Neutrosophic Sets and Systems

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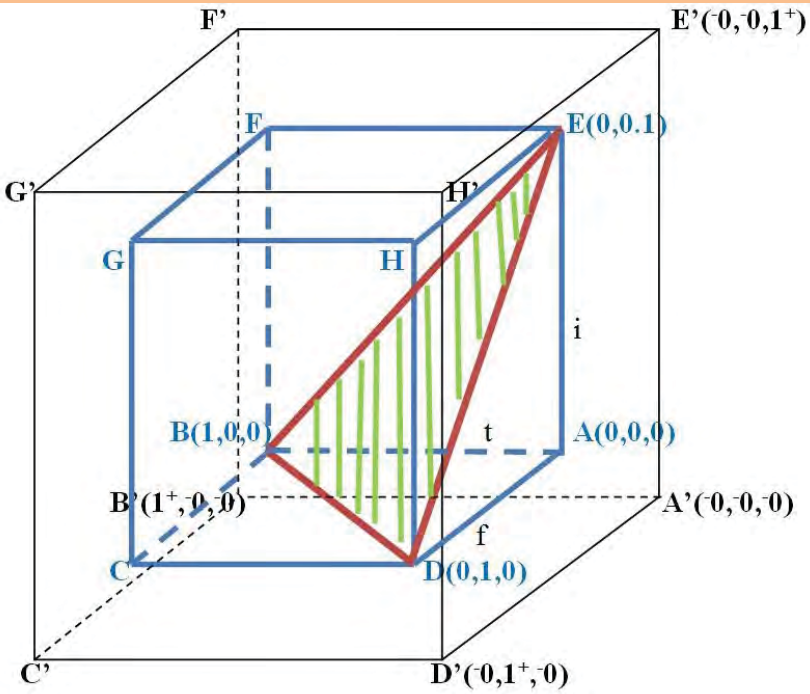

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Information for Authors and Subscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Neutrosophic Shortest Path Problem

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Abstract. Neutrosophic set theory provides a new tool to handle the uncertainties in shortest path problem (SPP). This paper introduces the SPP from a source node to a destination node on a neutrosophic graph in which a positive neutrosophic number is assigned to each edge as its edge cost. We define this problem as neutrosophic shortest path problem (NSSPP). A simple algorithm is also introduced to solve the NSSPP. The proposed algorithm finds the neutrosophic shortest path (NSSP) and its corresponding neutrosophic shortest path length (NSSPL) between source node and destination node. Our proposed algorithm is also capable to find crisp shortest path length (CrSPL) of the corresponding neutrosophic shortest path length (NSSPL) which helps the decision maker to choose the shortest path easily. We also compare our proposed algorithm with some existing methods to show efficiency of our proposed algorithm. Finally, some numerical experiments are given to show the effectiveness and robustness of the new model. Numerical and graphical results demonstrate that the novel methods are superior to the existing method.

Keywords: Trapezoidal neutrosophic fuzzy numbers; scoring, accuracy and certainty index, shortest path problem

1 Introduction

Let $G = (V, E)$ be a graph, where V is a set of all the nodes (or vertices) and E is a set of all the edges (or arcs). The aim of the shortest path problem (SPP) is to find a path between two nodes and optimizing the weight of the path. The SPP is known as one of the well-studied fields in the area operations research and mathematical optimization and it is commonly encountered in wide array of practical applications including road network [1], flow shop scheduling [2], routing problems [3], transportation planning [4], geographical information system(GIS) field [5], optimal path[6-7] and so on.

There are several methods for solving traditional SPP such as Dijkstra [8] algorithm or the label-correcting Bellman [9] algorithm. Due to uncertain factors in real-world problems, such as efficiency, expense, and path capacity variation, we must consider SPP with imprecise information. Under some circumstances, an approximate method applies fuzzy numbers to solve SPP, called Fuzzy-SPP (FSPP). Many researchers have focused on FSPP and intuitionistic FSPP (IFSPP) formulations and solution approaches. Dubois and Prade [10] first introduced FSPP. Later, different approaches were presented by various researchers/scientists to evaluate the FSPP. Some of them are as follows; Keshavarz and Khorram [11] used the highest reliability, Deng et al. [12] suggested extended Dijkstra Principle technique, Hassanzadeh et al. [13], and Syarif et al. [14] proposed a genetic algorithm model, Ebrahimnejad et al. [15] using the artificial bee colony model, Li et al.[16]; Zhong and Zhou [17] used neural networks for finding FSPP. Moreover, Motameni and Ebrahimnejad [18] considered constraint SPP, Mukherjee [19], Geetharamani and Jayagowri [20] and Biswas et al. [21] considered the IFSPP. In recent years, research on this subject has increased and that is of continuing interest such as Kristianto et al.[22], Zhang et al.[23], Mukherjee [24], Huang and Wang [25], Dey et al. [26], Niroomand et al. [27], Rashmanlou et al. [28], Mali and Gautam [29], Wang et al. [30], Yen and Cheng [31] and so on.

Recently, neutrosophic set (NS) theory is proposed by Smarandache [32-33], and this is generalised from the fuzzy set [34] and intuitionistic fuzzy set [35]. NS deals with uncertain, indeterminate and incongruous data where the indeterminacy is quantified explicitly. Moreover, falsity, indeterminacy and truth membership are completely independent. It overcomes some limitations of the existing methods in depicting uncertain decision information. Some extensions of NSs, including interval NS [36-38] , bipolar NS[39], single-valued NS [40-44], multi-valued NS [45-47], neutrosophic linguistic set [48-49], rough neutrosophic set [50-62], triangular fuzzy neutrosophic set [63], and neutrosophic trapezoidal set [64-67] have been proposed and applied to solve various problems. However, to the best of our knowledge, there are few methods which deal with NSSPP. Recently, Broumi et al.[68-71] proposed some models to solve SPP in the neutrosophic environment. Broumi et al. [68-71], proposed a new method for the TrNSSPP and TNSSPP. However, the mentioned methods [68-71] have some shortcomings and are not valid. In this paper, for finding NSSPP, the shortcomings of the mentioned models are pointed out, and a new method is proposed for the same.

2 Preliminaries

Definition 2.1: [72]: Let 1 is a special NS on the real number set R, whose truth-MF $\mu_{\tilde{a}}(x)$, indeterminacy-MF $\nu_{\tilde{a}}(x)$, and falsity-MF $\lambda_{\tilde{a}}(x)$ are given as follows:

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{T_{\tilde{a}}\left(x-\tilde{a}_T\right)}{\left(\tilde{a}_I-\tilde{a}_T\right)} & \tilde{a}_T\le x\le \tilde{a}_I,\\ T_{\tilde{a}} & \tilde{a}_I\le x\le \tilde{a}_P,\\ \dfrac{T_{\tilde{a}}\left(\tilde{a}_S-x\right)}{\left(\tilde{a}_S-\tilde{a}_P\right)} & \tilde{a}_P\le x\le \tilde{a}_S,\\ 0 & otherwise.\end{cases} \quad (1)$$

$$\nu_{\tilde{a}}(x)=\begin{cases}\dfrac{\left(\tilde{a}_I-x+I_{\tilde{a}}\left(x-\tilde{a}_T\right)\right)}{\left(\tilde{a}_I-\tilde{a}_T\right)} & \tilde{a}_T\le x\le \tilde{a}_I,\\ I_{\tilde{a}} & \tilde{a}_I\le x\le \tilde{a}_P,\\ \dfrac{\left(x-\tilde{a}_P+I_{\tilde{a}}\left(\tilde{a}_S-x\right)\right)}{\left(\tilde{a}_S-\tilde{a}_P\right)} & \tilde{a}_P\le x\le \tilde{a}_S,\\ 1 & otherwise.\end{cases} \quad (2)$$

$$\lambda_{\tilde{a}}(x)=\begin{cases}\dfrac{\left(\tilde{a}_I-x+F_{\tilde{a}}\left(x-\tilde{a}_T\right)\right)}{\left(\tilde{a}_I-\tilde{a}_T\right)} & \tilde{a}_T\le x\le \tilde{a}_I,\\ F_{\tilde{a}} & \tilde{a}_I\le x\le \tilde{a}_p,\\ \dfrac{\left(x-\tilde{a}_P+F_{\tilde{a}}\left(\tilde{a}_S-x\right)\right)}{\left(\tilde{a}_S-\tilde{a}_P\right)} & \tilde{a}_P\le x\le \tilde{a}_S,\\ 1 & otherwise.\end{cases} \quad (3)$$

The graphical representation of the TrNS number $\tilde{a}=\left\langle\left[\tilde{a}_T,\tilde{a}_I,\tilde{a}_P,\tilde{a}_S\right],\left(T_{\tilde{a}},I_{\tilde{a}},F_{\tilde{a}}\right)\right\rangle$ is shown in Fig. 1, where, the burgundy colour graph show truth-MF, the yellow colour graph shows indeterminacy-MF, and the red colour graph shows the falsity-MF. Blackline represent the truth value, the cyan line represents the indeterminacy value, and the blue line represents the falsity value (here, we consider $T_{\tilde{a}}>I_{\tilde{a}}>F_{\tilde{a}}$).

Additionally, when $\tilde{a}_T>0$, $\tilde{a}=\left\langle\left[\tilde{a}_T,\tilde{a}_I,\tilde{a}_P,\tilde{a}_S\right],\left(T_{\tilde{a}},I_{\tilde{a}},F_{\tilde{a}}\right)\right\rangle$ is called a positive TrNS number. Similarly, when $\tilde{a}_S\le 0$, $\tilde{a}=\left\langle\left[\tilde{a}_T,\tilde{a}_I,\tilde{a}_P,\tilde{a}_S\right],\left(T_{\tilde{a}},I_{\tilde{a}},F_{\tilde{a}}\right)\right\rangle$ becomes a negative TrNS number. When $0\le\tilde{a}_T\le\tilde{a}_I\le\tilde{a}_P\le\tilde{a}_S\le 1$ and $T_{\tilde{a}},I_{\tilde{a}},F_{\tilde{a}}\in[0,1]$, $\tilde{a}$ is called a normalised TrNS number. When $I_{\tilde{a}}=1-T_{\tilde{a}}-F_{\tilde{a}}$, the TrNS number is reduced to triangular intuitionistic fuzzy numbers (TrIFN). When $\tilde{a}_I=\tilde{a}_P,\tilde{a}=\left\langle\left[\tilde{a}_T,\tilde{a}_I,\tilde{a}_P,\tilde{a}_S\right],\left(T_{\tilde{a}},I_{\tilde{a}},F_{\tilde{a}}\right)\right\rangle$ transforming into a TNS number. When $I_{\tilde{a}}=0,F_{\tilde{a}}=0$, a TrNS number is reduced to generalised TrFN, $\tilde{a}=\left\langle\left[\tilde{a}_T,\tilde{a}_I,\tilde{a}_P,\tilde{a}_S\right],T_{\tilde{a}}\right\rangle$.

Definition 2.2: [40] : Let X be a space point or objects, with a genetic element in X denoted by *x.* A single-valued NS, V in X is characterised by three independent parts, namely truth-MF T_V, indeterminacy-MF I_V and falsity-MF F_V, such that $T_V:X\to[0,1], I_V:X\to[0,1],\ and\ F_V:X\to[0,1]$.
Now, V is denoted as $V=\left\{<x,\left(T_V(x),I_V(x),F_V(x)\right)>\ \mid x\in X\right\}$, satisfying $0\le T_V(x)+I_V(x)+F_V(x)\le 3$.

Definition 2.3: [72]: Let $\hat{r}^N=\left\langle\left[\hat{r}_T,\hat{r}_I,\hat{r}_M,\hat{r}_E\right],\left(T_{\hat{r}},I_{\hat{r}},F_{\hat{r}}\right)\right\rangle$ and $\hat{s}^N=\left\langle\left[\hat{s}_T,\hat{s}_I,\hat{s}_M,\hat{s}_E\right],\left(T_{\hat{s}},I_{\hat{s}},F_{\hat{s}}\right)\right\rangle$ be two arbitrary TrNSNs, and $\theta\ge 0$; then arithmetic operation on TrNS are as follows:

$$\hat{r}^N \oplus \hat{s}^N = \left\langle \left[\hat{r}_T + \hat{s}_T, \hat{r}_I + \hat{s}_I, \hat{r}_M + \hat{s}_M, \hat{r}_E + \hat{s}_E\right], \left(T_{\hat{r}} + T_{\hat{s}} - T_{\hat{r}}T_{\hat{s}}, I_{\hat{r}}I_{\hat{s}}, F_{\hat{r}}F_{\hat{s}}\right)\right\rangle$$

$$\hat{r}^N \otimes \hat{s}^N = \left\langle \left[\hat{r}_T \cdot \hat{s}_T, \hat{r}_I \cdot \hat{s}_I, \hat{r}_M \cdot \hat{s}_M, \hat{r}_E \cdot \hat{s}_E\right], \left(T_{\hat{r}} \cdot T_{\hat{s}}, I_{\hat{r}} + I_{\hat{s}} - I_{\hat{r}}I_{\hat{s}}, F_{\hat{r}} + F_{\hat{s}} - F_{\hat{r}}F_{\hat{s}}\right)\right\rangle$$

$$\theta\hat{r}^N = \left\langle \left[\theta\hat{r}_T, \theta\hat{r}_I, \theta\hat{r}_M, \theta\hat{r}_E\right], \left(1-\left(1-T_{\hat{r}}\right)^{\theta}, \left(I_{\hat{r}}\right)^{\theta}, \left(F_{\hat{r}}\right)^{\theta}\right)\right\rangle$$

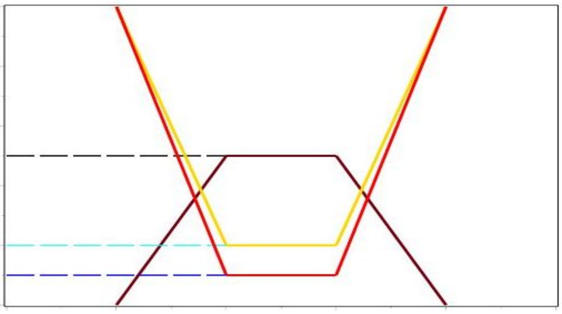

Figure 1. The graphical representation of the membership functions of TrNS.

Definition 2.4: [73]: (Comparison of any two random TrNS numbers): Let $\hat{r}^N = \left\langle \left[\hat{r}_T, \hat{r}_I, \hat{r}_M, \hat{r}_E\right], \left(T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}\right)\right\rangle$ be a TrNS number,and then the score and accuracy function is defined, as follow:

$$s\left(\hat{r}^N\right) = \frac{1}{12}\left[\hat{r}_T + \hat{r}_I + \hat{r}_M + \hat{r}_E\right] \times \left[2 + T_{\hat{r}} - I_{\hat{r}} - F_{\hat{r}}\right]$$

$$a\left(\hat{r}^N\right) = \frac{1}{12}\left[\hat{r}_T + \hat{r}_I + \hat{r}_M + \hat{r}_E\right] \times \left[2 + T_{\hat{r}} - I_{\hat{r}} + F_{\hat{r}}\right]$$

Let $\hat{r}^N = \left\langle \left[\hat{r}_T, \hat{r}_I, \hat{r}_M, \hat{r}_E\right], \left(T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}\right)\right\rangle$ and $\hat{s}^N = \left\langle \left[\hat{s}_T, \hat{s}_I, \hat{s}_M, \hat{s}_E\right], \left(T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}}\right)\right\rangle$ be two TrNS numbers, the ranking of $\hat{r}^N$ and $\hat{s}^N$ by score function is described as follows:

1. $\textit{if } s\left(\hat{r}^N\right) \prec s\left(\hat{s}^N\right) \textit{ then } \hat{r}^N \prec \hat{s}^N$
2. $\textit{if } s\left(\hat{r}^N\right) \approx s\left(\hat{s}^N\right) \textit{ and if}$
   a. $a\left(\hat{r}^N\right) \prec a\left(\hat{s}^N\right) \textit{ then } \hat{r}^N \prec \hat{s}^N$
   b. $a\left(\hat{r}^N\right) \succ a\left(\hat{s}^N\right) \textit{ then } \hat{r}^N \succ \hat{s}^N$
   c. $a\left(\hat{r}^N\right) \approx a\left(\hat{s}^N\right) \textit{ then } \hat{r}^N \approx \hat{s}^N$

Definition 2.5:[73]: (Comparison of any two random TNS numbers).Let $\hat{r}^{NS} = \left\langle \left[\hat{r}_T, \hat{r}_I, \hat{r}_P\right], \left(T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}\right)\right\rangle$ be a TNS number, and then the score and accuracy functions are defined as follows:

$$s\left(\hat{r}^{NS}\right) = \frac{1}{12}\left[\hat{r}_T + 2 \cdot \hat{r}_I + \hat{r}_P\right] \times \left[2 + T_{\hat{r}} - I_{\hat{r}} - F_{\hat{r}}\right]$$

$$a\left(\hat{r}^{NS}\right) = \frac{1}{12}\left[\hat{r}_T + 2 \cdot \hat{r}_I + \hat{r}_P\right] \times \left[2 + T_{\hat{r}} - I_{\hat{r}} + F_{\hat{r}}\right]$$

Let $\hat{r}^{NS} = \left\langle \left[\hat{r}_T, \hat{r}_I, \hat{r}_P\right], \left(T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}\right)\right\rangle$ and $\hat{s}^{NS} = \left\langle \left[\hat{s}_T, \hat{s}_I, \hat{s}_P\right], \left(T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}}\right)\right\rangle$ be two arbitrary TNSNs, the ranking of $\hat{r}^{NS}$ and $\hat{s}^{NS}$ by score function is defined as follows:

1. $\textit{if } s\left(\hat{r}^{NS}\right) \prec s\left(\hat{s}^{NS}\right) \textit{ then } \hat{r}^{NS} \prec \hat{s}^{NS}$
2. $\textit{if } s\left(\hat{r}^{NS}\right) \approx s\left(\hat{s}^{NS}\right) \textit{ and if}$
   a. $a\left(\hat{r}^{NS}\right) \prec a\left(\hat{s}^{NS}\right) \textit{ then } \hat{r}^{NS} \prec \hat{s}^{NS}$
   b. $a\left(\hat{r}^{NS}\right) \succ a\left(\hat{s}^{NS}\right) \textit{ then } \hat{r}^{NS} \succ \hat{s}^{NS}$
   c. $a\left(\hat{r}^{NS}\right) \approx a\left(\hat{s}^{NS}\right) \textit{ then } \hat{r}^{NS} \approx \hat{s}^{NS}$

Definition 2.6: [72]: Let $\hat{r} = [\hat{r}_T, \hat{r}_I, \hat{r}_M, \hat{r}_E]$ be a TrFN, and $\hat{r}_T \le \hat{r}_I \le \hat{r}_M \le \hat{r}_E$ then the centre of gravity (COG) of $\hat{r}$ can be defined as

$$COG(\hat{r}) = \begin{cases} \hat{r}, & if \quad \hat{r}_T = \hat{r}_I = \hat{r}_M = \hat{r}_E \\ \dfrac{1}{3}\left[\hat{r}_T + \hat{r}_I + \hat{r}_M + \hat{r}_E - \dfrac{\hat{r}_E\hat{r}_M - \hat{r}_I\hat{r}_T}{\hat{r}_E + \hat{r}_M - \hat{r}_I - \hat{r}_T}\right], & otherwise \end{cases}$$

Definition 2.7:[72]: (Comparison of any two random TrNS numbers).Let $\hat{s}^{NS} = \langle [\hat{s}_T, \hat{s}_I, \hat{s}_M, \hat{s}_E], (T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}}) \rangle$ be a TrNSNs,and then the score function, accuracy function, and certainty functions are defined as follows:

$$E(\hat{s}^N) = COG(\hat{r}) \times \frac{(2 + T_{\hat{s}} - I_{\hat{s}} - F_{\hat{s}})}{3},$$

$$A(\hat{s}^N) = COG(\hat{r}) \times (T_{\hat{s}} - F_{\hat{s}}),$$

$$C(\hat{s}^N) = COG(\hat{r}) \times (T_{\hat{s}})$$

Let $\hat{r}^{NS} = \langle [\hat{r}_T, \hat{r}_I, \hat{r}_P], (T_{\hat{r}}, I_{\hat{r}}, F_{\hat{r}}) \rangle$ and $\hat{s}^{NS} = \langle [\hat{s}_T, \hat{s}_I, \hat{s}_M, \hat{s}_E], (T_{\hat{s}}, I_{\hat{s}}, F_{\hat{s}}) \rangle$ be two arbitrary TrNSNs, the ranking of $\hat{r}^{NS}$ and $\hat{s}^{NS}$ by score function is defined as follows:

1. $if\ E(\hat{r}^{NS}) \succ E(\hat{s}^{NS})\ then\ \hat{r}^{NS} \succ \hat{s}^{NS}$
2. $if\ E(\hat{r}^{NS}) \approx E(\hat{s}^{NS})\ and\ if\ A(\hat{r}^{NS}) \succ A(\hat{s}^{NS}) then\ \hat{r}^{NS} \succ \hat{s}^{NS}$
3. $if\ E(\hat{r}^{NS}) \approx E(\hat{s}^{NS})\ and\ if\ A(\hat{r}^{NS}) \prec A(\hat{s}^{NS}) then\ \hat{r}^{NS} \prec \hat{s}^{NS}$
4. $if\ E(\hat{r}^{NS}) \approx E(\hat{s}^{NS})\ and\ if\ A(\hat{r}^{NS}) \prec A(\hat{s}^{NS})\ and\ C(\hat{r}^{NS}) \prec C(\hat{s}^{NS}) then\ \hat{r}^{NS} \prec \hat{s}^{NS}$
5. $if\ E(\hat{r}^{NS}) \approx E(\hat{s}^{NS})\ and\ if\ A(\hat{r}^{NS}) \approx A(\hat{s}^{NS})\ and\ C(\hat{r}^{NS}) \succ C(\hat{s}^{NS}) then\ \hat{r}^{NS} \succ \hat{s}^{NS}$
6. $if\ E(\hat{r}^{NS}) \approx E(\hat{s}^{NS})\ and\ if\ A(\hat{r}^{NS}) \approx A(\hat{s}^{NS})\ and\ C(\hat{r}^{NS}) \approx C(\hat{s}^{NS}) then\ \hat{r}^{NS} \approx \hat{s}^{NS}$

2.1 List of Abbreviation used throughout this paper.

SPP stands for "shortest path problem."
NSSPP stands for "neutrosophic shortest path problem."
NSSP stands for "neutrosophic shortest path."
NSSPL stands for "neutrosophic shortest path length."
CrSPL stands for "crisp shortest path length."
FSPP stands for "fuzzy shortest path problem."
IFSPP stands for "intuitionistic fuzzy shortest path problem."
NS stands for "neutrosophic set."
TrNSSPP stands for "trapezoidal neutrosophic shortest path."
TNSSPP stands for "triangular neutrosophic shortest path."
TrNS stands for "trapezoidal neutrosophic set."
TNS stands for "triangular neutrosophic set."
MF stands for "membership function."
TFN stands for "triangular fuzzy number."
TrFN stands for "trapezoidal fuzzy number."

3 The Proposed model

Before we start the main algorithm, we introduce a sub-section i.e., shortcoming and limitation of some of the existing models:

3.1 Discussion on shortcoming of some of the existing methods

At first, we discussed the shortcoming and limitation of the existing methods under two different type of NS environment.

Broumi et al. [68-69] first proposed a method to find the shortest path under TrNS environment. It is a very well known and propular paper in the field of neutrosophic set and system. However, the authors used some mathematical assumption to solve the problem which may be invalid in some cases. This has been discussed in detail in Example 3.1. and Example 3.2

Example 3.1**:** Broumi et al.[69]: Here authors have considered two arbitrary i.e., $\tilde{r}, \tilde{s}$ be the following TrNS numbers:

$$\tilde{r} = \langle (1,2,3,4);\ 0.4, 0.6, 0.7 \rangle,$$

$$\tilde{s} = \langle (1,5,7,9);\ 0.7, 0.6, 0.8 \rangle.$$

We observe that the authors used an invalid mathematical assumption to solve the problem i.e.,

$$S(\tilde{r} + \tilde{s}) = S(\tilde{r}) + S(\tilde{s})$$

Our objective is to show that above considered assumption is not valid such as

$$S(\tilde{r} + \tilde{s}) \neq S(\tilde{r}) + S(\tilde{s}).$$

Solution : According to the method of Broumi et al. [69] [see; iteration 4, page no 420, ref. Broumi et al.[69]] ,we have:

$$\tilde{r} + \tilde{s} = \langle (1,2,3,4);\ 0.4, 0.6, 0.7 \rangle \oplus \langle (1,5,7,9);\ 0.7, 0.6, 0.8 \rangle = \langle 2,\ 7,\ 10,\ 13);\ 0.82,\ 0.36,\ 0.56 \rangle.$$

Therefore, we get, $S(\tilde{r} + \tilde{s}) =$ 5.06 . but $S(\tilde{r}) + S(\tilde{s})$ = 3.3.
Hence, It is clear that $S(\tilde{r} + \tilde{s}) \neq S(\tilde{r}) + S(\tilde{s})$.
Therefore, we can say that the method of Broumi et al. [68-71] is not valid. So we think there is a still a scope of improvement. So to remove this limitation we proposed our new method.

3.2. Existing crisp model in SPP

In this section, we study the notation and existing crisp SPP and proposed neutrosophic SPPs.

Notations

Ω **:** Starting node
□ **:** Final destination node
$\underline{\pi}_{mk}$: The shortest distance from an m^{th} node to k^{th} node.
$\sum_{k=1}^{s} x_{mk}$: The total flow out of node *s*.
$\sum_{k=1}^{s} x_{mk}$: The total flow into node *s*.
RK_{mk} *:* the objective cost in crisp environment
According to Bazaraa et al. [74], The crisp SPP model is as follows :

$$Min = \sum_{m=1}^{s} \sum_{k=1}^{s} RK_{mk} \cdot x_{mk}$$

Subject to: (4)

$$\sum_{m=1}^{s} x_{mk} - \sum_{k=1}^{s} x_{km} = \underset{\sim}{\kappa}_m$$

for all $x_{mk} \in \Re$ and non-negative where $m, k = 1, 2, ..., s$ and:

$$\underset{\sim}{\kappa}_m = \begin{cases} 1 \ if & m = \Omega, \\ 0 \ if & m = \Omega + 1, \Omega + 2, ..., □ - 1 \\ -1 \ if & m = □ . \end{cases} \quad (5)$$

3.3. Transformation of crisp SPP model into nutrosophic SPP

If we replaced the parameter RK_{mk} into nutrosophic cost parameters, *i.e.* RK_{mk}^{N} , then the model is as follows:

$$Min = \sum_{m=1}^{s} \sum_{k=1}^{s} RK_{mk}^{N} .\ x_{mk}$$

Subject to: (6)

$$\sum_{m=1}^{s} x_{mk} - \sum_{k=1}^{s} x_{km} = \tilde{\kappa}_m \qquad m,k = 1,2,.....,s.$$

$x_{mk} \in \Re$ And are non-negative.

3.4. Algorithm: A novel approach for finding the SPP under TrNS and TNS environment

We consider a directed acyclic graph whose arc lengths are represented by neutrosophic number. Our proposed algorithm finds the shortest path from the source node *s* to the destination *d* of the graph. The steps of the algorithm are as follows:

Step 1: Let *m* be the total number of paths from *s* to *d*. Compute the score function of each arc length under the given network using the Definition 2.6-2.7.

Step 2: Find all possible paths A_i, and also find the path length of corresponding A_i, where $i = 1,2,3,.......,m$,for m possible number of paths. Now, each of m paths can be considered as an arc from *s* to *d* as shown in Fig. 2. Each of these arcs are represented by a neutrosophic number.

Step 3: Calculate the summation of the score function of each arc length corresponding to the path A_i, and set that, $E(\theta_i)$ where $i = 1,2,3,.......,m$.

Step 4: By ranking the score value obtained in Step 3 in ascending order, find the lowest rank which is the shortest route of the given network under neutrosophic environment.

End

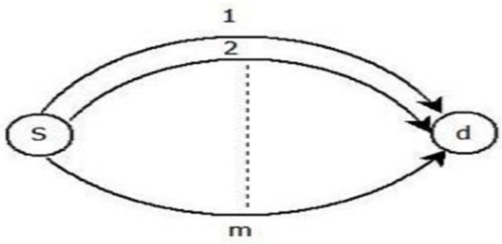

Figure. 2. m paths from source node s to destination node d are represented by m arcs

4 Example of real life application:

To justify our proposed algorithms, we consider a network shown in Fig. 3 [Broumi et al [68-71]]

Distribution network problems: In Example 4.1, and Example 4.2, we have considered a real-life problem of a distribution network for a soft drink company. Here we have considered a soft drink company which is having 6 distribution areas. This configuration is shown in Fig. 3. The time of delivery of the goods between the distribution centers can vary day to day due to many uncertain reasons such as road conjunction, driver bad health, vehicle break down, natural calamities such as flood, tsunami, earthquake and so on. Therefore, companies want to determine the range of cost per day in between the two consecutive locations but the problem is that the time is uncertain so the shortest distance will also be uncertain. This uncertainty can be avoided by predicting the shortest path using neutrosophic number and therefore we have considered TrNS and TNS numbers for our assumption where the neutrosophic cost between the two consecutive distribution centers is given in Table 1 and Table 3 respectively. The company wants to find the NSP on the basis of lowest cost of transportation for distribution between the geographical centers.

Example 4.1: Consider a network (Fig. 3), with six nodes and eight edges, where node 1 is the source node and node 6 is the destination node. The TrNS cost is given in Table 1.

Table 1. The conditions of Example 4.1.

T	H	TrNS cost	T	H	TrNS cost
1	2	<(1,2,3,4); 0.4,0.6,0.7>	3	4	<(2,4,8,9); 0.5,0.3,0.1>
1	3	<(2,5,7,8; 0.2,0.3,0.4>	3	5	<(3,4,5,10); 0.3,0.4,0.7>
2	3	<(3,7,8,9); 0.1,0.4,0.6>	4	6	<(7,8,9,10); 0.3,0.2,0.6>
2	5	<(1,5,7,9); 0.7,0.6,0.8>	5	6	<(2,4,5,7); 0.6,0.5,0.3>

Solution: Applying steps 1-4 in proposed Algorithm, we get the NSP as $1 \to 2 \to 5 \to 6$ with the lowest cost is 5.96, and the NSPL is <(4,11,15,20); 0.928, 0.18, 0.168>. It is clear that the range of NSPL is 4 to 20 and we get an optimal solution which lies inside the region. The final result is shown in Table 2.

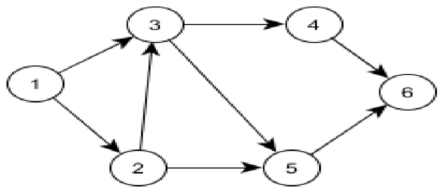

Figure 3.A network with six vertices and eight edges [Broumi et al. [68-71]]

Table 2. Final step obtained by proposed algorithm is as follows

Possible path	$E(\theta_i)$	Ranking
$A_1: 1 \to 2 \to 5 \to 6$	5.96	1
$A_2: 1 \to 3 \to 5 \to 6$	7.71	2
$A_3: 1 \to 2 \to 3 \to 5 \to 6$	8.33	3
$A_4: 1 \to 3 \to 4 \to 6$	10.97	4
$A_5: 1 \to 2 \to 3 \to 4 \to 6$	11.59	5

Example 4.2. Consider a network from Fig. 3, with six nodes and eight edges, where node 1 is the source node and node 6 is the destination node. The TNS cost is given in Table 3.

Table 3. The conditions of Example 4.2.

T	H	TNS cost	T	H	TNS cost
1	2	<(1,2,3); 0.4,0.6,0.7>	3	4	<(2,4,8); 0.5,0.3,0.1>
1	3	<(2,5,7); 0.2,0.3,0.4>	3	5	<(3,4,5); 0.3,0.4,0.7>
2	3	<(3,7,8); 0.1,0.4,0.6>	4	6	<(7,8,9); 0.3,0.2,0.6>
2	5	<(1,5,7); 0.7,0.6,0.8>	5	6	<(2,4,5); 0.6,0.5,0.3>

Solution: Applying steps 1-4 in proposed Algorithm, the NSP is $1 \to 2 \to 5 \to 6$ and the minimum cost is 4.811; so the NSPL is < (4,11, 15); 0.93, 0.18, 0.17>. It is clear that the range of NSPL is 4 to 15 and our objective value is 4.811. So we conclude that the crisp minimum cost is 4.811. The result is shown in Table 4

Table 4. Final result of proposed Algorithm for Example 4.2 is as follows:

Possible path	$E(\theta_i)$	Ranking
$A_1: 1 \to 2 \to 5 \to 6$	4.811	1
$A_2: 1 \to 3 \to 5 \to 6$	6.133	2
$A_3: 1 \to 2 \to 3 \to 5 \to 6$	6.733	3
$A_4: 1 \to 3 \to 4 \to 6$	9.6	4
$A_5: 1 \to 2 \to 3 \to 4 \to 6$	10.2	5

5. Result and Discussion

At first, we discussed the Example 4.1 and Example 4.2 which is considered by Broumi et al.[68-71]. We found that the proposed Algorithm gives the same shortest route as suggested by Broumi et al.[68-71]. However, our proposed methods predict the better crisp optimum cost value as compared with the mentioned existing methods.

Table 5. Logical Comparison of predicted crisp optimum cost values with the existing methods.

Example 4.1	Broumi et al. Method [68] $\succ$ our proposed method
Example 4.2	Broumi et al. Method [71] $\succ$ our proposed method

In Fig. 4 and Fig. 5 (Graphical comparison with existing methods) when we have compared our proposed method with the other existing methods, we have found that the objective value of our proposed method is smaller than to the existing methods. The best part about our proposed algorithms is that it gives the crisp optimum cost values as compared with the present existing method. This is shown in Table 5 (Logical Comparison with existing methods) and Table 6 (Numerical Comparison with existing methods) respectively. Also, we can say that the objective value obtained by our proposed algorithm lies within the neutrosophic region.

Table 6. Numerical Comparison of our proposed method with the existing methods.

Ex	The method's name	Proposed path	SVNSPP
4.1	Method 1 [69]		NA
	Method 2 [68]	$1 \to 2 \to 5 \to 6$	Crisp optimum cost: 10.75 NSPL: < (4,11,15,20); 0.93, 0.18, 0.17>.
	Method 3 [70]	-	NA
	Method 4 [71]	-	NA
	Proposed Algorithm	$1 \to 2 \to 5 \to 6$	Suggested crisp optimum cost: 5.96 NSSPL: < (4,11,15,20); 0.93, 0.18, 0.17>.
4.2	Method 1 [69]		NA
	Method 2 [68]		NA
	Method 3 [70]		NA
	Method 4 [71]	$1 \to 2 \to 5 \to 6$	Crisp optimum cost: 8.815 NSSPL: < (4,11,15); 0.93, 0.18, 0.17>.
	Proposed Algorithm	$1 \to 2 \to 5 \to 6$	Suggested crisp optimum cost: 4.811 NSSPL: < (4,11,15); 0.93, 0.18, 0.17>. .

Because of these capabilities, we can say that our proposed algorithms are superior to the existing methods.

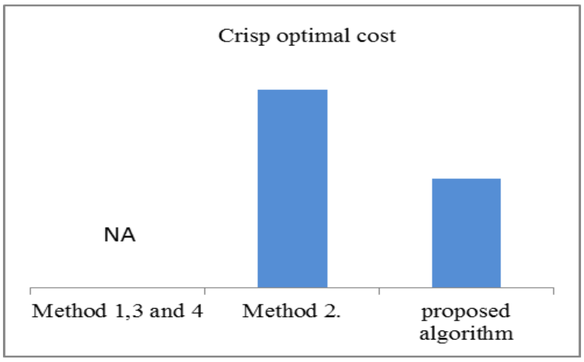

Figure 4. Comparison of crisp optimum cost value for Example 4.1 with different methods

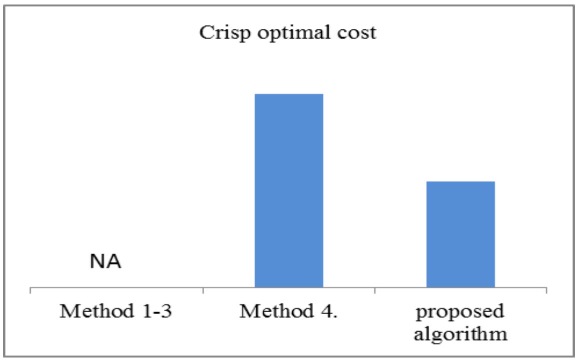

Figure 5. Comparison of crisp optimum cost values for Example 4.2 with different methods

Conclusion

In this paper, we have introduced an algorithm for solving the NSSPP. In this NSSPP, firstly we find all possible paths from source node to destination node and compute their corresponding path lengths in terms of neutrosophic number. Considering each path as an arc (from source node to destination node), we find rank of the path based on score function. The path corresponding to the lowest rank is the shortest path. An example graph is considered to demonstrate our proposed algorithm. These algorithms are not only suggested the NSSP but also able to predict the NSSPL and CrSPL. Moreover, the shortcomings of the existing algorithms are pointed out and to show the advantages of the proposed algorithms. For this purpose, we have considered NSSPP and compare with existing methods. The numerical results show that the new algorithms outperform the current methods. In the future, the proposed method can be applied to real-world problems in the field of minimum cost flow problem (MCFP), job scheduling, transportation, and so on.

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Intrusion Detection System and Neutrosophic Theory for MANETs: A Comparative Study

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Abstract. Mobile Ad hoc Network (MANET) is a system of wireless mobile nodes that dynamically self-organized in arbitrary and temporary network topologies without communication infrastructure. This network may change quickly and unforeseeable. The unique characteristics of MANET give an adversary the opportunity to launch numerous attacks against ad-hoc networks. So the security is an important role in MANETs. This article aims to present the concept of intrusion detection system (IDs) and surveys some of major intrusion detection approach against neutrosophic intrusion detection system in MANETs. Current IDS's corresponding to those architectures are also reviewed and compared. This paper introduces the accuracy rate and false alarm rate of four completely different classifiers to observe the percentage and the efficiency of the classifiers in detecting attacks in MANETs. Results show that Neutrosophic intelligent system based on genetic algorithm could facilitate significantly in detecting malicious activities in MANETs. Hence, neutrosophic techniques could be utilized to suit the ambiguity nature of the IDs.

Keywords: MANETs, Intrusion Detection, Neutrosophic, Security, Attacks.

1. Introduction

MANETs is a self-adapting network that's formed automatically by a group of mobile nodes without the needed of a hard and fast infrastructure or centralized control. Every node is supplied with a wireless transmitter and receiver, which permit it to contact with alternative nodes in its radio communication area. In order for a node to forward a packet to a node that's out of its radio area, the cooperation of alternative nodes within the network is needed; this can be referred to as a multi-hop communication. Therefore, every node must act as both a client and a router at the same time. The network topology often changes because of the mobility of nodes as they move at interval, into, or out of the network. A MANET with the features presented above was fit to military purposes.

In last years, MANETs have been expansion speedily and are progressively being used in several applications, starting from military to civilian and commercial uses, since forming such networks can be avoided the help of any infrastructure or interaction with a human. Several examples are: data collection, and virtual classrooms and conferences where laptops, or other mobile devices share wireless medium and connect to each other. As MANETs become widely spread, the security problem has become one of the fundamental attention. For example, a lot of the routing protocols designed for MANETs suppose that each node inside the network is cooperative and not pernicious [16, 31]. Therefore, just one node can cause the failure of the whole network. There are both passive and active attacks in MANETs. For passive attacks, packets containing secret data might be eavesdropped, that violates confidentiality. Active attacks, including sending data to incorrect destinations into the network, removing data, change the contents of data, and impersonating other nodes violate availability, integrity, authentication, and non-repudiation. Proactive approaches such as cryptography and authentication [21, 30, 29 and 8] were first brought into consideration, and a lots of techniques have been proposed and implemented. However, these applications don't seem to be sufficient. If the flexibility to discover the attack once it comes into the network is got, these attacks can be stopped from doing any harm to the system or any data. Here is where the intrusion detection system comes in.

Intrusion detection as declared is a method of monitoring activities in a system, which may be a computer or network system. The mechanism that is achieved is named an IDs. An IDs gather activity data and then analyses it to determine whether or not there are any actions that assaulted the Protection rules. Once an IDs decide that an upnormal behavior or an action which recognize to be an attack occurs, it then makes an alarm to warn the security administrator. Additionally, IDs also can start a correct response to the malicious activity. Although there are many IDs techniques built for wired networks these days, they're not appropriate for wireless networks because of the variations in their characteristics. Therefore, those techniques should be changed or new techniques should be developed to form intrusion detection work effectively in MANETs. Because of the variations in the MANETs characteristics our research tend to use Neutrosophic Technique as a novel solution. In previous articles research [7 and 8] a neutrosophic intelligent system based on genetic algorithm was proposed, which the novelty in that technique is used three dimension (MEMBERSHIP- INDETERMENANCY - NONMEMEBRSHIP) all pervious technique used only two dimension (MEMBERSHIP- NONMEMEBRSHIP).

The concept of Neutrosophic Set (NS) was first introduced by Smarandache [12, 13] which is a generalization of classical sets, fuzzy set, intuitionistic fuzzy set etc. Salama et al. Work [2, 3, 4, 5, 6, 7, 8 and 11] formulated a beginning to new fields of neutrosophic theory in computer discipline. The neutrosophic indeterminacy assumption is very significant in many of circumstances such as information fusion (collecting data from various sensors). Also, NS is a conceivable common traditional system that generalize the principle of the traditional set, Fuzzy Set (FS) [23] and Intuitionistic Fuzzy Set (IFS)) [17], etc. NS 'A' determined on universe U. $\boldsymbol{x} = \boldsymbol{x}(\mathbf{T}, \mathbf{I}, \mathbf{F}) \in$ A with $\mathbf{T}, \mathbf{I}$ and $\mathbf{F}$ are defined over the interval $]\mathbf{0}^{-}, \mathbf{1}^{+}[$. $\mathbf{T}$ is the truth-MEMEBERSHIP , $\mathbf{I}$ is the INDETERMINACY and F is the falsity-MEMEBERSHIP degrees on the set $\mathbf{A}$.

Designing a neutrosophic IDS is a proper solution in handling vague circumstances. The neutrosophic IDS is formed of two sub phases: the preprocessing stage and the network attacks classification stage. The preprocessing stage is concerned by formulating the network features in a format appropriate for the classification. The KDD network data [18] is reformatted into neutrosophic form $(\mathbf{x}, \boldsymbol{\mu}_{\mathbf{A}}(\mathbf{x}), \boldsymbol{\sigma}_{\mathbf{A}}(\mathbf{x}), \boldsymbol{\nu}_{\mathbf{A}}(\mathbf{x}))$ where x is the value of feature, $\boldsymbol{\mu}_{\mathbf{A}}(\mathbf{x})$ is the MEMEBERSHIP (MEM), $\boldsymbol{\sigma}_{\mathbf{A}}(\mathbf{x})$ is the INDETERMINACY (I) and $\boldsymbol{\nu}_{\mathbf{A}}(\mathbf{x})$ is the NONMEMEBERSHIP (NON_MEM) degrees of the x in the feature space. This article extends a previous work [14 and 15] that compares how effectively intrusions in MANETs are detected by totally different classification algorithms. This paper is arranged as follows. Section 2 includes the Wireless technology. Section 3 proposes the Security in MANETs. The Literature Review of IDs in MANETs and conclusion are shown respectively in section 4 and 5.

2. Wireless technology

Wireless networks use the open medium as communication channel and electromagnetic waves to send data between participants. Nodes in wireless networks able to communicate with each different node placed inside a particular distance, known as transmission range. Once a node desires to send a packet to a different node that doesn't belong in its one-hop neighbourhood then it has to swear to intermediate nodes to forward the packets to the final destination. Thus, efficient routing protocols are needed in order to optimize the communication ways. Security problems in wireless communication may have a significant impact in different types of network architectures since many network architectures use wireless channels. As an example, a design using the 802.11 standard (usually referred as Wi-Fi or WLAN networks) uses a stationary infrastructure that communicates with different networks using a wired connection, however connects with the neighbors of its own network using a wireless channel. This design needs all the nodes to be placed inside the transmission area of the fixed infrastructure (access point), and any drawback concerning this central purpose might have an effect on the whole network. Wireless ad hoc networks or Mobile ad hoc Networks (MANETs) do not use a fixed infrastructure and all the nodes inside to the network could also be mobile. There's no central node acting as an access point, and mobile nodes share the responsibility of the proper functionality of the network, since a cooperative behaviour is required.

3. Security in MANETs

The fundamental nature of MANETs excite the emanation of recent security risks, whereas some present vulnerabilities in wired networks are accentuated. The use of security technologies built for wired networks to protect wireless networks isn't direct also not easy to perform. Within the absence of a wire connecting the nodes, any bad node might access the network without physical restraints. So as to stop dishonorable outsiders entering the network, cryptographic algorithms are often used to authenticate the nodes. However, additional difficult issues arise once an inside benign node is compromised, that is, if any assailant impersonates the identity of a node that's authorized in the network. Since the functionality of the network is typically based on a whole confidence between the participants, a bad node impersonating a trusty node might cause a significant security bridge. Most of the attacks in mobile environments focuses on routing protocols. These protocols were first off designed to be efficient without taking under consideration the protection problems. They typically need the collaboration between the entrants and assume confidence between them. Even so, a bad node might change its assumed benign functionality disturbing the general manner of the protocol. Below, a list of attacks is presented.

- Packet dropping attack: during this attack, the assailant rejects Route Error packets leading the truth nodes to forward packets in broken links [9].
- Flooding attack: The bad node broadcasts solid Route Request packets at random to any or all nodes each a hundred ms so as to overload the network [26].
- Black hole attack [20]: during this attack a bad node announces itself as having the shortest route to different nodes of the network. Nevertheless, as presently because it receives packets forwarded to different nodes, it drops them rather than forwarding to the ultimate destination. In our simulation situation, on every occasion a malicious black hole node receives a Route Request packet it sends a Route Reply packet to the destination on faith if it extremely contains a path towards the chosen destination. Thus, the black-hole node is always the primary node that responds to a Route Request packet. Moreover, the bad node falling all Route Reply and data packets it receives if the packets are destined to different nodes.
- Forging attack [25]: A bad node changes and broadcasts to the prey node Route Error packets resulting in repeated link features.

4. Intrusion Detection in MANETs

Nearly All researchers have two classify of attacks on the MANETs. They described attacks to passive and active. The passive attacks usually inclose only eaves dropping of data, while the active attacks inclose actions achieved by foes like replication, modification and deletion of changed data. Particularly, Attacks in MANET will cause congestion, propagate incorrect routing data, stop services from operating properly or termination them fully (Sharma & Sharma, 2011; Blazevic, et al., 2001).

Intrusion Detection systems (IDs) are software or hardware tools (even a combination of both) that automatically scan and monitor events in a laptop or network, searching for intrusive evidence [27]. Once planning an IDS to be utilized in a MANET, some considerations should be taken under consideration. There are many variations in the method the detection engine should behave with regard to a wired network IDS. A rather complete survey regarding this subject will be found in [11], wherever Anjum et al. present the most challenges to secure wireless mobile networks.

A lot of recently, Sen and Clark [28] have introduced a survey regarding existing intrusion detection approaches for MANETs. Traditional anomaly-based IDSs use predefined ‘‘normality’’ models to discover anomalies within the network. This can be an approach that cannot be simply deployed in MANETs, since the mobility and flexibility of MANET nodes, build hard the definition of ‘‘normal’’ and ‘‘malicious’’ behaviour. Moreover, the mobility of nodes leads to changes of the network topology, increasing the complexity of the detection method. In addition, since the MANET nodes haven't any fixed location, there's no central management and/or monitoring point wherever an IDS might be placed. This means that the detection method could also be distributed into many nodes, as well as the collection and analysis of data. Consequently, IDS are categorized into cooperative or independent (non-collaborative) [28].

Independent IDs are consist of IDs agents setted in the nodes of the network and be accountable for observing all nodes inside the network and sending alarms whenever they detect any suspect activity. the main trouble of this design is determining the place of the IDs agents, since nodes are moveable, and several domains of the network might not be monitored (for example, if the node hosting an IDs agent of one domain moves to another, the first remains uncovered). Another drawback is that some resources such as bandwidth, central processing unit and/or power are scarce in these environments. Therefore, nodes hosting the IDs agents ought to be those having more resources and moreover, a bigger transmission vary. Maximizing the detection rate subject to resource limitation is a nondeterministic polynomial time (NP) complete problem and a few algorithms are planned to approximate the solution [11]. Several IDs architectures have been proposed to be used in mobile networks. The most updated IDs in the last three years ago are summarized.

Md Nasir Sulaiman, in 2015 [24] proposed a new classifier to enhance the abnormal attacks detection rate based on support vector machine (SVM) and genetic programming (GP). Depending on the experimental results, GPSVM classifier is controled to earn higher detection rate on the scarce abnormal attacks, without vital reduction on the general accuracy. This can be as a result of, GPSVM optimization mission is to confirm the accuracy is balanced between classes without reducing the generalization property of SVM, by an average accuracy of 88.51% .

Shankar Sriram V S, in 2017 [22], presented an adaptive, and a strong IDs technique using Hypergraph based Genetic algorithm (HG - GA) for parameter setting and feature selection in Support Vector Machine (SVM). Hyper – clique property of hypergraph was exploited for the generation of initial population to fasten the search for the optimum answer and to stop the trap at the local minima. HG-GA uses a weighted objective function to keep up the trade-off between increasing the detection rate and minimizing the false alarm rate, in conjunction with the optimum range of features. The performance of HG-GA SVM was evaluated using NSL-KDD intrusion dataset under two situations (i) All features and (ii) informative features obtained from HG – GA, with the Accuracy rate is 97.14% and the false rate is 0.83%.

Muder Almi'ani in 2018 [19], designed an intelligent IDs using clustered version of Self-Organized Map (SOM) network. The planned system consists of two subsequent stages: 1st, SOM network was designed, then a hierarchical agglomerative clustering using k-means was applied on SOM neurons. The proposed work in this research addressed the issues of sensitivity and time consumption for every connection record process. The proposed system was demonstrated using NSL-KDD benchmark dataset, wherever it's achieved superior sensitivity reached up to 96.66 you uninterested in less than 0.08 milliseconds per connection record.

The researches In 2017 and 2018 [14 and 15] tend to plan a novel approach for classifing MANETs attacks with a neutrosophic intelligent system based on genetic algorithm. Neutrosophic system could be a discipline that produces a mathematical formulation for the indeterminacy found in such complex situations. Neutrosophic rules compute with symbols rather than numeric values creating a good base for symbolic reasoning. These symbols ought to be carefully designed as they form the propositions base for the neutrosophic rules (NR) in the IDs. Every attack is set by MEM, NON_MEM and I degrees in neutrosophic system. The research proposed a MANETs attack inference by a hybrid framework of Self Organized Features Maps (SOFM) and the Genetic Algorithms (GA). The hybrid uses the unsupervised learning capabilities of the SOFM to determine the MANETs neutrosophic conditional variables. The neutrosophic variables along with the training information set are fed into the GA to find the most match neutrosophic rule set from a number of initial sub attacks according to the neutrosophic correlation coeffcient as a fitness function. This technique is designed to discover unknown attacks in MANETs. The simulation and experimental results are conducted on the KDD-99 network attacks data available in the UCI machine-learning repository for further process in knowledge discovery. The experiments cleared the feasibility of the proposed hybrid by an average accuracy of 99.3608 and false rate is 0.089.

It is clear that the neutrosophic IDs generated by GA takes highest precision percentage in comparision to all three classification based algorithms. Figure 1 refere to the corresponding chart for the result obtained in Table 1. Figure 2 shows the performance of existing and proposed neutrosophic IDs algorithm based on false alarm rate (FAR). Therefore our proposed neutrosophic IDs Algorithm [14 and 15] effectively detects attack with less false alarm rate.

System name	Accuracy%	false rate%
GPSVM	88.51	0.76
HG-GA SVM	97.14	0.83
Clustered SOM	96.66	0.08
neutrosophic intelligent system based on genetic algorithm	99.3608	0.089

Table 1: Performance of Neutrosophic genetic algorithm vs. existing algorithms

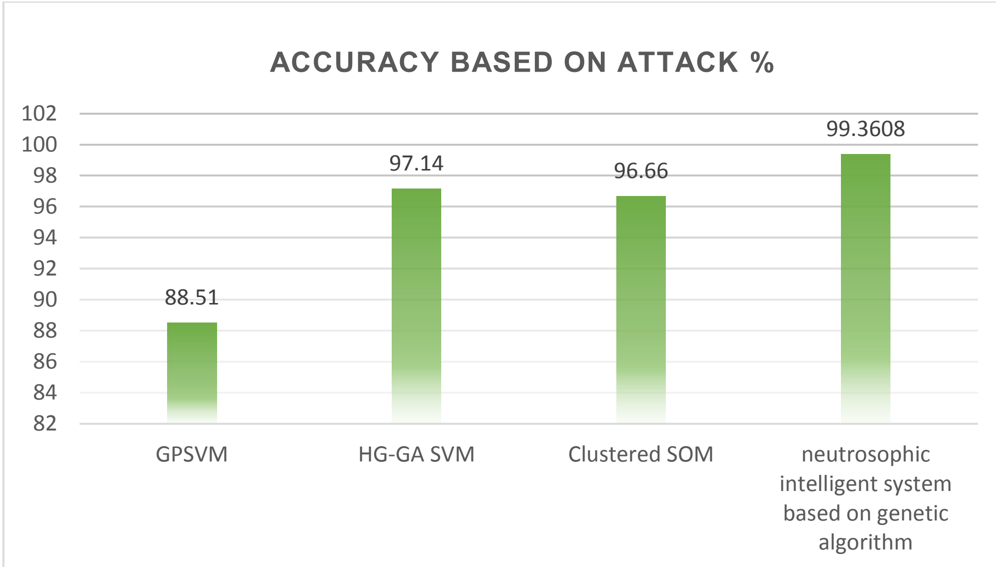

Figure 1: Results of neutrosophic genetic algorithm vs. existing algorithms

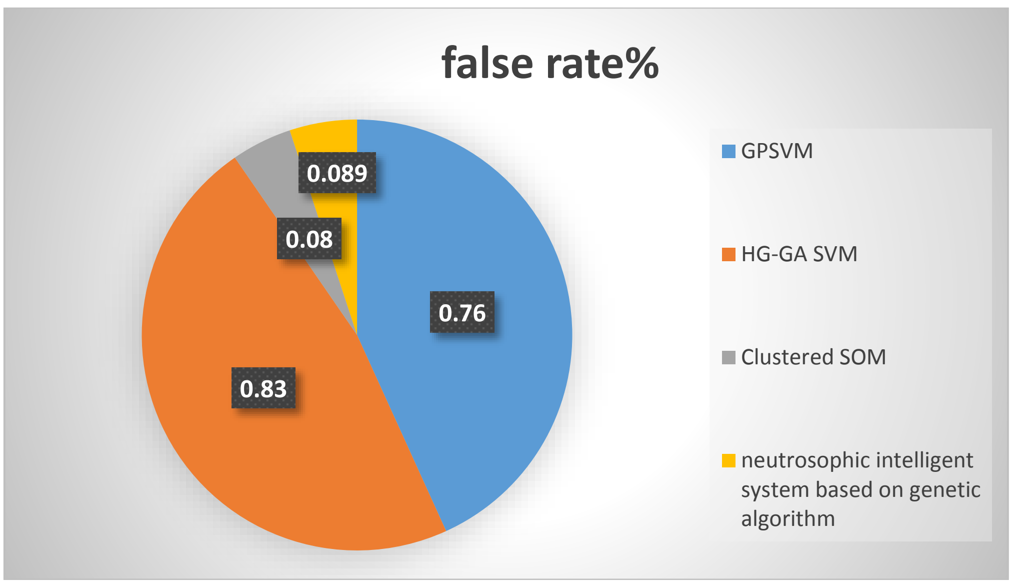

Figure 2. False alarm rate of proposed neutrosophic genetic algorithm vs. existing algorithms.

