN}^+ and I_{BSVN}^- are acceptable and unacceptable in future. F_{BSVN}^+ and F_{BSVN}^- are acceptable and unacceptable in present respectively.

3.2 Example : Let X={s_1,s_2,s_3}.Then a bipolar single-valued neutrosophic subset of *X* is

$$S=\begin{Bmatrix} < s_1,(0.1,-0.1),(0.2,-0.3),(0.3,-0.5) > \\ < s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5) > \\ < s_3,(0.2,-0.8),(0.6,-0.4),(0.7,-0.7) > \end{Bmatrix}$$

3.3 Definition : Let two bipolar single-valued neutrosophic sets $BSVN_1(S)$ and $BSVN_2(S)$ in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)):v\in X>$ and
$BSVN_2(S)=<v,(T_{BSVN}^+(2),T_{BSVN}^-(2)),(I_{BSVN}^+(2),I_{BSVN}^-(2)),(F_{BSVN}^+(2),F_{BSVN}^-(2)):v\in X>$. Then the operators are defined as follows:

(i) Complement

$BSVN^c(S)$ = {< v,(1-T_{BSVN}^+), (-1-T_{BSVN}^-), (1-I_{BSVN}^+), (-1-I_{BSVN}^-), (1-F_{BSVN}^+), (-1-F_{BSVN}^-): v∈X >}

(ii) Union of two BSVN

$BSVN_1(S)$ U $BSVN_2(S)$ =

$$\left\langle \begin{matrix} \max(T_{BSVN}^+(1),T_{BSVN}^+(2)),\min(I_{BSVN}^+(1),I_{BSVN}^+(2)),\min(F_{BSVN}^+(1),F_{BSVN}^+(2)) \\ \max(T_{BSVN}^-(1),T_{BSVN}^-(2)),\min(I_{BSVN}^-(1),I_{BSVN}^-(2)),\min(F_{BSVN}^-(1),F_{BSVN}^-(2)) \end{matrix} \right\rangle$$

(iii) Intersection of two BSVN

BSVN₁(S) ∩BSVN₂(S) =

$$\left\langle \begin{matrix} \min(T^{+}_{BSVN}(1),T^{+}_{BSVN}(2)),\max(I^{+}_{BSVN}(1),I^{+}_{BSVN}(2)),\max(F^{+}_{BSVN}(1),F^{+}_{BSVN}(2)) \\ \min(T^{-}_{BSVN}(1),T^{-}_{BSVN}(2)),\max(I^{-}_{BSVN}(1),I^{-}_{BSVN}(2)),\max(F^{-}_{BSVN}(1),F^{-}_{BSVN}(2)) \end{matrix} \right\rangle$$

3.4 Example : Let X={s_1,s_2,s_3}.Then the bipolar single-valued neutrosophic subsets S_1 and S_2 of X,

$$S_1 = \begin{Bmatrix} < s_1,(0.1,-0.1),(0.2,-0.3),(0.3,-0.5) > \\ < s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5) > \\ < s_3,(0.2,-0.8),(0.6,-0.4),(0.7,-0.7) > \end{Bmatrix} \text{ and } S_2 = \begin{Bmatrix} < s_1,(0.2,-0.1),(0.3,-0.5),(0.4,-0.5) > \\ < s_2,(0.3,-0.3),(0.3,-0.5),(0.4,-0.6) > \\ < s_3,(0.5,-0.3),(0.6,-0.3),(0.8,-0.7) > \end{Bmatrix}$$

(i) Complement of S_1 is $S_1^c = \begin{Bmatrix} < s_1,(0.9,-0.9),(0.8,-0.7),(0.7,-0.5 > \\ < s_2,(0.8,-0.7),(0.6,-0.6),(0.4,-0.5 > \\ < s_3,(0.8,-0.2),(0.4,-0.6),(0.3,-0.3 > \end{Bmatrix}$

(ii) Union of S_1 and S_2 is S_1US_2= $\begin{Bmatrix} < s_1,(0.2,-0.1),(0.2,-0.5),(0.3,-0.5 > \\ < s_2,(0.3,-0.3),(0.3,-0.5),(0.4,-0.6) > \\ < s_3,(0.5,-0.3),(0.6,-0.4),(0.7,-0.7) > \end{Bmatrix}$

(iii) Intersection of S_1 and S_2 is $S_1 \cap S_2$= $\begin{Bmatrix} < s_1,(0.1,-0.1),(0.3,-0.3),(0.4,-0.5) > \\ < s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5) > \\ < s_3,(0.2,-0.8),(0.6,-0.3),(0.8,-0.7) > \end{Bmatrix}$

3.5 Definition : Let two bipolar single-valued neutrosophic sets be BSVN₁(S) and BSVN₂(S) in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^{+}(1),T_{BSVN}^{-}(1)),(I_{BSVN}^{+}(1),I_{BSVN}^{-}(1)),(F_{BSVN}^{+}(1),F_{BSVN}^{-}(1)):v\in X>$ and
$BSVN_2(S)=<v, (T_{BSVN}^{+}(2),T_{BSVN}^{-}(2)),(I_{BSVN}^{+}(2),I_{BSVN}^{-}(2)), (F_{BSVN}^{+}(2),F_{BSVN}^{-}(2)):v\in X>$.

Then $S_1=S_2$ if and only if

$T_{BSVN}^{+}(1) = T_{BSVN}^{+}(2)$, $I_{BSVN}^{+}(1) = I_{BSVN}^{+}(2)$, $F_{BSVN}^{+}(1) = F_{BSVN}^{+}(2)$,
$T_{BSVN}^{-}(1) = T_{BSVN}^{-}(2)$, $I_{BSVN}^{-}(1) = I_{BSVN}^{-}(2)$, $F_{BSVN}^{-}(1) = F_{BSVN}^{-}(2)$ for all $v\in X$.

3.6 Definition : Let two bipolar single-valued neutrosophic sets be BSVN₁ and BSVN₂ in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^{+}(1),T_{BSVN}^{-}(1)),(I_{BSVN}^{+}(1),I_{BSVN}^{-}(1)),(F_{BSVN}^{+}(1),F_{BSVN}^{-}(1)):v\in X>$ and
$BSVN_2(S)=<v, (T_{BSVN}^{+}(2),T_{BSVN}^{-}(2)),(I_{BSVN}^{+}(2),I_{BSVN}^{-}(2)), (F_{BSVN}^{+}(2),F_{BSVN}^{-}(2)):v\in X>$.

Then $S_1 \subseteq S_2$ if and only if

$T_{BSVN}^{+}(1) \leq T_{BSVN}^{+}(2)$, $I_{BSVN}^{+}(1) \geq I_{BSVN}^{+}(2)$, $F_{BSVN}^{+}(1) \geq F_{BSVN}^{+}(2)$,
$T_{BSVN}^{-}(1) \leq T_{BSVN}^{-}(2)$, $I_{BSVN}^{-}(1) \geq I_{BSVN}^{-}(2)$, $F_{BSVN}^{-}(1) \geq F_{BSVN}^{-}(2)$ for all $v\in X$.

4. Bipolar single-valued Neutrosophic Topological space:

4.1 Definition : A bipolar single-valued neutrosophic topology on a non-empty set X is a τ of BSVN sets satisfying the axioms

(i) 0_{BSVN} , $1_{BSVN} \in \tau$
(ii) $S_1 \cap S_2 \in \tau$ for any $S_1,S_2 \in \tau$
(iii) $US_i \in \tau$ for any arbitrary family $\{S_i : i\in j\} \in \tau$

The pair (X,τ) is called BSVN topological space. Any BSVN set in τ is called as BSVN open set in X. The complement S^c of BSVN set in BSVN topological space (X,τ) is called a BSVN closed set.

4.2 Definition : Null or Empty bipolar single-valued neutrosophic set of a Bipolar single-valued Neutrosophic set S over X is said to be if <v, (0, 0), (0, 0), (0, 0)> for all v∈X and it is denoted by 0_{BSVN}.

4.3 Definition : Absolute Bipolar single-valued neutrosophic set denoted by 1_{BSVN} of a Bipolar single-valued Neutrosophic set S over X is said to be if <v, (1,-1), (1,-1), (1,-1)> for all v∈X.

4.4 Example : Let X={s_1,s_2,s_3} and τ= {0_{BSVN},1_{BSVN},P,Q,R,S}Then a bipolar single-valued neutrosophic subset of *X* is

$$P=\begin{Bmatrix} < s_1,(0.3,-0.5),(0.4,-0.2),(0.5,-0.3) > \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.4) > \\ < s_3,()0.2,-0.7,(0.4,-0.3),(0.4,-0.1) > \end{Bmatrix} \quad Q=\begin{Bmatrix} < s_1,(0.5,-0.2),(0.5,-0.2),(0.3,-0.2) > \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.2) > \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4) > \end{Bmatrix}$$

$$R=\begin{Bmatrix} < s_1,(0.5,-0.2),(0.4,-0.2),(0.3,-0.3) > \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.4) > \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4) > \end{Bmatrix} \quad S=\begin{Bmatrix} < s_1,(0.3,-0.5),(0.5,-0.2),(0.5,-0.2) > \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.2) > \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1) > \end{Bmatrix}$$

Then (X,τ) is called BSVN topological space on X.

4.5 Definition : Let (X,τ) be a BSVN topological space and
BSVN (S) =<v, (T_{BSVN}^+,T_{BSVN}^-), (I_{BSVN}^+,I_{BSVN}^-),(F_{BSVN}^+,F_{BSVN}^-):v∈X> be a BSVN set in X. Then the closure and interior of A is defined as

Int (S) = U {F: F is a BSVN open set (BSVNOs) in X and F⊆S}

Cl (S) = ∩ {F: F is a BSVN closed set (BSVNCs) in X and S⊆F}.

Here cl(S) is a BSVNCs and int (S) is a BSVNOs in X.

(a) S is a BSVNCs in X iff cl (S) =S.
(b) S is a BSVNOs in X iff int (S) =S.

4.6 Example : Let X={s_1,s_2,s_3} and τ={0_{BSVN},1_{BSVN},P,Q,R,S}. Then a bipolar single-valued neutrosophic subset of *X* is

$$P=\begin{Bmatrix} < s_1,(0.3,-0.5),(0.4,-0.2),(0.5,-0.3) > \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.4) > \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1) > \end{Bmatrix} Q=\begin{Bmatrix} < s_1,(0.5,-0.2),(0.5,-0.2),(0.3,-0.2) > \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.2) > \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4) > \end{Bmatrix}$$

$$R=\begin{Bmatrix} < s_1,(0.5,-0.2),(0.4,-0.2),(0.3,-0.3) > \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.4) > \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4) > \end{Bmatrix} S=\begin{Bmatrix} < s_1,(0.3,-0.5),(0.5,-0.2),(0.5,-0.2) > \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.2) > \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1) > \end{Bmatrix}$$

$$T=\begin{Bmatrix} < s_1,0.7,0.3,0.3,-0.5,-0.2,-0.4 > \\ < s_2,0.6,0.6,0.3,-0.3,-0.5,-0.5 > \\ < s_3,0.5,0.2,0.3,-0.5,-0.5,-0.6 > \end{Bmatrix}$$

Then int(T)=P and cl(T)=1_{BSVN}.

4.7 Proposition : Let BSVNTS of (X,τ) and S,T be BSVN's in X. Then the properties hold:

i. int (S) ⊆ S and S ⊆ cl(S)
ii. S⊆ T ⇒int(S) ⊆ int(T)
S⊆ T ⇒cl(S) ⊆ cl(T)

iii. $int(int(S))=int(S)$
$cl(cl(S))=cl(S)$
iv. $int(S\cap T)=int(S)\cap int(T)$
$cl(S\cup T)=cl(S)\cup cl(T)$
v. $int(1_{BSVN})=1_{BSVN}$
$cl(0_{BSVN})=0_{BSVN}$

Proof: The proof is obvious.

4.8 Proposition : Let BSVN sets of S_i's and T in X, then $S_i \subseteq T$ for each $i\in J \Rightarrow$ (a). $\cup S_i \subseteq T$ and (b). $T \subseteq \cap S_i$.
Proof: (a).Let $S_i \subseteq B$ (i.e) $S_1 \subseteq B$, $S_2 \subseteq B$,….., $S_n \subseteq B$.

$\Rightarrow \{T_{BSVN}^{+}(S_1) \leq T_{BSVN}^{+}(T)$, $T_{BSVN}^{-}(S_1) \leq T_{BSVN}^{-}(T)$, $I_{BSVN}^{+}(S_1) \geq I_{BSVN}^{+}(T)$, $I_{BSVN}^{-}(S_1) \geq I_{BSVN}^{-}(T)$,
$F_{BSVN}^{+}(S_1) \geq F_{BSVN}^{+}(T)$, $F_{BSVN}^{-}(S_1) \geq F_{BSVN}^{-}(T)$,$T_{BSVN}^{+}(S_2) \leq T_{BSVN}^{+}(T)$, $T_{BSVN}^{-}(S_2) \leq T_{BSVN}^{-}(T)$,
$I_{BSVN}^{+}(S_2)\geq I_{BSVN}^{+}(T)$,$I_{BSVN}^{-}(S_2)\geq I_{BSVN}^{-}(T)$,$F_{BSVN}^{+}(S_2)\geq F_{BSVN}^{+}(T)$,$F_{BSVN}^{-}(S_2)\geq F_{BSVN}^{-}(T)$…...................,
$T_{BSVN}^{+}(S_n)\leq T_{BSVN}^{+}(T)$, $T_{BSVN}^{-}(S_n) \leq T_{BSVN}^{-}(T)$, $I_{BSVN}^{+}(S_n) \geq I_{BSVN}^{+}(T)$, $I_{BSVN}^{-}(S_n) \geq I_{BSVN}^{-}(T)$,
$F_{BSVN}^{+}(S_n) \geq F_{BSVN}^{+}(T)$, $F_{BSVN}^{-}(S_n) \geq F_{BSVN}^{-}(T)\}$

$\Rightarrow \max\{(T_{BSVN}^{+}(S_1), T_{BSVN}^{+}(S_2) ,...,T_{BSVN}^{+}(S_n)),(T_{BSVN}^{-}(S_1) ,T_{BSVN}^{-}(S_2) ,..,T_{BSVN}^{-}(S_n))\}\leq(T_{BSVN}^{+}(T),T_{BSVN}^{-}(T))$
$\min\{(I_{BSVN}^{+}(S_1) ,I_{BSVN}^{+}(S_2) ,…,I_{BSVN}^{+}(S_n)), (I_{BSVN}^{-}(S_1) ,I_{BSVN}^{-}(S_2) ,…,I_{BSVN}^{-}(S_n))\} \geq (I_{BSVN}^{+}(T) ,I_{BSVN}^{-}(T))$
$\min\{(F_{BSVN}^{+}(S_1) ,F_{BSVN}^{+}(S_2) ,…,F_{BSVN}^{+}(S_n)),(F_{BSVN}^{-}(S_1),F_{BSVN}^{-}(S_2) ,…,F_{BSVN}^{-}(S_n))\} \geq (F_{BSVN}^{+}(T),F_{BSVN}^{-}(T))$

where $\cup A_i = <x,\ \max\{(T_{BSVN}^{+}(S_1), T_{BSVN}^{+}(S_2) ,...,T_{BSVN}^{+}(S_n)),(T_{BSVN}^{-}(S_1) ,T_{BSVN}^{-}(S_2) ,..,T_{BSVN}^{-}(S_n))\}$
$\min\{(I_{BSVN}^{+}(S_1) ,I_{BSVN}^{+}(S_2) ,…,I_{BSVN}^{+}(S_n)), (I_{BSVN}^{-}(S_1) ,I_{BSVN}^{-}(S_2) ,…,I_{BSVN}^{-}(S_n))\}$
$\min\{(F_{BSVN}^{+}(S_1) ,F_{BSVN}^{+}(S_2) ,…,F_{BSVN}^{+}(S_n)), (F_{BSVN}^{-}(S_1) ,F_{BSVN}^{-}(S_2) ,…,F_{BSVN}^{-}(S_n))\} >$
$\therefore \cup S_i \subseteq T$ .Hence proved.

(b)Let $T\subseteq S_i$ (i.e) $T\subseteq S_1$, $T\subseteq S_2$, … $T\subseteq S_i$.

$\Rightarrow< T_{BSVN}^{+}(T) \leq T_{BSVN}^{+}(S_1)$, $T_{BSVN}^{-}(T) \leq T_{BSVN}^{-}(S_1)$, $I_{BSVN}^{+}(T) \geq I_{BSVN}^{+}(S_1)$, $I_{BSVN}^{-}(T) \geq I_{BSVN}^{-}(S_1)$,
$F_{BSVN}^{+}(T) \geq F_{BSVN}^{+}(S_1)$, $F_{BSVN}^{-}(T) \geq F_{BSVN}^{-}(S_1)$, $T_{BSVN}^{+}(T) \leq T_{BSVN}^{+}(S_2)$, $T_{BSVN}^{-}(T) \leq T_{BSVN}^{-}(S_2)$,
$I_{BSVN}^{+}(T) \geq I_{BSVN}^{+}(S_2)$, $I_{BSVN}^{-}(T) \geq I_{BSVN}^{-}(S_2)$,$F_{BSVN}^{+}(T) \geq F_{BSVN}^{+}(S_2)$,$F_{BSVN}^{-}(T) \geq F_{BSVN}^{-}(S_2)$,………………,
$T_{BSVN}^{+}(T)\leq T_{BSVN}^{+}(S_n)$, $T_{BSVN}^{-}(T) \leq T_{BSVN}^{-}(S_n)$,$I_{BSVN}^{+}(T) \geq I_{BSVN}^{+}(S_n)$, $I_{BSVN}^{-}(T) \geq I_{BSVN}^{-}(S_n)$,
$F_{BSVN}^{+}(T) \geq F_{BSVN}^{+}(S_n)$, $F_{BSVN}^{-}(T) \geq F_{BSVN}^{-}(S_n) \}>$

$\Rightarrow(T_{BSVN}^{+}(T),T_{BSVN}^{-}(T)) \leq \min\{(T_{BSVN}^{+}(S_1),T_{BSVN}^{+}(S_2) ,...,T_{BSVN}^{+}(S_n)),(T_{BSVN}^{-}(S_1) ,T_{BSVN}^{-}(S_2) ,..,T_{BSVN}^{-}(S_n))\}$
$(I_{BSVN}^{+}(T) ,I_{BSVN}^{-}(T)) \geq \max\{(I_{BSVN}^{+}(S_1) ,I_{BSVN}^{+}(S_2) ,…,I_{BSVN}^{+}(S_n)), (I_{BSVN}^{-}(S_1) ,I_{BSVN}^{-}(S_2) ,…,I_{BSVN}^{-}(S_n))\}$
$(F_{BSVN}^{+}(T),F_{BSVN}^{-}(T))\geq \max\{(F_{BSVN}^{+}(S_1) ,F_{BSVN}^{+}(S_2) ,…,F_{BSVN}^{+}(S_n)),(F_{BSVN}^{-}(S_1) ,F_{BSVN}^{-}(S_2) ,…, F_{BSVN}(S_n))\}$

Where $\cap A_i= <x, \min\{(T_{BSVN}^{+}(S_1), T_{BSVN}^{+}(S_2) ,...,T_{BSVN}^{+}(S_n)),(T_{BSVN}^{-}(S_1) ,T_{BSVN}^{-}(S_2) ,..,T_{BSVN}^{-}(S_n))\}$
$\max\{(I_{BSVN}^{+}(S_1) ,I_{BSVN}^{+}(S_2) ,…,I_{BSVN}^{+}(S_n)), (I_{BSVN}^{-}(S_1) ,I_{BSVN}^{-}(S_2) ,…,I_{BSVN}^{-}(S_n))\}$
$\max\{(F_{BSVN}^{+}(S_1) ,F_{BSVN}^{+}(S_2) ,…,F_{BSVN}^{+}(S_n)), (F_{BSVN}^{-}(S_1) ,F_{BSVN}^{-}(S_2) ,…,F_{BSVN}^{-}(S_n))\} >$
$\therefore T \subseteq \cap S_i$ .Hence proved.

