Jozsef Sándor

GEOMETRIC THEOREMS AND ARITHMETIC FUNCTIONS
(COLLECTED PAPERS)

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On Smarandache's Podaire Theorem

Let A', B', C' be the feet of the altitudes of an acute-angled triangle ABC ($A' \in BC$, $B' \in AC$, $C' \in AB$). Let a', b', b' denote the sides of the podaire triangle $A'B'C'$. Smarandache's Podaire theorem [2] (see [1]) states that

$$\sum a'b' \leq \frac{1}{4} \sum a^2 \tag{1}$$

where a, b, c are the sides of the triangle ABC. Our aim is to improve (1) in the following form:

$$\sum a'b' \leq \frac{1}{3} \left(\sum a'\right)^2 \leq \frac{1}{12} \left(\sum a\right)^2 \leq \frac{1}{4} \sum a^2. \tag{2}$$

First we need the following auxiliary proposition.

Lemma. *Let p and p' denote the semi-perimeters of triangles ABC and $A'B'C'$, respectively. Then*

$$p' \leq \frac{p}{2}. \tag{3}$$

Proof. Since $AC' = b\cos A$, $AB' = c\cos A$, we get

$$C'B' = AB'^2 + AC'^2 - 2AB' \cdot AC' \cdot \cos A = a^2 \cos^2 A,$$

so $C'B' = a\cos A$. Similarly one obtains

$$A'C' = b\cos B, \quad A'B' = c\cos C.$$

Therefore

$$p' = \frac{1}{2} \sum A'B' = \frac{1}{2} \sum a\cos A = \frac{R}{2} \sum \sin 2A = 2R \sin A \sin B \sin C$$

(where R is the radius of the circumcircle). By $a = 2R\sin A$, etc. one has

$$p' = 2R\prod \frac{a}{2R} = \frac{S}{R},$$

where $S = area(ABC)$. By $p = \frac{S}{r}$ (r = radius of the incircle) we obtain

$$p' = \frac{r}{R}p. \tag{4}$$

Now, Euler's inequality $2r \le R$ gives relation (3).

For the proof of (2) we shall apply the standard algebraic inequalities

$$3(xy + xz + yz) \le (x + y + z)^2 \le 3(x^2 + y^2 + z^2).$$

Now, the proof of (2) runs as follows:

$$\sum a'b' \le \frac{1}{3}\left(\sum a'\right)^2 = \frac{1}{3}(2p')^2 \le \frac{1}{3}p^2 = \frac{1}{3}\frac{\left(\sum a\right)^2}{4} \le \frac{1}{4}\sum a^2.$$

Remark. Other properties of the podaire triangle are included in a recent paper of the author ([4]), as well as in his monograph [3].

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On the diophantine equation $a^2 + b^2 = 100a + b$

The numbers 1233 and 8833 have the curious property that $1233 = 12^2 + 33^2$ and $8833 = 88^2 + 33^2$. Let $\overline{xyzt}$ be a four-digit number satisfying this property, i.e. $\overline{xyzt} = \overline{xy}^2 + \overline{zt}^2$. By putting $a = \overline{xy}$, $b = \overline{zt}$, since $\overline{xyzt} = 100\overline{xy} + \overline{zt} = 100a + b$, we are led to the following diophantine equation:

$$a^2 + b^2 = 100a + b. \tag{5}$$

The above problem required a and b to have two digits, but we generally will solve this equation for all positive integers a and b.

By considering (1) as a quadratic equation in a, we can write

$$a_{1,2} = 50 \pm \sqrt{2500 + b - b^2}. \tag{6}$$

To have integer solutions, we must suppose that

$$2500 + b - b^2 = x^2 \tag{7}$$

for certain positive integer x, giving $a_{1,2} = 50 \pm x$.

By multiplying with 4 both sides of equation (3) we can remark that this transforms equation (3) into

$$(2x)^2 + (2b - 1)^2 = 10001. \tag{8}$$

It is well known that an equation of type $u^2 + v^2 = n$ ($n > 1$) has the number of solutions $4(\tau_1 - \tau_2)$, where τ_1 and τ_2 denote the number of divisors of n having the forms $4k+1$ and $4k+3$, respectively. Since $10001 = 137 \cdot 73$ and $137 = 4 \cdot 34 + 1$, $73 = 4 \cdot 18 + 1$, clearly $\tau_1 = 4$, $\tau_2 = 0$. Thus $u^2 + v^2 = 10001$ can have exactly $16 : 4 = 4$ positive solutions, giving two distinct solutions. Remarking that $73 = 3^2 + 8^2$, $137 = 11^2 + 4^2$, by the identities

$$(\alpha^2 + \beta^2)(u^2 + v^2) = (\alpha u - \beta v)^2 + (u\beta + \alpha v)^2 = (\beta u - \alpha v)^2 + (\alpha u + \beta v)^2,$$

we can deduce the relations $76^2 + 65^2 = 10001$, $100^2 + 1^2 = 10001$; implying $2x = 76$, $2b - 1 = 65$; $2x = 100$, $2b - 1 = 1$ respectively. For $x = 38$ and $b = 33$ we get the values $a_1 = 50 + 38 = 88$, $a_2 = 50 - 38 = 12$. For $x = 50$, $b = 1$ one has $a_1 = 100$, $a_2 = 0$. Therefore, all solutions in positive integers of equation (1) are $(a, b) = (12, 33)$; $(a, b) = (88, 33)$. These are exactly the numbers stated at the beginning of this note.

References

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On the least common multiple of the first n positive integers

1. A. Murthy [1] and F. Russo [2] recently have considered the sequence $(a(n))$, where $a(n) = [1, 2, \ldots, n]$ denotes the l.c.m. of the positive integers $1, 2, \ldots, n$.

We note that $a(n)$ has a long-standing and well known connection with the famous "prime-number theorem". Indeed, let Λ be the Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \ (p \text{ prime}) \\ 0, & \text{otherwise} \end{cases}$$

Put $\psi(x) = \sum_{n \le x} \Lambda(n)$, known as one of the Chebysheff's function. Now

$$\sum_{m \le n} \Lambda(m) = \sum_{p^k \le n} \log p = \log \prod_{p^k \le n} p.$$

Let k_p be the largest positive integer with $p^{k_p} \le n$. Then

$$\log \prod_{p \le n} p^{k_p} = \log a(n)$$

on the base of the known calculation of l.c.m. Therefore

$$a(n) = e^{\psi(n)} \tag{14}$$

where $e^x = \exp(x)$. By the equivalent formulation of the prime number-theorem one has $\frac{\psi(n)}{n} \to 1$ as $n \to \infty$, giving by (1):

$$\lim_{n \to \infty} \sqrt[n]{a(n)} = e. \tag{15}$$

Now, by Cauchy's test of convergence of series of positive terms, this gives immediately that

$$\sum_{n \ge 1} \frac{1}{a(n)} \quad \text{and} \quad \sum_{n \ge 1} \frac{a(n)}{n!} \tag{16}$$

are convergent series; the first one appears also as a problem in Niven-Zuckerman [3]. Problem 21.3.2 of [4] states that this series is irrational. A similar method shows that the second series is irrational, too.

2. Relation (2) has many interesting applications. For example, this is an important tool in the Apéry proof of the irrationality of $\zeta(3)$ (where ζ is the Riemann zeta function). For same methods see e.g. Alladi [7]. See also [8]. From known estimates for the function ψ, clearly one can deduce relations for $a(n)$. For example, Rosser and Schoenfeld [5] have shown that $\frac{\psi(x)}{x}$ takes its maximum at $x = 113$ and $\frac{\psi(x)}{x} < 1.03883$ for $x > 0$. Therefore $\left(\sqrt[n]{a(n)}\right)$ takes its greatest value for $n = 113$, and

$$\sqrt[n]{a(n)} < e^{1.03883} \quad \text{for all} \quad n \geq 1. \tag{17}$$

Costa Pereira [6] proved that $\frac{530}{531} < \frac{\psi(x)}{x}$ for $x \geq 70841$ and $\frac{\psi(x)}{x} < \frac{532}{531}$ for $x \geq 60299$; giving

$$e^{530/531} < \sqrt[n]{a(n)} < e^{532/531} \quad \text{for} \quad n \geq 70841. \tag{18}$$

A. Perelli [9] proved that if $N^{\theta+\varepsilon} < H \leq N$, then $\psi(x+H) - \psi(x) \sim H$ for almost all x ($\theta \in (0,1)$ is given), yielding:

$$\log \frac{a(n+H)}{a(n)} \sim H \quad \text{for almost all} \quad n, \tag{19}$$

for $N^{\theta+\varepsilon} < H \leq N$.

