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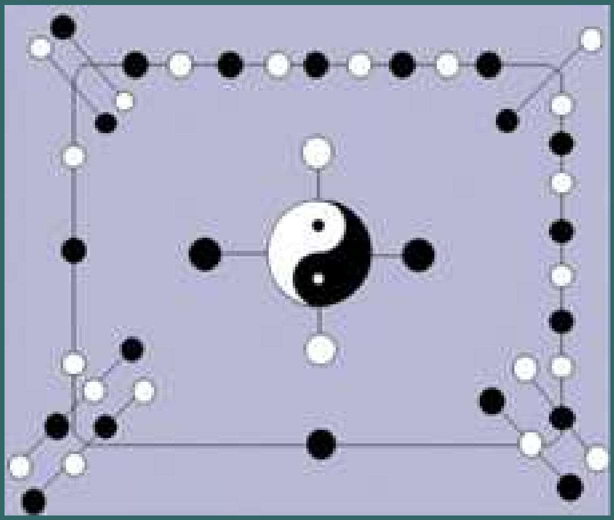

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The tragedy of the world is that those who are imaginative have but slight experience, and those who are experienced have feeble imaginations.

By Alfred North Whitehead, a British philosopher and mathematician.

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Quarter Symmetric Metric Connection on Generalized Semi Pseudo Ricci Symmetric Manifold

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Abstract: Object of this paper is to find some properties of generalized semi pseudo Ricci symmetric manifold (denoted by $G(SPRS)_n$) admitting quarter symmetric metric connection. At last we have given an example of this manifold.

Key Words: Generalized semi pseudo Ricci symmetric manifold, semi pseudo Ricci symmetric manifold, pseudo Ricci symmetric manifold,quarter symmetric metric connection.

AMS(2010): 53C25, 53B30, 53C15.

§1. Introduction

The notion of locally symmetric and Ricci symmetric Riemannian manifold began with work of Cartan [10] and Eisenhert[8] respectively. A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies the relation

$$\nabla R = 0, \tag{1}$$

where ∇ is the operator of covariant differentiation w.r.t. the metric tensor g. Again a Ricci symmetric manifold is a Riemannian manifold with the Ricci tensor S of type (0,2) satisfying

$$\nabla S = 0. \tag{2}$$

After them these notions have flowed in several branches such as recurrent manifold, Ricci-recurrent manifold, semi-symmetric manifold, pseudo-symmetric manifold[4], pseudo Ricci-symmetric manifold[6] and so on.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be pseudo Ricci symmetric manifold $((PRS)_n)$[5] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \tag{3}$$

[1]Received February 17, 2015, Accepted November 20, 2015.

where A is nonzero 1-form satisfying

$$g(X,U) = A(X) \tag{4}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be semi pseudo Ricci symmetric manifold $((SPRS)_n)$[2] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = A(Y)S(X,Z) + A(Z)S(X,Y), \tag{5}$$

where A is nonzero 1-form satisfying

$$g(X,U) = A(X) \tag{6}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be generalised semi pseudo Ricci symmetric manifold $(G(SPRS)_n)$[1] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = A(Y)S(X,Z) + B(Z)S(X,Y) \tag{7}$$

where A, B are nonzero 1-forms satisfying

$$g(X,V) = A(X) \tag{8}$$

$$g(X,W) = B(X) \tag{9}$$

for particular vector fields V, W respectively.

From the above definition we observe that when $\delta = A - B$ is identically zero, $G(SPRS)_n$ reduces $(SPRS)_n$.

Consider a non flat Riemannian manifold $(M^n, g), (n > 2)$with a Riemannian connection ∇. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + H(X,Y), \tag{10}$$

where H is a tensor field of type $(0, 2)$.

Now D is said to be quarter symmetric connection on M if the torsion tensor $\bar{T}$ with respect to D satisfies

$$\bar{T}(X,Y) = \eta(Y)QX - \eta(X)QY, \tag{11}$$

where Q is the symmetric endomorphism of the tangent space at each point of a $G(SPRS)_n$ corresponding to the Ricci tensor S.

D is said to be metric connection if

$$(D_X g)(Y,Z) = 0. \tag{12}$$

The above relations will be used in the followings.

§2. Preliminaries

Let a non flat Riemannian manifold $(M^n, g), (n > 2)$ which is $G(SPRS)_n$ and Q be the symmetric endomorphism of the tangent space at each point of M corresponding to the Ricci tensor S. Then M satisfies (7) and

$$g(QX, Y) = S(X, Y). \tag{13}$$

Again from (7) we can get

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = A(Y)S(X, Z) - A(X)S(Y, Z). \tag{14}$$

Contracting above with respect to Y and Z, we have

$$dr(X) = 2\bar{A}(X) - 2A(X)r, \tag{15}$$

where $\bar{A}(X) = A(QX)$.

Now let ∇ be the Riemannian connection and D be a quarter symmetric metric connection on M and $\bar{T}$ is the torsion tensor with respect to D. Then from (10) and (12), we have,

$$g(H(X, Y), Z) + g(H(X, Z), Y) = 0. \tag{16}$$

From (10) and (11) we can obtain,

$$H(X, Y) - H(Y, X) = A(Y)QX - A(X)QY. \tag{17}$$

Then (16) and (17) gives us the following,

$$H(X, Y) = A(Y)QX - S(X, Y)V, \tag{18}$$

where V is a particular vector field such that $g(X, V) = A(X)$. Then (10) can be written as

$$D_X Y = \nabla_X Y + A(Y)QX - S(X, Y)V. \tag{19}$$

§3. Curvature Tensor with Respect to a Quarter Symmetric Metric Connection on $G(SPRS)_n$

Let R, S, r be the curvature tensor, Ricci tensor, and scalar curvature respectively with respect to Reimannian connection ∇ on M. Again let $\bar{R}, \bar{S}, \bar{r}$ be the curvature tensor, Ricci tensor, and scalar curvature respectively with respect to a quarter symmetric metric connection D on M. Then

$$\bar{R}(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \tag{20}$$

Then (14), (19) and (20) gives us,

$$\begin{aligned}\bar{R}(X,Y,Z) =& R(X,Y,Z) + [(\nabla_X A)(Z) - A(X)A(Z) + \frac{1}{2}S(X,Z)A(V)]QY \\ & - [(\nabla_Y A)(Z) - A(Y)A(Z) + \frac{1}{2}S(Y,Z)A(V)]QX \\ & + S(X,Z)[\nabla_Y V - A(Y)V + \frac{1}{2}A(V)QY] \\ & - S(Y,Z)[\nabla_X V - A(X)V + \frac{1}{2}A(V)QX].\end{aligned} \tag{21}$$

Now let

$$\lambda(X,Z) = (\nabla_X A)(Z) - A(X)A(Z) + \frac{1}{2}S(X,Z)A(V) = g(LX,Z), \tag{22}$$

where L is the symmetric endomorphism corresponding to the Ricci tensor with respect to the quarter symmetric metric connection D on M.

Hence (21) can be reduced to

$$\bar{R}(X,Y,Z) = R(X,Y,Z) + \lambda(X,Z)QY - \lambda(Y,Z)QX + S(X,Z)LY - S(Y,Z)LX. \tag{23}$$

Thus we can state the followings.

Theorem 1 *If an $G(SPRS)_n$ admits a quarter symmetric metric connection D, then the curvature tensor with respect to D is of the form* (23).

Corollary 1 *On a $G(SPRS)_n$ admitting a quarter symmetric metric connection D, λ defined by* (22) *is symmetric iff A is closed.*

Corollary 2 *On a $G(SPRS)_n$ admitting a quarter symmetric metric connection D,the necessary and sufficient condition for $\bar{R} = R$ is that*

$$\lambda(X,Z)QY - \lambda(Y,Z)QX + S(X,Z)LY - S(Y,Z)LX = 0, \tag{24}$$

where λ is defined by (22).

§4. Ricci Tensor and Scalar Curvature with Respect to a Quarter Symmetric Metric Connection on $G(SPRS)_n$

Now contracting (23) with respect to X we get,

$$\bar{S}(Y,Z) = S(Y,Z) + \lambda(QY,Z) - r\lambda(Y,Z) + \lambda(Y,QZ) - aS(Y,Z), \tag{25}$$

where

$$a = tr.L = divA + \frac{r-2}{2}A(\rho). \tag{26}$$

Again contracting (25) with respect to Y, Z and using (22) and (15) we can get,

$$\bar{r} = r + (\nabla_{QX} A)(X) + (\nabla_X A)(QX) - dr(X)A(X) + 2A(X)A(X)r. \tag{27}$$

These give us the following theorem.

Theorem 2 *If an $G(SPRS)_n$ admits a quarter symmetric metric connection D, then the Ricci tensor and scalar curvature with respect to D is of the form* (25) *and* (27) *respectively.*

§5. Examples

Let us consider M^3 be an open subsets of R^3 with the basis $\{e_1, e_2, e_3\}$ where

$$e_1 = \frac{1}{x^1 x^3} \frac{\partial}{\partial x^1}, \; e_2 = \frac{\partial}{\partial x^2}, \; e_3 = \frac{\partial}{\partial x^3}. \tag{28}$$

Let us define the metric g as

$$g(e_i, e_j) = \begin{cases} 1, & \text{i = j} \\ 0, & \text{otherwise.} \end{cases}$$

Then form of the metric is

$$g = g_{ij} dx^i dx^j = (x^1 x^3)^2 (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad i, j = 1, 2, 3. \tag{29}$$

Obviously it is a Riemannian metric. Then the Ricci tensor is

$$S_{11} = (x^1)^2, S_{11,1} = 2x^1 \tag{30}$$

and all others vanish identically, where (,) denotes the covariant differentiation with respect to metric g.

Now we define

$$A_i(x) = \begin{cases} -\frac{1}{x^1}, & \text{i = 1} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \frac{3}{x^1}, & \text{i = 1} \\ 0, & \text{otherwise,} \end{cases}$$

for any point $x \in M$, Then

$$S_{11} = A_1 S_{11} + B_1 S_{11} \tag{31}$$

and all other forms vanish identically. The relation (31) implies that the above Riemannian manifold(M^3, g) is a $G(SPRS)_3$.

Now let the symmetric endomorphism Q defined by

$$Q(e_1) = (x^1)^2 e_1, \quad Q(e_2) = e_3, \quad Q(e_3) = e_2 \tag{32}$$

and let the vector field $V = -\frac{1}{x^1}e_1$ so that $A(X) = g(X, V)$. Then we get,

$$[e_i, e_j] = 0, \quad \forall \quad i, j = 1, 2, 3. \tag{33}$$

Then using Koszul's formula we have

$$\nabla_{e_i} e_j = 0, \quad \forall \quad i, j = 1, 2, 3, \tag{34}$$

where ∇ is Levi-Civita connection with respect to g.

Again using (19) we can define a connection D on M as follows:

$$D_{e_2} e_1 = -\frac{1}{x^1} e_3, D_{e_3} e_1 = -\frac{1}{x^1} e_2 \tag{35}$$

and all others vanish identically.

Using (11) we can find the torsion tensor with respect to D as follows:

$$T(e_1, e_2) = \frac{1}{x^1} e_3, T(e_1, e_3) = \frac{1}{x^1} e_2 \tag{36}$$

and all others vanish identically.

Using (12) we have

$$(D_{e_1} g)(e_2, e_3) = (D_{e_2} g)(e_1, e_3) = (D_{e_3} g)(e_2, e_1) = 0. \tag{37}$$

The above approves that D is a quarter symmetric metric connection on M. Thus (M^3, g) is a $G(SPRS)_3$ with a quarter symmetric metric connection D.

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A Note on (j, k)-Symmetric Harmonic Functions Defined by Sälägean Derivatives

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Abstract: In this paper, using the concept of (j, k)-symmetric functions and Sälägean operator we introduce the class $\mathcal{SH}_s^{j,k}(\beta, \lambda, b)$ of functions $f = h + \overline{g}$ which are harmonic in $\mathcal{U}$. Coefficients bound for functions to be in this class. We also shown that this coefficient bound is also necessary for the class of functions of the form $f = h_\lambda + \overline{g_\lambda}$, belonging to the class $\overline{\mathcal{SH}}_s^{j,k}(\beta, \lambda, b)$. Distortion bounds, extreme points and neighborhood for the class $\overline{\mathcal{SH}}_s^{j,k}(\beta, \lambda, b)$ are also obtained.