4.9 Proposition : Let S_i's and T are BSVN sets in X then (i). $(\cup S_i)^c = \cap S_i^c$, (ii). $(\cap S_i)^c = \cup S_i^c$ and
(iii). $(S^c)^c = S$.
Proof: (i) Let $\cup S_i = <x, \max\{(T_{BSVN}^{+}(S_1), T_{BSVN}^{+}(S_2) ,...,T_{BSVN}^{+}(S_n)),(T_{BSVN}^{-}(S_1) ,T_{BSVN}^{-}(S_2) ,..,T_{BSVN}^{-}(S_n))\}$
$\min\{(I_{BSVN}^{+}(S_1) ,I_{BSVN}^{+}(S_2) ,…,I_{BSVN}^{+}(S_n)), (I_{BSVN}^{-}(S_1) ,I_{BSVN}^{-}(S_2) ,…,I_{BSVN}^{-}(S_n))\}$
$\min\{(F_{BSVN}^{+}(S_1) ,F_{BSVN}^{+}(S_2) ,…,F_{BSVN}^{+}(S_n)),(F_{BSVN}^{-}(S_1) ,F_{BSVN}^{-}(S_2) ,..,F_{BSVN}^{-}(S_n))\} >$

$(\cup S_i)^c = <x,\min\{(1-T_{BSVN}^{+}(S_1),1-T_{BSVN}^{+}(S_2) ,..,1-T_{BSVN}^{+}(S_n)),(-1- T_{BSVN}^{-}(S_1) ,-1-T_{BSVN}^{-}(S_2) ,..,-1- T_{BSVN}^{-}(S_n))\}$
$\max\{(1-I_{BSVN}^{+}(S_1) ,1-I_{BSVN}^{+}(S_2) ,…,1-I_{BSVN}^{+}(S_n)), (-1-I_{BSVN}^{-}(S_1) ,-1-I_{BSVN}^{-}(S_2) ,…,-1-I_{BSVN}^{-}(S_n))\}$
$\max\{(1-F_{BSVN}^{+}(S_1) ,1-F_{BSVN}^{+}(S_2) ,..,1-F_{BSVN}^{+}(S_n)), (-1-F_{BSVN}^{-}(S_1) ,-1-F_{BSVN}^{-}(S_2) ,..,-1-F_{BSVN}^{-}(S_n)) >$
------------------->(1)

$S_i^c = <x,(1-T_{BSVN}^{+}(S_1),1- T_{BSVN}^{+}(S_2) ,...,1-T_{BSVN}^{+}(S_n)),(-1- T_{BSVN}^{-}(S_1) ,-1-T_{BSVN}^{-}(S_2) ,..,-1-T_{BSVN}^{-}(S_n))$
$(1-I_{BSVN}^{+}(S_1) ,1-I_{BSVN}^{+}(S_2) ,…,1-I_{BSVN}^{+}(S_n)), (-1-I_{BSVN}^{-}(S_1) ,-1-I_{BSVN}^{-}(S_2) ,…,-1-I_{BSVN}^{-}(S_n))$
$(1-F_{BSVN}^{+}(S_1) ,1-F_{BSVN}^{+}(S_2) ,…,1-F_{BSVN}^{+}(S_n)), (-1-F_{BSVN}^{-}(S_1) ,-1-F_{BSVN}^{-}(S_2) ,…,-1-F_{BSVN}^{-}(S_n)>$

$\cap S_i^c =$ <x,min $\{(1-T_{BSVN}^+(S_1),1-T_{BSVN}^+(S_2),...,1-T_{BSVN}^+(S_n)),(-1-T_{BSVN}^-(S_1),-1-T_{BSVN}^-(S_2),..,-1-T_{BSVN}^-(S_n))\}$
max$\{(1-I_{BSVN}^+(S_1),1-I_{BSVN}^+(S_2),\ldots,1-I_{BSVN}^+(S_n)),(-1-I_{BSVN}^-(S_1),-1-I_{BSVN}^-(S_2),\ldots,-1-I_{BSVN}^-(S_n))\}$
max$\{(1-F_{BSVN}^+(S_1),1-F_{BSVN}^+(S_2),..,1-F_{BSVN}^+(S_n)),(-1-F_{BSVN}^-(S_1),-1-F_{BSVN}^-(S_2),..,-1-F_{BSVN}^-(S_n)\}$>
------------------->(2)

From (1) and (2), $(US_i)^c = \cap S_i^c$ .Hence proved.

(ii). Similar as proof of (i).

(iii). Let S=< $(T_{BSVN}^+(S),T_{BSVN}^-(S)),(I_{BSVN}^+(S),I_{BSVN}^-(S)),(F_{BSVN}^+(S),F_{BSVN}^-(S))$> be a BSVN set in X, then
S^c = < $(1-T_{BSVN}^+(S),-1-T_{BSVN}^-(S)),(1-I_{BSVN}^+(S),-1-I_{BSVN}^-(S)),(1-F_{BSVN}^+(S),-1-F_{BSVN}^-(S)$>
$(S^c)^c$ = < $(T_{BSVN}^+(S),T_{BSVN}^-(S)),(I_{BSVN}^+(S),I_{BSVN}^-(S)),(F_{BSVN}^+(S),F_{BSVN}^-(S))$>
$(S^c)^c$ = S. Hence proved.

5. Bipolar single-valued Neutrosophic Number (BSVNN)

5.1 Definition : Let two bipolar single-valued neutrosophic number(BSVNN) be
$\widetilde{S}_1$=<$T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)$> and
$\widetilde{S}_2$=<$T_{BSVN}^+(2),T_{BSVN}^-(2)),(I_{BSVN}^+(2),I_{BSVN}^-(2)),(F_{BSVN}^+(2),F_{BSVN}^-(2)$> . Then the operations are

i. $\lambda\widetilde{S}_1$=<$1-(1-T_{BSVN}^+(1))^\lambda, -(-T_{BSVN}^-(1))^\lambda, (I_{BSVN}^+(1))^\lambda, -(-I_{BSVN}^-(1))^\lambda, (F_{BSVN}^+(1))^\lambda, -(1-(1-(-F_{BSVN}^-(1)))^\lambda)$>
ii. $\widetilde{S}_1^\lambda$ =<$(T_{BSVN}^+(1))^\lambda, -(1-(1-(-T_{BSVN}^-(1)))^\lambda), 1-(1-I_{BSVN}^+(1))^\lambda, -(-I_{BSVN}^-(1))^\lambda, 1-(1-F_{BSVN}^+(1))^\lambda, -(-F_{BSVN}^-(1))^\lambda$>
iii. $\widetilde{S}_1+\widetilde{S}_2$= <$T_{BSVN}^+(1)+T_{BSVN}^+(2)-T_{BSVN}^+(1)T_{BSVN}^+(2),-T_{BSVN}^-(1)T_{BSVN}^-(2)$,
$I_{BSVN}^+(1)I_{BSVN}^+(2), -(-I_{BSVN}^-(1)-I_{BSVN}^-(2)-I_{BSVN}^-(1)I_{BSVN}^-(2))$,
$F_{BSVN}^+(1)F_{BSVN}^+(2), -(-F_{BSVN}^-(1)-F_{BSVN}^-(2)-F_{BSVN}^-(1)F_{BSVN}^-(2))$>
iv. $\widetilde{S}_1.\widetilde{S}_2$ =< $T_{BSVN}^+(1)T_{BSVN}^+(2),-(-T_{BSVN}^-(1)-T_{BSVN}^-(2)-T_{BSVN}^-(1)T_{BSVN}^-(2))$,
$I_{BSVN}^+(1)+I_{BSVN}^+(2)-I_{BSVN}^+(1)I_{BSVN}^+(2),-I_{BSVN}^-(1)I_{BSVN}^-(2)$,
$F_{BSVN}^+(1)+F_{BSVN}^+(2)-F_{BSVN}^+(1)F_{BSVN}^+(2),-F_{BSVN}^-(1)F_{BSVN}^-(2)$>

5.2 Definition : Let a bipolar single-valued neutrosophic number(BSVNN) be
$\widetilde{S}_1$=<$T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)$>.Then

i. score function: $s(\widetilde{S}_1)=(T_{BSVN}^+(1)+1-I_{BSVN}^+(1)+1-F_{BSVN}^+(1)+1+T_{BSVN}^-(1)-I_{BSVN}^-(1)-F_{BSVN}^-(1))/6$
ii. accuracy function: $a(\widetilde{S}_1)=T_{BSVN}^+(1)-F_{BSVN}^+(1)+T_{BSVN}^-(1)-F_{BSVN}^-(1)$
iii. certainty function : $c(\widetilde{S}_1)=T_{BSVN}^+(1)-F_{BSVN}^+(1)$

5.3 Definition : The two bipolar single-valued neutrosophic numbers (BSVNN) are compared
$\widetilde{S}_1$=<$T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)$>
$\widetilde{S}_2$=< $T_{BSVN}^+(2),T_{BSVN}^-(2)),(I_{BSVN}^+(2),I_{BSVN}^-(2)),(F_{BSVN}^+(2),F_{BSVN}^-(2)$> can be defined as

i. If $s(\widetilde{S}_1)>s(\widetilde{S}_2)$, $\widetilde{S}_1$ is superior to $\widetilde{S}_2$,(i.e.) $\widetilde{S}_1$ is greater than $\widetilde{S}_2$ denoted as $\widetilde{S}_1>\widetilde{S}_2$.
ii. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}(\widetilde{S}_1)>\widetilde{S}(\widetilde{S}_2)$, $\widetilde{S}_1$is superior to $\widetilde{S}_2$,(i.e.) $\widetilde{S}_1$ is greater than $\widetilde{S}_2$ denoted as $\widetilde{S}_1<\widetilde{S}_2$.
iii. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}(\widetilde{S}_1)=\widetilde{S}(\widetilde{S}_2)$ and $c(\widetilde{S}_1)>c(\widetilde{S}_2)$, $\widetilde{S}_1$ is greater than $\widetilde{S}_2$,that is $\widetilde{S}_1$ is superior to $\widetilde{S}_2$,denoted as $\widetilde{S}_1>\widetilde{S}_1$.
iv. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}(\widetilde{S}_1)=\widetilde{S}(\widetilde{S}_2)$ and $c(\widetilde{S}_1)=c(\widetilde{S}_2)$, $\widetilde{S}_1$ is equal to $\widetilde{S}_2$,that is $\widetilde{S}_1$ is indifferent to $\widetilde{S}_2$,denoted as $\widetilde{S}_1=\widetilde{S}_1$.

5.4 Definition : Let a family of bipolar single-valued neutrosophic numbers(BSVNN) be $\widetilde{S}_j$=<$T_{BSVN}^+(j),T_{BSVN}^-(j)),(I_{BSVN}^+(j),I_{BSVN}^-(j)),(F_{BSVN}^+(j),F_{BSVN}^-(j)$>(j=1,2,3,…,n). A mapping $A_\omega:F_n\rightarrow F$ is called bipolar single-valued Neutrosophic weighted average ($BSVNWA_\omega$) operator if satisfies

$$A_\omega(\widetilde{s}_1, \widetilde{s}_2, .., \widetilde{s}_n) = \sum_{j=1}^{n} \omega_j \widetilde{s}_j = < 1-\prod_{j=1}^{n}(1-T_{BSVN}^{+}(j))^{\omega_j}, -\prod_{j=1}^{n}(-T_{BSVN}^{-}(j))^{\omega_j}, \prod_{j=1}^{n} I_{BSVN}^{+}(j)^{\omega_j},$$

$$-(1-\prod_{j=1}^{n}(1-(-I_{BSVN}^{-}))^{\omega_j}), \prod_{j=1}^{n} F_{BSVN}^{+}(j)^{\omega_j}, -(1-\prod_{j=1}^{n}(1-(-F_{BSVN}^{-}))^{\omega_j})>$$

Here ω_j is the weight of $\widetilde{s}_j$ (j=1,2,…n), $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j \in [0,1]$.

5.5 Definition : Let a family of bipolar single-valued neutrosophic numbers(BSVNN) be $\widetilde{s}_j = <T_{BSVN}^{+}(j), T_{BSVN}^{-}(j)),(I_{BSVN}^{+}(j),I_{BSVN}^{-}(j)),(F_{BSVN}^{+}(j),F_{BSVN}^{-}(j)>$(j=1,2,3,…,n) . A mapping $G_\omega:F_n \rightarrow F$ is called bipolar single-valued neutrosophic weighted geometric($BSVNWG_\omega$) operator if it satisfies

$$G_\omega(\widetilde{s}_1, \widetilde{s}_2, .., \widetilde{s}_n) = \prod_{j=1}^{n} \widetilde{s}_j^{\;\omega_j} = < \prod_{j=1}^{n} T_{BSVN}^{+}(j))^{\omega_j}, -(1-\prod_{j=1}^{n}(1-(-T_{BSVN}^{-}(j)))^{\omega_j}),$$

$$1-\prod_{j=1}^{n}(1-I_{BSVN}^{+}(j))^{\omega_j}, -\prod_{j=1}^{n}(-I_{BSVN}^{-}))^{\omega_j}, 1-\prod_{j=1}^{n}(1-F_{BSVN}^{+}(j)^{\omega_j}, -\prod_{j=1}^{n}(-F_{BSVN}^{-})^{\omega_j}>$$ where ω_j is the weight of $\widetilde{s}_j$ (j=1,2,…n), $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j \in [0,1]$.

5.6. Decision making problem:

Here, with bipolar single-valued neutrosophic data, we develop decision making problem based on A_ω operator Suppose the set of alternatives is S = {S_1,S_2,...S_m} and the set of all criterions (or attributes) are G= {G_1,G_2,….,G_n}.Let $\omega=(\omega_1,\omega_2,….\omega_n)^T$ be the weight vector of attributes such that $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j \geq 0$ (j=1,2,…n) and ω_j assign to the weight of attribute G_j.An alternative on criterions is calculated by the decision maker and the assess values are represented by the design of bipolar single-valued neutrosophic numbers.

Assume the decision matrix $(\widetilde{S}_{ij})_{m \times n} =(<T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij)),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij)),(F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij)>)_{mxn}$ contributed by the decision maker, for Alternative S_i with criterion G_j ,the bipolar single-valued neutrosophic number is $\widetilde{S}_{ij}$. The conditions are $T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij),F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij) \in [0,1]$ such that $0 \leq T_{BSVN}^{+}(ij) -T_{BSVN}^{-}(ij) +I_{BSVN}^{+}(ij) -I_{BSVN}^{-}(ij) +F_{BSVN}^{+}(ij) -F_{BSVN}^{-}(ij) \leq 6$ for i=1,2,3,…m and j=1,2,…n.

Algorithm:

STEP 1: Construct the decision matrix by the decision maker.

$$(\widetilde{S}_{ij})_{m \times n} =(<T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij)),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij)),(F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij)>)_{mxn}$$

STEP 2: Compute $\widetilde{S}_i=A_\omega(\widetilde{S}_{i1}, \widetilde{S}_{i2},… \widetilde{S}_{in})$ for each i=1,2,….m.

STEP 3: Using the set of overall bipolar single-valued neutrosophic number of $\widetilde{S}_i$ (i=1,2,…m),calculate the score values $\widetilde{S}(\widetilde{S}_i)$.

STEP 4: Rank all the structures of $\widetilde{S}_i$ (i=1,2,…m) according to the score values.

Example (5.7): A patient is intending to analyze which disease is caused to him. Four types of diseases S_i(i=1,2,3,4) are Cancer, Asthuma, Hyperactive, Typhoid. The set of symptoms are G_1=cough, G_2=Headache,G_3=stomach pain,G_4=blood cloting. To evaluate the 4 diseases (alternatives) S_i(i=1,2,3,4) under the above four symptoms(attributes) using the bipolar single-valued neutrosophic values.The weight vector of the attributes G_j (j=1, 2, 3, 4) is ω= (0.25,0.35,0.20,0.20)^T.

STEP 1: The decision matrix provided by the patient is constructed as below:

S_i /G_i	G_1	G_2	G_3	G_4
S_1	(0.3,-0.5)(0.4,-0.4)(0.4,-0.2)	(0.3,-0.3)(0.5,-0.2)(0.3,-0.4)	(0.6,-0.4)(0.4,-0.3)(0.3,-0.5)	(0.1,-0.3)(0.6,-0.4)(0.5,-0.3)
S_2	(0.3,-0.4)(0.7,-0.5)(0.4,-0.5)	(0.1,-0.3)(0.2,-0.4)(0.3,-0.5)	(0.3,-0.5)(0.2,-0.4)(0.1,-0.3)	(0.4,-0.2)(0.2,-0.3)(0.1,-0.2)
S_3	(0.3,-0.4)(0.4,-0.5)(0.5,-0.6)	(0.1,-0.2)(0.2,-0.3)(0.3,-0.4)	(0.5,-0.4)(0.4,-0.5)(0.5,-0.6)	(0.1,-0.3)(0.2,-0.4)(0.3,-0.6)
S_4	(0.3,-0.2)(0.2,-0.1)(0.1,-0.2)	(0.3,-0.1)(0.4,-0.2)(0.5,-0.3)	(0.2,-0.3)(0.4,-0.7)(0.7,-0.8)	(0.1,-0.3)(0.2,-0.5)(0.3,-0.7)

STEP 2: Compute $\widetilde{S}_i=A_\omega(\widetilde{S}_{i1}, \widetilde{S}_{i2}, \widetilde{S}_{i3}, \widetilde{S}_{i4})$ for each i=1,2,3,4;

$\widetilde{S}_1$=< (0.3,-0.4) (0.5,-0.3) (0.4,-0.4)>
$\widetilde{S}_2$=< (0.2,-0.3) (0.3,-0.4) (0.2,-0.4)>
$\widetilde{S}_3$=< (0.2,-0.3) (0.3,-0.4) (0.4,-0.5)>
$\widetilde{S}_4$=< (0.2,-0.2) (0.3,-0.4) (0.3,-0.5)>

STEP 3: The score value of $\widetilde{S}(\widetilde{S}_i)$ (i=1, 2, 3, 4) are computed for the set of overall bipolar single-valued neutrosophic number .

$\widetilde{S}(\widetilde{S}_1)=0.45$
$\widetilde{S}(\widetilde{S}_2)=0.53$
$\widetilde{S}(\widetilde{S}_3)=0.51$
$\widetilde{S}(\widetilde{S}_4)=0.55$

STEP 4: According to the score values rank all the software systems of S_i (i=1, 2, 3, and 4)

$\mathbf{S_4 > S_2 > S_3 > S_1}$

Thus S_4 is the most affected disease (alternative) . Typhoid(S_4) is affected to him.

Conclusion:

In this paper, bipolar single-valued neutrosophic sets were developed. Bipolar single-valued neutrosophic topological spaces were also introduced and characterized some of its properties. Further score function, certainty function and accuracy functions of the Bipolar single-valued neutrosophic were given. We proposed the average and geometric operators (A_ω and G_ω) for bipolar single-valued neutrosophic information. To calculate the integrity of alternatives on the attributes taken, a bipolar single-valued neutrosophic decision making approach using the score function, certainty function and accuracy function were refined.

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Novel Open Source Python Neutrosophic Package

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Abstract: Neutrosophic sets has gained wide popularity and acceptance in both academia and industry. Different fields have successfully adopted and utilized neutrosophic sets. However, there is no open-source implementation that provides the basic neutrosophic concepts. Open-source is a global movement that enables developers to share their source-code, as researchers do share their research ideas and results. Presented Open-source python neutrosophic package is the first of its kind. It utilizes object oriented concepts. It is based on one of the most popular multi-paradigm programming languages that is widely used in different academic and industry fields and activities; namely python. Presented package tends to dissolve the barriers and enable both researchers and developers to adopt neutrosophic sets and theory in research and applications. In this paper, the first Open-source, Object-Oriented, Python based Neutrosophic package is presented. This paper intensively scanned neutrosophic sets research attempting to reach the most widely accepted neutrosophic proofs, and then transforming them into source-code. Presented package implements most of the basic neutrosophic concepts. Presented neutrosophic package presents four different classes: Single Valued Neutrosophic Number, Single Valued Neutrosophic Sets, Interval Valued Neutrosophic Number, and Interval Valued Neutrosophic Sets. Presented package source code, test cases, usage examples, and updated documentation can be found online at `https://www.github.com/helghareeb/neutrosophic`. Presented neutrosophic package can be easily integrated into research and applications. This is an ongoing work and research, as neutrosophic theory is largely expanding, and there are lots of features to cover.

Keywords: Single Valued Neutrosophic Number, Neutrosophic Sets, Open Source, Python

1 Neutrosophic Theory

Neutrosophic sets have been introduced to the literature by Smarandache to handle incomplete, indeterminate, and inconsistent information [18]. In neutrosophic sets, indeterminacy is quantified explicitly through a new parameter I. Truth-membership (T), indeterminacy membership (I) and falsity-membership (F) are three independent parameters that are used to define a Neutrosophic Number.

$$\dot{N} = \left\{ < x; T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) >, x \in X \right\}$$

$$x \in X, \quad T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) \in [0,1] \tag{1.1}$$

This paper presents the first novel open source implementation of a Neutrosophic Package in Python programming language. Proposed implementation aims to facilitate Neutrosophic sets utilization in different Python based applications. Python programming language has been chosen exclusively for different reasons. Python is a high-level programming language, that is efficient, supports high-level data structures, and is highly utilized in different academic and industry disciplines; including big data analytics, artificial intelligence, and machine learning. There are no reasons that prevents porting the proposed Neutrosophic package implementation to other programming languages. Porting presented package is a step to take in the near future. However, Python will be the main focus of this research paper.

Presented characteristics and operations are implemented in the presented Open-Source Python Package can be found at `https://www.github.com/helghareeb/neutrosophic`. Presented Neutrosophic Package development covers the following neutrosophic objects:

- Single Valued Neutrorosphic Number (SVNN)
- Interval Valued Neutrosophic Number (IVNN)
- Single Valued Neutrosophic Sets (SVNS)
- Interval Valued Neutrosophic Sets (IVNS)

Rest of the paper goes as follows:

Section 2 presents a literature review on the most recent areas of applications that utilize neutrosophy theory and neutrosophic sets. Neutrosophic sets has been widely accepted among different disciplines, and the need for an open source neutrosophic package implementation has become a necessity.

Section 3 presents the core design methodology and concepts around the presented novel neutrosophic package. Presented package is object oriented based, that supports open-source concepts, and utilizes some Python magic to enhance the performance and functionality.

In the following sections, an introduction to the neutrosophic theory of the presented section is highlighted, followed by the equation that implements the presented operation. Implementation of the presented equation is presented immediately after the equation, so the reader can follow each section of the code and what it is actually responsible for. The complete source code is available at `https://www.github.com/helghareeb/neutrosophic`. This paper avoids mathematical proofs of the implemented equations, and includes external references that include the proofs of the implemented equations and calculations.

Section 4 presents the basic element and the most widely used neutrosophic number, that is the single valued neutrosophic number (SVNN). SVNNs operations and their implementation are presented in this section.

Section 5 introduces the Single Valued Neutrosophic Sets (SVNS). SVNS consists of multiple SVNNs. Aggregation operations are presented in this section. Implementation details simplified the calculation, and hopefully will act as an enabler for researchers in academia and developers in industry as a guidance and concrete implementation on how to adopt neutrosophic sets in real world applications.

Section 6 highlights one of the most important concepts in neutrosophic theory; that is Interval Valued Neutrosophic Numbers (IVNNs). Though IVNNs are really important in describing real world cases, they are tough to implement because they lack the crisp mathematical characteristics presented in SVNNs. This section presents a simplified way to convert mathematical concepts into concrete implementations.