M. Nair [10] has shown by a new method that $\sum_{n \leq x} \psi(n) \geq \alpha x^2$ for all $x \geq x_0$, where $\alpha = 0.49517\ldots$; thus:

$$\sum_{m \leq n} \log a(m) \geq \alpha n^2 \quad \text{for} \quad n \geq n_0. \tag{20}$$

Let $\Delta(x) = \psi(x) - x$. Assuming the Riemann hypothesis, it can be proved that $\Delta(x) = O\left(\sqrt{x}\log^2 x\right)$; i.e.

$$\log a(n) - n = O\left(\sqrt{n}\log^2 n\right). \tag{21}$$

This is due to von Koch [11]. Let

$$D(x) = \frac{1}{x}\int_1^x |\Delta(t)|dt.$$

By the Riemann hypothesis, Cramér [12] proved that $D(x) = O\left(\sqrt{x}\right)$ and S. Knaponski [13] showed that

$$D(x) > \sqrt{x}\exp\left(-c\frac{\log x}{\log\log x} \cdot \log\log\log x\right).$$

Without any hypothesis, J. Pintz [14] proved that

$$D(x) > \frac{\sqrt{x}}{2200} \quad \text{for} \quad x > 2. \tag{22}$$

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On certain limits related to prime numbers

1. Let p_n denote the nth prime number. The famous prime number theorem states (in equivalent form) that

$$\frac{p_n}{n\log n} \to 1 \text{ as } n \to \infty. \tag{23}$$

(In what follows, for simplicity we will note $x_n \to a$ when $\lim\limits_{n\to\infty} x_n = a$). There are some immediate consequences of this relation, for example:

$$\frac{p_{n+1}}{p_n} \to 1; \tag{24}$$

$$\frac{\log p_n}{\log n} \to 1. \tag{25}$$

Without logarithms, (1) and (3) have the form

$$n^{n/p_n} \to e; \tag{26}$$

$$p_n^{1/\log n} \to e. \tag{27}$$

From (2) easily follows

$$\sqrt[n]{p_n} \to 1; \tag{28}$$

while (1) and (2) imply

$$\frac{p_{n+1}-p_n}{n\log n} \to 0. \tag{29}$$

In paper [1] there were stated a number of 106 Conjectures on certain inequalities related to (p_n). The above limits, combined with Stolz-Cesaro's theorem, Stirling's theorem on $n!$, simple inequalities imply the following relations (see [7], [8]):

$$\frac{\log n}{\dfrac{1}{p_1}+\ldots+\dfrac{1}{p_n}} \to \infty; \tag{30}$$

$$\frac{p_1+p_2+\ldots+p_n}{\dfrac{n(n+1)}{2}\log n} \to 1; \tag{31}$$

$$\frac{p_{[\log n]}}{\log p_n} \to \infty; \tag{32}$$

$$\sqrt[p_n]{p_{n+1}p_{n+2}} \to 1; \tag{33}$$

$$\frac{\sqrt[n]{p_1p_2\cdots p_n}}{n!} \to 0; \tag{34}$$

$$\frac{p_{(n+1)!} - p_{n!}}{np_n} \to \infty; \tag{35}$$

$$\frac{p_{n!}}{(p_n)!} \to 0; \tag{36}$$

$$\frac{p_{n!}}{p_1p_2\cdots p_n} \to 0; \tag{37}$$

$$\frac{p_{(n+1)!} - p_{n!}}{(p_{n+1} - p_n)!} \to \infty; \tag{38}$$

$$\frac{p_1 + p_2 + \ldots + p_n}{p_{1!} + p_{2!} + \ldots + p_{n!}} \to 0; \tag{39}$$

$$\frac{\log\log p_{n+1} - \log\log p_n}{\log p_{n+1} - \log p_n} \to 0; \tag{40}$$

$$\frac{1}{p_n}\log\frac{e^{p_{n+1}} - e^{p_n}}{p_{n+1} - p_n} \to 1; \tag{41}$$

$$\limsup\left(\frac{p_{p_{n+1}} - p_{p_n}}{p_{n+1} - p_n}\right) = +\infty; \tag{42}$$

$$\liminf\left(\sqrt[3]{p_{n+1}p_{n+2}} - \sqrt[3]{p_np_{n+1}}\right) = 0; \tag{43}$$

$$\liminf\left(p_{[\sqrt{n+1}]} - p_{[\sqrt{n}]}\right) = 0; \tag{44}$$

$$\limsup\left(p_{[\sqrt{n+1}]} - p_{[\sqrt{n}]}\right) = \infty; \tag{45}$$

$$\liminf p_n^\lambda\left(\sqrt{p_{n+1}} - \sqrt{p_n}\right) = 0 \quad \left(\lambda \in \left(0, \frac{1}{2}\right)\right); \tag{46}$$

$$\limsup p_n^{1-\frac{1}{k}}\left(\sqrt[k]{p_{n+1}} - \sqrt[k]{p_n}\right) = +\infty \quad (k \geq 2,\ k \in \mathbb{N}), \quad \text{etc.} \tag{47}$$

With the use of these limits, a number of conjectures were shown to be false or trivial. On the other hand, a couple of conjectures are very difficult at present. Clearly, (24) implies

$$\liminf\frac{p_{n+1} - p_n}{\sqrt{p_n}} = 0. \tag{48}$$

A famous unproved conjecture of Cramér [3] states that

$$\liminf\frac{p_{n+1} - p_n}{(\log p_n)^2} = 1. \tag{49}$$

If this is true, clearly one can deduce that

$$\limsup\frac{p_{n+1} - p_n}{(\log p_n)^2} \leq 1. \tag{50}$$

Even

$$\limsup \frac{p_{n+1} - p_n}{(\log p_n)^2} < \infty \tag{51}$$

seems very difficult. A conjecture of Schinzel [2] states that between x and $x + (\log x)^2$ there is always a prime. This would imply $p_n < p_{n+1} < p_n + (\log p_n)^2$, so

$$\frac{p_{n+1} - p_n}{\sqrt{p_n}} \to 0. \tag{52}$$

Probably, this is true. A result of Huxley [4] says that with the notation $d_n = p_{n+1} - p_n$ one has $d_n < p_n^{\frac{7}{12}+\varepsilon}$ ($\varepsilon > 0$), and the Riemann hypothesis would imply $d_n < p_n^{\frac{1}{2}+\varepsilon}$. Even these statements wouldn't imply (30). Erdös and Turán [5] have proved that $\frac{d_{n+1}}{d_n} > 1$ for infinitely many n, while $\frac{d_{m+1}}{d_m} < 1$ for infinitely many m; probably

$$\limsup \frac{d_{n+1}}{d_n} = +\infty \tag{53}$$

is true.

2. In [12] it is shown that

$$\log p_n - \frac{p_n}{n} \to 1. \tag{54}$$

Therefore

$$\log p_{n+1} - \frac{p_{n+1}}{n+1} - \log p_n + \frac{p_n}{n} \to 0,$$

so by putting $x_n = \frac{p_{n+1}}{n+1} - \frac{p_n}{n}$, by $\log p_{n+1} - \log p_n \to 0$, we get

$$x_n \to 0. \tag{55}$$

Thus

$$|x_n| \to 0, \tag{56}$$

implying $|x_n| \leq 1/2$ for sufficiently large n. This settles essentially conjecture 81 of [1] (and clearly, improves it, for large n). Now, by a result of Erdös and Prachar [6] one has

$$c_1 \log^2 p_n < \sum_{m=1}^{n} |x_m| < c_2 \log^2 p_n$$

($c_1, c_2 > 0$ constants), so we obtain

$$\limsup \left(\frac{|x_1| + \ldots + |x_n|}{\log^2 p_n} \right) < \infty; \tag{57}$$

$$\liminf \left(\frac{|x_1| + \ldots + |x_n|}{\log^2 p_n} \right) > 0; \tag{58}$$

it would be interesting to obtain more precise results. By applying the arithmetic-geometric inequality, one obtains

$$\limsup \frac{n}{\log^2 p_n} |x_1 x_2 \ldots x_n|^{1/n} < \infty. \tag{59}$$

What can be said on lim inf of this expression?

3. In paper [11] there are stated ten conjectures on prime numbers. By the following limits we can state that the inequalities stated there are true for all sufficiently large values of n. By Huxley's result (for certain improvements, see [2]),

$$\frac{n^\alpha d_n}{p_{n+1} p_n} < \frac{n^\alpha}{n^{5/12-\varepsilon} (\log n)^{5/12-\varepsilon}} \to 0,$$

so if $\alpha < 5/12 - \varepsilon$, we have

$$\frac{p_{n+1} - p_n}{p_{n+1} + p_n} < n^{-\alpha} \tag{60}$$

for sufficiently large n. This is related to conjecture 2 of [11].

We now prove that

$$\frac{n^{\log p_{n+1}}}{(n+1)^{\log p_n}} \to 1, \tag{61}$$

this settles conjecture 7 for all large n, since $\frac{1}{2^{\log 2}} < 1$ and $\frac{30^{\log 127}}{31^{\log 113}} > 1$. In order to prove (39), remark that the expression can be written as $\left(\frac{n}{n+1}\right)^{\log p_n} \cdot \left(n^{\log p_{n+1} - \log p_n}\right)$. Now,

$$\left(\frac{n+1}{n}\right)^{\log p_n} = \left[\left(\frac{n+1}{n}\right)^{\log n}\right]^{\log p_n / \log n} \to 1^1 = 1,$$

since

$$\left(\frac{n+1}{n}\right)^{\log n} = \left[\left(1 + \frac{1}{n}\right)^n\right]^{\frac{\log n}{n}} \to e^0 = 1$$

and apply relation (3). Therefore, it is sufficient to prove

$$n^{\log p_{n+1} - \log p_n} \to 1. \tag{62}$$

By Lagrange's mean value theorm applied to the function $t \mapsto \log t$ on $t \in [p_n, p_{n+1}]$ we easily can deduce

$$\frac{p_{n+1} - p_n}{p_{n+1}} < \log p_{n+1} - \log p_n < \frac{p_{n+1} - p_n}{p_n}.$$

Therefore, it is sufficient to prove

$$n^{(p_{n+1}-p_n)/p_n} \to 1; \tag{63}$$

$$n^{(p_{n+1}-p_n)/p_{n+1}} \to 1. \tag{64}$$

By (2), (42) follows from (41). Now, for (41) it is enough to prove (by taking logarithms) that $\frac{p_{n+1}-p_n}{p_n}\log n \to 0$, or, by using (1); that

$$\frac{p_{n+1}-p_n}{n} \to 0. \tag{65}$$

This is stronger than (7), but it is true, and follows clearly e.g. by $d_n < n^{7/12+\varepsilon}$. This finishes the proof of (39).

Conjectures (8) and (10) of [11] are clearly valid for sufficiently large n, since

$$\frac{\sqrt{p_{n+1}}-\log p_{n+1}}{\sqrt{p_n}-\log p_n} \to 1 \tag{66}$$

and

$$\frac{\sqrt{p_n}-\log p_{n+1}}{\sqrt{p_{n+1}}-\log p_n} \to 1. \tag{67}$$

Indeed,

$$\frac{\sqrt{p_{n+1}}\left(1-\log p_{n+1}/\sqrt{p_{n+1}}\right)}{\sqrt{p_n}\left(1-\log p_n/\sqrt{p_n}\right)} \to 1\cdot\left(\frac{1-0}{1-0}\right)=1, \quad \text{etc.}$$

Now, conjecture (9) is true for large n, if one could prove that

$$\frac{(\log p_{n+1})^{\sqrt{p_n}}}{(\log p_n)^{\sqrt{p_{n+1}}}} \to 1. \tag{68}$$

Since this expression can be written as $\left(\frac{\log p_{n+1}}{\log p_n}\right)^{\sqrt{p_n}}(\log p_n)^{\sqrt{p_n+1}-\sqrt{p_n}}$, we will prove first that

$$(\log p_n)^{\sqrt{p_{n+1}}-\sqrt{p_n}} \to 1. \tag{69}$$

By logarithmation,

$$(\sqrt{p_{n+1}}-\sqrt{p_n})\log\log p_n = \frac{d_n}{\sqrt{p_n}+\sqrt{p_{n+1}}}\log\log p_n < \frac{p_n^{7/12+\varepsilon}}{2\sqrt{p_n}}\log\log p_n \to 0,$$

so indeed (47) follows.