Key Words: (j, k)-Symmetric functions, harmonic functions, Sälägean operator, distortion bounds, neighborhood.

AMS(2010): 30C45, 30C50, 30C55.

§1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \in \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$, (see [14]).

Denote by $\mathcal{SH}$ the class of functions $f(z) = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$, for which $f(0) = f_z(0) - 1 = 0$. Then for $f(z) = h + \overline{g} \in \mathcal{SH}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1}$$

Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the

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co-analytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

A function $f(z) = h + \overline{g}$ with h and g given by (1) is said to be harmonic starlike of order β for $(0 \leq \beta < 1$, for $|z| = r < 1$ if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \Im\left\{\frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f(re^{i\theta})}\right\} = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \beta.$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}^*_{\mathcal{H}}(\beta)$ and extensively studied by Jahangiri ([1]). The cases $\beta = 0$ and $b_1 = 1$ were studied by Silverman and Silvia ([2]) and Silverman ([6]).

Definition 1.1 *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of the odd univalent functions.

The notion of (j,k)-symmetrical functions $(k = 2, 3, \cdots \quad ; \; j = 0, 1, 2, \cdots, k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings ([12]).

Definition 1.2 *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, \cdots, k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \qquad z \in \mathcal{U}.$$

We note that the family of all (j,k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.3([12]) *For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there* *exists exactly the sequence of* (j,k)- *symmetrical functions* $f_{j,k}$,

$$f(z)=\sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^{v}z), \tag{3}$$

$$(f\in\mathcal{A};\ \ k=1,2,\cdots;j=0,1,2,\cdots,k-1)$$

From (3) we can get

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^{v}z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty}a_n(\varepsilon^{v}z)^n\right),$$

then

$$f_{j,k}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\quad a_1=1 \qquad ,\delta_{n,j}=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lk+j;\\ 0, & n\neq lk+j;\end{cases}, \tag{4}$$

Ahuja and Jahangiri ([3]) discussed the class $\mathcal{SH}(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and satisfying

$$\Im\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\}>\beta,\qquad 0\leq\beta<1.$$

The authors ([13]) introduced and discussed the class $\mathcal{SH}_s^(j,k)(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and satisfying

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\}>\beta,\qquad 0\leq\beta<1.$$

For $f=h+\overline{g}$ where h and g are given by (1), Jahangiri ([15]) defined the modified Sälägean operator of f as

$$D^{\lambda}f(z)=D^{\lambda}h(z)+(-1)^{\lambda}\overline{D^{\lambda}g(z)},\quad \lambda=0,1,2,\cdots, \tag{5}$$

where

$$D^{\lambda}h(z)=z+\sum_{n=2}^{\infty}n^{\lambda}a_nz^n,\qquad D^{\lambda}g(z)=\sum_{n=1}^{\infty}n^{\lambda}b_nz^n. \tag{6}$$

Now using Sälägean operator D^{λ} and the concepts of $(j,k)-$symmetric points we define the following.

Definition 1.4 *For* $0\leq\beta<1$ *and* $k=1,2,3,\cdots,\ j=0,1,\cdots,k-1,\lambda\in\mathbb{N}_0,\ b\neq 0,$

let $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$ *denote the class of harmonic functions* f *of the form (1) which satisfy the condition*

$$\Re\left\{1-\frac{2}{b}+\frac{2D^{\lambda+1}f(z)}{bD^{\lambda}f_{j,k}(z)}\right\}>\beta, \tag{7}$$

where

$$D^{\lambda}f_{j,k}(z)=\delta_{1,j}z+\sum_{n=2}^{\infty}n^{\lambda}\delta_{n,j}a_nz^n+(-1)^{\lambda}\overline{\sum_{n=1}^{\infty}n^{\lambda}\delta_{n,j}b_nz^n} \tag{8}$$

and $\delta_{n,j}$ *is defined by (4).*

Let $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ denote the subclass of $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$ consist of harmonic functions of $f_\lambda=h_\lambda+\overline{g_\lambda}$ such that h_λ and $\overline{g_\lambda}$ are of the form

$$h_\lambda(z)=z-\sum_{n=2}^{\infty}|a_n|z^n, \qquad g_\lambda(z)=(-1)^{\lambda}\sum_{n=1}^{\infty}|b_n|z^n. \tag{9}$$

Since

$$rrh_{j,k}=\delta_{1,j}z+\sum_{n=2}^{\infty}\delta_{n,j}a_nz^n, \qquad g_{j,k}(z)=\overline{\sum_{n=1}^{\infty}\delta_{n,j}b_nz^n}, \tag{10}$$

also, let $f_{\lambda,j,k}h_{\lambda,j,k}+\overline{g_{\lambda,j,k}}$ such that $h_{\lambda,j,k}$ and $\overline{g_{\lambda,j,k}}$ are of the form

$$h_{\lambda,j,k}(z)=\delta_{-1,j}z-\sum_{n=2}^{\infty}\delta n,j|a_n|z^n, \qquad g_{\lambda,j,k}(z)=(-1)^{\lambda}\sum_{n=1}^{\infty}\delta n,j|b_n|z^n, \tag{11}$$

where $\delta n,j$ is given by (4).

The following special cases are of interest

(1) $\mathcal{SH}_s^{1,k}(\beta,0,2)=\mathcal{SH}_s^{k}(\beta)$, the class introduced by AL-Shaqsi and Darus in [15];
(2) $\mathcal{SH}_s^{1,2}(\beta,0,2)=\mathcal{SH}(\beta)$, the class introduced byAhuja and Jahangiri in [3];
(3) $\mathcal{SH}_s^{1,1}(\beta,0,2)=\mathcal{S_H}^*(\beta)$ the class introduced by Jahangiri in [1];
(4) $\mathcal{SH}_s^{1,1}(0,0,2)=\mathcal{S_H}^*$ the class introduced by Silverman and Silvia in [2];
(5) $\mathcal{SH}_s^{1,1}(\beta,\lambda,2)=\mathcal{SH}(\beta,\lambda)$ the class introduced by Jahangiri in [5];
(6) $\mathcal{SH}_s^{j,k}(\beta,0,2)=\mathcal{SH}_s^{j,k}(\beta)$ the class introduced Fuad Alsarari and S.Latha ([13]).

§2. Coefficient Bounds

Theorem 2.1 *If* $f=h+\overline{g}$ *with* h *and* g *given by* (**??**) *and* $D^{\lambda}f_{j,k}$ *is defined by (8)*

$$\sum_{n=1}^{\infty}n^{\lambda}\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|+\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\leq 2, \tag{12}$$

where $b\neq 0, a_1=1,\ \ 0\leq\beta<1,\ \ \lambda\in\mathbb{N}_0$ *and* $\delta_{n,j}$ *is defined by (4), then* f *is sense-preserving, harmonic univalent in* $\mathcal{U}$*, and* $f\in\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$.

Proof If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|, \\
&= 1-\left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n(z_1^n-z_2^n)}\right|, \\
&> 1-\frac{\sum_{n=1}^{\infty}|b_n|}{1-\sum_{n=2}^{\infty}|a_n|}, \\
&> 1-\frac{\sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n|}{1-\sum_{n=2}^{\infty}\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n|} \geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense-preserving in $\mathcal{U}$. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1-\sum_{n=2}^{\infty}\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n| \\
&\geq \sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n| > \sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n||z|^{n-1} \\
&\geq \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.
\end{aligned}$$

Using the fact $\Re\{w\} > \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$ it suffices that. Let

$$\Re\left\{\frac{A(z)}{B(z)}\right\} = \Re\left\{\frac{(b-2)D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)}{bD^{\lambda}f_{j,k}(z)}\right\} > \beta, \tag{13}$$

it suffices to show that

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)| \geq 0.$$

Now we have

$$\begin{aligned}
|(1-\beta)B(z)+A(z)| &= \left|(1-\beta)bD^{\lambda}f_{j,k}(z)+(b-2)D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)\right| \\
&= \left|[b(2-\beta)-2]D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)\right| \\
&= \Bigg|(b(2-\beta)-2)[\delta_{1,j}z+\sum_{n=2}^{\infty} n^{\lambda}\delta_{n,j}a_n z^n+(-1)^{\lambda}\sum_{n=1}^{\infty} n^{\lambda}\delta_{n,j}b_n z^n] \\
&\quad + 2[z+\sum_{n=2}^{\infty} n^{\lambda+1}a_n z^n-(-1)^{\lambda}\sum_{n=1}^{\infty} n^{\lambda+1}b_n z^n]\Bigg|.
\end{aligned}$$

So

$$\begin{aligned}
|(1-\beta)B(z)+A(z)| \geq [(b(2-\beta)-2)\delta_{1,j}+2]|z|-\sum_{n=2}^{\infty}[2n+(b(2-\beta) \\
-2)\delta_{n,j}]n^{\lambda}|a_n||z|^n-\sum_{n=1}^{\infty}[2n-(b(2-\beta)-2)\delta_{n,j}]n^{\lambda}|b_n||z|^n.
\end{aligned} \tag{14}$$

Also

$$\begin{aligned}|(1+\beta)B(z)-A(z)| &= \left|(1+\beta)bD^{\lambda}f_{j,k}(z)-(b-2)D^{\lambda}f_{j,k}(z)-2D^{\lambda+1}f(z)\right| \\ &= \Big|(2+b\beta)[\delta_{1,j}z+\sum_{n=2}^{\infty}n^{\lambda}\delta_{n,j}a_nz^n+(-1)^{\lambda}\sum_{n=1}^{\infty}n^{\lambda}\delta_{n,j}b_nz^n] \\ &\quad -2[z+\sum_{n=2}^{\infty}n^{\lambda+1}a_nz^n-(-1)^{\lambda}\sum_{n=1}^{\infty}n^{\lambda+1}b_nz^n]\Big|.\end{aligned}$$

So

$$\begin{aligned}|(1+\beta)B(z)-A(z)| &\leq [(2+b\beta)\delta_{1,j}-2]|z|+\sum_{n=2}^{\infty}[2n-(2+b\beta)\delta_{n,j}]n^{\lambda}|a_n||z|^n+\sum_{n=1}^{\infty} \\ &\times [2n+(2+b\beta)\delta_{n,j}]n^{\lambda}|b_n||z|^n. \end{aligned} \tag{15}$$

By (14) and (15), we have

$$\begin{aligned}&|(1-\beta)B(z)+A(z)|-|(1+\beta)B(z)-A(z)| \\ &\geq 2\{[b(1-\beta)-2]\delta_{1,j}+2\}|z|-2\sum_{n=2}^{\infty}\{[b(1-\beta)-2]\delta_{n,j}+2n\}n^{\lambda}|a_n||z|^n \\ &\quad -2\sum_{n=1}^{\infty}\{[2-b(1-\beta)]\delta_{n,j}+2n\}n^{\lambda}|b_n||z|^n \\ &\geq 2\{[b(1-\beta)-2]\delta_{1,j}+2\} \\ &\quad\times\left\{1-\sum_{n=2}^{\infty}\frac{\{[b(1-\beta)-2]\delta_{n,j}+2n\}}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n|-\sum_{n=1}^{\infty}\frac{\{[2-b(1-\beta)]\delta_{n,j}+2n\}}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n|\right\}.\end{aligned}$$

This last expression is nonnegative by (12), and so the proof is complete. □

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{n,j}+2n\}n^{\lambda}}x_nz^n+\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[2-b(1-\beta)]\delta_{n,j}+2n\}n^{\lambda}}\overline{y_nz^n}, \tag{16}$$

where $b\neq 0,\ \ 0\leq\beta<1,\ \ \lambda\in\mathbb{N}_0$ and $\delta_{n,j}$ is defined by (4), and $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$ shows that the coefficient given by (12) is sharp, and the functions of form (160 are in $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$.

We next show that condition (12) is also necessary for functions in $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$.

Theorem 2.2 *Let $f_\lambda=h_\lambda+\overline{g_\lambda}$ be given by (9). Then $f_\lambda\in\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ if and only if*

$$\sum_{n=1}^{\infty}n^{\lambda}\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|+\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\leq 2, \tag{17}$$

where $b>0, a_1=1,\ \ 0\leq\beta<1,\ \ \lambda\in\mathbb{N}_0$ and $\delta_{n,j}$ is defined by (4).