Section 7 presents the Interval Valued Neutrosophic Sets (IVNSs). IVNS is an importnat neutrosophic concept that must be supported in neutrosophic packages. Averaging function of IVNNs is presented.

Section 8 concludes the paper, highlighting the main features and advantages of the presented open source neutrosophic package, and presents the future work. This is an ongoing work that needs continous development, and will grow exponentially as neutrosophic theory, sets, and systems keeps in growing and gaining wide popularity and acceptance. Paper ends with references.

2 Neutrosophic Sets in Applications and Disciplines

Neutrosophic has been widely adopted in important areas. – Here we need to include important references for important areas of neutrosophic applications, specially areas where our package can be utilized.

2.1 MADM and MCDM

Utilizing neutrosophic sets in Multi Attributed Decision Making (MADM) and Multi Criteria Decision Making (MCDM) has gained wide acceptance in research and academia.

Analytic Hierarchy Process (AHP) in neutrosophic environment has gained polpularity and achieved success in different cases. In some realistic situations, the decision makers might be unable to assign deterministic evaluation values to the comparison judgments due to limited knowledge or the differences of individual judgments in group decision making. To overcome these challenges, neutrosophic set theory to have been utilized to handle the AHP, where each pair-wise comparison judgment is represented as a triangular neutrosophic number (TNN). [3] presents such a utilization and applies it to a real life example based on expert opinions from Zagazig University, Egypt. The problem is solved to show the effectiveness of the proposed neutrosophic-AHP decision making model.

An Extension of Neutrosophic AHP–SWOT Analysis for Strategic Planning and Decision-Making is presented in [1]. Every organization seeks to set strategies for its development and growth and to do this, it must take into account the factors that affect its success or failure. The most widely used technique in strategic planning is SWOT analysis. SWOT examines strengths (S), weaknesses (W), opportunities (O) and threats (T), to select and implement the best strategy to achieve organizational goals. The chosen strategy should harness the advantages of strengths and opportunities, handle weaknesses, and avoid or mitigate threats. SWOT analysis does not quantify factors (i.e., strengths, weaknesses, opportunities and threats) and it fails to rank available alternatives. To overcome this drawback, [1] integrated it with the analytic hierarchy process (AHP). The AHP is able to determine both quantitative and the qualitative elements by weighting and ranking them via comparison matrices. Due to the vague and inconsistent information that exists in the real world. The proposed model have been applied in a neutrosophic environment in a real case study of Starbucks Company to validate the model.

A Hybrid Neutrosophic Group ANP-TOPSIS Framework for Supplier Selection Problems is presented in [2]. One of the most significant competitive strategies for organizations is sustainable supply chain management (SSCM). The vital part in the administration of a sustainable supply chain is the sustainable supplier selection, which is a multi-criteria decision-making issue, including many conflicting criteria. The valuation and selection of sustainable suppliers are difficult problems due to vague, inconsistent and imprecise knowledge of decision makers. In the literature on supply chain management for measuring green performance, the requirement for methodological analysis of how sustainable variables affect each other, and how to consider vague, imprecise and inconsistent knowledge, is still unresolved. [2] provides an incorporated multi-criteria decision-making procedure for sustainable supplier selection problems (SSSPs). An integrated framework is presented via interval-valued neutrosophic sets to deal with vague, imprecise and inconsistent information that exists usually in real world. The analytic network process (ANP) is employed to calculate weights of selected criteria by considering their interdependencies. For ranking alternatives and avoiding additional comparisons of analytic network processes, the technique for order preference by similarity to ideal solution (TOPSIS) is used. The proposed framework is turned to account for analyzing and selecting the optimal supplier. An actual case study of a dairy company in Egypt is examined within the proposed framework. Comparison with other existing methods is implemented to confirm the effectiveness and efficiency of the proposed approach.

An Integrated Neutrosophic-TOPSIS Approach and its Application to Personnel Selection as a New Trend in Brain Processing and Analysis is presented in [17]. Personnel selection is a critical obstacle that influences the success of enterprise. The complexity of personnel selection is to determine efficiently the proper applicant to fulfill enterprise requirements. The decision makers do their best to match enterprise requirements with the most suitable applicant. Unfortunately, the numerous criteria, alternatives, and goals make the process of choosing among several applicants very complex and confusing to decision makers. The environment of decision making is a MCDM surrounded by inconsistency and uncertainty. [17] contributes to support personnel selection process by integrating neutrosophic Analytical Hierarchy Process (AHP) with Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) to illustrate an ideal solution among different alternatives. A case study on smart village Cairo Egypt is developed based on decision maker's judgments recommendations. The proposed study applies neutrosophic analytical hierarchy process and TOPSIS to enhance the traditional methods of personnel selection to achieve the ideal solutions. By reaching to the ideal solutions, the smart village will enhance the resource management for attaining the goals to be a success enterprise. The proposed method demonstrates a great impact on the personnel selection process rather than the traditional decision making methods.

Neutrosophic AHP can be used to help decision makers to estimate the influential factors of IoT in enterprises. A study that combines AHP methods with neutrosophic techniques to estimate the influential factors for a successful enterprise is presented in [16].

2.2 Control Systems

The indeterminacy of parameters in actual control systems is inherent property because some parameters in actual control systems are changeable rather than constants in some cases, such as manufacturing tolerances, aging of main components, and environmental changes, which present an uncertain threat to actual control systems. Therefore, these indeterminate parameters can affect the control behavior and performance. [25] develops a new neutrosophic design method that introduces neutrosophic state space models and the neutrosophic controllability and observability in indeterminate linear systems. Then, establishes a neutrosophic state feedback design method for achieving a desired closed-loop state equation or a desired control ratio for single-input single-output (SISO) neutrosophic linear systems.

2.3 Image Processing and Segmentation

Segmentation is considered as an important step in image processing and computer vision applications, which divides an input image into various non-overlapping homogenous regions and helps to interpret the image more conveniently. [11] presents an efficient image segmentation algorithm using neutrosophic graph cut (NGC). An image is presented in neutrosophic set, and an indeterminacy filter is constructed using the indeterminacy value of the input image, which is defined by combining the spatial information and intensity information. The indeterminacy filter reduces the indeterminacy of the spatial and intensity information. A graph is defined on the image and the weight for each pixel is represented using the value after indeterminacy filtering. The segmentation results are obtained using a maximum-flow algorithm on the graph.

Medical field touches everyone's life, and it has benefited a lot from neutrosophic. Fully automated algorithm for image segmentation in medical field is presented in [19]. Such algorithm segments fluid-associated (fluid-filled) and cyst regions in optical coherence tomography (OCT) retina images of subjects with diabetic macular edema. The OCT image is segmented using a novel neutrosophic transformation and a graph-based shortest path method. An image g is transformed into three sets: T (true), I (indeterminate) that represents noise, and F (false). Fully automatic and accurate breast lesion segmentation that utilizes a novel phase feature to improve the image quality, and a novel neutrosophic clustering approach to detect the accurate lesion boundary is presented in [23].

An efficient scheme for unsupervised colour-texture image segmentation using neutrosophic set (NS) and non-subsampled contourlet transform (NSCT) is presented in [13]. First, the image colour and texture information are extracted via CIE Luv colour space model and NSCT, respectively. Then, the extracted colour and texture information are transformed into the NS domain efficiently by the authors' proposed approach. In the NS-based image segmentation, the indeterminacy assessment of the images in the NS domain is notified by the entropy concept. The lower quantity of indeterminacy in the NS domain, the higher confidence and easier segmentation could be achieved. Therefore, to achieve a better segmentation result, an appropriate indeterminacy reduction operation is proposed. Finally, the K -means clustering algorithm is applied to perform the image segmentation in which the cluster number K is determined by the cluster validity analysis.

2.4 Pattern Recognition and Machine Learning

Data clustering, or cluster analysis, is an important research area in pattern recognition and machine learning which helps the understanding of a data structure for further applications. The clustering procedure is generally handled by partitioning the data into different clusters where similarity inside clusters and the dissimilarity between different clusters are high.

New clustering algorithm, neutrosophic c-means (NCM) for uncertain data clustering, which is inspired from fuzzy c-means and the neutrosophic set framework. To derive such a structure, a novel suitable objective function is defined and minimized, and the clustering problem is formulated as a constrained minimization problem, whose solution depends on the objective function in [12].

The work presented in [12] has been extended by [5] via presenting a new clustering algorithm that is called Kernel Neutrosophic c-Means(KNCM), that has been evaluated through extensive experiments.

3 Proposed Novel Neutrosophic Package Design

Attempting informally to categorize neutrosophic research based implementations, it is clear from scanning the neutrosophic literature that spreadsheets are the most widely used tool. Though spreadsheets is an excellent tool for certain types of problems; such as Multi Criteria Decision Making (MCDM), it is not suitable for automated software systems and machine learning based solutions. The need for an open source neutrosophic package to be utilized via different programming languages is clear.

An attempt to utilize Object Oriented Programming to build a neutrosophic package was presented in [21] and [20], and an attempt to use it in e-Learning systems has been presented in [22]. This attempt suffered from different shortages, including not only:

- It was not even a close start to implementing the basic operations of any neutrosophic family subset
- It was not an open source based attempt, so its source code was never made available to neutrosophic research community to be tested, verified, validated, and utilized
- Though that solution utilized Object Oriented Concepts in CSharp, Programming was used only to calculate values to be exported to Microsoft Excel that was used for plotting the graphs. This approach suffers a lot when developers attempt to integrate it within software solutions.
- Presented OOP CSharp based package has not been tested, verified, or used in real world situations
- There is no clear documentation and illustration of the design of that package. When combined with the lack of software source code, utilizing such a package is almost impossible
- It is clear that it was a very early immature attempt that never made its way through implementation and adoption in neutrosophic community or real world examples and projects. There has been no updates or research papers related to this package since then

In this section, key decisions about the presented novel neutrosophic open source package are discussed. Those key decisions reflects the philosophy of the package author, and shapes the current state, and the future state of the neutrosophic package. Those decisions include:

- Programming Language used
- Open Source Choice (Source Code Availability)
- Neutrosophic Package Licensing
- Packaging choices and alternatives
- Programming Methodology

3.1 Python

Python is considered a multilanguage model, where a high-level language is used to interface libraries and software packages written in low-level languages. In a high-level scientific computing environment, this type of interoperability with software packages written in low-level languages (e.g., Fortran, C, or C++) is an important requirement. Python excels at this type of integration, and as a result, Python has become an interface for setting up and controlling computations that use code written in low-level programming languages for time-consuming number crunching. This is an important reason for why Python is a popular language for numerical computing. The multilanguage model enables rapid code development in a high-level language while retaining most of the performance of low-level languages. [14]

Table 1: MIT License Permissions, Limitations, and Conditions

Permissions	**Limitations**	**Conditions**
✓Commercial use	✗Liability	License and copyright notice
✓Modification	✗Warranty	
✓Distribution		
✓Private use		

3.2 Open Source

Open Source software is a form of intellectual gratification with an intrinsic utility similar to that of scientific discovery [8]. Emerging as it does from the university and research environment, the movement adopts the motivations of scientific research, transferring them into the production of technologies that have a potential commercial value. The process of scientific discovery involves the sharing of results, just as the dictates of the Open Source movement involve sharing source code. Sharing results enables researchers both to improve their results through feedback from other members of the scientific community and to gain recognition and hence prestige for their work. The same thing happens when source code is shared: other members of the group provide feedback that helps to perfect it, while the fact that the results are clearly visible to everyone confers a degree of prestige which expands in proportion to the size of the community.

3.3 Neutrosophic Package Licensing

Presented Neutrosophic Package is licensed under the MIT License. Table 1 presents the MIT License Permissions, Limitations, and Conditions. MIT License is a short and simple permissive license with conditions only requiring preservation of copyright and license notices. Licensed works, modifications, and larger works may be distributed under different terms and without source code. License and copyright notice condition means that a copy of the license and copyright notice must be included with the software. License can be found at `https://github.com/helghareeb/neutrosophic/blob/master/LICENSE`

3.4 Software Packaging

Effective reuse depends not only on finding and reusing components, but also on the ways those components are combined [24]. Software engineering provides a number of diverse styles for organizing software systems. These styles, or architectures, show how to compose systems from components; different styles expect different kinds of component packaging and different kinds of interactions between the components. Unfortunately, these styles and packaging distinctions are often implicit; as a consequence, components with appropriate functionality may fail to work together. Different packaging techniques have been presented in both academia and industry. Though there is no single agreed on packaging methodology; due to differences in features and programming languages, different guidelines are available to achieve successful packaging process. Presented Python Neutrosophic package will be packaged and shipped as a standard Python package.

3.5 Object Oriented Programming

Object oriented programming departs from conventional programming by emphasizing the relationship between consumers and suppliers of codes rather then the relationship between a programmer and code [10].

Main Object Oriented Concepts include [6]

- Inheritance: a mechanism by which object implementations can be organized to share descriptions
- Object: both a data carrier and executes actions. Object is something that has state, behavior, and identity
- Class: set of objects described by the same declaration and is the basic element of Object Oriented modeling
- Encapsulation: There are three primary conceptualizations of encapsulation in the literature.
  - First conceptualization: a process used to package data with the functions that act on the data
  - Second conceptualization: hides the details of the object's implementation so that clients access the object only via its defined external interface
  - Third conceptualization: information about an object, how that information is processed, kept strictly together, and separated from everything else
- Method: involves accessing, setting, or manipulating the object's data
- Message Passing: Message is merely a procedure call from one function to another. Message passing makes a request to one of object's methods
- Polymorphism: There are different conceptualization
  - First conceptualization: ability to hide different implementations behind a common interface
  - Second conceptualization: ability of different objects to respond to the same message and invoke different responses
  - Third conceptualization: ability of different classes to contain different methods of the same name, which appear to behave the same way in a given context; yet different objects can respond to the same message with their own behavior
  - Fourth conceptualization: refers to late binding or dynamic binding
  - Fifth conceptualization: ability of different classes to respond to the same message and each implement the method appropriately
- Abstraction:
  - First conceptualization: mechanism that allows representing a complex reality in terms of a simplified model so that irrelevant details can be suppressed in order to enhance understanding
  - Second conceptualization: the act of removing certain distinctions between objects so that we can see commonalities
  - Third conceptualization: the act of creating classes to simplify aspects of reality using distinctions inherent to the problem

4 Single Valued Neutrosophic Number

4.1 Constructor

Section 1 highlighted that each neutrosophic number consists of three elements: truth, indeterminacy, and false values. Listing 1 highlights the constructor function (the function that gets invoked automatically) when instantiating a new object instance from the Single Valued Neutrosophic Class. New Single Valued Neutrosophic Number validates the values of T, I, F to satisfy 1.1.

Listing 1: SVNN - Constructor

```
class SingleValuedNeutrosophicNumber:

    def __init__(self, id, truth, indeterminacy, falsehood):
        """Initialize neutrosophic element
        :truth:
        :indeterminacy:
        :falsehood:"""
        assert id is not None, 'provide id for element to be initialized'
        assert 0 <= truth <= 1, 'invalid truth value'
        assert 0 <= indeterminacy <= 1, 'invalid indeterminacy value'
        assert 0 <= falsehood <= 1, 'invalid falsehood value'
        assert 0 <= truth + falsehood + indeterminacy <= 3, 'invalid combined sum
            ↪ values'
        self._id = id
        self._truth = truth
        self._indeterminacy = indeterminacy
        self._falsehood = falsehood
```

4.2 SVNN Operations

Single Valued Nuetrosophic Numbers arithmetic operations are defined in [15] as follows: Let two single-valued neutrosophic numbers be

$$x = \langle T_x, I_x, F_x \rangle$$

$$y = \langle T_y, I_y, F_y \rangle$$

4.2.1 SVNN Complement

Calculating SVNN Complement is based on the Equation 4.1 and implemented in Listing 2

$$x^c = \langle F_x, 1 - I_x, T_x \rangle \tag{4.1}$$

Listing 2: SVNN - Complement

```
    def complement(self):
        """
        :return: SVNN object with the new TIF values
        """
        return SingleValuedNeutrosophicNumber (f'{self._id}_complement', self.
            ↪ _falsehood, 1 - self._indeterminacy, self._truth)
```

4.2.2 SVNN is subset of

Identifying either an SVNN is a subset of another SVNN is determined based on the Equation 4.2 and implemented in Listing 3. This method returns a `bool` value type with either `True` or `False`, if the current SVNN object is a subset of another SVNN.

$$x \subseteq y \iff T_x \leq T_y, I_x \geq I_y, F_x \geq F_y \tag{4.2}$$

Listing 3: SVNN - is subset of

```
        def is_subset_of(self, svnn):
                """Check if SVNN is a subset of another SVNN
                :param svnn: Single Value Neutrosohpic Number to compare with
                :return: True or False
                """
                if self._truth <= svnn._truth and self._indeterminacy >= svnn.
                    ↪ _indeterminacy and self._falsehood >= svnn._falsehood:
                        return True
                return False
```

4.2.3 SVNN Equal

Comparing two SVNN to detect if they are equal or not is calculated based on the Equation 4.3 and implemented in Listing 4. One of the advantages of Python magic is utilized in this function via implementing it as `__eq__` which gives the neutrosophic package capability of comparing two SVNN numbers via the equal sign operator.

$$x = y \iff x \subseteq y, y \subseteq x \tag{4.3}$$

Listing 4: SVNN - Equal

```
        def __eq__(self, svnn):
                if self.is_subset_of(svnn) and svnn.is_subset_of(self):
                        return True
                return False
```

4.2.4 SVNN Add

Two SVNNs can be added using the Equation 4.4 and implemented in Listing 5. Added SVNNs return a new SVNN. Using Python magic by implementing the add functionality through `__add__` enables us to utilize the plus operator $\oplus$ operator on SVNNs.

$$x \oplus y = \langle T_x + T_y - T_x T_y, I_x I_y, F_x F_y \rangle \tag{4.4}$$

Listing 5: SVNN Add

```
def __add__(self, svnn):
    return svnn(f'{self._id} + {svnn._id}', (self._truth + svnn._truth) - (self._truth * svnn._truth), self._indeterminacy * svnn._indeterminacy, self._falsehood * svnn._falsehood)
```

4.2.5 SVNN Multiply by SVNN

Two SVNNs can be multiplied by the Equation 4.5 and implemented in Listing 6. Using Python magic by implementing the multiply functionality through `__mul__` enables us to utilize the multiply operator $\otimes$ on SVNNs.

$$x \otimes y = \langle T_x T_y, I_x + I_y - I_x I_y, F_x + F_y - F_x F_y \rangle \tag{4.5}$$

Listing 6: SVNN Multiply by SVNN

```
    def __mul__(self, svnn):
        return svnn(f'{self._id} * {svnn._id}', self._truth * svnn._truth, svnn._indeterminacy - (self._indeterminacy * svnn._indeterminacy), (self._falsehood + svnn._falsehood) - (self._falsehood * svnn._falsehood))
```

4.2.6 SVNN Multiply by Alpha

SVNN can be multiplied by constant (alpha) using the Equation 4.6 and implemented in Listing 7. Multiplying by constant is an important operation that is very useful in scaling, that is crucial for computer graphics and image processing, among other fields.

$$\alpha x = \langle 1 - (1 - T_x)^{\alpha}, I_x^{\alpha}, F_x^{\alpha} \rangle \longleftarrow \alpha > 0 \tag{4.6}$$

Listing 7: Multiply by Number

```
    def multiply_by_alpha(self, alpha):
        assert alpha > 0, 'Alpha must be larger than zero'
        return svnn(f'{self._id}_multiplied_by_{alpha}', 1 - pow(1 - self._truth), alpha, pow(self._indeterminacy, alpha), pow(self._falsehood, alpha))
```

4.2.7 SVNN Score

Calculating SVNN Score is important for MCDM. Listing 8 presents the implementation of the SVNN Score function calculated in Equation 4.7

$$E(x) = \frac{(2 + T_x - I_x - F_x)}{3}, E(x) \in [0, 1] \tag{4.7}$$

Listing 8: SVNN Score

```
def score(self):
    return ( 2 + self._truth - self._indeterminacy - self._falsehood ) /
        ↪ 3
```

4.2.8 SVNN Accuracy

SVNN Accuracy also plays an important rule in MCDM. Examples include, not only: rule engines. Listing 9 highlights the Python code that calculates SVNN Accuracy presented in Equation 4.8

$$H(x) = T_x - F_x, H(x) \in [-1, 1] \tag{4.8}$$

Listing 9: SVNN Accuracy

```
def accuracy(self):
    return self._truth - self._falsehood
```

4.2.9 SVNN Ranking

The ranking method is based on both the score values of E(x) and E(y) [15] and the accuracy degrees of H(x) and H(y) has the following relations depicted in Equations 4.9, 4.10, 4.11 and implemented in Listing 10 as follows:

$$\text{if } E(x) > E(y) \text{ then } x \succ y \tag{4.9}$$

$$\text{if } E(x) = E(y) \text{ and } H(x) > H(y) \text{ then } x \succ y \tag{4.10}$$

$$\text{if } E(x) = E(y) \text{ and } H(x) > H(y) \text{ then } x = y \tag{4.11}$$

Listing 10: SVNN Rank

```
def ranking_compared_to(self, svnn):
    """
    :param svnn:
    :return: -1 -> Not Applicable, 0 -> equal ranking, 1 -> higher ranking
    """
    if self.score() > svnn.score():
```

```
                return 1
            if self.score() == svnn.score() and self.accuracy() > svnn.accuracy()
                ↪ :
                return 1
            if self.score() == svnn.score() and self.accuracy() == svnn.accuracy
                ↪ ():
                return 0
            return −1
```

4.2.10 SVNN Deneutrosophication / Score Function

Deneutrosophication can be defined as mapping a Single Valued Neutrosophic Number into a crisp output and is calculated in [7] as

$$\psi = 1 - \sqrt{\frac{(1 - T_x)^2 + I_x^2 + F_x^2}{3}} \tag{4.12}$$

Listing 11 presents the Python code required to implement the Equation 4.12

Listing 11: SVNN Deneutrosophy

```
    def deneutrosophy(self):
        from math import sqrt, pow
        return 1 − (sqrt (((pow(1 − self._truth),2) + pow(self._indeterminacy
            ↪ ,2) + pow(self._falsehood,2)) / 3))
```

4.3 SVNN Helper Methods

Those are additional methods required for coding, debugging, and documentation purposes. SVNN additional methods are listed in Listing 12

Listing 12: SVNN - Helper Methods

```
  def __str__(self):
      return f'ID: {self._id} − Truth: {self._truth} − Indeterminacy: {self.
          ↪ _indeterminacy} − Falsehood: {self._falsehood}'

  def __repr__(self):
      return f'ID: {self._id} − Truth: {self._truth} − Indeterminacy: {self.
          ↪ _indeterminacy} − Falsehood: {self._falsehood}'
```

5 Single Valued Neutrosophic Sets

The implementation in Listing 13 represents thinking of a Single Valued Neutrosophic Set (SVNS) as a Set of Single Valued Neutrosophic Numbers (SVNNs). Utilizing Object Oriented Concepts in proposed neutrosophic package, SVNN is presented as a class, and SVNS is presented as another class, and there is an association relationship between them. This justifies importing SVNN within SVNS class.

