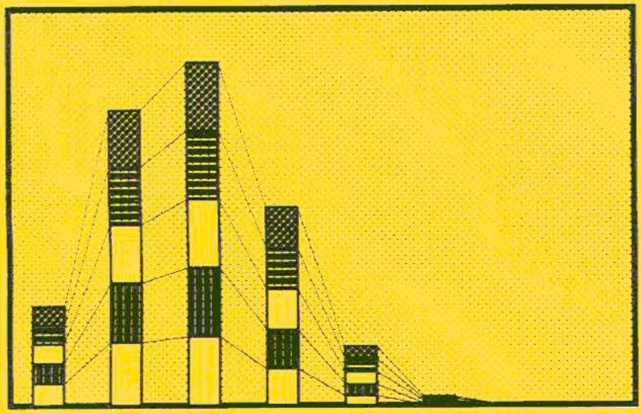
Surfing on the ocean of numbers

-a Few Smarandache Notions and Similar Topics

Henry Ibstedt



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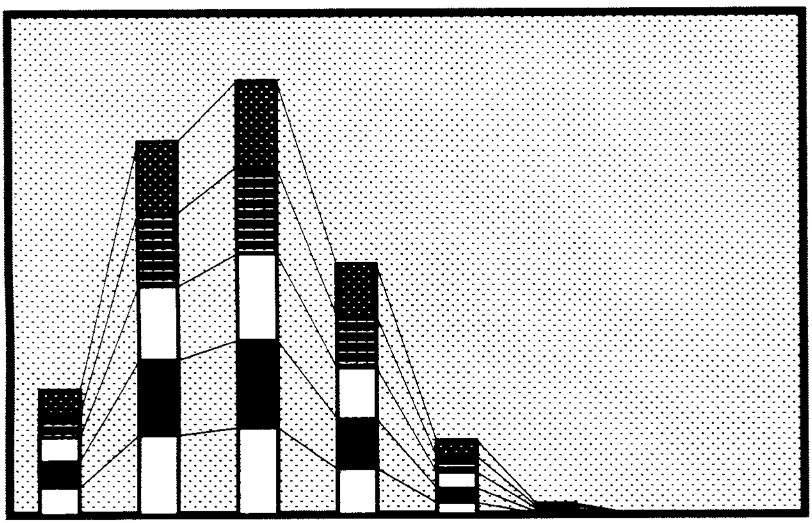

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The cover picture refers to the last article in this book. It illustrates cumulative statistics on the occurrence of square free integers with 1, 2, 3, etc. prime factors for successive intervals of integers. A detailed discussion is given in the article.

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To the Memory of

my Son

Carl-Magnus

Preface

Surfing on the Ocean of Numbers - why this title? Because this little book does not attempt to give theorems and rigorous proofs in the theory of numbers. Instead it will attempt to throw light on some properties of numbers, nota bene integers, through a study of the behaviour of large numbers of integers in order to draw some reasonably certain conclusions or support already made conjectures. But no matter how far we extend our search or increase our samples in these studies we are in fact, in spite of more and more powerful technologies, merely skimming the surface of the immense sea of numbers. - Hence the title.

Most books in Mathematics are used as reference books. I still consult my first Number Theory book which I bought in 1949 - *Elementary Number Theory* by Uspensky and Heaslet. It was when I was young and enthusiastic and dreamt about becoming a Mathematician. I am still enthusiastic but I became a Physicist instead.. However, I stayed on the theoretical side and avoided to have to much to do with things that can break. But even so experiments are a major source of knowledge and maybe this book shows a little of a Physicist's approach to Mathematics. Most results are presented or supported by tables and graphs. All calculations have been carried out on a Pentium 100 Mhz laptop using Ubasic as a programming language. Finally, the author has tried to make a book which should be easy and pleasant to read.

A word about the beauty of Mathematics and Number Theory in particular. The crystallized truth of a theorem, where a whole spectrum of mathematical thoughts come together to form an entity, is like a painting where designs and colours merge into a work of art. But sometimes it is not the finished result which is the most interesting - it could be the unsolved problem itself. Why? Maybe it is the challenge of getting somewhere with it or the hours and days of thinking and trying that occupy the mind in a positive sense different from the problems of our time. It all brings piece to the mind - it's like walking in the silence of the forest enjoying the trees, the sun and the blue sky , and should it happen that all the bits and pieces suddenly fall into place to give a solution then it is the most sublime experience for the human mind - eureka! But then the interest in the problem fades away unless solving the problem created new ones - and that is almost always the case.

Most topics in this book have been selected from *Only Problems, Not Solutions* by F. Smarandache. Others have bee suggested by Dr. R. Muller of Erhus University Press. A few problems which the author has found interesting originate from the Numbers Count Column of *Personal Computer World*. This journal has had great importance for the author as a source of recreational Mathematics and I take this opportunity to thank the Editor of this column Mike Mudge for all correspondence and encouragement he gave me in the past.

Illustrations, graphics, layout and final editing up to camera ready form has been done by the author. Tables have been created by direct transfer from computer files established at the time of computation to the manuscript so as to avoid typing errors.

This book has come into being thanks to R. Muller at Erhus University Press who has never failed an opportunity to give his support and encouragement. Rapid e-mail exchange between him in the USA and me in France has greatly facilitated our work. I also thank Dr. Muller's colleagues for their help. Many thanks are also due to my son Michael Ibstedt for his help and advice concerning computer equipment and software.

Last but not least my warm thanks to my dear wife Anne-Marie for her encouragement and endless patience with a husband who does not always listen because his mind is somewhere else.

February 1997
Henry Ibstedt

Contents

Chapter I

On Prime Numbers

This chapter deals with some computational observations on prime numbers and their distribution. These computations only skim the surface of the ocean of integers but they give an idea of the general behavior of primes and often support some of the many conjectures that are made concerning primes. Computer programs have been written in Ubasic with extensive use of some of the built in functions of this language. In some cases the use of these functions will be illustrated with a few lines of program code. Most results are given in tabular and/or graphical form.

1. On the Sequence $a \cdot p_n + b$

In his book *Only Problems, Not Solutions*[1] F. Smarandache asks the following question:
If $(a,b)=1$, how many primes does the progression $a \cdot p_n + b$, where p_n is prime and $n \varepsilon \{1,2,\ldots\}$, contain?

Already for a=1 and b=2 we run into the classical unsolved problem "*Are there infinitely many twin primes?*". The answer to <u>how many?</u> is certainly equally difficult for other sets of parameters a,b. However, some interesting information on how $a \cdot p_n + b$ behaves will be obtained for the first 10,000,000 primes p_n.

Let m be the number of primes produced by $a \cdot p_n + b$ for $n \leq N$, i.e. if $a \cdot p_n + b$ is prime we can write $a \cdot p_n + b = q_m$ where q_m is prime. The following Ubasic program lines have been used to determine whether $a \cdot p_n + b$ is prime or not:

```
while N<10000001
p=nxtprm(p)
inc N
c=a%*p+b%
if nxtprm(c-1)=c then inc m
wend
```

The program has been implemented for a set of values of the parameters a and b. The result is shown in table 1. It is interesting to visualize the result. Because of the logarithmic behaviour of the distribution of primes it is reasonable represent m/N as a

[1] Unsolved problem number 17.

function of $\log_{10}N$ rather than as a function of N. For this reason the value of m has been recorded during the computation for N=10, 10^2, 10^3, 10^4, 10^5, 10^6, and 10^7.

Table 1. Number of primes m in the progression $a \cdot p_n + b$ for n<N

a,b /N	10	10^2	10^3	10^4	10^5	10^6	10^7
1,2	5	25	174	1270	10250	86027	738597
2,1	5	25	166	1221	9667	82236	711153
3,2	8	47	290	2350	18919	160127	1392733
4,1	3	21	145	1108	9314	78676	685069
5,2	5	26	188	1492	12020	103010	903165
6,1	7	39	277	2175	18019	153925	1342255
7,2	4	23	167	1288	10634	91232	

The graphs in figure 1 show m/N (y-axis) as a function of $\log_{10}N$ (x-axis) where m is the number of primes of the form ap_n+b for n<N. Figure 1b is an enlargement of figure 1a for large values of N, ($N \le 10^7$). The eight curves correspond to the following sets of

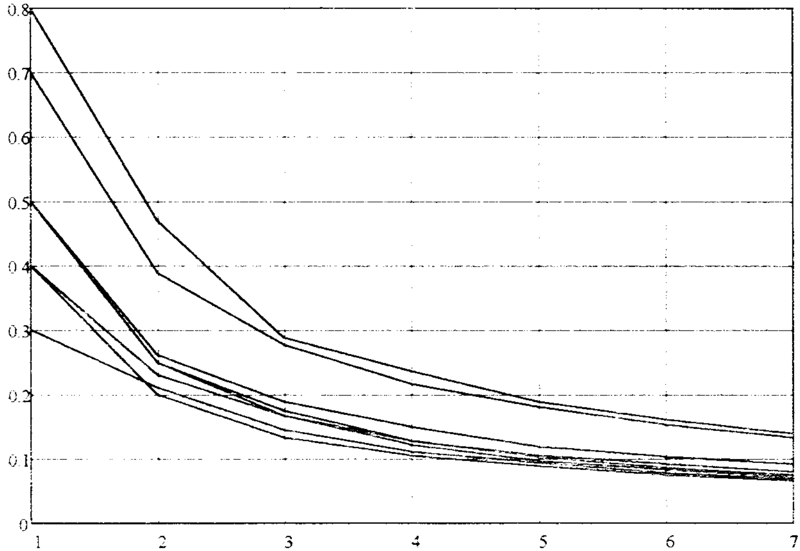

Figure 1a.

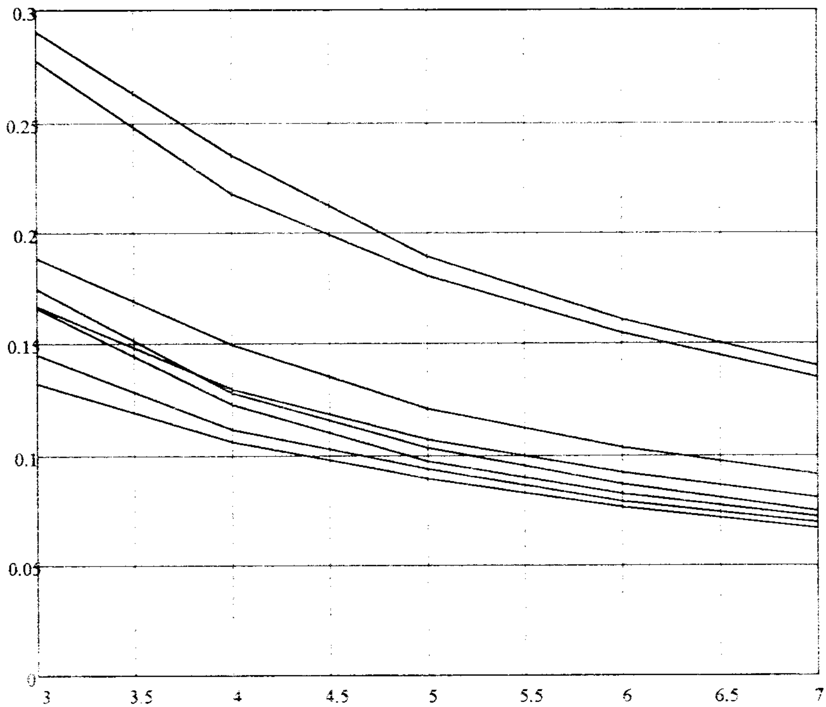

Figure 1b.

parameters listed in the order in which the curves appear from top to bottom in the right hand side of the two figures: (a,b)=(3,2), (6,1), (5,2), (7,2), (1,2) (2,1), (4,1) and (8,1)

Table 2. Number of primes m generated by p_n+b for $n \le N$

b / N	10	100	1000	10000	100000	1000000
2	5	25	174	1270	10250	86027
4	4	27	170	1264	10214	85834
6	7	48	344	2538	20472	170910
8	5	24	178	1303	10336	85866
10	5	34	231	1682	13653	114394
12	7	48	340	2515	20462	171618

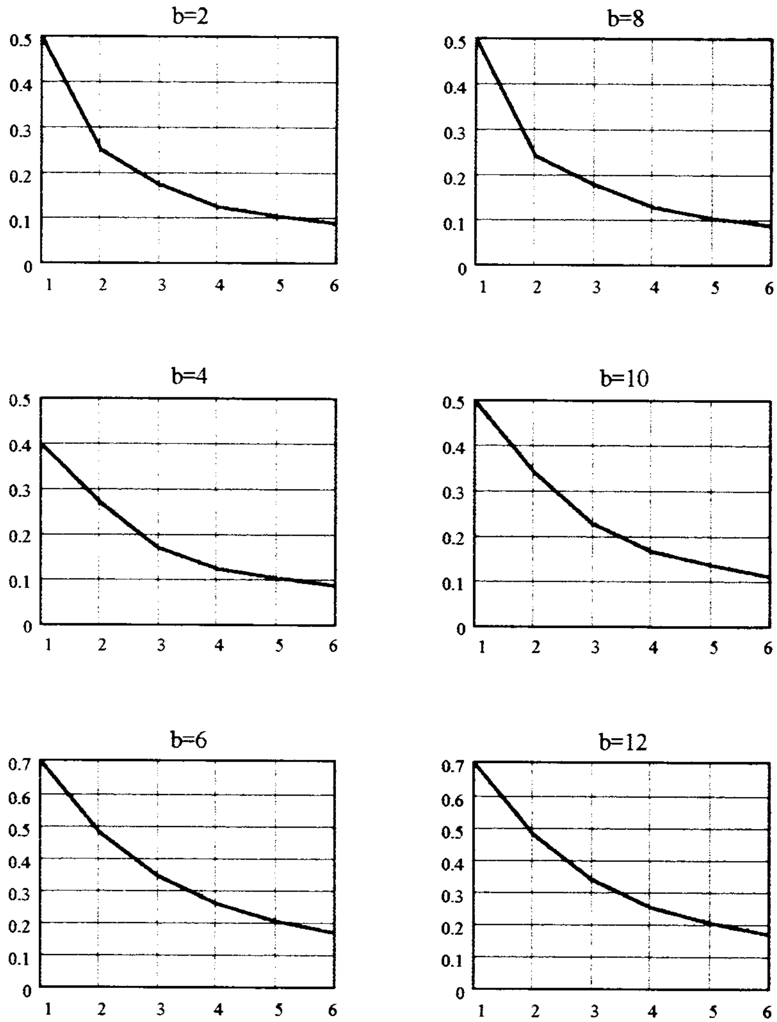

Figure 2. The ratio m/M plotted against logN for b=2,4,6,8,10,and 12

The number of primes m in the sequence $a \cdot p_n+b$ for n = 1, 2, ... N is illustrated in figure 2 for a = 1 and b = 2, 4, 6, 8, 10 and 12 where the ratio M=m/N is plotted against logN. The corresponding numerical results are given in table 2.

The general appearance of the graphs for various values of the parameters (a,b) in ap_n+b is very similar. In particular figure 1b shows an interesting picture of curves running parallel to one another and in particular to the one for (a,b)=(1,2), that is the curve for p+2 which corresponds to prime twins for which we have the classical conjecture that there are infinitely many. This makes the following conclusion reasonable.

Conjecture: The progression ap_n+b , (a,b)=1 contains infinitely many prime numbers.

2. Prime Number Gaps

Smarandache asked how many primes there is in the progression ap_n+b. For a=1 and b=2 the question is equivalent to 'how many twin primes are there?'. Since we have a very stable conjecture that there are infinitely many we now want to know something about their distribution and also about the distribution of other prime number gaps $g=p_{n+1}-p_n$. With a small change in the Ubasic program used in the previous section we can study the distribution of primes over gaps g=2, 4, 6, ...

```
p=3
while p<N
q=p
p=nxtprm(p)
u=(p-q)/2
inc f(u)                 'Count the number of gaps = p-q.
if p(u)=0 then p(u)=q    'Store the smallest prime for which the
wend                     'gap occurs in p().
```

This program was run for primes $p<N=2 \cdot 10^9$ The result is shown in table 3,where f is the number of gaps g and p the prime number for which the gap first occurs, $N<2 \cdot 10^7$. All gaps $g \leq 292$ except 264, 278 and 286 are represented in the table which is arranged so that gaps $g \equiv 2 \pmod 6$, $g \equiv 4 \pmod 6$ and $g \equiv 0 \pmod 6$ are found in separate columns . Gaps $g \equiv 0 \pmod 6$ occur much more frequently than the other two. This is illustrated in figure 3 which also shows that **ln f as a function of g** has a near linear behaviour for all three types. The "wild" behaviour fot gaps>250 would certainly correct itself if the range of primes in the study were extended. The area below the curves for g≡2 (mod 6) and g≡4 (mod 6) are equal as will be shown shortly. The curve for g≡2 (mod 6) behaves very well while the one for g≡4 (mod 6) shows an interesting ripple effect. In particular it shows a "bump" for g=70 which showed up already in the smallest sample $N<10^6$ for which g=70 first appeared. What causes this high frequency for g=70?