Proof Since $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b) \subset \mathcal{SH}_s^{j,k}(\beta,\lambda,b)$, we only need to prove the necessity part of the theorem. To this end, for functions f_λ and $f_{\lambda,j,k}$ of the form (9) and (11) respectively, we notice that the condition (7) is equivalent to

$$\Re\left\{\frac{[b(1-\beta)-2]D^\lambda f_{\lambda,j,k}(z)+2D^{\lambda+1}(z)}{bD^\lambda f_{\lambda,j,k}(z)}\right\} \geq 0$$

$$\Re\left\{\frac{[(b(1-\beta)-2)\delta_{1,j}+2]z-\sum_{n=2}^\infty[2n+(b(1-\beta)-2)\delta_{n,j}]n^\lambda|a_n|z^n-\sum_{n=1}^\infty[2n-(b(1-\beta)-2)\delta_{n,j}]n^\lambda|b_n|\overline{z^n}}{b[\delta_{n,1}z-\sum_{n=2}^\infty n^\lambda\delta_{n,j}|a_n|z^n+\sum_{n=1}^\infty n^\lambda\delta_{n,j}]|b_n|\overline{z^n}}\right\} \geq 0.$$

The above required in the above inequality must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we have

$$\frac{[(b(1-\beta)-2)\delta_{1,j}+2]-\sum_{n=2}^\infty[2n+(b(1-\beta)-2)\delta_{n,j}]n^\lambda|a_n|r^{n-1}-\sum_{n=1}^\infty[2n-(b(1-\beta)-2)\delta_{n,j}]n^\lambda|b_n|r^{n-1}}{b[\delta_{n,1}z-\sum_{n=2}^\infty n^\lambda\delta_{n,j}|a_n|r^{n-1}+\sum_{n=1}^\infty n^\lambda\delta_{n,j}|b_n|r^{n-1}]} \geq 0.$$

If the condition (7) does not hold, then the numerator in the above inequality is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0 \in (0,1)$ for which the quotient in the above inequality is negative. This contradicts the required condition for $f_\lambda \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ so the proof is complete. □

§3. Distortion Bounds

Theorem 3.1 *Let* $f_\lambda \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. *Then for* $|z| = r < 1$, *one has*

$$|f_\lambda(z)| \geq (1-|b_1|)r - r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right),$$

$$|f_\lambda(z)| \leq (1-|b_1|)r + r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right).$$

Proof Let $f_\lambda \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Taking the obsolete vale of f_λ, we obtain

$$\begin{aligned}
|f_\lambda(z)| &= \left|z-\sum_{n=2}^\infty a_nz^n+(-1)^\lambda\sum_{n=1}^\infty b_nz^n\right| \\
&\geq (1-|b_1|)r-\sum_{n=2}^\infty(|a_n|+|b_n|)r^n \geq (1-|b_1|)r-r^2\sum_{n=2}^\infty(|a_n|+|b_n|) \\
&\geq (1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(\sum_{n=2}^\infty\frac{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}{b(1-\beta)-2]\delta_{1,j}+2}(|a_n|+|b_n|)\right) \\
&\geq (1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^\infty n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|\right\} \\
&\quad -r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^\infty n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\} \\
&\geq (1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right).
\end{aligned}$$

Also

$$\begin{aligned}
|f_\lambda(z)| &= \left|z-\sum_{n=2}^{\infty}a_nz^n+(-1)^\lambda\sum_{n=1}^{\infty}b_nz^n\right| \\
&\leq (1-|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \leq (1-|b_1|)r+r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|) \\
&\leq (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(\sum_{n=2}^{\infty}\frac{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}{b(1-\beta)-2]\delta_{1,j}+2}(|a_n|+|b_n|)\right) \\
&\leq (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^{\infty}n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|\right\} \\
&\quad +r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^{\infty}n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\} \\
&\leq (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right).
\end{aligned}$$

□

The following covering result follows form left-hand inequality in Theorem 3.1.

Corollary 3.2 *Let $f_\lambda = h_\lambda + \overline{g_\lambda}$ be given by (9) are in $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$, then*

$$\left\{w:|w|<\frac{2^{\lambda+1}[4+\{b(1-\beta)-2\}\delta_{2,j}]-[b(1-\beta)-2]\delta_{1,j}+2}{2^\lambda[4+\{b(1-\beta)-2\}\delta_{2,j}]}\right.$$
$$\left.-\frac{2^{\lambda+1}[4+\{b(1-\beta)-2\}\delta_{2,j}-[2+\{2-b(1-\beta)\}\delta_{1,j}]]}{2^\lambda[4+\{b(1-\beta)-2\}\delta_{2,j}]}|b_1|\right\}\subset f_\lambda(\mathcal{U}).$$

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ denoted by $clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. $[2n+\{b(1-\beta)-2\}\delta_{n,j}], [b(1-\beta)-2]\delta_{1,j}+2, [2n+\{2-b(1-\beta)\}\delta_{n,j}]$.

Theorem 3.3 *Let $f_\lambda = h_\lambda + \overline{g_\lambda}$ be given by (9). Then $f_\lambda \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ if and only if*

$$f_\lambda(z)=\sum_{n=1}^{\infty}(X_nh_{\lambda_n}(z)+Y_ng_{\lambda_n}(z)),$$

where $h_{\lambda_1}=z$,

$$\begin{aligned}
h_{\lambda_n}(z) &= z-([b(1-\beta)-2]\delta_{1,j}+2)/n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]z^n,\ n=2,3,4,\cdots, \\
g_{\lambda_n}(z) &= z+(-1)^\lambda([b(1-\beta)-2]\delta_{1,j}+2)/n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]\overline{z}^n,\ n=1,2,3,\cdots
\end{aligned}$$

and

$$\sum_{n=1}^{\infty}(X_n+Y_n)=1,\ X_n\geq 0,\ Y_n\geq 0.$$

In particular, the extreme point of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ are $\{h_{\lambda_n}\}$ and $\{g_{\lambda_n}\}$.

Proof Since

$$\begin{aligned} f_\lambda(z) &= \sum_{n=1}^{\infty}(X_n h_{\lambda_n}(z)+Y_n g_{\lambda_n}(z)) \\ &= \sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]}X_n z^n \\ &\quad +(-1)^\lambda\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]}Y_n\overline{z}^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty}n^\lambda\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n| &+ \sum_{n=1}^{\infty}n^\lambda\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n| \\ &= \sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1, \end{aligned}$$

and so $f_\lambda \in clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$.

Conversely suppose that $f_\lambda \in clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Setting

$$X_n=\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|, \qquad 0\leq X_n\leq 1, \quad n=2,3,4,\cdots,$$

$$Y_n=\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|, \qquad 0\leq Y_n\leq 1, \quad n=1,2,3,\cdots,$$

and

$$X_1=1-\sum_{n=2}^{\infty}X_n-\sum_{n=2}^{\infty}Y_n.$$

Therefore f_λ can be written as

$$\begin{aligned} f_\lambda(z) &= z-\sum_{n=2}^{\infty}|a_n|z^n+(-1)^\lambda-\sum_{n=1}^{\infty}|b_n|\overline{z}^n \\ &= z-\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]}X_n z^n \\ &\quad +(-1)^\lambda-\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]}Y_n\overline{z}^n \\ &= z-\sum_{n=2}^{\infty}(h_{\lambda_n}(z)-z)X_n+(-1)^\lambda\sum_{n=1}^{\infty}(g_{\lambda_n}(z)-z)Y_n \\ &= \sum_{n=2}^{\infty}h_{\lambda_n}(z)X_n+\sum_{n=1}^{\infty}g_{\lambda_n}(z)Y_n+z\left(1-\sum_{n=2}^{\infty}X_n-\lambda\sum_{n=1}^{\infty}Y_n\right) \\ &= \sum_{n=1}^{\infty}\left(h_{\lambda_n}(z)X_n+g_{\lambda_n}(z)Y_n\right). \end{aligned}$$

□

§4. Neighborhood Result

In this section, we will prove that the functions in neighborhood of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ are starlike harmonic functions.

Using (4), we define the ϱ-neighborhood of function $f \in \tau H$

$$\mathcal{N}_\rho(f) = \left\{ F(z) = z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=1}^{\infty} B_n \overline{z}^n,\ \sum_{n=2}^{\infty} n[|a_n - A_n| + |b_n - B_n| + |b_1 - B_1| \leq \rho \right\},$$

where $\rho > 0$.

Theorem 4.1 *Let*

$$\rho = \frac{2^\lambda[\{2 + \{\frac{b}{2}(1-\beta) - 1\}\delta_{2,j}] - [b(1-\beta) - 2]\delta_{1,j} + 2 - \{2^\lambda[2 + \{\frac{b}{2}(1-\beta) - 1\}\delta_{2,j} - [2 + \{2 - b(1-\beta)\}\delta_{1,j}]]\}}{2^\lambda[\{2 + \{\frac{b}{2}(1-\beta) - 1\}\delta_{2,j}]}.$$

Then $N_\rho(\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)) \subset \tau H$.

Proof Suppose $f \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Let $F_\lambda = H_\lambda + \overline{G_\lambda} \in N_\rho(f_\lambda)$, where $H_\lambda = z - \sum_{n=2}^{\infty} A_n z^n$ and $G_\lambda = (-1)^\lambda \sum_{n=1}^{\infty} B_n \overline{z}^n$, we need to show that $F_\lambda \in \tau H$. In other words, it suffices to show that F_λ satisfies the condition $\tau(F) = \sum_{n=2}^{\infty} n[|A_n| + |B_n|] + |B_1| \leq 1$. We observe that

$$\begin{aligned}
\tau(F) &= \sum_{n=2}^{\infty} n[|A_n| + |B_n|] + |B_1| \\
&= \sum_{n=2}^{\infty} n[|A_n - a_n + a_n| + |B_n b_n + b_n|] + |B_1 - b_1 + b_1| \\
&= \sum_{n=2}^{\infty} n[|A_n - a_n| + |B_n - b_n|] + \sum_{n=2}^{\infty} n[|a_n| + |b_n|] + |B_1 - b_1| + |b_1| \\
&= \rho + |b_1| + \sum_{n=2}^{\infty} n[|a_n| + |b_n|] \\
&= \rho + |b_1| + \frac{[b(1-\beta) - 2]\delta_{1,j} + 2}{\{[b(1-\beta) - 2]\delta_{2,j} + 4\}2^{\lambda-1}} \left(\sum_{n=2}^{\infty} \frac{\{[b(1-\beta) - 2]\delta_{2,j} + 4\}2^{\lambda-1}}{b(1-\beta) - 2]\delta_{1,j} + 2} (|a_n| + |b_n|) \right) \\
&\leq \rho + |b_1| + \frac{[b(1-\beta) - 2]\delta_{1,j} + 2}{\{[b(1-\beta) - 2]\delta_{2,j} + 4\}2^{\lambda-1}} \sum_{n=2}^{\infty} n^\lambda \left\{ \frac{[2n + \{b(1-\beta) - 2\}\delta_{n,j}]}{[b(1-\beta) - 2]\delta_{1,j} + 2} |a_n| \right\} \\
&\quad + \frac{[b(1-\beta) - 2]\delta_{1,j} + 2}{\{[b(1-\beta) - 2]\delta_{2,j} + 4\}2^{\lambda-1}} \sum_{n=2}^{\infty} n^\lambda \left\{ \frac{[2n + \{b(1-\beta) - 2\}\delta_{n,j}]}{[b(1-\beta) - 2]\delta_{1,j} + 2} |b_n| \right\} \\
&\leq \rho + |b_1| + \frac{[b(1-\beta) - 2]\delta_{1,j} + 2}{\{[b(1-\beta) - 2]\delta_{2,j} + 4\}2^{\lambda-1}} \left(1 - \frac{[2 + \{2 - b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta) - 2]\delta_{1,j} + 2} |b_1| \right).
\end{aligned}$$

Now this last expression is never greater than one if

$$\rho \le 1-|b_1|-\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right)$$

$$=\frac{2^{\lambda}[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}]-[b(1-\beta)-2]\delta_{1,j}+2-\{2^{\lambda}[2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}-[2+\{2-b(1-\beta)\}\delta_{1,j}]]\}}{2^{\lambda}[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}]}.$$

□

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Neighborhood Total 2-Domination Number and Connectivity of Graphs

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Abstract: A subset S of V is called a dominating set in a graph G if every vertex in $V-S$ is adjacent to at least one vertex in S. A set $S \subseteq V$ is called the neighborhood total 2-dominating set (nt2d-set) of a graph G if every vertex in $V-S$ is adjacent to at least two vertices in S and the induced subgraph $\langle N(S)\rangle$ has no isolated vertices. The minimum cardinality of an nt2d-set of G is called the neighborhood total 2-domination number of G and is denoted by $\gamma_{2nt}(G)$. The connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we find an upper bound for the sum of the neighborhood total 2-domination number and connectivity of a graph and characterize the corresponding extremal graphs.