Now, the limit

$$\left(\frac{\log p_{n+1}}{\log p_n}\right)^{\sqrt{p_n}} \to 1 \tag{70}$$

seems difficult. By taking logarithms, $\sqrt{p_n}\log\left(\frac{\log p_{n+1}}{\log p_n}\right) \to 0$ will follow, if we suppose that

$$\log\left(\frac{\log p_{n+1}}{\log p_n}\right) < \frac{1}{n} \tag{71}$$

is true for sufficiently large n. This is exactly conjecture 6 of [11]. Now, by (49) we get (48), since clearly $\frac{\sqrt{p_n}}{n} \to 0$ (e.g. by (1)). Therefore one can say that conjecture 6 implies conjecture 9 in [11] (for large values of n).

4. I can prove that Conjecture 6 holds true for infinitely many n, in fact a slightly stronger result is obtainable. The logarithmic mean $L(a,b)$ of two positive numbers a, b is defined by

$$L(a,b) = \frac{b-a}{\log b - \log a}.$$

It is well-known that (see e.g. [13])

$$\sqrt{ab} < L(a,b) < \frac{a+b}{2}.$$

Thus

$$\log\left(\frac{\log p_{n+1}}{\log p_n}\right) = \log(\log p_{n+1}) - \log(\log p_n) < \frac{\log p_{n+1} - \log p_n}{\sqrt{\log p_n \log p_{n+1}}} <$$

$$< \frac{p_{n+1}-p_n}{\sqrt{p_n p_{n+1}\log p_n \log p_{n+1}}} < \frac{p_{n+1}-p_n}{\log p_n}\cdot\frac{1}{p_n} = \frac{b_n}{p_n}.$$

Now, if

$$b_n < \frac{p_n}{n}, \tag{72}$$

then Conjecture 6 is proved. The sequence (b_n) has a long history. It is known (due to Erdös) that $b_n < 1$ for infinitely many n. Since $\frac{p_n}{n} > 1$, clearly (50) holds for infinitely many n. It is not known that

$$\liminf b_n = 0, \tag{73}$$

but we know that

$$\limsup b_n = +\infty. \tag{74}$$

The relation

$$\frac{b_1 + b_2 + \ldots + b_n}{n} \to 1 \tag{75}$$

is due to L. Panaitopol, many other results are quoted in [9].

Remarks. 1) Conjecture 5, i.e. $\log d_n < n^{3/10}$ is true for large n by Huxley's result.

2) Conjectures 3 and 8 (left side) are completely settled by other methods ([10]).

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On a Generalized Bisector Theorem

In the book [1] by Smarandache (see also [2]) appears the following generalization of the well-known bisector theorem.

Let AM be a cevian of the triangle which forms the angles u and v with the sides AB and AC, respectively. Then

$$\frac{AB}{AC} = \frac{MB}{MC} \cdot \frac{\sin v}{\sin u}. \tag{76}$$

We wish to mention here that relation (1) **also** appeared in my book [3] on page 112, where it is used for a generalization of Steiner's theorem. Namely, the following result holds true (see Theorem 25 in page 112):

Let AD and AE be two cevians ($D, E \in (BC)$) forming angles α, β with the sides AB, AC, respectively. If $\widehat{A} \leq 90°$ and $\alpha \leq \beta$, then

$$\frac{BD \cdot BE}{CD \cdot CE} \leq \frac{AB^2}{AC^2}. \tag{77}$$

Indeed, by applying the area resp. trigonometrical formulas of the area of a triangle, we get

$$\frac{BD}{CD} = \frac{A(ABD)}{A(ACD)} = \frac{AB \sin \alpha}{AC \sin(A - \alpha)}$$

(i.e. relation (1) with $u = \alpha$, $v = \beta - \alpha$). Similarly one has

$$\frac{BE}{CE} = \frac{AB \sin(A - \beta)}{AC \sin \beta}.$$

Therefore

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin(A - \beta)}{\sin(A - \alpha)}. \tag{78}$$

Now, identity (3), by $0 < \alpha \leq \beta < 90°$ and $0 < A - \beta \leq A - \alpha < 90°$ gives immediately relation (2). This solution appears in [3]. For $\alpha = \beta$ one has

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \tag{79}$$

which is the classical Steiner theorem. When $D \equiv E$, this gives the well known bisector theorem.

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On certain conjectures by Russo

In a recent note [1] F. Russo published ten conjectures on prime numbers. Here we prove three of them. (For solutions of other conjectures for large n, see [2]).

Conjecture 3 is the following:

$$e^{\sqrt{\frac{n+1}{p_{n+1}}}}/e^{\sqrt{\frac{p_n}{n}}} < e^{\sqrt{\frac{3}{5}}}/e^{\sqrt{\frac{3}{2}}} \tag{80}$$

Written equivalently as

$$e^{\sqrt{\frac{n+1}{p_{n+1}}}+\sqrt{\frac{3}{2}}} < e^{\sqrt{\frac{p_n}{n}}+\sqrt{\frac{3}{5}}},$$

we have to prove that

$$\sqrt{\frac{n+1}{p_{n+1}}}+\sqrt{\frac{3}{2}} < \sqrt{\frac{p_n}{n}}+\sqrt{\frac{3}{5}}. \tag{81}$$

For $n \le 16$, (2) can be verified by calculations. Now, let $n \ge 17$. Then $p_n > 3n$. Indeed, $p_{17} = 53 > 3 \cdot 17 = 51$. Assuming this inequality to be valid for n, one has $p_{n+1} \ge p_n + 2 > 3n + 2$ so $p_{n+1} \ge 3n + 3 = 3(n+1)$. But $3(n+1)$ is divisible by 3, so $p_{n+1} > 3(n+1)$. Since $\frac{n+1}{p_{n+1}} \le \frac{1}{3}$, it is sufficient to prove that

$$\sqrt{3}+\sqrt{\frac{3}{5}} > \sqrt{\frac{3}{2}}+\sqrt{\frac{1}{3}},$$

i.e. $3+\frac{3}{\sqrt{5}} > \frac{3}{\sqrt{2}}+1$ or $2 > 3\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{5}}\right)$, i.e. $2\sqrt{10} > 3\left(\sqrt{5}-\sqrt{2}\right)$. This is easily seen to be true. Therefore (2), i.e. (1) is proved.

Remark. The proof shows that (2) is valid whenever a sequence (p_n) of positive integers satisfies $p_n > 3n$.

Conjecture 5 is

$$\log d_n - \log\sqrt{d_n} < \frac{1}{2}n^{3/10}, \quad \text{where} \quad d_n = p_{n+1} - p_n. \tag{82}$$

By $\log\sqrt{d_n} = \frac{1}{2}\log d_n$, (3) can be written as

$$\log d_n < n^{3/10}. \tag{83}$$

It is immediate that (4) holds for sufficiently large n since $d_n < p_n$ and $\log p_n \sim \log n$ $(n \to \infty)$ while $\log n < n^{3/10}$ for sufficiently large n. Such arguments appear in [2].

Now we completely prove the left side of **conjecture 8**. We will prove a stronger relation, namely

$$\frac{\sqrt{p_{n+1}} - \log p_{n+1}}{\sqrt{p_n} - \log p_n} > 1 \quad (n \geq 3) \tag{84}$$

Since $\dfrac{\sqrt{3} - \log 3}{\sqrt{2} - \log 2} < 1$, (5) will be an improvement. The logarithmic mean of two positive numbers is

$$L(a, b) = \frac{b - a}{\log b - \log a}.$$

It is well-known that $L(a, b) > \sqrt{ab}$ for $a \neq b$. Now let $a = p_{n+1}$, $b = p_n$. Then $\sqrt{ab} > \sqrt{a} + \sqrt{b}$ is equivalent to $\sqrt{p_{n+1}}\left(\sqrt{p_n} - 1\right) > \sqrt{p_n}$. If $\sqrt{p_n} - 1 \geq 1$, i.e. $p_n \geq 4$ $(n \geq 3)$, this is true. Now,

$$\frac{p_{n+1} - p_n}{\log p_{n+1} - \log p_n} > \sqrt{p_n p_{n+1}} > \sqrt{p_n} + \sqrt{p_{n+1}}$$

gives

$$\frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} > \log p_{n+1} - \log p_n,$$

i.e.

$$\sqrt{p_{n+1}} - \log p_{n+1} > \sqrt{p_n} - \log p_n.$$

This is exactly inequality (5). We can remark that (5) holds true for any strictly increasing positive sequence such that $p_n \geq 4$.

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On values of arithmetical functions at factorials I

1. The Smarandache function is a characterization of factorials, since $S(k!) = k$, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \geq 1 \text{ given}) \tag{85}$$

has $d(k!) - d((k-1)!)$ solutions, where $d(n)$ denotes the number of divisors of n. This follows from $\{x : S(x) = k\} = \{x : x|k!, \ x \nmid (k-1)!\}$. Thus, equation (1) always has at least a solution, if $d(k!) > d((k-1)!)$ for $k \geq 2$. In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions $\varphi, \sigma, d, \omega, \Omega$ at factorials. Here $\varphi(n)$ = Euler's arithmetical function, $\sigma(n)$ = sum of divisors of n, $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of total divisors of n. As it is well known, we have $\varphi(1) = d(1) = 1$, while $\omega(1) = \Omega(1) = 0$, and for $1 < \prod_{i=1}^{r} p_i^{a_i}$ ($a_i \geq 1$, p_i distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i,$$

$$d(n) = \prod_{i=1}^{r} (a_i + 1). \tag{86}$$

The functions φ, σ, d are multiplicative, ω is additive, while Ω is totally additive, i.e. φ, σ, d satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m, n) = 1$, while ω, Ω satisfy the equation $g(mn) = g(m) + g(n)$ for $(m, n) = 1$ in case of ω, and for all m, n is case of Ω (see [1]).