For a prime number $p \geq 5$ we have $p \equiv \pm 1 \pmod 6$. Let q and p be two consecutive primes forming the gap $g=p-q$. We distinguish between the following cases:

		Shift
1.	$q \equiv 1 \pmod 6$ and $p \equiv 1 \pmod 6 \Rightarrow g \equiv 0 \pmod 6$	++
2.	$q \equiv 1 \pmod 6$ and $p \equiv -1 \pmod 6 \Rightarrow g \equiv 4 \pmod 6$	+-
3.	$q \equiv -1 \pmod 6$ and $p \equiv 1 \pmod 6 \Rightarrow g \equiv 2 \pmod 6$	-+
4.	$q \equiv -1 \pmod 6$ and $p \equiv -1 \pmod 6 \Rightarrow g \equiv 0 \pmod 6$	--

A sequence of consecutive primes (with the first prime =5) can be characterized by the shifts:

-+-+ .... +-+--+-++++--++------++++++++------+-+-- .....

The longest sequence of consecutive primes ≡ 1 (mod 6) for $p<10^9$ is of length 18:

450988159, 450988177, 450988207, 450988231, 450988241, 450988261, 450988297, 450988333, 450988339, 450988381, 450988387, 450988399, 450988411, 450988423, 450988441, 450988471, 450988477, 450988567

and the longest sequence of consecutive primes ≡ -1 (mod 6) for $p<10^9$ is of length 22:

766319189, 766319201, 766319231, 766319237, 766319249, 766319261, 766319273, 766319291, 766319339, 766319357, 766319363, 766319369, 766319423, 766319441, 766319453, 766319483, 766319507, 766319549, 766319573, 766319579, 766319621, 766319627

Let the number of gaps≡2, 4 or 0 (mod 6) be f_2, f_4, and f_0 respectively for $p<N$. Then we will have $f_2=f_4$ if the last shift is +- otherwise we will have $f_2=f_4+1$.

In a simple model it would be reasonable to assume that, at an arbitrary point in the sequence of shifts, the probabilities of finding the next shift to be +-, ++, -+ or -- are equal, i.e. if we define $F_2=f_2/(f_0+f_2+f_4)$, $F_4=f_4/(f_0+f_2+f_4)$ and $F_0=f_0/(f_0+f_2+f_4)$ we would have $F_2=F_4=0.25$ and $F_0=0.5$. <u>This is not the case.</u>

Before looking into this let's first consider a related question: ***Do primes≡1 (mod 6) (notation f_+) occur with the same frequency as primes ≡ -1 (mod 6) (notation f_-) ?*** Table 4 shows a study of the number of primes f_- and f_+ congruent to -1 respectively 1 (mod 6) for primes less than 10^k for k=1, 2, 3, ... 9.

Within the range of this study we have $f_- > f_+$. However, the ratio $r = (f_- - f_+)/(f_- - f_+)$ is decreasing. Will eventually $f_- < f_+$?

We have proved that $f_2 = f_4$ (assuming the last shift to be +-). We will now study the relative frequencies defined through

$$F_2=f_2/(f_0+f_2+f_4) \qquad F_4=f_4/(f_0+f_2+f_4) \qquad F_0=f_0/(f_0+f_2+f_4)$$

Again we have $F_2 = F_4$ and of course $F_0=1 - 2\cdot F_2$. To study how F_2 varies as we increase the number of consecutive primes $p<10^k$ the execution of the program for gap statistics was stopped for k=1, 2, ... 9 to produce the data shown in table 5.

Table 4. Number of primes ≡ -1 respectively 1 (mod 6). $r=(f_- - f_+)/(f_- + f_+)$

k	1	2	3	4	5	6	7	8	9
f_-	1	12	85	616	4805	39264	332383	2880936	25424819
f_+	1	11	81	611	4785	39232	332194	2880517	25422713
$f_- - f_+$	0	1	4	5	20	32	189	419	2106
$f_- + f_+$	2	23	166	1227	9590	78496	664577	5761453	50847532
$r\cdot 10^4$	0	435	241	41	21	4	3	0.7	0.4

Table 5. Prime number gap distribution (mod 6) for primes $<10^k$

k	g≡2(mod6)	g≡4(mod6)	g≡0(mod6)	Total	F_2
1	2	1	0	3	0.5
2	9	8	7	24	0.354166667
3	58	56	53	167	0.341317365
4	379	378	471	1228	0.308224756
5	2870	2868	3853	9591	0.299134605
6	22839	22837	32821	78497	0.290941055
7	189285	189284	286009	664578	0.284819088
8	1616471	1616470	2528513	5761454	0.280566416
9	14107250	14107249	22633034	50847533	0.277442162

Table 3.

g=2	f	p	g=4	f	p	g=0	f	p
2	6388042	3	4	6386967	7	6	11407651	23
8	5069051	89	10	6568071	139	12	8472823	199
14	4690561	113	16	3527160	1831	18	6427670	523
20	3528810	887	22	3030348	1129	24	4600962	1669
26	2190452	2477	28	2386944	2971	30	4298663	4297
32	1359889	5591	34	1430231	1327	36	2341569	9551
38	1103677	30593	40	1308406	19333	42	1940894	16141
44	796213	15683	46	687135	81463	48	1190342	28229
50	678359	31907	52	511183	19609	54	856601	35617
56	444581	82073	58	383239	44293	60	789454	43331
62	247659	34061	64	253846	89689	66	466901	162143
68	191321	134513	70	272834	173359	72	277514	31397
74	137620	404597	76	122523	212701	78	233230	188029
80	119756	542603	82	85030	265621	84	176328	461717
86	63174	155921	88	65612	544279	90	133019	404851
92	44723	927869	94	40821	1100977	96	71864	360653
98	37946	604073	100	39504	396733	102	52752	1444309
104	24215	1388483	106	20996	1098847	108	36484	2238823
110	21894	1468277	112	17316	370261	114	26413	492113
116	11385	5845193	118	10863	1349533	120	23526	1895359
122	7408	3117299	124	7521	6752623	126	14443	1671781
128	5181	3851459	130	7111	5518687	132	8974	1357201
134	3881	6958667	136	3380	6371401	138	6567	3826019
140	3970	7621259	142	2393	10343761	144	4104	11981443
146	1776	6034247	148	1966	2010733	150	4022	13626257
152	1288	8421251	154	1561	4652353	156	2152	17983717
158	886	49269581	160	1012	33803689	162	1413	39175217
164	661	20285099	166	553	83751121	168	1271	37305713
170	607	27915737	172	430	38394127	174	729	52721113
176	332	38089277	178	292	39389989	180	638	17051707
182	238	36271601	184	235	79167733	186	342	147684137
188	124	134065829	190	205	142414669	192	219	123454691
194	109	166726367	196	112	70396393	198	221	46006769
200	91	378043979	202	71	107534587	204	129	112098817
206	44	232423823	208	56	192983851	210	141	20831323
212	35	215949407	214	38	253878403	216	50	202551667
In	21	327966101	220	36	47326693	222	31	122164747
224	18	409866323	226	15	519653371	228	21	895858039
230	17	607010093	232	3	525436489	234	23	189695659
236	10	216668603	238	8	673919143	240	15	391995431
242	8	367876529	244	5	693103639	246	7	555142061
248	6	191912783	250	8	387096133	252	8	630045137
254	3	1202442089	256	1	1872851947	258	2	1316355323
260	3	944192807	262	1	1649328997	270	2	1391048047
266	1	1438779821	268	1	1579306789	276	1	649580171
272	1	1851255191	274	1	1282463269	282	3	436273009
284	2	1667186459	280	2	1855047163	288	2	1294268491
290	1	1948819133	292	1	1453168141			

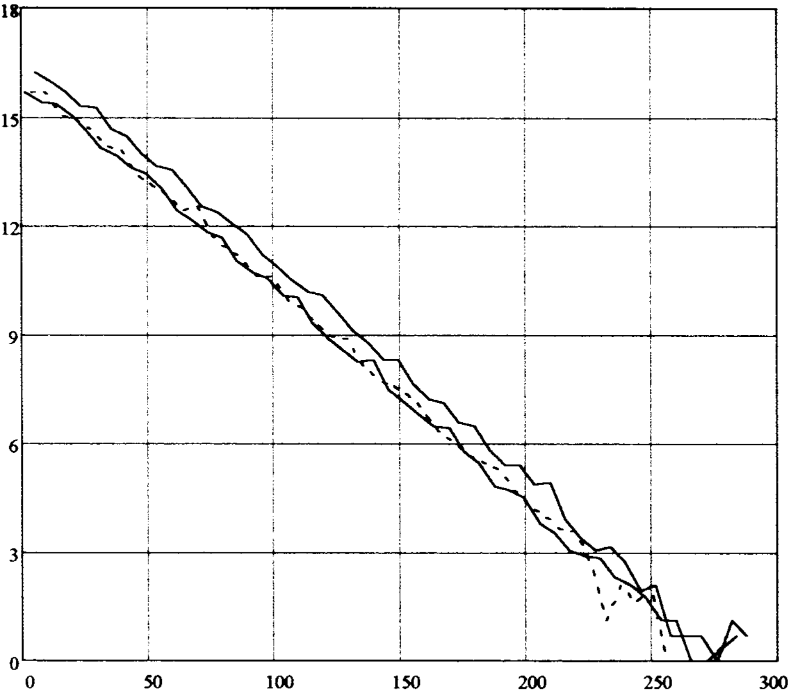

Figure 3. ln(f) as a function of g for $N<2\cdot10^9$. $g\equiv0 \pmod 6$ upper solid line, $g\equiv2 \pmod 6$ lower solid line and $g\equiv4 \pmod 6$ dashed line.

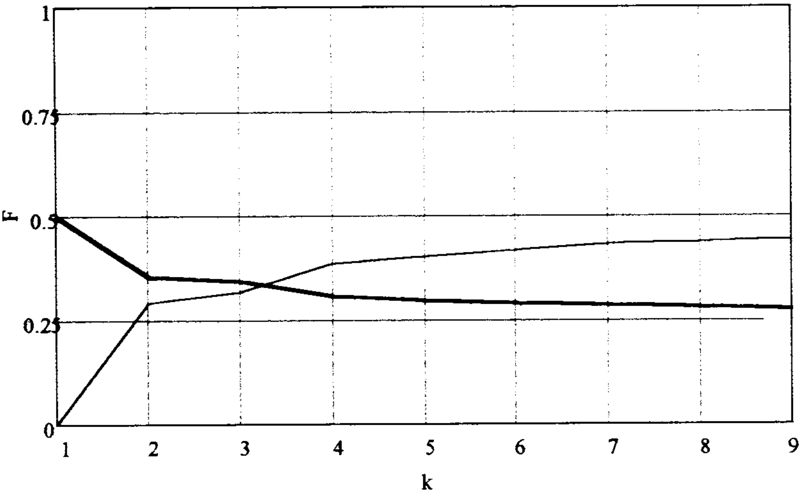

Figure 4. Relative frequency of prime gaps . $F_2 = F_4$ (thick line) and F_0 (thin line).

Conclusion: F_0 <0.45 and $F_2 = F_4$ >0.27 for primes <10^9.

The approach to the values 0.5 and 0.25 that one would have expected is very slow and is slowed down with increasing k, - one realizes that the interval $8<k<9$ is ten times as large as the interval $0<k<8$. The data used is cumulative but even if we consider only the interval between 10^8 and 10^9 we have F_2=12,659,767/45,680,669= 0.2771 compared to F_2=0.2774 for the whole interval between 0 and 10^9.

Question: Given an arbitrarily small number $\delta>0$, does a prime p_1 exist so that $F_2<0.25+\delta$ for all $p>p_1$?

Chapter II

On Smarandache Functions

1. Smarandache - Fibonacci Triplets

We recall the definition of the Smarandache Function S(n):

S(n) = the smallest positive integer such that S(n)! is divisible by n.

and the Fibonacci recurrence formula:

$$\mathbf{F_n = F_{n-1} + F_{n-2} \ (n \geq 2)}$$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

We will concern ourselves with isolated occurrences of triplets n, n-1, n-2 for which S(n)=S(n-1)+S(n-2) and pose the questions: Are there infinitely many such triplets? Is there a method of finding such triplets which would indicate that there are in fact infinitely many of them?

A straight forward search by applying the definition of the Smarandache Function to consecutive integers was used to identify the first eleven triplets [1] which are listed in table 1. As often in empirical number theory this merely scratched the surface of the ocean of integers. As can be seen from figure 1 the next triplet may occur for a value of n so large that a sequential search may be impractical and will not make us much wiser.

Table 1. The first 11 Smarandache-Fibonacci triplets

#	n	S(n)	S(n-1)	S(n-2)
1	11	11	5	2·3
2	121	2·11	5	17
3	4902	43	29	2·7
4	26245	181	18	163
5	32112	223	197	2·13
6	64010	173	2·23	127
7	368140	233	2·41	151
8	415664	313	2·73	167
9	2091206	269	2·101	67
10	2519648	1109	2·101	907
11	4573053	569	2·53	463

However, an interesting observation can be made from the triplets already found. Apart from n=26245 the Smarandache-Fibonacci Triplets have in common that one member is two times a prime number while the other two members are prime numbers. This observation leads to a method to search for Smarandache Fibonacci triplets in which the following two theorems play a rôle:

I. If $n = ab$ with $(a,b) = 1$ and $S(a) < S(b)$ then $S(n) = S(b)$.

II. If $n = p^a$ where p is a prime and $a \leq p$ then $S(p^a) = p$.

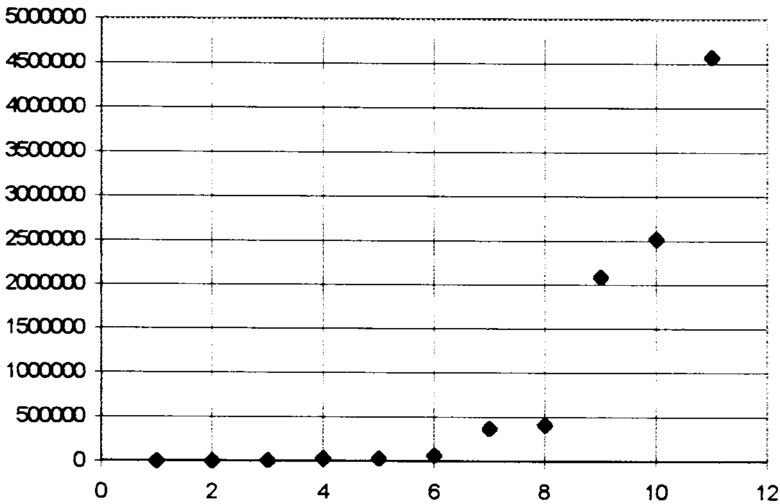

Figure 1. The values of n for which the first 11 Smarandache-Fibonacci triplets occur.