Key Words: Domination number, Smarandachely k-domination number, neighborhood total 2-domination number, connectivity.

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§1. Introduction

The graph $G=(V,E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges and with out isolated vertices. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by deg$u$.The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [2,3].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v]=N(v)\cup\{v\}$. If $S \subseteq V$ then $N(S)=\bigcup_{v\in S} N(v)$ for all $v \in S$ and $N[S]=N(S)\cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u,S]=\{v : N[v]\cap S=\{u\}\}$. $H(m_1,m_2,\cdots,m_n)$ denotes the graph obtained from the graph H by attaching m_i pendant edges to the vertex $v_i \in V(H), 1 \le i \le n$. The graph $K_2(m_1,m_2)$ is called bistar and it is also denoted by $B(m_1,m_2)$. $H(P_{m_1},P_{m_2},\cdots,P_{m_n})$ is

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the graph obtained from the graph H by attaching an end vertex of P_{m_i} to the vertex v_i in $H, 1 \leq i \leq n$.

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. Generally, a set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Clearly, a dominating set is nothing else but a Smarandachely 1-dominating set of G. The *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a number is called the *domination number* of G and denoted by $\gamma(G)$. C.Sivagnanam [5] introduced the concept of neighborhood total 2-domination in graphs. A set $S \subseteq V$ is called a neighborhood total 2-dominating set (nt2d-set) of a graph G if every vertex in $V - S$ is adjacent to at least two vertices in S, i.e., S is a Smarandachely 2-dominating set and the induced subgraph $\langle N(S) \rangle$ has no isolated vertices. The minimum cardinality of an nt2d-set of G is called the neighborhood total 2-domination number of G and is denoted by $\gamma_{2nt}(G)$.The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Several authors have studied the problem of obtaining an upper bound for the sum of a dominating parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J.Paulraj Joseph and S.Arumugam [4] proved that $\gamma(G) + \kappa(G) \leq n$ and characterized the corresponding extremal graphs. In this paper, we obtain a sharp upper bound for the sum of the neighborhood total 2-domination number and connectivity of a graph and characterize the corresponding extremal graphs.We use the following theorems.

Theorem 1.1 ([5]) *Let G be a connected graph on $n \geq 2$ vertices. Then $\gamma_{2nt}(G) \leq n$ and equality holds if and only if G is a star.*

Theorem 1.2 ([1]) *For any graph G, $\kappa(G) \leq \delta(G)$.*

§2. **Main Results**

Theorem 2.1 *For any graph G, $\gamma_{2nt}(G) + \kappa(G) \leq 2n - 1$ and equality holds if and only if G is isomorphic to K_2.*

Proof $\gamma_{2nt}(G) + \kappa(G) \leq n + \delta \leq n + n - 1 = 2n - 1$. Let $\gamma_{2nt}(G) + \kappa(G) = 2n - 1$. Then $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 1$ which gives G is a star as well as a complete graph on n vertices. Hence G is isomorphic to K_2. The converse is obvious. □

Theorem 2.2 *For any graph G, $\gamma_{2nt}(G) + \kappa(G) = 2n - 2$ if and only if G is isomorphic to either K_3 or $K_{1,2}$.*

Proof Let $\gamma_{2nt} + \kappa(G) = 2n - 2$. Then there are two cases to consider: (1) $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 2$ and (2) $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 1$.

Case 1. $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 2$.

Then G is a star and hence $\kappa(G) = 1$ which gives $n = 3$. Thus G is isomorphic to $K_{1,2}$.

Case 2. $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 1$.

Then G is a complete graph. This gives $\gamma_{2nt}(G) = 2$. Then $n = 3$ and hence G is isomorphic to K_3. The converse is obvious. □

Theorem 2.3 *For any graph G, $\gamma_{2nt}(G) + \kappa(G) = 2n - 3$ if and only if G is isomorphic to C_4 or $K_{1,3}$ or K_4.*

Proof Let $\gamma_{2nt}(G) + \kappa(G) = 2n - 3$. Then there are three cases to consider: (1) $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 3$; (2) $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 2$; (3) $\gamma_{2nt}(G) = n - 2$ and $\kappa(G) = n - 1$.

Case 1. $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 3$.

Then G is a star and hence $\kappa(G) = 1$ which gives $n = 4$. Thus G is isomorphic to $K_{1,3}$.

Case 2. $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 2$.

Then $n - 2 \leq \delta(G)$. If $\delta = n - 1$ then G is a complete graph which is a contradiction. Hence $\delta(G) = n - 2$. Then G is isomorphic to $K_n - Y$ where Y is a matching in K_n. Then $\gamma_{2nt}(G) \leq 3$. If $\gamma_{2nt}(G) = 3$ then $n = 4$ and hence G is isomorphic to C_4. If $\gamma_{2nt}(G) = 2$ then $n = 3$ and hence G is isomorphic to $K_{1,2}$ which is a contradiction.

Case 3. $\gamma_{2nt}(G) = n - 2$ and $\kappa(G) = n - 1$.

Then G is a complete graph on n vertices. Since $\gamma_{2nt} = 2$ we have $n = 4$. Hence G is isomorphic to K_4. The converse is obvious. □

Theorem 2.4 *For any graph G, $\gamma_{2nt}(G) + \kappa(G) = 2n - 4$ if and only if G is isomorphic to P_4 or K_5 or $K_4 - e$ or $K_{1,4}$ or $K_3(1, 0, 0)$.*

Proof Let $\gamma_{2nt}(G) + \kappa(G) = 2n - 4$. Then there are four cases to consider: (1) $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 4$; (2) $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 3$; (3) $\gamma_{2nt}(G) = n - 2$ and $\kappa(G) = n - 2$; (4) $\gamma_{2nt}(G) = n - 3$ and $\kappa(G) = n - 1$.

Case 1. $\gamma_{2nt}(G) = n$ and $\kappa(G) = n - 4$.

Then G is a star and hence $\kappa(G) = 1$ which gives $n = 5$. Thus G is isomorphic to $K_{1,4}$.

Case 2. $\gamma_{2nt}(G) = n - 1$ and $\kappa(G) = n - 3$.

Then $n - 3 \leq \delta$. If $\delta = n - 1$ then G is a complete graph which is a contradiction. If $\delta = n - 2$ then G is isomorphic to $K_n - Y$ where Y is matching in K_n. Then $\gamma_{2nt}(G) = 2$ or 3. If $\gamma_{2nt}(G) = 3$ then $n = 4$. Hence G is either $K_4 - e$ or C_4. For these two graphs $\kappa(G) = 2 \neq n - 3$ which is a contradiction. If $\gamma_{2nt}(G) = 2$ then $n = 3$ which is a contradiction to $\kappa(G) = n - 3$. Hence $\delta = n - 3$. Let $X = \{v_1, v_2, \cdots, v_{n-3}\}$ be the vertex cut of G and let $V - X = \{x_1, x_2, x_3\}$.

Subcase 2.1 $\langle V - X \rangle = \overline{K_3}$.

Then every vertex of $V - X$ is adjacent to all the vertices in X. If $|X| = 1$ then G is a star which is a contradiction. If $|X| = 2$ then G is isomorphic to either $K_{2,3}$ or the graph obtained from $K_{2,3}$ by joining the vertices of degree 3 by an edge. But for these two graphs $\gamma_{2nt} \leq 3$ which is a contradiction.Hence $|X| \geq 3$. Then $\{v_1, v_2, x_1, x_2\}$ is a nt2d-set of G. Hence $\gamma_{2nt}(G) \leq 4$. Then $n = 5$ which is a contradiction.

Subcase 2.2 $\langle V - X \rangle = K_1 \cup K_2$.

Let $x_1x_2 \in E(G)$. Then x_3 is adjacent to all the vertices in X and x_1, x_2 are not adjacent to at most one vertex in X. If degx_1 or degx_2 is $n-2$ then $\{x_1, x_2, x_3\}$ is a nt2d-set of G and hence $\gamma_{2nt} \leq 3$. Then $n \leq 4$ which gives $n = 4$. Hence G is isomorphic to P_4 or $K_3(1, 0, 0)$.

Suppose degx_1 = deg$x_2 = n-3$. If $N(x_1) = N(x_2)$ then there is a vertex $v_1 \in X$ such that v_1 is not adjacent to both x_1 and x_2. Then v_1 is adjacent to all the vertices in X. It is clear that $\{v_1, x_1, x_2, x_3\}$ is a nt2d-set of G. Hence $\gamma_{2nt} \leq 4$. Thus $n \leq 5$ which gives $n = 5$. Then G is isomorphic to the graph obtained from two copies of C_3 by merging one vertex of a copy of C_3 to a vertex of another copy of C_3. For this graph $\kappa(G) = 1$ which is a contradiction.If $N(x_1) \neq N(x_2)$ then there are at least two vertices v_1 and v_2 such that v_1 is not adjacent to x_1 but adjacent to x_2 and v_2 is not adjacent to x_2 but adjacent to x_1. Then $\{x_1, x_2, x_3\}$ is a nt2d-set of G and hence $n \leq 4$ which is a contradiction.

Case 3. $\gamma_{2nt}(G) = n-2$ and $\kappa(G) = n-2$.

Then $n - 2 \leq \delta(G)$. If $\delta = n-1$ then G is a complete graph which is a contradiction. Hence $\delta(G) = n-2$. Then G is isomorphic to $K_n - Y$where Y is a matching in K_n. Then $\gamma_{2nt}(G) \leq 3$. If $\gamma_{2nt}(G) = 3$ then $n = 5$. But $\gamma_{2nt}(K_5 - Y) = 2 \neq n-2$ which is a contradiction. If $\gamma_{2nt}(G) = 2$ then $n = 4$. Hence G is isomorphic to $K_4 - e$.

Case 4. $\gamma_{2nt}(G) = n-3$ and $\kappa(G) = n-1$.

Then G is a complete graph on n vertices. Since $\gamma_{2nt}(G) = n-3$ we have $n = 5$. Hence G is isomorphic to K_5. The converse is obvious. □

Notation 2.5 We use the following notations in this paper:

(i) G^* is a graph obtained from $C_5 + e$ by joining two non adjacent vertices one has degree two and another has degree three by an edge.

(ii) H^* is a graph obtained from K_4 by subdividing an edge once.

(iii) The set $A = \{G^*, H^*, P_5, C_5, C_5+e, K_6, K_{1,5}, K_{2,3}, K_{3,3}, K_3(2,0), B(2,1), K_5 - Y$, where Y is a matching in $K_5\}$.

Theorem 2.6 *For any connected graph G, $\gamma_{2nt}(G) + \kappa(G) = 2n-5$ if and only if $G \in A$.*

Proof Let $\gamma_{2nt}(G) + \kappa(G) = 2n-5$. Then there are five cases to consider: (1) $\gamma_{2nt}(G) = n$ and $\kappa(G) = n-5$; (2) $\gamma_{2nt}(G) = n-1$ and $\kappa(G) = n-4$; (3) $\gamma_{2nt}(G) = n-2$ and $\kappa(G) = n-3$; (4) $\gamma_{2nt}(G) = n-3$ and $\kappa(G) = n-2$; (5) $\gamma_{2nt}(G) = n-4$ and $\kappa(G) = n-1$.

Case 1. $\gamma_{2nt}(G) = n$ and $\kappa(G) = n-5$.

Then G is a star and hence $\kappa(G) = 1$ which gives $n = 6$. Thus G is isomorphic to $K_{1,5}$.

Case 2. $\gamma_{2nt}(G) = n-1$ and $\kappa(G) = n-4$.

