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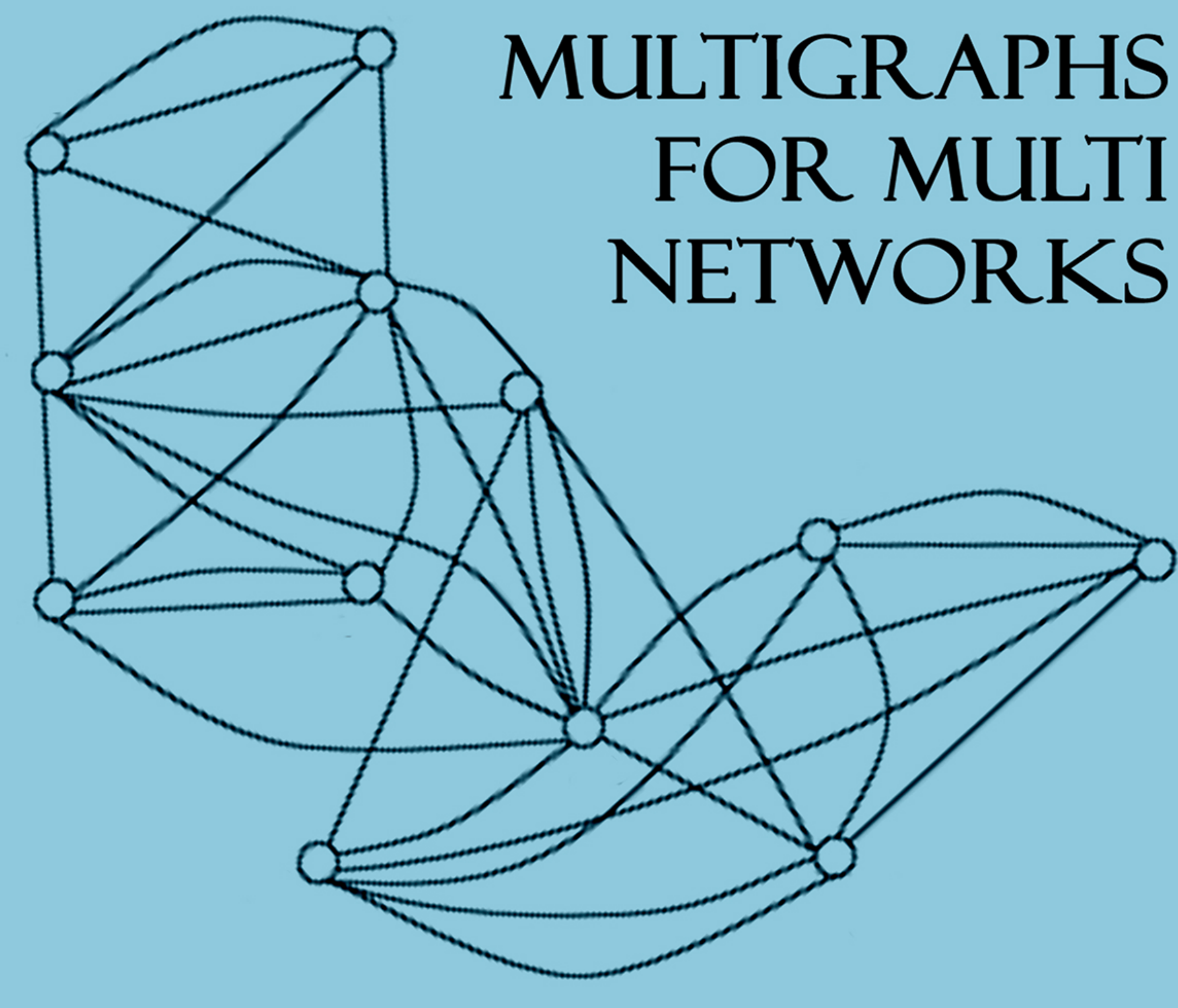

MULTIGRAPHS FOR MULTI NETWORKS

Multigraphs for Multi Networks

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PREFACE

Graph theory ensues to be a major tool in the study and construction of networks. There are several exhaustive books on graph theory. Despite the increased need for multi-structures like multi-line networks, multiple networks or multi networks, there are no books solely dedicated to the systematic study of multigraphs. This book is an attempt to fill that gap.

In this book any network which can be represented as a multigraph is referred to as a multi network. Several properties of multigraphs have been described and developed in this book. When multi path or multi walk or multi trail is considered in a multigraph, it is seen that there can be many multi walks, and so on between any two nodes and this makes multigraphs very different.

Another interesting feature of this book is we have defined unconditional line graphs of both graphs and multigraphs. These multi networks whose underlying structures are multigraphs can be used for transportation networks, computer networks and social networks. Several open problems are suggested for researchers.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

W.B.VASANTHA KANDASAMY
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Chapter One

INTRODUCTION

The knowledge of graph theory dates to early 17^{th} century with the advent of Euler settling the famous unsolved problem called in his time as Konigsberg Bridge's problem in the year 1736. Since then there is a consistent development in graph theory. One can have the history and development of graph theory from any e-source. However, the notion of multigraphs though used in social networks and transportation networks we choose to call them as multinetworks. We do not find any systematic analysis of multigraphs in the form of book or any other source.

We get only definition of multigraphs and some properties associated with them. Hence authors felt it essential to have a book on multigraph theory which gives some of the properties of multigraphs and in several situations, we are forced to define and develop the properties of multigraphs by defining some additional notions to derive the properties as in case of usual graphs. For existing literature of multigraphs refer [10].

We in this chapter define three forms of multigraphs and develop the notion of submultigraphs (or mutisubgraphs) multiclique, multiwalk, multipath etc. The adjacency matrix is different in some cases. We further define the notion of edge labeled and vertex labeled multigraphs.

We just give examples of multigraphs.

Example 1.1. Let G be a multigraph given by the following figure.

G =

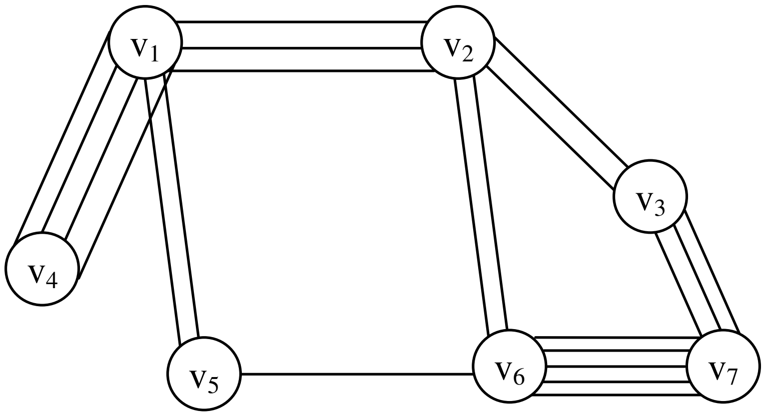


Figure 1.1

G is an undirected multigraph with seven vertices and the maximum number of edges between any two vertices v_i and v_j is five and in this case $v_i = v_6$ and $v_j = v_7$.

We call this nonuniform multigraph G as the number of edges between any pair of vertices is different. This type of multigraphs are defined in [10].

We assume throughout this book that any multigraph has only finite number of edges and finite number of vertices and has no loops.

We give one example of a uniform multigraph.

Example 1.2. Let H be a multigraph given by the following figure.

H =

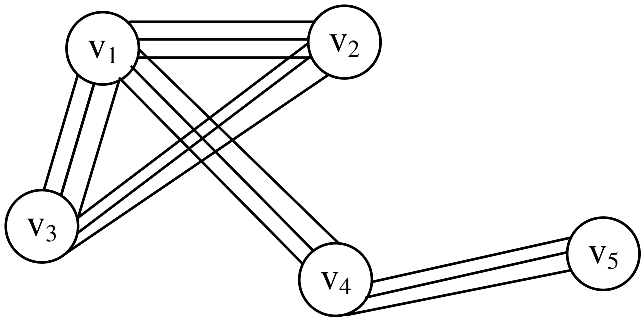


Figure 1.2

We call H a uniform multigraph as the number of edges between any two appropriate vertices is only 3.

This is the difference between uniform multigraphs and non-uniform multigraphs.

Now we give an example of directed multigraph.

Example 1.3. Let K be a multigraph given by the following figure.

K =

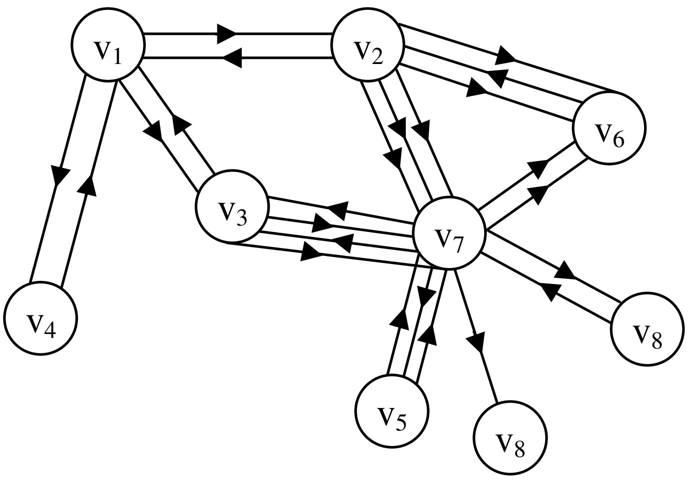


Figure 1.3

K is a directed multigraph.

Now we provide examples of edge labeled and vertex labeled multigraphs for both not directed and directed multigraphs.

We see all multigraphs given in examples 1.1 to 1.3 are vertex labeled.

Example 1.4. Let G be a multigraph given by the following figure.

G =

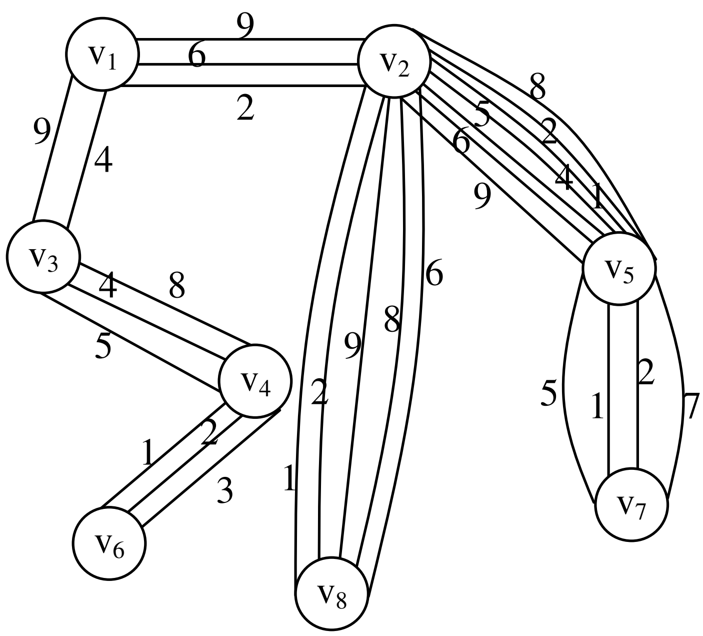


Figure 1.4

G is a edge labeled and vertex labeled multigraph. The maximum number of edges can be 9 however in this G we have only 7 connecting v_2 to v_5. The edges can only be labeled from 1 to 9 from any pair of appropriate vertex sets. This multigraph is undirected.

We now provide an example of a directed multigraph for which both edges and vertices are labeled.

Example 1.5. Let H be a directed multigraph which is both edge and vertex labeled given by the following figure.

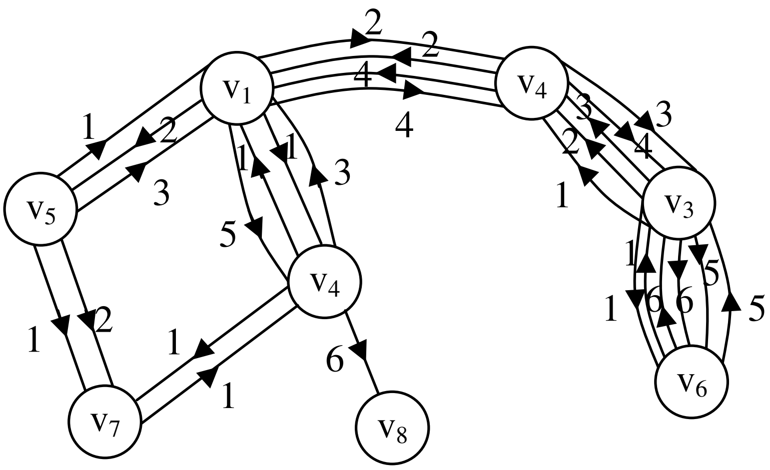


Figure 1.5

We see H is a directed multigraph which is both edge and vertex labeled. Authors in this book deal with 3 types of multisubgraphs.

We say a multisubgraph is a vertex multisubgraph H of G if we take some vertex sets from the vertex sets of G, and the edges remain the same as that of G in H also.

We say an edge multisubgraph if one or more edges in G are removed. In the edge multisubgraph a vertex will be removed if all the edges connecting them are removed.

A multisubgraph in general need not always be a vertex multisubgraph or an edge multisubgraph.

Infact finding the number of edges in multisubgraphs happens to be a challenging problem.

We just indicate a few of the applications of multigraphs: Multigraph approach to social network analysis [21] in which Florentin network is illustrated. However [4, 5, 7, 11] have researched on social structure from multiple networks which we choose to call as multi network. Godeharbt [8] has made a study on Probability model for Random Multigraphs with application in Cluster Analysis. Strategies for multigraph edge colouring [6] has been discussed and it also described several interesting features about multigraphs.

However, this book only develops the concept of multigraphs and in several places to achieve or develop properties akin to graphs we circumvent the properties or definitions to best suit the multigraphs structure. Multiwalks, multipath, multitrials etc. and concepts like multiclique behave entirely in a different way in case of multigraphs.

Throughout this book we take only the following definition for multigraph.

Definition 1.1: *A graph G is called a multigraph if there are many edges or line connection any two adjacent vertices of G.*

For more details refer [10].

Chapter Two

MULTIGRAPHS

The study of graphs is very common and there are several books on graph theory. These graphs are basically not multigraphs and lots of results relating to them are also present in these books. However, some of these books just mention about multigraphs and pseudo graphs but no related properties about them are analysed.

To the best of authors knowledge there is no book which separately deals with multigraphs theory.

But it has become the present-day demand that multistructures play a vital role. So, it is important that a systematic analysis of multigraphs are given separately which will be boon to nonmathematicians or to be more precise a person with least knowledge in multigraph theory and for researchers who work on multistructures like multinetwork and so on.

Here multigraphs are introduced and also the limitations of several operations done on graphs are brought out and also methods to overcome these impediments are given.

Throughout this book we take only multigraphs with finite number of edges and finite number of vertices.

It is pertinent to record that finite number of vertices does not imply finite number of edges as in case of usual graphs. In multigraphs unless mentioned the number of edges are finite even with finite number of vertices we can have infinite number of edges. Thus, in this book we assume all multigraphs have finite number of vertices and finite number of edges.

Just we express even with two vertices we can have a finite number of edges in case of multigraphs. If our multigraph under consideration is directed, we may have many more directed multigraphs.

How in case of graphs we may have an empty graph with two vertices or a unique graph in case it is not directed and two directed graphs which are described by the following figures.

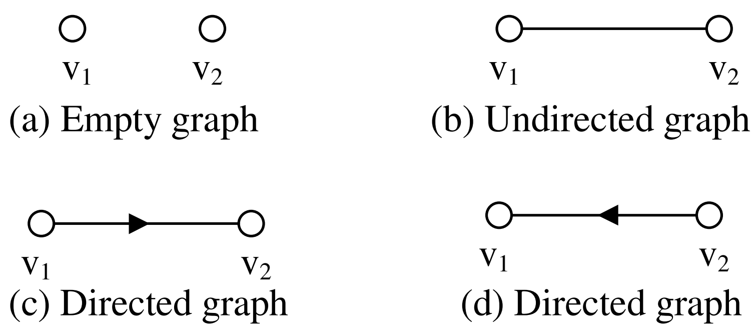


Figure 2.1

Clearly (a) is an empty dyad (b) is a dyad, both way reciprocating dyad. (c) and (d) are one way directed or non-reciprocating dyads.

There are only four graphs using two vertices. However, this is not the case with multigraphs with two vertices and say just three edges.

We list out the multigraphs both directed and undirected with two vertices and maximum of 3 edges in the following.

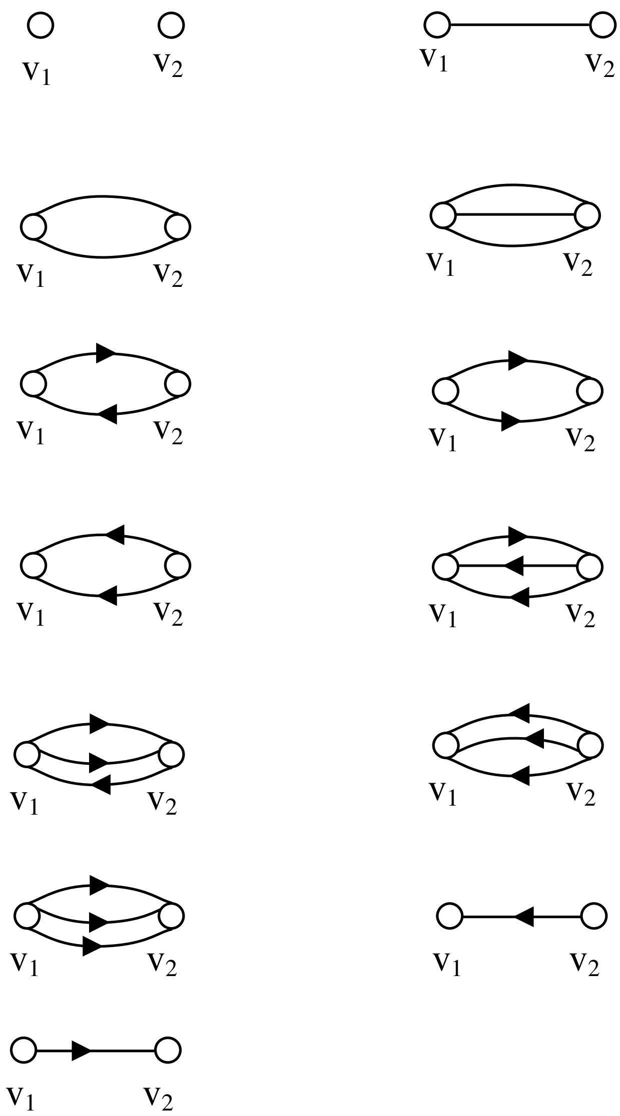


Figure 2.2

There are 13 multigraphs with three edges and two vertices 8 of them are directed one way or two ways reciprocating multidyads. We choose to call multigraphs with two vertices as multidyads. Next, we find all two vertex multigraphs with maximum of four edges given by the following figures.

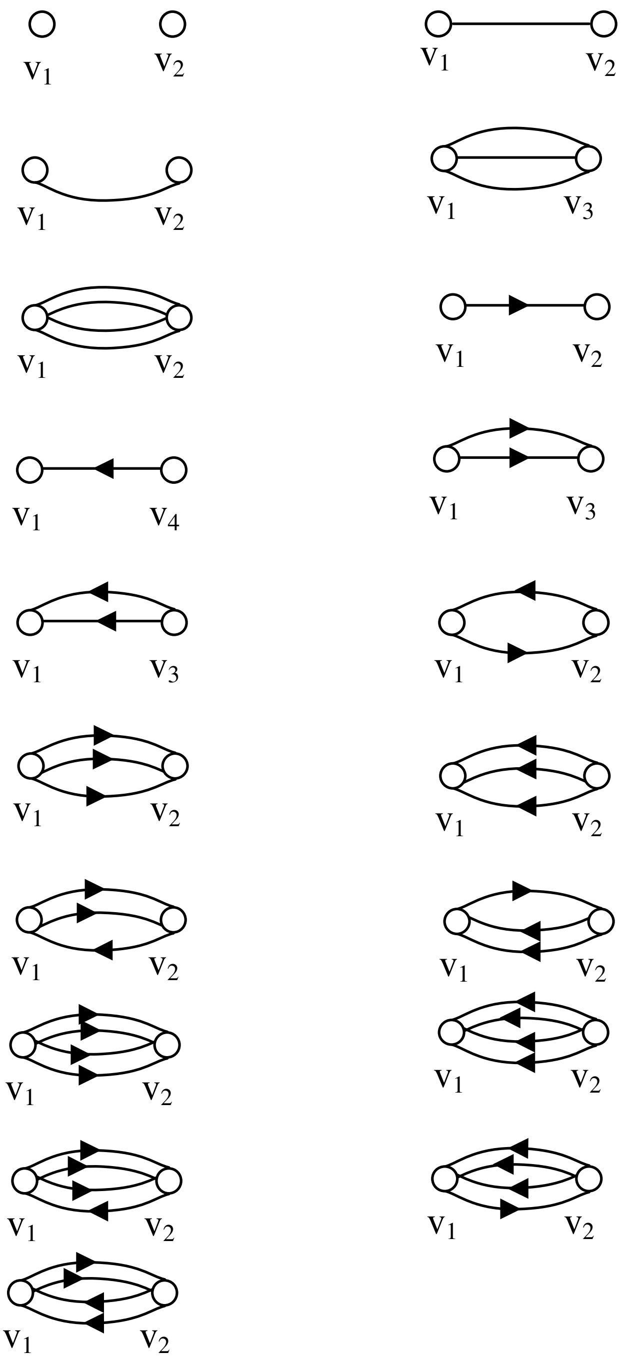


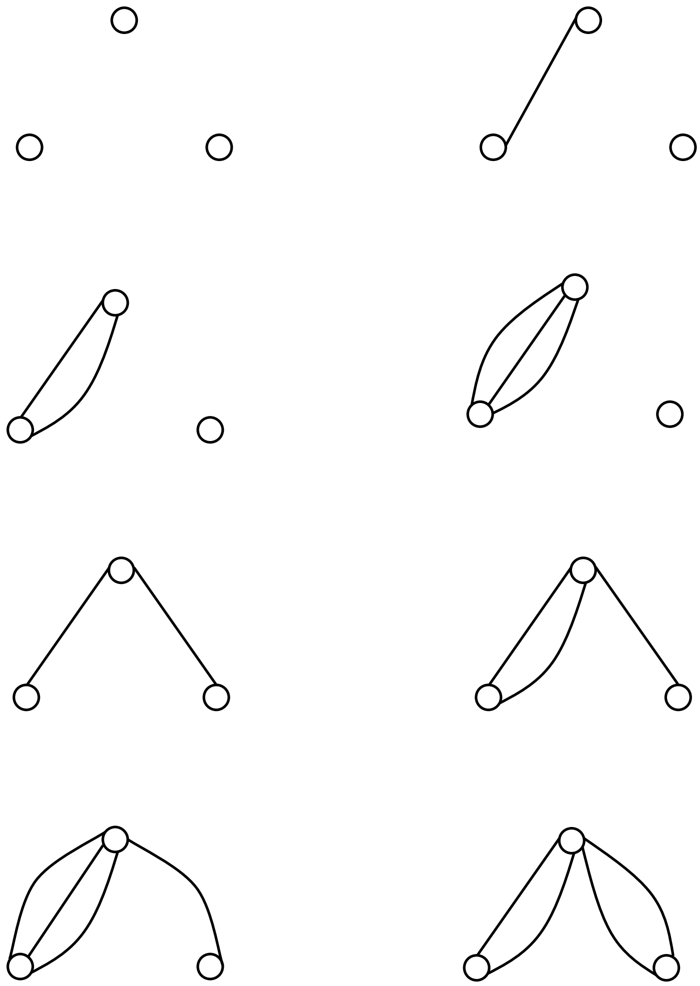
Figure 2.3

There are 19 multigraphs with four edges all of them are labeled.

We see if the multigraphs are directed, we have a greater number of them.

We give one more example of 3-edges multigraphs with three vertices.

The 3-edges multigraphs with three vertices which are not directed and not labeled is given below by the following figure.



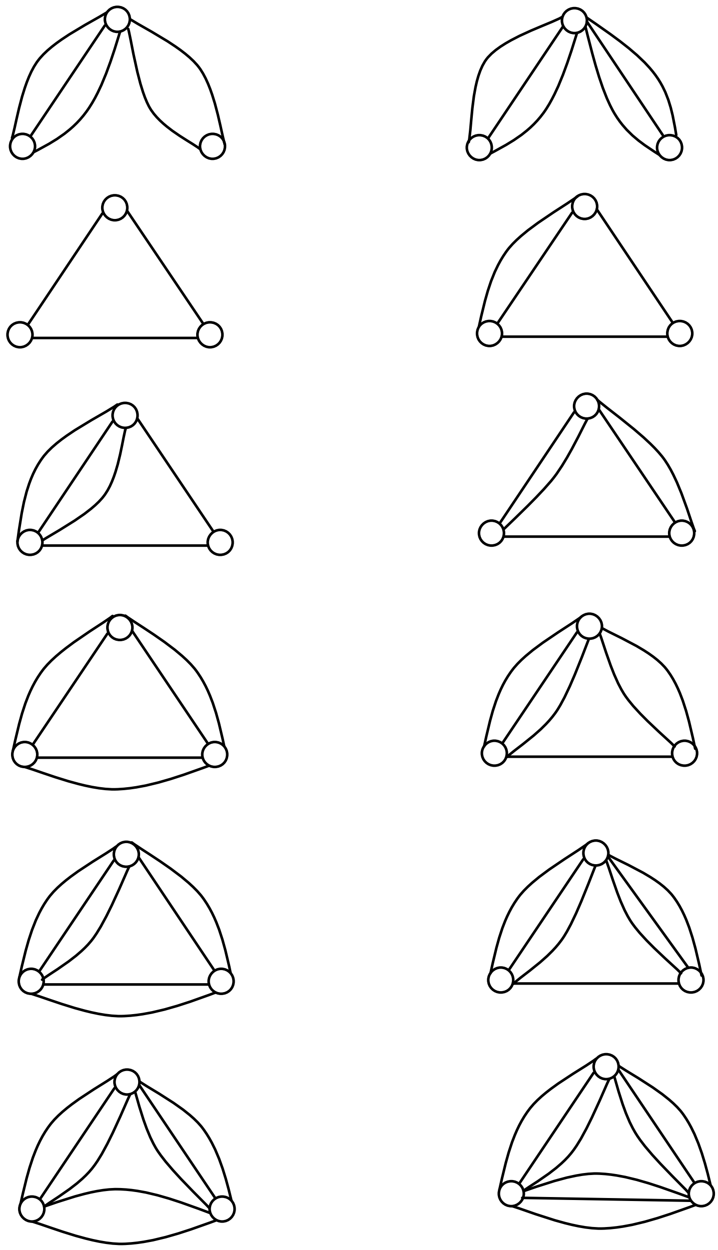
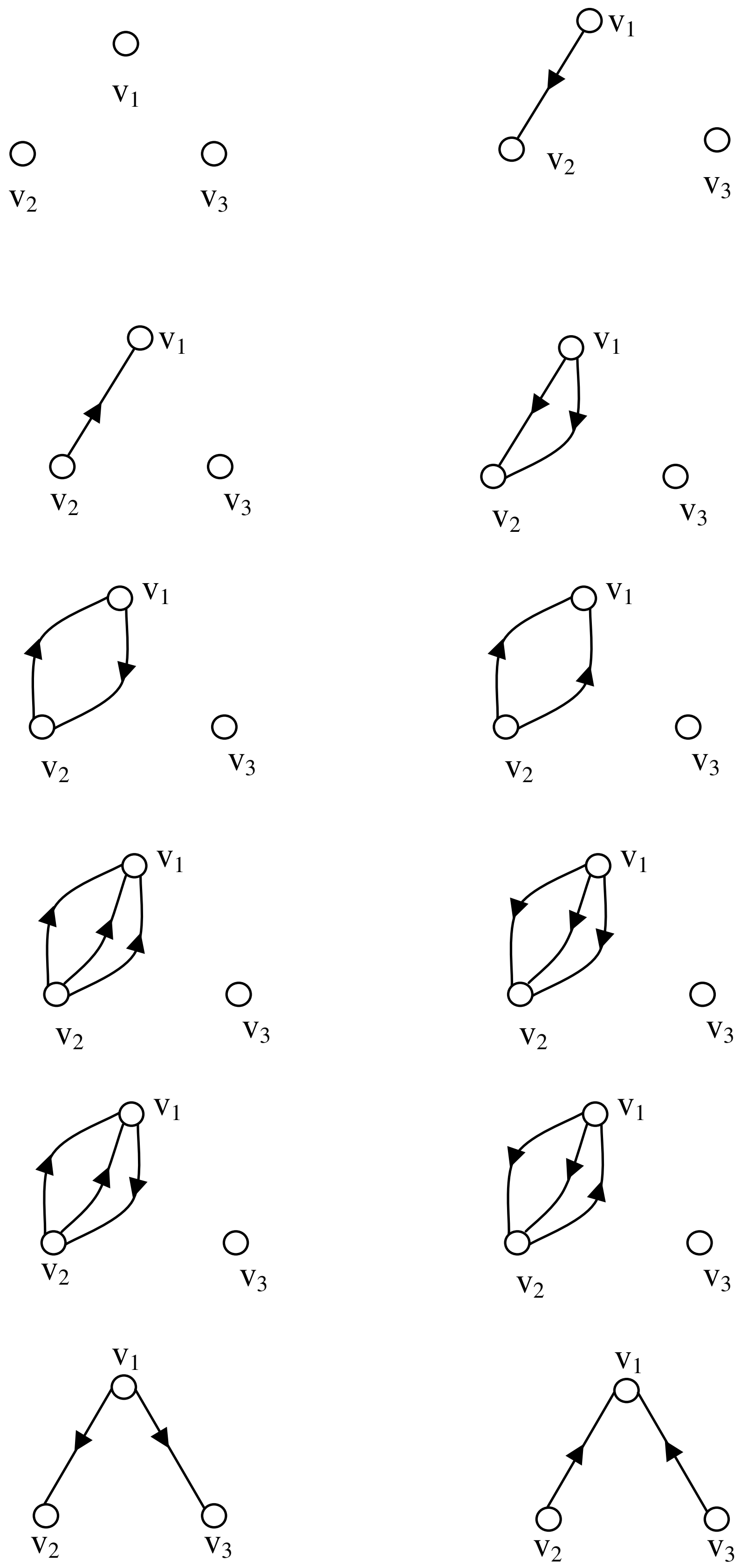


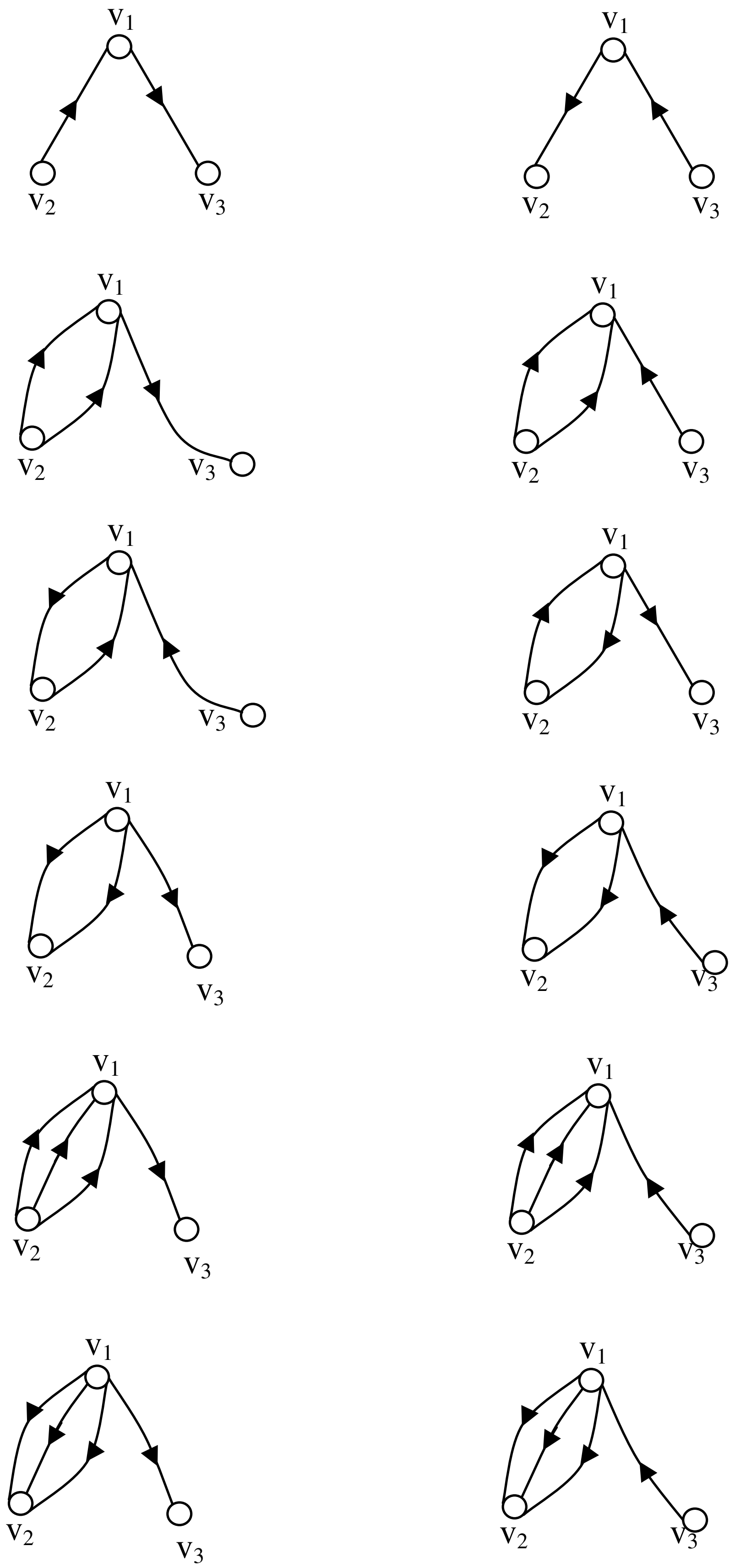
Figure 2.4

If the multigraphs of figure 2.4 have labeled vertices we will have a greater number of multigraphs.

If vertices are labeled, we will have 52 of them or 32 more of them given in figure 2.4.

Now consider all directed 3-edges multigraphs with 3 vertices. If the multigraphs are to be directed, we must label them.

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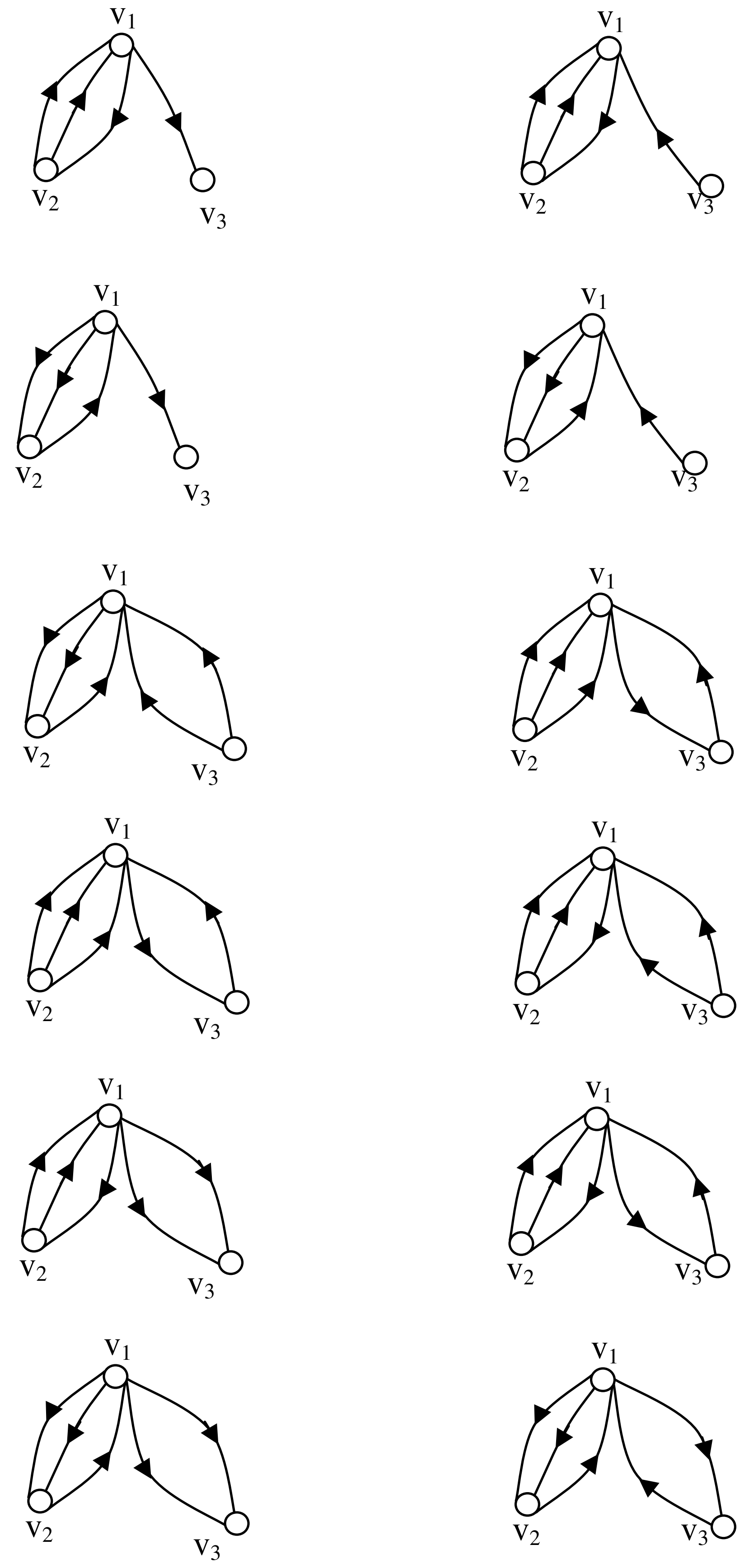
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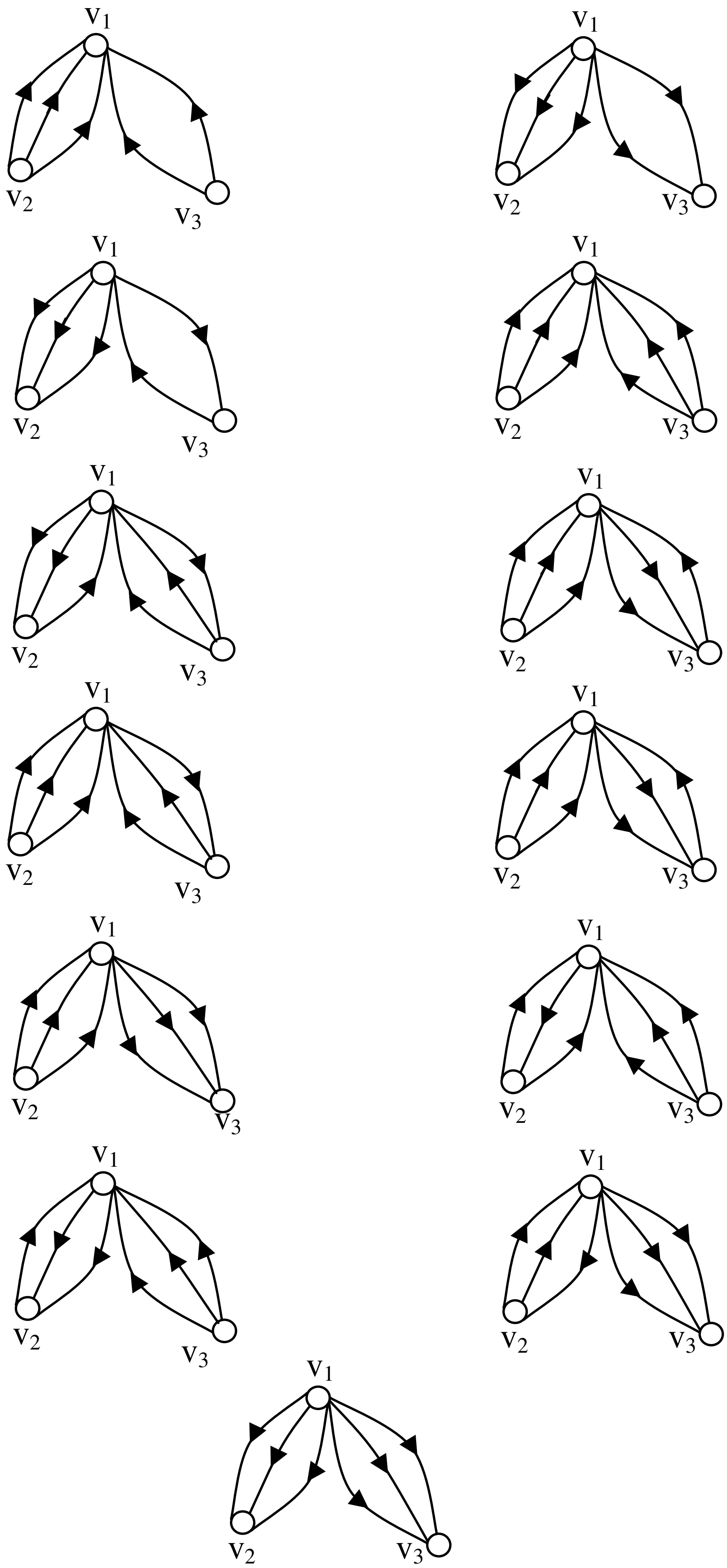
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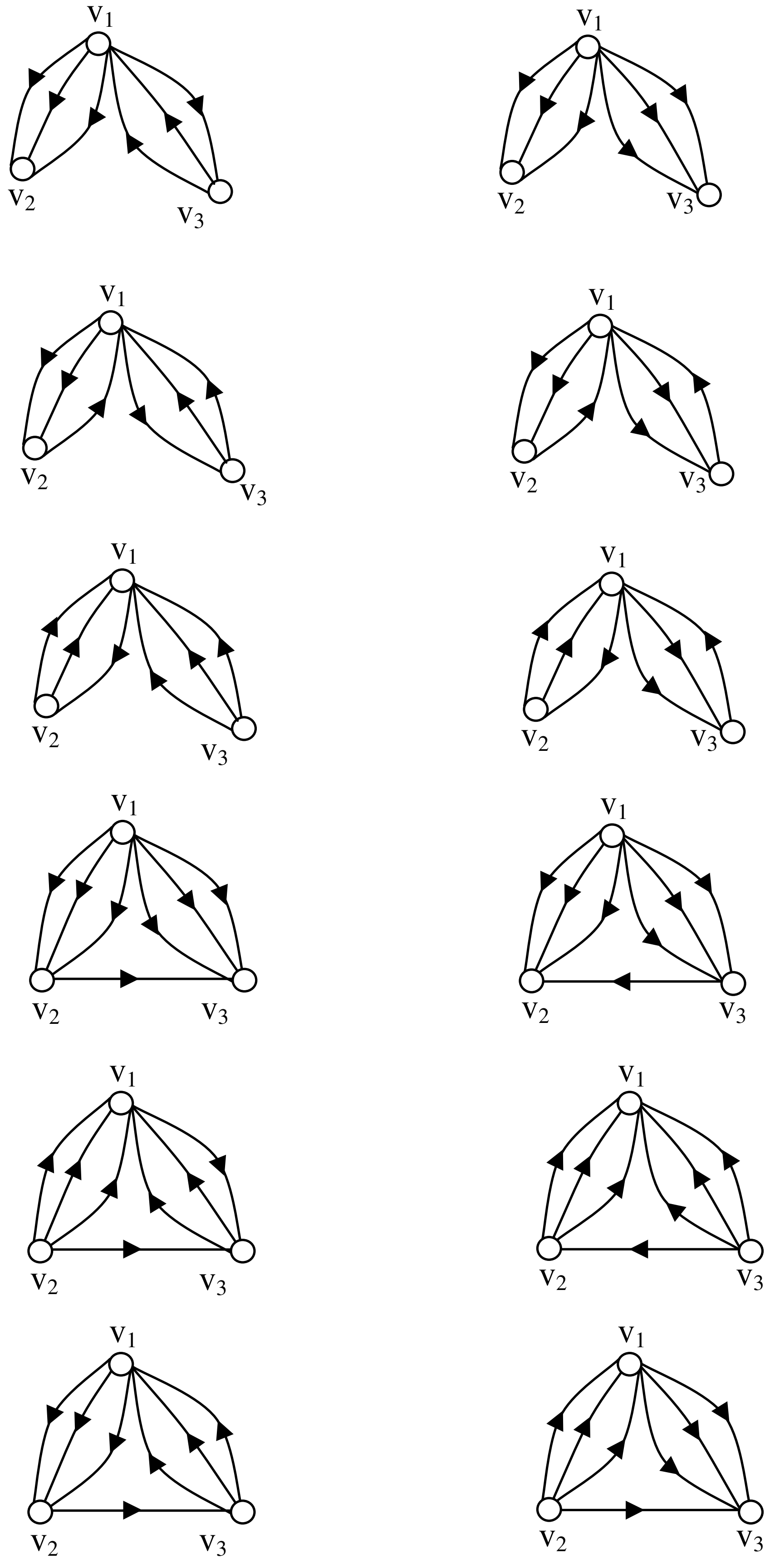
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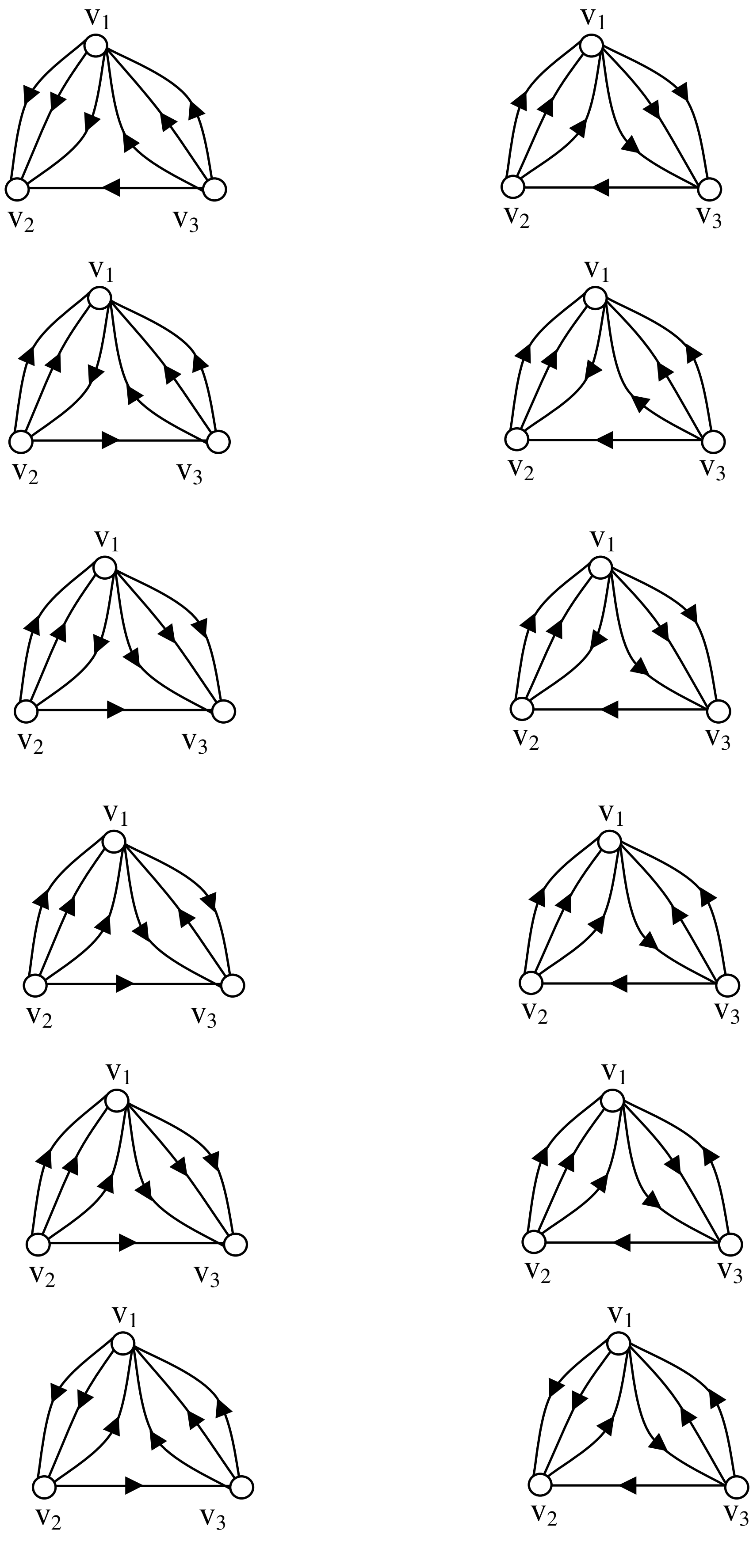
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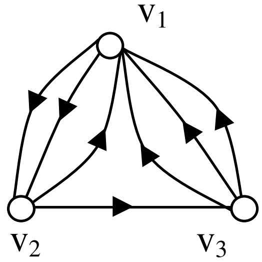
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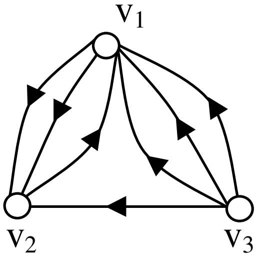
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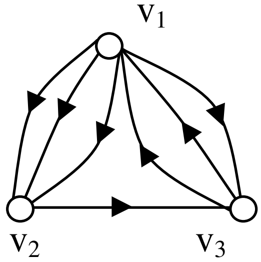
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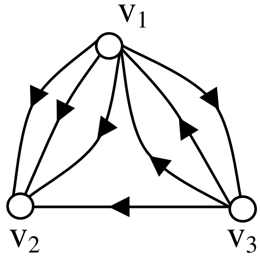
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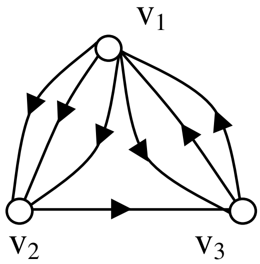
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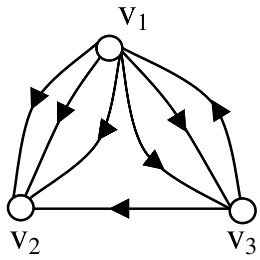
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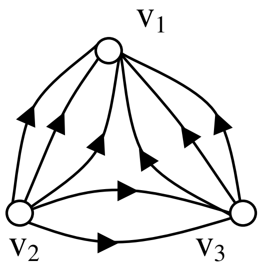
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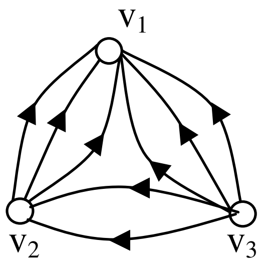
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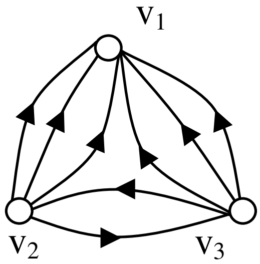
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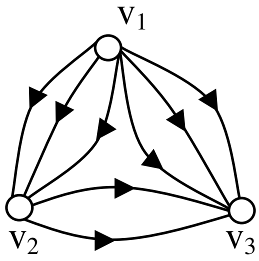
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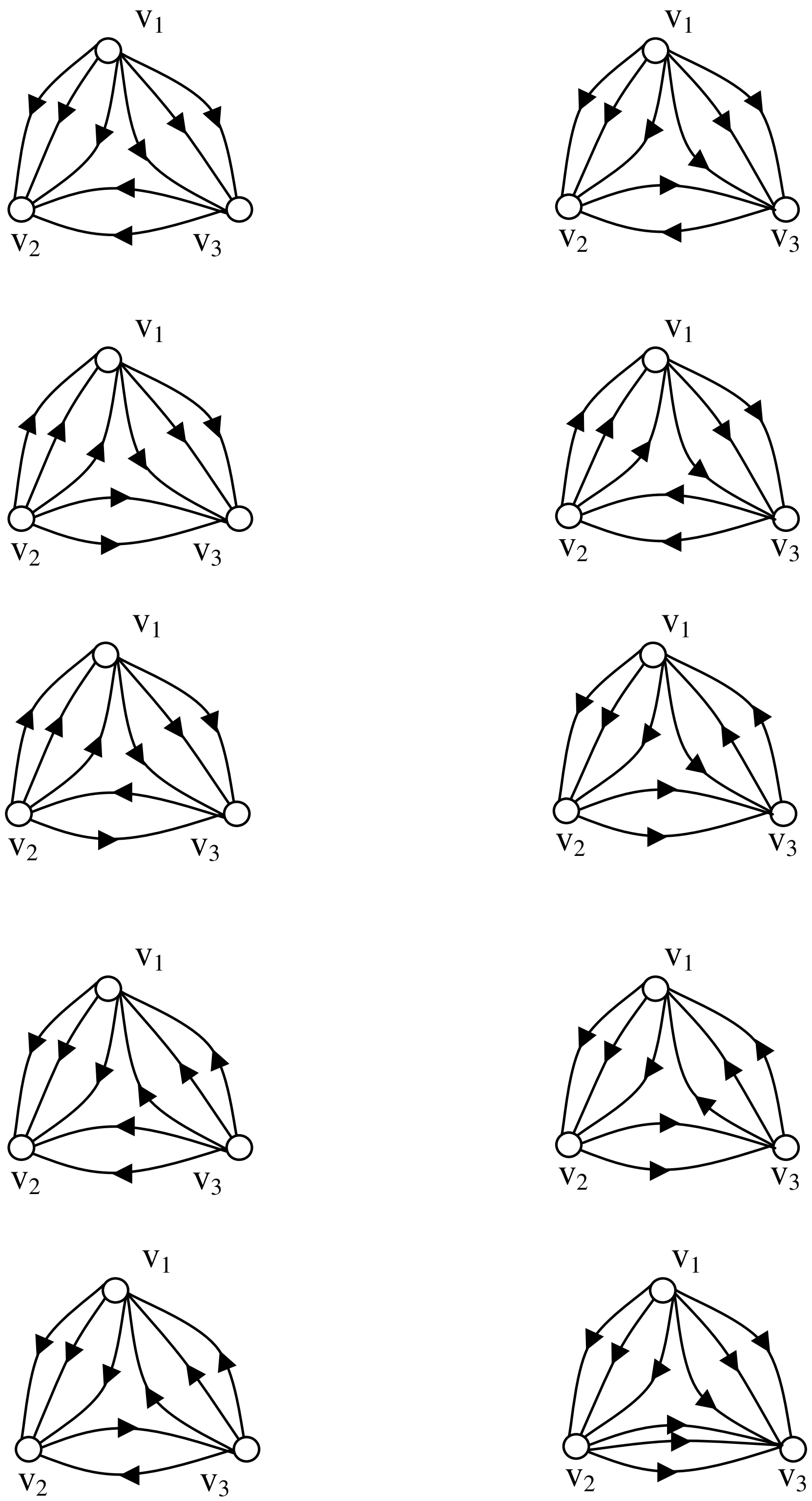
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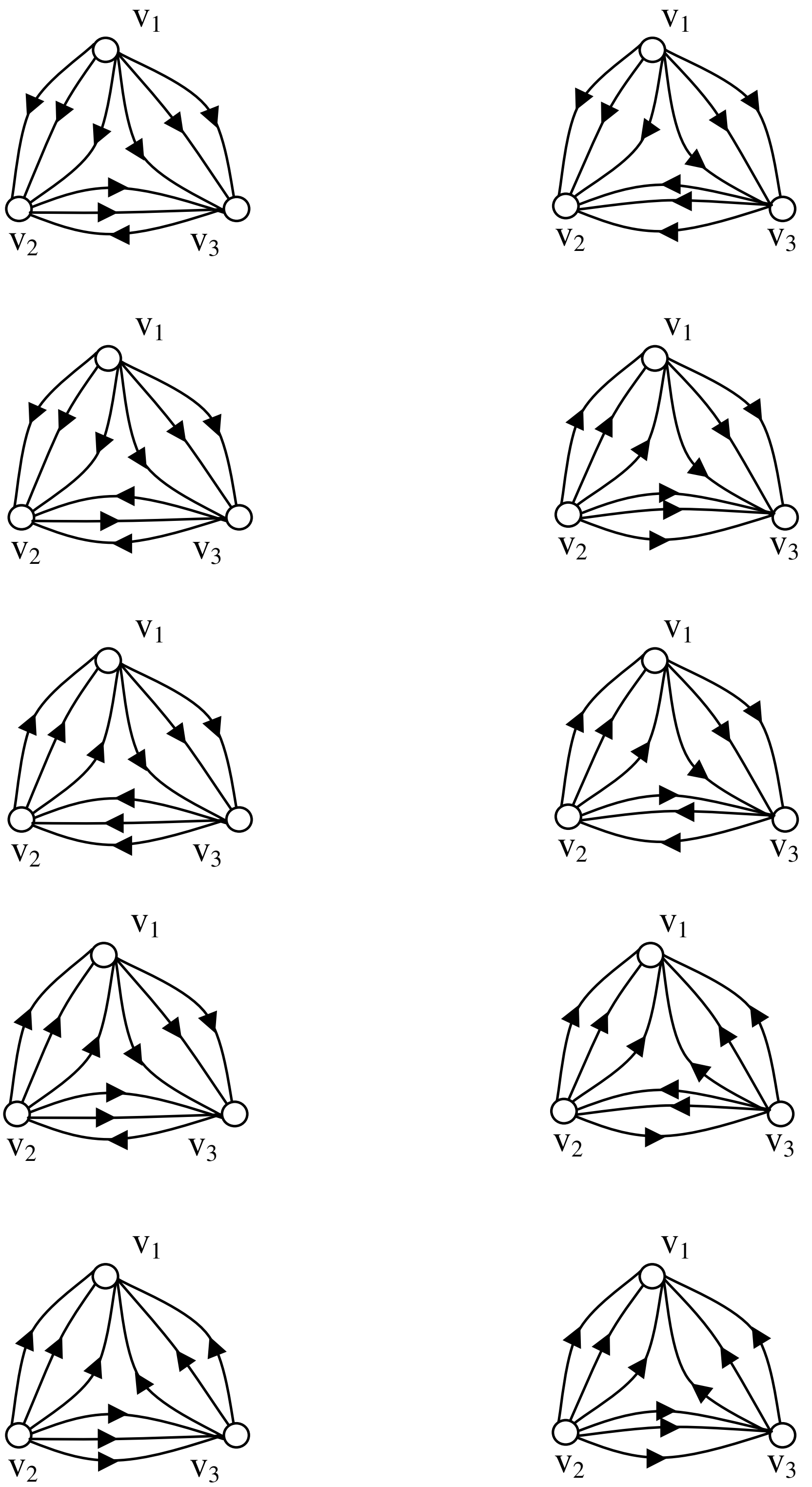
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Figure 2.5

Thus for a small value like 3-edges multigraphs with 3-vertices we see there are nearly 100 such 3-edges multigraphs with 3-vertices.

Thus study of directed n-edges multigraphs happens to more unwieldy than the undirected n-edges multigraphs. So in this chapter authors study only undirected n-edges multigraphs with m vertices. Clearly when n = 1 we get the classical undirected graph with m vertices.

In general if the number edges are not specified then the multigraphs can have infinite number of edges also between any vertices be it directed or otherwise.

Thus for a given set of 2 vertices we can have infinite number of multigraphs directed or otherwise.

So we are not interested in analyzing multigraphs with any number of edges. Just as we have put a condition on the number of vertices to be finite for any general graph, we put forth the condition in case of multigraphs the number of edges between any two vertices is always finite and we just say maximum number of edges we can have between any two vertices is ‘m’; m a fixed positive integer for any multigraph. Further we assume that the number of edges between any two vertices in a multigraph can vary from 0 to m.

We call such multigraphs as m-edges multigraphs. The number of multisubgraphs of a multigraph with arbitrary number of edges is also infinite.

So in this book arbitrary number of edges multigraphs are not analyzed only m-edges multigraphs are defined and developed.

We also introduced a special type of uniform m-edges multigraphs.

We call a m-edges multigraph to be a uniform m-edges multigraph if whenever two vertices are adjacent, they have m-edges connecting them.

We first illustrate this situation by some examples.

Example 2.1. Let G be a 4-uniform multigraph with 6 vertices given by the following figure.

G =

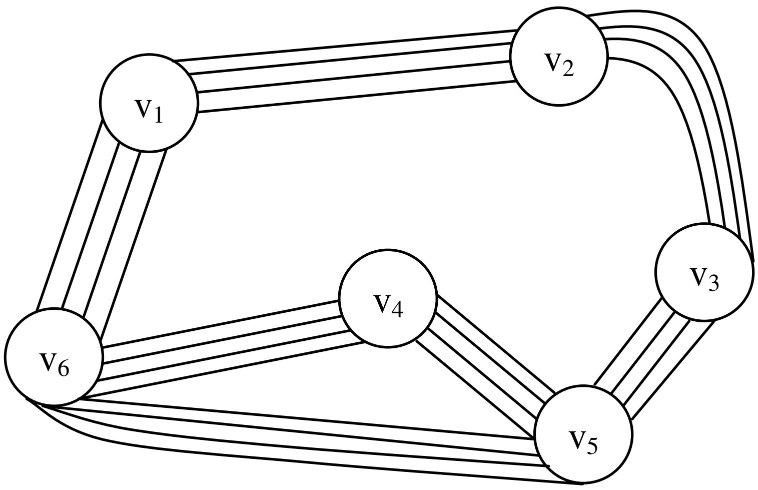


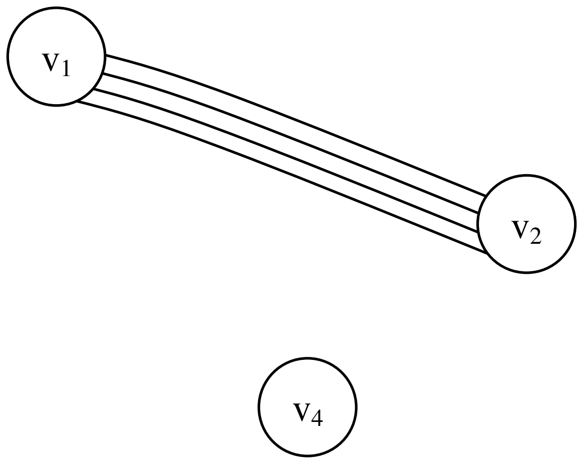
Figure 2.6

G is a 4-uniform multigraph with 6 vertices.

Clearly G has $6C_2$ + $6C_3$ + $6C_4$ + $6C_5$ number of 4-uniform multisubgraph by neglect of singleton multisubgraphs and the multigraph G itself.

Some 4-uniform multisubgraphs are given by the following figures.

H_1 =



H_2 =

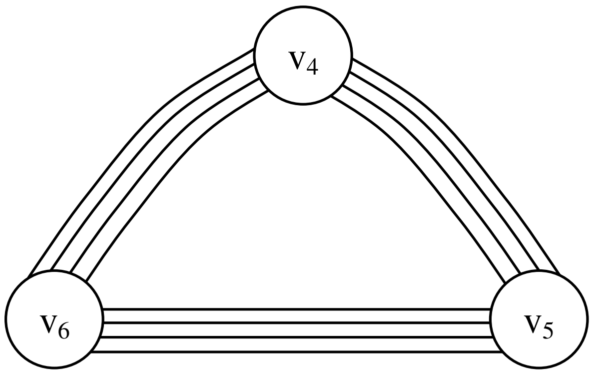


Figure 2.7

We see H_1 is only a disconnected multisubgraph whereas H_2 is the complete 4-edges multitriad.

H_3 =

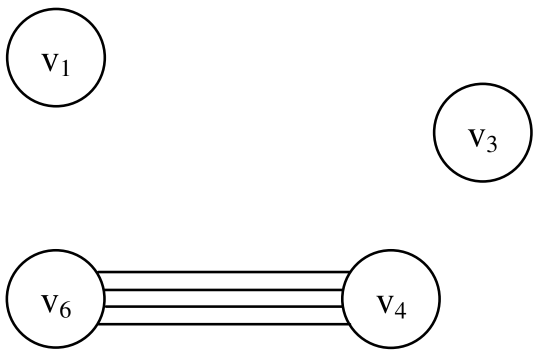


Figure 2.8

Clearly H_3 is only a disconnected 4-edges multisubgraph of order four.

H_4 =

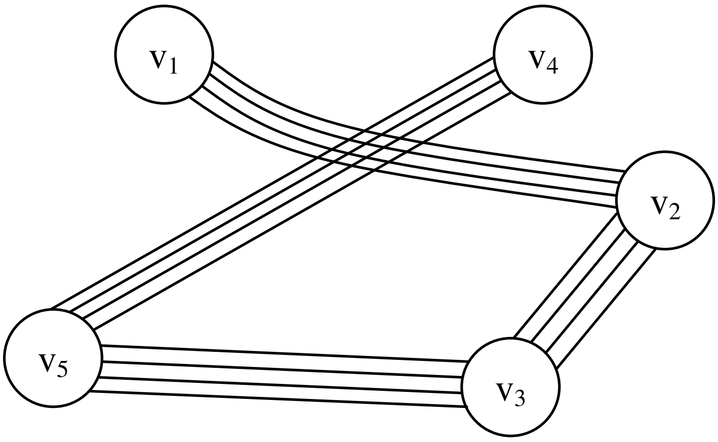


Figure 2.9

Thus H_4 is a connected 4-edges multisubgraph.

The main observation from this study is that if G is a m-uniform edges multigraph then it behaves more like the usual graph as the only difference being that between any two relevant vertices the edges are m in case of m-uniform edges multigraph and one edge in case of usual graphs.

However m-uniform edges multigraphs will be useful in case even if one of the links between any two vertices fails yet one can have some other edges to function but in case of the usual classical graphs this is an impossibility.

We just give one or two examples of m-uniform edges directed multigraphs.

Example 2.2. Let G be a 3-uniform edge directed multigraph given by the following figure.

G =

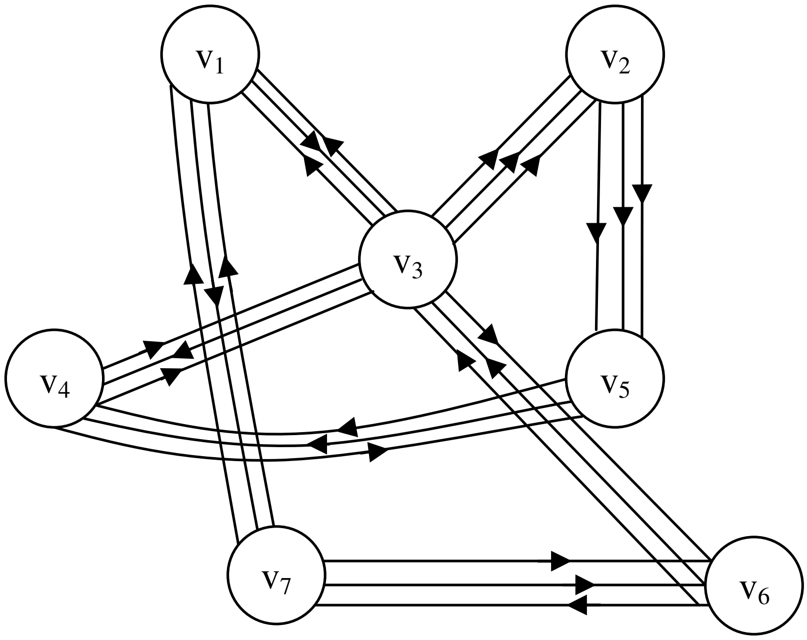


Figure 2.10

We see directions are mixed in each of the uniform multiedges whenever edge exist.

There are only $7C_2 + 7C_3 + 7C_4 + 7C_5 + 7C_6$ number of proper 3-uniform edges directed multisubgraphs.

We give in the following figure the 3-uniform multiedges directed subgraph of G (say)H_1

H_1 =

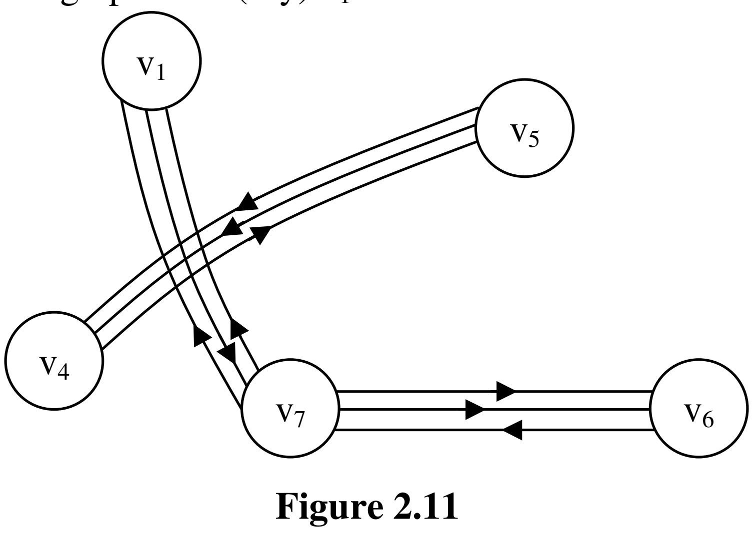


Figure 2.11

Clearly this multisubgraph is not connected.

In this chapter we do not venture to describe m-edges multigraphs which are directed. Only study of not directed m-edges multigraphs are carried out.

We also define a new class of multigraphs as n-edges multigraphs. These are usual multigraphs G in which the number of edges between any two relevant vertices (that is the term relevant is used to say if a edge exists between two vertices than it will in this context be relevant vertices) is at most less than or equal to n. That is no two vertices in the multigraph G can have more than n edges connecting them.

We will first illustrate them by examples.

Example 2.3. Let G be a 5-multigraph given by the following figure.

G =

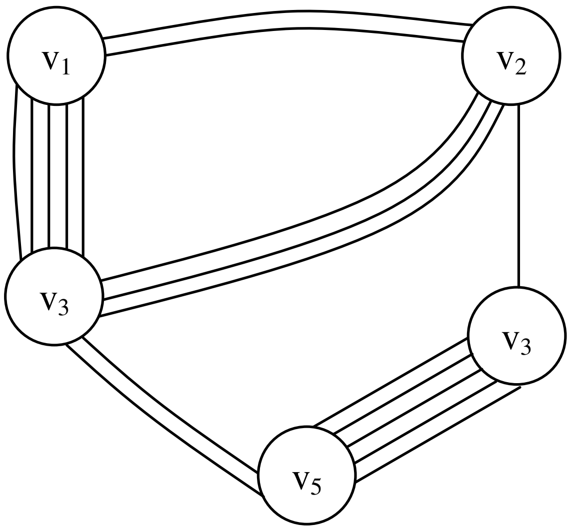


Figure 2.12

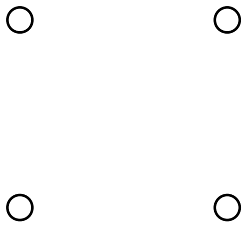
G is a connected 5-edge multigraph.

Clearly G is 5-edges multigraph with 5 vertices.

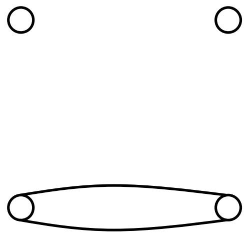
Finding the number of 5-edges multigraphs with 5 vertices is itself a difficult problem

Example 2.4. We denote the set of all 3-edged multigraphs with 4 vertices.