The search for Smarandache-Fibonacci triplets will be restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \leq p \text{ and } S(x) < ap \quad (1)$$

$$n-1 = yq^b \text{ with } b \leq q \text{ and } S(y) < bq \quad (2)$$

$$n-2 = zr^c \text{ with } c \leq r \text{ and } S(z) < cr \quad (3)$$

Table 2a. Smarandache-Fibonacci triplets

#	N	S(N)		S(N-1)		S(N-2)		t
1	4	4	*	3		2	*	0
2	11	11		5		6	*	0
3	121	22	*	5		17		0
4	4902	43		29		14	*	-4
5	32112	223		197		26	*	-1
6	64010	173		46	*	127		-1
7	368140	233		82	*	151		-1
8	415664	313		167		146	*	-8
9	2091206	269		202	*	67		-1
10	2519648	1109		202	*	907		0
11	4573053	569		106	*	463		-3
12	7783364	2591		202	*	2389		0
13	79269727	2861		2719		142	*	10
14	136193976	3433		554	*	2879		-1
15	321022289	7589		178	*	7411		5
16	445810543	1714	*	761		953		-1
17	559199345	1129		662	*	467		-5
18	670994143	6491		838	*	5653		-1
19	836250239	9859		482	*	9377		1
20	893950202	2213		2062	*	151		0
21	937203749	10501		10223		278	*	-9
22	1041478032	2647		1286	*	1361		-1
23	1148788154	2467		746	*	1721		3
24	1305978672	5653		1514	*	4139		0
25	1834527185	3671		634	*	3037		-5
26	2390706171	6661		2642	*	4019		0
27	2502250627	2861		2578	*	283		-1
28	3969415464	5801		1198	*	4603		-2
29	3970638169	2066	*	643		1423		-6
30	4652535626	3506	*	3307		199		0
31	6079276799	3394	*	2837		557		-1
32	6493607750	3049		1262	*	1787		5
33	6964546435	2161		1814	*	347		-4
34	11329931930	3023		2026	*	997		-4
35	11695098243	12821		1294	*	11527		2
36	11777879792	2174	*	1597		577		6

Table 2b. Smarandache-Fibonacci triplets

#	N	S(N)		S(N-1)		S(N-2)	t
37	13429326313	4778	*	1597		3181	1
38	13849559620	6883		2474	*	4409	1
39	14298230970	2038	*	1847		191	7
40	14988125477	3209		2986	*	223	2
41	17560225226	4241		3118	*	1123	-2
42	18704681856	3046	*	1823		1223	4
43	23283250475	4562	*	463		4099	-10
44	25184038673	5582	*	1951		3631	-2
45	29795026777	11278	*	8819		2459	0
46	69481145903	6301		3722	*	2579	3
47	107456166733	10562	*	6043		4519	-1
48	107722646054	8222	*	6673		1549	-1
49	122311664350	20626	*	10463		10163	0
50	126460024832	6917		2578	*	4339	11
51	155205225351	8317		4034	*	4283	-5
52	196209376292	7246	*	3257		3989	-5
53	210621762776	6914	*	1567		5347	11
54	211939749997	16774	*	11273		5501	0
55	344645609138	7226	*	2803		4423	9
56	484400122414	16811		12658	*	4153	-1
57	533671822944	21089		18118	*	2971	0
58	620317662021	21929		20302	*	1627	0
59	703403257356	13147		10874	*	2273	-2
60	859525157632	14158	*	3557		10601	-5
61	898606860813	19973		13402	*	6571	1
62	972733721905	10267		10214	*	53	-4
63	1185892343342	18251		12022	*	6229	-2
64	1225392079121	12202	*	9293		2909	-4
65	1294530625810	17614	*	5807		11807	-3
66	1517767218627	11617		8318	*	3299	-8
67	1905302845042	22079		21478	*	601	-1
68	2679220490034	11402	*	7459		3943	11
69	3043063820555	14951		12202	*	2749	5
70	6098616817142	24767		20206	*	4561	2
71	6505091986039	31729		19862	*	11867	2
72	13666465868293	28099		16442	*	11657	7

p,q and r are primes. We then have S(n)=ap,S(n-1)=bq and S(n-2)=cr. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \tag{4}$$

$$xp^a - yq^b = 1 \tag{5}$$

$$yq^b - zr^c = 1 \tag{6}$$

The greatest common divisor $(p^a,q^b)=1$ obviously divides the right hand side of (5). This is the condition for (5) to have infinitely many solutions for each solution (p,q) to (4). Such solutions are found using Euclid's algorithm and can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and (x_0, y_0) is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-11 \le t \le 11$. The result is shown in table 2. Since the correctness of these calculations are easily verified from factorizations of S(n), S(n-1) and S(n-2) these are given in table 3 for two large solutions taken from an extension of table 2.

Table 3. Factorization of two Smarandache-Fibonacci triplets

n =	$16{,}738{,}688{,}950{,}356=2^2\cdot 3\cdot 31\cdot 193\cdot \underline{15{,}269}^2$	S(n)=	$\underline{2\cdot 15{,}269}$
n-1 =	$16{,}738{,}688{,}950{,}355=5\cdot 197\cdot 1{,}399\cdot 1{,}741\cdot \underline{6{,}977}$	S(n-1)=	$\underline{6{,}977}$
n-2 =	$16{,}738{,}688{,}950{,}354=2\cdot 7^2\cdot 19\cdot 23\cdot 53\cdot 313\cdot \underline{23{,}561}$	S(n-2)=	$\underline{23{,}561}$
n =	$19{,}448{,}047{,}080{,}036=2^2\cdot 3^2\cdot 43^2\cdot \underline{17{,}093}^2$	S(n)=	$\underline{2\cdot 17{,}093}$
n-1 =	$19{,}448{,}047{,}080{,}035=5\cdot 7\cdot 19\cdot 37\cdot 61\cdot 761\cdot \underline{17{,}027}$	S(n-1)=	$\underline{17{,}027}$
n-2 =	$19{,}448{,}047{,}080{,}034=2\cdot 97\cdot 1{,}609\cdot 3{,}631\cdot \underline{17{,}159}$	S(n-2)=	$\underline{17{,}159}$

Conjecture:

There are infinitely many triplets n, n-1, n-2 such that S(n)=S(n-1)+S(n-2).

Questions:

1. It is interesting to note that there are only 7 cases in table 2 where S(n-2) is two times a prime number and that they all occur for relatively small values of n. Which is the next case?
2. The solution for n=26245 stands out as a very interesting one. Is it a unique case or is it a member of a family of Smarandache-Fibonacci triplets different from those studied here?

References:

C. Ashbacher and M. Mudge, *Personal Computer World*, October 1995, page 302.

2. Radu's Problem

For a positive integer n, the Smarandache function S(n) is defined as the smallest positive integer such that S(n)! is divisible by n. Radu [1] noticed that for nearly all values of n up to 4800 there is always at least one prime number between S(n) and S(n+1) including possibly S(n) and S(n+1). The exceptions are n=224 for which S(n)=8 and S(n+1)=10 and n=2057 for which S(n)=22 and S(n+1)=21. Radu conjectured that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1). The conjecture does not hold if there are infinitely many solutions to the following problem.

Find consecutive integers n and n+1 for which two consecutive primes p_k and p_{k+1} exist so that $p_k < Min(S(n),S(n+1))$ and $p_{k+1} > Max(S(n),S(n+1))$.

Consider

$$n+1 = xp_r^s \tag{1}$$

and

$$n = yp_{r-1}^s \tag{2}$$

where p_r and p_{r-1} are consecutive prime numbers. Subtract (2) from (1).

$$xp_r^s - yp_{r-1}^s = 1 \tag{3}$$

The greatest common divisor $(p_r^s, p_{r-1}^s) = 1$ divides the right hand side of (3) which is the condition for this diophantine equation to have infintely many solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

$$S(n+1) = sp_r,\ \text{i.e. } S(x) < sp_r \tag{4}$$

$$S(n) = sp_{r-1},\ \text{i.e. } S(y) < sp_{r-1} \tag{5}$$

In addition we require that the interval

$sp_r^s < q < sp_{r-1}^s$ is prime free, i.e. that q is not a prime.

Euclid's algorithm is used to obtain principal solutions (x_0, y_0) to (3). The general set of solutions to (3) is given by

$$x = x_0 + p_{r-1}^{st}, \qquad y = y_0 - p_r^s t \tag{7}$$

with t an integer.

These algorithms were implemented for different values of the parameters $d=p_{r-1}-p_r$, s and t. The result was a very large number of solutions. Table 4 shows the 20 smallest (in respect of n) solutions found. There is no indication that the set would be finite. One pair of primes may produce several solutions.

Within the limits set by the design of the program the largest prime difference for which a solution was found was d=42 and the largest exponent which produced solutions was s=4. Some numerically large examples illustrating these facts are given in table 5.

To see the relation between these large numbers and the corresponding values of the Smarandache function in table 5 the factorizations of these numbers are given below:

$118229366471522957848301 8 = 2\cdot3\cdot89\cdot193\cdot431\cdot1612781^2$
$118229366471522957848301 7 = 509\cdot3253\cdot1612823^2$
$111579064978581002637386836 34 = 2\cdot7\cdot37^2\cdot56671\cdot55333^3$
$111579064978581002637386836 35 = 3\cdot5\cdot11\cdot19^2\cdot16433\cdot55337^3$
$175498652132211624135022362 27 = 3\cdot11^2\cdot307\cdot12671\cdot55333^3$
$175498652132211624135022362 26 = 2\cdot23\cdot37\cdot71\cdot419\cdot743\cdot55337^3$
$27032997592120525363470705182284857039131 4 = 2\cdot3^3\cdot47\cdot1289\cdot2017\cdot119983\cdot167441^4$
$27032997592120525363470705182284857039131 3 = 37\cdot23117\cdot24517\cdot38303\cdot167443^4$

Table 4. The 20 smallest solutions which occurred for s=2 and d=2

#	n	S(n)	S(n+1)	p_1	p_2	t
1	265225	206	202	199	211	0
2	843637	302	298	293	307	0
3	6530355	122	118	113	127	-1
4	24652435	926	922	919	929	0
5	35558770	1046	1042	1039	1049	0
6	40201975	142	146	139	149	1
7	45388758	122	118	113	127	-4
8	46297822	1142	1138	1129	1151	0
9	67697937	214	218	211	223	0
10	138852445	1646	1642	1637	1657	0
11	157906534	1718	1714	1709	1721	0
12	171531580	1766	1762	1759	1777	0
13	299441785	2126	2122	2113	2129	0
14	551787925	2606	2602	2593	2609	0
15	1223918824	3398	3394	3391	3407	0
16	1276553470	3446	3442	3433	3449	0
17	1655870629	3758	3754	3739	3761	0
18	1853717287	3902	3898	3889	3907	0
19	1994004499	3998	3994	3989	4001	0
20	2256222280	4166	4162	4159	4177	0

Table 5. Four numerically large solutions

Pairs of consecutive integers	S( )	d	s	t	p_k, p_{k+1}
1182293664715229578483018	3225562	42	2	-2	1612781
1182293664715229578483017	3225646				1612823
11157906497858100263738683634	165999	4	3	0	55333
11157906497858100263738683635	166011				55337
17549865213221162413502236227	165999	4	3	-1	55333
17549865213221162413502236226	166011				55337
270329975921205253634707051822848570391314	669764	2	4	0	167441
270329975921205253634707051822848570391313	669772				167443

It is also interesting to see which are the nearest smaller P_k and nearest bigger P_{k-1} primes to S_1 = Min(S(n),S(n+1)) and S_2 = Max(S(n),S(n+1)) respectively. This is shown in table 6 for the above examples.

Table 6. $P_k < S_1 < S_2 < P_{k+1}$

P_k	S_1	S_2	P_{k+1}	$G = P_{+1} - P_k$
3225539	3225562	3225646	3225647	108
165983	165999	166011	166013	30
669763	669764	669772	669787	24

Conclusion:
There are infinitely many intervals {Min(S(n),S(n+1)),Max(S(n),S(n+1))} which are prime free.

References:

I.M. Radu, *Mathematical Spectrum, Sheffield University, UK,* Vol. 27, No. 2, 1994/5, p. 43.

3. The Smarandache Ceil Function

Definition: For a positive integer n the Smarandache ceil function of order k is defined through [1]

$S_k(n) = m$ where m is the smallest positive integer for which n divides m^k.

In the study of this function we will make frequent use of the ceil function defined as follows:

$\lceil x \rceil$ = the smallest integer not less than x.

The following properties follow directly from the above definitions:

1. $S_1(n) = n$
2. $S_k(p^\alpha) = p^{\lceil \alpha/k \rceil}$ for any prime number p.
3. For distinct primes p, q, …r we have $S_k(p^\alpha q^\beta \cdots r^\delta) = p^{\lceil \alpha/k \rceil} q^{\lceil \beta/k \rceil} \cdots r^{\lceil \delta/k \rceil}$.

Theorem I. $S_k(n)$ is a multiplicative function.

[1] This function has a great resemblance to the Smarandache function. It's definition was proposed by K. Kashihara (Japan) and conveyed to the author by R. Muller.

A function f(n) is said to be multiplicative if for $(n_1,n_2) = 1$ if it is true that $f(n_1 n_2) = f(n_1)f(n_2)$. In our case it follows directly from (3) that if $(n_1,n_2) = 1$ then $S_k(n_1 n_2) = S_k(n_1)S_k(n_2)$.

However, consider $n=n_1 n_2$ when $(n_1,n_2)\neq 1$. In a simple case let $n_1=m_1\cdot p^\alpha$ and $n_2=m_2\cdot p^\beta$ with $(m_1,m_2)=1$ we then have $S_k(n) = S_k(m_1)S_k(m_2)\cdot p^{\lceil(\alpha+\beta)/k\rceil}$ which differs from $S_k(n_1)S_k(n_2)$ whenever $\lceil(\alpha+\beta)/k\rceil \neq \lceil\alpha/k\rceil + \lceil\beta/k\rceil$. In fact one easily proves that $\lceil(\alpha+\beta)/k\rceil = \lceil\alpha/k\rceil + \lceil\beta/k\rceil$ or $\lceil(\alpha+\beta)/k\rceil = \lceil\alpha/k\rceil + \lceil\beta/k\rceil - 1$.

Theorem II. $S_{k+1}(n)$ divides $S_k(n)$

Express n in prime factor form $n=p^\alpha q^\beta \cdots r^\delta$ and apply (3). We then see that all prime powers in $S_{k+1}(n)$ are less than or equal to those of $S_k(n)$, i.e. $S_{k+1}(n) \mid S_k(n)$.

Theorem III. For sufficiently large values of k we have $S_k(n)=\prod p_i$ where the product is taken over all distinct primes p_i of n.

By extending the argument in theorem II we have that, if $j=\max(\alpha, \beta, \ldots \delta)$ then $S_k(n)=pq\cdots r$ for $k \geq j$.

Corollary 1. $S_k(p) = p$ for any prime number p.

Corollary 2. If n is square free then $S_2(n) = n$.

Theorem IV. k exists so that $S_k(n!) = p\#$, where p is the largest prime dividing n.

p# denotes the product of all primes less than or equal to p. Let's write n! in prime factor form. $n!=2^\alpha 3^\beta \cdots p^\gamma$, where $\alpha>\beta>\ldots>\gamma$. In order to apply theorem III we need to find α. Consider $1\cdot2\cdot3\cdot4\cdot5\cdot6 \;\ldots\; n$. This product contains $[n!/2]$ even integers, $[n!/4]$ multiples of 4, etc ....and finally $[n!/2^\delta]$ multiples of 2^δ, where $2^\delta \leq n! < 2^{\delta+1}$. δ is determined by $\delta=[\log n/\log 2]$. From this we find that $S_k(n!) = p\#$ for

$$k = \alpha = \sum_{r=1}^{\delta} [n!/2^r]$$

Table 7. The Smarandache ceil function

n	S2	S3	S4	S5	S6	S7
4	2					
8	4	2				
9	3					
12	6					
16	4	4	2			
18	6					
20	10					
24	12	6				
25	5					
27	9	3				
28	14					
32	8	4	4	2		
36	6					
40	20	10				
44	22					
45	15					
48	12	12	6			
49	7					
50	10					
52	26					
54	18	6				
56	28	14				
60	30					
63	21					
64	8	4	4	4	2	
68	34					
72	12	6				
75	15					
76	38					
80	20	20	10			
81	9	9	3			
84	42					
88	44	22				
90	30					
92	46					
96	24	12	12	6		
98	14					
99	33					
100	10					
104	52	26				
108	18	6				
112	28	28	14			
116	58					
117	39					
120	60	30				
121	11					
124	62					
125	25	5				
126	42					
128	16	8	4	4	4	2
132	66					

n	S2	S3	S4	S5	S6	S7	S8
135	45	15					
136	68	34					
140	70						
144	12	12	6				
147	21						
148	74						
150	30						
152	76	38					
153	51						
156	78						
160	40	20	20	10			
162	18	18	6				
164	82						
168	84	42					
169	13						
171	57						
172	86						
175	35						
176	44	44	22				
180	30	1					
184	92	46					
188	94						
189	63	21					
192	24	12	12	12			
196	14						
198	66						
200	20	10					
204	102						
207	69						
208	52	52	26				
212	106						
216	36	6					
220	110						
224	56	28	28	14			
225	15						
228	114						
232	116	58					
234	78						
236	118						
240	60	60	30				
242	22						
243	27	9	9	3			
244	122						
245	35						
248	124	62					
250	50	10					
252	42						
256	16	8	4	4	4	4	2
260	130						
261	87						
264	132	66					

Calculations. Calculation of $S_k(n)$ for n < 1000 were carried out in Ubasic, which has a built in ceil function. The result is shown in table 7. Since $S_2(n) = n$ for square free numbers these have been excluded from the table. When $S_k(n)$ is square free the entries for larger values of k become repetitive. Instead of repeating these values the corresponding spaces in the table have been left blank.

4. The Smarandache Pseudo Function Z(n)

Definition[2]: Z(n) is the smallest positive integer m such that 1+2+...+m is divisible by n

Alternative formulation: For a given positive integer n, Z(n) equals the smallest positive integer m such that m(m+1)/2n is an integer.