Then $n-4 \leq \delta(G)$. If $\delta(G) = n-1$ then G is a complete graph which is a contradiction. If $\delta(G) = n-2$ then G is isomorphic to $K_n - Y$ where Y is a matching in K_n. Then $\gamma_{2nt}(G) \leq 3$ and hence $n \leq 4$ which is a contradiction to $\kappa(G) = n-4$. Suppose $\delta(G) = n-3$. Let $X = \{v_1, v_2, \cdots, v_{n-4}\}$ be the vertex cut of G and let $V - X = \{x_1, x_2, x_3, x_4\}$. If $\langle V - X\rangle$ contains an isolated vertex then $\delta(G) \leq n-4$ which is a contradiction. Hence $\langle V - S\rangle$ is isomorphic to $K_2 \cup K_2$. Also every vertex of $V - X$ is adjacent to all the vertices of X. Then $\gamma_{2nt}(G) = 3$. Hence $n = 4$ which is a contradiction. Thus $\delta(G) = n-4$.

Subcase 2.1 $\langle V - X\rangle = \overline{K_4}$.

Then every vertex of $V - X$ is adjacent to all the vertices in X. If $|X| = 1$ then G is a star which is a contradiction. If $|X| = 2$ then G is isomorphic to either $K_{2,4}$ or the graph obtained from $K_{2,4}$ by joining the vertices of degree 4 by an edge. But for these two graphs $\gamma_{2nt} \leq 3$ which is a contradiction. Hence $|X| \geq 3$. Then $\{v_1, v_2, x_1, x_2\}$ is a nt2d-set of G. Hence $\gamma_{2nt} \leq 4$. Then $n \leq 5$ which is a contradiction.

Subcase 2.2 $\langle V - X\rangle = P_3 \cup K_1$.

Let x_1 be the isolated vertex and (x_2, x_3, x_4) be a path in $\langle V - X\rangle$. Then x_1 is adjacent to all the vertices in X and x_2, x_4 are not adjacent to at most one vertex in X. Let $v_1 \in X - N(x_2)$. If $N(x_3) \cap X = \emptyset$ then $\delta(G) = 2$ and hence $n = 6$ then $\{x_1, x_2, x_3, x_4\}$ or $\{x_1, x_2, x_3, v_2\}$ is a nt2d-set of G which is a contradiction to $\gamma_{2nt}(G) = n-1$. If $N(x_3) \cap X \neq \emptyset$ then $\{x_1, x_2, x_4, v_1\}$ is a nt2d-set of G and hence $\gamma_{2nt} \leq 4$. Thus $n = 5$. Then G is isomorphic to $B(2,1)$ or $K_3(1,1,0)$ or the graph G_1 where G_1 is obtained from $K_4 - e$ by attaching a pendant edge to the vertex of degree 3. But $\gamma_{2nt}(G_1) = \gamma_{2nt}(K_3(1,1,0)) = 3 \neq n-1$ which is a contradiction. Hence G is isomorphic to $B(2,1)$.

Suppose $N(x_2) = N(x_4) = X$. Then $\{x_1, x_2, x_4, v_1\}$ is a nt2d-set of G. Hence $\gamma_{2nt}(G) \leq 4$ which gives $n = 5$. Then G is isomorphic to either G_1 or $C_4(1,0,0,0)$. But $\gamma_{2nt}(G_1) = \gamma_{2nt}(C_4(1,0,0,0)) = 3 \neq n-1$ which is a contradiction.

Subcase 2.3 $\langle V - X\rangle = K_3 \cup K_1$.

Let x_1 be the isolated vertex in $\langle V - X\rangle$ and $\langle\{x_2, x_3, x_4\}\rangle$ be the complete graph. Then x_1 is adjacent to all the vertices in X and x_2, x_3, x_4 are not adjacent to at most two vertices in X. Then $\{x_1, x_2, x_3, v_1, v_2\}$ where $v_1, v_2 \in X - [N(x_2) \cup N(x_3)]$ is a nt2d-set of G and hence $n = 5$ or 6. Suppose $n = 5$. Then G is isomorphic to $K_4(1,0,0,0)$ or $K_3(P_3, P_1, P_1)$ or the graph obtained from $K_4 - e$ by attaching a pendant egde to the vertex of degree two. For these graphs $\gamma_{2nt}(G) + \kappa(G) \neq 2n-5$. Suppose $n = 6$. Then $\{x_1, x_2, x_3, v_1\}$ or $\{x_1, x_2, x_3, v_2\}$ or $\{x_2, x_3, v_1, v_2\}$ is a nt2d-set of G which is a contradiction to $\gamma_{2nt} = n-1$.

Subcase 2.4 $\langle V - X\rangle = K_2 \cup \overline{K_2}$.

Let $x_1x_2 \in E(G)$ and $x_3x_4 \in E(\overline{G})$. Then each $x_i, i = 1$ or 2 is non adjacent to at most one vertex in X and each $x_j, j = 3$ or 4 is adjacent to all the vertices in X. Then $\{x_1, x_3, x_4, v_1\}$ where $v_1 \in N(x_2) \cap X$ is a nt2d-set of G and hence $n = 5$. Then G is isomorphic to $B(2,1)$ or $K_3(2,0)$.

Subcase 2.5. $\langle V - X\rangle = K_2 \cup K_2$.

Let $x_1x_2, x_3x_4 \in E(G)$. Since $\delta(G) = n-4$ each x_i is non adjacent to at most one vertex in X. Then at most one vertex say $v_1 \in X$ such that $|N(v_1) \cap (V-X)| = 1$. If all $v_i \in X$ such that $|N(v_i) \cap (V-X)| \geq 2$ then $\{x_1, x_2, x_3, x_4\}$ is a nt2d-set of G and hence $n = 5$. Then $X = \{v_1\}$. If $|N(v_1) \cap (V-X)| = 2$ then G is isomorphic to P_5. If $|N(v_1) \cap (V-X)| \geq 3$ then $\gamma_{2nt}(G) = 3 \neq n-1$ which is a contradiction. Suppose $|N(v_1) \cap (V-X)| = 1$ and $|N(v_i) \cap (V-X)| \geq 2$ for $i \neq 1$ then $\{x_1, x_2, x_3, x_4, v_1\}$ is a nt2d-set of G and hence $n = 6$. For this graph $\gamma_{2nt}(G) + \kappa(G) \neq 2n-5$.

Case 3. $\gamma_{2nt}(G) = n-2$ and $\kappa(G) = n-3$.

Then $n-3 \leq \delta$. If $\delta = n-1$ then G is a complete graph which is a contradiction. If $\delta = n-2$ then G is isomorphic to $K_n - Y$ where Y is any matching in K_n. Then $\gamma_{2nt}(G) = 2$ or 3. If $\gamma_{2nt}(G) = 3$ then $n = 5$ which gives a contradiction. If $\gamma_{2nt}(G) = 2$ then $n = 4$. Hence G is either $K_4 - e$ or C_4. For these two graphs $\kappa(G) = 2 \neq n-3$ which is a contradiction. Hence $\delta = n-3$. Let $X = \{v_1, v_2, ..., v_{n-3}\}$ be the vertex cut of G and let $V - X = \{x_1, x_2, x_3\}$.

Subcase 3.1 $\langle V - X \rangle = \overline{K_3}$

Then every vertex of $V-X$ is adjacent to all the vertices in X. If $|X| = 1$ then G is a star which is a contradiction. If $|X| = 2$ then G is isomorphic to either $K_{2,3}$ or the graph H_1 which is obtained from $K_{2,3}$ by joining the vertices of degree three by an edge. But $\gamma_{2nt}(H_1) = 2$. Hence G is isomorphic to $K_{2,3}$. Suppose $|X| \geq 3$. Then $\{v_1, v_2, x_1, x_2\}$ is a nt2d-set of G. Hence $\gamma_{2nt}(G) \leq 4$. Then $n = 6$. Thus G is isomorphic to $K_{3,3}$ or $K_{3,3} + e$ or $P_3 + \overline{K_3}$. But $\gamma_{2nt}(K_{3,3} + e) = \gamma_{2nt}(P_3 + \overline{K_3}) = 3$ which is a contradiction. Hence G is isomorphic to $K_{3,3}$.

Subcase 3.2 $\langle V - X \rangle = K_1 \cup K_2$.

Let $x_1x_2 \in E(G)$. Then x_3 is adjacent to all the vertices in X and x_1, x_2 are not adjacent to at most one vertex in X. If degx_1 or degx_2 is $n-2$ then $\{x_1, x_2, x_3\}$ is a nc2d-set of G and hence $\gamma_{2nt}(G) \leq 3$. Then $n \leq 5$. If $n = 4$ then G is isomorphic to P_4 or $K_3(1,0,0)$. But for these graphs $\gamma_{2nt} \neq n-2$. Suppose $n = 5$. Let $X = \{v_1, v_2\}$ and let $v_1v_2 \in E(G)$. If deg$x_1 = 3$ and deg$x_2 = 2$ then G is isomorphic to G^*. If deg$x_1 = 3$ and deg$x_2 = 3$ then G is isomorphic to a graph H_2 which is obtained from $K_4 \cup K_1$ by joining any vertices of K_4 to the vertex of K_1 by the edges. But $\gamma_{2nt}(H_2) = 2 \neq n-2$ which is a contradiction. If $v_1v_2 \notin E(G)$ then G is isomorphic to $C_5 + e$ or H^*. Suppose degx_1 = deg$x_2 = n-3$. If $N(x_1) = N(x_2)$ then there is a vertex $v_1 \in X$ such that v_1 is not adjacent to both x_1 and x_2. Then v_1 is adjacent to all the vertices in X. If $|X| \geq 4$ then $\{x_1, x_2, x_3, v_1\}$ is a nt2d-set of G and hence $n \leq 6$ which is a contradiction. If $|X| = 3$ then $\{v_1, v_2, v_3\}$ is a nt2d-set of G and hence $n \leq 5$ which is a contradiction. If $N(x_1) \neq N(x_2)$ then two vertices say v_1 and v_2 such that v_1 is not adjacent to x_1 but adjacent to x_2 and v_2 is not adjacent to x_2 but adjacent to x_1. Then $\{x_1, x_2, x_3\}$ is a nt2d-set of G and hence $n \leq 5$. Then G is isomorphic to C_5 or $C_5 + e$.

Case 4. $\gamma_{2nt}(G) = n-3$ and $\kappa(G) = n-2$.

Then $n-2 \leq \delta(G)$. If $\delta = n-1$ then G is a complete graph which is a contradiction. Hence $\delta(G) = n-2$. Then G is isomorphic to $K_n - Y$ where Y is a matching in K_n. Then $\gamma_{2nt}(G) \leq 3$. If $\gamma_{2nt}(G) = 3$ then $n = 6$. But $\gamma_{2nt}(K_6 - Y) = 2 \neq n-3$ which is a contradiction. If $\gamma_{2nt}(G) = 2$ then $n = 5$. Hence G is isomorphic to $K_5 - Y$ where Y is any matching in K_5.

Case 5. $\gamma_{2nt}(G) = n - 4$ and $\kappa(G) = n - 1$.

Then G is a complete graph. Since $\gamma_{2nt}(G) = n - 4$ we have $n = 6$. Hence G is isomorphic to K_6. The converse is obvious. □

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Transformation Graph G^{xy} with $xy = +-$

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Abstract: For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The transformation graph G^{+-} of G is the graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident with it in G. In this paper, we obtain structural properties and eccentricity properties of G^{+-}. We establish characterization of graphs whose G^{+-} are Eulerian. In addition, we obtain middle graphs, total graphs and quasi-total graphs of G, which are isomorphic to G^{+-}.

Key Words: Eccentricity, transformation graph, Smarandachely transformation graph, middle graph, total graph, quasi-total graph.

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§1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. For graph theoretic terminology, we refer to [3].

The eccentricity of a vertex $u \in V(G)$ is defined as $e_G(u) = \max\{d_G(u,v) : v \in V(G)\}$, where $d_G(u,v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of a vertex v in G as $e(v)$ and use $d(u,v)$ to denote the distance between two vertices u, v in G. The minimum and maximum eccentricities are the radius $r(G)$ and diameter $diam(G)$ of G, respectively.