2. Let $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $n = \prod_{i=1}^{r} p_i^{\beta_i}$ ($\alpha_i, \beta_i \geq 0$) be the canonical factorizations of m and n. (Here some α_i or β_i can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r} (\beta_i + 1)$$

with equality only if $\alpha_i = 0$ for all i. Thus:

$$d(mn) \geq d(n) \tag{87}$$

for all m, n, with equality only for $m = 1$.

Since $\prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \leq \prod_{i=1}^{r}(\alpha_i + 1)\prod_{i=1}^{r}(\beta_i + 1)$, we get the relation

$$d(mn) \leq d(m)d(n) \tag{88}$$

with equality only for $(n, m) = 1$.

Let now $m = k$, $n = (k-1)!$ for $k \geq 2$. Then relation (3) gives

$$d(k!) > d((k-1)!) \text{ for all } k \geq 2, \tag{89}$$

thus proving the assertion that equation (1) always has at least a solution (for $k = 1$ one can take $x = 1$).

With the same substitutions, relation (4) yields

$$d(k!) \leq d((k-1)!)d(k) \text{ for } k \geq 2 \tag{90}$$

Let $k = p$ (prime) in (6). Since $((p-1)!, p) = 1$, we have equality in (6):

$$\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime}. \tag{91}$$

3. Since $S(k!)/k! \to 0$, $\frac{S(k!)}{S((k-1)!)} = \frac{k}{k-1} \to 1$ as $k \to \infty$, one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

$$\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty. \tag{92}$$

In fact, this follows from $\sigma(k) = \sum_{d|k} d = \sum_{d|k} \frac{k}{d}$, so

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,$$

as it is known.

From the known inequality ([1]) $\varphi(n)\sigma(n) \leq n^2$ it follows

$$\frac{n}{\varphi(n)} \geq \frac{\sigma(n)}{n},$$

so $\frac{n!}{\varphi(n!)} \to \infty$, implying

$$\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{93}$$

Since $\varphi(n) > d(n)$ for $n > 30$ (see [2]), we have $\varphi(n!) > d(n!)$ for $n! > 30$ (i.e. $n \geq 5$), so, by (9)

$$\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{94}$$

In fact, much stronger relation is true, since $\frac{d(n)}{n^\varepsilon} \to 0$ for each $\varepsilon > 0$ $(n \to \infty)$ (see [1]). From $\frac{d(n!)}{n!} < \frac{\varphi(n!)}{n!}$ and the above remark on $\sigma(n!) > n! \log n$, it follows that

$$\limsup_{n\to\infty} \frac{d(n!)}{n!} \log n \leq 1. \tag{95}$$

These relations are obtained by very elementary arguments. From the inequality $\varphi(n)(\omega(n)+1) \geq n$ (see [2]) we get

$$\omega(n!) \to \infty \text{ as } n \to \infty \tag{96}$$

and, since $\Omega(s) \geq \omega(s)$, we have

$$\Omega(n!) \to \infty \text{ as } n \to \infty. \tag{97}$$

From the inequality $nd(n) \geq \varphi(n) + \sigma(n)$ (see [2]), and (8), (9) we have

$$d(n!) \to \infty \text{ as } n \to \infty. \tag{98}$$

This follows also from the known inequality $\varphi(n)d(n) \geq n$ and (9), by replacing n with $n!$. From $\sigma(mn) \geq m\sigma(n)$ (see [3]) with $n = (k-1)!$, $m = k$ we get

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \geq k \quad (k \geq 2) \tag{99}$$

and, since $\sigma(mn) \leq \sigma(m)\sigma(n)$, by the same argument

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \leq \sigma(k) \quad (k \geq 2). \tag{100}$$

Clearly, relation (15) implies

$$\lim_{k\to\infty} \frac{\sigma(k!)}{\sigma((k-1)!)} = +\infty. \tag{101}$$

From $\varphi(m)\varphi(n) \le \varphi(mn) \le m\varphi(n)$, we get, by the above remarks, that

$$\varphi(k) \le \frac{\varphi(k!)}{\varphi((k-1)!)} \le k, \quad (k \ge 2) \tag{102}$$

implying, by $\varphi(k) \to \infty$ as $k \to \infty$ (e.g. from $\varphi(k) > \sqrt{k}$ for $k > 6$) that

$$\lim_{k\to\infty} \frac{\varphi(k!)}{\varphi((k-1)!} = +\infty. \tag{103}$$

By writing $\sigma(k!) - \sigma((k-1)!) = \sigma((k-1)!) \left[\frac{\sigma(k!)}{\sigma((k-1)!)} - 1\right]$, from (17) and $\sigma((k-1)!) \to \infty$ as $k \to \infty$, we trivially have:

$$\lim_{k\to\infty} [\sigma(k!) - \sigma((k-1)!)] = +\infty. \tag{104}$$

In completely analogous way, we can write:

$$\lim_{k\to\infty} [\varphi(k!) - \varphi((k-1)!)] = +\infty. \tag{105}$$

4. Let us remark that for $k = p$ (prime), clearly $((k-1)!, k) = 1$, while for $k =$ composite, all prime factors of k are also prime factors of $(k-1)!$. Thus

$$\omega(k!) = \begin{cases} \omega((k-1)!k) = \omega((k-1)!) + \omega(k) & \text{if } k \text{ is prime} \\ \omega((k-1)!) & \text{if } k \text{ is composite } (k \ge 2). \end{cases}$$

Thus

$$\omega(k!) - \omega((k-1)!) = \begin{cases} 1, & \text{for } k = \text{prime} \\ 0, & \text{for } k = \text{composite} \end{cases} \tag{106}$$

Thus we have

$$\begin{aligned} &\limsup_{k\to\infty}[\omega(k!) - \omega((k-1)!)] = 1 \\ &\liminf_{k\to\infty}[\omega(k!) - \omega((k-1)!)] = 0 \end{aligned} \tag{107}$$

Let p_n be the nth prime number. From (22) we get

$$\frac{\omega(k!)}{\omega((k-1)!)} - 1 = \begin{cases} \dfrac{1}{n-1}, \text{if } k = p_n \\ 0, \text{if } k = \text{composite}. \end{cases}$$

Thus, we get

$$\lim_{k\to\infty} \frac{\omega(k!)}{\omega((k-1)!)} = 1. \tag{108}$$

The function Ω is totally additive, so

$$\Omega(k!) = \Omega((k-1)!k) = \Omega((k-1)!) + \Omega(k),$$

giving

$$\Omega(k!) - \Omega((k-1)!) = \Omega(k). \tag{109}$$

This implies

$$\limsup_{k\to\infty} [\Omega(k!) - \Omega((k-1)!)] = +\infty \tag{110}$$

(take e.g. $k = 2^m$ and let $m \to \infty$), and

$$\liminf_{k\to\infty} [\Omega(k!) - \Omega((k-1)!)] = 2$$

(take $k =$ prime).

For $\Omega(k!)/\Omega((k-1)!)$ we must evaluate

$$\frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}.$$

Since $\Omega(k) \le \dfrac{\log k}{\log 2}$ and by the theorem of Hardy and Ramanujan (see [1]) we have

$$\sum_{n\le x} \Omega(n) \sim x \log\log x \quad (x \to \infty)$$

so, since $\dfrac{\log k}{(k-1)\log\log(k-1)} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \tag{111}$$

5. Inequality (18) applied for $k = p$ (prime) implies

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!)} = 1. \tag{112}$$

This follows by $\varphi(p) = p - 1$. On the other hand, let $k > 4$ be composite. Then, it is known (see [1]) that $k|(k-1)!$. So $\varphi(k!) = \varphi((k-1)!k) = k\varphi((k-1)!)$, since $\varphi(mn) = m\varphi(n)$ if $m|n$. In view of (28), we can write

$$\lim_{k\to\infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \tag{113}$$

For the function σ, by (15) and (16), we have for $k = p$ (prime) that $p \leq \frac{\sigma(p!)}{\sigma((p-1)!)} \leq \sigma(p) = p + 1$, yielding

$$\lim_{p \to \infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p-1)!)} = 1. \tag{114}$$

In fact, in view of (15) this implies that

$$\liminf_{k \to \infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k-1)!)} = 1. \tag{115}$$

By (6) and (7) we easily obtain

$$\limsup_{k \to \infty} \frac{d(k!)}{d(k)d((k-1)!)} = 1. \tag{116}$$

In fact, inequality (6) can be improved, if we remark that for $k = p$ (prime) we have $d(k!) = d((k-1)!) \cdot 2$, while for k = composite, $k > 4$, it is known that $k|(k-1)!$. We apply the following

Lemma. *If $n|m$, then*

$$\frac{d(mn)}{d(m)} \leq \frac{d(n^2)}{d(n)}. \tag{117}$$

Proof. Let $m = \prod p^{\alpha} \prod q^{\beta}$, $n = \prod p^{\alpha'}$ $(\alpha' \leq \alpha)$ be the prime factorizations of m and n, where $n|m$. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha + \alpha' + 1) \prod(\beta + 1)}{\prod(\alpha + 1) \prod(\beta + 1)} = \prod \left(\frac{\alpha + \alpha' + 1}{\alpha + 1} \right).$$

Now $\frac{\alpha + \alpha' + 1}{\alpha + 1} \leq \frac{2\alpha' + 1}{\alpha' + 1} \Leftrightarrow \alpha' \leq \alpha$ as an easy calculations verifies. This immediately implies relation (33).