The following properties follow directly from the definition:

1. Z(1)=1
2. Z(2)=3
3. For any odd prime number p, Z(p)=p-1
4. By extension of (3) we have $Z(p^k)=p^k-1$
5. In the special case $n=2^k$ we have $Z(2^k)=2^{k+1}-1$

Calculation of Z(n)

We need to find m so that m(m+1)=2nk has a positive integer solution for the smallest possible positive value of k.

$$m = \frac{-1+\sqrt{1+8kn}}{2}$$

For a given value of n the smallest square 1+8kn is found by executing the following program lines in Ubasic where effective use of the ISQRT(x) has been made:

[2] Definition by K.Kashihara (Japan) conveyed to the author by R. Muller, Erhus University Press, USA

```
10 INPUT "n ";n
20 k=0
30 inc k
40 x=1+8*k*n
50 if x<>(isqrt(x))^2 then goto 30
etc - to evaluate m
```

The complete program has been implemented for $n \le 1000$. The result is displayed in table 8.

Theorem: If n=pq, where p and q are two distinct primes with g=q-p, then

$$Z(n)=\mathrm{Min}(p(qk+1)/g \text{ where } pk+1\equiv 0 \pmod g),\ q(pk-1)/g \text{ where } pk-1\equiv 0 \pmod g))$$

Proof:
We have to consider three cases:

1. $p \mid m$ and $q \mid (m+1)$ which, since we assume q>p, we distinguish from
2. $p \mid (m+1)$ and $q \mid m$
3. $pq \mid (m+1)$

Case 1. Consider px=m and qy=m+1 which together with g=q-p gives

$$p(x-y)=gy-1 \tag{1}$$

Since we must have $p \mid (gy-1)$ we can put gy-1=pk where $k \mid (x-y)$. Our solution for y then becomes

$$y=(pk+1)/g \text{ with } pk+1\equiv 0 \bmod g) \tag{2}$$

Inserting this in (1) results in

$$p(x-(pk+1)/g)=pk$$

from which

$$x=(qk+1)/g \tag{3}$$

which we insert in m=px to obtain

$$m=p(qk+1)/g \text{ where k is determined through } pk+1\equiv 0 \pmod g)$$

Table 8a. Z(n) for n ≤ 1000, n non-prime

	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1	1	58	28	114	56	164	40	215	85
2	3	60	15	115	45	165	44	216	80
4	7	62	31	116	87	166	83	217	62
6	3	63	27	117	26	168	48	218	108
8	15	64	127	118	59	169	168	219	72
9	8	65	25	119	34	170	84	220	55
10	4	66	11	120	15	171	18	221	51
12	8	68	16	121	120	172	128	222	36
14	7	69	23	122	60	174	87	224	63
15	5	70	20	123	41	175	49	225	99
16	31	72	63	124	31	176	32	226	112
18	8	74	36	125	124	177	59	228	56
20	15	75	24	126	27	178	88	230	115
21	6	76	56	128	255	180	80	231	21
22	11	77	21	129	42	182	91	232	144
24	15	78	12	130	39	183	60	234	116
25	24	80	64	132	32	184	160	235	94
26	12	81	80	133	56	185	74	236	176
27	26	82	40	134	67	186	92	237	78
28	7	84	48	135	54	187	33	238	84
30	15	85	34	136	16	188	47	240	95
32	63	86	43	138	23	189	27	242	120
33	11	87	29	140	55	190	19	243	242
34	16	88	32	141	47	192	128	244	183
35	14	90	35	142	71	194	96	245	49
36	8	91	13	143	65	195	39	246	123
38	19	92	23	144	63	196	48	247	38
39	12	93	30	145	29	198	44	248	31
40	15	94	47	146	72	200	175	249	83
42	20	95	19	147	48	201	66	250	124
44	32	96	63	148	111	202	100	252	63
45	9	98	48	150	24	203	28	253	22
46	23	99	44	152	95	204	119	254	127
48	32	100	24	153	17	205	40	255	50
49	48	102	51	154	55	206	103	256	511
50	24	104	64	155	30	207	45	258	128
51	17	105	14	156	39	208	64	259	111
52	39	106	52	158	79	209	76	260	39
54	27	108	80	159	53	210	20	261	116
55	10	110	44	160	64	212	159	262	131
56	48	111	36	161	69	213	71	264	32
57	18	112	63	162	80	214	107	265	105

Table 8b . Z(n) for n ≤ 1000, n non-prime

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
266	56	318	159	366	60	416	64	469	133
267	89	319	87	368	160	417	138	470	140
268	200	320	255	369	81	418	76	471	156
270	80	321	107	370	184	420	104	472	176
272	255	322	91	371	105	422	211	473	43
273	77	323	152	372	216	423	188	474	236
274	136	324	80	374	187	424	159	475	75
275	99	325	25	375	125	425	50	476	119
276	23	326	163	376	47	426	71	477	53
278	139	327	108	377	116	427	182	478	239
279	62	328	287	378	27	428	320	480	255
280	160	329	140	380	95	429	65	481	221
282	47	330	44	381	126	430	215	482	240
284	71	332	248	382	191	432	351	483	69
285	75	333	36	384	255	434	216	484	120
286	143	334	167	385	55	435	29	485	194
287	41	335	134	386	192	436	327	486	243
288	63	336	63	387	171	437	114	488	304
289	288	338	168	388	96	438	72	489	162
290	115	339	113	390	39	440	175	490	195
291	96	340	119	391	68	441	98	492	287
292	72	341	154	392	48	442	51	493	203
294	48	342	152	393	131	444	111	494	208
295	59	343	342	394	196	445	89	495	44
296	111	344	128	395	79	446	223	496	31
297	54	345	45	396	143	447	149	497	70
298	148	346	172	398	199	448	384	498	83
299	91	348	87	399	56	450	99	500	375
300	24	350	175	400	224	451	164	501	167
301	42	351	26	402	200	452	112	502	251
302	151	352	319	403	155	453	150	504	63
303	101	354	59	404	303	454	227	505	100
304	95	355	70	405	80	455	90	506	252
305	60	356	88	406	28	456	95	507	168
306	135	357	84	407	110	458	228	508	127
308	55	358	179	408	255	459	135	510	84
309	102	360	80	410	40	460	160	511	146
310	124	361	360	411	137	462	132	512	1023
312	143	362	180	412	103	464	319	513	189
314	156	363	120	413	118	465	30	514	256
315	35	364	104	414	207	466	232	515	205
316	79	365	145	415	165	468	143	516	128

Table 8c. Z(n) for n ≤ 1000, n non-prime

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
517	187	565	225	616	175	667	115	715	65
518	111	566	283	618	308	668	167	716	536
519	173	567	161	620	279	669	222	717	239
520	64	568	496	621	161	670	200	718	359
522	116	570	75	622	311	671	121	720	224
524	392	572	143	623	266	672	63	721	308
525	125	573	191	624	351	674	336	722	360
526	263	574	287	625	624	675	324	723	240
527	186	575	275	626	312	676	168	724	543
528	32	576	512	627	132	678	339	725	174
529	528	578	288	628	471	679	97	726	120
530	159	579	192	629	221	680	255	728	272
531	117	580	144	630	35	681	227	729	728
532	56	581	83	632	79	682	340	730	219
533	246	582	96	633	210	684	152	731	85
534	267	583	264	634	316	685	274	732	183
535	214	584	511	635	254	686	343	734	367
536	335	585	90	636	159	687	228	735	195
537	179	586	292	637	195	688	128	736	575
538	268	588	48	638	87	689	52	737	66
539	98	589	247	639	71	690	275	738	287
540	80	590	59	640	255	692	519	740	184
542	271	591	197	642	107	693	98	741	38
543	180	592	480	644	160	694	347	742	371
544	255	594	296	645	129	695	139	744	464
545	109	595	34	646	152	696	144	745	149
546	104	596	447	648	80	697	204	746	372
548	136	597	198	649	176	698	348	747	332
549	243	598	91	650	299	699	233	748	407
550	99	600	224	651	62	700	175	749	321
551	57	602	300	652	488	702	324	750	375
552	207	603	134	654	108	703	37	752	704
553	237	604	151	655	130	704	384	753	251
554	276	605	120	656	287	705	140	754	116
555	74	606	303	657	72	706	352	755	150
556	416	608	512	658	140	707	202	756	216
558	216	609	174	660	120	708	176	758	379
559	129	610	60	662	331	710	284	759	230
560	160	611	234	663	51	711	315	760	95
561	33	612	135	664	415	712	623	762	380
562	280	614	307	665	189	713	92	763	217
564	47	615	164	666	36	714	84	764	191

Table 8d. Z(n) for n ≤ 1000, n non-prime

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
765	135	813	270	864	512	912	95	960	255
766	383	814	296	865	345	913	165	961	960
767	117	815	325	866	432	914	456	962	259
768	512	816	255	867	288	915	60	963	107
770	55	817	171	868	216	916	687	964	240
771	257	818	408	869	395	917	392	965	385
772	192	819	90	870	144	918	135	966	252
774	171	820	40	871	402	920	160	968	847
775	124	822	411	872	544	921	306	969	152
776	96	824	720	873	387	922	460	970	484
777	111	825	99	874	436	923	142	972	728
778	388	826	412	875	125	924	231	973	139
779	246	828	207	876	72	925	74	974	487
780	39	830	415	878	439	926	463	975	299
781	142	831	276	879	293	927	206	976	671
782	68	832	767	880	319	928	319	978	488
783	377	833	391	882	440	930	155	979	88
784	735	834	416	884	272	931	342	980	440
785	314	835	334	885	59	932	232	981	108
786	131	836	208	886	443	933	311	982	491
788	591	837	216	888	111	934	467	984	287
789	263	838	419	889	126	935	220	985	394
790	79	840	224	890	355	936	143	986	203
791	112	841	840	891	242	938	335	987	140
792	143	842	420	892	223	939	312	988	208
793	182	843	281	893	94	940	375	989	344
794	396	844	632	894	447	942	156	990	44
795	105	845	169	895	179	943	368	992	960
796	199	846	188	896	511	944	767	993	330
798	56	847	363	897	207	945	189	994	496
799	187	848	159	898	448	946	43	995	199
800	575	849	282	899	434	948	552	996	248
801	89	850	424	900	224	949	364	998	499
802	400	851	184	901	424	950	75	999	296
803	219	852	71	902	164	951	317	1000	624
804	200	854	244	903	42	952	272		
805	69	855	170	904	112	954	423		
806	155	856	320	905	180	955	190		
807	269	858	143	906	452	956	239		
808	303	860	215	908	680	957	87		
810	80	861	41	909	404	958	479		
812	231	862	431	910	104	959	273		

Case 2. Consider $px=m+1$ and $qy=m$ together with $g=q-p$. Repeating the calculation in case 1 results in:

$$m=q(pk-1)/g \text{ where k is determined through } pk-1\equiv 0 \pmod g$$

Case 3. Consider $m=pq-1$. We must have $pq-1<p(qk+1)/g$ to beat case 1. This inequality can be written $p(q(g-k)-1)<g$ where $g>k$ and $q\geq 2$ from which we see that this inequality is impossible. The argument is similar to reject case 3 in comparison with case 2. m is therefore given by the minimum value for m as calculated in cases 1 and 2, i.e.

$$Z(n)=\mathrm{Min}(p(qk+1)/g \text{ where } pk+1\equiv 0 \pmod g,\ q(pk-1)/g \text{ where } pk-1\equiv 0 \pmod g)$$

Corollary: $Z(n)$ is not a multiplicative function.

The above theory has been used to calculate $Z(pq)$ for a few prime number gaps $g=q-p$. The result is shown in table 9.

Table 9a. Z(n) for the first 10 prime gaps g=10 and g=30

Gap: q-p=10				Gap: q-p=30			
p	q	n	Z(n)	p	q	n	Z(n)
139	149	20711	2085	4297	4327	18593119	8056874
181	191	34571	3438	4831	4861	23483491	782621
241	251	60491	6024	5351	5381	28793731	10557522
283	293	82919	24904	5749	5779	33223471	12182131
337	347	116939	35047	6491	6521	42327811	15519980
409	419	171371	17178	6917	6947	48052399	11212457
421	431	181451	18102	7253	7283	52823599	22890468
547	557	304679	91348	7759	7789	60434851	22159704
577	587	338699	101551	7963	7993	63648259	14850994
631	641	404471	40383	8389	8419	70626991	25896843

Table 9b. Z(n) for the first 10 prime gaps g=20 and g=40

Gap: q-p=20				Gap: q-p=40			
p	q	n	Z(n)	p	q	n	Z(n)
887	907	804509	120631	19333	19373	374538209	28090849
1637	1657	2712509	949460	20809	20849	433846841	97615018
3089	3109	9603701	4321510	22573	22613	510443249	38283808
3413	3433	11716829	1757695	25261	25301	639128561	303586698
3947	3967	15657749	2348464	33247	33287	1106692889	470345309
5717	5737	32798429	11479736	38461	38501	1480786961	703374768
5903	5923	34963469	12236918	45013	45053	2027970689	152098927
5987	6007	35963909	5394286	48907	48947	2393850929	179537596
6803	6823	46416869	16245563	52321	52361	2739579881	68488188
7649	7669	58660181	26396698	60169	60209	3622715321	815109442

Chapter III

Loops and Invariants

In many practical as well as theoretical processes we repeat the same operation on an object again and again in order to arrive at a final result, or sustain a certain state or maybe simply to see what is going to happen. Each time a hammer hits a nail the nails sinks a bit deeper until it can sink no further. This repeated operation has resulted in an invariant state. A different situation occurs in an engine when energy is used to make a piston perform cycles or loops. Many similar phenomena occur when repeating processes on numbers through iterations. Before going into the problems of this section let us consider the iteration process itself.

Let $I(n)$ define an operation to be carried out on n. If we apply this operation to $I(n)$ itself we say that we perform an iteration and could write this as $I(I(n))$. After a number of iterations we could have something like $I(I(\ldots I(n)\ldots))$, which we will write $I_{(k)}(n)$ to indicate the *kth* iteration. Alternatively we can use n_k to denote the result of the *kth* iteration and n_0 to denote the starting value. We then have

$$n_1 = I(n_0), \qquad n_2 = I(n_1), \qquad \ldots\ldots \qquad n_{k+1} = I(n_k)$$

Let us apply this to a simple case where $I(n)$ is defined through $I(n) = 1/n$. If we take $n_0=1$ then $n_k=1$ for all k, i.e. the result of the iteration is invariant. If we take $n_0=2$ then $n_k=1/2$ for odd values of k and $n_k=2$ for even values of k, i.e. we have an iteration resulting in a loop of length 2. If we apply the iteration process to $I(n)=n^2$ then the result will be forever increasing and we say that the iteration is divergent.

Since we are dealing with a very important concept which has attracted a lot attention some of the topics in this section have been dealt with before. In particular Perfect Digital Invariants [1], which has recently been reactivated under another name "Steinhaus' problem" [2]. However, all results presented here have been generated in recent studies carried out by the author and have been retained even though some of them may duplicate earlier results. This proved necessary in order to arrive at a consistent presentation and some new results. J. S. Madachy has made me aware of more literature on this interesting topic some of which is listed in the references.

1. Perfect Digital Invariants and Related Loops

For an arbitrary positive integer

$$N_k = a_n 10^n + a_{n-1} 10^{n-1} + \dots a_1 10 + a_0$$

we define $N_{k+1} = I(N_k)$ through

$$I(N_k) = a_n{}^q + a_{n-1}{}^q + \dots + a_1{}^q + a_0{}^q, \text{ where } q \in N,\ q \geq 2.$$

For a given positive integer n_0 the iteration process $n_1 = I(n_0)$, $n_2 = I(n_1)$, ... is continued until one of the following situations is reached:

I.	$I(n_k) = n_k$	Perfect Digital Invariant (PDI)
II.	$I(n_{k-L}) = n_k$	Loop of length L
III.	$I(n_k) \rightarrow \infty$	Divergent

Case III is impossible. It follows from the following determination of the upper search limit for each value of q. Let us consider the largest possible n-digit number $N = 999...99 = 10^n-1$. We have $I(N) = n \cdot 9^q$ and need to determine N_{max} so that $N-I(N)>0$ for all $N>N_{max}$. Let u be the largest value of n for which $10^n-1-n \cdot 9^q < 0$, i.e.

$$10^u - 1 - u \cdot 9^q < 0 \quad \text{while} \quad 10^{u+1} - 1 - (u+1) \cdot 9^q > 0$$

then there is a smallest positive integer a, $1 \leq a \leq 9$, such that

$$a \cdot 10^u + 10^u - 1 - a^q - u \cdot 9^q > 0$$

This gives $N_{max} = (a+1) \cdot 10^u - 1$ as an upper limit for solutions. N_{max} could be improved by looking for a smaller value than 9 for the second most significant figure but this would give more complications than benefits in computer implementation. That N_{max} exists proves that the iteration process does not diverge since after a number of iterations larger than N_{max} a previously assumed value must be repeated completing a loop or collapsing on an invariant.