G_1 =



G_2 =



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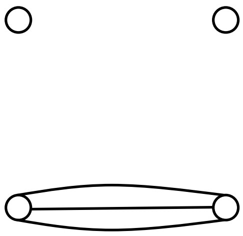

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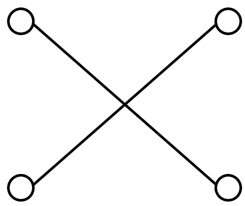

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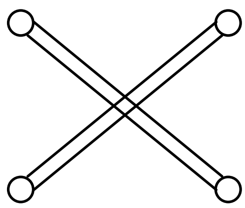

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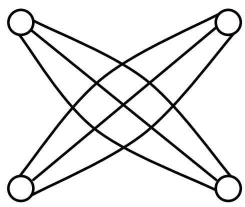

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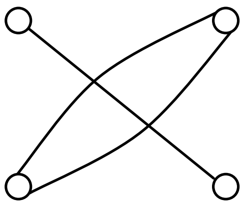

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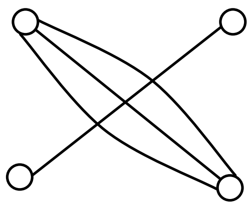

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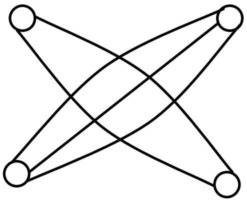
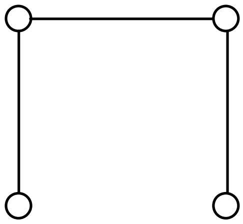


G_{10} =



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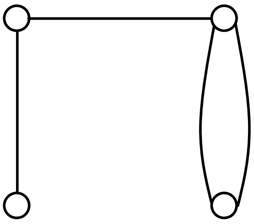

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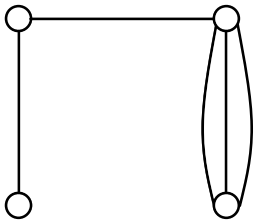
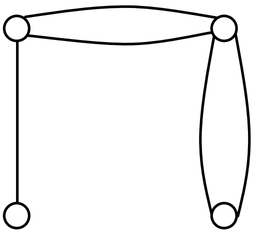
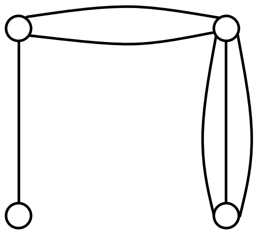


G_{13} =



G_{14} =



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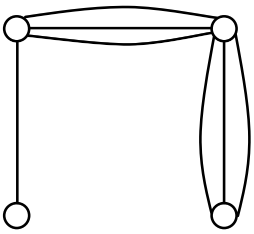

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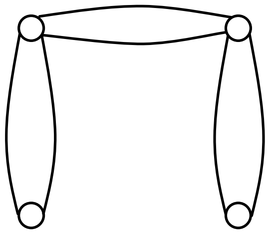

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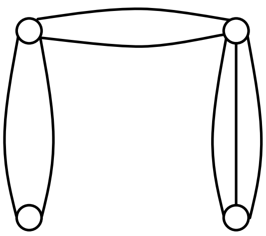

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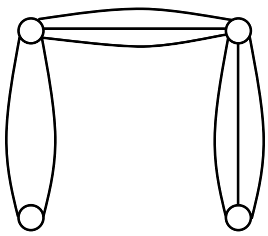

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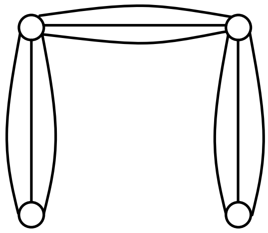

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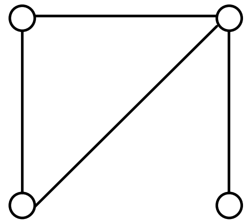

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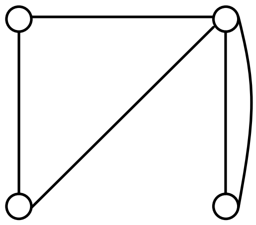

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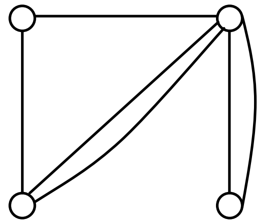

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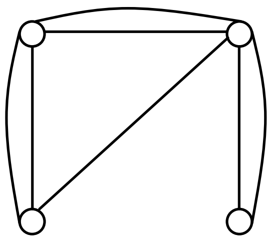

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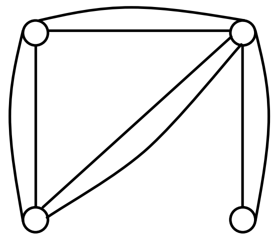

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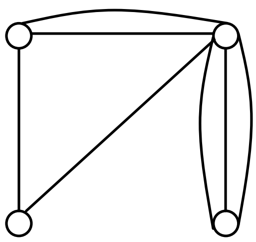

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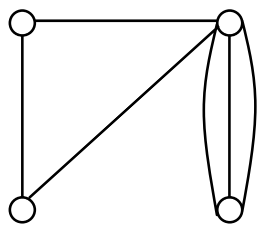

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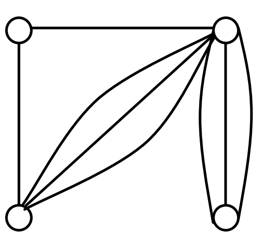

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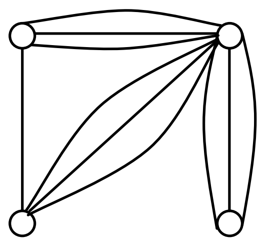

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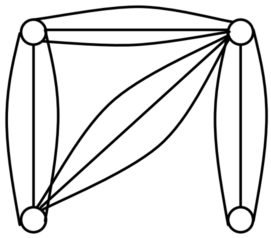

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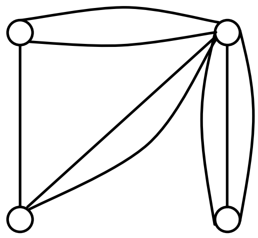

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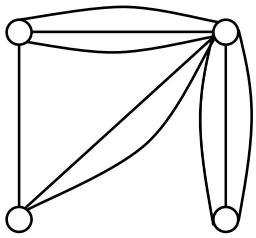

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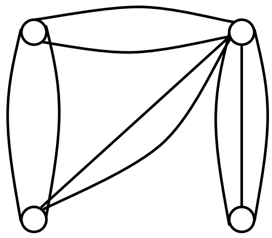

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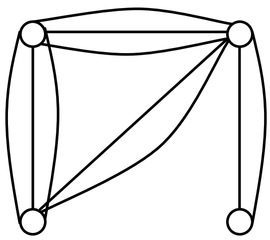

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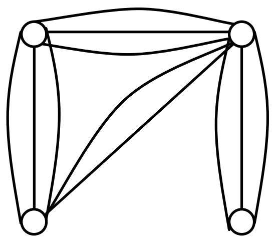

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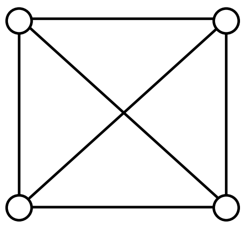

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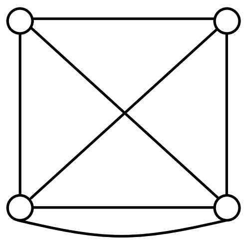

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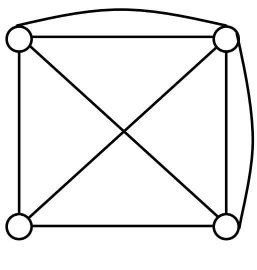

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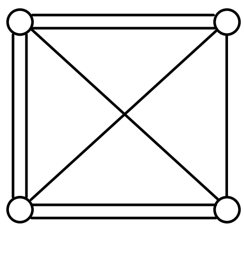
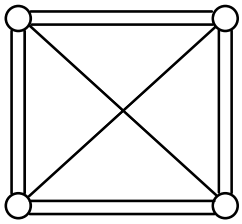
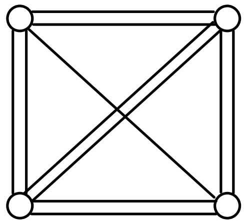
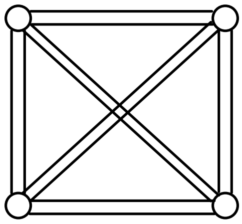
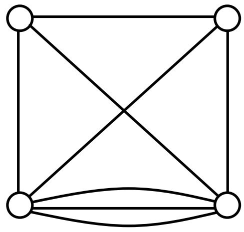
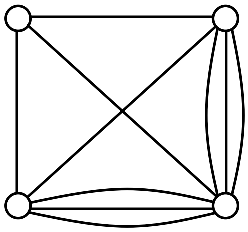


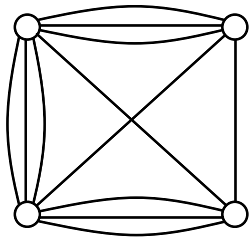
$G_{39} =$ 

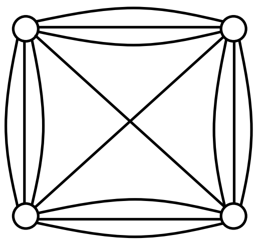
$G_{40} =$ 

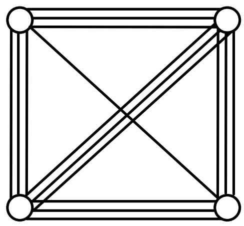
$G_{41} =$ 

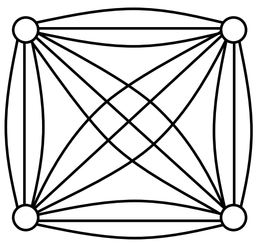
$G_{42} =$ 

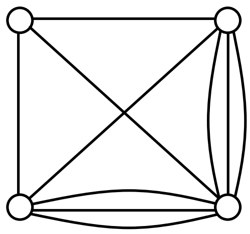
$G_{43} =$ 

$G_{44} =$ 

$G_{45} =$ 

$G_{46} =$ 

$G_{47} =$ 

$G_{48} =$ 

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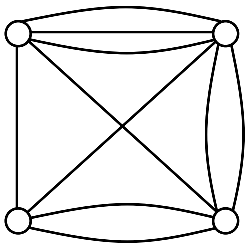

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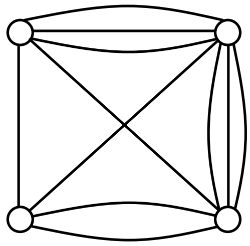

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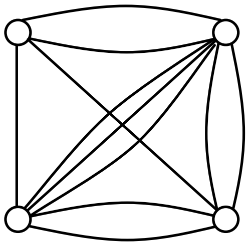

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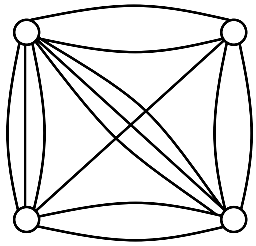

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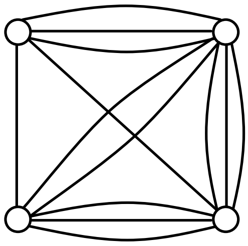

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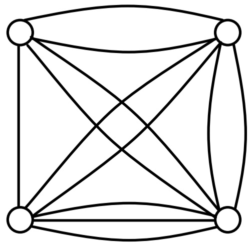

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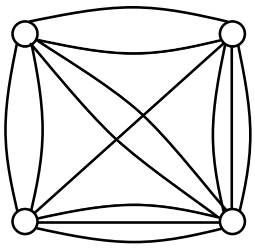

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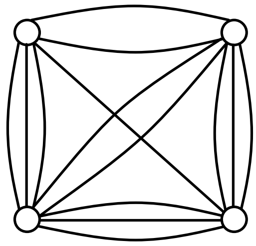

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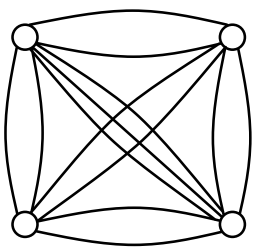

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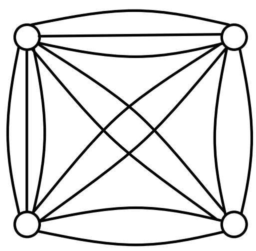

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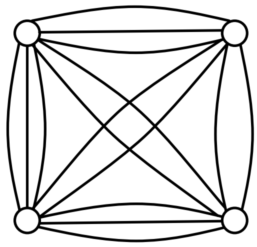

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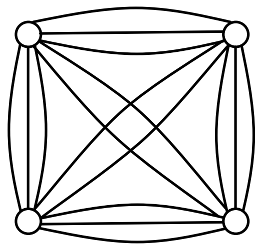

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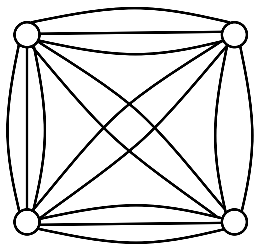

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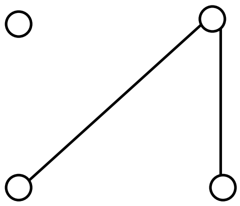

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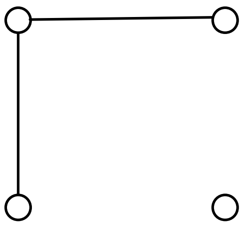

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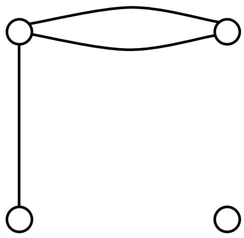

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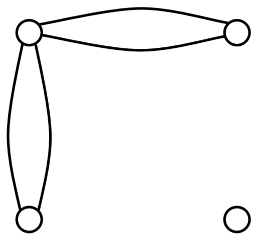

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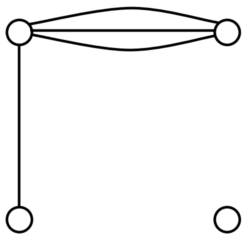

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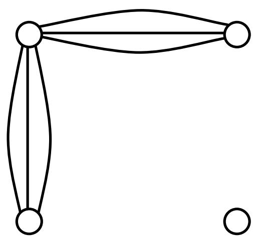

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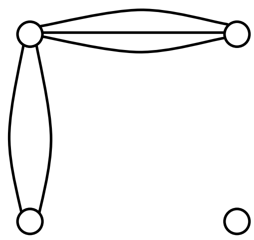

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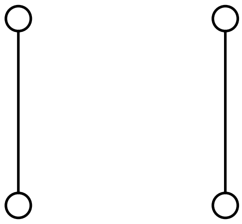

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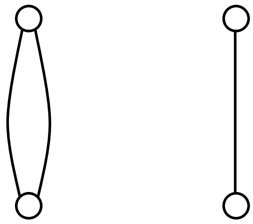

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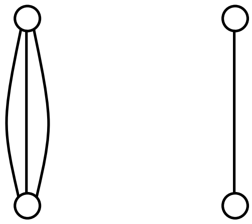

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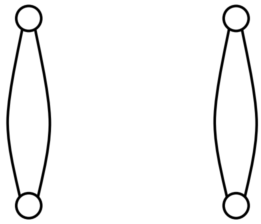

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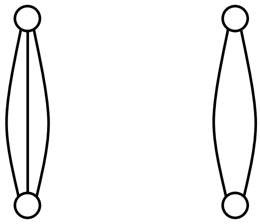

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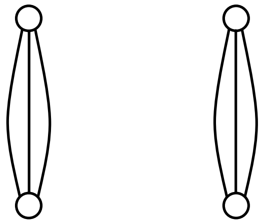
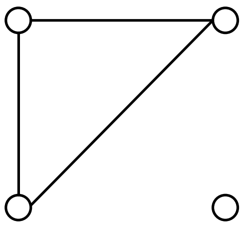
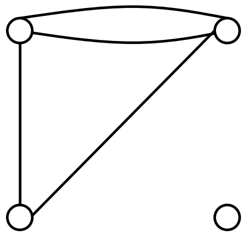


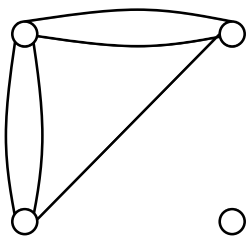
G_{75} =



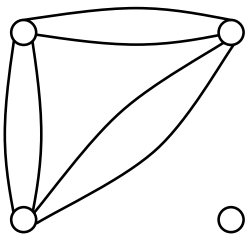
G_{76} =

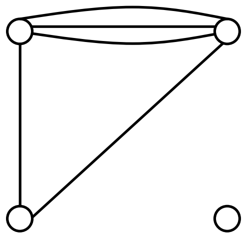


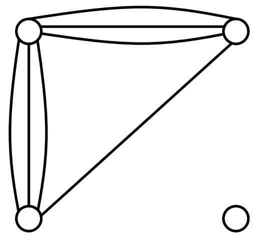
G_{77} =

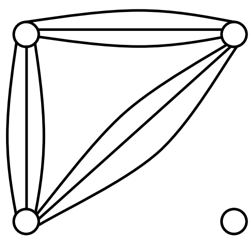


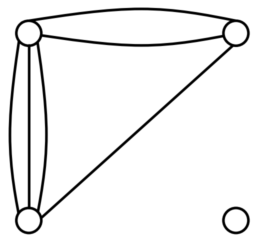
G_{78} =

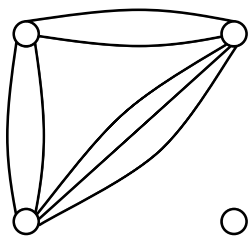


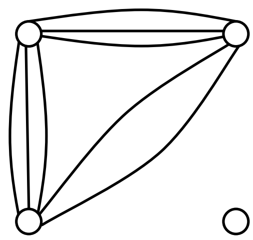
$G_{79} =$ 

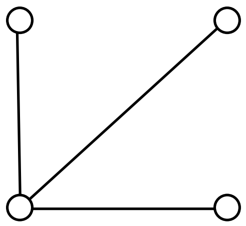
$G_{80} =$ 

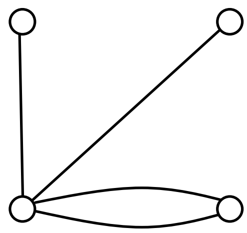
$G_{81} =$ 

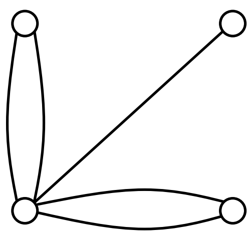
$G_{82} =$ 

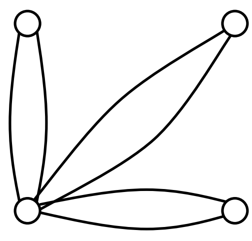
$G_{83} =$ 

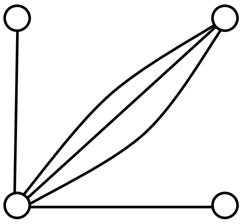
$G_{84} =$ 

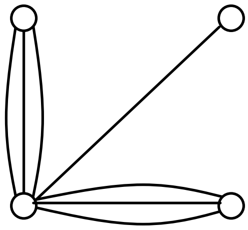
$G_{85} =$ 

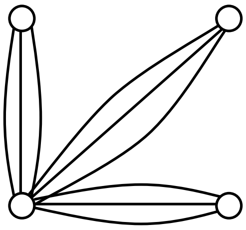
$G_{86} =$ 

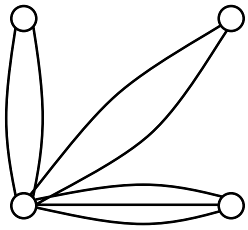
$G_{87} =$ 

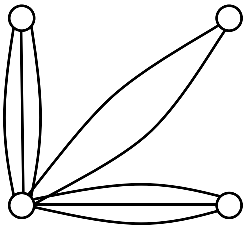
$G_{88} =$ 

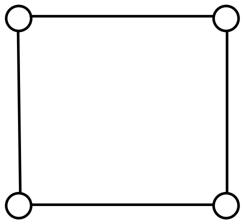
$G_{89} =$ 

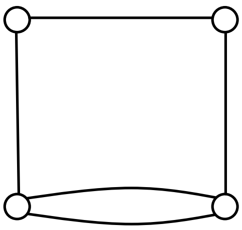
$G_{90} =$ 

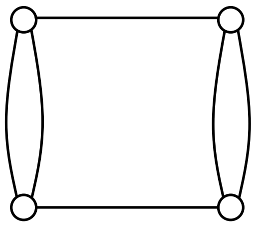
$G_{91} =$ 

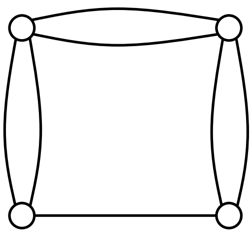
$G_{92} =$ 

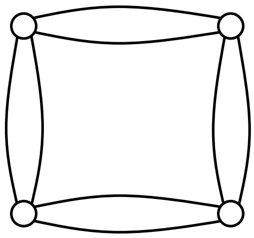
$G_{93} =$ 

$G_{94} =$ 

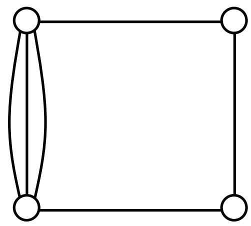
$G_{95} =$ 

$G_{96} =$ 

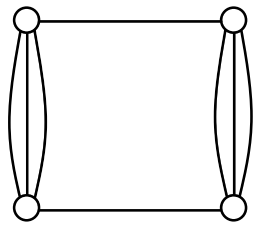
$G_{97} =$ 

$G_{98} =$ 

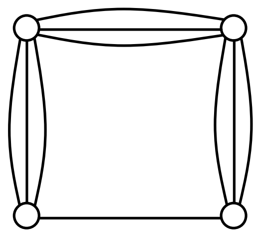
G_{99} =



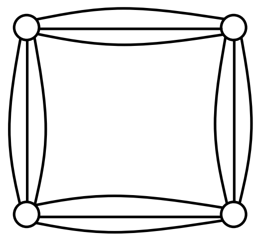
G_{100} =



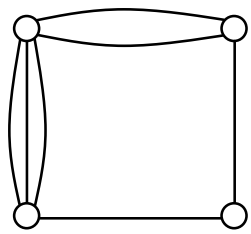
G_{101} =



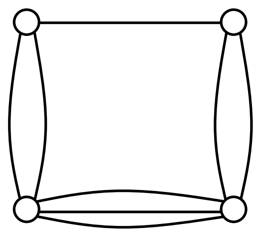
G_{102} =



G_{103} =



G_{104} =



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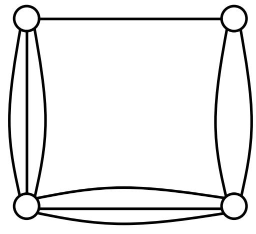

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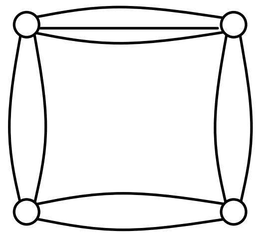
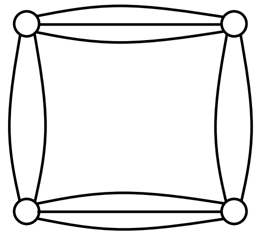


G_{107} =



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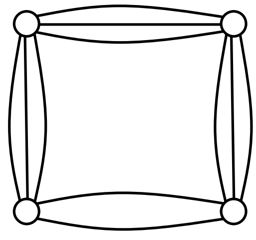
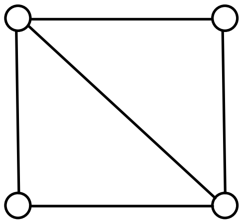
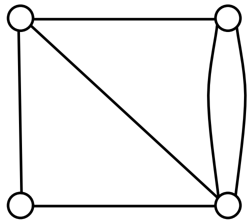
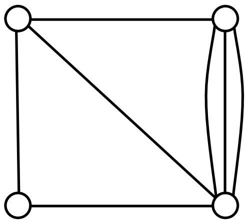
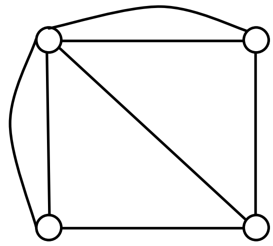
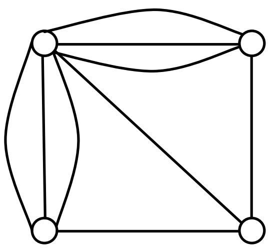


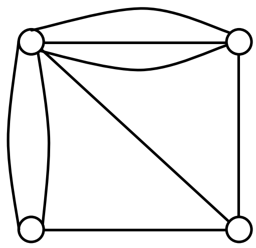
$G_{109} =$ 

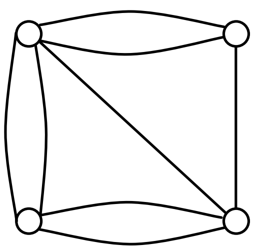
$G_{110} =$ 

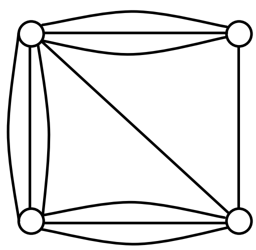
$G_{111} =$ 

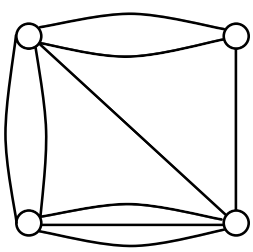
$G_{112} =$ 

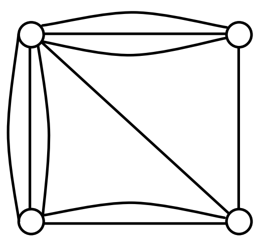
$G_{113} =$ 

$G_{114} =$ 

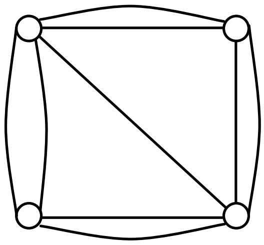
$G_{115} =$ 

$G_{116} =$ 

$G_{117} =$ 

$G_{118} =$ 

G_{119} =



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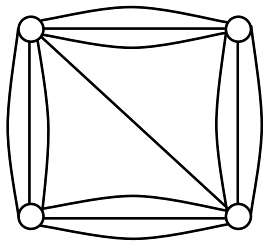
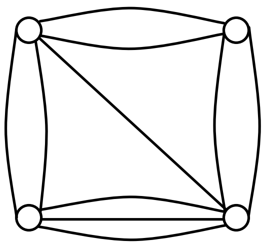
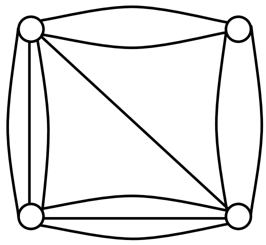


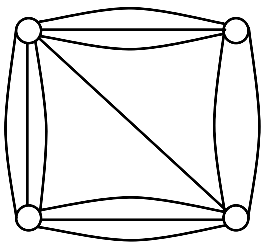
G_{121} =



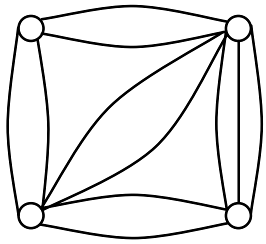
G_{122} =



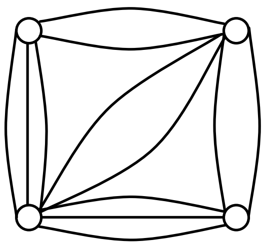
G_{123} =



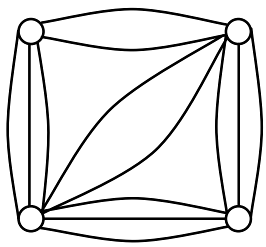
G_{124} =



G_{125} =



G_{126} =



G_{127} =

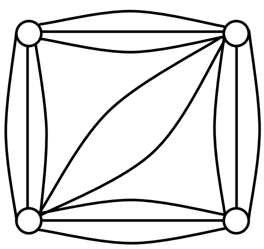


Figure 2.13

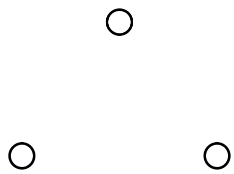
It is pertinent to keep on record that there are only 11 usual graphs with four vertices which are non-isomorphic. However, in case of 3 edges multigraphs with four vertices we see there are over 130 such multigraphs.

So one can imagine the number of multigraphs with four vertices and n-edges $n \geq 4$ will have a larger number of such multigraphs and so on.

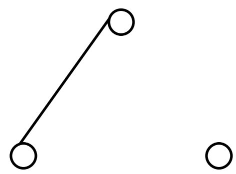
One of these problems will be encountered when we study the social networks with associated freeman index. For calculating freeman index happens to be a challenging problem.

Now we find out the number of case, of atleast 4 edges multigraph with 3 vertices in the followings.

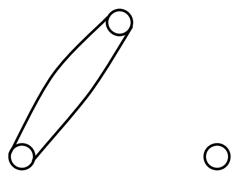
H_1 =



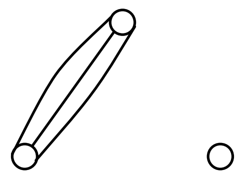
H_2 =



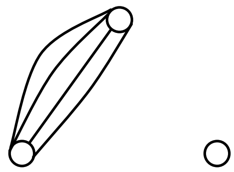
H_3 =



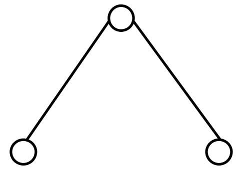
H_4 =



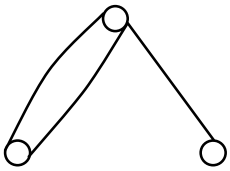
H_5 =



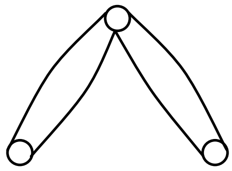
H_6 =



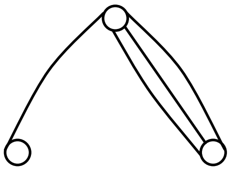
H_7 =



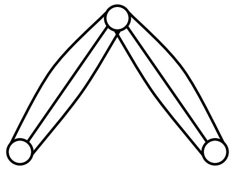
H_8 =



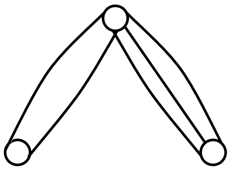
H_9 =



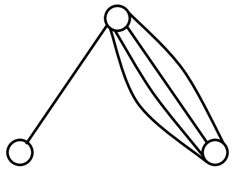
H_{10} =



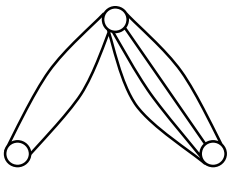
H_{11} =



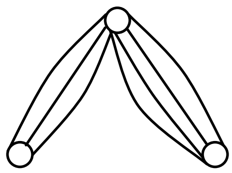
H_{12} =



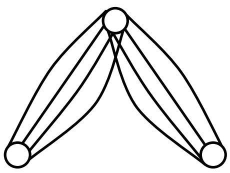
H_{13} =



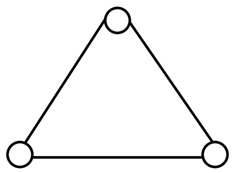
H_{14} =



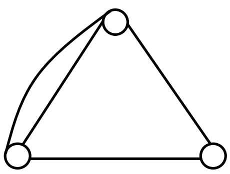
H_{15} =



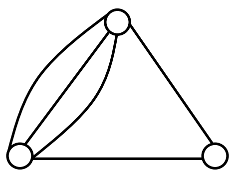
H_{16} =



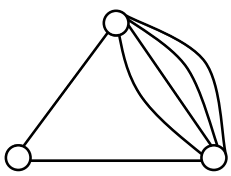
H_{17} =



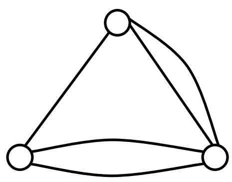
H_{18} =

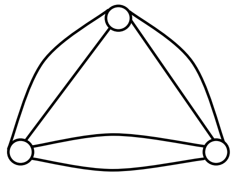


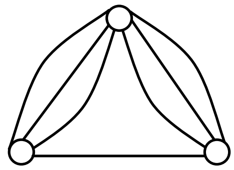
H_{19} =

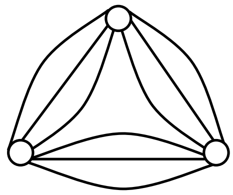


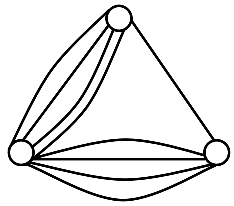
H_{20} =

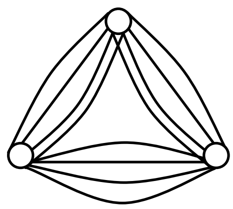


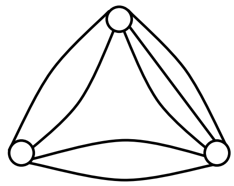
$H_{21} =$ 

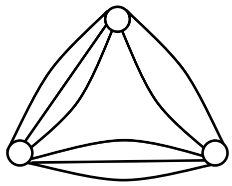
$H_{22} =$ 

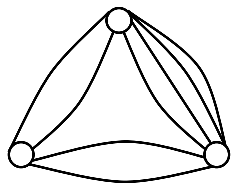
$H_{23} =$ 

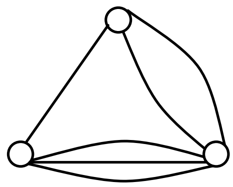
$H_{24} =$ 

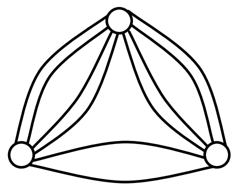
$H_{25} =$ 

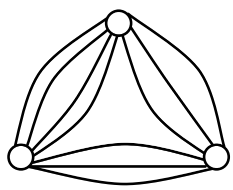
$H_{26} =$ 

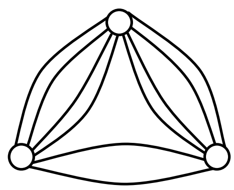
$H_{27} =$ 

$H_{28} =$ 

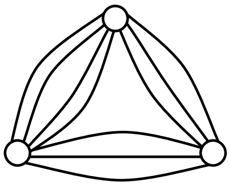
$H_{29} =$ 

$H_{30} =$ 

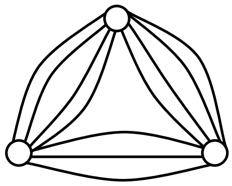
$H_{31} =$ 

$H_{32} =$ 

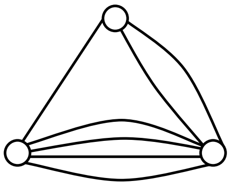
H_{33} =



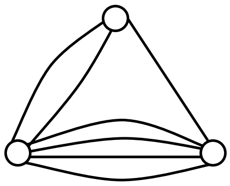
H_{34} =



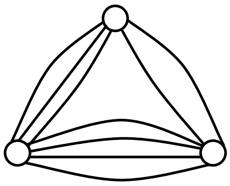
H_{35} =



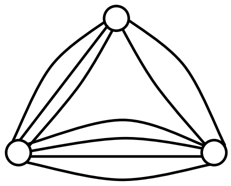
H_{36} =



H_{37} =



H_{38} =



Figures 2.14

Now we say in case of 3 vertex graphs there are only four such graphs given by the following figures.

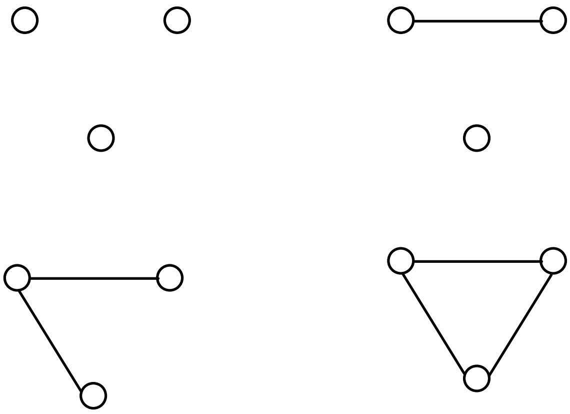
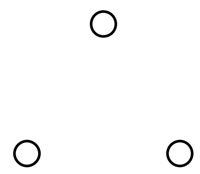
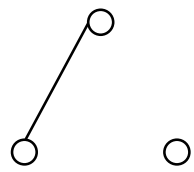


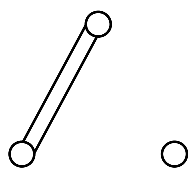
Figure 2.15

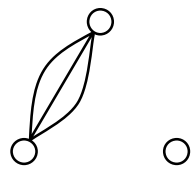
However there are 36 such 4-edges multigraphs with 3 vertices, in contrast with 4 usual graphs with 3 vertices.

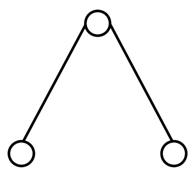
Now we find all 3-edges multigraphs with 3 vertices in the following.

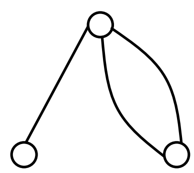
K_1 = 

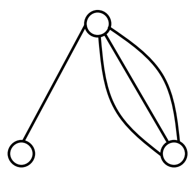
K_2 = 

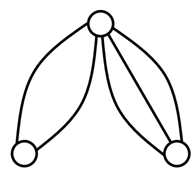
K_3 = 

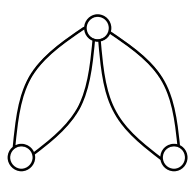
K_4 = 

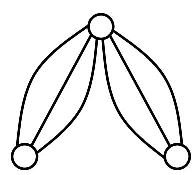
K_5 = 

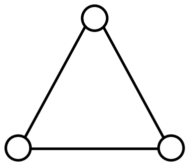
K_6 = 

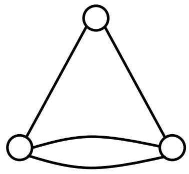
K_7 = 

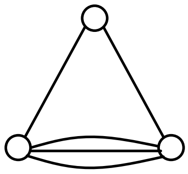
K_8 = 

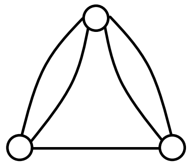
K_9 = 

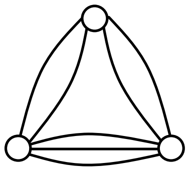
K_{10} = 

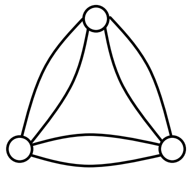
K_{11} = 

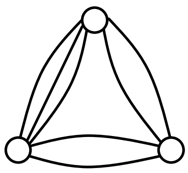
K_{12} = 

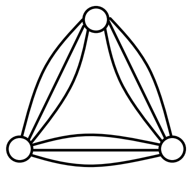
K_{13} = 

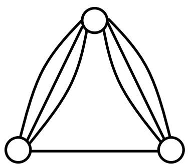
K_{14} = 

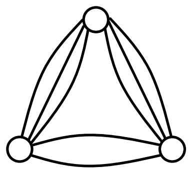
K_{15} = 

K_{16} = 

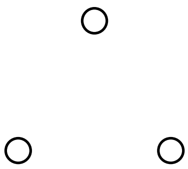
K_{17} = 

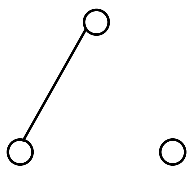
K_{18} = 

K_{19} = 

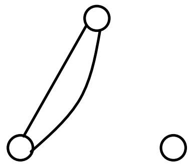
K_{20} = 

Now we enlist all 5-edges multigraphs with 3 vertices in the following.

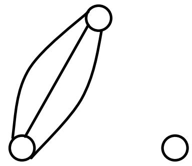
B_1 = 

B_2 = 

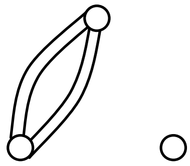
B_3 =



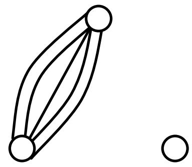
B_4 =



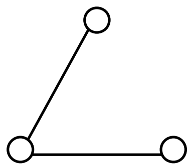
B_5 =



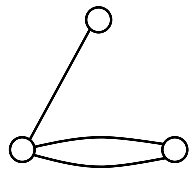
B_6 =



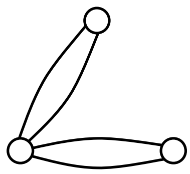
B_7 =



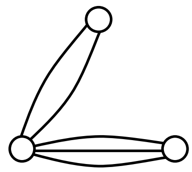
B_8 =



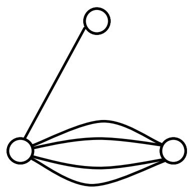
B_9 =



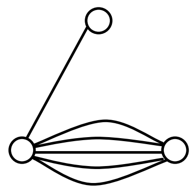
B_{10} =



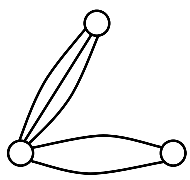
B_{11} =



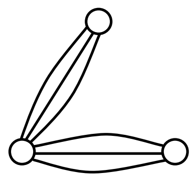
B_{12} =

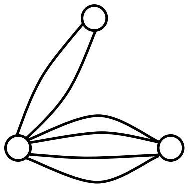


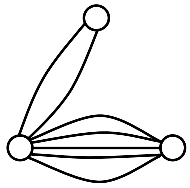
B_{13} =

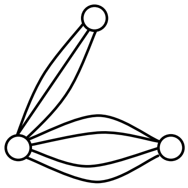


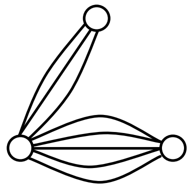
B_{14} =

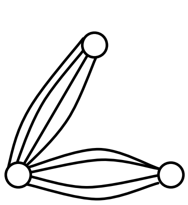


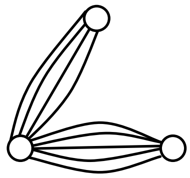
$B_{15} =$ 

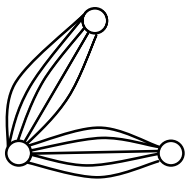
$B_{16} =$ 

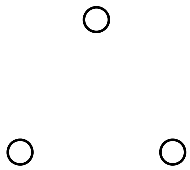
$B_{17} =$ 

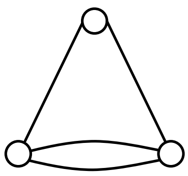
$B_{18} =$ 

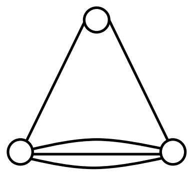
$B_{19} =$ 

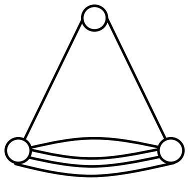
$B_{20} =$ 

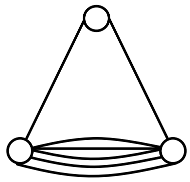
$B_{21} =$ 

$B_{22} =$ 

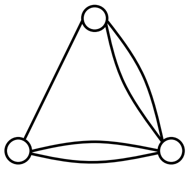
$B_{23} =$ 

$B_{24} =$ 

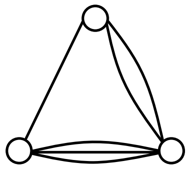
$B_{25} =$ 

$B_{26} =$ 

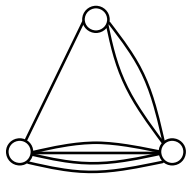
B_{27} =



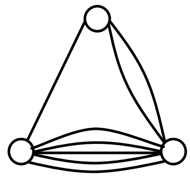
B_{28} =



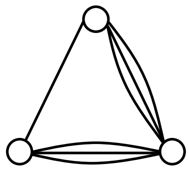
B_{29} =



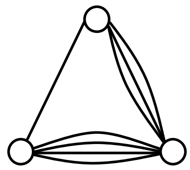
B_{30} =



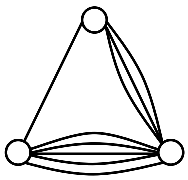
B_{31} =



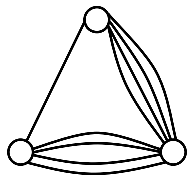
B_{32} =



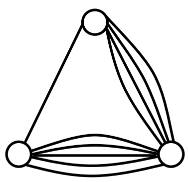
B_{33} =



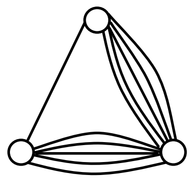
B_{34} =



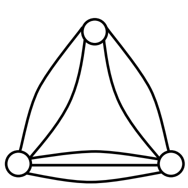
B_{35} =



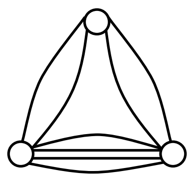
B_{36} =



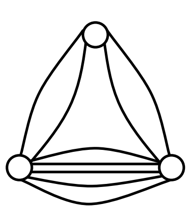
B_{37} =



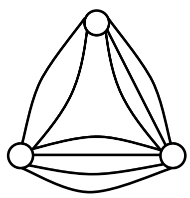
B_{38} =



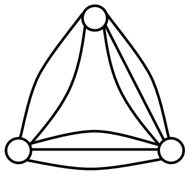
B_{39} =



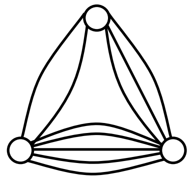
B_{40} =



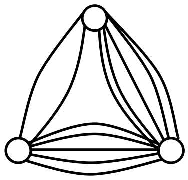
B_{41} =



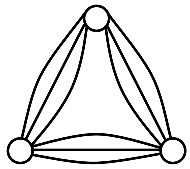
B_{42} =



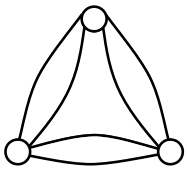
B_{43} =



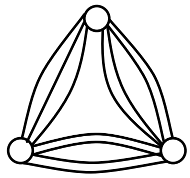
B_{44} =



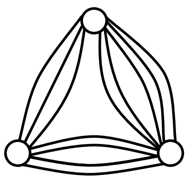
B_{45} =



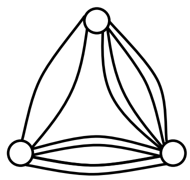
B_{46} =



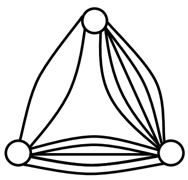
B_{47} =



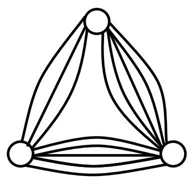
B_{48} =



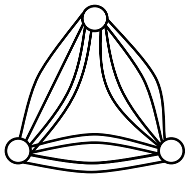
B_{49} =



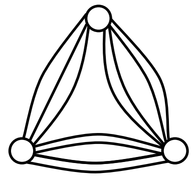
B_{50} =



B_{51} =



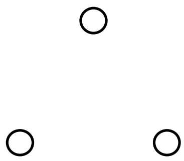
B_{52} =

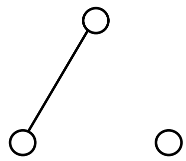


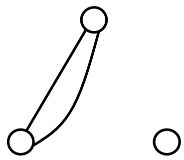
Figures 2.16

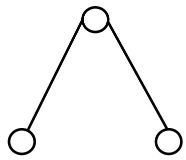
Now we just for the sake of providing the table give the number of triads which are 3-edges multigraphs and 2-edges multigraphs all of them are not directed in the following.

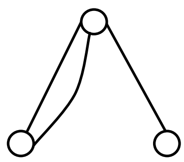
The list of 2 edges multigraphs is as follows.

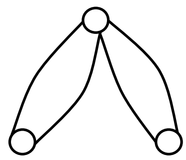
P_1 = 

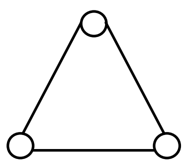
P_2 = 

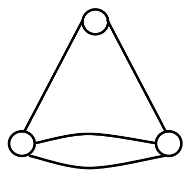
P_3 = 

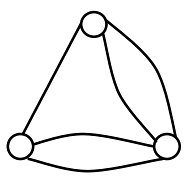
P_4 = 

P_5 = 

P_6 = 

P_7 = 

P_8 = 

P_9 = 

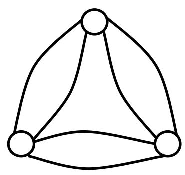
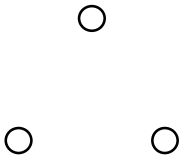
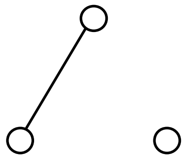
P_{10} = 

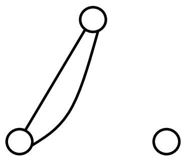
Figure 2.17

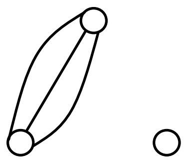
There are 10; 2-edges multigraphs with 3 vertices.

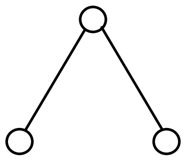
Now in the following we list out the 3-edges multigraphs with 3 – vertices.

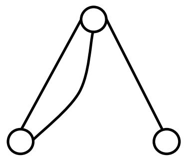
D_1 = 

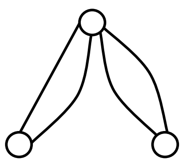
D_2 = 

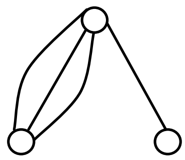
D_3 = 

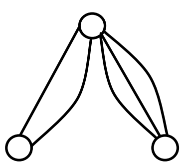
D_4 = 

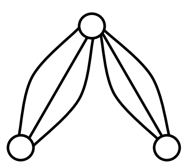
D_5 = 

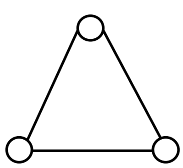
D_6 = 

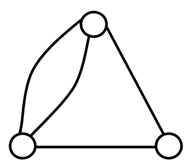
D_7 = 

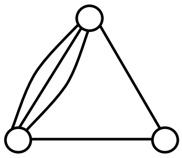
D_8 = 

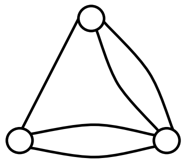
D_9 = 

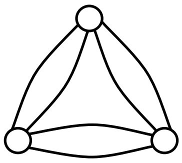
D_{10} = 

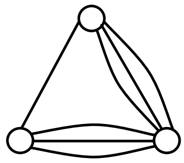
D_{11} = 

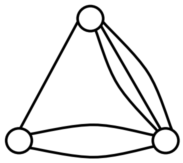
D_{12} = 

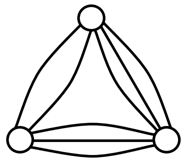
D_{13} = 

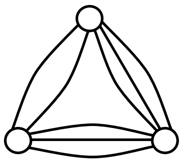
D_{14} = 

D_{15} = 

D_{16} = 

D_{17} = 

D_{18} = 

D_{19} = 

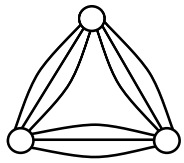
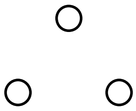
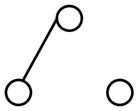
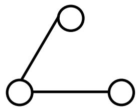
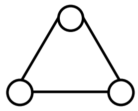
D_{20} = 

Figure 2.18

Now we proceed on the give the table for number of n-edges multigraphs with 3 vertices, ($n < \infty$); n = 1, 2, 3, 4 and 5.

Table for n-edges multigraphs with three vertices.

n				
n = 1	1	1	1	1
n = 2	1	2	3	4
n = 3	1	3	6	10
n = 4	1	4	10	20
n = 5	1	5	15	35
n = 6	1	6	21	56
n = 7	1	7	28	84
n = 8	1	8	30	120
n = 9	1	9	45	165
n = 10	1	10	55	220
n = 11	1	11	66	286
n = 12	1	12	78	364
n = 13	1	13	91	455
n = 14	1	14	105	560
n = 15	1	15	120	680
n = 16	1	16	136	816
n = 17	1	17	153	969
n = 18	1	18	171	1140

For any we have the following

n	1	n	$\frac{n(n+1)}{2}$	$\frac{n(n+1)}{2} + E$

where E is the total number of (n – 1) edges multigraphs for a complete graph of order 3.

Thus we try to find the number of n-edges multigraphs with k vertices.

This happens to be one of the challenging problems.

For n = 3 we can give a method of finding them by knowing the number previous (n – 1) - edge multigraph for

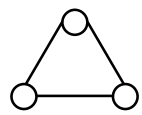


Figure 2.19

Further the sum of each row yields the number of (n – 1) edge multigraphs.

Proposition 2.1. Let G be a n-edge multigraph with two vertices ($1 \leq n < \infty$). For each n there are n and only n dyads.

Proof. Follows from the fact number of edges connecting the vertices of dyad G can be 1 or 2 or 3 or … n. So there can be only n such n-edges multigraph dyads given by the following figure.

Figure 2.20

(n – 1) edges n – edges

Figure 2.21

Barring the empty n-edges multigraph.

We have a such n - edges multigraph.

Now finding for any arbitrary number of vertices say m and say n-edges happens to be a challenging problem.

More problem is encountered with when we go for applying these concepts to social information network for calculating the Freeman index.

For we see in case of a order 3 graph or a triad there are only 4 of them but in case of same triad if we take the 10 edges multigraph we have 296 such multigraphs.

If we consider the 18 edges multigraph with 3 set of vertices there are 1 + 18 + 171 + 1140 = 1330 in number. So calculating Freeman index happens to be a very difficult task.

However it is very vital for multistructures, play a vital role, so multigraphs and multinetworks have become mandatory as tools to analyse them.

Now we provide a few examples of them and means and methods to find matrix representation of them in the following.

Example 2.5. Let G be a 6-edges multigraph given by the following figure.

G =

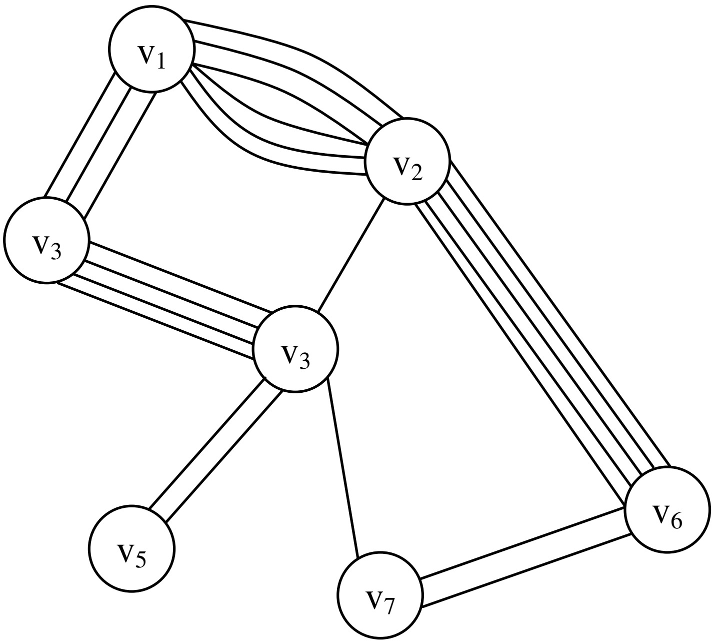


Figure 2.22

The adjacency matrix associated with the 6-edges multigraph can be any one of the following.

If multiedges are treated as usual we can suggest as

$$A_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 0 & 6 & 3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 1 & 0 & 5 & 0 \\ 3 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \end{pmatrix} \end{array}.$$

We list out the drawbacks of using the matrix A_1 as the associated matrix of the 6-edge multigraph G.

i) The matrix A_1 by observations can give the 6-multigraph and vice versa.

For instance we give an example of an adjacency matrix and obtain the corresponding multigraph and comment upon them.

Let M be the adjacency matrix given in the following.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 4 & 0 & 2 \\ 3 & 0 & 5 & 0 & 6 & 0 \\ 0 & 5 & 0 & 0 & 1 & 3 \\ 4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 6 & 1 & 5 & 0 & 2 \\ 2 & 0 & 3 & 0 & 2 & 0 \end{bmatrix} \end{array}.$$

Let G be the multigraph given by the following figure.

G =

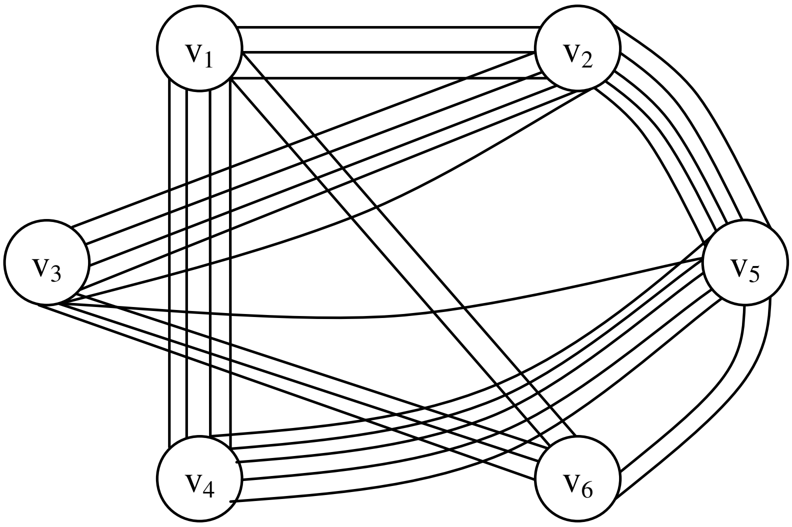


Figure 2.23

Unless it is clearly stated that the matrix represents a multigraph one will not be in a position to get it, as there is also a probability that it is misunderstood with weighted graphs. Further the highest entry say M in the given matrix will give the m-edges multigraph.

So we can say for a given a symmetric n × n matrix with no diagonal entries with highest value m we will get a m-edges multigraphs G with n nodes and vice versa.

We will illustrate by an example a 7 edges multigraph with 5 edges and find some of its multisubgraphs.

Example2.6. Let G be a 7-edges multigraphs with 5 vertices given by the following figure.

G =

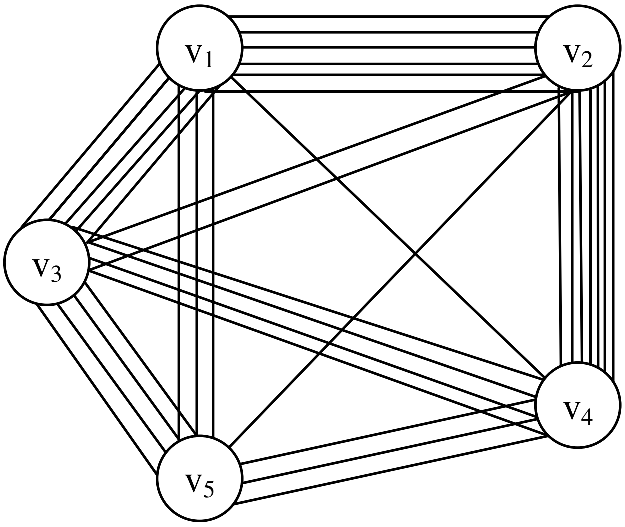


Figure 2.24

The adjacency matrix M associated with G is as follows.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 6 & 5 & 1 & 3 \\ 6 & 0 & 2 & 7 & 1 \\ 5 & 2 & 0 & 4 & 4 \\ 1 & 7 & 4 & 0 & 3 \\ 3 & 1 & 4 & 3 & 0 \end{pmatrix} \end{array}.$$

We see M is a symmetric matrix. As no a_{ij} is zero if $i \neq j$ we know G is a complete 7-edges multigraph.

Infact all 7-edges multisubgraphs are also complete.

The multisubgraphs which we enlist is edge preserving multisubgraphs.

Infact there are $5C_2 + 5C_3 + 5C_4 = 25$ number of 7-edges multisubgraphs of G barring the single point multisubgraphs.

We give a few of them in the following.

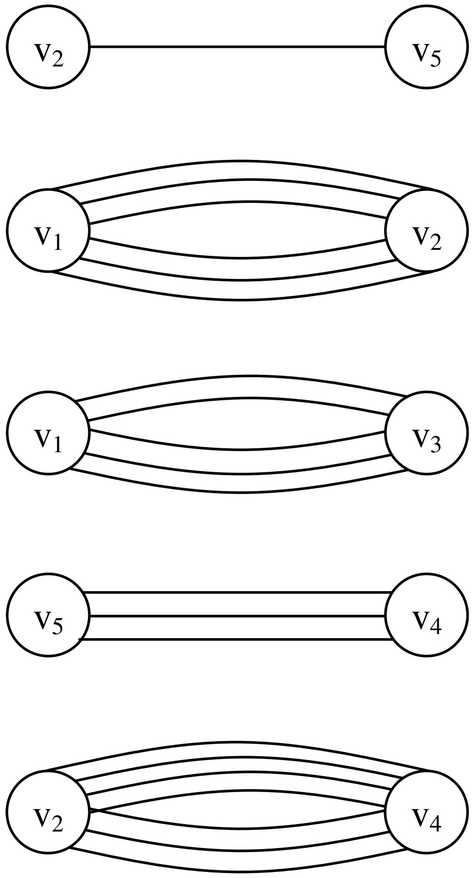
$H_1 =$

$H_2 =$

$H_3 =$

$H_4 =$

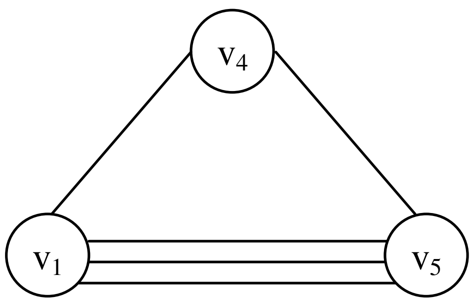
$H_5 =$



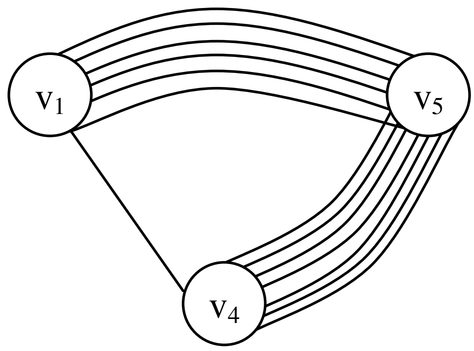
and so on.

Figure 2.25

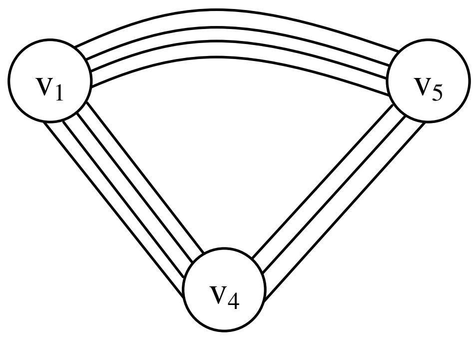
$H_6 =$



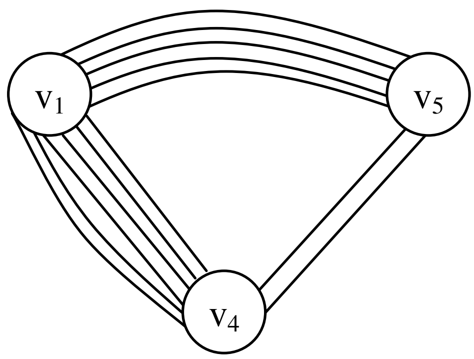
$H_7 =$



$H_8 =$



$H_9 =$



and so on.

Figure 2.26

We see all these 7-edges multisubgraphs are triads or complete multisubgraphs of order three.

Next we proceed onto give a few 7-edge multisubgraphs with 4 vertices in the following.

H_{10} =

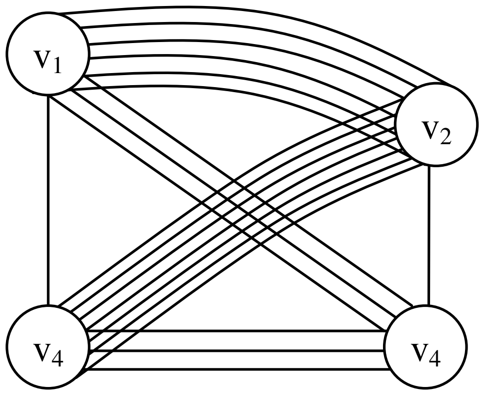
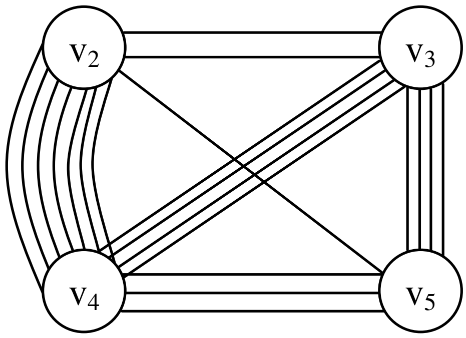


Figure 2.27

We see H_{10} is 7-edge multisubgraph with 4 vertices which is complete.

Infact all 7-edges multisubgraphs with four vertices are complete multisubgraphs.

H_{11} =



H_{12} =

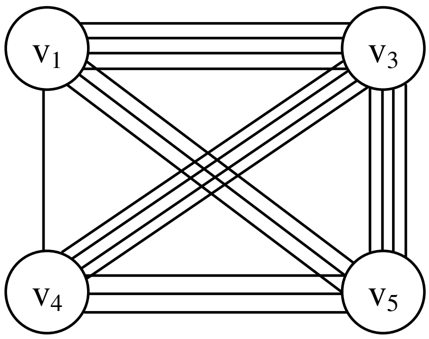


Figure 2.28

Clearly all 7-edges multigraphs with four vertices which will be 5 in number are all complete.

We can find also the corresponding matrices for these subgraphs. What is difficult in this case is defining clique or quasi clique. For even if we have complete 7-edges multisubgraphs with four vertices the notion of completeness happens to be little unsatisfactory as all the multiedges may not be of the same number. For instance take the complete 7-edges multisubgraph H_{12} with four vertices clearly one edge is between (v_1) and (v_4) and 3 edges between the vertices v_4 and v_5 and v_1 and v_5.

Likewise four edges between the vertices v_4 and v_3 and v_3 and v_5. Thus the number of edges vary as 1, 3, 4 and 5. Still we choose to call them as complete 7-edges multisubgraphs with four vertices.

Thus in case we have complete n-edges multigraph G we will have all the n-edges multisubgraph of that complete n-edge multigraph will only be complete n-edge multigraph.

This property of complete n-edge multigraphs are very vital.

Next we find the n-edges multigraphs of different types.

Example 2.7. Let G be a 6-edges multigraph with 8 vertices given by the following figure.

G =

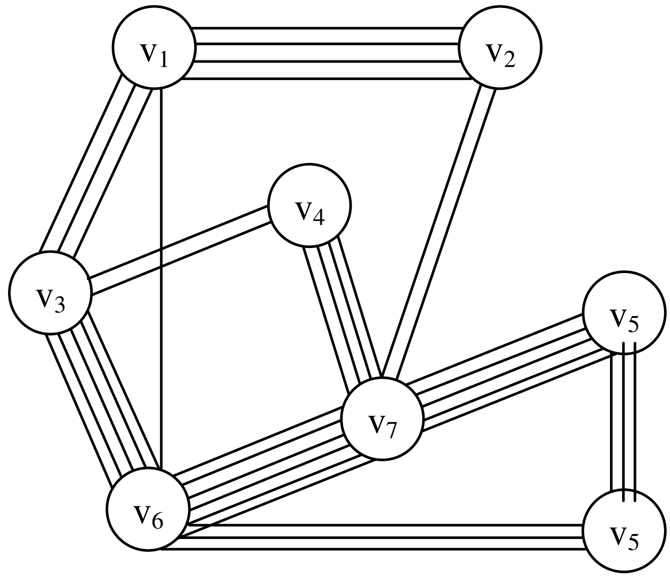


Figure 2.29

There are $8C_2 + 8C_3 + 8C4 + 8C_5 + 8C_6 + 8C_7$ number of 6 edges multisubgraphs.

Clearly G is not a complete 6 - edges multigraph.

Now we enlist a few of the 6-edges multisubgraphs in the following.

H_1 =

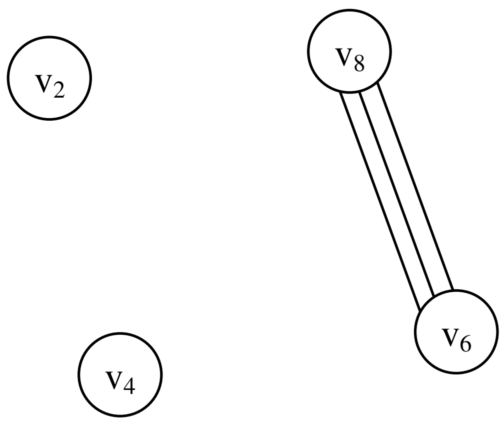


Figure 2.30

The 6-edges multisubgraph H_1 of G is only a disconnected 6-edges multisubgraph.

Let H_2 be the 6-edge multisubgraph given by the following figure.

H_2 =

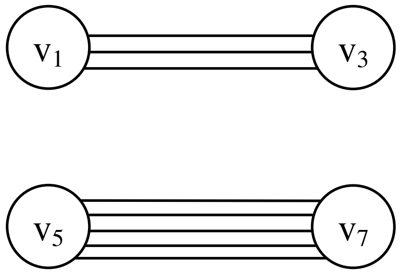


Figure 2.31

Let H_3 be the 6-edge multisubgraph of G given by the following figure.

H_3 =

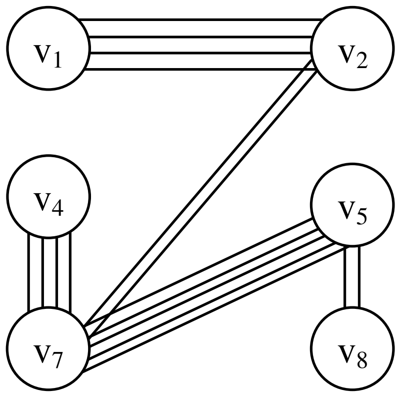


Figure 2.32

H_3 is a connected 6-edges multisubgraph with 6 edges.

Let H_4 be the 6-edges multisubgraph with 5 vertices given by the following figure.

H_4 =

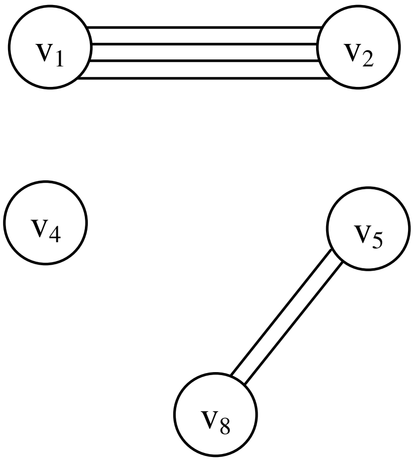


Figure 2.33

is a disconnected 6-edges multisubgraph with 5 vertices.

Now we consider a 6-edges multisubgraph H_5 with 7-vertices given by the following figure.

H_5 =

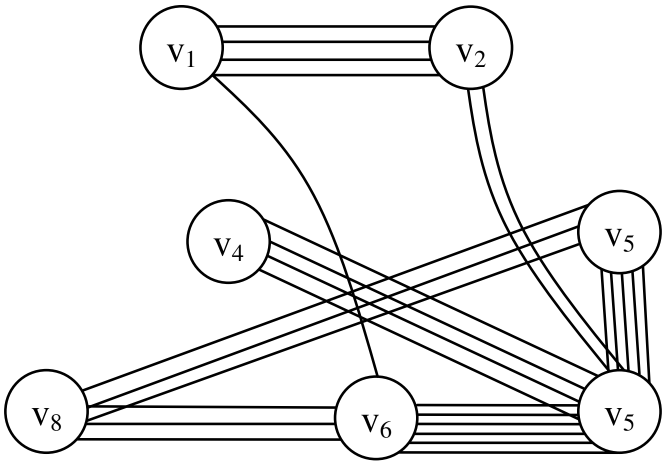


Figure 2.34

Clearly H_5 is a connected 6-multisubgrpah with 7 vertices.

Infact all 6-edges multisubgraphs with 7 vertices are connected.

However all 6-edges multisubgraphs with 6 vertices are not connected only some of them are connected.

Let H_6 be the 6-edges multisubgraph given by the following figure.

$H_6 =$

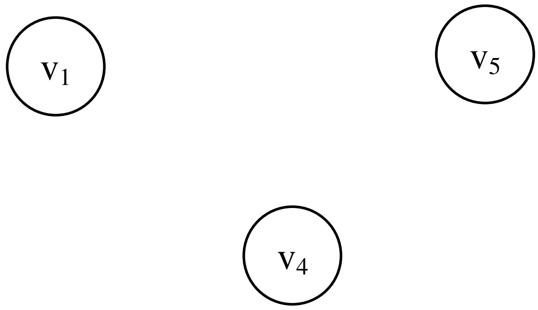


Figure 2.35

We see H_6 is a 6-edges empty multisubgraph with three vertices.