By selecting now $n = k$, $m = (k-1)!$, $k > 4$ composite we can deduce from (33):

$$\frac{d(k!)}{d((k-1)!)} \leq \frac{d(k^2)}{d(k)}. \tag{118}$$

By (4) we can write $d(k^2) < (d(k))^2$, so (34) represents indeed, a refinement of relation (6).

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On the Irrationality of Certain Constants Related to the Smarandache Function

1. Let $S(n)$ be the Smarandache function. Recently I. Cojocaru and S. Cojocaru [2] have proved the irrationality of $\sum_{n=1}^{n} \frac{S(n)}{n!}$.

The author of this note [5] showed that this is a consequence of an old irrationality criteria (which will be used here once again), and proved a result implying the irrationality of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

E. Burton [1] has studied series of type $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$, which has a value $\in \left(e - \frac{5}{2}, \frac{1}{2}\right)$. He showed that the series $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$ is convergent for all $r \in \mathbf{N}$. I. Cojocaru and S. Cojocaru [3] have introduced the "third constant of Smarandache" namely $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)}$, which has a value between $\frac{71}{100}$ and $\frac{97}{100}$. Our aim in the following is to prove that the constants introduced by Burton and Cojocaru-Cojocaru are all irrational.

2. The first result is in fact a refinement of an old irraionality criteria (see [4] p.5):

Theorem 1. *Let (x_n) be a sequence of nonnegative integers having the properties:*

(1) there exists $n_0 \in \mathbf{N}^*$ such that $x_n \le n$ for all $n \ge n_0$;

(2) $x_n < n-1$ for infinitely many n;

(3) $x_m > 0$ for an infinity of m.

Then the series $\sum_{n=1}^{\infty} \frac{x_n}{n!}$ is irrational.

Let now $x_n = S(n-1)$. Then

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} = \sum_{n=3}^{\infty} \frac{x_n}{n!}.$$

Here $S(n-1) \le n-1 < n$ for all $n \ge 2$; $S(m-1) < m-2$ for $m > 3$ composite, since by $S(m-1) < \frac{2}{3}(m-1) < m-2$ for $m > 4$ this holds true. (For the inequality $S(k) < \frac{2}{3}k$ for $k > 3$ composite, see [6]). Finally, $S(m-1) > 0$ for all $m \ge 1$. This proves the irrationality of $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$.

Analogously, write

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} = \sum_{m=r+2}^{\infty} \frac{S(m-r)}{m!}.$$

Put $x_m = S(m-r)$. Here $S(m-r) \le m-r < m$, $S(m-r) \le m-r < m-1$ for $r \ge 2$, and $S(m-r) > 0$ for $m \ge r+2$. Thus, the above series is irrational for $r \ge 2$, too.

3. The third constant of Smarandache will be studied with the following irrationality criterion (see [4], p.8):

Theorem 2. *Let $(a_n), (b_n)$ be two sequences of nonnegative integers satisfying the following conditions:*

(1) $a_n > 0$ for an infinity of n;

(2) $b_n \ge 2$, $0 \le a_n \le b_n - 1$ for all $n \ge 1$;

(3) there exists an increasing sequence (i_n) of positive integers such that

$$\lim_{n\to\infty} b_{i_n} = +\infty, \quad \lim_{n\to\infty} a_{i_n}/b_{i_n} = 0.$$

Then the series $\sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n}$ is irrational.

Corollary. *For $b_n \ge 2$, (b_n positive integers), (b_n) unbounded the series $\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \dots b_n}$ is irrational.*

Proof. Let $a_n \equiv 1$. Since $\limsup_{n\to\infty} b_n = +\infty$, there exists a sequence (i_n) such that $b_{i_n} \to \infty$. Then $\frac{1}{b_{i_n}} \to 0$, and the three conditions of Theorem 2 are verified.

By selecting $b_n \equiv S(n)$, we have $b_p = S(p) = p \to \infty$ for p a prime, so by the above Corollary, the series $\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)\dots S(n)}$ is irrational.

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On certain generalizations of the Smarandache function

1. The famous Smarandache function is defined by $S(n) := \min\{k \in \mathbf{N}: \mathbf{n}|\mathbf{k}!\}$, $n \geq 1$ positive integer. This arithmetical function is connected to the number of divisors of n, and other important number theoretic functions (see e.g. [6], [7], [9], [10]). A very natural generalization is the following one: Let $f : \mathbf{N}^* \to \mathbf{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbf{N}^*$ there exists at least a $k \in \mathbf{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbf{N}^* \to \mathbf{N}^*$ defined by

$$F_f(n) = \min\{k \in \mathbf{N}: \mathbf{n}|\mathbf{f}(\mathbf{k})\}. \tag{1}$$

Since every subset of natural numbers is well-ordered, the definition (1) is correct, and clearly $F_f(n) \geq 1$ for all $n \in \mathbf{N}^*$.

Examples. 1) Let $id(k) = k$ for all $k \geq 1$. Then clearly (P_1) is satisfied, and

$$F_{id}(n) = n. \tag{2}$$

2) Let $f(k) = k!$. Then $F_!(n) = S(n)$ - the Smarandache function.

3) Let $f(k) = p_k!$, where p_k denotes the kth prime number. Then

$$F_f(n) = \min\{k \in \mathbf{N}^*: \mathbf{n}|\mathbf{p_k}!\}. \tag{3}$$

Here (P_1) is satisfied, as we can take for each $n \geq 1$ the least prime greater than n.

4) Let $f(k) = \varphi(k)$, Euler's totient. First we prove that (P_1) is satisfied. Let $n \geq 1$ be given. By Dirichlet's theorem on arithmetical progressions ([1]) there exists a positive integer a such that $k = an + 1$ is prime (in fact for infinitely many a's). Then clearly $\varphi(k) = an$, which is divisible by n.

We shall denote this function by F_φ. (4)

5) Let $f(k) = \sigma(k)$, the sum of divisors of k. Let k be a prime of the form $an - 1$, where $n \geq 1$ is given. Then clearly $\sigma(n) = an$ divisible by n. Thus (P_1) is satisfied. One obtains the arithmetical function F_σ. (5)

2. Let $A \subset \mathbf{N}^*$, $A \neq \emptyset$ a nonvoid subset of $\mathbf{N}$, having the property:

(P_2) For each $n \geq 1$ there exists $k \in A$ such that $n|k!$.

Then the following arithmetical function may be introduced:

$$S_A(n) = \min\{k \in A : \ n|k!\}. \tag{6}$$

Examples. 1) Let $A = \mathbf{N}^*$. Then $S_{\mathbf{N}}(n) \equiv S(n)$ - the Smarandache function.

2) Let $A = \mathbf{N_1}$ = set of odd positive integers. Then clearly (P_2) is satisfied. (7)

3) Let $A = \mathbf{N_2}$ = set of even positive integers. One obtains a new Smarandache-type function. (8)

4) Let $A = P$ = set of prime numbers. Then $S_P(n) = \min\{k \in P : \ n|k!\}$. We shall denote this function by $P(n)$, as we will consider more closely this function. (9)

3. Let $g : \mathbf{N}^* \to \mathbf{N}^*$ be a given arithmetical function. Suppose that g satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$. (10)

Let the function $G_g : \mathbf{N}^* \to \mathbf{N}^*$ be defined as follows:

$$G_g(n) = \max\{k \in \mathbf{N}^* : \ \mathbf{g(k)|n}\}. \tag{11}$$

This is not a generalization of $S(n)$, but for $g(k) = k!$, in fact one obtains a "dual"-function of $S(n)$, namely

$$G_!(n) = \max\{k \in \mathbf{N}^* : \ \mathbf{k!|n}\}. \tag{12}$$

Let us denote this function by $S_*(n)$.

There are many other particular cases, but we stop here, and study in more detail some of the above stated functions.

4. The function $P(n)$

This has been defined in (9) by: the least prime p such that $n|p!$. Some values are: $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11, \ldots$

Proposition 1. For each prime p one has $P(p) = p$, and if n is squarefree, then $P(n)$ = greatest prime divisor of n.

Proof. Since $p|p!$ and $p \nmid q!$ with $q < p$, clearly $P(p) = p$. If $n = p_1 p_2 \dots p_r$ is squarefree, with $p_1, \dots, p_r$ distinct primes, if $p_r = \max\{p_1, \dots, p_r\}$, then $p_1 \dots p_r | p_r!$. On the other hand, $p_1 \dots p_r \nmid q!$ for $q < p_r$, since $p_r \nmid q!$. Thus p_r is the least prime with the required property.

The calculation of $P(p^2)$ is not so simple but we can state the following result:

Proposition 2. One has the inequality $P(p^2) \geq 2p + 1$. If $2p + 1 = q$ is prime, then $P(p^2) = q$. More generally, $P(p^m) \geq mp + 1$ for all primes p and all integers m. There is equality, if $mp + 1$ is prime.

Proof. From $p^2|(1 \cdot 2 \dots p)(p+1) \dots (2p)$ we have $p^2|(2p)!$. Thus $P(p^2) \geq 2p+1$. One has equality, if $2p+1$ is prime. By writing $p^m|\underbrace{1 \cdot 2 \dots p}\underbrace{(p+1) \dots 2p} \dots \underbrace{[(m-1)p+1] \dots mp}$, where each group of p consecutive terms contains a member divisible by p, one obtains $P(p^m) \geq mp + 1$.

Remark. If $2p + 1$ is not a prime, then clearly $P(p^2) \geq 2p + 3$.

It is not known if there exist infinitely many primes p such that $2p + 1$ is prime too (see [4]).

Proposition 3. The following double inequality is true:

$$2p + 1 \leq P(p^2) \leq 3p - 1 \tag{13}$$

$$mp + 1 \leq P(p^m) \leq (m+1)p - 1 \tag{14}$$

if $p \geq p_0$.

Proof. We use the known fact from the prime number theory ([1], [8]) tha for all $a \geq 2$ there exists at least a prime between $2a$ and $3a$. Thus between $2p$ and $3p$ there is at least a prime, implying $P(p^2) \leq 3p - 1$. On the same lines, for sufficiently large p, there is a prime between mp and $(m+1)p$. This gives the inequality (14).