Only a small subset of all integers $< N_{max}$ needs to be used as input numbers in a search program. The following two input criteria greatly reduce the computer execution time.

1. The order in which digits occur in an input number is of no importance. N_0 = 2337 will give the same result as N_0 = 3732. A number whose digits are a permutation of an already used input number will therefore be rejected. As an example take q=5 for which N_{max} = 299999. In this case an input number needs to have maximum 6 digits of which there can be at most 6 ones or twos and a maximum of 5 of the other digits.

2. Input criterion number 1 is used so that the search always proceeds from a smaller to a larger input number. If for any input number N_0 we have $I(N_0)<N_0$ then the iteration process is aborted since $I(N_0)$ has either been dealt with before or doesn't meet criterion number 1.

Complete solutions were calculated for q = 2,3,4, ... 15. Apart from the trivial case I(N)=1 these solutions are given in table 1 together with the upper search limit for each q. The longest loop is of length 381 for q = 14. There are no Perfect Digital Invariants (PDIs), i.e. solutions to N = I(N) for q=2, q=12 and q=15.

References:

1. Lionel E. Deimel Jr. and Michael T. Jones, *Journal of Recreational Mathematics*, pgs 87-108, Vol. 14 .2
2. *Personal Computer World*, page 333, January 1996
3. Dean Morrow, *Journal of Recreational Mathematics*, pgs 9-12, Vol. 27.1

Table 1a. PDIs and related loops

q	N_{max} $S=\Sigma N$	Length of Loop	Smallest term	Largest term	N
2	199/4	8	4	145	3
3	2999	1	153		17
		1	370		16
	S=76	1	371		19
		1	407		2
		2	136	244	1
		2	919	1459	11
		3	55	250	6
		3	160	352	3
4	29999	1	1634		1
		1	8208		8
	S=153	1	9474		2
		2	2178	6514	6
		7	1138	13139	135
5	299999	1	4150		1
		1	4151		1
	S=345	1	54748		10
		1	92727		1
		1	93084		1
		1	194979		1
		2	58618	76438	23
		2	89883	157596	1
		4	10933	73318	9
		6	8299	150898	13
		10	8294	183635	24
		10	9044	133682	33
		12	24584	180515	93
		22	9045	167916	112
		28	244	213040	21
6	3999999	1	548834		2
		2	63804	313625	5
	S=401	3	282595	845130	71
		4	93531	650550	5
		10	239459	1083396	167
		30	17148	1758629	150

Table 1b. PDIs and related loops

q	N_{max} S=Σ N	Length of Loop	Smallest term	Largest term	N
7	$4\cdot10^7$-1	1	1741725		2
		1	4210818		7
	S=1012	1	9800817		10
		1	9926315		12
		1	14459929		1
		2	2755907	6586433	1
		2	8139850	9057586	30
		3	2767918	8807272	46
		6	2191663	9646378	21
		12	1152428	14349038	70
		14	922428	16417266	28
		21	982108	14600170	93
		27	253074	18575979	141
		30	376762	19210003	225
		56	86874	19134192	114
		92	80441	14628971	210
8	$4\cdot10^8$-1	1	24678050		19
		1	24678051		188
	S=1544	1	88593477		12
		3	7973187	77124902	14
		25	8616804	149277123	828
		154	6822	153362052	482
9	$4\cdot10^9$-1	1	146511208		34
		1	472335975		22
	S=5058	1	534494836		53
		1	912985153		34
		2	144839908	1043820406	45
		2	277668893	756738746	3
		3	180975193	951385123	8
		3	409589079	1339048071	13
		4	52666768	574062013	171
		8	20700388	1212975109	545
		10	62986925	931168247	164
		10	180450907	857521513	87
		19	42569390	1001797253	403
		24	52687474	708260569	373
		28	322219	1298734342	262
		30	41179919	1202877221	1434
		80	32972646	1724515947	1133
		93	2274831	1430717631	273

Table 1c. PDIs and related loops

q	N_{max} $S=\Sigma N$	Length of Loop	Smallest term	Largest term	N
10	$4 \cdot 10^{10}-1$	1	4679307774		1
		2	304162700	344050075	13
	S=7408	6	123148627	7540618502	159
		7	1139785743	9131926726	1084
		17	62681428	13957953853	706
		81	20818070	15434111703	711
		123	192215803	14230723551	4733
11	$4 \cdot 10^{11}-1$	1	32164049650		30
		1	32164049651		63
	S=7628	1	40028394225		5
		1	42678290603		6
		1	44708635679		7
		1	49388550606		11
		1	82693916578		21
		1	94204591914		1
		2	4370652168	11346057072	58
		3	2491335968	71768229638	75
		3	4517134494	33424168842	2
		3	6666140097	36704410767	3
		5	416528075	103153306403	54
		5	2181207047	28167146357	43
		7	9005758176	71727610926	50
		10	3967417642	98110415227	100
		18	12650989279	128870085703	486
		20	2075164239	127554589656	1205
		42	195493746	106744983639	1075
		48	101858747	134844138593	1015
		117	739062760	169812860326	1566
		118	872080538	165906857819	609
		181	8922100	176062673167	1142
12	$4 \cdot 10^{12}-1$	5	98840282759	785119716404	128
		40	2700216437	1157645923834	1557
	S=9466	94	4876998775	1281243062759	1883
		133	16068940818	1200615480166	5897

Table 1d. PDIs and related loops

q	N_{max} $S=\Sigma N$	Length of Loop	Smallest term	Largest term	N
13	$4\cdot10^{13}-1$	1	564240140138		10
		3	3396705890823	6294418483143	240
	S=35177	5	2310198454262	10830899644190	50
		5	3656948275943	6405584099531	1601
		8	688271525816	5647840775906	8846
		9	1113928853354	5840462013812	503
		13	672769417360	6842950405261	240
		16	662218816395	10829615029038	3403
		18	100555910945	7735617503306	404
		22	221192011997	7722486519974	821
		31	759039131807	12916172647637	625
		100	98183216073	10719727658703	6794
		146	111240729637	11270705761846	11753
14	$4\cdot10^{14}-1$	1	28116440335967		60
		4	6674458190799	74548238736415	9
	S=40180	4	27510477911590	47800729611562	428
		5	841332967215	57268678554888	154
		6	5911230616470	55918604360271	441
		6	22955961974580	94220062144011	311
		7	5833130055708	97267770775241	35
		14	1447300158177	92876091448633	5119
		15	4423275478678	78789825354783	1698
		65	931899363208	118800319349172	3517
		96	157270047611	92354596687594	9448
		381	78924999008	142415666495594	18959
15	$4\cdot10^{15}-1$	3	255349823145519	447090882837630	17
		8	76058866219899	729415146625008	10157
	S=48785	11	70433029388274	657638038056753	518
		12	41845353013296	632509424234352	1261
		15	14745577655668	619115571288208	177
		19	14746637048586	895875047683584	362
		30	1096193499692	1030397118565900	941
		46	10059036985836	662536011243090	1302
		75	593343453343	864468451964287	18270
		216	92683904599	1065173414568230	2400
		362	595476555320	1145004602355100	13379

2. The Squambling Function

The squambling function U(N) originates from *The Penguin Dictionary of Curious and Interesting Numbers*, page 169. In its present form the problem was formulated by G. Samson in the November 1995 issue of *Personal Computer World:*

For a given integer $N = a_0 + a_1 10 + \ldots + a_n 10^n > 10^n$, $a_n \neq 0$, the squambling function is given by

$$U(N) = a_0^{n+2} + a_1^{n+1} + \ldots + a_n^2.$$

Iterating the squambling function will result in an invariant or a loop. As in the previous problem there are no starting values for which the process diverges. To see this consider

$$U(N) = a_0^{n+2} + a_1^{n+1} + \ldots + a_n^2 < 9^2 + 9^3 + \ldots + 9^{n-2} = 81(9^{n-1}-1)/8$$

If N_0 exists for which $U(N) < N_0$ for all $N > N_0$ then the squambling function must return to one of its previous values after a sufficiently large number of iterations, i.e. the process converges (or goes into a loop). N_0 exists and is estimated to be less than 10^{43} from the inequality below

$$U(N) < 81\cdot(9^{n-1}-1)/8 < 10^n < N$$

More detailed considerations may considerably lower this upper limit for N_0, but in any case the upper limit is so large that a complete search for invariants and loops is hardly feasible. The result of a search for squam-invariants and squam-loops for integers up to 10^6 is summarized in table 2, where L=the total number of numbers in the loop (=length of the loop, which equals 1 for an invariant), M=the smallest number in the loop, N=the largest number in the loop and Q=the number of loops found for initiating integers less than 10^6.

Table 2. Squam-loops and squam-invariants

L	1	1	1	1	1	3	8	105
M	1	43	63	47016	542186	126529	579	5
N	1	43	63	47016	542186	4787463	59830	43055027
Q	165421	613722	4617	125	6	13	2077	214018

If there are more invariants and loops then they must have a smallest element greater than 10^6, or more precisely a smallest element which is larger than the upper limit for a previous search. This has been used in an extended search for starting integers up to 50,000,000. No new loops were found. **Question:** Are there any?

Let us finally consider what happens if we reduce the powers to which each digit is raised by one. This makes the process à fortiori convergent as can be seen from the arguments used previously. The iteration process was examined for starting integers up to 10^5. No loops were found. Apart from iterations ending on one-digit invariants a number of other invariants were found.

Table 3. The number of start integers Q below 10^5 resulting in an invariant I

I	89	135	175	518	598	1306	1676	2427
Q	347	492	54	319	128	102	20	27

Question: Are there no loops in this case?

3. Wondrous Numbers

This study deals with the extended Wondrous Numbers Conjecture stated as follows by B.C. Wiggin [1]:

> *Any integer* $n \geq (D-1)$ *may be directed through a series of iterations as a function* $n \equiv R \pmod{D}$. *Starting with* n_0 *the series of iterated Wondrous Numbers are* $n_1, \quad n_2, \dots n_k, \dots$
> *Definition: Let* $n_k \equiv R_k \pmod{D}$
>
> *if* $R_k=0$ *then* $n_{k+1}=n_k/D$
> *if* $1 \leq R_k \leq D-2$ *then* $n_{k+1}=n_k(D+1)-R_k$
> *if* $R_k=D-1$ *then* $n_{k+1}=n_k(D+1)+1$
>
> *It is conjectured that the series ultimately converges to $n<D$.*

This study will show that the above conjecture does not hold.

To examine the behaviour of the above process assume that for a given n_k we have

$$n_k \equiv 0 \pmod{D}$$

in which case

$$n_{k+1} = n_k/D; \qquad n_k = Dn_{k+1}$$

Let

$$n_{k+1} \equiv R_{k+1} \pmod D$$

We have to consider three cases

Case I

$$R_{k+1} = 0$$

then

$$n_{k+2} = n_{k+1}/D; \qquad n_{k+2} = n_k/D^2$$

Case II

$$0 < R_{k+1} \leq D-2$$

then

$$n_{k+2} = n_{k+1}(D+1) - R_{k+1}$$

From this we see that $n_{k+2} \equiv 0 \pmod D$ and substituting from (1) that

$$n_{k+2} < n_k(1 + 1/D)$$

If R_{k+2i-1} stays within the above limits for I=1, 2, … m then

$$n_{k+2m} < n_k(1 + 1/D)^m$$

Case III

$$R_{k+1} = D-1$$

As in case II $n_{k+2} \equiv 0 \pmod D$ but

$$n_{k+2} < n_k(1 + 1/D) + 1 \tag{4}$$

For the series to diverge the case R=0 must occur with a frequency which is much higher than the expected 1/D (on the average). In a worst case scenario let us assume that $R \neq 0$ so many times that it balances the effect of the occurrence of case I. Ignoring (4) which can be considered as compensated by ignoring R_{k+1} in (3) we then have.

$$n_k(1 + 1/D)^m/D^2 = n_k$$

For D = 11 this gives $m \approx 56$ just to hold the balance. If remainders R are evenly distributed then m would vary around 10. It is very safe to say that the series does not diverge. However, it does not always converge to $n < D$ as conjectured. Brendan Woods [2] reported that she failed to get termination with (D,N) = (13,70), (14,75), (58,59), (82,83) and (198,199). The reason is that these cases produce loops as shown in table 4. $N_0 = 199$, D = 198 produces a very long loop of length 2279 requiring 2499 iterations.

Charles Ashbacher [3] investigated all divisors in the interval $3 \leq D \leq 12$ for all initial values $n_0 < 14000000$. Even with the most effective programming and up to date equipment this is a very impressive piece of work. In all cases iterations terminated on a one-digit number. He listed those cases for D=11 and D=12 where more than 1000 iterations were required before the terminal value was reached and made the following observation *"...for certain series of ascending n the number of cycles descended in steps of 2 contrary to the expected behaviour that the number of cycles rises with the size of the number"*. He adds that no justification has been found for this and challenges readers to further explore this behaviour. The remainder of this section will be devoted to an explanation of this mystery though an analysis of the case D = 11. A similar analysis to the one below has been carried out for D = 12 with similar results.

Since all iterations result in a one-digit terminal value all cases with $n_0 < 14000000$ which require more than 1000 iterations can be classified according to their terminal value. This is done in the column marked p in table 6, which contains Ashbacher's table as a subset. There are only three different terminal values 3, 7 and 8.

Table 5 shows the 180 first iterations for $n_0 = 1345680$, which is the first entry in Ashbacher's table. Table 5 confirms the above theory that at least every second R is zero. This explains the step 2 decrease with ascending n since this causes the terms to oscillate. In general every second term was out of bounds of the investigation which was limited to $n_0<14000000$. These cases with long cycles are due to long "freak" oscillate. In general every second term was out of bounds of the investigation which was limited to $n_0<14000000$. These cases with long cycles are due to long "freak" sequences if iterations with remainders R=0. Let's apply (3') to the first occurrence of R=0 for even k in table 5. $n_1 = 16148154$, $n_{167} = 167493499$.

$$n_{167} < 16148154(1 + 1/11)^{82}/121 \approx 167494096$$

This, as is seen, is a very good approximation.

Table 4. Loops instead of anticipated convergence

n	n_0	Loop starts for :	First term of loop	Loop finishes for:	Length of loop
13	70	92	1911	162	71
14	75	13	166	91	79
58	59	19	4002	555	537
82	83	153	13120	925	773
198	199	221	61380	2499	2279

We also see from table 5 that it contains a sequence of numbers which also occur in table 6, i.e. the initiating value $n_0 = 1345680$ is the "grandparent" of a whole family of larger initiating values having the same terminal value 8. However, table 6 also contains initiating values with terminal value 8 which do not occur in table 5. At some stage, however, these will merge with the iteration process for the "grandparent". These cases have been identified and labeled in column c (c="child") in table 6. These "children" have been used as starting numbers for iteration processes to see at which point they will join the "grandparent" iteration. The result is shown in tables 7 and 8, where K-c is the number of iterations for the child (which may have a number of children of its own as can be seen from table 6), and K-p is the number of iterations for the grandparent before merging occurs. It is amazing how soon this happens. Only the 8-family child 11700624 makes it on its own almost to the end. N-max is the maximum value for the iteration process which occurs after K-m iterations. The 3-family is childless.