$H_7 =$

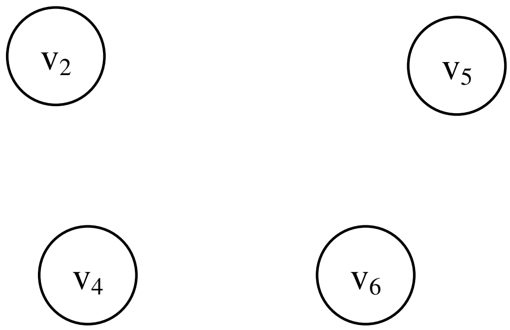


Figure 2.36

H_7 is also a 6-edges empty multisubgraph with 4 vertices.

Now we give an example of a 8-edges multigraph with 8-vertices.

Example 2.8. Let G be a 8 - edges star multigraph with 8 vertices given by the following figure.

G =

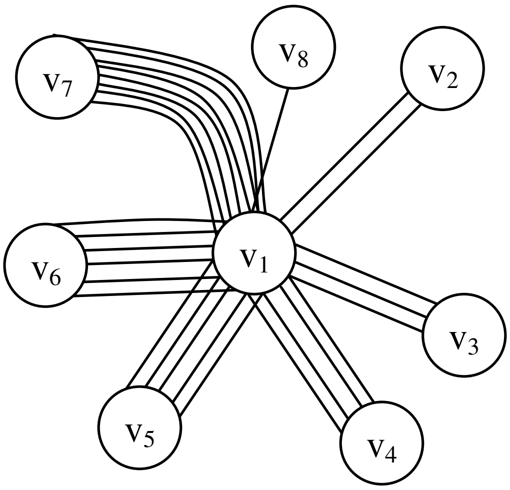


Figure 2.37

All 8-edges multisubgraphs which do not include the vertex v_1 are empty 8-edges multisubgraphs.

All 8-edges multisubgraphs which include vertex v_1 are star 8-edges multisubgraphs.

We now describe those 8-edges multisubgraphs.

K_1 =

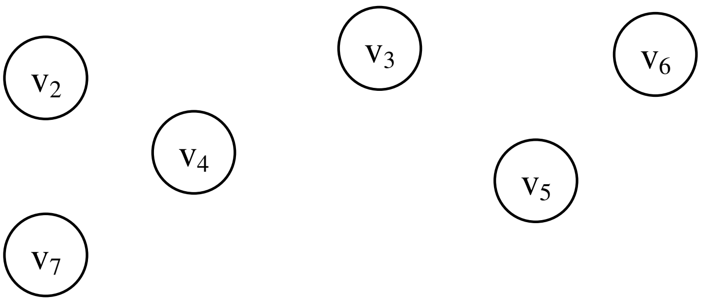


Figure 2.38

K_1 is just a 8-edges empty multisubgraph.

We now describe the 8-edges multisubgraph K_2 given by the following figure.

K_2 =

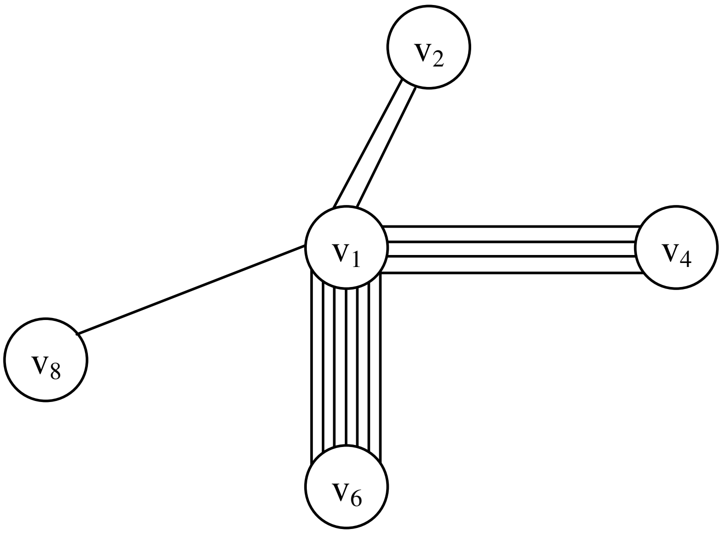


Figure 2.39

In view of all this we have the following theorem.

Theorem 2.1. *Let G be a n-edges star multigraph with m vertices. Let v_1 be the central node.*

i) *All n-edges multisubgraphs which contain v_1 as one of its vertices are n-edges star multisubgraphs.*
ii) *All n-edges multisubgraphs which do not contain the vertex v_1 are only n-edges empty multisubgraphs.*

Proof is direct and hence left as an exercise to the reader.

The following theorem can be proved by the reader.

Theorem 2.2. *Let G be a n-edges multigraph which is complete.*

Every n-edges multisubgraph of G is complete.

Now we provide an example of a circle n-edge multigraph.

Example 2.9. Let G be a 6-edges circles multigraph with 9 vertices given by the following figure.

G =

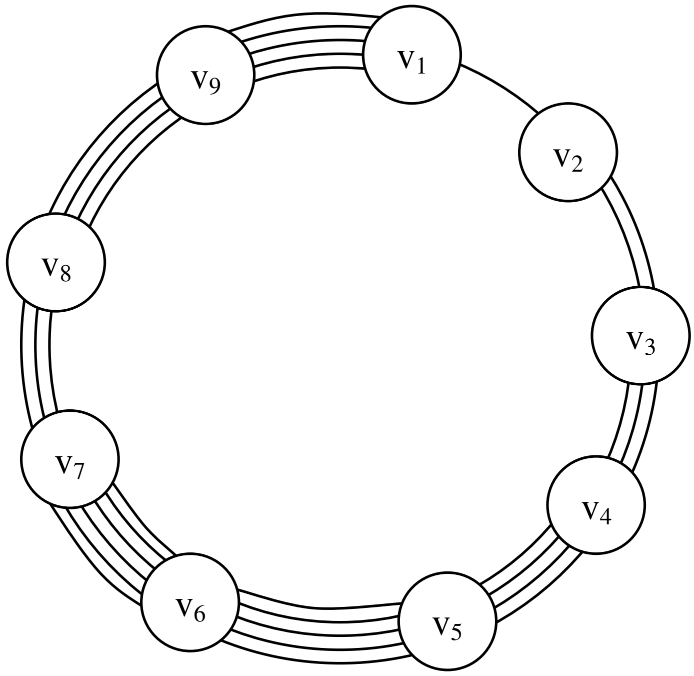


Figure 2.40

We give a few 6-edges multisubgraph P_1 of G with 5 vertices given by the following figure

P_1 =

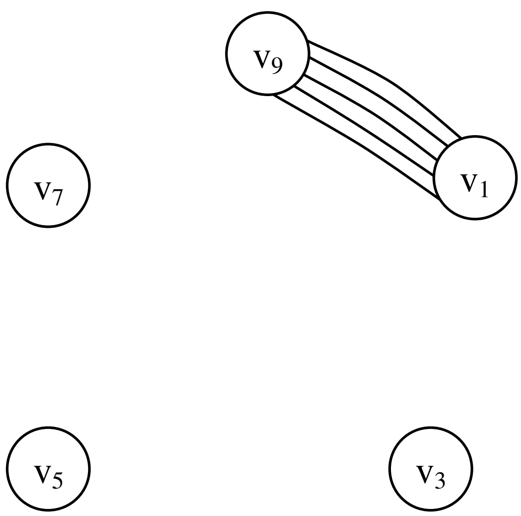


Figure 2.41

Clearly P_1 is a disconnected 6-edges multisubgraph which is disconnected.

Let P_2 be the 6-edges multisubgraph of G given by the following figure.

$P_2 =$

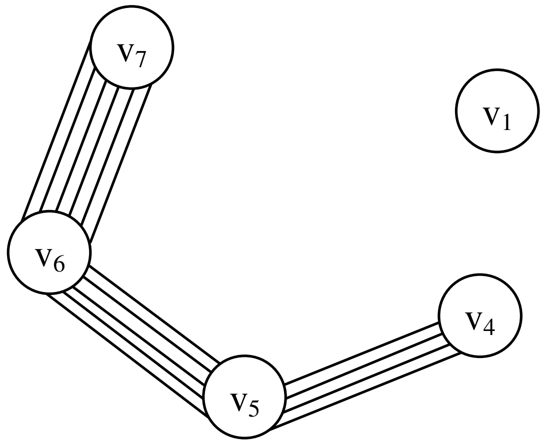


Figure 2.42

We see P_2 is also a disconnected 6-edges mutlisubgraph of G.

Let P_3 be the 6-edges multisubgraph of G given by the following figure.

$P_3 =$

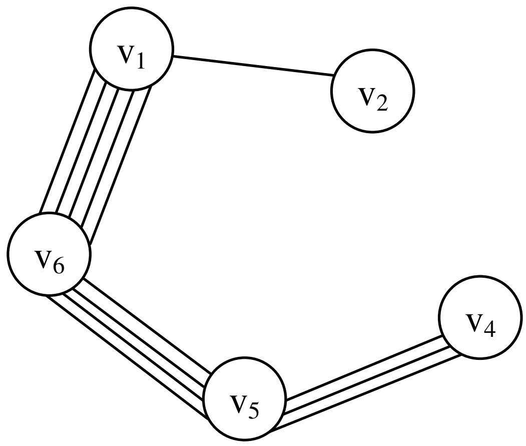


Figure 2.43

Clearly P_3 is a connected 6-edges multisubgraph which is not a circle.

Hence no 6-edges multisubgraph of G will be circle. In view of this we have the following theorem.

Theorem 2.3. *Let G be n-edges circle multigraph with m vertices.*

i) *None of the n-edges multisubgraphs are circles.*

ii) *H_i the n-edges multisubgraph is empty if the nodes are picked alternatively.*

iii) *H_j's the n-edges multisubgraphs are disconnected if the vertex set / nodes are not continuously indexed.*

iv) *H_j's the n-edges multisubgraphs are connected if the vertex set / nodes are continuously indexed.*

Proof. Given G is a circle n-edges multigraph with $v_1, v_2, \ldots, v_m$ as vertices and we see there exists t-edges only between v_i and v_{i+1}; $1 \leq i \leq m - 1$ and a t-edges between v_1 and v_m; $1 \leq t \leq n$. Thus G is a circle n-edges multigraph.

Proof of (i). If H is any proper n-edges multisubgraph then clearly H is not a circle graph.

Proof of (ii). If the set $v_2, v_4, v_6, \ldots, v_{2t}$, $2t < m$ are taken clearly that collection is a empty n-edges multisubgraph of G for varying t; $2 < 2t < m$.

Likewise $v_1, v_3, v_5, \ldots, v_{2t+1}$; $2t+1 < m$ are taken clearly that collection is a empty n-edges multisubgraph of G for varying t, $2 < 2t + 1 < m$.

Proof of (iii). If we have an arbitrary collection of vertex subset of the vertex set G such that if in that collection.

v_i or v_{i+1} only occurs and v_{i-1} and v_{i+2} has occurred (or in the mutually exclusively sense), then the resulting n-edges multisubgraph is disconnected.

Proof of (iv). If we have $\{v_1, v_{i+1}, \ldots, v_t\}$ a subset of vertex sets which forms the consecutive indexing $i < 1 + 1 < \ldots < t$ then the resulting n-edges multisubgraph is connected. Hence the claim.

Finally we proceed onto study n-edges multigraphs which are trees or to be specific multitrees.

Example2.10. Let G be a 9-edges multigraph which is a tree given by the following figure.

G =

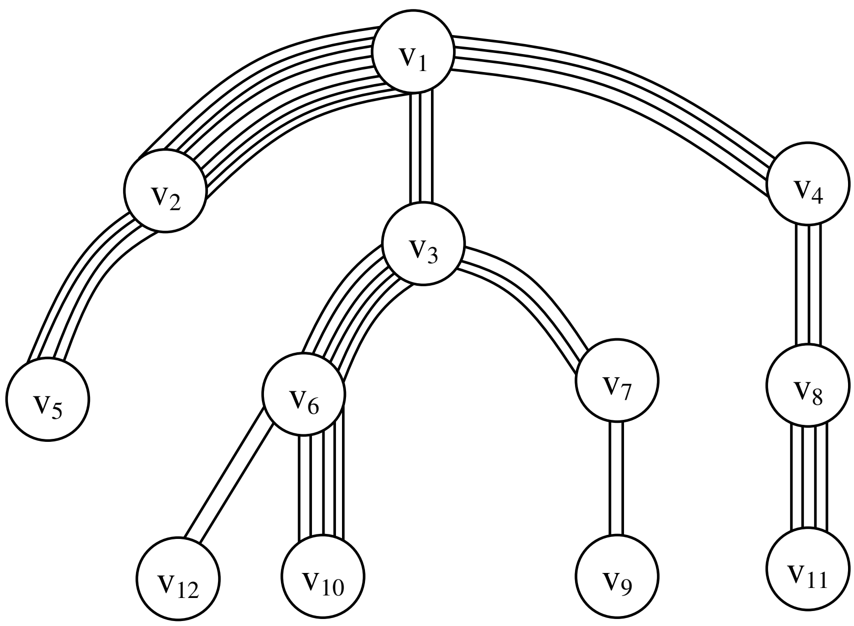


Figure 2.44

Clearly G is a multitree with maximum 9 edges.

We see all connected 9-edges multisubgrahs are only 9-edges multitrees. However if they are disconnected the gaint component will be a multitrees and the rest will be singleton nodes or in the extreme case the 9-edges multisubgraph can also be empty.

To this effect we will provide multisubgraphs for this G in the following.

Let H_1 be the 9-edges multisubgraph given by the following figure.

H_1 =

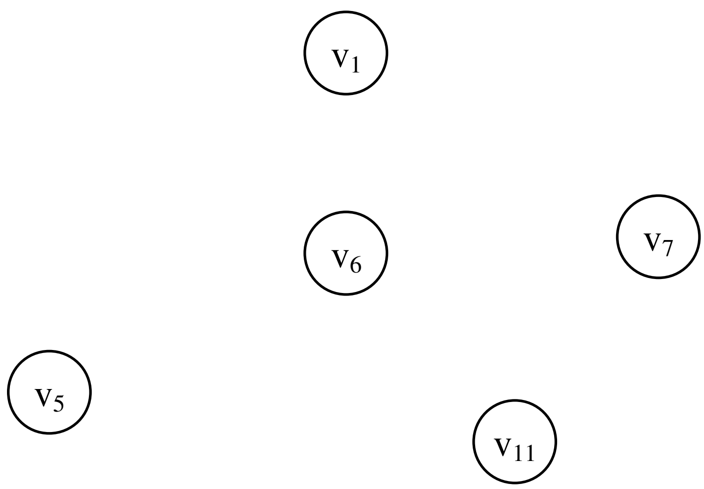


Figure 2.45

Clearly H_1 is a 9-edges multisubgraph which is empty.

Let H_2 be 9-edges multisubgraph given by the following figure.

H_2 =

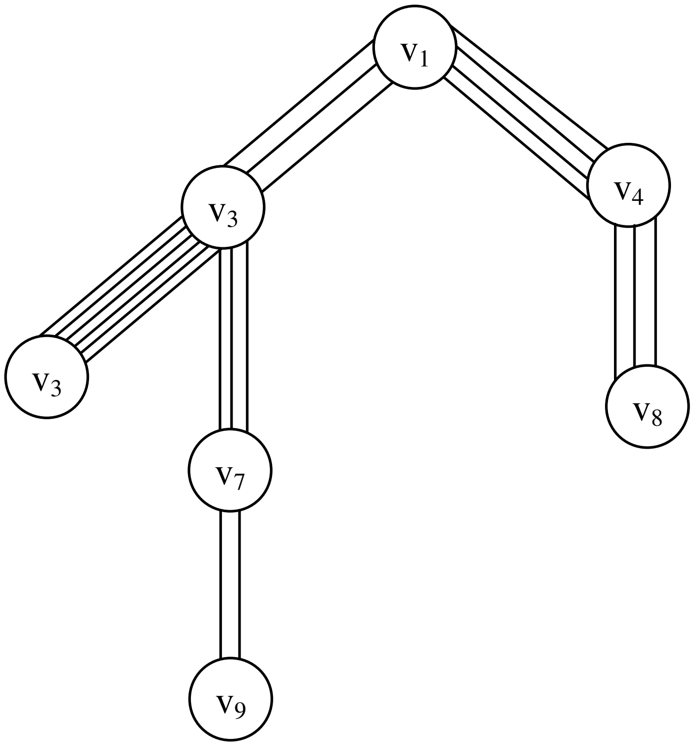


Figure 2.46

Clearly H_2 is a connected 9-edges multisubgraph which is a multitree.

Let H_3 be a 9-edges multisubgraph given by the following figure.

H_3 =

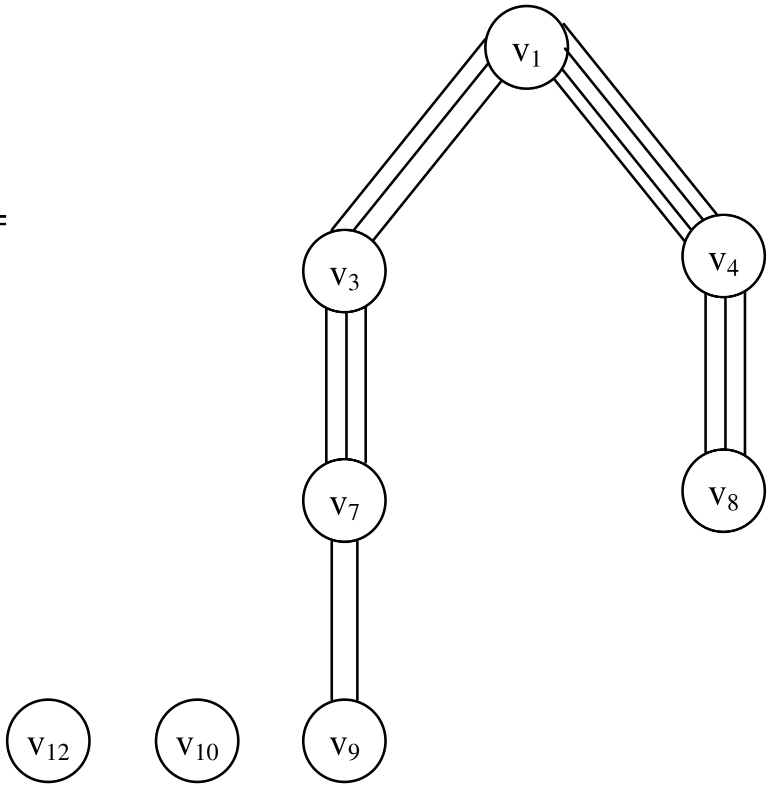


Figure 2.47

Clearly H_3 is a 9-edges multisubgraph which is disconnected the giant component is a multisubgraph with 2 singleton nodes v_{12} and v_{10}.

Let H_4 be a 9-edges multisubgraph given by the following figure.

H_4 =

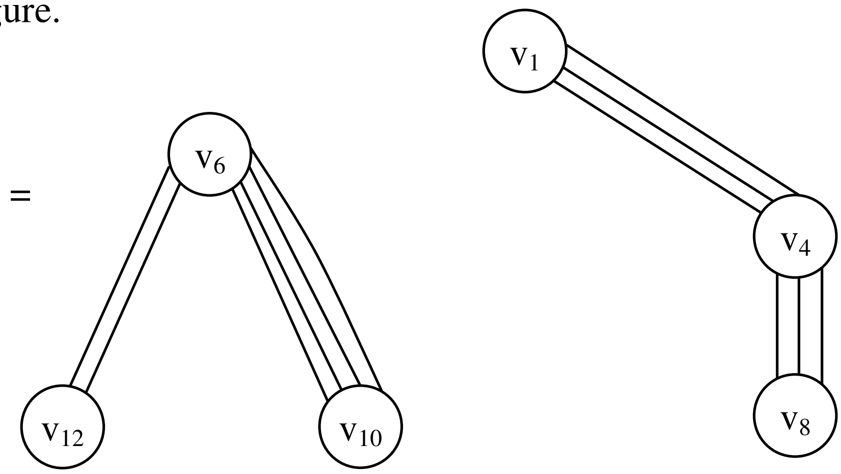


Figure 2.48

Clearly H_4 is a 9-edges multisubgraph which is disconnected and has 2 components both are 9-edges multitrees.

In view of all these we have the following theorem.

Theorem 2.4. *Let G be a n-edges multigraph with m-vertices which is a tree. The n-edges multisubgraphs of G are either multitrees or empty multisubgraphs or disconnected with components as multitrees or disconnected with components as multitrees and rest singletons.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe a n-edges bipartite multigraph by examples.

Example 2.11. Let G be a 6-edges. Bipartite multigraph given by the following figure.

G =

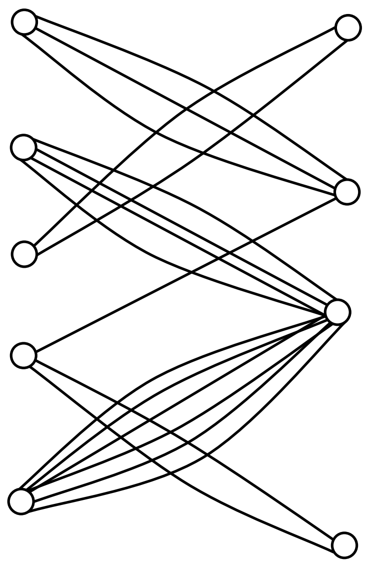


Figure 2.49

G is a bipartitite 6-edges multigraph with 9 vertices.

However G is not a complete bipartite 6-edges multigraph.

We give an example of a complete bipartite 4-edges multigraph.

Example 2.12. Let G be a complete bipartite 4-edges multigraph given b y the following figure.

G =

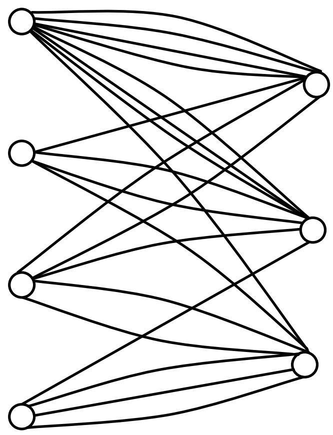


Figure 2.50

G is a complete 4-edges bipartite multigraph with 7 vertices.

It is important and interesting to observe not all multiedges from two relevant vertices have 4 edges between them. They may have one or two or three or four in case of complete n-edges bipartite multigraphs and no edges in case the n-edges bipartite multigraph is not complete.

We will now provide some illustrations n-edges mutlisubgraphs of a bipartite n-edges multigraph in the following.

Example 2.13. Let G be a 3-edges bipartite multigraph given b y the following figure.

G =

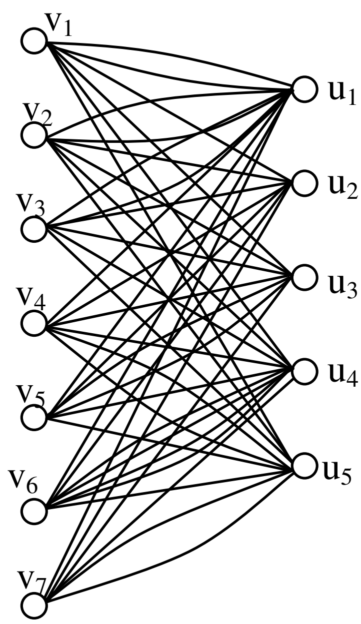


Figure 2.51

We give in the following some 3-edges multisubgraphs of G. Infact G is a complete bipartite 3-edges multigraph.

Let H_1 be the 3-edges multisubgraph of G given by the following figure.

H_1 =

Figure 2.52

Clearly H_1 is a 3-edges multisubgraph of G which is empty.

H_2 be the 3-edges multisubgraph of G given by the following figure.

H_2 =

Figure 2.53

H_2 is also a empty 3-edges multisubgraph of G.

It is interesting to observe in case of bipartite n-edges complete multigraphs we can have empty n-edges multisubgraphs. This property is deviant from the usual n-edges complete multigraphs.

Now let H_3 be the 3-edges multisubgraph of G given by the following figure.

H_3 =

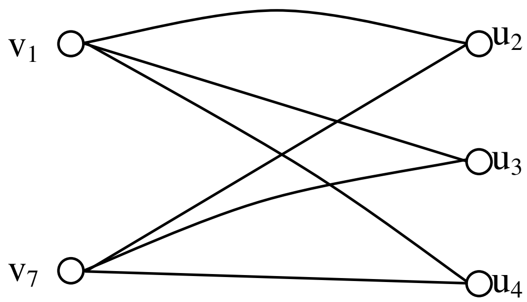


Figure 2.54

Clearly H_3 is a complete bipartite 3-edges multisubgraph of G which is does not contain any multiedges.

Let H_4 be the 3-edges bipartite multisubgraph of G given by the following figure.

H_4 =

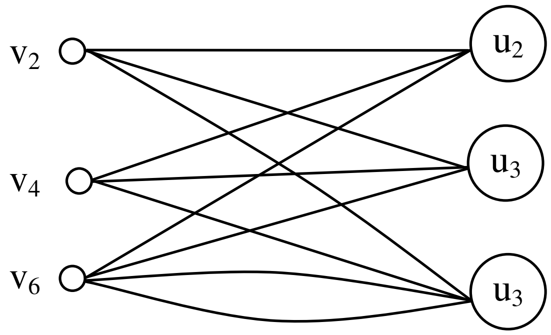


Figure 2.55

H_3 is a 3-edges bipartite complete multisubgraph of G.

Let H_5 be the 3-edges multisubgraph of G given by the following figure.

H_5 =

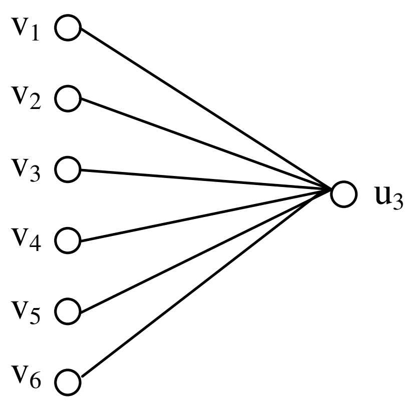


Figure 2.56

Clearly H_5 is a complete bipartite 3-edges multisubgraph of G.

Let H_6 be the 3-edges bipartite 3-edges multisubgraph given by the following figure.

H_6 =

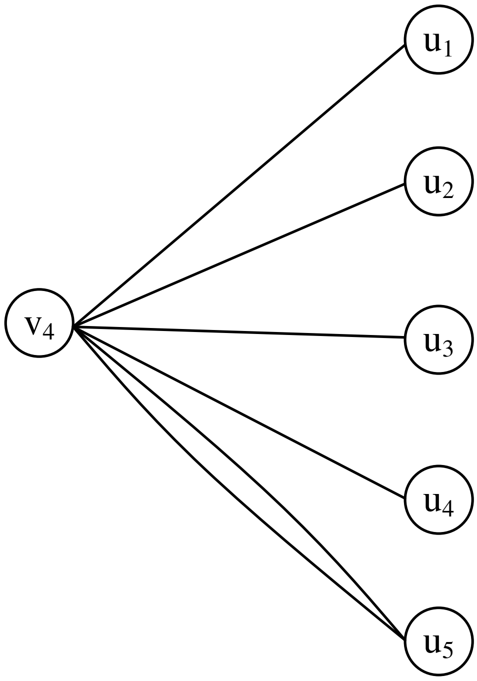


Figure 2.57

H_6 is also a 3-edges bipartite complete multisubgraph of G.

In view of all these we have the following result.

Theorem 2.5. *Let G be a complete n-edges bipartite multigraph.*

n-edges bipartite mutlisubgraph of G can be either.

i) empty n-edges bipartite multisubgraph or
ii) complete n-edges bipartite multisubgraph.

Proof is direct and hence left as an exercise the reader.

Now we proceed onto give examples of n-edges multisubgraphs of G.

Example 2.14. Let G be a bipartite 4-edges multigraph given by the following figure.

G =

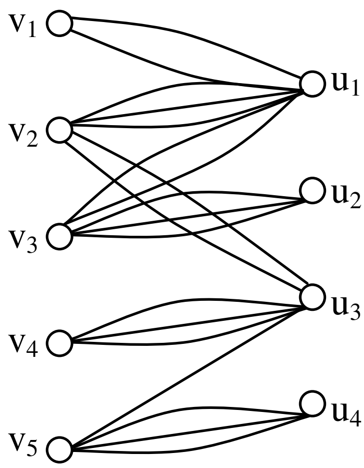


Figure 2.58

G is not a complete bipartite 4-edges multigraph. It is only a bipartite 4-edges multigraph which is not complete.

We will give for this bipartite 4-edges multigraph some multisubgraph in the following.

Let K_1 be the 4-edges multisubgraph of G given b y the following figure.

K_1 =

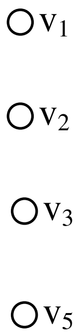


Figure 2.59

K_1 is a empty 4-edges multisubgraph of G.

Let K_2 be a 4-edges multisubgraph of G given by the following figure.

K_2 =

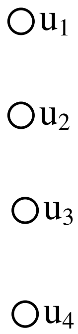


Figure 2.60

Clearly K_2 is also a empty 4-edges multisubgraph of G.

Let K_3 be a 4-edges multisubgraph of G given by the following figure.

K_3 =

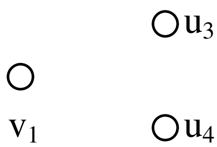


Figure 2.61

We see K_3 is also only a empty 4-edges multisubgraph of G.

Let K_4 be the 4 edges multisubgraph of G given by the following figure.

K_4 =

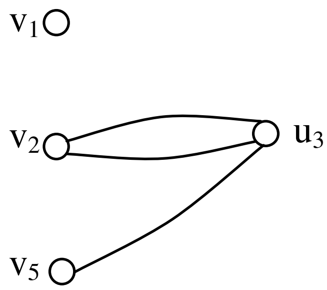


Figure 2.62

K_4 is a 4-edges multisubgraph of G which is disconnected. It has four vertices.

Let K_5 be the 4-edges multisubgraph given by the following figure.

K_5 =

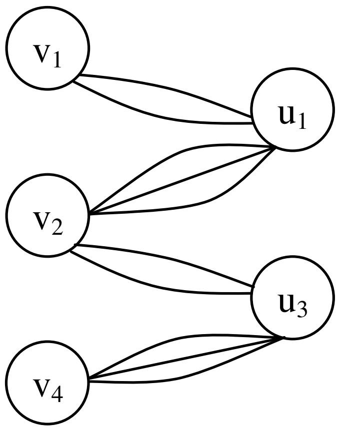


Figure 2.63

Clearly K_5 is a bipartite 4-edges multisubgraph which is connected.

In view of all these we enlist the following properties.

Theorem 2.6. *Let G be a bipartite n-edges multigraph with two sets of disjoint vertices* $(v_1, \ldots, v_t)$ *and* $(u_1, u_2, \ldots, u_s)$, $s + t = m$ *which is not a complete bipartite n-edges multigraph.*

i) *G has n-edges multisubgraphs which are empty.*

ii) *G has n-edges multisubgraphs which are disconnected with the component multisubgraphs to be bipartite multisubgraphs and (or) singleton.*

iii) *G has n-edges multisubgraphs which are connected.*

Proof is direct and hence left as an exercise to the reader.

We can also have n-partite m-edges multigraphs. We will illustrate by examples for few values of n in the following.

Example 2.15. Let G be a 3-partite 5-edges multigraph given by the following figure.

G =

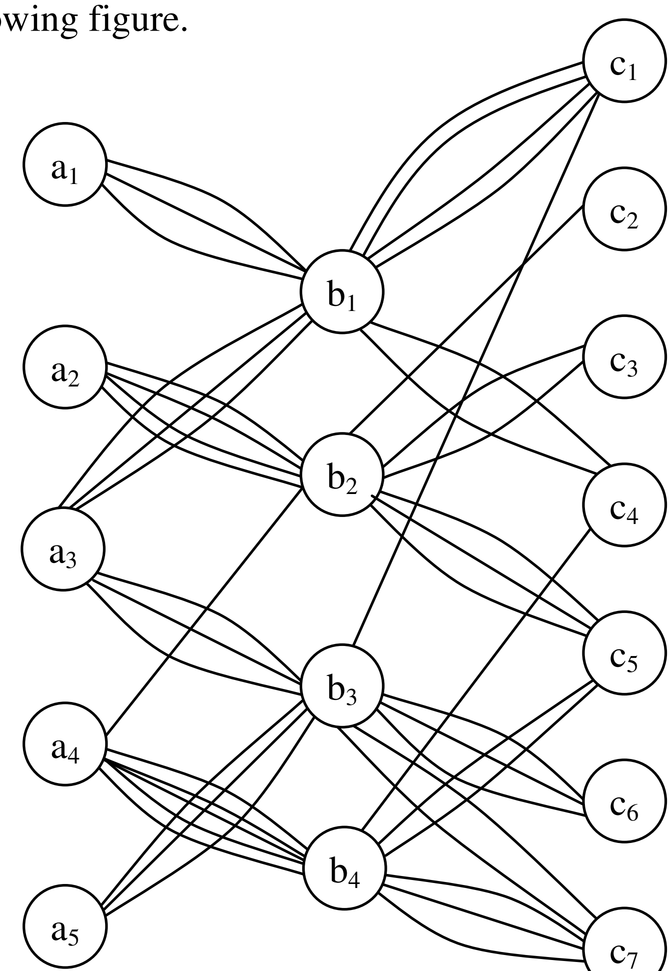


Figure 2.64

G is a 3-partite 5-edges multigraph. We now give some examples of these 3-partite 5-edges multisubgraph in the following.

K_1 =

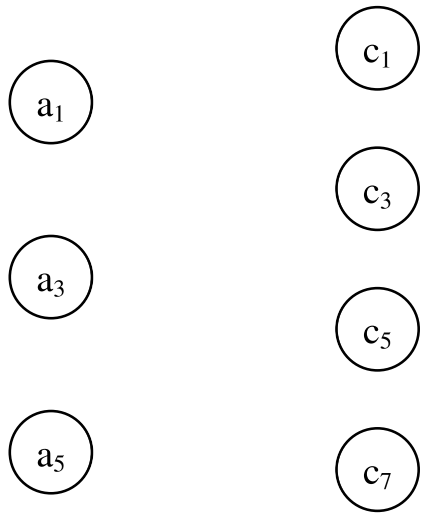


Figure 2.65

K_1 is a 3-partite 5-edges multisubgraph which is empty.

Let K_2 be the 3-partite 5-edges multisubgraph of G given by the following figure.

K_2 =

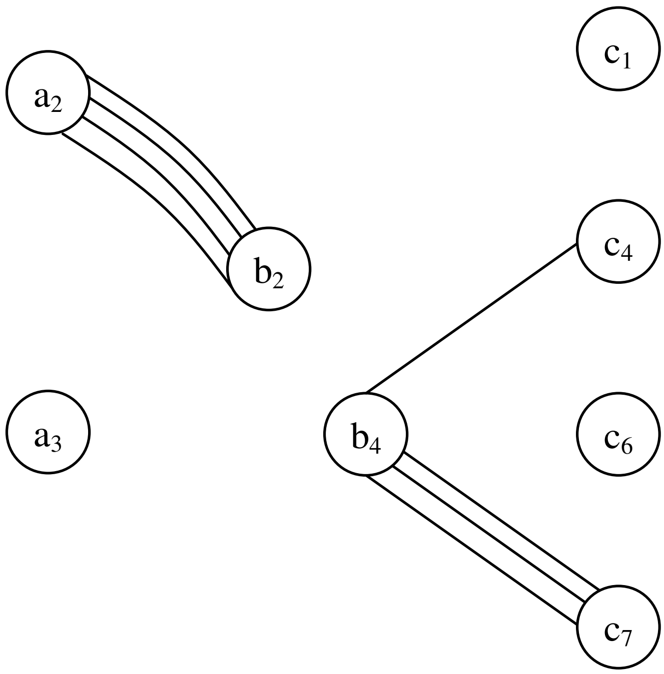


Figure 2.66

We see K_2 is a disconnected tripartite 5-edges multisubgraph of G which is nonempty.

Let K_3 be the 3-partite 5-edges multisubgraph given by the following figure.

K_3 =

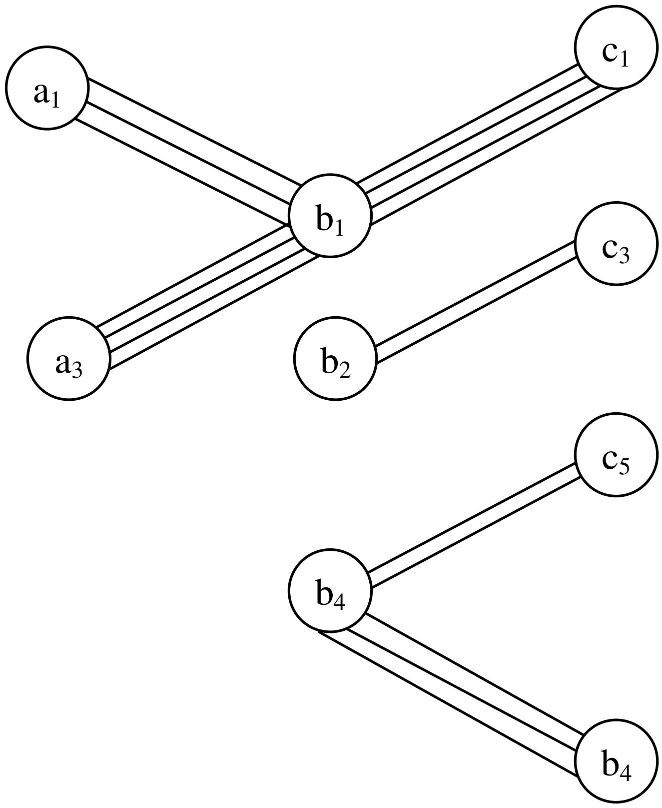


Figure 2.67

K_3 is a 3-partite 5-edges multisubgraph which is disconnected has 3-components.

Let K_4 be the 3-partite 5-edges multisubgraph of G given by the following figure.

K_4 =

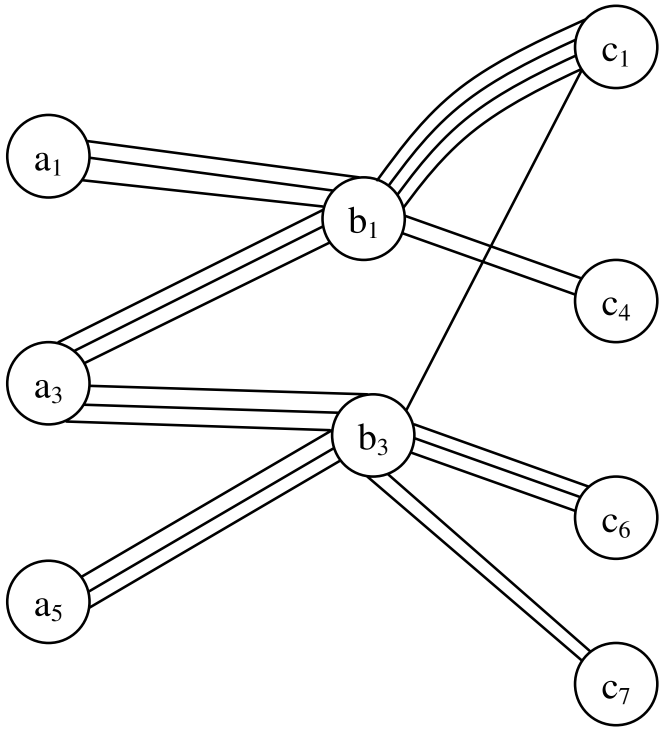


Figure 2.68

K_4 is a connected 3-partite 5-edges multisubgraph of G.

Thus we have seen examples of 3 partite multisubgraphs which are empty or disconnected or connected.

Next we proceed onto describe a n-edges multigraph G plus or minus a point or a line by some examples.

Example 2.16. Let G be a 3-edges multigraph given by the following figure.

G =

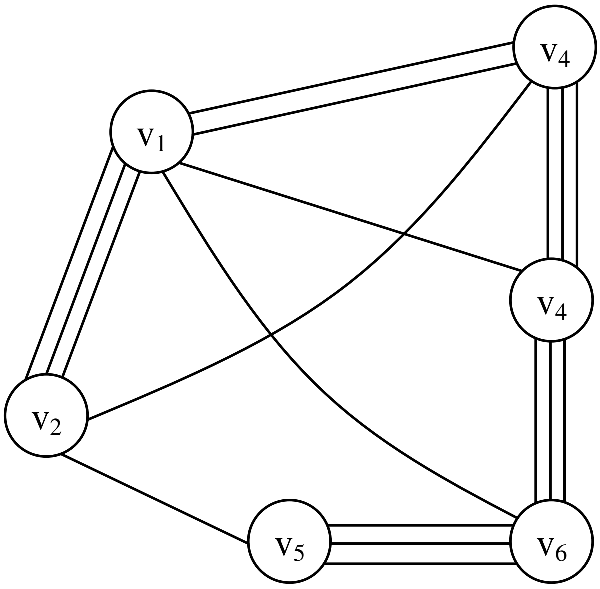


Figure 2.69

We first find G \ v_1 the 3-edges multisubgraph H_1 of G got by taking away using v_1.

H_1 =

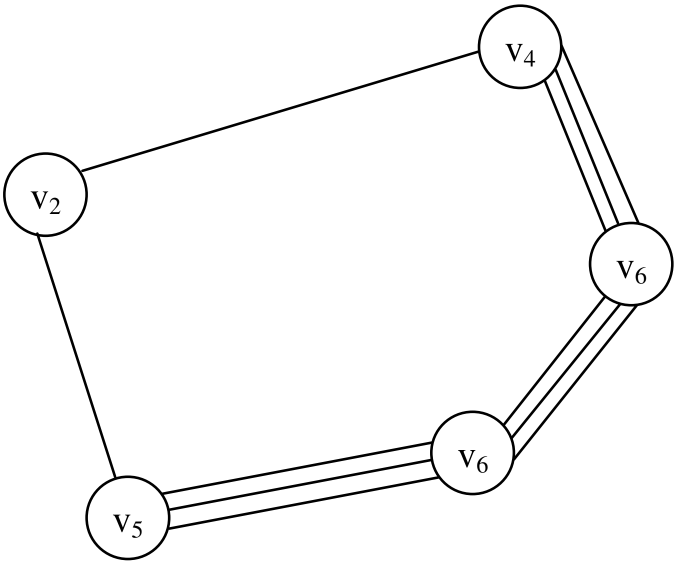


Figure 2.70

We see H_1 is a circle 3-edges multisubgraph of G.

We now find H_2 the G \ v_2 3-edges multigraph of G by v_2 from G.

H_2 =

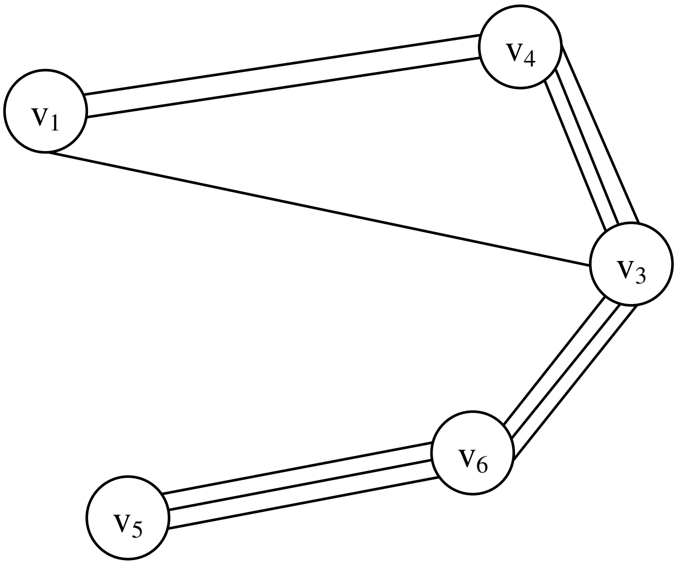


Figure 2.71

We see H_2 the 3-edges multisubgraph of $G \setminus v_2$ is connected and is not a circle contains a triad.

Let H_3 be the 3-edges multisubgraph of $G \setminus v_3$ given by the following figure.

H_3 =

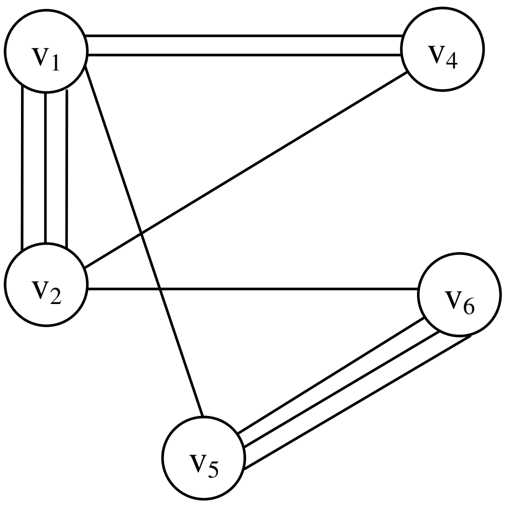


Figure 2.72

Clearly H_3 is also a connected 3-edges multisubgraph has only one triad.

Let H_4 be the 3-edges multisubgraph of G got by removing v_4 from G.

H_4 is given by the following figure.

H_4 =

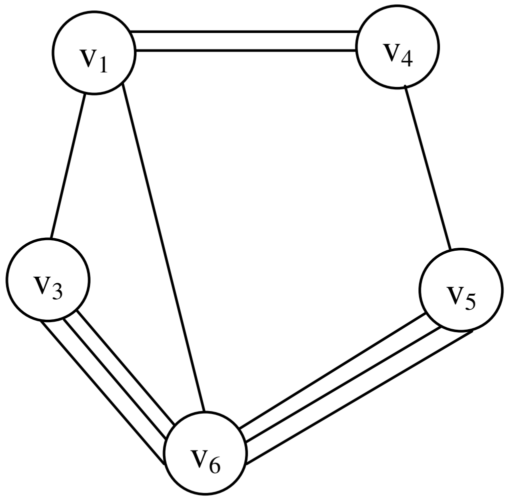


Figure 2.73

H_4 is also a connected 3-edges multisubgraph which has a triad.

Let H_5 be the 3-edges multisubgraph for from G by removing the vertex v_5. H_5 is given by the following figure.

H_5 =

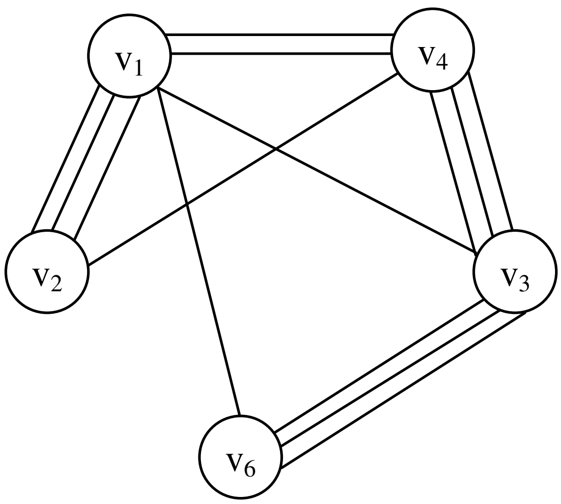


Figure 2.74

H_5 is also connected 3-edges multisubgraph of G.

We see H_5 has 3 triads and the 3-edges multisubgraphs H_2 or H_3 or H_4 has only one triad.

However G also has only 3 triads.

Now we proceed on the describe the 3-edges multisubgraph H_6 of G got by removing the vertex v_6. The grap of H_6 is as follows.

H_6 =

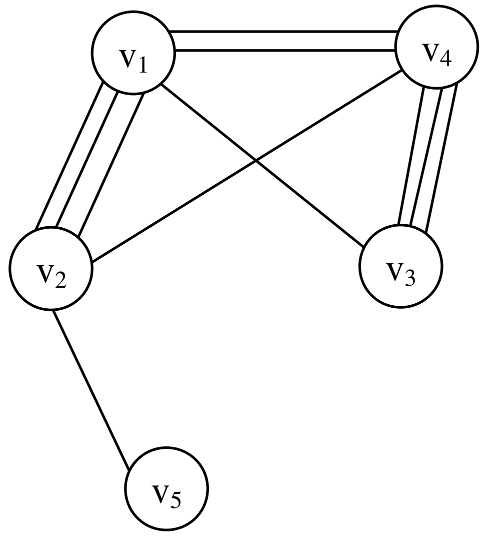


Figure 2.75

H_6 is a 3-edges multisubgraph of G which has only two triads.

In view of these results we proceed onto prove the following result.

Now we study complete n-edges multigraph G with m vertices for the number of triads in $G \setminus v_i$; $1 \leq i \leq m$ by examples.

Example 2.17. Let G be 5 edges multigraph with 5 vertices which is complete given by the following figure.

G =

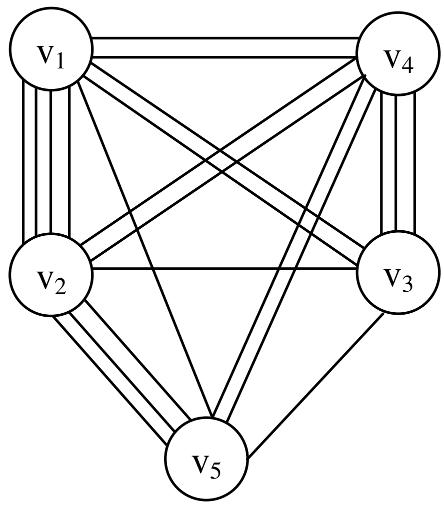


Figure 2.76

This 5-edges complete multigraph has 10 triads.

Now we find the number of triads of $G \setminus v_1$.

Let P_1 be the 5-edges multisubgraph of G given by the following figure.

P_1 =

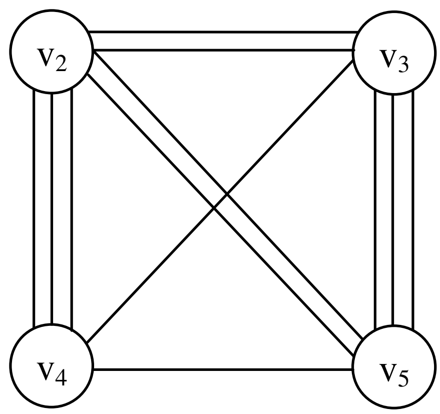


Figure 2.77

Clearly P_1 is a complete 5-edges multisubgraph of G.

Infact by removing any vertex this complete 5-edges multigraph G gives only a complete 5-edges multigraph of size 4.

So we can put forth the following result.

Theorem 2.7. *Let G be a complete n-edges multigraph with m-vertices; $\{v_1, \ldots, v_m\}$. By removing any vertex from the multigraph G say $G \backslash v_i$ we get the resulting structure to be only a n-edges multisubgraph with (m – 1) vertices which is also complete ($1 \leq i \leq m$).*

Proof is direct and hence left as an exercise to the reader.

Now we characterize these complete n-edges multigraphs in terms of triads. We are aware of the fact that any complete graph with n-vertices has nC_3 =

$$\frac{n(n-1)(n-2)}{2.3} = \frac{n(n-1)(n-2)}{6}$$

number of triads.

In view of this we have the following theorem.

Theorem 2.8. *Let G be a complete n-edges multigraph with m vertices; $v = \{v_1, v_2, \ldots, v_m\}$.*

Every n-edges multisubgraph of G given by the $G \setminus v_i$ (by removing a vertex v_i from the vertex set v of G; $1 \leq i \leq m$) has $\frac{(m-1)(m-2)(m-3)}{6}$ number of triads.

Proof is direct and hence left as an exercise to the reader.

Now when we are analyzing n-edges multigraphs G with m vertices; which are not complete the study becomes difficult.

However we try to study the problem in case G has t number of triads.

$$1 \leq t < \frac{m(m-1)(m-2)}{6}.$$

To this end first we study only some examples of graphs with one triad, 2 triads and so on with some fixed number of vertices.

Example 2.18. Let G be a 5-edges multigraph with 9 vertices given by the following figure.

G =

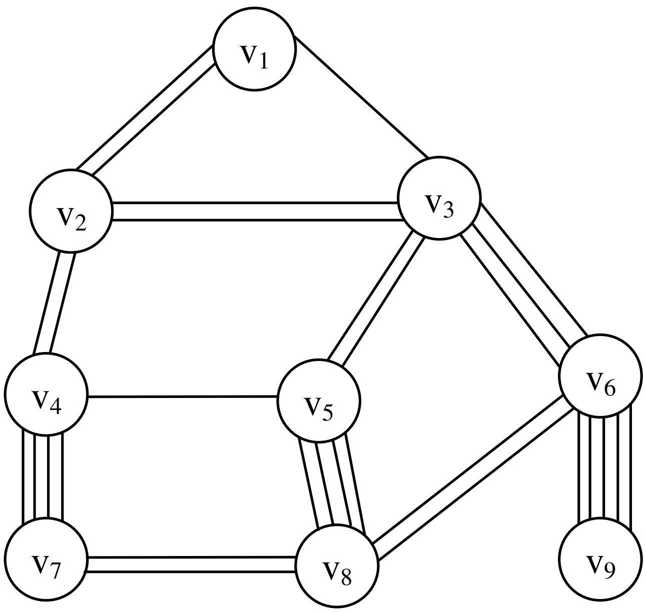


Figure 2.78

Clearly G has only one triad given by the vertex set $\{v_1, v_2, v_3\}$.

We see by removing any of the vertices {v_4 or v_5 or v_6 or v_7 or v_8 or v_9} from G the resulting 5-edges multisubgraph will always contain the one triad undisturbed the 5-edges multisubgraph may become disconnected at times however they will always contain the triad {v_1, v_2, v_3}.

Clearly H_1 has no triad but it is a connected 5-edges multisubgraph of G which is got by removing the vertex v_1 from G.

H_1 =

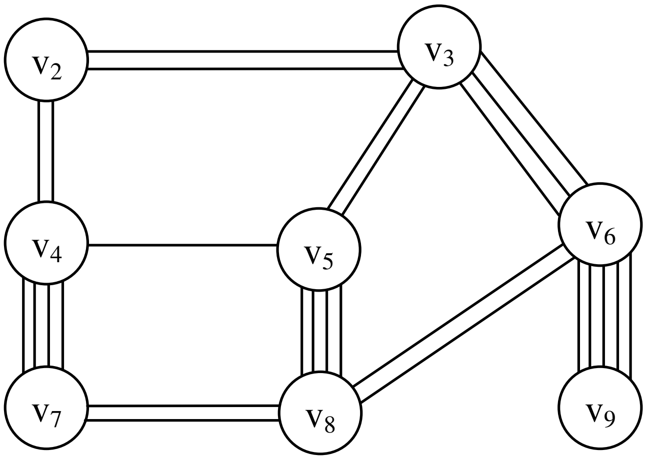


Figure 2.79

Consider H_2 the 5-edges multisubgraph of G by removing the vertex v_2 from G. The figure of H_2 is as follows.

H_2 =

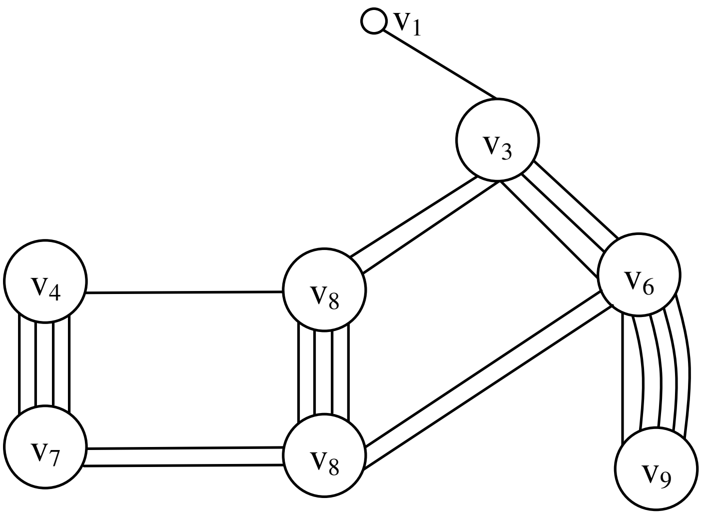


Figure 2.80

The 5-edges multisubgraph H_2 has no triads however it is connected.

Let H_3 be the 5-edges multisubgrpah of G got by removing the vertex v_3 from the graph G.

The figure of H_3 is as follows.

H_3 =

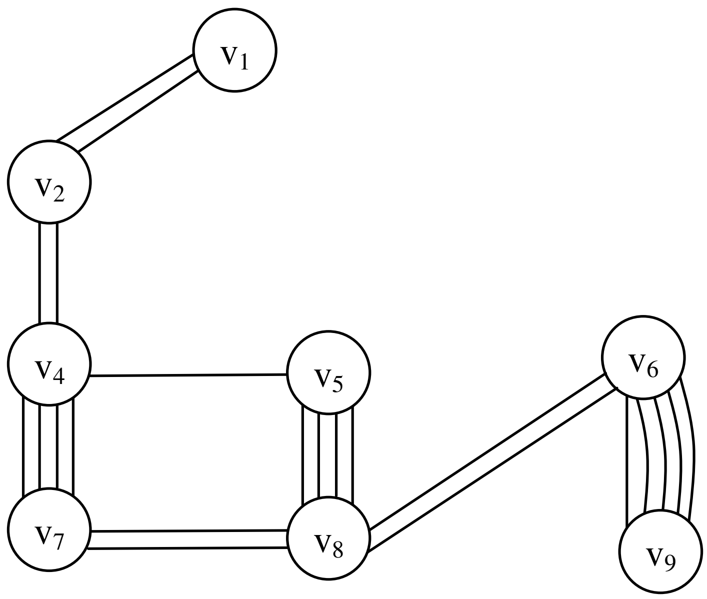


Figure 2.81

The 5-edges multisubgraph H_3 has no triads, H_3 also is connected.

In view of all these we have the following theorem.

Theorem 2.9. *Let G be a n-edges multigraph with vertex set $v = \{v_1, \ldots, v_m\}$. G has only one triad contributed by the triplet say $\{v_i, v_j, v_k\}$; $(i \neq j, i \neq k, j \neq k)$; $1 \leq i,j, k \leq m$.*

i) *All n-edges multisubgraphs H_t got by removing the vertex v_t from G has one triad contributed b y $\{v_i, v_j, v_k\}$. $(t \neq i, t \neq j, t \neq k)$, $1 \leq t \leq m$.*

ii) If t = i or j or k (or used in the mutually exclusive sense) then H_i or H_j or H_k does not contain any triad.

Proof is direct and hence left as an exercise to the reader.

The next natural question would be what is the situation if this n-edges multigraph G has two triads.

To this effect we provide 3 examples of n-edges multigraphs and arrive at a result.

Example 2.19. Let G be a 4-edge multigraph with 12 vertices given by the following figure.

G =

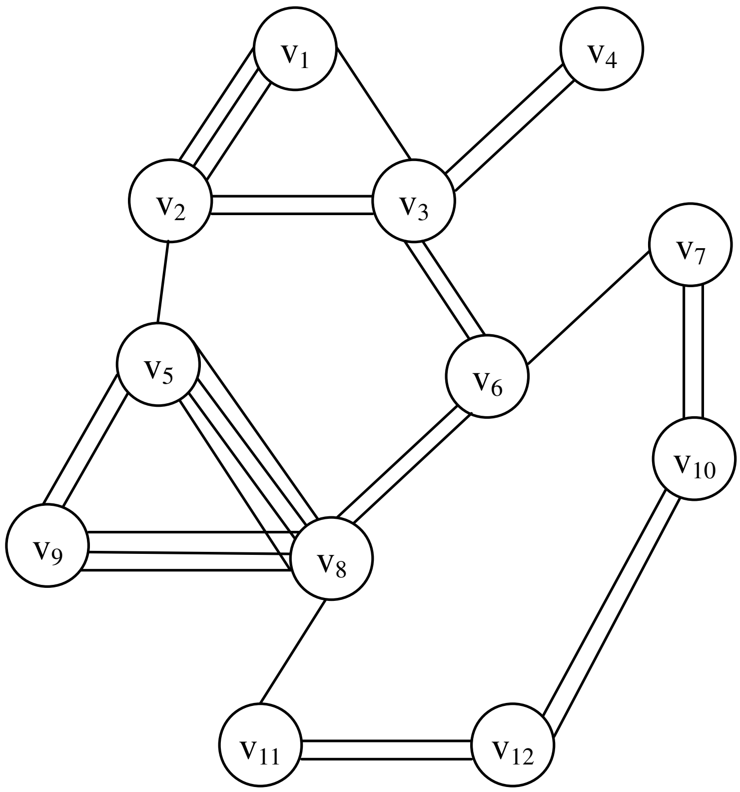


Figure 2.82

G is a 4-edges multigraphs with two triads given by the triplets $\{v_1, v_2, v_3\}$ and $\{v_5, v_9, v_8\}$. G is a connected 4-edges multigraph with two triads.

We see G \ {any one of v_4 or v_6 or v_7 or v_{10} or v_{12} or v_{12}} gives way to a 4-edges multisubgraphs which as two triads. In all cases these multisubgraphs are connected.

Now if the node v_1 is removed from G we get a multisubgraph H_1 given by the following figure.

H_1 =

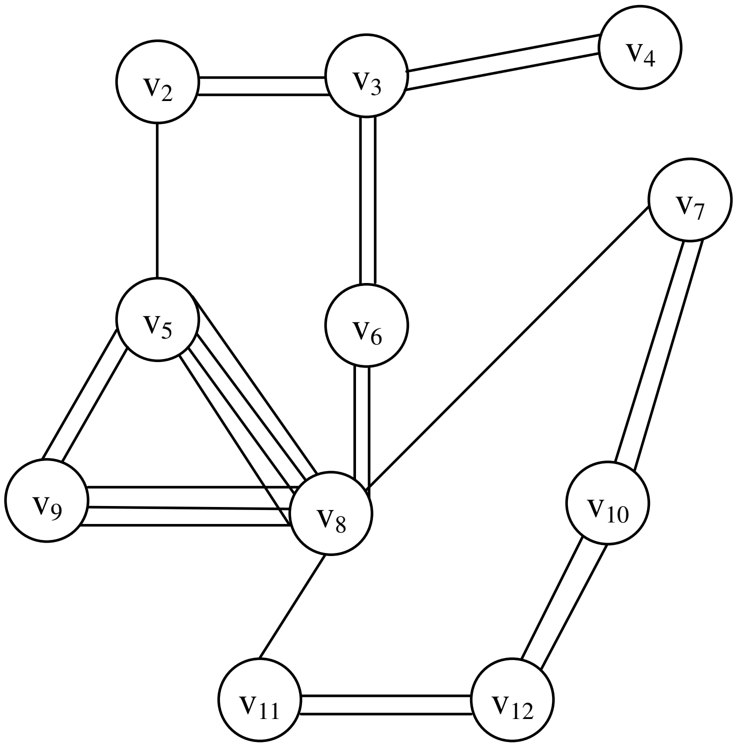


Figure 2.83

We get only one triad in H_1 given by the triple $\{v_5, v_8, v_9\}$.

Likewise if node v_2 or v_3 is removed the resultant 5-edges multisubgraph H_2 or H_3 has only one triad given by the triplet $\{v_5, v_9, v_8\}$.

Now on similar lines if the node v_5 is removed from G we get the resultant 5-edges multisubgraph H_5 which is as follows.

H_5 =

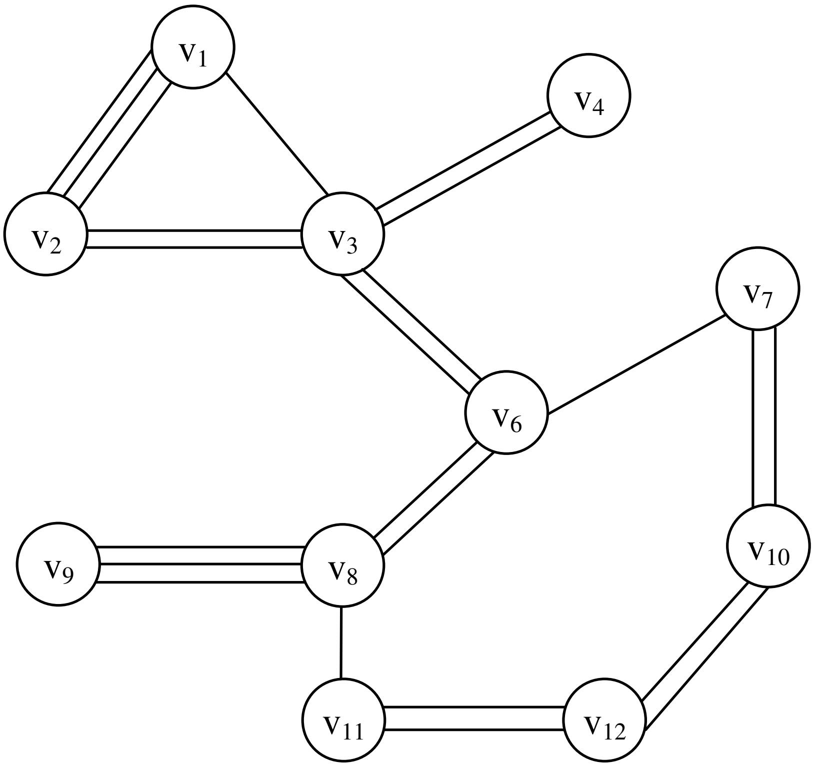


Figure 2.84

Clearly the 5-edges multisubgraph H_5 has only one triad given by the triple $\{v_1, v_2, v_3\}$.

In view of all these we can say if two triads are disjoint that is they do not enjoy a common node or a two nodes and an edge then we call them disjoint triads.

We putforth the following result.

Theorem 2.10. *Let G be a n-edges multigraph with vertex set* $V = \{v_1, v_2, \ldots, v_m\}$*. If G has two disjoint triads given by the following triples* $\{v_i, v_j, v_k\}$ *and* $\{v_t, v_s, v_r\}$ *where all the six vertices* $v_i, v_j, v_k, v_r, v_s, v_t \in V$ *are distinct.*

i) *All n-edges multisubgraph H_l of G got from removing a vertex $v_l \in V \setminus \{v_i, v_j, v_k, v_r, v_s, v_t\}$ contain two triads with triplets $\{v_i, v_j, v_k\}$ and $\{v_t, v_s, v_r\}$ associated with them.*

ii) *All n-edges multisubgraphs P_i of G got from removing $v_p \in \{v_i, v_j, v_k, v_r, v_s, v_t\}$ contain only one triad with triplet $\{v_i, v_j, v_k\}$ or $\{v_t, v_s, v_r\}$ according as $v_p \in \{v_t, v_s, v_r\}$ or $v_p \in \{v_i, v_j, v_k\}$ respectively.*

Proof is left as an exercise for the reader.

Next we proceed onto give an example of a n-edges multigraph which has two triads enjoying a common vertex.

Example 2.20. Let G be a 4-edges multigraph which has only two triads such that the two triads have a common point given by the following figure.

G =

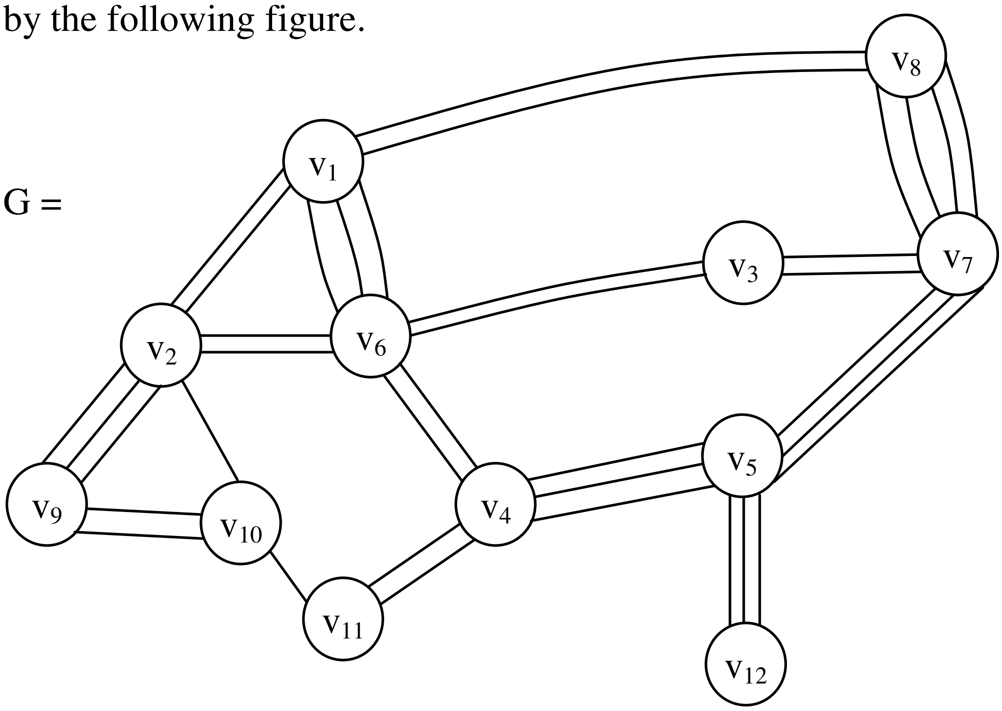


Figure 2.85

G is a 4-edges multigraph with 12 vertices and having two triads which have the vertex v_2 to be the common node.

All 4-edges multisubgraphs H_j got by removing the v_j from G that is $G \setminus v_j$, where $v_j \in \{v_{11}, v_4, v_5, v_{12}, v_3, v_7, v_8\}$ gives a 4-edges multisubgraph which has two triads.

We illustrate this in case of the node $v_j = v_8$, H_8 given by $G\setminus v_8$ is as follows.

H_8 =

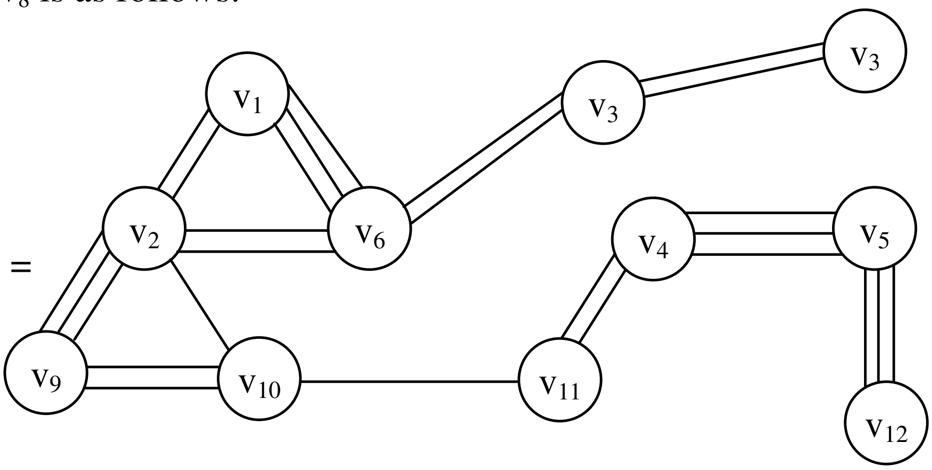


Figure 2.86

Clearly H_8 is the 4-edges multisubgraph which has two triads.

Now let P_i denote the 4-edges multisubgraph of G got by removing the node v_i from G where $v_i \in \{v_1, v_6, v_9, v_{10}\}$. Clearly these 4-edges multisubgraphs contain only one triad. We present this in case $v_i = v_6$ and $v_i = v_{10}$.

When P_6 results from using $G\setminus v_6$ we have the following figure.

P_6 =

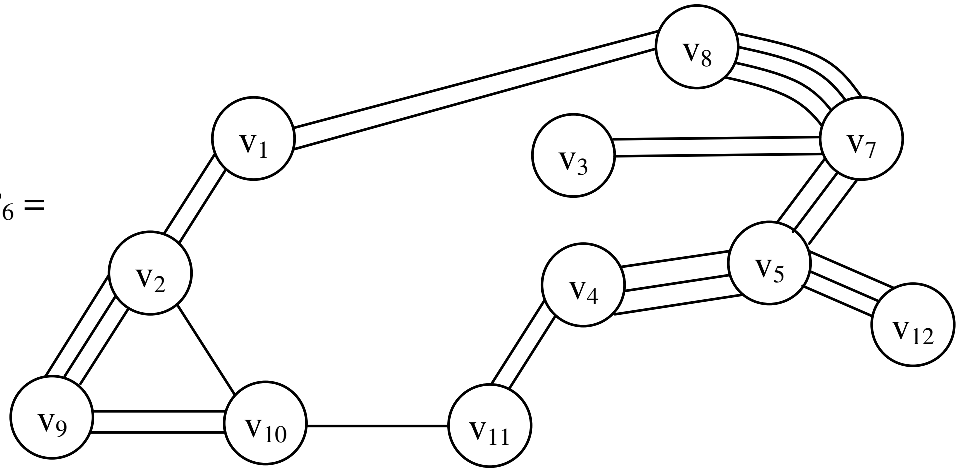


Figure 2.87

We see P_6 is a 4-edges multisubgraphs with only one triad given by the triple $\{v_2, v_9, v_{10}\}$.