Listing 13: SVNS - Constructor

```
from svnn import SingleValuedNeutrosophicNumber

class SVNSet:
"""This class has association relationship with SVNN
"""

    def __init__(self):
        # List of SVNNs
        self._items = []
        # Index variable - used for iteration over SVNNs in SVNS
        self.__idx = -1
```

5.1 SVNS Hybrid Arithmetic Operators

where $\wedge$ is the t-norm, and $\vee$is the t-conorm. Hybrid arithmetic and geometric aggregation operators are defined in [15] as follows

- Single Valued Neutrosophic Number Weighted Arithmetic Average (SVNNWAA)
- Single Valued Neutrosophic Number Weighted Geometric Average (SVNNWGA)
- Single Valued Neutrosophic Number Ordered Weighted Arithmetic Average (SVNNOWAA)
- Single Valued Neutrosophic Number Ordered Weighted Geometric Average (SVNNOWGA)

5.1.1 Single Valued Neutrosophic Number Weighted Arithmetic Average (SVNNWAA)

Listing 14 presents the Python code that calculates SVNWAA as presented in Equation 5.1

$$SVNNWAA(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} w_j z_j = \langle 1 - \prod_{j=1}^{n} (1 - T_j)^{w_j}, \prod_{j=1}^{n} (U_j)^{w_j}, \prod_{j=1}^{n} (V_j)^{w_j} \rangle \quad (5.1)$$

Listing 14: SVNNWAA

```
def weighted_arithmetic_average(self, weights):
    """
    single-valued neutrosophic number weighted arithmetic average (SVNNWAA)
    weights: List of weights of each item - list length must be equal to the length of the items
    For more information: Google weighted arithmetic average
    or watch https://www.youtube.com/watch?reload=9&v=IuuBU6fwtNo
    :return: Three values: T, U, V
    """
    assert len(weights) == len(self._items), 'Weights List Length Does Not
        ↪ Match Collection SVNN Items'
```

```
        weights_sum = 0.0
        for weight in weights:
            weights_sum += weight
        assert weights_sum == 1, 'Weight\'s sum does not equal 1'
        truth_total = 1.0
        indetermenacy_total = 1.0
        falsehood_total = 1.0
        for item, weight in zip(self._items, weights):
            truth_total *= pow(1 - item._truth, weight)
            indetermenacy_total *= pow(item._indeterminacy, weight)
            falsehood_total *= pow(item._falsehood, weight)
        return 1 - truth_total, indetermenacy_total, falsehood_total
```

5.1.2 Single Valued Neutrosophic Number Weighted Geometric Average (SVNNWGA)

Listing 15 implements Equation 5.2.

$$SVNNWGA(z_1, z_2, \ldots, z_n) = \prod_{j=1}^{n} z_j^{w_j} = \langle \prod_{j=1}^{n} (T_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - U_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - V_j)^{w_j} \rangle \quad (5.2)$$

Listing 15: SVNNWGA

```
    def weighted_geometric_average(self, weights):
        """single-valued neutrosophic number weighted geometric average
        """
        weights_sum = 0.0
        for weight in weights:
            weights_sum += weight
        assert weights_sum == 1, 'Weight\'s sum does not equal 1'
        truth_total = 1.0
        indetermenacy_total = 1.0
        falsehood_total = 1.0
        weights.sort()
        for item, weight in zip(self._items, weights):
            truth_total *= pow(item._truth, weight)
            indetermenacy_total *= pow(1 - item._indeterminacy, weight)
            falsehood_total *= pow(1 - item._falsehood, weight)
        return truth_total, 1 - indetermenacy_total, 1 - falsehood_total
```

5.2 SVNS Geometric Aggregation Operators

5.2.1 Single Valued Neutrosophic Number Ordered Weighted Arithmetic Average (SVNNOWAA)

Listing 16 presents Python implementation of Equation 5.3.

$$SVNNOWAA(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} \zeta_j z_{p(j)} = \langle 1 - \prod_{j=1}^{n} (1 - T_{p(j)})^{\zeta_j}, \prod_{j=1}^{n} (U_{p(j)})^{\zeta_j}, \prod_{j=1}^{n} (V_{p(j)})^{\zeta_j} \rangle \quad (5.3)$$

Listing 16: SVNNOWAA

```
def ordered_weighted_arithmetic_average(self, weights, ordered_by_position =
    ↪ False):
        assert len(weights) == len(self._items), 'Weights List Length Does Not
            ↪  Match Collection SVNN Items'
        weights_sum = 0.0
        for weight in weights:
                weights_sum += weight
        assert weights_sum == 1, 'Weight\'s sum does not equal 1'
        truth_total = 1.0
        indetermenacy_total = 1.0
        falsehood_total = 1.0
        for item, weight in zip(self._items, weights):
                truth_total *= pow(1 - item._truth, weight)
                indetermenacy_total *= pow(item._indeterminacy, weight)
                falsehood_total *= pow(item._falsehood, weight)
        return 1 - truth_total, indetermenacy_total, falsehood_total
```

5.2.2 Single Valued Neutrosophic Number Ordered Weighted Geometric Average (SVNNOWGA)

Listing 17 depicts the implementation of Equation 5.4. Python provides efficient ways that helps in building such complicated calculations. Here, Weights are sorted to be used for the calculation. Python utilizes efficient builtin methods, for example like the one presented for sorting.

$$SVNNOWGA(z_1, z_2, \ldots, z_n) = \prod_{j=1}^{n} z_{p(j)}^{\zeta_j} = \langle \prod_{j=1}^{n} (T_{p(j)})^{\zeta_j}, 1 - \prod_{j=1}^{n} (1 - U_{p(j)})^{\zeta_j}, 1 - \prod_{j=1}^{n} (1 - V_{p(j)})^{\zeta_j} \rangle \quad (5.4)$$

Listing 17: SVNNOWGA

```
def ordered_weighted_geometric_average(self, weights):
        """single-valued neutrosophic number weighted geometric average
        """
        weights_sum = 0.0
```

```
        for weight in weights:
            weights_sum += weight
        assert weights_sum == 1, 'Weight\'s sum does not equal 1'
        truth_total = 1.0
        indetermenacy_total = 1.0
        falsehood_total = 1.0
        # The following line is the main difference
        weights.sort()
        for item, weight in zip(self._items, weights):
            truth_total *= pow(item._truth, weight)
            indetermenacy_total *= pow(1 - item._indeterminacy, weight)
            falsehood_total *= pow(1 - item._falsehood, weight)
        return truth_total, 1 - indetermenacy_total, 1 - falsehood_total
```

5.3 SVNS Helper Methods

Additional helper methods are needed for supporting basic SVNS operations, such as

5.3.1 Add SVNN

Supports adding SVNN to SVNS, as depicted in Listing 18

Listing 18: SVNS - Add SVNN

```
    def add_svnn(self, svnn):
    # TODO: Prevent Duplication
        self._items.append(svnn)
```

5.3.2 Delete SVNN

Supports removing SVNN from SVNS, as implemented in Listing 19

Listing 19: SVNS - Delete SVNN

```
    def delete_svnn(self, svnn):
        #TODO: Notify user about Exception handling
        self._items.remove(svnn)
```

5.3.3 Retrieve All SVNNs

Retrieving a list of all SVNNs in SVNS is a crucial task. Returned list is an iterable one that can be used for further processing. Listing 20 presents such functionality implementation.

Listing 20: SVNS - Retrieve All SVNNs

```
    def get_all_svnns(self):
        return self._items
```

5.3.4 Count All SVNNs in SVNS

Counting all SVNNs in SVNS is a primitive task. Listing 21 presents the code to implement such a functionality. Using Python magic, via utilizing the `__len__` enables us to use the `len()` function syntax over SVNN object.

Listing 21: SVNS - Count All SVNNs

```
    def __len__(self):
        return len(self._items)
```

5.3.5 SVNS: is_empty

Though checking either SVNS is empty or not can be achieved via `__len__()` function, it is important to enable proposed neutrosophic package to check `is_empty()` in conditionals. Listing 22 depicts such functionality.

Listing 22: SVNS - is_empty

```
    def is_empty(self):
        if len(self) == 0:
            return True
        return False
```

5.3.6 SVNS - Iteration

Iteration is a general term for taking each item of something, one after another. While using a loop for example, going over a group of items is called iteration. An iterable object is an object that has an `__iter__` method which returns an iterator. `__getitem__` method can take sequential indexes starting from zero (and raises an IndexError when the indexes are no longer valid). An iterator is an object with a `__next__` method.

Providing iteration functionality within our proposed neutrosophic package is critical, so later users can either loop, or apply map functionalities over SVNNs within SVNSs. Such characteristic is an important feature for future use cases. Listing 23 depicts how iteration functionality is implemented in SVNS.

Listing 23: SVNS - Iteration

```
    def __iter__(self):
        return self

    def __next__(self):
        self.__idx += 1
        try:
            return self._items[self.__idx]
        except IndexError:
            self.__idx = 0
            raise StopIteration

    def __getitem__(self, id):
        try:
```

```
            return self._items[id]
        except IndexError:
            raise StopIteration
```

6 Interval Valued Neutrosophic Number

Given the following definitions, operational laws can be applied as defined in [9]

$$\dot{N}_1 = \left\{ < x : \left[T^L_{\dot{N}_1}, T^U_{\dot{N}_1} \right], \left[I^L_{\dot{N}_1}, I^U_{\dot{N}_1} \right], \left[F^L_{\dot{N}_1}, F^U_{\dot{N}_1} \right] >, x \in X \right\}$$

$$\dot{N}_2 = \left\{ < x : \left[T^L_{\dot{N}_2}, T^U_{\dot{N}_2} \right], \left[I^L_{\dot{N}_2}, I^U_{\dot{N}_2} \right], \left[F^L_{\dot{N}_2}, F^U_{\dot{N}_2} \right] >, x \in X \right\}$$

Listing 24 presents the IVNN Class Declaration and Constructor. In the presented implementation, `t_lower` ↪ and `t_upper` for example represents the following mathematical symbols respectively

$$T^L_{\dot{N}_1}, T^U_{\dot{N}_1}$$

Listing 24: IVNN Class Declaration and Constructor

```
class IVNN:

    def __init__(self, id, t_lower, t_upper, i_lower, i_upper, f_lower, f_upper):

        assert 0 <= t_lower <= 1
        assert 0 <= t_upper <= 1
        assert 0 <= i_lower <= 1
        assert 0 <= i_upper <= 1
        assert 0 <= f_lower <= 1
        assert 0 <= f_upper <= 1

        assert 0 <= t_lower + i_lower + f_lower <= 3

        self._id = id
        self._t_lower = t_lower
        self._t_upper = t_upper
        self._i_lower = i_lower
        self._i_upper = i_upper
        self._f_lower = f_lower
        self._f_upper = f_upper
```

6.1 IVNN Operations

6.1.1 IVNN Complement

The complement of an interval valued neutrosophic number

$$A = \langle [T_A^l, T_A^r], [I_A^l, I_A^r], [F_A^l, F_A^r] \rangle$$

is defined by [4] as

$$A^c = \langle [F_A^l, F_A^r], [I_A^l, I_A^r], [T_A^l, T_A^r] \rangle \tag{6.1}$$

Listing 25: IVNN - Complement

```
    def complement(self):
        return IVNN(f'{self._id}_complement',
        self._f_lower,
        self._f_upper,
        self._i_lower,
        self._i_upper,
        self._t_lower,
        self._t_upper)
```

6.1.2 IVNN Add

Two SVNNs can be added using the Equation 6.2 and implemented in Listing 26. Added IVNNs return a new IVNN. Again, using Python magic by implementing the add functionality through __add__ enables us to utilize the plus operator ⊕ operator on IVNNs.

$$\dot{N}_1 \oplus \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} + T^L_{\dot{N}_2} - T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} + T^U_{\dot{N}_2} - T^U_{\dot{N}_1} T^U_{\dot{N}_2} \right], \left[I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} I^U_{\dot{N}_2} \right], \left[F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} F^U_{\dot{N}_2} \right] \right\rangle \tag{6.2}$$

Listing 26: IVNN - Add Two IVNNs

```
    def __add__(self, ivnn):
        return IVNN(f'{self._id} + {ivnn._id}',
        self._t_lower + ivnn._t_lower − self._t_lower ∗ ivnn._t_lower,
        self._t_upper + ivnn._t_upper − self._t_upper ∗ ivnn._t_upper,
        self._i_lower ∗ ivnn._i_lower,
        self._i_upper ∗ ivnn._i_upper,
        self._f_lower ∗ ivnn._f_lower,
        self._f_upper ∗ ivnn._f_upper)
```

6.1.3 IVNN Multiply by IVNN

Two IVNNs can be multiplied by the Equation 6.3 and implemented in Listing 27. Using Python magic by implementing the multiply functionality through __mul__ enables us to utilize the multiply operator $\otimes$ on IVNNs.

$$\dot{N}_1 \otimes \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_1} + I^L_{\dot{N}_2} - I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} + I^U_{\dot{N}_2} - I^U_{\dot{N}_1} I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_1} + F^L_{\dot{N}_2} - F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} + F^U_{\dot{N}_2} - F^U_{\dot{N}_1} F^U_{\dot{N}_2}\right] \right\rangle \quad (6.3)$$

Listing 27: IVNN - Multiply Two IVNNs

```
def __mul__(self, ivnn):
    return IVNN(f'P{self._id} * {ivnn._id}',
    self._t_lower * ivnn._t_lower,
    self._t_upper * ivnn._t_upper,
    self._i_lower + ivnn._i_lower - self._i_lower * ivnn._i_lower,
    self._i_upper + ivnn._i_upper - self._i_upper * ivnn._i_upper,
    self._f_lower + ivnn._f_lower - self._f_lower * ivnn._f_lower,
    self._f_upper + ivnn._f_upper - self._f_upper * ivnn._f_upper)
```

6.1.4 IVNN Multiply by Alpha

IVNN can be multiplied by constant (alpha) using the Equation 6.4 and implemented in Listing 28.

$$\delta \dot{N} = \left\langle \left[1 - \left(1 - T^L_N\right)^\delta, 1 - \left(1 - T^U_N\right)^\delta\right], \left[\left(T^L_N\right)^\delta, \left(T^U_N\right)^\delta\right], \left[\left(F^L_N\right)^\delta, \left(F^U_N\right)^\delta\right] \right\rangle \quad (6.4)$$

Listing 28: IVNN - Multiply by Alpha

```
def multiply_by(self, alpha):
    return IVNN(f'{alpha} * {self._id}',
    1 - pow((1 - self._t_lower),alpha),
    1 - pow((1 - self._t_upper),alpha),
    pow(self._i_lower, alpha),
    pow(self._i_upper, alpha),
    pow(self._f_lower, alpha),
    pow(self._i_upper, alpha))
```

7 Interval Valued Neutrosophic Sets

7.1 IVNS - Weighted Average

Interval Neutrosophic Number Weighted Average Operator (INNWA) defined by [26] Let

$$A_j = \langle T_{A_j}, I_{A_j}, F_{A_j} \rangle (j = 1, 2, ..., n)$$

be a collection of IVNNs, and let

$$INNWA : INN^n \rightarrow INN$$

$$\begin{array}{c} INNWA_w(A_1, A_2, \ldots, A_n) \\ = \langle [1 - \prod_{i=1}^n (1 - \inf T_{A_i})^{w_i}, 1 - \prod_{i=1}^n (1 - \sup T_{A_i})^{w_i}], \\ [\prod_{i=1}^n \inf I_{A_i}^{w_i}, \prod_{i=1}^n \sup I_{A_i}^{w_i}], \\ [\prod_{i=1}^n \inf F_{A_i}^{w_i}, \prod_{i=1}^n \sup F_{A_i}^{w_i}] \rangle, \end{array} \tag{7.1}$$

Listing 29 presents Python implementation of 7.1

Listing 29: INNWA

```
    def weighted_average(self, weights):
        """
        :return: IVNN
        """
        weights_sum = 0
        for weight in weights:
            assert 0 <= weight <= 1
            weights_sum += weight
        assert weights_sum == 1

        t_lower_dot_product = 1.0
        t_upper_dot_product = 1.0
        i_lower_dot_product = 1.0
        i_upper_dot_product = 1.0
        f_lower_dot_product = 1.0
        f_upper_dot_product = 1.0

        for ivnn, weight in zip(self._ivnns, weights):
            t_lower_dot_product *= pow(1 - ivnn._t_lower, weight)
            t_upper_dot_product *= pow(1 - ivnn._t_upper, weight)
            i_lower_dot_product *= pow(ivnn._i_lower, weight)
            i_upper_dot_product *= pow(ivnn._i_upper, weight)
            f_lower_dot_product *= pow(ivnn._f_lower, weight)
            f_upper_dot_product *= pow(ivnn._i_upper, weight)

        return IVNN(1 - t_lower_dot_product, 1 - t_upper_dot_product,
           ↪ i_lower_dot_product, i_upper_dot_product, f_lower_dot_product,
           ↪ f_upper_dot_product)
```

7.2 IVNS Helper Methods

Additional IVNS helper method is presented in Listing 30. IVNS is a collection of IVNNs, and thus the method add_ivnn is presented.

Listing 30: IVNN - Helper Methods

```
    def add_ivnn(self, ivnn):
        self._ivnns.append(ivnn)
```

8 Conclusion and Future Work

Neutrosophic sets has gained wide popularity and acceptance in different disciplines. This paper presented an Open Source Python Neutrosophic package. Presented package utilizes Object Oriented Design and implementation concepts. Presented package is licensed under MIT License. Licensing was chosen carefully to support and enable both open source and neutrosophic community. Python was chosen for this package as a result of Python's wide applicability in different paradigms, including mainly Big Data Analytics, Machine Learning, and Artificial Intelligence. Presented package is an open source one, so developers and researchers in different disciplines can adopt it effectively. Presented Neutrosophic package is a work on progress, as Neutrosophic sets and theory becomes more popular and gets utilized in different fields. Presented package presented support for: Single Valued Neutrosophic Numbers, Single Valued Neutrosophic Sets, Interval Valued Neutrosophic Numbers, and Interval Valued Neutrosophic Sets. Different operations were presented. Presented package can be found at `https://www.github.com/helghareeb/neutrosophic`.

The main challenge was the multiple definitions and proofs for the same operation, with different calculation methods. Example of such a challenge is the Score Function. There are numerous deneutrosophy functions for the same neutrosophic number, each with its own proof. Future Work includes uploading the presented Neutrosophic package to one of the most widely utilized Python Package servers. Besides, porting the presented neutrosophic package into different Programming Languages. Implementing Different Deneutrosophication / Score Functions, highlighting the differences between them is another step to take. Support of Triangular and Trapezoidal Neutrosophic Numbers is another challenge to tackle.

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Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras

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Abstract: In the present paper, we discuss the Neutrosophic quadruple q-ideals and (regular) neutrosophic quadruple ideals and investigate their related properties. Also, for any two nonempty subsets U and V of a BCI-algebra S, conditions for the set $NQ(U, V)$ to be a (regular) neutrosophic quadruple ideal and a neutrosophic quadruple q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$ are discussed. Furthermore, we prove that let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Keywords: neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, (regular) neutrosophic quadruple ideal, neutrosophic quadruple q-ideal.

1 Introduction

To deal with incomplete, inconsistent and indeterminate information, Smarandache introduced the notion of neutrosophic sets (see ([1], [2] and [3]). In fact, neutrosophic set is a useful mathematical tool which extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory has useful applications in several branches (see for e.g., [4], [5], [6] and [7]).

In [8], Smarandache considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [9] and [10]. Recently, neutrosophic set theory has been applied to the BCK/BCI-algebras on various aspects (see for e.g., [11], [12] [13], [14], [15], [16], [17], [18], [19] and [20].) Using the notion of neutrosophic quadruple numbers based on a set, Jun et al. [21] constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered ideal and positive implicative ideal in neutrosophic quadruple BCK-algebra, and closed ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, they considered sets $NQ(U,V)$ which consists of neutrosophic quadruple BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(U,V)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(U,V)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra. Muhiuddin et al. [22] discussed several properties and (implicative) neutrosophic quadruple ideals in (implicative) neutrosophic quadruple BCK-algebras.