5. Conclusions

Detecting bad activities in MANETs might be a complicated mission because of the inherent features of those networks, like the mobility of the nodes, the shortage of a fixed design as well as the severe resource constraints. There's a pressing need to safeguard these communication networks and to propose efficient mechanisms in order to discover malicious behaviour. This article offers a comparison of the effectiveness of various classifiers which will use as intrusion detection algorithms in MANETs. Results show that Neutrosophic intrusion detection system relyied on GA could also be a good paradigm to use once the goal is to detect and to point that is the specific attack launched. The analysis of the classifiers is performed considering that the intrusion detection process is totally distributed and each node of the network hosts an independent intrusion detection agent.

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Neutrosophic Soft Cubic Set in Topological Spaces

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Abstract: This research article lays the foundation to propose the new concept of neutrosophic soft cubic topology. Here we focus on the systematic study of neutrosophic soft cubic sets and deduce various properties which are induced by them. This enables us to introduce some equivalent characterizations and brings out the inter relations among them.

Keywords: Neutrosophic Soft Cubic Set, Neutrosophic Soft Cubic Topological Space.

1 Introduction

Molodtsov [11]proposed the concept of soft set theory in 1999 which is an entirely new approach for modeling various forms of vagueness and uncertainty of real life situations. Soft set theory has a rich potential to penetrate itself in several direction which is clearly figured out in Molodtsov's pioneer work[11]. Zadeh [19] in 1965 came out with a novel concept of fuzzy set which deals with the degree of membership function between [0,1]. In [9] Maji et al.intiated the concept of fuzzy soft sets with some properties regarding fuzzy soft union , intersection, complement of fuzzy soft set. Moreover in [10] Maji et al. extended the idea of soft sets to Neutrosophic setting. Neutrosophic Logic has been proposed by Florentine Smarandache[15] which is based on nonstandard analysis that was given by Abraham Robinson[14] in 1960s.

Neutrosophic Logic was developed to represent mathematical model of uncertainty, vagueness, ambiguity, imprecision undefined, incompleteness, inconsistency,redundancy, contradiction. The neutrosophic logic is formal frame to measure truth, indeterminacy and falsehood values . In Neutrosophic set, indeterminacy is quantified explicitly and the truth membership, indeterminacy membership and falsity membership are independent. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors.

Jun et al.[17]presented the concept of cubic set by combining the fuzzy sets and interval valued fuzzy set. Y.B.Jun et al.[18] extended this idea under neutrosophic environment and named it Neutrosophic cubic sets. Wang et al.[16] introduced the concept of interval neutrosophic set and studied some of its properties.Chinnadurai [7] investigated some characterizations of its properties of Neutrosophic cubic soft set. Following him Pramanik et al.[13] introduced further operations and brought some of the properties of Neutrosophic cubic soft set.

As a further development Anitha et al.[1] proposed the notion of neutrosophic soft cubic set and defined internal neutrosophic soft cubic set, external neutrosophic soft cubic set and studied some new type of internal neutrosophic cubic set (INSCS) and external neutrosophic cubic set (ENSCS) namely, $\frac{1}{3}$ INSCS or $\frac{2}{3}$, ENSCS , $\frac{2}{3}$ INSCS or $\frac{1}{3}$ ENSCS.Anitha et al.[2-3] has discussed various operations on Neutrosophic soft cubic sets and investigated several related properties. In [4] the author has presented an application of Neutrosophic soft Cubic set in pattern recognition. Neutrosophic soft Cubic set theory was applied in BCI/BCK algebra[5]. In this paper we define neutrosophic soft cubic topological space and we discuss some of its properties.

2 Preliminaries

Definition 2.1. [19]
Let E be a universe. Then a fuzzy set over E is defined by $X = \{\mu_x(x)/x : x\epsilon E\}$ where μ_x is called membership function of X and defined by $\mu_x : E \to [0, 1]$. For each $x\epsilon E$, the value $\mu_x(x)$ represents the degree of x belonging to the fuzzy set X.

Definition 2.2. [17]
Let X be a non-empty set. By a cubic set, we mean a structure $\Xi = \{\langle A(x), \mu(x)\rangle | x\epsilon X\}$ in which A is an interval valued fuzzy set (IVF) and $\mu_(x)$ is a fuzzy set. It is denoted by $\langle A, \mu\rangle$.

Definition 2.3. [8]
Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let P(U) denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the soft neutrosophic set over U, where F is a mapping given by$F : A \to P(U)$.

Definition 2.4. [15]
Let X be an universe. Then a neutrosophic set(NS) λ is an object having the form $\lambda = \{\langle x : T(x), I(x), F(x)\rangle : x\varepsilon X\}$ where the functions $T, I, F : X \to]0, 1^+[$ define respectively the degree of Truth, the degree of indeterminacy, and the degree of falsehood of the element $x\varepsilon X$ to the set λ with the condition
$-0 \leq T(x) + I(x) + F(x) \leq 3+$.

Definition 2.5. [16]
Let X be a non-empty set. An interval neutrosophic set (INS) A in X is characterized by the truth-membership function A_T, the indeterminacy-membership function A_I and the falsity-membership function A_F. For each point $x\epsilon X, A_T(x), A_I(x), A_F(x) \subseteq [0, 1]$.

Definition 2.6. [1]
Let U be an the initial universal set. Let $NC(U)$ denote the set of all neutrosophic cubic sets and E be a set of parameters. Let M $\subseteq$ E then
$(P, M) = \{P(e) = \{\langle x, A_e(x), \lambda_e(x)\rangle : x\epsilon U\}e\epsilon M\}$
where $A_e(x) = \{\langle x, A^T_e(x), A^I_e(x), A^F_e(x)\rangle : x\epsilon U\}$ is an interval neutrosophic set and
$\lambda_e(x) = \{\langle x, \lambda^T_e(x), \lambda^I_e(x), \lambda^F_e(x)\rangle : x\epsilon U\}$ is an neutrosophic set.
Let P(U) denote the set of all neutrosophic cubic sets of U. The collection (P, M) is termed to be the neutrosophic soft cubic set over U, where F is a mapping given by F : A $\longrightarrow$ P(U).
The neutrosophic soft cubic set is denoted by NSCset / NSCS. The collection of all neutrosophic soft cubic set over U is denoted by NSCS(U).

3 Some Results On Neutrosophic Soft Cubic Set

Definition 3.1. Let (P , E) be neutrosophic soft cubic set over U.

(i) (P , E) is called absolute or universal neutrosophic soft cubic set U if $P(e) = \hat{1} = \{(\widetilde{1}, \widetilde{1}, \widetilde{1}), (1, 1, 1)\}$ for all $e \in$ E. We denote it by U.

(ii) (P , E) is called null or empty neutrosophic soft cubic set U if $P(e) = \hat{0} = \{(\widetilde{0}, \widetilde{0}, \widetilde{0})(0, 0, 0)\}$ for all $e \in$ E. We denote it by Φ.
Obviously $\Phi^c = U$ and $U^c = \Phi$.

Definition 3.2. A neutrosophic soft cubic set (P , M) is said to be a subset of a neutrosophic soft cubic set (Q , N) if $M \subseteq N$ and $P(e) \subseteq Q(e)$ ∀ e ∈ M , x ∈ U if and only if $A_e(x) \subseteq B_e(x)$ and $\lambda_e(x) \subseteq \mu_e(x)$ ∀ e ∈ M , u ∈ U . We denote it by $(P, M)\widetilde{\subseteq}$ (Q , N). where $P(e) \subseteq Q(e)$ ∀ e ∈ M , u ∈ U if and only if $A_e(x) \subseteq B_e(x)$ and $\lambda_e(x) \subseteq \mu_e(x)$,

$$\begin{aligned} A_e(x) \subseteq B_e(x) \quad &\Longrightarrow \quad A_e^{-T}(x) \leq B_e^{-T}(x), \\ &\quad\quad A_e^{+T}(x) \leq B_e^{+T}(x), \\ &\quad\quad A_e^{-I}(x) \geq B_e^{-I}(x), \\ &\quad\quad A_e^{+I}(x) \geq B_e^{+I}(x), \\ &\quad\quad A_e^{-F}(x) \geq B_e^{-F}(x), \\ &\quad\quad A_e^{+F}(x) \geq B_e^{+F}(x) \\ \text{and } \lambda_e(x) \subseteq \mu_e(x) \quad &\Longrightarrow \quad \lambda_e^{T}(x) \leq \mu_e^{T}(x), \\ &\quad\quad \lambda_e^{I}(x) \geq \mu_e^{I}(x), \\ &\quad\quad \lambda_e^{F}(x) \geq \mu_e^{F}(x), \end{aligned}$$

Definition 3.3. The complement of neutrosophic soft cubic set (P , M) is denoted by $(P, M)^c$ and defined as $(P, M)^c = (P^c, \neg M)$ where $P^c : \neg A \longrightarrow P(U)$ is a mapping given by

$$\begin{aligned} (P, M)^c &= \{\langle x, \tilde{A}_e^c(x), \lambda_e^c(x)\rangle : x\epsilon U\}e\epsilon M\} \\ &= \{\langle x, 1 - \tilde{A}_e^T(x), 1 - \tilde{A}_e^I(x), 1 - \tilde{A}_e^F(x), \\ &\quad 1 - \lambda_e^T(x), 1 - \lambda_e^I(x), 1 - \lambda_e^F(x)\rangle : x\epsilon U\}e\epsilon M\} \\ &= \{\langle x, 1 - A_e^{+T}, 1 - A_e^{-T}](x), [1 - A_e^{+I}, 1 - A_e^{-I}](x), \\ &\quad [1 - A_e^{-F}, 1 - A_e^{+F}](x), 1 - \lambda_e^T(x), 1 - \lambda_e^I(x), \\ &\quad 1 - \lambda_e^F(x)\rangle : x\epsilon U\}e\epsilon M\} \end{aligned}$$

Remark 3.4. The complement of a neutrosophic soft cubic set (P , M) can also be defined as $(P, M)^c = U \backslash P(e)$ for all $e \in M$.

Definition 3.5. The union of two neutrosophic soft cubic sets (P, M)=$\{\langle x, A_e(x), \lambda_e(x)\rangle : x\epsilon U\}e\epsilon M\}$ and (Q, N)=$\{\langle x, A_e(x), \lambda_e(x)\rangle : x\epsilon U\}e\epsilon N\}$ over (U,E) is neutrosophic soft cubic set where C = M ∪ N ,$\forall e \in C$

$$H(e) = \begin{cases} P(e) & \text{if e} \in A - B \\ Q(e) & \text{if e} \in B - A \\ P(e) \cup Q(e) & \text{if e} \in A \cap B \end{cases}$$

and is written as (P , M)∪(Q , N)= (H,C).
where $H(e) = \langle x, max(\tilde{A}_e(x), \tilde{B}_e(x)), max(\lambda_e(x), \mu_e(x))\rangle$.

Definition 3.6. The intersection of two neutrosophic soft cubic sets (P , M)and (Q , N) over (U,E) is neutrosophic soft cubic set where C = M ∩ N ,$\forall e \in C$
$H(e) = P(e) \cap Q(e)$ and is written as (P , M)∩(Q , N)= (H,C).
where $H(e) = \langle x, min(\tilde{A}_e(x), \tilde{B}_e(x)), min(\lambda_e(x), \mu_e(x))\rangle$.

Definition 3.7. If (P , M) and (Q , N) be two neutrosophic soft cubic sets then (P , M) AND (Q , N) is a NSCS denoted by
(P , M) ∧ (Q , N) and is defined by (P , M) ∧ (Q , N) = (H,M× N), where (H,A× B)= $P(\alpha_i) \cap F(\beta_i)$ the truth membership, indeterminacy membership and the falsity membership of (H,A× B) are as follows:
$H^T(a_i, b_i)(h) = min(\tilde{A}^T(a)(h), \tilde{A}^T(b)(h)), min(\lambda^T(a)(h), \lambda^T(b)(h))$
$H^I(a_i, b_i)(h) = min(\tilde{A}^I(a)(h), \tilde{A}^I(b)(h)), min(\lambda^I(a)(h), \lambda^I(b)(h))$ and
$H^F(a_i, b_i)(h) = min(\tilde{A}^F(a)(h), \tilde{A}^F(b)(h)), min(\lambda^F(a)(h), \lambda^F(b)(h))$
$forall(a_i, b_i)\epsilon M \times N.$

Definition 3.8. [2] If (P , M) and (Q , N) be two Neutrosophic soft cubic sets then (P , M)OR (Q , N) is a NSCS denoted by
(P , M) ∨ (Q , N) and is defined by (P , M) ∨ (Q , N) = (H,A× B), where the truth membership, indeterminacy membership and the falsity membership of (H,A× B) are as follows:
$H^T(a_i, b_i)(h) = max(\tilde{A}^T(a)(h), \tilde{A}^T(b)(h)), max(\lambda^T(a)(h), \lambda^T(b)(h))$
$H^I(a_i, b_i)(h) = max(\tilde{A}^I(a)(h), \tilde{A}^I(b)(h)), max(\lambda^I(a)(h), \lambda^I(b)(h))$ and
$H^F(a_i, b_i)(h) = max(\tilde{A}^F(a)(h), \tilde{A}^F(b)(h)), max(\lambda^F(a)(h), \lambda^F(b)(h))$
$forall(a_i, b_i)\epsilon M \times N.$

Proposition 3.9. Let U be an initial universal set and E be a set of parameters.Let (P , E) and (Q , E) be NSCS over U. Then the following are true.

(i) (P , E)⊆(Q , E) iff (P , E)∩(Q , E)= (P , E)

(ii) (P , E)⊆(Q , E) iff (P , E)∪(Q , E)= (Q , E)

Proof:

(i) Suppose that (P , E) ⊆(Q , E) then P(e)⊆Q(e) for all $e \in E$.
Let (P , E)∩(Q , E)=(H , E). Since H(e)=P(e)∩ Q(e) = P(e)for all $e \in E$ then (H , E)= (P , E).
Suppose that (P , E)∩ (Q , E) = (P , E) and let (P , E)∩(Q , E)= (H , E).
Since H(e)= P(e) ∩ Q(e) for all $e \in E$, we know that P(e)⊆ Q(e) for all $e \in E$.
Hence (P , E)⊆(Q , E).

(ii) Suppose that (P , E) ⊆(Q , E) then P(e)⊆Q(e) for all $e \in E$.
Let (P , E)∪(Q , E)=(H , E). Since H(e)=P(e)∪ Q(e) = Q(e)for all $e \in E$ then (H , E)= (Q , E).
Suppose that (P , E)∪ (Q , E) = (Q , E) and let (P , E)∪(Q , E)= (H , E).
Since H(e)= P(e) ∪ Q(e) for all $e \in E$, we know that P(e)⊆ Q(e) for all $e \in E$.
Hence (P , E)⊆(Q , E)..

Proposition 3.10. Let U be an initial universal set and E be a set of parameters.Let (P , E), (Q , E), (H , E) and (K , E) be NSCS over U. Then the following are true.

(i) If (P , E) ∩ (Q , E) = Φ, then (P , E)⊆(Q , E)^c

(ii) If (P , E) ⊆ (Q , E) and (Q , E) ⊆ (H , E) then (P , E) ⊆ (H , E)

(iii) If (P , E) ⊆ (Q , E) and (H , E) ⊆ (K , E) then (P , E) ∩ (H , E)⊆ (Q , E) ∩ (K , E)

(iv) (P , E) ⊆ (Q , E) iff (Q , E)^c ⊆ (P , E)^c

Proof:

(i) Suppose that $(P, E)\cap(Q, E) = \Phi$, then P(e)∩ Q(e) = Φ.
So $P(e) \subseteq U \setminus Q(e) = Q^c(e)$ for all $e \in E$.
Therefore we have $(P, E)\subseteq(Q, E)^c$.

(ii) (P , E) ⊆ (Q , E) and (Q , E) ⊆ (H , E)
⇒ P(e)⊆ Q(e) and Q(e)⊆ H(e)for all $e \in E$
⇒ P(e)⊆ Q(e)⊆ H(e)for all $e \in E$
⇒ P(e)⊆ H(e) for all $e \in E$.

(iii) (P , E) ⊆ (Q , E) and (H , E) ⊆ (K , E)for all $e \in E$.
⇒ P(e)⊆ Q(e) and H(e)⊆ K(e)for all $e \in E$
⇒ (P , E) ∩ (H , E) ⊆ (Q , E) ∩ (K , E)for all $e \in E$..

(iv) (P , E) ⊆ (Q , E) ⇔ P(e)⊆ Q(e) for all $e \in E$
⇔ (Q(e))^c ⊆ (P(e))^c for all $e \in E$
⇔ Q ^{c}(e) ⊆ P ^{c}(e) for all $e \in E$
⇔ (Q , E)^c ⊆ (P , E)^c.

Definition 3.11. Let I be an arbitrary indexed set and $\{(P_i, E)\}_{i\in I}$ be a subfamily of NSCS over U with parameter set E.

(i) The union of these NSCS is the NSCS (H , E) where $H(e) = \bigcup_{i\in I} P_i(e)$ for each $e \in E$. We write $\bigcup_{i\in I}(P_i, E) = (H, E)$.

(ii) The intersection of these NSCS is the NSCS (K , E) where $K(e) = \bigcap_{i\in I} P_i(e)$ for each $e \in E$. We write $\bigcap_{i\in I}(P_i, E) = (K, E)$.

Proposition 3.12. Let I be an arbitrary indexed set and $\{(P_i, E)\}_{i\in I}$ be a subfamily of NSCS over U with parameter set E. Then

(i) $\left(\bigcup_{i\in I}(P_i, E)\right)^c = \bigcap_{i\in I}(P_i, E)^c$

(ii) $\left(\bigcap_{i\in I}(P_i, E)\right)^c = \bigcup_{i\in I}(P_i, E)^c$

Proof:

(i) Let $\left(\bigcup_{i\in I}(P_i,E)\right)=(H,E)$ implies $\left(\bigcup_{i\in I}(P_i,E)\right)^c=(H,E)^c$.
Then $H^c(e)=U\setminus H(e)=U\setminus\bigcup_{i\in I}P_i(e)=\bigcap_{i\in I}(U\setminus P_i(e))$ for all $e\in E$ ——(1).
On the otherhand, $\bigcap_{i\in I}(P_i,E)^c=(K,E)$, by definition
$K(e)=\bigcap_{i\in I}P_i^c(e)=\bigcap_{i\in I}(U\setminus P_i(e))$ for all $e\in E$ ——(2).
From (1) and (2) we have the result.

(ii) Let $\left(\bigcap_{i\in I}(P_i,E)\right)=(H,E)$ implies $\left(\bigcap_{i\in I}(P_i,E)\right)^c=(H,E)^c$.
Then $H^c(e)=U\setminus H(e)=U\setminus\bigcap_{i\in I}P_i(e)=\bigcup_{i\in I}(U\setminus P_i(e))$ for all $e\in E$ ——(1).
On the otherhand, $\bigcup_{i\in I}(P_i,E)^c=(K,E)$, by definition
$K(e)=\bigcup_{i\in I}P_i^c(e)=\bigcup_{i\in I}(U\setminus P_i(e))$ for all $e\in E$ ——(2).
From (1) and (2) we have the result.

Proposition 3.13. Let U be an initial universal set and E be a set of parameters.

(i) $(\Phi,E)^c=(U,E)$.

(ii) $(U,E)^c=(\Phi,E)$.

Proof:

(i) Let $(\Phi,E)=(P,E)$, then for all $e\in E$,

$$\begin{aligned}
P(e) &= \{(x,\tilde{A}_e^T(x),\tilde{A}_e^I(x),\tilde{A}_e^F(x)),\\
&\quad \lambda_e^T(x),\lambda_e^I(x),\lambda_e^F(x):x\in U\}\\
&= \{x,(\tilde{0},\tilde{0},\tilde{0})(0,0,0);x\in U\}\\
(\Phi,E)^c &= (P,E)^c, \text{then for all} e\in E\\
&= \{(x,\tilde{A}_e^T(x),\tilde{A}_e^I(x),\tilde{A}_e^F(x)),\\
&\quad \lambda_e^T(x),\lambda_e^I(x),\lambda_e^F(x):x\in U\}^c\\
&= \{x,1-\tilde{A}_e^T(x),1-\tilde{A}_e^I(x),\\
&\quad 1-\tilde{A}_e^F(x),1-\lambda^T(e)(x),\\
&\quad 1-\lambda_e^I(x),1-\lambda_e^F(x):x\in U\}\\
&= \{x,[1-A_e^{+T},1-A_e^{-T}](x),\\
&\quad [1-A_e^{+I},1-A_e^{-I}](x),\\
&\quad [1-A_e^{-F},1-A_e^{+F}](x),\\
&\quad 1-\lambda^T(e)(x),1-\lambda_e^I(x),\\
&\quad 1-\lambda_e^F(x):x\in U\}\\
&= \{(x,(\hat{1},\hat{1},\hat{1}),(1,1,1));x\in U\}.
\end{aligned}$$

$Thus(\Phi,E)^c=(U,E).$

(ii) Let $(U, E) = (P, E)$, then for all $e \in E$,

$$\begin{aligned} P(e) &= \{(x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \\ &\quad \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x) : x \in U\} \\ &= \{x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1); x \in U\} \\ (U, E)^c &= (P, E)^c, then for all e \in E \\ &= \{(x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \\ &\quad \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x) : x \in U\}^c \\ &= \{x, 1 - \tilde{A}_e^T(x), 1 - \tilde{A}_e^I(x), \\ &\quad 1 - \tilde{A}_e^F(x), 1 - \lambda^T(e)(x), \\ &\quad 1 - \lambda_e^I(x), 1 - \lambda_e^F(x) : x \in U\} \\ &= \{x, [1 - A_e^{+T}, 1 - A_e^{-T}](x), \\ &\quad [1 - A_e^{+I}, 1 - A_e^{-I}](x), \\ &\quad [1 - A_e^{-F}, 1 - A_e^{+F}](x), \\ &\quad 1 - \lambda^T(e)(x), 1 - \lambda_e^I(x), \\ &\quad 1 - \lambda_e^F(x) : x \in U\} \\ &= \{(x, (\hat{0}, \hat{0}, \hat{0}), (0,0,0)); x \in U\}. \\ Thus (U, E)^c = (\Phi, E) & \end{aligned}$$

Proposition 3.14. Let U be an initial universal set and E be a set of parameters.

(i) $(P, E) \cup (\Phi, E) = (P, E)$.

(ii) $(P, E) \cup (U, E) = (U, E)$.

Proof:

(i) $(P, E) = \{(x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)) : x \in U\}\ \forall e \in E$
$(\Phi, E) = \{(x, (\tilde{0}, \tilde{0}, \tilde{0})(0,0,0)) : x \in U\}\ \forall e \in E$
$(P, E) \cup (\Phi, E)$
$= \{(x, max(\tilde{A}_e^T(x), \tilde{0}), max(\tilde{A}_e^I(x), \tilde{0}), max(\tilde{A}_e^F(x), \tilde{0}), max(\lambda_e^T(x), 0), max(\lambda_e^I(x), 0), max(\lambda_e^F(x), 0) :$
$x \in U\}\ \forall e \in E$
= $\{(x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)); x \in U\}\ \forall e \in E$
= (P, E).
Thus $(P, E) \cup (\Phi, E) = (P, E)$

(ii) $(P, E) = \{(x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)) : x \in U\}\ \forall e \in E$
$(\Phi, E) = \{(x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)) : x \in U\}\ \forall e \in E$
$(P, E) \cup (U, E)$
$= \{(x, max(\tilde{A}_e^T(x), \tilde{1}), max(\tilde{A}_e^I(x), \tilde{1}), max(\tilde{A}_e^F(x), \tilde{1}), max(\lambda_e^T(x), 1), max(\lambda_e^I(x), 1), max(\lambda_e^F(x), 1)) :$
$x \in U\}\ \forall e \in E$
= $\{(x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)) : x \in U\}\ \forall e \in E$
= (U, E).
Thus $(P, E) \cup (U, E) = (U, E)$

Proposition 3.15. Let U be an initial universal set and E be a set of parameters.

(i) $(P, E) \cap (\Phi, E) = (\Phi, E)$.

(ii) $(P, E) \cap (U, E) = (P, E)$.

Proof:

(i) $(P, E) = \{e, (x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)) : x \in U\} \forall e \in E$
$(\Phi, E) = \{e, (x, (\tilde{0}, \tilde{0}, \tilde{0})(0, 0, 0)); x \in U\} \forall e \in E$
$(P, E) \cap (\Phi, E) = \{e, (x, min(\tilde{A}_e^T(x), \tilde{0}), min(\tilde{A}_e^I(x), \tilde{0}), min(\tilde{A}_e^F(x), \tilde{0}),$
$min(\lambda_e^T(x), 0), min(\lambda_e^I(x), 0), min(\lambda_e^F(x), 1) : x \in U\} \forall e \in E$
= $\{e, (x, (\tilde{0}, \tilde{0}, \tilde{0})(0, 0, 0)); x \in U\} \forall e \in E$ = (Φ, E).
Thus $(P, E) \cap (\Phi, E) = (\Phi, E)$

(ii) $(P, E) = \{e, (x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)) : x \in U\} \forall e \in E$
$(U, E) = \{e, (x, (\tilde{1}, \tilde{1}, \tilde{1})(1, 1, 1)); x \in U\} \forall e \in E$
$(P, E) \cap (U, E) = \{e, (x, min(\tilde{A}_e^T(x), \tilde{1}), min(\tilde{A}_e^I(x), \tilde{1}), min(\tilde{A}_e^F(x), \tilde{1}),$
$min(\lambda_e^T(x), 1), min(\lambda_e^I(x), 1), min(\lambda_e^F(x), 1) : x \in U\} \forall e \in E$
=$\{e, (x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)) : x \in U\} \forall e \in E$.
Thus $(P, E) \cap (U, E) = (P, E)$

Proposition 3.16. Let U be an initial universal set , E be a set of parameters and $M, N \subseteq E$.

(i) $(P, M) \cup (\Phi, N) = (P, M)$ iff $N \subseteq M$.

(ii) $(P, M) \cup (U, N) = (U, N)$ iff $M \subseteq N$.

Proof:

(i) For (P , M) we have
$P(e) = \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\}\ \forall e \in M$
Also let $(\Phi, N) = (Q, N)$, then
$Q(e) = \{\langle x, (\tilde{0}, \tilde{0}, \tilde{0})(0, 0, 0)\rangle; x \in U\} \forall e \in N$
Let $(P, M) \cup (\Phi, N)$ = $(P, M) \cup (Q, N)$ = (H, C)
where $C = M \cup N$ and for all $e \in C$

$$H(e) = \begin{cases} P(e) & \text{if e} \in M - N \\ Q(e) & \text{if e} \in N - M \\ P(e) \cap Q(e) & \text{if e} \in M \cap N \end{cases}$$

$$H(e) = \begin{cases} \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), & \\ \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\}, & \text{if e} \in M - N \\ \{\langle x, \tilde{B}_e^T(x), \tilde{B}_e^I(x), \tilde{B}_e^F(x), & \\ \mu_e^T(x), \mu_e^I(x), \mu_e^F(x)\rangle : x \in U\} & \text{if e} \in N - M \\ \{\langle x, max(\tilde{A}_e(x), \tilde{B}_e(x)), & \\ max(\lambda_e(x), \mu_e(x))\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

$$H(e)=\begin{cases}\{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x),\\ \lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\} & \text{if e} \in M-N\\ \{\langle x,(\tilde{0},\tilde{0},\tilde{0})(0,0,0)(x)\rangle : x\in U\} & \text{if e} \in N-M\\ \{\langle x,max(\tilde{A}^T_e(x),\tilde{0}),\\ max(\tilde{A}^I_e(x),\tilde{0}),\\ max(\tilde{A}^F_e(x),\tilde{0}),\}\\ \{max(\lambda_e(x),0),max(\lambda_e(x),0),\\ max(\lambda_e(x),0\rangle : x\in U\} & \text{if e} \in M\cap N\end{cases}$$

$$H(e)=\begin{cases}\{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x),\\ \lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\}, & \text{if e} \in M-N\\ \{\langle x,(\tilde{0},\tilde{0},\tilde{0})(0,0,0)(x)\rangle : x\in U\} & \text{if e} \in N-M\\ \{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x),\\ \lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\} & \text{if e} \in M\cap N\end{cases}$$

Let $N\subseteq M$, then

$$H(e)=\begin{cases}\{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x),\\ \lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\}, & \text{if e} \in M-N\\ \{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x),\\ \lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\} & \text{if e} \in M\cap N\end{cases}$$

=P(e),$\forall e\in M$.
Conversely, Let $(P,M)\cup(\Phi,N)=(P,M)$
Then $M=M\cup N\Rightarrow N\subseteq M$

(ii) For (P , M) we have
$P(e)=\{\langle x,\tilde{A}^T_e(x),\tilde{A}^I_e(x),\tilde{A}^F_e(x)),\lambda^T_e(x),\lambda^I_e(x),\lambda^F_e(x)\rangle : x\in U\}\ \forall e\in M$
Also let $(U,N)=(Q,N)$, then
$Q(e)=\{\langle x,(\tilde{1},\tilde{1},\tilde{1})(1,1,1)\rangle ; x\in U\}\forall e\in N$
Let $(P,M)\cup(U,N)$ = $(P,M)\cup(Q,N)$ = (H,C)
where $C=M\cup N$ and for all $e\in C$

$$H(e)=\begin{cases}P(e) & \text{if e} \in M-N\\ Q(e) & \text{if e} \in N-M\\ P(e)\cup Q(e) & \text{if e} \in M\cap N\end{cases}$$

$$H(e) = \begin{cases} \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \\ \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\}, & \text{if e} \in M - N \\ \{\langle x, \tilde{B}_e^T(x), \tilde{B}_e^I(x), \tilde{B}_e^F(x), \\ \mu_e^T(x), \mu_e^I(x), \mu_e^F(x)\rangle : x \in U\} & \text{if e} \in N - M \\ \{\langle x, max(\tilde{A}_e(x), \tilde{B}_e(x)), \\ max(\lambda_e(x), \mu_e(x))\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

$$H(e) = \begin{cases} \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \\ \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\} & \text{if e} \in M - N \\ \{\langle x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)(x)\rangle : x \in U\} & \text{if e} \in N - M \\ \{\langle x, max(\tilde{A}_e^T(x), \tilde{1}), \\ max(\tilde{A}_e^I(x), \tilde{1}), \\ max(\tilde{A}_e^F(x), \tilde{1}), \} \\ \{max(\lambda_e(x), 1), max(\lambda_e(x), 1), \\ max(\lambda_e(x), 1\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

$$H(e) = \begin{cases} \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \\ \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\}, & \text{if e} \in M - N \\ \{\langle x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)(x)\rangle : x \in U\} & \text{if e} \in N - M \\ \{\langle x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)(x)\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

Let $M \subseteq N$, then

$$H(e) = \begin{cases} \{\langle x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)(x)\rangle : x \in U\}, & \text{if e} \in N - M \\ \{\langle x, (\tilde{1}, \tilde{1}, \tilde{1})(1,1,1)(x)\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

=(U, N),$\forall e \in M$.
Conversely, Let $(P, M) \cup (U, N) = (U, N)$
Then $N = M \cup N \Rightarrow M \subseteq N$

Proposition 3.17. Let U be an initial universal set , E be a set of parameters and $M, N \subseteq E$.

(i) $(P, M) \cap (\Phi, N) = (\Phi, M \cap N)$.

(ii) $(P, M) \cap (U, N) = (P, M \cap N)$.

Proof:

(i) For (P , M) we have
$P(e) = \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x) \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \epsilon U\}\ e \epsilon M$
Also let $(\Phi, N) = (Q, N)$, then
$Q(e) = \{x, (\tilde{0}, \tilde{0}, \tilde{0}), (0, 0, 0); x \in U\}\ \forall e \in N$
Let $(P, M) \cap (\Phi, N)$ = $(P, M) \cap (Q, N)$ = (H, C)
where $C = M \cap N$ and for all $e \in C$

$H(e) = \left\{ \langle x, min(\tilde{A}_e(x), \tilde{B}_e(x)), min(\lambda_e(x), \mu_e(x)) \rangle : x \in U \right\}$
$= \{(x, min(\tilde{A}_e^T(x), \tilde{0}), min(\tilde{A}_e^I(x), \tilde{0}), min(\tilde{A}_e^F(x), \tilde{0}),$
$min(\lambda_e^T(x), 0), min(\lambda_e^I(x), 0), min(\lambda_e^F(x), 0)) : x \in U\}$
$= \{(x, (\tilde{0}, \tilde{0}, \tilde{0})(0, 0, 0)); x \in U\}$ for all $e \in C$.
= (Q , C) = (Φ, C).
Thus $(P, M)\cap(\Phi, N) = (\Phi, M \cap N)$.

(ii) For (P , M) we have
$P(e) = \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)\lambda_e^T(x), \lambda_e^I(x), \lambda_e^P(x)\rangle : x\epsilon U\}\ e\epsilon M$
Also let $(U, N) = (Q, N)$, then
$Q(e) = \{x, (\tilde{1}, \tilde{1}, \tilde{1}), (1, 1, 1); x \in U\}\ \forall e \in N$
Let $(P, M)\cap(U, N)$ = $(P, M)\cap(Q, N)$ = (H, C)
where $C = M \cap N$ and for all $e \in C$
$H(e) = \left\{ \langle x, min(\tilde{A}_e(x), \tilde{B}_e(x)), min(\lambda_e(x), \mu_e(x)) \rangle : x \in U \right\}$
$= \{(x, min(\tilde{A}_e^T(x), \tilde{1}), min(\tilde{A}_e^I(x), \tilde{1}), min(\tilde{A}_e^F(x), \tilde{1}),$
$min(\lambda_e^T(x), 1), min(\lambda_e^I(x), 1), min(\lambda_e^F(x), 1)) : x \in U\}$
= $\{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x)\lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x\epsilon U\}$ for all $e \in C$
= $P(e) = (P, C)$.
Thus $(P, M)\cap(U, N) = (P, M \cap N)$.

Proposition 3.18. Let U be an initial universal set, E be a set of parameters and $M, N \subseteq E$.

(i) $((P, M)\cup(Q, N))^c \subseteq (P, M)^c \cup (Q, N)^c$.

(ii) $(P, M)^c \cap (Q, N)^c \subseteq ((P, M)\cap(Q, N))^c$.

Proof:

(i) Let $(P, M)\cup(Q, N) = (H, C)$ where $C = M \cup N$ and $\forall e \in C$

$$H(e) = \begin{cases} \{\langle x, \tilde{A}_e^T(x), \tilde{A}_e^I(x), \tilde{A}_e^F(x), \lambda_e^T(x), \lambda_e^I(x), \lambda_e^F(x)\rangle : x \in U\}, & \text{if e} \in M - N \\ \{\langle x, \tilde{B}_e^T(x), \tilde{B}_e^I(x), \tilde{B}_e^F(x), \mu_e^T(x), \mu_e^I(x), \mu_e^F(x)\rangle : x \in U\} & \text{if e} \in N - M \\ \{\langle x, max(\tilde{A}_e(x), \tilde{B}_e(x)), max(\lambda_e(x), \mu_e(x))\rangle : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

Thus $((P, M)\cup(Q, N))^c = (H, C)^c$, where $C = M \cup N$ and $\forall e \in C$

$$(H(e))^c = \begin{cases} \{(P(e))^c, & \text{if e} \in M - N \\ \{(Q(e))^c & \text{if e} \in N - M \\ \{(P(e) \cup Q(e))^c & \text{if e} \in N \cap M \end{cases}$$

$$= \begin{cases} \{(x, 1 - \tilde{A}_e^T(x), 1 - \tilde{A}_e^I(x), 1 - \tilde{A}_e^F(x), \\ 1 - \lambda^T(e)(x), 1 - \lambda_e^I(x), 1 - \lambda_e^F(x) : x \in U\} & \text{if e} \in M - N \\ \{(x, 1 - \tilde{B}_e^T(x), 1 - \tilde{B}_e^I(x), 1 - \tilde{B}_e^F(x), \\ 1 - \mu^T(e)(x), 1 - \mu_e^I(x), 1 - \mu_e^F(x) : x \in U\} & \text{if e} \in N - M \\ \{(x, 1 - max(\tilde{A}_e^T(x), \tilde{B}_e^T(x)), 1 - max(\tilde{A}_e^I(x), \tilde{B}_e^I(x)), 1 - max(\tilde{A}_e^F(x), \tilde{B}_e^F(x)), \\ 1 - max(\lambda_e^T(x), \mu_e^T(x)), 1 - max(\lambda_e^I(x), \mu_e^I(x)), 1 - max(\lambda_e^F(x), \mu_e^F(x)) : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

Again $(P,M)^c \cup (Q,N)^c = (I,J)$ say J = M ∪ N and $\forall e \in J$

$$I(e) = \begin{cases} \{(P(e))^c, & \text{if e} \in M-N \\ \{(Q(e))^c & \text{if e} \in N-M \\ \{(P(e) \cup Q(e))^c & \text{if e} \in M \cap N \end{cases}$$

$$= \begin{cases} \{(x, 1-P_e^T(x), 1-P_e^I(x), 1-P_e^F(x)) : x \in U\}, & \text{if e} \in M-N \\ \{(x, 1-Q_e^T(x), 1-Q_e^I(x), 1-Q_e^F(x)) : x \in U\} & \text{if e} \in N-M \\ \{(x, 1-max(\tilde{A}_e^T(x), \tilde{B}_e^T(x)), 1-max(\tilde{A}_e^I(x), \tilde{B}_e^I(x)), 1-max(\tilde{A}_e^F(x), \tilde{B}_e^F(x)), & \\ 1-max(\lambda_e^T(x), \mu_e^T(x)), 1-max(\lambda_e^I(x), \mu_e^I(x)), 1-max(\lambda_e^F(x), \mu_e^F(x)) : x \in U\} & \text{if e} \in M \cap N \end{cases}$$

C ⊆ J $\forall e \in J$. $(H(e))^c \subseteq I(e)$. Thus $((P,M) \cup (Q,N))^c \subseteq (P,M)^c \cup (Q,N)^c$

(ii) Let $(P,M) \cap (Q,N) = (H,C)$ where $C = M \cap N$ and $\forall e \in C$ and
H(e)= P(e)∩ Q(e) =
$\{(x, min(P_e^T(x), Q_e^T(x)), min(P_e^I(x), Q_e^I(x)), \min(P_e^F(x), Q_e^F(x)))\}$
where
$min(P_e^T(x), Q_e^T(x))= min(\tilde{A}_e^T(x), \tilde{B}_e^T(x)), min(\lambda_e^T(x), \mu_e^T(x)),$
$min(P_e^I(x), Q_e^I(x)) = min(\tilde{A}_e^I(x), \tilde{B}_e^I(x)), min(\lambda_e^I(x), \mu_e^I(x))$
$min(P_e^F(x), Q_e^F(x)))=min(\tilde{A}_e^F(x), \tilde{B}_e^F(x)), min(\lambda_e^F(x), \mu_e^F(x))$
Thus $((P,M) \cap (Q,N))^c = (H,C)^c$, where $C = M \cap N$ and $\forall e \in C$
$(H(e))^c = \{(x, min(P_e^T(x), Q_e^T(x)), min(P_e^I(x), Q_e^I(x)), min(P_e^F(x), Q_e^F(x)))\}^c$
$= \{(x, 1-min(P_e^T(x), Q_e^T(x)), 1-min(P_e^I(x), Q_e^I(x)), 1-min(P_e^F(x), Q_e^F(x)))\}$
Again $(P,M)^c \cap (Q,N)^c = (I,J)$ say J = M ∩ N and $\forall e \in J$
$I(e) = (P(e))^c \cap (Q(e))^c$
$= \{(x, 1-min(P_e^T(x), Q_e^T(x)), 1-min(P_e^I(x), Q_e^I(x)), 1-min(P_e^F(x), Q_e^F(x)))\}$
We see that C = J and $\forall e \in J$, I(e)⊆ $(H(e))^c$.
Thus $(P,M)^c \cap (Q,N)^c \subseteq ((P,M) \cup (Q,N))^c$.

Proposition 3.19. (De Morgan's Law) For neutrosophic soft cubic sets (P , E)and (Q , E) over the same universe U with parameter set E, we have the following.

(i) $((P,E) \cup (Q,E))^c = (P,E)^c \cap (Q,E)^c$.

(ii) $((P,E) \cap (Q,E))^c = (P,E)^c \cup (Q,E)^c$.

Proof:

(i) Let $(P,E) \cup (Q,E) = (H,E)$ where $\forall e \in E$
$H(e) = P(e) \cup Q(e)$
$= \{(x, max(P_e^T(x), Q_e^T(x)), max(P_e^I(x), Q_e^I(x)), max(P_e^F(x), Q_e^F(x)))\}$
Thus $((P,E) \cup (Q,E))^c = (H,E)^c$ where $\forall e \in E$

$H(e)^c = (P(e) \cup Q(e))^c$
$=\{(x, max(P_e^T(x), Q_e^T(x)), max(P_e^I(x), Q_e^I(x)), max(P_e^F(x), Q_e^F(x)))\}^c$
$= \{(x, 1-max(P_e^T(x), Q_e^T(x)), 1-max(P_e^I(x), Q_e^I(x)), 1-max(P_e^F(x), Q_e^F(x)))\}$
Again $(P,E)^c \cap (Q,E)^c = (I,E)$ say $\forall e \in E$

$I(e) = (P(e))^c \cap (P(e))^c$
$= \{(x, min(1 - P_e^T(x), 1 - Q_e^T(x)), min(1 - P_e^I(x), 1 - Q_e^I(x)), min(1 - P_e^F(x), 1 - Q_e^F(x)))\}^c$
$= \{(x, 1 - max(P_e^T(x), Q_e^T(x)), 1 - max(P_e^I(x), Q_e^I(x)), 1 - max(P_e^F(x), Q_e^F(x))\}^c$
Thus $((P,E) \cup (Q,E))^c = (P,E)^c \cap (Q,E)^c$.

(ii) Let $(P,E) \cap (Q,E) = (H,E)$ where $\forall e \in E$
$H(e) = P(e) \cap Q(e)$
$= \{(x, min(P_e^T(x), Q_e^T(x)), min(P_e^I(x), Q_e^I(x)), min(P_e^F(x), Q_e^F(x)))\}$
Thus $((P,E) \cap (Q,E))^c = (H,E)^c$ where $\forall e \in E$
$H(e)^c = (P(e) \cap Q(e))^c$
$=\{(x, min(P_e^T(x), Q_e^T(x)), min(P_e^I(x), Q_e^I(x)), min(P_e^F(x), Q_e^F(x)))\}^c$
$= \{(x, 1 - min(P_e^T(x), Q_e^T(x)), 1 - min(P_e^I(x), Q_e^I(x)), 1 - min(P_e^F(x), Q_e^F(x)))\}$
Again $(P,E)^c \cup (Q,E)^c = (I,E)$ say $\forall e \in E$
$I(e) = (P(e))^c \cup (P(e))^c$
$= \{(x, max(1 - P_e^T(x), 1 - Q_e^T(x)), max(1 - P_e^I(x), 1 - Q_e^I(x)), max(1 - P_e^F(x), 1 - Q_e^F(x)))\}^c$
$= \{(x, 1 - min(P_e^T(x), Q_e^T(x)), 1 - min(P_e^I(x), Q_e^I(x)), 1 - min(P_e^F(x), Q_e^F(x))\}^c$
Thus $((P,E) \cap (Q,E))^c = (P,E)^c \cup (Q,E)^c$.

Proposition 3.20. Let U be an initial universal set , E be a set of parameters and $A, B \subseteq E$.

(i) $((P,M) \wedge (Q,N))^c = (P,M)^c \vee (Q,N)^c$.

(ii) $((P,M) \vee (Q,N))^c = (P,M)^c \wedge (Q,N)^c$.

Proof:

(i) Let $(P,M) \wedge (Q,N) = (H, M \times N)$ where
$H(m,n) =$
$\{(x, min(P_e^T(x), Q_e^T(x)), min(P_e^I(x), Q_e^I(x)), min(P_e^F(x), Q_e^F(x)))\}$
$\forall m \in M$ and $\forall n \in N$
$((P,M) \wedge (Q,N))^c = (H, M \times N)^c \; \forall (a,b) \in M \times N$
$(H(m,n))^c =$
$\{(x, 1 - min(P_e^T(x), Q_e^T(x)), 1 - min(P_e^I(x), Q_e^I(x)), 1 - min(P_e^F(x), Q_e^F(x)))\}^c \; \forall m \in M$ and $\forall n \in N$
Let $(P,M)^c \vee (Q,N)^c = (R, M \times N)$ where
$R(m,n) =$
$\{(x, max(1 - P_e^T(x), 1 - Q_e^T(x)),$
$max(1 - P_e^I(x), 1 - Q_e^I(x)), max(1 - P_e^F(x), 1 - Q_e^F(x)))\} \; \forall m \in M$ and $\forall n \in N$
$= \{(x, 1 - min(P_e^T(x), Q_e^T(x)), 1 - min(P_e^I(x), Q_e^I(x)), 1 - min(P_e^F(x), Q_e^F(x)))\} \; \forall m \in M$ and $\forall n \in N$.
Thus $((P,M) \wedge (Q,N))^c = (P,M)^c \vee (Q,N)^c$

(ii) Let $(P,M) \vee (Q,N) = (H, M \times N)$ where
$H(m,n) =$
$\{(x, max(P_e^T(x), Q_e^T(x)), max(P_e^I(x), Q_e^I(x)), max(P_e^F(x), Q_e^F(x)))\}$
$\forall m \in M$ and $\forall n \in N$
$((P,M) \vee (Q,N))^c = (H, M \times N)^c \; \forall (a,b) \in M \times N$
$(H(m,n))^c =$

$\{(x, 1-max(P_e^T(x), Q_e^T(x)), 1-max(P_e^I(x), Q_e^I(x)), 1-max(P_e^F(x), Q_e^F(x)))\}^c$ $\forall m \in M$ and $\forall n \in N$
Let $(P, M)^c \wedge (Q, N)^c = (R, M \times N)$ where
$R(m, n) =$
$\{(x, min(1 - P_e^T(x), 1 - Q_e^T(x)),$
$min(1 - P_e^I(x), 1 - Q_e^I(x)), min(1 - P_e^F(x), 1 - Q_e^F(x)))\}$ $\forall m \in M$ and $\forall n \in N$
$= \{(x, 1 - max(P_e^T(x), Q_e^T(x)), 1 - max(P_e^I(x), Q_e^I(x)), 1 - max(P_e^F(x), Q_e^F(x)))\}$ $\forall m \in M$ and $\forall n \in N$.
Thus $((P, M) \vee (Q, N))^c = (P, M)^c \wedge (Q, N)^c$

4 Neutrosophic soft cubic topological spaces

In this section, we give the definition of neutrosophic soft cubic topological spaces with some examples and results

Let U be an universe set, E be the set of parameters,$\wp(U)$ be the set of all subsets of U, NSCS(U) be the set of all neutrosophic soft cubic sets in U and NSCS(U;E) be the family of all neutrosophic soft cubic sets over U via parameters in E.

Definition 4.1. Let(X, E)be an element of $NSCS(U; E)$, $\wp(X; E)$ be the collection of all neutrosophic soft cubic subsets of(X, E) .A sub family τ of $\wp(X; E)$is called neutrosophic soft cubic topology (in short NSCS-topology) on (X,E) if the following conditions hold.

(i) $(\Phi, E), (X, E) \in \tau$.

(ii) (P , E), (Q , E) $\in \tau$ implies $(P, E) \cap (Q, E) \in \tau$.

(iii) $\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau$ implies $\bigcup \{(P_\alpha, E); \alpha \in \Gamma\} \in \tau$

The triplet (X, τ, E) is called a neutrosophic soft cubic topological space (in short NSCTS) over (X,E).
Every member of τ is called a neutrosophic soft cubic open set in (X,E)(in short NSCOP(X)).
$\Phi : A \longrightarrow NSCS(U)$ is defined as
$\Phi(e) = \{x, ([0, 0], [0, 0], [0, 0]), (0, 0, 0) : x \in X\} \forall e \in A \subseteq E.$
A neutrosophic soft cubic subset (P , E) of (X,E) is called a neutrosophic soft cubic closed set in (X,E)(in short NSCCS(X)) if $(P, E) \in \tau^c$ where $\tau^c = \{(P, E)^c : (P, E) \in \tau\}$

Example 4.2. Let $X = \{x_1, x_2, x_3\}, E = \{e_1, e_2, e_3, e_4\}, A = \{e_1, e_2, e_3\}$. The tabular representations which are shown in Table1-6.

X	e_1	e_2	e_3
x_1	$([0.5, 0.8], [0.3, 0.5], [0.2, 0.7])(0.8, 0.4, 0.5)$	$([0.4, 0.7], [0.2, 0.3], [0.1, 0.3])(0.7, 0.7, 0.5)$	$([0.3, 0.9], [0, 0.1], [0, 0.2])(0.7, 0.5, 0.6)$
x_2	$([0.5, 1], [0, 0.1], [0.3, 0.6])(0.5, 0.4, 0.5)$	$([0.6, 0.8], [0.2, 0.4], [0.1, 0.3])(0.6, 0.7, 0.6)$	$([0.4, 0.9], [0.1, 0.3], [0.2, 0.4])(0.6, 0.5, 0.7)$
x_3	$([0.4, 0.7], [0.3, 0.4], [0.1, 0.2])(0.9, 0.5, 0.6)$	$([0.6, 0.9], [0.1, 0.2], [0.10.2])(0.8, 0.8, 0.5)$	$([0.4, 0.8], [0.1, 0.2], [0, 0.5])(0.8, 0.6, 0.6)$

Table 1: The tabular representation of (X, E).

Here the sub-family $\tau_1 = \{(\Phi, E), (X, E), (P, E), (Q, E), (H, E), (L, E)\}$ of $\wp(X, E)$ is a neutrosophic soft cubic topology on (X,E), as it satisfies the necessary three axioms of topology and (X, τ, E) is a NSCTS. But the sub-family $\tau_2 = \{(\Phi, E), (X, E), (P, E), (Q, E)\}$ of $\wp(X, E)$ is not a neutrosophic soft cubic topology on (X,E), as the union $(P, E) \cup (Q, E)$ does not belong to τ_2

X	e_1	e_2	e_3
x_1	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$
x_2	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$
x_3	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$	$([0,0],[0,0],[0,0])(0,0,0)$

Table 2: The tabular representation of (Φ, E).

X	e_1	e_2	e_3
x_1	$([0.1,0.7],[0.4,0.8],[0.3,1])(0.8,0.4,0.5)$	$([0.1,0.3],[0.4,0.6],[0.2,0.6])(0.4,0.5,0.3)$	$([0.2,0.5],[0.8,0.9],[0.4,0.9])(0.5,0.5,0.4)$
x_2	$([0.4,0.8],[0.6,0.7],[0.6,0.9])(0.5,0.4,0.5)$	$([3,0.4],[0.4,0.7],[0.2,0.8])(0.5,0.3,0.3)$	$([0.1,0.3],[0.6,0.8],[0.3,0.7])(0.4,0.5,0.4)$
x_3	$([0.1,0.3],[0.6,0.7],[0.2,0.8])0.7,0.7,0.3$	$([0,0.5],[0.5,0.8],[0.4,1])(0.6,0.7,0.4)$	$([0,0.3],[0.6,0.9],[0.1,0.7])(0.6,0.5,0.5)$

Table 3: The tabular representation of (P, E).

X	e_1	e_2	e_3
x_1	$([0.4,0.7],[0.5,0.7],[0.4,0.9])(0.5,0.4,0.6)$	$([0.2,0.3],[0.4,0.5],[0.7,0.9])(0.5,0.4,0.4)$	$([0.3,0.7],[0.5,0.8],[0.1,0.2])(0.4,0.5,0.4)$
x_2	$([0.3,0.9],[0.1,0.2],[0.6,0.7])(0.6,0.7,0.3)$	$([5,0.6],[0.6,0.7],[0.3,0.4])(0.3,0.4,0.5)$	$([2,0.6],[0.3,0.5],[0.5,0.8])(0.4,0.7,0.5)$
x_3	$([0.3,0.5],[0.4,0.8],[0.1,0.4])(0.5,0.4,0.5)$	$([4,0.6],[0.3,0.5],[0.2,0.5])(0.4,0.6,0.3)$	$([0.1,0.3],[0.3,0.5],[0.6,0.8])(0.3,0.5,0.4)$

Table 4: The tabular representation of (Q, E).

Let $(H, E) = (P, E) \cap (Q, E)$ The tabular representation of (H, E) is given by

X	e_1	e_2	e_3
x_1	$([0.1,0.7],[0.4,0.7],[0.3,0.9])(0.5,0.4,0.5)$	$([0.1,0.3],[0.4,0.5],[0.2,0.6])(0.4,0.4,0.3)$	$([0.2,0.5],[0.5,0.8],[0.1,0.2])(0.4,0.5,0.4)$
x_2	$([0.3,0.8],[0.1,0.2],[0.6,0.7])(0.5,0.4,0.3)$	$([0.3,0.4],[0.4,0.7],[0.2,0.4])(0.3,0.3,0.3)$	$([0.1,0.3],[0.3,0.5],[0.3,0.7])(0.4,0.5,0.4)$
x_3	$([0.1,0.3],[0.4,0.7],[0.1,0.4])(0.5,0.4,0.3)$	$([0,0.5],[0.3,0.5],[0.2,0.5])(0.4,0.6,0.3)$	$([0.0,0.3],[0.3,0.5],[0.1,0.7])(0.3,0.5,0.4)$

Table 5: The tabular representation of (H, E)

Let $(L, E) = (P, E) \cup (Q, E)$ The tabular representation of (L, E) is given by

X	e_1	e_2	e_3
x_1	$([0.4,0.7],[0.5,0.8],[0.4,1])(0.8,0.4,0.6)$	$([0.2,0.3],[0.4,0.6],[0.7,0.9])(0.5,0.5,0.4)$	$([0.3,0.7],[0.8,0.9],[0.4,0.9])(0.5,0.5,0.4)$
x_2	$([0.4,0.9],[0.6,0.7],[0.6,0.9])0.6,0.7,0.5$	$([0.5,0.6],[0.6,0.7],[0.3,0.8])(0.5,0.4,0.5)$	$([0.2,0.6],[0.6,0.8],[0.5,0.8])(0.4,0.7,0.5)$
x_3	$([0.3,0.5],[0.6,0.8],[0.3,0.8])(0.7,0.7,0.5)$	$([0.4,0.6],[0.5,0.8],[0.4,1])(0.6,0.7,0.4)$	$([0.1,0.3],[0.6,0.9],[0.6,0.8])(0.3,0.5,0.4)$

Table 6: The tabular representation of (L, E)

Definition 4.3. As every NSC-topology on (X, E) must contain the sets $(\Phi, E), (X, E) \in \tau$ so the family $\tau = \{(\Phi, E), (X, E)\}$ forms a NSC-topology on (X, E). The topology is called indiscrete NSC-topology and the triplet(X, τ, E) is called an indiscrete neutrosophic soft cubic topological space (or simply indiscrete NSC-topological space).