Now we take the node v_{10} and find the resulting 4-edges multisubgraph given by $G \setminus v_{10}$.

Let P_{10} be the 4-edges multisubgraph given by the following figure.

P_{10} =

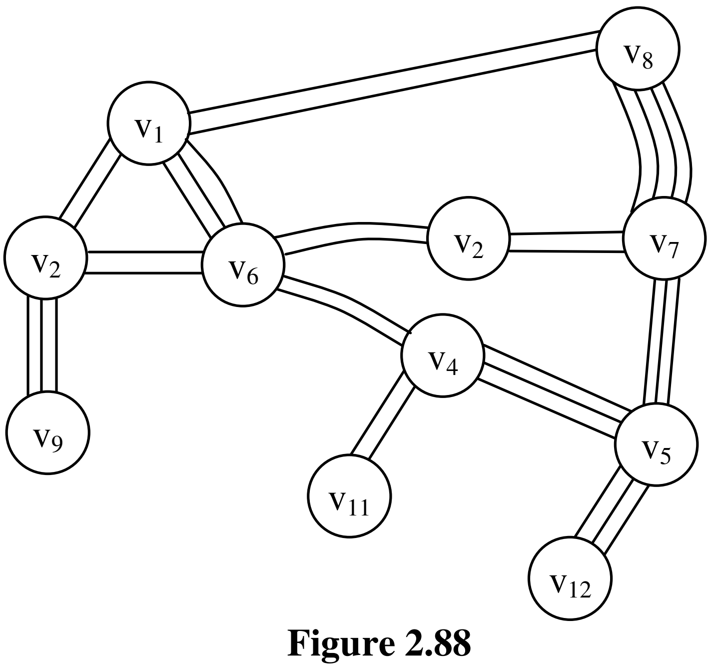


Figure 2.88

Clearly P_{10} is a 4-edges multisubgraph with one triad in it given by the triple $\{v_1, v_2, v_6\}$.

We see in case of removal of other nodes v_1 or v_9 we get 4-edges multisubgraph have only one triad given by triplets $\{v_2, v_9, v_{10}\}$ or $\{v_1, v_2, v_6\}$ respectively.

Now let L be the 4-edges multisubgraph got by removing the mode v_2 from G.

The resulting figure L is given by the following figure.

L =

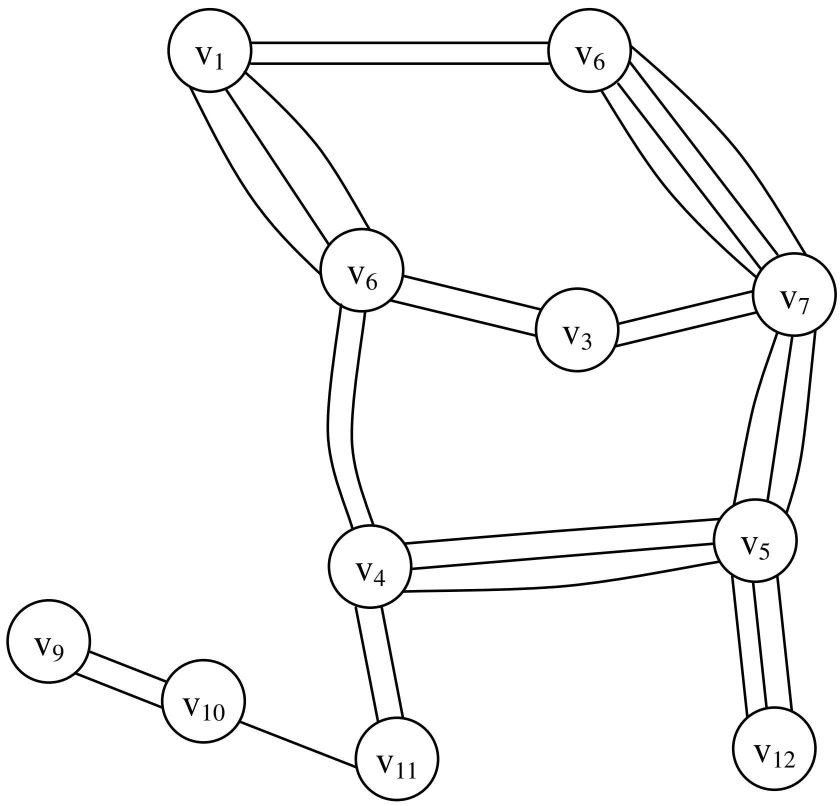


Figure 2.89

L is a 4-edges multisubgraph with no triads. We call v_2 as the vital triad dismantling node or vertex.

Now we just study those triads which has one edge in common.

We see this graph G is a 4-edges multigraph

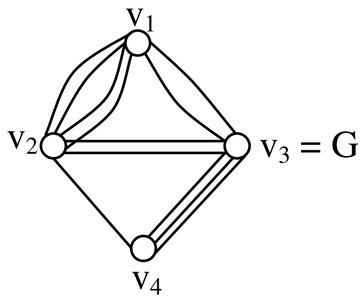


Figure 2.90

which has two adjacent triads associated with it as shown in the figure.

Now we study those n-edges multigraphs G with two adjacent triads as given in the figure by some examples.

Example 2.21. Let G be a 5-edges multigraph given b y the following figure.

G =

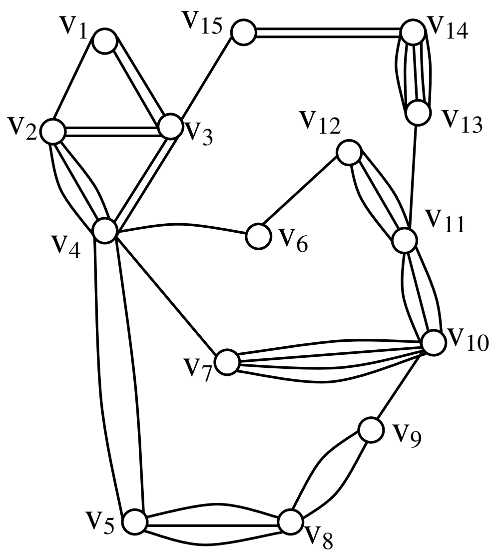


Figure 2.91

Now G is a 5-edges multigraph which has an adjacent triad given by

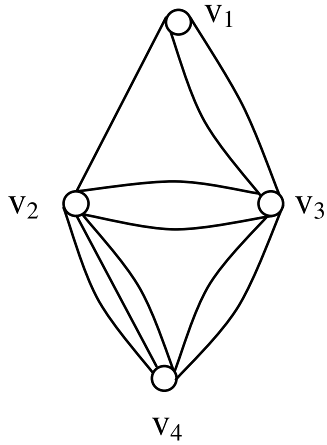


Figure 2.92

Now let H_i be a 5-edges multisubgraph of G got by removing the vertex v_i from G; that is $G \backslash v_i$ where $v_i \in \{v_5, v_6, \ldots v_{15}\}$.

The resulting 5-edges multisubgraphs H_i will continue to contain this adjacent triad.

We just take $G \backslash v_9$; let H_9 denote the 5-edges multisubgraph of $G \backslash v_9$ given by the following figure.

$H_9 =$

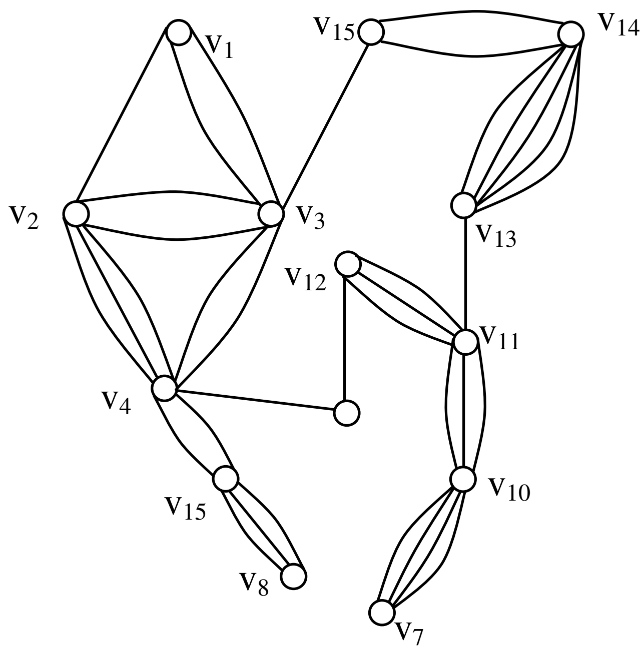
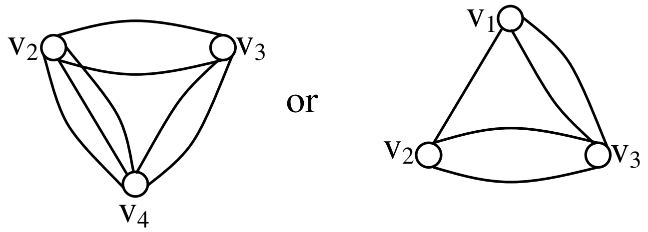


Figure 2.93

All H_i mentioned above will contain this adjacent triad given by the set $\{v_1, v_2, v_3, v_4\}$.

Clearly as the two triads are adjacent we will have only four nodes.

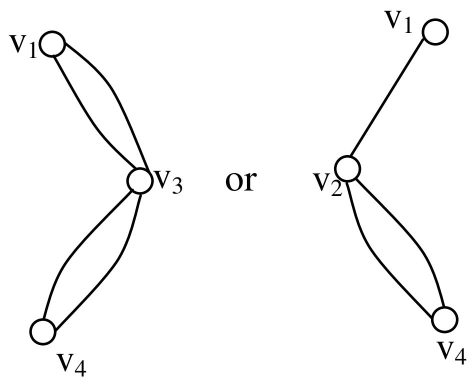
Now we proceed on to discuss how the adjacent triads become dismantled and in case v_1 or v_4 is removed from G we get a triad as



respectively.

Figure 2.94

However if v_2 or v_3 is removed, the adjacent triads reduce to the following forms, other vertices are not shown in this figure.



respectively.

Figure 2.95

Thus when certain nodes are removed. The resultant subgraphs do not contain any triads.

Consider the 5-edges multisubgraphs B_1 got by removing from G, the vertex v_2 given by G \ v_2. B_1 is given by the following figure.

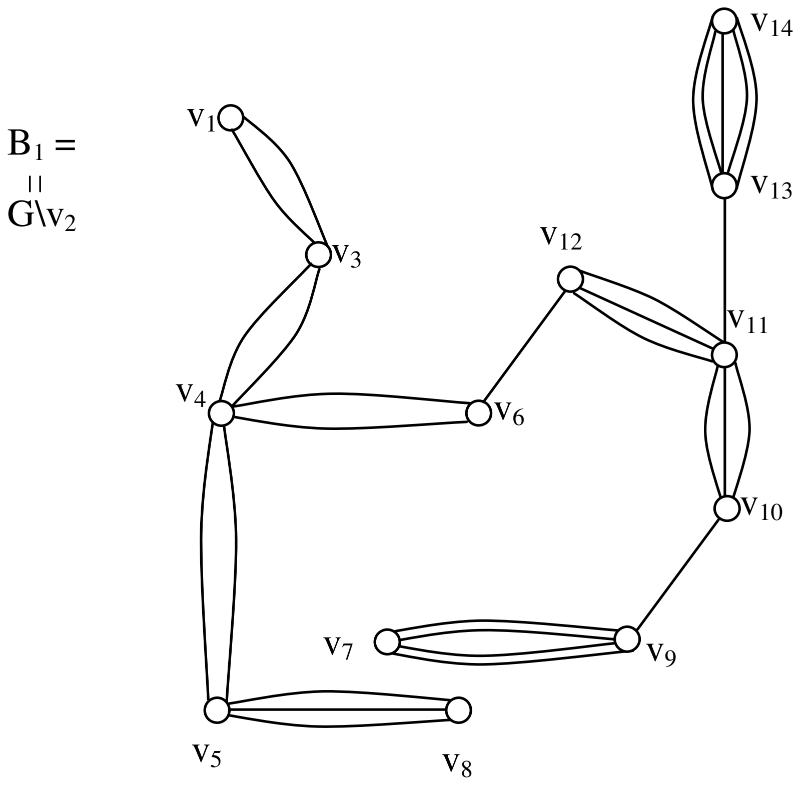


Figure 2.96

Clearly B_1 has no triads.

Consider the 5-edge multisubgraph B_2 given by G \ v_3 where the vertex v_3 is removed from G given by the following figure.

$B_2 =$

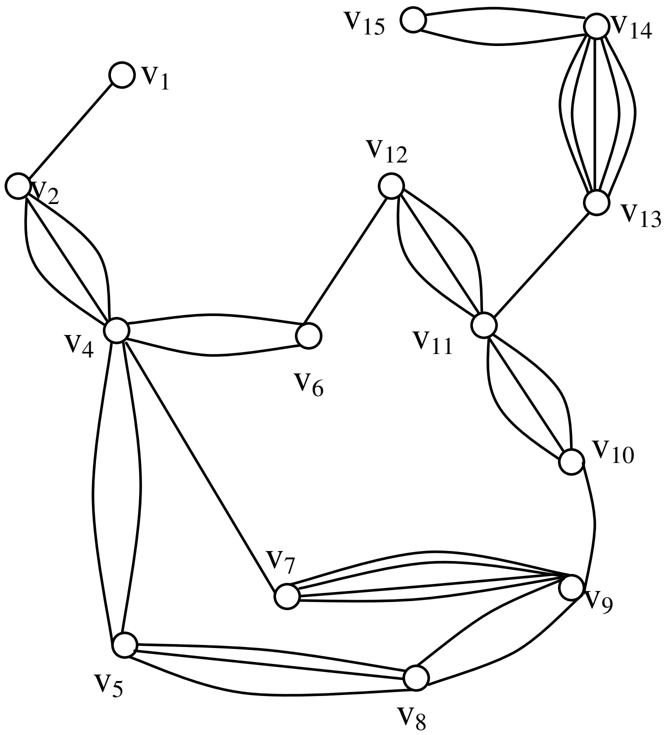


Figure 2.97

Clearly B_2 also does not contain any triad.

Let D_1 be the 5-edges multisubgraph got from removing the vertex v_4 from G, that is given in the following figure.

D_1 =

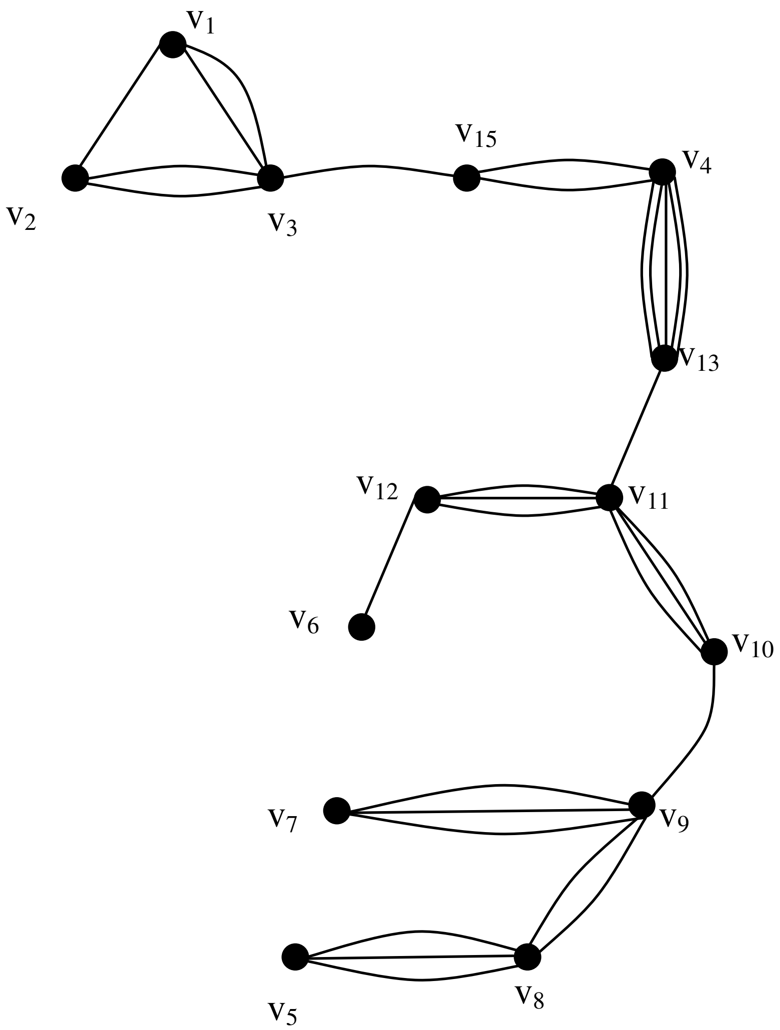


Figure 2.98

We see D_1 the 5-edges multigraph has only one triad given by the triplet $\{v_1, v_2, v_3\}$.

Likewise by removing vertex v_1 from G we get a 5-edges multisubgraph which contains a triad given by the triple $\{v_2, v_3, v_4\}$.

Thus if v_{11} vertex is removed from G we get a 5-edges multisubgraph K_1 given by the following figure.

K_1 =

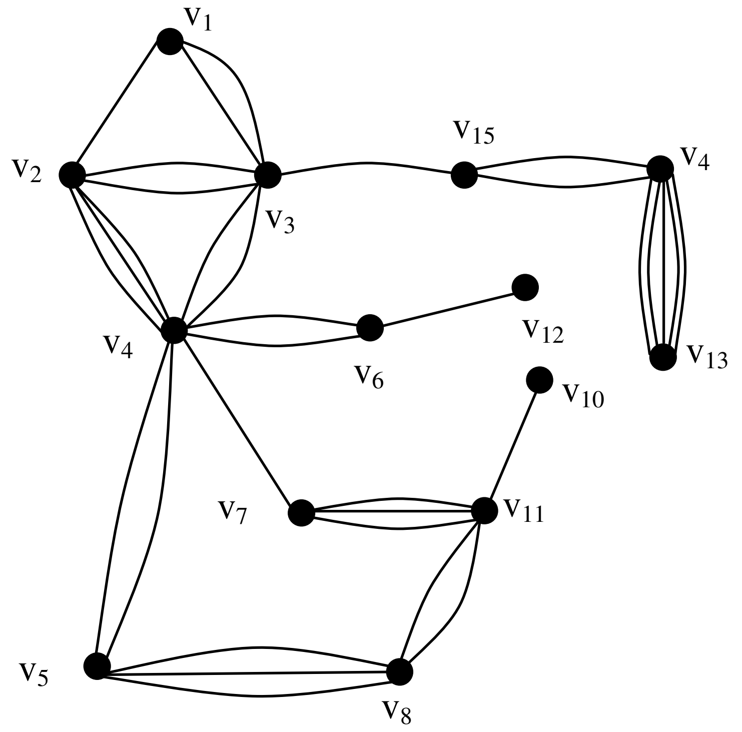


Figure 2.99

K_1 contains both the triads, that is the adjacent triads given by $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$.

We can have 3 triads where two by two are adjacent given by the following figure.

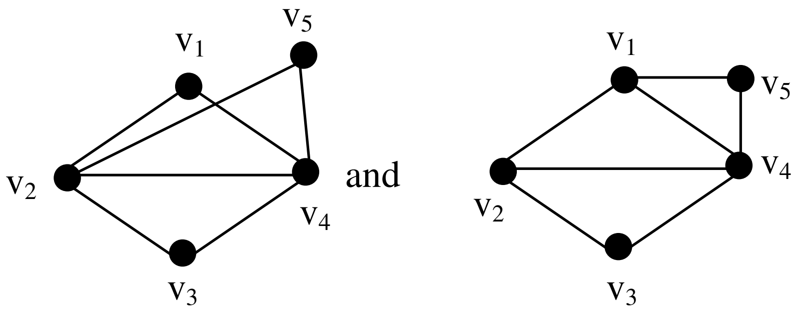


Figure (a) Figure (b)

Figure 2.100

We now observe the two figures, which are the adjacent three triplets given by the figures have only 5 nodes v_1, v_2, v_3, v_4

and v_5 and have 7 edges, however in case of figure (a) the adjacent triads are $\{v_1, v_2, v_4\}$ is adjacent with $\{v_5\ v_4\ v_1\}$ and $\{v_2, v_5, v_4\}$.

$\{v_5, v_4, v_1\}$ is adjacent with $\{v_1, v_2, v_4\}$ and $\{v_2, v_5, v_4\}$.

$\{v_2, v_5, v_4\}$ is adjacent with $\{v_1, v_2, v_4\}$ and $\{v_5, v_4, v_1\}$.

We call such triads are super adjacent triads.

If we observe the figure (b) we observe the triad $\{v_1, v_2, v_4\}$ is adjacent with $\{v_2, v_3, v_4\}$ and $\{v_1, v_5, v_4\}$ but $\{v_1, v_4, v_5\}$ is adjacent only with $\{v_1, v_2, v_4\}$ and $\{v_1, v_4, v_5\}$ is not adjacent with $\{v_2, v_3, v_4\}$. The triad $\{v_2, v_3, v_4\}$ is adjacent only with triad $\{v_1, v_2, v_4\}$ and not with the triad $\{v_1, v_4, v_5\}$.

We call these super adjacent triads as mutually adjacent triads.

Infact we can get mutually adjacent triads infact from ths sample geometric property of a circle graph and the triangles on the diameter of it.

We see all the triads are adjacent. We illustrate this situation by an example.

Example 2.22. Let G be a multigraph given by the following figure.

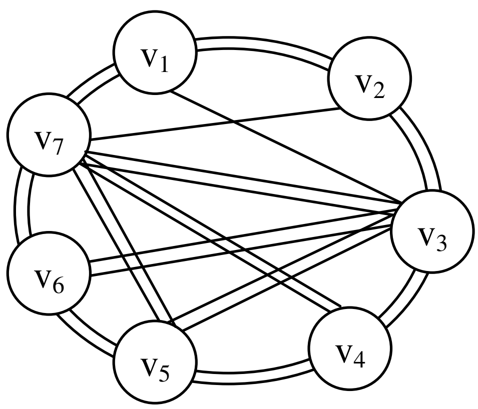


Figure 2.101

There are several adjacent triads but not all of them are adjacent. We see the triad {v_1, v_2, v_7} is not adjacent withi {v_5, v_7, v_3} and {v_3, v_5, v_4}.

Thus the following 3-edges multigraph with n-vertices is such that every triad is adjacent with all the rest of the triads.

First we will illustrate this situation by an example for n = 10.

Example 2.23. Let G be a 3-edges multigraph with 10 vertices $v_1, v_2, \ldots, v_{10}$.

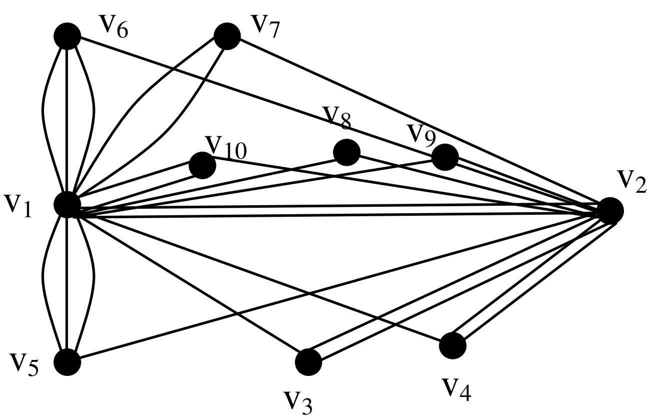


Figure 2.102

We see there are 8 triads and every triad is adjacent with the rest of the 7 triads.

In view of this we have the following theorem.

Theorem 2.11*. Let G be a m-edges multigraph with n vertices $v_1, v_2, \ldots, v_n$ with a m-edge from v_1 to v_n and each v_i ($i \neq 1$ and n) is adjacent with both v_1 and v_n. Then*

i) there are (n – 2) number of multitriads.

ii) Every triad is adjacent with the rest of the (n – 3) of the triads.

Proof: Given G is a m-edges multigraph with n-vertices v_1, v_2, …, v_n. There is a m-edge connecting v_1 and v_n.

Further every vertex v_i ($i \neq 1$ and n) are connected (or has edge between v_1 and v_i and v_n and v_i) $2 \leq i \leq n - 1$.

We see every $\{v_i, v_1, v_n\}$ is a triad $2 \leq i \leq n - 1$. Any other triplet from the n vertices if they contain v_1 or v_n say $\{v_i, v_j, v_1\}$ or $\{v_i, v_j, v_n\}$; $2 \leq i, j \leq n - 1$ ($i \leq j$) form a triplet which is not complete. If we take $\{v_i, v_j, v_K\}$ $2 \leq i, j, k \leq n - 2$; $i \leq j$, $j \leq k$ and $k \neq i$ the triplet forms only an empty graph.

Thus we can say there are (n – 2) number of multitriads.

Hence (i) is true.

Now for the proof of (ii) we see every (n – 2) triad is such that they have v_1 and v_n to be a common multiedges hence we see every triad is adjacent with (n – 3) triads.

Now we see while pruning the edges if the edge v_1 to v_n is cut then the resultant will become a multigraph with no multitriads. If any other multiedge is removed it will only destroy one triad. So in terms of seed or clique only community will be destroyed. Hence we call these as strongly knitted adjacent super multitriads.

Likewise we can construct the notion of strongly knitted complete multigraphs of adjacent cliques of size 4 or 5 or 6 and so on.

Before we proceed onto work with them we illustrate them by examples.

Example 2.24. Let G be a graph with 10 vertices given by the following figure.

G =

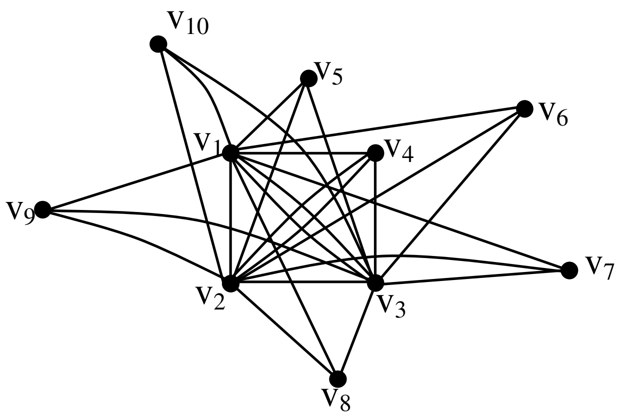


Figure 2.103

We see there are 7 complete graphs of size for there are 7 cliques of size four. There are given by $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5\}$, $\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_3, v_8\}$, $\{v_1, v_2, v_3, v_9\}$ and $\{v_1, v_2, v_3, v_{10}\}$.

All of them have the common triad $\{v_1, v_2, v_3\}$.

We have the similar one in case of multigraphs. We can call G as the strongly knitted adjacent clique of order four.

We can characterize the strong knitted adjacent cliques of order four by the following theorem.

Theorem 2.12. *Let G be a m-edges multigraph with n vertices $v_1, v_2, \ldots, v_n$. If using the multitriad $\{v_1, v_2, v_3\}$ a node v_i $4 \leq i \leq n$ are joined to form a clique of size 4; such that there are multiedges v_1v_i, v_2v_i and v_3v_i then the resulting multigraph with* *vertices {v_1, v_2, v_3, v_i}; $4 \leq i \leq n$ is a clique of size 4 given by the following figure.*

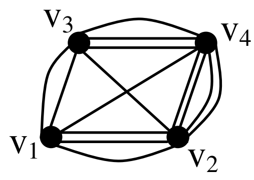


Figure 2.104

If the same procedure is done for all i; $4 \leq i \leq n$

i) We have in the multigraph G there are (n – 3) such cliques of size 4.

ii) Every clique of size 4 is adjacent with the other (n – 3) cliques.

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto discuss about cliques of size and 5 and 6 with examples before we generalize this to a clique of size t; $3 \leq t < \infty$. We can also defined the notion of strongly knitted cliques of size t.

Example 2.25. Let G be a graph or m edges mutligraph with v_1, v_2, …, v_{12} vertices given by the following figure.

G =

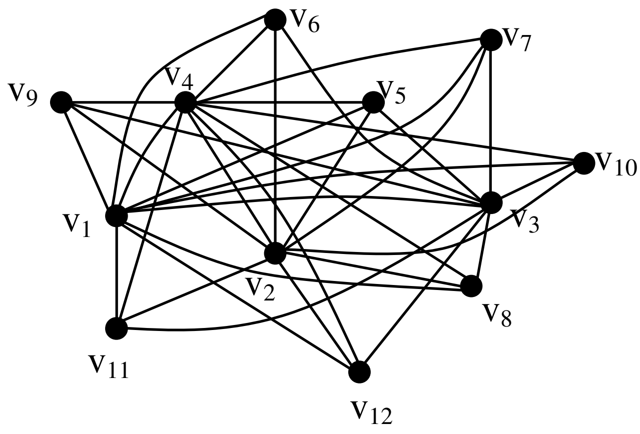


Figure 2.105

We see $\{v_1, v_2, v_3, v_4, v_5\}$ is a clique or complete group of size 5; $\{v_1, v_2, v_3, v_4, v_i\}$; $5 \leq i \leq 12$ are all cliques of order 5. Infact all these 8 cliques are adjacent with each other.

The common nodes are $\{v_1, v_2, v_3, v_4\}$ and the common subgraph being the complete subgraph of size 4.

It is interesting to note that in case of strongly knitted super adjacent triads we have only one common dyad which serves as the base to maintain the adjacency of rest of the triads.

In case of strongly knitted adjacent cliques of size 4 we have one and only one triad which as the base to make all the cliques of size 4 to be adjacent with each other.

We have also observed from the above example for the strongly knitted cliques of size 5 we have the base cliques of size four acting as the base clique of size four to make all the cliques of size 5 to be adjacent with each other.

We will provide one more example before we proceed onto generalize the general situation.

Example 2.26. Let G be a graph with $v_1, v_2, \ldots, v_{13}$ as vertices given by the following figure.

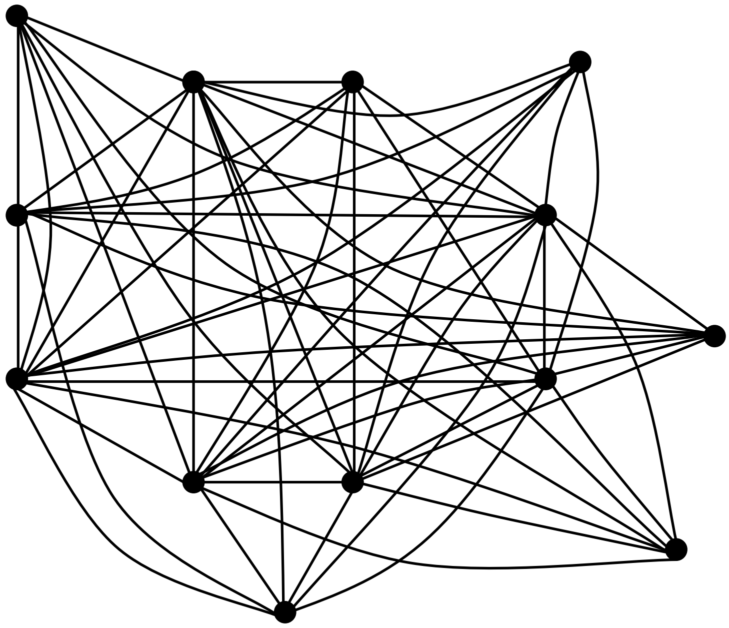
G = 

Figure 2.106

We see for these clique of size 8, $\{v_1, v_2, \ldots, v_8\}$ has all cliques $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_i\}$ $8 \leq i \leq 13$ are cliques of size 8.

For all these cliques of size 8 the base clique $\{v_1, v_2, \ldots, v_7\}$ of size 7 is the one which keeps all the other cliques of size 8 to be adjacent.

In view of this we have the following theorem.

Theorem 2.13. *Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. Suppose there is a clique of size t with vertex set $W = \{v_1, v_2, \ldots, v_{t-1}, v_t\}$. For any $v_i \notin W$ we have edges $v_i v_j \in W \setminus v_t$; $1 \leq j \leq t - 1$.*

Then each of the t-tuple $\{v_1, v_2, \ldots, v_{t-1}, v_i\}$, $t \leq i \leq n$ is a clique of size t.

Further every clique of size t in G has the clique of size $t-1$ formed by $\{v_1, v_2, \ldots, v_{t-1}\}$ as a subgraph. In particular every clique of size t in G is adjacent with every other clique of size t and the clique of size $t-1$ contributed by $\{v_1, v_2, \ldots, v_{t-1}\}$ serves as the common clique to make them adjacent.

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto suggest a few problems for the interested reader.

Problems

1. What are the special features enjoyed by multigraphs? Enlist them.

2. Compare the multigraphs with the usual or classical graphs.

3. Can one say multigraphs perform better than usual graphs under special situations? Justify your claim.

4. Give atleast two models where multigraphs do better than usual graphs.

5. How many 5-edges multigraphs can be drawn using four vertices?

6. How many complete 6-edges multigraphs can be drawn using 6 vertices?

7. How many complete 9-edges multigraphs can be drawn using 3-vertices?

8. Compare the number complete 6-edges multigraphs of
   i) Dyads
   ii) Complete with triads
   iii) Complete with four vertices
   iv) Complete with five vertices
   v) Complete with six vertices
   and so on. Complete n-vertices with 6-edges.
   vi) Does the numbers associated with (i), (ii), (iii), (iv) and so on form a pattern in the numbers? (do they form a proper sequence? Justify)

9. G is a multigraph given by the following figure.

G =

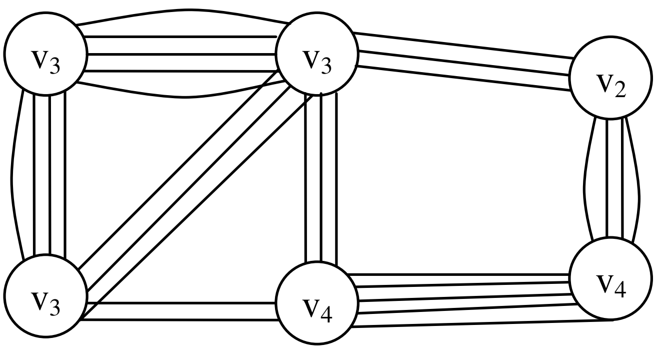


Figure 2.107

   i) How many multisubgraphs of G are connected?
   ii) How many multisubgraphs of G are disconnected?
   iii) Can G have empty multisubgraphs?
   iv) How many multitriads can be constructed using G?
   v) Does G contain adjacent multitriads?

10. Describe a 8-edge multigraph which is complete?

11. Let G be a 5-edges multigraph with 6 vertices given by the following figure.

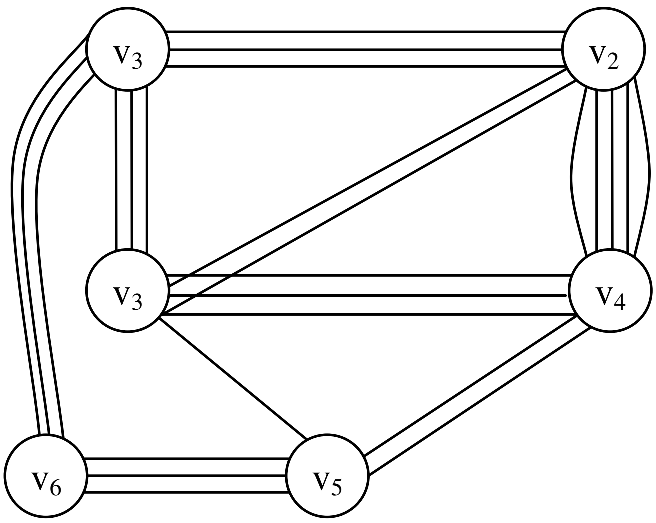


Figure 2.108

i) Find all 5-edges multisubgraph with 3 vertices.

ii) How many adjacent triads are there in G?

iii) How many disconnected 5-edges multisubgraphs can G contain with four vertices?

iv) How many connected 5-edges multisubgraphs can be constructed using G?

12. Given the vertex set V = {v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9} how many 3-edges multigraphs can be built using V.

13. How many of the 3-edges multigraphs using V in problem 12 as the vertex set V are disconnected?

14. Find the longest walk and path for the following 5-edges multigraph G with 9 vertices given by the following figure.

G =

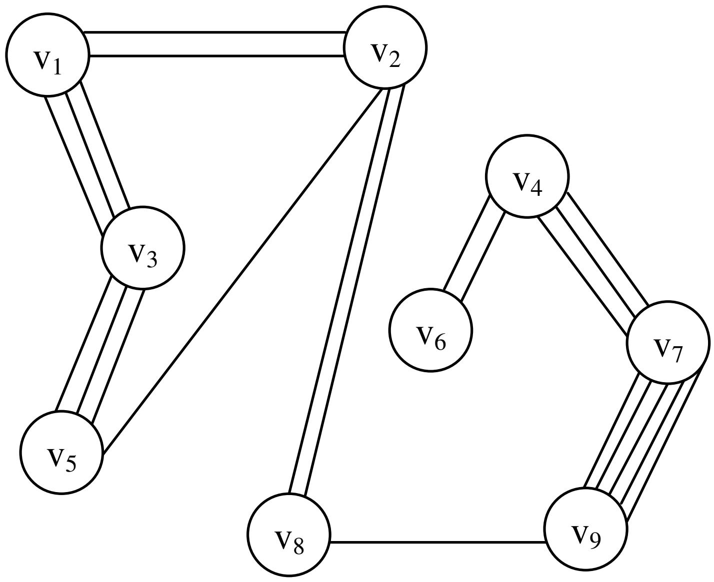


Figure 2.109

15. Enumerate a few possible applications of m-edges multigraph in multinetworks.

16. Show by an real world problem where multigraphs are better suited than classical graphs.

17. Show by an example of real world problem that the classical graphs are better than multigraphs.

18. Give an example of a 8-edges multigraph with 15 vertices which contributes to adjacent triads.

19. For the 3-edges multigraph given by the following figure.

G =

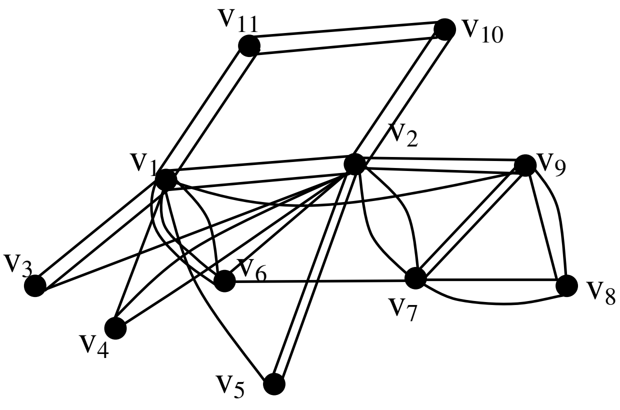


Figure 2.110

a) Find the maximum number adjacent triads that can be got using G.

b) Find all adjacent triads which enjoy the edge (i) v_1 v_2, (ii) v_7v_9 and (iii) $v_2v_{5.}$

20. For the 4-edges multigraph G find all adjacent cliques of size four.

G =

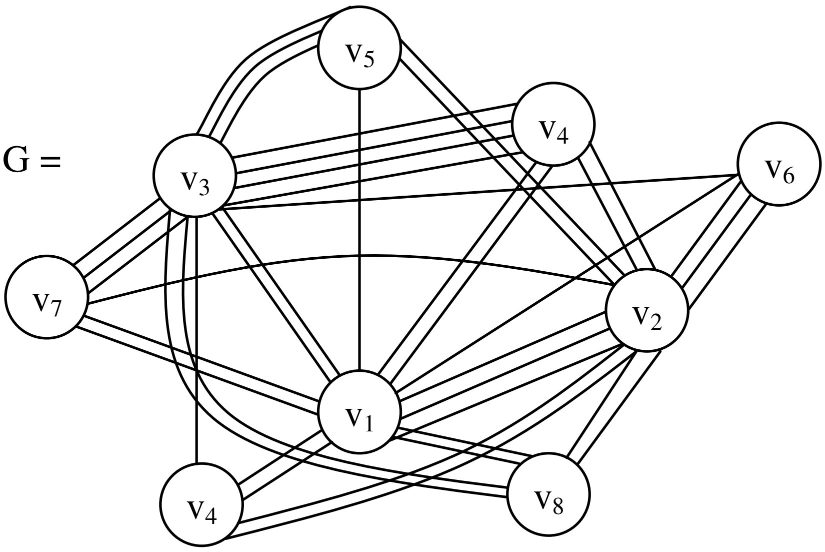


Figure 2.111

a) Does G contain adjacent multitriads which are not adjacent multicliques of order 4.

21. Let G be a 6-edges multiwheel given by the following figure.

G =

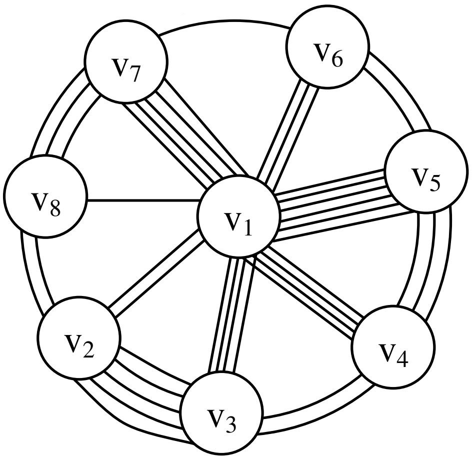


Figure 2.112

i) Find all the pairs of adjacent multitriads.

ii) Find any disconnected multigraph of order 5.

22. Let G be a 4-edges multigraph given by the following figure.

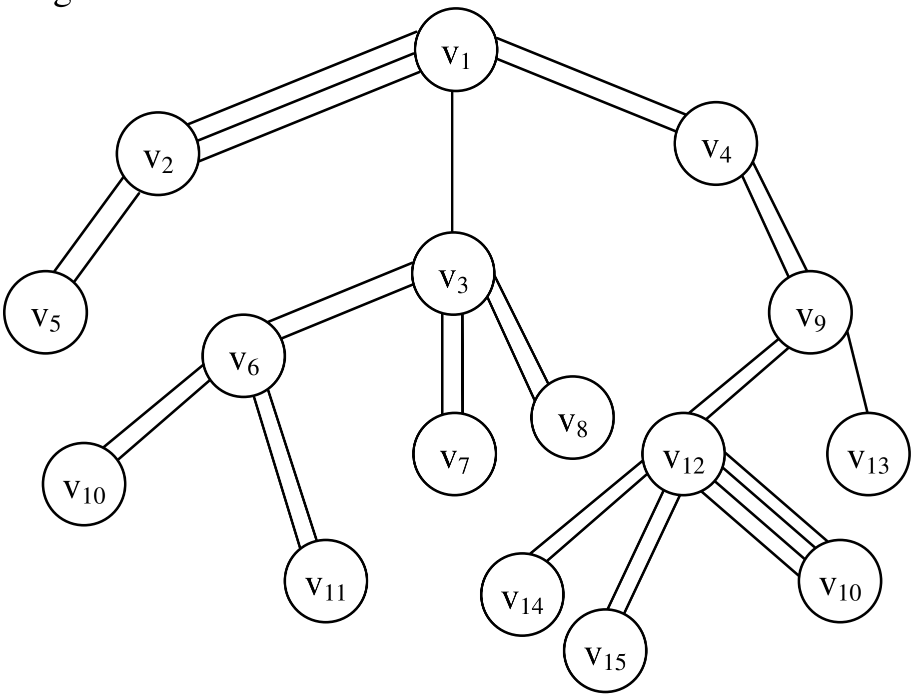
G = 

Figure 2.113

i) Will all 4-edges multisubgraphs of G with 6 vertices be a tree?

a) Justify your claim.

b) Find all such 4-edges multisubgraph trees.

ii) Find all disconnected 4-edges multisubgraphs of G.

iii) Can there by a 4-edges multisubgraph with 12 vertices which is not a multitree? Justify.

23. For the 3-edges multigraph G given by the following figure prove all multisubgraphs which contain the vertex v_1 are also 3-edges multistar subgraphs.

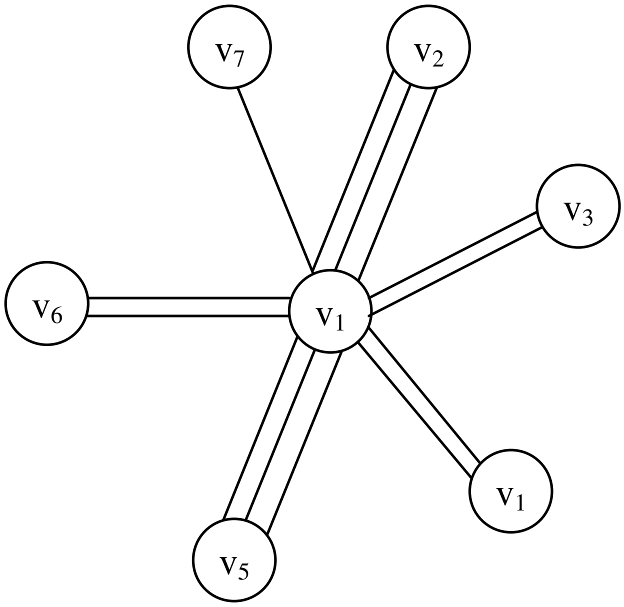
G = 

Figure 2.114

i) Find all empty 3-edges multisubgraphs.

ii) Prove no empty 3-edges multisubgraphs can contain the vertex v_1.

24. Prove that for a m-edges circle multigraph none of its m-edges mutlisubgraphs will be a circle multigraph.

25. Prove or disprove mutlisubgraph of a complete multigraph is complete.

26. Prove multisubgraphs of a line 9-edges multigraph G are multiline subgraphs which can be disconnected or connected.

27. Enlist all special features enjoyed by m-edges multigraphs in general?

28. Compare a complete 4-edges multigraphs with 6 vertices with a 3-edges multigraphs with 6-vertices.

a) Find the number of them associated with 4-edges and 3-edges.

29. Let G_1 = {collection of 5-edges multitriads (complete) with 3 vertices}.

G_2 = {collection of 4-edges multitriads with 3 vertices} and so on.

G_n = {collection of (n + 2) edges multitriads with 3 vertices}

i) Prove $|G_1| < |G_2| < \ldots < |G_n|$.
ii) Find $|G_i| = o(G_i)$; i = 1, 2, …, n.

30. Let G be the collection of all 3-edges multigraphs with 7 vertices.

i) Find o(G).

ii) How many 3-edges multigraphs which are complete with 7 vertices?

iii) How many 3-edges multigraphs with 7 vertices are ring or circle?

iv) How many 3-edges multigraphs with 7-vertices are star graphs?

v) How many 3-edges multigraphs with 7-vertices are disconnected?

vi) How many 3-edges complete multigraphs are there from the collection of all 3-edges multigraphs of 7 vertices?

31. For the 4-edges multigraph with 6 vertices G given below find all its multisubgraphs.

G =

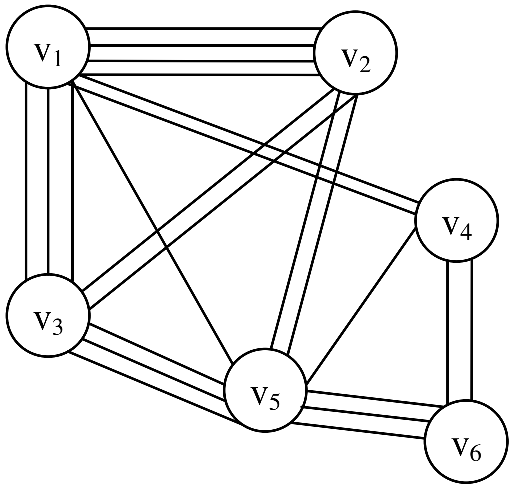


Figure 2.115

i) How many 4-edges multisubgraphs are disconnected?

ii) How many complete 4-edges multisubgraphs of size 3 and 4 are in G?

iii) Does G contain any adjacent multitriad?

32. Enumerate the advantage of using m-edges multitrees in the place of usual or classical trees.

33. Will multigraphs play a vital role in multinetworks?

34. Let G be a 5-edges circle multigraph on 9 vertices given by the following figure.

G =

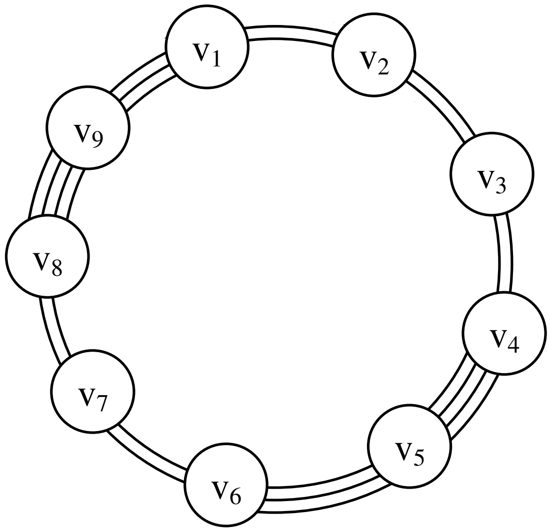


Figure 2.116

i) Find the number of 5-edges multisubgraphs which are (a) connected (b) disconnected

ii) Prove none of the 5-edges multisubgraph can be circle.

35. Find all distinct special features between usual graphs and m-edges multigraphs on a fixed number of n-vertices.

36. Let G be a 6-edges multigraph with 6 vertices given by the following figure.

G =

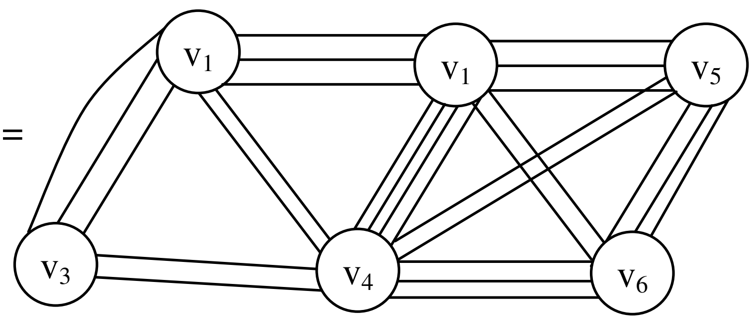


Figure 2.117

i) Find all 6-edges multisubgraphs which yield adjacent multitriads.

ii) Is G a adjacent complete 4-vertices multigraph? Justify your claim.

iii) Find all disconnected 6-edges multisubgraphs of G?

iv) Find the number of connected 6-edges multisubgraphs of G on 4-edges.

37. Prove that in case of a star 8-edges multigraphs with 8 vertices either the multisubgraphs are empty or they are only star 8-edges multisubgraphs with less than 8 vertices.

38. Prove the n-edges multisubgrpahs of a tree is either a tree or an empty multigraph or is a collection of disconnected line graphs for any m number of vertices.

39. Prove that in case of complete m-edges multigraphs with n-vertices ($n > 3$) if two vertex is removed the m-edges multisubgraphs continue to be connected.

Hence or otherwise show in case of $n = 4$ is a multiline and in case of $n = 5$ it is a multitriad.

Can we say if $n = t$ and 2 nodes of removed the multisubgraph will be complete?

40. Let G be a 3-edges multigraph with 16 vertices given by the following figure.

G =

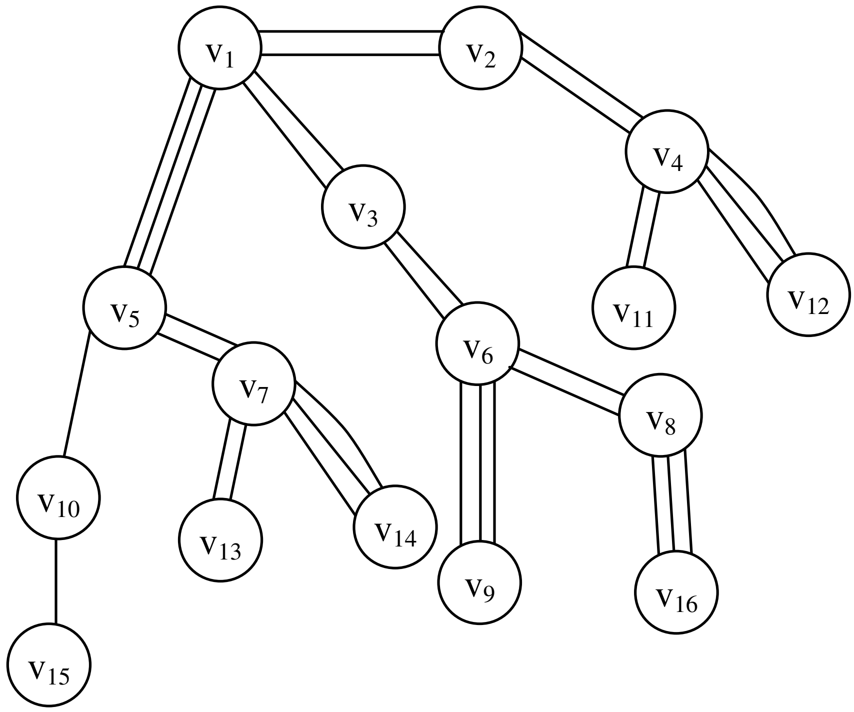


Figure 2.118

i) Find all 3-edges multisubgraphs of G which are trees

ii) How many empty multisubgraphs are there in G?

iii) Find all disconnected multisubgraphs of G.

41. Obtain any special applications of multitrees (m-edges multigraphs with n-vertices).

42. Can these m-edges multigraph be useful in the social network? Justify your claim by an application to a social network problem.

43. Let G be a 3-edges multigraph with 18 vertices given by the following figure.

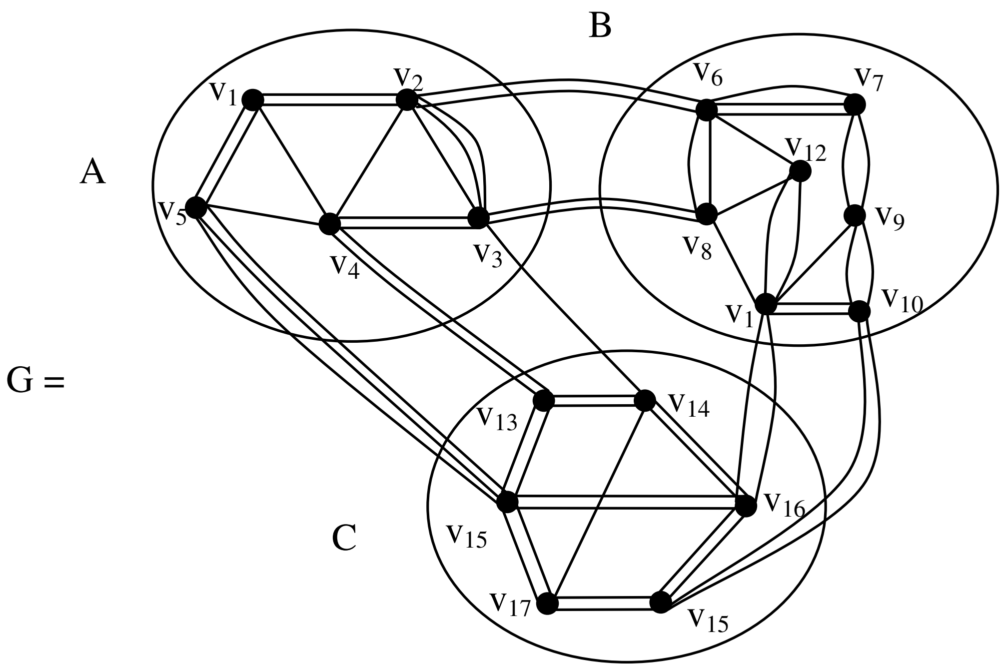


Figure 2.119

Study this multigraph as a social network of three groups and derive the conclusions based on your study.

44. Can a n-edges multigraph with 6 vertices be implemented in the place of a simple graph visualization of the RDF document?

    What are advantages and drawbacks of this?

45. Study the similarities and differences between n-edges directed multigraphs and a multivariate directed graph.

46. Dyads are the base or the cornerstone of social network. If two actors enjoy a feature we usually denote the dyad as

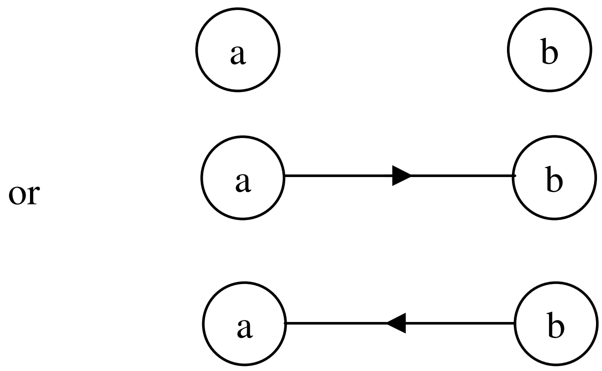


Figure 2.120

But it may so happen that it is not based on one relation the two actors are related.

Take for instance the two actors as two firms A and B. In usual graphs we can only say

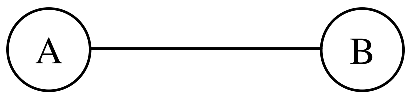


Figure 2.121

A relational tie exists between A and B.

Suppose

i) A supplies raw material to B

ii) B gives a finished goods for sale.

iii) A has profits / loss shared with B

iv) All sales in A is from finished goods B and not svice versa.

47. How best a classical dyad can represent the material in problem 46 this?

i) How will a multigraph (multidyad) represent this?

ii) Which representation is more realistic?

iii) Which representation is more specific?

iv) Describe using multidyads

a) Two friends

b) Two educational institution which enjoys some ties

c) Two banks which has some forced ties.

47. Describe 1-mode network and 2-mode network using multigraphs.

48. Can multigraphs play a role in affiliation networks?

(For instance two institutions A and B are such that the students of A get registered to courses from B.

Teachers in B are registered as Ph.D students in the institution A and so on).

49. Bring out the advantages / disadvantages of using multigraphs in affiliation networks.

50. Can multigraphs be used in n-mode networks (n > 3).

51. Can one categorically say multigraphs are better suited in n-mode networks than usual graphs as they are deal easily with more number of different kinds of social entities?

52. Can multigraphs be used in ego centric social networks? Justify your claim!

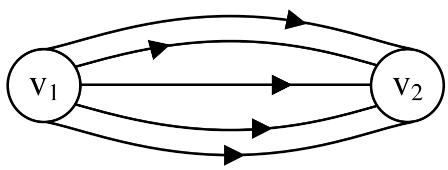
Chapter Three

DIRECTED MULTIGRAPHS AND THEIR PROPERTIES

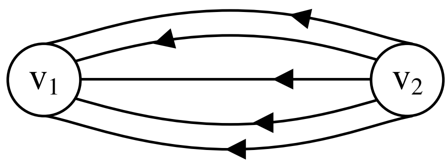
In this chapter we proceed onto define describe and develop the notion of directed multigraphs in a systematic way. However the notion of directed multigraphs is just illustrated by examples in Chapter II but we have not developed this notion is a systematic way.

First for the sake of completeness we describe directed multigraphs by some examples.

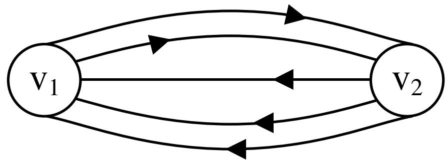
Example 3.1. Let G be the directed multigraph given by the following figure with 5 edges.



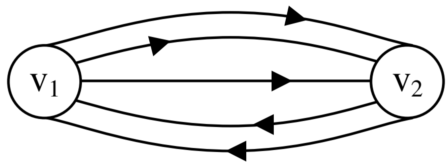
(a)



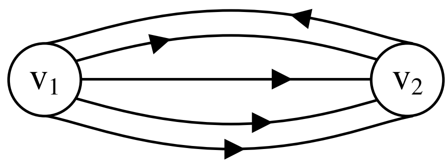
(b)



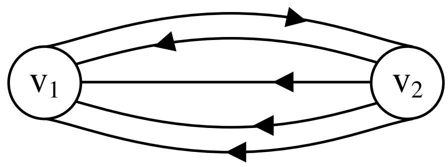
(c)



(d)



(e)



(f)

Figures 3.1

There are 6 multigraphs with five edges from vertex sets v_1 and v_2. All the 6 multigraphs are distinct. We see the cases of (a) and (b) the flow in the case (a) is from v_1 to v_2 all edges point from v_1 to v_2 and in case of (b) all edges point from v_2 to v_1.

In cases of (c) and (d) we see in case of (c) there are only two edges from v_1 to v_2 and three edges from v_2 to v_1. In case of (d) there are only two edges from v_2 to v_1 and three edges from v_1 to v_2.

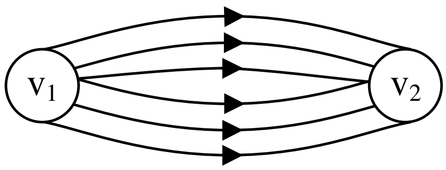
Finally in case of multigraphs (e) and (f) we see there is only one edge from v_2 to v_1 in case of (e) and four edges from v_1 to v_2 in case of (f).This is reversed in case of multigraph f as there is four edges from v_2 to v_1 and only one edge from v_1 to v_2.

All the 6 directed multigraphs are dyads or directed multiedge dyads, enjoying some special features.

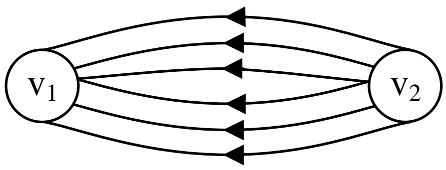
Since dyads are the basic units of a network multigraph so these directed multigraphs we study and characterize them in a very special way.

We give yet another example of directed multigraphs with 6 edges.

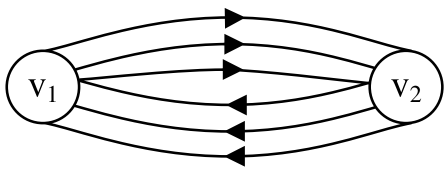
Example 3.2. Let G be a directed multigraph with 6 edges and two vertices given by the following



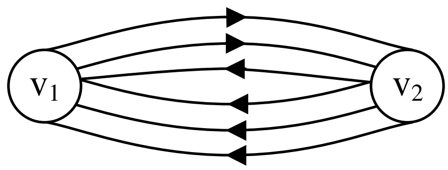
(a)



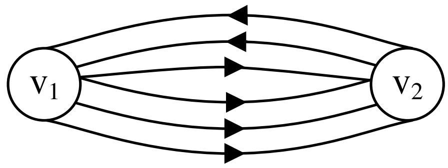
(b)



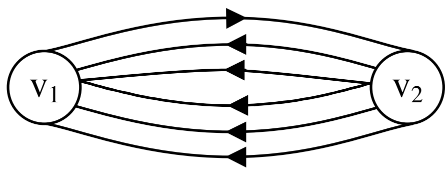
(c)



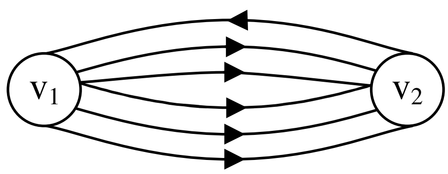
(d)



(e)



(f)



(g)

Figures 3.2

We see there are 7 directed multigraphs with 6 edges. All the 7 directed multidyads are distinct. We see in the multigraph (a) all the edges are from v_1 to v_2 and that of in (b) all edges are from v_2 to v_1. Both happen to be opposite of each other whereas in case of (c).

We see the number of edges from v_1 to v_2 is three and that of v_2 to v_1 is also three we call such directed multigraph as balanced directed multidyads.

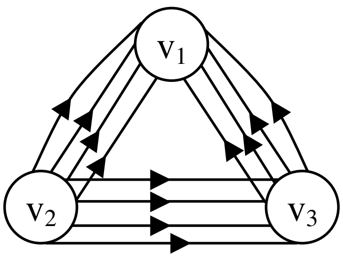
We will be finding conditions on multidyads to be balanced. Study in this direction is interesting and innovative.

We recall the directed multidyads in (d) and (f). In (d) v_2 is more attracted towards v_1 and exactly the reverse happens in case of (f) v_1 is more attracted towards v_2.

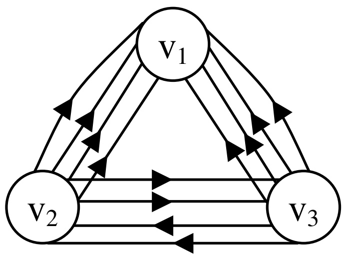
Similarly, in directed multigraphs (g) and (h) we see they are directed multidyads, they work in a just opposite way or (g) can be realized as the complement (interms of edges) of (h) and vice versa. These pairs of directed multidyads we define as edge complemented multidyad pairs.

Next, we proceed onto describe directed multigraphs with four edges and 3 vertices or in short directed multitriads.

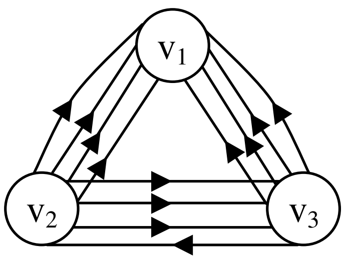
Example 3.3. Let G be a directed multitriads with four edges described by the following figures.



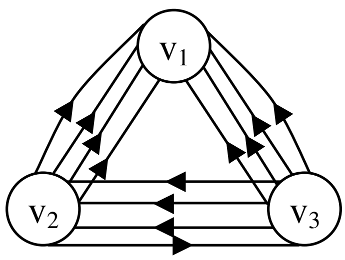
(a)



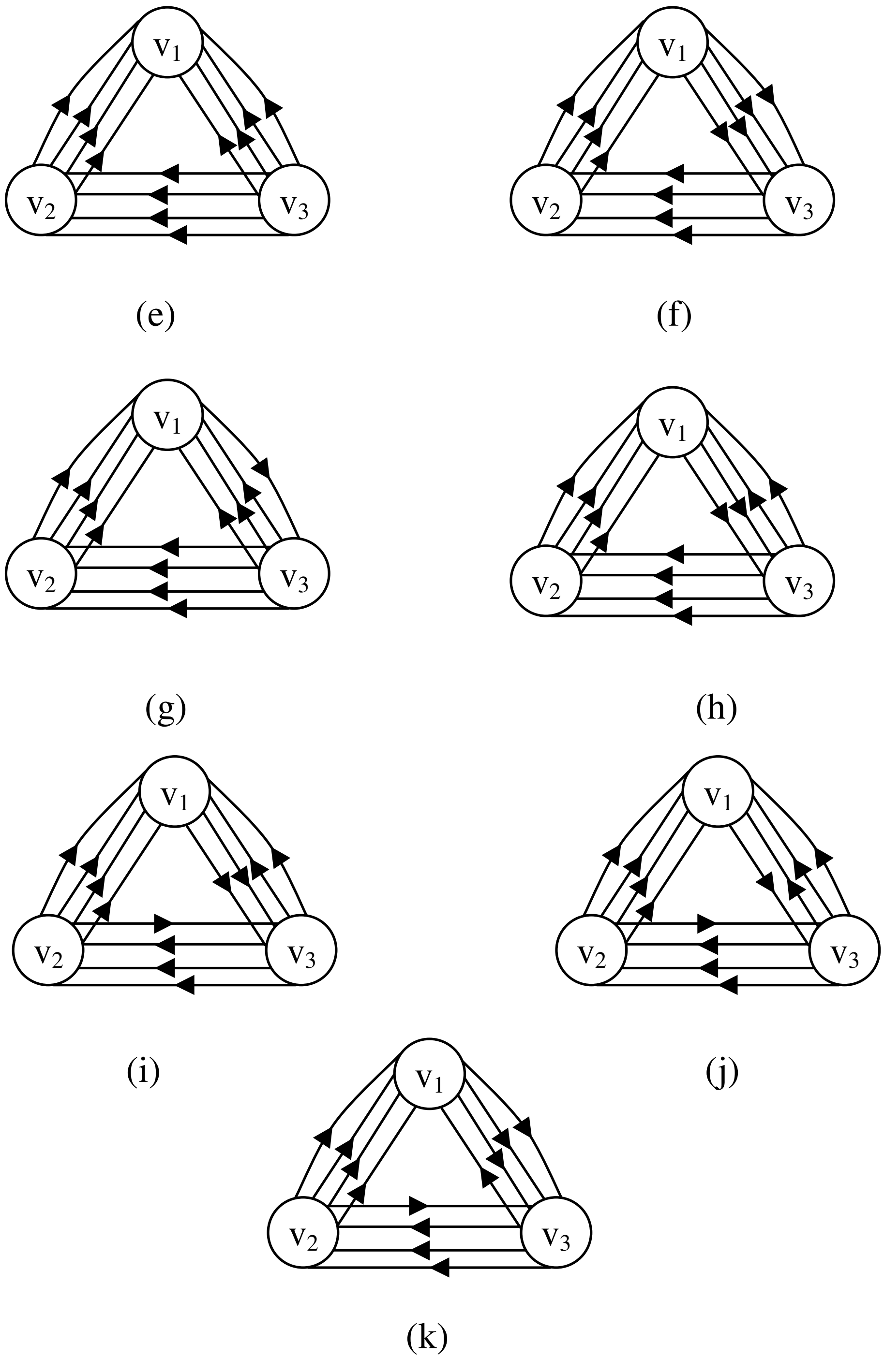
(b)



(c)



(d)



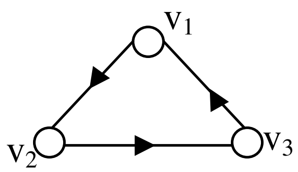
Figures 3.3

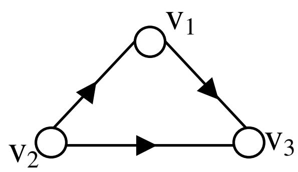
and so on. Thus it is an interesting problem to find the number directed multitriads with four edges. If we assume it has less than or equal to four edges and all of them are multitriads the number of them will increase drastically.

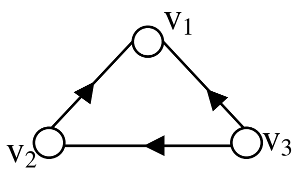
Here we only try to find all directed multitriads with less than or equal to 2 edges.

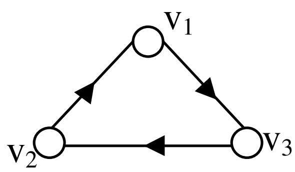
Example 3.4. Let v_1, v_2, v_3 be the three vertices which are associated with directed multitriads which has less than or equal to two multiedges (need not necessarily be a uniform directed multitriad.

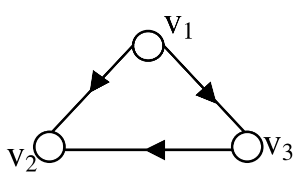
The figures are as follows.