In this paper, we introduce the notions of (regular) neutrosophic quadruple ideal and neutrosophic quadruple q-ideal in neutrosophic quadruple BCI-algebras, and investigate related properties. Given nonempty subsets A and B of a BCI-algebra S, we consider conditions for the set $NQ(U,V)$ to be a (regular) neutrosophic quadruple ideal of $NQ(S)$ and a neutrosophic quadruple q-ideal of $NQ(S)$.

2 Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set S with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x, y, z \in S$ the following conditions hold ([23] and [24]):

(I) $(((x*y)*(x*z))*(z*y)=0)$,

(II) $((x*(x*y))*y=0)$,

(III) $(x*x=0)$,

(IV) $(x*y=0,\, y*x=0 \;\Rightarrow\; x=y)$.

If a BCI-algebra S satisfies the following identity:

(V) $(\forall x \in S)\,(0*x=0)$,

then S is called a *BCK-algebra.* Define a binary relation $\leq$ on X by letting $x*y=0$ if and only if $x \leq y$. Then $(S,\leq)$ is a partially ordered set.

Theorem 2.1. *Let S be a BCK/BCI-algebra. Then following conditions are hold:*

$$(\forall x \in S)\,(x*0=x)\,, \tag{2.1}$$
$$(\forall x,y,z \in S)\,(x \leq y \;\Rightarrow\; x*z \leq y*z,\; z*y \leq z*x)\,, \tag{2.2}$$
$$(\forall x,y,z \in S)\,((x*y)*z=(x*z)*y)\,, \tag{2.3}$$
$$(\forall x,y,z \in S)\,((x*z)*(y*z) \leq x*y) \tag{2.4}$$

where $x \leq y$ if and only if $x*y=0$.

Any BCI-algebra S satisfies the following conditions (see [25]):

$$(\forall x,y \in S)(x*(x*(x*y))=x*y), \tag{2.5}$$
$$(\forall x,y \in S)(0*(x*y)=(0*x)*(0*y)), \tag{2.6}$$
$$(\forall x,y \in S)(0*(0*(x*y))=(0*y)*(0*x)). \tag{2.7}$$

A nonempty subset A of a BCK/BCI-algebra S is called a *subalgebra* of S if $x * y \in A$ for all $x, y \in A$. A subset I of a BCK/BCI-algebra S is called an *ideal* of S if it satisfies:

$$0 \in I, \tag{2.8}$$
$$(\forall x \in S)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.9}$$

An ideal I of a BCI-algebra S is said to be *regular* (see [26]) if it is also a subalgebra of S.

It is clear that every ideal of a BCK-algebra is regular (see [26]).

A subset I of a BCI-algebra S is called a *q-ideal* of S (see [27]) if it satisfies (2.8) and

$$(\forall x, y, z \in S)(x * (y * z) \in I,\, y \in I \;\Rightarrow\; x * z \in I). \tag{2.10}$$

We refer the reader to the books [25, 28] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 2.2 ([21])**.** Let S be a set. A *neutrosophic quadruple S-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in S$ and T, I, F have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple S-numbers is denoted by $NQ(S)$, that is,

$$NQ(S) := \{(a, xT, yI, zF) \mid a, x, y, z \in S\},$$

and it is called the *neutrosophic quadruple set* based on S. If S is a BCK/BCI-algebra, a neutrosophic quadruple S-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(S)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let S be a BCK/BCI-algebra. We define a binary operation $\circledast$ on $NQ(S)$ by

$$(a, xT, yI, zF) \circledast (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(S)$. Given $a_1, a_2, a_3, a_4 \in S$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(S)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(S)$.

Theorem 2.3 ([21])**.** *If S is a BCK/BCI-algebra, then $(NQ(S); \circledast, \tilde{0})$ is a BCK/BCI-algebra.*

We say that $(NQ(S); \circledast, \tilde{0})$ is a *neutrosophic quadruple BCK/BCI-algebra,* and it is simply denoted by $NQ(S)$.

Let S be a BCK/BCI-algebra. Given nonempty subsets A and B of S, consider the set

$$NQ(U, V) := \{(a, xT, yI, zF) \in NQ(S) \mid a, x \in U \;\&\; y, z \in V\},$$

which is called the *neutrosophic quadruple (U, V)-set.*

The set $NQ(U, U)$ is denoted by $NQ(U)$, and it is called the *neutrosophic quadruple U-set.*

3 (Regular) neutrosophic quadruple ideals

Definition 3.1. Given nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a (regular) ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U, V)$ is a *(regular) neutrosophic quadruple ideal* of $NQ(S)$.

Question 1. *If U and V are subalgebras of a BCI-algebra S, then is the neutrosophic quadruple (U, V)-set $NQ(U, V)$ a neutrosophic quadruple ideal of $NQ(S)$?*

The answer to Question 1 is negative as seen in the following example.

Example 3.2. Consider a BCI-algebra $S = \{0, 1, a, b, c\}$ with the binary operation $*$, which is given in Table 1. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 625 elements. Note that $U = \{0, a\}$ and $V = \{0, b\}$

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

are subalgebras of S. The neutrosophic quadruple (U, V)-set $NQ(U, V)$ consists of the following elements:

$$NQ(U, V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, bF)$, $\tilde{2} = (0, 0T, bI, 0F)$,
$\tilde{3} = (0, 0T, bI, bF)$, $\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, bF)$,
$\tilde{6} = (0, aT, bI, 0F)$, $\tilde{7} = (0, aT, bI, bF)$, $\tilde{8} = (a, 0T, 0I, 0F)$,
$\tilde{9} = (a, 0T, 0I, bF)$, $\tilde{10} = (a, 0T, bI, 0F)$, $\tilde{11} = (a, 0T, bI, bF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, bF)$,
$\tilde{14} = (a, aT, bI, 0F)$, $\tilde{15} = (a, aT, bI, bF)$.

If we take $(1, aT, bI, 0F) \in NQ(S)$, then $(1, aT, bI, 0F) \notin NQ(U, V)$ and

$$(1, aT, bI, 0F) \circledast \tilde{9} = \tilde{15} \in NQ(U, V).$$

Hence the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is not a neutrosophic quadruple ideal of $NQ(S)$.

We consider conditions for the neutrosophic quadruple (U, V)-set $NQ(U, V)$ to be a regular neutrosophic quadruple ideal of $NQ(S)$.

Lemma 3.3 ([21])**.** *If U and V are subalgebras (resp., ideals) of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple subalgebra (resp., ideal) of $NQ(S)$.*

Theorem 3.4. *Let U and V be subalgebras of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \text{ (resp., } V),\ y \notin U \text{ (resp., } V) \ \Rightarrow\ y * x \notin U \text{ (resp., } V)). \tag{3.1}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Proof. By Lemma 3.3, $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$. Hence it is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(S)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(S)$ be such that $\tilde{y} \circledast \tilde{x} \in NQ(U, V)$ and $\tilde{x} \in NQ(U, V)$. Then $x_i \in U$ and $x_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Also,

$$\begin{aligned} \tilde{y} \circledast \tilde{x} &= (y_1, y_2T, y_3I, y_4F) \circledast (x_1, x_2T, x_3I, x_4F) \\ &= (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in NQ(U, V), \end{aligned}$$

and so $y_1 * x_1 \in U$, $y_2 * x_2 \in U$, $y_3 * x_3 \in V$ and $y_4 * x_4 \in V$. If $\tilde{y} \notin NQ(U, V)$, then $y_i \notin A$ or $y_j \notin B$ for some $i = 1, 2$ and $j = 3, 4$. It follows from (3.1) that $y_i * x_i \notin U$ or $y_j * x_j \notin V$ for some $i = 1, 2$ and $j = 3, 4$. This is a contradiction, and so $\tilde{y} \in NQ(U, V)$. Thus $NQ(U, V)$ is a neutrosophic quadruple ideal of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.5. *Let U be a subalgebra of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U,\ y \notin U \ \Rightarrow\ y * x \notin U). \tag{3.2}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Theorem 3.6. *Let U and V be subsets of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U, V)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U, V)$ for all $\tilde{x} \in NQ(U, V)$, then $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. For any $\tilde{x}, \tilde{y} \in NQ(U, V)$, we have

$$(\tilde{x} \circledast \tilde{y}) \circledast \tilde{x} = (\tilde{x} \circledast \tilde{x}) \circledast \tilde{y} = \tilde{0} \circledast \tilde{y} \in NQ(U, V).$$

Since $NQ(U, V)$ is an ideal of $NQ(S)$, it follows that $\tilde{x} \circledast \tilde{y} \in NQ(U, V)$. Hence $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.7. *Let U be a subset of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U)$ for all $\tilde{x} \in NQ(U)$, then $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Theorem 3.8. *If U and V are ideals of a finite BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. By Lemma 3.3, $NQ(U, V)$ is a neutrosophic quadruple ideal of $NQ(S)$. Since S is finite, $NQ(S)$ is also finite. Assume that $|NQ(S)| = n$. For any element $\tilde{x} \in NQ(U, V)$, consider the following $n+1$ elements:

$$\tilde{0}, \tilde{0} \circledast \tilde{x}, (\tilde{0} \circledast \tilde{x}) \circledast \tilde{x}, \cdots, (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{n\text{-times}}.$$

Then there exist natural numbers p and q with $p > q$ such that

$$(\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p\text{-times}} = (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q\text{-times}}.$$

Hence

$$\begin{aligned}
\tilde{0} &= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}} \in NQ(U, V).
\end{aligned}$$

Since $NQ(U, V)$ is an ideal of $NQ(S)$, it follows that $\tilde{0} \circledast \tilde{x} \in NQ(U, V)$. Therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$ by Theorem 3.6. □

Corollary 3.9. *If U is an ideal of a finite BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

4 Neutrosophic quadruple q-ideals

Definition 4.1. Given nonempty subsets U and V of S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U, V)$ is a *neutrosophic quadruple q-ideal* of $NQ(S)$.

Example 4.2. Consider a BCI-algebra $S = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 81 elements. If we take $U = \{0, 1\}$ and $V = \{0, 1\}$, then

$$NQ(U, V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

is a neutrosophic quadruple q-ideal of $NQ(S)$ where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$,
$\tilde{3} = (0, 0T, 1I, 1F)$, $\tilde{4} = (0, 1T, 0I, 0F)$, $\tilde{5} = (0, 1T, 0I, 1F)$,
$\tilde{6} = (0, 1T, 1I, 0F)$, $\tilde{7} = (0, 1T, 1I, 1F)$, $\tilde{8} = (1, 0T, 0I, 0F)$,
$\tilde{9} = (1, 0T, 0I, 1F)$, $\tilde{10} = (1, 0T, 1I, 0F)$, $\tilde{11} = (1, 0T, 1I, 1F)$,

Table 2: Cayley table for the binary operation "$*$"

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, 1F)$,
$\tilde{14} = (1, 1T, 1I, 0F)$, $\tilde{15} = (1, 1T, 1I, 1F)$.

Theorem 4.3. *For any nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, then it is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$.*

Proof. Assume that $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. Since $\tilde{0} \in NQ(U, V)$, we have $0 \in U$ and $0 \in V$. Let $x, y, z \in S$ be such that $x * (y * z) \in U \cap V$ and $y \in U \cap V$. Then $(y, yT, yI, yF) \in NQ(U, V)$ and

$$\begin{aligned}&(x, xT, xI, xF) \circledast ((y, yT, yI, yF) \circledast (z, zT, zI, zF))\\ &= (x, xT, xI, xF) \circledast (y * z, (y * z)T, (y * z)I, (y * z)F)\\ &= (x * (y * z), (x * (y * z))T, (x * (y * z))I, (x * (y * z))F) \in NQ(U, V).\end{aligned}$$

Since $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, it follows that

$$(x * z, (x * z)T, (x * z)I, (x * z)F) = (x, xT, xI, xF) \circledast (z, zT, zI, zF) \in NQ(U, V).$$

Hence $x * z \in U \cap V$, and therefore U and V are q-ideals of S. Since every q-ideal is both a subalgebra and an ideal, it follows from Lemma 3.3 that $NQ(U, V)$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. □

The converse of Theorem 4.3 is not true as seen in the following example.

Example 4.4. Consider a BCI-algebra $S = \{0, a, b, c\}$ with the binary operation $*$, which is given in Table 3.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 256 elements. If we take $A = \{0\}$ and $B = \{0\}$, then $NQ(U,V) = \{\tilde{0}\}$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. If we take $\tilde{x} := (c, bT, 0I, aF)$, $\tilde{z} := (a, bT, 0I, cF) \in NQ(S)$, then

$$\begin{aligned}\tilde{x} \circledast (\tilde{0} \circledast \tilde{z}) &= (c, bT, 0I, aF) \circledast (\tilde{0} \circledast (a, bT, 0I, cF)) \\ &= (c, bT, 0I, aF) \circledast (c, bT, 0I, aF) = \tilde{0} \in NQ(U,V).\end{aligned}$$

But

$$\begin{aligned}\tilde{x} \circledast \tilde{z} &= (c, bT, 0I, aF) \circledast (a, bT, 0I, cF) \\ &= (c * a, (b * b)T, (0 * 0)I, (a * c)F) \\ &= (b, 0T, 0I, bF) \notin NQ(U,V).\end{aligned}$$

Therefore $NQ(U,V)$ is not a neutrosophic quadruple q-ideal of $NQ(S)$.

We provide conditions for the neutrosophic quadruple (U,V)-set $NQ(U,V)$ to be a neutrosophic quadruple q-ideal.

Theorem 4.5. *If U and V are q-ideals of a BCI-algebra S, then the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Suppose that U and V are q-ideals of a BCI-algebra S. Obviously, $\tilde{0} \in NQ(U,V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U,V)$ and $\tilde{y} \in NQ(U,V)$. Then $y_i \in A$, $y_j \in B$ for $i = 1, 2$ and $j = 3, 4$, and

$$\begin{aligned}&\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) = (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\ &= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\ &= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\ &\in NQ(U,V),\end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.10) that $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U,V), \tag{4.1}$$

and therefore $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.6. *If A is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Corollary 4.7. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideals U and V of S.*

Corollary 4.8. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideal U of S.*

Theorem 4.9. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \cap V \ \Rightarrow \ (x * y) * z \in U \cap V). \tag{4.2}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. It is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U, V)$ and $\tilde{y} \in NQ(U, V)$. Then $y_1, y_2 \in U$, $y_3, y_4 \in V$ and

$$\begin{aligned} \tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) &= (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\ &= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\ &= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\ &\in NQ(U, V), \end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in V$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.3) and (4.2) that $(x_i * z_i) * y_i = (x_i * y_i) * z_i \in U$ and $(x_j * z_j) * y_j = (x_j * y_j) * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Since U and V are ideals of S, we have $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U, V), \tag{4.3}$$

and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.10. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \ \Rightarrow \ (x * y) * z \in U). \tag{4.4}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.11. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \cap V \ \Rightarrow \ x * y \in U \cap V). \tag{4.5}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ for all $x, y, z \in S$. Note that

$$\begin{aligned} ((x * y)) * (0 * z)) * (x * (y * z)) &= ((x * y) * (x * (y * z))) * (0 * z) \\ &\leq ((y * z) * y) * (0 * z) \\ &= (0 * z) * (0 * z) = 0 \in U \cap V \end{aligned}$$

Thus $(x * y) * (0 * z) \in U \cap V$ since U and V are ideals of S. It follows from (4.9) that $(x * y) * z \in U \cap V$. Using Theorem 4.9, $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.12. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \ \Rightarrow \ x * y \in U). \tag{4.6}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.13. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \cap U \ \Rightarrow \ x * y \in U \cap V). \tag{4.7}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ and $y \in U \cap V$ for all $x, y, z \in S$. Using (2.3) and (4.7), we get $(x * z) * (y * z) = (x * (y * z)) * z \in U \cap V$ and $y * z \in U \cap V$. Since U and V are ideals of S, it follows that $x * z \in U \cap V$. Hence U and V are q-ideals of S, and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.5. □

Corollary 4.14. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \ \Rightarrow \ x * y \in U). \tag{4.8}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.15. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Let $x, y, z \in S$ be such that $x * (0 * y) \in U \cap V$. Then

$$(x * (x * (0 * y))) * (0 * y) = (x * (0 * y)) * (x * (0 * y)) = 0 \in I \cap J$$

by (2.3) and (III). Since I and J are q-ideals of S, it follows from (2.3) and (2.10) that

$$(x * y) * (x * (0 * y)) = (x * (x * (0 * y))) * y \in I \cap J \subseteq U \cap V$$

Since U and V are ideals of S, we have $x * y \in U \cap V$. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.11. □

Corollary 4.16. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$. If I is a q-ideal of S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Theorem 4.17. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \cap J \ \Rightarrow \ (x * y) * z \in I \cap J). \tag{4.9}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Let $x, y, z \in S$ be such that $x * (y * z) \in I \cap J$ and $y \in I \cap J$. Then

$$(x * z) * y = (x * y) * z \in I \cap J$$

by (2.3) and (4.9). Since I and J are ideals of S, it follows that $x * z \in I \cap J$. This shows that I and J are q-ideals of S. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.18. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \ \Rightarrow \ (x * y) * z \in I). \tag{4.10}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.19. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x \in I \cap J \;\Rightarrow\; x * y \in I \cap J). \tag{4.11}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. By the proof of Theorem 4.13, we know that I and J are q-ideals of S. Hence $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.20. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x \in I \;\Rightarrow\; x * y \in I). \tag{4.12}$$

Then the neutrosophic quadruple A-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.21. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \cap J \;\Rightarrow\; x * y \in I \cap J). \tag{4.13}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x*(y*z) \in I \cap J$ For all $x, y, z \in S$. Then $(x*y)*z \in I \cap J$ by the proof of Theorem 4.11. It follows from Theorem 4.17 that neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.22. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \;\Rightarrow\; x * y \in I). \tag{4.14}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Future Work: Using the results of this paper, we will aply it to another algebraic structures, for example, MV-algebras, BL-algebras, MTL-algebras, R_0-algebras, hoops, (ordered) semigroups and (semi, near) rings etc.

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Some similarity and entropy measurements of bipolar neutrosophic soft sets

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Abstract: In this paper, we proposed a different approach on bipolar neutrosophic soft sets and discussed their properties with examples which was initially introduced by Mumtaz Ali et al.[15]. Also we defined some similarity and entropy measurements between any two bipolar neutrosophic soft sets. Further, we proposed the representation of a 2-D digital image in bipolar neutrosophic soft domain. Finally, based on similarity measurements, we propose a decision making process of real-time problem in image analysis.

Keywords: Neutrosophic set, Bipolar Neutrosophic set, similarity, entropy, Digital image.

1 Introduction

In our physical world, many real life situations don't have an exact solution. For that problems, we cannot use conventional method to determine the solution. To avoid those difficulties in dealing with uncertainities, we apply the concepts of Neutrosophy. Neutrosophy is the branch of philosophy which was introduced by Florentin Smarandache [10]. Neutrosophy deals with three components truth-membership, indeterminacy-membership and falsity-membership. Apparently, in the case of uncertainty, we have different solution methods like fuzzy theory, rough theory, vague theory etc. Since Neutrosophy is the extension of fuzzy theory, it is one of the efficient method among those. By using Neutrosophy, we can analyze the origin, nature and scope of the neutralities. Neutrosophy is the base for neutrosophic sets. Neutrosophic set was introduced by Smarandache which has three components called Truth-membership, Indeterminacy-membership and Falsity-membership ranges in the non-standard interval $]^-0, 1^+[$.

But for engineering and real life problems we prefer specific solution. Since it will be difficult to apply in real life problems, Wang et al. [11] introduced the concept of single valued neutrosophic set (SVNS) which is the immediate result of neutrosophic set by taking standard interval [0,1] instead of non-standard interval $]^-0, 1^+[$. Single valued neutrosophic theory is useful in modeling uncertain imprecision. Yanhui et al. [8] proposed image segmentation through neutrosophy whereas A. A. Salama et al. [7] proposed a neutrosophic approach to grayscale images. Majundar et al. [5, 6] introduced some measures of similarity and entropy of neutrosophic sets (as well as SVNS). Aydogdu [4] proposed these similarity and entropy to Interval valued neutrosophic sets (IVNS). Also ahin and Kk [1] proposed the concepts similarity and entropy to neutrosophic soft sets.

In 2015, Deli et al. [2] introduced the concepts of bipolar neutrosophic sets (BNS) as an extension of neutrosophic sets. In 2016, Uluay et al. [3] proposed some measures of similarities of bipolar neutrosophic sets. In 2017, Mumtaz Ali et al.[15] introduced the concepts of bipolar neutrosophic soft sets which is a combined version of bipolar neutrosophic set and neutrosophic soft set. Neutrosophic set concepts are very useful in decision making problem. Abdel-Basset et al.[18, 19, 20] proposed some decision making algorithms for problems in engineering and medical fields.

In this paper, we proposed slightly different approach on bipolar neutrosophic soft sets(BNSS). Section 2 contains important preliminary definitions. In section 3, we propose different approach on bipolar neutrosophic soft set which was introduced by Ali et al.[15] and also we discuss their properties with examples. In section 4, we define entropy measurement to calculate the indeterminacy. In section 5, we defined various distances between any two BNSSs to calculate the similarity between them. In section 6, we propose the representation of 2-D digital image in bipolar neutrosophic soft domain. In section 7, we propose the decision making process of image based on similarity measurements for a real-time problem in image analysis. Finally, section 8 contains conclusion of our work.