A set of vertices which covers all the edges of a graph G is called a vertex cover for G, while a set of edges which covers all the vertices is an edge cover. The smallest number of vertices in any vertex cover for G is called its vertex covering number and is denoted by $\alpha_0(G)$ or α_0. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of edges in any edge cover of G and is called its edge covering number. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_0(G)$ or β_0. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number

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$\beta_1(G)$ or β_1.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. The adjacency relation between two vertices or two edges and incidence relationship between vertices and edges define new structure from the given graph.

A connected graph G is said to be geodetic, if a unique shortest path joins any two of its vertices. The shortest path between two vertices u and v is called geodetic path between u and v. A subset S of V is called a dominating set if $N[S] = V$. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ ([4]).

The line graph of G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$ with two vertices adjacent in $L(G)$ whenever the corresponding edges of G are adjacent ([3]). The middle graph $M(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in $M(G)$ whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it ([1]). Clearly, $E(M(G)) = E(T(G)) \setminus E(G)$. Sampathkumar and Chikkodimath also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G ([5]). The total graph $T(G)$ of G has vertex set $V(G) \cup E(G)$ and vertices of $T(G)$ are adjacent whenever they are neighbors in G ([2]). The quasi-total graph $P(G)$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident with it in G ([6]).

Let $G = (V, E)$ be a graph and α, β be elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise $-$. Let xy be a 2-permutation of the set $\{+, -\}$. We say that α and β correspond to the first term x of xy if both α and β are in $E(G)$. We say that α and β correspond to the second term y of xy if one of α and β is in $V(G)$ and the other in $E(G)$. The transformation graph G^{xy} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xy} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xy. Since there are four distinct 2-permutation of $\{+, -\}$, we obtain four graph transformations of G namely G^{++}, G^{+-}, G^{-+}, G^{--}. It is interesting to see that G^{++} is exactly the middle graph $M(G)$ of G.

We define the transformation graph G^{+-} as follows: G^{+-} of G is the graph with vertex set $V(G) \bigcup E(G)$ and two vertices are adjacent if and only if they correspond to adjacent edges of G or to a vertex and an edge not incident with it in G. The vertex v_i (e'_i) of G^{+-} corresponding to a vertex v_i (*edge* e_i) of G and is referred to as point (line) vertex. Generally, a *Smarandachely transformation graph* G^{+-}_S for a subset $S \subset (V(G) \cup E(G))$ is defined to be a graph with vertex set $V(G) \bigcup E(G)$ and two vertices are adjacent if and only if they correspond to adjacent edges of G or to a vertex and an edge not incident with it in $\langle S \rangle_G$. Thus, if $S = V(G) \bigcup E(G)$, then $G^{+-}_S = G^{+-}$.

§2. Main Results

Observations Let G be a (p, q) graph. Then

1. The line graph $L(G)$ of G is an induced subgraph of G^{+-};

2. The number of vertices of G^{+-} is $p+q$;

3. The degree of a point vertex v in G^{+-} is $q - deg_G(v)$;

4. For every edge $e = uv$ of G the degree of a line vertex e' in G^{+-} is $p + deg_G(u) + deg_G(v) - 4$.

Theorem 2.1 *Let G be a (p, q) graph. Then the number of edges $q^{'}$ of G^{+-} is $q^{'} = \frac{1}{2}\sum d_i^2 + q(p-3)$, where $d_i = deg_G(v_i)$.*

Proof By the definition of G^{+-}, each line vertex is adjacent to $p-2$ point vertices. Since the number of edges in line graph is given by $-q + \frac{1}{2}\sum d_i^2$, it follows that

$$q^{'} = \frac{1}{2}\sum d_i^2 + q(p-3). \qquad \square$$

In the following we characterize disconnected G^{+-}.

Theorem 2.2 *G^{+-} is disconnected if and only if G is one of the following graphs: $2K_2$ and $K_{1,n} \cup lK_1$, $n, l \geq 0$.*

Proof Let G be a (p, q) graph. Assume that G^{+-} is disconnected. We consider the following cases.

Case 1. $q = 0$ and $p \geq 1$. Then $G \cong lK_1$, $l \geq 1$. It follows that G is totally disconnected, so that G^{+-} is totally disconnected.

Case 2. $q = 1$ and $p \geq 2$. Then $G \cong K_2 \cup lK_1$, $l \geq 0$.

Case 3. $q = 2$ and $p \geq 3$. Then $G \cong 2K_2 \cup lK_1$ or $G \cong K_{1,2} \cup lK_1$. But for $G \cong 2K_2 \cup lK_1$, $l \geq 1$, G^{+-} is connected.

Case 4. $q \geq 3$ and $p \geq 3$, we consider the following subcases.

Subcase 4.1 $q = 3$ and $p = 3$. Then $G = C_3$, for which G^{+-} is connected.

Subcase 4.2 $q \geq 3$ and $p \geq 4$. Then G can have at most one vertex of degree greater than or equal three, and all other vertices are either isolated vertices or pendant vertices, otherwise G^{+-} is connected. Hence, $G \cong K_{1,n} \cup lK_1$, $n \geq 3$, $l \geq 0$.

From all the above cases, it follows that, if G^{+-} is disconnected, then $G \cong 2K_2$ or $G \cong K_{1,n} \cup lK_1$, $n, l \geq 0$.

The converse is obvious. $\square$

Corollary 2.3 *G^{+-} is connected if and only if G is none of the following graphs: $2K_2$ and $K_{1,n} \cup lK_1$, $n, l \geq 0$.*

Proof Proof follows from above theorem. $\square$

Theorem 2.4 *Let G be a connected graph, and v be a cut vertex of G incident with $q-1$ edges. Then G^{+-} has a cut vertex such that v is pendant vertex.*

Proof Assume that G is a connected graph having a cut vertex v. Then $G - v$ is disconnected. Since v is incident with $q-1$ edges in G, there exists an edge e of G which is not incident with v in G. By the definition of G^{+-}, the point vertex v is adjacent with only one vertex e' in G^{+-}. Hence the line vertex e' is a cut vertex of G^{+-}, and hence $G^{+-} - e'$ is disconnected graph and v is pendant vertex. □

Corollary 2.5 *Let G be any graph, such that G^{+-} is disconnected. Then G^{+-} contains a cut vertex if and only if G is any graph of $2K_2$, $K_{1,2} \cup lK_1$, $l \geq 0$ and $K_2 \cup lK_l$, $l \geq 2$.*

In the following, we find the girth of G^{+-}.

Theorem 2.6 *For any connected (p, q) graph G with $p \geq 4$, the girth of G^{+-} is* 3.

Proof If G contains a triangle or $K_{1,3}$, then the line graph $L(G)$ of G contains triangle. Since $L(G)$ is a subgraph of G^{+-}, it follows that girth of G^{+-} is 3. Assume that G is triangle-free and $K_{1,3}$ - free. Then $\beta_0 \geq 2$. Since $p \geq 4$, let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be edges in G such that (u_1, v_1, u_2, v_2) is a path of length 3 in G. Let $e_3 = (v_1, u_2)$. Then the subgraph induced by the vertices e'_1, e'_3, v_2 and e'_1 is a triangle in G^{+-}. Thus, the girth of G^{+-} is 3. □

In the following theorem, we find the graph G for which G^{+-} is geodetic.

Theorem 2.7 *Let G be a (p, q) graph such that G^{+-} is connected. Then G^{+-} is geodetic if and only if G is one of the following graphs: P_4, $K_3 \cup K_1$, $nK_2 \cup K_1$, $n \geq 2$.*

Proof Since G^{+-} is connected, it follows from Theorem 2.2 that, G cannot be any of the following graphs $2K_2$ or $K_{1,n} \cup lK_1$, $n, l \geq 0$. Assume that G^{+-} is geodetic. We consider the following cases.

Case 1. $p \leq 3$. In this case, by Theorem 2.2, G^{+-} is disconnected, a contradiction.

Case 2. $p \geq 5$. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$. If e_1 is adjacent to e_2, then $v_1 = u_2$ and (u_1, v_1, v_2) is P_3. Since $p \geq 5$, there exist at least two more vertices, say v_3 and v_4. Both v_3 and v_4 cannot be adjacent to v_1, and both cannot be isolated vertices in G, by Theorem 2.2. Now, we consider the following subcases.

Subcase 2.1 $e_3 = (v_2, v_3)$. If v_4 is isolated or $e_4 = (v_2, v_4)$ or $e_4 = (v_3, v_4)$, then $\langle e'_2, u_1, e'_3, v_4 \rangle$ is $K_4 - e$, where e is an edge of K_4, so that G^{+-} is not geodetic.

Subcase 2.2 $e_3 = (v_3, v_4)$. Then $\langle e'_1, v_3, e'_2, v_4 \rangle$ is $K_4 - e$, and G^{+-} is not geodetic.

In all the above cases, we have a contradiction to the fact that G^{+-} is geodetic. Hence, no two edges are adjacent in G. Thus, for $p \geq 5$, $G \cong nK_2 \cup K_1$, $n \geq 2$.

Case 3: $p = 4$. Then G is connected or $G \cong K_3 \cup K_1$, since otherwise G^{+-} is disconnected by Theorem 2.2. Also, $G \ncong K_{1,3}$, again by Theorem 2.2. So, G is one of the following graphs: P_4, $K_3 \cup K_1$, C_4, $K_3 \bullet K_2$, $K_4 - e$ or K_4.

If $G \cong C_4$, $K_3 \bullet K_2$, $K_4 - e$ or K_4, then G^{+-} contains C_4 or $K_4 - e$ as induced subgraph and hence not geodetic. So, $G \cong P_4$ or $G \cong K_3 \cup K_1$, for which G^{+-} is geodetic.

The converse is obvious. □

Theorem 2.8 *G^{+-} contains $K_{1,3}$ as an induced subgraph if and only if G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph.*

Proof Assume G^{+-} contains $K_{1,3}$ as an induced subgraph.

(1) If the center vertex of $K_{1,3}$ in G^{+-} is a line vertex then G contains $K_2 \cup 3K_1$ as an induced subgraph.

(2) If the center vertex of $K_{1,3}$ in G^{+-} is a point vertex then G contains $3K_2 \cup K_1$ as an induced subgraph.

Hence G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph.

The converse is obvious. □

Corollary 2.9 *If G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph then G^{+-} cannot be the line graph of any graph.*

Proof Suppose G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph, then by above Theorem, G^{+-} contains $K_{1,3}$ as an induced subgraph. Since $K_{1,3}$ is forbidden induced subgraph for line graphs, the result follows. □

In the following, we find the domination number of G^{+-}.

Theorem 2.10 *For any (p, q) graph G, which is not totally disconnected,* $\gamma(G^{+-}) \leq 3$.

Proof Clearly G has at least one edge, let e' be a line vertex of G^{+-} corresponding to an edge $e = (u, v)$ of G. Since each line vertex of G^{+-} is adjacent to $p - 2$ point vertices, e' is adjacent to all point vertices except u and v. By the definition of G^{+-}, any line vertex in G^{+-} other than e' is adjacent to u or v or both and hence $\{u, v, e'\}$ forms a dominating set of G^{+-}. But for $G \cong 2K_2$, we have $\gamma(G^{+-}) = 2$. Thus, $\gamma(G^{+-}) \leq 3$. □

Now, we establish a criterion for G^{+-} to be Eulerian.

Theorem 2.11 *Let G be a (p, q) graph such that G^{+-} is connected, then G^{+-} is Eulerian if and only if one of the following holds:*

(1) p is even, q is odd and $deg_G(u)$ is odd for all $u \in V(G)$;

(2) p is even, q is even and $deg_G(u)$ is even for all $u \in V(G)$.

Proof Suppose that G^{+-} is Eulerian. Then the degree of each vertex in G^{+-} is even. By Observation 3, $deg_{G^{+-}}(u) = q - deg_G(u)$, for every point vertex u of G^{+-}. So, q and $deg_G(u)$ are both even or both odd. Also by Observation 4, $deg_{G^{+-}}(e') = deg_G(u) + deg_G(v) + p - 4$ for every line vertex e' corresponding to $e = (u, v)$ of G. Since $u, v \in V(G^{+-})$, $deg_G(u)$ and $deg_G(v)$ are both even or both odd. So, $deg_G(u) + deg_G(v) + p - 4$ is even, and hence p is even.