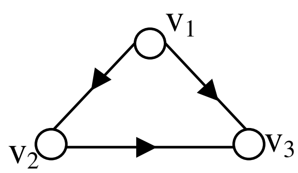


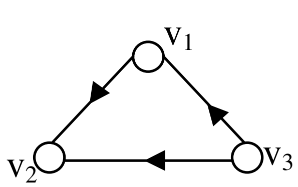


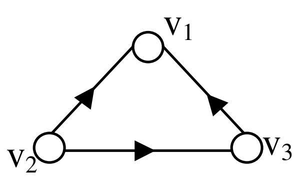


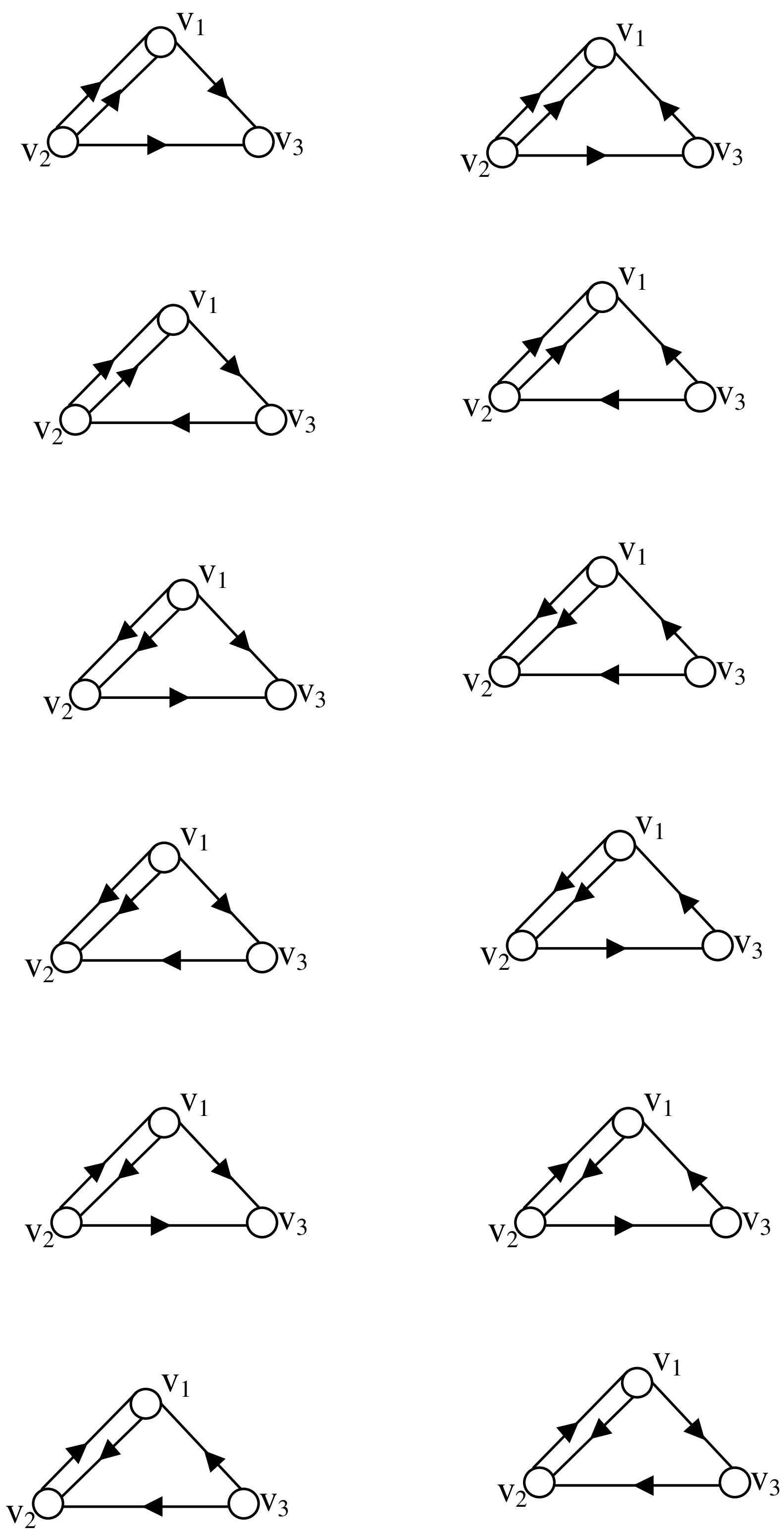






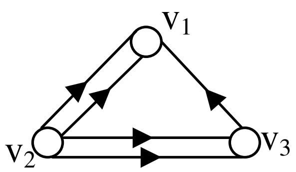


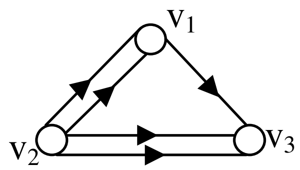


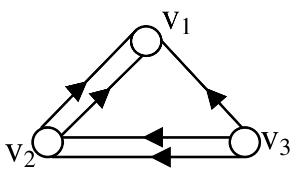


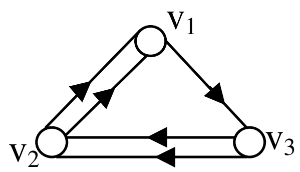
Figures 3.4

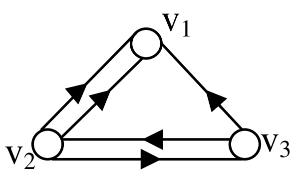
There are 24 such directed triads when v_1 v_3 has two edges and v_2 v_3 has two edge each contributing to 12 such multitriads. Now we give other directed multitriads.

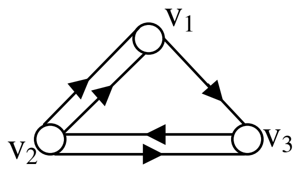


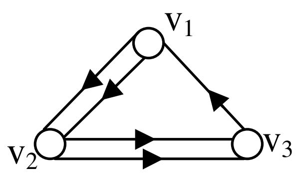


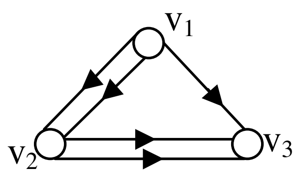


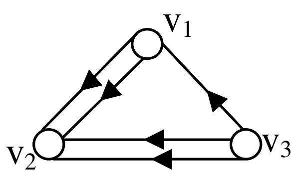


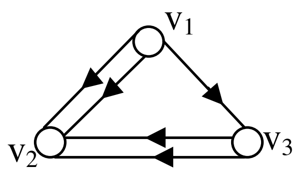


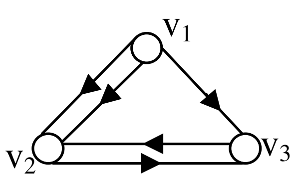


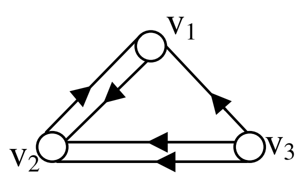


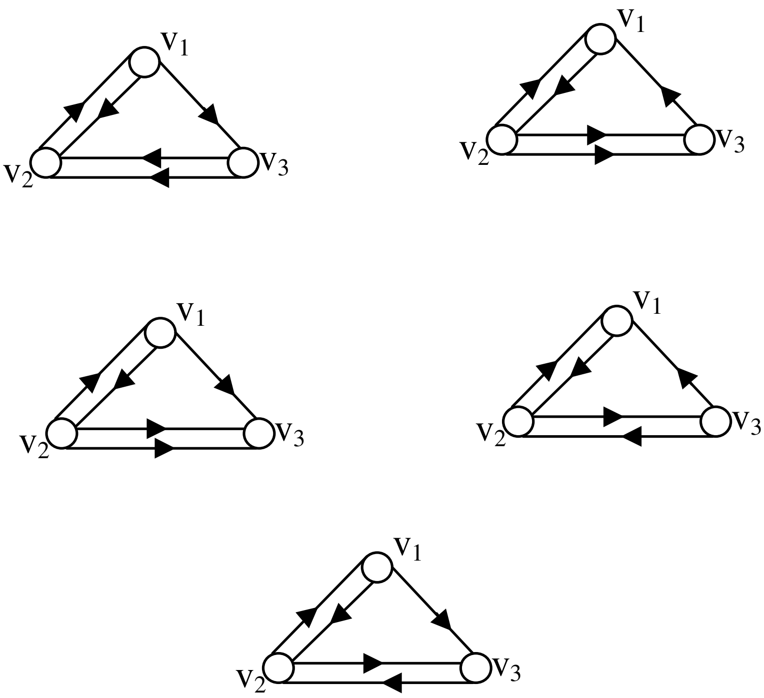












Figures 3.5

There are 36 directed multigraphs which are multitriads this type which has only one edge connecting v_1v_3 and v_3v_1.

Each edges v_1v_3 will contribute 18 such multitriads and v_2v_3 will contribute to 18 such multitraids.

Finally, if all the three edges v_1v_2, v_2v_3 and v_3v_1 are connected by two edges then we get the following uniform complete directed multitriads given by the following figures.

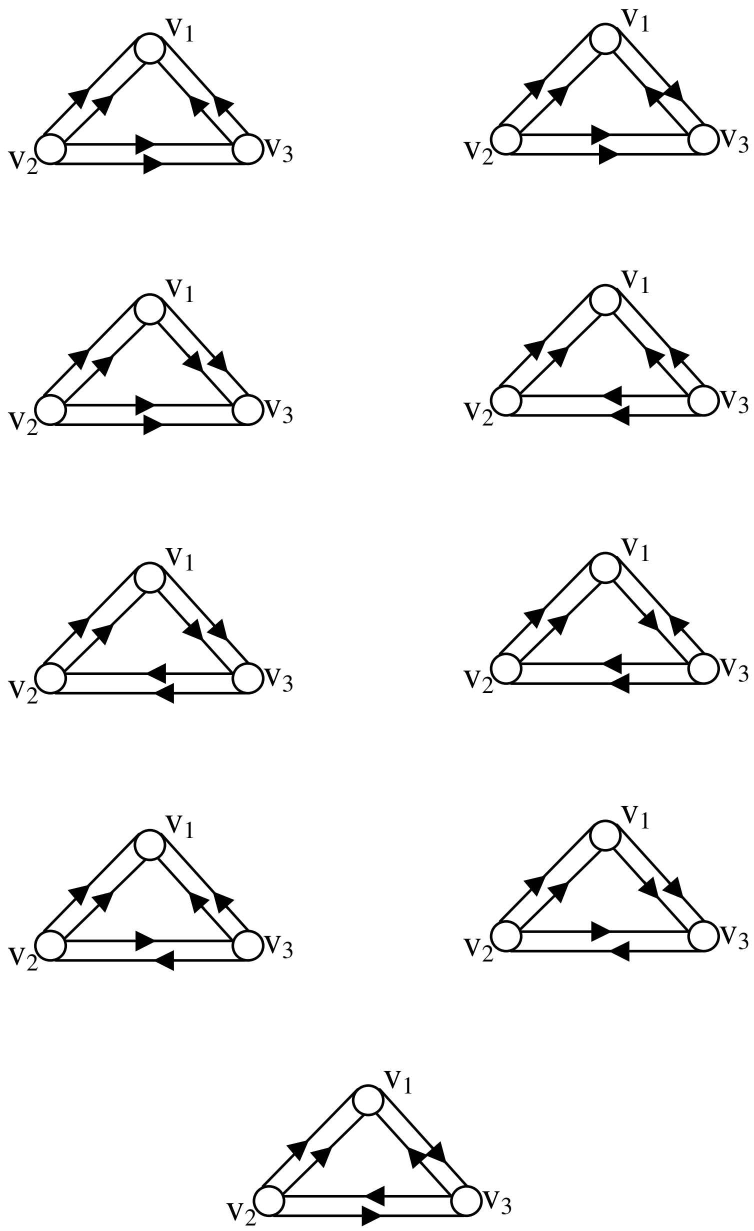


Figure 3.6

There will be 9 such directed multitriads if the direction from $\overrightarrow{v_2 v_1}$ is changed to $\overrightarrow{v_1 v_2}$.

Hence in all 18 multitriads exist. Similarly, for the multiedges

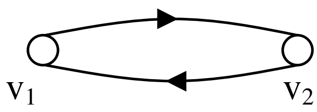


Figure 3.7

there will be 9 more such directed multitriads. So the total number of multitriads using of the vertices v_1 and v_2 alone is 27.

This total directed multitraids are

$8 + 36 + 54 + 27 = 125.$

However, based on this we propose the following conjecture.

Conjecture 3.1. Let v_1, v_2 and v_3 be three vertices. How many directed complete multitriads with number of edges n or less than n exist.

i) For n = 2 does the exists 25 such directed multitriads? (proved easily)

ii) Find for n = 3 (Find the number, easy to calculate).

Does the number of them for each n = 2, 3, …, m; ($m < \infty$) follow any common number theoretic formula?

We proceed onto give more examples of directed multigraphs and then their multisubgraphs.

Example 3.5. Let G be a directed multigraph given by the following figure which has a maximum of 5 edges.

G =

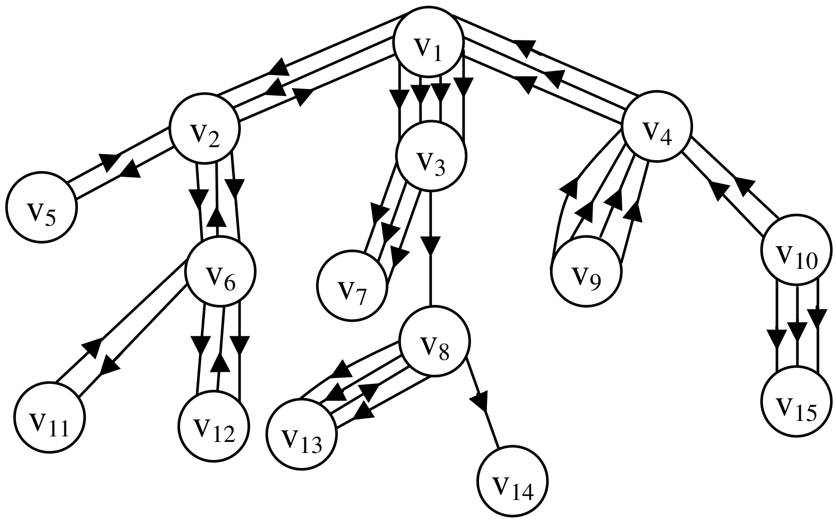


Figure 3.8

Clearly G is a directed special type of multitree. We see the directions are mixed. Every directed multisubgraph is not a tree in general, some may be empty multisubgraphs.

We list out some of the directed multisubgraphs of G in the following.

H_1 =

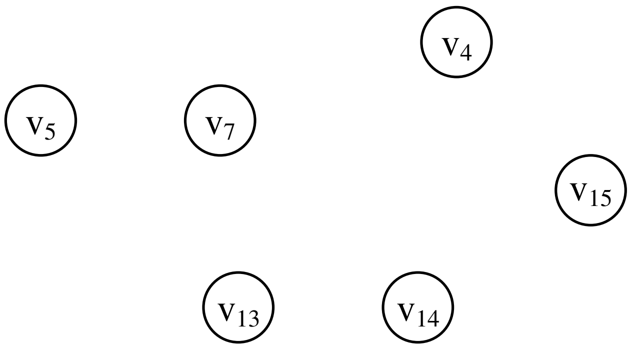


Figure 3.9

H_1 is the empty multisubgraph of G which is not a tree.

H_2 =

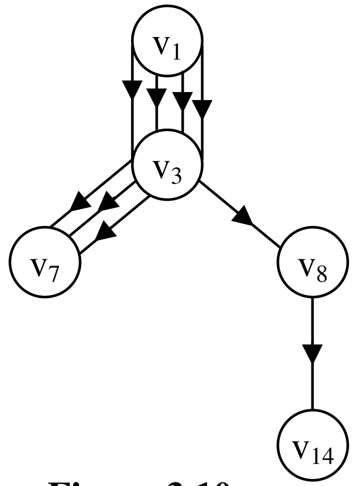


Figure 3.10

H_2 is a multisubgraph which is a multisubtree of G.

We see G has mixed directions, however. H_2 is only one way directed from root to leaves.

H_2 is also a connected multisubtree of G. Consider H_3 the multisubgraph given by the following figure.

H_3 =

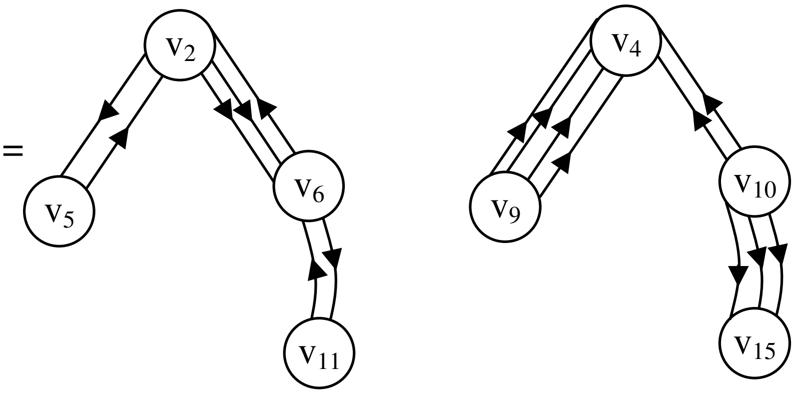


Figure 3.11

Both of them are multitrees, however this multisubgraph is disconnected.

Let H_4 be the multisubgraph of G given by the following figure.

H_4 =

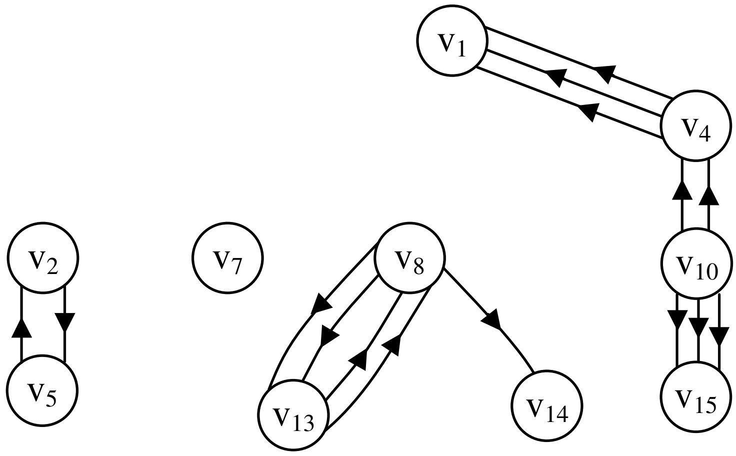


Figure 3.12

We see H_4 is a disconnected one with three multisubtrees and a node.

Next we proceed onto describe directed star multigraph which is directed by an example.

Example 3.6. Let G be a star multigraph with maximum six edges given by the following figure.

G =

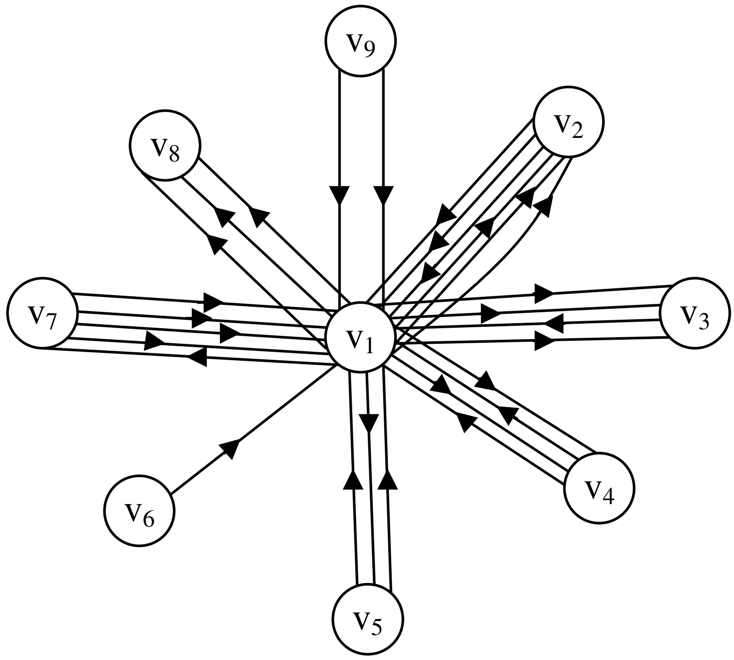


Figure 3.13

We enlist some of the multisubgraphs of the star multigraph G.

Let H_1 be the star multisubgraph of G given by the following figure.

H_1 =

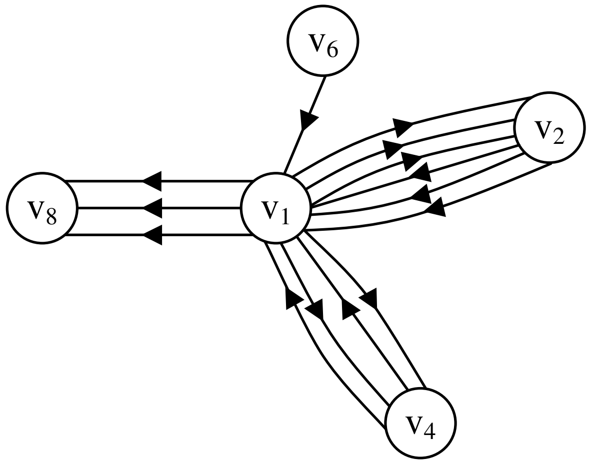


Figure 3.14

Clearly H_1 is again a directed star multisubgraph of G.

Let H_2 be the multisubgraph of G given by the following figure.

H_2 =

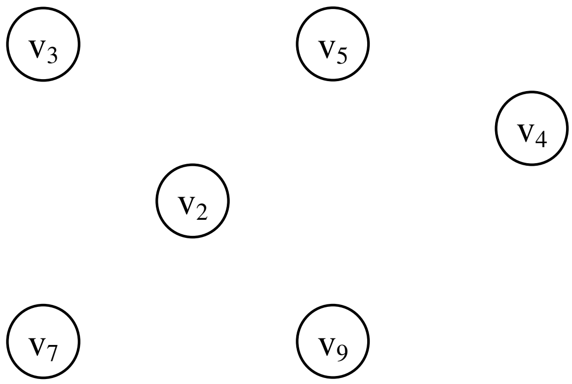


Figure 3.15

Clearly H_2 is an empty multisubgraph of G.

In view of all these we just make the following observations in case of star multigraphs.

If G is a star multigraph given in figure, if the vertex v_1 is not present in any of the multisubgraphs then all those multisubgraphs will only be empty multisubgraphs.

We put forth the following theorem.

Theorem 3.1. *Let G be a directed star multigraph with v_1 as the central vertex node which is connected to all other nodes.*

i) *Every multisubgraph which does not contain the vertex set v_1 is an empty multisubgraph of G.*

ii) *Every multisubgraph of G which contains the vertex v_1 is again a star multisubgraph of G.*

Proof follows from logical arguments and is left as an exercise to the reader.

Next we proceed onto give an example of a directed circle multigraph and find their multisubgraphs.

Example 3.7. Let G be a directed circle multigraph with maximum 5 edges given by the following figure.

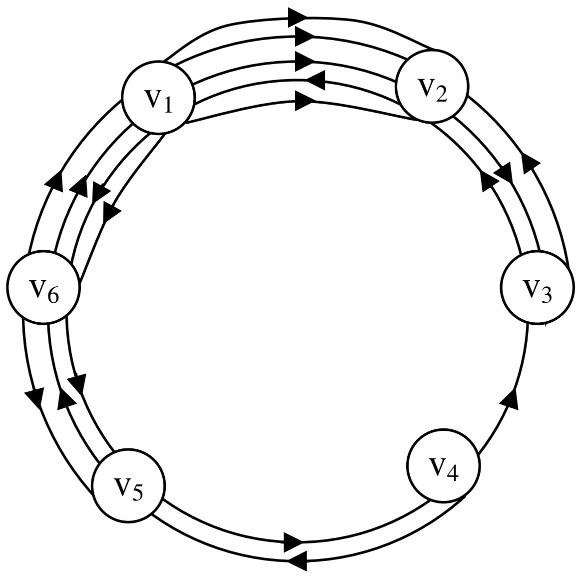
G = 

Figure 3.16

We see G is a circle multigraph with six vertices. The maximum edges which this G contains is 5.

Now we find a few of the multisubgraphs of this G.

Let H_1 be a multisubgraph of G given by the following figure.

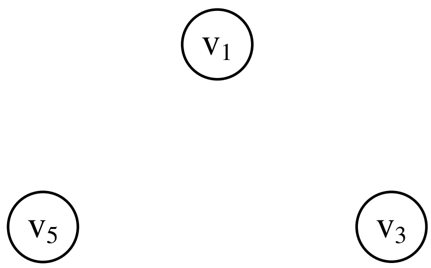
H_1 = 

Figure 3.17

Clearly H_1 is a empty multisubgraph of G.

Let H_2 be a multisubgraph of G given by the following figure.

H_2 =

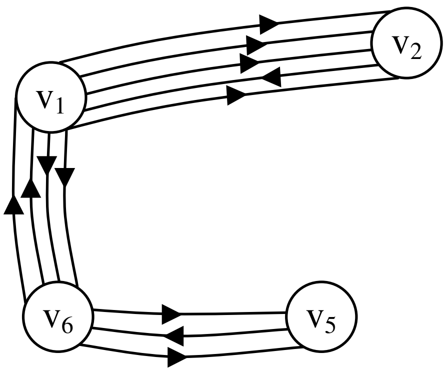


Figure 3.18

Clearly H_2 is a connected multisubgraph of G which is not a circle multisubgraph.

Let H_3 be the multisubgraph of G given by the following figure.

H_3 =

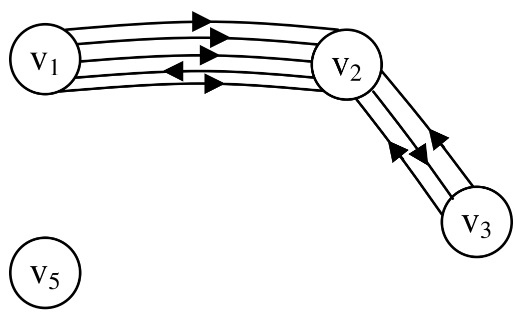


Figure 3.19

Clearly H_3 is a disconnected multisubgraph of G which is clearly not a circle.

In view of this we have the following theorem.

Theorem 3.2. *Let G be a directed circle mutigraph with maximum n edges.*

i) *None of the multisubgraphs of G will be a circle multigraph.*

ii) *G can have disconnected, or connected or empty multisubgraphs.*

Proof is direct and hence left as an exercise to the reader.

Next interesting feature is to study line multigraphs.

We will first illustrate this situation by some examples.

Example 3.8. Let G be a line multigraph given by the following figure with maximum six edges given by the following figure.

G =

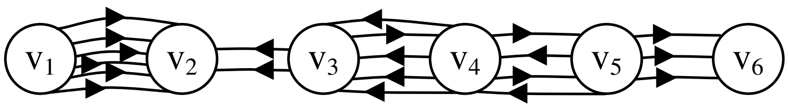


Figure 3.20

We now enlist a few of the multisubgraphs of G. It is to be noted that G is a line multigraph with maximum edges 6.

Let H_1 be the multisubgraph of G given by the following figure.

H_1 =

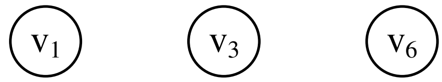


Figure 3.21

Clearly H_1 is a empty multisubgraph of G.

Let H_2 be the multisubgraph of G given by the following figure.

H_2 =

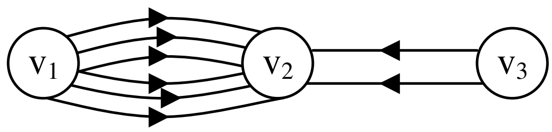


Figure 3.22

Clearly H_2 is a multisubgraph of G which is connected and is also a line multisubgraph of G.

Let H_3 be the multisubgraph of G given by the following figure.

H_3 =

Figure 3.23

Clearly this multisubgraph H_3 of G is a disconnected one which has two components which are line multisugraphs.

Let H_4 be the multisubgraph of G given by the following figure.

H_4 =

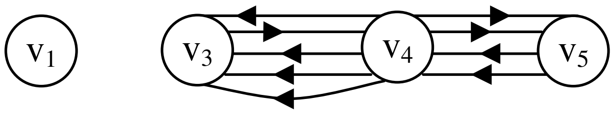


Figure 3.24

Clearly H_4 is a disconnected multisubgraph with 2 components one is a line subgraph other is just a node.

Now we proceed onto study the line multigraph of a multigraph G.

We first describe them by example.

Example 3.9. Let G be a multidyad given by the following figure.

G =

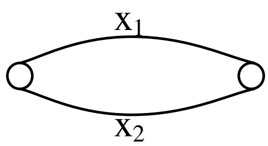


Figure 3.25

The multiline graph of G is multidyad given by

l(G) =

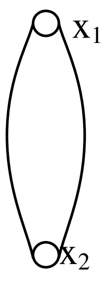


Figure 3.26

We see G = l(G) in this case.

Consider the multidyad G given by the following figure.

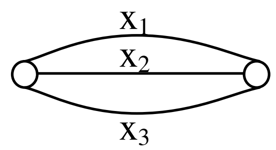
G = 

Figure 3.27

The multiline graph of G;

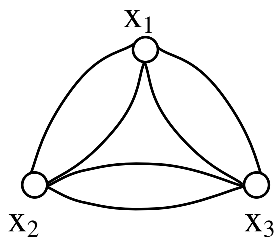
l(G) = 

Figure 3.28

Clearly l(G) is a multitriad.

Let G be the multidyad given by the following figure.

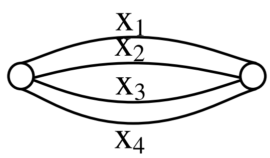
G = 

Figure 3.29

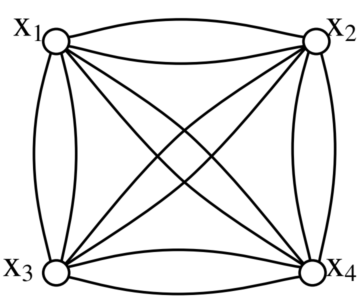
l(G) = 

Figure 3.30

The line multigraph of G,l(G) is a complete uniform multigraph of order 4.

Let G be a multidyad given by the following figure;

G =

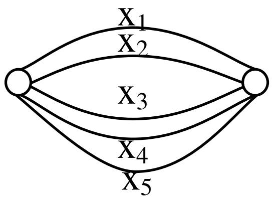


Figure 3.31

The line multigraph of G

l(G) =

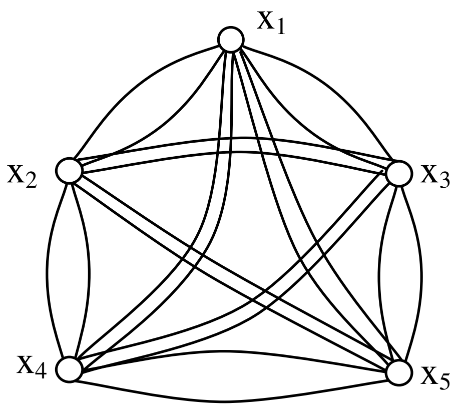


Figure 3.32

We see the resultant line multigraph l(G) of G is a uniform complete multigraph with edges two each.

In view of this we record the following theorem.

Theorem 3.3. *Let G be a multidyad with n-edges ($n \geq 3$).*

G =

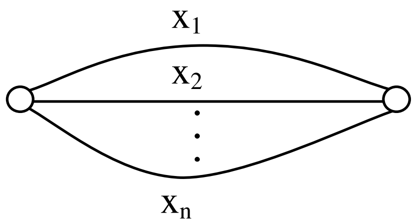


Figure 3.33

The line multigraph of G; l(G) is a uniform complete multigraph of order n with two edges.

Proof is direct and hence left as an exercise to the reader.

However this theorem is very interesting for a n-edges multigraphs line multigraph happens to be a uniform complete multigraph with n vertices.

Now we proceed onto analyse the line graphs of a triad.

Example 3.10. Let G_1 be a triad which is incomplete given by the following figure.

G_1 =

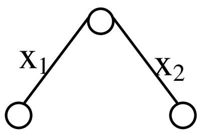


Figure 3.34

The line graph of G_1 is

l(G) =

Figure 3.35

is a dyad.

Let G_2 be a triad given by

G_2 =

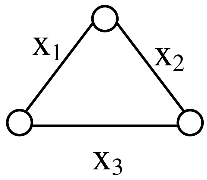


Figure 3.36

The line graph of G_2 is

l(G) =

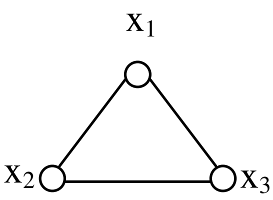


Figure 3.37

We see $G_2 = l(G_2)$.

Consider the incomplete multitriad given by

G_3 =

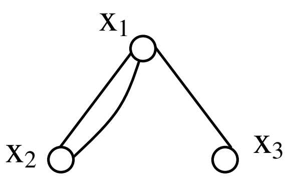


Figure 3.38

The multi line graph associated with G_3 is

$l(G_3)$ =

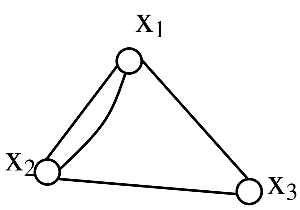


Figure 3.39

$l(G_3)$ is a not uniform complete multitriad.

Let G_4 =

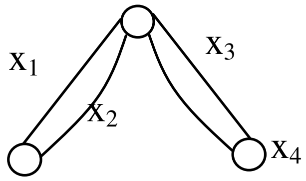


Figure 3.40

be an incomplete mutlitriad. The line multigraph of G_4 is

$l(G_4)$ =

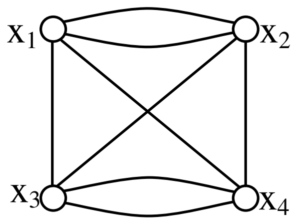


Figure 3.41

The line multigraph of G_4 is a complete not uniform multigraph of order four.

Let G_5 be a multi incomplete triad given by the following figure.

G_5 =

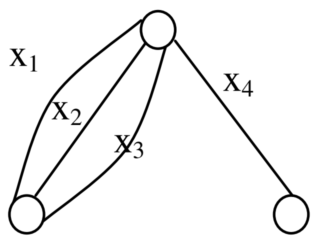


Figure 3.42

The line multigraph associated with G_5 is as follows

$l(G_5)$ =

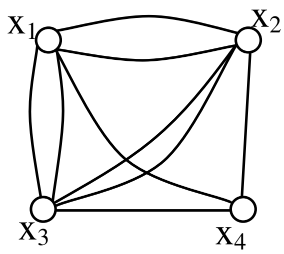


Figure 3.43

Clearly G_5 is a nonuniform complete multigraph of order four.

We see $l(G_5)$ and $l(G_4)$ are different.

Consider G_6 an incomplete multitriad given by the following figure.

G_6 =

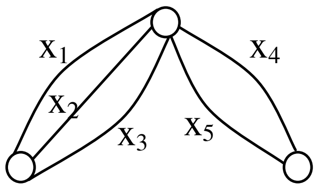


Figure 3.44

The multiline graph $l(G_6)$ associated with G_6 is as follows.

$l(G_6)$ =

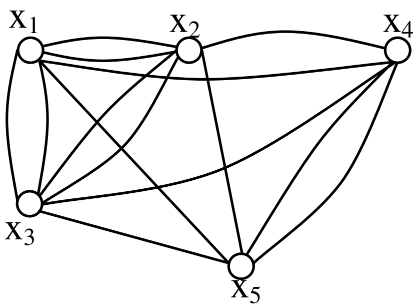


Figure 3.45

Clearly l(G_6) is a non uniform complete multigraph of order S.

Consider G_7 a multiincomplete triad given by the following figure.

G_7 =

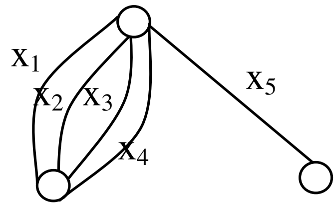


Figure 3.46

The multiline graph l(G_7) is given by the following figure.

l(G_7) =

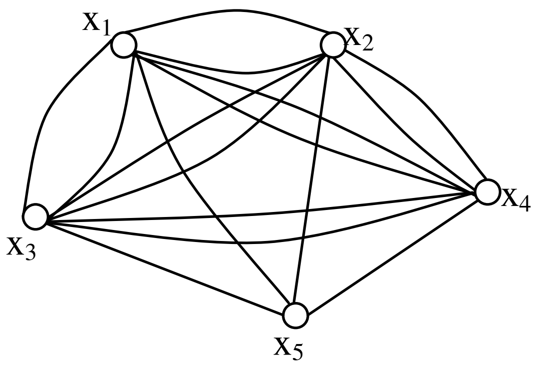


Figure 3.47

l(G_7) is also a non uniform complete multigraph of order 5.

Clearly l(G_6) and l(G_7) are different.

Let G_8 be a incomplete multitriad given by the following figure.

G_8 =

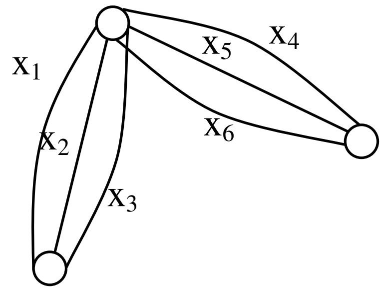


Figure 3.48

The multiline graph $l(G_8)$ of the multigraph G_8 is s follows.

$l(G_8)$ =

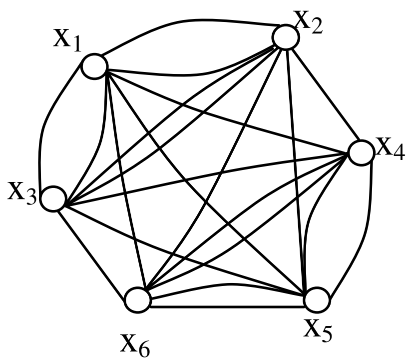


Figure 3.49

Clearly $l(G_8)$ is a non uniform complete multigraph of order six.

Let G_9 be a nonuniform complete multitriad given by the following figure.

G_9 =

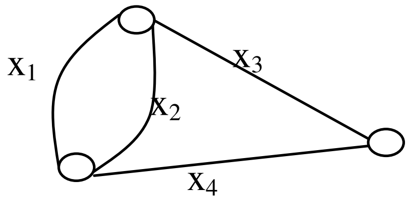


Figure 3.50

Let l(G_9) be the multiline graph given by the following figure.

l(G_9) =

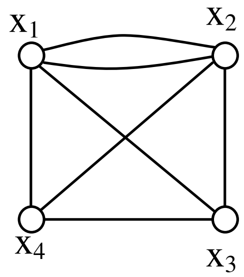


Figure 3.51

l(G_9) is a non uniform complete multigraph of order four but it is different from l(G_9) and l(G_5).

Let G_{10} be a multitriad given by the following figure

G_{10} =

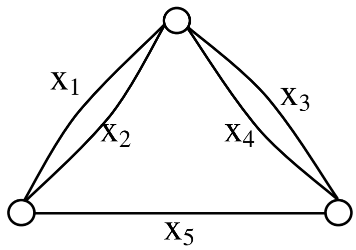


Figure 3.52

The multiline graph of G_{10} is as follows.

l(G_{10}) =

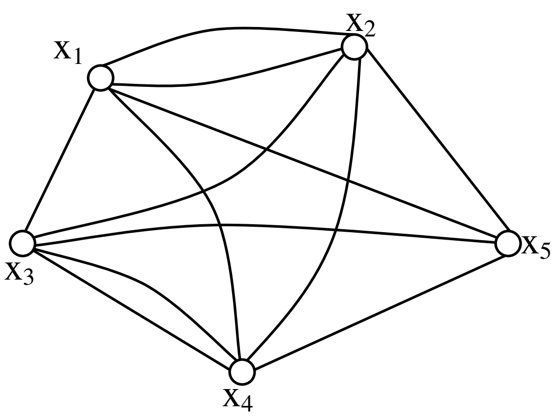


Figure 3.53

Clearly $l(G_{10})$ is a non uniform complete graph of order 5. But $l(G_{10})$ is different from $l(G_6)$ and $l(G_7)$.

Let G_{11} be a multitriad given by the following figure.

G_{11} =

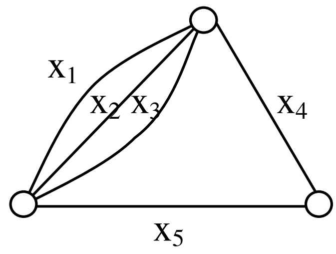


Figure 3.54

Let $l(G_{11})$ be the line multigraph associated with G_{11} given b y the following figure.

$l(G_{11})$ =

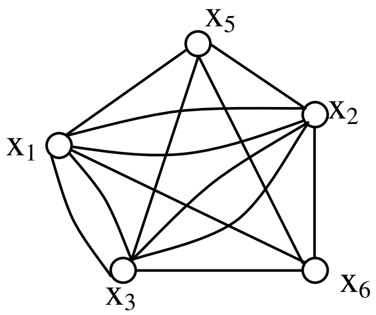


Figure 3.55

Clearly $l(G_{11})$ is a nonmultigraph which is complete and is different from $l(G_{10})$, $l(G_7)$ and $l(G_6)$.

Let G_{12} be a multitriad given by the following figure.

G_{12} =

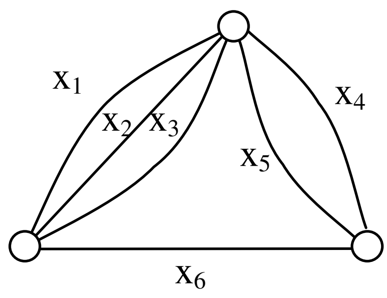


Figure 3.56

Let $l(G_{12})$ be the multiline graph of G_{12} given by the following figure.

$l(G_{12})$ =

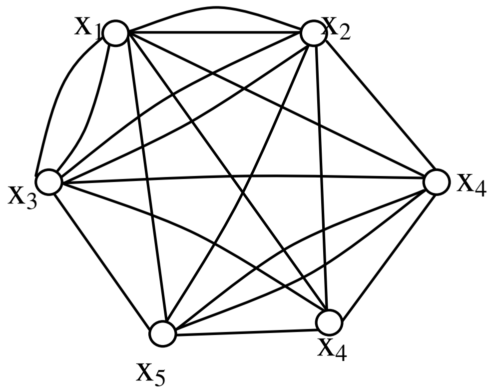


Figure 3.57

Clearly $l(G_{12})$ is a non uniform complete multigraph of order 6.

Let G_{13} be the multitriad given by the following figure

G_{13} =

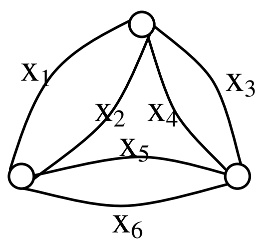


Figure 3.58

Let $l(G_{13})$ be the line graph of G_{13} given by the following figure.

$l(G_{13})$ =

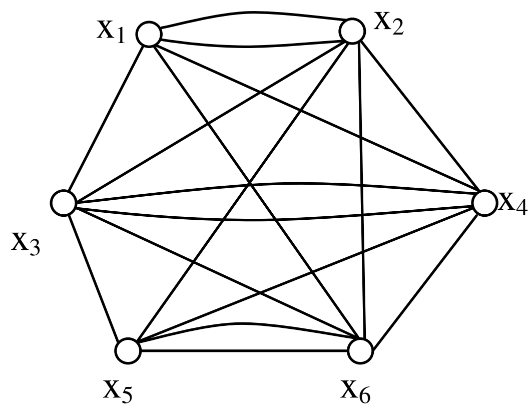


Figure 3.59

Clearly $l(G_{13})$ is a non uniform complete multigraph of order 6 different from $l(G_{12})$ and $l(G_8)$.

Let G_{14} be a multitriad given by the following figure.

G_{14} =

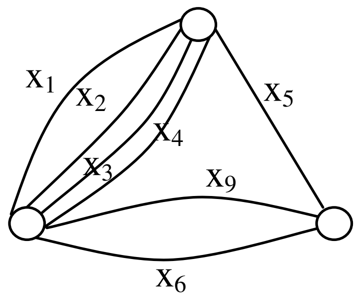


Figure 3.60

The line multigraph $l(G_{14})$ is as follows.

$l(G_{14})$ =

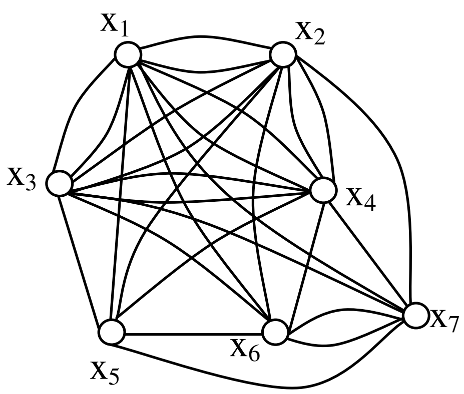


Figure 3.61

$l(G_{14})$ is a complete nonuniform multigraph which is different from $l(G_{13})$, $l(G_{12})$ and $l(G_8)$.

This has a uniform complete multigraph of order four with maximum 2-edges whereas $l(G_{12})$ has only one uniform complete multitriad and $l(G_8)$ has two nonadjacent uniform complete multitriads. $l(G_{13})$ has no complete uniform multitriads or complete uniform multigraph of order four.

Consider the multitriad given by the following figure.

G_{15} =

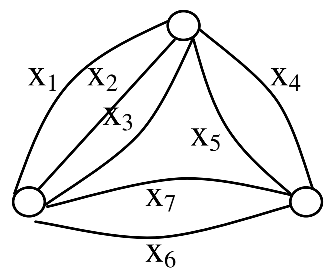


Figure 3.62

The line complete multigraph of G_{15}; $l(G_{15})$ is as follows.

$l(G_{15})$ =

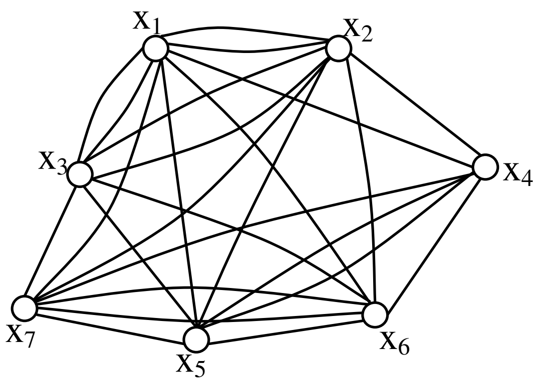


Figure 3.63

Thus $l(G_{15})$ is a complete nonuniform multigraph of order 7.

We just give one more example of multigraph G and its line multigraph l(G).

Example 3.11. Let G be a star multigraph given by the following figure.

G =

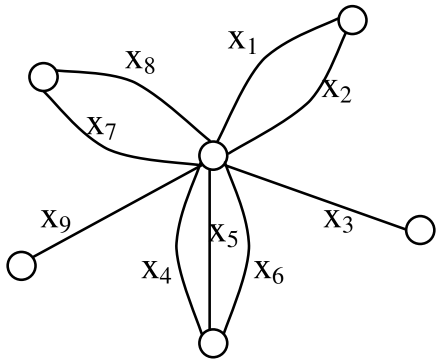


Figure 3.64

Now we find the line multigraph l(G) of G in the following.

l(G) =

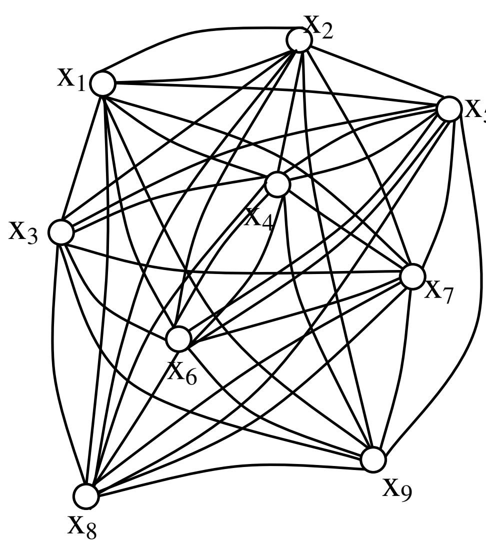


Figure 3.65

We see l(G) is a multiline graph of G which is nonuniform and complete of order 9.

Now we first proceed on to define the special type of line graph of a graph G as well as any multigraph G.

In the first place we wish to keep on record that the line graphs / multigraphs are built unconditionally so only we choose to call them as unconditional line graphs (or multigraphs) of a graph G.

When we say unconditional we mean that the line graph of the graph G does not depend on the intersection graph r (X) of G where X is the set of lines of a graph G. So in general the unconditional line graph of a graph G which we define here in general is different from the line graph l(G) defined in the classical way (Haryer).

Definition 3.1. *Let G be a graph or a multigraph with edges labeled as $x_1, x_2, \ldots, x_n$. The unconditional line graph l(G) of the graph or the multigraph G is got by taking the lines $x_1, x_2, \ldots, x_n$ as vertices and two vertices x_i and x_j (lines x_i and x_j) are adjacent if x_i and x_j intersect in G $i \neq j$; $1 \leq i, j \leq n$ or in short the lines x_i and x_j $(i \neq j)$ are adjacent, $1 \leq i, j \leq n$.*

It is to be noted in case of L(G) the line graphs defined in the classical way such properties are possible for in case of the unconditional line graphs l(G) every graph / multigraph G which is nonempty has a l(G) to exist.

For instance L(G) is forbidden in case of

G_1 =

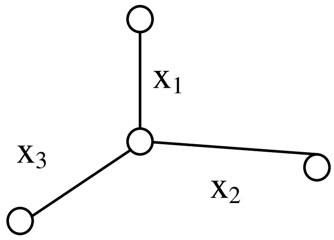


Figure 3.66

But l(G_1) =

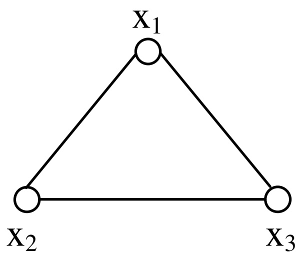


Figure 3.67

in the case of unconditional line graph which is a complete triad.

Let

G_2 =

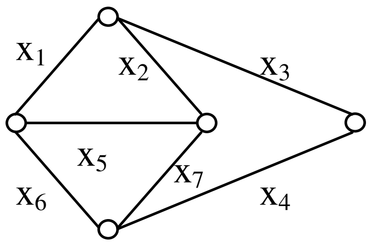


Figure 3.68

be a graph. This is a forbidden subgraph for classical line graphs; but in case of unconditional line graph $l(G_2)$ is as follows.

$l(G_2)$ =

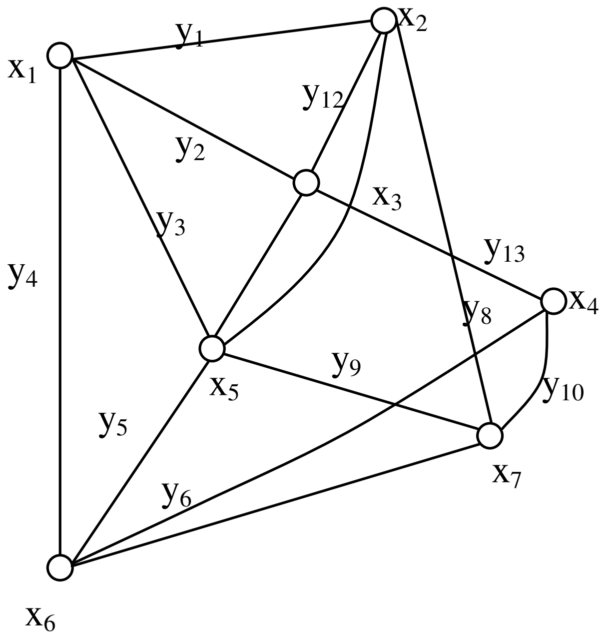


Figure 3.69

We now find $l(G_3)$ where G_3 =

G_3 =

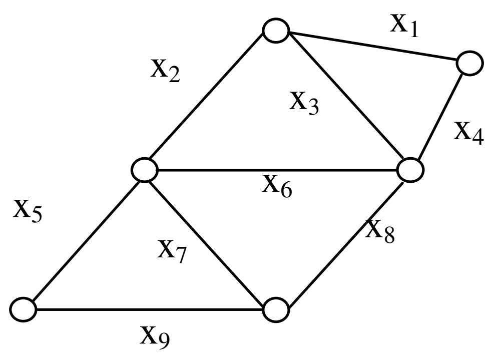


Figure 3.70

$l(G_8)$ when G_8 is a subgraph is a forbidden line graph.

$l(G_3)$ =

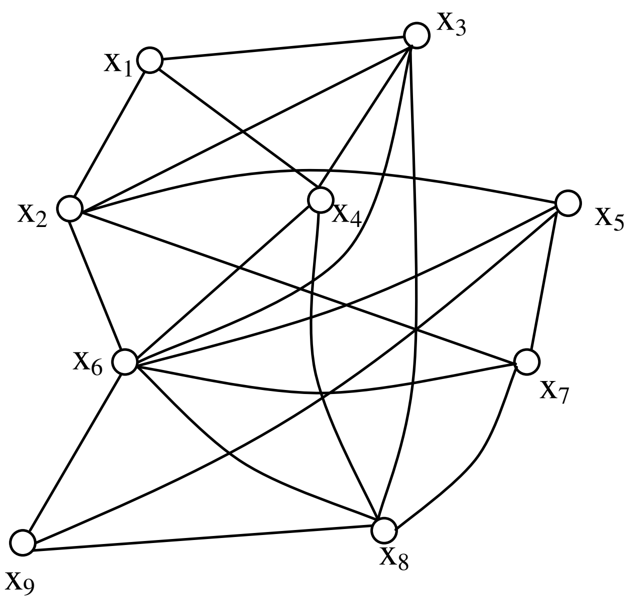


Figure 3.71

We see G_3 has adjacent triads.

Finding adjacent triads of $l(G_3)$ is left as an exercise to the reader.

Now for G_1 =

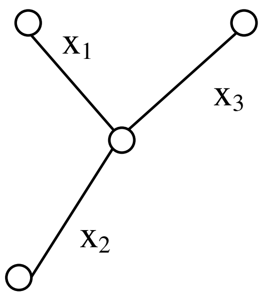


Figure 3.72

$l(G)$ =

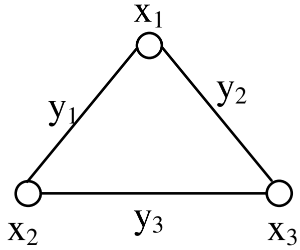


Figure 3.73

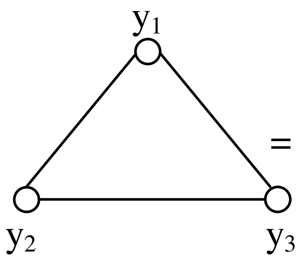
$l(l(G_1)) =$  $= l(G_1)$

Figure 3.74

We leave the following problems as open conjectures.

Conjecture 3.2. Find all graphs G such that

i) $l(G) = G$

ii) $l(l(G)) = l(G)$ or G

iii) $l(l(l(G))) = l(G)$ or G

iv) $l^n(G) = l^t(G)$ or $l(G)$ or G $2 \leq t < n$.

Let G_4 be the forbidden subgraph for a line graph given by the following figure.

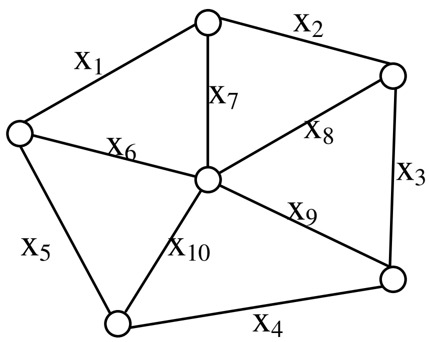
$G_4 =$ 

Figure 3.75

The unconditional line graph $l(G_4)$ has 10 vertices given by the following figure.

$l(G_4)$ =

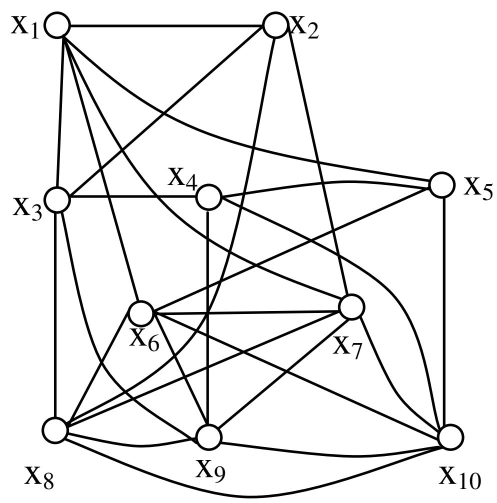


Figure 3.76

We see the edges outside when taken as vertex of $l(G_4)$ have four edges incident to it whereas the central edges when taken as a vertex have 6 edges incident to them.

Thus the edges x_1, x_2, x_3, x_4 and x_5 taken as vertices of $l(G_4)$ each has only four edges incident to them, where as the edges x_6, x_7, x_8, x_9 and x_{10} taken as vertices of $l(G_4)$ each has six edges adjacent to them.

Thus degree of each x_i is either 4 or 6, $1 \leq i \leq 10$.

Next we find the unconditional line graph $l(G)$ of G given by the following figure.

G =

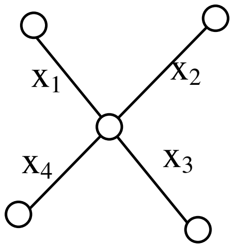


Figure 3.77

l(G) =

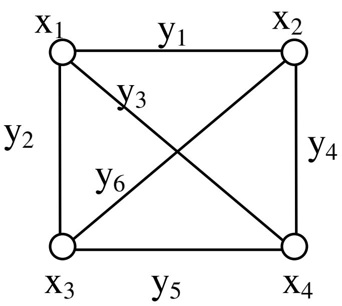


Figure 3.78

l(G) is a complete graph. The line graph of G is complete.

l(l(G)) =

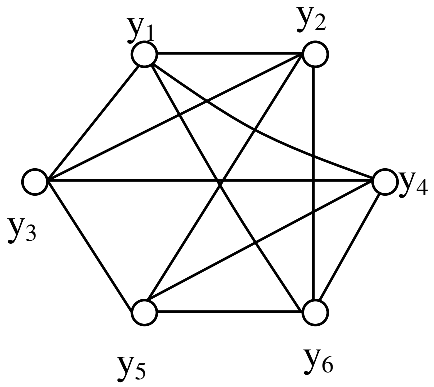


Figure 3.79

l(l(G)) is not a complete graph and has six vertices. Interested reader can find l(l(l(G)) and so on!

Now in view of all these results we putforth the fundamental theorem about the unconditional line graph of a star graph G.

Theorem 3.4. *Let G be a star graph with n edges. The line graph l(G) of G (the star graph) is a complete graph with n vertices.*

Proof is direct and hence left as an exercise to the reader.

We now study by examples.

Further we wish to record we will not be using the term unconditional line graph but only line graph l(G) as by the very notation we use only (l(G)) small l of G.

Example 3.12. Let G be a uniform star multigraph given by the following figure

G_1 =

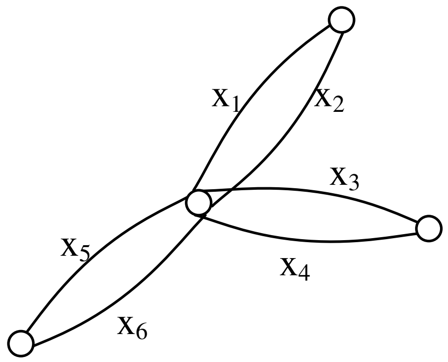


Figure 3.80

The line multigraph of G_1, $l(G_1)$ is as follows.

$l(G_1)$ =

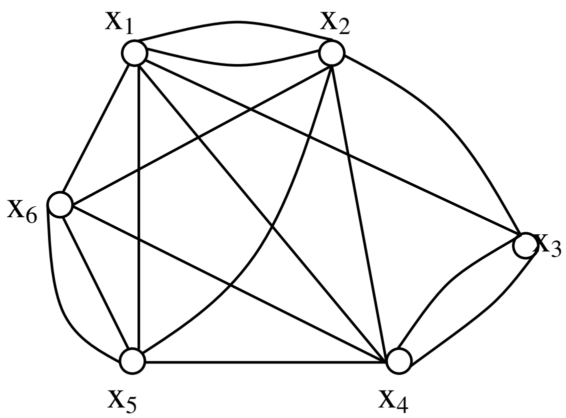


Figure 3.81

We see $l(G_1)$ is only a non uniform complete multigraph with six vertices.

Consider the following star multigraph G_2 given by the following figure.

G_2 =

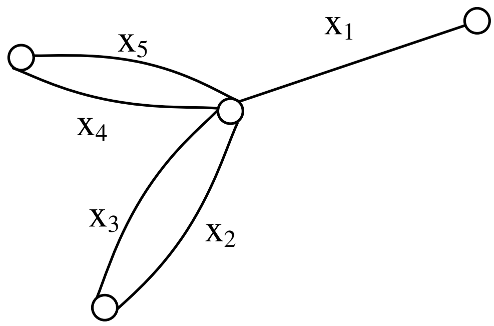


Figure 3.82

The line multigraph $l(G_2)$ is as follows.

$l(G_2)$ =

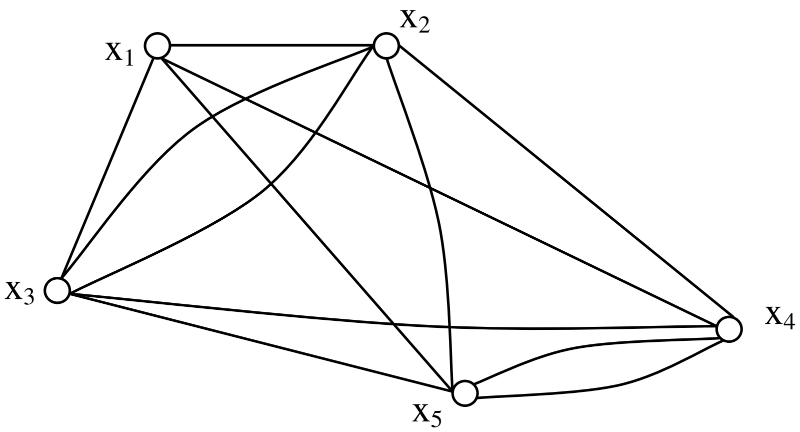


Figure 3.83

Again $l(G_2)$ is a non uniform complete multigraph with 5 edges. The number of vertices in $l(G_2)$ components to number edges in G_2.

Consider the star multigraph G_3 given by the following figure.

G_3 =

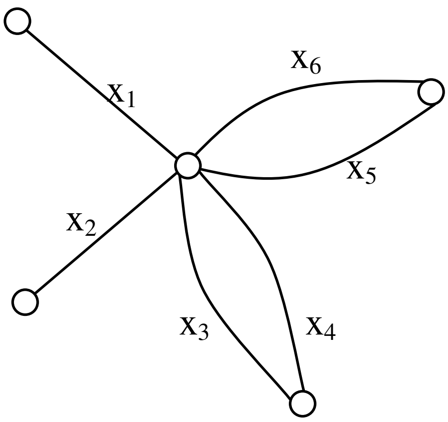


Figure 3.84

The multiline graph of $l(G_3)$is as follows;

l(G_3) =

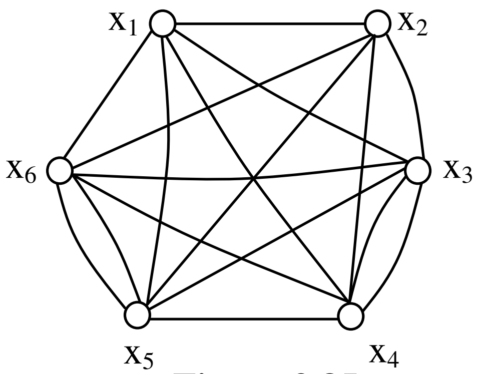


Figure 3.85

l(G_3) is a non uniform complete multigraph with 6 vertices and is different from l(G_1) which is also a nonuniform complete multigraph.

The multigraph G_1 and G_3 are also different.

Let G_4 be a star multigraph given by the following figure.

G_4 =

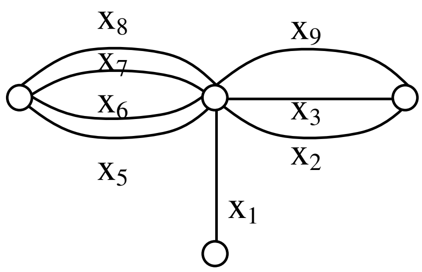


Figure 3.86

Let l(G_4) be the line multigraph of G_4 given by the following figure.

l(G_4) =

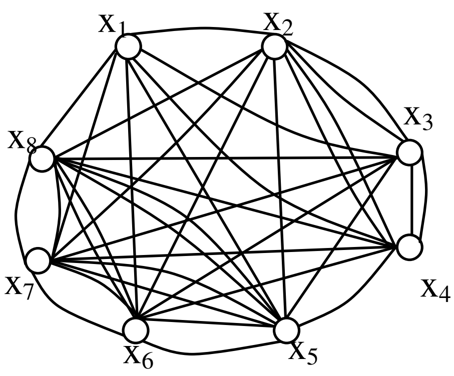


Figure 3.87

Clearly l(G_4) is a non uniform complete multigraph with 8 vertices which corresponds to 8 edges of G_4.

Let G_5 be a star multigraph given by the following figure.

G_5 =

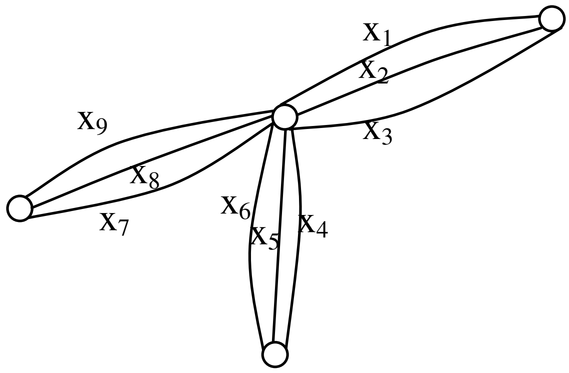


Figure 3.88

Let l(G_5) be the line multigraph of G_5 given by the following figure.

$l(G_5)$ =

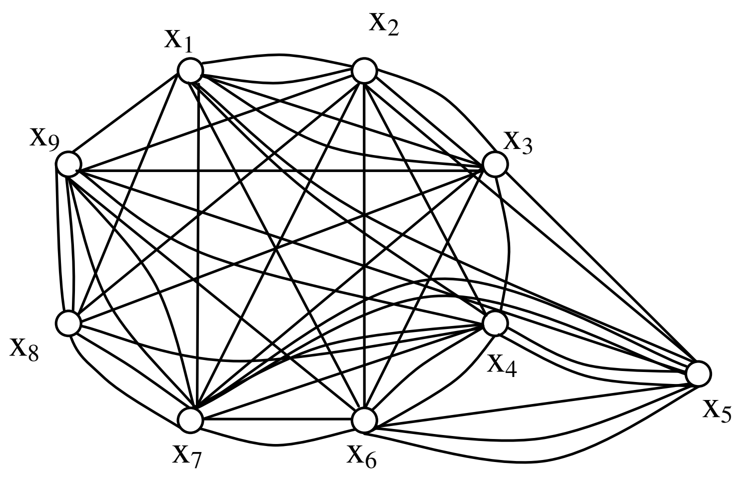


Figure 3.89

$l(G_5)$ is a non uniform complete multigraph with nine vertices which are the 9 edges of G_5.

In view of all these we have the following theorem.

Theorem 3.5. *Let G be a star multigraph with n edges. The line multigraph l(G) is a non uniform complete mutligraph of order n.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto describe the line multigraphs of a circle multigraphs by some examples.

Example 3.13. Let G be a circle multigraph given by the following figure.

G =

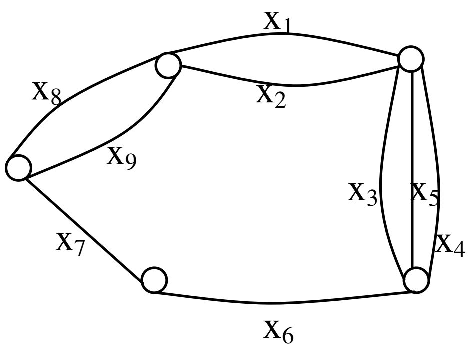


Figure 3.90

We see G has 9 edges and 5 vertices. We now find the unconditional multilinegraph l(G) of G. Clearly l(G) has nine vertices and is given below.

l(G) =

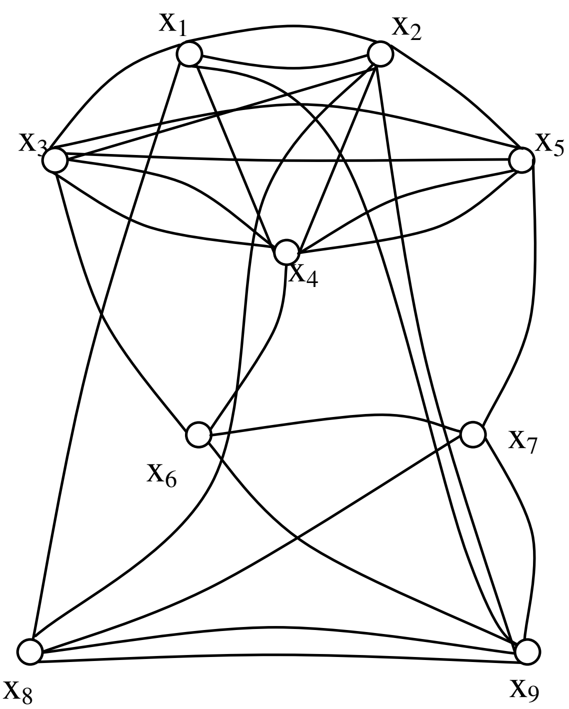


Figure 3.91

Clearly l(G) is not a complete or nonuniform complete multigraph.

Consider H the circle graph given by the following figure.

H =

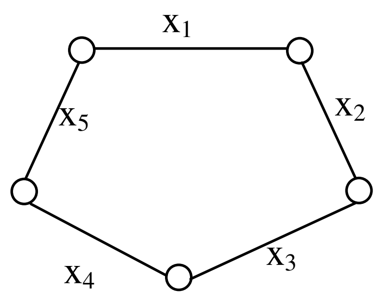


Figure 3.92

The line graph l(H) of H is as follows.

l(H) =

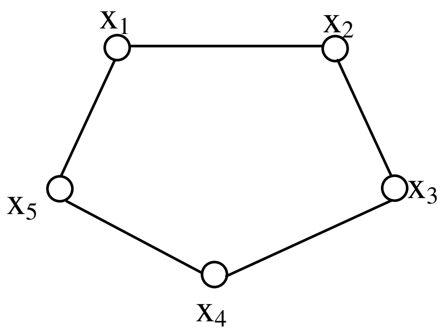


Figure 3.93

Clearly l(H) = H.

Consider the circle graph K given by the following figure.

K =

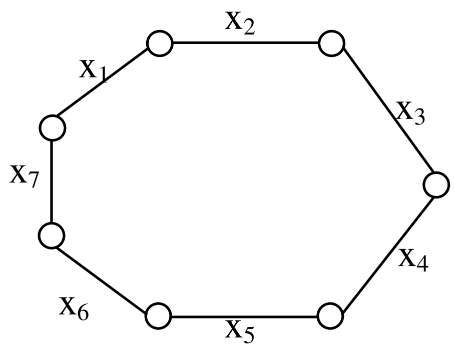


Figure 3.94

Let l(K) be the line graph of K given by the following figure.

l(K) has seven vertices and K has 7 vertices and edges.

l(K) =

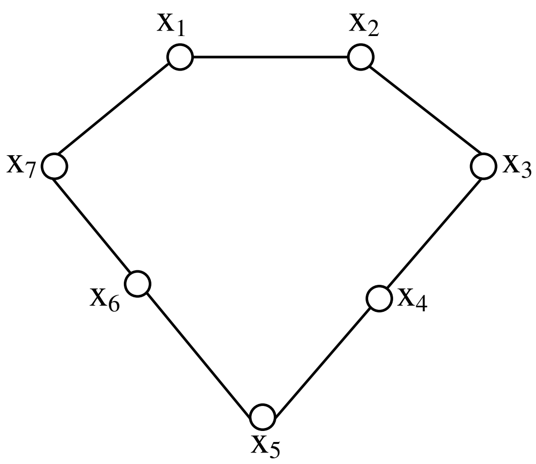


Figure 3.95

We see l(K) = K so we have the following theorem.

Theorem 3.6. *Let K be a circle graph. The unconditional line graph l(K) of K is such that l(K) = K.*

Proof is direct and hence left as an exercise to the reader.

Now we see the result in case of multicircle graph.

Example 3.14. Let M be a circle multigraph given by the following figure.