2 Preliminaries

Definition 2.1. [12]
Let X be a universal set which contains arbitrary points x. A Neutrosophic set A is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

where $T_A(x), I_A(x), F_A(x)$ referred as truth-membership function, indeterminacy-membership function and falsity-membership function respectively.
Here

$$T_A(x), I_A(x), F_A(x) : X \to]^{-}0, 1^{+}[.$$

Further it satisfies the condition

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}.$$

Example 2.2. Let $X = \{x_1, x_2, x_3\}$ be the universal set. Here, x_1, x_2, x_3 represents capacity, trustworthiness and price of a machine, respectively. Then $T_A(x), I_A(x), F_A(x)$ gives the degree of 'good service', degree of indeterminacy, degree of 'poor service' respectively. The neutrosophic set is defined by
$A = \{\langle x_1, 0.3, 0.4, 0.5\rangle, \langle x_2, 0.5, 0.2, 0.3\rangle, \langle x_3, 0.7, 0.2, 0.2\rangle\}$
where $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$

Definition 2.3. [11]
Neutrosophic set(NS) is defined over the non-standard unit interval $]^{-}0, 1^{+}[$ whereas single valued neutrosophic set is defined over standard unit interval [0,1].
It means a single valued neutrosophic set A is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

where

$$T_A(x), I_A(x), F_A(x) : X \to [0, 1]$$

such that

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 2.4. [13, 16]
A pair (F, A) is a soft set over X if

$$F : A \to P(x)$$

That means the soft set is a parameterized family of subsets of the set X.
For any parameter $e \in A$, $F(e) \subseteq X$ is the set of e-approximation elements of the soft set (F, A).

Example 2.5. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of 2-dimensional images and let $A = \{e_1, e_2, e_3\}$ be set of parameters. where e_1=contrast, e_2=saturation and e_3=sharpness.
suppose that

$$F(e_1) = \{x_1, x_2\}$$
$$F(e_2) = \{x_1, x_3\}$$
$$F(e_3) = \{x_2, x_4\}.$$

Then, the set

$$F(A) = \{F(e_1), F(e_2), F(e_3)\}$$

is the parameterized family of subsets of X.

Definition 2.6. [14]
A neutrosophic soft set (F_A, E) over X is defined by the set

$$(F_A, E) = \big\{\langle e, F_A(e)\rangle : e \in E, F_A(e) \in NS(X)\big\}$$

where $F_A : E \longrightarrow NS(x)$ such that $F_A(e) = \varphi$ if $e \notin A$.
Also, since $F_A(e)$ is a neutrosophic set over X is defined by

$$F_A(e) = \big\{\big\langle x, u_{F_A(e)}(x), v_{F_A(e)}(x), w_{F_A(e)}(x)\big\rangle : x \in X\big\}$$

where $u_{F_A(e)}(x), v_{F_A(e)}(x), w_{F_A(e)}(x)$ represents truth-membership degree of x which holds the parameter e, indeterminacy-membership degree of x which holds the parameter e and falsity-membership degree of x which holds the parameter e.

Example 2.7. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of houses under consideration. Let $A = \{e_1, e_2, e_3\}$ be set of parameters where e_1, e_2, e_3 represents beautiful, wooden and costly, respectively.
Then we define

$$(F_A, E) = \big\{\langle e_1, F_A(e_1)\rangle, \langle e_2, F_A(e_2)\rangle, \langle e_3, F_A(e_3)\rangle\big\}$$

Here

$$F_A(e_1) = \Big\{ \langle x_1, 0.4, 0.3 \rangle, \langle x_2, 0.5, 0.6, 0.7 \rangle, \langle x_3, 0.5, 0.6, 0.7 \rangle, \langle x_4, 0.5, 0.6, 0.7 \rangle \Big\}$$

$$F_A(e_1) = \Big\{ \langle x_1, 0.5, 0.6, 0.3 \rangle, \langle x_2, 0.4, 0.7, 0.6 \rangle, \langle x_3, 0.6, 0.2, 0.3 \rangle, \langle x_4, 0.7, 0.2, 0.3 \rangle \Big\}$$

$$F_A(e_2) = \Big\{ \langle x_1, 0.6, 0.3, 0.5 \rangle, \langle x_2, 0.7, 0.4, 0.3 \rangle, \langle x_3, 0.8, 0.1, 0.2 \rangle, \langle x_4, 0.7, 0.1, 0.3 \rangle \Big\}$$

$$F_A(e_3) = \Big\{ \langle x_1, 0.7, 0.4, 0.3 \rangle, \langle x_2, 0.6, 0.1, 0.2 \rangle, \langle x_3, 0.7, 0.2, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle \Big\}$$

Hence (F_A, E) is a neutrosophic soft set.

Definition 2.8. [2, 3]
Let X be the universal set which contains arbitrary points x. A bipolar neutrosophic set (BNS) A is defined by

$$A = \Big\{ \langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \rangle : x \in X \Big\}$$

where

$$T^+, I^+, F^+ : E \to [0,1] \text{ (positive membership-degrees)}$$
$$T^-, I^-, F^- : E \to [-1,0] \text{ (negative membership-degrees)}$$

such that

$$0 \leq T^+(x) + I^+(x) + F^+(x) \leq 3 \text{ , } -3 \leq T^-(x) + I^-(x) + F^-(x) \leq 0.$$

Example 2.9. Let $X = \{x_1, x_2, x_3\}$ be the universal set. A bipolar neutrosophic set (BNS) is defined by

$$\begin{aligned} A = \big\{ & \langle x_1, 0.3, 0.4, 0.5, -0.2, -0.4, -0.1 \rangle, \\ & \langle x_2, 0.5, 0.2, 0.3, -0.2, -0.7, -0.5 \rangle, \\ & \langle x_3, 0.7, 0.2, 0.2, -0.5, -0.4, -0.5 \rangle \big\} \end{aligned}$$

where $0 \leq T_A^+(x) + I_A^+(x) + F_A^+(x) \leq 3$ and $-3 \leq T_A^-(x) + I_A^-(x) + F_A^-(x) \leq 0$.
Also $T_A^+(x), I_A^+(x), F_A^+(x) \to [0,1]$ and $T_A^-(x), I_A^-(x), F_A^-(x) \to [-1,0]$.

3 Different approach on bipolar neutrosophic soft set

In this section, we propose a slightly different approach on bipolar neutrosophic soft sets which is the combined version of neutrosophic soft set and bipolar neutrosophic set and this was initially introduced by Mumtaz Ali et al.[15]. He defined a bipolar neutrosophic soft set associated with the whole parameter set E.

In our approach, we define a bipolar neutrosophic soft set associated with only subset of a parameter set E. Because, there is a possibility to exist different bipolar neutrosophic soft sets associated with different subsets of E.

Ali et al.[15] definition is given below.

Definition 3.1. Let U be a universe and E be a set of parameters that are describing the elements of U. A bipolar neutrosophic soft set $\mathbb{B}$ in U is defined as:

$$\mathbb{B} = \left\{(e, \left\{(u, T^{+}(u), I^{+}(u), F^{+}(u), T^{-}(u), I^{-}(u), F^{-}(u) : u \in U\right\} : e \in E\right\}$$

where $T^{+}, I^{+}, F^{+} \to [0,1]$ and $T^{-}, I^{-}, F^{-} \to [-1,0]$. The positive membership degree $T^{+}(u), I^{+}(u), F^{+}(u)$, denotes the truth membership, indeterminate membership and false membership of an element corresponding to a bipolar neutrosophic soft set $\mathbb{B}$ and the negative membership degree $T^{-}(u), I^{-}(u), F^{-}(u)$ denotes the truth membership, indeterminate membership and false membership of an element $u \in U$ to some implicit counter-property corresponding to a bipolar neutrosophic soft set $\mathbb{B}$.

Our approach is given below.

Definition 3.2. Let X be the universe and E be the parameter set. Let A be subset of the parameter set E.

A bipolar neutrosophic soft set $\mathcal{B}$ over X is defined by

$$\mathcal{B}=(F_A, E) = \left\{\langle e, F_A(e)\rangle : e \in E, F_A(e) \in BNS(X)\right\}$$

Here

$$F_A(e) = \left\{\left\langle x, u^{+}_{F_A(e)}(x), v^{+}_{F_A(e)}(x), w^{+}_{F_A(e)}(x), u^{-}_{F_A(e)}(x), v^{-}_{F_A(e)}(x), w^{-}_{F_A(e)}(x)\right\rangle : x \in X\right\}.$$

where $u^{+}_{F_A(e)}(x), v^{+}_{F_A(e)}(x), w^{+}_{F_A(e)}(x)$ represents positive truth-membership degree , positive indeterminacy-membership degree and positive falsity-membership degree of x which holds the parametrer e, and similarly $u^{-}_{F_A(e)}(x), v^{-}_{F_A(e)}(x), w^{-}_{F_A(e)}(x)$ represents negative truth-membership degree , negative indeterminacy-membership degree and negative falsity-membership degree of x which holds the parameter e .

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4\}$ be a universal set and let $E = \{e_1, e_2, e_3\}$ be the parameter set. Also, let $A = \{e_1, e_2\} \subseteq E$ and $B = \{e_3\} \subseteq E$ be two subsets of E.

Then we define

$$\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(e)\rangle : e \in E, F_A(e) \in BNS(X)\}$$
$$\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(e)\rangle : e \in E, G_B(e) \in BNS(X)\}$$

where,

$$F_A(e_1) = \Big\{\langle x_1, 0.5, 0.4, 0.3, -0.02, -0.4, -0.5\rangle, \langle x_2, 0.4, 0.7, 0.6, -0.3, -0.5, -0.02\rangle, \langle x_3, 0.4, 0.3, 0.5, -0.6, -0.4, -0.2\rangle, \langle x_4, 0.4, 0.6, 0.3, -0.6, -0.2, -0.3\rangle\Big\}$$

$$F_A(e_2) = \Big\{ \langle x_1, 0.6, 0.3, 0.2, -0.4, -0.5, -0.04 \rangle, \langle x_2, 0.5, 0.2, 0.3, -0.1, -0.3, -0.6 \rangle, \langle x_3, 0.3, 0.4, 0.2, -0.3, -0.4, -0.7 \rangle, \langle x_4, 0.8, 0.2, 0.01, -0.4, -0.5, -0.1 \rangle \Big\}$$

$$G_B(e_3) = \Big\{ \langle x_1, 0.6, 0.3, 0.4, -0.4, -0.5, -0.3 \rangle, \langle x_2, 0.4, 0.5, 0.1, -0.2, -0.6, -0.4 \rangle, \langle x_3, 0.2, 0.3, 0.1, -0.4, -0.4, -0.2 \rangle, \langle x_4, 0.3, 0.4, 0.4, -0.5, -0.3, -0.2 \rangle \Big\}$$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are the parameterized family of bipolar neutrosophic soft sets over X.

3.1 Properties of Bipolar Neutrosophic soft sets

In this section, we have discussed some basic properties of Bipolar neutrosophic soft sets.

3.1.1 Subsets and Eqiuvalent sets

Let X be universal set and E be a parameter set. Let $A, B \subseteq E$. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bipolar neutrosophic soft sets. Then $\mathcal{B}_1 \subseteq \mathcal{B}_2$ if and only if $A \subseteq B$ and
$u^+_{F_A(e)}(x) \leq u^+_{G_B(e)}(x), v^+_{F_A(e)}(x) \geq v^+_{G_B(e)}(x), w^+_{F_A(e)}(x) \geq w^+_{G_B(e)}(x)$ and
$u^-_{F_A(e)}(x) \geq u^-_{G_B(e)}(x), v^-_{F_A(e)}(x) \leq v^-_{G_B(e)}(x), w^-_{F_A(e)}(x) \leq w^-_{G_B(e)}(x)$.

Also $\mathcal{B}_1$ and $\mathcal{B}_2$ are called equivalent sets only if $A = B$ and all the parameters of $\mathcal{B}_1$ and $\mathcal{B}_2$ are corresponding to each other.

Example 3.4. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bipolar neutrosophic soft sets associated with $A = \{e_2\}$ and $B = \{e_1, e_2\}$.
Let $\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(e)\rangle : e \in E\}$ and $\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(e)\rangle : e \in E\}$
Here,

$$F_A(e_2) = \Big\{ \langle x_1, 0.4, 0.3, 0.9, -0.2, -0.3, -0.4 \rangle, \langle x_2, 0.5, 0.6, 0.7, -0.3, -0.4, -0.6 \rangle \Big\}$$

$$G_B(e_1) = \Big\{ \langle x_1, 0.5, 0.4, 0.3, -0.6, -0.2, -0.4 \rangle, \langle x_2, 0.6, 0.3, 0.2, -0.5, -0.3, -0.2 \rangle \Big\}$$

$$G_B(e_2) = \Big\{ \langle x_1, 0.6, 0.4, 0.2, -0.5, -0.1, -0.1 \rangle, \langle x_2, 0.7, 0.6, 0.3, -0.4, -0.2, -0.3 \rangle \Big\}$$

This implies $\mathcal{B}_1 \subseteq \mathcal{B}_2$.

3.1.2 Union and Intersection

The union is defined by

$$\mathcal{B}_1 \cup \mathcal{B}_2 = (F_A \bigcup G_B) = \Big\{ \Big\langle max(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \frac{v^+_{F_A(e)}(x) + v^+_{G_B(e)}(x)}{2}, min(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), min(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)), \frac{v^-_{F_A(e)}(x) + v^-_{G_B(e)}(x)}{2}, max(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)) \Big\rangle \Big\}$$

The intersection is defined by

$$\mathcal{B}_1 \cap \mathcal{B}_2 = (F_A \bigcap G_B, E) = \left\{ \left\langle min(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \frac{v^+_{F_A(e)}(x) + v^+_{G_B(e)}(x)}{2}, max(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), max(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)), \frac{v^-_{F_A(e)}(x) + v^-_{G_B(e)}(x)}{2}, min(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)) \right\rangle \right\}$$

Example 3.5. Suppose

$$\mathcal{B}_1 = (F_A, E) = \{\langle x_1, 0.4, 0.3, 0.9, -0.5, -0.2, -0.1\rangle, \langle x_2, 0.5, 0.6, 0.7, -0.3, -0.4, -0.6\rangle\}$$

$$\mathcal{B}_2 = (G_B, E) = \{\langle x_1, 0.5, 0.4, 0.3, -0.6, -0.3, -0.4\rangle, \langle x_2, 0.6, 0.3, 0.2, -0.5, -0.3, -0.2\rangle\}$$

be two bipolar neutrosophic sets. Then the union is

$$\mathcal{B}_1 \cup \mathcal{B}_2 = (F_A \bigcup G_B, E) = \{\langle x_1, 0.5, 0.35, 0.3, -0.6, -0.25, -0.1\rangle, \langle x_2, 0.6, 0.45, 0.2, -0.5, -0.35, -0.2\rangle\}$$

the intersection is

$$\mathcal{B}_1 \cap \mathcal{B}_2 = (F_A \bigcap G_B, E) = \{\langle x_1, 0.4, 0.35, 0.9, -0.3, -0.25, -0.4\rangle, \langle x_2, 0.5, 0.45, 0.7, -0.3, -0.35, -0.6\rangle\}$$

3.1.3 The complement

The complement of a BNSS is

$$\mathcal{B}^c = (F_A, E)^c = (F^c_A, \neg E) = \left\langle w^+_{F_A(e)}(x), 1 - v^+_{F_A(e)}(x), u^+_{F_A(e)}(x), w^-_{F_A(e)}(x), -1 - v^-_{F_A(e)}(x), u^-_{F_A(e)}(x) \right\rangle$$

Example 3.6. Let $\mathcal{B}$ be a bipolar neutrosophic soft set.

$$\mathcal{B} = (F_A, E) = \{\langle x_1, 0.4, 0.3, 0.9, -0.5, -0.2, -0.1\rangle, \langle x_2, 0.5, 0.6, 0.7, -0.3, -0.4, -0.6\rangle\}$$

Then the complement is defined by

$$\mathcal{B}^c = (F_A, E)^c = \{\langle x_1, 0.9, 0.7, 0.4, -0.1, -0.8, -0.5\rangle, \langle x_2, 0.7, 0.4, 0.5, -0.6, -0.6, -0.3\rangle\}$$

3.1.4 Complete BNSS and null BNSS

The complete bipolar neutrosophic soft set $comp - \mathcal{B}$ is defined by
$comp - \mathcal{B} = \{e, \rangle x_i, 1, 0, 0, 0, -1, -1\rangle : e \in E; x \in X\}$

The null bipolar neutrosophic soft set is defined by
$null - \mathcal{B} = \{e, \rangle x_i, 0, 1, 1, -1, 0, 0\rangle : e \in E; x \in X\}$

The following propositions were given by Ali et al. for bipolar neutrosophic soft set associated with the whole parameter set. These propositions are also suitable for our approach.

Proposition 3.7. *Let X be a universe and E be a parameter set. Also, $A, B, C \in E$. Let $\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(E)\rangle : e \in E, F_A(E) \in BNS(X)\}$, $\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(E)\rangle : e \in E, G_B(E) \in BNS(X)\}$, $\mathcal{B}_3 = (H_C, E) = \{\langle e, H_C(E)\rangle : e \in E, H_C(E) \in BNS(X)\}$ be three bipolar neutrosophic soft sets over X. Then,*

1. $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}_2 \cup \mathcal{B}_1$
2. $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_2 \cap \mathcal{B}_1$
3. $\mathcal{B}_1 \cup (\mathcal{B}_2 \cup \mathcal{B}_3) = (\mathcal{B}_1 \cup \mathcal{B}_2) \cup \mathcal{B}_2$
4. $\mathcal{B}_1 \cap (\mathcal{B}_2 \cap \mathcal{B}_3) = (\mathcal{B}_1 \cap \mathcal{B}_2) \cap \mathcal{B}_2$

Proof. This proof is obvious. □

Proposition 3.8. *Let X be a universe and E be a parameter set. Also, $A, B \in E$. Let $\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(E)\rangle : e \in E, F_A(E) \in BNS(X)\}$, $\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(E)\rangle : e \in E, G_B(E) \in BNS(X)\}$ be two bipolar neutrosophic soft sets over X. Then the following De Morgan's laws are valid.*

1. $(\mathcal{B}_1 \cup \mathcal{B}_2)^c = (\mathcal{B}_1)^c \cap (\mathcal{B}_1)^c$
2. $(\mathcal{B}_1 \cap \mathcal{B}_2)^c = (\mathcal{B}_1)^c \cup (\mathcal{B}_1)^c$

Proof. Let $\mathcal{B}_1 = \left\{e, \left\langle x, u^+_{F_A(e)}(x), v^+_{F_A(e)}(x), w^+_{F_A(e)}(x), u^-_{F_A(e)}(x), v^-_{F_A(e)}(x), w^-_{F_A(e)}(x)\right\rangle : e \in E\right\}$
$\mathcal{B}_2 = \left\{e, \left\langle x, u^+_{G_B(e)}(x), v^+_{G_B(e)}(x), w^+_{G_B(e)}(x), u^-_{G_B(e)}(x), v^-_{G_B(e)}(x), w^-_{G_B(e)}(x)\right\rangle : e \in E\right\}$
Then,

$$\begin{aligned}
(\mathcal{B}_1 \cup \mathcal{B}_2)^c = \Big\{e, \Big\langle x, & max(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), min(v^+_{F_A(e)}(x), v^+_{G_B(e)}(x)), min(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), \\
& min(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)), max(v^-_{F_A(e)}(x), v^-_{G_B(e)}(x)), max(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x))\Big\rangle : e \in E\Big\}^c \\
= \Big\{e, \Big\langle x, & min(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), 1 - min(v^+_{F_A(e)}(x), v^+_{G_B(e)}(x)), max(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \\
& max(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)), -1 - max(v^-_{F_A(e)}(x), v^-_{G_B(e)}(x)), min(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x))\Big\rangle : e \in E\Big\} \\
= \Big\{e, \Big\langle x, & min(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), max(1 - v^+_{F_A(e)}(x), 1 - v^+_{G_B(e)}(x)), max(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \\
& max(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)), min(-1 - v^-_{F_A(e)}(x), -1 - v^-_{G_B(e)}(x)), min(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x))\Big\rangle : e \in E\Big\} \\
= & \Big\{e, \Big\langle x, w^+_{F_A(e)}(x), 1 - v^+_{F_A(e)}(x), u^+_{F_A(e)}(x), w^-_{F_A(e)}(x), -1 - v^-_{F_A(e)}(x), u^-_{F_A(e)}(x)\Big\rangle : e \in E\Big\} \\
\cap & \Big\{e, \Big\langle x, w^+_{G_B(e)}(x), 1 - v^+_{G_B(e)}(x), u^+_{G_B(e)}(x), w^-_{G_B(e)}(x), -1 - v^-_{G_B(e)}(x), u^-_{G_B(e)}(x)\Big\rangle : e \in E\Big\} \\
= & (\mathcal{B}_1)^c \cap (\mathcal{B}_2)^c
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{B}_1 \cap \mathcal{B}_2)^c = \Big\{ e, \Big\langle & x, min(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), max(v^+_{F_A(e)}(x), v^+_{G_B(e)}(x)), max(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), \\
& max(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)), min(v^-_{F_A(e)}(x), v^-_{G_B(e)}(x)), min(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)) \Big\rangle : e \in E \Big\}^c \\
= \Big\{ e, \Big\langle & x, max(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), 1 - max(v^+_{F_A(e)}(x), v^+_{G_B(e)}(x)), min(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \\
& min(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)), -1 - min(v^-_{F_A(e)}(x), v^-_{G_B(e)}(x)), max(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)) \Big\rangle : e \in E \Big\} \\
= \Big\{ e, \Big\langle & x, max(w^+_{F_A(e)}(x), w^+_{G_B(e)}(x)), min(1 - v^+_{F_A(e)}(x), 1 - v^+_{G_B(e)}(x)), min(u^+_{F_A(e)}(x), u^+_{G_B(e)}(x)), \\
& min(w^-_{F_A(e)}(x), w^-_{G_B(e)}(x)), max(-1 - v^-_{F_A(e)}(x), -1 - v^-_{G_B(e)}(x)), max(u^-_{F_A(e)}(x), u^-_{G_B(e)}(x)) \Big\rangle : e \in E \Big\} \\
= & \Big\{ e, \Big\langle x, w^+_{F_A(e)}(x), 1 - v^+_{F_A(e)}(x), u^+_{F_A(e)}(x), w^-_{F_A(e)}(x), -1 - v^-_{F_A(e)}(x), u^-_{F_A(e)}(x) \Big\rangle : e \in E \Big\} \\
& \cup \Big\{ e, \Big\langle x, w^+_{G_B(e)}(x), 1 - v^+_{G_B(e)}(x), u^+_{G_B(e)}(x), w^-_{G_B(e)}(x), -1 - v^-_{G_B(e)}(x), u^-_{G_B(e)}(x) \Big\rangle : e \in E \Big\} \\
& = (\mathcal{B}_1)^c \cup (\mathcal{B}_2)^c
\end{aligned}
$$

□

Proposition 3.9. *Let X be a universe and E be a parameter set. Also, $A, B, C \in E$. Let $\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(E)\rangle : e \in E, F_A(E) \in BNS(X)\}$, $\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(E)\rangle : e \in E, G_B(E) \in BNS(X)\}$, $\mathcal{B}_3 = (H_C, E) = \{\langle e, H_C(E)\rangle : e \in E, H_C(E) \in BNS(X)\}$ be three bipolar neutrosophic soft sets over X. Then,*

1. $\mathcal{B}_1 \cap (\mathcal{B}_2 \cup \mathcal{B}_3) = (\mathcal{B}_1 \cap \mathcal{B}_2) \cup (\mathcal{B}_1 \cap \mathcal{B}_3)$
2. $\mathcal{B}_1 \cup (\mathcal{B}_2 \cap \mathcal{B}_3) = (\mathcal{B}_1 \cup \mathcal{B}_2) \cap (\mathcal{B}_1 \cup \mathcal{B}_3)$

Proof. This proof is obvious.