Conversely, suppose that (1) holds. Since $deg_{G^{+-}}(u) = q - deg_G(u)$, it follows that for all $u \in V(G)$, $deg_{G^{+-}}(u)$ is even. Also, for every line vertex e' in G^{+-} corresponding to an edge $e = (x, y)$ of G, we have $deg_{G^{+-}}(e') = deg_G(x) + deg_G(y) + p - 4$. Since $deg_G(x)$ and $deg_G(y)$

are odd, and p is even, it follows that $deg_{G^{+-}}(e')$ is even. So, every vertex of G^{+-} has even degree. Thus, G^{+-} is Eulerian. The proof is similar if (2) holds. □

Theorem 2.12 *For any connected graph G with at least three vertices, such that G^{+-} is connected, diam(G^{+-}) is at most* 4.

Proof Let G be a connected graph with at least three vertices such that G^{+-} is connected. We consider the following cases.

Case 1. Let e_1' and e_2' be line vertices of G^{+-}. If e_1 and e_2 are adjacent in G, then $d_{G^{+-}}(e_1', e_2') = 1$. If e_1 and e_2 are not adjacent in G, then there exists an edge e in G adjacent to both e_1 and e_2 in G or there exists a vertex in G not incident with both e_1 and e_2, since otherwise G^{+-} would be disconnected. In both cases, $d_{G^{+-}}(e_1', e_2') = 2$, so that, the distance between any two line vertices in G^{+-} is at most 2.

Case 2. Let u and v be point vertices of G^{+-}. We consider the following subcases.

Subcase 2.1 u and v are not adjacent in G and e is an edge in G not incident with both u and v. Then, (u, e', v) is geodetic path in G^{+-}, and hence $d_{G^{+-}}(u, v) = 2$.

Subcase 2.2 u and v are not adjacent in G and e is an edge in G incident with u but not incident with v. Since G is connected, u and v are connected by path. Let $(u, e, v_1, e_1, v_2, e_2, v_3 \ldots, e_k, v)$ be the $u - v$ geodetic path in G. If $k = 1$, then (u, e_1', e', v) is geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 3$. If $k \geq 2$, then (u, e_k', e_{k-1}', v) is geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 3$.

Subcase 2.3 u and v are adjacent in G. Let $e = (u, v)$. Since $p \geq 3$ and G is connected, there exists a vertex w in G such that (u, v, w) is a path in G. Since G^{+-} is also connected, we consider the following subcases.

Subcase 2.3.1 There exists a vertex x in G such that x is adjacent to w, let $e_1 = (w, x)$. Then (u, e_1', v) is a path in G^{+-} and $d_{G^{+-}}(u, v) = 2$.

Subcase 2.3.2 There exists a vertex x in G such that x is adjacent to u, let $e_2 = (x, u)$. Then (u, e_1', e', e_2', v) is a geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 4$.

Case 3. Let u and e' be point vertex and line vertex, respectively of G^{+-}. If u is not incident with e in G, then $d_{G^{+-}}(u, e') = 1$. If u is incident with e in G, then let $e = (u, v)$. Since $p \geq 3$, and G is connected, it follows that there exists an edge $e_1 = (v, w)$, say. Then $d_{G^{+-}}(u, e') = 2$.

Hence, from all the above cases, $diam(G^{+-}) \leq 4$. □

In the following theorem, we obtain graphs whose middle graph and G^{+-} are isomorphic.

Theorem 2.13 *For any $G(p, q)$ graph, G^{+-} and middle graph $M(G)$ are isomorphic if and only if G is one of the following graphs $\overline{K}_p$, $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4.*

Proof Assume that $G^{+-} \cong M(G)$. If $q = 0$, then $G \cong G^{+-} \cong M(G) \cong \overline{K}_p$. If $q \neq 0$, then $|E(G^{+-})| = |E(M(G))|$. That is $|E(L(G))| + q(p-2) = |E(L(G))| + 2q$, so that, $q(p-2) = 2q$. Then $p = 4$ and G is one of the graphs $K_2 \cup 2K_1$, $2K_2$, P_4, $K_3 \cup K_1$, C_4, $P_3 \cup K_1$, $K_{1,3}$, $K_3 \bullet K_2$,

$K_4 - e$ or K_4. Among these graphs, if G is $K_2 \cup 2K_1$ or $K_{1,3}$ or $K_3 \cup K_1$ or $K_3 \bullet K_2$, G^{+-} and $M(G)$ are not isomorphic. Hence, G is one of the graphs: $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4.

Conversely, if G is one of the following graphs $\overline{K_p}$, $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4, then clearly G^{+-} and middle graph $M(G)$ are isomorphic. □

In the following theorem, we obtain graphs whose total graph and G^{+-} are isomorphic.

Theorem 2.14 *For any graph $G(p,q)$, G^{+-} and total graph $T(G)$ are isomorphic if and only if $G \cong \overline{K_p}$.*

Proof Assume that $G^{+-} \cong T(G)$. If $q = 0$, then $G \cong G^{+-} \cong T(G) \cong \overline{K_p}$. If $q \neq 0$, then $|E(G^{+-})| = |E(T(G))|$. That is, $|E(L(G)| + q(p-2) = |E(G)| + |E(L(G))| + 2q$, so that $q(p-2) = |E(G)| + 2q$. Hence, $p = 5$. If G has at least one edge then G^{+-} and $T(G)$ are non isomorphic, a contradiction.

Conversely, clearly if $G \cong \overline{K_p}$, then G^{+-} and total graph $T(G)$ are isomorphic. □

In the following theorem, we obtain graphs whose quasi-total graph and G^{+-} are isomorphic.

Theorem 2.15 *Let G be any connected (p,q) graph such that $p \geq 5$. Then G^{+-} and quasi-total graph $P(G)$ are non isomorphic.*

Proof Assume that G^{+-} and $P(G)$ are isomorphic. Then

$$\begin{aligned} |E(L(G))| + q(p-2) &= |E(\overline{G})| + |E(L(G))| + 2q \\ &\Longrightarrow q(p-2) = \frac{p(p-1)}{2} - q + 2q \\ &\Longrightarrow \frac{p(p-1)}{2} = q(p-3) \Longrightarrow p(p-1) = 2q(p-3). \end{aligned} \tag{1}$$

We consider the following cases.

Case 1. $p = q$. Then G is unicyclic. From Equation (1), $p = q = 5$. But $G^{+-} \ncong P(G)$.

Case 2. $q < p$. Since G is connected, $q = p - 1$. Then G is a tree. From Equation (1), $p = 6$ and $q = 5$. It can be verified that $G^{+-} \ncong P(G)$.

Case 3. $q > p$. Then from Equation (1), $p \leq 4$. But $p \geq 5$. So, $G^{+-} \ncong P(G)$.

Hence, from all the above cases, G^{+-} and $P(G)$ are non isomorphic. □

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A New Type of Group Action Through the Applications of Fuzzy Sets and Neutrosophic Sets

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Abstract: Fuzzy sets are the most significant tools to handle uncertain data while neutrosophic sets are the generalizations of fuzzy sets in the sense to handle uncertain, incomplete, inconsistent, indeterminate, false data. In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivty in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces.

Key Words: Fuzzy set, neutrosophic set, group action, G-space, fuzzy subspace, neutrosophic subspace.

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§1. Introduction

The theory of fuzzy set was first proposed by Zadeh in the seminal paper [22] in 1965. The concept of fuzzy set is used successfully to modelling uncertain information in several areas of real life. A fuzzy set is defined by a membership function μ with the range in unit interval $[0, 1]$. The theory and applications of fuzzy sets and logics have been studied extensively in several aspects in the last few decades such as control, reasoning, pattern recognition, and computer vision etc. The mathematical framework of fuzzy sets become an important area for the research in several phenomenon such as medical diagnosis, engineering, social sciences etc. Literature on fuzzy sets can be seen in a wide range in $[7, 24, 25, 26]$.

The degree of membership of an element in a fuzzy set is single value between 0 and 1. Thus it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there is some kind of hesitation degree. Therefore, in 1986, Atanassov [1] introduced an extension of fuzzy sets called intuitionistic fuzzy set. An intuitionistic fuzzy sets incorporate the hesitation degree called hesitation margin and

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this hesitation margin is defining as 1 minus the sum of membership and non-membership degree. Therefore the intuitionistic fuzzy set is defined by a membership degree μ as well as a non-membership function υ with same range $[0,1]$. The concpet of Intuitionistic fuzzy sets have been applied successfully in several fields such as medical diagnosis, sale analysis, product marketing, financial services, psychological investigations, pattern recognition, machine learning decision making etc.

Smarandache [14] in 1980, introduced a new theory called Neutrosophy, which is basically a branch of philosophy that focus on the origin, nature, and scope of neutralities and their interactions with different ideational spectra. On the basis of neutrosophy, he proposed the concept of neutrosophic set which is characterized by a degree of truth membership T , a degree of indeterminacy membership I and a degree falsehood membership F. A neutrosophic set is powerful mathematical tool which generalizes the concept of classical sets, fuzzy sets [22], intuitionistic fuzzy sets [2], interval valued fuzzy sets [15], paraconsistent sets [14], dialetheist sets [14], paradoxist sets [14], and tautological sets [14]. Neutrosophic sets can handle the indeterminate, imprecise and inconsistent information that exists around our daily life. Wang et al. [17] introduced single valued neutrosophic sets in order to use them easily in scientific and engineering areas that gives an extra possibility to represent uncertain, incomplete, imprecise, and inconsistent information. Hanafy *et.al* further studied the correlation coefficient of neutrosophic sets [5, 6]. Ye [18] defined the correlation coefficient for single valued neutrosophic sets. Broumi and Smaradache conducted study on the correlation coefficient of interval neutrosophic set in [2]. Salama et al. [12] focused on neutrosophic sets and netrosophic topological spaces. Some more literature about neutrosophic set is presented in [4, 8, 10, 11, 13, 16, 19, 20, 23].

The notions of a G-spaces [3] were introduced as a consequence of an action of a group on an ordinary set under certain rulers and conditions. Over the passed history of Mathematics and Algebra, the theory of group action [3] has proven to be an applicable and effective mathematical framework for the study of several types of structures to make connection among them. The applications of group action can be found in different areas of science such as physics, chemistry, biology, computer science, game theory, cryptography etc which has been worked out very well. The abstraction provided by group actions is an important one, because it allows geometrical ideas to be applied to more abstract objects. Several objects and things have found in mathematics which have natural group actions defined on them. Specifically, groups can act on other groups, or even on themselves. Despite this important generalization, the theory of group actions comprise a wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several other fields.

§2. Literature Review and Basic Concepts

Definition 2.1([22]) *Let X be a space of points and let $x \in X$. A fuzzy set A in X is characterized by a membership function μ which is defined by a mapping $\mu : X \to [0,1]$. The fuzzy set can be represented as*

$$A = \{\langle x, \mu(x)\rangle : x \in X\}.$$

Definition 2.2([14]) *Let X be a space of points and let $x \in X$. A neutrosophic set A in X is characterized by a truth membership.*

function T , an indeterminacy membership function I , and a falsity membership function F. T, I, F are real standard or non-standard subsets of $]0^-, 1^+[$, and $T, I, F : X \to]0^-, 1^+[$. The neutrosophic set can be represented as

$$A = \{\langle x, T(x), I(x), F(x)\rangle : x \in X\}.$$

There is no restriction on the sum of T, I, F, so $0^- \leq T + I + F \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. Thus it is necessary to take the interval $[0, 1]$ instead of $]0^-, 1^+[$ for technical applications. It is difficult to apply $]0^-, 1^+[$ in the real life applications such as engineering and scientific problems.

Definition 2.3([3]) *Let Ω be a non empty set and G be a group. Let $\upsilon : \Omega \times G \longrightarrow \Omega$ be a mapping. Then υ is called an action of G on Ω if for all $\omega \in \Omega$ and $g, h \in G$, there are*

(1) $\upsilon(\upsilon(\omega, g), h) = \upsilon(\omega, gh)$

(2) $\upsilon(\omega, 1) = \omega$, *where* 1 *is the identity element in* G.

Usually we write ω^g instead of $\upsilon(\omega, g)$. Therefore (1) and (2) becomes as

(1) $(\omega^g)^h = (\omega)^{gh}$. For all $\omega \in \Omega$ and $g, h \in G$.

(2) $\omega^1 = \omega$.