M =

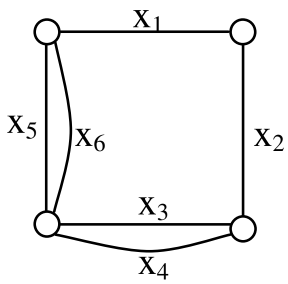


Figure 3.96

Now let l(M) be the line multigraph of M given by the following figure.

l(M) =

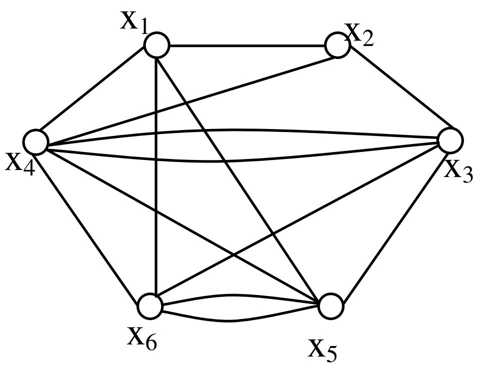


Figure 3.97

Clearly l(M) is not a circle multigraph. It has 6 vertices and l(M) is not a uniform or nonuniform complete multigraph. The number of vertices of the line multigraph l(M) is equal to the number of edges the multigraph M has.

In view of this we have the following theorem.

Theorem 3.7. *Let M be a circle multigraph with m vertices (m > 3) and n edges. The line multigraph l(M) has n vertices* *and is not a circle multigraph or a nonuniform complete multigraph.*

The proof is left as an exercise to the reader.

Consider the following circle multigraphs with three vertices.

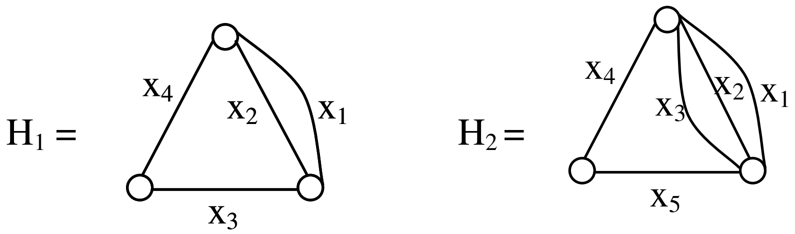


Figure 3.98

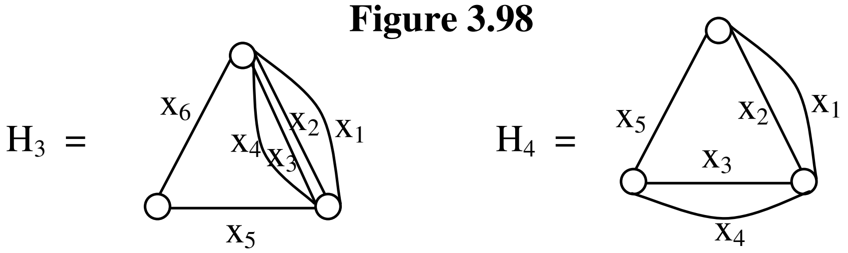


Figure 3.99

Now we find the line multigraphs of H_i; i = 1, 2, 3, 4.

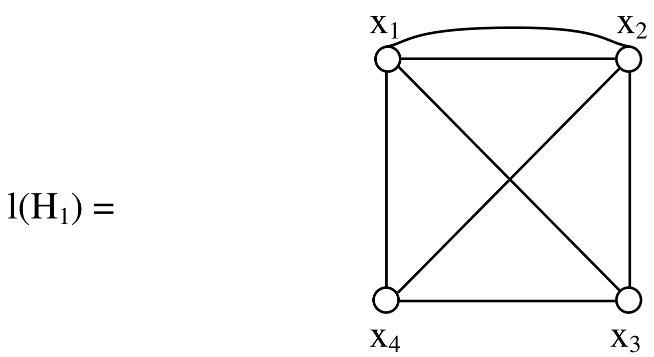


Figure 3.100

$l(H_1)$ is a nonuniform complete multigraph with four vertices.

$l(H_2)$ =

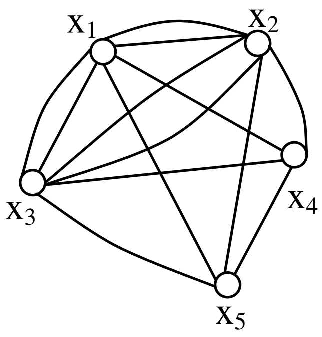


Figure 3.101

Clearly $l(H_2)$ is also a nonuniform complete multigraph with 5 vertices.

Now the following figure gives $l(H_3)$, the multiline graph of H_3.

$l(H_3)$ =

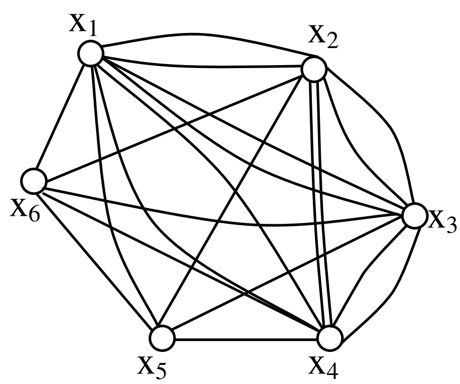


Figure 3.102

Clearly $l(H_3)$ is a nonuniform complete multigraph.

Consider $l(H_4)$ the multiline graph of H_4 given by the following figure.

$l(H_4)$ =

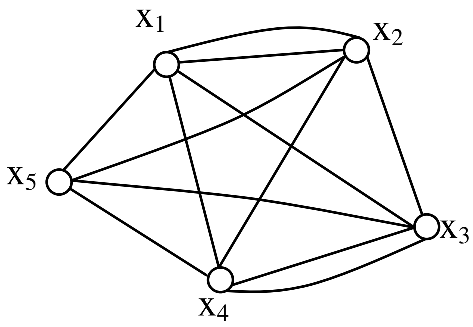


Figure 3.103

$l(H_4)$ is also a nonuniform complete multigraph.

Infact in view of all these we propose the following theorem.

Theorem 3.8. *Let G be a multitriad with n edges.*

The unconditional line multigraph of G, l(G) is a nonuniform complete multigraph with n vertices.

Proof. Given G is a multitriad with n edges. Let $x_1, x_2, \ldots, x_n$ be the vertices of l(G). The line multigraph of G will be such that any two vertices x_i, x_j $i \neq j$ are always adjacent hence the claim.

Next we proceed onto study the line graph of trees by examples.

Example 3.15. Let G be the binary tree given by the following figure.

G =

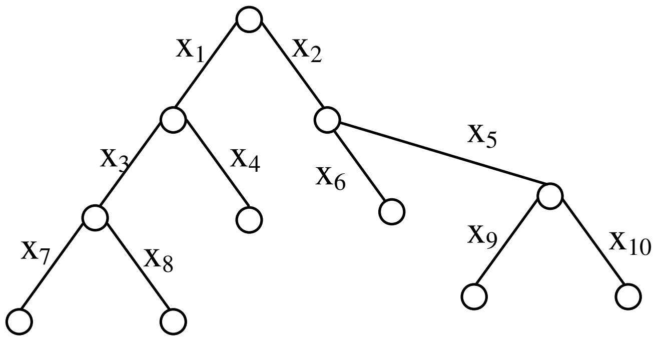


Figure 3.104

Now we find l(G); l(G) has n vertices, l(G) is given by the following figure.

l(G) =

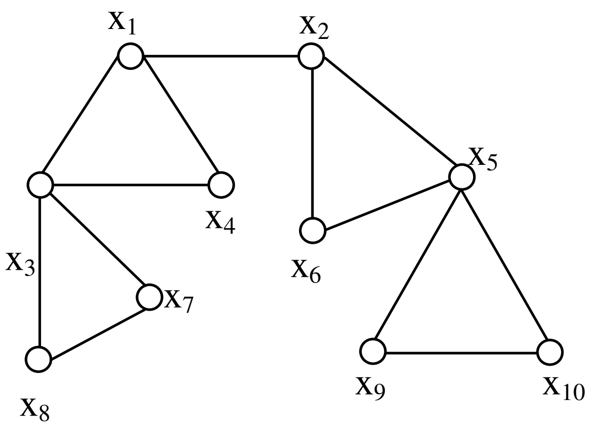


Figure 3.105

We see the line graph of a binary tree is only a graph which is not a tree has four triads none of them are adjacent triads but a connected one.

Consider the tree H given by the following figure.

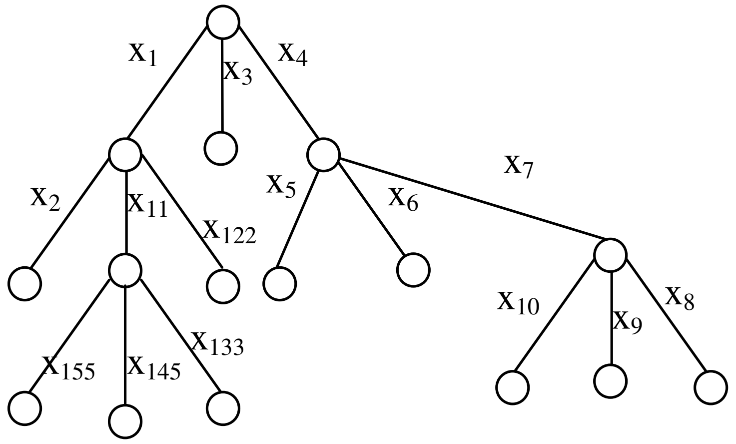
H = 

Figure 3.106

The line graph of H given by l(H) has 15 vertices we now give the graph of l(H) in the following.'

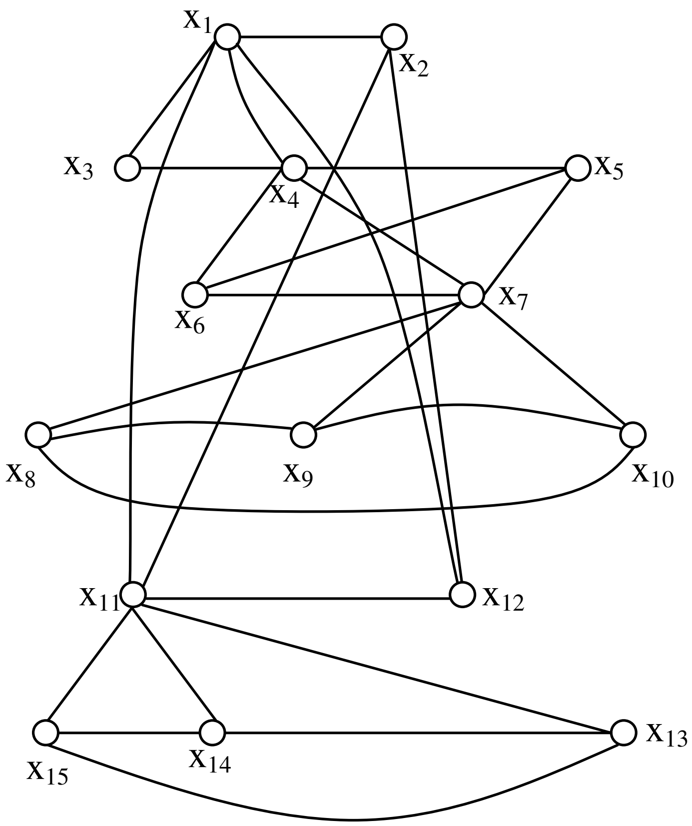
l(H) = 

Figure 3.107

For the 3-ary tree also we get triads but the triads are adjacent triads; there is no triad which is not adjacent with atleast one of them.

Next we study by an example the line graph of the tree K given by the following figure.

K =

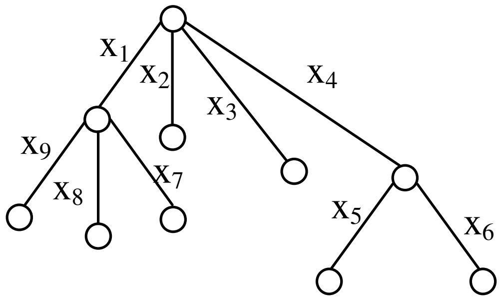


Figure 3.108

We now find l(K) the line graph of K.

l(K) =

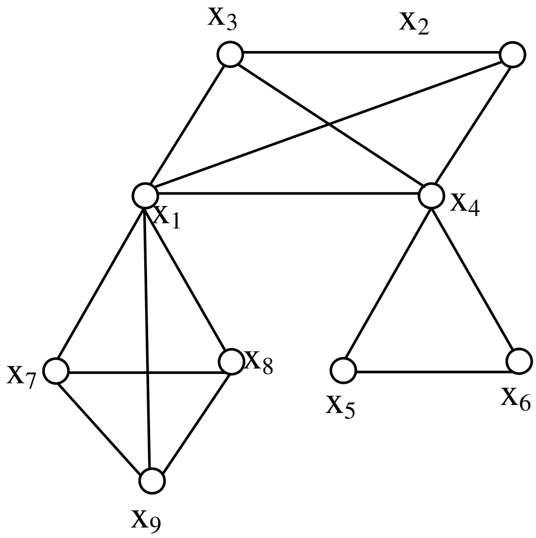


Figure 3.109

We see this tree is a mixed one and the root node has four edges incident to it the line graph of that component yields a complete graph of order four.

Thus if n edges are incident to any node then certainly that will be yield a complete graph of order four.

Thus we can say the line graph a n-ary tree will have a complete graph of order n.

A tree M of the form given by the following figure.

M =

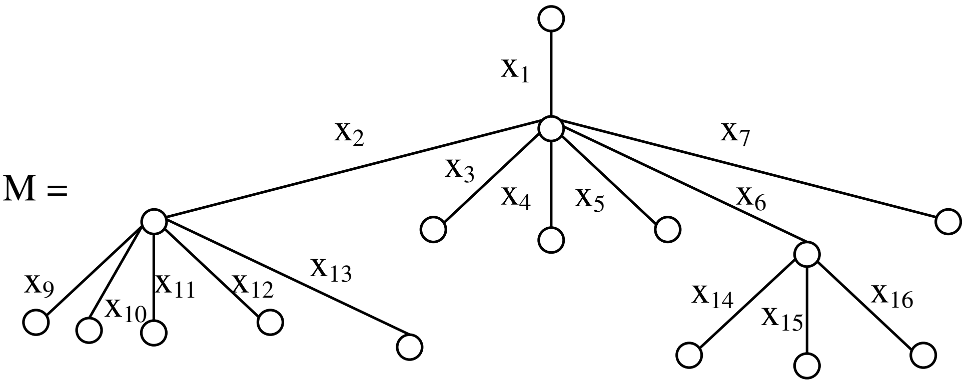


Figure 3.110

We can say l(M) the line graph of M will have the following complete graphs with vertex set $\{x_6, x_{14}, x_{15}, x_{16}\}$, $\{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$, $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ as complete graphs of order four, six and seven respectively.

The order seven complete graph and order four complete graph have a common vertex x_6. The figure of l(M) is as follows.

l(M) =

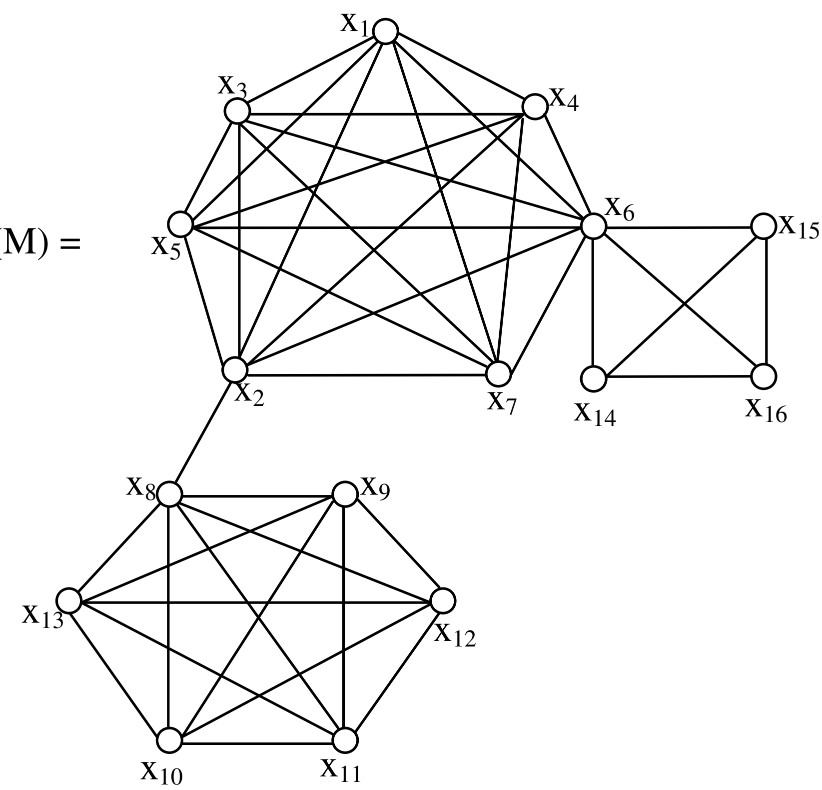


Figure 3.111

Thus in case of simple graphs which are trees. The line graphs has only complete subgraph components.

This property can have some nice applications and interested researchers are left with this task of finding them.

Now we analyse this in case of multitrees or multigraphs which are trees.

We first illustrate this situation by some examples.

Example 3.16. Let T be a multitree (multigraph which is a tree) given by the following figure

T =

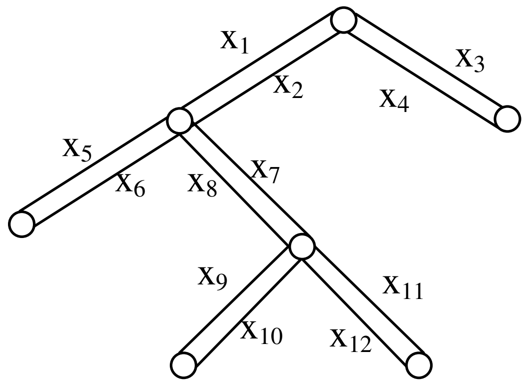
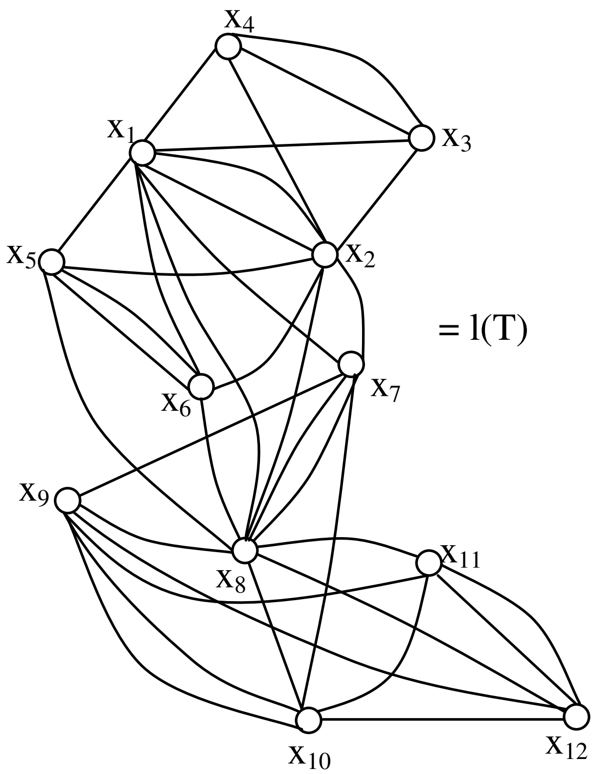


Figure 3.112

Now l(T) be the line graph of T given by the following figure.



= l(T)

Figure 3.113

We see the line multigraph l(T) of T is not a multitree it consists of nonuniform complete multigraphs of order four and three.

Let S be the multitree given by the following figure.

S =

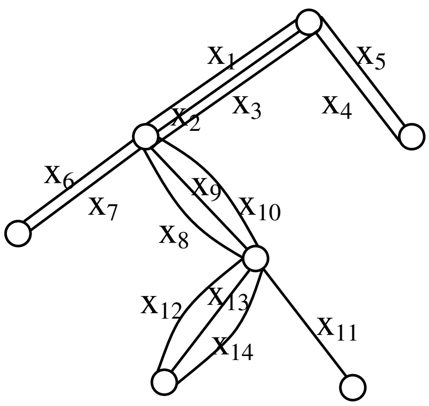


Figure 3.114

Let l(S) be the line graph of S associated with S which is as follows.

l(S) =

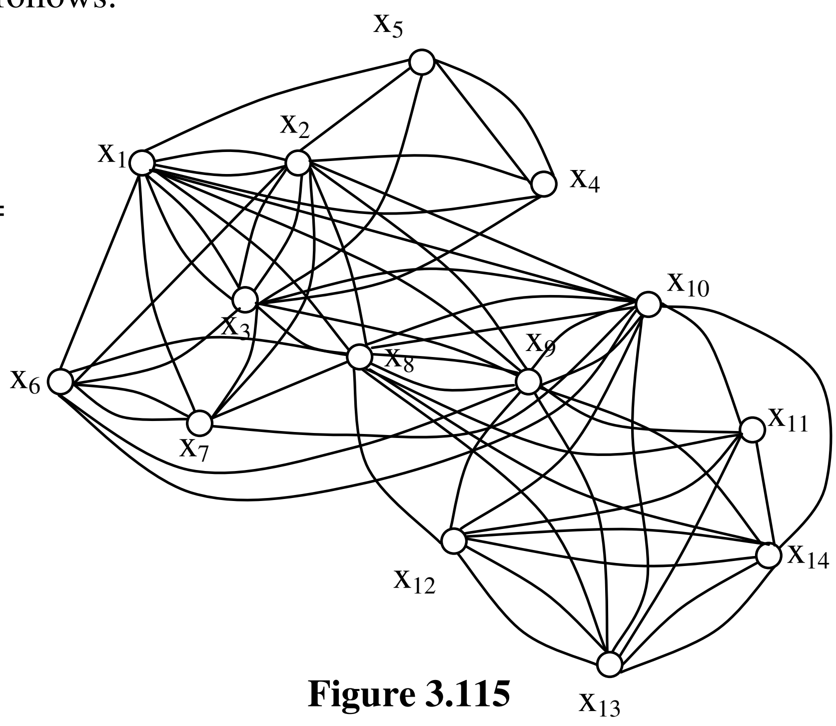


Figure 3.115

We see this line multigraph l(S) has several non uniform complete multisubgraphs.

Consider the multigraph P given by the following figure.

P =

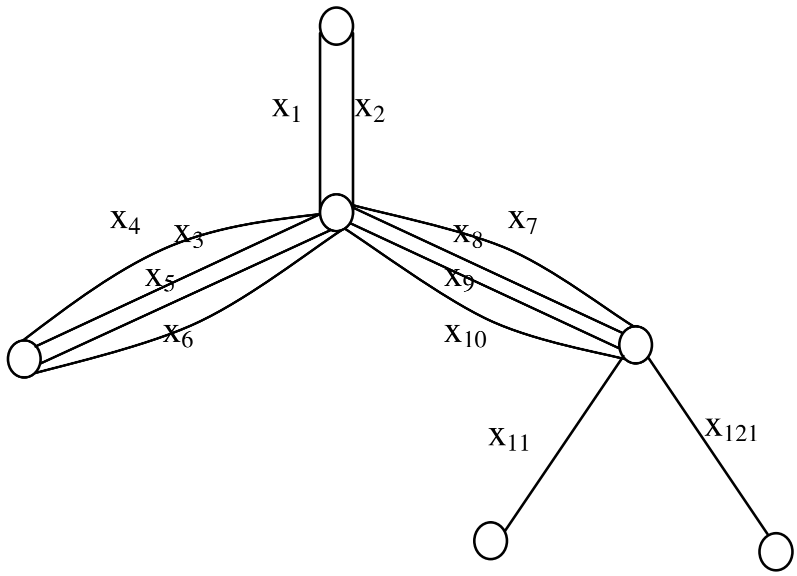


Figure 3.116

Clearly P is a multitree with 12 edges.

So the line multigraph l(P) of P will have 12 vertices given by the following figure.

l(P) =

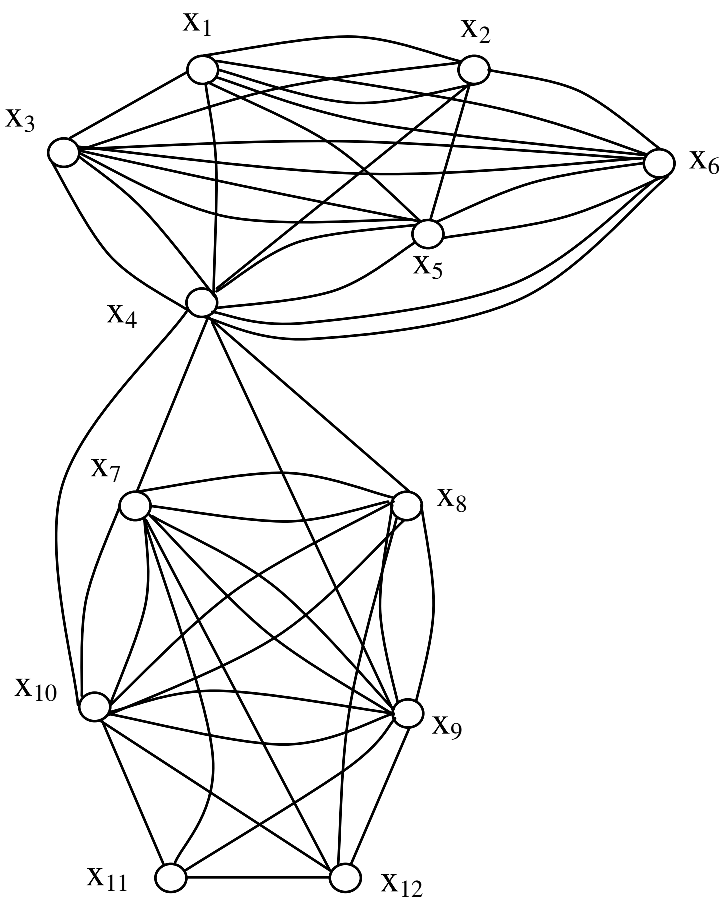


Figure 3.117

We see l(P) be the line graph of P. l(P) has several complete nonuniform multisubgraphs.

It is left as an exercise for the reader to find whether l(P) can have any multitrees.

Let M be a multitrere given by the follow figure.

M =

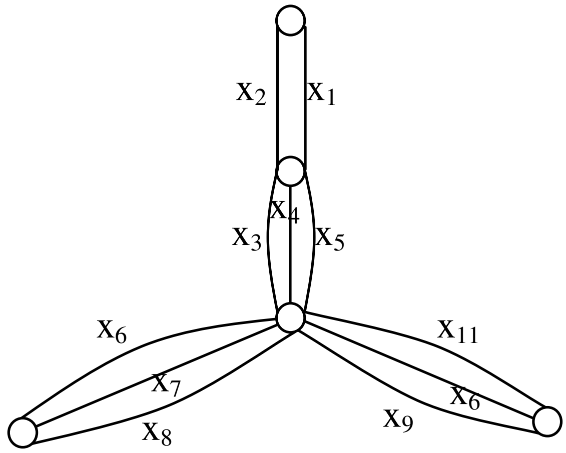


Figure 3.118

Let l(M) be the multiline graph of the multitree M. The figure of l(M) is as follows.

l(M) =

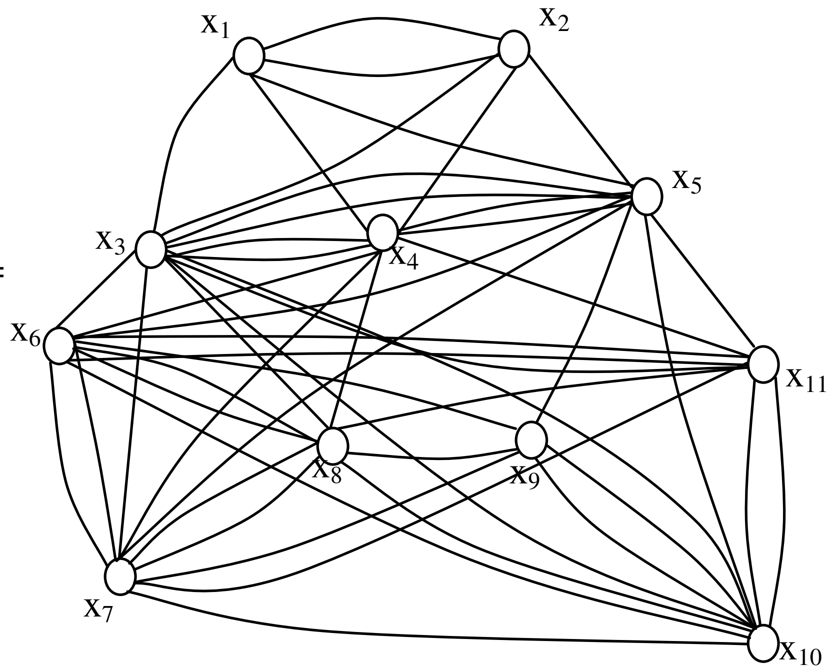


Figure 3.119

There is a nonuniform complete multigraph of order 9 as 9 of the edges are incident at a node. Also there is a non uniform complete multigraph with 5 vertices.

There are also triads which are non uniform as well as line subgraphs like

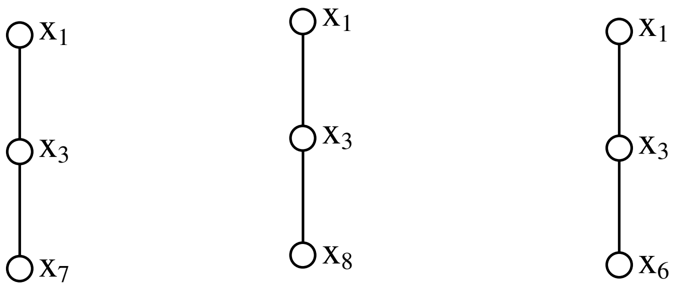


Figure 3.120

and so on. There is no line subgraph with more than three vertices.

Thus we see in case of multitrees also the line graph of them are such that they contain non uniform complete multisubgraphs. The highest order of these uniform complete multisubgraph being the highest number of edges that is incident to a vertex.

If one of the vertex has m-edges incident to it and that happens to be the maximum, then the line multigraph will have a nonuniform complete multisubgraph of order m.

Next we find line multigraphs of some multigraphs in the following examples.

Example 3.17. Let G be a multigraph given by the following figure.

G =

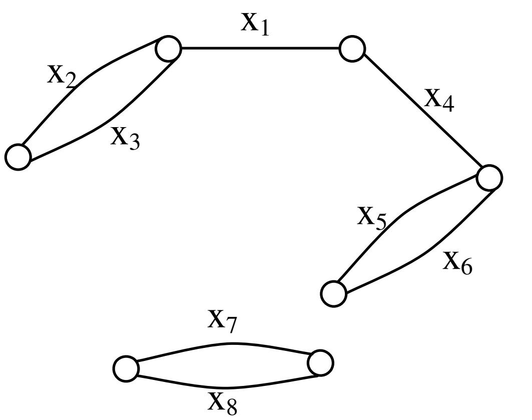


Figure 3.121

Now we see the line multigraph l(G) has 8 vertices given by the following figure.

l(G) =

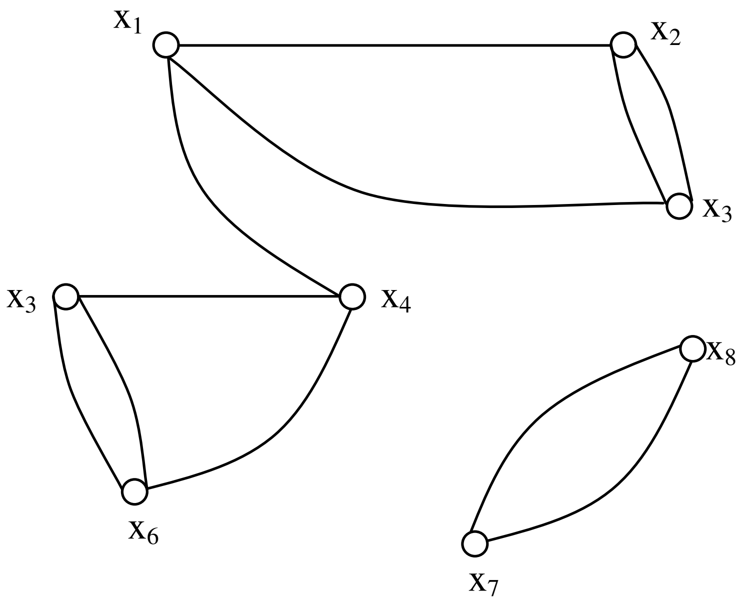


Figure 3.122

Clearly l(G) the multiline graph is also disconnected we see G is also a disconnected multigraph so is l(G).

In view of this we can say general the multiline graph or line multigraph l(H) of any disconnected multigraph H is also disconnected.

Having discussed about line multigraphs of a multigraph G we now proceed onto discuss few line graphs of a bipartite graphs and tripartite graphs by examples.

Example 3.18. Let G be a bipartite graph given by the following figure.

G =

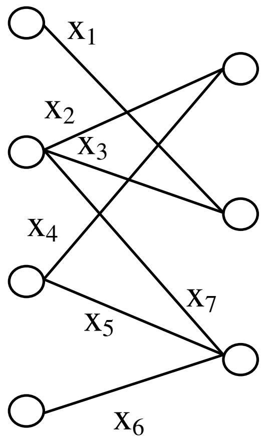


Figure 3.123

The line graph l(G) of G is as follows.

l(G) =

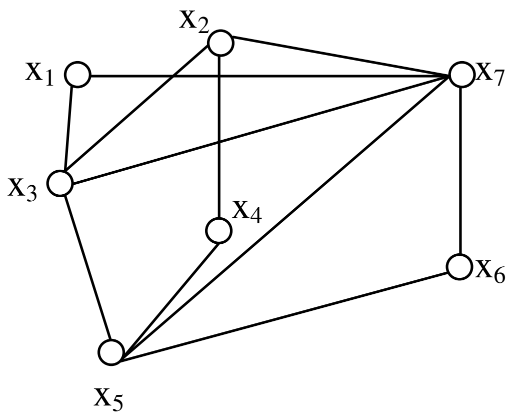


Figure 3.124

Clearly l(5) is not a bipartite graph infact.

This sort of conversion may be helpful in the study of mathematical models were a bipartite graph is converted into a non partite graph by the line graph. The FRM model can be converted into FCM model. this study is innovative and interesting.

Let G be a bipartite graph given by the following figure.

G =

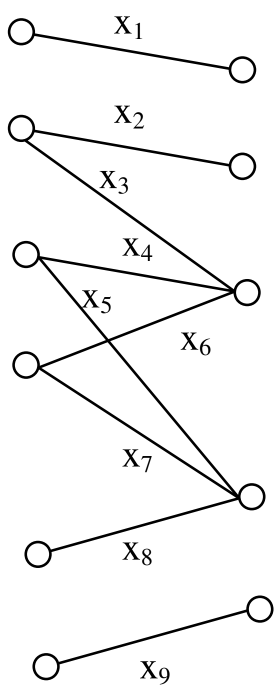


Figure 3.125

Let l(G) be the line graph of G given by the following figure.

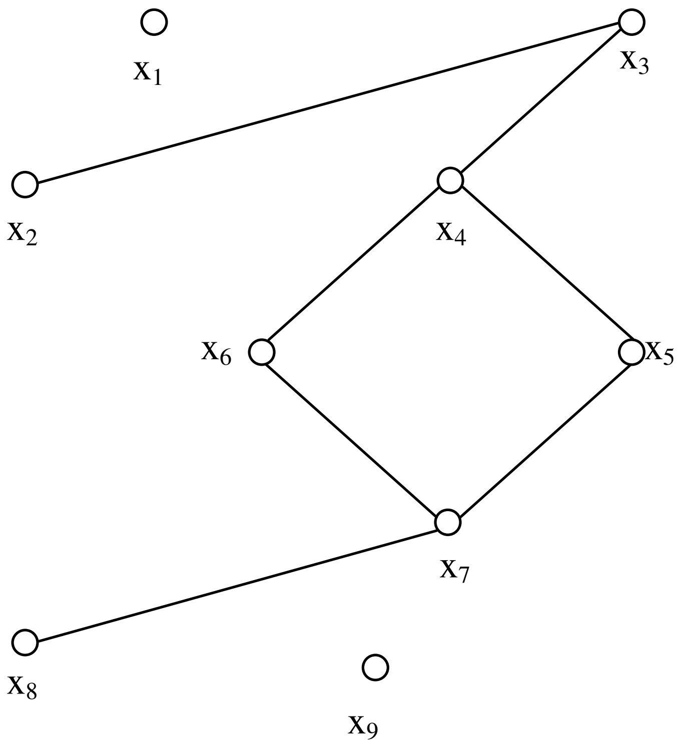


Figure 3.126

We see the bipartite graph is disconnected hence the line graph of it is also disconnected.

Now we give examples of multibigraphs and their line graphs in the following.

Example 3.19. Let G be a multibigraph given by the following figure.

G =

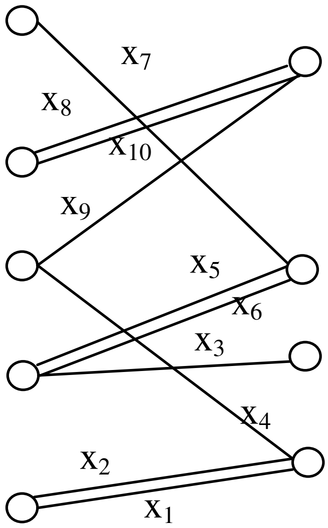


Figure 3.127

The multiline bigraph l(G) of G is as follows.

l(G) =

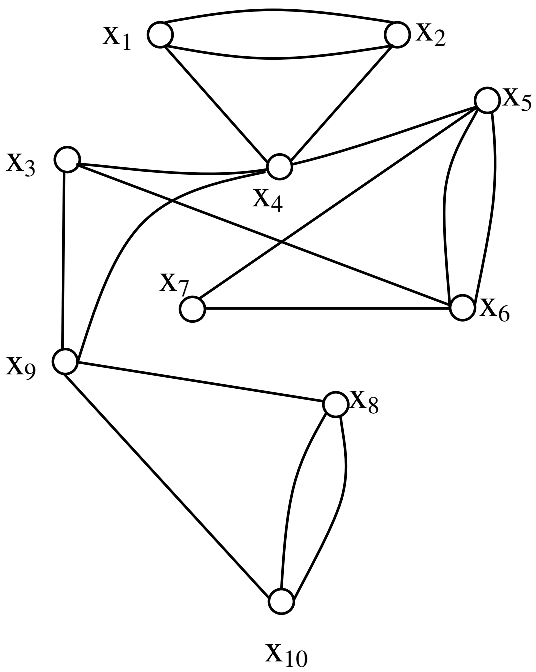


Figure 3.128

l(G) the multiline graph of the bipartite multigraph has some triads.

Next we proceed onto find the line graph l(G) of a line graph G for a few line graphs G.

Example 3.20. Let G be the line graph given by the following figure.

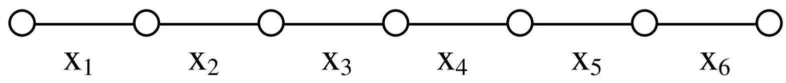
G =


Figure 3.129

Clearly l(G) will have six vertices. The graph of l(G) is as follows.

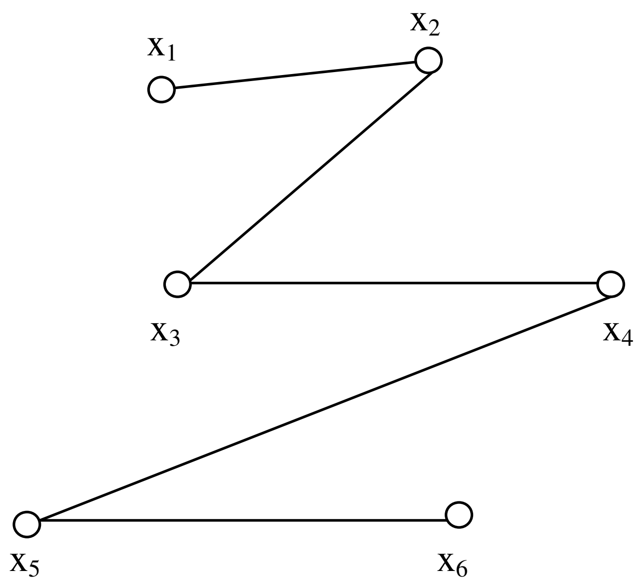
l(G) =


Figure 3.130

We see l(G) is also a line graph with six vertices.

Thus if G is also line graph so is l(G) but the number of vertices will be less by one.

In view of this we have the following theorem.

Theorem 3.9. *Let G be a line graph with n vertices. The line graph l(G) of G is again a line graph with n – 1 vertices.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto study the line multigraphs l(G) of a line multigraph G by examples.

Example 3.21. Let G be a line multigraph given by the following figure.

G =

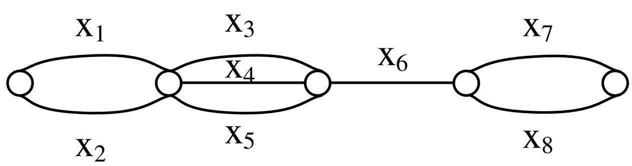


Figure 3.131

G is a line multigraph with 5 vertices.

l(G) the line graph of G has 8 vertices given by the following figure.

l(G) =

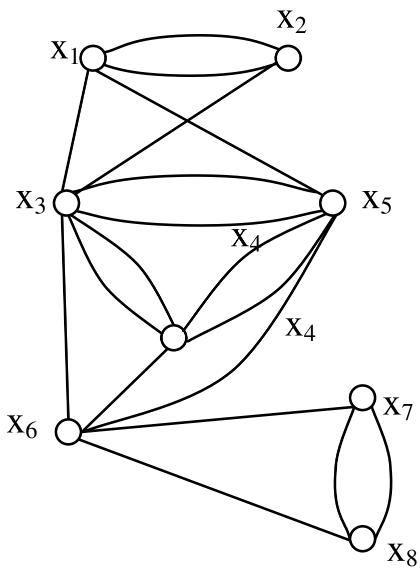


Figure 3.132

Clearly the unconditional line multigraph l(G) of a multigraph G is not a line multigraph in the usual sense.

Example 3.22. Let G be a multiline graph given by the following figure.

G =

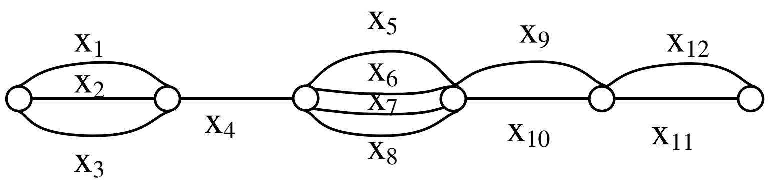


Figure 3.133

G is a line multigraph with 12 vertices, the unconditional line multigraph l(G) of G is given by the following figure which has 12 vertices where as G has only 6 vertices.

l(G) =

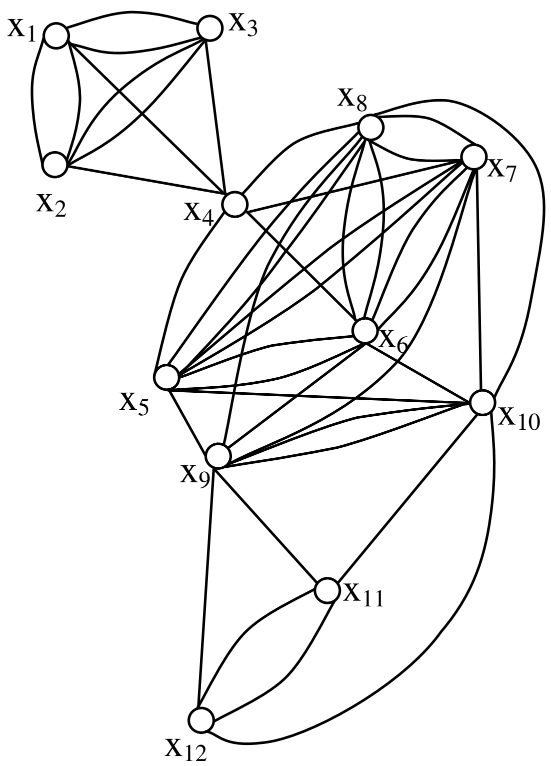


Figure 3.134

Clearly l(G) is not a line multigraph; infact it has multisubgraphs which are non uniform complete.

Example 3.23. Let G be a uniform line multigraph given by the following figure.

G =

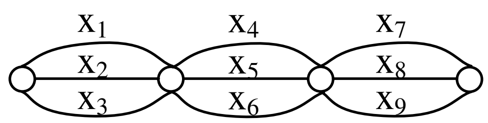


Figure 3.135

The line multigraph l(G) is as follows

l(G) =

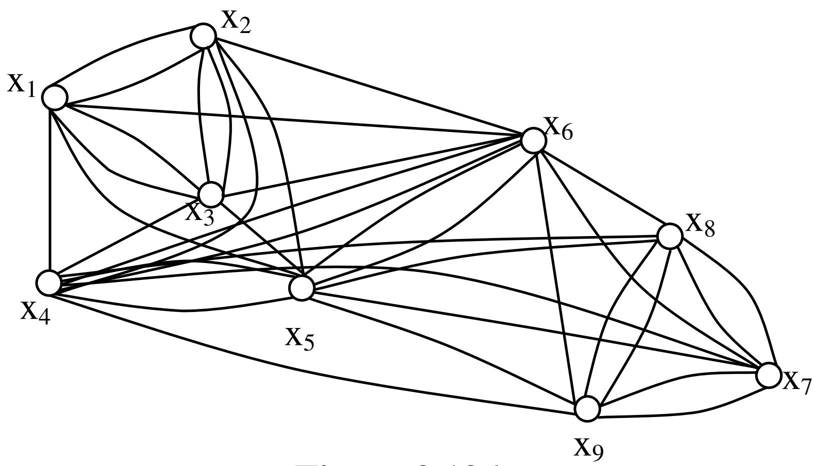


Figure 3.136

Clearly l(G) has non uniform complete multisubgraph of order 6 five and three.

In view of all these we have the following theorem.

Theorem 3.10. *Let G be a line multigraph with n vertices and m edges (m > n);*

i) *l(G) the multiline graph of G is not a multiline graph*

ii) *Number of vertices are always greater in case of l(G) than that of G*

iii) *If t edges passes through a vertex of G there exists a nonuniform complete multisubgraph of order t (t < m).*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto suggest a few problems.

Problems

1. Draw a directed multigraph G with six vertices and 15 edges.

2. How many such directed multigraph can be draw n under the conditions of problem 1?

3. Find all the multisubgraphs of the following multigraph G.

G =

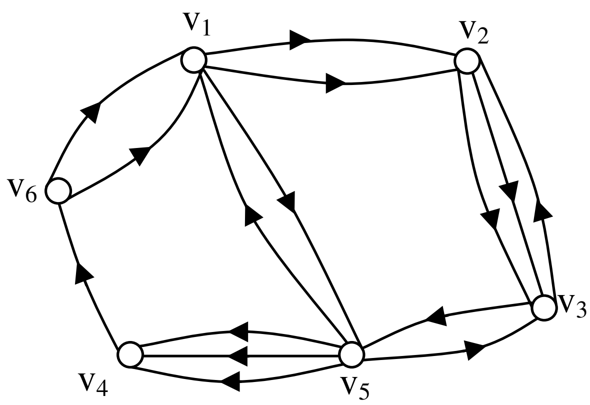


Figure 3.137

4. Obtain any special features associated with directed multigraphs.

5. What are the main features enjoyed by directed multigraphs in comparison with usual directed graphs?

6. Can multigraphs be applied in fuzzy congnitive maps models?

7. Give any other applications of directed multigraphs to practical problems.

8. Can directed multigraphs be better suited in case of transportation problems?

9. Given a directed multigraph G find its line directed multigraph where

G =

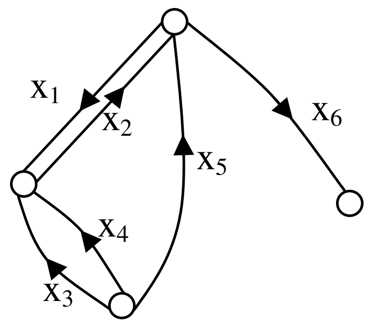


Figure 3.138

10. Can there be a multigraph tree whose line multigraph is a multitree? Justify your claim.

11. Find the line multliraph l(G) of the multitree G given by the following figure.

G =

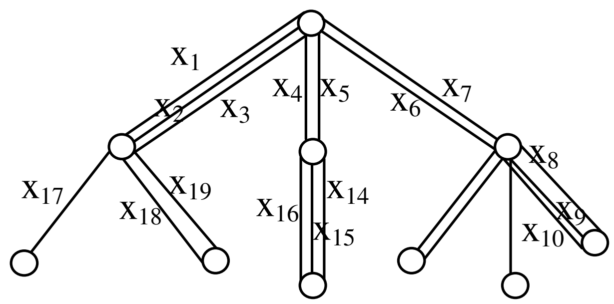


Figure 3.139

a) Find the largest nonuniform complete multisubgraph of l(G).

b) Find all nonuniform complete multisubgraphs of l(G).

c) Can l(G) contain a complete multisubgraph which is uniform?

d) Obtain any other special feature associated with l(G).

12. Given 5 vertices and the maximum number of edges being six.

i) Find how many multigraphs can be drawn with 5 vertices and maximum six edges.

ii) How many of these yield uniform complete multigraphs of order 5?

iii) How many of these yield nonuniform complete multigraph of order 5?

iv) How many multigraphs has multisubgraphs that have nonuniform complete multisubgraphs of order four?

v) How many of these multigraphs contain adjacent nonuniform multitriads?

vi) Can we have trees?

vii) Find all multigraphs which are circle multigraphs using 5 vertices and maximum 6 edges.

viii) Find all star multigraphs using the properties given.

12. Let G be a multigraph given by the following figure.

G =

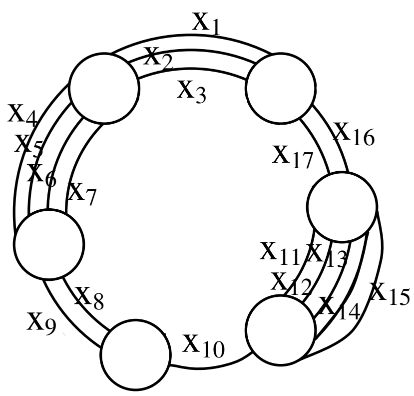


Figure 3.140

i) Find all multisubgraphs of G.

ii) Prove none of them is a circle multisubgraph of G.

iii) Find l(G) the line multigraph of G.

iv) Prove any other interesting property associated with l(G), the line multigraph of the circle multigraph G.

13. Let G be a nonuniform star multigraph given by the following figure.

G =

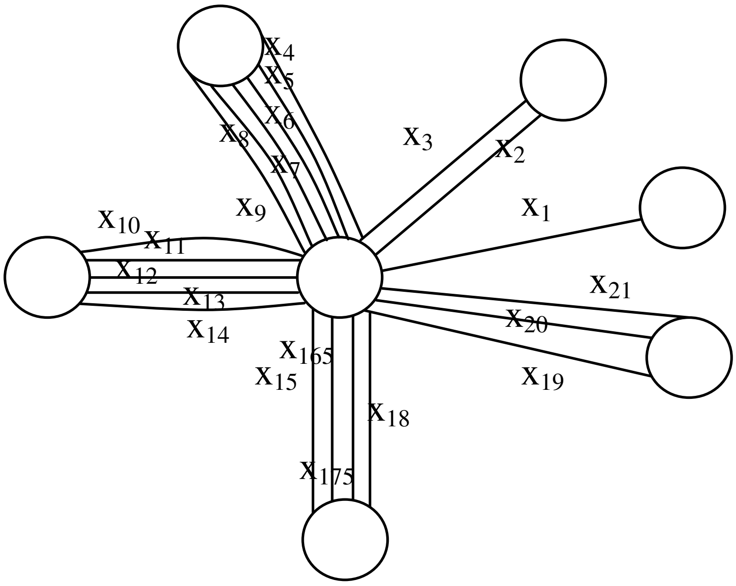


Figure 3.141

i) Find the line multigraph l(G) of G.

ii) Prove all multisubgraphs of G are either star multigraph or a empty multisubgraph.

iii) Is l(G) a nonuniform complete multigraph of order 21.

iv) Find any other special feature enjoyed by l(G).

14. Let G be a uniform star multigraph given by the following figure.

G =

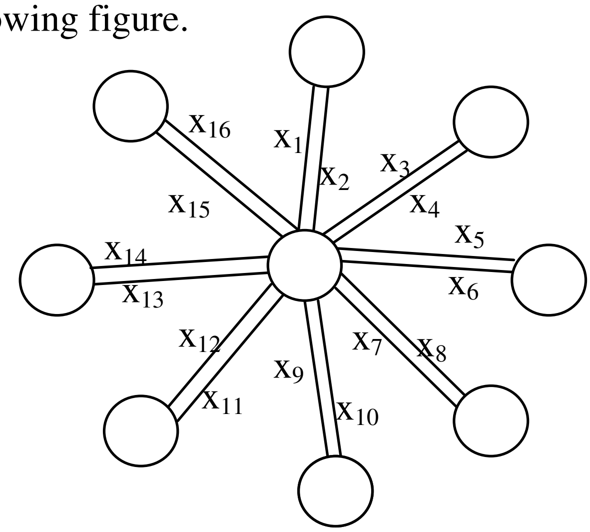


Figure 3.142

i) Is l(G) a uniform complete line multigraph?

ii) Show l(G) has multisubgraphs which are also complete multigraphs

iii) Can l(G) have multisubgraphs which are not complete multisubgraphs? Justify your claim.

15. Let G be a multigraph given by the following figure.

G =

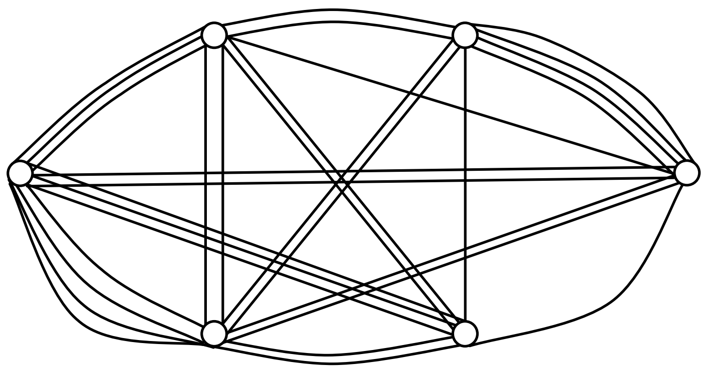


Figure 3.143

i) Find the number of edges of G and label the edges.

ii) What is the structure enjoyed by the line multigraph l(G)?

iii) Prove all multisubgraphs of G are nonuniform complete multigraphs.

iv) Obtain any other special feature enjoyed by G and l(G).

v) Can l(G) have multisubgraphs which are star multigraphs? Justify your claim.

16. Find the number of multigraphs with four vertices and a maximum of five edges.

17. Find the number of directed multigraphs with four vertices and a maximum of five edges.

18. Compare the multigraphs in problem (16) and (17) and prove the number of multigraphs got in problem (17) is a larger collection than the number of multigraphs got in problem (16).

19. Let G be a multigraph given by the following figure.

G =

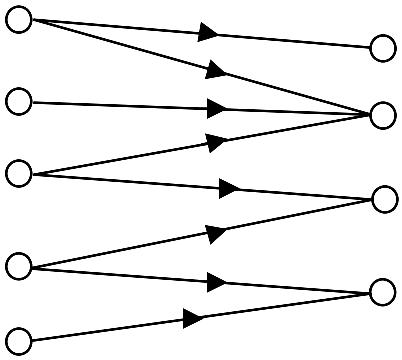


Figure 3.144

i) Find l(G) the line multigraph of G.

ii) Is l(G) a directed multigraph which is bipartite?

iii) Is l(G) a nonuniform complete multigraph?

iv) If in G the directions are ignored will l(G) be a nonuniform complete multigraph?

v) Obtain any other special feature enjoyed by G and l(G).

20. Let G be a graph given below.

G =

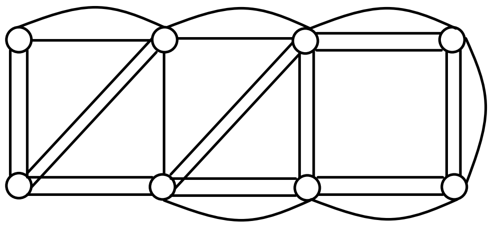


Figure 3.145

i) Find the line multigraph l(G) of G.

ii) Find all multisubgraph of G.

iii) How many vertices does l(G) have?

21. Can this concept of finding l(G) be used in stegenograph?

22. Given l(G); is it possible to find the related G?

23. Can atleast in case of l(G) comprising only of nonuniform complete multisubgraphs as its components represent only a multitree?

24. Characterize those multigraphs G for which

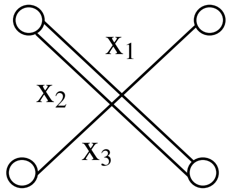
i) l(G) = G.

ii) l(l(G)) = G or l(G).

iii) l(l(l(G)) = l(l(G)).

25. Can we say the that there are multigraphs for which the number of vertices of l(G) is less than that of G?

26. Let

G = 

be a multigraph.

Figure 3.146

Is the claim in problem 25 true or false? Justify.

27. Let

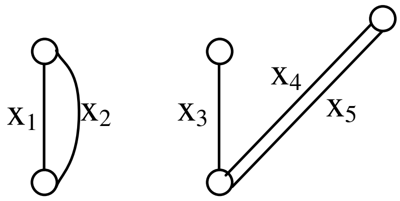
G = 

Figure 3.147

be a disconnected multigraph. Is the line multigraph l(G) of a G disconnected one.

28. Prove or disprove if G is a disconnected multigraph, then the line multigraph l(G) of G is disconnected.

29. Prove or disprove that if G is a line graph with n vertices then the line graph l(G) of G has only (n – 1) vertices.

30. Let G be a line multigraph given by the following figure.

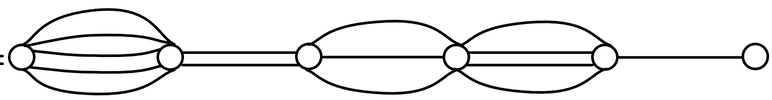
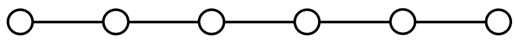
G = 

Figure 3.148

i) Find l(G) the unconditional line multigraph of G.

ii) What are the special features enjoyed by l(G)?

iii) Find l(H) of the line graph,

H = 

and compare it with l(G).

iv) Can this method of associating the multiline graph l(G) with a multiline graph G be used in hidden data storage system?

v) Can this method help in saving space?

31. Find some special applications of unconditional line multigraphs l(G) of a line multigraph G.

32. Let G be a multigraph given by the following figure.

G =

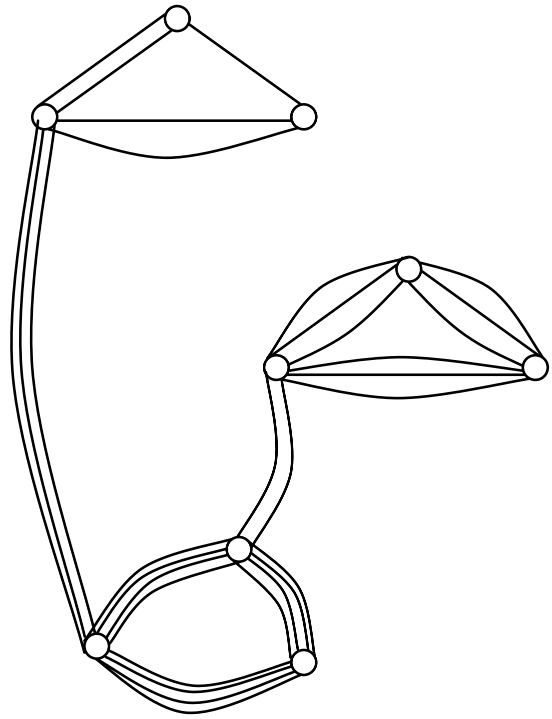


Figure 3.149

i) Find l(G), the unconditional line multigraph of G.

ii) Does l(G) contain a nonuniform complete multisubgraph of of order 11?

iii) How many nonuniform complete multisubgraphs are there in l(G)?

iv) Find the length of the multiline subgraph of l(G).

v) Can l(G) have multistar subgraphs?

vi) Can l(G) have circle multisubgraphs?

vii) Enumerate any other special feature enjoyed by the unconditional line multigraph l(G).

33. Let G be the mutligraph given by the following figure.

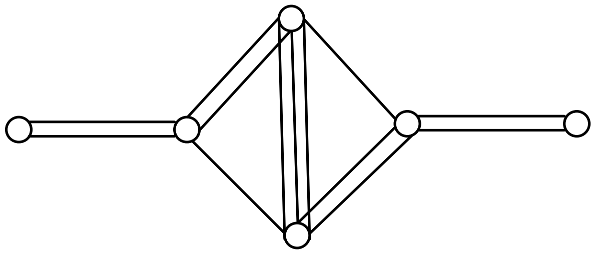
G = 

Figure 3.150

i) Find l(G), the line multigraph of G.

ii) Prove or disprove l(G) has a nonuniform complete multisubgraph of order 6.

iii) Clearly G has no multiclique of order greater than 3 but l(G) has a nonuniform multiclique of order 6 (prove or disprove).

iv) Can one say l(G) will result in more nonuniform cliques than in G?

v) Can we say l(G) is a richer structure than G in this case?

vi) Can we extend this result (v) for any G?

vii) Prove if (i) is to be true than the vertices in G must be lesser than the edges?

34. Let G be a multigraph given by the following figure.

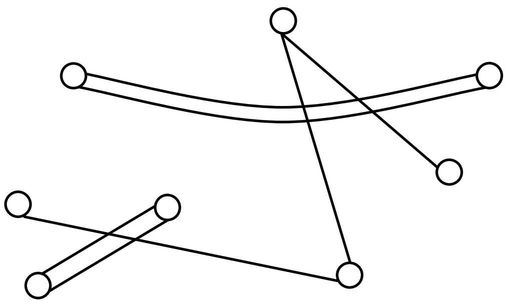
G = 

Figure 3.151

i) Find the line multigraph l(G) of G.

ii) Compare G with l(G).

iii) Can l(G) have nonuniform complete cliques?

iv) Is l(G) connected?

v) Does G and l(G) have multisubgraphs which are identical or isomorphic?

35. The following multigraph is the unconditional line graph of a graph G.

l(G) =

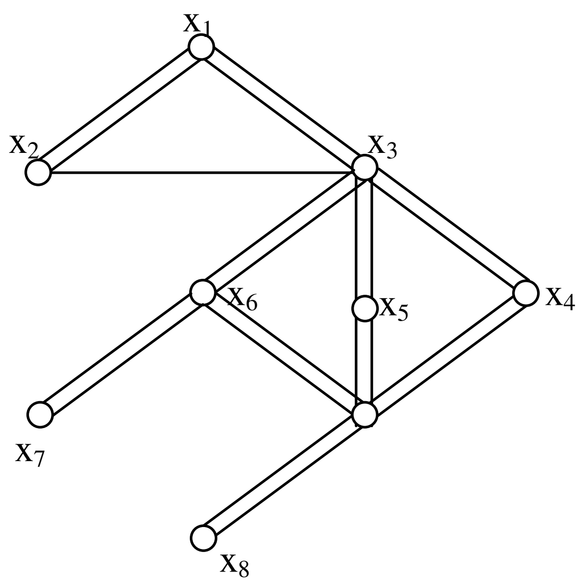


Figure 3.152

i) Does a G exist in a unique way?

ii) What will be the order of that G?

iii) Is it possible to find G given l(G)?

36. Given l(G) the unconditional multiline graph of a graph G in the following figure.

l(G) =

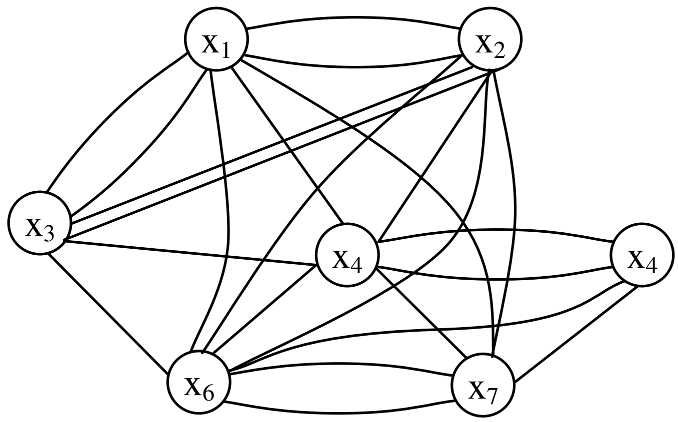


Figure 3.153

i) Find G for this l(G)

ii) Is G unique?

iii) Is it possible to find G given l(G)?

iv) Can one start with a multigraph with a node which has 7 edges incident to it? Justify your claim.

37. It is left as an open conjecture to find G given l(G) where G is a multigraph and l(G) the unconditional line multigraph of G.

38. For a given multigraph G find L(G) and l(G).

i) Compare l(G) with L(G).

ii) How best L(G) can be found in multigraph in general?

39. How many directed multigraphs with maximum 5 edges can be got using

    i) 3 vertices ?

    ii) 4 vertices ?

    iv) n-vertices ?

    Derive a formula for fixing the number of directed multigraph with a maximum number of edges t.

40. Show in case of multipartite (or n-partite) graphs G the line multigraphs are not multibipartite (or n-partite).

Chapter Four

MULTIGRAPHS AND THEIR SPECIAL FEATURES

In this chapter we mainly discuss about the concept of walk, length of walk, girth of a graph G, cycle, shortest cycle, longest cycle, no cycles, trail, geodesic and diameter in case of multigraphs.

Several concepts mentioned above cannot be got as a matter of routine, they need modifications and tailoring properly. Multigraphs enjoy several special features which are distinctly different from the usual graphs.

The situation, the changes one has to observe is given in the following example.

Example 4.1. Let G be a multigraph given by the following figure.

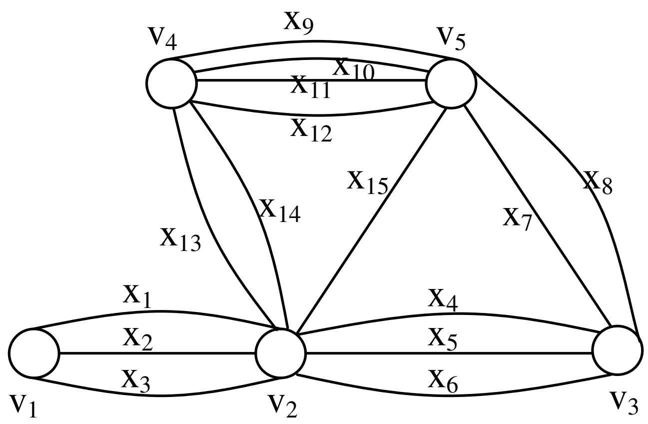
G = 

Figure 4.1

We see $v_1\ v_2\ v_5\ v_2\ v_3$ is a walk but we have 3 ways to reach from v_1 to v_2 one way to reach from v_2 to v_5, and v_5 to v_2 but there is again 3 ways to reach v_2 to v_3. So one has 9 ways in which the walk $v_1\ v_2\ v_5\ v_2\ v_3$ can be defined. Hence with multigraphs we also have to indicate the number of ways the walk $v_1\ v_2\ v_5\ v_2\ v_3$ is defined.

Now if we are interested to know the trail which is not a path we have $v_1\ v_2\ v_5\ v_4\ v_2\ v_3$.

The number of trails for $v_1\ v_2\ v_5\ v_4\ v_2\ v_3$ is as follows.

The number of ways one reaches from v_1 to v_2 is 3, the number of ways one reaches from v_2 to v_5 is 1, the number of ways one can go from v_5 to v_4 is 4, the number of ways to reach from v_4 to v_2 is 2 and finally the number of ways to reach from v_2 to v_3 is 3.

Thus the total number of trails one can have is

3. 1. 4. 2. 3 = 72.

Hence we see the concept of trail which is not a path is very large in case of the multigraph G.

Thus we in case of multigraphs only define multiwalk, multitrail, multipath, multicycle and so on.

Now we see v_1 v_2 v_5 v_4 is a path which is not a cycle.

The number of paths that exist which is not a cycle is as follows. The number of ways to reach from v_1 to v_2 is 3, the number of ways to reach from v_2 to v_5 is one and the number to ways to reach v_5 to v_4 is 4. Thus we have 3.1.4 = 12 ways.

Hence a multipath is the one which has multiple routes say 12 routes for the path v_1 v_2 v_5 v_4. Finally v_2 v_4 v_5 v_2 is a cycle. How many routes exist for this cycle?

To reach from v_2 to v_4 is two, from v_4 to v_5 is 4 and from v_5 to v_2 is one. So the number of cycles in this case is 2.4.1 = 8.

Thus we call this as a multicycle.

From the associated number we see all the four concepts multipath, multicycle, multiwalk and multitrail have different values as 12, 8, 9 and 72 respectively. The number associated with them are also distinct.

This sort of property can be applied to transportation problems, problems of networking of any form of multi networks. Multistructure happens to be one important criteria in the study of multichannel networks, communication multi networks, multiroute problems and so on.

Consider the following example of a multigraph given by the following figure.

Example 4.2. Let G be a edge labeled multigraph with vertex sets $v_1, v_2, \ldots, v_7$.

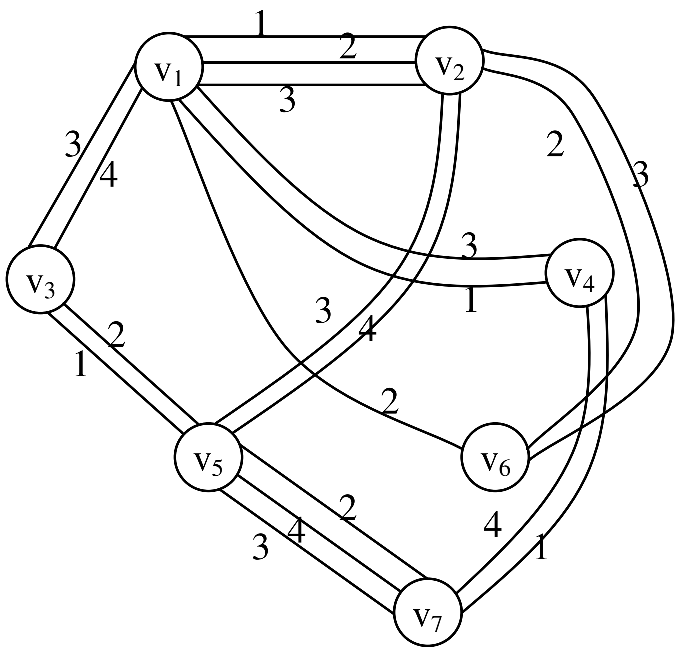


Figure 4.2

We see there is no path connecting v_1 and v_6 through edge 1. However there is a path from v_1 to v_6 by edge 2.

There is a path from vertex v_1 to v_7 only through edge 3. There is a path from vertex v_1 to v_7 through edge 1. There are two paths from v_2 to v_7 through edges 3 and 4. So we say there are two distinct paths from v_2 to v_7.

This sort of defining walk or path or trial where the same numbered edge must be taken will be defined as same edge multipath or multiwalk or multitrail or multicycle. However if one could reach from v_i to v_j, via any edge than we call it as mixed edge multipath or multitrail or multiwalk or multicycle.

First we illustrate this situation by an example and then proceed on to define these concepts abstractly.

Example 4.3. Let G be a multigraph given by the following figure with 8 vertices and a maximum of 6 edges and we find walks from v_1 to v_8.