Proposition 4. For all $n, m \geq 1$ one has:

$$S(n) \leq P(n) \leq 2S(n) - 1 \tag{15}$$

and

$$P(nm) \leq 2[P(n) + P(m)] - 1 \tag{16}$$

where $S(n)$ is the Smarandache function.

Proof. The left side of (15) is a consequence of definitions of $S(n)$ and $P(n)$, while the right-hand side follows from Chebyshev's theorem on the existence of a prime between a and $2a$ (where $a = S(n)$, when $2a$ is not a prime).

For the right side of (16) we use the inequality $S(mn) \leq S(n) + S(m)$ (see [5]): $P(nm) \leq 2S(nm) - 1 \leq 2[S(n) + S(m)] - 1 \leq 2[P(n) + P(m)] - 1$, by (15).

Corollary.

$$\lim_{n\to\infty} \sqrt[n]{P(n)} = 1. \tag{17}$$

This is an easy consequence of (15) and the fact that $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$. (For other limits, see [6]).

5. The function $S_*(n)$

As we have seen in (12), $S_*(n)$ is in certain sense a dual of $S(n)$, and clearly $(S_*(n))!|n|(S(n))!$ which implies

$$1 \leq S_*(n) \leq S(n) \leq n \tag{18}$$

thus, as a consequence,

$$\lim_{n\to\infty} \sqrt[n]{\frac{S_*(n)}{S(n)}} = 1. \tag{19}$$

On the other hand, from known properties of S it follows that

$$\liminf_{n\to\infty} \frac{S_*(n)}{S(n)} = 0, \quad \limsup_{n\to\infty} \frac{S_*(n)}{S(n)} = 1. \tag{20}$$

For odd values n, we clearly have $S_*(n) = 1$.

Proposition 5. For $n \geq 3$ one has

$$S_*(n! + 2) = 2 \tag{21}$$

and more generally, if p is a prime, then for $n \geq p$ we have

$$S_*(n! + (p-1)!) = p - 1. \tag{22}$$

Proof. (21) is true, since $2|(n!+2)$ and if one assumes that $k!|(n!+2)$ with $k \geq 3$, then $3|(n!+2)$, impossible, since for $n \geq 3$, $3|n!$. So $k \leq 2$, and remains $k = 2$.

For the general case, let us remark that if $n \geq k+1$, then, since $k|(n!+k!)$, we have $S_*(n!+k!) \geq k$.

On the other hand, if for some $s \geq k+1$ we have $s!|(n!+k!)$, by $k+1 \leq n$ we get $(k+1)|(n!+k!)$ yielding $(k+1)|k!$, since $(k+1)|n!$. So, if $(k+1)|k!$ is not true, then we have

$$S_*(n!+k!) = k. \tag{23}$$

Particularly, for $k = p-1$ (p prime) we have $p \nmid (p-1)!$.

Corollary. For infinitely many m one has $S_*(m) = p-1$, where p is a given prime.

Proposition 6. For all $k, m \geq 1$ we have

$$S_*(k!m) \geq k \tag{24}$$

and for all $a, b \geq 1$,

$$S_*(ab) \geq \max\{S_*(a), S_*(b)\}. \tag{25}$$

Proof. (24) trivially follows from $k!|(k!m)$, while (25) is a consequence of $(S_*(a))!|a \Rightarrow (S_*(a))!|(ab)$ so $S_*(ab) \geq S_*(a)$. This is true if a is replaced by b, so (25) follows.

Proposition 7. $S_*[x(x-1)\dots(x-a+1)] \geq a$ for all $x \geq a$ (x positive integer).(26)

Proof. This is a consequence of the known fact that the product of $\underline{a}$ consecutive integers is divisible by $a!$.

We now investigate certain properties of $S_*(a!b!)$. By (24) or (25) we have $S_*(a!b!) \geq \max\{a, b\}$. If the equation

$$a!b! = c! \tag{27}$$

is solvable, then clearly $S_*(a!b!) = c$. For example, since $3! \cdot 5! = 6!$, we have $S_*(3! \cdot 5!) = 6$. The equation (27) has a trivial solution $c = k!$, $a = k!-1$, $b = k$. Thus $S_*(k!(k!-1)!) = k$.

In general, the nontrivial solutions of (27) are not known (see e.g. [3], [1]).

We now prove:

Proposition 8. $S_*((2k)!(2k+2)!) = 2k+2$, if $2k+3$ is a prime; (28)

$S_*((2k)!(2k+2)!) \geq 2k+4$, if $2k+3$ is not a prime. (29)

Proof. If $2k+3 = p$ is a prime, (28) is obvious, since $(2k+2)!|(2k)!(2k+2)!$, but

$(2k+3)! \nmid (2k)!(2k+2)!$. We shall prove first that if $2k+3$ is not prime, then

$$(2k+3)|(1 \cdot 2 \dots (2k)) \qquad (*)$$

Indeed, let $2k+3 = ab$, with $a, b \geq 3$ odd numbers. If $a < b$, then $a < k$, and from $2k+3 \geq 3b$ we have $b \leq \frac{2}{3}k + 1 < k$. So $(2k)!$ is divisible by ab, since a, b are distinct numbers between 1 and k. If $a = b$, i.e. $2k+3 = a^2$, then $(*)$ is equivalent with $a^2|(1 \cdot 2 \dots a)(a+1)\dots(a^2-3)$. We show that there is a positive integer k such that $a+1 < ka \leq a^2-3$ or. Indeed, $a(a-3) = a^2-3a < a^2-3$ for $a > 3$ and $a(a-3) > a+1$ by $a^2 > 4a+1$, valid for $a \geq 5$. For $a = 3$ we can verifiy $(*)$ directly. Now $(*)$ gives

$$(2k+3)!|(2k)!(2k+2)!, \text{ if } 2k+3 \neq \text{prime} \qquad (**)$$

implying inequality (29).

For consecutive odd numbers, the product of factorials gives for certain values

$$S_*(3! \cdot 5!) = 6, \quad S_*(5! \cdot 7!) = 8, \quad S_*(7! \cdot 9!) = 10,$$

$$S_*(9! \cdot 11!) = 12, \quad S_*(11! \cdot 13!) = 16, \quad S_*(13! \cdot 15!) = 16, \quad S_*(15! \cdot 17!) = 18,$$

$$S_*(17! \cdot 19!) = 22, \quad S_*(19! \cdot 21!) = 22, \quad S_*(21! \cdot 23!) = 28.$$

The following conjecture arises:

Conjecture. $S_*((2k-1)!(2k+1)!) = q_k - 1$, where q_k is the first prime following $2k+1$.

Corollary. From $(q_k-1)!|(2k-1)!(2k+1)!$ it follows that $q_k > 2k+1$. On the other hand, by $(2k-1)!(2k+1)!|(4k)!$, we get $q_k \leq 4k-3$. Thus between $2k+1$ and $4k+2$ there is at least a prime q_k. This means that the above conjecture, if true, is stronger than Bertrand's postulate (Chebyshev's theorem [1], [8]).

6. Finally, we make some remarks on the functions defined by (4), (5), other functions of this type, and certain other generalizations and analogous functions for further study, related to the Smarandache function.

First, consider the function F_φ of (4), defined by

$$F_\varphi = \min\{k \in \mathbf{N}^* : \ \mathbf{n}|\varphi(\mathbf{k})\}.$$

First observe that if $n+1 =$ prime, then $n = \varphi(n+1)$, so $F_\varphi(n) = n+1$. Thus

$$n+1 = \text{prime} \Rightarrow F_\varphi(n) = n+1. \quad (30)$$

This is somewhat converse to the φ-function property

$$n+1 = \text{prime} \Rightarrow \varphi(n+1) = n.$$

Proposition 9. Let ϕ_n be the nth cyclotomic polynomial. Then for each $a \geq 2$ (integer) one has

$$F_\varphi(n) \leq \phi_n(a) \text{ for all } n. \quad (31)$$

Proof. The cyclotomic polynomial is the irreducible polynomial of grade $\varphi(n)$ with integer coefficients with the primitive roots of order n as zeros. It is known (see [2]) the following property:

$$n|\varphi(\phi_n(a)) \text{ for all } n \geq 1, \text{ all } a \geq 2. \quad (32)$$

The definition of F_φ gives immediately inequality (31).

Remark. We note that there exist in the literature a number of congruence properties of the function φ. E.g. it is known that $n|\varphi(a^n - 1)$ for all $n \geq 1$, $a \geq 2$. But this is a consequence of (32), since $\phi_n(a)|a^n - 1$, and $u|v \Rightarrow \varphi(u)|\varphi(v)$ implies (known property of φ) what we have stated.

The most famous congruence property of φ is the following

Conjecture. (D.H. Lehmer (see [4])) If $\varphi(n)|(n-1)$, then $n =$ prime.

Another congruence property of φ is contained in Euler's theorem: $m|(a^{\varphi(m)} - 1)$ for $(a, m) = 1$. In fact this implies

$$S_*[a^{\varphi(m!)} - 1] \geq m \text{ for } (a, m!) = 1 \quad (33)$$

and by the same procedure,

$$S_*(\varphi(a^{n!} - 1)] \geq n \text{ for all } n. \quad (34)$$

As a corollary of (34) we can state that

$$\limsup_{k \to \infty} S_*[\varphi(k)] = +\infty. \quad (35)$$

(It is sufficient to take $k = a^{n!} - 1 \to \infty$ as $n \to \infty$).

7. In a completely similar way one can define $F_d(n) = \min\{k : n|d(k)\}$, where $d(k)$ is the number of distinct divisors of k. Since $d(2^{n-1}) = n$, one has

$$F_d(n) \leq 2^{n-1}. \tag{36}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be the canonical factorization of the number n. Then Smarandache ([9]) proved that $S(n) = \max\{S(p_1^{\alpha_1}), \dots, S(p_r^{\alpha_r})\}$.

In the analogous way, we may define the functions $S_\varphi(n) = \max\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S_\sigma(n) = \max\{\sigma(p_1^{\alpha_1}), \dots, \sigma(p_r^{\alpha_r})\}$, etc.