□

4 Entropy measure of bipolar neutrosophic soft sets

Generally Entropy measures are used to calculate indeterminacy of sets. In this section, we define entropy measurement for bipolar neutrosophic soft sets.

Definition 4.1. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a universe of discourse set and $E = \{e_1, e_2, \ldots, e_n\}$ be subset of a parameter set A. Let $\mathcal{B}_1 = (F_A, E)$ and $\mathcal{B}_2 = (G_A, E)$ be two bipolar neutrosophic soft sets. The mapping $\mathcal{E} : BNSS(X) \to \mathbb{R}^+ \cup \{0\}$ is called an entropy on bipolar neutrosophic soft sets if $\mathcal{E}$ satisfies the following conditions.

1. $\mathcal{E}(B) = 0$ if and only if $B \in IFSS(X)$ (Intiutionistic fuzzy soft set)

2. $\mathcal{E}(B)$ is maximum if and only if $u^{+}_{F_A(e)}(x) = v^{+}_{F_A(e)}(x) = w^{+}_{F_A(e)}(x)$ and $u^{-}_{F_A(e)}(x) = v^{-}_{F_A(e)}(x) = w^{-}_{F_A(e)}(x)$ for all $e \in E$ and $x \in X$

3. $\mathcal{E}(B) = \mathcal{E}(B^c)$ for all $B \in BNSS(X)$

4. $\mathcal{E}(B_1) \leq \mathcal{E}(B_2)$ if $B_2 \subseteq B_1$.

Definition 4.2. Let $\mathcal{B}$ be a bipolar neutrosophic soft set. Then, entropy of $\mathcal{B}$ is denoted by $\mathcal{E}(\mathcal{B})$ and defined as follows:

$$\mathcal{E}(\mathcal{B}) = 1 - \frac{1}{2mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[\Big(u^{+}_{\mathcal{B}(e_j)}(x_i) + w^{+}_{\mathcal{B}(e_j)}(x_i)\Big)\cdot\Big|v^{+}_{\mathcal{B}(e_j)}(x_i) - v^{+}_{\mathcal{B}^c(e_j)}(x_i)\Big| - \Big(u^{-}_{\mathcal{B}(e_j)}(x_i) + w^{-}_{\mathcal{B}(e_j)}(x_i)\Big)\cdot\Big|v^{-}_{\mathcal{B}(e_j)}(x_i) - v^{-}_{\mathcal{B}^c(e_j)}(x_i)\Big|\Bigg]$$

Example 4.3. Let $X = \{x_1, x_2, x_3, x_4\}$ be a universal set and let $E = \{e_1, e_2, e_3\}$ be the parameter set. Let $A = \{e_1, e_2\}$ be a subset of E.

1. Define $\mathcal{B}_1 = (F_A, E) = \{\langle e_1, F_A(e_1)\rangle, \langle e_2, F_A(e_2)\rangle\}$
   where,

$$F_A(e_1) = \Big\{\langle x_1, 0.6, 0, 0.4, -0.3, 0, -0.7\rangle, \langle x_2, 0.3, 0, 0.7, -0.2, 0, -0.8\rangle, \langle x_3, 0.4, 0, 0.6, -0.6, 0, -0.4\rangle, \langle x_4, 0.1, 0, 0.9, -0.5, 0, -0.5\rangle\Big\}$$

$$F_A(e_2) = \Big\{\langle x_1, 0.5, 0, 0.5, -0.4, 0, -0.6\rangle, \langle x_2, 0.2, 0, 0.8, -0.1, 0, -0.9\rangle, \langle x_3, 0.3, 0, 0.7, -0.7, 0, -0.3\rangle, \langle x_4, 0.8, 0, 0.2, -0.4, 0, -0.6\rangle\Big\}$$

   Since all the indeterminacy degrees are zero, $\mathcal{B}_1$ becomes intituitionistic fuzzy soft set(IFSS).
   By Definition 4.2, $\mathcal{E}(\mathcal{B}_1) = 0$

2. Define $\mathcal{B}_2 = (F_A, E) = \{\langle e_1, F_A(e_1)\rangle, \langle e_2, F_A(e_2)\rangle\}$ where,

$$F_A(e_1) = \Big\{\langle x_1, 0.5, 0.5, 0.5, -0.9, -0.9, -0.9\rangle, \langle x_2, 0.3, 0.3, 0.3, -0.8, -0.8, -0.8\rangle, \langle x_3, 0.4, 0.4, 0.4, -0.5, -0.5, -0.5\rangle, \langle x_4, 0.5, 0.5, 0.5, -0.5, -0.5, -0.5\rangle\Big\}$$

$$F_A(e_2) = \Big\{ \langle x_1, 0.4, 0.4, 0.4, -0.4, -0.4, -0.4 \rangle, \langle x_2, 0.5, 0.5, 0.5, -0.1, -0.1, -0.1 \rangle, \langle x_3, 0.3, 0.3, 0.3, -0.5, -0.5, -0.5 \rangle, \langle x_4, 0.8, 0.8, 0.8, -0.2, -0.2, -0.2 \rangle \Big\}$$

Since truth-membership, indeterminacy and falsity-membership degrees are equal,
By Definition 4.2, $\mathcal{E}(\mathcal{B}_1) = 1$ (i.e maximum).

3. Define $\mathcal{B}_3 = (F_A, E) = \{\langle e_1, F_A(e_1)\rangle, \langle e_2, F_A(e_2)\rangle\}$ where,

$$F_A(e_1) = \Big\{ \langle x_1, 0.5, 0.4, 0.7, -0.2, -0.5, -0.7 \rangle, \langle x_2, 0.4, 0.7, 0.3, -0.6, -0.2, -0.1 \rangle, \langle x_3, 0.4, 0.6, 0.2, -0.5, -0.3, -0.7 \rangle, \langle x_4, 0.6, 0.3, 0.2, -0.7, -0.5, -0.3 \rangle \Big\}$$

$$F_A(e_2) = \Big\{ \langle x_1, 0.6, 0.3, 0.7, -0.4, -0.2, -0.4 \rangle, \langle x_2, 0.4, 0.7, 0.3, -0.7, -0.3, -0.4 \rangle, \langle x_3, 0.3, 0.5, 0.1, -0.5, -0.7, -0.3 \rangle, \langle x_4, 0.8, 0.3, 0.1, -0.5, -0.2, -0.4 \rangle \Big\}$$

Then,
$(\mathcal{B}_3)^c = (F_A^c, \neg E) = \{\langle e_1, F_A^c(e_1)\rangle, \langle e_2, F_A^c(e_2)\rangle\}$

where,

$$F_A^c(e_1) = \Big\{ \langle x_1, 0.7, 0.6, 0.5, -0.7, -0.5, -0.2 \rangle, \langle x_2, 0.3, 0.3, 0.4, -0.1, -0.8, -0.6 \rangle, \langle x_3, 0.2, 0.4, 0.4, -0.7, -0.7, -0.5 \rangle, \langle x_4, 0.2, 0.7, 0.6, -0.3, -0.5, -0.7 \rangle \Big\}$$

$$F_A^c(e_2) = \Big\{ \langle x_1, 0.7, 0.7, 0.6, -0.4, -0.8, -0.4 \rangle, \langle x_2, 0.3, 0.3, 0.4, -0.4, -0.7, -0.7 \rangle, \langle x_3, 0.1, 0.5, 0.7, -0.3, -0.3, -0.5 \rangle, \langle x_4, 0.1, 0.7, 0.8, -0.4, -0.8, -0.5 \rangle \Big\}$$

Since the sum of indeterminacy and its complement is one and complement of truth-membership becomes falsify-membership and vice versa,
By Definition 4.2, $\mathcal{E}(B) = \mathcal{E}(B^c)$ for any BNSS.

4. Let $\mathcal{B}_1 = (F_A, E) = \{\langle e, F_A(e)\rangle : e \in E\}$ and $\mathcal{B}_2 = (G_B, E) = \{\langle e, G_B(e)\rangle : e \in E\}$
Here,

$$F_A(e_2) = \Big\{ \langle x_1, 0.4, 0.3, 0.9, -0.2, -0.3, -0.4 \rangle,\ \langle x_2, 0.5, 0.6, 0.7, -0.3, -0.4, -0.6 \rangle \Big\}$$

$$G_B(e_1) = \left\{ \langle x_1, 0.5, 0.4, 0.3, -0.6, -0.2, -0.4 \rangle, \ \langle x_2, 0.6, 0.3, 0.2, -0.5, -0.3, -0.2 \rangle \right\}$$

$$G_B(e_2) = \left\{ \langle x_1, 0.6, 0.4, 0.2, -0.5, -0.1, -0.1 \rangle, \ \langle x_2, 0.7, 0.6, 0.3, -0.4, -0.2, -0.3 \rangle \right\}$$

Here $\mathcal{B}_1 \subseteq \mathcal{B}_2$.
By Definition 4.2,

$$\mathcal{E}(\mathcal{B}_1) = 0.705$$
$$\mathcal{E}(\mathcal{B}_2) = 0.6725$$

Hence

$$\mathcal{E}(\mathcal{B}_2) \leq \mathcal{E}(\mathcal{B}_1) \text{ if } \mathcal{B}_1 \subseteq \mathcal{B}_2$$

5 Distance between bipolar neutrosophic soft sets

In this section, we will define some distance measures of bipolar neutrosophic soft sets. Let X be a universe, E be a parameter set and let A, B be two subsets of E.
Let $\mathcal{B}_1 = (F_A, E)$ and $\mathcal{B}_2 = (G_B, E)$ be two bipolar neutrosophic soft sets.

Here
$F_A(e) = \left\{ \left\langle x, u^+_{F_A(e)}(x), v^+_{F_A(e)}(x), w^+_{F_A(e)}(x), u^-_{F_A(e)}(x), v^-_{F_A(e)}(x), w^-_{F_A(e)}(x) \right\rangle : x \in X \right\}$
$G_B(e) = \left\{ \left\langle x, u^+_{G_B(e)}(x), v^+_{G_B(e)}(x), w^+_{G_B(e)}(x), u^-_{G_B(e)}(x), v^-_{G_B(e)}(x), w^-_{G_B(e)}(x) \right\rangle : x \in X \right\}$

Definition 5.1. Consider the two Bipolar neutrosophic soft sets $\mathcal{B}_1 = (F_A, E)$ and $\mathcal{B}_2 = (G_B, E)$ defined above. Let d be a mapping defined as $d : BNSS(x) \times BNSS(x) \to \mathbb{R}^+ \cup \{0\}$ and it satisfies the following conditions.
$i)\ d(\mathcal{B}_1, \mathcal{B}_2) \geq 0$
$ii)\ d(\mathcal{B}_1, \mathcal{B}_2) = d(\mathcal{B}_2, \mathcal{B}_1)$
$iii)\ d(\mathcal{B}_1, \mathcal{B}_2) = 0 iff \mathcal{B}_1 = \mathcal{B}_2$
$iv)\ d(\mathcal{B}_1, \mathcal{B}_2) + d(\mathcal{B}_2, \mathcal{B}_3) \geq d(\mathcal{B}_1, \mathcal{B}_3)\ (for\ any\ \mathcal{B}_3)$

Then, $d(\mathcal{B}_1, \mathcal{B}_2)$ is called a distance measure between two bipolar neutrosopihic soft sets $\mathcal{B}_1$ and $\mathcal{B}_2$.

Definition 5.2. A real function $\mathcal{S} : BNSS(X) \times BNSS(X) \to [0, 1]$ is called a similarity measure between two bipolar neutrosophic soft sets $\mathcal{B}_1 = [a_{ij}]_{m \times n}\ and\ \mathcal{B}_2 = [b_{ij}]_{m \times n}$ if $\mathcal{S}$ satisfies the following conditions.
$i) \mathcal{S}(\mathcal{B}_1, \mathcal{B}_2) \in [0, 1]$
$ii) \mathcal{S}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{S}(\mathcal{B}_2, \mathcal{B}_1)$
$iii) \mathcal{S}(\mathcal{B}_1, \mathcal{B}_2) = 1 iff [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$
$iv) \mathcal{S}(\mathcal{B}_1, \mathcal{B}_3) \leq \mathcal{S}(\mathcal{B}_1, \mathcal{B}_2) + \mathcal{S}(\mathcal{B}_2, \mathcal{B}_3)\ if\ \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3\ (for\ any\ \mathcal{B}_3)$

5.1 Hamming distance between two bipolar neutrosophic soft sets

$$d^H_{BNSS}(\mathcal{B}_1,\mathcal{B}_2)=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{|\Delta_{ij}u(x)|+|\nabla_{ij}u(x)|+|\Delta_{ij}v(x)|+|\nabla_{ij}v(x)|+|\Delta_{ij}w(x)|+|\nabla_{ij}w(x)|}{6}.$$

where

$$\Delta_{ij}u(x)=u^+_{\mathcal{B}_1(e_j)}(x_i)-u^+_{\mathcal{B}_2(e_j)}(x_i)$$
$$\nabla_{ij}u(x)=u^-_{\mathcal{B}_1(e_j)}(x_i)-u^-_{\mathcal{B}_2(e_j)}(x_i)$$

Proof. i) Since $|\Delta_{ij}u(x)|,|\nabla_{ij}u(x)|,|\Delta_{ij}v(x)|,|\nabla_{ij}v(x)|,|\Delta_{ij}w(x)|,|\nabla_{ij}w(x)|$ are all positive, $d^H_{BNSS}(\mathcal{B}_1,\mathcal{B}_2)\geq 0$
ii) Since $\left|u^+_{\mathcal{B}_1(e_j)}(x_i)-u^+_{\mathcal{B}_2(e_j)}(x_i)\right|=\left|u^+_{\mathcal{B}_2(e_j)}(x_i)-u^+_{\mathcal{B}_1(e_j)}(x_i)\right|$,
$|\Delta_{ij}u(X)|$ is same for both $d^H_{BNSS}(\mathcal{B}_1,\mathcal{B}_2)$ and $d^H_{BNSS}(\mathcal{B}_2,\mathcal{B}_1)$.
Also this is true for all membership degrees.
Hence $d^H_{BNSS}(\mathcal{B}_1,\mathcal{B}_2)=d^H_{BNSS}(\mathcal{B}_2,\mathcal{B}_1)$
iii) Since $\Delta_{ij}u(X)=u^+_{\mathcal{B}_1(e_j)}(x_i)-u^+_{\mathcal{B}_2(e_j)}(x_i)$ and $\nabla_{ij}u(X)=u^-_{\mathcal{B}_1(e_j)}(x_i)-u^-_{\mathcal{B}_2(e_j)}(x_i)=0$ are both zero for $\mathcal{B}_1=\mathcal{B}_2$,
$d^H_{BNSS}(\mathcal{B}_1,\mathcal{B}_2)=0$ if $\mathcal{B}_1=\mathcal{B}_2$.
iv) Let

$$d^H_{BNSS}(B_1,B_2)=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{|\Delta_{ij}u_1(x)|+|\nabla_{ij}u_1(x)|+|\Delta_{ij}v_1(x)|+|\nabla_{ij}v_1(x)|+|\Delta_{ij}w_1(x)|+|\nabla_{ij}w_1(x)|}{6}$$
$$d^H_{BNSS}(B_2,B_3)=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{|\Delta_{ij}u_2(x)|+|\nabla_{ij}u_2(x)|+|\Delta_{ij}v_2(x)|+|\nabla_{ij}v_2(x)|+|\Delta_{ij}w_2(x)|+|\nabla_{ij}w_2(x)|}{6}$$

$$d^H_{BNSS}(B_1,B_2)+d^H_{BNSS}(B_2,B_3)$$
$$=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\left|u^+_{\mathcal{B}_1(e_j)}(x_i)-u^+_{\mathcal{B}_2(e_j)}(x_i)\right|+\left|u^+_{\mathcal{B}_2(e_j)}(x_i)-u^+_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|u^-_{\mathcal{B}_1(e_j)}(x_i)-u^-_{\mathcal{B}_2(e_j)}(x_i)\right|+}{6}$$
$$\overline{\left|u^-_{\mathcal{B}_2(e_j)}(x_i)-u^-_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|v^+_{\mathcal{B}_1(e_j)}(x_i)-v^+_{\mathcal{B}_2(e_j)}(x_i)\right|+\left|v^+_{\mathcal{B}_2(e_j)}(x_i)-v^+_{\mathcal{B}_3(e_j)}(x_i)\right|+}$$
$$\overline{\left|v^-_{\mathcal{B}_1(e_j)}(x_i)-v^-_{\mathcal{B}_2(e_j)}(x_i)\right|+\left|v^-_{\mathcal{B}_2(e_j)}(x_i)-v^-_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|w^+_{\mathcal{B}_1(e_j)}(x_i)-w^+_{\mathcal{B}_2(e_j)}(x_i)\right|+}$$
$$\overline{\left|w^+_{\mathcal{B}_2(e_j)}(x_i)-w^+_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|w^-_{\mathcal{B}_1(e_j)}(x_i)-w^-_{\mathcal{B}_2(e_j)}(x_i)\right|+\left|w^-_{\mathcal{B}_2(e_j)}(x_i)-w^-_{\mathcal{B}_3(e_j)}(x_i)\right|}$$
$$\geq\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\left|u^+_{\mathcal{B}_1(e_j)}(x_i)-u^+_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|v^+_{\mathcal{B}_1(e_j)}(x_i)-v^+_{\mathcal{B}_3(e_j)}(x_i)\right|+\left|w^+_{\mathcal{B}_1(e_j)}(x_i)-w^+_{\mathcal{B}_3(e_j)}(x_i)\right|}{6}$$

This implies

$$d^H_{BNSS}(B_1, B_2) + d^H_{BNSS}(B_2, B_3) \geq d^H_{BNSS}(B_1, B_3)$$

□

5.2 Normalized Hamming distance

$$d^{nH}_{BNSS}(B_1, B_2) = \frac{d^H_{BNSS}(B_1, B_2)}{mn}$$

Proof. Since $d^H_{BNSS}(B_1, B_2)$ satisfies definition 5.1, for any positive m, n

$$d^{nH}_{BNSS}(B_1, B_2) = \frac{d^H_{BNSS}(B_1, B_2)}{mn}$$

also satisfies definition 5.1 □

5.3 Euclidean distance between two BNSS

$$d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = \left[\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{(\Delta_{ij}u(x))^2 + (\nabla_{ij}u(x))^2 + (\Delta_{ij}v(x))^2 + (\nabla_{ij}v(x))^2 + (\Delta_{ij}w(x))^2 + (\nabla_{ij}w(x))^2}{6}\right]^{\frac{1}{2}}$$

where

$$\Delta_{ij}u(x) = u^+_{\mathcal{B}_1(e_j)}(x_i) - u^+_{\mathcal{B}_2(e_j)}(x_i)$$
$$\nabla_{ij}u(x) = u^-_{\mathcal{B}_1(e_j)}(x_i) - u^-_{\mathcal{B}_2(e_j)}(x_i)$$

Proof. i) Since $(\Delta_{ij}u(x))^2, (\nabla_{ij}u(x))^2, (\Delta_{ij}v(x))^2, (\nabla_{ij}v(x))^2, (\Delta_{ij}w(x))^2, (\nabla_{ij}w(x))^2$ are all positive, $d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) \geq 0$

ii) Since $(u^+_{\mathcal{B}_1(e_j)}(x_i) - u^+_{\mathcal{B}_2(e_j)}(x_i))^2 = (u^+_{\mathcal{B}_2(e_j)}(x_i) - u^+_{\mathcal{B}_1(e_j)}(x_i))^2$, $(\Delta_{ij}u(X))^2$ is same for both $d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2)$ and $d^E_{BNSS}(\mathcal{B}_2, \mathcal{B}_1)$.
Also this is true for all membership degrees.
Hence $d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = d^E_{BNSS}(\mathcal{B}_2, \mathcal{B}_1)$

iii) Since $\Delta_{ij}u(X) = u^+_{\mathcal{B}_1(e_j)}(x_i) - u^+_{\mathcal{B}_2(e_j)}(x_i)$ and $\nabla_{ij}u(X) = u^-_{\mathcal{B}_1(e_j)}(x_i) - u^-_{\mathcal{B}_2(e_j)}(x_i) = 0$ are both zero for $\mathcal{B}_1 = \mathcal{B}_2$,
$d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = 0$ if $\mathcal{B}_1 = \mathcal{B}_2$.

iv) Let

$$d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = \left[\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{(\Delta_{ij}u_1(x))^2 + (\nabla_{ij}u_1(x))^2 + (\Delta_{ij}v_1(x))^2 + (\nabla_{ij}v_1(x))^2 + (\Delta_{ij}w_1(x))^2 + (\nabla_{ij}w_1(x))^2}{6}\right]^{\frac{1}{2}}$$

$$d^E_{BNSS}(\mathcal{B}_2, \mathcal{B}_3) = \left[\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{(\Delta_{ij}u_2(x))^2 + (\nabla_{ij}u_2(x))^2 + (\Delta_{ij}v_2(x))^2 + (\nabla_{ij}v_2(x))^2 + (\Delta_{ij}w_2(x))^2 + (\nabla_{ij}w_2(x))^2}{6}\right]^{\frac{1}{2}}$$

By the definition of Euclidean norm, we take
$d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = \|\mathcal{B}_1 - \mathcal{B}_2\|_2$
$d^E_{BNSS}(\mathcal{B}_2, \mathcal{B}_3) = \|\mathcal{B}_2 - \mathcal{B}_3\|_2$
Then, $\|\mathcal{B}_1 - \mathcal{B}_3\|_2 = \|\mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_2 - \mathcal{B}_3\|_2$
By Triangle inequality,
$\|\mathcal{B}_1 - \mathcal{B}_3\|_2 \leq \|\mathcal{B}_1 - \mathcal{B}_2\|_2 + \|\mathcal{B}_2 - \mathcal{B}_3\|_2$
Hence $d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) + d^E_{BNSS}(\mathcal{B}_2, \mathcal{B}_3) \geq d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_3)$

□

5.4 Normalized Euclidean distance

$$d^{nE}_{BNSS}(B_1, B_2) = \frac{d^E_{BNSS}(B_1, B_2)}{\sqrt{mn}}$$

Proof. Since, $d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2)$ satisfies Definition 5.1,

$$d^{nE}_{BNSS}(\mathcal{B}_1, \mathcal{B}_2) = \frac{d^E_{BNSS}(\mathcal{B}_1, \mathcal{B}_2)}{\sqrt{mn}}$$

also satisfies Definition 5.1 for all m, n. □

Note 5.3. From the above measurements, we conclude the following conditions.