References:

1. B.C. Wiggin, *Journal of Recreational Mathematics*, pgs 52-56, Vol. 20.1
2. M. Mudge, *Personal Computer World*, page 335, Dec. 1995
3 C. Ashbacher, *Journal of Recreational Mathematics*, pgs. 12-15, Vol. 24.1

Table 5. The first 180 iterations for D=11, N=1345680

k	R	n	k	R	n	k	R	n	k	R	n
1	0	16148154	2	9	1468014	91	0	810226780	92	1	73656980
3	0	17616159	4	1	1601469	93	0	883883759	94	5	80353069
5	0	19217627	6	4	1747057	95	0	964236823	96	4	87657893
7	0	20964680	8	9	1905880	97	0	1051894712	98	8	95626792
9	0	22870551	10	9	2079141	99	0	1147521496	100	8	104320136
11	0	24949683	12	8	2268153	101	0	1251841624	102	6	113803784
13	0	27217828	14	8	2474348	103	0	1365645402	104	7	124149582
15	0	29692168	16	9	2699288	105	0	1489794977	106	2	135435907
17	0	32391447	18	10	2944677	107	0	1625230882	108	2	147748262
19	0	35336125	20	1	3212375	109	0	1772979142	110	2	161179922
21	0	38548499	22	7	3504409	111	0	1934159062	112	7	175832642
23	0	42052901	24	7	3822991	113	0	2109991697	114	10	191817427
25	0	45875885	26	6	4170535	115	0	2301809125	116	10	209255375
27	0	50046414	28	8	4549674	117	0	2511064501	118	2	228278591
29	0	54596080	30	3	4963280	119	0	2739343090	120	1	249031190
31	0	59559357	32	1	5414487	121	0	2988374279	122	1	271670389
33	0	64973843	34	10	5906713	123	0	3260044667	124	10	296367697
35	0	70880557	36	8	6443687	125	0	3556412365	126	8	323310215
37	0	77324236	38	3	7029476	127	0	3879722572	128	10	352702052
39	0	84353709	40	1	7668519	129	0	4232424625	130	10	384765875
41	0	92022227	42	3	8365657	131	0	4617190501	132	2	419744591
43	0	100387881	44	10	9126171	133	0	5036935090	134	8	457903190
45	0	109514053	46	9	9955823	135	0	5494838272	136	6	499530752
47	0	119469867	48	3	10860897	137	0	5994369018	138	9	544942638
49	0	130330761	50	8	11848251	139	0	6539311647	140	10	594482877
51	0	142179004	52	1	12925364	141	0	7133794525	142	6	648526775
53	0	155104367	54	3	14100397	143	0	7782321294	144	10	707483754
55	0	169204761	56	5	15382251	145	0	8489805049	146	1	771800459
57	0	184587007	58	5	16780637	147	0	9261605507	148	3	841964137
59	0	201367639	60	4	18306149	149	0	10103569641	150	6	918506331
61	0	219673784	62	9	19970344	151	0	11022075966	152	10	1002006906
63	0	239644119	64	10	21785829	153	0	12024082873	154	8	1093098443
65	0	261429949	66	1	23766359	155	0	13117181308	156	1	1192471028
67	0	285196307	68	3	25926937	157	0	14309652335	158	6	1300877485
69	0	311123241	70	5	28283931	159	0	15610529814	160	1	1419139074
71	0	339407167	72	10	30855197	161	0	17029668887	162	2	1548151717
73	0	370262365	74	6	33660215	163	0	18577820602	164	5	1688892782
75	0	403922574	76	1	36720234	165	0	20266713379	166	0	1842428489
77	0	440642807	78	1	40058437	167	8	167493499	168	0	2009921980
79	0	480701243	80	6	43700113	169	5	182720180	170	0	2192642155
81	0	524401350	82	5	47672850	171	6	199331105	172	0	2391973254
83	0	572074195	84	10	52006745	173	0	217452114	174	10	19768374
85	0	624080941	86	8	56734631	175	0	237220489	176	10	21565499
87	0	680815564	88	10	61892324	177	0	258785989	178	2	23525999
89	0	742707889	90	8	67518899	179	0	282311986	180	10	25664726

Table 6. Some Wondrous Numbers and their Genetic Relations, D=11

#	n	k	p	c	#	n	k	p	c	#	n	k	p	c
1	1345680	1127	8	0	42	5308496	1148	7	0	83	10648288	1132	7	0
2	1468014	1125	8	0	43	5414487	1095	8	0	84	10860897	1079	8	0
3	1601469	1123	8	0	44	5572804	1203	7	1	85	10971881	1191	8	4
4	1747057	1121	8	0	45	5684068	1150	8	1	86	11178458	1187	7	2
5	1905880	1119	8	0	46	5791086	1146	7	0	87	11178462	1187	7	1
6	2038440	1170	7	0	47	5906713	1093	8	0	88	11311510	1230	3	0
7	2079141	1117	8	0	48	6079422	1201	7	1	89	11401647	1134	8	1
8	2223752	1168	7	0	49	6200801	1148	8	1	90	11401651	1134	8	3
9	2268153	1115	8	0	50	6317548	1144	7	0	91	11401653	1134	8	2
10	2425911	1166	7	0	51	6443687	1091	8	0	92	11616314	1130	7	0
11	2474348	1113	8	0	52	6632096	1199	7	1	93	11700624	1077	8	5
12	2597542	1168	8	1	53	6764510	1146	8	1	94	11735002	1242	7	3
13	2646448	1164	7	0	54	6891870	1142	7	0	95	11735026	1242	7	4
14	2699288	1111	8	0	55	7029476	1089	8	0	96	11848251	1077	8	0
15	2833682	1166	8	1	56	7235013	1197	7	1	97	11969324	1189	8	4
16	2887034	1162	7	0	57	7379465	1144	8	1	98	12194681	1185	7	2
17	2944677	1109	8	0	58	7518403	1140	7	0	99	12194685	1185	7	1
18	3091289	1164	8	1	59	7668519	1087	8	0	100	12339829	1228	3	0
19	3149491	1160	7	0	60	7892741	1195	7	1	101	12438160	1132	8	1
20	3212375	1107	8	0	61	8050325	1142	8	1	102	12438164	1132	8	3
21	3372315	1162	8	1	62	8050330	1142	8	2	103	12438167	1132	8	2
22	3435808	1158	7	0	63	8201894	1138	7	0	104	12438192	1132	8	6
23	3504409	1105	8	0	64	8365657	1085	8	0	105	12672342	1128	7	0
24	3606876	1213	7	1	65	8610263	1193	7	1	106	12764317	1075	8	5
25	3678889	1160	8	1	66	8782172	1140	8	1	107	12801820	1240	7	3
26	3748154	1156	7	0	67	8782176	1140	8	3	108	12801846	1240	7	4
27	3822991	1103	8	0	68	8782178	1140	8	2	109	12925364	1075	8	0
28	3934773	1211	7	1	69	8947520	1136	7	0	110	13057444	1187	8	4
29	4013333	1158	8	1	70	9126171	1083	8	0	111	13303288	1183	7	2
30	4088895	1154	7	0	71	9393014	1191	7	1	112	13303292	1183	7	1
31	4170535	1101	8	0	72	9580551	1138	8	1	113	13461631	1226	3	0
32	4292479	1209	7	1	73	9580555	1138	8	3	114	13568901	1130	8	1
33	4378181	1156	8	1	74	9580557	1138	8	2	115	13568906	1130	8	3
34	4460612	1152	7	0	75	9760931	1134	7	0	116	13568909	1130	8	2
35	4549674	1099	8	0	76	9955823	1081	8	0	117	13568928	1130	8	7
36	4682704	1207	7	1	77	10057558	1193	8	4	118	13568936	1130	8	6
37	4776197	1154	8	1	78	10246920	1189	7	2	119	13824373	1126	7	0
38	4866122	1150	7	0	79	10246924	1189	7	1	120	13924709	1073	8	5
39	4963280	1097	8	0	80	10451510	1136	8	1	121	13965621	1238	7	3
40	5108404	1205	7	1	81	10451514	1136	8	3	122	13965648	1238	7	5
41	5210396	1152	8	1	82	10451516	1136	8	2	123	13965650	1238	7	4

Table 7. The 7-family. Grandparent = 2038440

#	N-child	N-child one one step before merging	N-parent one step before merging	K - c	K - p	N- merger	N-max	K - m
1	3606876	293535231	2223752	46	3	26685021	8601368512990	562
2	1024692	381088411	2887034	28	9	34644401	8601368512990	538
3	1173500	764423308	5791086	97	25	69493028	8601368512990	591
4	1173502	293535231	2223752	75	3	26685021	8601368512990	591
5	1396564	293535231	2223752	71	3	26685021	8601368512990	587

Table 8. The 8-family. Grandparent = 1345680

#	N-child	N-child one one step before merging	N- one before merging	K - c	K - p	N- merger	N-max	K - m
1	259754	504634735	3822991	66	25	45875885	22512799837	489
2	805033	326613848	2474348	30	15	29692168	22512799837	463
3	878217	388697375	2944677	32	19	35336125	22512799837	461
4	100575	504634735	3822991	91	25	45875885	22512799837	514
5	117006	38236	290	1048	1098	3476	43996530071824	348
6	124381	177629694	1345680	6	1	16148154	22512799837	453
7	135689	211393897	1601469	8	5	19217627	22512799837	451

4. Iterating d(n) and σ(n) - Two problems proposed by F. Smarandache

(a) *Let n be a positive integer and d(n) the number of positive divisors of n including 1 and n. Find the (smallest)*[1] *k for which d(d(...d(n)...))=2 after k iterations, i.e. find k so that* $d_{(k)}(n) = 2$.

d(n) is an important arithmetic function. We will look at its most important properties. The factors of p^α, where p is a prime, are 1, p, p^2, … p^α. Consequently $d(p^\alpha) = 1+\alpha$. The number of factors in $n = p^\alpha p^\beta$ is easily seen to be $(1+\alpha)(1+\beta)$ from which the following important theorem follows:

[1] In fact k is a (single-valued) function of n.

If n_1 and n_2 have no common divisor, i.e. are relatively prime which we write $(n_1,n_2)=1$, then $d(n_1 n_2)=d(n_1)d(n_2)$. We say that the *arithmetic function $d(n)$ is multiplicative.*

With n written in standard form $n=p_1^{\alpha}p_2^{\beta}\dots p_r^{\tau}$ we have

$$d(n) = (1+\alpha)(1+\beta)\dots(1+\tau) \tag{1}$$

We can now state:

$d(n)<n$ for all n.

$d(n)=1$ if and only if $n=1$.

$d(n)=2$ if and only if n is a prime number.

From the above properties we see that d(n) is a measure of how far n is from being a prime number. The larger the number of factors of n the larger is d(n), d(n) being equal to 2 only when n is a prime. This makes it interesting to try to answer Smarandache's question [1]: *How many iterations k are required in $d_{(k)}(n)=2$?*

Before looking at this problem let's make an important observation:

Given an arbitrarily large positive integer k we can always construct infinitely many integers n for which $d_{(i)}(n)>2$ for all $i<k$ and $d_{(k)}(n)=2$.

Here is how: Let $p_1, p_2, \dots p_k$ be odd primes (not necessarily distinct). Make the following series of constructions:

$$n_1 = p_1 \qquad d(n_1) = 2$$

$$n_2 = p_2^{p_1-1} \qquad d(n_2) = p_1$$

$$n_3 = p_3^{p_2^{p_1-1}-1} = p_3^{n_2-1} \qquad d(n_3) = p_2^{p_1-1}$$

$$n_4 = p_4^{n_3-1} \qquad d(n_4) = n_3$$

....

$$n_k = p_k^{n_{k-1}-1} \qquad d(n_k) = n_{k-1}$$

So that for $n = n_k$ we have $d_{(k)}(n) = 2$ while $d_{(i)}(n) > 2$ for $i < k$. Since we can choose our primes anyway we like as long as $p_i \neq 2$ this construction can be carried out in infinitely many ways. If we ask for *the smallest k for which $d_{(k)}(n) = 2$ for all n* then the answer is that such a k does not exist.

Now to the problem. Factorizations and applications of (1) have been used to calculate k as a function of n for $n \leq 100$, table 9. This does not tell us much about the general behaviour of $d_{(k)}(n)$. Table 10 provides some interesting cumulative statistics for $n \leq 10^6$. No more than 6 iterations are required for any $n \leq 10^6$. It seems strange that the column for $k = 7$ remains empty for $n \leq 10^5$ and $n \leq 10^6$ in particular in view of the regular behaviour in the column for $k = 3$ and our previous observation that k can be arbitrarily large for properly chosen sufficiently large n. This calls for further study.

Table 9. K as a function of n for $n \leq 100$

n	k	n	k	n	k	n	k	n	k	n	k	n	k	n	k	n	k	n	k
1	1	11	1	21	3	31	1	41	1	51	3	61	1	71	1	81	2	91	3
2	1	12	4	22	3	32	4	42	4	52	4	62	3	72	5	82	3	92	4
3	1	13	1	23	1	33	3	43	1	53	1	63	4	73	1	83	1	93	3
4	2	14	3	24	4	34	3	44	4	54	4	64	2	74	3	84	5	94	3
5	1	15	3	25	2	35	3	45	4	55	3	65	3	75	4	85	3	95	3
6	3	16	2	26	3	36	3	46	3	56	4	66	4	76	4	86	3	96	5
7	1	17	1	27	3	37	1	47	1	57	3	67	1	77	3	87	3	97	1
8	3	18	4	28	4	38	3	48	4	58	3	68	4	78	4	88	4	98	4
9	2	19	1	29	1	39	3	49	2	59	1	69	3	79	1	89	1	99	4
10	3	20	4	30	4	40	4	50	4	60	5	70	4	80	4	90	5	100	3

Let p# denote the product of all prime numbers less than or equal to p and consider the largest number r of distinct prime numbers which are needed to construct any integer $\leq 10^s$ i.e. $p_r\# \leq 10^s < p_{r+1}\#$. With these primes consider all possible constructions

$$2^{\alpha}3^{\beta}\ldots p_r^{\tau} \leq 10^s \tag{2}$$

This does not mean constructing all $n \leq 10^s$ but it does mean arriving at a structure into which all prime factorizations of $n \leq 10^s$ fits. This will be so because any number $\leq 10^s$ not produced by (2) will have fewer prime factors and smaller powers than one or more of the integers produced by (2). To illustrate this let's look at the case $n \leq 100$. We have $2 \cdot 3 \cdot 5 < 100 < 2 \cdot 3 \cdot 5 \cdot 7$. We will therefore consider all possible construction $2^{\alpha} \cdot 3^{\beta} \cdot 5^{\varepsilon} \leq 100$. These are obtained for $\alpha \leq 6$, $\beta \leq 4$ and $\varepsilon \leq 2$ resulting in table 11.

Table 10. Number of iteration k required to arrive at $d_{(k)}(n)=2$

n ≤	k=1	k=2	k=3	k=4	k=5	k=6	k=7
10	4	2	3				
100	25	7	34	28	5		
1000	168	16	348	323	144		
10000	1229	33	3444	3181	2108	4	
100000	9592	79	34429	30466	24839	594	
1000000	78498	189	344238	292460	271971	12643	

Table 11. All possible prime factorization combinations C for n ≤ 100

#	C	d	n	#	C	d	n	#	C	d	n	#	C	d	n
1	0 0 0	1	1	11	1 0 0	2	2	21	2 0 2	9	100	31	4 1 0	10	48
2	0 0 1	2	5	12	1 0 1	4	10	22	2 1 0	6	12	32	5 0 0	6	32
3	0 0 2	3	25	13	1 0 2	6	50	23	2 1 1	12	60	33	5 1 0	12	96
4	0 1 0	2	3	14	1 1 0	4	6	24	2 2 0	9	36	34	6 0 0	7	64
5	0 1 1	4	15	15	1 1 1	8	30	25	3 0 0	4	8				
6	0 1 2	6	75	16	1 2 0	6	18	26	3 0 1	8	40				
7	0 2 0	3	9	17	1 2 1	12	90	27	3 1 0	8	24				
8	0 2 1	6	45	18	1 3 0	8	54	28	3 2 0	12	72				
9	0 3 0	4	27	19	2 0 0	3	4	29	4 0 0	5	16				
10	0 4 0	5	81	20	2 0 1	6	20	30	4 0 1	10	80				

Any number ≤ 100 corresponds to one or more of these structures, for example $77 = 7 \cdot 11$ corresponds to 1 1 0, 1 0 1 and 0 1 1. This means that d(n) can only assume values listed in table 11 for n ≤ 100. The above scheme has been computer implemented for $n \leq 10^{12}$. The result is shown on table 12 and figure 1, which gives a clear picture of the overall behaviour of d(n).