But we can define $S_\varphi^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, etc. For an arithmetical function f one can define

$$\Delta_f(n) = l.c.m.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}$$

and

$$\delta_f(n) = g.c.d.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}.$$

For the function $\Delta_\varphi(n)$ the following divisibility property is known (see [8], p.140, Problem 6).

If $(a, n) = 1$, then

$$n|[a^{\Delta_\varphi(n)} - 1]. \tag{37}$$

These functions and many related others may be studied in the near (or further) future.

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On an inequality for the Smarandache function

1. In paper [2] the author proved among others the inequality $S(ab) \leq aS(b)$ for all a, b positive integers. This was refined to

$$S(ab) \leq S(a) + S(b) \tag{119}$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1 a_2 \ldots a_n) \leq S(a_1) + S(a_2) + \ldots + S(a_n) \tag{120}$$

for all integers $a_i \geq 1$ $(i = 1, \ldots, n)$. When $a_1 = \ldots = a_n = n$ we obtain

$$S(a^n) \leq nS(a). \tag{121}$$

For three applications of this inequality, remark that

$$S((m!)^n) \leq nS(m!) = nm \tag{122}$$

since $S(m!) = m$. This is inequality 3) part 1. from [3]. By the same way, $S((n!)^{(n-1)!}) \leq (n-1)!S(n!) = (n-1)!n = n!$, i.e.

$$S((n!)^{(n-1)!}) \leq n! \tag{123}$$

Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by $S(n^2) \leq 2S(n) \leq n$ for n even (see [3], inequality 1), $n > 4$, we have obtained a refinement of $S(n^2) \leq n$:

$$S(n^2) \leq 2S(n) \leq n \tag{124}$$

for $n > 4$, even.

3. Let m be a divisor of n, i.e. $n = km$. Then (1) gives $S(n) = S(km) \leq S(m) + S(k)$, so we obtain:

If $m|n$, then

$$S(n) - S(m) \leq S\left(\frac{n}{m}\right). \tag{125}$$

As an application of (7), let $d(n)$ be the number of divisors of n. Since $\prod_{k|n} k = n^{d(n)/2}$, and $\prod_{k \leq n} k = n!$ (see [3]), and by $\prod_{k|n} k | \prod_{k \leq n} k$, from (7) we can deduce that

$$S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \geq n. \tag{126}$$

This improves our relation (10) from [3].

4. Let $S(a) = u$, $S(b) = v$. Then $b|v!$ and $u!|x(x-1)\dots(x-u+1)$ for all integers $x \geq u$. But from $a|u!$ we have $a|x(x-1)\dots(x-u+1)$ for all $x \geq u$. Let $x = u + v + k$ $(k \geq 1)$. Then, clearly $ab(v+1)\dots(v+k)|(u+v+k)!$, so we have $S[ab(v+1)\dots(v+k)] \leq u+v+k$. Here $v = S(b)$, so we have obtained that

$$S[ab(S(b)+1)\dots(S(b)+k)] \leq S(a) + S(b) + k. \tag{127}$$

For example, for $k = 1$ one has

$$S[ab(S(b)+1)] \leq S(a) + S(b) + 1. \tag{128}$$

This is not a consequence of (2) for $n = 3$, since $S[S(b)+1]$ may be much larger than 1.

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On multiplicatively deficient and abundant numbers

Definition 28 of [1] introduces the so-called "impotent numbers" n whose proper divisors product is less than n. It is mentioned there that the sequence of these numbers contains terms with the forms p and p^2, where p is a prime.

Let $T(n)$ denote the product of all divisors of n. Then $T(n) = n^2$ iff n is a multiplicatively-perfect (or shortly m-perfect) number. In a recent paper [2] we have studied these numbers or, for example, numbers satisfying equations of type $T(T(n)) = n^2$ (called m-superperfect numbers). Clearly, the above impotent numbers satisfy the inequality

$$T(n) < n^2 \tag{129}$$

i.e. they are multiplicatively deficient (or "m-deficient") numbers. Therefore it is not necessary to introduce a new terminology in this case.

First remark, that all m-deficient numbers can be written in the forms $1, p, p^2, pq, p^2q$, where p, q are distinct primes. Indeed, if $d_1, d_2, \ldots, d_s$ are all divisors of n, then

$$\{d_1, \ldots, d_s\} = \left\{\frac{n}{d_1}, \ldots, \frac{n}{d_s}\right\},$$

implying that

$$d_1 d_2 \ldots d_s = \frac{n}{d_1} \cdot \frac{n}{d_2} \cdots \frac{n}{d_s},$$

i.e.

$$T(n) = n^{s/2} \tag{130}$$

where $s = d(n)$ denotes the number of distinct divisors of n. Therefore inequality (1) is satisfied only when $d(n) < 4$, implying $n \in \{1, p, p^2, pq, p^2q\}$. Clearly, n is m-abundant when

$$T(n) > n^2 \tag{131}$$

implying $d(n) > 4$. Since for $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ one has $d(n) = (\alpha_1 + 1) \ldots (\alpha_r + 1)$, in the case $r = 1$, (3) is true only for $\alpha_1 > 3$; when $r = 2$ for $\alpha_1 = 1$ we must have $\alpha_2 \geq 2$, while for $\alpha_1 \geq 2$, $\alpha_2 \geq 2$ this is always valid; for $r \geq 3$, (3) always holds true. Therefore, all m-abundant numbers are of the forms $n = p^\alpha$ $(\alpha \geq 4)$; pq^β $(\beta \geq 2)$, $p^\alpha q^\beta$ $(\alpha, \beta \geq 2)$;

$w(n) \geq 3$ (where p, q are distinct primes and $w(n)$ denotes the number of distinct prime divisors of n).

On the other hand, let us remark that for $n \geq 2$ one has $d(n) \geq 2$, so

$$T(n) \geq n \tag{132}$$

with equality, only for $n =$ prime. If $n \neq$ prime, then $d(n) \geq 3$ gives

$$T(n) \geq n^{3/2} \quad (n \neq \text{prime}). \tag{133}$$

Now, relations (4) and (5) give together

$$T(T(n)) \geq n^{9/4} \quad \text{for} \quad n \neq \text{prime} \tag{134}$$

Since $9/4 > 2$, we have obtained that for all composite numbers we have $T(T(n)) > n^2$, i.e. all composite numbers are m-super abundant. Since $T(T(p)) = p < p^2$, all prime numbers are m-super deficient. Therefore we can state the following "primality criterion".

Theorem 1. *The number $n > 1$ is prime if and only if it is m-super deficient.*

In fact, by iteration from (6) we can obtain

$$\underbrace{T(T(\ldots T(n)\ldots))}_{k} \geq n^{3^k/2^k}, \quad n \neq \text{prime}.$$

Since $3^k > 2^k \cdot k$ for all $k \geq 1$, we have the following generalization.

Theorem 2. *The number $n > 1$ is prime if and only if it is m-k-super deficient.*

(n is m-k-super deficient if $\underbrace{T(T(\ldots T(n)\ldots))}_{k} < n^k$).

For related results see [2].

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On Certain Arithmetic Functions

In the recent book [1] there appear certain arithmetic functions which are similar to the Smarandache function. In a recent paper [2] we have considered certain generalization or duals of the Smarandache function $S(n)$. In this note we wish to point out that the arithmetic functions introduced in [1] all are particular cases of our function F_f, defined in the following manner (see [2] or [3]).

Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbb{N}^* \to \mathbb{N}*$ defined by

$$F_f(n) = \min\{k \in \mathbb{N}^* : n|f(k)\} \tag{135}$$

In Problem 6 of [1] it is defined the "ceil function of t-th order" by $S_t(n) = \min\{k : n|k^t\}$. Clearly here one can select $f(m) = m^t$ $(m = 1, 2, \ldots)$, where $t \geq 1$ is fixed. Property (P_1) is satisfied with $k = n^t$. For $f(m) = \dfrac{m(m+1)}{2}$, one obtains the "Pseudo-Smarandache" function of Problem 7. The Smarandache "double-factorial" function

$$SDF(n) = \min\{k : n|k!!\}$$

where

$$k!! = \begin{cases} 1 \cdot 3 \cdot 5 \ldots k & \text{if } k \text{ is odd} \\ 2 \cdot 2 \cdot 6 \ldots k & \text{if } k \text{ is even} \end{cases}$$

of Problem 9 [1] is the particular case $f(m) = m!!$. The "power function" of Definition 24, i.e. $SP(n) = \min\{k : n|k^k\}$ is the case of $f(k) = k^k$. We note that the Definitions 39 and 40 give the particular case of S_t for $t = 2$ and $t = 3$.

In our paper we have introduced also the following "dual" of F_f. Let $g : \mathbb{N}^* \to \mathbb{N}*$ be a given arithmetical function, which satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$.

Let $G_g : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$G_g(n) = \max\{k \in \mathbb{N}^* : g(k)|n\}. \tag{136}$$

Since $k^t|n$, $k!!|n$, $k^k|n$, $\frac{k(k+1)}{2}|n$ all are verified for $k = 1$, property (P_3) is satisfied, so we can define the following duals of the above considered functions:

$$S_t^*(n) = \max\{k : k^t|n\};$$

$$SDF^*(n) = \max\{k : k!!|n\};$$

$$SP^*(n) = \max\{k : k^k|n\};$$

$$Z^*(n) = \max\left\{k : \frac{k(k+1)}{2}|n\right\}.$$

These functions are particular cases of (2), and they could deserve a further study, as well.

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On a dual of the Pseudo-Smarandache function

1 Introduction

In paper [3] we have defined certain generalizations and extensions of the Smarandache function. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetic function with the following property: for each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$. Let

$$F_f : \mathbb{N}^* \to \mathbb{N}^* \text{ defined by } F_f(n) = \min\{k \in \mathbb{N}^* : \ n|f(k)\}. \tag{137}$$

This function generalizes many particular functions. For $f(k) = k!$ one gets the Smarandache function, while for $f(k) = \frac{k(k+1)}{2}$ one has the Pseudo-Smarandache function Z (see [1], [4-5]). In the above paper [3] we have defined also dual arithmetic functions as follows: Let $g : \mathbb{N}^* \to \mathbb{N}^*$ be a function having the property that for each $n \geq 1$ there exists at least a $k \geq 1$ such that $g(k)|n$.