Definition 4.4. Let ξ denote the family of all NSC-subsets of (X, E). Then we observe that ξ satisfies all the axioms of topology on(X, E). This topology is called discrete neutrosophic soft cubic topology and the triplet (X, ξ, E) is called discrete discrete neutrosophic soft cubic topological space (or simply discrete NSCTS).

Definition 4.5. Let (X, τ, E) be an NSC-topological space over (X, E). A neutrosophic soft cubic subset (P, E) of (X, E) is called neutrosophic soft cubic set (in short NSC-closed set) if its complement $(P, E)^c$ is a member of τ.

Example 4.6. Let us consider Example 4.2 then the NSC-closed set in (X, τ_1, E) are shown in Table7-12.

X	e_1	e_2	
x_1	$([0.5, 0.8], [0.3, 0.5], [0.2, 0.7])(0.8, 0.4, 0.5)$	$([0.4, 0.7], [0.2, 0.3], [0.1, 0.3])(0.7, 0.7, 0.5)$	$([0.3, 0.9], [0, 0.1], [0, 0.2])(0.7, 0.5, 0.6)$
x_2	$([0.5, 1], [0, 0.1], [0.3, 0.6])(0.5, 0.4, 0.5)$	$([0.6, 0.8], [0.2, 0.4], [0.1, 0.3])(0.6, 0.7, 0.6)$	$([0.4, 0.9], [0.1, 0.3], [0.2, 0.4])(0.6, 0.5, 0.7)$
x_3	$([0.4, 0.7], [0.3, 0.4], [0.1, 0.2])(0.9, 0.5, 0.6)$	$([0.6, 0.9], [0.1, 0.2], [0.1, 0.2])(0.8, 0.8, 0.5)$	$([0.4, 0.8], [0.1, 0.2], [0, 0.5])(0.8, 0.6, 0.6)$

Table 7: The tabular representation of $(X, E)^c$

X	e_1	e_2	e_3
x_1	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$
x_2	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$	$([1, 1], [, 1], [1, 1])(1, 1, 1)$	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$
x_3	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$	$([1, 1], [1, 1], [1, 1])(1, 1, 1))$	$([1, 1], [1, 1], [1, 1])(1, 1, 1)$

Table 8: The tabular representation of $(\Phi, E)^c$

X	e_1	e_2	
x_1	$([0.3, 0.9], [0.2, 0.6], [0, 0.7])(0.2, 0.6, 0.5)$	$([0.7, 0.9], [0.4, 0.6], [0.4, 0.8])(0.6, 0.5, 0.7)$	$([0.5, 0.8], [0.1, 0.2], [0.1, 0.6])(0.5, 0.5, 0.6)$
x_2	$([0.2, 0.6], [3, 0.4], [0.1, 0.4])(0.5, 0.6, 0.5)$	$([0.6, 0.7], [0.3, 0.6], [0.2, 0.8])(0.5, 0.7, 0.7)$	$([0.7, 0.9], [0.2, 0.4], [0.3, 0.7])(0.6, 0.5, 0.6)$
x_3	$([0.7, 0.9], [0.3, 0.4], [0.2, 0.8])(0.3, 0.3, 0.7)$	$([0.5, 1], [0.2, 0.5], [0, 0.6])(0.4, 0.3, 0.6)$	$([0.7, 1], [0.1, 0.4], [0.3, 0.9])(0.4, 0.6, 0.6)$

Table 9: The tabular representation of $(P, E)^c$

Proposition 4.7. Let (X, τ_1, E) and (X,τ_2,E) be two neutrosophic soft cubic topological spaces. Denote $\tau_1 \cap \tau_2 = \{(P, E) : (P, E) \in \tau_1 \; and \; (P, E) \in \tau_2\}$. Then $\tau_1 \cap \tau_2$ is a neutrosophic soft cubic topology.
Proof:

(i) Since (X, τ_1, E) is a neutrosophic soft cubic topological space then $(\Phi, E) \in \tau_1$,
(X, τ_2, E) is a neutrosophic soft cubic topological space then $(\Phi, E) \in \tau_2$.
Therefore $(\Phi, E) \in \tau_1 \cap \tau_2$
Since (X, τ_1, E) is a neutrosophic soft cubic topological space then $(X, E) \in \tau_1$,
(X, τ_2, E) is a neutrosophic soft cubic topological space then $(X, E) \in \tau_2$.
Therefore $(X, E) \in \tau_1 \cap \tau_2$

(ii Let (P , E), (Q , E) $\in \tau_1 \cap \tau_2$. Then (P , E), (Q , E) $\in \tau_1$ and (P , E), (Q , E) $\in \tau_2$, τ_1 and τ_2 are two neutrosophic soft cubic topologies on X. Then $(P, E) \cap (Q, E) \in \tau_1$ and $(P, E) \cap (Q, E) \in \tau_2$. Hence $(P, E) \cap (Q, E) \in \tau_1 \cap \tau_2$.

(iii Let $\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_1 \cap \tau_2$. Then $\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_1$ and $\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_2$. Since τ_1 and τ_2 are two neutrosophic soft cubic topologies on X. Then $\bigcup\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_1$ and $\bigcup\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_2$.
Thus $\bigcup\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau_1 \cap \tau_2$.

Theorem 4.8. Let $\{\tau_i : i \in I\}$ be any collection of NSC-topology on (X, E). Then their intersection $\bigcap\limits_{i\in I} \tau_i$ is also a NSC-topology on (X, E).
Proof:

(i) Since$(\Phi, E), (X, E) \in \tau_i foreachi \in I$. Hence $(\Phi, E), (X, E) \in \bigcap\limits_{i\in I} \tau_i$.

(ii) Let $\{(P_\alpha, E); \alpha \in \Gamma\}$ be an arbitrary family of neutrosophic soft cubic sets where $\{(P_\alpha, E); \in \bigcap\limits_{i\in I} \tau\}$ for each $\alpha \in \Gamma$. Then for each $i \in I$ $\{(P_\alpha, E); \in \tau$ for $\alpha \in \Gamma\}$ and since for each $i \in I$ is a NSC-topology, therefore for each $\bigcup\limits_{i\in I}(P_\alpha, E); \in \tau_i$for each $i \in I$. Hence $\bigcup\limits_{i\in I}(P_\alpha, E); \in \bigcap\limits_{i\in I} \tau_i$.

(iii) Let $(P, E), (Q, E)X \in \bigcap\limits_{i\in I} \tau_i$, then $(P, E), (Q, E)X \in \tau_i$ for each$i \in I$. Since for each$i \in I$,τ_i is an NSC-topology, therefore $(P, E) \cap (Q, E) \in \tau_i$ for each $i \in I$. Hence $(P, E) \cap (Q, E)X \in \bigcap\limits_{i\in I} \tau_i$. Thus $\bigcap\limits_{i\in I} \tau_i$ satisfies all the axioms of topology. Hence $\bigcap\limits_{i\in I} \tau_i$ forms a NSC-topology. But union of NSC-topologies need not be a NSC-topology. Let us show this with the following example

Remark 4.9. If τ_1 and τ_2 be two neutrosophic soft cubic topologies on (X,E).

(i) $\tau_1 \vee \tau_2 = \{(P, E)\cup(Q, E) : (P, E) \in \tau_1 \ and \ (Q, E) \in \tau_2\}$.

(ii) $\tau_1 \wedge \tau_2 = \{(P, E)\cap(Q, E) : (P, E) \in \tau_1 \ and \ (Q, E) \in \tau_2\}$.

Example 4.10. Let (P , E) and (Q , E) be neutrosophic soft cubic set as in Example 28.
Define $\tau_1 = \{(\Phi, E), (X, E), (P, E)\}$, $\tau_2 = \{(\Phi, E), (X, E), (Q, E)\}$.
Then $\tau_1 \cap \tau_2 = \{(\Phi, E), (X, E)\}$ is neutrosophic soft cubic topology on X.
But $\tau_1 \cup \tau_2 = \{(\Phi, E), (X, E), (P, E), (Q, E)\}$,
$\tau_1 \vee \tau_2 = \{(\Phi, E), (X, E), (P, E), (Q, E), (P, E)\cup(Q, E)\}$,
$\tau_1 \wedge \tau_2 = \{(\Phi, E), (X, E), (P, E), (Q, E), (P, E)\cap(Q, E)\}$ are not neutrosophic soft cubic topology on(X,E).

Definition 4.11. Let (X, τ, E) be a neutrosophic soft cubic topological space on $(X, E)$and $(Y, E) \in \wp(X, E)$. Then the collection $\tau_Y = \{(Y, E)\cap(Q, E) : (Q, E) \in \tau\}$ is called a neutrosophic soft cubic subspace topology on (X,E). Hence (Y, τ_Y, E) is called a neutrosophic soft cubic topological subspace of (X, τ, E)

Theorem 4.12. Let (X, τ, E) be a neutrosophic soft cubic topological space and $e \in E$, $\tau(e) = \{P(e) : (P, E) \in \tau\}$ is a neutrosophic soft cubic topology on (X,E).
Proof: Let $e \in E$

X	e_1	e_2	e_3
x_1	$([0.3, 0.6], [0.3, 0.5], [0.1, 0.6])(0.5, 0.6, 0.4)$	$([0.7, 0.8], [0.5, 0.6], [0.1, 0.4])(0.5, 0.6, 0.6)$	$([0.3, 0.7], [0.2, 0.5], [0.8, 0.9])(0.6, 0.5, 0.6)$
x_2	$([0.1, 0.7], [0.8, 0.9], [0.3, 0.4])(0.4, 0.3, 0.7)$	$([0.4, 0.5], [0.3, 0.4], [0.6, 0.7])(0.7, 0.6, 0.5)$	$([0.4, 0.8], [0.5, 0.7], [0.2, 0.5])(0.6, 0.3, 0.5)$
x_3	$([0.5, 0.7], [0.2, 0.6], [0.6, 0.9])(0.5, 0.6, 0.5)$	$([0.4, 0.6], [0.5, 0.7], [0.5, 0.8])(0.6, 0.4, 0.7)$	$([0.7, 0.9], [0.5, 0.7], [0.2, 0.4])(0.7, 0.5, 0.6)$

Table 10: The tabular representation of $(Q, E)^c$

(i) Let (Φ, E),$(X, E) \in \tau$, (Φ, E), $(X, E) \in \tau(e)$.

(ii) Let V, W $\in \tau$, then there exist (P , E), (Q , E) $\in \tau$ such that V = P(e)and W = Q(e).
Since τ is a neutrosophic soft cubic topology on X, (P , E)$\cap$(Q , E)$\in \tau$.
Put (H , E)= (P , E)$\cap$(Q , E). Then (H , E) $\in \tau$.
We have V $\cap$ W = P(e)$\cap$ Q(e)=H(e) and $\tau(e) = \{P(e) : (P, E) \in \tau\}$.
Then V $\cap$ W $\in \tau$.

(iii) Let $\{(V_\alpha, E); \alpha \in \Gamma\} \in \tau(e)$ .Then for every $\alpha \in \Gamma$, there exist $(P_\alpha, E) \in \tau$ such that $V_\alpha = P_\alpha(e)$.
Since τ is a neutrosophic soft cubic topological space on X, $\bigcup\{(P_\alpha, E); \alpha \in \Gamma\} \in \tau$.
Put $(P, E) = \bigcup\{(P_\alpha, E); \alpha \in \Gamma\}$ then (P , E) $\in \tau$.
Note that $\bigcup\limits_{\alpha\in\Gamma} V_\alpha = \bigcup\{P_\alpha, (e); \alpha \in \Gamma\}$ = P(e) and $\tau(e) = \{P(e) : (P, E) \in \tau\}$. Then $\bigcup\limits_{\alpha\in\Gamma} V_\alpha \in \tau(e)$.
Therefore $\tau(e) = \{P(e) : (P, E) \in \tau\}$ is a neutrosophic soft cubic topology on X.

Definition 4.13. Let (X, τ, E) be a neutrosophic soft cubic topological space over (X,E) and $\mathcal{B} \subseteq \tau$. $\mathcal{B}$ is a basis on τ if for each $(Q, E) \in \tau$, there exist $\mathcal{B}' \subseteq \mathcal{B}$ such that $(Q, E) = \cup\mathcal{B}'$

Example 4.14. Let τ be a neutrosophic soft cubic topology as in Example 28. Then
$\mathcal{B} = \{(P, E), (Q, E), (L, E), (\Phi, E), (X, E)\}$ is a basis for τ

Theorem 4.15. Let $\mathcal{B}$ be a basis for neutrosophic soft cubic topology τ. Denote $\mathcal{B}_e = \{P(e) : (P, E) \in \mathcal{B}\}$ and $\tau(e) = \{P(e) : (P, E) \in \tau\}$ for any $e \in E$. Then $\mathcal{B}_e$ is a basis for neutrosophic soft cubic topology $\tau(e)$.
Proof: Let $e \in E$. For any V $\in \tau(e)$, V = Q(e) for $(Q, E) \in \tau$. Here $\mathcal{B}$ is a basis for τ. Then there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $(Q, E) = \cup\mathcal{B}'$. So $V = \cup\mathcal{B}'_e$ where $\mathcal{B}'_e = \{P(e) : (P, E) \in \mathcal{B}'\} \subseteq \mathcal{B}_e$. Thus $\mathcal{B}_e$ is a basis for neutrosophic soft cubic topology $\tau(e)$ for any $e \in E$

Definition 4.16. Let (X, τ, E) be a neutrosophic soft cubic topological space and let (P , E) be a neutrosophic soft cubic set over (X, E). Then the interior and closure of (P , E) denoted respectively by int(P , E) and cl(P , E) are defined as follows. $int(P, E) = \cup\{(Q, E) \in \tau : (Q, E)\subseteq(P, E)\}$
ie., $int(P, E) = \cup\{(Q, E) : (Q, E)\subseteq(P, E)$ and (Q, E) is $NSCOS\}$
$cl(P, E) = \cap\{(Q, E) \in \tau^c : (P, E)\subseteq(Q, E)\}$ ie., $cl(P, E) = \cap\{(Q, E) : (P, E)\subseteq(Q, E)$ and (Q, E) is $NSCCS\}$

Example 4.17. We consider the Example 4.2 and take NSCS (G,E) as shown in Table 13.
$int(G, E) = (P, E)$ and $cl(G, E) = (P, E)^c$.

Theorem 4.18. Let (X, τ, E) be a neutrosophic soft cubic topological space. Then the following properties hold.

(i) $(\Phi, E), (X, E) \in \tau$

(ii) The intersection of any number of neutrosophic soft cubic closed sets is a neutrosophic soft cubic closed set over X.

X	e_1	e_2	e_3
x_1	$([0.3, 0.9], [0.3, 0.6], [0.1, 0.7])(0.5, 0.6, 0.5)$	$([0.4, 0.7], [0.2, 0.3], [0.1, 0.3])(0.7, 0.7, 0.5)$	$([0.3, 0.9], [0, 0.1], [0, 0.2])(0.7, 0.5, 0.6)$
x_2	$([0.5, 1], [0, 0.1], [0.3, 0.6])(0.5, 0.6, 0.7)$	$([0.6, 0.8], [0.2, 0.4], [0.1, 0.3])(0.6, 0.7, 0.6)$	$([0.4, 0.9], [0.1, 0.3], [0.2, 0.4])(0.6, 0.5, 0.7)$
x_3	$([0.4, 0.7], [0.3, 0.4], [0.6, 0.9])(0.5, 0.6, 0.7)$	$([0.6, 0.9], [0.1, 0.2], [0.1, 0.2])(0.8, 0.8, 0.5)$	$([0.4, 0.8], [0.1, 0.2], [0, 0.5])(0.8, 0.6, 0.6)$

Table 11: The tabular representation of $(L, E)^c$

(iii) The union of any two neutrosophic soft cubic closed sets is a neutrosophic soft cubic closed set over X.

Proof:

(i) Since $(\Phi, E), (X, E) \in \tau$,therefore $(\Phi, E)^c, (X, E)^c$ are NSC-closed set.

(ii) Let $\{(P_\alpha, E); \alpha \in \Gamma\}$ be an arbitrary family of NSC-closed sets in (X, τ,E) and let $(P, E) = \bigcap_{\alpha\in\Gamma} (P_\alpha, E)$. Now $(P, E)^c = (\bigcap_{\alpha\in\Gamma} (P_\alpha, E))^c = \bigcup_{\alpha\in\Gamma} (P_\alpha, E)^c$ and $(P_\alpha, E)^c \in \tau$ for each $\alpha \in \Gamma$, so $\bigcup_{\alpha\in\Gamma} (P_\alpha, E)^c \in \tau$. Hence $(P_\alpha, E)^c \in \tau$. Thus $(P_\alpha, E)^c$ is NSC-closed set.

(iii) Let $\{(P_i, E) : i = 1, 2, 3, ...n\}$ be the family of NSCCS

Theorem 4.19. Let (X, τ,E) be a neutrosophic soft cubic topology on X and let (P , E) be neutrosophic soft cubic set over (X,E). Then the following properties hold.

(i) $int(P, E)\subseteq(P, E)$.

(ii) $(Q, E)\subseteq(P, E) \Rightarrow int(Q, E)\subseteq int(P, E)$.

(iii) (P, E) is a neutrosophic soft cubic open set $\Leftrightarrow int(P, E) = (P, E)$.

(iv) $int(int(P, E)) = int(P, E)$.

(v) $int((\Phi, E)) = (\Phi, E)$, $int((X, E)) = (X, E)$.

Proof: (i) and (v) follows from definition [4.16].

(ii) $int(Q, E) = \cup\{(K, E) : (K, E)\subseteq(Q, E)$ and (K, E) is NCSOS in X$\}$
$int(P, E) = \cup\{(S, E) : (S, E)\subseteq(P, E)$ and (S, E) is NCSOS in X$\}$
Now $int(Q, E)\subseteq(Q, E)\subseteq(P, E) \Rightarrow int(Q, E)\subseteq(P, E)$.
Since $int(P, E)$ is the largest NCSOS contained in (P , E). Therefore $int(Q, E)\subseteq int(P, E)$

(iii) Let (P , E) be a neutrosophic soft cubic open set. Then it is the largest neutrosophic soft cubic open set contained in (P , E) and hence (P , E)= int(P , E). Conversely let (P , E)= int(P , E), since int(P , E)is the union of neutrosophic soft cubic open sets which is neutrosophic soft cubic open set.Hence (P , E) is neutrosophic soft cubic open set.

(iv) $int(P, E) = \cup\{(S, E) : (S, E)\subseteq int(P, E)$ and (S, E) is NCSOS in X$\}$
Since $int(P, E)$is the largest neutrosophic soft cubic open set contained in int (P , E).
Therefore$int(int(P, E)) = int(P, E)$.

Theorem 4.20. Let (X,τ,E) be a neutrosophic soft cubic topological space and (P , E) be a neutrosophic soft cubic set over (X,E). Then the following properties hold.

X	e_1	e_2	e_3
x_1	$([0.5, 0.8], [0.3, 0.5], [0.2, 0.7])(0.8, 0.4, 0.5)$	$([0.4, 0.7], [0.2, 0.3], [0.1, 0.3])(0.7, 0.7, 0.5)$	$([0.3, 0.9], [0, 0.1], [0, 0.2])(0.7, 0.5, 0.6)$
x_2	$([0.5, 1], [0, 0.1], [0.3, 0.6])(0.5, 0.4, 0.5)$	$([0.6, 0.8], [0.2, 0.4], [0.1, 0.3])(0.6, 0.7, 0.6)$	$([0.4, 0.9], [0.1, 0.3], [0.2, 0.4])(0.6, 0.5, 0.7)$
x_3	$([0.4, 0.7], [0.3, 0.4], [0.1, 0.2])(0.9, 0.5, 0.6)$	$([0.6, 0.9], [0.1, 0.2], [0.1, 0.2])(0.8, 0.8, 0.5)$	$([0.4, 0.8], [0.1, 0.2], [0, 0.5])(0.8, 0.6, 0.6)$

Table 12: The tabular representation of $(H, E)^c$

(i) $(P, E) \subseteq cl(P, E)$.

(ii) $(Q, E) \subseteq (P, E) \Rightarrow cl(Q, E) \subseteq cl(P, E)$.

(iii) (P, E) is a neutrosophic soft cubic closed set $\Leftrightarrow cl(P, E) = (P, E)$.

(iv) $cl(cl(P, E)) = cl(P, E)$.

(v) $cl((\Phi, E)) = (\Phi, E)$, $cl((X, E)) = (X, E)$.

Proof:

(i) From the definition [4.16] $(P, E) \subseteq cl(P, E)$.

(ii) $cl(Q, E) = \cap\{(K, E) : (Q, E) \subseteq (K, E)$ and (K, E) is NCSCS in X$\}$
$cl(P, E) = \cap\{(S, E) : (P, E) \subseteq (S, E)$ and (S, E) is NCSCS in X$\}$
Since $(Q, E) \subseteq cl(Q, E)$ and $(P, E) \subseteq cl(P, E) \Rightarrow (Q, E) \subseteq (P, E) \subseteq cl(P, E) \Rightarrow (Q, E) \subseteq cl(P, E)$.
Since $cl(P, E)$ is the smallest neutrosophic soft cubic closed set containing (P, E). Hence $cl(Q, E) \subseteq cl(P, E)$.

(iii) Let (P, E) be neutrosophic soft cubic closed set. Then it is the smallest neutrosophic soft cubic closed set containing itself and hence $(P, E) = cl(P, E)$.
Conversely let $(P, E) = cl(P, E)$, since $cl(P, E)$ being the intersection of neutrosophic soft cubic closed sets is neutrosophic soft cubic closed set. Hence $(P, E)$is neutrosophic soft cubic closed set.

(iv) $cl(cl(P, E)) = \cap\{(S, E) : (P, E) \subseteq (S, E)$ and (S, E) is NCSCS in X$\}$
Since $cl(P, E)$ is the smallest closed neutrosophic soft closed set containing cl (P , E).
Therefore $cl(cl(P, E)) = cl(P, E)$.

(v) $cl((\Phi, E)) = (\Phi, E)$, $cl((X, E)) = (X, E)$ are follows from definition[4.16].

Theorem 4.21. Let (X,τ,E) be neutrosophic soft cubic topological space and let (P , E) and (Q , E) are neutrosophic soft cubic sets over (X,E). Then the following properties hold.

(i) $int(Q, E) \cap int(P, E) = int((Q, E) \cap (P, E))$

(ii) $int(Q, E) \cup int(P, E) \subseteq int((Q, E) \cup (P, E))$

(iii) $cl(Q, E) \cup cl(P, E) = cl((Q, E) \cup (P, E))$

(iv) $cl((Q, E) \cap (P, E)) \subseteq cl(Q, E) \cap cl(P, E)$

(v) $[int(Q, E)]^c = cl[(Q, E)^c]$

(vi) $[cl(Q, E)]^c = int[(Q, E)^c]$

X	e_1	e_2	e_3
x_1	$([0.2, 0.8], [0.3, 0.6], [0.2, 0.8])(0.8, 0.3, 0.4)$	$([0.2, 0.4], [0.4, 0.6], [0.2, 0.4])(0.6, 0.4, 0.3)$	$([0.2, 0.6], [0.7, 0.8], [0.3, 0.4])(0.6, 0.5, 0.4)$
x_2	$([0.5, 0.8], [0.5, 0.6], [0.5, 0.8])(0.6, 0.3, 0.5)$	$([0.1, 0.4], [0.4, 0.6], [0.1, 0.5])(0.7, 0.3, 0.3)$	$([0.0.2, 0.5], [0.5, 0.8], [0.2, 0.4])(0.5, 0.5, 0.4)$
x_3	$([0.1, 0.6], [0.4, 5], [0.2, 0.7])(0.7, 0.4, 0.3)$	$([0.2, 0.6], [0.5, 0.7], [0.1, 0.7])(0.7, 0.6, 0.4)$	$([0.1, 0.4], [0.2, 0.5], [0.1, 0.5])(0.7, 0.4, 0.4)$

Table 13: The tabular representation of (G, E)

Proof:

(i) Since $(Q,E)\cap(P,E)\subseteq(Q,E)$ and $(Q,E)\cap(P,E)\subseteq(Q,E)$,
by Theorem 4.19(ii), $(Q,E)\subseteq(P,E) \Rightarrow int(Q,E)\subseteq int(P,E)$,
then $int((Q,E)\cap(P,E))\subseteq int(Q,E)$ and $int((Q,E)\cap(P,E))\subseteq int(P,E)$
$\Rightarrow int((Q,E)\cap(P,E))\subseteq int(Q,E)\cap int(P,E)$.
Now int(Q , E) and int(P , E) are NCSOSs
$\Rightarrow int(Q,E)\cap int(P,E)$ is NCSOS,
then $int(Q,E)\subseteq(Q,E)$ and $int(P,E)\subseteq(P,E) \Rightarrow int(Q,E)\cap int(P,E)\subseteq int((Q,E)\cap(P,E))$.
Therefore $int(Q,E)\cap int(P,E) = int((Q,E)\cap(P,E))$.

(ii) $(Q,E)\subseteq(P,E)\cup(Q,E)$ and $(P,E)\subseteq(P,E)\cup(Q,E)$
by Theorem 4.19(ii), $(Q,E)\subseteq(P,E) \Rightarrow int(Q,E)\subseteq int(P,E)$,
then $int(Q,E)\subseteq int((P,E)\cup(Q,E))$ and $int(P,E)\subseteq int((P,E)\cup(Q,E))$.
Hence $int(Q,E)\cup int(P,E)\subseteq int((Q,E)\cup(P,E))$

(iii) Since $(Q,E)\subseteq cl(Q,E)$ and $(P,E)\subseteq cl(P,E)$.
We have $(Q,E)\cup(P,E) = cl(Q,E)\cup cl(P,E)$
$\Rightarrow cl((Q,E)\cup(P,E)) = cl(Q,E)\cup cl(P,E)$ .....(1)
$cl(Q,E)\cup cl(P,E) = cl((Q,E)\cup(P,E))$ And since $(Q,E)\subseteq cl(Q,E)$ and $(P,E)\subseteq cl(P,E)$
so $cl(Q,E)\subseteq cl((Q,E)\cup(P,E))$ and $cl(P,E)\subseteq cl((Q,E)\cup(P,E))$ ......(2)
Therefore$cl(Q,E)\cup cl(P,E)\subseteq cl((Q,E)\cup(P,E))$
From (1) and (2) $cl(Q,E)\cup cl(P,E) = cl((Q,E)\cup(P,E))$.

(iv) Since $(Q,E)\cap(P,E)\subseteq(Q,E)$ and $(Q,E)\cap(P,E)\subseteq(P,E)$
and so $cl((Q,E)\cap(P,E))\subseteq cl(Q,E)$ and $cl((Q,E)\cap(P,E))\subseteq\cap cl(P,E)$
Hence $cl((Q,E)\cap(P,E))\subseteq cl(Q,E)\cap cl(P,E)$

(v) $[int(Q,E)]^c = [\cup\{(K,E) : (K,E)\subseteq(Q,E)$ and (K,E) is NCSOS in X$\}]^c$
$= \cap\{(K,E)^c : (Q,E)^c\subseteq(K,E)^c$ and $(K,E)^c$ is NCSCS in X$\}]\}$
$= cl[(Q,E)^c]$.

(vi) $[cl(Q,E)]^c = [\cap\{(K,E) : (K,E)\subseteq(Q,E)$ and (K,E) is NCSCS in X$\}]^c$
$= \cup\{(K,E)^c : (K,E)^c\subseteq(Q,E)^c$ and $(K,E)^c$ is NCSOS in X$\}]^c\}$
$= int[(Q,E)^c]$..

5 Conclusions

This paper lays the foundation for the further study on different separation axioms via this sets. Further various types of relation between neutrosophic soft cubic topological sets can be analysed under various mappings.

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Validation of the pedagogical strategy for the formation of the competence entrepreneurshipin high education through the use of neutrosophic logic and Iadov technique

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Abstract.The objective of this work is to validate the implementation of the pedagogical strategy for the development of the competence to undertake as a contribution to the comprehensive education in senior high students in "10 de Octubre" borough in Havana, Cuba.The research seeks to increase scientific knowledge; so it is necessary to objectify the requirements of validity and reliability on which it is based. Validity is understood as the consistency and stability shown by the results of research when applying different demonstration methods, based on the assumption that these are conceived and structured with the capacity to determine and measure.Reliability allows establishing that the conclusive results of the research are balanced and refers to the degree to which the same action submitted to a measurement by the same investigative or different subject produces similar results. For the authors, investigative reliability constitutes the degree of stability that when applying the validation they tend not to vary.In order to sustain the derivations of the development of this strategy, a survey instrument was applied to training and recipients whose results were evaluated through a complex methodology, which integrates the Iadov technique and the neutrosophic logic, determining the transcendences and strategic training repercussions and its consequences on the performers of senior high education.

Keywords:Iadov, Neutrosophic, Indeterminacy, Group Satisfaction Index

1 Introduction

The systematization of several definitions led the authors to define the entrepreneurship competence as the complex and systemic set of knowledge, abilities, skills, attitudes and values that interact synergistically and make viable the autonomous and effective performance of the individual, by providing it with tools to create, manage, interpret, understand and transform the social environment with a critical, proactive and innovative vision, sustaining a life model, personal development in present and in future[1].

The formation of competence to undertake makes the learner a protagonist of the community context by enabling the application of knowledge through the selection of methods, procedures and disjunctive proposals, which mobilize the cognitive and attitudinal structures developed during the process, complex arrangements that pay tribute to the integral development of student[2].

In order to contribute to the development of the comprehensive education of the student of senior high education, this pedagogical strategy for the development of entrepreneurship competition was conceived and applied. This moment is concretized in the strategic performance of modeling of ventures, based on proposals resulting from the student's identification of possible solutions to problems that arise from the social demands of the surrounding environment[3].

In the provisions of stages and phases, students are encouraged in the search of alternatives, of conformation of factical schemes, which allows the application, in the social context, of knowledge, skills, attitudes,and values.

The determination of the actors' assessment of the strategy's impact makes up a significant indicator of the validity of the strategy. This action needs to validate the results by the investigation and with this purpose, the Iadov technique is applied. Iadov constitutes an indirect way to study of satisfaction, in this case, the actors' developers and evaluators of the process and the addressees[4].
Iadov's technique uses, as suggested by the original method[5], the related criteria of answers to intercalated questions whose relationship the subject does not know, at the same time the unrelated or complementary questions serve as an introduction and support of objectivity to the respondent who uses them to locate and contrast the answers.The results of these questions interact through what is called the "Iadov Logical Table"[6, 7]. In this paper, the satisfaction of emitting actors (teaching staff and training activity) and those who are beneficiaries of the development strategy, the receiving actors are combined.User criterion techniques should be used as a way to assess results in those cases in which the evaluators are users of what is proposed, that is, in addition to having control over the problem being studied, they are "contextualized", immersed in the context in which is the application of the result[7].
The degree of satisfaction-dissatisfaction is a psychological state that manifests itself in people as an expression of the interaction of a set of affective experiences that move between the positive and negative poles insofar as in the activity that the subject develops, the object, responds to their needs and corresponds to their motives and interests[8]. The relationship between indeterminacy and user importance has not yet been clarified and include in Iadov.
Recently a new theory has been introduced in decision making which is known as neutrosophiclogic and set developed by Florentin Smarandache in 1995[9]. The term neutrosophy means knowledge of neutral thought and thisneutral represents the principal distinction between fuzzy and intuitionistic fuzzy logicand sets [10]. With neutrosophy theory, a new logic is introduced in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F)[11]. Many extensions of classical decision-making methods have been proposed for dealing with indeterminacy based on neutrosophy theory like DEMATEL [12] AHP [13], VIKOR[14]and TOPSIS [15].
The original proposal of the Iadov method does not allow an adequate management of the indetermination nor the management of the importance of the users[11].The introduction of the neutrosophic estimation seeks to solve the problems of indeterminacy that appear universally in the evaluations of the surveys and other instruments, by taking advantage of not only the opposing positions but also the neutral or ambiguous ones[16]. Under the principle that every idea <A> tends to be neutralized, diminished, balanced by other ideas, in clear rupture with binary doctrines in the explanation and understanding of phenomena[17].
This work continues as follows: Section 2 is about some important concepts about neutrosophy and Iadov . A case study is presented and discussed in section 3. The paper ends with conclusions and some recommendation for future work.

2 Materials and methods

In Iadov technique the questionnaire used to determine the degree of user satisfaction with the proposed system of indicators to predict, design and measure the impact of the researcher's strategy has a total of seven questions, three of which are closed and four open, whose relationship is ignored by the subject[18].These three closed questions are related through the "Iadov logical table", which is presented adapted to the present investigation. The resulting number of the interrelation of the three questions indicates the position of each subject in the satisfaction scale, that is, your individual satisfaction This satisfaction scale is expressed by SVN numbers[19]. The original definition of true value in the neutrosophic logic is shown below [20]:
Be $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$a neutrosophical valuation is a mapping of a group of proportional formulas to$N$, and for each p sentence we have:

$$v(p) = (T, I, F) \quad (1)$$

In order to ease the practical application to a decision making and engineering problems, it was carried out the proposal of single valued neutrosophic sets (SVNS) this allows the use of linguistic variables[21, 22], this increase the interpretation of models of recommendation and the usage of the indetermination.

Be X an universe of discourse. A SVNS Aon $X$is an object of the form.

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\} \quad (2)$$

where,$u_A(x): X \rightarrow [0,1]$, $r_A(x), : X \rightarrow [0,1]$and $v_A(x): X \rightarrow [0,1]$ with $0 \leq u_A(x) + r_A(x) + v_A(x) :\leq 3$ for all $x \in X$. The intervals $u_A(x), r_A(x)$and $v_A(x)$ denote the memberships to true, indeterminate and false of x in A, respectively. For convenience reasons, an SVN number will be expressed as$A$ = (a, b, c), where a, b, c∈ [0, 1], y a+ b + c≤ 3.

In order to analyze the results, it is established a scoring function. To order the alternatives it is used a score function[23]adapted :

$$s(\mathrm{V}) = \mathrm{T} - \mathrm{F} - \mathrm{I} \quad (3)$$

In the event that the assessment corresponds to indeterminacy(not defined) (I) a process of de-neutrosophication developed as proposed by Salmerón and Smarandache[24]. In this case, I ∈ [-1,1].Finally, we work with the average of the extreme values$I$ ∈ [0,1]to obtain a single one.

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (4)$$

Subsequently, the results are aggregated and the weighted average aggregation operator is used to calculate the group satisfaction index (GSI). The weighted average (WA) is one of the most mentioned aggregation operators in the literature[25, 26]. A WA operator has associated a vector of weights,V, with$v_i \in [0,1]$ and$\sum_1^n v_i = 1$, having the following form:

$$WA(a_1, .., a_n) = \sum_1^n v_i a_i \quad (5)$$

Where v_i represented the importance of the source. This proposal allow to fill a gap in the literature of the Iadov techniques extending it to deal with indeterminacy and importance of user due to expertise or any other reason [27].

3 Survey of teachers and methodologists of senior high education:

The case study was developed for the validation of a pedagogical strategy for the development of the competence to undertake as in “10 de Octubre” borough in Havana, CubaA scale with individual satisfaction and its corresponding score value was used (Table 1).

Expression	Number SVN	Scoring
Clearly pleased	(1, 0, 0)	1
More pleased than unpleased	(1, 0.25, 0.25)	0.5
Not defined	I	0
More unpleased than pleased	(0.25, 0.25, 1)	-0.5
Clearly unpleased	(0,0,1)	-1
Contradictory	(1,0,1)	0

Table 1. Individual satisfaction scale.

A sample of 21 teachers and methodologists from senior high education were surveyed. The survey was elaborated with 7 questions, three closed questions interspersed in four open questions; of which 1 fulfilled the introductory function and three functioned as reaffirmation and sustenance of objectivity to the respondent.

<table>
<tr><td rowspan="2"></td><td colspan="9">Would you consider postponing the development of the competence to undertake as a contribution to the comprehensive education of the student of senior high education?</td></tr>
<tr><td colspan="3">No</td><td colspan="3">I don´t know</td><td colspan="3">yes</td></tr>
<tr><td rowspan="2">Do your expectations meet the application of the strategy for the development of the competence to undertake as a contribution to the comprehensive education of the student of senior high education?</td><td colspan="9">If you could choose freely, a strategy for the formation of competencies in students of senior high education would you choose one with similar characteristics to the one used for the development of the competence to undertake?</td></tr>
<tr><td>yes</td><td>I don´t know</td><td>No</td><td>yes</td><td>I don´t know</td><td>No</td><td>yes</td><td>I don´t know</td><td>No</td></tr>
</table>

Very pleased.	1 (14)	2 (2)	6	2	2	6	6	6	6
Parcially pleased.	2 (2)	2 (2)	3	2 (1)	3	3	6	3	6
It's all the same to me	3	3	3	3	3	3	3	3	3
More unpleased than pleased.	6	3	6	3	4	4	3	4	4
Not pleased	6	6	6	6	4	4	6	4	5
I don´t know what to say	2	3	6	3	3	3	6	3	4

Table 2. The logical picture of the Iadov technique for teachers and methodologists from senior high education.

In this case,the following results are as follows:

Expression	Total	%
Clearly pleased	14	66
More pleased than unpleased	7	33
Not defined	0	0
More unpleased than pleased	0	0
Clearly unpleased	0	0
Contradictory	0	0

Table 3.Results of the application to teachers and methodologists.

The calculation of the score is carried out and it is determined by I. in this case, it was given the same value to each user. The final result of the index of group satisfaction (GSI) that the method portrays, in this case, is: GSI =0.82

This shows a high level of satisfaction according to the satisfaction scale.

For the students, a survey similar to that of the teachers was prepared, and a total of 101 senior high students were interviewed who received the training program with the following results:

	Can you do without undertaking and achieve your professional realization?								
	No			I don´t know			yes		
Are you satisfied with the way in which the program was applied to develop your skills and knowledge to learn to undertake?	Would you like to be an entrepreneur and take on the challenges in your future personal performance?								
	yes	I don´t know	No	yes	I don´t know	No	yes	I don´t know	No
Very pleased.	1 (71)	2 (2)	6	2	2	6	6	6	6
Parcially pleased.	2 (23)	2 (1)	3	2 (1)	3	3	6	3 (1)	6
Its all the same to me.	3 (1)	3	3	3	3 (1)	3	3	3	3
more unpleased than pleased.	6	3	6	3 (1)	4	4	3	4	4
Not pleased	6	6	6	6	4	4	6	4	5
I don´t know what to say	2	3	6	3	3	3	6	3	4

Table 4. Logical picture of Iadov students of senior high education.

In this case, the following results are obtained:

Expression	Total	%
Clearly pleased	71	70.3
More pleased than unpleased	27	26.7
Not defined	3	2.97
More unpleased than pleased	0	0
Clearly unpleased	0	0
Contradictory	0	0

Table 5.Results of the application of the students in senior high.

The calculation of the score is carried out. In this case, it was given the same value of importance to each user. The final result of the index of group satisfaction (GSI) that the method portrays, in this case, is: GSI=0.837 this shows a high level of satisfaction according to the satisfaction scale.

By locating the values reached in the satisfaction scale

- Actors developers: 0.809
- Recipients – actors: 0.837

In both cases, the results are positive, which certifies the effectiveness of the implementation of the strategy, as shown in the graph.

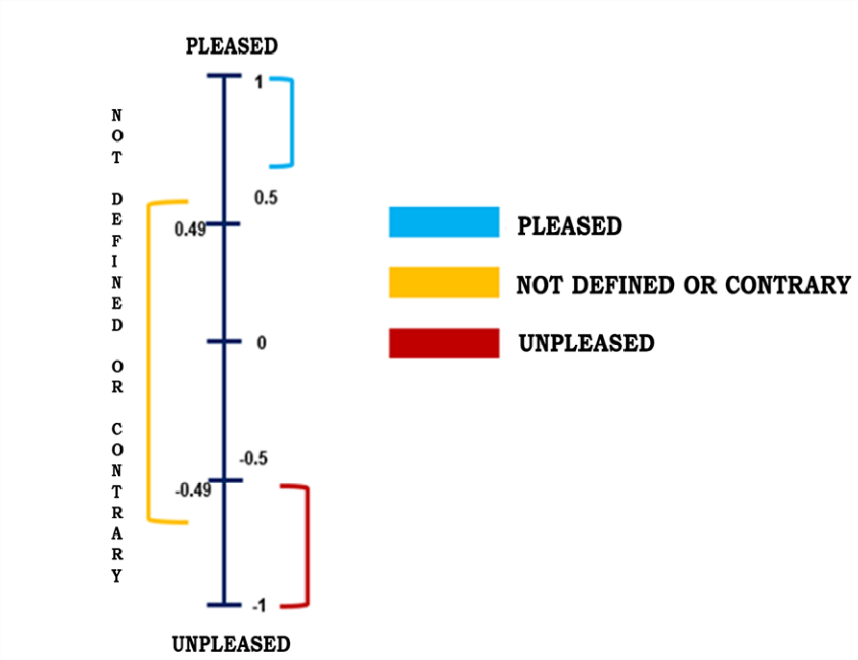

Figure 1. Scale with group satisfaction index

The proposal of extending Iadov method with neutrosophyresults to be easy to use and practical in real-world application. The inclusions of indeterminacy allow a more powerful way to represent information compared with the classical application of the technique. The inclusion of the aggregation operator extends the traditionalIadov method including the importanceof information sources[28]. The application in the real world of the proposal validates the pedagogical strategy for the formation of the competence entrepreneurship in high education.

Conclusions

In this paper, the Iadov method was extended allowing an adequate management of the indeterminacy and the management of the importance of users. Iadov's method with the inclusion of neutrosophic analysis showed its applicability and ease of use in a case study. Among the advantages with respect to the original approach is that it can incorporate the indetermination and contradiction more naturally. Another advantage is that it allows the use of aggregation operators which makes it possible to express, in this case, the importance or expertise of users according to experience or some other criteria.

The validation process using the neutrosophic Iadov technique in users of the implementation of the pedagogical strategy for the formation of the competence entrepreneurship in high education in “10 de Octubre” borough in Havana, Cuba confirmed its feasibility of use. The results were expressed quantitatively in a high Group Satisfaction Index in the two applications presented in the case study.

Future works will concentrate on the uses of the 2-tuplelinguistic model for giving a linguistic output and the use of different aggregation operator. The development of a software tool supporting the proposal is another area of future research.

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Neutrosophic Soft Normed Linear Spaces

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Abstract: In this paper, the neutrosophic norm has been defined on a soft linear space which is hereafter called neutrosophic soft normed linear space (NSNLS). Several characteristics of sequences defined in this space have been investigated here. Moreover, the notion of convexity and the metric in NSNLS have been introduced and some of their properties are established.

Keywords: Neutrosophic Soft Norm; Neutrosophic Soft Normed Linear Space (NSNLS); Convergent Sequence; Cauchy Sequence; Convexity; Metric in NSNLS; Neutrosophic Soft Metric Space (NSMS).

1 Introduction

The concept of Neutrosophic Set (NS) was first introduced by Smarandache [4, 5] which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Zadeh's [11] classical concept of fuzzy set is a strong mathematical tool to deal with the complexity generally arising from uncertainty in the form of ambiguity in real life scenario. For different specialized purposes, there are suggestions for nonclassical and higher order fuzzy sets since from the initiation of fuzzy set theory. Among several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [10] have been found to be very useful and applicable. But each of these theories has it's different difficulties as pointed out by Molodtsov [3]. The basic reason for these difficulties is inadequacy of parametrization tool of the theories.

Molodtsov [3] presented soft set theory as a completely generic mathematical tool which is free from the parametrization inadequacy syndrome of different theory dealing with uncertainty. Molodtsov successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. Now, soft set theory and it's applications are progressing rapidly in different fields. The concept of soft point was provided by so many authors but more authentic definition was given in [19]. There is a progressive development of norm linear spaces and inner product spaces over fuzzy set, intuitionistic fuzzy set and soft set by different researchers for instance Dinda and Samanta [1], Felbin [2], Yazar et al. [12], Issac and Maya K. [13], Saadati and Vaezpour [16], Cheng and Mordeson [17], Vijayablaji et al. [18], Das et al. [19-22], Samanta and Jebril [25], Bag and Samanta [26-29], Beaula and Priyanga [30] and many others.

In 2013, Maji [14] has introduced a combined concept Neutrosophic soft set (NS_s). Accordingly, several mathematicians have produced their research works in different mathematical structures for instance Deli [6, 7], Deli and Broumi [8, 9], Maji [15], Broumi et al. [23, 24], Bera and Mahapatra [31-35]. Later, this concept has been modified by Deli and Broumi [9].

In the present study, our aim is to define the neutrosophic norm on a soft linear space and investigate it's several characteristics. Section 2 gives some preliminary necessary definitions which will be used in rest of this paper. The notion of neutrosophic norm over soft linear space and the sequence of soft points in an NSNLS have been introduced in Section 3. Then, there is a study on Cauchy sequence in an NSNLS in Section 4. The concept of convexity and the metric in NSNLS have been developed in Section 5 and Section 6, respectively. Finally, the conclusion of the present work is briefly stated in Section 7.

2 Preliminaries

We recall some basic definitions related to fuzzy set, soft set, neutrosophic soft set for the sake of completeness.

2.1 Definition [33]

A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t - norm if $*$ satisfies the following conditions :
(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a$, $\forall a \in [0,1]$.
(iv) $a * b \leq c * d$ if $a \leq c, b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2.2 Definition [33]

A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm (s-norm) if $\diamond$ satisfies the following conditions :
(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a$, $\forall a \in [0,1]$.
(iv) $a \diamond b \leq c \diamond d$ if $a \leq c, b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

2.3 Definition [4]

Let X be a space of points (objects), with a generic element in X denoted by x. A NS B on X is characterized by a truth-membership function T_B, an indeterminacy-membership function I_B and a falsity-membership function F_B where $T_B(x)$, $I_B(x)$ and $F_B(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$ i.e., $T_B, I_B, F_B : X \rightarrow]^-0, 1^+[$. Thus the NS B over X is defined as : $B = \{< x, (T_B(x), I_B(x), F_B(x)) > | x \in X\}$.

There is no restriction on the sum of $T_B(x), I_B(x), F_B(x)$ and so, $^-0 \leq \sup T_B(x) + \sup I_B(x) + \sup F_B(x) \leq 3^+$. Here $1^+ = 1 + \epsilon$, where 1 is it's standard part and ϵ it's non-standard part. Similarly $^-0 = 0 - \epsilon$, where 0 is it's standard part and ϵ it's non-standard part.

From philosophical point of view, a NS takes the value from real standard or nonstandard subsets of $]^-0, 1^+[$. But to practice in real scientific and engineering areas, it is difficult to use NS with value from real standard or nonstandard subset of $]^-0, 1^+[$. Hence we consider the NS which takes the value from the subset of [0,1].

2.4 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \rightarrow P(U)$ is a mapping.

2.5 Definition [14]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the power set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NS_s over U, where $F : A \rightarrow NS(U)$ is a mapping.

This concept has been redefined by Deli and Broumi [9] as given below.

2.6 Definition [9]

Let U be an initial universe set and E be a set of parameters that describes the elements of U. Let $NS(U)$ denote the power set of all NSs over U. Then, a NS_s N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \rightarrow NS(U)$ where f_N is called approximate function of N. In other words, the NS_s N is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs i.e., $N = \{(e, f_N(e)) : e \in E\}$ where $f_N(e) = \{< x, (T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)) > | x \in U\}$. Here $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$ are respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2.6.1 *Example*

Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1(\text{beautiful}), e_2(\text{wooden}), e_3(\text{costly})\}$ be a set of parameters with respect to which the nature of houses are described. Let,

$f_N(e_1) = \{< h_1, (0.5, 0.6, 0.3) >, < h_2, (0.4, 0.7, 0.6) >, < h_3, (0.6, 0.2, 0.3) >\}$;
$f_N(e_2) = \{< h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.1, 0.2) >\}$;
$f_N(e_3) = \{< h_1, (0.7, 0.4, 0.3) >, < h_2, (0.6, 0.7, 0.2) >, < h_3, (0.7, 0.2, 0.5) >\}$;

Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is a NS_s over (U, E). The tabular representation of the NS_s N is as :

Table 1 : Tabular form of NS_s N

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.1,0.2)	(0.7,0.2,0.5)

2.6.2 Definition [9]

Let N_1 and N_2 be two NS_ss over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as :

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) > | x \in U\}) | e \in E\}$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)$, $F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$.

Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) > | x \in U\}) | e \in E\}$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x), I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x)$, $F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$.

2.7 Definition [25]

An intuitionistic fuzzy norm on a linear space $V(F)$ is an object of the form $A = \{< (x, t), \mu(x, t), \nu(x, t) > | (x, t) \in V \times R^+\}$, where μ, ν are fuzzy functions on $V \times R^+$, μ denotes the degree of membership and ν denotes the degree of non-membership $(x, t) \in V \times R^+$ satisfying the following conditions :

(i) $\mu(x, t) + \nu(x, t) \leq 1, \forall (x, t) \in V \times R^+$.
(ii) $\mu(x, t) > 0$.
(iii) $\mu(x, t) = 1$ iff $x = \theta$.

(iv) $\mu(cx,t) = \mu(x, \frac{t}{|c|})$, $\forall c(\neq 0) \in F$.
(v) $\mu(x,s) * \mu(y,t) \leq \mu(x+y, s+t)$.
(vi) $\mu(x,\cdot)$ is non-decreasing function of R^+ and $\lim_{t\to\infty} \mu(x,t) = 1$.
(vii) $\nu(x,t) < 1$.
(viii) $\nu(x,t) = 0$ iff $x = \theta$.
(ix) $\nu(cx,t) = \nu(x, \frac{t}{|c|})$, $\forall c(\neq 0) \in F$.
(x) $\nu(x,s) \diamond \nu(y,t) \geq \nu(x+y, s+t)$.
(xi) $\nu(x,\cdot)$ is non-increasing function of R^+ and $\lim_{t\to\infty} \nu(x,t) = 0$.

2.8 Definition [19]

A soft set (F,E) over X is said to be a soft point if there is exactly one $e \in E$, such that $F(e) = \{x\}$ and $F(e') = \phi, \forall e' \in E - \{e\}$. It is denoted by x_e. Two soft points $x_e, y_{e'}$ are said to be equal if $e = e'$ and $x = y$.

2.9 Definition [22]

Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over V so that $G(\lambda)$ is a vector subspace of V, $\forall \lambda \in A$. Then G is called a soft vector space or soft linear space of V over K.

2.10 Definition [12]

Let $SV(\widetilde{X})$ be a soft vector space. Then a mapping $||\cdot|| : SV(\widetilde{X}) \to R^+(E)$ is said to be a soft norm on $SV(\widetilde{X})$, if $||\cdot||$ satisfies the following conditions :
(1) $||x_e|| \geq \tilde{0},\ \forall x_e \in SV(\widetilde{X})$ and $||x_e|| = \tilde{0} \Leftrightarrow x_e = \theta_0$.
(2) $||\tilde{r}x_e|| = |\tilde{r}|\,||x_e||,\ \forall x_e \in SV(\widetilde{X})$ and for every soft scalar $\tilde{r}$.
(3) $||x_e + y_{e'}|| \leq ||x_e|| + ||y_{e'}||,\ \forall x_e, y_{e'} \in SV(\widetilde{X})$.
(4) $||x_e.y_{e'}|| = ||x_e||\,||y_{e'}||,\ \forall x_e, y_{e'} \in SV(\widetilde{X})$.

The soft vector space $SV(\tilde{X})$ with a soft norm $||\cdot||$ on $\widetilde{X}$ is said to be a soft normed linear space and is denoted by $(\widetilde{X}, ||\cdot||)$.

2.11 Definition [32]

Let A be a NS over the universal set X. The (α,β,γ) -cut of A is a crisp subset $A_{(\alpha,\beta,\gamma)}$ of the neutrosophic set A and is defined as $A_{(\alpha,\beta,\gamma)} = \{x \in X : T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$ where $\alpha,\beta,\gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. This $A_{(\alpha,\beta,\gamma)}$ is called (α,β,γ)-level set or (α,β,γ)-cut set of the neutrosophic set A.

3 Neutrosophic soft norm

In this section, we have defined NSNLS with suitable examples, the convergence of sequence in NSNLS and have studied some related basic properties.

Unless otherwise stated, $V(K)$ is a vector space over the field K and E is treated as the parametric set through out this paper, $e \in E$ an arbitrary parameter.

3.1 Definition

Let $\widetilde{V}$ be a soft linear space over the field K and $R(E)$, $\Delta_{\widetilde{V}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{V}$. Then, a neutrosophic subset N over $\Delta_{\widetilde{V}} \times R(E)$ is called a neutrosophic soft norm on $\widetilde{V}$ if for $x_e, y_{e'} \in \Delta_{\widetilde{V}}$ and $\tilde{c} \in K$ ($\tilde{c}$ being soft scalar), the following conditions hold.

(i) $0 \leq T_N(x_e, \tilde{t}),\ I_N(x_e, \tilde{t}),\ F_N(x_e, \tilde{t}) \leq 1,\ \forall \tilde{t} \in R(E)$.
(ii) $0 \leq T_N(x_e, \tilde{t}) + I_N(x_e, \tilde{t}) + F_N(x_e, \tilde{t}) \leq 3,\ \forall \tilde{t} \in R(E)$.
(iii) $T_N(x_e, \tilde{t}) = 0$ with $\tilde{t} \leq \tilde{0}$.
(iv) $T_N(x_e, \tilde{t}) = 1$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(v) $T_N(\tilde{c}x_e, \tilde{t}) = T_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(vi) $T_N(x_e, \tilde{s}) * T_N(y_{e'}, \tilde{t}) \leq T_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t})\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(vii) $T_N(x_e, \cdot)$ is a continuous non-decreasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} T_N(x_e, \tilde{t}) = 1$.
(viii) $I_N(x_e, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(ix) $I_N(x_e, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(x) $I_N(\tilde{c}x_e, \tilde{t}) = I_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(xi) $I_N(x_e, \tilde{s}) \diamond I_N(y_{e'}, \tilde{t}) \geq I_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(xii) $I_N(x_e, \cdot)$ is a continuous non-increasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} I_N(x_e, \tilde{t}) = 0$.
(xiii) $F_N(x_e, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(xiv) $F_N(x_e, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_e = \tilde{\theta}$, the null soft vector.
(xv) $F_N(\tilde{c}x_e, \tilde{t}) = F_N(x_e, \frac{\tilde{t}}{|\tilde{c}|}),\ \forall \tilde{c}(\neq \tilde{0}),\ \tilde{t} > \tilde{0}$.
(xvi) $F_N(x_e, \tilde{s}) \diamond F_N(y_{e'}, \tilde{t}) \geq F_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}),\ \forall \tilde{s}, \tilde{t} \in R(E)$.
(xvii) $F_N(x_e, \cdot)$ is a continuous non-increasing function for $\tilde{t} > \tilde{0}$ and $\lim_{\tilde{t} \to \infty} F_N(x_e, \tilde{t}) = 0$.

Then $(\widetilde{V}(K), N, *, \diamond)$ is a NSNLS.