G =

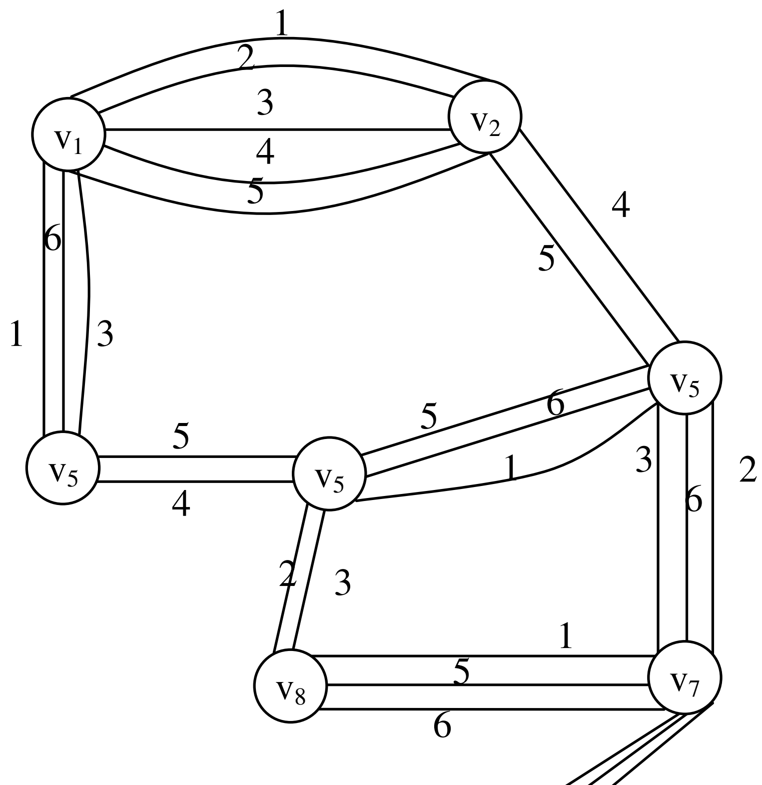


Figure 4.3

i) $v_1\ v_3\ v_4\ v_6\ v_7\ v_8$

ii) $v_1\ v_2\ v_5\ v_4\ v_5\ v_7\ v_8$

These two walks from v_1 to v_8 is mixed edge multipaths.

Now we study the possibility of finding a same edge multiwalk from v_2 to v_8. We see there does not exist a same edge multiwalk or path or trail from v_2 to v_8.

However, we can find mixed edge multipaths and mixed edge multiwalk and mixed edge multitrail.

It is an interesting problem to find the existence of same edge multiwalk or multipath or multitrail in case of connected multigraphs.

First we give an example of a multigraph in which it has always a same multiedge between any two vertices.

Example 4.4. Let G be a edge and vertex labeled multigraph given by the following figure.

G =

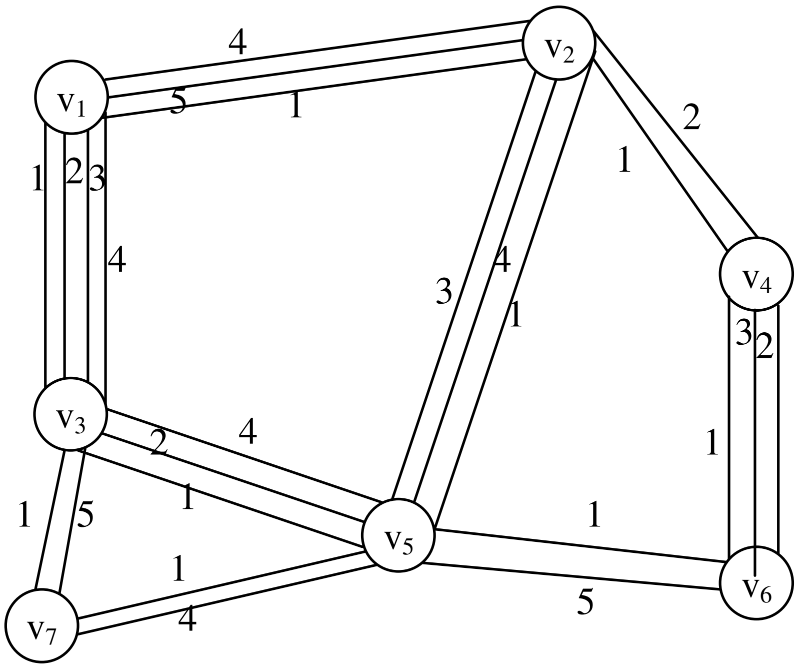


Figure 4.4

It is easily verified that there exist same multipath, same multiwalk, same multitrail from every pair of vertices in G via the edge 1.

We see there is a same multipath between vertices v_1, v_2, v_5, v_7 and v_3 (however no edge connecting v_7 to v_3 or v_3 to v_7). We have same multicycle using edges (1, 4),

v_1 v_2 v_5 v_3 v_1.

Thus we see not all multigraphs can have same multiwalk or multipath or multitrail, so finding the conditions for same multiedge paths to exist will be described.

Further we wish to keep on record that we can find same multipaths (walks or trials) for all multigraphs; we illustrate this by example.

Example 4.5. Let G be a directed edge and vertex labeled multigraph given by the following figure.

G =

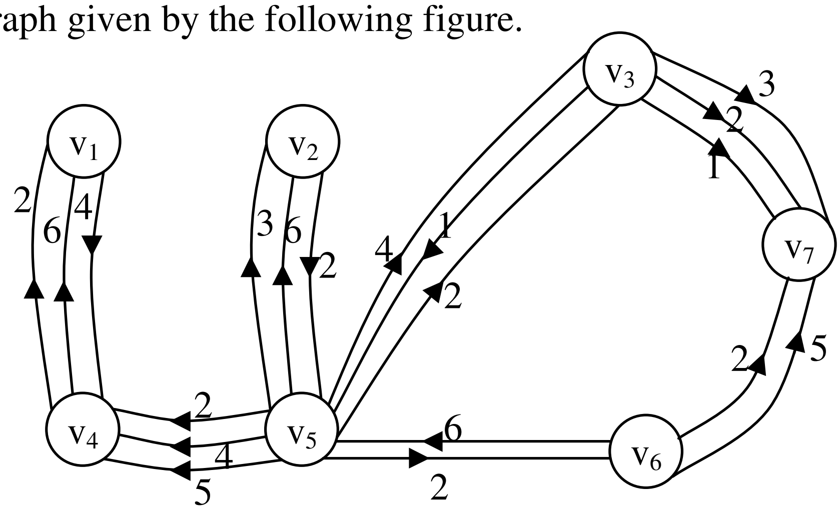


Figure 4.5

We see because the multigraph is directed we cannot find easily same multipaths. However, we may have mixed multiwalks.

Study in this direction is both innovative and interesting.

Example 4.6. Let G be a multigraph given by the following figure.

G =

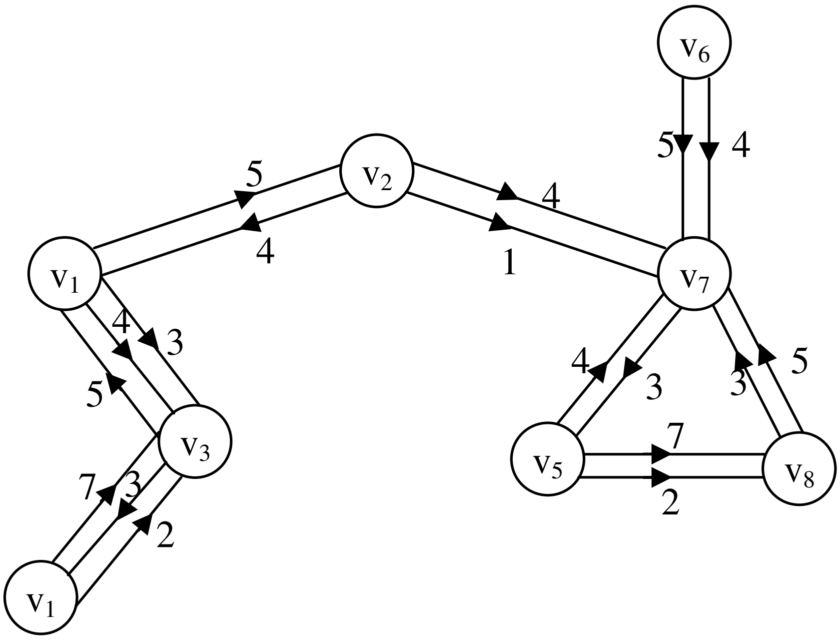


Figure 4.6

The reader is left with the task of finding multiwalks, multipaths and multitrail.

Can there be a multicycle with vertices v_4 and v_6? Justify your claim.

It is important to note in case of directed multigraphs it is not very easy to find either same multiwalk or mixed multiwalks. The situation is very different and only if direction happens to be the same then at least for some edges we can have multipaths or multiwalk or multitrails.

It is a difficult problem to get the multipath or multiwalk be it same or mixed in case of directed multigraphs.

Next we proceed onto describe the adjacency matrices of both multigraphs and directed multigraphs in the following by examples.

Example 4.7. Let G be a multigraph given by the following figure.

G =

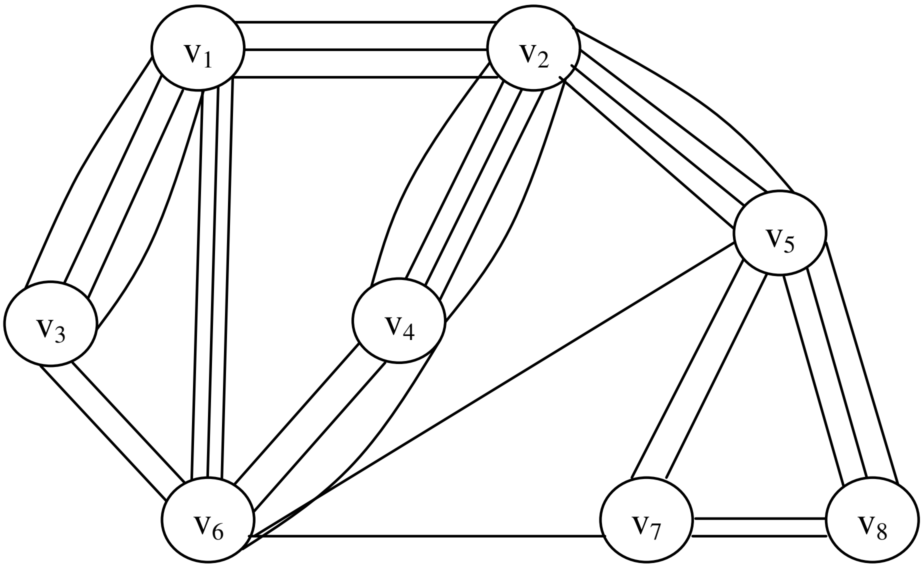


Figure 4.7

The adjacency matrix M associated with G is as follows.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \left[\begin{array}{cccccccc} 0 & 3 & 4 & 0 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 5 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 \end{array}\right] \end{array}.$$

Thus from the adjacency matrix M of the multigraph G we can easily find the maximum number of edges in the multigraph as well as the pair of vertices which has the maximum number of edges.

However there are no loops in our multigraph so we see all times the diagonal elements are zero.

We provide yet another example of the same.

Example 4.8. Let G be a multigraph given by the following figure. Clearly G is multitree which is vertex labeled.

G =

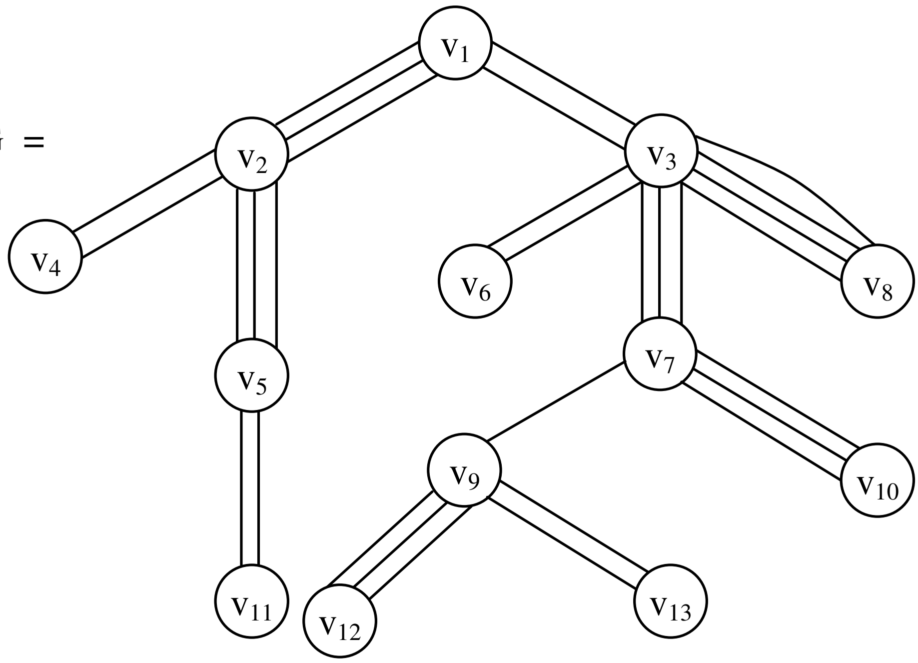


Figure 4.8

We find the adjacency matrix P of this multitree G in the following.

P =

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
v_1	0	3	2	0	0	0	0	0	0	0	0	0	0
v_2	3	0	0	2	3	0	0	0	0	0	0	0	0
v_3	2	0	0	0	0	2	3	4	0	0	0	0	0
v_4	0	2	0	0	0	0	0	0	0	0	0	0	0
v_5	0	3	0	0	0	0	0	0	0	0	2	0	0
v_6	0	0	2	0	0	0	0	0	0	0	0	0	0
v_7	0	0	3	0	0	0	0	0	1	3	0	0	0
v_8	0	0	4	0	0	0	0	0	0	0	0	0	0
v_9	0	0	0	0	0	0	1	0	0	0	0	3	2
v_{10}	0	0	0	0	0	0	3	0	0	0	0	0	0
v_{11}	0	0	0	0	2	0	0	0	0	0	0	0	0
v_{12}	0	0	0	0	0	0	0	0	3	0	0	0	0
v_{13}	0	0	0	0	0	0	0	0	2	0	0	0	0

In view of all these we have the following theorem.

Theorem 4.1. *Let G be a multigraph with n vertices. The adjacency matrix G of M is a n × n symmetric matrix with zeros in the diagonal and the highest entry in M gives the maximum number of edges connecting those pairs of vertices.*

Proof. Given G is a multigraph with n vertices. So we have the adjacency matrix M of G which is a n × n symmetric matrix with diagonal entries as zero so G is not a directed multigraph. If the vertices v_i, v_j is such that the edge v_iv_j is the highest value in the matrix M then as M is the symmetric matrix we have the maximum number of edges connecting v_i and v_j (and v_j and v_i). Hence the claim of the theorem.

Next we give by example of the adjacency matrix of the directed multigraph.

Example 4.9. Let G be a directed multigraph given by the following figure.

G =

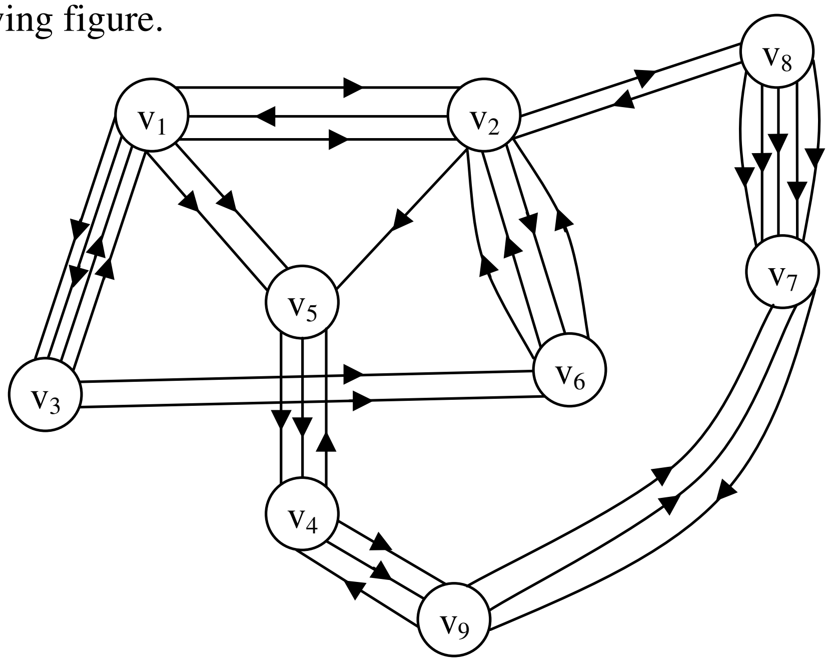


Figure 4.9

The adjacency matrix N of the directed multigraph G is as follows.

N =

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	0	2	2	0	2	0	0	0	0
v_2	1	0	0	0	1	1	0	1	0
v_3	2	0	0	0	0	2	0	0	0
v_4	0	0	0	0	1	0	0	0	2
v_5	0	0	0	2	0	0	0	0	0
v_6	0	3	0	0	0	0	0	0	0
v_7	0	0	0	0	0	0	0	0	1
v_8	0	1	0	0	0	0	5	0	0
v_9	0	0	0	1	0	0	2	0	0

.

We make the following observations.

Clearly N is not a symmetric matrix. It is a 9 × 9 square matrix.

The diagonal elements for this matrix N is also zero. But the sum of the values of $v_iv_j + v_jv_i$ in N will give the maximum number of edges. We see $v_8v_7 = 5$ but $v_7v_8 = 0$ so that maximum number of edges in this graph G is 5 for $v_8v_7 + v_7 v_8 = 5 + 0 = 5$.

The next higher value for edges is $v_1v_3 + v_3v_1 = 2 + 2 = 4$

and $v_6v_2 + v_2v_6 = 3 + 1 = 4$.

We give one more example of the adjacency matrix of a directed multigraph in the following.

Example 4.10. Let G be a multigraph given by the following figure.

G =

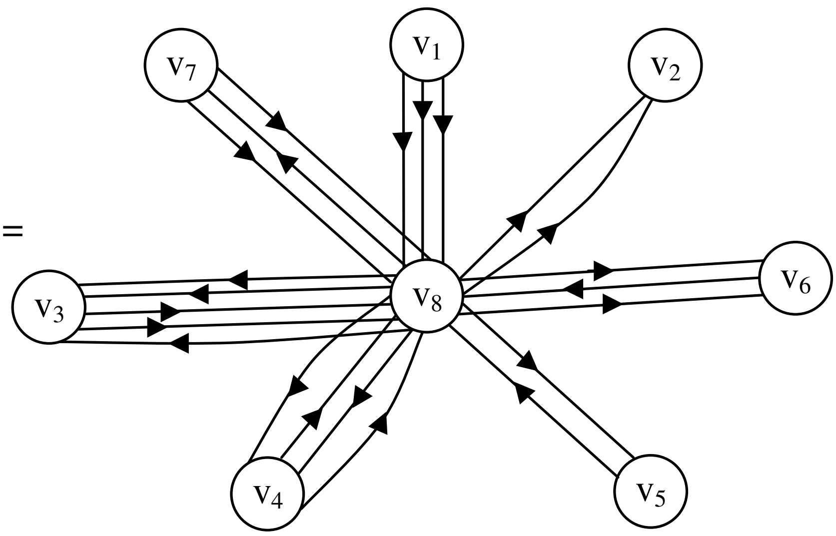


Figure 4.10

G is a star multigraph which is directed G has 8 vertices. Let B be the adjacency matrix associated with G.

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix} \end{array}.$$

We see this adjacency matrix of the directed star multigraph is non symmetric with zero diagonal entries but if v_8 is the egocentric node then the entries in the matrix B are only $v_i\, v_8$ and $v_8 v_i$, $1 \leq i \leq 8$.

So in case of directed star multigraphs if v_1 is the ego centric vertex then the matrix will have only a column and row given by $v_i v_1$ and $v_1 v_i$ respectively, $1 \leq i \leq n$.

In view of all these we have the following theorem.

Theorem 4.2. *Let G be a directed multigraph with n vertices. Let N be a adjacency $n \times n$ matrix. i) The diagonal elements of N are zero. ii) N is not symmetric, iii) The number of edges between a pair of vertices v_i and v_j of the graph is given by $v_iv_j + v_jv_i$ ($i \neq j$; $1 \leq i, j \leq n$). iv) The maximum number of edges is given by the maximum value of $v_iv_j + v_jv_i$; $i \neq j$; $1 \leq i, j \leq n$.*

Proof is left as an exercise for the reader.

We provide yet another example of a directed multigraph.

Example 4.11. Let G be a circle directed multigraph given by the following figure.

G =

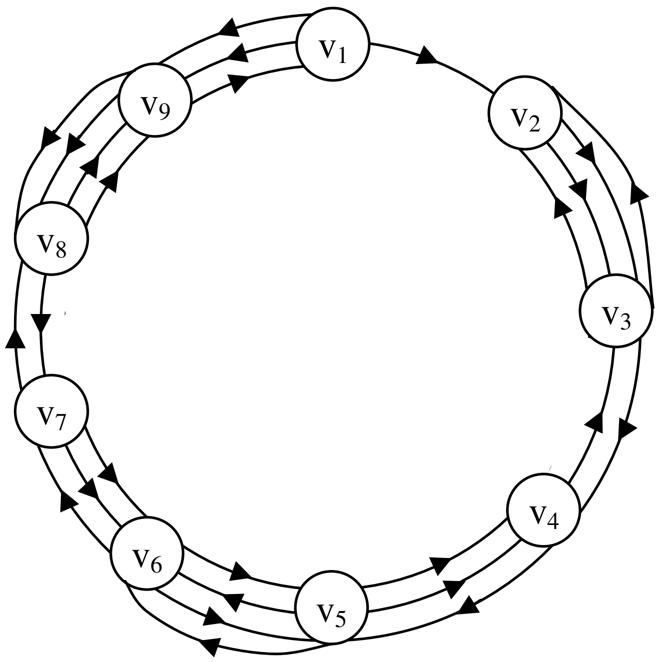


Figure 4.11

Let D be the adjacency matrix associated with the multiring G, given in the following.

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \end{array}.$$

We see the adjacency matrix is a special type the non symmetric matrix with diagonal entries zero and the upper and the lower diagonal has some nonzero elements and the elements a_1 a_n and or $a_n a_1$ are non zero if the ring starts from a_1 and ends at a_n, the ring is formed as a_1 to a_2 a_2 to a_3, a_3 to a_4 so on a_{n-1} to a_n and a_n to a_1.

In case of undirected multigraph which is a ring we see the resulting adjacency matrix is a symmetrical one with diagonal elements zero and a_1 a_n and or a_n a_1 is non zero. The upper diagonal and the lower diagonal elements are non zero. All other elements are zero.

Next we proceed onto describe the directed matrix of the multigraphs both directed and undirected by some examples.

Example 4.12. Let G be a multigraph given by the following figure.

G =

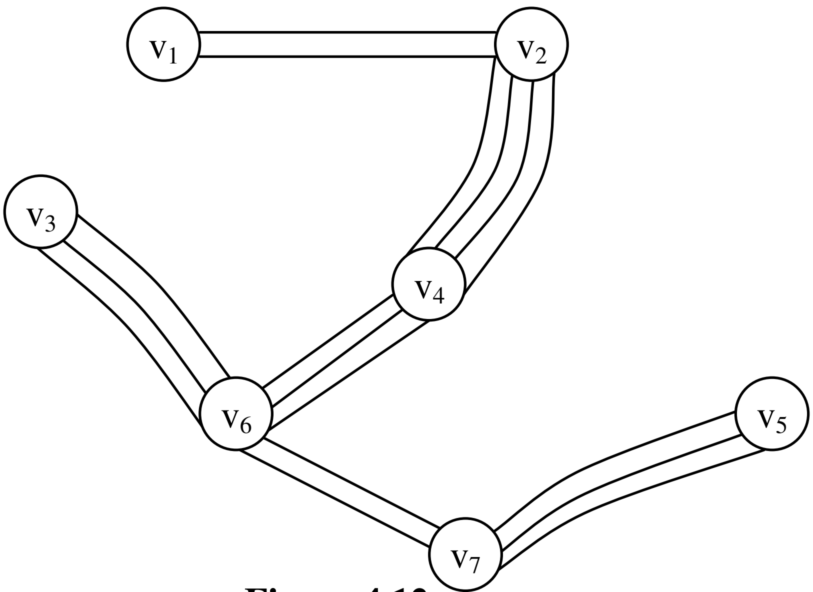


Figure 4.12

Let D be the distance matrix of the multigraph G is given in the following:

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{bmatrix} 0 & 2 & 72 & 8 & 144 & 24 & 48 \\ 2 & 0 & 36 & 4 & 72 & 12 & 24 \\ 72 & 36 & 0 & 9 & 18 & 3 & 6 \\ 8 & 4 & 9 & 0 & 18 & 3 & 6 \\ 144 & 72 & 18 & 18 & 0 & 6 & 3 \\ 24 & 12 & 3 & 3 & 6 & 0 & 2 \\ 48 & 24 & 6 & 6 & 3 & 2 & 0 \end{bmatrix} \end{array}.$$

Next we proceed onto give the distance matrix of a directed multigraph by an example.

Example 4.13. Let G be a directed multigraph given by the following figure.

G =

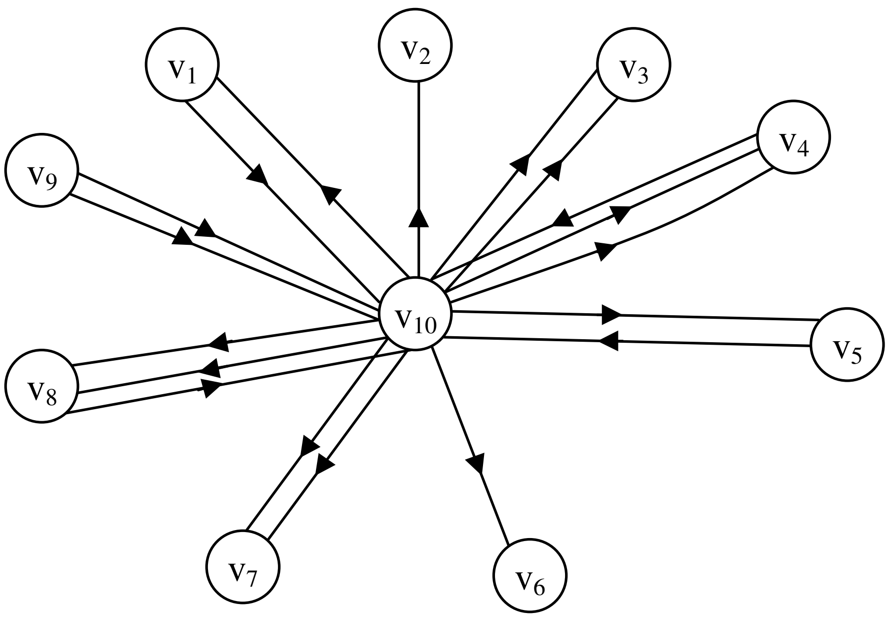


Figure 4.13

Let D be the distance matrix of the multigraph G given by the following figure.

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \end{array} \\ \begin{bmatrix} 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 \\ \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & \infty & 1 \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty & 1 \\ \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & 2 \\ 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & \infty & 0 \end{bmatrix} \end{array}.$$

We give yet another example of a distance matrix of a directed multiring in the following.

G =

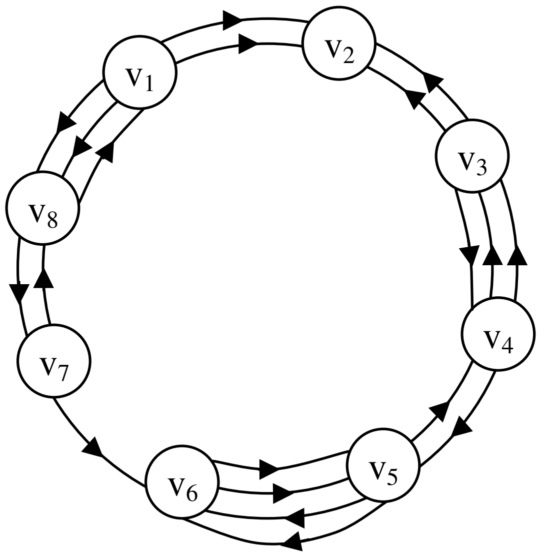


Figure 4.14

Let D be the distance matrix of the multidigraph G.

$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 2 & \infty & \infty & \infty & \infty & 2 & 2 \\ \infty & 0 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 2 & 0 & 1 & 1 & 3 & \infty & \infty \\ \infty & 4 & 2 & 0 & 1 & 2 & \infty & \infty \\ \infty & 4 & 2 & 1 & 0 & 2 & \infty & \infty \\ \infty & 8 & 4 & 2 & 2 & 0 & \infty & \infty \\ 1 & 8 & 4 & 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 4 & 2 & 2 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Interested reader is left with the task of finding detour matrix for both multigraphs and directed multigraphs.

Finally we see multisubgraphs of a multigraph will enjoy adjacency matrix which will be a submatrix in a restricted sense. That is the entries will not change in case of adjacency submatrix of a adjacency matrix for a multigraph. We will illustrate this situation by some examples.

Example 4.14. Let G be a multigraph given by the following figure.

G =

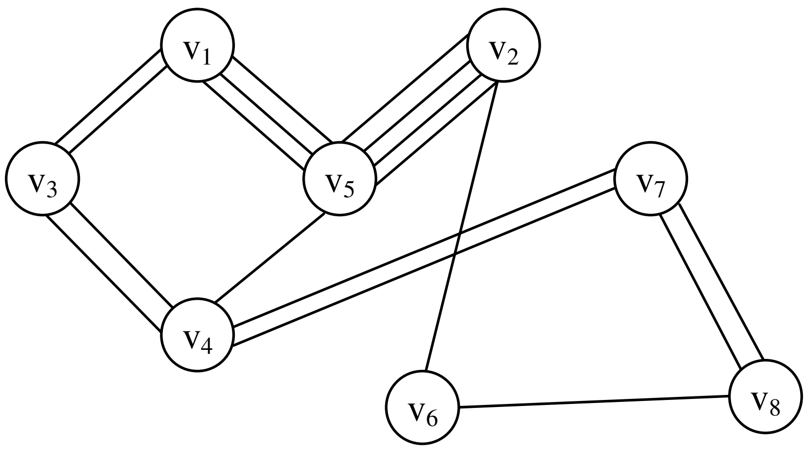


Figure 4.15

Let M be the adjacency matrix associated with G.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ 3 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix} \end{array}.$$

Now we consider the multisubgraph H_1 given by the following figure.

H_1 =

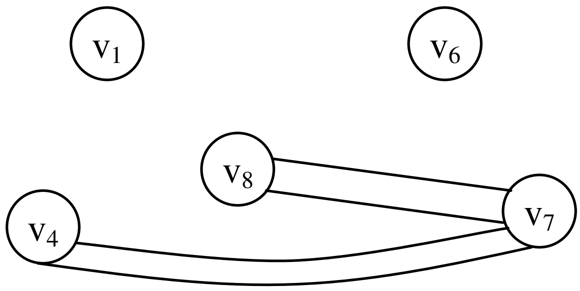


Figure 4.16

The adjacency matrix N_1 of the submultigraph is as follows.

$$N_1 = \begin{array}{c} \\ v_1 \\ v_4 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_4 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \end{array}.$$

We call this N_1 as a special type of submatrix of M.

Thus we see any v_iv_j in both M and N_1 has the same value. Under this fact we call a matrix N_1 of the matrix M to be a submatrix if value of v_iv_j if v_i and v_j are in the multisubgraph H_1 of G then the value of v_iv_j in G are the same in the matrices M and $N_{1.}$

Let H_2 be the multisubgraph of the multigraph G given by the following figure.

H_2 =

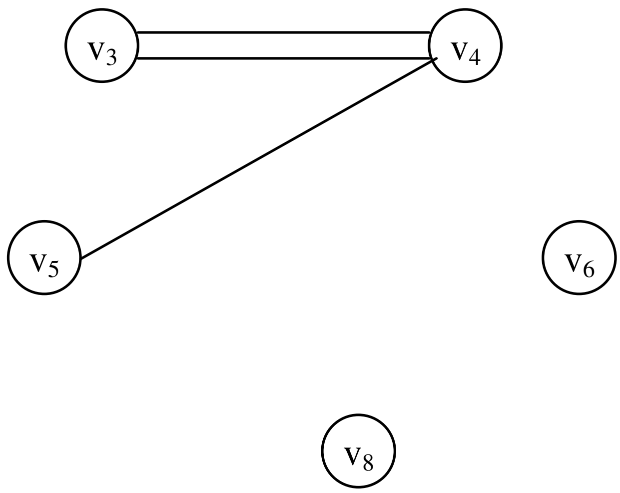


Figure 4.17

The adjacency submatrix N_2 of the matrix M associated with H_2 is as follows.

$$N_2 = \begin{array}{c} \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_8 \end{array} \begin{array}{c} \begin{array}{ccccc} v_3 & v_4 & v_5 & v_6 & v_8 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Clearly N_2 is a submatrix of the matrix M as v_3v_4, v_3v_5, v_3v_6, …, v_8v_7 take the same values both in N_2 and M.

Now we just give one example to show that the distance matrix P of the multigraph G and the distance matrix T_1 of a multisubgraph H_1 of G need not be submatrices in general.

The distance matrix P of the multigraph G is as follows.

$$P = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 12 & 2 & 3 & 3 & 12 & 6 & 12 \\ 12 & 0 & 8 & 4 & 4 & 1 & 2 & 1 \\ 2 & 8 & 0 & 2 & 2 & 8 & 4 & 8 \\ 3 & 4 & 2 & 0 & 1 & 4 & 2 & 4 \\ 3 & 4 & 2 & 1 & 0 & 4 & 2 & 4 \\ 12 & 1 & 8 & 4 & 4 & 0 & 2 & 1 \\ 6 & 2 & 4 & 2 & 2 & 2 & 0 & 2 \\ 12 & 1 & 8 & 4 & 4 & 1 & 2 & 0 \end{bmatrix} \end{array}.$$

When we work for distance matrices in case of these multigraphs we are in the problem of ambiguity for v_3 to v_5 v_3 v_1 v_5 and v_3 v_4 v_5 we see the path is 6 and 2 respectively. We choose only 2. On similar lines v_3 to v_2 is v_3 v_4 v_5 v_2 and v_3 v_1 v_5 v_2 and the lengths are 8 and 24 respectively. Thus we say that only small numerical values will only be taken.

See v_3 to v_8 has a path in G but it has no path in H_2 as its value is ∞.

Thus the concept of submatrix of a distance matrix in general in case of multisubgraphs has no relevance.

Now we proceed onto define subgraphs of a graph by removing an edge or some edges. For when we talk of a subgraph we can define vertex removed subgraphs and edge removed subgraphs.

Throughout our book we have only defined or described a submultigraph of a multigraph by taking the substructure when some vertices are removed. That is why we say if a multigraph has n vertices then we have $nC_2 + nC_3 + \ldots, + nC_{n-1}$ number of multisubgraphs barring the singleton vertices and the multigraph itself.

We have submultigraphs of three types (i) submultigraphs formed by taking subsets of vertices and the edges formed as that of the parent graph.

Multisubgraphs obtained by removing edges in which cases the vertices be removed automatically. Other type of multisubgraphs are those which are formed using subsets of the vertex set we get the edges from the parent or original multigraph.

Other type is multisubgraphs in which edges are removed consequently the vertices (relevant) are removed. Other type is arbitrary multisubgraphs just formed arbitrarily from the multigraph.

We provide examples of edge multisubgraphs. These are multisubgraphs which are got from the multigraphs by removing edges.

Example 4.15. Let G be a edge and vertex labeled multigraph given by the following figure.

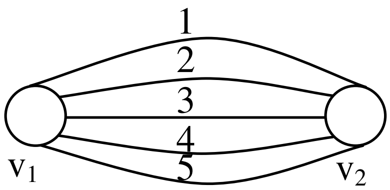
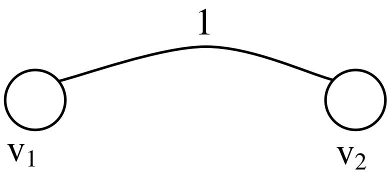
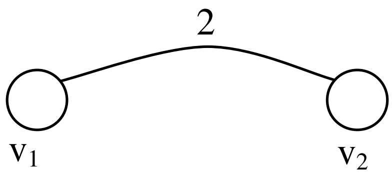
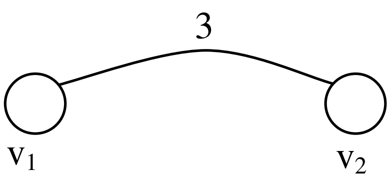
G = 

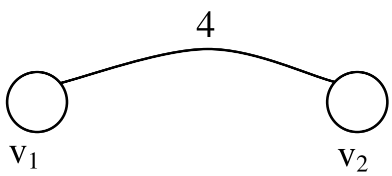
Figure 4.18

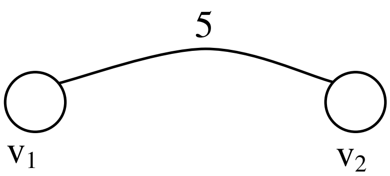
The edge multisubgraphs of G are

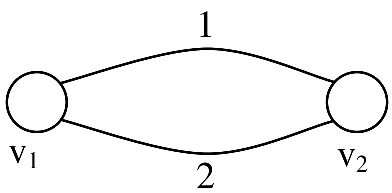
H_1 = 

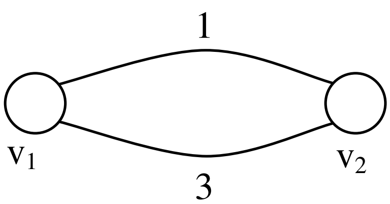
H_2 = 

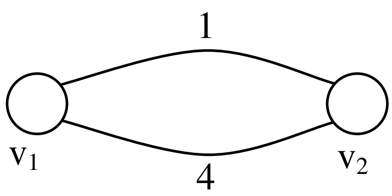
H_3 = 

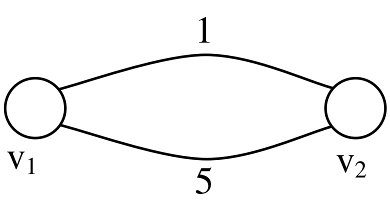
H_4 = 

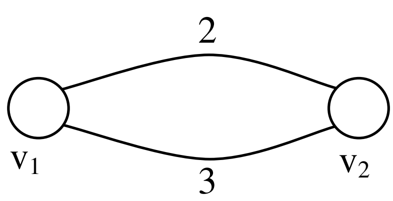
H_5 = 

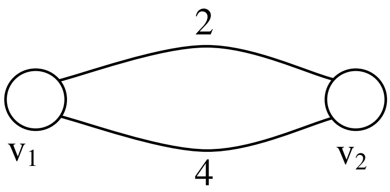
H_6 = 

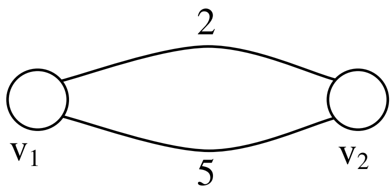
H_7 = 

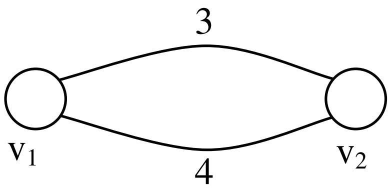
H_8 = 

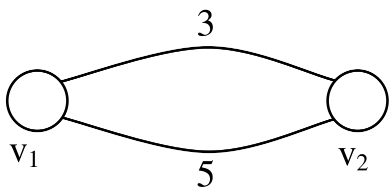
H_9 = 

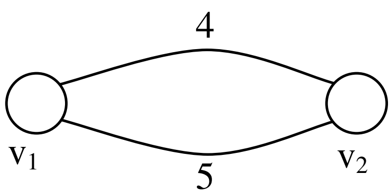
H_{10} = 

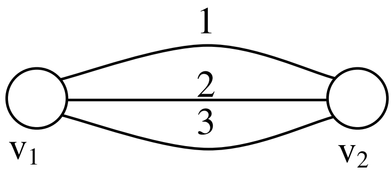
$H_{11} =$ 

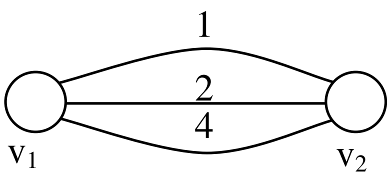
$H_{12} =$ 

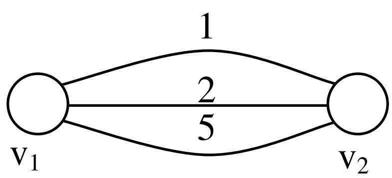
$H_{13} =$ 

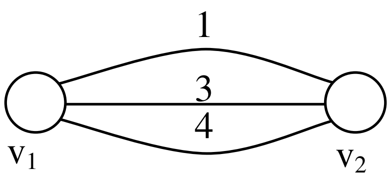
$H_{14} =$ 

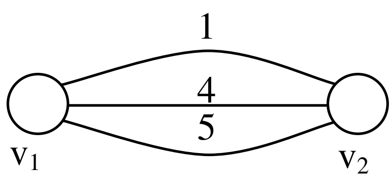
$H_{15} =$ 

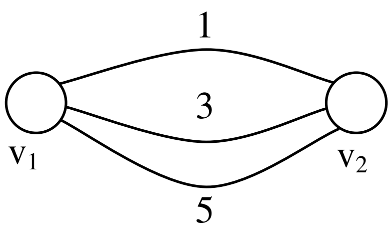
$H_{16} =$ 

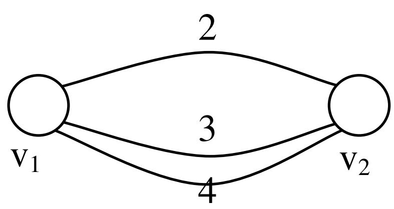
$H_{17} =$ 

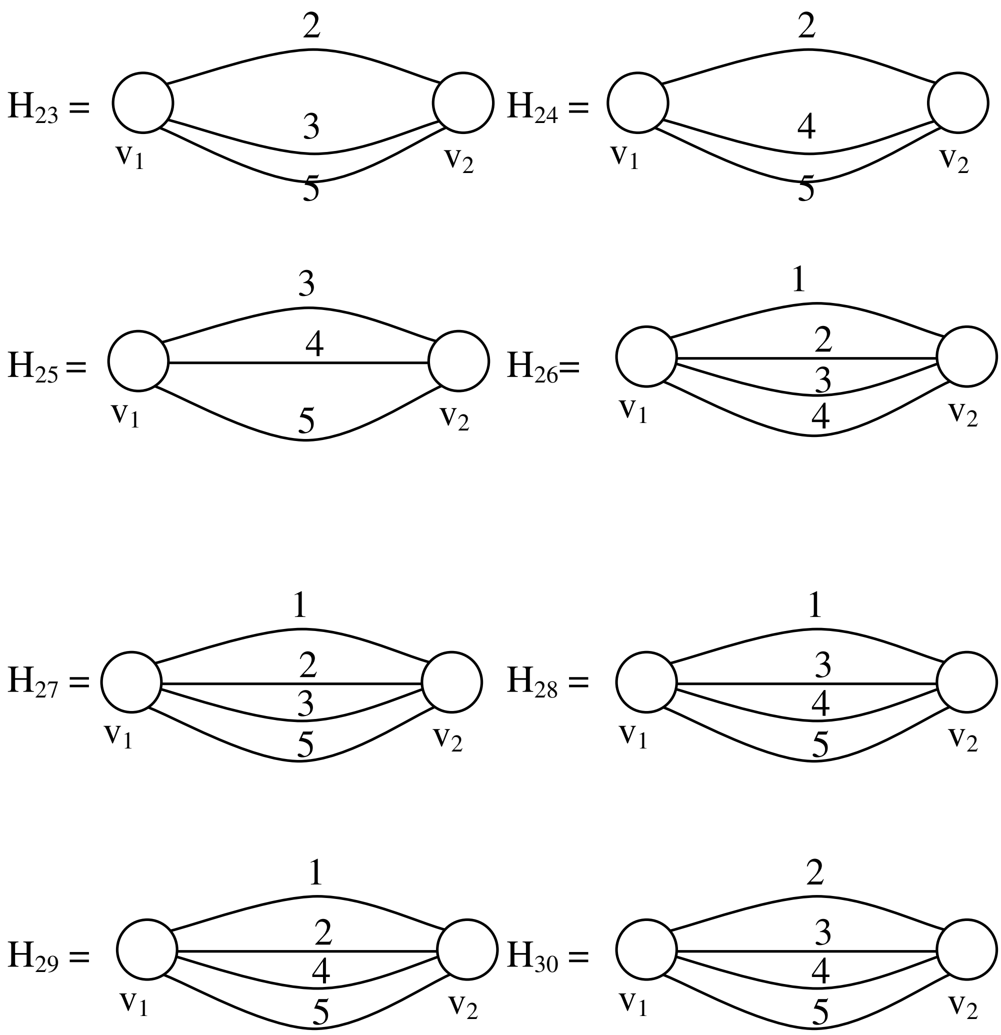
$H_{18} =$ 

$H_{19} =$ 

$H_{20} =$ 

$H_{21} =$ 

$H_{22} =$ 



Figures 4.19

We have given 30 such edge multisubgraphs of the multigraph G.

We see a multidyad with 5 edges can have 30 such multidyads. Thus the edge multisubgraphs of a multigraph are many in comparison with vertex multisubgraphs.

So we see the notion of edges multisubgraph yields several multisubgraphs. This concept will be useful in case of retrievable multi networks or multichannels.

We provide yet another example.

Example 4.16. Let G be a edge and vertex labeled multigraph given by the following figure.

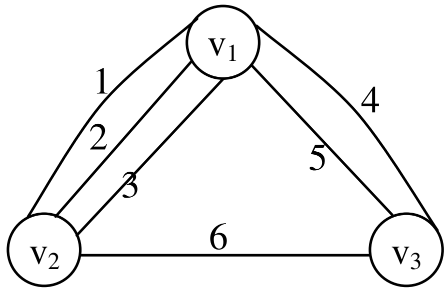
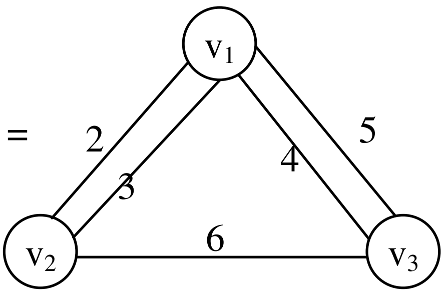


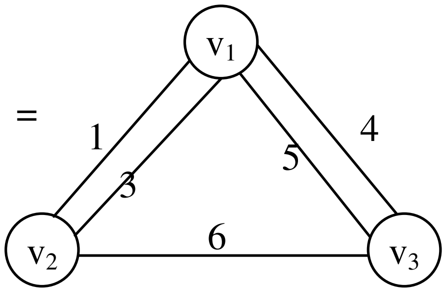
Figure 4.20

This multitriad is a nonuniform complete multitriad. We give the edge multisubgraphs of G in the following.

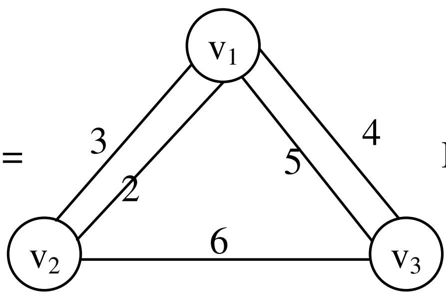
H_1 =



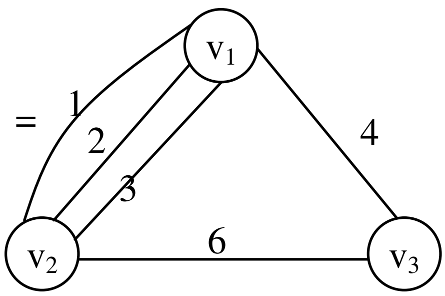
H_2 =

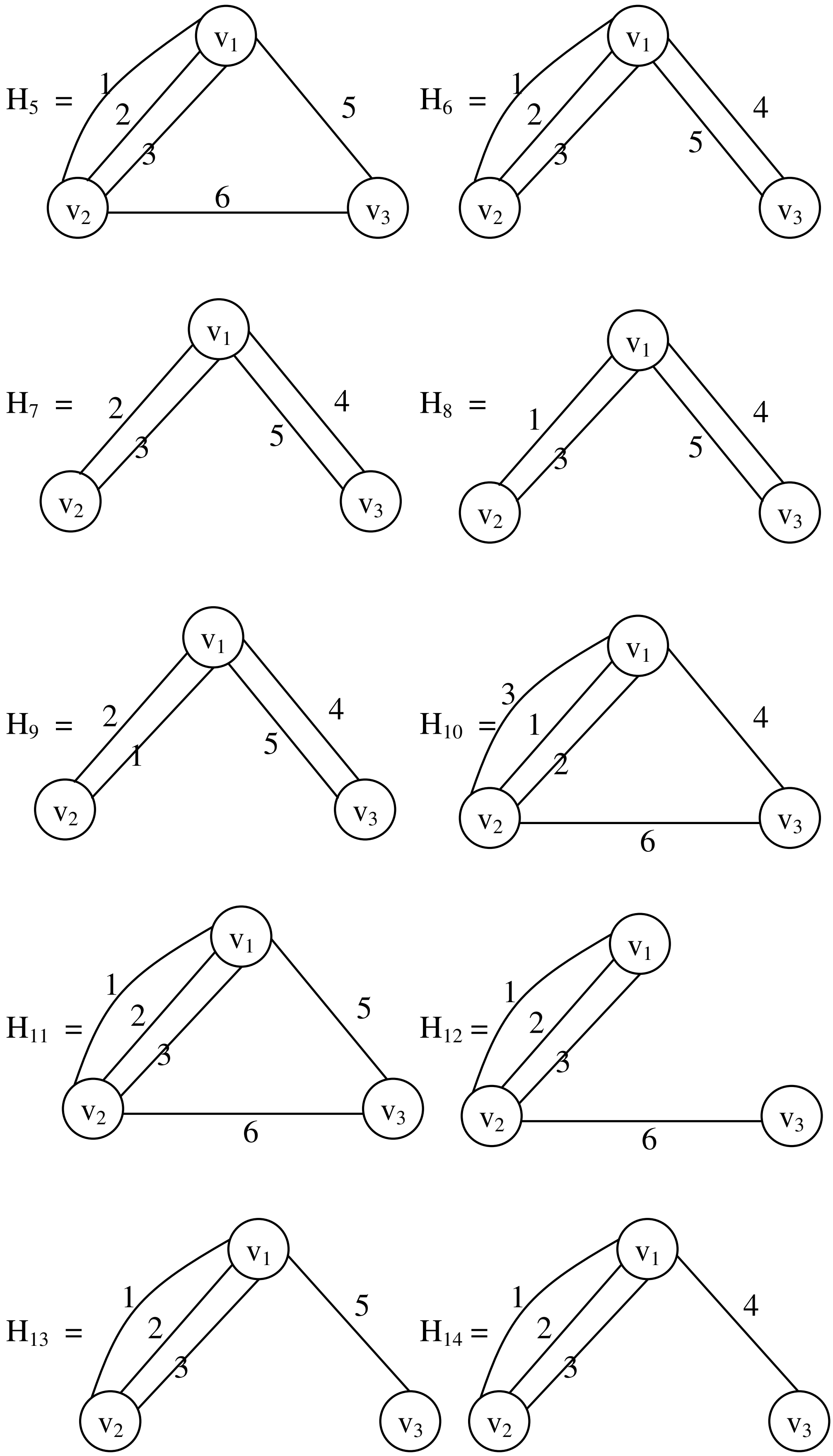


H_3 =



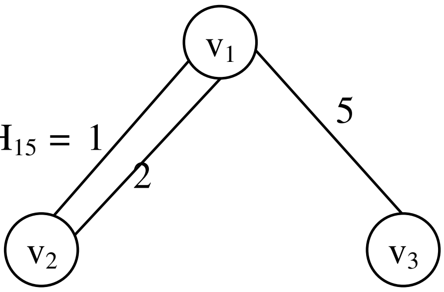
H_4 =



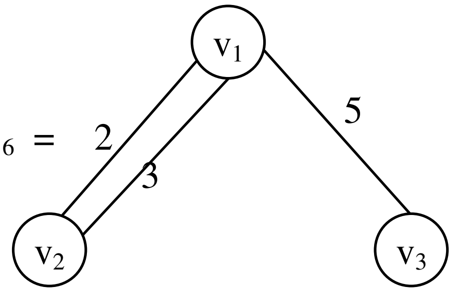


H_5 =
H_6 =
H_7 =
H_8 =
H_9 =
H_{10} =
H_{11} =
H_{12} =
H_{13} =
H_{14} =

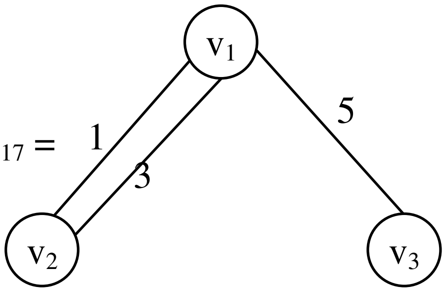
H_{15} =



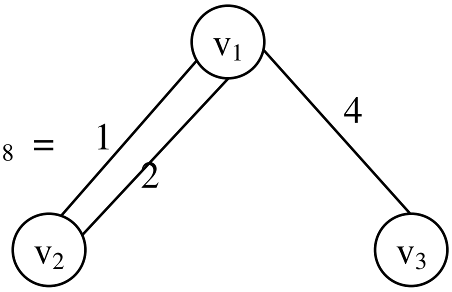
H_{16} =



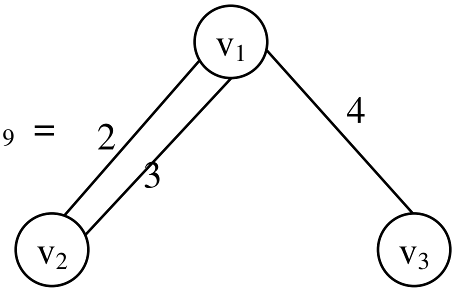
H_{17} =



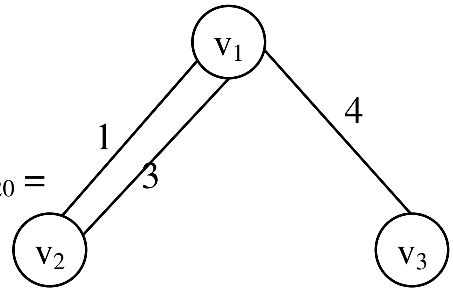
H_{18} =



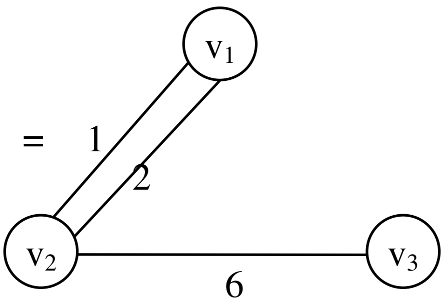
H_{19} =



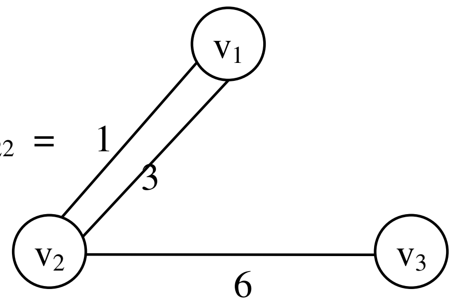
H_{20} =



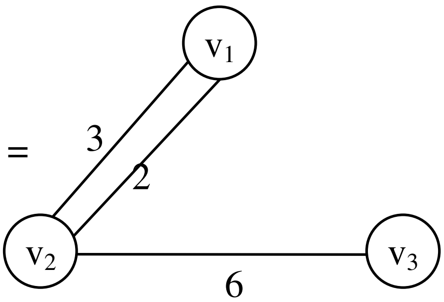
H_{21} =



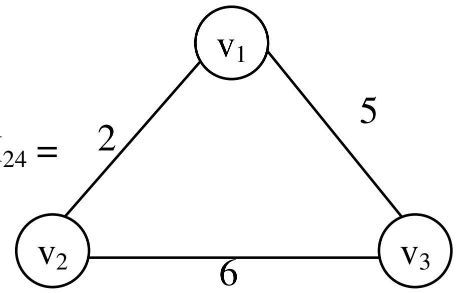
H_{22} =



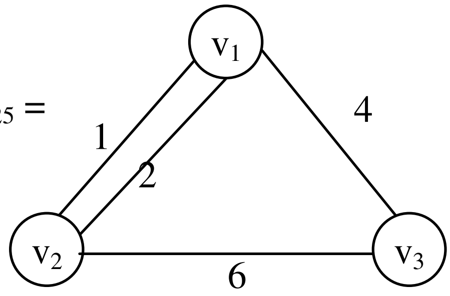
H_{23} =



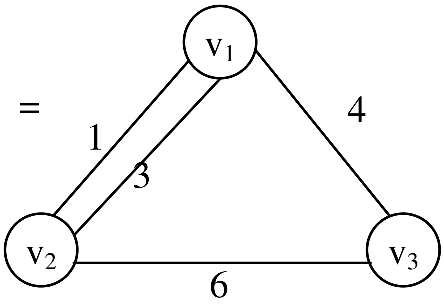
H_{24} =



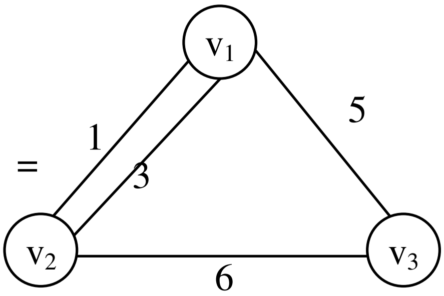
H_{25} =



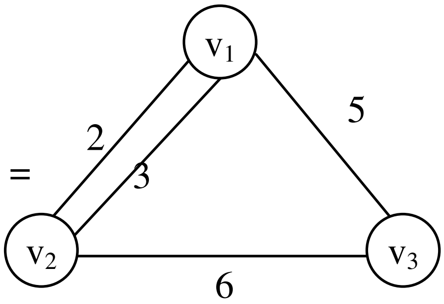
H_{26} =



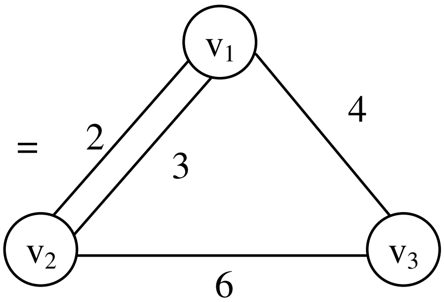
H_{27} =



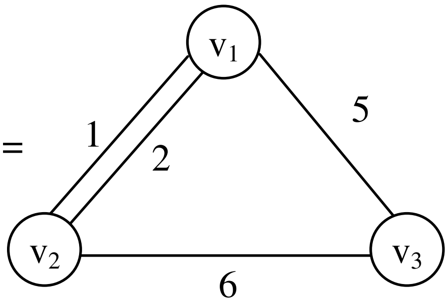
H_{28} =



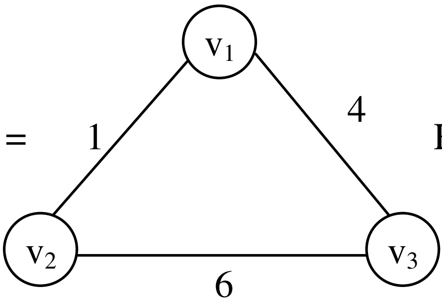
H_{29} =



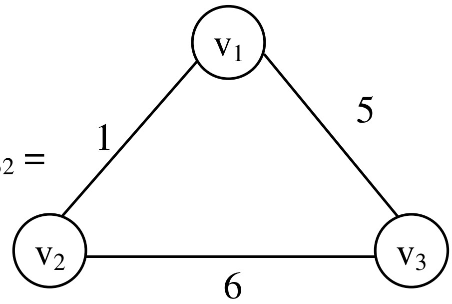
H_{30} =



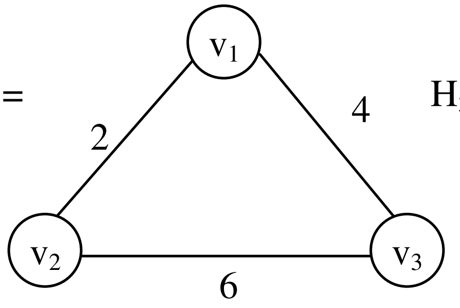
H_{31} =



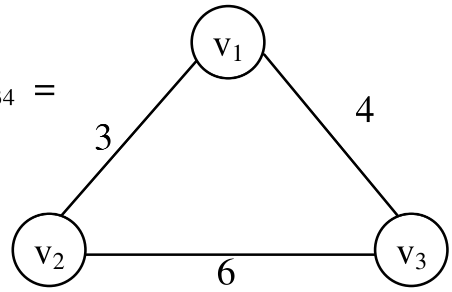
H_{32} =



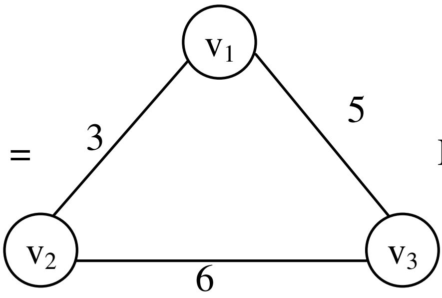
H_{33} =



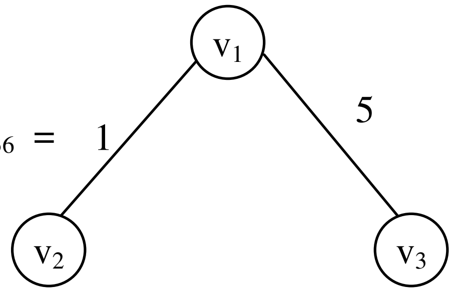
H_{34} =



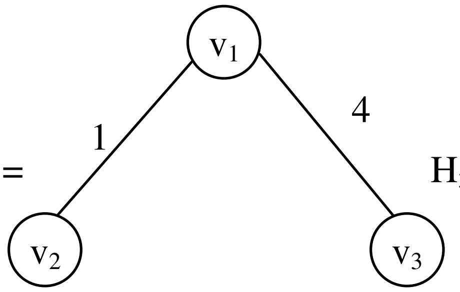
H_{35} =



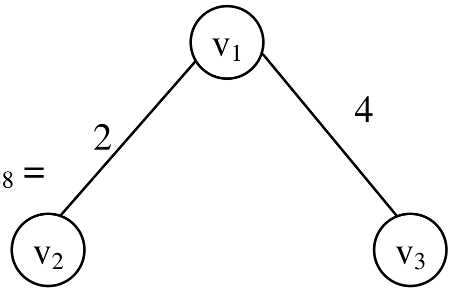
H_{36} =



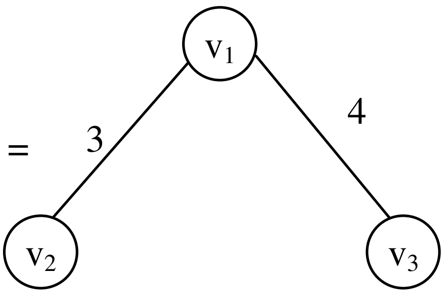
H_{37} =



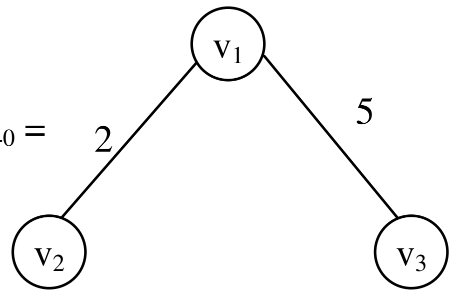
H_{38} =



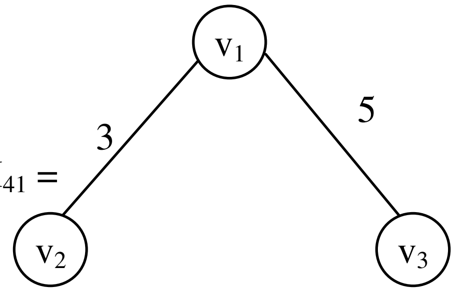
H_{39} =



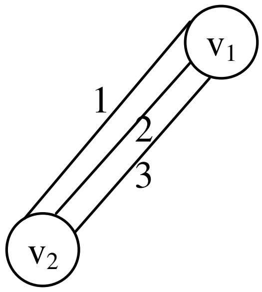
H_{40} =



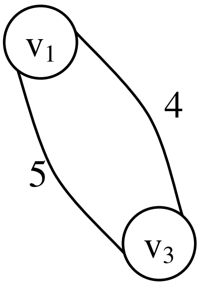
H_{41} =



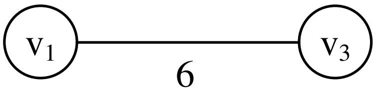
H_{42} =



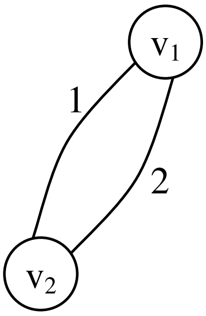
H_{43} =



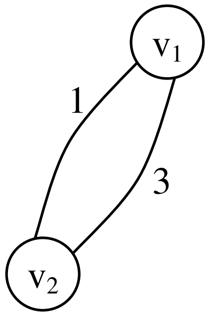
H_{44} =



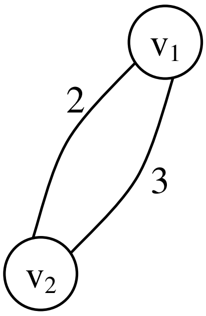
H_{45} =



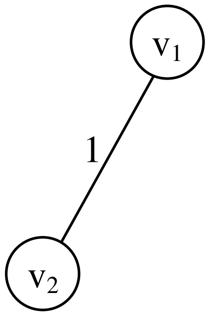
H_{46} =



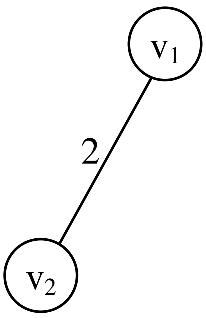
H_{47} =



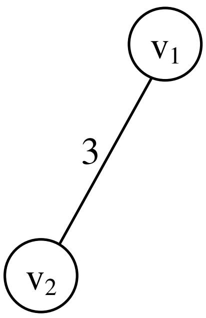
H_{48} =



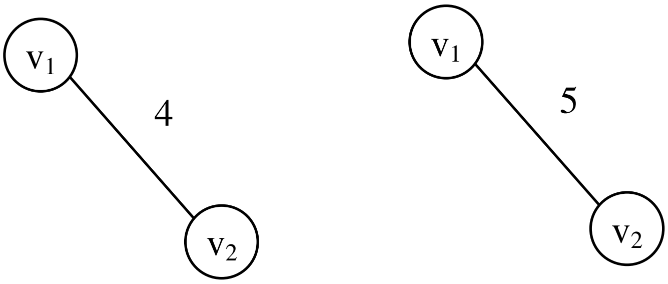
H_{49} =



H_{50} =



H_{51} =



Figures 4.21

and so on. Infact it is a difficult job to find the number of such edge multisubgraphs of the multigraph G.

Now we proceed onto describe another example for edge multisubgraphs of a multigraph.

Example 4.17. Let G be a edge and vertex labeled multigraph given by the following figure.

G =

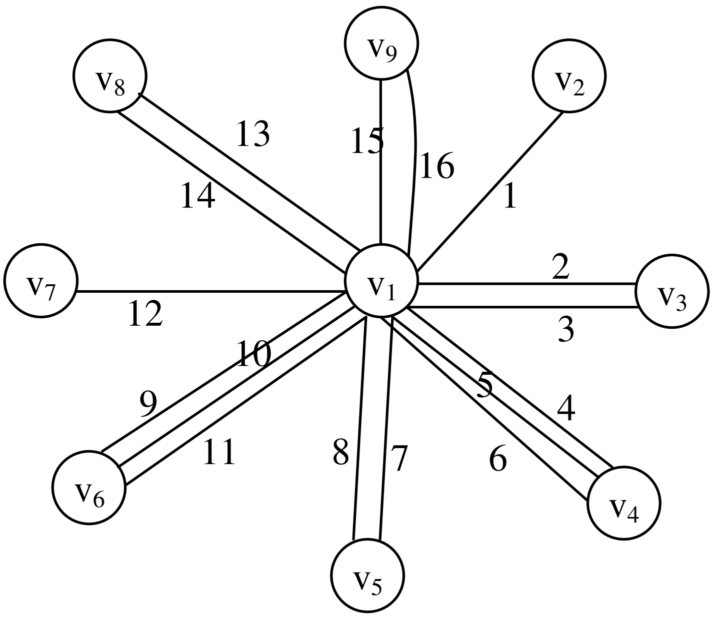


Figure 4.22

$H_1 =$

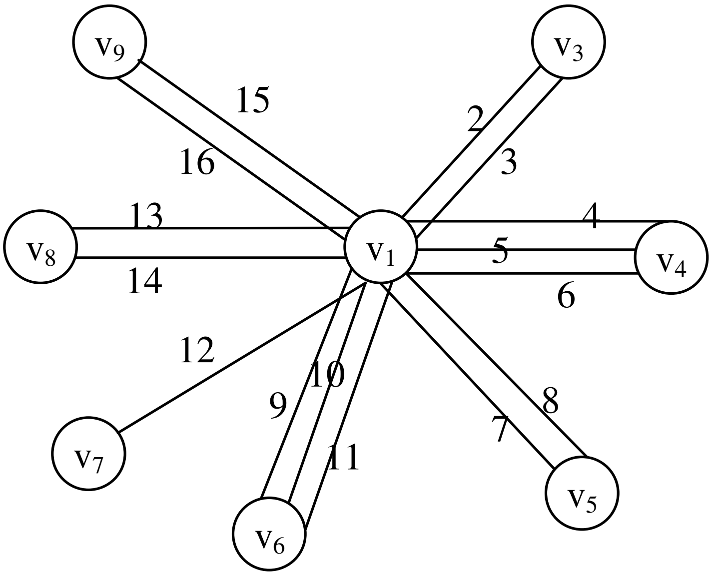


Figure 4.23

Removal of edge 1 has resulted in the removal of the vertex $v_{2.}$

Let H_2 be a edge multisubgraph of G given by the following figure.

$H_2 =$

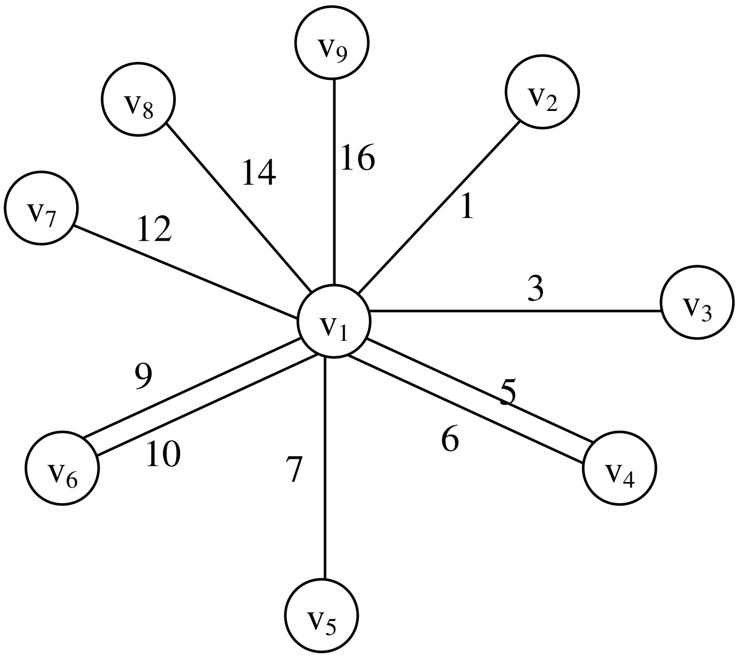


Figure 4.24

Clearly H_2 is a edge multisubgraph for which the edges 2, 4, 8, 11, 13 and 15 are removed. H_2 has all the nine vertices, though so many edges are removed.

Consider H_3 the multisubgraph given by the following figure.