$i)$ $0 \leq d^H_{BNSS}(B_1, B_2) \leq mn$ [Obviously true]
$ii)$ $0 \leq d^{nH}_{BNSS}(B_1, B_2) \leq 1$ [from *i)*]
$iii)$ $0 \leq d^E_{BNSS}(B_1, B_2) \leq \sqrt{mn}$ [Obvious from *i)*]
$iv)$ $0 \leq d^{nE}_{BNSS}(B_1, B_2) \leq 1$ [from *iii)*]

Based on these distance measures, we can calculate the similarity between two BNSSs using the following measures.
$i)$ $\mathcal{S}^H_{BNSS}(B_1, B_2) = \dfrac{1}{1 + d^H_{BNSS}(B_1, B_2)}$
$ii)$ $\mathcal{S}^E_{BNSS}(B_1, B_2) = \dfrac{1}{1 + d^E_{BNSS}(B_1, B_2)}$
$iii)$ $\mathcal{S}^n H_{BNSS}(B_1, B_2) = \dfrac{1}{1 + d^n H_{BNSS}(B_1, B_2)}$
$iv)$ $\mathcal{S}^n E_{BNSS}(B_1, B_2) = \dfrac{1}{1 + d^n E_{BNSS}(B_1, B_2)}$

6 Representation of image in bipolar neutrosophic soft Domain

In this section, we convert 2-dimensional digital image into bipolar neutrosophic set. A digital image contains many pixels. According to pixel intensity values, we classified digital image as foreground image and background image.

we define bipolar neutrosophic soft set as parameterization of family of subsets which contains positive mebership degrees and negative membership degrees. Here we assign positive membership degrees to foreground image and negative membership degree to background image.

For example, Let us consider a 2-dimensional digital image as $X = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Here x_1, x_2, x_3 represents foreground pixels and y_1, y_2, y_3 represents background pixels. Let $A = \{e_1, e_2, e_3\}$ be set of parameters, where e_1, e_2, e_3 denotes contrast, brightness and sharpness of given image respectively.

Define $\mathcal{B} = (F_A, E) = \langle e, F_A(e) \rangle : e \in E, F_A(e) \in BNS(X)$
Here

$$F_A(e_1) = \Bigg\{ \Bigg\langle x_1, u^+_{F_A(e_1)}(x_1), v^+_{F_A(e_1)}(x_1), w^+_{F_A(e_1)}(x_1), u^-_{F_A(e_1)}(x_1), v^-_{F_A(e_1)}(x_1), w^-_{F_A(e_1)}(x_1) \Bigg\rangle, \Bigg\langle x_2, u^+_{F_A(e_1)}(x_2), v^+_{F_A(e_1)}(x_2), w^+_{F_A(e_1)}(x_2), u^-_{F_A(e_1)}(x_2), v^-_{F_A(e_1)}(x_2), w^-_{F_A(e_1)}(x_2) \Bigg\rangle, \Bigg\langle x_3, u^+_{F_A(e_1)}(x_3), v^+_{F_A(e_1)}(x_3), w^+_{F_A(e_1)}(x_3), u^-_{F_A(e_1)}(x_3), v^-_{F_A(e_1)}(x_3), w^-_{F_A(e_1)}(x_3) \Bigg\rangle \Bigg\}$$

$$F_A(e_2) = \Bigg\{ \Bigg\langle x_1, u^+_{F_A(e_2)}(x_1), v^+_{F_A(e_2)}(x_1), w^+_{F_A(e_2)}(x_1), u^-_{F_A(e_2)}(x_1), v^-_{F_A(e_2)}(x_1), w^-_{F_A(e_2)}(x_1) \Bigg\rangle, \Bigg\langle x_2, u^+_{F_A(e_2)}(x_2), v^+_{F_A(e_2)}(x_2), w^+_{F_A(e_2)}(x_2), u^-_{F_A(e_2)}(x_2), v^-_{F_A(e_2)}(x_2), w^-_{F_A(e_2)}(x_2) \Bigg\rangle, \Bigg\langle x_3, u^+_{F_A(e_2)}(x_3), v^+_{F_A(e_2)}(x_3), w^+_{F_A(e_2)}(x_3), u^-_{F_A(e_2)}(x_3), v^-_{F_A(e_2)}(x_3), w^-_{F_A(e_2)}(x_3) \Bigg\rangle \Bigg\}$$

$$F_A(e_3) = \Bigg\{ \Bigg\langle x_1, u^+_{F_A(e_3)}(x_1), v^+_{F_A(e_3)}(x_1), w^+_{F_A(e_3)}(x_1), u^-_{F_A(e_3)}(x_1), v^-_{F_A(e_3)}(x_1), w^-_{F_A(e_3)}(x_1) \Bigg\rangle, \Bigg\langle x_2, u^+_{F_A(e_3)}(x_2), v^+_{F_A(e_3)}(x_2), w^+_{F_A(e_3)}(x_2), u^-_{F_A(e_3)}(x_2), v^-_{F_A(e_3)}(x_2), w^-_{F_A(e_3)}(x_2) \Bigg\rangle, \Bigg\langle x_3, u^+_{F_A(e_3)}(x_3), v^+_{F_A(e_3)}(x_3), w^+_{F_A(e_3)}(x_3), u^-_{F_A(e_3)}(x_3), v^-_{F_A(e_3)}(x_3), w^-_{F_A(e_3)}(x_3) \Bigg\rangle \Bigg\}$$

where $u^+_{F_A(e)}(x), v^+_{F_A(e)}(x), w^+_{F_A(e)}(x)$ represents positive truth-membership degree , positive indeterminacy-membership degree and positive falsity-membership degree of a pixel x which holds the parametrer e, and similarly $u^-_{F_A(e)}(x), v^-_{F_A(e)}(x), w^-_{F_A(e)}(x)$ represents negative truth-membership degree , negative indeterminacy-membership degree and negative falsity-membership degree of a pixel x which holds the parameter e .

Remark 6.1. We assume the pixels are already classified as foreground and background pixels based on their intensity values. This assumption leads us to the following conditions.

For absolute foreground pixels,
$u^+(x) = [0, 1] \qquad u^-(x) = 0$
$v^+(x) = [0, 1] \qquad v^-(x) = -1$
$w^+(x) = [0, 1] \qquad w^-(x) = -1$
For absolute background pixels,

$u^+(x) = 0 \qquad u^-(x) = [-1, 0]$
$v^+(x) = 1 \qquad v^-(x) = [-1, 0]$
$w^+(x) = 1 \qquad w^-(x) = [-1, 0]$

6.1 Pixels in BNSS domain

Digital images are just array of pixels; each and every pixel has particular intensity values. Initially, Yanhui et al.,[8, 17] proposed the technique to transform image into neutrosophic domain. In this subsection, we extend this technique to bipolar neutrosophic domain.

We allocate membership values for each pixel according to their attributes. For foreground pixels $u^+(i,j), v^+(i,j), w^+(i,j)$ named as positive truth-membership, positive indeterminacy, positive falsity-membership respectively and for background pixels $u^-(i,j), v^-(i,j), w^-(i,j)$ named as negative truth-membership, negative indeterminacy, negative falsity-membership respectively.

An arbitrary pixel can be represented as follows:
$P_{BNS}(i,j) = \{u^+(i,j), v^+(i,j), w^+(i,j), u^-(i,j), v^-(i,j), w^-(i,j)\}$.
Here

$$u^+(i,j) = \frac{\bar{g}(i,j) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \qquad v^+(i,j) = \frac{\delta(i,j) - \delta_{min}}{\delta_{max} - \delta_{min}}$$
$$w^+(i,j) = 1 - u^+(i,j) = \frac{\bar{g}_{max} - \bar{g}(i,j)}{\bar{g}_{max} - \bar{g}_{min}}$$
$$u^-(i,j) = \frac{\hat{g}_{min} - \hat{g}(i,j)}{\hat{g}_{max} - \hat{g}_{min}} \qquad v^-(i,j) = \frac{\delta_{min} - \delta(i,j)}{\delta_{max} - \delta_{min}}$$
$$w^-(i,j) = -1 - u^-(i,j) = \frac{\hat{g}(i,j) - \hat{g}_{max}}{\hat{g}_{max} - \hat{g}_{min}}$$

where $\bar{g}(i,j)$ represents mean intensity of foreground pixel in some neighbourhoods W and $\hat{g}(i,j)$ represents the mean intensity of background pixel in some neighbourhoods W^*.
Here

$$\bar{g}(i,j) = \frac{1}{W \times W} \sum_{m=i-w/2}^{i+w/2} \sum_{n=j-w/2}^{j+w/2} g(m,n)$$
$$\hat{g}(i,j) = \frac{1}{W^* \times W^*} \sum_{m=i-w^*/2}^{i+w^*/2} \sum_{n=j-w^*/2}^{j+w^*/2} g(m,n)$$
$$\delta(i,j) = |g(i,j) - \bar{g}(i,j)|$$
$$\delta(i,j) = |g(i,j) - \hat{g}(i,j)|$$
$$\delta_{max} = max\delta(i,j) \qquad \delta_{min} = min\delta(i,j)$$

Example 6.2. Let $X = \{f_1, f_2, b_1, b_2\}$ be pixel set of a 2-D image. Also let $E = \{e_1, e_2, e_3\}$ be the subset of the parameter set A with parameters e_1, e_2, e_3 as contrast, brightness and sharpness, respectively.

Now we define
$(F_A, E) = \{\langle e, F_A(e)\rangle : e \in E, F_A(e) \in BNS(X)\}$.
Here

$$F(e_1) = \Big\{ \langle f_1, 0.5, 0.4, 0.3, 0, -1, -1\rangle, \langle f_2, 0.4, 0.7, 0.6, 0, -1, -1\rangle, \langle f_3, 0.4, 0.3, 0.5, 0, -1, -1\rangle,$$
$$\langle b_1, 0, 1, 1, -0.6, -0.2, -0.3\rangle, \langle b_2, 0, 1, 1, -0.7, -0.1, -0.3\rangle, \langle b_3, 0, 1, 1, -0.4, -0.2, -0.3\rangle \Big\}$$
$$F(e_2) = \Big\{ \langle f_1, 0.6, 0.3, 0.2, 0, -1, -1\rangle, \langle f_2, 0.5, 0.2, 0.3, 0, -1, -1\rangle, \langle f_3, 0.3, 0.4, 0.2, 0, -1, -1\rangle,$$
$$\langle b_1, 0, 1, 1, -0.4, -0.5, -0.1\rangle, \langle b_2, 0, 1, 1, -0.6, -0.2, -0.3\rangle, \langle b_3, 0, 1, 1, -0.4, -0.5, -0.1\rangle \Big\}$$
$$F(e_3) = \Big\{ \langle f_1, 0.6, 0.3, 0.4, 0, -1, -1\rangle, \langle f_2, 0.4, 0.5, 0.1, 0, -1, -1\rangle, \langle f_3, 0.2, 0.3, 0.1, 0, -1, -1\rangle$$
$$\langle b_1, 0, 1, 1, -0.5, -0.3, -0.2\rangle, \langle b_2, 0, 1, 1, -0.5, -0.4, -0.2\rangle, \langle b_3, 0, 1, 1, -0.7, -0.9, -0.1\rangle \Big\}$$

Then (F_A, E) is a bipolar neutrosophic soft set which is the parameterized family of soft subsets of X.

7 Decision making process based on similarity measurements

Since neutrosophic set theory deals with uncertainities, it is useful for decision making problems. Due to lack of parametrization tools in neutrosophic sets alone, we have some difficulties while making decisions. There fore, neutrosophic set along with parameters are more favorable for decision making problems.

In this evaluation criteria, we have two types of membership degrees as positive and negative membership degrees. So we consider positive membership degrees for foreground pixels and negative membership degrees for background pixels. This means, we expect maximum positive truth-membership value and minimum negative truth-membership value for foreground pixels while maximum negative truth-membership value and minimum positive truth-membership value for background pixels.

So we define ideal neutrosophic values for our criteria in the following way.

$$[f_{ij}] = \Big\{ e_j, \Big\langle max(u^+_{F(e_j)}(x_i)), min(v^+_{F(e_j)}(x_i)), min(w^+_{F(e_j)}(x_i)), max(u^-_{F(e_j)}(x_i)), min(v^-_{F(e_j)}(x_i)),$$
$$min(w^-_{F(e_j)}(x_i)) \Big\rangle : e_j \in E; x_i \in X \Big\}$$
$$[b_{ij}] = \Big\{ e_j, \Big\langle min(u^+_{F(e_j)}(x_i)), max(v^+_{F(e_j)}(x_i)), max(w^+_{F(e_j)}(x_i)), min(u^-_{F(e_j)}(x_i)), max(v^-_{F(e_j)}(x_i)),$$
$$max(w^-_{F(e_j)}(x_i)) \Big\rangle : e_j \in E; x_i \in X \Big\}$$

So our aim is to select the most relevant foreground and background set of pixels by their brightness, contrast level and sharpness level from the image samples of a particular image. The different types of lena images and their corresponding neutrosophic values are given below.

(a) Blur image (b) Noisy image (c) Low resolution

Figure 1: Different types of Lena images

$\mathcal{B}_1$	Brightness(e_1)	Contrast(e_2)	Sharpness(e_3)
f_1	(0.5,0.4,0.3,-0.2,-0.3,-0.9)	(0.8,0.2,0.4,-0.3,-0.4,-0.8)	(0.4,0.7,0.6,-0.2,-0.3,-0.9)
f_2	(0.2,0.3,0.7,-0.1,-0.4,-0.3)	(0.6,0.3,0.3,-0.6,-0.3,-0.5)	(0.5,0.6,0.3,-0.4,-0.6,-0.8)
b_1	(0.7,0.2,0.4,-0.5,-0.6,-0.9)	(0.5,0.6,0.2,-0.7,-0.3,-0.2)	(0.2,0.1,0.3,-0.7,-0.5,-0.5)
b_1	(0.4,0.6,0.8,-0.7,-0.3,-0.3)	(0.6,0.6,0.8,-0.7,-0.2,-0.2)	(0.3,0.4,0.3,-0.9,-0.1,-0.2)

Table 1:Neutrosophic values of (a) Blur image.

$\mathcal{B}_2$	Brightness(e_1)	Contrast(e_2)	Sharpness(e_3)
f_1	(0.6,0.5,0.4,-0.1,-0.2,-0.8)	(0.7,0.1,0.3,-0.4,-0.5,-0.9)	(0.3,0.6,0.5,-0.3,-0.4,-0.9)
f_2	(0.8,0.3,0.5,-0.4,-0.5,-0.8)	(0.4,0.5,0.1,-0.8,-0.4,-0.3)	(0.4,0.3,0.5,-0.5,-0.3,-0.5)
b_1	(0.5,0,0.2,-0.7,-0.4,-0.7)	(0.3,0.4,0.4,-0.8,-0.4,-0.3)	(0.4,0.3,0.5,-0.5,-0.3,-0.3)
b_2	(0.2,0.4,0.6,-0.9,-0.1,-0.1)	(0.4,0.4,0.8,-0.5,-0.2,-0.2)	(0.2,0.2,0.3,-0.5,-0.1,-0.4)

Table 2:Neutrosophic values of (b) Noisy image.

$\mathcal{B}_3$	Brightness(e_1)	Contrast(e_2)	Sharpness(e_3)
f_1	(0.4,0.5,0.7,-0.9,-0.8,-0.2)	(0.3,0.8,0.7,-0.6,-0.5,-0.1)	(0.7,0.4,0.5,-0.7,-0.6,-0.1)
f_2	(0.2,0.7,0.5,-0.6,-0.5,-0.2)	(0.6,0.5,0.9,-0.2,-0.6,-0.7)	(0.6,0.7,0.5,-0.5,-0.7,-0.5)
b_1	(0.5,0.4,0.8,-0.3,-0.6,-0.3)	(0.7,0.4,0.6,-0.2,-0.6,-0.7)	(0.6,0.7,0.5,-0.5,-0.7,-0.7)
b_2	(0.8,0.6,0.4,-0.1,-0.9,-0.9)	(0.6,0.6,0.2,-0.5,-0.8,-0.8)	(0.8,0.8,0.7,-0.5,-0.9,-0.6)

Table 3:Neutrosophic values of (c) Low resolution image.

Following table shows that the neutrosophic values of absolute foreground and background pixels.

$model - \mathcal{B}$	Brightness(e_1)	Contrast(e_2)	Sharpness(e_3)
f	(1,0,0,0,-1,-1)	(1,0,0,0,-1,-1)	(1,0,0,0,-1,-1)
b	(0,1,1,-1,0,0)	(0,1,1,-1,0,0)	(0,1,1,-1,0,0)

By our criteria, we define ideal neutrosobhic values as follows.

$\mathcal{B}$	Brightness(e_1)	Contrast(e_2)	Sharpness(e_3)
f_1	(0.6,0.4,0.3,-0.1,-0.8,-0.9)	(0.8,0.1,0.3,-0.6,-0.5,-0.9)	(0.7,0.4,0.5,-0.2,-0.6,-0.9)
f_2	(0.8,0.3,0.5,-0.1,-0.5,-0.9)	(0.6,0.5,0.9,-0.2,-0.3,-0.3)	(0.6,0.3,0.3,-0.4,-0.7,-0.8)
b_1	(0.5,0.4,0.8,-0.7,-0.4,-0.3)	(0.3,0.6,0.6,-0.8,-0.3,-0.2)	(0.2,0.7,0.5,-0.7,-0.3,-0.3)
b_2	(0.2,0.6,0.8,-0.9,-0.1,-0.1)	(0.4,0.6,0.8,-0.7,-0.2,-0.2)	(0.2,0.8,0.7,-0.9,-0.1,-0.2)

Now we compute the Hamming distance between our ideal bipolar neutrosophic soft set and the bipolar neutrosophic set of each images to find the similarity.

$$d^H_{BNSS}(\mathcal{B},\mathcal{B}_1) = 1.9$$
$$d^H_{BNSS}(\mathcal{B},\mathcal{B}_2) = 1.7667$$
$$d^H_{BNSS}(\mathcal{B},\mathcal{B}_3) = 3.6$$

Then the similarity values are,

$$\mathcal{S}^H_{BNSS}(\mathcal{B},\mathcal{B}_1) = \frac{1}{1+d^H_{BNSS}(\mathcal{B},\mathcal{B}_1)} = 0.3448$$
$$\mathcal{S}^H_{BNSS}(\mathcal{B},\mathcal{B}_2) = \frac{1}{1+d^H_{BNSS}(\mathcal{B},\mathcal{B}_2)} = 0.3614$$
$$\mathcal{S}^H_{BNSS}(\mathcal{B},\mathcal{B}_3) = \frac{1}{1+d^H_{BNSS}(\mathcal{B},\mathcal{B}_3)} = 0.2174$$

Based on these similarity scores, we choose $\mathcal{B}_2$ as the reliable bipolar neutrosophic soft set. This means among these three types of image samples, second image is more favorable to our criteria.

8 Conclusion and Future work

In this paper, we proposed a different approach on bipolar neutrosophic soft sets and discussed their properties which was initially introduced by Ali et al. Further we defined some distance measures between any two bipolar neutrosophic soft sets to check similarity between them. And also we defined entropy measure to calculate indeterminacy. In section 6, we gave the representation of 2-D image in bipolar neutrosophic domain. Finally, the proposed similarity measurements have been applied to decision making problem in image analysis. Our future work will include more decision making methods based upon different similarity measurements.

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