3.1.1 *Example*

Let $(\widetilde{V}, ||\cdot||)$ be a soft normed linear space. Take $a * b = ab$ and $a \diamond b = a + b - ab$. Define,

$$T_N(x_e, \tilde{t}) = \begin{cases} \frac{\tilde{t}}{\tilde{t} \oplus ||x_e||} & \text{if } \tilde{t} > ||x_e|| \\ 0 & \text{otherwise .} \end{cases} \quad I_N(x_e, \tilde{t}) = \begin{cases} \frac{||x_e||}{\tilde{t} \oplus ||x_e||} & \text{if } \tilde{t} > ||x_e|| \\ 1 & \text{otherwise .} \end{cases} \quad F_N(x_e, \tilde{t}) = \begin{cases} \frac{||x_e||}{\tilde{t}} & \text{if } \tilde{t} > ||x_e|| \\ 1 & \text{otherwise .} \end{cases}$$

Then $(\widetilde{V}(K), N, *, \diamond)$ is an NSNLS.

Proof. All the conditions are well satisfied. We shall only verify the conditions (vi), (xi), (xvi) for $\tilde{s}, \tilde{t} > \tilde{0}$ because these are obvious for $\tilde{s}, \tilde{t} \leq \tilde{0}$. Now,

$$\begin{aligned} & T_N(x_e \oplus y_{e'}, \tilde{s} \oplus \tilde{t}) - T_N(x_e, \tilde{s}) * T_N(y_{e'}, \tilde{t}) \\ = & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus ||x_e \oplus y_{e'}||} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} \\ \geq & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||)} \\ = & \{(\tilde{s} \oplus \tilde{t})(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||) - \tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||)\}/B \\ & \text{where } B = (\tilde{s} \oplus \tilde{t} \oplus ||x_e|| \oplus ||y_{e'}||)(\tilde{s} \oplus ||x_e||)(\tilde{t} \oplus ||y_{e'}||) \\ = & \{\tilde{t}^2||x_e|| \oplus \tilde{s}^2||y_e'|| \oplus (\tilde{s} \oplus \tilde{t})||x_e y_e'||\}/B \geq 0. \end{aligned}$$

Hence, $T_N(x_e,\tilde{s}) * T_N(y_{e'},\tilde{t}) \leq T_N(x_e \oplus y_{e'}, \tilde{s}\oplus\tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. Next,

$$\begin{aligned}
& I_N(x_e,\tilde{s}) \diamond I_N(y_{e'},\tilde{t}) - I_N(x_e \oplus y_{e'}, \tilde{s}\oplus\tilde{t}) \\
= & \frac{||x_e||}{\tilde{s}\oplus||x_e||} \oplus \frac{||y_{e'}||}{\tilde{t}\oplus||y_{e'}||} - \frac{||x_e y_{e'}||}{(\tilde{s}\oplus||x_e||)(\tilde{t}\oplus||y_{e'}||)} - \frac{||x_e\oplus y_{e'}||}{||x_e\oplus y_{e'}||\oplus\tilde{s}\oplus\tilde{t}} \\
= & \frac{||x_e y_{e'}||\oplus\tilde{t}||x_e||\oplus\tilde{s}||y_{e'}||}{(\tilde{s}\oplus||x_e||)(\tilde{t}\oplus||y_{e'}||)} - \frac{||x_e\oplus y_{e'}||}{||x_e\oplus y_{e'}||\oplus\tilde{s}\oplus\tilde{t}} \\
= & \{(||x_e\oplus y_{e'}||\oplus\tilde{s}\oplus\tilde{t})(\tilde{t}||x_e||\oplus\tilde{s}||y_{e'}||\oplus||x_e y_{e'}||) \\
& -||x_e\oplus y_{e'}||(\tilde{s}\oplus||x_e||)(\tilde{t}\oplus||y_{e'}||)\}/D \\
& \text{where } D=(\tilde{s}\oplus\tilde{t}\oplus||x_e\oplus y_{e'}||)(\tilde{s}\oplus||x_e||)(\tilde{t}\oplus||y_{e'}||) \\
= & \{(\tilde{s}\oplus\tilde{t})(\tilde{t}||x_e||\oplus\tilde{s}||y_{e'}||\oplus||x_e y_{e'}||)-\tilde{s}\tilde{t}||x_e\oplus y_{e'}||\}/D \\
\geq & \{(\tilde{s}\oplus\tilde{t})(\tilde{t}||x_e||\oplus\tilde{s}||y_{e'}||\oplus||x_e y_{e'}||)-\tilde{s}\tilde{t}(||x_e||\oplus||y_{e'}||)\}/D \\
= & \{\tilde{t}^2||x_e||\oplus\tilde{s}^2||y_{e'}||\oplus(\tilde{s}\oplus\tilde{t})||x_e y_{e'}||\}/D \geq 0.
\end{aligned}$$

Hence, $I_N(x_e,\tilde{s}) \diamond I_N(y_{e'},\tilde{t}) \geq I_N(x_e \oplus y_{e'}, \tilde{s}\oplus\tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. Finally,

$$\begin{aligned}
& F_N(x_e,\tilde{s}) \diamond F_N(y_{e'},\tilde{t}) - F_N(x_e \oplus y_{e'}, \tilde{s}\oplus\tilde{t}) \\
= & \frac{||x_e||}{\tilde{s}} \oplus \frac{||y_{e'}||}{\tilde{t}} - \frac{||x_e y_{e'}||}{\tilde{s}\tilde{t}} - \frac{||x_e\oplus y_{e'}||}{\tilde{s}\oplus\tilde{t}} \\
= & \frac{\tilde{t}||x_e||\oplus\tilde{s}||y_{e'}||-||x_e y_{e'}||}{st} - \frac{||x_e\oplus y_{e'}||}{\tilde{s}\oplus\tilde{t}} \\
\geq & \{\tilde{s}^2||y_{e'}||\oplus\tilde{t}^2||x_e||-(\tilde{s}\oplus\tilde{t})||x_e y_{e'}||\}/\tilde{s}\tilde{t}(\tilde{s}\oplus\tilde{t}) \\
= & \{\tilde{s}||y_{e'}||(\tilde{s}-||x_e||)\oplus\tilde{t}||x_e||(\tilde{t}-||y_{e'}||)\}/\tilde{s}\tilde{t}(\tilde{s}\oplus\tilde{t}) \geq 0 \text{ (as } \tilde{s}>||x_e||,\ \tilde{t}>||y_{e'}||).
\end{aligned}$$

Thus, $F_N(x_e,\tilde{s}) \diamond F_N(y_{e'},\tilde{t}) \geq F_N(x_e \oplus y_{e'}, \tilde{s}\oplus\tilde{t}),\ \forall \tilde{s},\tilde{t} \in R(E)$. This completes the proof.

3.2 Definition

Let $\{x^n_{e_n}\}$ be a sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$. Then the sequence converges to a soft point $x_e \in \widetilde{V}$ iff for given $r \in (0,1), \tilde{t} > \tilde{0}$ there exists $n_0 \in \mathbf{N}$ (the set of natural numbers) such that
$T_N(x^n_{e_n} - x_e, \tilde{t}) > 1-r, I_N(x^n_{e_n} - x_e, \tilde{t}) < r, F_N(x^n_{e_n} - x_e, \tilde{t}) < r, \forall n \geq n_0$. Or,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \tilde{t}) = 1,\ \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \tilde{t}) = 0,\ \lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \tilde{t}) = 0$ as $\tilde{t}\to\infty$.
Then the sequence $\{x^n_{e_n}\}$ is called a convergent sequence in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

3.3 Theorem

If the sequence $\{x^n_{e_n}\}$ in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is convergent, then the point of convergence is unique.

Proof. Let $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} x^n_{e_n} = y_{e_k}$ for $x_{e_j} \neq y_{e_k}$. Then for $\tilde{s},\tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0$ as $\tilde{s}\to\infty$ and
$\lim_{n\to\infty} T_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - y_{e_k}, \tilde{t}) = 0$ as $\tilde{t}\to\infty$.

Now, $\quad T_N(x_{e_j} - y_{e_k}, \tilde{s}\oplus\tilde{t}) = T_N(x_{e_j} - x^n_{e_n} \oplus x^n_{e_n} - y_{e_k}, \tilde{s}\oplus\tilde{t}) \geq T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) * T_N(x^n_{e_n} - y_{e_k}, \tilde{t})$

Taking limit as $n\to\infty$ and for $\tilde{s},\tilde{t}\to\infty$,

$T_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq 1 * 1 = 1$ i.e., $T_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 1$ (1)

Further, $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = I_N(x_{e_j} - x^n_{e_n} \oplus x^n_{e_n} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) \diamond I_N(x^n_{e_n} - y_{e_k}, \tilde{t})$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$,
$I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq 0 \diamond 0 = 0$ i.e., $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (2)
Similarly, $F_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (3)
Hence, $x_{e_j} = y_{e_k}$ and this completes the proof.

3.4 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} y^n_{e_n} = y_{e_k}$ then $\lim_{n\to\infty}(x^n_{e_n} \oplus y^n_{e_n}) = x_{e_j} \oplus y_{e_k}$.

Proof. Here, for $\tilde{s}, \tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 1, \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0, \lim_{n\to\infty} F_N(x^n_{e_n} - x_{e_j}, \tilde{s}) = 0$ as $\tilde{s} \to \infty$ and $\lim_{n\to\infty} T_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 1, \lim_{n\to\infty} I_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 0, \lim_{n\to\infty} F_N(y^n_{e_n} - y_{e_k}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$. Now,

$$\begin{aligned}
&\lim_{n\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = \lim_{n\to\infty} T_N[(x^n_{e_n} - x_{e_j}) \oplus (y^n_{e_n} - y_{e_k}), \tilde{s} \oplus \tilde{t}] \\
&\geq \lim_{n\to\infty} T_N(x^n_{e_n} - x_{e_j}, \tilde{s}) * \lim_{n\to\infty} T_N(y^n_{e_n} - y_{e_k}, \tilde{t}) \quad \text{[by (vi) in Definition 3.1]} \\
&= 1 * 1 = 1 \text{ as } \tilde{s}, \tilde{t} \to \infty.
\end{aligned}$$

Hence, $\lim_{n\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 1$ as $\tilde{s}, \tilde{t} \to \infty$. Again,

$$\begin{aligned}
&\lim_{n\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = \lim_{n\to\infty} I_N[(x^n_{e_n} - x_{e_j}) \oplus (y^n_{e_n} - y_{e_k}), \tilde{s} \oplus \tilde{t}] \\
&\leq \lim_{n\to\infty} I_N(x^n_{e_n} - x_{e_j}, \tilde{s}) \diamond \lim_{n\to\infty} I_N(y^n_{e_n} - y_{e_k}, \tilde{t}) \quad \text{[by (xi) in Definition 3.1]} \\
&= 0 \diamond 0 = 0 \text{ as } \tilde{s}, \tilde{t} \to \infty.
\end{aligned}$$

So, $\lim_{n\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 0$ as $\tilde{s}, \tilde{t} \to \infty$.
Similarly, $\lim_{n\to\infty} F_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x_{e_j} \oplus y_{e_k}), \tilde{s} \oplus \tilde{t}] = 0$ as $\tilde{s}, \tilde{t} \to \infty$ and this ends the theorem.

3.5 Theorem

If $\lim_{n\to\infty} x^n_{e_n} = x_e$ and $\tilde{0} \neq \tilde{c} \in K$, then $\lim_{n\to\infty} \tilde{c}x^n_{e_n} = \tilde{c}x_e$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

Proof. Here,
$\lim_{n\to\infty} T_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 1$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
$\lim_{n\to\infty} I_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 0$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
$\lim_{n\to\infty} F_N(\tilde{c}x^n_{e_n} - \tilde{c}x_e, \tilde{t}) = \lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{|\tilde{c}|}) = 0$, as $\frac{\tilde{t}}{|\tilde{c}|} \to \infty$.
Thus the theorem is proved.

4 Cauchy Sequence in NSNLS (Fundamental Sequence in NSNLS)

Here, we have defined the Cauchy sequence in NSNLS, Complete NSNLS and have investigated their several structural characteristics.

4.1 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be bounded for $r \in (0,1)$ and $\tilde{t} > \tilde{0}$ if the followings hold :

$T_N(x^n_{e_n}, \tilde{t}) > 1-r, I_N(x^n_{e_n}, \tilde{t}) < r, F_N(x^n_{e_n}, \tilde{t}) < r$, $\forall n \in$ **N** (the set of natural numbers).

4.2 Definition

1. A sequence $\{x^n_{e_n}\}$ of soft points in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be a Cauchy sequence if given $r \in (0,1), \tilde{t} > \tilde{0}$ there exists $n_0 \in$ **N** (the set of natural numbers) such that

$T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) > 1-r, I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) < r, F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) < r, \ \ \forall m, n \geq n_0$. Or,

$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1$, $\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$, $\lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

2. Let $\{x^n_{e_n}\}$ be a Cauchy sequence of soft points in a soft normed linear space $(\widetilde{V}, ||\cdot||)$. Then $\lim_{n,m\to\infty} ||x^n_{e_n} - x^m_{e_m}|| = 0$ hold.

4.2.1 *Example*

For $\tilde{t} > \tilde{0}$, let $T_N(x_e, \tilde{t}) = \frac{\tilde{t}}{\tilde{t} \oplus ||x_e||}$, $I_N(x_e, \tilde{t}) = \frac{||x_e||}{\tilde{t} \oplus ||x_e||}$, $F_N(x_e, \tilde{t}) = \frac{||x_e||}{\tilde{t}}$.

Then $(\widetilde{V}(K), N, *, \diamond)$ is an NSNLS. Now,

$$\lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = 1, \ \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = 0, \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t}} = 0.$$

$$\Rightarrow \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1, \ \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0 \text{ as } \tilde{t} \to \infty.$$

This shows that $\{x^n_{e_n}\}$ is a Cauchy sequence in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

4.3 Theorem

Every convergent sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is a Cauchy sequence.

Proof. Let $\{x^n_{e_n}\}$ be a convergent sequence of soft points in a NSNLS $(\widetilde{V}(K), N, *, \diamond)$ so that $\lim_{n\to\infty} x^n_{e_n} = x_e$. Then for $\tilde{t} > \tilde{0}$,

$$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m} \oplus x_e - x_e, \tilde{t}) = \lim_{n,m\to\infty} T_N[(x^n_{e_n} - x_e) \oplus (x_e - x^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) * \lim_{m\to\infty} T_N(x_e - x^m_{e_m}, \frac{\tilde{t}}{2}) \quad \text{[by (vi) in Definition 3.1]}$$
$$= \lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) * \lim_{m\to\infty} T_N(x^m_{e_m} - x_e, \frac{\tilde{t}}{2}) \quad \text{[by (v) in Definition 3.1]}$$
$$= 1 * 1 = 1 \ \text{ as } \ \tilde{t} \to \infty.$$

So, $\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1$. Again,

$$\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m} \oplus x_e - x_e, \tilde{t}) = \lim_{n,m\to\infty} I_N[(x^n_{e_n} - x_e) \oplus (x_e - x^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) \diamond \lim_{m\to\infty} I_N(x_e - x^m_{e_m}, \frac{\tilde{t}}{2}) \quad \text{[by (xi) in Definition 3.1]}$$

$= \lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \frac{\tilde{t}}{2}) \diamond \lim_{m\to\infty} I_N(x^m_{e_m} - x_e, \frac{\tilde{t}}{2})$ [by (x) in Definition 3.1]
$= 0 \diamond 0 = 0$ as $\tilde{t} \to \infty$.

So, $\lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ and similarly, $\lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$.
Hence, $\{x^n_{e_n}\}$ is a Cauchy sequence.

4.3.1 *Example*

The following example will clarify that the inverse of the Theorem 4.3 may not be true.
Let $R_1 = \{\frac{1}{n} | n \in \mathbf{N}\}$ (**N** being the set of natural numbers) be a subset of real numbers and $||x|| = |x|$. With respect to the neutrosophic norm defined in [4.2.1], obviously $(\widetilde{R}_1(R), N, *, \diamond)$ is an NSNLS. Now,

$$\lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = \lim_{n,m\to\infty} \frac{\tilde{t}}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{m}|} = 1, \ \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t} \oplus ||x^n_{e_n} - x^m_{e_m}||} = \lim_{n,m\to\infty} \frac{|\frac{1}{n} - \frac{1}{m}|}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{m}|} = 0,$$
$$\text{and } \lim_{n,m\to\infty} \frac{||x^n_{e_n} - x^m_{e_m}||}{\tilde{t}} = \lim_{n,m\to\infty} \frac{|\frac{1}{n} - \frac{1}{m}|}{\tilde{t}} = 0.$$

Thus $\{x^n_{e_n}\}$ is a Cauchy sequence of soft point in the NSNLS $(\widetilde{R}_1(R), N, *, \diamond)$. But,
$\lim_{n\to\infty} I_N(x^n_{e_n} - x^k_{e_k}, \tilde{t}) = \lim_{n\to\infty} \frac{|\frac{1}{n} - \frac{1}{k}|}{\tilde{t} \oplus |\frac{1}{n} - \frac{1}{k}|} \neq 0.$
This shows that the Cauchy sequence $\{x^n_{e_n}\}$ is not convergent in that NSNLS.

4.4 Theorem

Every Cauchy sequence is bounded in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ if $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for any two real numbers $a, b \in [0, 1]$.

Proof. Let $\{x^n_{e_n}\}$ be a Cauchy sequence. Then given a fixed $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_0 such that $T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') > 1 - r_0, \ \forall m, n \geq n_0$.
Since $\lim_{\tilde{t}\to\infty} T_N(x_e, \tilde{t}) = 1$, for each $x^i_{e_i}$ there exists $\tilde{t}_i > \tilde{0}$ such that $T_N(x^i_{e_i}, \tilde{t}) > 1 - r_0, \ \forall \tilde{t} \geq \tilde{t}_i, \ i = 1, 2, 3, \cdots$; Let $\tilde{t}_0 = \tilde{t}' + \max\{\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_{n_0}\}$. Then,

$T_N(x^n_{e_n}, \tilde{t}_0) > T_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}_{n_0})$ [by (vii) in Definition (3.1) (i.e., the monotonicity property)]
$= T_N(x^n_{e_n} - x^{n_0}_{e_{n_0}} \oplus x^{n_0}_{e_{n_0}}, \tilde{t}' \oplus \tilde{t}_{n_0}) \geq T_N(x^n_{e_n} - x^{n_0}_{e_{n_0}}, \tilde{t}') * T_N(x^{n_0}_{e_{n_0}}, \tilde{t}_{n_0})$ [by (vi) in Definition (3.1)]
$> (1 - r_0) * (1 - r_0) = 1 - r_0, \ \forall n \geq n_0$.

Thus $T_N(x^n_{e_n}, \tilde{t}_0) > 1 - r_0, \ \forall n \geq n_0$ and further $T_N(x^n_{e_n}, \tilde{t}_0) \geq T_N(x^n_{e_n}, \tilde{t}_n) > 1 - r_0, \ \forall n = 1, 2, \cdots, n_0$.
Hence as a whole, $T_N(x^n_{e_n}, \tilde{t}_0) > 1 - r_0$. (4)

Next, for $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_1 such that $I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') < r_0, \ \forall m, n \geq n_1$. Since $\lim_{\tilde{t}\to\infty} I_N(x_e, \tilde{t}) = 0$, for each $x^i_{e_i}$ there exists $\tilde{t}'_i > 0$ such that $I_N(x^i_{e_i}, \tilde{t}) < r_0, \ \forall \tilde{t} \geq \tilde{t}'_i, \ i = 1, 2, 3, \cdots$; Let $\tilde{t}'_0 = \tilde{t}' + \max\{\tilde{t}'_1, \tilde{t}'_2, \cdots \tilde{t}'_{n_1}\}$. Then,

$I_N(x^n_{e_n}, \tilde{t}'_0) < I_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}'_{n_1})$ [by (xii) in Definition (3.1) (i.e., the monotonicity property)]
$= I_N(x^n_{e_n} - x^{n_1}_{e_{n_1}} \oplus x^{n_1}_{e_{n_1}}, \tilde{t}' \oplus \tilde{t}'_{n_1}) \leq I_N(x^n_{e_n} - x^{n_1}_{e_{n_1}}, \tilde{t}') \diamond I_N(x^{n_1}_{e_{n_1}}, \tilde{t}'_{n_1})$ [by (xi) in Definition (3.1)]
$< r_0 \diamond r_0 = r_0, \ \forall n \geq n_1$.

Thus $I_N(x^n_{e_n}, \tilde{t}'_0) < r_0,\ \forall n \geq n_1$ and further $I_N(x^n_{e_n}, \tilde{t}'_0) \leq I_N(x^n_{e_n}, \tilde{t}'_n) < r_0 \forall n = 1, 2, \cdots, n_1$.
Hence as a whole, $I_N(x^n_{e_n}, \tilde{t}'_0) < r_0$. (5)

Finally, for $r_0 \in (0, 1)$ and $\tilde{t}' > \tilde{0}$, there exists a natural number n_2 such that $F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}') < r_0,\ \forall m, n \geq n_2$. Since $\lim_{\tilde{t}\to\infty} F_N(x_e, \tilde{t}) = 0$, for each $x^i_{e_i}$ there exists $\tilde{t}''_i > 0$ such that $F_N(x^i_{e_i}, \tilde{t}) < r_0,\ \forall \tilde{t} \geq \tilde{t}''_i,\ i = 1, 2, 3, \cdots$; Let $\tilde{t}''_0 = \tilde{t}' + \max\{\tilde{t}''_1, \tilde{t}''_2, \cdots \tilde{t}''_{n_2}\}$. Then,

$$F_N(x^n_{e_n}, \tilde{t}''_0) < F_N(x^n_{e_n}, \tilde{t}' \oplus \tilde{t}'_{n_2}) \text{ [by (xvii) in Definition (3.1) (i.e., the monotonicity property)]}$$
$$= F_N(x^n_{e_n} - x^{n_2}_{e_{n_2}} \oplus x^{n_2}_{e_{n_2}}, \tilde{t}' \oplus \tilde{t}''_{n_2}) \leq F_N(x^n_{e_n} - x^{n_2}_{e_{n_2}}, \tilde{t}') \diamond F_N(x^{n_2}_{e_{n_2}}, \tilde{t}''_{n_2}) \text{ [by (xvi) in Definition (3.1)]}$$
$$< r_0 \diamond r_0 = r_0\ \ \forall n \geq n_2$$

Thus $F_N(x^n_{e_{n_2}}, \tilde{t}''_0) < r_0,\ \forall n \geq n_2$ and further $F_N(x^n_{e_n}, \tilde{t}''_0) \leq F_N(x^n_{e_n}, \tilde{t}''_n) < r_0,\ \forall n = 1, 2, \cdots, n_2$.
Hence as a whole, $F_N(x^n_{e_n}, \tilde{t}''_0) < r_0$. (6)

This completes the theorem.

4.5 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if $\{x^n_{e_n}\}$, $\{y^n_{e_n}\}$ are Cauchy sequences of soft vectors and $\{\tilde{\lambda}_n\}$ is a Cauchy sequence of soft scalars in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, then $\{x^n_{e_n} \oplus y^n_{e_n}\}$ and $\{\tilde{\lambda}_n y^n_{e_n}\}$ are also Cauchy sequences in NSNLS $(\widetilde{V}(K), N, *, \diamond)$.

Proof. For $\tilde{t} > \tilde{0}$, we have,
$\lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 1, \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(x^n_{e_n} - x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$ and
$\lim_{n,m\to\infty} T_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 1, \lim_{n,m\to\infty} I_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 0, \lim_{n,m\to\infty} F_N(y^n_{e_n} - y^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

$$\lim_{n,m\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = \lim_{n,m\to\infty} T_N[(x^n_{e_n} - x^m_{e_m}) \oplus (y^n_{e_n} - y^m_{e_m}), \tilde{t}]$$
$$\geq \lim_{n,m\to\infty} T_N(x^n_{e_n} - x^m_{e_m}, \frac{\tilde{t}}{2}) * \lim_{n,m\to\infty} T_N(y^n_{e_n} - y^m_{e_m}, \frac{\tilde{t}}{2}) = 1 * 1 = 1 \text{ as } \tilde{t} \to \infty.$$

Hence, $\lim_{n,m\to\infty} T_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 1$ as $\tilde{t} \to \infty$.

$$\lim_{n,m\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = \lim_{n,m\to\infty} I_N[(x^n_{e_n} - x^m_{e_m}) \oplus (y^n_{e_n} - y^m_{e_m}), \tilde{t}]$$
$$\leq \lim_{n,m\to\infty} I_N(x^n_{e_n} - x^m_{e_m}, \frac{\tilde{t}}{2}) \diamond \lim_{n,m\to\infty} I_N(y^n_{e_n} - y^m_{e_m}, \frac{\tilde{t}}{2}) = 0 \diamond 0 = 0 \text{ as } \tilde{t} \to \infty$$

So, $\lim_{n,m\to\infty} I_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$.
Similarly, $\lim_{n,m\to\infty} F_N[(x^n_{e_n} \oplus y^n_{e_n}) - (x^m_{e_m} \oplus y^m_{e_m}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$.

This ends the first part. For the next part,

$$\lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = \lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}) \oplus (\tilde{\lambda}_m y^n_{e_n} - \tilde{\lambda}_m y^n_{e_n}), \tilde{t}]$$
$$= \lim_{m,n\to\infty} T_N[\tilde{\lambda}_m (y^m_{e_m} - y^n_{e_n}) \oplus y^n_{e_n}(\tilde{\lambda}_m - \tilde{\lambda}_n), \tilde{t}] \geq \lim_{m,n\to\infty} T_N[(y^m_{e_m} - y^n_{e_n}), \frac{\tilde{t}}{2|\tilde{\lambda}_m|}] * T_N(y^n_{e_n}, \frac{\tilde{t}}{2|\tilde{\lambda}_m - \tilde{\lambda}_n|})$$

Since $|\tilde{\lambda}_m - \tilde{\lambda}_n| \to \tilde{0}$ as $m, n \to \infty$, so $|\tilde{\lambda}_m - \tilde{\lambda}_n| \neq \tilde{0}$. Again $\{y^n_{e_n}\}$ being Cauchy sequence is bounded.

Hence, $\lim_{m,n\to\infty} T_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 1$ as $\tilde{t} \to \infty$. Further,

$$\lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = \lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}) \oplus (\tilde{\lambda}_m y^n_{e_n} - \tilde{\lambda}_m y^n_{e_n}), \tilde{t}]$$
$$= \lim_{m,n\to\infty} I_N[\tilde{\lambda}_m (y^m_{e_m} - y^n_{e_n}) \oplus y^n_{e_n}(\tilde{\lambda}_m - \tilde{\lambda}_n), \tilde{t}] \le \lim_{m,n\to\infty} I_N[(y^m_{e_m} - y^n_{e_n}), \frac{\tilde{t}}{2|\tilde{\lambda}_m|}] \diamond I_N(y^n_{e_n}, \frac{\tilde{t}}{2|\tilde{\lambda}_m - \tilde{\lambda}_n|})$$

By similar argument, $\lim_{m,n\to\infty} I_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$ and finally, $\lim_{m,n\to\infty} F_N[(\tilde{\lambda}_m y^m_{e_m} - \tilde{\lambda}_n y^n_{e_n}), \tilde{t}] = 0$ as $\tilde{t} \to \infty$. Hence, the 2nd part is completed.

4.6 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a NSNLS and $\Delta_{\widetilde{V}}$ be the collection of all soft points on $\widetilde{V}$. Then $(\widetilde{V}(K), N, *, \diamond)$ is said to be a complete NSNLS if every Cauchy sequence of soft points in $\Delta_{\widetilde{V}}$ converges to a soft point of $\Delta_{\widetilde{V}}$.

4.7 Theorem

In an NSNLS $(\widetilde{V}(K), N, *, \diamond)$, if every Cauchy sequence has a convergent subsequence then $(\widetilde{V}(K), N, *, \diamond)$ is a complete NSNLS.

Proof. Let $\{x^{n_k}_{e_{n_k}}\}$ be a convergent subsequence of a Cauchy sequence $\{x^n_{e_n}\}$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ such that $\{x^{n_k}_{e_{n_k}}\} \to x_e \in \widetilde{V}$. Since $\{x^n_{e_n}\}$ be a Cauchy sequence in $(\widetilde{V}(K), N, *, \diamond)$, given $\tilde{t} > \tilde{0}$
$\lim_{n,k\to\infty} T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 1, \lim_{n,k\to\infty} I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0, \lim_{n,k\to\infty} F_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$
Again since $\{x^{n_k}_{e_{n_k}}\}$ converges to x_e, then
$\lim_{k\to\infty} T_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 1, \lim_{k\to\infty} I_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 0, \lim_{k\to\infty} F_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$.
Now, $T_N(x^n_{e_n} - x_e, \tilde{t}) = T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}} \oplus x^{n_k}_{e_{n_k}} - x_e, \tilde{t}) \ge T_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) * T_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2})$.
It implies $\lim_{n\to\infty} T_N(x^n_{e_n} - x_e, \tilde{t}) = 1$.
Further, $I_N(x^n_{e_n} - x_e, \tilde{t}) = I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}} \oplus x^{n_k}_{e_{n_k}} - x_e, \tilde{t}) \le I_N(x^n_{e_n} - x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) \diamond I_N(x^{n_k}_{e_{n_k}} - x_e, \frac{\tilde{t}}{2})$.
It implies $\lim_{n\to\infty} I_N(x^n_{e_n} - x_e, \tilde{t}) = 0$. Similarly, $\lim_{n\to\infty} F_N(x^n_{e_n} - x_e, \tilde{t}) = 0$.
This shows that $\{x^n_{e_j}\}$ converges to $x_{e_j} \in \widetilde{V}$ and thus the theorem is proved.

5 Convexity of NSNLS

Here, the notion of convex NSNLS has been introduced along with the development of some basic theorems.

5.1 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a neutrosophic soft normed linear space and $x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$. Then the set of all soft points of the form $z_{e_k} = \tilde{c}x_{e_i} \oplus (\tilde{1} - \tilde{c})y_{e_j}$ such that $\tilde{c}(e) \in (0, 1), \forall e \in E$ is called the line segment joining the soft points x_{e_i}, y_{e_j}. A soft set $(\tilde{\phi} \ne \widetilde{W}(K) \subset \widetilde{V}(K))$ is said to form a convex NSNLS with respect to the same neutrosophic soft norm as defined on $\widetilde{V}$ if all the line segments joining any two soft points of $\widetilde{W}$ are contained in $\widetilde{W}$ and satisfy all the neutrosophic soft norm axioms.

5.2 Definition

A soft subset $\widetilde{W}$ of $\widetilde{V}$ in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ is said to be bounded if for given $r \in (0,1)$ and $\tilde{t} > \tilde{0}$, the following inequalities hold.

$$T_N(x_e, \tilde{t}) > 1 - r, I_N(x_e, \tilde{t}) < r, F_N(x_e, \tilde{t}) < r, \ \forall x_e \in \widetilde{W}\,. \tag{7}$$

5.3 Definition

Let $(\widetilde{V}(K), N, *, \diamond)$ be a NSNLS and $\tilde{t} \in R^+(E)$ (the set of all non-negative soft real numbers). Then an open ball and a closed ball with centre at x_e and radius $r \in (0,1)$ are as follows :

$$OB(x_e, r, \tilde{t}) = \{y_{e'} \in \Delta_{\widetilde{V}} | T_N(x_e - y_{e'}, \tilde{t}) > 1 - r, I_N(x_e - y_{e'}, \tilde{t}) < r, F_N(x_e - y_{e'}, \tilde{t}) < r\}. \tag{8}$$

$$CB[x_e, r, \tilde{t}] = \{y_{e'} \in \Delta_{\widetilde{V}} | T_N(x_e - y_{e'}, \tilde{t}) \geq 1 - r, I_N(x_e - y_{e'}, \tilde{t}) \leq r, F_N(x_e - y_{e'}, \tilde{t}) \leq r\}. \tag{9}$$

5.4 Theorem

Every open ball (closed ball) in an NSNLS is convex and bounded if $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for any two real numbers $a, b \in [0, 1]$.

Proof. Let $OB(x_e, r, \tilde{t})$ be an open ball with centre x_e and radius r in an NSNLS $(\widetilde{V}(K), N, *, \diamond)$. Suppose $y_{e_j}, z_{e_k} \in OB(x_e, r, \tilde{t})$. Then,

$T_N(x_e - y_{e_j}, \tilde{t}) > 1 - r,\ I_N(x_e - y_{e_j}, \tilde{t}) < r,\ F_N(x_e - y_{e_j}, \tilde{t}) < r$ and

$T_N(x_e - z_{e_k}, \tilde{t}) > 1 - r,\ I_N(x_e - z_{e_k}, \tilde{t}) < r,\ F_N(x_e - z_{e_k}, \tilde{t}) < r$.

Now, for $\tilde{c} \in (\tilde{0}, \tilde{1})$ ($\tilde{c}$ being a soft scalar),

$$T_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}] = T_N[(\tilde{c} \oplus \tilde{1} - \tilde{c})x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}]$$

$$= T_N[\tilde{c}(x_e - y_{e_j}) \oplus (\tilde{1} - \tilde{c})(x_e - z_{e_k}), \tilde{t}] \geq T_N[\tilde{c}(x_e - y_{e_j}), \frac{\tilde{t}}{2}] * T_N[(\tilde{1} - \tilde{c})(x_e - z_{e_k}), \frac{\tilde{t}}{2}]$$

$$= T_N[(x_e - y_{e_j}), \frac{\tilde{t}}{2|\tilde{c}|}] * T_N[(x_e - z_{e_k}), \frac{\tilde{t}}{2|(\tilde{1} - \tilde{c})|}] > (1 - r) * (1 - r) = 1 - r.$$

$$I_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}] = I_N[(\tilde{c} \oplus \tilde{1} - \tilde{c})x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}]$$

$$= I_N[\tilde{c}(x_e - y_{e_j}) \oplus (\tilde{1} - \tilde{c})(x_e - z_{e_k}), \tilde{t}] \leq I_N[\tilde{c}(x_e - y_{e_j}), \frac{\tilde{t}}{2}] \diamond I_N[(\tilde{1} - \tilde{c})(x_e - z_{e_k}), \frac{\tilde{t}}{2}]$$

$$= I_N[(x_e - y_{e_j}), \frac{\tilde{t}}{2|\tilde{c}|}] \diamond I_N[(x_e - z_{e_k}), \frac{\tilde{t}}{2|(\tilde{1} - \tilde{c})|}] < r \diamond r = r.$$

Similarly, $F_N[x_e - (\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}), \tilde{t}] < r$.

This shows that $[\tilde{c}y_{e_j} \oplus (\tilde{1} - \tilde{c})z_{e_k}] \in B(x_e, r, \tilde{t})$ with respect to the neutrosophic soft norm N. Hence, the 1st part is completed.

For the 2nd part, let $y_{e_j} \in B(x_e, r, \tilde{t})$ an arbitrary soft point. Then $T_N(x_e - y_{e_j}, \tilde{t}) > 1 - r, I_N(x_e - y_{e_j}, \tilde{t}) < r$, $F_N(x_e - y_{e_j}, \tilde{t}) < r$. Now,

$$T_N(y_{e_j}, \tilde{t}) = T_N(y_{e_j} - x_e \oplus x_e, \tilde{t}) \geq T_N(y_{e_j} - x_e, \frac{\tilde{t}}{2}) * T_N(x_e, \frac{\tilde{t}}{2}) \text{ [by (vi) in Definition 3.1]}$$

$$= T_N(x_e - y_{e_j}, \frac{\tilde{t}}{2}) * T_N(x_e, \frac{\tilde{t}}{2}) \text{ [by (v) in Definition 3.1]}$$

$$> (1 - r) * T_N(x_e, \frac{\tilde{t}}{2}).$$

Since $\lim_{\tilde{t}\to\infty} T_N(x_e,\tilde{t}) = 1$, $\exists\,\tilde{t}_0 > \tilde{0}$ so that $T_N(x_e,\tilde{t}) \geq 1-r,\ \forall\tilde{t} \geq \tilde{t}_0$. Thus, $T_N(y_{e_j},\tilde{t}) \geq (1-r)*(1-r) = (1-r),\ \forall\tilde{t} \geq \tilde{t}_0$. Next,

$$I_N(y_{e_j},\tilde{t}) = I_N(y_{e_j} - x_e \oplus x_e,\tilde{t}) \leq I_N(y_{e_j} - x_e,\frac{\tilde{t}}{2}) \diamond I_N(x_e,\frac{\tilde{t}}{2}) \text{ [by (xi) in Definition 3.1]}$$
$$= I_N(x_e - y_{e_j},\frac{\tilde{t}}{2}) \diamond I_N(x_e,\frac{\tilde{t}}{2}) \text{ [by (x) in Definition 3.1]}$$
$$< r \diamond I_N(x_e,\frac{\tilde{t}}{2}).$$

Since $\lim_{\tilde{t}\to\infty} I_N(x_e,\tilde{t}) = 0$, $\exists\,\tilde{t}_1 > \tilde{0}$ so that $I_N(x_e,\tilde{t}) \leq r,\ \forall\tilde{t} \geq \tilde{t}_1$. Thus, $I_N(y_{e_j},\tilde{t}) \leq r \diamond r = r$, $\forall\tilde{t} \geq \tilde{t}_1$. Similarly, $F_N(y_{e_j},\tilde{t}) \leq r$, $\forall\tilde{t} \geq \tilde{t}_2$.
Hence, $T_N(y_{e_j},\tilde{t}) \geq 1-r,\ I_N(y_{e_j},\tilde{t}) \leq r,\ F_N(y_{e_j},\tilde{t}) \leq r,\ \forall y_{e_j} \in OB(x_e,r,\tilde{t})$ and $\forall\tilde{t} \geq \max\{\tilde{t}_0,\tilde{t}_1,\tilde{t}_2\}$ and this ends the 2nd part.

5.5 Theorem

The intersection of an arbitrary number of convex soft sets is also convex in an NSNLS.

Proof. Let $\{\widetilde{W}_i | i \in \Gamma\}$ be a collection of convex soft sets in the NSNLS $(\widetilde{V}(K), N, *, \diamond)$ such that each $\widetilde{W}_i \subset \widetilde{V}$. Then $\cap_i \widetilde{W}_i = \widetilde{W}$ (say) is obviously convex. Let $x_{e_i} = [\tilde{c}y_{e_j} \oplus (\tilde{1}-\tilde{c})z_{e_k}] \in \widetilde{W}$ for $y_{e_j}, z_{e_k} \in \widetilde{W}$ and $\tilde{c} \in (\tilde{0},\tilde{1})$. Since $\widetilde{W} \subset \widetilde{V}$, so $(\widetilde{W}(K), N, *, \diamond)$ is a convex NSNLS and this proves the theorem.

6 Metric in NSNLS

The metric of NSNLS is defined in this section. Some related theorems are developed also.

6.1 Definition

The set of all mappings $T_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$, $I_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$ and $F_N : \Delta_{\widetilde{V}} \times \Delta_{\widetilde{V}} \times R(E) \to [0,1]$ together is said to form a neutrosophic soft metric on the soft linear space $\widetilde{V}$ if $\{T_N, I_N, F_N\}$ satisfies the following axioms :
(i) $0 \leq T_N(x_{e_i},y_{e_j},\tilde{t}),\ I_N(x_{e_i},y_{e_j},\tilde{t}),\ F_N(x_{e_i},y_{e_j},\tilde{t}) \leq 1$, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$ and $\forall\tilde{t} \in R(E)$.
(ii) $T_N(x_{e_i},y_{e_j},\tilde{t}) + I_N(x_{e_i},y_{e_j},\tilde{t}) + F_N(x_{e_i},y_{e_j},\tilde{t}) \leq 3$, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$ and $\tilde{t} \in R(E)$.
(iii) $T_N(x_{e_i},y_{e_j},\tilde{t}) = 0$ with $\tilde{t} \leq \tilde{0}$.
(iv) $T_N(x_{e_i},y_{e_j},\tilde{t}) = 1$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(v) $T_N(x_{e_i},y_{e_j},\tilde{t}) = T_N(y_{e_j},x_{e_i},\tilde{t})$ with $\tilde{t} > \tilde{0}$.
(vi) $T_N(x_{e_i},y_{e_j},\tilde{s}) * T_N(y_{e_j},z_{e_k},\tilde{t}) \leq T_N(x_{e_i},z_{e_k},\tilde{s}\oplus\tilde{t}),\ \forall\,\tilde{s},\tilde{t} > \tilde{0};\ x_{e_i}, y_{e_j}, z_{e_k} \in \Delta_{\widetilde{V}}$.
(vii) $T_N(x_{e_i},y_{e_j},\cdot) : [0,\infty) \to [0,1]$ is continuous, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$.
(viii) $\lim_{t\to\infty} T_N(x_{e_i},y_{e_j},\tilde{t}) = 1,\ \forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$
(ix) $I_N(x_{e_i},y_{e_j},\tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(x) $I_N(x_{e_i},y_{e_j},\tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(xi) $I_N(x_{e_i},y_{e_j},\tilde{t}) = I_N(y_{e_j},x_{e_i},\tilde{t})$ with $\tilde{t} > \tilde{0}$.
(xii) $I_N(x_{e_i},y_{e_j},\tilde{s}) \diamond I_N(y_{e_j},z_{e_k},\tilde{t}) \geq I_N(x_{e_i},z_{e_k},\tilde{s}\oplus\tilde{t}),\ \forall\,\tilde{s},\tilde{t} > \tilde{0};\ x_{e_i}, y_{e_j}, z_{e_k} \in \Delta_{\widetilde{V}}$.
(xiii) $I_N(x_{e_i},y_{e_j},\cdot) : [0,\infty) \to [0,1]$ is continuous, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$.

(xiv) $\lim_{t\to\infty} I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0, \ \ \forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$
(xv) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1$ with $\tilde{t} \leq \tilde{0}$.
(xvi) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0$ with $\tilde{t} > \tilde{0}$ iff $x_{e_i} = y_{e_j}$
(xvii) $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = F_N(y_{e_j}, x_{e_i}, \tilde{t})$ with $\tilde{t} > \tilde{0}$.
(xviii) $F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}), \ \forall \tilde{s}, \tilde{t} > \tilde{0}; \ x_{e_i}, y_{e_j}, z_{e_k} \in \Delta_{\widetilde{V}}$.
(xix) $F_N(x_{e_i}, y_{e_j}, \cdot) : [0,\infty) \to [0,1]$ is continuous, $\forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$.
(xx) $\lim_{t\to\infty} F_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0, \ \forall x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}, \tilde{t} > \tilde{0}$

Then $(\widetilde{V}(K), \{T_N, I_N, F_N\}, *, \diamond)$ is a neutrosophic soft metric space (NSMS).

6.1.1 *Example*

Let $(\widetilde{X}, d)$ be a soft metric space. Define $a * b = ab$, $a \diamond b = a + b - ab$ and $\forall x_{e_i}, y_{e_j} \in \widetilde{X}$, $\tilde{t} > \tilde{0}$,
$T_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{\tilde{t}}{\tilde{t} \oplus d(x_{e_i}, y_{e_j})}$, $I_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{d(x_{e_i}, y_{e_j})}{\tilde{t} \oplus d(x_{e_i}, y_{e_j})}$, $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = \frac{d(x_{e_i}, y_{e_j})}{\tilde{t}}$;

Then $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is an NSMS.

Proof. We shall only verify the axioms (vi), (xii), (xviii). Others are straight forward.

$$\begin{aligned}
& T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) - T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \\
= \ & \frac{\tilde{s} \oplus \tilde{t}}{\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})} - \frac{\tilde{s}\tilde{t}}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} \\
= \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})\}/G \\
& \text{where } G = (\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k}))(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) \\
= \ & \{\tilde{s}\tilde{t}[d(x_{e_i}, y_{e_j}) \oplus d(y_{e_j}, z_{e_k})] \oplus \tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \\
& \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k}) - \tilde{s}\tilde{t} d(x_{e_i}, z_{e_k})\}/G \\
\geq \ & \{\tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})\}/G \geq 0
\end{aligned}$$

Hence, $T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$.

$$\begin{aligned}
& I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) - I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \\
= \ & \frac{d(x_{e_i}, y_{e_j})}{\tilde{s} \oplus d(x_{e_i}, y_{e_j})} \oplus \frac{d(y_{e_j}, z_{e_k})}{\tilde{t} \oplus d(y_{e_j}, z_{e_k})} - \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} - \frac{d(x_{e_i}, z_{e_k})}{(\tilde{s} \oplus \tilde{t}) \oplus d(x_{e_i}, z_{e_k})} \\
= \ & \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k}) \oplus \tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k})}{(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k}))} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k})} \\
= \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k}) \oplus d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t} d(x_{e_i}, z_{e_k})\}/H \\
& \text{where } H = (\tilde{s} \oplus \tilde{t} \oplus d(x_{e_i}, z_{e_k}))(\tilde{s} \oplus d(x_{e_i}, y_{e_j}))(\tilde{t} \oplus d(y_{e_j}, z_{e_k})) \\
\geq \ & \{(\tilde{s} \oplus \tilde{t})(\tilde{t} d(x_{e_i}, y_{e_j}) \oplus \tilde{s} d(y_{e_j}, z_{e_k}) \oplus d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})) - \tilde{s}\tilde{t}[d(x_{e_i}, y_{e_j}) \oplus d(y_{e_j}, z_{e_k})]\}/H \\
= \ & \{\tilde{t}^2 d(x_{e_i}, y_{e_j}) \oplus \tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})\}/H \geq 0
\end{aligned}$$

Hence, $I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$. Finally,

$$\begin{aligned}
& F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) - F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \\
= \ & \frac{d(x_{e_i}, y_{e_j})}{\tilde{s}} \oplus \frac{d(y_{e_j}, z_{e_k})}{\tilde{t}} - \frac{d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})}{\tilde{s}\tilde{t}} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\tilde{t}d(x_{e_i}, y_{e_j}) \oplus \tilde{s}d(y_{e_j}, z_{e_k}) - d(x_{e_i}, y_{e_j})d(y_{e_j}, z_{e_k})}{st} - \frac{d(x_{e_i}, z_{e_k})}{\tilde{s} \oplus \tilde{t}} \\
&\geq \{\tilde{s}^2 d(y_{e_j}, z_{e_k}) \oplus \tilde{t}^2 d(x_{e_i}, y_{e_j}) - (\tilde{s} \oplus \tilde{t}) d(x_{e_i}, y_{e_j}) d(y_{e_j}, z_{e_k})\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \\
&= \{\tilde{s}d(y_{e_j}, z_{e_k})(\tilde{s} - d(x_{e_i}, y_{e_j})) \oplus \tilde{t}d(x_{e_i}, y_{e_j})(\tilde{t} - d(y_{e_j}, z_{e_k}))\}/\tilde{s}\tilde{t}(\tilde{s} \oplus \tilde{t}) \geq 0 \\
&\quad [\text{as } F_N \in [0,1] \text{ so } \tilde{s} \geq d(x_{e_i}, y_{e_j}),\ \tilde{t} \geq d(y_{e_j}, z_{e_k})]
\end{aligned}
$$

Thus, $F_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond F_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq F_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t})$. This completes the proof.

6.2 Theorem

Every NSNLS is a NSMS.

Proof. Define a neutrosophic soft metric $\{T_N, I_N, F_N\}$ over an NSNLS $(\widetilde{V}(K), N, *, \diamond)$ as follows.
$T_N(x_{e_i}, y_{e_j}, \tilde{t}) = T_N(x_{e_i} - y_{e_j}, \tilde{t})$, $I_N(x_{e_i}, y_{e_j}, \tilde{t}) = I_N(x_{e_i} - y_{e_j}, \tilde{t})$, $F_N(x_{e_i}, y_{e_j}, \tilde{t}) = F_N(x_{e_i} - y_{e_j}, \tilde{t})$ for each of $x_{e_i}, y_{e_j} \in \Delta_{\widetilde{V}}$. We shall verify here the metric axioms (v), (vi) only. Rest axioms are satisfied in well manner.

(v) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = T_N(x_{e_i} - y_{e_j}, \tilde{t}) = T_N(y_{e_j} - x_{e_i}, \dfrac{\tilde{t}}{|-1|}) = T_N(y_{e_j} - x_{e_i}, \tilde{t}) = T_N(y_{e_j}, x_{e_i}, \tilde{t})$

(vi) $T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) = T_N(x_{e_i} - z_{e_k}, \tilde{s} \oplus \tilde{t}) = T_N(x_{e_i} - y_{e_j} \oplus y_{e_j} - z_{e_k}, \tilde{s} \oplus \tilde{t})$
$\geq T_N(x_{e_i} - y_{e_j}, \tilde{s}) * T_N(y_{e_j} - z_{e_k}, \tilde{t}) = T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t})$

The four metric axioms (xi), (xii), (xvii), (xviii) can be similarly verified.

6.3 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be a convergent sequence and converges to x_e if

$\lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \tilde{t}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \tilde{t}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_e, \tilde{t}) = 0$ as $\tilde{t} \to \infty$.

6.4 Theorem

The limit of a convergent sequence $\{x^n_{e_n}\}$ in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is unique.

Proof. If possible $\lim_{n\to\infty} x^n_{e_n} = x_{e_j}$ and $\lim_{n\to\infty} x^n_{e_n} = y_{e_k}$ for $x_{e_j} \neq y_{e_k}$. Then for $\tilde{s}, \tilde{t} > \tilde{0}$,
$\lim_{n\to\infty} T_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_{e_j}, \tilde{s}) = 0$ as $\tilde{s} \to \infty$ and $\lim_{n\to\infty} T_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 1$, $\lim_{n\to\infty} I_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 0$, $\lim_{n\to\infty} F_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$. Now,

$$T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq T_N(x_{e_j}, x^n_{e_n}, \tilde{s}) * T_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = T_N(x^n_{e_n}, x_{e_j}, \tilde{s}) * T_N(x^n_{e_n}, y_{e_k}, \tilde{t})$$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$, $T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \geq 1 * 1 = 1$.
It implies $T_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) = 1$. (10)

Next, $I_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq I_N(x_{e_j}, x^n_{e_n}, \tilde{s}) \diamond I_N(x^n_{e_n}, y_{e_k}, \tilde{t}) = I_N(x^n_{e_n}, x_{e_j}, \tilde{s}) \diamond I_N(x^n_{e_n}, y_{e_k}, \tilde{t})$

Taking limit as $n \to \infty$ and for $\tilde{s}, \tilde{t} \to \infty$, $I_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) \leq 0 \diamond 0 = 0$.
This shows $I_N(x_{e_j} - y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$. (11)
Similarly, $F_N(x_{e_j}, y_{e_k}, \tilde{s} \oplus \tilde{t}) = 0$ (12)

Hence, $x_{e_j} = y_{e_k}$ and this completes the proof.

6.5 Definition

A sequence $\{x^n_{e_n}\}$ of soft points in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 1$, $\lim_{n,m\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$, $\lim_{n,m\to\infty} F_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$ as $\tilde{t} \to \infty$ and $\forall x^n_{e_n}, x^m_{e_m} \in \widetilde{X}$.

6.6 Theorem

Every convergent sequence is a Cauchy sequence in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$.

Proof. Let $\{x^n_{e_n}\}$ be a convergent sequence in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ with $\lim_{n\to\infty} x^n_{e_n} = x_e$. Then for $\tilde{t} > \tilde{0}$,

$$\lim_{n\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) \geq \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) * \lim_{n\to\infty} T_N(x_e, x^m_{e_m}, \frac{\tilde{t}}{2})$$
$$= \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) * \lim_{n\to\infty} T_N(x^m_{e_m}, x_e, \frac{\tilde{t}}{2}) = 1 * 1 = 1$$

So, $\lim_{n\to\infty} T_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 1$. Next,

$$\lim_{n\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) \leq \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) \diamond \lim_{n\to\infty} I_N(x_e, x^m_{e_m}, \frac{\tilde{t}}{2})$$
$$= \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \frac{\tilde{t}}{2}) \diamond \lim_{n\to\infty} I_N(x^m_{e_m}, x_e, \frac{\tilde{t}}{2}) = 0 \diamond 0 = 0$$

So, $\lim_{n\to\infty} I_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$ and similarly, $\lim_{n\to\infty} F_N(x^n_{e_n}, x^m_{e_m}, \tilde{t}) = 0$.
Hence, $\{x^n_{e_n}\}$ is a Cauchy sequence.

6.7 Definition

A NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ is said to be complete if every Cauchy sequence of soft points in $\widetilde{X}$ converges to a soft point of $\widetilde{X}$.

6.8 Theorem

In a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$, if every Cauchy sequence has a convergent subsequence then the NSMS is complete.

Proof. Let $\{x^{n_k}_{e_{n_k}}\}$ be a subsequence of a Cauchy sequence $\{x^n_{e_n}\}$ in a NSMS $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$ such that $\{x^{n_k}_{e_{n_k}}\} \to x_e \in \widetilde{X}$. Since $\{x^n_{e_n}\}$ be a Cauchy sequence in $(\widetilde{X}, \{T_N, I_N, F_N\}, *, \diamond)$, given $\tilde{t} > \tilde{0}$,
$\lim_{n,k\to\infty} T_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 1$, $\lim_{n,k\to\infty} I_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$, $\lim_{n,k\to\infty} F_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) = 0$ as $\tilde{t} \to \infty$.
Since $\{x^{n_k}_{e_{n_k}}\}$ converges x_e, then as $\tilde{t} \to \infty$,
$\lim_{k\to\infty} T_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 1$, $\lim_{k\to\infty} I_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 0$, $\lim_{k\to\infty} F_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) = 0$.
Now, $T_N(x^n_{e_n}, x_e, \tilde{t}) \geq T_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) * T_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) \Rightarrow \lim_{n\to\infty} T_N(x^n_{e_n}, x_e, \tilde{t}) = 1$.

Next, $I_N(x^n_{e_n}, x_e, \tilde{t}) \leq I_N(x^n_{e_n}, x^{n_k}_{e_{n_k}}, \frac{\tilde{t}}{2}) \diamond I_N(x^{n_k}_{e_{n_k}}, x_e, \frac{\tilde{t}}{2}) \Rightarrow \lim_{n\to\infty} I_N(x^n_{e_n}, x_e, \tilde{t}) = 0.$
Similarly, $\lim_{n\to\infty} F_N(x^n_{e_n}, x_e, \tilde{t}) = 0.$

This shows that $\{x^n_{e_n}\}$ converges to $x_e \in \widetilde{X}$ and thus the theorem is proved.

6.9 *Remark*

The using of $\tilde{0}, \tilde{1}$ instead of 0, 1 in some equalities is meaning that the left side of equality represents a set of soft points, not a set of neutrosophic components.

6.10 Theorem

In a NSMS $(\widetilde{V}(K), \{T_N, I_N, F_N\}, *, \diamond)$, define

$$||x_{e_i} - y_{e_j}||^1_\alpha = \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} \tag{13}$$
$$||x_{e_i} - y_{e_j}||^2_\beta = \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} \tag{14}$$
$$||x_{e_i} - y_{e_j}||^3_\gamma = \sup\{\tilde{t}|F_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \gamma, \gamma \in (0,1)\} \tag{15}$$

Then $\{||\cdot||^1_\alpha, ||\cdot||^2_\beta, ||\cdot||^3_\gamma\}$ are ascending family of norms on $\widetilde{V}$ if $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for any two real numbers $a, b \in [0, 1]$.