H_3 =

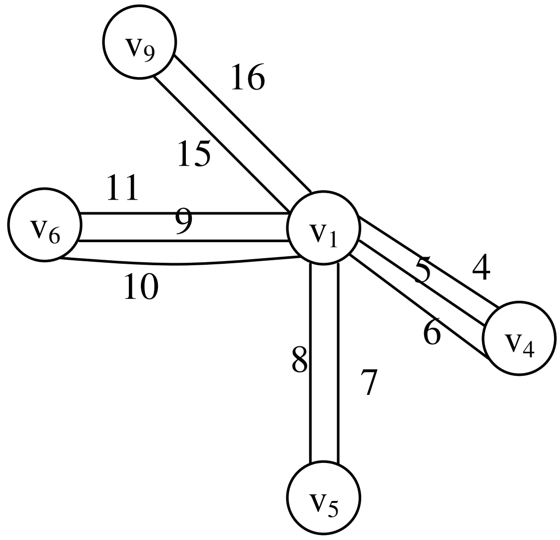


Figure 4.25

We have removed only 6 edges but the resulting edge multisubgraph has only 5 vertices.

Thus we cannot say the removal of a fixed number of edges will affect the number of vertices.

For six edges are removed for the following multisubgraph H_4 given by the following figure.

H_4 =

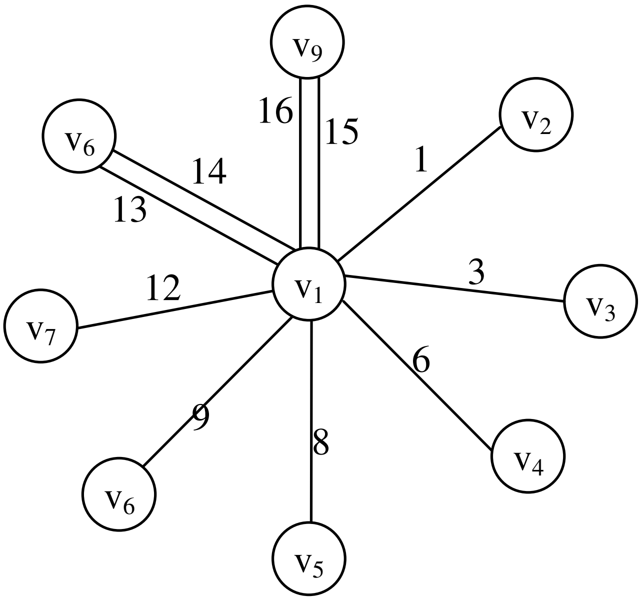


Figure 4.26

H_4 is a multisubgraph and it has all its vertices intact inspite of removing six edges.

We give yet another example of a multigraph and its edge multisubgraphs.

Example 4.18. Let G be a edge and vertex labeled multigraph given by the following figure.

G =

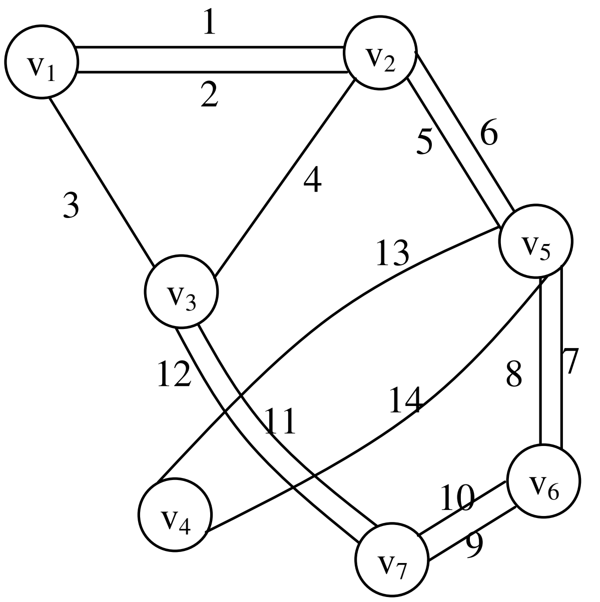


Figure 4.27

Let H_1 be the edge multisubgraph of G given by the following figure.

H_1 =

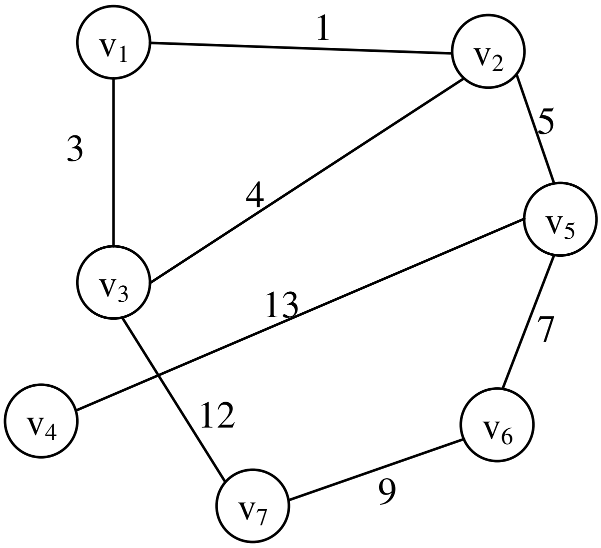


Figure 4.28

We have removed six edges and the resulting multisubgraph is a usual simple graph.

Let H_2 be the edge multisubgraph given by the following figure.

H_2 =

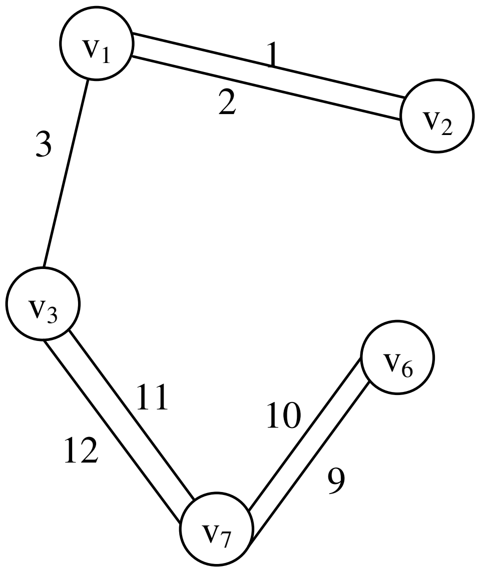


Figure 4.29

We see H_2 is a multisubgraph for which four edges is removed and automatically two vertices are removed.

Let H_3 be the edge multisubgraph given by the following figure.

H_3 =

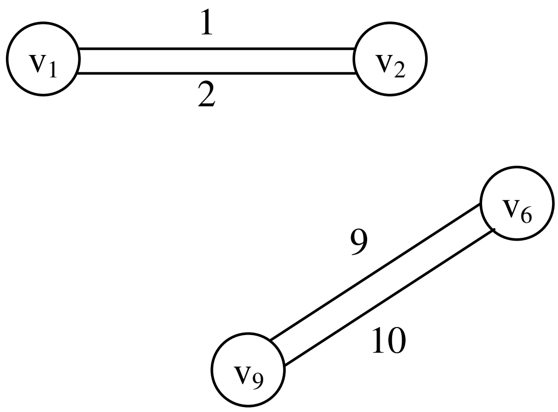


Figure 4.30

H_3 is again a edge multisubgraph.

We provide vertex multisubgraph given by the following figures P_1, P_2 and P_3 respectively.

P_1 =

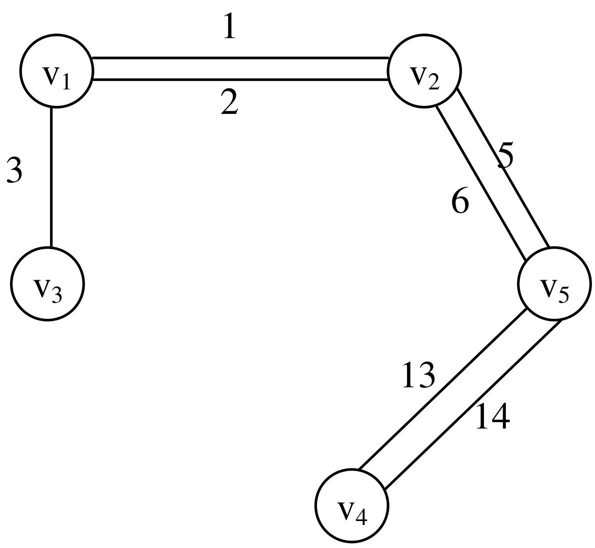


Figure 4.31

P_2 =

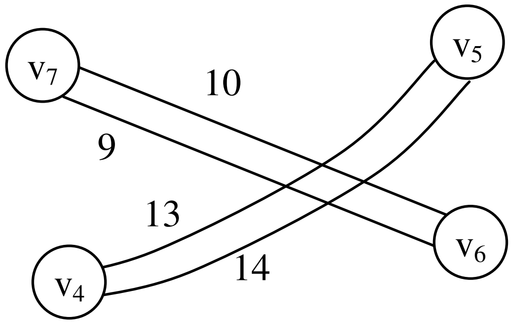


Figure 4.32

P_3 =

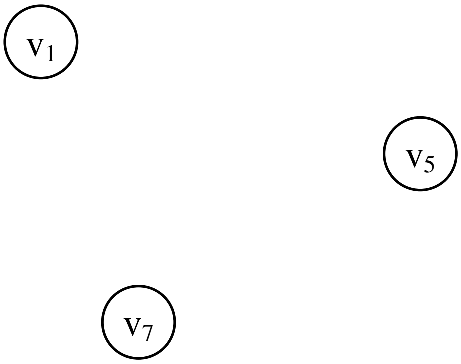


Figure 4.33

We see P_3 is a empty multisubgraph of G.

Now we give 3 multisubgraph which are not edge multisubgraphs or vertex multisubgraph given by the figures Q_1, Q_2 and Q_3.

Q_1 =

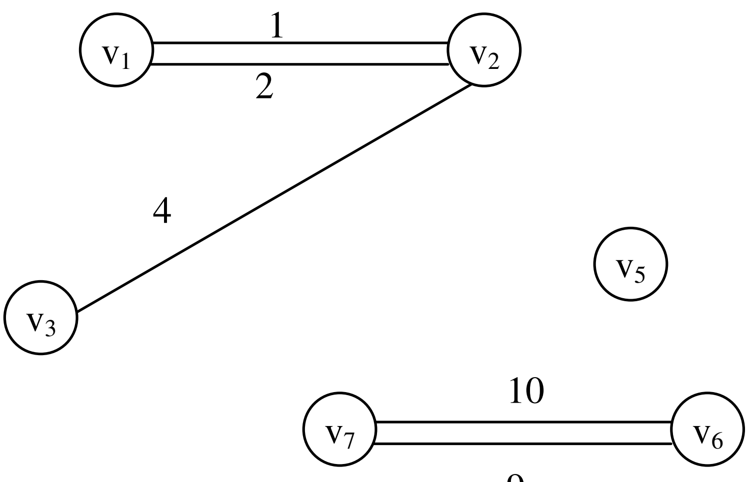


Figure 4.34

Clearly Q_1 is not a edge multisubgraph or a vertex multigraph.

Let Q_2 be the multisubgraph given by the following figure.

Q_2 =

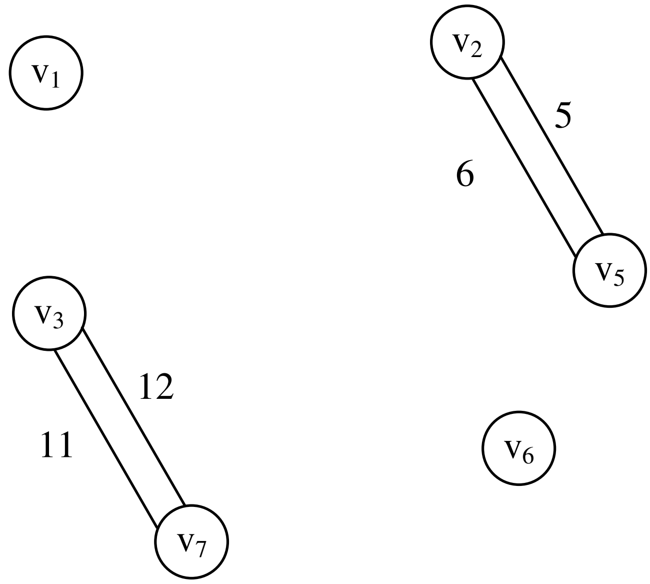


Figure 4.35

Q is again not a edge multisubgraph for if the edges 1 and 2 are removed then v_1 may not exist as the edge 3 is also removed. Q is not a vertex multisubgraph as the edges are not presented for the existing vertices.

Next we proceed onto give example of edge multisubgraphs of directed multigraphs.

Example 4.19. Let G be a directed edge and vertex labeled multigraph given by the following figure.

G =

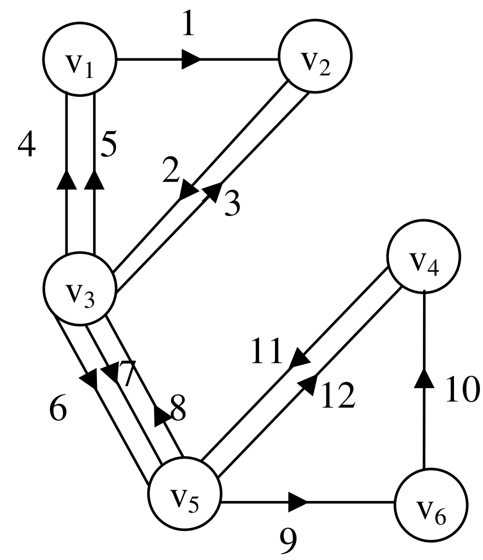


Figure 4.36

Let H_1 be a edge multisubgraph given by the following figure.

H_1 =

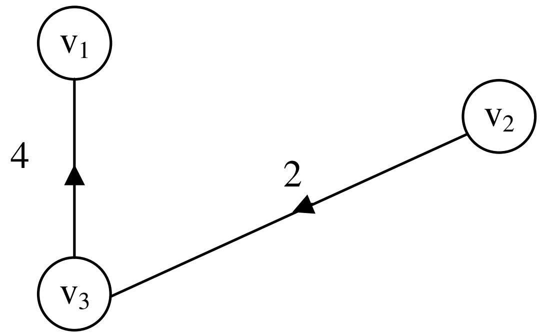


Figure 4.37

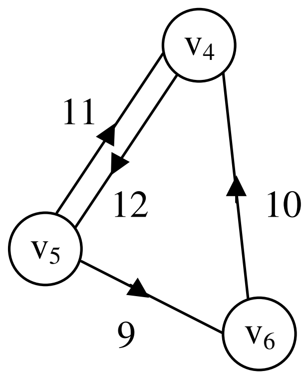


Figure 4.38

Clearly H_1 is disjoint and the most connecting vertex v_3 is removed as the related edges are removed.

Clearly this multisubgraph is not a vertex multisubgraph of G.

Let H_2 be a edge multisubgraph given by the following figure.

H_2 =

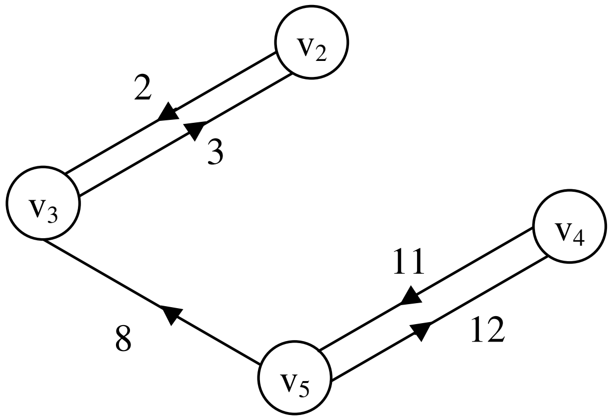


Figure 4.39

H_2 is a connected multisubgraph. The edges connecting v_1 with v_3 and v_2 and that of v_6 connecting with v_4 and v_5 are removed.

Consider the vertex multisubgraph B_1 with vertex set v_1, v_3, v_2 and v_5 given by the following figure.

B_1 =

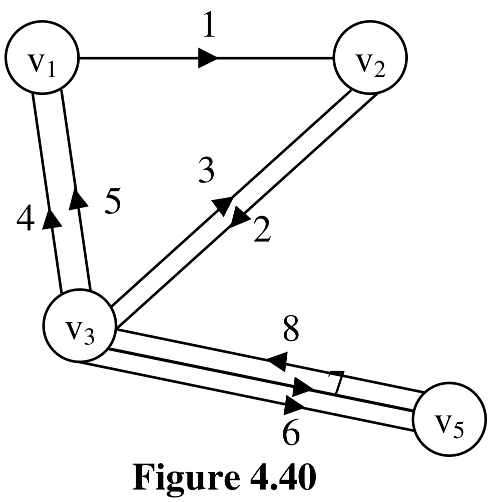


Figure 4.40

Let D be the multisubgraph which is neither edge multisubgraph nor a vertex multisubgraph of G given by the following figure.

D =

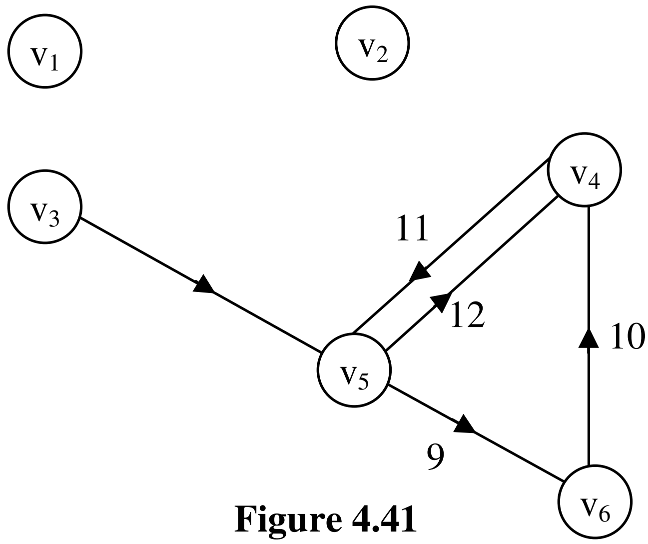


Figure 4.41

Clearly D is not a edge multisubgraph or a vertex multisubgraph of G.

Here it is important to note that multipath or multiwalk or multitrail of the multisubgraphs H of G are in general different in case of H from that of G.

Study in this direction is important and innovative.

We propose the following conjecture.

Conjecture 4.1. Let G be a multigraph with n vertices and m edges. Find the number of edge multisubgraphs of G.

We see finding edge multisubgraph of G even for very small multigraph happens to be a challenging one.

In case of multigraphs we have to describe uniform clique and nonuniform clique of a multigraph by some examples.

Example 4.20. Let G be a multigraph given by the following figure.

G =

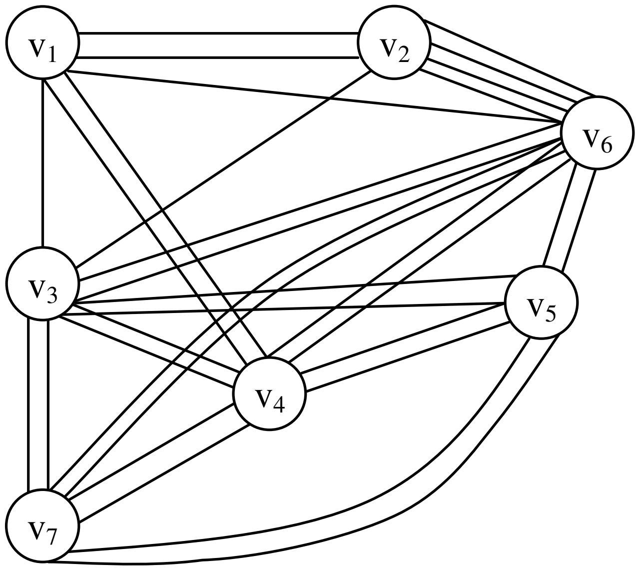


Figure 4.42

We see the multiclique H in G is given by the following figure.

H =

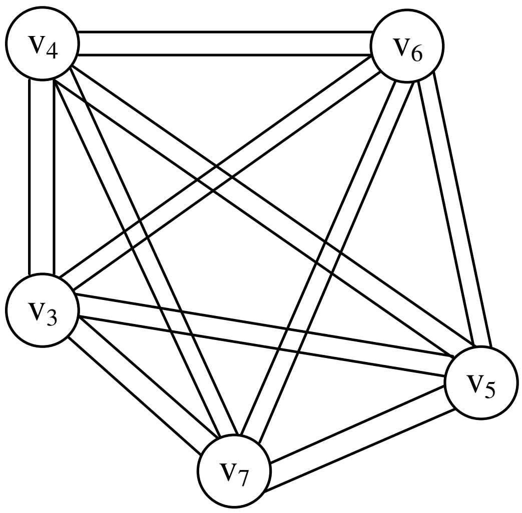


Figure 4.43

We see H is a complete multigraph of order 5 and every pair of vertices has only 2 edges. So we call H a uniform complete multigraph.

Thus G the multigraph has a uniform clique of order 5.

Example 4.21. Let G be a multigraph given by the following figure.

G =

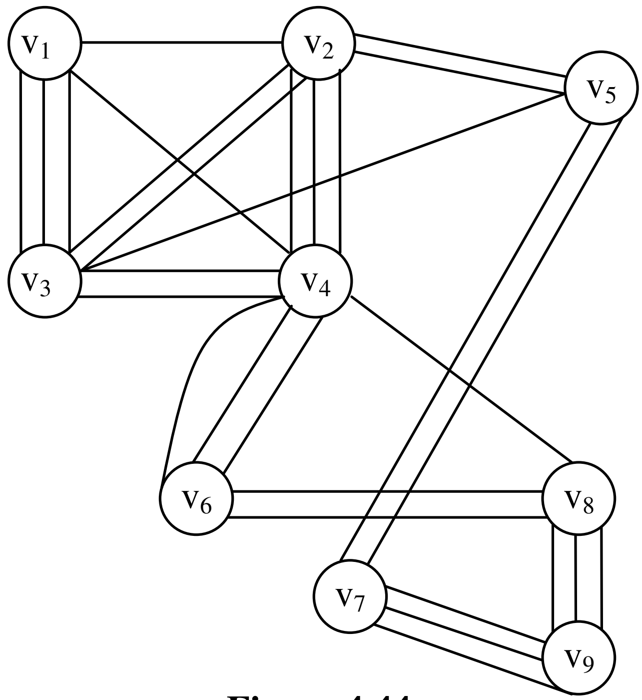


Figure 4.44

We see G has a multisubgraph given by K;

K =

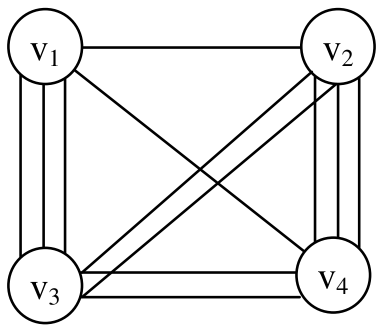


Figure 4.45

We see every two vertices of K is adjacent however the number of edges in every vase is not the same. We call such multigraphs as non uniform complete multigraphs as there is atleast an edge between a v_iv_j and every v_i is adjacent with v_j.

Thus this G has a nonuniform multiclique of order four.

In view of all these we define the following types of complete multigraphs.

Definition 4.1. *Let G be a multigraph with vertex set v_1, v_2, …, v_m. We say G is a complete uniform multigraph if every v_i is adjacent with v_j and the number of edges connecting them is a fixed number t; $2 \leq t < \infty$. $1 \leq i, j \leq n$; $i \neq j$. If t = 1 we have the classical or usual complete graph.*

We say G to be a nonuniform complete multigraph if every v_iv_j is adjacent but the number of edges connecting are different and vary from 1 to m; $1 < m < \infty$.

We will first provide examples of them.

Example 4.22. Let G be a multigraph given by the following figure.

G =

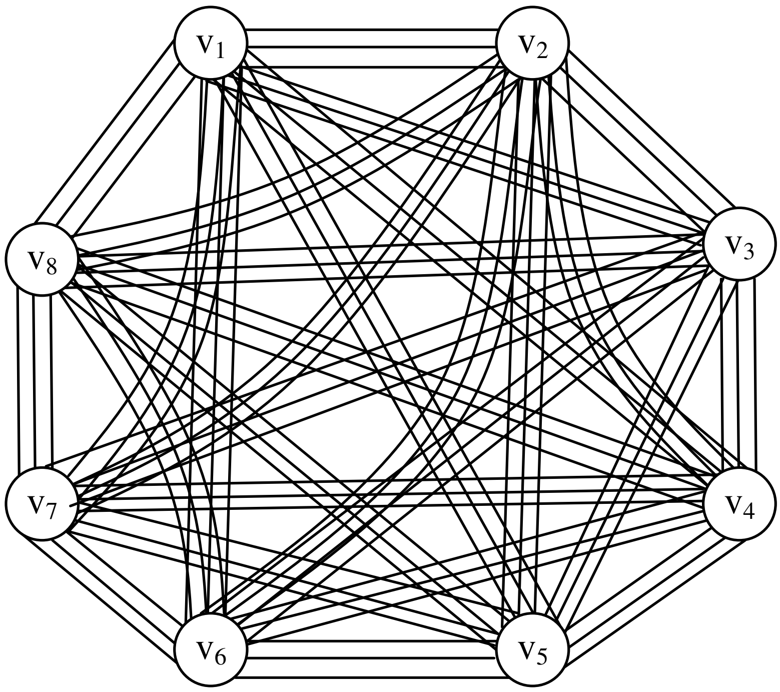


Figure 4.46

G is a uniform complete multigraph with 3 edges here t = 3 in the definition and n = 8.

We give an example of a non uniform complete multigraph.

Example 4.23. Let G be a multigraph given by the following figure.

G =

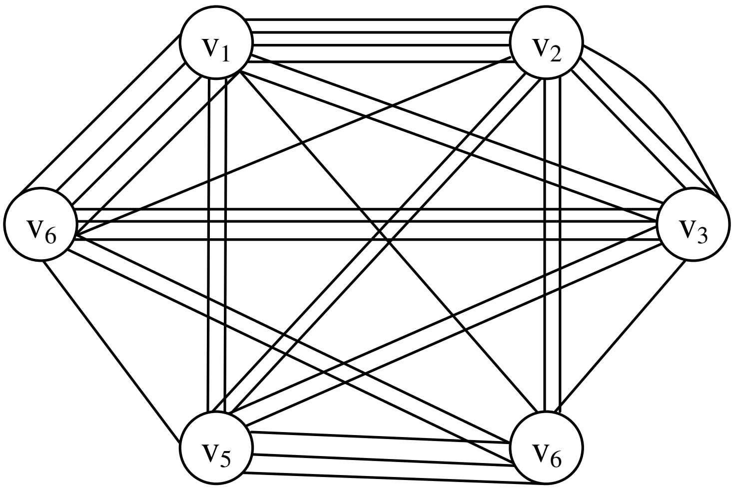


Figure 4.47

G is a non uniform complete multigraph with six vertices.

Number of edges from v_1 to v_2 is 3 that are v_1 to v_3 is 2, v_1 to v_4 is 1 v_1 to v_5 is 2 and v_1 to v_6 is 4.

Thus the number of edges in case of this G varies from 1 to 4. Now having defined both non uniform and uniform complete multigraphs now we proceed on to define uniform multi clique of a multigraph and a non uniform multiclique of a multigraph in the following.

Definition 4.2. *Let G be a multigraph. If G has a multisubgraph H which is such that H is a uniform multi complete multi subgraph of G and H happens to be the largest such one then we define H to be a uniform multi clique of G.*

It is interesting to note that all multigraphs may not in general contain a uniform multiclique.

We will substantiate our claim by the following examples.

Example 4.24. Let G be a multigraph given by the following figure.

G =

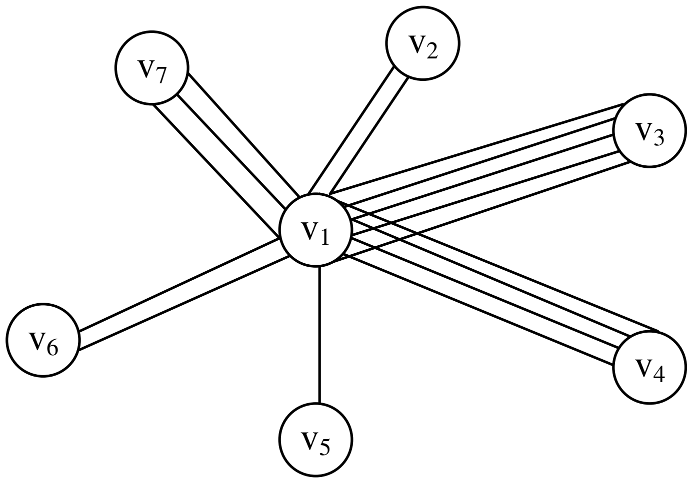


Figure 4.48

Clearly G is a star multigraph which has no multi cliques be it uniform or nonuniform.

Example 4.25. Let G be a multigraph which is a multiring given by the following figure.

G =

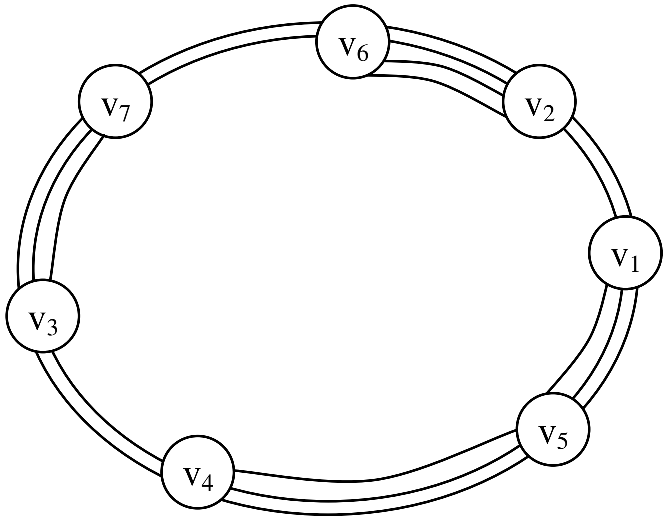


Figure 4.49

This multigraph G can never have a uniform multi clique or non uniform multi clique.

Example 4.26. Let G be a multigraph given by the following figure. G is a multitree.

G =

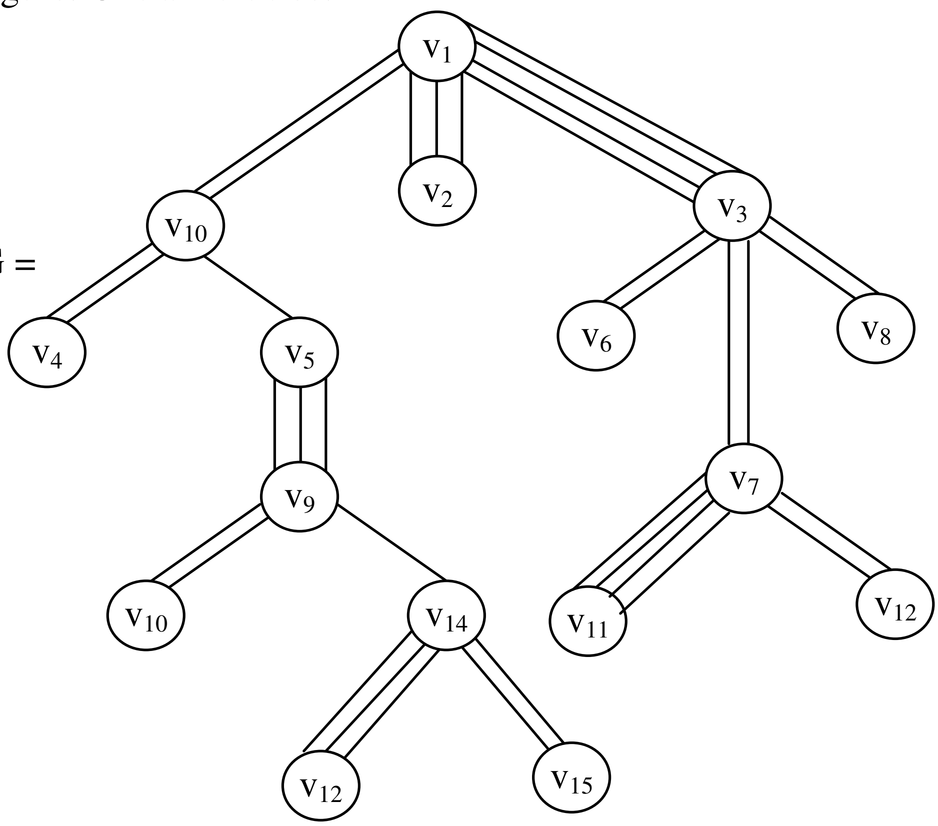


Figure 4.50

Clearly this multitree cannot have a uniform clique or a non uniform clique.

In view of all these we have the following theorem.

Theorem 4.3. *Multitrees, multistar graphs and multi ring graphs do not contain uniform multiclique or nonuniform multiclique.*

Proof is left as an exercise to the reader.

We now describe non uniform multi clique graph and uniform multiclique graph first by examples.

Example 4.27. Let G be a multigraph given by the following figure.

G =

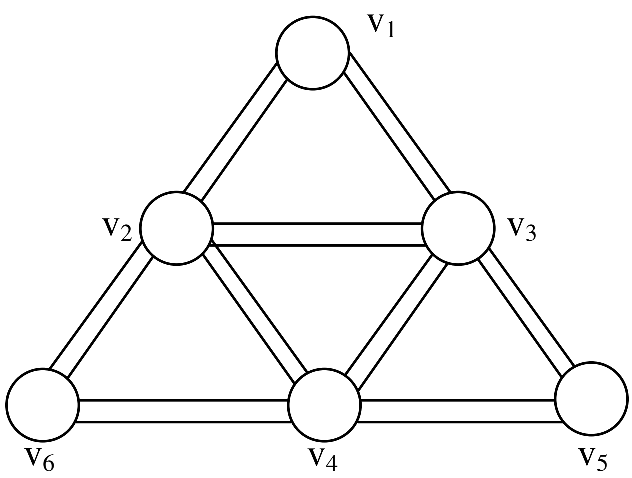


Figure 4.51

A multigraph G is a uniform multiclique graph if and only if it contains a family F of uniform complete multisubgraphs whose union is G, such that every pair of such uniform complete multigraphs is some subfamily F′ have a nonempty intersection, the intersection of all members of F′ is not empty.

We also call this as adjacent multitriads taken in a special way.

For every uniform multitriad is adjacent only with the uniform multitriad with vertex set $\{v_2, v_3, v_4\}$.

Can the multigraph G has uniform clique multigraph given by H?

H =

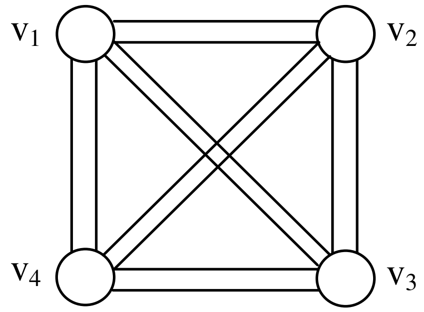


Figure 4.52

Is $H \approx K_4$?

K_4 =

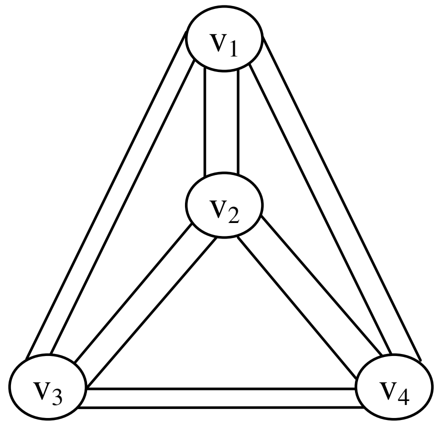


Figure 4.53

On similar lines we can define non uniform clique multigraph. This task is left as an exercise for the reader.

We leave it for the reader to define or develop new notions in multigraphs. Most of the results for the graphs can be developed to multigraphs with appropriate modifications. Finally we wish to state if in a multigraph the edges are marked then how to represent the adjacency matrix of the same.

We will illustrate this situation by some examples.

Example 4.28. Let G be a edge and vertex labeled multigraph with maximum 5 edges and 6 vertices given by the following figure. The edges are numbered for each number edge has a channel only through that edge and we represent the adjacency matrix M of that G.

G =

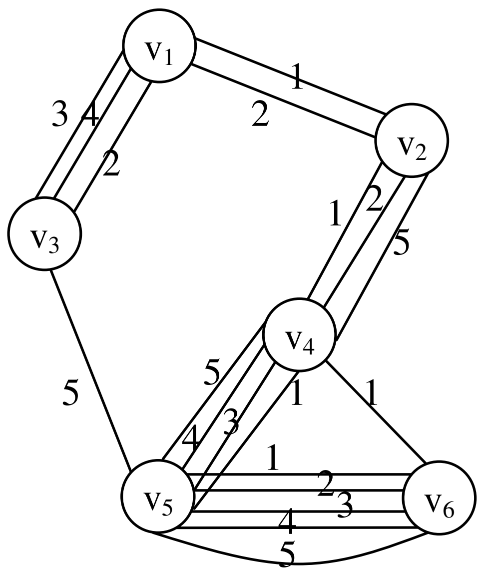


Figure 4.54

We now give the adjacency matrix M of G.

Adjacency matrix of this multigraph has its entries from the vector set $(x_1, x_2, x_3, x_4, x_5)$; $x_i \in \{0, 1\}$; $1 \leq i \leq 5$.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ (0,0,0,0,0) & (1,1,0,0,0) & (0,1,1,1,0) & (0,0,0,0,0) \\ (1,1,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (1,1,0,0,1) \\ (0,1,1,1,0) & (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (1,1,0,0,1) & (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,1) & (1,0,1,1,1) \\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) & (1,0,0,0,0) \end{array}$$

$$\begin{array}{cc} v_5 & v_6 \\ (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,1) & (0,0,0,0,0) \\ (1,0,1,1,1) & (1,0,0,0,0) \\ (0,0,0,0,0) & (1,1,1,1,1) \\ (1,1,1,1,1) & (0,0,0,0,0) \end{array}.$$

Clearly the diagonal entries are the zero vector (0,0,0,0,0) only.

However it is interesting to note that if the edges or the channels are not numbered we get the usual adjacency matrix N of the multigraph which is as follows;

$$N = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 0 & 2 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}.$$

We see both the adjacency matrices M and N are symmetric 6 × 6 matrices whose diagonal entries are zero. However they are related in this manner. We see the entry of v_1v_3 is (0, 1, 1, 1, 0) in M and that of in N is 3, thus sum of the non zero components in M is also 3. This is the way M and N are related.

Now we have two types of multiwalk, multipath, multitrot and multitrail with same route will be defined if and only if the multipath or multitort or multitrail take the same numbered edge.

In case of mixed number edge we may have multipath and sometimes same number edge multipath may not exist.

Thus we give the generalized result or definition of row adjacency matrix when the edges are numbered or labeled.

Let G be a multigraph with n vertices $v_1, v_2, \ldots, v_n$ and let the maximum number of edges be m. Then what is the form of the adjacency matrix we associate with it.

Let B_r denote the vector elements with row matrix adjacency associated with the multigraph G.

i) B_r is a symmetric n × n matrix with rows and the diagonal elements are zero vectors.

ii) Elements of B_r are $(x_1^{ij}, x_2^{ij}, \ldots, x_n^{ij})$ such that $x_t^{ij} \in \{0, 1\}$; $1 \leq t \leq m$.

iii)

$$
B_r = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}
\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \cdots & v_{n-1} \quad\quad v_n \end{array} \\
\left[\begin{array}{ccccc}
(x_1^{11}\; x_2^{11} \ldots x_m^{11}) & (x_1^{12}\; x_2^{12} \ldots x_m^{12}) & \cdots & (x_1^{1(n-1)}\; x_2^{1(n-1)} \ldots x_m^{1(n-1)} & (x_1^{1n}\; x_2^{1n} \ldots x_m^{1n}) \\
(x_i^{21}\; x_2^{21} \ldots x_m^{21}) & (x_1^{22}\; x_2^{22} \ldots x_m^{22}) & & (x_1^{2(n-1)}\; x_2^{2(n-1)} \ldots x_m^{2(n-1)} & (x_1^{2n}\; x_2^{2n} \ldots x_m^{2n}) \\
(x_1^{31}\; x_2^{31} \ldots x_m^{31}) & (x_1^{32}\; x_2^{32} \ldots x_m^{32}) & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(x_1^{n1}\; x_2^{n1} \ldots x_m^{n1}) & (x_1^{n2}\; x_2^{n2} \ldots x_m^{n2}) & \cdots & (x_1^{(n(n-1)}\; x_2^{n(n-1)} \ldots x_m^{n(n-1)}) & (x_1^{nn}\; x_2^{nn} \ldots x_m^{nn})
\end{array}\right]
\end{array}
$$

where the (ij) will correspond to i^{th} row and j^{th} column and the elements in B_r will be of the form $(x_1^{ij}, x_2^{ij}, \ldots x_m^{ij})$ where $x_k^{ij} \in \{0, 1\}$, $i \neq j$, and $1 \leq i, j \leq n$; $1 \leq k \leq m$ if $i = j$ then $x_k^{ii} = 0$; for $1 \leq k \leq m$. $1 \leq i \leq n$.

This is the most general form.

In case of the directed multigraph with ordered edges we give an example.

Example 4.29. In the first place we say maximum number edges in direction is 3 so that the number of edges any two adjacent nodes can have is at most 6.

Let G be a multigraph with maximum 3 edges the figure is given in the following.

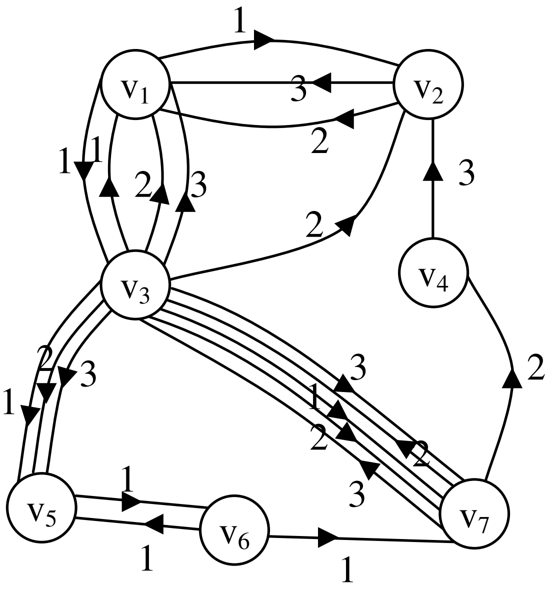
G = 

Figure 4.55

We see the maximum number of edges in G is 5 however the edges are only 3 barring direction. So it is enough we if take (x_1, x_2, x_3), with $x_i \in \{0, 1\}$. The adjacency matrix associated with this multigraph G is as follows.

D_r =

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	(000)	(100)	(100)	(000)	(000)	(000)	(000)
v_2	(011)	(000)	(000)	(000)	(000)	(000)	(000)
v_3	(111)	(010)	(000)	(000)	(111)	(000)	(111)
v_4	(000)	(001)	(000)	(000)	(000)	(000)	(000)
v_5	(000)	(000)	(000)	(000)	(000)	(100)	(000)
v_6	(000)	(000)	(000)	(000)	(100)	(000)	(100)
v_7	(000)	(000)	(011)	(010)	(000)	(000)	(000)

We see D_r the adjacency matrix of G is not symmetric or skew symmetric. In view of this we can describe the adjacency matrix of a directed multigraph whose edges are also labeled.

Let G be a directed multigraph with maximum m edges labeled as 1, 2, …, m.

Let G have the vertices $v_1, v_2, \ldots, v_n$. The row adjacency matrix associated with G is as follows.

$$D_r =$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \left[\begin{array}{cc} v_1 & v_2 \\ (x_1^{11}, x_2^{11}, \ldots, x_m^{11}) & (x_1^{12}, \ldots, x_m^{12}) \\ (x_1^{21}, x_2^{21}, \ldots, x_m^{21}) & (x_1^{22}, x_2^{22}, \ldots, x_m^{22}) \\ \vdots & \vdots \\ (x_1^{(n-1)1}, x_2^{(n-1)1}, \ldots, x_m^{(n-1)1}) & (x_1^{(n-1)} x_2^{(n-1)2}, \ldots, x_m^{(n-1)2}) \\ (x_1^{n1}, x_2^{n1}, \ldots, x_m^{n1}) & (x_1^{n2}, x_2^{n2}, \ldots, x_m^{n2}) \end{array} \right.$$

$$\left. \begin{array}{ccc} & v_{n-1} & v_n \\ \cdots & (x_1^{1(n-1)}, x_2^{1(n-1)}, \ldots, x_m^{(1(n-1)}) & (x_1^{1n}, x_2^{1n}, \ldots, x_m^{1n})) \\ \cdots & (x_1^{2(n-1)}, x_2^{2(n-1)}, \ldots, x_m^{2(n-1)}) & (x_1^{2n}, x_2^{2n}, \ldots, x_m^{2n}) \\ \vdots & \vdots & \vdots \\ & (x_1^{(n-1),(n-1)}, \ldots, x_{2,\ldots,x_m}^{(n-1)(n-1)}) & (x_1^{(n-1)n}, x_2^{(n-1)n}, \ldots, x_m^{(n-1)n}) \\ & (x_1^{n(n-1)}, x_2^{n(n-1)}, \ldots, x_m^{n(n-1)}) & (x_1^{nn}, x_2^{nn}, \ldots, x_n^{nn}) \end{array} \right]$$

We see $(x_1^{ii}, x_2^{ii}, \ldots, x_m^{ii}) = (0\ 0\ 0\ 0\ \ldots\ 0)$ for all $1 \leq i \leq n$. $x_k^{ij} \in \{0, 1\}$; $i \neq j$; $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

Clearly in general $(x_1^{ij}, x_2^{ij}, \ldots, x_m^{ij}) \neq (x_1^{ji}, x_2^{ii}, \ldots, x_m^{ii})$; $(i \neq j)$ $1 \leq i, j \leq n$.

In this way we obtain a special way of defining the adjacency matrix in case of multigraphs which are such that both edges and vertices are labeled.

This sort of networking (multigraphs) will be a boon in case of mathematical models which built using Plithogenic set and Neutrosophic sets which is under press.

Finally we can for these multigraphs find the multipath, multiwalk etc. This task is left as an exercise to the reader.

Next we describe and define cliques for bipartite multigraphs.

Example 4.30. Let G be the bipartite multigraph given by the following figure.

G =

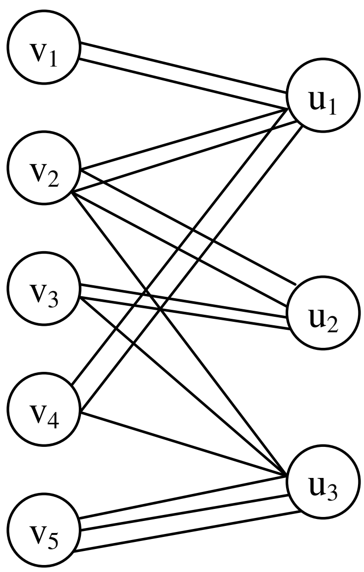


Figure 4.56

The adjacency matrix M of the multibipartite graph G is as follows;

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \left[\begin{array}{ccc} 2 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right] \end{array}.$$

We see it is 5×3 matrix. Each entry denotes the number of edges between v_i and v_j. $1 \leq i \leq 5$ and $1 \leq j \leq 3$.

Now we give the adjacency matrix of a edge and vertex labeled bipartite multigraph by the following example.

Just before we provide the justification for using numbered edges. We need numbered edges as we basically use these multigraphs as multinetwork in which case each vertex will serve as concepts so a vertex may be associated with some say t concepts t a positive non zero integer. If some r concepts r $\geq$ t are associated with each of these concepts.

When we have from a vertex v_i to v_j ($i \neq j$) there may be some m number of concepts which are common between v_i and v_j. Then in that case we number them as 1 to m. Some v_k ($k \neq i$, $k \neq j$) may have some p number of concepts common between v_i and v_k so we number them from same 1 to m only.

This is the vital reason for one to have the labeled edges. When we say labeled edges unlike vertices they may be named the same in many cases. The numbering may repeat itself from 1 to a fixed number say m.

Thus the maximum number of edges which any two vertices can enjoy will be less than m if the multinetwork is built using m concepts and each vertex v_i can take less than m and only one vertex can take m for the vertices are assumed to be distinct and labeled distinctly (only in case of non bipartite multigraphs.

Now we illustrate this for bipartite multigraph G by an example.

Example 4.31. Let G be a bipartite multigraph given by the following figure.

G =

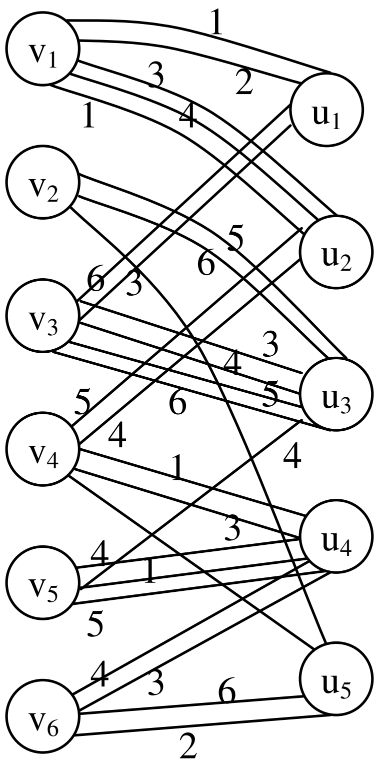


Figure 4.57

Here the vertex sets v_1, v_2, …, v_6 and u_1, u_2, …, u_5 take some concepts from the set $\{c_1, c_2, c_3, c_4, c_5, c_6\} = C$, so that each vertex takes a subset from C. Thus we can at most have only 6 edges between any vertices. For both u_i or v_j can take 6 edges. This is the special case of bipartite multigraphs for general multigraphs which are not bipartite this is not true for such an assumption will make two of the vertices identical.

Now we give the adjacency matrix M of G.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \left[\begin{array}{ccccc} (1,1,0,0,0,0) & (1,0,1,1,0,0) & (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,0,1,1) & (0,0,0,0,0,0) & (0,0,0,0,0,1) \\ (0,0,1,0,0,1) & (0,0,0,0,0,0) & (0,0,1,1,1,1) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (0,0,0,1,1,0) & (0,0,0,0,0,0) & (1,0,1,0,0,0) & (0,0,0,1,0,0) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,1,0,0) & (1,1,0,0,1,0) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,1,1,0,0) & (0,1,0,0,0,1) \end{array}\right] \end{array}$$

This is the way adjacency matrix of a labeled edge and labeled vertex multigraphs which is bipartite are carried out.

We suggest a few problems for the reader.

Problems

1. For the multigraph G given by the following figure.

G =

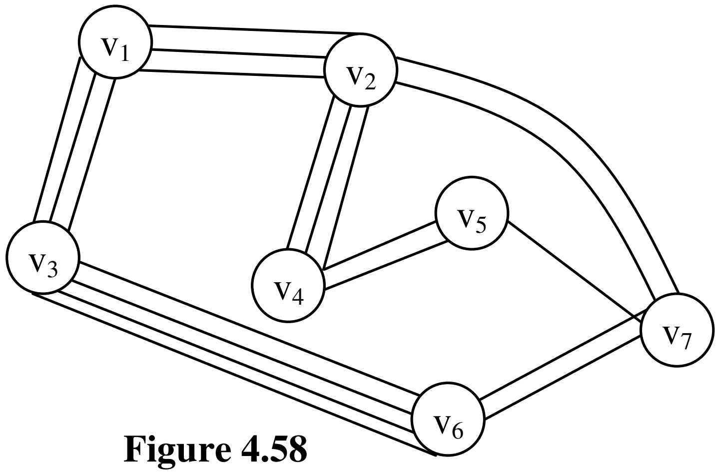


Figure 4.58

i) Find the multipaths from v_1 to v_7.

ii) Is it unique?

iii) Find a multiwalk from v_1 to v_7

iv) Find a multitrail from v_1 to v_7. Compare all the four.

v) Can G have a multicycle? Justify.

2. Let G be a multidirected graph given by the following figure;

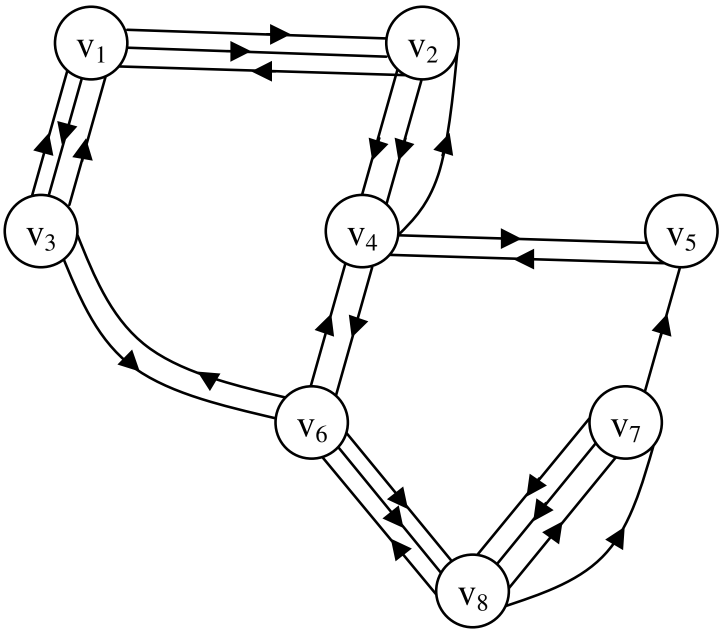


Figure 4.59

(Note for a multiwalk or multipath of multitrail or a multicycle we need they should follow the same direction).

i) Does there exists a multiwalk from v_1 to v_7

ii) Does there exist a multiwalk from v_2 to v_5?

iii) Does there exist a multipath from v_2 to v_5?

iv) Is there a unique multitrail from v_1 to v_7?

v) Find a shortest multipath from v_3 to v_5.

vi) Does there exist a multicycle from v_3 routing v_4 and v_7?

vii) Does there exist a multipath from v_1 to v_5?

viii) Does there exist a multiwalk?

ix) Obtain all multiwalks which are multitrails in G.

3. Let G be directed edge and vertex labeled multigraph with maximum 5 edges and the edges marked from 1 to 5 given by the following figure.

G =

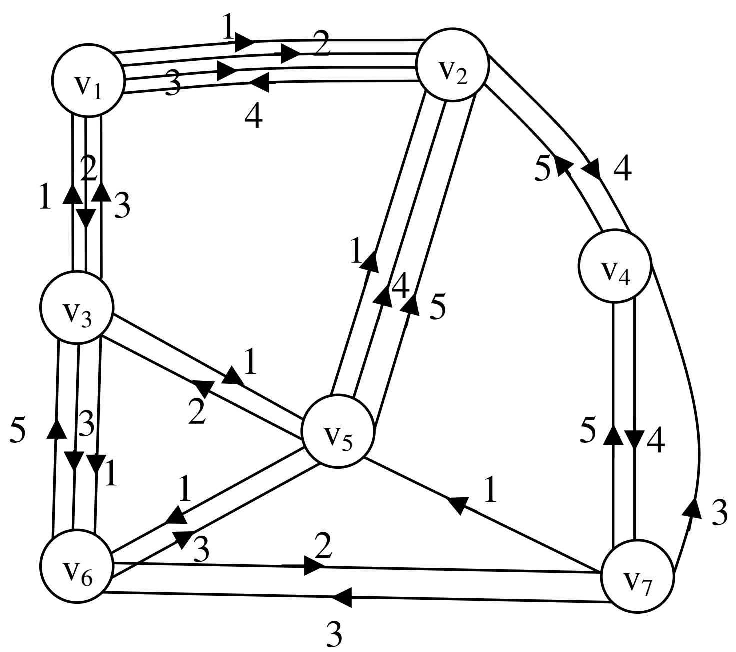


Figure 4.60

i) Does there exist a same multipath from v_1 to v_7?

ii) Does there exist a same multiwalk from v_1 to v_7?

iii) Does there exist a mixed multiwalk from v_1 to v_7?

iv) Find all same multipaths from node v_iv_j, $i \neq j$; which has atmost 2 edges.

v) Find all same multicycles from v_7 to v_7.

4. Let G be a multitree which is directed given by the following figure.

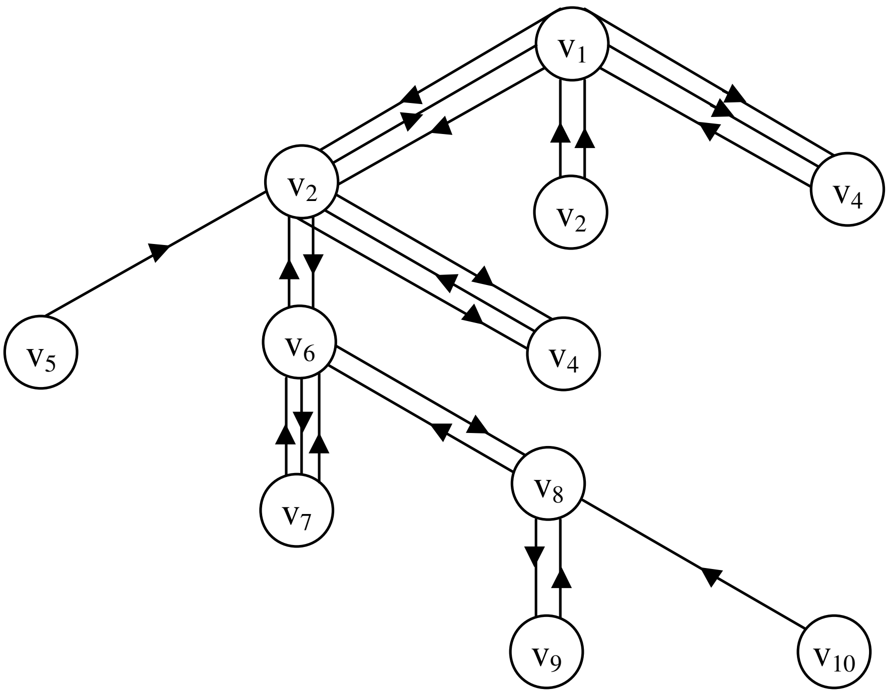


Figure 4.61

i) Does there exist a multi path from v_4 to v_9?

ii) Does there exist a multiwalk from v_2 to v_7?

iii) Does there exist a multitrail from v_4 to v_{10}?

iv) Does the multitrail from v_4 to v_{10} coincide with the multipath and multiwalk?

v) Show there is no multicycle in G (Justify your claim).

vi) Can there be a multitrot from v_3 to v_{10}?

5. Let G be a multigraph given b the following figure.

G =

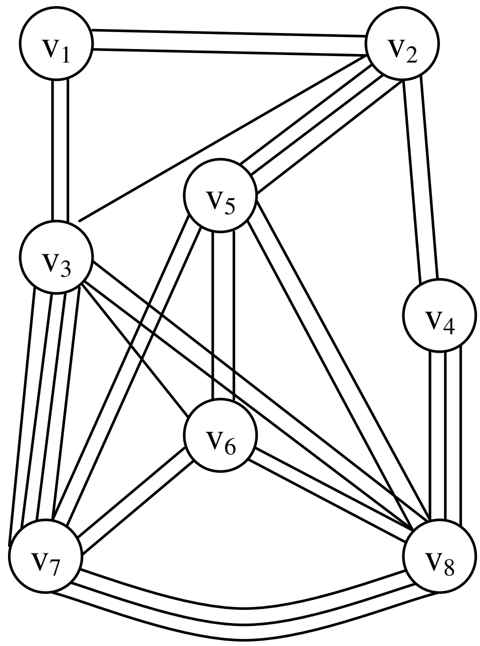


Figure 4.62

i) Does G contain uniform multiclique of order 5?

ii) Find all complete uniform multisubgraphs of order 3, 4 and 5.

iii) Does G contain nonuniform multisubgraphs of order 3, 4 and 5?

iv) Does G contain a non uniform star multisubgraph?

v) Does G contain a non uniform multiring subgraph?

vi) Find the number uniform multitriads.

vii) Find the number of non uniform multitriads.

viii) Find the adjacency matrix of G.

ix) Find the distance matrix of G.

6. Let G be a multigraph which is a multiwheel given by the following figure;

G =

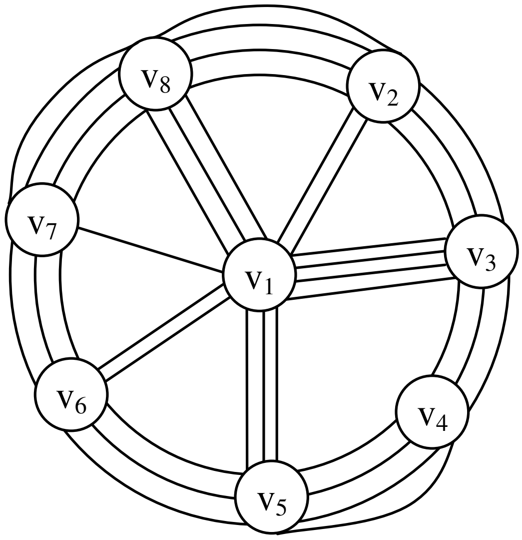


Figure 4.63

Study questions (i) to (ix) of problem 5. (We call G a non uniform multiwheel).

7. Let G be a multibipartite graph given by the following figure.

G =

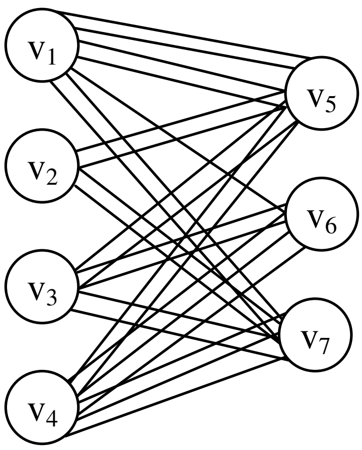


Figure 4.64

i) Is G a uniform or non uniform complete multigraph?

ii) Can G have multisubgraphs which are uniform complete?

iii) Give the adjacency matrix of this bipartite multigraph G.

8. Let G be a directed multigraph which is uniform complete given by the following figure.

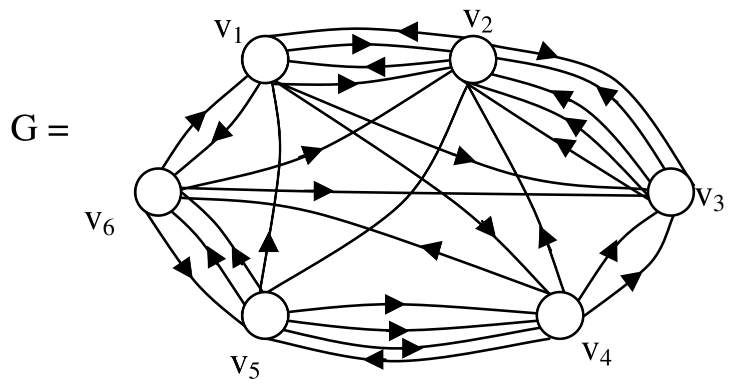


Figure 4.65

i) Find the adjacency matrix M of G.

ii) Is M symmetric?

iii) What is the maximum number edges between v_i and v_j; $i \neq j$; $1 \leq i,j \leq 6$?

iv) Give the distance matrix of this multigraph G.

v) Can this G have uniform multiclique?

(Note the give multigraph is only nonuniform complete so if this G has a multisubgraph which is a uniform complete multisubgraph H then we say that multisubgraph H of G as that multisubgraph H of G as the uniform multiclique).

9. Let G be a multigraph given by the following figure.

G =

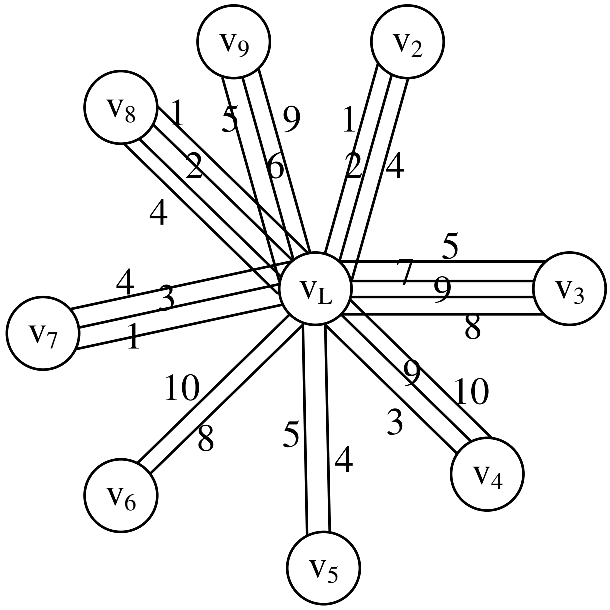


Figure 4.66

i) Find the adjacency matrix of the multistargraph G.

ii) Can G have a distance matrix? Justify your claim.

iii) Find all edge - multisubgraphs of G

iv) Find all vertex - multisubgraphs of G.

v) Give all multisubgraphs of G which are neither vertex multisubgraphs nor edge removed multisubgraphs.

10. Let G be a multigraph given by the following figure

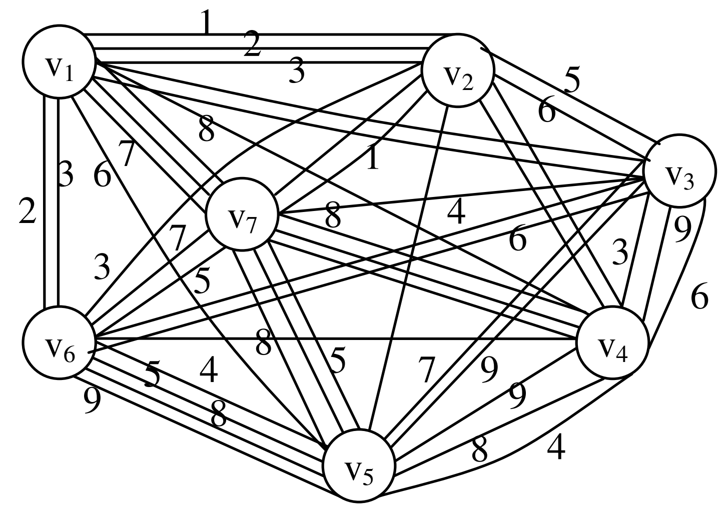


Figure 4.67

i) Study questions (i) to (v) problem 9 for this G.

ii) Is the adjacency matrix of G symmetric? Justify!

iii) How many uniform multitriad are there in G?

iv) Does G contain uniform complete multisubgraphs of order four?

v) Does G contain uniform complete multisubgraphs of order five and order?

vi) Does there exist a same multiedge path from v_1 to v_7?

vii) Find the number of same edge multiwalk in G.

11. Let G be a multigraph given by the following figure;

G =

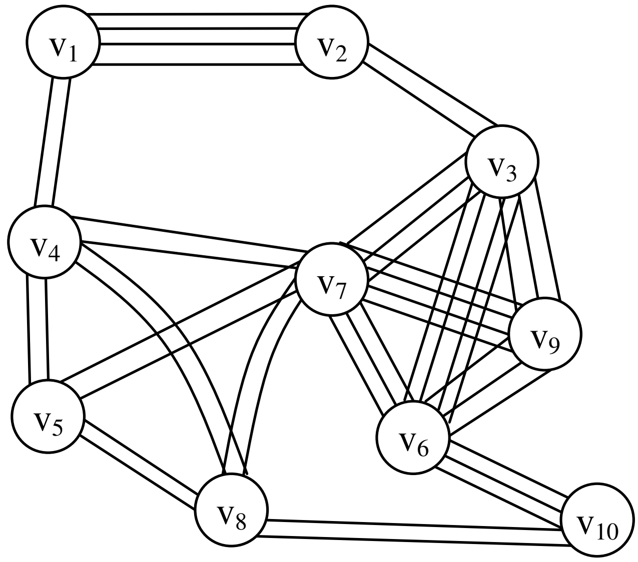


Figure 4.68

i) Find all vertex multisubgraphs of G.

ii) Find all edge multisubgraphs (edge multisubgraphs are those got from the multigraph G) of G.

iii) Find all multisubgraphs which are not vertex multisubgraphs or edge multisubgraphs of G.

iv) Find all uniform multicliques of G.

v) Does G contain nonuniform cliques?

vi) Find the adjacency matrix of G.

vii) Find the distance matrix of G

viii) Obtain any other special feature enjoyed by G.

12. Let G be a multigraph given by the following figure.

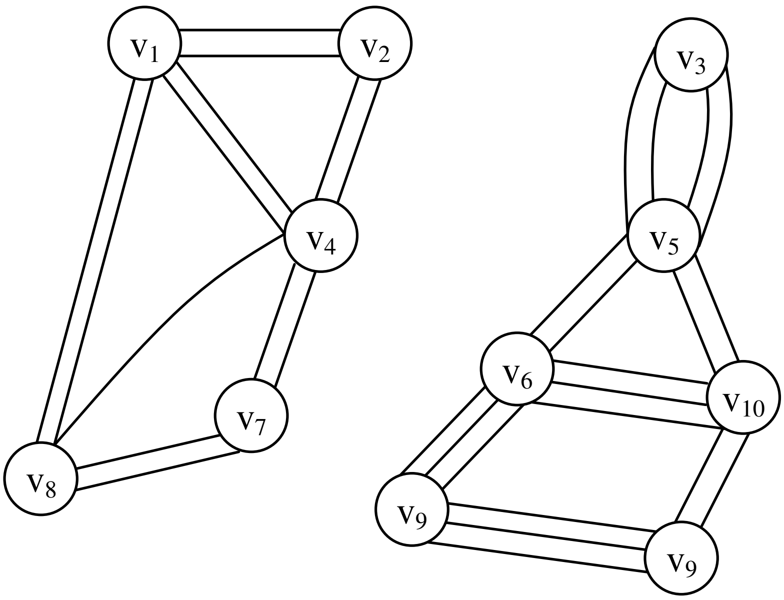
G = 

Figure 4.69

i) Study questions (i) to (vii) of problem 11.

ii) Will all vertex multisubgraphs of G be disconnected?

iii) Will all edge multisubgraphs of G be disconnected?

iv) Can multisubgraphs of G have more than two components?

v) Obtain any other special feature associated with this G.

13. Give some applications of multigraphs as multi networks.

14. Show by a suitable example (real world) where multigraphs are used in social information networks.

15. Can we characterize those multigraphs which has always a uniform multiclique?

16. Characterize those multigraphs which cannot have uniform or non-uniform multicliques.

17. Show by a real world example multigraphs are best suited for mathematical models which has multi networks.

18. Let G be a multigraph given by the following figure.

G =

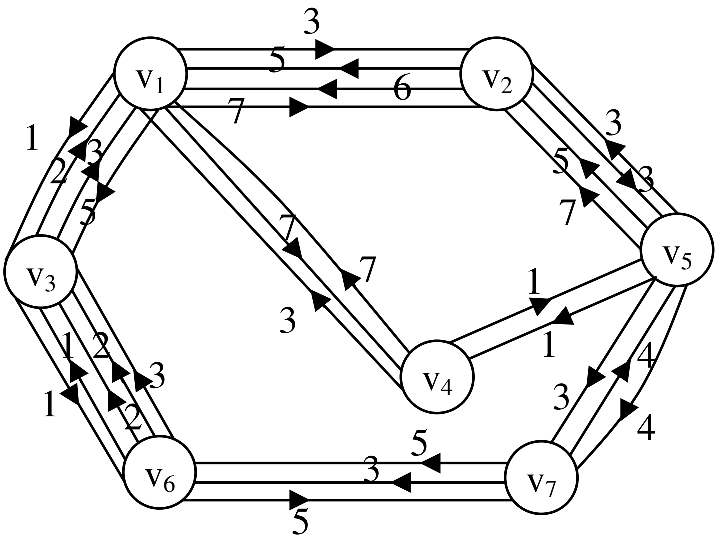


Figure 4.70

i) Find the adjacency matrix M of the multigraph G and prove M is not symmetric.

ii) Find the ring multisubgraph of G.

iii) Prove G has not uniform or nonuniform cliques.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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Study of multigraphs happen to be a challenging one for several concepts which are easily derivable in graphs cannot be carried out in case of multigraphs. Research on unconditional line graphs and line multigraphs will yield interesting results. Multigraphs find applications in transportation networks, communication networks and social information networks.

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