Finally we will be able to say something about $d_{(k)}(n)$. For $n \leq 10^{12}$ we have $d(n) \leq 6720 < 10^4$. From table 10 it is seen that $k \leq 6$ for $n \leq 10^4$ and we therefore conclude that

$$k \leq 7 \text{ for } n \leq 10^{12}.$$

Table 12. Largest value of d(n) for $n \le 10^k$, $k \le 12$

k	Largest d	Number of d values	Number of combinations	Prime comb. for largest d	Corresponding n	log(d)
1	4	4	4	11	6	1.3863
2	12	11	34	121	90	2.4849
3	32	22	141	3111	840	3.4657
4	64	38	522	311110	9240	4.1589
5	128	60	1848	3311010	98280	4.8520
6	240	94	6179	4211101	942480	5.4806
7	448	135	20198	63111100	8648640	6.1048
8	768	190	42950	33211110	91891800	6.6438
9	1344	266	133440	621111110	931170240	7.2034
10	2304	359	399341	5312111100	9777287520	7.7424
11	4032	481	783061	6322111100	97772875200	8.3020
12	6720	626	2309712	64211111100	963761198400	8.8128

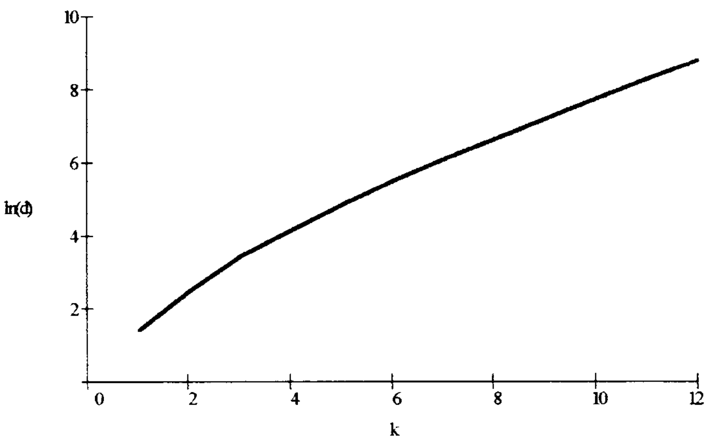

Diagram 1. Largest value of ln(d(n)) for $n < 10^k$.

(b) *Let* $\sigma(n) = \sum_{d|n, d>0} d$ *and m a given positive integer. Find the smallest k for which* $\sigma(\sigma(\ldots\sigma(2)\ldots)) \geq m$ *after k iterations, i.e.* $\sigma_{(k)}(2) \geq m$.

Clearly k is a function of m. It is a stepwise increasing function. The first six iterations have been used to illustrate this for the interval $2 \leq m \leq 25$ in diagram2.

However, a far more interesting function to study is the inverse of k(m). This function m(k) is growing so rapidly that numerical results are difficult to interpret and represent. A better picture of the behavior of $\sigma_K(2)$ is obtained by studying the function

$$f(k) = \ln(m)/k$$

This function is represented in diagram 3 for the interval $1 \leq k \leq 100$. After going through an interesting minimum for small values of k the curve flattens out. It seems to remain downwards convex. Does it approach a horizontal asymptote?

Finally, a few iteration results (k,m): (1,3), (2,4), (3,7), (4,8), (5,15), (6,24), (7,60), ...
(100,2972648508456959686477689735325484246606843303655482359755571200)

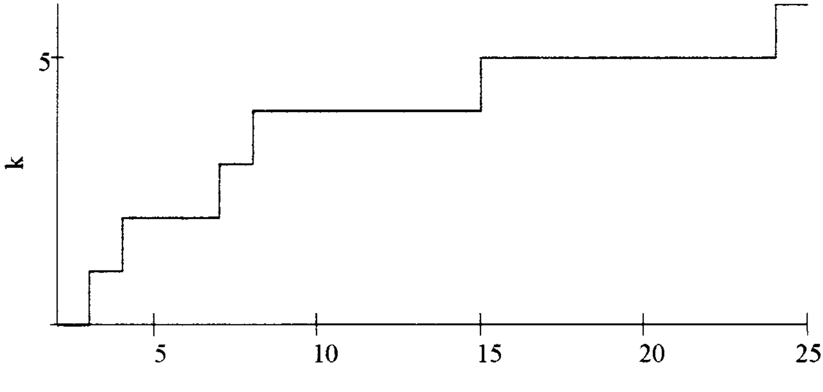

Diagram 2. k as a function of m.

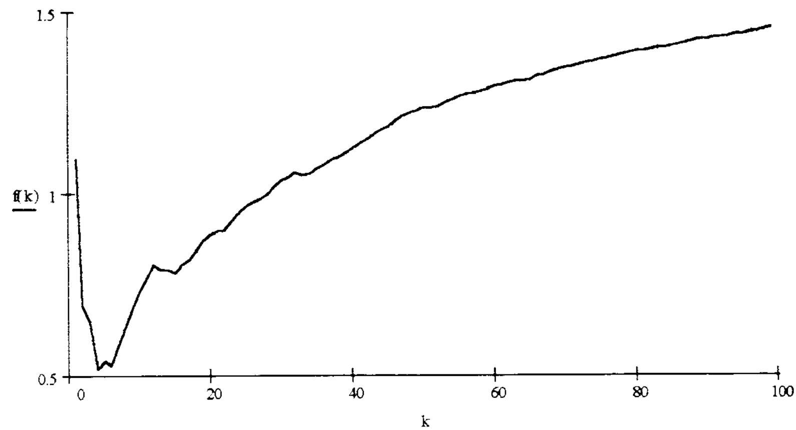

Diagram 3. f(k)=ln(m)/k.

References:

1. F. Smarandache, Unsolved problem: 52, *Only Problems, Not Solutions*, ISBN 1-879585-00-6, Xiquan Publishing House, 1993

Chapter IV

Diophantine Equations

1. Some Thoughts on the Equation $|y^p - x^q| = k$

In his book *Only Problems, Not Solutions* Smarandache formulated unsolved problem #20 as a conjecture rather than a problem:

> Let k be a non-zero integer. There are only a finite number of solutions in integers p, q, x, y, each greater than 1, to the equation $x^p - y^q = k$.[1]

In this study we will only consider the equation for $p \neq q$. By writing the equation in the form $|x^p - y^q| = k$ we only need to consider cases where $p > q$.

For a given set of parameters (p,q,k) it would then be desirable to list this finite number of solutions (x,y). However, if this were possible if would probably already have been done. So Smarandache's statement is likely to be based on statistical evidence rather than analytical reasoning. It is mainly from the statistical point of view we will study this equation. The parameters will be restricted to $k \leq 200$ and $p \leq 9$. As in most studies of this nature Ubasic provides the most effective computer language. All solutions where $x<100$ and $y<100$ can be churned out in a couple of seconds. To go any further we need a general approach to avoid running through meaningless search intervals. Consider

$$x = \sqrt[p]{y^q \pm k}$$

For sufficiently large y only $x = \lceil \sqrt[p]{y^q} \rceil$ or $x = \lfloor \sqrt[p]{y^q} \rfloor$ can produce solutions corresponding to

[1] Smarandache adds that "For k=1 this was conjectured by Cassels (1953) and proved by Tijdeman 1976)." (Gamma 2/1986)

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[1] Smarandache adds that "For k=1 this was conjectured by Cassels (1953) and proved by Tijdeman 1976)." (Gamma 2/1986)

$$k = \lceil \sqrt[p]{y^q} \rceil^p - y^q \tag{1a}$$

or

$$k = y^q - \lfloor \sqrt[p]{y^q} \rfloor^p \tag{1b}$$

It is easy to imagine from (1a) and (1b) that the number of solutions thin out rapidly as we increase y. Let's illustrate this for p=3 and q=2 by looking at the number of squares s which fit into the interval between $(n-1)^3$ and $(n+1)^3$, i.e. integers s for which

$$(n-1)^3 < s^2 < (n+1)^3 \tag{2}$$

Let s_{min} be the smallest and s_{max} the largest s which satisfies (2). We can then calculate the ratio f between the number of squares between $(n-1)^3$ and $(n+1)^3$ and the difference between these cubes, i.e.

$$f = \frac{s_{\max} - s_{\min}}{(n+1)^3 - (n-1)^3}$$

To have a solution we must have an s such that $|s^2 - n^3| = k$. The smaller f is the smaller is the chance for this to happen for our very limited range for k. Let s_1^2 be the largest square which is smaller than n^3 and s_2^2 be the smallest square which is larger than n^3 then s_1 and s_2 give an indication of the behaviour of $|s^2 - n^3|$. These two ways of looking at the problem are displayed in table 1 for a sequence of values of n which have been chosen so that n is neither square nor cube.

Table 1. Frequency of squares

n	s^2-n^3	n^3-s^2	f
5	19	4	$3.95\cdot10^{-2}$
50	316	391	$1.40\cdot10^{-3}$
$5\cdot10^2$	14761	7600	$4.40\cdot10^{-5}$
$5\cdot10^3$	430916	276191	$1.41\cdot10^{-6}$
$5\cdot10^4$	2515600	19845079	$4.47\cdot10^{-8}$
$5\cdot10^5$	287598881	419507900	$1.41\cdot10^{-9}$
$5\cdot10^6$	11203852544	11156827231	$4.47\cdot10^{-11}$
$5\cdot10^7$	513527672836	193579108351	$1.41\cdot10^{-12}$
$5\cdot10^8$	1151976475001	21208703299996	$4.47\cdot10^{-14}$

Table 2. Largest y

k	p	q	y	x
24	3	2	8158	736844
199	4	2	10	99
28	4	3	15	37
60	5	2	76	50354
24	5	3	4	10
13	5	4	3	4
127	6	2	4	63
37	6	3	2	3
104	6	4	3	5
32	6	5	2	2
95	7	2	6	529
10	7	3	3	13
95	7	4	6	23
96	7	5	2	2
64	7	6	2	2
161	8	2	3	80
40	8	3	2	6
175	8	4	2	3
13	8	5	2	3
192	8	6	2	2
128	8	7	2	2
83	9	2	3	140
169	9	3	2	7
113	9	4	2	5

Although the search for solutions was extended to y=10000 no solutions were found for the following values of k:

6, 14, 21, 29, 34, 42, 43, 50, 52, 58, 59, 62, 66, 69, 70, 75, 78, 82, 85, 86, 91, 102, 110, 111, 114, 123, 125, 130, 133, 134, 146, 149, 150, 158, 160, 165, 173, 176, 177, 178, 182, 187, 189, 195.

No solutions were found for (p,q)=(9,5), (9,6), (9,7) and (9,8).

Computer search: Calculations were carried out in Ubasic for $y \le 10000$, $k \le 200$ and $p \le 9$. Most solutions occur for small values of y for which we cannot use the CEIL ($\lceil\ \rceil$) and the FLOOR ($\lfloor\ \rfloor$) functions. The ROUND function has been used instead. Calculations are carried out with real numbers to 19 decimal places (POINT 9 in the Ubasic language).To safegard against "near integer" solutions a "proposed" solution is recalculated with integers in a subroutine. A simple version of the program is given below:

```
10  point 9
20  Y1=10000
30  for R%=3 to 9
40  for S%=2 to R%-1
50  for K%=1 to 200
60  Y=2
70  while Y<Y1
80  if (Y^R%-K%)<1 then X1=(K%-Y^R%)^(1/S%):goto 100
90  X1=(Y^R%-K%)^(1/S%)
100  X=round(X1)
110  if abs(X-X1)<10^(-30) then gosub 210
120  endif
130  X1=(Y^R%+K%)^(1/S%)
140  X=round(X1)
150  if abs(X-X1)<10^(-30) then gosub 210
160  endif
170  inc Y
180  wend
190  next:next:next
200  end
210  if abs(Y^R%-X^S%)<>K% then goto 240
220  if X<=1 then goto 240
230  print R%,S%,Y,X,K%
240  return
```

The number of solutions for each set of parameters is given in table 3. The largest value of y which occurs in a solution for each parameter set (p,q) is given in table 2, which confirms the indications for the rarity of solutions for large y given in table 1.

The largest number of solutions (11) occur for k=17. Several of these are due to the fact that no distinction is made between $(x^a)^b$ and $(x^b)^a$. These solutions are displayed in table 4. A limited search ($y \leq 100$) was carried out for $10 \leq p \leq 20$, $k \leq 200$. Only two solutions with $y \neq 2$ were found. These results are shown in table 5.

Conclusion: Smarandache's conjecture is well supported by the numerical results obtained in this study. The number of solutions diminish rapidly with increasing y, p and q.

Table 4. Solutions for k=17

Sol. #	p	q	y	x
1	3	2	2	5
2	3	2	4	9
3	3	2	8	23
4	3	2	43	282
5	3	2	52	375
6	4	2	3	8
7	4	3	3	4
8	5	2	2	7
9	6	2	2	9
10	6	4	2	3
11	9	2	2	23

Table 5. Solutions for $10 \le p \le 20$

k	p	q	y	x
63	10	2	2	31
65	10	2	2	33
124	10	2	2	30
132	10	2	2	34
183	10	2	2	29
24	10	3	2	10
23	11	2	2	45
68	11	2	2	46
94	11	2	3	421
112	11	2	2	44
161	11	2	2	47
199	11	2	2	43
149	11	3	2	13
139	11	7	2	3
127	12	2	2	63
129	12	2	2	65
89	13	2	2	91
92	13	2	2	90
192	13	3	2	20
7	15	2	2	181
37	15	2	3	3788

Table 3a. The number of solutions to $|y^p - x^q| = k$

p	3	4	4	5	5	5	6	6	6	6	7	7	7	7	7	8	8	8	8	8	8	9	
k/q	2	2	3	2	3	4	2	3	4	5	2	3	4	5	6	2	3	4	5	6	7	2	Sum
1	1																						1
2	1																						1
3												1											1
4	2			1																			3
5					1																		1
7	1	1		2							1												5
8	2		1																				3
9	3	1																					4
10												1											1
11	2		1	1																			4
12	1	1																					2
13	1			1		1													1				4
15	1						1																2
16				1		1					1												3
17	5	1	1	1			1		1													1	11
18	2			1																			3
19	2	1		1																			4
20	1	1																					2
22	1										1												2
23	1			1																			2
24	2				2																		4
25	1																						1
26	1			1																			2
27					1																		1
28	4		1	1			1				1											1	9
30	1																						1
31		1														1							2
32		1		2	1					1													5
33		2														1							3
35	1		1																				2
36	2						1																3
37	1			1				1															3
38	1																						1
39	3						1																4
40	2	1	1														1						5
41	1										1												2
44	2		1																				3

Table 3b. The number of solutions to $|y^p - x^q| = k$

p	3	4	4	5	5	5	6	6	6	6	7	7	7	7	7	8	8	8	8	8	8	9	
k/q	2	2	3	2	3	4	2	3	4	5	2	3	4	5	6	2	3	4	5	6	7	2	Sum
45	1	1																					2
46				1																			1
47	3			1							1		1										6
48	2	1	1				1		1														6
49	1	1		1		1																	4
51		1																					1
53	2						1																3
54	2		1																				3
55	3						2																5
56	2	1						1															4
57	2						1																3
60	1	1		1			1									1							5
61	1							1															2
63		1		1																			2
64	1										1	1			1							1	5
65	1	2		1																			4
67	1																						1
68		1		1							1					1							4
71	2	1									1											1	5
72	1	1																					2
73	4	1	1																				6
74	1			1																			2
76	1																						1
77		1																					1
79	2										1												3
80	2						1																3
81	1			1																			2
83	1																					1	2
84		1																					1
87	1	1	1													1	1						5
88		1										1											2
89	3			1																			4
92	1										1												2
93				1	1																		2
94	1																						1
95	1										1		1										3
96		1												1									2
97	1	1									1												3
98	1																						1
99		1		1																			2

Table 3c. The number of solutions to $|y^p - x^q| = k$

p	3	4	4	5	5	5	6	6	6	6	7	7	7	7	7	8	8	8	8	8	8	9	9	9	
k/q	2	2	3	2	3	4	2	3	4	5	2	3	4	5	6	2	3	4	5	6	7	2	3	4	Sum
100	6			1	1																				8
101	1											1													2
103											1														1
104	3	1	1				1		1																7
105	1	2					1									1									5
106	2																								2
107	1																								1
108	1																								1
109	1		1																						2
112	2	1		1			1				1		1			1						1			9
113	4		1																			1		1	7
115		1												1											2
116	3																								3
117	1										1														2
118	1			1	1																				3
119	1										1														2
120												1													1
121	2																								2
122				1																					1
124				2							1														3
126	1																								1
127	1	1					1																		3
128	2	1									1		1								1				6
129	1	1					1																		3
131	1		1														1								3
132	1			1			1																		3
135	3	1	1													1									6
136	1																								1
137				1																					1
138	1																								1
139	1																								1
140		1																							1
141	1	1																							2
142	1																								1
143	1			1																					2
144		2		1												1									4
145	2			1																					3

Table 3d. The number of solutions to $|y^p - x^q| = k$

p	3	4	4	5	5	5	6	6	6	6	7	7	7	7	7	8	8	8	8	8	8	9	9	9	
k/q	2	2	3	2	3	4	2	3	4	5	2	3	4	5	6	2	3	4	5	6	7	2	3	4	Sum
147	2																								2
148	1	1																							2
151	1																					1			2
152	3							1																	4
153	1	1					1																		3
154	1																								1
155	1																								1
156	1	1														1									3
157				1																					1
159	1	1																							2
161	2	1					1				1					1									6
162				1		1					1														3
163	1	1														1									3
164	2			1																		1			4
166				1																					1
167	1																								1
168	1																								1
169	2			1																			1		4
170	1																								1
171	1						1																		2
172	2																								2
174	1																								1
175	1	2														1		1							5
179				1	1					1															3
180	2	1																							3
183				1																					1
184	2	1			1																				4
185		1														1									2
186	1																								1
188	2																					1			3
190	1																								1
191	1																								1
192	1	2	1				1		1							1	1			1					9
193				1																					1
194				1																					1
196	1										1														2
197	1																								1
198	2			1																		1			4
199	2	1																							3
200	3	1	1				1																		6

2. The Equation $7(p^4+q^4+r^4+s^4+t^4) - 5(p^2+q^2+r^2+s^2+t^2)^2 + 90pqrst = 0$

This equation first appeared in the Numbers Count column of the *Personal Computer World* in March 1986. At that time the author found two until then unknown solutions (p,q,r,s,t)=(87,42,6,3,3) and (p,q,r,s,t)=(99,97,39,13,2). This took approximately 75 hrs on an 8086 processor running a 4.7 Mhz. Now, in 1996, these results were reproduced using the same program in 21 minutes on a Pentium 100 Mhz. The method used is reproduced below together with a number of new solutions.