Proof. For $||\cdot||^1_\alpha$, we have

(i) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0, \ \forall \tilde{t} \leq \tilde{0}$ [by (iii) in Definition 6.1]
$\Rightarrow \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{0}$
$\Rightarrow \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{0}$
$\Rightarrow ||x_{e_i} - y_{e_j}||^1_\alpha = \tilde{0}$

(ii) $T_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1, \ \forall \tilde{t} \geq \tilde{0}$ iff $x_{e_i} = y_{e_j}$
$\Rightarrow \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{1}$
$\Rightarrow \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = \tilde{1}$
$\Rightarrow ||x_{e_i} - y_{e_j}||^1_\alpha = \tilde{1}$

(iii) $||x_{e_i} - y_{e_j}||^1_\alpha = \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\}$
$= \inf\{\tilde{t}|T_N(y_{e_j}, x_{e_i}, \tilde{t}) \geq \alpha, \alpha \in (0,1)\} = ||y_{e_j} - x_{e_i}||^1_\alpha$

(iv) $||x_{e_i} - y_{e_j}||^1_\alpha + ||y_{e_j} - z_{e_k}||^1_\alpha, \ \alpha \in (0,1)$
$= \inf\{\tilde{s}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) \geq \alpha\} + \inf\{\tilde{t}|T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha\}$
$= \inf\{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) \geq \alpha, T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha\}$
$= \inf\{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{s}) * T_N(y_{e_j}, z_{e_k}, \tilde{t}) \geq \alpha * \alpha\}$
$\leq \{\tilde{s} \oplus \tilde{t}|T_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \geq \alpha\}$ [by (vi) in Definition 6.1]
$= ||x_{e_i} - z_{e_k}||^1_\alpha$

Thus, $||\cdot||^1_\alpha$ is an $\alpha-$ norm induced by the fuzzy soft metric T_N on $\widetilde{V}$.

Finally, for $0 < \alpha_1 < \alpha_2$,

$$\begin{aligned} & \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_2\} \subset \{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_1\} \\ \Rightarrow\ & \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_1\} \leq \inf\{\tilde{t}|T_N(x_{e_i}, y_{e_j}, \tilde{t}) \geq \alpha_2\} \\ \Rightarrow\ & ||x_{e_i} - y_{e_j}||^1_{\alpha_1} \leq ||x_{e_i} - y_{e_j}||^1_{\alpha_2} \end{aligned}$$

Hence, $|| \cdot ||^1_\alpha$ is an ascending norm on $\widetilde{V}$. Next, for $|| \cdot ||^2_\beta$, we have

$$\begin{aligned} \text{(v)} \quad & I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 1,\ \forall \tilde{t} \leq \tilde{0} \text{ [by (ix) in Definition 6.1]} \\ \Rightarrow\ & \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{1} \\ \Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{1} \\ \Rightarrow\ & ||x_{e_i} - y_{e_j}||^1_\beta = \tilde{1} \\ \text{(vi)} \quad & I_N(x_{e_i}, y_{e_j}, \tilde{t}) = 0,\ \forall \tilde{t} \geq \tilde{0} \iff x_{e_i} = y_{e_j} \\ \Rightarrow\ & \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{0} \\ \Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = \tilde{0} \\ \Rightarrow\ & ||x_{e_i} - y_{e_j}||^2_\beta = \tilde{0} \\ \text{(vii)} \quad & ||x_{e_i} - y_{e_j}||^2_\beta = \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta, \beta \in (0,1)\} \\ =\ & \sup\{\tilde{t}|I_N(y_{e_j}, x_{e_i}, \tilde{t}) \leq \beta, \beta \in (0,1)\} = ||y_{e_j} - x_{e_i}||^2_\beta \\ \text{(viii)} \quad & ||x_{e_i} - y_{e_j}||^2_\beta + ||y_{e_j} - z_{e_k}||^2_\beta,\ \beta \in (0,1) \\ =\ & \sup\{\tilde{s}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \leq \beta\} + \sup\{\tilde{t}|I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta\} \\ =\ & \sup\{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \leq \beta, I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta\} \\ =\ & \sup\{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{s}) \diamond I_N(y_{e_j}, z_{e_k}, \tilde{t}) \leq \beta \diamond \beta\} \\ \geq\ & \{\tilde{s} \oplus \tilde{t}|I_N(x_{e_i}, z_{e_k}, \tilde{s} \oplus \tilde{t}) \leq \beta\} \text{ [by (xi) in Definition 6.1]} \\ =\ & ||x_{e_i} - z_{e_k}||^2_\beta \end{aligned}$$

Thus, $|| \cdot ||^2_\beta$ is a $\beta-$ norm induced by the fuzzy soft metric I_N on $\widetilde{V}$. Finally, for $0 < \beta_1 < \beta_2$,

$$\begin{aligned} & \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_2\} \supset \{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_1\} \\ \Rightarrow\ & \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_1\} \leq \sup\{\tilde{t}|I_N(x_{e_i}, y_{e_j}, \tilde{t}) \leq \beta_2\} \\ \Rightarrow\ & ||x_{e_i} - y_{e_j}||^2_{\beta_1} \leq ||x_{e_i} - y_{e_j}||^2_{\beta_2} \end{aligned}$$

Hence, $|| \cdot ||^2_\beta$ is an ascending norm on $\widetilde{V}$.
In a similar manner, $|| \cdot ||^3_\gamma$ is also an ascending norm on $\widetilde{V}$ and this ends the theorem.

7 Conclusion

The motivation of the present paper is to define a neutrosophic norm on a soft linear space. The convergence of sequence, characteristics of Cauchy sequence, the concept of convexity and the metric in NSNLS have been introduced here. These are illustrated by suitable examples. Their several related properties and structural characteristics have been investigated. We expect, this paper will promote the future study on neutrosophic soft normed linear spaces and many other general frameworks.

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MBJ-neutrosophic structures and its applications in BCK/BCI-algebras

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Abstract: Smarandache (F. Smarandache. Neutrosophy, neutrosophic probability, set, and logic, ProQuest Information & Learning, Ann Arbor,Michigan, USA, 105 p., 1998) initiated neutrosophic sets which can be used as a mathematical tool for dealing with indeterminates and inconsistent information. As a generalization of a neutrosophic set, the notion of MBJ-neutrosophic sets is introduced, and it is applied to BCK/BCI-algebras. The concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras is introduced, and related properties are investigated. A characterization of MBJ-neutrosophic subalgebra is provided. Using an MBJ-neutrosophic subalgebra of a BCI-algebra, a new MBJ-neutrosophic subalgebra is established. Homomorphic inverse image of MBJ-neutrosophic subalgebra is considered. Translation of MBJ-neutrosophic subalgebra is discussed. Conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic subalgebra are provided.

Keywords: MBJ-neutrosophic set; MBJ-neutrosophic subalgebra; MBJ-neutrosophic S-extension.

1 Introduction

In many practical situations and in many complex systems like biologial, behavioral and chemical etc., different types of uncertainties are encountered. The fuzzy set was introduced by L.A. Zadeh [19] in 1965 to handle uncertainties in many real applications, and the intuitionistic fuzzy set on a universe X was introduced by K. Atanassov in 1983 as a generalization of fuzzy set. The notion of neutrosophic set is developed by Smarandache ([14], [15] and [16]), and is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [1], [2], [6], [7], [8], [9], [10], [12], [13], [17] and [18]. We know that there are many generalizations of Smarandache's neutrosophic sets. The aim of this article is to consider another generalization of a neutrosophic set. In the neutrosophic set, the truth, false and indeterminate membership functions are fuzzy sets. In considering a generalization of neutrosophic set, we use the interval valued fuzzy set as the indeterminate membership function because interval valued fuzzy set is a generalization of a fuzzy set. We introduce the notion of MBJ-neutrosophic sets, and we apply it to BCK/BCI-algebras. We introduce the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigate related properties. We provide a characterization of MBJ-neutrosophic subalgebra, and establish a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI-algebra. We consider the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discuss translation of MBJ-neutrosophic subalgebra. We provide conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic subalgebra.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Is´eki (see [4] and [5]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \le y\ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. Any BCI-algebra X satisfies the following conditions (see [3]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{2.5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{2.6}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned} &\mathrm{rmin}\,\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\mathrm{rmax}\,\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\ &\tilde{a}_1 \succeq \tilde{a}_2\ \Leftrightarrow\ a_1^- \ge a_2^-,\ a_1^+ \ge a_2^+, \end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [15]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

We refer the reader to the books [3, 11] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3 MBJ-neutrosophic structures with applications in BCK/BCI-algebras

Definition 3.1. Let X be a non-empty set. By an *MBJ-neutrosophic set* in X, we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

In an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, if we take

$$\tilde{B}_A : X \to [I],\ x \mapsto [B_A^-(x), B_A^+(x)]$$

with $B_A^-(x) = B_A^+(x)$, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a neutrosophic set in X.

Definition 3.2. Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an *MBJ-neutrosophic subalgebra* of X if it satisfies:

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(x * y) \geq \min\{M_A(x), M_A(y)\}, \\ \tilde{B}_A(x * y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ J_A(x * y) \leq \max\{J_A(x), J_A(y)\}. \end{array} \right) \tag{3.1}$$

Example 3.3. Consider a set $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$(X; *, 0)$ is a BCK-algebra (see [11]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X.

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.3, 0.8]$	0.2
a	0.3	$[0.1, 0.5]$	0.6
b	0.1	$[0.3, 0.8]$	0.4
c	0.5	$[0.1, 0.5]$	0.7

Example 3.4. Consider a BCI-algebra $(\mathbb{Z}, -, 0)$ and let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $\mathbb{Z}$ defined by

$$M_A : \mathbb{Z} \to [0, 1],\ x \mapsto \begin{cases} 0.6 & \text{if } x \in 4\mathbb{Z}, \\ 0.4 & \text{if } x \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \\ 0.3 & \text{otherwise,} \end{cases}$$

$$\tilde{B}_A : \mathbb{Z} \to [I],\ x \mapsto \begin{cases} [0.6, 0.8] & \text{if } x \in 6\mathbb{Z}, \\ [0.4, 0.5] & \text{if } x \in 3\mathbb{Z} \setminus 6\mathbb{Z}, \\ [0.2, 0.3] & \text{otherwise,} \end{cases}$$

$$J_A : \mathbb{Z} \to [0, 1],\ x \mapsto \begin{cases} 0.2 & \text{if } x \in 8\mathbb{Z}, \\ 0.4 & \text{if } x \in 4\mathbb{Z} \setminus 8\mathbb{Z}, \\ 0.5 & \text{otherwise.} \end{cases}$$

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of $(\mathbb{Z}, -, 0)$.

In what follows, let X be a BCK/BCI-algebra unless otherwise specified.

Proposition 3.5. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X, then $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\begin{aligned} \tilde{B}_A(0) &= \tilde{B}_A(x * x) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} \\ &= \text{rmin}\{[B_A^-(x), B_A^+(x)], [B_A^-(x), B_A^+(x)]\} \\ &= [B_A^-(x), B_A^+(x)] = \tilde{B}_A(x), \end{aligned}$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

This completes the proof. □

Proposition 3.6. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} M_A(x_n) = 1,\ \lim_{n\to\infty} \tilde{B}_A(x_n) = [1, 1] \text{ and } \lim_{n\to\infty} J_A(x_n) = 0, \tag{3.2}$$

then $M_A(0) = 1$, $\tilde{B}_A(0) = [1, 1]$ and $J_A(0) = 0$.

Proof. Using Proposition 3.5, we know that $M_A(0) \geq M_A(x_n)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x_n)$ and $J_A(0) \leq J_A(x_n)$ for every positive integer n. Note that

$$\begin{aligned} &1 \geq M_A(0) \geq \lim_{n\to\infty} M_A(x_n) = 1, \\ &[1, 1] \succeq \tilde{B}_A(0) \succeq \lim_{n\to\infty} \tilde{B}_A(x_n) = [1, 1], \\ &0 \leq J_A(0) \leq \lim_{n\to\infty} J_A(x_n) = 0. \end{aligned}$$

Therefore $M_A(0) = 1$, $\tilde{B}_A(0) = [1, 1]$ and $J_A(0) = 0$. □

Theorem 3.7. *Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, if (M_A, J_A) is an intuitionistic fuzzy subalgebra of X, and B_A^- and B_A^+ are fuzzy subalgebras of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X.*

Proof. It is sufficient to show that $\tilde{B}_A$ satisfies the condition

$$(\forall x, y \in X)(\tilde{B}_A(x * y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}). \tag{3.3}$$

For any $x, y \in X$, we get

$$\begin{aligned}\tilde{B}_A(x * y) &= [B_A^-(x * y), B_A^+(x * y)] \\ &\succeq [\min\{B_A^-(x), B_A^-(y)\}, \min\{B_A^+(x), B_A^+(y)\}] \\ &= \text{rmin}\{[B_A^-(x), B_A^+(x)], [B_A^-(y), B_A^+(y)] \\ &= \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}.\end{aligned}$$

Therefore $\tilde{B}_A$ satisfies the condition (3.3), and so $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. □

If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X, then

$$\begin{aligned}[B_A^-(x * y), B_A^+(x * y)] &= \tilde{B}_A(x * y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ &= \text{rmin}\{[B_A^-(x), B_A^+(x), [B_A^-(y), B_A^+(y)]\} \\ &= [\min\{B_A^-(x), B_A^-(y)\}, \min\{B_A^+(x), B_A^+(y)\}]\end{aligned}$$

for all $x, y \in X$. It follows that $B_A^-(x * y) \geq \min\{B_A^-(x), B_A^-(y)\}$ and $B_A^+(x * y) \geq \min\{B_A^+(x), B_A^+(y)\}$. Thus B_A^- and B_A^+ are fuzzy subalgebras of X. But (M_A, J_A) is not an intuitionistic fuzzy subalgebra of X as seen in Example 3.3. This shows that the converse of Theorem 3.7 is not true.

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned}&U(M_A; t) := \{x \in X \mid M_A(x) \geq t\}, \\ &U(\tilde{B}_A; [\delta_1, \delta_2]) := \{x \in X \mid \tilde{B}_A(x) \succeq [\delta_1, \delta_2]\}, \\ &L(J_A; s) := \{x \in X \mid J_A(x) \leq s\}\end{aligned}$$

where $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$.

Theorem 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is an MBJ-neutrosophic subalgebra of X if and only if the non-empty sets $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are subalgebras of X for all $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. Let $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$ be such that $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are non-empty. For any $x, y, a, b, u, v \in X$, if $x, y \in U(M_A; t)$, $a, b \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $u, v \in L(J_A; s)$, then

$$\begin{aligned}&M_A(x * y) \geq \min\{M_A(x), M_A(y)\} \geq \min\{t, t\} = t, \\ &\tilde{B}_A(a * b) \succeq \text{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\} \succeq \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2], \\ &J_A(u * v) \leq \max\{J_A(u), J_A(v)\} \leq \min\{s, s\} = s,\end{aligned}$$

and so $x * y \in U(M_A; t)$, $a * b \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $u * v \in L(J_A; s)$. Therefore $U(M_A; t)$, $U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are subalgebras of X.

Conversely, assume that the non-empty sets $U(M_A;t)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;s)$ are subalgebras of X for all $t,s\in[0,1]$ and $[\delta_1,\delta_2]\in[I]$. If $M_A(a_0*b_0)<\min\{M_A(a_0),M_A(b_0)\}$ for some $a_0,b_0\in X$, then $a_0,b_0\in U(M_A;t_0)$ but $a_0*b_0\notin U(M_A;t_0)$ for $t_0:=\min\{M_A(a_0),M_A(b_0)\}$. This is a contradiction, and thus $M_A(a*b)\geq\min\{M_A(a),M_A(b)\}$ for all $a,b\in X$. Similarly, we can show that $J_A(a*b)\leq\max\{J_A(a),J_A(b)\}$ for all $a,b\in X$. Suppose that $\tilde{B}_A(a_0*b_0)\prec\text{rmin}\{\tilde{B}_A(a_0),\tilde{B}_A(b_0)\}$ for some $a_0,b_0\in X$. Let $\tilde{B}_A(a_0)=[\lambda_1,\lambda_2]$, $\tilde{B}_A(b_0)=[\lambda_3,\lambda_4]$ and $\tilde{B}_A(a_0*b_0)=[\delta_1,\delta_2]$. Then

$$[\delta_1,\delta_2]\prec\text{rmin}\{[\lambda_1,\lambda_2],[\lambda_3,\lambda_4]\}=[\min\{\lambda_1,\lambda_3\},\min\{\lambda_2,\lambda_4\}],$$

and so $\delta_1<\min\{\lambda_1,\lambda_3\}$ and $\delta_2<\min\{\lambda_2,\lambda_4\}$. Taking

$$[\gamma_1,\gamma_2]:=\tfrac{1}{2}\left(\tilde{B}_A(a_0*b_0)+\text{rmin}\{\tilde{B}_A(a_0),\tilde{B}_A(b_0)\}\right)$$

implies that

$$\begin{aligned}[\gamma_1,\gamma_2]&=\tfrac{1}{2}\left([\delta_1,\delta_2]+[\min\{\lambda_1,\lambda_3\},\min\{\lambda_2,\lambda_4\}]\right)\\&=\left[\tfrac{1}{2}(\delta_1+\min\{\lambda_1,\lambda_3\},\tfrac{1}{2}(\delta_2+\min\{\lambda_2,\lambda_4\}\right].\end{aligned}$$

It follows that

$$\min\{\lambda_1,\lambda_3\}>\gamma_1=\tfrac{1}{2}(\delta_1+\min\{\lambda_1,\lambda_3\}>\delta_1$$

and

$$\min\{\lambda_2,\lambda_4\}>\gamma_2=\tfrac{1}{2}(\delta_2+\min\{\lambda_2,\lambda_4\}>\delta_2.$$

Hence $[\min\{\lambda_1,\lambda_3\},\min\{\lambda_2,\lambda_4\}]\succ[\gamma_1,\gamma_2]\succ[\delta_1,\delta_2]=\tilde{B}_A(a_0*b_0)$, and therefore $a_0*b_0\notin U(\tilde{B}_A;[\gamma_1,\gamma_2])$. On the other hand,

$$\tilde{B}_A(a_0)=[\lambda_1,\lambda_2]\succeq[\min\{\lambda_1,\lambda_3\},\min\{\lambda_2,\lambda_4\}]\succ[\gamma_1,\gamma_2]$$

and

$$\tilde{B}_A(b_0)=[\lambda_3,\lambda_4]\succeq[\min\{\lambda_1,\lambda_3\},\min\{\lambda_2,\lambda_4\}]\succ[\gamma_1,\gamma_2],$$

that is, $a_0,b_0\in U(\tilde{B}_A;[\gamma_1,\gamma_2])$. This is a contradiction, and therefore $\tilde{B}_A(x*y)\succeq\text{rmin}\{\tilde{B}_A(x),\tilde{B}_A(y)\}$ for all $x,y\in X$. Consequently $\mathcal{A}=(M_A,\tilde{B}_A,J_A)$ is an MBJ-neutrosophic subalgebra of X. □

Using Proposition 3.5 and Theorem 3.8, we have the following corollary.

Corollary 3.9. *If $\mathcal{A}=(M_A,\tilde{B}_A,J_A)$ is an MBJ-neutrosophic subalgebra of X, then the sets $X_{M_A}:=\{x\in X\mid M_A(x)=M_A(0)\}$, $X_{\tilde{B}_A}:=\{x\in X\mid\tilde{B}_A(x)=\tilde{B}_A(0)\}$, and $X_{J_A}:=\{x\in X\mid J_A(x)=J_A(0)\}$ are subalgebras of X.*

We say that the subalgebras $U(M_A;t)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;s)$ are *MBJ-subalgebras* of $\mathcal{A}=(M_A,\tilde{B}_A,J_A)$.

Theorem 3.10. *Every subalgebra of X can be realized as MBJ-subalgebras of an MBJ-neutrosophic subalgebra of X.*

Proof. Let K be a subalgebra of X and let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by

$$M_A(x) = \begin{cases} t & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A(x) = \begin{cases} [\gamma_1, \gamma_2] & \text{if } x \in K, \\ [0,0] & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \tag{3.4}$$

where $t \in (0,1]$, $s \in [0,1)$ and $\gamma_1, \gamma_2 \in (0,1]$ with $\gamma_1 < \gamma_2$. It is clear that $U(M_A;t) = K$, $U(\tilde{B}_A;[\gamma_1,\gamma_2]) = K$ and $L(J_A;s) = K$. Let $x, y \in X$. If $x, y \in K$, then $x * y \in K$ and so

$$\begin{aligned} &M_A(x*y) = t = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A(x*y) = [\gamma_1,\gamma_2] = \text{rmin}\{[\gamma_1,\gamma_2],[\gamma_1,\gamma_2]\} = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ &J_A(x*y) = s = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If any one of x and y is contained in K, say $x \in K$, then $M_A(x) = t$, $\tilde{B}_A(x) = [\gamma_1,\gamma_2]$, $J_A(x) = s$, $M_A(y) = 0$, $\tilde{B}_A(y) = [0,0]$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x*y) \geq 0 = \min\{t, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A(x*y) \succeq [0,0] = \text{rmin}\{[\gamma_1,\gamma_2],[0,0]\} = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ &J_A(x*y) \leq 1 = \max\{s, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If $x, y \notin K$, then $M_A(x) = 0 = M_A(y)$, $\tilde{B}_A(x) = [0,0] = \tilde{B}_A(y)$ and $J_A(x) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x*y) \geq 0 = \min\{0, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A(x*y) \succeq [0,0] = \text{rmin}\{[0,0],[0,0]\} = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ &J_A(x*y) \leq 1 = \max\{1, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. □

Theorem 3.11. *For any non-empty subset K of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in (3.4). If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X, then K is a subalgebra of X.*

Proof. Let $x, y \in K$. Then $M_A(x) = t = M_A(y)$, $\tilde{B}_A(x) = [\gamma_1,\gamma_2] = \tilde{B}_A(y)$ and $J_A(x) = s = J_A(y)$. Thus

$$\begin{aligned} &M_A(x*y) \geq \min\{M_A(x), M_A(y)\} = t, \\ &\tilde{B}_A(x*y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} = [\gamma_1,\gamma_2], \\ &J_A(x*y) \leq \max\{J_A(x), J_A(y)\} = s, \end{aligned}$$

and therefore $x * y \in K$. Hence K is a subalgebra of X. □

Using an MBJ-neutrosophic subalgebra of a BCI-algera, we establish a new MBJ-neutrosophic subalgebra.

Theorem 3.12. *Given an MBJ-neutrosophic subalgebra $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of a BCI-algebra X, let $\mathcal{A}^* = (M_A^*, \tilde{B}_A^*, J_A^*)$ be an MBJ-neutrosophic set in X defined by $M_A^*(x) = M_A(0*x)$, $\tilde{B}_A^*(x) = \tilde{B}_A(0*x)$ and $J_A^*(x) = J_A(0*x)$ for all $x \in X$. Then $\mathcal{A}^* = (M_A^*, \tilde{B}_A^*, J_A^*)$ is an MBJ-neutrosophic subalgebra of X.*

Proof. Note that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. We have

$$\begin{aligned} M_A^*(x * y) &= M_A(0 * (x * y)) = M_A((0 * x) * (0 * y)) \\ &\geq \min\{M_A(0 * x), M_A(0 * y)\} \\ &= \min\{M_A^*(x), M_A^*(y)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A^*(x * y) &= \tilde{B}_A(0 * (x * y)) = \tilde{B}_A((0 * x) * (0 * y)) \\ &\succeq \text{rmin}\{\tilde{B}_A(0 * x), \tilde{B}_A(0 * y)\} \\ &= \text{rmin}\{\tilde{B}_A^*(x), \tilde{B}_A^*(y)\} \end{aligned}$$

and

$$\begin{aligned} J_A^*(x * y) &= J_A(0 * (x * y)) = J_A((0 * x) * (0 * y)) \\ &\leq \max\{J_A(0 * x), J_A(0 * y)\} \\ &= \max\{J_A^*(x), J_A^*(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A}^* = (M_A^*, \tilde{B}_A^*, J_A^*)$ is an MBJ-neutrosophic subalgebra of X. □

Theorem 3.13. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is an MBJ-neutrosophic subalgebra of Y, then $f^{-1}(\mathcal{B}) = (f^{-1}(M_B), f^{-1}(\tilde{B}_B), f^{-1}(J_B))$ is an MBJ-neutrosophic subalgebra of X, where $f^{-1}(M_B)(x) = M_B(f(x))$, $f^{-1}(\tilde{B}_B)(x) = \tilde{B}_B(f(x))$ and $f^{-1}(J_B)(x) = J_B(f(x))$ for all $x \in X$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(M_B)(x * y) &= M_B(f(x * y)) = M_B(f(x) * f(y)) \\ &\geq \min\{M_B(f(x)), M_B(f(y))\} \\ &= \min\{f^{-1}(M_B)(x), f^{-1}(M_B)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\tilde{B}_B)(x * y) &= \tilde{B}_B(f(x * y)) = \tilde{B}_B(f(x) * f(y)) \\ &\succeq \text{rmin}\{\tilde{B}_B(f(x)), \tilde{B}_B(f(y))\} \\ &= \text{rmin}\{f^{-1}(\tilde{B}_B)(x), f^{-1}(\tilde{B}_B)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(J_B)(x * y) &= J_B(f(x * y)) = J_B(f(x) * f(y)) \\ &\leq \max\{J_B(f(x)), J_B(f(y))\} \\ &= \max\{f^{-1}(J_B)(x), f^{-1}(J_B)(y)\}. \end{aligned}$$

Hence $f^{-1}(\mathcal{B}) = (f^{-1}(M_B), f^{-1}(\tilde{B}_B), f^{-1}(J_B))$ is an MBJ-neutrosophic subalgebra of X. □

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in a set X. We denote

$$\begin{aligned}\top &:= 1 - \sup\{M_A(x) \mid x \in X\},\\ \Pi &:= [1,1] - \operatorname{rsup}\{\tilde{B}_A(x) \mid x \in X\},\\ \bot &:= \inf\{J_A(x) \mid x \in X\}.\end{aligned}$$

For any $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \bot]$, we define $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ by $M_A^p(x) = M_A(x) + p$, $\tilde{B}_A^{\tilde{a}}(x) = \tilde{B}_A(x) + \tilde{a}$ and $J_A^q(x) = J_A(x) - q$. Then $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ is an MBJ-neutrosophic set in X, which is called a $(p, \tilde{a}, q)$-*translative MBJ-neutrosophic set* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

Theorem 3.14. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X, then the $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is also an MBJ-neutrosophic subalgebra of X.*

Proof. For any $x, y \in X$, we get

$$\begin{aligned}M_A^p(x * y) &= M_A(x * y) + p \geq \min\{M_A(x), M_A(y)\} + p\\ &= \min\{M_A(x) + p, M_A(y) + p\} = \min\{M_A^p(x), M_A^p(y)\},\end{aligned}$$

$$\begin{aligned}\tilde{B}_A^{\tilde{a}}(x * y) &= \tilde{B}_A(x * y) + \tilde{a} \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} + \tilde{a}\\ &= \operatorname{rmin}\{\tilde{B}_A(x) + \tilde{a}, \tilde{B}_A(y) + \tilde{a}\} = \operatorname{rmin}\{\tilde{B}_A^{\tilde{a}}(x), \tilde{B}_A^{\tilde{a}}(y)\},\end{aligned}$$

and

$$\begin{aligned}J_A^q(x * y) &= J_A(x * y) - q \leq \max\{J_A(x), J_A(y)\} - q\\ &= \max\{J_A(x) - q, J_A(y) - q\} = \max\{J_A^q(x), J_A^q(y)\}.\end{aligned}$$

Therefore $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ is an MBJ-neutrosophic subalgebra of X. □

We consider the converse of Theorem 3.14.

Theorem 3.15. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X such that its $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set is an MBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \bot]$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X.*

Proof. Assume that $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ is an MBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \bot]$. Let $x, y \in X$. Then

$$\begin{aligned}M_A(x * y) + p = M_A^p(x * y) &\geq \min\{M_A^p(x), M_A^p(y)\}\\ &= \min\{M_A(x) + p, M_A(y) + p\}\\ &= \min\{M_A(x), M_A(y)\} + p,\end{aligned}$$

$$\begin{aligned}\tilde{B}_A(x * y) + \tilde{a} = \tilde{B}_A^{\tilde{a}}(x * y) &\succeq \operatorname{rmin}\{\tilde{B}_A^{\tilde{a}}(x), \tilde{B}_A^{\tilde{a}}(y)\}\\ &= \operatorname{rmin}\{\tilde{B}_A(x) + \tilde{a}, \tilde{B}_A(y) + \tilde{a}\}\\ &= \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} + \tilde{a},\end{aligned}$$

and

$$\begin{aligned} J_A(x*y)-q = J_A^q(x*y) &\leq \max\{J_A^q(x), J_A^q(y)\} \\ &= \max\{J_A(x)-q, J_A(y)-q\} \\ &= \max\{J_A(x), J_A(y)\}-q. \end{aligned}$$

It follows that $M_A(x*y) \geq \min\{M_A(x), M_A(y)\}$, $\tilde{B}_A(x*y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}$ and $J_A(x*y) \leq \max\{J_A(x), J_A(y)\}$ for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. □

Definition 3.16. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ and $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ be MBJ-neutrosophic sets in X. Then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is called an *MBJ-neutrosophic S-extension* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ if the following assertions are valid.

(1) $M_B(x) \geq M_A(x)$, $\tilde{B}_B(x) \succeq \tilde{B}_A(x)$ and $J_B(x) \leq J_A(x)$ for all $x \in X$,

(2) If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X, then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is an MBJ-neutrosophic subalgebra of X.

Theorem 3.17. *Given $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \perp]$, the $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ of an MBJ-neutrosophic subalgebra $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic S-extension of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.*

Proof. Straightforward. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} &U_p(M_A; t) := \{x \in X \mid M_A(x) \geq t - p\}, \\ &U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2]) := \{x \in X \mid \tilde{B}_A(x) \succeq [\delta_1, \delta_2] - \tilde{a}\}, \\ &L_q(J_A; s) := \{x \in X \mid J_A(x) \leq s + q\} \end{aligned}$$

where $t, s \in [0,1]$, $[\delta_1, \delta_2] \in [I]$, $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \perp]$ such that $t \geq p$, $[\delta_1, \delta_2] \succeq \tilde{a}$ and $s \leq q$.

Theorem 3.18. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Given $p \in [0, \top]$, $\tilde{a} \in [[0,0], \Pi]$ and $q \in [0, \perp]$, the $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X if and only if $U_p(M_A; t)$, $U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2])$ and $L_q(J_A; s)$ are subalgebras of X for all $t \in Im(M_A)$, $[\delta_1, \delta_2] \in Im(\tilde{B}_A)$ and $s \in Im(J_A)$ with $t \geq p$, $[\delta_1, \delta_2] \succeq \tilde{a}$ and $s \leq q$.*

Proof. Assume that the $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. Let $x, y \in U_p(M_A; t)$. Then $M_A(x) \geq t - p$ and $M_A(y) \geq t - p$, which imply that $M_A^p(x) \geq t$ and $M_A^p(y) \geq t$. It follows that

$$M_A^p(x*y) \geq \min\{M_A^p(x), M_A^p(y)\} \geq t$$

and so that $M_A(x*y) \geq t - p$. Hence $x*y \in U_p(M_A; t)$. If $x, y \in U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2])$, then $\tilde{B}_A(x) \succeq [\delta_1, \delta_2] - \tilde{a}$ and $\tilde{B}_A(y) \succeq [\delta_1, \delta_2] - \tilde{a}$. Hence

$$\tilde{B}_A^{\tilde{a}}(x*y) \succeq \text{rmin}\{\tilde{B}_A^{\tilde{a}}(x), \tilde{B}_A^{\tilde{a}}(y)\} \succeq [\delta_1, \delta_2],$$

and so $\tilde{B}_A(x*y) \succeq [\delta_1, \delta_2] - \tilde{a}$. Thus $x*y \in U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2])$. Let $x, y \in L_q(J_A; s)$. Then $J_A(x) \leq s + q$ and $J_A(y) \leq s + q$. It follows that

$$J_A^q(x*y) \leq \max\{J_A^q(x), J_A^q(y)\} \leq s,$$

that is, $J_A(x*y) \leq s + q$. Thus $x*y \in L_q(J_A; s)$. Therefore $U_p(M_A; t)$, $U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2])$ and $L_q(J_A; s)$ are subalgebras of X.

Conversely, suppose that $U_p(M_A; t)$, $U_{\tilde{a}}(\tilde{B}_A; [\delta_1, \delta_2])$ and $L_q(J_A; s)$ are subalgebras of X for all $t \in Im(M_A)$, $[\delta_1, \delta_2] \in Im(\tilde{B}_A)$ and $s \in Im(J_A)$ with $t \geq p$, $[\delta_1, \delta_2] \succeq \tilde{a}$ and $s \leq q$. Assume that $M_A^p(a*b) < \min\{M_A^p(a), M_A^p(b)\}$ for some $a, b \in X$. Then $a, b \in U_p(M_A; t_0)$ and $a*b \notin U_p(M_A; t_0)$ for $t_0 = \min\{M_A^p(a), M_A^p(b)\}$. This is a contradiction, and so $M_A^p(x*y) \geq \min\{M_A^p(x), M_A^p(y)\}$ for all $x, y \in X$. If $\tilde{B}_A^{\tilde{a}}(x_0*y_0) \prec \mathrm{rmin}\{\tilde{B}_A^{\tilde{a}}(x_0), M_A^{\tilde{a}}(y_0)\}$ for some $x_0, y_0 \in X$, then there exists $\tilde{b} \in [I]$ such that $\tilde{B}_A^{\tilde{a}}(x_0*y_0) \prec \tilde{b} \preceq \mathrm{rmin}\{\tilde{B}_A^{\tilde{a}}(x_0), M_A^{\tilde{a}}(y_0)\}$. Hence $x_0, y_0 \in U_{\tilde{a}}(\tilde{B}_A; \tilde{b})$ but $x_0 * y_0 \notin U_{\tilde{a}}(\tilde{B}_A; \tilde{b})$, which is a contradiction. Thus $\tilde{B}_A^{\tilde{a}}(x*y) \succeq \mathrm{rmin}\{\tilde{B}_A^{\tilde{a}}(x), M_A^{\tilde{a}}(y)\}$ for all $x, y \in X$. Suppose that $J_A^q(a*b) > \max\{J_A^q(a), J_A^q(b)\}$ for some $a, b \in X$. Taking $s_0 := \max\{J_A^q(a), J_A^q(b)\}$ implies that $J_A(a) \leq s_0 + q$ and $J_A(b) \leq s_0 + q$ but $J_A(a*b) > s_0 + q$. This shows that $a, b \in L_q(J_A; s_0)$ and $a*b \in L_q(J_A; s_0)$. This is a contradiction, and therefore $J_A^q(x*y) \leq \max\{J_A^q(x), J_A^q(y)\}$ for all $x, y \in X$. Consequently, the $(p, \tilde{a}, q)$-translative MBJ-neutrosophic set $\mathcal{A}^T = (M_A^p, \tilde{B}_A^{\tilde{a}}, J_A^q)$ of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic subalgebra of X. □

4 Conclusion

This paper is written during the third author visit Shahid Beheshti University. In the study of Smarandache's neutrosophic sets, the authors of this article had a periodical research meeting three times per week, and tried to get a generalization of Smarandache's neutrosophic sets. In the neutrosophic set, the truth, false and indeterminate membership functions are fuzzy sets. In considering a generalization of neutrosophic set, we used the interval valued fuzzy set as the indeterminate membership function because interval valued fuzzy set is a generalization of a fuzzy set, and we called it MBJ-neutrosophic set where "MBJ" is the initial of authors's surname, that is, Mohseni, Borzooei and Jun, respectively. We also use M_A, $\tilde{B}_A$ and J_A as the truth membership function, the indeterminate membership function and the false membership function, respectively. We know that there are many generalizations of Smarandache's neutrosophic sets. In this article, we have made up a generalization of neutrosophic set, called an MBJ-neutrosophic set, and have applied it to BCK/BCI-algebras. We have introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. We have provided a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI-algebra. We have considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. We also have found conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic subalgebra.

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Algebraic Structure of Neutrosophic Duplets in Neutrosophic Rings $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

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Abstract: The concept of neutrosophy and indeterminacy I was introduced by Smarandache, to deal with neutralies. Since then the notions of neutrosophic rings, neutrosophic semigroups and other algebraic structures have been developed. Neutrosophic duplets and their properties were introduced by Florentin and other researchers have pursued this study.In this paper authors determine the neutrosophic duplets in neutrosophic rings of characteristic zero. The neutrosophic duplets of $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$; the neutrosophic ring of integers, neutrosophic ring of rationals and neutrosophic ring of reals respectively have been analysed. It is proved the collection of neutrosophic duplets happens to be infinite in number in these neutrosophic rings. Further the collection enjoys a nice algebraic structure like a neutrosophic subring, in case of the duplets collection $\{a - aI | a \in Z\}$ for which $1 - I$ acts as the neutral. For the other type of neutrosophic duplet pairs $\{a - aI, 1 - dI\}$ where $a \in R^+$ and $d \in R$, this collection under component wise multiplication forms a neutrosophic semigroup. Several other interesting algebraic properties enjoyed by them are obtained in this paper.

Keywords: Neutrosophic ring; neutrosophic duplet; neutrosophic duplet pairs; neutrosophic semigroup; neutrosophic subring

1 Introduction

The concept of indeterminacy in the real world data was introduced by Florentin Smarandache [1, 2] as Neutrosophy. Existing neutralities and indeterminacies are dealt by the neutrosophic theory and are applied to real world and engineering problems [3, 4, 5]. Neutrosophic algebraic structures were introduced and studied by [6]. Since then several researchers have been pursuing their research in this direction [7, 8, 9, 10, 11, 12]. Neutrosophic rings [9] and other neutrosophic algebraic structures are elaborately studied in [6, 7, 8, 10].

Related theories of neutrosophic triplet, neutrosophic duplet, and duplet set was studied by Smarandache [13]. Neutrosophic duplets and triplets have interested many and they have studied [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Neutrosophic duplet semigroup [18], the neutrosophic triplet group [12], classical group of neutrosophic triplet groups[22] and neutrosophic duplets of $\{Z_{pn}, \times\}$ and $\{Z_{pq}, \times\}$ [23] have been recently studied.

Here we mainly introduce the concept of neutrosophic duplets in case of of neutrosophic rings of characteristic zero and study only the algebraic properties enjoyed by neutrosophic duplets, neutrals and neutrosophic duplet pairs.

In this paper we investigate the neutrosophic duplets of the neutrosophic rings $\langle Z \cup I\rangle$, $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$. We prove the duplets for a fixed neutral happens to be an infinite collection and enjoys a nice algebraic structure. In fact the collection of neutrals for fixed duplet happens to be infinite in number and they too enjoy a nice algebraic structure.

This paper is organised into five sections , section one is introductory in nature. Important results in this paper are given in section two of this paper. Neutrosophic duplets of the neutrosophic ring $\langle Z \cup I\rangle$, and its properties are analysed in section three of this paper. In the forth section neutrosophic duplets of the rings $\langle Q\cup I\rangle$ and $\langle R\cup I\rangle$; are defined and developed and several theorems are proved. In the final section discussions, conclusions and future research that can be carried out is described.

2 Results

The basic definition of neutrosophic duplet is recalled from [12]. We just give the notations and describe the neutrosophic rings and neutrosophic semigroups [9].

Notation: $\langle Z \cup I\rangle = \{a + bI|a, b \in Z, I^2 = I\}$ is the collection of neutrosophic integers which is a neutrosophic ring of integers. $\langle Q \cup I\rangle = \{a + bI|a, b \in Q, I^2 = I\}$ is the collection of neutrosophic rationals and $\langle R \cup I\rangle = \{a + bI|a, b \in R, I^2 = I\}$ is the collection of neutrosophic reals which are neutrosophic ring of rationals and reals respectively.

Let S be any ring which is commutative and has a unit element 1. Then $\langle S \cup I\rangle = \{a + bI|a, b \in S, I^2 = I, +, \times\}$ be the neutrosophic ring. For more refer [9].

Consider U to be the universe of discourse, and D a set in U, which has a well-defined law #.

Definition 2.1. Consider $\langle a, neut(a)\rangle$, where a, and $neut(a)$ belong to D. It is said to be a neutrosophic duplet if it satisfies the following conditions:

1. $neut(a)$ is not same as the unitary element of D in relation with the law # (if any);
2. $a\#\ neut(a)$ = neut(a) # a = a;
3. $anti(a) \notin$ D for which a # anti(a) = anti(a) # a = neut(a).

The results proved in this paper are

1. All elements of the form $a - aI$ and $aI - a$ with $1 - I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.
2. In fact $B = \{a - aI/a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.
3. Let $S = \{\langle Q\cup I\rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q\setminus\{0\}$ we have $a+bI \in \langle Q\cup I\rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is a neutrosophic duplet.
4. The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.

5. For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.

6. The ordered pair of neutrosophic duplets $B = \{(nI, m - (m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.

7. The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ forms a neutrosophic semigroup under product taken component wise.

3 Neutrosophic duplets of $\langle Z \cup I \rangle$ and its properties

In this section we find the neutrosophic duplets in $\langle Z \cup I \rangle$. Infact we prove there are infinite number of neutrals for any relevant element in $\langle Z \cup I \rangle$. Several interesting results are proved.

First we illustrate some of the neutrosophic duplets in $\langle Z \cup I \rangle$.

Example 3.1. Let S = $\langle Z \cup I \rangle = \{a + bI | a, b \in I, I^2 = I\}$ be the neutrosophic ring. Consider any element $x = 9I \in \langle Z \cup I \rangle$; we see the element $16 - 15I \in \langle Z \cup I \rangle$ is such that $9I \times 16 - 15I = 144I - 135I = 9I = x$. Thus $16 - 15I$ acts as the neutral of $9I$ and $\{9I, 16 - 15I\}$ is a neutrosophic duplet.

Cconsider $15I = y \in \langle Z \cup I \rangle$; $15I \times 16 - 15I = 15I = y$. Thus $\{15I, 16 - 15I\}$ is again a neutrosophic duplet. Let $-9I = s \in \langle Z \cup I \rangle$; $-9I \times 16 - 15I = -144I + 135I = -9I = s$, so $\{-9I, 16 - 15I\}$ is a neutrosophic duplet. Thus $\{\pm 9I, 16 - 15I\}$ happens to be neutrosophic duplets.

Further $nI \in \langle Z \cup I \rangle$ is such that $nI \times 16 - 15I = 16nI - 15nI = nI$. Similarly $-nI \times 16 - 15I = -16nI + 15nI = -nI$. So $\{nI, 16 - 15I\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$. Another natural question which comes to one mind is will $16I - 15$ act as a neutral for nI; $n \in Z \setminus \{0\}$, the answer is yes for $nI \times (16I - 15) = 16nI - 15nI = nI$. Hence the claim.

We call $0I = 0$ as the trivial neutrosophic duplet as $(0, x)$ is a neutrosophic duplet for all $x \in \langle Z \cup I \rangle$.

In view of this example we prove the following theorem.

Theorem 3.2. *Let $S = \langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be a neutrosophic ring. Every $\pm nI \in S; n \in Z \setminus \{0\}$ has infinite number of neutrals of the form*

- $mI - (m-1) = x$
- $m - (m-1)I = y$
- $(m-1) - mI = -x$
- $(m-1)I - mI = -y$

where $m \in Z^+ \setminus \{1, 0\}$.

Proof. Consider $nI \in \langle Z \cup I \rangle$ we see

$$nI \times x = nI[mI - (m-1)] = nnI - nmI + nI = nI.$$

Thus $\{nI, mI - (m-1)\}$ form an infinite collection of neutrosophic duplets for a fixed n and varying $m \in Z^+ \setminus \{0, 1\}$. Proof for other parts (ii), (iii) and (iv) follows by a similar argument. □

Thus in view of the above theorem we can say for any $nI; n \in Z \setminus \{0\}$, n is fixed; we have an infinite collection of neutrals paving way for an infinite collection of neutrosophic duplets contributed by elements $x, y, -x$ and $-y$ given in the theorem. On the other hand for any fixed x or y or $-x$ or $-y$ given in the theorem we have an infinite collection of elements of the form $nI; n \in Z \setminus \{0\}$ such that $\{n, x,$or y or $-x$ or $-y\}$ is a neutrosophic duplet.

Now our problem is to find does these neutrals collection $\{x, y, -x, -y\}$ in theorem satisfy any nice algebraic structure in $\langle Z \cup I \rangle$.

We first illustrate this using some examples before we propose and prove any theorem.

Example 3.3. Let $S = \langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be the ring. $\{S, \times\}$ is a commutative semigroup under product []. Consider the element $x = 5I - 4 \in \langle Z \cup I \rangle$. $5I - 4$ acts as neutral for all elements $nI \in \langle Z \cup I \rangle, n \in Z \setminus \{0\}$. Consider $x \times x = 5I - 4 \times 5I - 4 = 25I - 20I - 20I + 16 = -15I + 16 = x^2$. Now $-15I + 16 \times nI = -15nI + 16nI = nI$. Thus if $\{nI, x\}$ a neutrosophic duplet so is $\{nI, x^2\}$. Consider

$$x^3 = x^2 \times x = (-15I + 16) \times (5I - 4)$$

$$= -75I + 80I + 60I - 64 = 65I - 64 = x^3$$

$$nI \times x^3 = 65nI - 64nI = nI$$

So $\{nI, 65I - 64\} = \{nI, x^3\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$ Consider

$$x^4 = x^3 \times x = 65I - 64 \times 5I - 4$$

$$= 325I - 320I - 260I + 256 = -255I + 256 = x^4$$

Clearly

$$nI \times x^4 = nI \times (-255I + 25) = -255nI + 256nI = nI.$$

So $\{nI, x^4\}$ is a neutrosophic duplet. In fact one can prove for any $nI \in \langle Z \cup I \rangle; n \in Z \setminus \{0\}$ then $x = m - (m-1)I$ is the neutral of nI then $\{nI, x\}, \{nI, x^2\}, \{nI, x^3\}, \ldots, \{nI, x^r\}, \ldots, \{nI, x^t\}; t \in Z^+ \setminus \{0\}$ are all neutrosophic duplets for nI. Thus for any fixed nI there is an infinite collection of neutrals. We see if x is a neutral then the cyclic semigroup generated by x denoted by $\langle x \rangle = \{x, x^2, x^3, \ldots\}$ happens to be a collection of neutrals for $nI \in S$.

Now we proceed onto give examples of other forms of neutrosophic duplets using $\langle Z \cup I \rangle$.

Example 3.4. Let $S = \{\langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring. We see $x = 1 - I \in S$ such that

$$(1 - I)^2 = 1 - I \times 1 - I = 1 - 2I + I^2 (\because I^2 = I)$$

$$= 1 - I = x.$$

Thus x is an idempotent of S. We see $y = 5 - 5I$ such that

$$y \times x = (5 - 5I) \times (5 - 5I) = 5 - 5I - 5I + 5I = 5 - 5I = y$$

Thus $\{5 - 5I, 1 - I\}$ is a neutrosophic duplets and $1 - I$ is the neutral of $5 - 5I$.

$$y^2 = 5 - 5I \times 5 - 5I = 25 - 25I - 25I + 25I = 25 - 25I$$

We see $\{y^2, 1-I\}$ is again a neutrosophic duplet.

$$y^3 = y \times y^2 = 5 - 5I \times (25 - 25I) = 125 - 125I - 125I + 125I$$

$$= 125 - 125I = y^3$$

Once again $\{y^3, 1-I\}$ is a neutrosophic duplet. In fact we can say for the idempotent $1-I$ the cyclic semigroup $B = \{y, y^2, y^3, \ldots\}$ is such that for every element in B, $1-I$ serves as the neutral.

In view of all these we prove the following theorem.

Theorem 3.5. *Let $S = \{\langle Z \cup I \rangle, +, \times\}$ be the neutrosophic ring.*

1. *$1-I$ is an idempotent of S.*
2. *All elements of the form $a - aI$ and $aI - a$ with $1-I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.*
3. *In fact $B = \{a - aI / a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.*

Proof. 1. Let $x = 1 - I \in S$ to show x is an idempotent of S, we must show $x \times x = x$. We see $1 - I \times 1 - I = 1 - 2I + I^2$ as $I^2 = I$, we get $1 - I \times 1 - I = 1 - I$; hence the claim.

2. Let $a - aI \in S; a \in Z$. $1-I$ is the neutral of $a - aI$ as $a - aI \times 1 - I = a - aI - aI + aI = a - aI$. Thus $\{a - aI, 1 - I\}$ is a neutrosophic duplet. On similar lines $aI - a$ will also yield a neutrosophic duplet with $1 - I$. Hence the result (ii).

3. Given $B = \{a - aI | a \in Z\}$. To prove B is a group under +. Let $x = a - aI$ and $y = b - bI \in B$; $x + y = a - aI + b - bI = (a + b) - (a + b)I$ as $a + b \in Z$; $a + b - (a + b)I \in B$. So B is closed under the operation +. When $a = 0$ we get $0 - 0I = \in B$ and $a - aI + 0 = a - aI$. 0 acts as the additive identity of B. For every $a - aI \in B$ we have

$$-(a - aI) = (-a) - (-a)I = -a + aI \in B$$

is such that $a - aI + (-a) + aI = 0$ so every $a - aI$ has an additive inverse. Now we show $\{B, \times\}$ is a semigroup under product $\times$.

$$(a - aI) \times (b - bI) = ab - abI - baI + abI = ab - abI \in B.$$

Thus B is a semigroup under product. Clearly $1 - I \in B$. Now we test the distributive law. let $x = a - aI, y = b - bI$ and $z = c - cI \in B$.

$$(a - aI) \times [b - bI + c - cI] = a - aI \times [(b + c) - (b + c)I$$

$$= a(b + c) - aI(b + c) - (b + c)aI + a(b + c)I = a(b + c) - aI(b + c) \in B$$

Thus $\{B, +, \times\}$ is a neutrosophic subring of S. Finally we prove $\langle Z \cup I \rangle$ has neutrosophic duplets of the form $\{a - aI, 1 + dI\}; d \in Z \setminus \{0\}$.

□

Theorem 3.6. *Let $S = \{\langle Z \cup I\rangle = \{a + bI|a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring $a + bI \in S$ contributes to a neutrosophic duplet if and only if $a = -b$.*

Proof. Let $a + bI \in S(a \neq 0, b \neq 0)$ be an element which contributes a neutrosophic duplet with $c + dI \in S$. If $\{a + bI, c + dI\}$ is a neutrosophic duplet then $(a + bI) \times (c + dI) = a + bI$, this implies

$$ac + (bd + ad + bc)I = a + bI.$$

This implies $ac = a$ and $bd + ad + bc = b$. $ac = a$ implies $a(c - 1) = 0$ since $a \neq 0$ we have $c = 1$. Now in $bd + ad + bc = b$ substitute $c = 1$; it becomes $bd + ad + b = b$ which implies $bd + ad = 0$ that is $(b + a)d = 0$; $d \neq 0$ for if $d = 0$ then $c + dI = 1$ acts as a neutral, for all $a + bI \in S$ which is a trivial neutrosophic duplet. Thus $d \neq 0$, which forces $a + b = 0$ or $a = -b$. hence $a + bI = a - aI$. Now we have to find d. We have $(a - aI)(1 + dI) = a - aI + adI - adI = a - aI$.

This is true for any $d \in Z \setminus \{0\}$. Proof of the converse is direct. □

Next we proceed on to study neutrosophic duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$

4 Neutrosophic Duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$

In this section we study the neutrosophic duplets of the neutrosophic rings $\langle Q \cup I\rangle = \{a+bI|a, b \in Q, I^2 = I\}$; where Q the field of rationals and $\langle R \cup I\rangle = \{a + bI|a, b, \in R, I^2 = I\}$; where R is the field of reals. We obtain several interesting results in this direction. It is important to note $\langle Z \cup I\rangle \subset \langle Q \cup I\rangle \subset \langle R \cup I\rangle$. Hence all neutrosophic duplets of $\langle Z \cup I\rangle$ will continue to be neutrosophic duplets of $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$. Our analysis pertains to the existence of other neutrosophic duplets as Z is only a ring where as Q and R are fields. We enumerate many interesting properties related to them.

Example 4.1. Let $S = \{\langle Q \cup I\rangle = \{a + bI|a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals. Consider for any nI ∈S we have the neutral

$$x = \frac{-7I}{9} + \frac{16}{9} \in S,$$

such that

$$nI \times x = nI\left(\frac{-7I}{9} + \frac{16}{9}\right) = nI.$$

Thus for the element nI the neutral is

$$\frac{-7I}{9} + \frac{16}{9} \in S.$$

We make the following observation

$$\frac{-7}{9} + \frac{16}{9} = 1.$$

In fact all elements of the form $a + bI$ in $\langle Q \cup I\rangle$ with $a + b = 1$; $a, b \in Q \setminus \{0\}$ can act as neutrals for nI. Suppose

$$x = \frac{8I}{9} + \frac{1}{9} \in \langle Q \cup I\rangle$$

then for $nI = y$ we see

$$x \times y = nI \times \left(\frac{8I}{9} + \frac{1}{9}\right) = \frac{8In}{9} + \frac{nI}{9} = nI.$$

Take $x = -9I + 10$ we see

$$x \times y = -9I + 0 \times nI = -9In + 10nI = nI$$

and so on.

However we have proved in section 3 of this paper for any $nI \in \langle Z \cup I \rangle$ the collection of all elements $a + bI \in \langle Z \cup I \rangle$ with $a + b = 1; a, b \in Z \setminus \{0\}$ will act as neutrals of nI.

In view of all these we put forth the following theorem.

Theorem 4.2. *Let $S = \{\langle Q \cup I \rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q \setminus \{0\}$ we have $a + bI \in \langle Q \cup I \rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is neutrosophic duplet.*

Proof. Given $nI \in \langle Q \cup I \rangle; n \in Q \setminus \{0\}$, we have to show $a + bI$ is a neutral where $a + b = 1, a, b, \in Q \setminus \{0\}$. consider

$$nI \times (a + bI) = anI + bnI = (a + b)nI = nI$$

as $a+b = 1$. Hence for any fixed $nI \in \langle Q \cup I \rangle$ we have an infinite collection of neutrals. Further the number of such neutrosophic duplets are infinite in number for varying n and varying $a, b \in Q \setminus \{0\}$ with $a + b = 1$. Thus the number of neutrosophic duplets in case of neutrosophic ring $\langle Q \cup I \rangle$ contains all the neutrosophic duplets of $\langle Z \cup I \rangle$ and the number of neutrosophic duplets in $\langle Q \cup I \rangle$ is a bigger infinite than that of the neutrosophic duplets in $\langle Z \cup I \rangle$. Further all $a + bI$ where $a, b \in Q \setminus Z$ with $a + b = 1$ happens to contribute to neutrosophic duplets which are not in $\langle Z \cup I \rangle$. □

Now we proceed on to give other types of neutrosopohic duplets in $\langle Q \cup I \rangle$ using $1 - I$ the idempotent which acts as neutral. Consider

$$x = \frac{5}{3} - \frac{5I}{3} \in \langle Q \cup I \rangle$$

let $y = 1 - I$, we find

$$x \times y = \frac{5}{3} - \frac{5I}{3} \times 1 - I = \frac{5}{3} - \frac{5I}{3} - \frac{5I}{3} + \frac{5I}{3} = \frac{5}{3} - \frac{5I}{3} = x.$$

In view of this we propose the following theorem.