Let

$$G_g(n) = \max\{k \in \mathbb{N}^* : \ g(k)|n\}. \tag{138}$$

For $g(k) = k!$ we obtain a dual of the Smarandache function. This particular function, denoted by us as S_* has been studied in the above paper. By putting $g(k) = \frac{k(k+1)}{2}$ one obtains a dual of the Pseudo-Smarandache function. Let us denote this function, by analogy by Z_*. Our aim is to study certain elementary properties of this arithmetic function.

2 The dual of yhe Pseudo-Smarandache function

Let

$$Z_*(n) = \max\left\{m \in \mathbb{N}^* : \ \frac{m(m+1)}{2}|n\right\}. \tag{139}$$

Recall that

$$Z(n) = \min\left\{k \in \mathbb{N}^* : \ n|\frac{k(k+1)}{2}\right\}. \tag{140}$$

First remark that

$$Z_*(1) = 1 \quad \text{and} \quad Z_*(p) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{141}$$

where p is an arbitrary prime. Indeed, $\frac{2 \cdot 3}{2} = 3|3$ but $\frac{m(m+1)}{2}|p$ for $p \neq 3$ is possible only for $m = 1$. More generally, let $s \geq 1$ be an integer, and p a prime. Then:

Proposition 1.

$$Z_*(p^s) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{142}$$

Proof. Let $\frac{m(m+1)}{2}|p^s$. If $m = 2M$ then $M(2M+1)|p^s$ is impossible for $M > 1$ since M and $2M+1$ are relatively prime. For $M = 1$ one has $m = 2$ and $3|p^s$ only if $p = 3$. For $m = 2M - 1$ we get $(2M-1)M|p^k$, where for $M > 1$ we have $(M, 2M-1) = 1$ as above, while for $M = 1$ we have $m = 1$.

The function Z_* can take large values too, since remark that for e.g. $n \equiv 0(\text{mod}6)$ we have $\frac{3 \cdot 4}{2} = 6|n$, so $Z_*(n) \geq 3$. More generally, let a be a given positive integer and n selected such that $n \equiv 0(\text{mod}a(2a+1))$. Then

$$Z_*(n) \geq 2a. \tag{143}$$

Indeed, $\frac{2a(2a+1)}{2} = a(2a+1)|n$ implies $Z_*(n) \geq 2a$.

A similar situation is in

Proposition 2. Let q be a prime such that $p = 2q - 1$ is a prime, too. Then

$$Z_*(pq) = p. \tag{144}$$

Proof. $\frac{p(p+1)}{2} = pq$ so clearly $Z_*(pq) = p$.

Remark. Examples are $Z_*(5 \cdot 3) = 5$, $Z_*(13 \cdot 7) = 13$, etc. It is a difficult open problem that for infinitely many q, the number p is prime, too (see e.g. [2]).

Proposition 3. For all $n \geq 1$ one has

$$1 \leq Z_*(n) \leq Z(n). \tag{145}$$

Proof. By (3) and (4) we can write $\frac{m(m+1)}{2}|n|\frac{k(k+1)}{2}$, therefore $m(m+1)|k(k+1)$. If $m > k$ then clearly $m(m+1) > k(k+1)$, a contradiction.

Corollary. One has the following limits:

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 1. \tag{146}$$

Proof. Put $n = p$ (prime) in the first relation. The first result follows by (6) for $s = 1$ and the well-known fact that $Z(p) = p$. Then put $n = \frac{a(a+1)}{2}$, when $\frac{Z_*(n)}{Z(n)} = 1$ and let $a \to \infty$.

As we have seen,

$$Z\left(\frac{a(a+1)}{2}\right) = Z_*\left(\frac{a(a+1)}{2}\right) = a.$$

Indeed, $\frac{a(a+1)}{2} \Big| \frac{k(k+1)}{2}$ is true for $k = a$ and is not true for any $k < a$. In the same manner, $\frac{m(m+1)}{2} \Big| \frac{a(a+1)}{2}$ is valied for $m = a$ but not for any $m > a$. The following problem arises: What are the solutions of the equation $Z(n) = Z_*(n)$?

Proposition 4. All solutions of equation $Z(n) = Z_*(n)$ can be written in the form $n = \frac{r(r+1)}{2}$ $(r \in \mathbb{N}^*)$.

Proof. Let $Z_*(n) = Z(n) = t$. Then $n | \frac{t(t+1)}{2} | n$ so $\frac{t(t+1)}{2} = n$. This gives $t^2 + t - 2n = 0$ or $(2t+1)^2 = 8n+1$, implying $t = \frac{\sqrt{8n+1}-1}{2}$, where $8n+1 = m^2$. Here m must be odd, let $m = 2r+1$, so $n = \frac{(m-1)(m+1)}{8}$ and $t = \frac{m-1}{2}$. Then $m - 1 = 2r$, $m + 1 = 2(r+1)$ and $n = \frac{r(r+1)}{2}$.

Proposition 5. One has the following limits:

$$\lim_{n\to\infty} \sqrt[n]{Z_*(n)} = \lim_{n\to\infty} \sqrt[n]{Z(n)} = 1. \tag{147}$$

Proof. It is known that $Z(n) \le 2n - 1$ with equality only for $n = 2^k$ (see e.g. [5]). Therefore, from (9) we have

$$1 \le \sqrt[n]{Z_*(n)} \le \sqrt[n]{Z(n)} \le \sqrt[n]{2n-1},$$

and by taking $n \to \infty$ since $\sqrt[n]{2n-1} \to 1$, the above simple result follows.

As we have seen in (9), upper bounds for $Z(n)$ give also upper bounds for $Z_*(n)$. E.g. for $n =$ odd, since $Z(n) \le n - 1$, we get also $Z_*(n) \le n - 1$. However, this upper bound is too large. The optimal one is given by:

Proposition 6.

$$Z_*(n) \le \frac{\sqrt{8n+1}-1}{2} \text{ for all } n. \tag{148}$$

Proof. The definition (3) implies with $Z_*(n) = m$ that $\frac{m(m+1)}{2}|n$, so $\frac{m(m+1)}{2} \le n$, i.e. $m^2 + m - 2n \le 0$. Resolving this inequality in the unknown m, easily follows (12). Inequality (12) cannot be improved since for $n = \frac{p(p+1)}{2}$ (thus for infinitely many n) we have equality. Indeed,

$$\left(\sqrt{\frac{8(p+1)p}{2}+1}-1\right)/2 = \left(\sqrt{4p(p+1)+1}-1\right)/2 = [(2p+1)-1]/2 = p.$$

Corollary.

$$\varliminf_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \quad \varlimsup_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = \sqrt{2}. \tag{149}$$

Proof. While the first limit is trivial (e.g. for n = prime), the second one is a consequence of (12). Indeed, (12) implies $Z_*(n)/\sqrt{n} \le \sqrt{2}\left(\sqrt{1+\frac{1}{8n}} - \sqrt{\frac{1}{8n}}\right)$, i.e. $\varlimsup_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} \le \sqrt{2}$. But this upper limit is exact for $n = \frac{p(p+1)}{2}$ $(p \to \infty)$.

Similar and other relations on the functions S and Z can be found in [4-5].

An inequality connecting $S_*(ab)$ with $S_*(a)$ and $S_*(b)$ appears in [3]. A similar result holds for the functions Z and Z_*.

Proposition 7. For all $a, b \ge 1$ one has

$$Z_*(ab) \ge \max\{Z_*(a), Z_*(b)\}, \tag{150}$$

$$Z(ab) \ge \max\{Z(a), Z(b)\} \ge \max\{Z_*(a), Z_*(b)\}. \tag{151}$$

Proof. If $m = Z_*(a)$, then $\frac{m(m+1)}{2}|a$. Since $a|ab$ for all $b \ge 1$, clearly $\frac{m(m+1)}{2}|ab$, implying $Z_*(ab) \ge m = Z_*(a)$. In the same manner, $Z_*(ab) \ge Z_*(b)$, giving (14).

Let now $k = Z(ab)$. Then, by (4) we can write $ab|\frac{k(k+1)}{2}$. By $a|ab$ it results $a|\frac{k(k+1)}{2}$, implying $Z(a) \le k = Z(ab)$. Analogously, $Z(b) \le Z(ab)$, which via (9) gives (15).

Corollary. $Z_*(3^s \cdot p) \ge 2$ for any integer $s \ge 1$ and any prime p. (16)

Indeed, by (14), $Z_*(3^s \cdot p) \ge \max\{Z_*(3^s), Z(p)\} = \max\{2, 1\} = 2$, by (6).

We now consider two irrational series.

Proposition 8. The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{n!}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}Z_*(n)}{n!}$ are irrational.

Proof. For the first series we apply the following irrationality criterion ([6]). Let (v_n) be a sequence of nonnegative integers such that

(i) $v_n < n$ for all large n;

(ii) $v_n < n-1$ for infinitely many n;

(iii) $v_n > 0$ for infinitely many n.

Then $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Let $v_n = Z_*(n)$. Then, by (12) $Z_*(n) < n-1$ follows from $\frac{\sqrt{8n+1}-1}{2} < n-1$, i.e. (after some elementary fact, which we omit here) $n > 3$. Since $Z_*(n) \geq 1$, conditions (i)-(iii) are trivially satisfied.

For the second series we will apply a criterion from [7]:

Let $(a_k), (b_k)$ be sequences of positive integers such that

(i) $k | a_1 a_2 \dots a_k$;

(ii) $\frac{b_{k+1}}{a_{k+1}} < b_k < a_k$ $(k \geq k_0)$. Then $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{b_k}{a_1 a_2 \dots a_k}$ is irrational.

Let $a_k = k$, $b_k = Z_*(k)$. Then (i) is trivial, while (ii) is $\frac{Z_*(k+1)}{k+1} < Z_*(k) < k$. Here $Z_*(k) < k$ for $k \geq 2$. Further $Z_*(k+1) < (k+1)Z_*(k)$ follows by $1 \leq Z_*(k)$ and $Z_*(k+1) < k+1$.

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