Consider the function

$$F(p,q,r,s,t) = 7(p^4+q^4+r^4+s^4+t^4) - 5(p^2+q^2+r^2+s^2+t^2)^2 + 90pqrst$$

This function is invariant under exchange of variables. This makes it useful to study the function while all variables except one is kept constant. Denote this variable x. After some elementary algebraic manipulations we have

$$F(\ldots,x+1,\ldots) = F(\ldots,x,\ldots) + G(x) - 10C\cdot H(x) + D/x$$

where

$$G(x) = 28x^3 + 22x^2 + 8x + 2$$
$$H(x) = 2x + 1$$
$C = p^2+q^2+r^2+s^2+t^2$ with one of the constants replaced by x
$D = 90pqrst$ with the same constant as above replaced by x

This permits us to calculate F(…,x+1,…), G(x), H(x), C and D. When, in the next step, one of the unknowns p,q,r,s,t is increased by one the following replacements must be made

$$C := C + H(x)$$
$$D := D(x+1)/x$$

Without imposing any restrictions we can assume $p \geq q \geq r \geq s \geq t$. For a given value of p the search will be conducted for descending values of the other unknowns. For given values of p, q, r, s consider the function

$$g(t) = 7(e+t^4) - 5(a + t^2)^2 + bt$$

where $e = p^4+q^4+r^4+s^4$, $a = p^2+q^2+r^2+s^2$ and $b = 90pqrs$

We have

$$g'(t) = 8t^3 - 20at + b$$
$$g''(t) = 24t^2 - 20a$$
$$g''(t) = 0 \text{ for } t_m = \sqrt{(5a/6)} \text{ which gives } g'_{min} = b - 16t_m^3$$

Case 1: $g'_{min} > 0$. Since $g'(0)>0$ it follows that $g'(t)>0$ for $t>0$. This means that $g(t)$ is an increasing function for $t>0$ If we have found t_1 such that $g(t_1)>0$ then $g(t)>0$ for all $t>t_1$ and the search can be interrupted for $t=t_1$.

Case 2: $g'_{min}\leq 0$. For $t>t_m$ the function $g(t)$ is convex and a value $t=t_1$ will be found for which the function is positive and increasing. The search is stopped for $t>t_1>t_m$.

The above method has been used in a computer program written in Ubasic. Implementation on Pentium 100 Mhz computer has revealed a few new solutions. A complete list of all solutions for $p\leq 400$ is given in table 4.

Table 4. Solutions for p≤400

#	p	q	r	s	t
1	1	1	1	1	1
2	2	1	1	1	1
3	2	2	1	1	1
4	3	3	2	1	1
5	4	2	1	1	1
6	6	3	2	1	1
7	7	7	4	2	1
8	17	7	7	1	1
9	59	47	19	7	2
10	87	42	6	3	3
11	99	97	39	13	2
12	124	63	42	17	1
13	127	47	34	2	1
14	189	87	27	3	3
15	286	154	11	11	11

3. The Equation $y = 2 \cdot x_1 x_2 \ldots x_k + 1$

Conjecture:

Let $k \geq 2$ be a positive integer. The diophantine equation: $y = 2 \cdot x_1 x_2 \ldots x_k + 1$ has a infinitely many solutions in distinct primes $y, x_1, x_2, \ldots x_k$.

This is unsolved problem number 11 in Smarandache's book *Only Problems, Not Solutions*. The word distinct has been added by the author. The purpose of this study is to see 'how stable' this conjecture is. This is done through a computer analysis of all possible solutions for $y \leq 10^9$. (A very thin layer when surfing the ocean of integers but big enough to take a bit of time on the computer when it comes to calculations -in fact more than 100 hrs). From the computational point of view there is no reason to exclude k=1. The interval $0<y<10^9$ is divided into 10 sub-intervals of equal length;

Interval #1: $0<y<10^8$
Interval #2: $10^8<y<2 \cdot 10^8$
Interval #3: $2 \cdot 10^8<y<3 \cdot 10^8$
...
Interval #10 $9 \cdot 10^8<y<10^9$

The endpoints are excluded since these do not contribute to the number of solutions.

Consider

$$t=(y-1)/2=\cdot x_1 x_2 \ldots x_k \quad (1)$$

The task is to identify sequentially all square free numbers $n<10^9$. For each square free number with k distinct prime factors we calculate the corresponding number y and test whether it is prime or not. The number of primes is denoted m_k and the number of square free numbers is denoted n_k. m_k and n_k are recorded for each interval and the frequency of solutions $F_k=m_k/n_k$ is calculated. The result is shown in table 5.

The same result are shown in diagram 1, which conveys a good visualization of a result obtained through surfing on a small area of the ocean of integers.

Let's make a few observations:

Why the irregularities for k=8 and k=9? The smallest square free integer with k prime factors is $p_k\#$ where p_k is the *kth* prime. For k=8 and k=9 this means there can be no solutions for $y<2 \cdot 19\#+1=19399381$ and $y<2 \cdot 23\#+1=446185741$ respectively. The samples for k=8 and k=9 are therefore too small to give a true picture - randomness takes precedence over mass behaviour.

Table 5. Frequency of solutions

#	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
1	0.0765	0.0893	0.1061	0.1282	0.1561	0.1933	0.2435	0.3419	
2	0.0701	0.0817	0.0967	0.1159	0.1407	0.1735	0.2167	0.2607	
3	0.0683	0.0794	0.0937	0.1118	0.1356	0.1665	0.2075	0.2810	
4	0.0672	0.0778	0.0919	0.1095	0.1323	0.1612	0.1991	0.2622	
5	0.0662	0.0769	0.0903	0.1080	0.1301	0.1580	0.1944	0.2252	0.0000
6	0.0655	0.0760	0.0895	0.1063	0.1280	0.1562	0.1944	0.2298	1.0000
7	0.0653	0.0752	0.0887	0.1054	0.1266	0.1543	0.1898	0.2496	1.0000
8	0.0646	0.0748	0.0878	0.1046	0.1256	0.1529	0.1846	0.2225	0.5000
9	0.0641	0.0744	0.0873	0.1036	0.1244	0.1508	0.1883	0.2359	0.6667
10	0.0639	0.0738	0.0867	0.1030	0.1238	0.1498	0.1846	0.2120	0.4000

Frequency of solutions

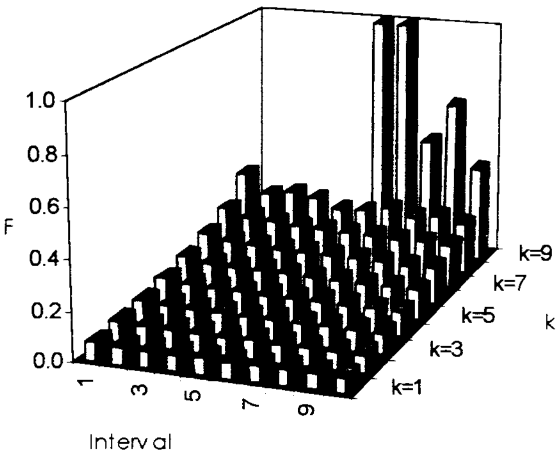

Diagram 1 .Frequency of solutions

With the previous observation in mind let's compare the frequencies for the first and the last interval. Diagram 2 shows F_k for the intervals $0<y<10^8$ and $9 \cdot 10^8<y<10^9$.

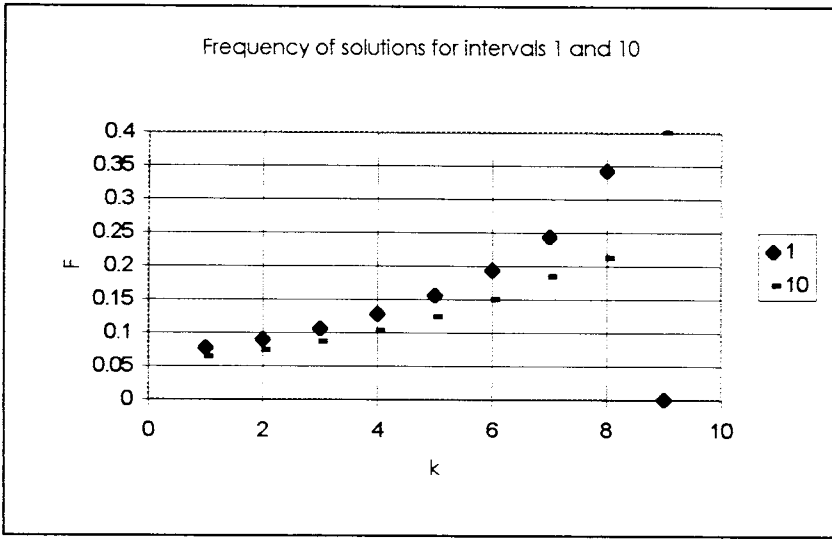

Diagram 2

Observation: When not too close to $p_k\#$ the frequency smoothly increases with increasing k. This is a good support for Samrandache's conjecture.

Let's now take a closer look at how the frequency of solutions behave for distinct values of k. Diagram 3 shows that the frequency decreases slightly for all k as larger integers as included. Is there an asymptote for each k?. The frequency of solutions increases as we increase k. It should be noted that it is the ratio between the number of solutions and the number of square free numbers in an interval determined by t in (1) which depicted. This is of course different from the number of square free numbers in the interval for y because when y runs through the interval $a<y<b$ then we consider square free numbers s which obey

$$(a-1)/2<s<(b-1)/2$$

As a bi-product to this study we have information on how many square free integers with 1, 2, ..., 9 prime factors there are in our different intervals. This is shown in table 6, where, as we have seen, we now only have five intervals:.

Interval #1: $0<s<10^8$
Interval #2: $10^8<s<2\cdot10^8$
Interval #3: $2\cdot10^8<s<3\cdot10^8$
Interval #4: $3\cdot10^8<s<4\cdot10^8$
Interval #5: $4\cdot10^8<s<5\cdot10^8$

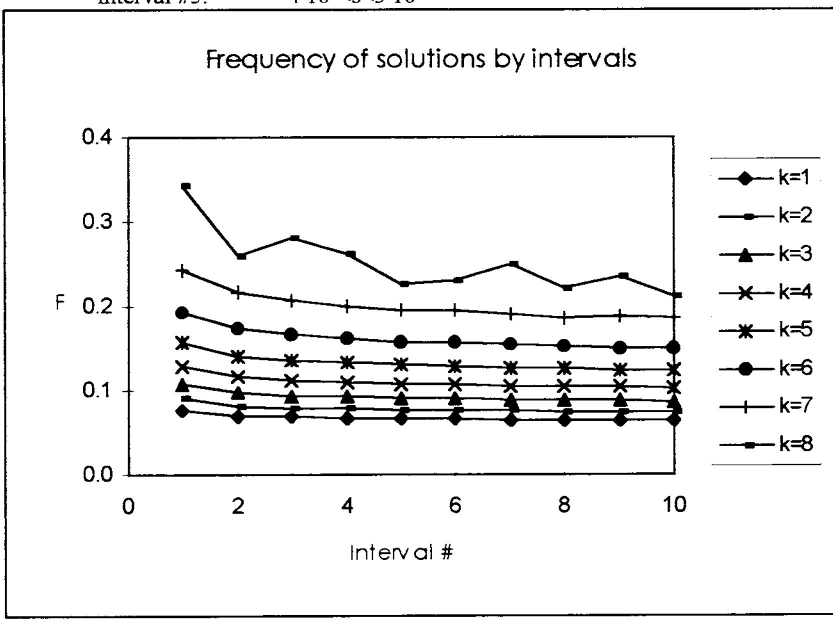

Diagram 3.

Table 6. Number of square free numbers

#	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
1	5761455	17426029	20710806	12364826	3884936	605939	38186	516	0
2	5317482	16565025	20539998	13029165	4476048	799963	63642	1409	0
3	5173395	16270874	20457818	13243252	4689541	879765	76114	2060	2
4	5084001	16085983	20402004	13374830	4825914	932513	84968	2560	6
5	5019541	15951738	20359052	13468885	4926227	972398	91767	3005	8

Diagram 4 shows a cumulative representation of the number of square free numbers with k=1, 2, 3, ... 9 prime factors. A square free number with only one prime factor is indeed a prime number so the first column in table 6 shows the number of prime numbers in the interval.

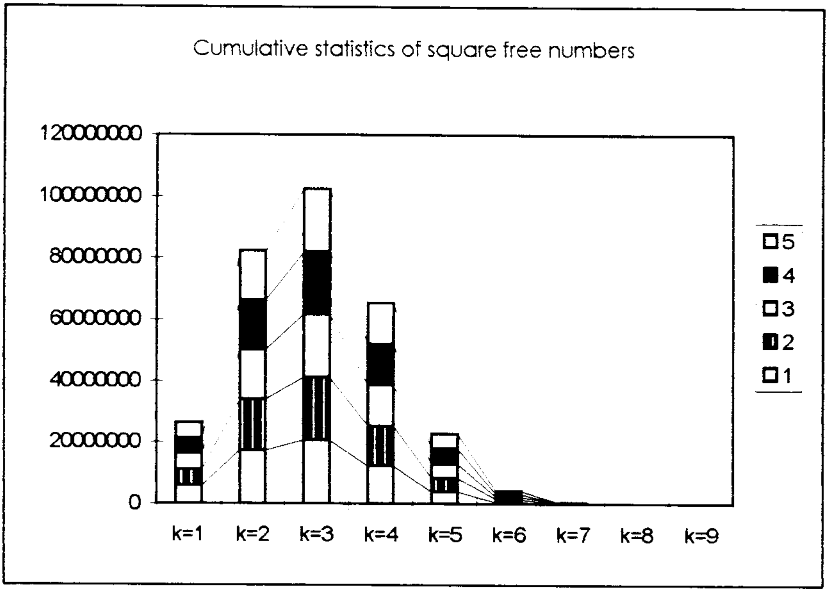

Diagram 4.

The illustration on the cover is another version of the above diagram for $n<5\cdot 10^7$.

Finally, a problem often leads to other problems. Let's go back to iterations. In the introduction to chapter III it was said that iterations result in an invariant, a loop or divergence. Is there really **no other possibility?** Let's look at this:

> We define $u_{k+1} = 2\cdot u_k+1$ where u_1 is a prime number. u_{k+1} will have the property of being a prime number or not being a prime number.

In this case we can give an explicit formula for u_{k+1} in terms of u_1. One easily **finds** $u_{k+1} = 2^k(u_1+1)-1$.

How do we characterize the result of iterations in relation to the property in which we are interested? Indefinite?

How many times can we iterate $u_{k+1} = 2 \cdot u_k + 1$ preserving primality? For u_1=305192579 it is eight times resulting in the following series of nine primes:

305192579, 610385159, 1220770319, 2441540639, 4883081279, 9766162559, 19532325119, 39064650239, 78129300479

Which is the first series with 10 terms - or 11? Maybe we need some deep sea diving!

The Smarandache Ceil Function

Definition: For a positive integer n the Smarandache ceil function of order k is defined through [1]

$S_k(n) = m$ where m is the ~~smallest positive~~ integer for which n divides m^k.

In the study of this function we will make frequent use of the ceil function defined as follows:

$\lceil x \rceil$ = the smallest integer not less than x.

The following properties follow directly from the above definitions:

1. $S_1(n) = n$
2. $S_k(p^\alpha) = p^{\lceil \alpha/k \rceil}$ for any prime number p.
3. For distinct primes p, q, ...r we have $S_k(p^\alpha q^\beta \cdots r^\delta) = p^{\lceil \alpha/k \rceil} q^{\lceil \beta/k \rceil} \cdots r^{\lceil \delta/k \rceil}$.

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