Theorem 4.3. *Let $S = \{\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals.*

1. *The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.*

2. *For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.*

Proof. Consider any $a - aI = x \in \langle Q \cup I \rangle; a \in Q$ we see for $y = 1 - I$ the idempotent in $\langle Q \cup I \rangle$.

$$x \times y = a - aI \times 1 - I = a - aI - aI + aI = a - aI = x.$$

Thus $1 - I$ acts as the neutral for $a - aI$; in fact $\{a - aI, 1_I\}$ is a neutrosophic duplet; for all $a \in Q$. Now consider $s = p - pI$ where $p \in Q$ and $r = 1 - dI \in \langle Q \cup I \rangle; d \in Q$.

$$S \times r = p - pI \times 1 - dI = p - pI - pdI + pdI = p - pI = s$$

Thus $\{p - pI, 1 - dI\}$ are neutrosophic duplets for all $p \in Q$ and $d \in Q$. The collection of neutrosophic duplets which are in $\langle Q \cup I \rangle \setminus \{\langle Z \cup I \rangle\}$ is in fact is of infinite cardinality. □

Next we search of other types of neutrosophic duplets in $\{\langle Q \cup I \rangle\}$. Suppose $a + bI \in \langle Q \cup I \rangle$ and let $c + dI$ be the possible neutral for it, we arrive the conditions on a, b, c and d

$$(a + bI) \times (c + dI) = a + bI$$

$$ac + bc + adI + bdI = a + bI$$

$ac = a$ which is possible if and only if $c = 1$. Hence

$$b + ad + bd = b$$

$$ad + bd = 0$$

$$d(a + b) = 0$$

as $d \neq 0$;

$$a = -b.$$

Thus $a + bI = a - aI$ are only possible elements in $\langle Q \cup I \rangle$which can contribute to neutrosophic duplets and the neutrals associated with them is of the form $1 \pm dI$ and $d \in Q \setminus \{0\}$. Thus we can say even in case of R the field of reals and for the associated neutrosophic ring $\langle R \cup I \rangle$. All results are true in case $\langle Q \cup I \rangle$ and $\langle Z \cup I \rangle$; expect $\langle R \cup I \rangle \setminus \langle Q \cup I \rangle$ has infinite duplets and $\langle R \cup I \rangle$ has infinitely many more neutrosophic duplets than $\langle Q \cup I \rangle$.

The following theorem on real neutrsophic rings is both innovative and intersting.

Theorem 4.4. *Let $S = \langle R \cup I \rangle$ be the real neutrosophic ring. The neutrosophic duplets are contributed only by elements of the form nI and $a - aI$ where $n \in R$ and $a \in R^+$ with neutrals $m - (m-1)I$ and $1 - dI; m, d \in R$ respectively.*

Proof. Consider $\{nI, m(m-1)I\}$ the pair

$$nI \times m - (m-1)I = nmI$$

$$-nmI + nI = nI$$

for all $n, m \in R \setminus \{1, 0\}$. Thus $\{nI, m - (m-1)I\}$ is an infinite collection of neutrosophic duplets. We define $(nI, m - (m-1)I)$ as a neutrosophic duplet pair. Consider the pair $\{(a - aI), (1 - dI)\}; a \in R^+, d \in R$. We see

$$a - aI \times 1 - dI = a - aI - daI + adI = a - aI$$

Thus $\{(a - aI), (1 - dI)\}$ forms an infinite collection of neutrosophic duplets. We call $((a - aI), (1 - dI))$ as a neutrosophic duplet pair. Hence the theorem. □

Theorem 4.5. *Let $S = \langle R \cup I \rangle$ be the neutrosophic ring*

1. *The ordered pair of neutrosophic duplets $B = \{(nI, m-(m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.*
2. *The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ form a neutrosophic semigroup under product taken component wise.*

Proof. Given $B = \{(nI, m-(m-1)I|n \in R, m \in (R \setminus \{1\})\} \cup (nI, 0) \subseteq (\{\langle R \cup I \rangle\}, \{\langle R \cup I \rangle\})$. To prove B is a neutrosophic semigroup of $(\langle R \cup I \rangle, \langle R \cup I \rangle)$.. For any $x = (nI, (m-(m-1)I)$ and $y = (sI, t - 9t - 1)I) \in B$ we prove $xy = yx \in B$

$$x \times y = xy = (nI, m - (m-1)I \times (sI, t - (t-1)I)$$

$$= (nsI, [m - (m-1)I] \times [t - (t-1)I])$$

$$(nsI, mt - t(m-1)I - m(t-1)I + (m-1)(t-1)I)$$

$$= (nsI, mt - (mt-1)I) \in B$$

It is easily verified $xy = yx$ for all $x, y \in B$. Thus $\{B, \times\}$ is a neutrosophic semigroup of neutrosophic duplet pairs. Consider $x, y \in D$; we show $x \times y \in D$. Let $x = (a - aI, 1 - dI)$ and $y = (b - bI, 1 - cI) \in D$

$$x \times y = (a - aI, 1 - dI) \times (b - bI, 1 - cI)$$

$$= (a - aI \times b - bI, (-aI \times 1 - cI)$$

$$= (ab - abI - abI + abI, 1 - dI - cI + cdI)$$

$$= (ab - abI, 1 - (d + c - cd)I) \in D$$

as $x \times y$ is also in the form of x and y. Hence D the neutrosophic duplet pairs forms a neutrosophic semigroup under component wise product. □

5 Discussions and Conclusions

In this paper the notion of duplets in case neutrosophic rings, $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$, have been introduced and analysed. It is proved that the number of neutrosophic duplets in all these three rings happens to be an infinite collection. We further prove there are infinitely many elements for which $1 - I$ happens to be the neutral. Here we establish the duplet pair $\{a - aI, 1 - dI\}; a \in R^+$ and $d \in R$ happen to be a neutrosophic semigroup under component wise product. The collection $\{a - aI\}$ forms a neutrosophic subring $a \in Z$ or Q or R. For future research we want to analyse whether these neutrosophic rings can have neutrosophic triplets and if that collections enjoy some nice algebraic property. Finally we leave it as an open problem to find some applications of these neutrosophic duplets which form an infinite collection.

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On Neutrosophic Crisp Topology via *N*-Topology

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Abstract. In this paper, we extend the neutrosophic crisp topological spaces into N–neutrosophic crisp topological spaces (N_{nc}-topological space). Moreover, we introduced new types of open and closed sets in N–neutrosophic crisp topological spaces. We also present N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets and N_{nc}-α-open (closed) sets and investigate their basic properties.

Keywords: N_{nc}-topology, N–neutrosophic crisp topological spaces, N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets, N_{nc}-α-open (closed) sets, $N_{nc}int(A)$, $N_{nc}cl(A)$.

Introduction

The concept of non-rigid (fuzzy) sets introduced in 1965 by L. A. Zadeh [11] which revolutionized the field of logic and set theory. Since the need for supplementing the classical two-valued logic with respect to notions with rigid extension engendered the concept of fuzzy set. Soon after its advent, this notion has been utilized in different fields of research such as, decision-making problems, modelling of mental processes, that is, establishing a theory of fuzzy algorithms, control theory, fuzzy graphs, fuzzy automatic machine etc., and in general topology. Three years after the presence of the concept of fuzzy set, Chang [3] introduced and developed the theory of fuzzy topological spaces. Many researchers focused on this theory and they developed it further in different directions. Then another new notion called intuitionistic fuzzy set was established by Atanassov [2] in 1983. Coker [4] introduced the notion of intuitionistic fuzzy topological space. F. Smarandache introduced the concepts of neutrosophy and neutrosophic set ([7], [8]). A. A. Salama and S. A. Alblowi [5] introduced the notions of neutrosophic crisp set and neutrosophic crisp topological space. In 2014, A.A. Salama, F. Smarandache and V. Kroumov [6] presented the concept of neutrosophic crisp topological space (NCTS). W. Al-Omeri [1] also investigated neutrosophic crisp sets in the context of neutrosophic crisp topological Spaces. The geometric existence of N -topology was given by M. Lellis Thivagar et al. [10], which is a nonempty set equipped with N-arbitrary topologies. The notion of N_n-open (closed) sets and N-neutrosophic topological spaces are introduced by M. Lellis Thivagar, S. Jafari,V. Antonysamy and V. Sutha Devi. [9]

In this paper, we explore the possibility of expanding the concept of neutrosophic crisp topological spaces into N-neutrosophic crisp topological spaces (N_{nc}-topological space). Further, we develop the concept of open (closed) sets, semiopen (semiclosed) sets, preopen (preclosed) sets and α-open (α-closed) sets in the context of N-neutrosophic crisp topological spaces and investigate some of their basic properties.

1.Preliminaries

In this section, we discuss some basic definitions and properties of N -topological spaces and neutrosophic crisp topological spaces which are useful in sequel.

Definition 1.1. [6] Let X be a non-empty fixed set. A neutrosophic crisp set (NCS) A is an object having the form $A = \{A_1, A_2, A_3\}$, where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

Definition 1.2. [6] Types of $NCSs$ ϕ_N and X_N in X are as follows:

1. ϕ_N may be defined in many ways as an NCS as follows:

   1. $\phi_N = (\phi, \phi, X)$ or
   2. $\phi_N = (\phi, X, X)$ or
   3. $\phi_N = (\phi, X, \phi)$ or
   4. $\phi_N = (\phi, \phi, \phi)$.

2. X_N may be defined in many ways as an NCS, as follows:

   1. $X_N = (X, \phi, \phi)$ or
   2. $X_N = (X, X, \phi)$ or
   3. $X_N = (X, X, X)$.

Definition 1.3. [6] Let X be a nonempty set, and the NCSs A and B be in the form $A = \{A_1, A_2, A_3\}, B = \{B_1, B_2, B_3\}$. Then we may consider two possible definitions for subset $A \subseteq B$ which may be defined in two ways:

1. $A \subseteq B \Leftrightarrow A_1 \subseteq B_1,\ A_2 \subseteq B_2\ and\ B_3 \subseteq A_3$.
2. $A \subseteq B \Leftrightarrow A_1 \subseteq B_1,\ B_2 \subseteq A_2\ and\ B_3 \subseteq A_3$.

Definition 1.4. [6] Let X be a non-empty set and the NCSs A and B in the form $A = \{A_1, A_2, A_3\},\ B = \{B_1, B_2, B_3\}$. Then:

1. $A \cap B$ may be defined in two ways as an NCS as follows:

$i)\quad A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3)$
$ii)\quad A \cap B = (A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3)$.

2. $A \cup B$ may be defined in two ways as an NCS, as follows:

$i)\quad A \cup B = (A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3)$
$ii)\quad A \cup B = (A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3)$.

Definition 1.5. [6] A neutrosophic crisp topology (NCT) on a non-empty set X is a family Γ of neutrosophic crisp subsets in X satisfying the following axioms:

1. $\phi_N, X_N \in \Gamma$.
2. $A_1 \cap A_2 \in \Gamma$, for any A_1 and $A_2 \in \Gamma$.
3. $\bigcup A_j \in \Gamma, \forall \{A_j : j \in J\} \subseteq \Gamma$.

The pair (X, Γ) is said to be a neutrosophic crisp topological space (NCTS) in X. Moreover, the elements in Γ are said to be neutrosophic crisp open sets (NCOS). A neutrosophic crisp set F is closed (NCCS) if and only if its complement F^c is an open neutrosophic crisp set.

Definition 1.6. [6] Let X be a non-empty set, and the NCSs A be in the form

$A = \{A_1, A_2, A_3\}$. *Then* A^c may be defined in three ways as an *NCS*:

$i)\ A^c = < A_1^c, A_2^c, A_3^c >$ or
$ii)\ A^c = < A_3, A_2, A_1 >$ or
$iii)\ A^c = < A_3, A_2^c, A_1 >$.

2.N_nc-Topological Spaces

In this section, we introduce N–neutrosophic crisp topological spaces (N_{nc}-topological space) and discuss their basic properties. Moreover, we introduced new types of open and closed sets in the context of N_{nc}-topological spaces.

Definition 2.1: Let X be a non-empty set. Then ${}_{nc}\tau_1$, ${}_{nc}\tau_2$, ..., ${}_{nc}\tau_N$ are *N*-arbitrary crisp topologies defined on X and the collection

$$N_{nc}\tau = \{G \subseteq X : G = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i) \in N_{nc}\tau, \quad A_i, B_i \in_{nc} \tau_i\}$$

is called N_{nc}-topology on X if the following axioms are satisfied:

1. $\phi_N, X_N \in N_{nc}\tau$.

2. $\bigcup_{i=1}^{\infty} G_i \in N_{nc}\tau \quad \textit{for all } \{G_i\}_{i=1}^{\infty} \in N_{nc}\tau$.

3. $\bigcap_{i=1}^{n} G_i \in N_{nc}\tau \quad for\ all\ \{G_i\}_{i=1}^{n} \in N_{nc}\tau.$

Then (X, $N_{nc}\tau$) is called N_{nc}-topological space on X. The elements of $N_{nc}\tau$ are known as

N_{nc}-open (N_{nc}- OS) sets on X and its complement is called N_{nc}-closed (N_{nc}- CS) sets on X. The elements of X are known asN_{nc}-sets (N_{nc}- S) on X.

Remark 2.2: Considering N = 2 in Definition 2.1, we get the required definition of bi-neutrosophic crisp topology on X. The pair (X, $2_{nc}\tau$) is called a bi-neutrosophic crisp topological space on X.

Remark 2.3: Considering N = 3 in Definition 2.1, we get the required definition of tri-neutrosophic crisp topology on X. The pair (X, $3_{nc}\tau$) is called a tri-neutrosophic crisp topological space on X.

Example 2.4:

$X = \{1,2,3,4\},\ {}_{nc}\tau_1 = \{\phi_N, X_N, \mathrm{A}\},\ {}_{nc}\tau_2 = \{\phi_N, X_N, \mathrm{B}\},{}_{nc}\tau_3 = \{\phi_N, X_N\}$
$\mathrm{A} =< \{3\},\{2,4\},\{1\} >, B =< \{1\},\{2\},\{2,3\} >,$
$\mathrm{A} \cup \mathrm{B} =< \{1,3\},\{2,4\},\varnothing >, A \cap B =< \varnothing,\{2\},\{1,2,3\} >,$ Then we get

$3_{nc}\tau = \{\varnothing_N, X_N, A, B, A \cup B, A \cap B\}$

which is a tri-neutrosophic crisp topology on *X*. The pair (*X, $3_{nc}\tau$*) is called a tri-neutrosophic crisp topological space on *X*.

Example 2.5:

$X = \{1,2,3,4\},\ {}_{nc}\tau_1 = \{\phi_N, X_N, \mathrm{A}\},\ {}_{nc}\tau_2 = \{\phi_N, X_N, \mathrm{B}\}$
$\mathrm{A} =< \{3\},\{2,4\},\{1\} >, B =< \{1\},\{2\},\{2,3\} >,$
$\mathrm{A} \cup \mathrm{B} =< \{1,3\},\{2,4\},\varnothing >, A \cap B =< \varnothing,\{2\},\{1,2,3\} >,$ Then

$2_{nc}\tau = \{\varnothing_N, X_N, A, B, A \cup B, A \cap B\}$

which is a bi-neutrosophic crisp topology on X. The pair $(X, 2_{nc}\tau)$ is called a bi-neutrosophic crisp topological space on X.

Definition 2.6: Let $(X, N_{nc}\tau)$ be a N_{nc}-topological space on X and A be an N_{nc}-set on X then the $N_{nc}int(A)$ and $N_{nc}cl(A)$ are respectively defined as

(i) $N_{nc}int(A) = \cup \{G : G \subseteq A$ and G is a N_{nc}-open set in $X\}$.

(ii) $N_{nc}cl(A) = \cap \{F: A \subseteq F$ and F is a N_{nc}-closed set in $X\}$.

Proposition 2.7: Let $(X, N_{nc}\tau)$ be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in $(X, N_{nc}\tau)$, so the N_{nc}-closure operator satisfies the following properties:

(i) $A \subseteq N_{nc}cl(A)$.

(ii) $A \subseteq B \Rightarrow N_{nc}cl(A) \subseteq N_{nc}cl(B)$.

(iii) $N_{nc}cl(A \cup B) = N_{nc}cl(A) \cup N_{nc}cl(B)$.

Proof

(i) $N_{nc}cl(A) = \cap \{G : G$ is a N_{nc}-closed set in X and $A \subseteq G\}$. Thus, $A \subseteq N_{nc}cl(A)$.

(ii) $N_{nc}cl(B) = \cap \{G : G$ is a N_{nc}-closed set in X and $B \subseteq G\} \supseteq \cap \{G :$ G is a N_{nc}-closed set in X and $A \subseteq G\} \supseteq N_{nc}cl(A)$. Thus, $N_{nc}cl(A)$ $\subseteq N_{nc}cl(B)$.

(iii) $N_{nc}cl(A \cup B) = \cap\{G : G$ is a N_k-closed set in X and $A \cup B \subseteq G\} =$ $(\cap\{G : G$ is a N_{nc}-closed set in X and $A \subseteq G\}) \cup (\cap\{G : G$ is a N_{nc}-closed set in X and $B \subseteq G\}) = N_{nc}cl(A) \cup N_{nc}cl(B)$. Thus, $N_{nc}cl(A \cup B) = N_{nc}cl(A) \cup N_{nc}cl(B)$.

Proposition 2.8: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in (X, $N_{nc}\tau$), then the $N_{nc}int(A)$ operator satisfies the following properties:

(i) $N_{nc}int(A) \subseteq A$.

(ii) $A \subseteq B \Rightarrow N_{nc}int(A) \subseteq N_{nc}int(B)$.

(iii) $N_{nc}int(A \cap B) = N_{nc}int(A) \cap N_{nc}int(B)$.

(iv) $(N_{nc}cl(A))^c = N_{nc}int(A)^c$.

(v) $(N_{nc}int(A))^c = N_{nc}cl(A)^c$.

Proof

(i) $N_{nc}int(A) = \cup$ {G: G is an N_{nc}-open set in X and $G \subseteq A$ }. Thus, $N_{nc}int(A) \subseteq A$.

(ii) $N_{nc}int(B) = \cup$ {G: G is a N_{nc}-open set in X and $G \subseteq B$ }$\supseteq \cup${ G : G is an N_{nc}-open set in X and $G \subseteq A$ } $\supseteq N_{nc}int(A)$. Thus, $N_{nc}int(A) \subseteq N_{nc}int(B)$.

(iii) $N_{nc}int(A \cap B) = \cup\{G : G$ is an N_{nc}-open set in X and $A \cap B \supseteq G\}$ $= (\cup\{G : G$ is a N_{nc}-open set in X and $A \supseteq G\}) \cap (\cup\{G : G$ is an N_{nc}-open set in X and $B \supseteq G\}) = N_{nc}int(A) \cap N_{nc}int(B)$. Thus, $N_{nc}int(A \cap B) = N_{nc}int(A) \cap N_{nc}int(B)$.

(iv) $N_{nc}cl(A) = \cap$ $\{G$: G is an N_{nc}-closed set in X and $A \subseteq G\}$, $(N_{nc}cl(A))^c = \cup\{G^c : G^c$ is an N_{nc}-open set in X and $A^c \supseteq G^c\} = N_{nc}int(A)^c$. Thus, $(N_{nc}cl(A))^c = N_{nc}int(A)^c$.

(v) $N_{nc}int(A) = \cup$ $\{G$: G is an N_{nc}-open set in X and $A \supseteq G\}$, $(N_{nc}int(A))^c = \cap\{G^c : G^c$ is *a* N_{nc}-closed set in X and $A^c \supseteq G^c\} = N_{nc}cl(A)^c$. Thus, $(N_{nc}int(A))^c = N_{nc}cl(A)^c$.

Proposition 2.9:

Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A is a N_{nc}-sets in (X, $N_{nc}\tau$), the following properties are true:

(i) $N_{nc}cl(A) = A$ iff A is a N_{nc}-closed set.

(ii) $N_{nc}int(A) = A$ iff A is a N_{nc}-open set.

(iii) $N_{nc}cl(A)$ is the smallest N_{nc}-closed set containing A.

(iv) $N_{nc}int(A)$ is the largest N_{nc}-open set contained in A.

Proof: (i), (ii), (iii) and (iv) are obvious.

3.New open setes *in* N_{nc}-Topological Spaces

Definition 3.1: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. Let A be an N_{nc}-set in (X, $N_{nc}\tau$). Then A is said to be:

(i) A N_{nc}-preopen set (N_{nc}-P-OS) if $A\subseteq N_{nc}int(N_{nc}cl(A))$. The complement of an N_{nc}-preopen set is called an N_{nc}-preopen set in X. The family of all N_{nc}-P-OS (resp. N_{nc}-P-CS) of X is denoted by (N_{nc}POS(X)) (resp. N_{nc}PCS).

(ii) An N_{nc}-semiopen set (N_{nc}-S-OS) if $A\subseteq N_{nc}cl(N_{nc}int(A))$. The complement of a N_{nc}-semiopen set is called a N_{nc}-semiopen set in X. The family of all N_{nc}-S-OS (resp. N_{nc}-S-CS) of X is denoted by (N_{nc}POS(X)) (resp. N_{nc}PCS).

(iii) A N_{nc}-α-open set (N_{nc}-α-OS) if $A\subseteq N_{nc}int(N_{nc}cl(N_{nc}int(A)))$. The complement of a N_{nc}-α-open set is called a N_{nc}-α-open set in X. The family of all N_{nc}-α-OS (resp. N_{nc}-α-CS) of X is denoted by ($N_{nc}\alpha$OS(X)) (resp. $N_{nc}\alpha$CS).

Example 3.2:

$X=\{a,b,c,d\}$, ${}_{nc}\tau_1=\{\phi_N, X_N, \text{A}\}$, ${}_{nc}\tau_2=\{\phi_N, X_N, \text{B}\}$

$A =< \{a\},\{b\},\{c\} >, B =< \{a\},\{b,d\},\{c\} >$, then we have $2_{nc}\tau = \{\varnothing_N, X_N, A, B\}$

which is a bi-neutrosophic crisp topology on *X*. Then the pair (*X*, $2_{nc}\tau$) is a bi-neutrosophic crisp topological space on *X*. If $H =< \{a,b\},\{c\},\{d\} >$,then H is a N_{nc}-P-OS but not N_{nc}-α-OS. It is clear that H^c is a N_{nc}-P-CS. A is a N_{nc}-S-OS. It is clear that A^c is a N_{nc}-S-CS. A is a N_{nc}-α-OS. It is clear that A^c is a N_{nc}-α-CS.

Definition 3.3: Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A be a N_{nc}-set on X then

(i) N_{nc}-P-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-P-OS in X}.

(ii) N_{nc}-P-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-P-CS in X}.

(iii) N_{nc}-S-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-S-OS in X}.

(iv) N_{nc}-S-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-S-CS in X}.

(v) N_{nc}-α-int(A) = ∪ {G: G ⊆ A and G is a N_{nc}-α-OS in X}.

(vi) N_{nc}-α-cl(A) = ∩ {F: A ⊆ F and F is a N_{nc}-α-CS in X}.

In Proposition 3.4 and Proposition 3.5, by the notion N_{nc}-k-cl(A)(N_{nc}-k-int(A)), we mean N_{nc}-P-cl(A)(N_{nc}-P-int(A)) (if k = p), N_{nc}-S-cl(A)(N_{nc}-S-int(A)) (if k = S) and N_{nc}-α-cl(A)(N_{nc}-α-int(A)) (if k = α).

Proposition 3.4: Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A and B are any two N_{nc}-sets in (X, $N_{nc}\tau$), then the N_{nc}-S-closure operator satisfies the following properties:

(i) A ⊆ N_{nc}-k-cl(A).

(ii) N_{nc}-k-int(A) ⊆ A.

(iii) A ⊆ B ⇒ N_{nc}-k-cl(A)⊆ N_{nc}-k-cl(B).

(iv) $A \subseteq B \Rightarrow N_{nc}$-k-int(A) $\subseteq N_{nc}$-k-int(B).

(v) N_{nc}-k-cl $(A \cup B) = N_{nc}$-k-cl(A)$\cup N_{nc}$-k-cl(B).

(vi) N_{nc}-k-int $(A \cap B) = N_{nc}$-k-int(A) $\cap N_{nc}$-k-int(B).

(vii) $(N_{nc}$-k-cl$(A))^c = N_{nc}$-k-cl$(A)^c$.

(viii) $(N_{nc}$-k-int$(A))^c = N_{nc}$-k-int$(A)^c$.

Proposition 3.5:

Let (X, $N_{nc}\tau$) be any N_{nc}-topological space. If A is an N_{nc}-sets in (X, $N_{nc}\tau$). Then the following properties are true:

(i) N_{nc}-k-cl(A)= A iff A is a N_{nc}-k-closed set.

(ii) N_{nc}-k-int(A)= A iff A is a N_{nc}-k-open set.

(iii) N_{nc}-k-cl(A) is the smallest N_{nc}-k-closed set containing A.

(iv) N_{nc}-k-int(A) is the largest N_{nc}-k-open set contained in A.

Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 3.6:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X. Then the following statements hold in whcih the equality of each statement are not true:

(i) Every N_{nc}- OS (resp. N_{nc}- CS) is a N_{nc}-α-OS (resp. N_{nc}-α-CS).

(ii) Every N_{nc}-α-OS (resp. N_{nc}-α-CS) is a N_{nc}-S-OS (resp. N_{nc}-S-CS).

(iii) Every N_{nc}-α-OS (resp. N_{nc}-α-CS) is a N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proposition 3.7:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X, then the following statements hold, and the equality of each statement are not true:

(i) Every N_{nc}-OS (resp. N_{nc}-CS) is a N_{nc}-S-OS (resp. N_{nc}-S-CS).

(ii) Every N_{nc}-OS (resp. N_{nc}-CS) is a N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proof.

(i) Suppose that A is a N_{nc}-OS. Then A=$N_{nc}int(A)$, and so $A \subseteq N_{nc}cl$(A)= $N_{nc}cl(N_{nc}int(A))$. so that A is a N_{nc}-S-OS.

(ii) Suppose that A is a N_{nc}-OS. Then A=$N_{nc}int(A)$, and since $A \subseteq N_{nc}cl$(A) so $A=N_{nc}int(A) \subseteq N_{nc}int(N_{nc}cl$(A)). so that A is a N_{nc}-P-OS.

Proposition 3.8:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A a N_{nc}-set on X. Then A is an N_{nc}-α-OS (resp. N_{nc}-α-CS) iff A is a N_{nc}-S-OS (resp. N_{nc}-S-CS) and N_{nc}-P-OS (resp. N_{nc}-P-CS).

Proof. The necessity condition follows from the Definition 3.1. Suppose that A is both a N_{nc}-S-OS and a N_{nc}-P-OS. Then A$\subseteq$ $N_{nc}cl(N_{nc}int(A))$, and hence $N_{nc}cl$(A)$\subseteq$ $N_{nc}cl(N_{nc}cl(N_{nc}int(A)))$= $N_{nc}cl(N_{nc}int(A))$.

It follows that A$\subseteq$ $N_{nc}int(N_{nc}cl(A))\subseteq$ $N_{nc}int(N_{nc}cl(N_{nc}int(A)))$, so that A is a N_{nc}-α-OS.

Proposition 3.9:

Let (X, $N_{nc}\tau$) be an N_{nc}-topological space on X and A an N_{nc}-set on X. Then A is an N_{nc}-α-CS iff A is an N_{nc}-S-CS and N_{nc}-P-CS.

Proof. The proof is straightforward.

Theorem 3.10:

Let (X, $N_{nc}\tau$) be a N_{nc}-topological space on X and A a N_{nc}-set on X. If B is a N_{nc}-S-OS such that B $\subseteq$A$\subseteq$ N_{nc}int(N_{nc}cl (A)), then A is a N_{nc}-α-OS .

Proof. Since B is a N_{nc}-S-OS, we have B $\subseteq$ N_{nc}int(N_{nc}cl (A)) . Thus, A $\subseteq$ N_{nc}int(N_{nc}cl (B))$\subseteq$ N_{nc}int(N_{nc}cl (N_{nc}cl (N_{nc}int((B))))$\subseteq$ N_{nc}int(N_{nc}cl (N_{nc}int((B))))

$\subseteq N_{nc}int(N_{nc}cl\ (N_{nc}int(\ (A))))$ and therefore *A is a* N_{nc}-α-OS.

Theorem 3.11:

Let (X, $N_{nc}\tau$) be an N_{nc}-topological space on X and A be an N_{nc}-set on X. Then $A \in N_{nc}\alpha OS(X))$ iff there exists an N_{nc}- OS H such that $H \subseteq A \subseteq N_{nc}int\ (N_{nc}cl\ (A))$.

Proposition 3.12:

The union of any family of $N_{nc}\alpha OS(X)$ is a $N_{nc}\alpha OS(X)$.

Proof. The proof is straightforward.

Remark 3.13:

The following diagram shows the relations among the different types of weakly neutrosophic crisp open sets that were studied in this paper:

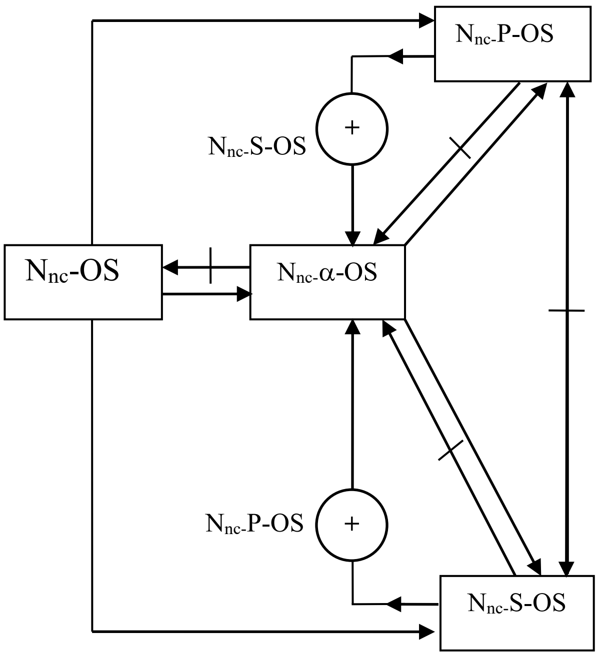

Diagram (3.1)

Conclusion

In this work, we have introduced some new notions of N-neutrosophic crisp open (closed) sets called N_{nc}-semi (open) closed sets, N_{nc}-preopen (closed) sets, and N_{nc}-α-open (closed) sets and studied some of their basic properties in the context of neutrosophic crisp topological spaces. The neutrosophic crisp semi-α-closed sets can be used to derive a new decomposition of neutrosophic crisp continuity.

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Performance Evaluation of Mutual Funds Via Single Valued Neutrosophic Set (SVNS) Perspective: A Case Study in Turkey

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Abstract.
The aim of this study was to use the Single-Valued Neutrosophic Set (SVNS) to analyze 58 mutual funds, traded at the Istanbul Stock Exchange, under incomplete, indeterminate and inconsistent information. To this end, the performance of the funds was first evaluated using the most commonly preferred criteria like the Morningstar rating, Sharpe ratio, Treynor ratio, and Jensen ratio. Following these criteria, SVNS based entropy was used to rank the funds. The results of the entropy weights revealed Morningstar rating to be the most important evaluation criterion followed by Treynor, Sharpe and Jensen ratios respectively. Yapı Kredi Asset Management Foreign Technology Sector Equity Fund was found to be the most successful fund, while İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive) Fund was the least successful fund.

Keywords: Performance Evaluation, Mutual Funds, Neutrosophic Set, Single valued neutrosophic set based entropy

1 Introduction

A feedback based on the performance evaluation of an investment is quite significant for the success of any investment. It is important to determine whether the portfolios created for the purpose of risk distribution, either by institutional or individual investors, have the capacity to create the desired benefit or if the there exists a healthy risk-return relationship.

Evaluations of the performance of investment funds, managed by professionals, and composed of various securities (just as a portfolio), will have an impact on the success of the investment fund. When fund managers determine the assets that they intend to invest in, it is extremely important to measure the fund performance by considering the return and risk on the assets alongside other indicators. When determining the fund performance, another important point is to determine whether or not the performance realized is the result of any chance factor. The right decisions by the fund managers will lead to the selection of the right securities for the fund as well as the inclusion to the fund of securities that move in tandem with an emerging market. They are also able to make changes to the securities making up the fund in time by including securities that change slower than the market in case of declining markets.

2 Fund Performance Evaluation

When determining fund performance, the basic emphasis should be on risk and return. The total return expected from the fund should be compared to the level of risk that the fund is exposed to. The first thing that should be done therefore is the determination of the risk and return of the fund. The risk and return of the fund will vary according to the risks and returns of the assets in the fund pool.

2.1 Calculating the Morningstar Return

The Morningstar system is fundamentally made of two parts; while the first part works out the 'Morningstar Return', the second part determines the 'Morningstar Risk'. In calculating the Morningstar Return in the first stage, the monthly returns used should be calculated from the monthly closing price per share of selected funds. This simple calculation of the return can be expressed as:

$$Rp = (Vt - Vt\text{-}1) / (Vt\text{-}1) \quad (1)$$

Where Rp is the monthly return of the fund, Vt is the monthly closing price per share, V-t is the monthly closing price per share for the previous month.

After calculating of the monthly returns of the fund, Morningstar obtains the value for 'adjusted return' for each month by subtracting, from the monthly returns, the monthly costs charged on the fund, such as commissions, expenses, and management fees among others. The monthly excess return for each fund is thus calculated based on the adjusted returns. In other words, the additional earnings by the fund above the risk-free rate. Since Morningstar always presents investors with investment alternatives to the risk-free assets, the excess return earned by a fund is now understood better as either above or below the risk-free rate. Consequently, the return of the fund is calculated by subtracting the risk-free interest rate for that month from the adjusted monthly return.

$$ERp = \text{Adjusted Fund Return} - Rf \quad (2)$$

Where ERp is the monthly excess return of the fund and Rf is risk-free rate.

In the next stage, Morningstar divides the funds into categories. When Morningstar initially put forward this system, it grouped the funds into the four basic asset classes, as mentioned earlier, and classified (assigned) the funds structures into one of these asset classes according to the investment strategies. However, these asset classes were later revised to allow better evaluation and to avoid the comparison of apples and oranges. In 1996, Morningstar introduced categories that grouped funds into narrower classes. However, these categories were not integrated into the Morningstar Star Rating System until mid-2002. With the initial being 48 categories, this number rose to 64, and 81 by August 2009, and finally to 122 in April 2016. After the fund categories have been defined, the monthly ‘category average return’ for every fund category is calculated. To do this, the adjusted monthly returns of all the funds in the category are added and divided by the number of funds in the category, resulting into the determination of the monthly category average return for that fund category. After this is done, Morningstar compares the category average return to the risk-free rate, resulting in the Morningstar Return.

$$\text{Morningstar Return} = ERp / (\text{category average return} - Rf) \text{ or } Rf \quad (3)$$

As can be seen, the ERp obtained by subtracting the risk-free rate-based return from the adjusted monthly return of the fund forms the numerator of the equation and is divided by the greater of the (category average return - Rf) or Rf. Also, it will be noticed that while the numerator essentially shows the excess return of the fund, the denominator shows a comparison of the average excess return of the category in which the fund is found and the risk-free rate. Thus, the denominator of this equation may change from month to month, i.e., it may be the category average return – Rf expression in some months and only made of the risk-free return for other months. Morningstar divides the result by one of these two variables to avoid distortions due to low or negative average excess returns in the denominator of equation [7].

2.2 Calculating the Morningstar Risk

Once the Morningstar return has been calculated, the fund's Morningstar risk should be determined. In this regard, it should be noted that Morningstar is not based on the risk values obtained using measures of risk such as the standard deviation or beta coefficient, but it is determined on the basis of the downward risk to investors, which is the financial risk associated with losses, and which is believed to be investors’ biggest fear. For this operation, first, the fund's adjusted monthly return and the risk-free interest rate are compared. At this point, months with reported negative monthly excess returns are identified. These negative returns are then summed up and divided by the total number of months in the period. The aim here is to determine the opportunity cost incurred by the investors in terms of monthly average (the monthly average loss) as a result of not investing at the risk-free rate. The same method is then applied to the fund category.

The average return of the category is compared with the risk-free interest rate, the months in which the monthly excess returns of the category in which the fund is found are reported as negative are determined, and these negative returns are summed and divided by the total number of months in the period. The average monthly loss of the category of funds, i.e. category risk, is calculated. The Morningstar risk of the fund is expressed as follows:

$$\text{Morningstar Risk} = AMLp / AMLc \quad (4)$$

Where AMLp is the Average Monthly Loss of the fund and AMLc is the Average Monthly Loss of the category in which the fund is found.

2.3 Calculation of Raw Return

After obtaining the required data on Morningstar return and the Morningstar risk the next step is the calculation of the Morningstar raw return for each fund. This value is obtained by subtracting the Morningstar risk from the Morningstar return of the fund.

Raw Return = Morningstar return - Morningstar risk (5)

2.4 Treynor Ratio

The Treynor Ratio (Index) introduced by Jack Treynor in 1965 measures portfolio performance using the beta coefficient, which measures the systematic risk, instead of the standard deviation which measures the total risk. The beta coefficient indicates how sensate the securities or portfolio are to the market. Treynor argued that although non-systematic risks could be eliminated through portfolio diversification, systematic risk cannot be removed in any way [8]. Treynor thus divided the investment risk on a diversifiable portfolio into two parts: general market fluctuations, and fluctuations in securities within the portfolio. Treynor argued that the first risk is valid for all stocks and that it cannot be eliminated while the second risk can be eliminated or reduced by appropriately diversifying the portfolio. Just like in the Sharpe Ratio, the risk premium is also calculated here. However, unlike the Sharpe Ratio, the premium calculated here is based on the beta coefficient and not standard deviation. In this way, there is a residual return for every unit of systematic risk assumed, i.e. earning above the risk-free rate of return. The Treynor ratio can be expressed as follow [11]:

$$T_n = (R_n - R_f)/\beta_n \qquad (6)$$

Where T_n= Treynor Ratio

R_f= Risk Free Rate

R_n = Return of Fund

B_n = Beta coefficient of the fund.

Treynor explained his portfolio performance based on the Security Market Line (SML), not the Capital Market Line (CML) used by Sharpe [24]. If the evaluated portfolio is above this level, then it has performed worse than the market, a performance below this line implies it is better than the market.

2.5 Sharpe Ratio

The Sharpe Ratio which measures portfolio performance based on the total risk was developed by William Forsyth Sharpe in 1966. This is one of the most commonly used and simplest methods for measuring portfolio performance. The idea that it is necessary for investors to invest in the index funds through the market portfolio in order to avoid the non-systematic risks that have been forecast by the Sharpe Ratio rests on the assumption that it would be erroneous to try to obtain more returns from the market, as stocks in active markets always reflect the prices correctly [22].

In his study, Sharpe tried to subject the measures presented by Treynor to the empirical test by evaluating their predictive powers. He also sought to advance Treynor's work, as well as make more explicit the relationship between the recent developments in capital theory and alternative models of investment fund performance and then subjecting these alternative models to more empirical tests. He analysed the annual return rates of 34 open-end investment funds between 1954 and 1963. In the study, the performance of 23 investment funds was found to be lower than the Dow Jones Industrial Average (DJIA), which was considered as the indicator. On the other hand, when relatively poorly diversified portfolios are considered, the Treynor Index, due to its diversification weakness and failure to address some of the variables, may lead the results to vary considerably. This led the conclusion by Sharpe that, whereas it may be a good measure for predicting future performance, the same reasons make it a poor measure for past performance [19].

The model introduced by Sharpe shows the extra return over the risk-free rate that the investor seeks to achieve for the amount of total risk undertaken, i.e. the additional return expected for every unit of total risk. The Sharpe Ratio is based on the Capital Market Line (CML) and assumes that the investment fund affects the portfolio [26]. The Capital Market Line can thus be said to be the indicator. When the Sharpe Ratio calculated for the portfolio that an investor holds is found to be greater than the Sharpe Ratio of the market portfolio, it can be concluded that the portfolio has performed better than the market [4].

The calculation of the Sharpe Ratio which is based on the total portfolio risk can be expressed as follows [22]:

$$S = (R_n + R_f)\ /\sigma_n \qquad (7)$$

Where S = Sharpe Ratio,

R_p = Return of Fund

R_f = Risk-Free Rate

σ_p = Standard Deviation of the Fund (Total Risk of the Fund)

The difference between the return on the portfolio and the risk-free interest rate, the numerator of the equation, is known as the risk premium. This premium shows the reward or the residual return (excess return, return on risk-free interest rate) that the investor gets for undertaking risk. The denominator of the function indicates the total risk made of both systematic and non-systematic risks.

2.6 Jensen Ratio

This ratio was developed by Michael Cole Jensen in 1968 and uses just a single value in measuring portfolio performance. The Jensen Ratio measures the deviation of any portfolio from the Securities Market Line [18].

The Ratio based on the Capital Assets Pricing Model (CAPM) is the difference between the realized return of the portfolio and the expected return (with assumptions) above the Securities Market Line. This ratio also acts as an examination of the skills of the portfolio manager in formulating the portfolio. Since Jensen expressed the difference mentioned in terms of Alpha Coefficient, the ratio is also known as the Jensen Alpha. This ratio can also be defined as the supernormal return above the expected return according to the CAPM.

In his study, Jensen asserted that he had created a method (criterion) that measures the predictive skills of portfolio managers as well as their contribution to the returns of the funds known as the Jensen Alpha. He sought to measure the predictive skills of 115 mutual fund managers between 1945 and 1964. He noted that the performance of the 115 mutual funds, and by extension the skills of the managers in predicting the stock prices did not exceed the average by much [10].

The Jensen Ratio is expressed as follows [1]:

$$\alpha_n = R_n - [R_f + \beta_n \ (R_m - R_f)] \qquad (8)$$

Where α_n= Jensen (Alpha) Ratio of the fund

R_p = Return of the Fund

R_m = Return of the Indicator Index (Market)

R_f = Risk Free Rate

β_n = Beta coefficient of the fund.

3 Methodology

3.1 Neutrosophic Set

Neutrosophic Sets (NS) is proposed by Smarandache (1998) having with degree of truth, indeterminacy and falsity membership functions in which all of them are totally independent [20,21]. Let U be a universe of discourse and $x \in U$. The neutrosophic set (NS) N can be expressed by a truth membership function $T_N(x)$, an indeterminacy membership function $I_N(x)$ and a falsity membership function $F_N(x)$, and is represented as $N = \{< x : T_N(x), I_N(x), F_N(x) >, x \in U\}$. Also the functions of $T_N(x)$, $I_N(x)$ and $F_N(x)$ are real standard or real nonstandard subsets of $]0^-, 1^+[$, and can be presented as $T, I, F : U \to]0^-, 1^+[$. There is not any restriction on the sum of the functions of $T_N(x)$, $I_N(x)$ and $F_N(x)$, so:

$$0^- \leq \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \leq 3^+ \qquad (9)$$

The complement of a NS N is represented by N^C and described as below [6]:

$$T_N^C(x) = 1^+ \ominus T_N(x) \qquad (10)$$

$$I_N^C(x) = 1^+ \ominus I_N(x) \quad (11)$$

$$F_N^C(x) = 1^+ \ominus F_N(x) \quad \text{for all } x \in U \quad (12)$$

There are applications of the neutrosophic set such as MCDM problems of supplier selection [1], strategic planning [2], logistic center location selection [17], teacher recruitment in higher education [13] and school choice [14].

3.2 Single Valued Neutrosophic Sets (SVNS)

Single-Valued Neutrosophic Set (SVNS) which is a case of NS was developed by Smarandache (1998) and Wang, Smarandache, Zhang, and Sunderraman (2010) in order to deal with indeterminate, inconsistent and incomplete information. The interval $[0,1]$ was considered rather than $]0^-,1^+[$ for better representation and application to real-world problems. Let U be a universe of discourse and $x \in U$. A single-valued neutrosophic set B in U is described by a truth membership function $T_B(x)$, an indeterminacy membership function $I_B(x)$ and a falsity membership function $F_B(x)$. When U is continuous, an SVNS B is depicted as $B = \int_x \frac{< T_B(x), I_B(x), F_B(x) >}{x} : x \in U$. When U is discrete an SVNS B can be represented as $B = \sum_{i=1}^{n} \frac{< T_B(x_i), I_B(x_i), F_B(x_i)}{x_i} : x_i \in U$ [15]. The functions of $T_B(x)$, $I_B(x)$ and $F_B(x)$ are real standard subsets of $[0,1]$ that is $T_B(x): U \to [0,1]$, $I_B(x): U \to [0,1]$ and $F_B(x): U \to [0,1]$. Also, the sum of $T_B(x)$, $I_B(x)$ and $F_B(x)$ are in $[0,3]$ that $0 \le T_B(x) + I_B(x) + F_B(x) \le 3$ [5].

For simplicity two SVNSs such as $B_1 = (t_1, \imath_1, f_1)$ and $B_2 = (t_2, \imath_2, f_2)$ then summation between B_1 and B_2 can be described as below:

$$B_1 \oplus B_2 = (t_1 + t_2 - t_1 t_2, \imath_1 \imath_2, f_1 f_2) \quad (13)$$

Two SVNSs such as $B_1 = (t_1, \imath_1, f_1)$ and $B_2 = (t_2, \imath_2, f_2)$ then multiplication between B_1 and B_2 can be described as below:

$$B_1 \otimes B_2 = (t_1 t_2, \imath_1 + \imath_2 - \imath_1 \imath_2, f_1 + f_2 - f_1 f_2) \quad (14)$$

For an SVNS as $B = (t, \imath, f)$ and $\lambda \in \Re$ an arbitrary positive real number then,

$$\lambda B = \left(1 - (1-t)^{\lambda}, \imath^{\lambda}, f^{\lambda}\right), \lambda > 0 \quad (15)$$

The complement of an SVNS B is represented by $_C(B)$ and is described as follow [9]:

$$T_C(B)(x) = F(B)(x) \quad (16)$$

$$I_C(B)(x) = 1 - I(B)(x) \quad (17)$$

$$F_C(B)(x) = T(B)(x) \quad \text{for all } x \in U \quad (18)$$

The union of two SVNS namely B_1 and B_2 is an SVNS B_3 denoted by $B_3 = B_1 \cup B_2$ and its truth, indeterminacy and falsity membership functions are shown below [23]:

$$T(B_3)(x) = \max\left(T(B_1)(x), T(B_2)(x)\right) \quad (19)$$

$I(B_3)(x) = \min(I(B_1)(x), I(B_2)(x))$ (20)

$F(B_3)(x) = \min(F(B_1)(x), F(B_2)(x))$ for all $x \in U$ (21)

The intersection of two SVNS namely B_1 and B_2 is an SVNS B_3 denoted by $B_3 = B_1 \cap B_2$ and its truth, indeterminacy and falsity membership functions are shown below [12]:

$T(B_3)(x) = \min\left(T(B_1)(x), T(B_2)(x)\right)$ (22)

$I(B_3)(x) = \max(I(B_1)(x), I(B_2)(x))$ (23)

$F(B_3)(x) = \max(F(B_1)(x), F(B_2)(x))$ for all $x \in U$ (24)

3.3 Single Valued Neutrosophic Sets (SVNS) Entropy Based Decision Making

A new single-valued neutrosophic sets (SVNS) entropy based multi-attribute decision making (MADM) was proposed by Nirmal and Bhatt (2016) and composed of steps seen as follows [16]:

1-Type of decision problem (ranking, evaluation, sorting etc.) is identified in the first step.

2-Then alternatives with regard to criteria having qualitative or quantitative values are identified.

3-Decision matrix involving criteria and alternatives with respect to decision-making problem is constructed.

4-Qualitative information is transformed into fuzzy numbers by means of matrix normalization techniques shown as Table 1:

Normalization technique	Normalized beneficial value	Normalized non-beneficial value
Linear scale transformation max method	$N_{ij} = \frac{x_{ij}}{x_{i\max}}$	$N_{ij} = \frac{x_{i\min}}{x_{ij}}$
Linear scale transformation max-min method	$N_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}}$	$N_{ij} = \frac{\max x_{ij} - x_{ij}}{\max x_{ij} - \min x_{ij}}$
Linear scale transformation sum method	$N_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_i}$	$N_{ij} = 1 - \frac{x_{ij}}{\sum_{i=1}^{m} x_i}$
Vector normalization method	$N_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$	$N_{ij} = 1 - \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$

Table 1. Matrix normalization techniques

5-Elements of input matrix in the classic or fuzzy set are conversed to single-valued neutrosophic sets by means of conversion rule for beneficial and non-beneficial criteria explained as below:

a) For beneficial criteria: Positive ideal solution (PIS) is constructed as $< T^*_{\max}(x), I^*_{\min}(x), F^*_{\min}(x) >$. Normalized input matrix beneficial criteria are created as the degree of truthness $T_L(x)$, the degree of indeterminacy and degree of falsehood are considered as $I_L(x) = F_L(x) = 1 - T_L(x)$ respectively.

b) For non-beneficial criteria: Negative ideal solution (NIS) is constructed as $< T^*_{\min}(x), I^*_{\max}(x), F^*_{\max}(x) >$.
Normalized input matrix non-beneficial criteria are created as the degree of indeterminacy and falsehood as $I_L(x) = F_L(x)$, the degree of truthness is considered as $T_L(x) = 1 - I_L(x) = 1 - F_L(x)$.

c)Entropy value for the jth attribute is calculated according to Eq.(25) as shown below:

$$E_j = 1 - \frac{1}{m}\left[\sum_{i=1}^{m}\left(T_{ij}(x_i) + F_{ij}(x_i)\right)\left|2\left(I_{ij}(x_i)\right) - 1\right|\right] \quad (25)$$

6-Entropy weight for the jth attribute is calculated as below [24]:

$$W_j = \frac{1 - E_j}{\sum_{j=1}^{n}\left(1 - E_j\right)} \quad (26)$$

Weight vector $W = (w_1, w_2, w_3, \ldots, w_n)^T$ of attributes, $K = \{K_j, j = 1,2,\ldots,n\}$ with $W_j \geq 0$ and $\sum_{j=1}^{n} W_j = 1$.

7-Value of each alternative is calculated as follows:

$$L_w = \sum_{j=1}^{n} W_j * \left((T_{ij}(x) * T^*_{ij}(x)) + (I_{ij}(x) * I^*_{ij}(x)) + (F_{ij}(x) * F^*_{ij}(x))\right) \quad (27)$$

Where for beneficial attribute PIS=$< T^*_{\max}(x), I^*_{\min}(x), F^*_{\min}(x) >$=<1,0,0>, and for non-beneficial attribute NIS=$< T^*_{\min}(x), I^*_{\max}(x), F^*_{\max}(x) >$=<0,1,1>.

8- Each alternative is ranked according to the descending order of L_w.

4 Data Set

This study investigated mutual funds which operated continuously and without merging with any other funds for the five-year period between 2012 and 2016, and found security funds that could be grouped under 58 “equity umbrella funds” according to the established criterion. Data from the relevant pages of the Turkish Capital Markets Board website were utilized in the determination of these funds. The unit share price, and consequently the total portfolio value of one of the funds- Alkhair Portfolio Participating Equity Fund (Equity intensive fund)- on 31.07.2013 was found to be zero. Since this had the potential to affect the return and risk of the fund, this fund was removed from the study, leaving only 57 funds for analysis. These crisp data are converted to neutrosophic values because of the superiority of neutrosophy over crisp ones.

5 Analysis

A decision matrix was constructed for crisp data drawn from the equity funds and within the framework of the four criteria (Morningstarrating, Sharpe ration, Treynor ratio, and Jensen ratio) as seen in Table 2.

Equity Funds	**Morningstar weighted point**	**Sharpe weighted point**	**Treynor weighted point**	**Jensen weighted point**
Ak Asset Management America Foreign Equity Fund	0,692148	0,228627	-0,0363502	0,0096924
Ak Asset Management Europe Foreign Equity Fund	-0,24477	0,113154	-0,0522	0,004424

Ak Asset Management Asia Foreign Equity Fund	-1,447455	-0,001768	-0,000950	0,0000765
Ak Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,683878	0,008855	0,000668	0,000955
Ak Asset Management BIST Banks Index Equity Fund (Equity Intensive Fund)	-3,82955	-0,03114	-0,00174	0,00126
Ak Asset Management BRIC Countries Foreign Equity Fund	-1,730383	0,035271	0,0076763	0,0019280
Ak Asset Management Equity Fund (Equity Intensive Fund)	-3,50801	-0,01854	-0,00094	-0,00049
Ak Asset Management Foreign Equity Fund	-0,120508	0,169739	-0,093322	0,0071872
Ata Portfolio First Equity Fund (Equity Intensive Fund)	-3,113982	0,089787	0,0056179	0,005439
Azimut PYŞ First Equity Fund (Equity Intensive Fund)	-2,553792	0,045385	0,0035996	0,002600
Bizim Portfolio Energy Sector Participation Equity Fund (Equity Intensive Fund)	-2,432883	-0,095570	-0,010140	-0,0036231
Bizim Portfolio Construction Industry Participation Equity Fund (Equity Intensive Fund)	-3,161174	0,039539	0,0027570	0,0024940
Deniz Portfolio BIST 100 Index Equity Fund (Equity Intensive Fund)	-3,694172	0,004020	0,000370	0,0006699
Deniz Portfolio Equity Fund (Equity Intensive Fund)	-3,517794	0,010906	0,000791	0,0009917
Finans Portfolio BIST 30 Index Equity Intensive Fund Exchange Traded Fund	-3,907524	0,021992	0,001424	0,001816
Finans Asset Management First Equity Fund	-3,296607	0,052358	0,0034596	0,0032420
Finans Asset Management Dow Jones İstanbul 20 (Equity Intensive) Exchange Traded Fund	-3,799676	0,0257993	0,0016396	0,0019916
Finans Asset Management Second Equity Fund	-3,479966	0,0508895	0,0031856	0,0032607
Finans Asset Management Turkey Large-Cap Banks (Equity Intensive) Exchange Traded Fund	-5,244110	-0,0010259	0,0000074	0,000465
Fokus Asset Management Equity Fund (Equity Intensive Fund)	-2,486771	0,041656	0,0028337	0,0020580
Garanti Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,877312	-0,000583	0,0000974	0,000431
Garanti Asset Management Equity Fund (Equity Intensive Fund)	-3,378613	0,040647	0,0025761	0,0026067
Gedik Asset Management First Equity Fund (Equity Intensive Fund)	-3,148063	-0,010759	-0,000589	-0,0001846
Gedik Asset Management G-20 Countries Foreign Securities (Equity Intensive Fund)	-0,604129	0,107282	0,120414	0,0041053
Gedik Asset Management Second Equity Fund (Equity Intensive Fund)	-1,946579	0,121667	0,0103163	0,0058834
Global MD Asset Management First Equity Fund (Equity Intensive Fund)	-3,418733	-0,042653	-0,0025318	-0,0016431
Global MD Asset Management Second Equity Fund (Equity Intensive Fund)	-3,266048	-0,063557	-0,0037824	-0,0024167
Halk Asset Management Equity Fund (Equity Intensive Fund)	-3,673393	0,0238852	0,0015849	0,0018096
HSBC Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,819283	-0,0068613	-0,000271	0,000007
HSBC Asset Management Equity Fund (Equity Intensive Fund)	-3,818957	0,036008	0,0022634	0,0026858
ING Asset Management First Equity Fund (Equity Intensive Fund)	-3,510678	-0,0138847	-0,000641	-0,00026
İş Asset Management Dividend Paying Corporations Share Fund (Equity Intensive Fund)	-3,380190	-0,012627	-0,000613	-0,000216
İstanbul Portfolio Equity Intensive Fund (Equity Intensive Fund)	-1,818028	-0,092841	-0,006290	-0,0022918
İş Asset Management BIST-30 Index Share Fund (Equity Intensive Fund)	-3,798493	-0,0012180	0,0000549	0,0003845

İş Asset Management BIST 30 equity intensive Exchange investment fund	-3,871759	0,0208728	0,00137201	0,0017460
İş Asset Management BIST Bank Index Share Fund (Equity Intensive Fund)	-4,291139	-0,0092844	-0,000475	-0,0001311
İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive Fund)	-0,875815	0,283976	0,034661	0,0166801
İş Asset Management Share Fund (Equity Intensive Fund)	-3,407564	-0,014655	-0,000716	-0,000307
İş Asset Management İş Bank Subsidiaries Fund (Equity Intensive Fund)	-2,531887	0,0536240	0,0037697	0,0025575
İş Asset Management Participation Share Fund (Equity Intensive Fund)	-1,323817	-0,0959215	-0,0099110	-0,0022049
İş Portfolio Banking Private Equity Fund (Equity Intensive Fund)	-3,334418	-0,0018027	0,0000289	0,000307
Kare Asset Management Equity Fund (Equity Intensive Fund)	-2,874745	0,143354	0,0088840	0,0087556
Qinvest Asset Management Equity Fund (Equity Intensive Fund)	-2,518660	0,0033631	0,000253	0,000378
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	-2,653906	0,117171	0,0081913	0,006719
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	-0,999745	0,074355	0,0089824	0,0021746
Şeker Asset Management Equity Fund (Equity Intensive Fund)	-2,965969	0,005848	0,000374	0,00060018
Tacirler Asset Management Equity Fund (Equity Intensive Fund)	-1,524716	0,041657	0,00270103	0,0011679
TEB Asset Management Equity Fund (Equity Intensive Fund)	-3,568576	0,024865	0,00166	0,0018599
Vakıf Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,749163	-0,0009460	0,0000723	0,000391
Yapı Kredi Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,688064	0,0097042	0,000707	0,000994
Yapı Kredi Asset Management BIST 100 Index Equity Fund (Equity Intensive Fund)	-3,602030	0,00260078	0,000302	0,000586
Yapı Kredi Asset Management First Equity Fund (Equity Intensive Fund)	-3,445713	0,0078333	0,000610	0,000827
Yapı Kredi Asset Management Koc Holding Affiliate and Equity Fund (Equity Intensive Fund)	-2,185636	0,1336988	0,0085835	0,0066485
Yapı Kredi Asset Management Foreign Technology Sector Equity Fund	0,152437	0,239112	-1,015735	0,0091293
Ziraat Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	-3,734672	0,0036299	0,000338	0,000647
Ziraat Asset Management Equity Fund (Equity Intensive Fund)	-3,091296	-0,008640	-0,000496	0,0000372
Ziraat Asset Management Dividend Paying Corporations Equity Fund (Equity Intensive Fund)	-2,095050	0,0000948	0,0000245	0,00027349

Table 2. Decision matrix for crisp data in terms of equity funds

After the decision matrix for crisp data, vector normalization process was implemented to obtain the normalized decision matrix. After that, the normalized decision matrix was transformed into the SVNS decision matrix comprised of the degree of truthness $T_L(x)$, indeterminacy $I_L(x)$, and falsehood $F_L(x)$ using the conversion rule for beneficial and non-beneficial criteria. This step is shown in Table 3.

Equity Funds	**Morningstar weighted point**	**Sharpe weighted point**	**Treynor weighted point**	**Jensen weighted point**
Ak Asset Management America Foreign Equity Fund	(0.0299,0.97,0.97)	(0.37,0.6299,0.6299)	(-0.035,1.035,1.035)	(0.323,0.676,0.676)
Ak Asset Management Europe Foreign Equity Fund	(-0.01,1.010,1.01	(0.183,0.816,0.816)	(-0.05,1.05,1.05)	(0.147,0.852,0.852)

	0)			
Ak Asset Management Asia Foreign Equity Fund	(-0.062,1.062,1.062)	(-0.002,1.002,1.002)	(-0.00092,1.00092,1.00092)	(-0.0025,1.0025,1.0025)
Ak Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.159,1.159,1.159)	(0.014,0.985,0.985)	(0.00064,0.999,0.999)	(0.0318,0.968,0.968)
Ak Asset Management BIST Banks Index Equity Fund (Equity Intensive Fund)	(-0.165,1.165,1.165)	(-0.05,1.05,1.05)	(-0.00169,1.00169,1.00169)	(-0.042,1.042,1.042)
Ak Asset Management BRIC Countries Foreign Equity Fund	(-0.07484,1.074,1.074)	(0.057,0.942,0.942)	(0.00745,0.992,0.992)	(0.0642,0.935,0.935)
Ak Asset Management Equity Fund (Equity Intensive Fund)	(-0.151,1.151,1.151)	(-0.03,1.03,1.03)	(-0.00091,1.00091,1.00091)	(-0.016,1.016,1.016)
Ak Asset Management Foreign Equity Fund	(0.00521,1.005,1.005)	(0.274,0.725,0.725)	(-0.0905,1.0905,1.0905)	(0.239,0.76,0,.76)
Ata Portfolio First Equity Fund (Equity Intensive Fund)	(-0.134,1.134,1.134)	(0.145,0.854,0.854)	(0.005,0.994,0.994)	(0.181,0.818,0.818)
Azimut PYŞ First Equity Fund (Equity Intensive Fund)	(-0.11,1.11,1.11)	(0.07,0.926,0.926)	(0.003,0.996,0.996)	(0.086,0.913,0.913)
Bizim Portfolio Energy Sector Participation Equity Fund (Equity Intensive Fund)	(-0.105,1.105,1.105)	(-0.154,1.154,1.154)	(-0.0098,1.0098,1.0098)	(-0,12,1.12,1.12)
Bizim Portfolio Construction Industry Participation Equity Fund (Equity Intensive Fund)	(-0.136,1.136,1.136)	(0.063,0.936,0.936)	(0.0026,0.997,0.997)	(0.083,0.916,0.916)
Deniz Portfolio BIST 100 Index Equity Fund (Equity Intensive Fund)	(-0.159,1.159,1.159)	(0.006,0.993,0.993)	(0.00035,0.999,0.999)	(0.022,0.977,0.977)
Deniz Portfolio Equity Fund (Equity Intensive Fund)	(-0.152,1.152,1.152)	(0.017,0.982,0.982)	(0.00076,0.999,0.999)	(0.033,0.966,0.966)
Finans Portfolio BIST 30 Index Equity Intensive Fund Exchange Traded Fund	(-0.169,1.169,1.169)	(0.035,0.964,0.964)	(0.0013,0.998,0.998)	(0.06,0.939,0.939)
Finans Asset Management First Equity Fund	(-0.142,1.142,1.142)	(0.084,0.915,0.915)	(0.0033,0.996,0.996)	(0.108,0.891,0.891)
Finans Asset Management Dow Jones İstanbul 20 (Equity Intensive) Exchange Traded Fund	(-0.164,1.164,1.164)	(0.041,0.958,0.958)	(0.0015,0.998,0.998)	(0.066,0.933,0.933)
Finans Asset Management Second Equity Fund	(-0.15,1.15,1.15)	(0.082,0.917,0.917)	(0.003,0.996,0.996)	(0.108,0.891,0.891)
Finans Asset Management Turkey Large-Cap Banks (Equity Intensive) Exchange Traded Fund	(-0.226,1.226,1.226)	(-0.001,1.001,1.001)	(0.00000722,0.999,0.999)	(0.015,0.984,0.984)
Fokus Asset Management Equity Fund (Equity Intensive Fund)	(-0.107,1.107,1.107)	(0.067,0.932,0.932)	(0.0027,0.997,0.997)	(0.068,0.931,0.931)
Garanti Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.167,1.167,1.167)	(-0.00095,1.0009,1.0009)	(0.0000946,0.999,0.999)	(0.0144,0.985,0.985)
Garanti Asset Management Equity Fund (Equity Intensive Fund)	(-0.146,1.146,1.146)	(0.065,0.934,0.934)	(0.0025,0.997,0.997)	(0.086,0.913,0.913)
Gedik Asset Management First Equity Fund (Equity Intensive Fund)	(-0.136,1.136,1.136)	(-0.017,1.017,1.017)	(-0.00057,1.00057,1.00057)	(-0.006,1.006,1.006)
Gedik Asset Management G-20 Countries Foreign Securities (Equity Intensive Fund)	(-0.026,1.026,1.0	(0.173,0.826,0.826)	(0.116,0.883,0.883)	(0.136,0.863,0.863)

	26)			
Gedik Asset Management Second Equity Fund (Equity Intensive Fund)	(-0.0841,1.084,1.084)	(0.196,0.803,0.803)	(0.01,0.989,0.989)	(0.196,0.803,0.803)
Global MD Asset Management First Equity Fund (Equity Intensive Fund)	(-0.147,1.147,1.147)	(-0.069,1.069,1.069)	(-0.0024,1.0024,1.0024)	(-0.054,1.054,1.054)
Global MD Asset Management Second Equity Fund (Equity Intensive Fund)	(-0.141,1.141,1.141)	(-0.102,1.102,1.102)	(-0.0036,1.0036,1.0036)	(-0.08,1.08,1.08)
Halk Asset Management Equity Fund (Equity Intensive Fund)	(-0.158,1.158,1.158)	(0.038,0.961,0.961)	(0.0015,0.998,0.998)	(0.0603,0.939,0.939)
HSBC Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.165,1.165,1.165)	(-0.011,1.011,1.011)	(-0.00026,1.00026,1.00026)	(0.002,0.997,0.997)
HSBC Asset Management Equity Fund (Equity Intensive Fund)	(-0.165,1.165,1.165)	(0.058,0.941,0.941)	(0.00219,0.997,0.997)	(0.089,0.91,0.91)
ING Asset Management First Equity Fund (Equity Intensive Fund)	(-0.151,1.151,1.151)	(-0.022,1.022,1.022)	(-0.00062,1.00062,1.00062)	(-0.008,1.008,1.008)
İş Asset Management Dividend Paying Corporations Share Fund (Equity Intensive Fund)	(-0.146,1.146,1.146)	(-0.0204,1.0204,1.0204)	(-0.00059,1.00059,1.00059)	(-0.0072,1.0072,1.0072)
İstanbul Portfolio Equity Intensive Fund (Equity Intensive Fund)	(-0.078,1.078,1.078)	(-0.15,1.15,1.15)	(-0.0061,1.0061,1.0061)	(-0.076,1.076,1.076)
İş Asset Management BIST-30 Index Share Fund (Equity Intensive Fund)	(-0.164,1.164,1.164)	(-0.0019,1.0019,1.0019)	(0.0000533,0.999,0.999)	(0.0128,0.987,0.987)
İş Asset Management BIST 30 equity intensive Exchange investment fund	(-0.167,1.167,1.167)	(0.033,0.966,0.966)	(0.00133,0.998,0.998)	(0.058,0.941,0.941)
İş Asset Management BIST Bank Index Share Fund (Equity Intensive Fund)	(-0.185,1.185,1.185)	(-0.015,1.015,1.015)	(-0.00046,1.00046,1.00046)	(-0.0043,1.0043,1.0043)
İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive Fund)	(-0.037,1.037,1.037)	(0.459,0.54,0.54)	(0.033,0.966,0.966)	(0.556,0.443,0.443)
İş Asset Management Share Fund (Equity Intensive Fund)	(-0.147,1.147,1.147)	(-0.023,1.023,1.023)	(-0.00069,1.00069,1.00069)	(-0.01,1.01,1.01)
İş Asset Management İş Bank Subsidiaries Fund (Equity Intensive Fund)	(-0.109,1.109,1.109)	(0.086,0.913,0.913)	(0.0036,0.996,0.996)	(0.0852,0.914,0.914)
İş Asset Management Participation Share Fund (Equity Intensive Fund)	(-0.057,1.057,1.057)	(-0.155,1.155,1.155)	(-0.0096,1.0096,1.0096)	(-0.0735,1.0735,1.0735)
İş Portfolio Banking Private Equity Fund (Equity Intensive Fund)	(-0.144,1.144,1.144)	(-0.0029,1.0029,1.0029)	(0.0000281,0.999,0.999)	(0.01,0.989,0.989)
Kare Asset Management Equity Fund (Equity Intensive Fund)	(-0.124,1.124,1.124)	(0.232,0.767,0.767)	(0.0086,0.991,0.991)	(0.291,0.708,0.708)
Qinvest Asset Management Equity Fund (Equity Intensive Fund)	(-0.108,1.108,1.108)	(0.0054,0.994,0.994)	(0.00024,0.999,0.999)	(0.012,0.987,0.987)
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	(-0.114,1.114,1.114)	(0.189,0.81,0.81)	(0.0079,0.992,0.992)	(0.224,0.775,0.775)
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	(-0.043,1.043,1.043)	(0.12,0.879,0.879)	(0.0087,0.991,0.991)	(0.0725,0.927,0.927)

Şeker Asset Management Equity Fund (Equity Intensive Fund)	(-0.128,1.128,1.128)	(0.0094,0.99,0.99)	(0.00036,0.999,0.999)	(0.02,0.979,0.979)
Tacirler Asset Management Equity Fund (Equity Intensive Fund)	(-0.065,1.065,1.065)	(0.067,0.932,0.932)	(0.00262,0.997,0.997)	(0.0389,0.961,0.961)
TEB Asset Management Equity Fund (Equity Intensive Fund)	(-0.154,1.154,1.154)	(0.04,0.959,0.959)	(0.00161,0.998,0.998)	(0.062,0.937,0.937)
Vakıf Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.162,1.162,1.162)	(-0.00153,1.00153,1.00153)	(0.0000702,0.999,0.999)	(0.013,0.986,0.986)
Yapı Kredi Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.159,1.159,1.159)	(0.015,0.984,0.984)	(0.00068,0.999,0.999)	(0.033,0.966,0.966)
Yapı Kredi Asset Management BIST 100 Index Equity Fund (Equity Intensive Fund)	(-0.155,1.155,1.155)	(0.004,0.995,0.995)	(0.00029,0.999,0.999)	(0.019,0.98,0.98)
Yapı Kredi Asset Management First Equity Fund (Equity Intensive Fund)	(-0.149,1.149,1.149)	(0.012,0.987,0.987)	(0.00059,0.999,0.999)	(0.0275,0.972,0.972)
Yapı Kredi Asset Management Koc Holding Affiliate and Equity Fund (Equity Intensive Fund)	(-0.094,1.094,1.094)	(0.216,0.783,0.783)	(0.0083,0.991,0.991)	(0.221,0.778,0.778)
Yapı Kredi Asset Management Foreign Technology Sector Equity Fund	(0.0065,0.993,0.993)	(0.387,0.612,0.612)	(-0.986,1.986,1.986)	(0.304,0.695,0.695)
Ziraat Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	(-0.161,1.161,1.161)	(0.005,0.994,0.994)	(0.00032,0.999,0.999)	(0.021,0.978,0.978)
Ziraat Asset Management Equity Fund (Equity Intensive Fund)	(-0.133,1.133,1.133)	(-0.013,1.013,1.013)	(-0.00048,1.00048,1.00048)	(0.0012,0.998,0.998)
Ziraat Asset Management Dividend Paying Corporations Equity Fund (Equity Intensive Fund)	(-0.09,1.09,1.09)	(0.000154,0.999,0.999)	(0.0000238,0.999,0.999)	(0.009,0.99,0.99)

Table 3. SVNS Decision Matrix

After constructing SVNS decision matrix, entropy values (E_j) and weights (W_j) for each criterion werecalculated as shown in Table 4.

Criteria	Entropy value (E_j)	Entropy weight (W_j)
Morningstar weighted point	1.243136	0.307845
Sharpe weighted point	0.891585	0.220789
Treynor weighted point	1.033546	0.255943
Jensen weighted point	0.869916	0.215422

Table 4. Entropy values (E_j) and weights (W_j) for each evaluation criteria

The findings presented in Table 4 above show that the entropy weights based on the different evaluation criteria were close to each other. The weights based on the Morningstar rating system were found to be the highest leading to the assumption of Morningstar as the most significant evaluation criteria. Jensen ratio was found to have the least weights prompting assumption of its weakness as evaluation criteria for equity funds. Finally, the value of each equity fund (L_w) was computed and ranked as highlighted in Table 5 below.

Equity Funds	Value (L_w)	Ranking
Ak Asset Management America Foreign Equity Fund	1,402621	56
Ak Asset Management Europe Foreign Equity Fund	1,471126	53
Ak Asset Management Asia Foreign Equity Fund	1,516942	45

Ak Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,568346	20
Ak Asset Management BIST Banks Index Equity Fund (Equity Intensive Fund)	1,587197	4
Ak Asset Management BRIC Countries Foreign Equity Fund	1,507607	46
Ak Asset Management Equity Fund (Equity Intensive Fund)	1,574065	11
Ak Asset Management Foreign Equity Fund	1,465676	54
Ata Portfolio First Equity Fund (Equity Intensive Fund)	1,523178	44
Azimut PYŞ First Equity Fund (Equity Intensive Fund)	1,526375	42
Bizim Portfolio Energy Sector Participation Equity Fund (Equity Intensive Fund)	1,57504	10
Bizim Portfolio Construction Industry Participation Equity Fund (Equity Intensive Fund)	1,54372	36
Deniz Portfolio BIST 100 Index Equity Fund (Equity Intensive Fund)	1,570483	15
Deniz Portfolio Equity Fund (Equity Intensive Fund)	1,563529	25
Finans Portfolio BIST 30 Index Equity Intensive Fund Exchange Traded Fund	1,56876	17
Finans Asset Management First Equity Fund	1,542249	38
Finans Asset Management Dow Jones İstanbul 20 (Equity Intensive) Exchange Traded Fund	1,564664	23
Finans Asset Management Second Equity Fund	1,547307	35
Finans Asset Management Turkey Large-Cap Banks (Equity Intensive) Exchange Traded Fund	1,611662	2
Fokus Asset Management Equity Fund (Equity Intensive Fund)	1,527103	41
Garanti Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,576842	8
Garanti Asset Management Equity Fund (Equity Intensive Fund)	1,54881	34
Gedik Asset Management First Equity Fund (Equity Intensive Fund)	1,562302	28
Gedik Asset Management G-20 Countries Foreign Securities (Equity Intensive Fund)	1,41159	55
Gedik Asset Management Second Equity Fund (Equity Intensive Fund)	1,483426	51
Global MD Asset Management First Equity Fund (Equity Intensive Fund)	1,580536	6
Global MD Asset Management Second Equity Fund (Equity Intensive Fund)	1,583647	5
Halk Asset Management Equity Fund (Equity Intensive Fund)	1,562333	27
HSBC Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,577772	7
HSBC Asset Management Equity Fund (Equity Intensive Fund)	1,56098	29
ING Asset Management First Equity Fund (Equity Intensive Fund)	1,57242	13
İş Asset Management Dividend Paying Corporations Share Fund (Equity Intensive Fund)	1,568706	18
İstanbul Portfolio Equity Intensive Fund (Equity Intensive Fund)	1,554046	32
İş Asset Management BIST-30 Index Share Fund (Equity Intensive Fund)	1,575108	9
İş Asset Management BIST 30 equity intensive Exchange investment fund	1,568286	21

İş Asset Management BIST Bank Index Share Fund (Equity Intensive Fund)	1,590875	3
İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive Fund)	1,384472	57
İş Asset Management Share Fund (Equity Intensive Fund)	1,570112	16
İş Asset Management İş Bank Subsidiaries Fund (Equity Intensive Fund)	1,524021	43
İş Asset Management Participation Share Fund (Equity Intensive Fund)	1,543436	37
İş Portfolio Banking Private Equity Fund (Equity Intensive Fund)	1,563628	24
Kare Asset Management Equity Fund (Equity Intensive Fund)	1,495966	49
Qinvest Asset Management Equity Fund (Equity Intensive Fund)	1,54147	39
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	1,501294	48
Strateji Asset Management Second Equity Fund (Equity Intensive Fund)	1,479225	52
Şeker Asset Management Equity Fund (Equity Intensive Fund)	1,55172	33
Tacirler Asset Management Equity Fund (Equity Intensive Fund)	1,504807	47
TEB Asset Management Equity Fund (Equity Intensive Fund)	1,559298	31
Vakıf Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,573768	12
Yapı Kredi Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,568154	22
Yapı Kredi Asset Management BIST 100 Index Equity Fund (Equity Intensive Fund)	1,568682	19
Yapı Kredi Asset Management First Equity Fund (Equity Intensive Fund)	1,562843	26
Yapı Kredi Asset Management Koc Holding Affiliate and Equity Fund (Equity Intensive Fund)	1,485724	50
Yapı Kredi Asset Management Foreign Technology Sector Equity Fund	1,816721	1
Ziraat Asset Management BIST 30 Index Equity Fund (Equity Intensive Fund)	1,571666	14
Ziraat Asset Management Equity Fund (Equity Intensive Fund)	1,559821	30
Ziraat Asset Management Dividend Paying Corporations Equity Fund (Equity Intensive Fund)	1,531766	40

Table 5. Value of each equity funds (L_w) and ranking

According to the findings in Table 5, Yapı Kredi Asset Management Foreign Technology Sector Equity Funds had the highest L_w value and ranked in the first position followed, respectively, by Finans Asset Management Turkey Large-Cap Banks (Equity Intensive) Exchange Traded Fund and İş Asset Management BIST Bank Index Share Fund (Equity Intensive Fund). On the other hand, İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive) Fund had the lowest L_w value and ranked in the last position.

Conclusion

In this study, SVNS entropy-based decision making is used to rank equity funds traded in Turkey under incomplete, indeterminate and inconsistent information by using four evaluation criteria. To the best of our knowledge, it is the first study in evaluating equity funds traded in Turkey from the neutrosophic set perspective. According to the entropy weights results, Morningstar rating was found to be the most significant evaluation criteria. This indicates the importance of Morningstar weighted scores for equity funds traded in Turkey. On the contrary, Jensen-based weighting was found to be the least significant evaluation criteria for equity funds. Yapı Kredi Asset

Management Foreign Technology Sector Equity Fund had the highest L_w value and ranked first, followed by Finans Asset Management Turkey Large-Cap Banks (Equity Intensive) Exchange Traded Fund and İş Asset Management BIST Bank Index Share Fund (Equity Intensive Fund). On the other hand, İş Asset Management BIST Technology Capped Index Share Fund (Equity Intensive) Fund had the lowest L_w value and ranked in the last position.

In future studies we suggest the use of other neutrosophic logic-based techniques and normalization methods (out of vector normalization) to analyze this concept. SVNS based entropy technique may also be applied in evaluating equity funds of different countries by considering other criteria. This methodology may also be applied to different sectors.

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Positive implicative BMBJ-neutrosophic ideals in BCK-algebras

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Abstract: The concepts of a positive implicative BMBJ-neutrosophic ideal is introduced, and several properties are investigated. Conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal are provided. Relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal are discussed. Characterizations of positive implicative BMBJ-neutrosophic ideal are displayed.

Keywords: MBJ-neutrosophic set; BMBJ-neutrosophic ideal; positive implicative BMBJ-neutrosophic ideal.

1 Introduction

In 1965, L.A. Zadeh [18] introduced the fuzzy set in order to handle uncertainties in many real applications. In 1983, K. Atanassov introdued the notion of intuitionistic fuzzy set as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([13], [14] and [15]). Neutrosophic set is applied to many branchs of sciences. In the aspect of algebraic structures, neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [1], [3], [4], [5], [6], [11], [12], [16] and [17]. In [9], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, and it is applied to BCK/BCI-algebras. Mohseni et al. [9] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. Jun and Roh [7] applied the notion of MBJ-neutrosophic sets to ideals of BCK/BI-algebras, and introduced the concept of MBJ-neutrosophic ideals in BCK/BCI-algebras.

In this article, we introduce the concepts of a positive implicative BMBJ-neutrosophic ideal, and investigate several properties. We provide conditions for an MBJ-neutrosophic set to be a (positive implicative) BMBJ-neutrosophic ideal, and discussed relations between BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal. We consider characterizations of positive implicative BMBJ-neutrosophic ideal.

2 Preliminaries

By a BCI-*algebra*, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0,$

(IV) $x * y = 0,\, y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\,(0 * x = 0),$

then X is called a BCK-*algebra.*

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \le y \;\Rightarrow\; x * z \le y * z,\; z * y \le z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* of X (see [8]) if it satisfies (2.5) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\; y * z \in I \;\Rightarrow\; x * z \in I)\,. \tag{2.7}$$

Note from [8] that a subset I of a BCK-algebra X is a positive implicative ideal of X if and only if it is an ideal of X which satisfies the condition

$$(\forall x, y \in X)((x * y) * y \in I \;\Rightarrow\; x * y \in I). \tag{2.8}$$

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\tilde{a}_1 \succeq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \ge a_2^-,\; a_1^+ \ge a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [14]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

We refer the reader to the books [2, 8] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X (see [9]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a ***BMBJ-neutrosophic ideal*** of X (see [10]) if it satisfies

$$(\forall x \in X)(M_A(x) + B_A^-(x) \leq 1,\, B_A^+(x) + J_A(x) \leq 1), \tag{2.9}$$

$$(\forall x \in X)\begin{pmatrix} M_A(0) \geq M_A(x) \\ B_A^-(0) \leq B_A^-(x) \\ B_A^+(0) \geq B_A^+(x) \\ J_A(0) \leq J_A(x) \end{pmatrix}, \tag{2.10}$$

and

$$(\forall x, y \in X) \begin{pmatrix} M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} \\ B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} \\ J_A(x) \leq \max\{J_A(x * y), J_A(y)\} \end{pmatrix}. \tag{2.11}$$

3 Positive implicative BMBJ-neutrosophic ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a *positive implicative BMBJ-neutrosophic ideal* of X if it satisfies (2.9), (2.10) and

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x * z) \geq \min\{M_A((x * y) * z), M_A(y * z)\} \\ B_A^-(x * z) \leq \max\{B_A^-((x * y) * z), B_A^-(y * z)\} \\ B_A^+(x * z) \geq \min\{B_A^+((x * y) * z), B_A^+(y * z)\} \\ J_A(x * z) \leq \max\{J_A((x * y) * z), J_A(y * z)\} \end{pmatrix}. \tag{3.1}$$

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1. Let $\mathcal{A} = (M_A, B\tilde{\ }_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.71	$[0.04, 0.09]$	0.22
1	0.61	$[0.03, 0.08]$	0.55
2	0.51	$[0.02, 0.06]$	0.55
3	0.41	$[0.01, 0.03]$	0.77
4	0.31	$[0.02, 0.05]$	0.99

$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X.

Theorem 3.3. *Every positive implicative BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal.*

Proof. The condition (2.11) is induced by taking $z = 0$ in (3.1) and using (2.1). Hence every positive implica-tive BMBJ-neutrosophic ideal is a BMBJ-neutrosophic ideal. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4.

Table 4: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.6	$[0.04, 0.09]$	0.3
1	0.5	$[0.03, 0.08]$	0.7
2	0.5	$[0.03, 0.08]$	0.7
3	0.3	$[0.01, 0.03]$	0.5

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. But it is not a positive implicative MBJ-neutrosophic ideal of X since

$$M_A(2 * 1) = 0.5 < 0.6 = \min\{M_A((2 * 1) * 1), M_A(1 * 1)\},$$

Lemma 3.5. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$(\forall x, y \in X) \left(x \leq y \ \Rightarrow \begin{cases} M_A(x) \geq M_A(y), B_A^-(x) \leq B_A^-(y), \\ B_A^+(x) \geq B_A^+(y), J_A(x) \leq J_A(y) \end{cases} \right). \tag{3.2}$$

Proof. Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y),$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} = \max\{B_A^-(0), B_A^-(y)\} = B_A^-(y),$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} = \min\{B_A^+(0), B_A^+(y)\} = B_A^+(y),$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y).$$

This completes the proof. □

We provide conditions for a BMBJ-neutrosophic ideal to be a positive implicative BMBJ-neutrosophic ideal.

Theorem 3.6. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if it is a BMBJ-neutrosophic ideal of X and satisfies the following condition.*

$$(\forall x, y \in X)\begin{pmatrix} M_A(x*y) \geq M_A((x*y)*y) \\ B_A^-(x*y) \leq B_A^-((x*y)*y) \\ B_A^+(x*y) \geq B_A^+((x*y)*y) \\ J_A(x*y) \leq J_A((x*y)*y) \end{pmatrix}. \tag{3.3}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. If z is replaced by y in (3.1), then

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*y), M_A(y*y)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^-(x*y) &\leq \max\{B_A^-((x*y)*y), B_A^-(y*y)\} \\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y), \end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+((x*y)*y), B_A^+(y*y)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*y), J_A(y*y)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of X satisfying the condition (3.3). Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned} &M_A((x*y)*z) \leq M_A(((x*z)*z)*(y*z)),\\ &B_A^-((x*y)*z) \geq B_A^-(((x*z)*z)*(y*z)),\\ &B_A^+((x*y)*z) \leq B_A^+(((x*z)*z)*(y*z)),\\ &J_A((x*y)*z) \geq J_A(((x*z)*z)*(y*z)) \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$. Using (3.3), (2.11) and (3.4), we have

$$\begin{aligned} M_A(x*z) &\geq M_A((x*z)*z) \geq \min\{M_A(((x*z)*z)*(y*z)), M_A(y*z)\}\\ &\geq \min\{M_A((x*y)*z), M_A(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^-(x*z) &\leq B_A^-((x*z)*z) \leq \max\{B_A^-(((x*z)*z)*(y*z)), B_A^-(y*z)\}\\ &\leq \max\{B_A^-((x*y)*z), B_A^-(y*z)\}, \end{aligned}$$

$$\begin{aligned} B_A^+(x*z) &\geq B_A^+((x*z)*z) \geq \min\{B_A^+(((x*z)*z)*(y*z)), B_A^+(y*z)\}\\ &\geq \min\{B_A^+((x*y)*z), B_A^+(y*z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*z) &\leq J_A((x*z)*z) \leq \max\{J_A(((x*z)*z)*(y*z)), J_A(y*z)\}\\ &\leq \max\{J_A((x*y)*z), J_A(y*z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} &U(M_A;t) := \{x \in X \mid M_A(x) \geq t\},\\ &L(B_A^-;\alpha^-) := \{x \in X \mid B_A^-(x) \leq \alpha^-\},\\ &U(B_A^+;\alpha^+) := \{x \in X \mid B_A^+(x) \geq \alpha^+\},\\ &L(J_A;s) := \{x \in X \mid J_A(x) \leq s\} \end{aligned}$$

where $t, s, \alpha^-, \alpha^+ \in [0, 1]$.

Lemma 3.7 ([10]). *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A;t)$, $L(B_A^-;\alpha^-)$, $U(B_A^+;\alpha^+)$ and $L(J_A;s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Theorem 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative BMBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A;t)$, $L(B_A^-;\alpha^-)$, $U(B_A^+;\alpha^+)$ and $L(J_A;s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3. It follows from Lemma 3.7 that the non-empty sets $U(M_A;t)$, $L(B_A^-;\alpha^-)$, $U(B_A^+;\alpha^+)$ and $L(J_A;s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Let

$x, y, a, b, c, d, u, v \in X$ be such that $(x * y) * y \in U(M_A; t)$, $(a * b) * b \in L(B_A^-; \alpha^-)$, $(c * d) * d \in U(B_A^+; \alpha^+)$ and $(u * v) * v \in L(J_A; s)$. Using Theorem 3.6, we have

$$\begin{aligned}
&M_A(x * y) \geq M_A((x * y) * y) \geq t, \text{ that is, } x * y \in U(M_A; t),\\
&B_A^-(a * b) \leq B_A^-((a * b) * b) \leq \alpha^-, \text{ that is, } a * b \in L(B_A^-; \alpha^-),\\
&B_A^+(c * d) \geq B_A^+((c * d) * d) \geq \alpha^+, \text{ that is, } c * d \in U(B_A^+; \alpha^+),\\
&J_A(u * v) \leq J_A((u * v) * v) \leq s, \text{ that is, } u * v \in L(J_A; s).
\end{aligned}$$

Therefore $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$.

Conversely, suppose that the non-empty sets $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are positive implicative ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. Then $U(M_A; t)$, $L(B_A^-; \alpha^-)$, $U(B_A^+; \alpha^+)$ and $L(J_A; s)$ are ideals of X for all $t, s, \alpha^-.\alpha^+ \in [0, 1]$. It follows from Lemma 3.7 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Assume that $M_A(x_0 * y_0) < M_A((x_0 * y_0) * y_0) = t_0$ for some $x_0, y_0 \in X$. Then $(x_0 * y_0) * y_0 \in U(M_A; t_0)$ and $x_0 * y_0 \notin U(M_A; t_0)$, which is a contradiction. Thus $M_A(x * y) \geq M_A((x * y) * y)$ for all $x, y \in X$. Similarly, we have $B_A^+(x * y) \geq B_A^+((x * y) * y)$ for all $x, y \in X$. If there exist $a_0, b_0 \in X$ such that $J_A(a_0 * b_0) > J_A((a_0 * b_0) * b_0) = s_0$, then $(a_0 * b_0) * b_0 \in L(J_A; s_0)$ and $a_0 * b_0 \notin L(J_A; s_0)$. This is impossible, and thus $J_A(a * b) \leq J_A((a * b) * b)$ for all $a, b \in X$. By the similar way, we know that $B_A^-(a * b) \leq B_A^-((a * b) * b)$ for all $a, b \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Theorem 3.9. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is positive implicative if and only if it satisfies the following condition.*

$$(\forall x, y, z \in X) \begin{pmatrix} M_A((x * z) * (y * z)) \geq M_A((x * y) * z), \\ B_A^-((x * z) * (y * z)) \leq B_A^-((x * y) * z), \\ B_A^+((x * z) * (y * z)) \geq B_A^+((x * y) * z), \\ J_A((x * z) * (y * z)) \leq J_A((x * y) * z). \end{pmatrix} \tag{3.5}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.3, and satisfies the condition (3.3) by Theorem 3.6. Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$\begin{aligned}
&M_A((x * y) * z) \leq M_A(((x * (y * z)) * z) * z),\\
&B_A^-((x * y) * z) \geq B_A^-(((x * (y * z)) * z) * z),\\
&B_A^+((x * y) * z) \leq B_A^+(((x * (y * z)) * z) * z),\\
&J_A((x * y) * z) \geq J_A(((x * (y * z)) * z) * z)
\end{aligned} \tag{3.6}$$

for all $x, y, z \in X$. Using (2.3), (3.3) and (3.6), we have

$$\begin{aligned} M_A((x*z)*(y*z)) &= M_A((x*(y*z))*z) \\ &\geq M_A(((x*(y*z))*z)*z) \\ &\geq M_A((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^-((x*z)*(y*z)) &= B_A^-((x*(y*z))*z) \\ &\leq B_A^-(((x*(y*z))*z)*z) \\ &\leq B_A^-((x*y)*z), \end{aligned}$$

$$\begin{aligned} B_A^+((x*z)*(y*z)) &= B_A^+((x*(y*z))*z) \\ &\geq B_A^+(((x*(y*z))*z)*z) \\ &\geq B_A^+((x*y)*z), \end{aligned}$$

and

$$\begin{aligned} J_A((x*z)*(y*z)) &= J_A((x*(y*z))*z) \\ &\leq J_A(((x*(y*z))*z)*z) \\ &\leq J_A((x*y)*z). \end{aligned}$$

Hence (3.5) is valid.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of X which satisfies the condition (3.5). If we put $z = y$ in (3.5) and use (III) and (2.1), then we obtain the condition (3.3). Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X by Theorem 3.6. □

Theorem 3.10. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X if and only if it satisfies the condition (2.9), (2.10) and*

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x*y) \geq \min\{M_A(((x*y)*y)*z), M_A(z)\}, \\ B_A^-(x*y) \leq \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\}, \\ B_A^+(x*y) \geq \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\}, \\ J_A(x*y) \leq \max\{J_A(((x*y)*y)*z), J_A(z)\}. \end{pmatrix} \tag{3.7}$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3), and so the conditions (2.9) and (2.10) are valid. Using (2.11), (III), (2.1), (2.3) and (3.5), we have

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*z), M_A(z)\} \\ &= \min\{M_A(((x*z)*y)*(y*y)), M_A(z)\} \\ &\geq \min\{M_A(((x*z)*y)*y), M_A(z)\} \\ &= \min\{M_A(((x*y)*y)*z), M_A(z)\}, \end{aligned}$$

$$\begin{aligned}B_A^-(x*y) &\leq \max\{B_A^-((x*y)*z), B_A^-(z)\}\\ &= \max\{B_A^-(((x*z)*y)*(y*y)), B_A^-(z)\}\\ &\leq \max\{B_A^-(((x*z)*y)*y), B_A^-(z)\}\\ &= \max\{B_A^-(((x*y)*y)*z), B_A^-(z)\},\end{aligned}$$

$$\begin{aligned}B_A^+(x*y) &\geq \min\{B_A^+((x*y)*z), B_A^+(z)\}\\ &= \min\{B_A^+(((x*z)*y)*(y*y)), B_A^+(z)\}\\ &\geq \min\{B_A^+(((x*z)*y)*y), B_A^+(z)\}\\ &= \min\{B_A^+(((x*y)*y)*z), B_A^+(z)\},\end{aligned}$$

and

$$\begin{aligned}J_A(x*y) &\leq \max\{J_A((x*y)*z), J_A(z)\}\\ &= \max\{J_A(((x*z)*y)*(y*y)), J_A(z)\}\\ &\leq \max\{J_A(((x*z)*y)*y), J_A(z)\}\\ &= \max\{J_A(((x*y)*y)*z), J_A(z)\}\end{aligned}$$

for all $x, y, z \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies conditions (2.9), (2.10) and (3.7). Then

$$M_A(x) = M_A(x*0) \geq \min\{M_A(((x*0)*0)*z), M_A(z)\} = \min\{M_A(x*z), M_A(z)\},$$

$$B_A^-(x) = B_A^-(x*0) \leq \max\{B_A^-(((x*0)*0)*z), B_A^-(z)\} = \max\{B_A^-(x*z), B_A^-(z)\},$$

$$B_A^+(x) = B_A^+(x*0) \geq \min\{B_A^+(((x*0)*0)*z), B_A^+(z)\} = \min\{B_A^+(x*z), B_A^+(z)\},$$

and

$$J_A(x) = J_A(x*0) \leq \max\{J_A(((x*0)*0)*z), J_A(z)\} = \max\{J_A(x*z), J_A(z)\}$$

for all $x, z \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. Taking $z = 0$ in (3.7) and using (2.1) and (2.10) imply that

$$\begin{aligned}M_A(x*y) &\geq \min\{M_A(((x*y)*y)*0), M_A(0)\}\\ &= \min\{M_A((x*y)*y), M_A(0)\} = M_A((x*y)*y),\end{aligned}$$

$$\begin{aligned}B_A^-(x*y) &\leq \max\{B_A^-(((x*y)*y)*0), B_A^-(0)\}\\ &= \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y),\end{aligned}$$

$$\begin{aligned} B_A^+(x*y) &\geq \min\{B_A^+(((x*y)*y)*0), B_A^+(0)\} \\ &= \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A(((x*y)*y)*0), J_A(0)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. □

Proposition 3.11. *Every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$x*y \leq z \;\Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ B_A^-(x) \leq \max\{B_A^-(y), B_A^-(z)\}, \\ B_A^+(x) \geq \min\{B_A^+(y), B_A^+(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases} \tag{3.8}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then

$$M_A(x*y) \geq \min\{M_A((x*y)*z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*z), B_A^-(z)\} = \max\{B_A^-(0), B_A^-(z)\} = B_A^-(z),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*z), B_A^+(z)\} = \min\{B_A^+(0), B_A^+(z)\} = B_A^+(z),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*z), J_A(z)\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \geq \min\{M_A(x*y), M_A(y)\} \geq \min\{M_A(y), M_A(z)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x*y), B_A^-(y)\} \leq \max\{B_A^-(y), B_A^-(z)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x*y), B_A^+(y)\} \geq \min\{B_A^+(y), B_A^+(z)\},$$

and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\} \leq \max\{J_A(y), J_A(z)\}.$$

This completes the proof. □

We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in BCK/BCI-algebras.

Theorem 3.12. *Every MBJ-neutrosophic set in X satisfying* (2.9), (2.10) *and* (3.8) *is a BMBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X satisfying (2.9), (2.10) and (3.8). Note that $x * (x * y) \leq y$ for all $x, y \in X$. It follows from (3.8) that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\},$$

$$B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\}.$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Theorem 3.13. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a BMBJ-neutrosophic ideal of X if and only if (M_A, B_A^-) and (B_A^+, J_A) are intuitionistic fuzzy ideals of X.*

Proof. Straightforward. □

Theorem 3.14. *Given an ideal I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by*

$$M_A(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad B_A^-(x) = \begin{cases} \alpha^- & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$
$$B_A^+(x) = \begin{cases} \alpha^+ & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in I, \\ 1 & \text{otherwise,} \end{cases}$$

where $t, \alpha^+ \in (0, 1]$ and $s, \alpha^- \in [0, 1)$ with $t + \alpha^- \leq 1$ and $s + \alpha^+ \leq 1$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X such that $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$.

Proof. It is clear that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ satisfies the condition (2.9) and $U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I$. Let $x, y \in X$. If $x * y \in I$ and $y \in I$, then $x \in I$ and so

$$\begin{aligned} M_A(x) &= t = \min\{M_A(x * y), M_A(y)\} \\ B_A^-(x) &= \alpha^- = \max\{B_A^-(x * y), B_A^-(y)\}, \\ B_A^+(x) &= \alpha^+ = \min\{B_A^+(x * y), B_A^+(y)\}, \\ J_A(x) &= s = \max\{J_A(x * y), J_A(y)\}. \end{aligned}$$

If any one of $x * y$ and y is contained in I, say $x * y \in I$, then $M_A(x * y) = t$, $B_A^-(x * y) = \alpha^-$, $J_A(x * y) = s$, $M_A(y) = 0$, $B_A^-(y) = 1$, $B_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned}
&M_A(x) \geq 0 = \min\{t, 0\} = \min\{M_A(x * y), M_A(y)\} \\
&B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\
&B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\
&J_A(x) \leq 1 = \max\{s, 1\} = \max\{J_A(x * y), J_A(y)\}.
\end{aligned}$$

If $x * y \notin I$ and $y \notin I$, then $M_A(x * y) = 0 = M_A(y)$, $B_A^-(x * y) = 1 = B_A^-(y)$, $B_A^+(x * y) = 0 = B_A^+(y)$ and $J_A(x * y) = 1 = J_A(y)$. It follows that

$$\begin{aligned}
&M_A(x) \geq 0 = \min\{M_A(x * y), M_A(y)\} \\
&B_A^-(x) \leq 1 = \max\{B_A^-(x * y), B_A^-(y)\}, \\
&B_A^+(x) \geq 0 = \min\{B_A^+(x * y), B_A^+(y)\}, \\
&J_A(x) \leq 1 = \max\{J_A(x * y), J_A(y)\}.
\end{aligned}$$

It is obvious that $M_A(0) \geq M_A(x)$, $B_A^-(0) \leq B_A^-(x)$, $B_A^+(0) \geq B_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X. □

Lemma 3.15. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X, then I is an ideal of X.*

Proof. Obviously, $0 \in I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then $M_A(x * y) = t = M_A(y)$, $B_A^-(x * y) = \alpha^- = B_A^-(y)$, $B_A^+(x * y) = \alpha^+ = B_A^+(y)$ and $J_A(x * y) = s = J_A(y)$. Thus

$$\begin{aligned}
&M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = t, \\
&B_A^-(x) \leq \max\{B_A^-(x * y), B_A^-(y)\} = \alpha^-, \\
&B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \alpha^+, \\
&J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = s,
\end{aligned}$$

and hence $x \in I$. Therefore I is an ideal of X. □

Theorem 3.16. *For any non-empty subset I of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which is given in Theorem 3.14. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then I is a positive implicative ideal of X.*

Proof. If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X and satisfies (3.3) by Theorem 3.6. It follows from Lemma 3.15 that I is an ideal of X. Let $x, y \in X$ be such that $(x * y) * y \in I$. Then

$$\begin{aligned}
&M_A(x * y) \geq M_A((x * y) * y) = t, B_A^-(x * y) \leq B_A^-((x * y) * y) = \alpha^-, \\
&B_A^+(x * y) \geq B_A^+((x * y) * y) = \alpha^+, J_A(x * y) \leq J_A((x * y) * y) = s,
\end{aligned}$$

and so $x * y \in I$. Therefore I is a positive implicative ideal of X. □

Proposition 3.17. *Every positive implicative BMBJ-neutrosophic ideal* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *of* X *satisfies the following condition.*

$$(((x * y) * y) * a) * b = 0 \;\Rightarrow \begin{cases} M_A(x * y) \geq \min\{M_A(a), M_A(b)\}, \\ B_A^-(x * y) \leq \max\{B_A^-(a), B_A^-(b)\}, \\ B_A^+(x * y) \geq \min\{B_A^+(a), B_A^+(b)\}, \\ J_A(x * y) \leq \max\{J_A(a), J_A(b)\} \end{cases} \tag{3.9}$$

for all $x, y, a, b \in X$.

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative BMBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X (see Theorem 3.3). Let $a, b, x, y \in X$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$M_A(x * y) \geq M_A((x * y) * y) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x * y) \leq \tilde{B}_A((x * y) * y) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x * y) \geq B_A^+((x * y) * y) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and $J_A(x * y) \leq J_A((x * y) * y) \leq \max\{J_A(a), J_A(b)\}$ by Theorem 3.6 and Proposition 3.11. Hence (3.9) is valid. □

Theorem 3.18. *If an MBJ-neutrosophic set* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *in* X *satisfies the conditions* (2.9) *and* (3.9), *then* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *is a positive implicative BMBJ-neutrosophic ideal of* X.

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies the conditions (2.9) and (3.9). It is clear that the condition (2.10) is induced by the condition (3.9). Let $x, a, b \in X$ be such that $x * a \leq b$. Then $(((x * 0) * 0) * a) * b = 0$, and so

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(a), M_A(b)\},$$

$$B_A^-(x) = B_A^-(x * 0) \leq \max\{B_A^-(a), B_A^-(b)\},$$

$$B_A^+(x) = B_A^+(x * 0) \geq \min\{B_A^+(a), B_A^+(b)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1) and (3.9). Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of X by Theorem 3.12. Since $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ for all $x, y \in X$, we have

$$M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),$$

$$B_A^-(x*y) \leq \max\{B_A^-((x*y)*y), B_A^-(0)\} = B_A^-((x*y)*y),$$

$$B_A^+(x*y) \geq \min\{B_A^+((x*y)*y), B_A^+(0)\} = B_A^+((x*y)*y),$$

and

$$J_A(x*y) \leq \max\{J_A((x*y)*y), J_A(0)\} = J_A((x*y)*y)$$

by (3.9). It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

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A new approach for multi-attribute decision-making problems in bipolar neutrosophic sets

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Abstract: In this study, we give a new outranking approach for multi-attribute decision-making problems in bipolar neutrosophic environment. To do this, we firstly propose some outranking relations for bipolar neutrosophic number based on ELECTRE, and the properties in the outranking relations are further discussed in detail. Also, we developed a ranking method based on the outranking relations of for bipolar neutrosophic number. Finally, we give a real example to illustrate the practicality and effectiveness of the proposed method.

Keywords: Single valued neutrosophic sets, single valued bipolar neutrosophic sets, outranking relations, multi-attribute decision making.

1 Introduction

As a generalization of fuzzy set [100] and intuitionistic fuzzy set [1] and so on, neutrosophic set was presented by Smarandache [67, 68] to capture the incomplete, indeterminate and inconsistent information. The neutrosophic set have three completely independent parts, which are truth-membership degree, indeterminacy-membership degree and falsity-membership degree, therefore it is applied to many different areas, such as deci-sion making problems [2, 16, 24]. In additionally, since the neutrosophic sets are hard to be apply in some real problems because of the truth-membership degree, indeterminacy-membership degree and falsity-membership degree lie in $]^{-}0,\ 1^{+}[$, single valued neutrosophic set, as a example of the neutrosophic set introduced by Wang et al. [73].

Recently, Lee [27, 28] proposed notation of bipolar fuzzy set and their operations based on fuzzy sets. A bipolar fuzzy set have a $T^{+} \rightarrow [0,1]$ and $T^{-} \rightarrow [-1,0]$ is called positive membership degree and negative membership degree $T^{-}(u)$. Also the bipolar fuzzy models have been studied by many authors both theory and application in [17, 20, 30, 69, 98]. After the definition of Smarandache's neutrosophic set, neutrosophic sets and neutrosophic logic have been applied in many real applications to handle uncertainty. The neutrosophic set uses one single value in $]^{-}0, 1^{+}[$ to represent the truth-membership degree, indeterminacy-membership degree and falsity-membership degree of a element in the universe X. Then, Deli et al. [23] introduced the concept of bipolar neutrosophic sets, as an extension of neutrosophic sets. In the bipolar neutrosophic sets, the positive membership degree $T^{+}(x)$, $I^{+}(x)$, $F^{+}(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^{-}(x)$, $I^{-}(x)$, $F^{-}(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Similarity measure is an important tool in constructing multi-criteria decision making methods in many areas such as medical diagnosis, pattern recognition, clustering analysis, decision making and so on. Similarity measures under all sorts of fuzzy environments including single-valued neutrosophic environments have been studied by many researchers in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 26, 31, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 79, 80, 81, 82, 83, 84, 85, 86, 85, 86, 87, 88, 89, 90, 65, 91, 92, 93, 94, 95, 96]. Also, S¸ahin et al.[72] presented a similarity measure on bipolar neutrosophic sets based on Jaccard vector similarity measure of neutrosophic set and applied to a decision making problem.

This paper is constructed as follows. In Sect. 2, some basic definitions of neutrosophic sets and bipolar neutrosophic sets are introduced. In Sect. 3, we propose the outranking relations of bipolar neutrosophic sets and investigate its several proprieties. In Sect. 4, an outranking approach for MCDM with simplified bipolar neutrosophic information is given. In Sect. 5, Illustrative examples is given. In Sect. 6, the conclusions are summarized.

2 Preliminary

In the subsection, we give some concepts related to neutrosophic sets and bipolar neutrosophic sets.

Definition 2.1. [67] Let E be a universe. A neutrosophic sets A over E is defined by

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle : x \in E\}.$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are called truth-membership function, indeterminacy-membership function and falsity-membership function, respectively. They are respectively defined by

$$T_A : E \to]^{-}0, 1^{+}[, \quad I_A : E \to]^{-}0, 1^{+}[, \quad F_A : E \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$.

Definition 2.2. [73] Let E be a universe. An single valued neutrosophic set (SVN-set) over E is a neutrosophic set over E, but the truth-membership function, indeterminacy-membership function and falsity-membership function are respectively defined by

$$T_A : E \to [0, 1], \quad I_A : E \to [0, 1], \quad F_A : E \to [0, 1]$$

such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.3. [23] A bipolar neutrosophic set A in X is defined as an object of the form

$$A = \{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\}.$$

where

$$T^+, I^+, F^+ : E \to [0,1], T^-, I^-, F^- : X \to [-1,0].$$

The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.4. [23] Let $A_1 = \langle x, T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x)\rangle$ and $A_2 = \langle x, T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x)\rangle$ be two bipolar neutrosophic sets in a universe of discourse X, then the following operations are defined as follows:

1. $A_1 = A_2$ if and only if $T_1^+(x) = T_2^+(x), I_1^+(x) = I_2^+(x), F_1^+(x) = F_2^+(x)$ and $T_1^-(x) = T_2^-(x), I_1^-(x) = I_2^-(x), F_1^-(x) = F_2^-(x)$.

2. $$\begin{aligned} A_1 \cup A_2 = \ & \{\langle x, max(T_1^+(x), T_2^+(x)), \tfrac{I_1^+(x)+I_2^+(x)}{2}, min(F_1^+(x), F_2^+(x)), \\ & min(T_1^-(x), T_2^-(x)), \tfrac{I_1^-(x)+I_2^-(x)}{2}, max(F_1^-(x), F_2^-(x))\rangle\} \end{aligned}$$
   $\forall x \in X$.

3. $$\begin{aligned} A_1 \cap A_2 = \ & \{\langle x, min(T_1^+(x), T_2^+(x)), \tfrac{I_1^+(x)+I_2^+(x)}{2}, max(F_1^+(x), F_2^+(x)), \\ & max(T_1^-(x), T_2^-(x)), \tfrac{I_1^-(x)+I_2^-(x)}{2}, min(F_1^-(x), F_2^-(x))\rangle\} \end{aligned}$$
   $\forall x \in X$.

4. $$A^c = \ \{\langle x, 1 - T_A^+(x), 1 - I_A^+(x), 1 - F_A^+(x), 1 - T_A^-(x), 1 - I_A^-(x), 1 - F_A^-(x)\rangle\}$$

5. $A_1 \subseteq A_2$ if and only if $T_1^+(x) \le T_2^+(x), I_1^+(x) \le I_2^+(x), F_1^+(x) \ge F_2^+(x)$ and $T_1^-(x) \ge T_2^-(x), I_1^-(x) \ge I_2^-(x), F_1^-(x) \le F_2^-(x)$.

Definition 2.5. [23] Let $\tilde{a}_1 = \langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-\rangle$ and $\tilde{a}_2 = \langle T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-\rangle$ be two bipolar neutrosophic number . Then the operations for BNNs are defined as below;

$i.$ $\lambda\tilde{a}_1 = \langle 1-(1-T_1^+)^\lambda, (I_1^+)^\lambda, (F_1^+)^\lambda, -(-T_1^-)^\lambda, -(-I_1^-)^\lambda, -(1-(1-(-F_1^-))^\lambda)\rangle$

$ii.$ $\tilde{a}_1^\lambda = \langle (T_1^+)^\lambda, 1-(1-I_1^+)^\lambda, 1-(1-F_1^+)^\lambda, -(1-(1-(-T_1^-))^\lambda), -(-I_1^-)^\lambda, -(-F_1^-)^\lambda\rangle$

$iii.$ $\tilde{a}_1 + \tilde{a}_2 = \langle T_1^+ + T_2^+ - T_1^+T_2^+, I_1^+I_2^+, F_1^+F_2^+, T_1^-T_2^-, -(-I_1^- - I_2^- - I_1^-I_2^-), -(-F_1^- - F_2^- - F_1^-F_2^-)\rangle$

$iv.$ $\tilde{a}_1 + \tilde{a}_2 = \langle T_1^+T_2^+, I_1^+ + I_2^+ - I_1^+I_2^+, F_1^+ + F_2^+ - F_1^+F_2^+, -(-T_1^- - T_2^- - T_1^-T_2^-, -I_1^-I_2^-, -F_1^-F_2^-\rangle$

where $\lambda > 0$.

Definition 2.6. [32] Let $A = \langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and

$B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ be any two SVNSs, then the normalized Euclidean distance between A and B can be defined as follows:

$$d(A,B) = \sqrt{\frac{1}{3n}(|\tilde{T}_A - \tilde{T}_B|^2 + |\tilde{I}_A - \tilde{I}_B|^2 + |\tilde{F}_A - \tilde{F}_B|^2}.$$

3 The outranking relations of Bipolar Neutrosophic Sets

In this section, The binary relations between two bipolar neutrosophic sets that are based on ELECTRE are now defined.

Definition 3.1. Let $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$ and $B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ be two BNSs in the set $X = \{x_1, x_2, ..., x_n\}$. Then,then the strong dominance relation, weak dominance relation, and indifference relation of BNSs can be defined as follows:

1. If $T_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- > F_B^-$ or $T_A^+ > T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- < T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$,then A strongly dominates B (B is strongly dominated by A), denoted by $A \succ_s B$.

2. If $T_A^+ \geq T_B^+, I_A^+ \geq I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- \leq I_B^-, F_A^- > F_B^-$ or $T_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ \geq F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- \leq F_B^-$,then A weakly dominates B (B is weakly dominated by A), denoted by $A \succ_w B$.

3. If $T_A^+ = T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- = T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$,then A is indifferent to B, denoted by$A \sim_l B$.

4. If none of the relations mentioned above exist between A and B for any $x \in X$, then A and B are incomparable, denoted by $A \perp B$.

Proposition 3.2. *Let $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$ and $B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ be two BNSs in the set $X = \{x_1, x_2, ..., x_n\}$, then the following properties can be obtained:*

1. *If $B \subset A$,then $A \succ_s B$;*

2. *If $A \succ_s B$, then $B \subseteq A$;*

3. *$A \sim_l B$ if and only if $A = B$.*

Proof. 1. If $B \subset A$,then $T_B^+ < T_A^+, I_B^+ > I_A^+, F_A^+ > F_B^+, T_B^- > T_A^-, I_A^- < I_B^-, F_B^- < F_A^-$. $A \succ_s B$ is definitely validated according to the strong dominance relation in Definition 3.1.

2. $A \succ_s B$, then based on Definition 3.1,$T_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- > F_B^-$ or $T_A^+ > T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- < T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$ are realized. From Definition 2.4.

3. Necessity:$A \sim_l B \Rightarrow A = B$. According to the indifference relation in Definition 3.1 it is known that it is known that$T_A^+ = T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- = T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$. Clearly $A \subseteq B$ and $B \subseteq A$ are achieved, then $A = B$.

   Sufficiency:$A = B \Rightarrow A \sim_l B$. If $A = B$,then it is known that $A \subseteq B$ and $B \subseteq A$, which means $T_A^+ \leq T_B^+, I_A^+ \geq I_B^+, F_A^+ \geq F_B^+, T_A^- \geq T_B^-, I_A^- \leq I_B^-, F_A^- \leq F_B^-$ and $T_A^+ \geq T_B^+, I_A^+ \leq I_B^+, F_A^+ \leq F_B^+, T_A^- \leq T_B^-, I_A^- \geq I_B^-, F_A^- \geq F_B^-$; then $T_A^+ = T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- = T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$, are obtained. Due to the indifference relation in Definition 3.1, $A \sim_l B$ is definitely validated.

□

Proposition 3.3. *Let* $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$, $B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ *and* $C = \langle T_C^+(x_i), I_C^+(x_i), F_C^+(x_i), T_C^-(x_i), I_C^-(x_i), F_C^-(x_i)\rangle$ *and be three BNSs in the set* $X = \{x_1, x_2, ..., x_n\}$*, if* $A \succ_s B$ *and* $B \succ_s C$*,then* $A \succ_s C$.

Proof. According to the strong dominance relation in Definition3.1,if $A \succ_s B$, then $T_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- > F_B^-$ or $T_A^+ > T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- < T_B^-, I_A^- = I_B^-, F_A^- = F_B^-$. If $B \succ_s C$, then $T_B^+ \geq T_C^+, I_B^+ < I_C^+, F_B^+ < F_C^+, T_B^- \leq T_C^-, I_B^- > I_C^-, F_B^- > F_C^-$ or $T_B^+ > T_C^+, I_B^+ = I_C^+, F_B^+ = F_C^+, T_B^- < T_C^-, I_B^- = I_C^-, F_B^- = F_C^-$. Therefore the further derivations are:

$$IfT_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- > F_B^- \quad (1)$$

$$T_B^+ \geq T_C^+, I_B^+ < I_C^+, F_B^+ < F_C^+, T_B^- \leq T_C^-, I_B^- > I_C^-, F_B^- > F_C^- \quad (2)$$

from (1) and (2)

$$T_A^+ \geq T_C^+, I_A^+ < I_C^+, F_A^+ < F_C^+, T_A^- \leq T_C^-, I_A^- > I_C^-, F_A^- > F_C^-$$

then based on Definition 3.1$A \succ_s C$ is realized.

$$IfT_A^+ \geq T_B^+, I_A^+ < I_B^+, F_A^+ < F_B^+, T_A^- \leq T_B^-, I_A^- > I_B^-, F_A^- > F_B^- \quad (3)$$

$$T_B^+ > T_C^+, I_B^+ = I_C^+, F_B^+ = F_C^+, T_B^- < T_C^-, I_B^- = I_C^-, F_B^- = F_C^- \quad (4)$$

from (3) and (4)

$$T_A^+ > T_C^+, I_A^+ = I_C^+, F_A^+ = F_C^+, T_A^- < T_C^-, I_A^- = I_C^-, F_A^- = F_C^-$$

then based on Definition 3.1$A \succ_s C$ is achieved.

$$IfT_A^+ > T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- < T_B^-, I_A^- = I_B^-, F_A^- = F_B^- \quad (5)$$

$$T_B^+ \geq T_C^+, I_B^+ < I_C^+, F_B^+ < F_C^+, T_B^- \leq T_C^-, I_B^- > I_C^-, F_B^- > F_C^- \quad (6)$$

from (5) and (6)

$$T_A^+ > T_C^+, I_A^+ < I_C^+, F_A^+ < F_C^+, T_A^- < T_C^-, I_A^- > I_C^-, F_A^- > F_C^-$$

then based on Definition 3.1$A \succ_s C$ is obtained.

$$IfT_A^+ > T_B^+, I_A^+ = I_B^+, F_A^+ = F_B^+, T_A^- < T_B^-, I_A^- = I_B^-, F_A^- = F_B^- \quad (7)$$

$$T_B^+ \geq T_C^+, I_B^+ = I_C^+, F_B^+ = F_C^+, T_B^- \leq T_C^-, I_B^- = I_C^-, F_B^- = F_C^- \quad (8)$$

from (7) and (8)

$$T_A^+ > T_C^+, I_A^+ = I_C^+, F_A^+ = F_C^+, T_A^- < T_C^-, I_A^- = I_C^-, F_A^- = F_C^-$$

then based on Definition 3.1$A \succ_s C$ is realized. Therefore, if $A \succ_s B$ and $B \succ_s C$,then $A \succ_s C$. □

Proposition 3.4. *Let* $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$,
$B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ *and* $C = \langle T_C^+(x_i), I_C^+(x_i), F_C^+(x_i),$
$T_C^-(x_i), I_C^-(x_i), F_C^-(x_i)\rangle$ *and be three BNSs in the set* $X = \{x_1, x_2, ..., x_n\}$*, if* $A \sim_l B$ *and* $B \sim_l C$*,then* $A \sim_l C$.

Proof. Clearly,if $A \sim_l B$ and $B \sim_l C$,then $A \sim_l C$ is surely validated. □

Proposition 3.5. *Let* $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$,
$B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ *and* $C = \langle T_C^+(x_i), I_C^+(x_i), F_C^+(x_i),$
$T_C^-(x_i), I_C^-(x_i), F_C^-(x_i)\rangle$ *and be three BNSs in the set* $X = \{x_1, x_2, ..., x_n\}$*, then the following results can be achieved.*

(1).*The strong dominance relations are categorized into:*

$$\begin{aligned}&1.irreflexivity: \forall A \in BNSs, A \not\succ_s A;\\&2.asymmetry: \forall A, B \in BNSs, A \succ_s B \Rightarrow B \not\succ_s A;\\&3.transitivity: \forall A, B, C \in BNSs, A \succ_s B, B \succ_s C \Rightarrow A \succ_s C.\end{aligned}$$

(2).*The weak dominance relations are categorized into:*

$$\begin{aligned}&4.irreflexivity: \forall A \in BNSs, A \not\succ_w A;\\&5.asymmetry: \forall A, B \in BNSs, A \succ_w B \Rightarrow B \not\succ_w A;\\&6.non-transitivity \exists A, B, C \in BNSs, A \succ_w B, B \succ_w C \Rightarrow A \succ_w C.\end{aligned}$$

(3).*The indifference relations are categorized into:*

$$\begin{aligned}&7.reflexivity: \forall A \in BNSs, A \sim_l A;\\&8.symmetry: \forall A, B \in BNSs, A \sim_l B \Rightarrow B \sim_l A;\\&9.transitivity \exists A, B, C \in BNSs, A \sim_l B, B \sim_l C \Rightarrow A \sim_l C.\end{aligned}$$

According to Definition 3.1,it is clear that $3, 7, 8$ *and* 9 *are true, and* $1, 2, 4, 5$ *and* 6 *need to be proven.*

Example 3.6. $1, 2, 4, 5$ and 6 are exemplified as follows.

1. If $A = \langle 0.5, 0.3, 0.1, -0.6, -0.4, -0.2\rangle$ is a BNSs, then $A \not\succ_s A$ can be obtained.

2. If $A = \langle 0.7, 0.4, 0.2, -0.5, -0.2, -0.1\rangle$ and $B = \langle 0.6, 0.5, 0.3, -0.4, -0.3, -0.2\rangle$ are two BNSs, then $A \succ_s B$,but $B \not\succ_s A$ is achieved.

3. If $A = \langle 0.5, 0.3, 0.1, -0.6, -0.4, -0.2\rangle$ is a BNSs, then $A \not\succ_w A$ is realized.

4. If $A = \langle 0.8, 0.5, 0.2, -0.6, -0.3, -0.3\rangle$ and $B = \langle 0.5, 0.5, 0.3, -0.4, -0.4, -0.2\rangle$ are two BNSs, then $A \succ_w B$ is obtained, however $B \not\succ_w A$.

5. If $A = \langle 0.8, 0.5, 0.4, -0.6, -0.5, -0.3\rangle$, $B = \langle 0.7, 0.2, 0.5, -0.5, -0.3, -0.2\rangle$ and $C = \langle 0.7, 0.4, 0.4, -0.3, -0.3, -0.1\rangle$ are three BNSs, then $A \succ_w B$ and $B \succ_w C$ are achieved, however $A \perp C$.

Proposition 3.7. *Let x_1 and x_2 be two actions, the performances for actions x_1 and x_2 be in the form of BNSs, and $P = s \cup w \cup l$ mean that " x_1 is at least as good as x_2",then four situations may arise:*

1. *x_1Px_2 and not x_2Px_1,that is $x_1 \succ_s x_2$ or $x_1 \succ_w x_2$;*
2. *x_2Px_1 and not x_1Px_2,that is $x_2 \succ_s x_1$ or $x_2 \succ_w x_1$;*
3. *x_1Px_2 and x_2Px_1,that is $x_1 \sim_l x_2$*
4. *not x_1Px_2 and not x_2Px_1,that is $x_1 \perp x_2$.*

Definition 3.8. Let $A = \langle T_A^+(x_i), I_A^+(x_i), F_A^+(x_i), T_A^-(x_i), I_A^-(x_i), F_A^-(x_i)\rangle$, and $B = \langle T_B^+(x_i), I_B^+(x_i), F_B^+(x_i), T_B^-(x_i), I_B^-(x_i), F_B^-(x_i)\rangle$ and and be two BNSs, then the normalized Euclidean distance between A and B can be defined as follows:

$$d(A,B) = \sqrt{\frac{1}{6n}[(|T_A^+ - T_B^+|^2 + |I_A^+ - I_B^+|^2 + |F_A^+ - F_B^+|^2) - (|T_A^- - T_B^-|^2 + |I_A^- - I_B^-|^2 + |F_A^- - F_B^-|^2)]}.$$

Proposition 3.9. *Let $d(A,B)$ be a normalized Euclidean distance between bipolar neutrosophic sets A and B. Then, we have*

1. *$0 \leq d(A,B) \leq 1$;*
2. *$d(A,B) = d(B,A)$;*
3. *$d(A,B) = 1$ for $A = B$ i.e.,$T_A^+(x_i) = T_B^+(x_i), I_A^+(x_i) = I_B^+(x_i), F_A^+(x_i) = F_B^+(x_i), T_A^-(x_i) = T_B^-(x_i), I_A^-(x_i) = I_B^-(x_i), F_A^-(x_i) = F_B^-(x_i)(i = 1,2...,n)\ \forall x_i(i = 1,2,...,n) \in X$.*

Proof. 1. It is clear from Definition 2.3.
2.

$$\begin{aligned} d(A,B) &= \sqrt{\tfrac{1}{6n}[(|T_A^+ - T_B^+|^2 + |I_A^+ - I_B^+|^2 + |F_A^+ - F_B^+|^2) - (|T_A^- - T_B^-|^2 + |I_A^- - I_B^-|^2 + |F_A^- - F_B^-|^2)]} \\ &= \sqrt{\tfrac{1}{6n}[(|T_B^+ - T_A^+|^2 + |I_B^+ - I_A^+|^2 + |F_A^+ - F_A^+|^2) - (|T_B^- - T_A^-|^2 + |I_B^- - I_A^-|^2 + |F_B^- - F_A^-|^2)]} \\ &= d(B,A) \end{aligned}$$

3. Since $T_A^+(x_i) = T_B^+(x_i), I_A^+(x_i) = I_B^+(x_i), F_A^+(x_i) = F_B^+(x_i), T_A^-(x_i) = T_B^-(x_i), I_A^-(x_i) = I_B^-(x_i), F_A^-(x_i) = F_B^-(x_i)(i = 1,2...,n)\ \forall x_i(i = 1,2,...,n) \in X$, we have $d(A,B) = 1$.
The proof is completed. □

4 An outranking approach for MCDM with simplified bipolar neutrosophic information

Definition 4.1. The MCDM ranking/selection problems with simplified BNSs information consist of a group of alternatives, denoted by $U = (u_1, u_2, ..., u_n)$ be a set of alternatives, $A = (a_1, a_2, ..., a_m)$ be the set of attributes, $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of the attributes $C_j(j = 1,2,...,n)$ such that $w_j \geq 0$

and $\sum_{j=1}^{n} = 1$ and $b_{ij} = \langle T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}, T_{ij}^{-}, I_{ij}^{-}, F_{ij}^{-} \rangle$ be the decision matrix in which the rating values of the alternatives. Then,

$$[b_{ij}]_{m\times n} = \begin{matrix} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{matrix} \begin{matrix} \begin{matrix} u_1 & u_2 & \cdots & u_n \end{matrix} \\ \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \end{matrix}$$

is called an NB-multi-attribute decision making matrix of the decision maker.

This method is an integration of BNSs and the outranking method to manage the MCDM problems mentioned above. In general, there are benefit criteria and cost criteria in MCDM problems and the cost-type criterion values can be transformed into benefit-type criterion values as follows:

$$\beta_{ij} = \begin{cases} b_{ij} & for\ benefit\ criterion\ a_j, \quad (i = 1, 2, ..., m; j = 1, 2, .., n) \\ (b_{ij})^c & for\ cost\ criterion\ a_j, \quad ...(9) \end{cases}$$

here $(b_{ij})^c$ is complement of b_{ij} as defined in Definition 2.4.

The analysis given above indicates that both c_{ik} and d_{ik} include the weights of the criteria and the outranking relations among the alternatives. However, they measure different aspects of the relations, and the concordance indices and discordance indices are therefore not complementary.

To rank all alternatives, the net dominance index of b_k

$$c_k = \sum_{i=1; i\neq k}^{n} c_{ik} - \sum_{i=1; i\neq k}^{n} c_{ki}, ...(10)$$

and the net disadvantage index of b_k is

$$d_k = \sum_{i=1; i\neq k}^{n} d_{ik} - \sum_{i=1; i\neq k}^{n} d_{ki}, ...(11)$$

Here, c_k is the sum of the concordance indices between b_k and $b_k(i \neq k)$ minus the sum of the concordance indices between $b_k(i \neq k)$ and b_k, and reflects the dominance degree of the alternative b_k among the relevant alternatives. Meanwhile, d_k reflects the disadvantage degree of the alternative b_k among the relevant alternatives. Therefore, b_k obtains a greater dominance over the other alternatives that are being compared as c_k increases and d_k decreases.

Definition 4.2. The ranking rules of two alternatives are

$$\begin{array}{rl} i. & if\ c_i < c_k\ and\ d_i > d_k,\ then\ b_k\ is\ superior\ to\ b_i,\ as\ denoted\ by\ b_k \succ b_i; \\ ii. & if\ c_i = c_k\ and\ d_i = d_k,\ then\ b_k is\ indifferent\ to b_i, as\ denoted\ by\ b_k \sim b_i; \\ iii. & if\ the\ relation\ between\ b_k\ and\ b_i\ does\ not\ belong\ to\ (i)\ or\ (ii), \\ & then\ b_k\ and\ b_i\ are\ incomparable,\ as\ denoted\ by b_k \perp b_i. \end{array}$$

A ranking of alternatives obtained by the rules defined above may be only a partial ranking, and greater detail is discussed by Wu and Chen [76]

It is now feasible to develop a new approach for the MCDM problems mentioned above.

Algorithm:

Step 1. Give the decision-making matrix $[b_{ij}]_{m\times n}$; for decision; The BNSs decision matrix $R = [b_{ij}]_{m\times n}$ can be transformed into a normalized BNSs decision matrix $R = [\beta_{ij}]_{m\times n}$ based on Eq. (9).

Step 2. Determine the weighted normalized matrix. According to the weight vector for the criteria, the weighted normalized decision matrix can be constructed using the following formula:

$$\gamma_{ij} = \beta_{ij} w_j, \qquad i = 1, 2, ..., m; j = 1, 2, ..., n.$$

where w_j is the weight of the j th criterion with $\sum_{j=1}^{n} w_j = 1$.

Step 3. Determine the concordance and discordance set of subscripts. The concordance set of subscripts, which should satisfy the constraint $b_{ij} P b_{kj}$, is represented as:

$$O_{ik} = \{j | b_{ij} P b_{kj} \quad (i, k = 1, 2, ..., m).$$

$b_{ij} P b_{kj}$ represents $b_{ij} >_s b_{kj}$ or $b_{ij} >_w b_{kj}$ or $b_{ij} \sim b_{kj}$.

The discordance set of subscripts for criteria is the complementary subset, therefore:

$$D_{ik} = J - O_{ik}.$$

Step 4. Determine the concordance and discordance matrix. By using the weight vector w that is associated with the criteria, the concordance index $C(b_i, b_k)$ is represented as:

$$C(b_i, b_k) = \sum_{j \in O_{ik}} w_j.$$

Thus, the concordance matrix C is:

$$C = \begin{pmatrix} - & c_{12} & \cdots & c_{1n} \\ c_{21} & - & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & - \end{pmatrix}$$

The discordance index $D(b_i, b_k)$ is represented as:

$$D_{ik} = \frac{\max_{j \in D_{ik}} \{d(b_{ij}, b_{kj})\}}{\max_{j \in J} \{d(b_{ij}, b_{kj})\}}$$

here $d(b_{ij}, b_{kj})$ denotes the normalized Euclidean distance between b_{ij} and b_{kj} as defined in Definition 3.8.

Thus, the discordance matrix D is:

$$D = \begin{pmatrix} - & d_{12} & \cdots & d_{1n} \\ d_{21} & - & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & - \end{pmatrix}$$

Step 5. Calculate the net dominance index of each alternative c_i (i=1,2,...,m) based on Formula (10), , and the net disadvantage index of each alternative d_i (i=1,2,...,m) based on Formula (11).

Step 6. Formulate the ranking of all alternatives in light of the rules given by Definition 4.2

5 Illustrative examples

In this section, an example for a MCDM problem with simplified bipolar neutrosophic information.

Example 5.1. ([37]).There is an investment company, which wants to invest a sum of money in the best option. This company has set up a panel which has to choose between four possible alternatives for investing the money:(1) b_1 is a car company; (2) b_2 is a food company; (3) b_3 is a computer company; and (4) b_4 is an arms company. The investment company must make a decision using the following three criteria: (1) a_1 is the risk; (2) a_2 is the growth; and (3) a_3 is the customer satisfaction; these are all benefit type criteria. The weight vector of the criteria is represented by $w = \{0.45, 0.15, 0.4\}$. The four possible alternatives are to be evaluated under the above three criteria in the form of BNNs for each decision-maker, as shown in the following simplified bipolar neutrosophic decision matrix R:

$$R = \begin{pmatrix} \langle 0.7, 0.5, 0.3, -0.3, -0.4, -0.5\rangle & \langle 0.8, 0.6, 0.1, -0.5, -0.3, -0.2\rangle & \langle 0.4, 0.6, 0.5, -0.2, -0.6, -0.4\rangle \\ \langle 0.6, 0.1, 0.4, -0.4, -0.3, -0.6\rangle & \langle 0.6, 0.1, 0.3, -0.4, -0.3, -0.1\rangle & \langle 0.5, 0.7, 0.3, -0.1, -0.2, -0.5\rangle \\ \langle 0.8, 0.6, 0.8, -0.3, -0.2, -0.1\rangle & \langle 0.9, 0.4, 0.5, -0.5, -0.3, -0.6\rangle & \langle 0.3, 0.4, 0.5, -0.2, -0.3, -0.4\rangle \\ \langle 0.8, 0.3, 0.1, -0.4, -0.2, -0.1\rangle & \langle 0.6, 0.1, 0.4, -0.3, -0.2, -0.3\rangle & \langle 0.6, 0.5, 0.6, -0.3, -0.4, -0.6\rangle \end{pmatrix}$$

The procedures for obtaining the best alternative are now outlined.

Step 1. Transform the decision matrix.

Since all the criteria are of the benefit type,$R' = R$ can be obtained.

Step 2. Determine the weighted normalized matrix.

$$R' = \begin{pmatrix} \langle 0.4128, 0.7320, 0.5817, -0.5817, -0.6621, -0.2679\rangle \\ \langle 0.3378, 0.3548, 0.6621, -0.6621, -0.5817, -0.3378\rangle \\ \langle 0.5153, 0.7946, 0.9044, -0.5817, -0.4846, -0.0463\rangle \\ \langle 0.5153, 0.5817, 0.3548, -0.6621, -0.4846, -0.0463\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle 0.2144, 0.9262, 0.7079, -0.9012, -0.8347, -0.0329\rangle \\ \langle 0.1284, 0.7079, 0.8347, -0.8715, -0.8347, -0.0156\rangle \\ \langle 0.2920, 0.8715, 0.9012, -0.9012, -0.8347, -0.1284\rangle \\ \langle 0.1284, 0.7079, 0.8715, -0.8347, -0.7855, -0.0521\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle 0.1848, 0.8151, 0.7578, -0.5253, -0.8151, -0.1848\rangle \\ \langle 0.2421, 0.8670, 0.6178, -0.3981, -0.5253, -0.2421\rangle \\ \langle 0.1329, 0.6931, 0.7578, -0.5253, -0.6178, -0.1848\rangle \\ \langle 0.3068, 0.7578, 0.8151, -0.6178, -0.6931, -0.3068\rangle \end{pmatrix}$$

Step 3. Determine the concordance and discordance set of subscripts.

The concordance set of subscripts is obtained as follows:

$$O_{12} = \{1, 2\}; O_{21} = \{3\}; O_{31} = \{2\}; O_{41} = \{1, 3\}; O_{13} = \{3\}; O_{23} = \{3\};$$

$$O_{32} = \{\}; O_{42} = \{1, 2, 3\}; O_{14} = \{2\}; O_{24} = \{2\}; O_{34} = \{\}; O_{43} = \{1, 2, 3\}.$$

The discordance set of subscripts is obtained as follows:

$$D_{12} = \{3\}; D_{21} = \{1, 2\}; D_{31} = \{1, 3\}; D_{41} = \{2\}; D_{13} = \{1, 2\}; D_{23} = \{1, 2\};$$

$$D_{32} = \{1, 2, 3\}; D_{42} = \{\}; D_{14} = \{1, 3\}; D_{24} = \{1, 3\}; D_{34} = \{1, 2, 3\}; D_{43} = \{\}.$$

where $\{\}$ denotes "empty".

Step 4. Determine the concordance and discordance matrix.

With regard to the weight vector w associated with the criteria, the concordance index is represented as follows:

$$C = \begin{pmatrix} - & 0.60 & 0.40 & 0.15 \\ 0.40 & - & 0.40 & 0.15 \\ 0.15 & 0 & - & 0 \\ 0.85 & 1 & 1 & - \end{pmatrix}$$

The discordance index can be calculated as follows. For example,

$$D_{21} = \frac{max\{d(b_{21}, b_{11}), d(b_{22}, b_{12})\}}{max\{d(b_{21}, b_{11}), d(b_{22}, b_{12}), d(b_{23}, b_{13})\}} = \frac{0.10334}{0.31501} = 0.3280$$

Here:

$$\begin{aligned} d(b_{21}, b_{11}) &= (\tfrac{1}{6}|((0.3378-0.4182)^2+(0.3548-0.7320)^2+(0.6621-0.5817)^2) \\ &-((-0.6621-(-0.5817))^2+(-0.5817-(-0.6621))^2+(-0.3378-(-0.2679))^2)|)^{\frac{1}{2}} \\ &= 0.08973 \end{aligned}$$

$$\begin{aligned} d(b_{22}, b_{12}) &= (\tfrac{1}{6}|((0.1284-0.2144)^2+(0.7079-0.9262)^2+(0.8347-0.7079)^2) \\ &-((-0.8715-(-0.9012))^2+(-0.8347-(-0.8347))^2+(-0.0156-(-0.0329))^2)|)^{\frac{1}{2}} \\ &= 0.10334; \end{aligned}$$

and

$$\begin{aligned} d(b_{23}, b_{13}) &= (\tfrac{1}{6}|((0.2421-0.1848)^2+(0.8670-0.8151)^2+(0.6178-0.7578)^2) \\ &-((-0.3981-(-0.5253))^2+(-0.5253-(-0.8151))^2+(-0.2421-(-0.1848))^2)|)^{\frac{1}{2}} \\ &= 0.31501; \end{aligned}$$

Therefore, the discordance index matrix is as follows:

$$D = \begin{pmatrix} - & 0,6230 & 1 & 1 \\ 0.3280 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 0 & 0 & - \end{pmatrix}$$

Step 5. Based on Formulae (10) and (11), the net dominance index of each alternative c_i (i=1,2,3,4) and the net disadvantage index of each alternative d_i (i=1,2,3,4) can be obtained as shown below:

$$c_1 = -0.25, c_2 = -0.65, c_3 = -1.65 \; and \; c_4 = 2.55, \;\Rightarrow\; c_3 < c_2 < c_1 < c_4;$$

$$d_1 = 0.295, d_2 = 0.705, d_3 = 1 \; and \; d_4 = -3 \;\Rightarrow\; d_3 > d_2 > d_1 > d_4.$$

Step 6. According to the rules of Definition 4.2, the final ranking is $b_4 \succ b_1 \succ b_2 \succ b_3$, and the best alternative is b_4.

6 Conclusions

This paper developed a multi-criteria decision making method for bipolar neutrosophic set is developed based on these given the outranking relations. The contribution of this study is that the proposed approach is simple and convenient with regard to computing,and effective in decreasing the loss of evaluative information. More effective decision methods of this proposes a new outranking approach will be investigated in the near future and applied these concepts to engineering, game theory, multi-agent systems, decision-making and so on.

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Sinos River basin social-environmental prospective assessment of water quality management using fuzzy cognitive maps and neutrosophic AHP-TOPSIS

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Abstract.The Sinos River basin (SRB) is one of the most polluted river basins in Brazil, leading to considerable efforts to mitigate the impacts and achieve their recovery, is possible through adequate integral management. Aiming at the need for water quality management through the analysis of the interrelationships among the different factors, which can be difficult given the multiple connections between the variables involved. In this article, the authors presented a tool of multi-criteria decision method using Neutrosophic elements in AHP-TOPSIS models and linked to Fuzzy Cognitive Maps, which can contribute to better environmental management to be carried out by the Basin Management Committee of the Sino River. This method, it is possible for modeling the complex system and variables involved inthe determination of water quality, according to the Water Quality IndexandusingNeutrosophic Analytical Hierarchical Process (NAHP) with Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) for ranking scenarios. The methodology exposed in this research shows an improved method to be used by the SRB Committee when planning decisions. The applicability of the framework has been demonstrated during the case study presented.

Keywords:FCM, Neutrosophic AHP, TOPSIS,Sino River Basin, Scenario Analysis

1 Introduction

The complexity of socio-environmental management in a water basin, strongly impacted by anthropic actions, is manifested in a considerable number of environmental problems that affect the health and well-being of the populations of the region. Additionally,altered the biological diversity the abiotic components of the ecosystem are eroded [1].

The present work focuses on the managing of water quality through the analysis of the interrelations between the different factors. Considering the variables that compose the Water Quality Index (WQI) on the SinosRiver basin(SRB)[2] and how they are influenced by actions such asincrease of industrial and domestic wastewater treatment, improve legal systems and law enforcement, conservation or recovery of the gallery forest, wetlands and swamplands areas. Moreover, the degree of the impacts to the biota, the people health,and the regional economy is determined.

The proposal consistsusingtheFuzzy Cognitive Maps (FCM) [3] as a tool to understand the complex nature of environmental management, making easier the analysis of existent interrelations, the discussion, and understanding of the problems complexity by the stakeholders, contributing to making the basin management process more democratic. Additionally, the combination with the Neutrosophic Analytical Hierarchical Process(NAHP)[4-6] and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [7] multi-criteria method allows making prospective management when analyzing and ranking of different scenarios.

Most noteworthy, this is the first study, to our knowledge, that integrates FCM with Neutrosophic NAHP-TOPSIS for water management. All these factsallow the analysis of different alternatives, ranking them and selecting the best one, optimizing the decision-making processes into the social-environmental management by the SRB Committee.

The paper continues as follows: Section 2 is about the SRB and his environmental issue and some important concepts about fuzzy cognitive maps, AHP and TOPSIS. Methodological aspects are detailed in section 3. A case studyisdiscussed and presented insection 4. Thisarticle ends with inferences and some recommendation for future work.

2 Preliminary

In this section the SRB and its environmental problems are presented then FCM fundamentals are discussed. Additionally necessary concepts about neutrosophic AHP and TOPSIS are presented.

2.1 The SRB social-environmental water problems

The SRB is one of the most polluted water basins in Brazil [8] which leads to tremendous efforts for its recovery through adequate integral management. The SRB Committee is responsible for the environmental management but, due to the complex nature of the interrelations between the different factors involved in environmental quality management becomes intricate and therefore requires the use of tools that facilitate decision making.

The SRB (Fig. 1),positioned in the eastern portion of the Rio Grande do Sul State. It has an area of approximately 3.696 km², equivalent to 1.3% of the total area of the Rio Grande do Sul State and 4.4% of the Guaíba Hydrographic Region [9], providing fresh water to nearly 1.3 million people in 32 municipalities [10].

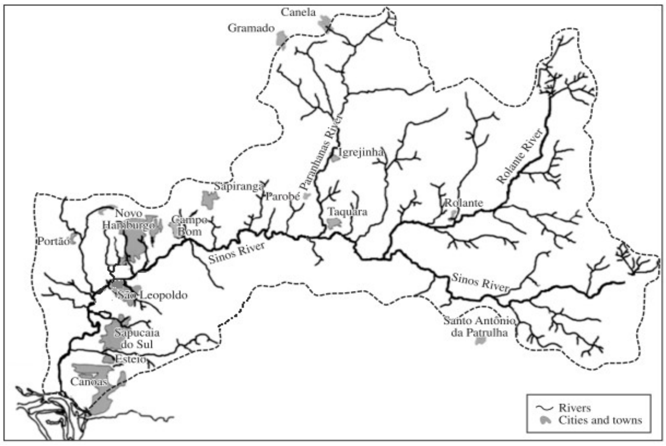

Figure 1: Sinos Rivers Basin

The SRBis frequentlycited as a highly degraded watershed due to the process of substantial economic development disjoined from environmental conservation concerns [11]. The deficiency of urban planning proper zoning hasstrong consequent in urbanization observed for the municipalities within the water basins [11].

The growth of towns and villages without following the guidelines of urban and territorial planning threaten the basin ecosystem biota. Another factor threatening isthe occupation of flooding areas by people, the riparian forest deforestation. Additionally,the domestic sewage with inadequate treatment threw into the water body, contributes to the surface and ground waters degradation. However, the city grew along the river also brought about an increase of industrial facilities, which pour, since the beginning until today, their rubbish into the streams of the river basin. So the primary sources of pollution of the SRB are two: the industrial wastewater and the domestic sewage [12].

The insufficient capacity of domestic wastewater and industrial effluents treatment has a negative influence on the ecosystem life, increasing the concentrations of pollutants in the water, killing thousands of fish and other types of life. [12, 13] .It is also responsible for waterborne diseases such as hepatitis, enteritis,and diarrhea[14], [15].

The industrial waste pumped into the streams of the basin is the source of illness due to many substances like chrome, nickel, iron, mercury, lead,and cyanide. These materials were found with values beyond the limits accepted by Brazilian legislation[16, 17]. Furthermore, organic compounds were found, such asDiethyl phthalate; Fluorene; Dibenzofuran; Nitrobenzene; 4-Bromodiphenyl ether; Hexachlorobenzene; Phenanthrene; Carbazole; Di-n- butyl phthalate e Benzyl butyl phthalate[18].

Another pollution source of the SRB is the diffused pollution linked with the increasing vehicular traffics, industrial air pollution and soil pollution by agricultural runoff [19]. The current situation shows the deficiencies and the inability of the watershed committee to reach the goals and have a proactive action into the social-environmental management. The SRB Committee need for new analysis tools that support decision making in this situation.

2.2 Fuzzy Cognitive Maps Fundamental

Cognitive maps were firstintroduced by Axelrod [20], where arcs indicate either positive or negative causal relations between nodes. fuzzy cognitive map (FCM)[3] extends cognitive maps with fuzzy values in arcs in the [-1,1] interval. Recently FCMs have gained considerable research interest and are mainly to analyze causal systems especially in system control and decision making [21-23]. When neutrosophic is included in arcs weights a neutrosophic cognitive is obtained [24].

In FCM there are three types of causal relations between nodes in the matrix: negative, positive andnone. The matrix representation of FCM allows the making of causal inferences. In FCM the dynamic analysis begins with the design of the initial vector state, which represents the initial value of each node. The value of a concept is calculated in each simulation step using the following calculation rule:

$$A_i^{(t+1)} = f\left(A_i^t + \Sigma_{j=1,\, j\neq i}^{n} A_i^t \times W_{ji}\right) \quad (1)$$

Where $A_i^{(t+1)}$ is the state of the node i at the instant t+1 , W_{ji} is the weight of the influence of j node over the i node, and f(x) is the activation function. The hyperbolic tangent activation function is defined as follows [25]:

$$S_i(C_{it}) = \tanh(\lambda C_{it}) \quad (2)$$

The calculation halts if an equilibrium state is reached. The final vector reflects the state of the FCM nodes after the system intervention [26, 27].

The interestin the use of FCM iscrescent, in the most recent years, as a participatory method for understanding social-ecological systems [28]. FCM has been used in a different set of contexts reaching from invasive species management [29] to agricultural policy design and communication [30]. The FCM is due mainly to its transparent graphical models of complex systems useful for decision making, the ability to illuminate the core presumptions of environmental stakeholders and to structure environmental problems for scenario development.

2.3Neutrosophic AHP

Smarandache[31] suggested the concept of a neutrosophic set, which usesthe truth-membership function, indeterminacy-membership function,and falsity-membership function. Neutrosophic set theory should be utilized to rationalize uncertainty associated with ambiguity in a manner analogous to a human in the decision-making process [32]. A single value neutrosophic number $\tilde{a} = \langle (a_1,\ a_2,\ a_3);\ \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ express a quantity approximately equal to a[33].

In this paper with the calculation of the weights through the analytical hierarchal process (AHP) using triangular neutrosophic numbers[34].

In AHP the relative priorities are assigned to different criteria using a scale for comparison by pairs (Table 1).

Saaty Scale	Explication	Neutrosophic Triangular Scale
1	Equally influential	$\tilde{1} = \langle (1,\ 1,\ 1); 0.50, 0.50,\ 0.50 \rangle$
3	Slightly influential	$\tilde{3} == \langle (2,\ 3,\ 4); 0.30, 0.75,\ 0.70 \rangle$
5	Strongly influential	$\tilde{5} = \langle (4,\ 5,\ 6); 0.80, 0.15,\ 0.20 \rangle$
7	Very strongly influential	$\tilde{7} = \langle (6,\ 7,\ 8); 0.90, 0.10,\ 0.10 \rangle$
9	Influential	$\tilde{9} = \langle (9,\ 9,\ 9); 1.00, 0.00,\ 0.00 \rangle$

Table 1. Priority scale of AHP criteria for pairwise comparison using triangular neutrosophic numbers [4].

Let be $\tilde{a} = \langle (a_1,\ a_2,\ a_3);\ \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ the neutrosophic comparison matrix it converted to its crisp form by using score degree of ã[4]:

$$S(\tilde{a}) = {}^{1}/_{8}\,[a_1 + a_2 + a_3] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (3)$$

and the accuracy degree of ã[4]:

$$A(\tilde{a}) = {}^{1}/_{8}\,[a_1 + a_2 + a_3] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}) \quad (4)$$

NAHP has the same advantages of classical AHP for example user with a richer structure framework than the classical AHP, fuzzy AHP,and intuitionistic fuzzy AHP. Describe the preference judgment values of the decision maker efficiently handling vagueness and uncertainty over fuzzy AHP and intuitionistic fuzzy AHP because it considers three different grades “membership degree, indeterminacy degree and non-membership degree [33, 35].

2.4 TOPSIS

Decision-making at environmental projects requires consideration of trade-offs between sociopolitical, environmental and economic impacts, making multi-criteria decision analysis (MCDA) a valuable methodology in this situations.TOPSIS is MCDA method to do rank alternative from a finite set of one’s [36]. The chosen alternative should have the farthest distance from the negative ideal solution and the shortest distance from the positive ideal solution [37]. Some extensions for TOPSIS have been developed based on neutrosophic[38].

The algorithm for TOPSIS is as follows

Step 1: Determine the normalized decision matrix (R). The raw decision matrix (D) is normalized for criteria comparability:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\Sigma_{i=1}^{m} x_{ij}^2}} \quad (5)$$

Step 2: Compute the weighted normalized decision matrix (V) with weights obtained from Neutrosophic-AHP. The weighted normalized value of can be computed by

$$v_{ij} = r_{ij} \cdot w_j \quad (6)$$

where w_j is the weight of the jth criterion and $\Sigma_{j=1}^{m} w_j = 1$.

Step 3: State the positive-ideal (A^+) and negative-ideal (A^-) alternatives. The values of the criteria in the positive-ideal and the negative-ideal alternative correspond to bethe best level and the worst level respectively [39]:

$$A^+ = \{(\max_{i=1}^{n} | j \in I^+ |), (\min_{i=1}^{n} | j \in I^- |)\} = [v_1^+, v_2^+, \dots, v_n^+],$$

and

$$A^- = \{(\min_{i=1}^{n} | j \in I^+ |), (\max_{i=1}^{n} | j \in I^- |)\} = [v_1^-, v_2^-, \dots, v_n^-],$$

where I^+ and I^- are the criteria sets of benefit and cost type, respectively.

Step 4: Compute the distance measures with the Euclidean distance. The separation to the positive-ideal alternative is:

$$d_i^+ = \sqrt{\Sigma_{j=1}^{m} (v_{ij} - v_j^+)},\ i = 1, \dots, n \quad (7)$$

Additionally, the distance to the negative-ideal alternative is denoted as:

$$d_i^- = \sqrt{\Sigma_{j=1}^m (v_{ij} - v_j^-)},\ i = 1,\ldots,n \quad (8)$$

Step 5: Compute the relative closeness to the ideal alternative and rank the preference order. The relative closeness of the ith to the ideal alternative concerning the ideal alternative is as follows:

$$C_i^+ = \frac{d_i^-}{d_i^+ + d_i^-} \quad (9)$$

A set of alternatives that can be preference ranked according to the descending order of C_i^+ ; then larger means a better alternative.

3Proposed Method

We propose an approach to support decision making in water management, made of some steps that range from indicator selection to scenario comparison and ranking support decision making.

1. Select relevant indicators

Relevant indicators are selected, and the FCM representing causality is modeled. The data source or expert(s) could be used in this step. Several methodologies could be used in order to reach a consensus within a group of participant experts [40].

2. Static Analysis

The concept in which the model can be categorized into one of three ways based on analysis: as driving components, receiving components or ordinary components [41].

The following measures are calculated with the absolute values of the FCM adjacency matrix:

Outdegree $od(v_i)$ is sum the of absolute values in the row of a variable in the adjacency matrix. It shows the cumulative strengths of connections (a_{ij}) departing the variable.

Indegree $id(v_i)$ is the sum of the absolute values in the colum of a variable. It shows the cumulative strength of variables incoming the variable.

The centrality measureof a variable is the summation of its indegree and outdegree

$$td(v_i) = od(v_i) + id(v_i) \quad (10)$$

Later variables could be classified according to the following rules and be selected in scenario development[42]:

a) Transmitter variables have a positive or indeterminacy outdegree, and zero indegrees.

b) Receiver variables have a positive indegree or indeterminacy and zero outdegree.

c) Ordinary variables can be more or less a receiver or transmitter variables, based on the relation of thei-routdegrees and indegrees measures.

3- Identify future scenarios

Scenarios are identified,and initial stimuli vector for each one are defined. A Stimulus vector is designed for each scenario representing the initial value of each node. The simulation of the scenarios with the FCM is run with the outcome in the form of concepts being 'activated' at different levels after reaching equilibrium [38].

5- Rank and evaluate the different scenarios.

The Neutrosophic AHP-TOPSIS method is a combination of the NAHP method with the TOPSIS method. In this case, the weights are calculated in the NAHP. At the first stage, NAHP is used to weight the relative importance of NODES when compared to each other. According to this, the positive-ideal scenario (PIS) and the negative-ideal one (NIS) are defined. Moreover, alternatives are ranked according to the TOPSIS algorithm [43].

4Results

Understanding the complexity of water pollution sources and their mitigation using the fuzzy cognitive maps modeling to supporting decision making.

In Step 1 relevant indicators are selected (Table 2).

Concept	Description
WQI	The WQI was created to assess the quality of raw water in the public supply treatment systems. This indicator has limitations because not analyzeessential parameters such as toxic substances, pathogenic viruses and protozoa,and others substances.
OD	This indicator shows the level of free oxygen present in the water body. It iscrucial for all life in the water.
Coliforms T	The quantity of Thermotolerant Coliform bacteria is an indicator of domestic sewage pollu-

	tion in the water body.
pH	pH is a measurement of water acidity or alkalinity, determined by hydrogen ions in the water
Water Temp	This indicator is determined by solar radiation,orphysical-chemical processes and the variability of the indicator could modify many water parameters such as surface tension or viscosity,which can affect the growth, reproduction or life of aquatic organisms?
Nitrogen Total	This indicator reflects the total quantity of nitrogen existing in the water from different sources.
Phosphorus Total	This indicator reflects the total amount of phosphorus present in the water from various sources. High phosphorus levels in water are the leading causes of eutrophication.
Turbidity	Turbidity indicates the degree of attenuation caused by the particles in suspension undergoing a ray of light passing through the water
Total Solids	The total residue is the remaining material after evaporation, calcination or drying of the water sample for a determine time and temperature.
DBO5	The BDO5 is the total of oxygen necessary to oxidize the organic substance present in the water through aerobic microbial decomposition for five days.
Domestic Wastewater	This concept is wastewater from residential towns and services, such as houses, restaurants, hotels; and which come up from toilets, bathrooms,and kitchens.
Industrial Liquid Water	This wastewater can be the result of any process, industrial activity or commercial activity, of the transformation of any natural resource or of operations with animals, such as feedlots, chicken coops or dairies.
Diffuse Pollution	Is the nonpoint source pollution, this term refers to the difficulty of adequately determining the origin of the pollutant.
Health Impact	It is the combination of methods, procedures,and tools through which can determine the relationship of certain phenomena and their effects on the health of people or animals, as well as the spatial or temporal distribution of these effects.
Economic Impact	It is the combination of methods, procedures,and tools with which can determine the relationship of certain phenomena and their effects on the economy of a region, as well as the distribution over time of these effects.
Biota impact	It is the combination of methods, procedures,and tools with which can determine the relationship of certain phenomena and their effects on the life of an ecosystem, as well as the spatial or temporal distribution of these effects.
Wastewater Treatment Plant	It is a process of elimination of pollutants from industrial and domestic wastewater. The physical, chemical and biological processes are included that make it easier to eliminate these pollutants and produce a safer discharge for the environment.
Law enforcement	This indicator reflects as the system using which the members of society act in an organized way to enforce the law, dissuading, rehabilitating or punishing people who violate the rules and regulations that govern that society.
wetlands conservation	The objective of this indicator is to measure the degree of protection and preservation of areas such as swamps, marshes,and wetlands.
Riparian forest conservation	This indicator measures the conservation of riparian forests, given their importance for forming a complex ecosystem and the occurrence of interrelationships between species of terrestrial and aquatic organisms, as well as the relationships between the biotic and abiotic components.

Table 2.CFM Relevant indicators (Nodes) and their meanings.

In Step 1 an FCM based on expert is developed. Figure 2 shows an FCM model with obtained with 20 nodes and 63 edges.

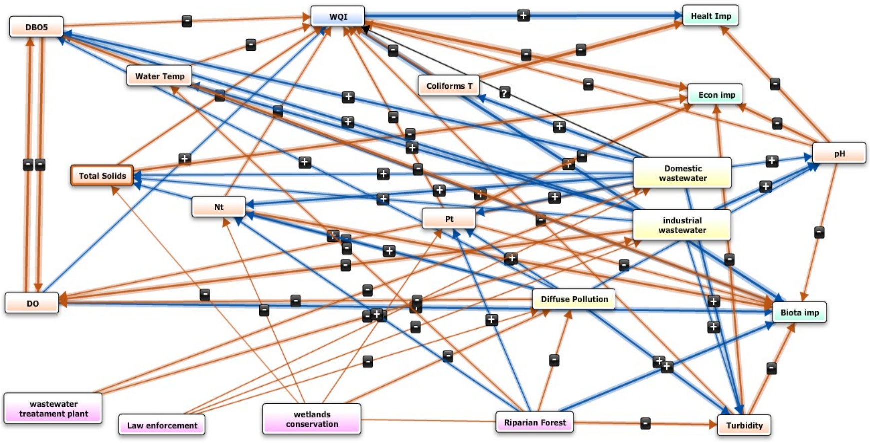

Figure 2: FCM model

FCM Model of the WQI relationships. Blue lines show positive relationships and red lines point to negative relationships and the fatness of the line represents the strength of the relationship.

In step 2 a static analysis is performed on centered on studying the features of the weighted directed graph that represent the model, using graph theory metrics (Figure 3).

Concept	Indegree	Outdegree	Centrality
WQI	2.91	2.86	5.77
OD	2.44	2.03	4.47
Coliforms T	0.81	1.36	2.17
pH	1.26	1.89	3.15
Water Temp	0.78	0.59	1.37
Nitrogen Total	1.39	1	2.39
Phosphorus Total	1.9	1.06	2.15
Turbidity	2.25	1.5	3.75
Total Solids	1.38	1.16	2.64
DBO5	2.69	1.86	4.55
Domestic Wastewater	0.34	4.53	4.87
Industrial Liquid Water	0.26	3.04	3.3
Diffuse Pollution	0.43	3.19	3.62
Health Impact	2.81	0	2.81
Economic Impact	3.39	0	3.39
Biota impact	4.7	0	4.7
Wastewater Treatment Plant	0	0.58	0.58
Law enforcement	0	0.03	0.03
Wetlands con-	0	0.45	0.45

servation			
Riparian forest conservation	0	1.8	1.8

Table 3: Static Analysis

The most central nodes are WQI, Domestic Wastewater,and Biota impact. Receiver variables are Health Impact, Economic Impact,and Biota impact. Transmitter variables are Law enforcement, Wetlands conservation, Riparian forest conservation.Scenarios are identified and simulated (Table 4).

In step 3 scenarios are identified,and initial stimuli vector for each one are defined.

Scenario	Description	Initial Stimulation
S1	The actual capacity of wastewater treatment in the basin of Sinos River	WTP 5%
S2	increase the quantity and capacity of the wastewater treatment system	WTP 35%
S3	Increasing of natural or artificial Wetlands areas	Wetlands 35%
S4	Increase the law enforcement	Law Enforcement 35%
S5	Increasing Wetlands areas and Law enforcement	Wetland 25% Law 30%
S6	Increasing of wastewater treatment plant, Wetlands areas,and Law enforcement	WTP 45% Law 25% Wetland 30%d

Table 4..Scenarios identification and stimulated.

Scenario planners calculate the FCM model for different input vectors that represent probable or desirable combinations of concept states.[44] In this case, the hyperbolic tangent activation function is used[25].

The scenarios are further investigated the next step (Step 4) for ranking.Ending scenarios are exemplified graphically in Figure 4.

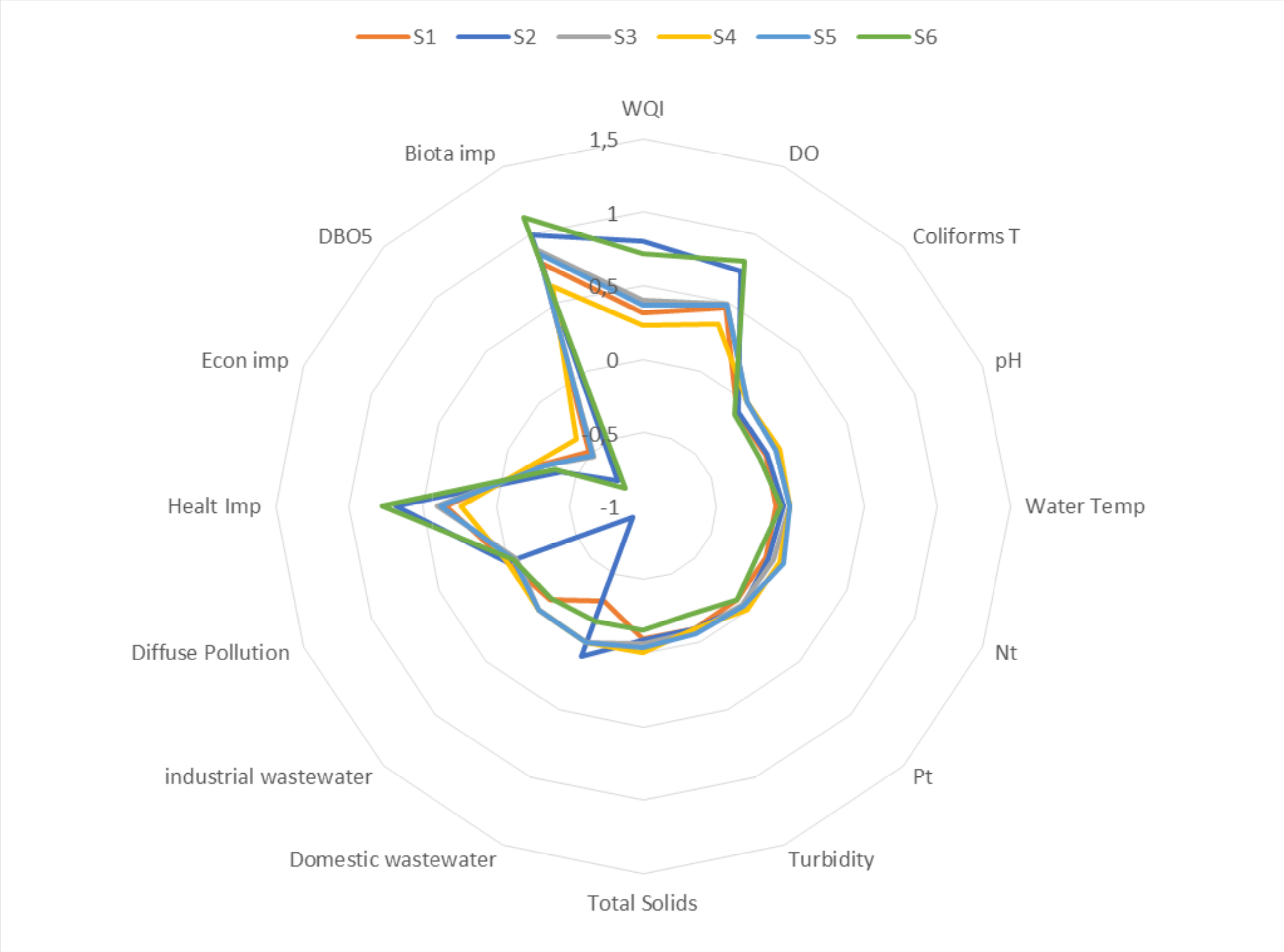

Figure 4: Scenarios' results

Scenarios ranked with NAHP-TOPSIS. Pairwise comparison matrix was obtained using triangular neutrosophic numbers. Nodes are weightedaccordingto the AHP method as follows, see table 4.

Indicators	Weights
WQI	0,15
OD	0,10
Coliforms T	0,08
Ph	0,08
Water Temp	0,07
Nt	0,06
Pt	0,06
Turbidity	0,06
Total Solids	0,06
Domestic Wastewater	0,05
Industrial Wastewater	0,05
Diffuse Pollution	0,04

Health Imp	0,02
Economic Imp	0,02
DBO5	0,05
Biota Imp	0,01
Wastewater treatment plant	0,02
Law enforcement	0,02
Wetlands Conservation	0,01
Riparian Forest	0,01

Table 5: Weights results

WQI and OD are the most important nodes. Theyjointly comprisethe 25% of the total weights. Additionally the least important are Biota Imp,Wetlands Conservation and Riparian Forest comprising only 3% of the total weights.

Then the weighted normalized decision matrix (V) is computed. The authorsadopt that all the nodes are classified as benefit (higher scores are better). After TOPSIS procedureoutcomesare displayed,and the scenarios are ranked (Table 5).

Scenario	Distance to Ideal	Distance to Anti-Ideal	Relative Degree Closeness	Rank
S1	0.96	0,93	0,49	5
S2	1.04	1,01	0,49	5
S3	0.72	1,11	0,61	2
S4	1.05	1,13	0,52	4
S5	0.75	1,11	0,60	3
S6	0.67	1,29	0,66	1

Table 6: TOPSIS results

S6 rank as the best scenario and S1 is the less desirable scenario. Increasing of wastewater treatment plant, Wetlands areas,and Law enforcement id the best scenario according to the method.The final ranking of scenarios is as follows: $S6 \succ S3 \succ S5 \succ S4 \succ S1$ ≈S2.This result coincide with experts opinions consulted.

In this study, we presented a methodology based in FCM to obtain a better comprehension about the complex relation of variables involved in water quality. The representation of relationships variables by FCM allows the SRB Committee do participation exercisesand all of the stakeholders gain in knowledge aboutthe challenges of social-environmental management and the water management specifically. Additionally scenarios are ranked taking into account importance of the factor involved with the NAHP and the TOPSIS methods.

Conclusion

In this paper, the authors present a Neutrosophic AHP TOPSIS multi-criteria decision method tool that can contribute for a better choice of environmental management in the watershed by the Basin Management Committee. With this technique, it is possible to use FCM to model the complex system of variables involved in the determination of water quality, according to the WQI and TOPSIS for ranking scenarios. The methodology exposed in this research shows an improved method to be used by the SRB Committee when planning decisions. The applicability of the framework has been demonstrated during the case study presented above.

The originality of the approach shown in this document is to use the built scenarios, their evaluation and their classification for water quality management. Future work should focus on extending the auxiliary proposal into a neutrosophic decision map to work out the decision-making dependency of multiple criteria and feedback problems. The development of a full neutrosophic proposal is another area of future research.

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