A set Ω with an action of some group G on it is called a G-space or a G-set. It basically means a triplet (Ω, G, υ).

Definition 2.4([3]) *Let Ω be a G-space and $\Omega_1 \neq \phi$ be a subset of Ω. Then Ω_1 is called a G-subspace of Ω if $\omega^g \in \Omega_1$ for all $\omega \in \Omega_1$ and $g \in G$.*

Definition 2.5([3]) *Let Ω be a G-space. We say that Ω is transitive G-space if for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$.*

§3. Fuzzy Subspace

Definition 3.1 *Let Ω be a G-space. Let $\mu : \Omega \to [0, 1]$ be a mapping. Then μ is called a fuzzy subspace of Ω if $\mu(\omega^g) \geq \mu(\omega)$ and $\mu\left(\omega^{g^{-1}}\right) \leq \mu(\omega)$ for all $\omega \in \Omega$ and $g \in G$.*

Example 3.1 Let $\Omega = (\mathbb{Z}_4, +)$ and $G = \{0, 2\} \leq \mathbb{Z}_4$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. Then Ω is a G-space. We define $\mu : \Omega \to [0, 1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a fuzzy subspace of Ω.

Definition 3.2 *Let Ω_μ be a fuzzy subspace of the G-space Ω. Then μ is called transitive fuzzy subspace if for any α, β from Ω, there exist $g \in G$ such that $\mu(\alpha^g) = \mu(\beta)$.*

Example 3.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $\mu : \Omega \to [0,1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a transitive fuzzy subspace of Ω.

Theorem 3.1 *If Ω is transitive G-space, then μ is also transitive fuzzy subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking μ on both sides, we get $\mu(\alpha^g) = \mu(\beta)$ for all $\alpha, \beta \in \Omega$. Hence by definition μ is a transitive fuzzy subspace of Ω. □

Definition 3.3 *A transitive fuzzy subspace of Ω is called fuzzy orbit.*

Example 3.3 Consider above Example, clearly μ is a fuzzy orbit of Ω.

Theorem 3.2 *Every fuzzy orbit is trivially a fuzzy subspace but the converse may not be true.*

For converse, see the following Example.

Example 3.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $\mu : \Omega \to [0,1]$ be defined as $\mu(e) = \mu(y) = \mu(x) = \mu(x^2) = \mu(xy) = \mu(x^2y) = \frac{2}{5}$. Thus μ is a fuzzy subspace of Ω but μ is not a transitive fuzzy subspace of Ω as μ has the following fuzzy orbits:

$$\begin{aligned} \mu_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, \\ \mu_2 &= \left\{\mu(x) = \mu(x^2) = \frac{2}{5}\right\}, \\ \mu_3 &= \left\{\mu(xy) = \mu(x^2y) = \frac{2}{5}\right\}. \end{aligned}$$

Definition 3.4 *Let Ω be a G-space and Ω_μ be a fuzzy subspace. Let $\alpha \in \Omega$. The fuzzy stabilizer is denoted by $G_{\mu(\alpha)}$ and is defined to be $G_{\mu(\alpha)} = \{g \in G : \mu(\alpha^g) = \mu(\alpha)\}$.*

Example 3.5 Consider the above Example. Then

$$G_{\mu(e)} = G_{\mu(y)} = G_{\mu(x)} = G_{\mu(x^2)} = G_{\mu(xy)} = G_{\mu(x^2y)} = \{e\}.$$

Theorem 3.3 *If G_α is G-stabilizer, then $G_{\mu(\alpha)}$ is a fuzzy stabilizer.*

Theorem 3.4 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G_\alpha$.*

Remark 3.1 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G$.*

§4. Neutrosophic Subspaces

Definition 4.1 *Let Ω be a G-space. Let $A : \Omega \to [0,1]^3$ be a mapping. Then A is called a neutrosophic subspace of Ω if The following conditions are hold.*

(1) $T(\omega^g) \geq T(\omega)$ and $T\left(\omega^{g^{-1}}\right) \leq T(\omega)$,

(2) $I(\omega^g) \leq I(\omega)$ and $I\left(\omega^{g^{-1}}\right) \geq I(\omega)$ and

(3) $F(\omega^g) \leq F(\omega)$ and $F\left(\omega^{g^{-1}}\right) \geq F(\omega)$ for all $\omega \in \Omega$ and $g \in G$.

Example 4.1 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω which is defined by $\omega^g = \omega + g$. Then Ω is a G-space under this action of G. Let $A : \Omega \to [0,1]^3$ be a mapping which is defined by

$$T(0) = 0.5,\ T(1) = T(2) = T(3) = 1,$$

$$I(0) = 0.3 \text{ and } I(1) = I(2) = I(3) = 0.1,$$

and

$$F(0) = 0.4 \text{ and } F(1) = F(2) = F(3) = 0.2.$$

Thus clearly A is a neutrosophic subspace as A satisfies conditions (1), (2) and (3).

Theorem 4.1 *A neutrosophic subspace is trivially the generalization of fuzzy subspace.*

Definition 4.2 *Let A be a neutrosophic subspace of the G-space Ω. Then A is called fuzzy transitive subspace if for any α, β from Ω, there exist $g \in G$ such that*

$$\begin{aligned} F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta). \end{aligned}$$

Example 4.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $A : \Omega \to [0,1]^3$ by

$$\begin{aligned} T(0) &= \frac{1}{2} \text{ and } T(1) = T(2) = T(3) = 1, \\ I(0) &= \frac{1}{3} \text{ and } I(1) = I(2) = I(3) = 1, \\ F(0) &= \frac{1}{4} \text{ and } F(1) = F(2) = F(3) = 1. \end{aligned}$$

Then clearly A is a neutrosophic transitive subspace of Ω.

Theorem 4.2 *If Ω is transitive G-space, then A is also neutrosophic transitive subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking T on both sides, we get $T(\alpha^g) = T(\beta)$ for all $\alpha, \beta \in \Omega$. Similarly, we can prove it for the other two components I and F. Hence by definition A is a neutrosophic transitive subspace of Ω. □

Definition 4.3 *A neutrosophic transitive subspace of Ω is called neutrosophic orbit.*

Example 4.3 Consider above Example 4.2, clearly A is a neutrosophic orbit of Ω.

Theorem 4.3 *All neutrosophic orbits are trivially the generalization of fuzzy orbits.*

Theorem 4.4 *Every neutrosophic orbit is trivially a neutrosophic subspace but the converse may not be true.*

For converse, see the following Example.

Example 4.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $A : \Omega \to [0, 1]$ be defined as

$$\begin{aligned} T(e) &= T(y) = T(x) = T(x^2) = T(xy) = T(x^2y) = \frac{2}{5}, \\ I(e) &= I(y) = I(x) = I(x^2) = I(xy) = I(x^2y) = \frac{3}{7}, \\ F(e) &= F(y) = F(x) = F(x^2) = F(xy) = F(x^2y) = \frac{4}{9}. \end{aligned}$$

Thus A is a neutrosophic subspace of Ω but A is not a neutrosophic transitive subspace of Ω as A has the following neutrosophic orbits:

$$\begin{aligned} T_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, I_1 = \left\{I(e) = I(y) = \frac{3}{7}\right\}, F_1 = \left\{F(e) = F(y) = \frac{4}{9}\right\}, \\ \mu_2 &= \left\{\mu(x) = \mu(x^2) = \frac{2}{5}\right\}, I_2 = \left\{I(x) = I(x^2) = \frac{3}{7}\right\}, F_2 = \left\{F(x) = F(x^2) = \frac{4}{9}\right\}, \\ \mu_3 &= \left\{\mu(xy) = \mu(x^2y) = \frac{2}{5}\right\}, I_3 = \left\{I(xy) = I(x^2y) = \frac{3}{7}\right\}, F_3 = \left\{F(xy) = F(x^2y) = \frac{4}{9}\right\}. \end{aligned}$$

Definition 4.4 *Let Ω be a G-space and A be a neutrosophic subspace. Let $\alpha \in \Omega$. The neutrosophic stablizer is denoted by $G_{A(\alpha)}$ and is defined to be*

$$G_{A(\alpha)} = \{g \in G : T(\alpha^g) = T(\alpha), I(\alpha^g) = I(\alpha), F(\alpha^g) = F(\alpha)\}.$$

Example 4.5 Consider the above Example 4.4. Then

$$G_{A(e)} = G_{A(y)} = G_{A(x)} = G_{A(x^2)} = G_{A(xy)} = G_{A(x^2y)} = \{e\}.$$

Theorem 4.5 *If G_α is G-stabilizer, then $G_{A(\alpha)}$ is a neutrosophic stabilizer.*

Theorem 4.6 *Every neutrosophic stabilizer is a generalization of fuzzy stabilizer.*

Theorem 4.7 *Let* $G_{A(\alpha)}$ *be a neutrosophic stabilizer. Then* $G_{A(\alpha)} \leq G_{\alpha}$.

Remark 4.1 *Let* $G_{A(\alpha)}$ *be a neutrosophic stabilizer. Then* $G_{A(\alpha)} \leq G$.

§5. Conclusion

In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivity in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces. In the near future, we are applying these concepts in the field of physics, chemistry and other related fields to find the uncertainty in symmetries.

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Almost Pseudo Ricci Symmetric Viscous Fluid Spacetime

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Abstract: The object of the present paper is to investigate the application of almost pseudo Ricci symmetric manifolds to the General Relativity and Cosmology. Also we study the space time when the anisotropic pressure tensor in energy momentum tensor of type (0,2) takes the different form.

Key Words: Pseudo Ricci symmetric manifold, almost pseudo Ricci symmetric manifold, scalar curvature, Viscous fluid space time, energy momentum tensor.

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§1. Introduction

The study of Riemannian symmetric manifolds began with the work of Cartan [1]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan if its curvature tensor R satisfy the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifolds by Walker [10], semi symmetric manifold by Szabó [8], pseudo symmetric manifold by Chaki [2], generalized pseudo symmetric manifold by Chaki [3], and weakly symmetric manifold by Támassy and Binh [9]. In 1988 Chaki [4] introduced the notion of pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is said to be pseudo Ricci symmetric if its Ricci tensor S of type (0, 2) is not identically zero and satisfy the relation

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X),$$

where α is a non zero 1-form such that $g(X, \rho) = \alpha(X)$ for every vector field X. Such an n - dimensional manifold is denoted by $(PRS)_n$.

Again, M. C. Chaki and T. Kawaguchi [5] introduced the notion of almost pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifolds if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [\alpha(X) + \beta(X)]S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X), \tag{1.1}$$

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where α and β are nowhere vanishing 1-forms such that $g(X,\rho)=\alpha(X)$ and $g(X,\mu)=\beta(X)$ for all X and ρ, μ are called the basic vector fields of the manifold. The 1-forms α and β are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If, in particular, $\beta=\alpha$ then it reduces a pseudo Ricci symmetric manifolds.

In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is determined from the physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modelled by some Lorenzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determine the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non flat.

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three different phases, namely, the initial phase, the intermediate phase, and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid. The study of $A(PRS)_4$ is important because such spacetime represents the intermediate phase in the evolution of the universe. Consequently the investigations of $A(PRS)_4$ help us to have a deeper understanding of the global character of the universe including the topology, because the nature of the singularities can be defined from a differential geometric standpoint.

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4,g) with Lorenz metric g with signature (-, +, +, +). The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this casuality that the Lorentz manifold becomes a convenient choice for the study of general relativity. Also we study the space time when the anisotropic pressure tensor in energy momentum tensor of type (0, 2) takes the different form. Here we consider a special type of spacetime which is called almost pseudo Ricci symmetric spacetime.

§2. **Preliminaries**

Let L be the symmetric endomorphism of the tangent space at each point of (M^n,g) corresponding to the Ricci tensor S. Then $g(LX,Y)=S(X,Y)$ for all vector fields X, Y. Let $\{e_i : i=1,2,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold.

Then setting $Y = Z = e_i$ in (1.1) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$dr(X) = r[\alpha(X) + \beta(X)] + 2\alpha(LX), \tag{2.1}$$

where r is the scalar curvature of the manifold. Again from (1.1) we get

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \beta(X)S(Y, Z) - \beta(Y)S(X, Z). \tag{2.2}$$

Setting $Y = Z = e_i$ in (2.2) then taking summation over i, $1 \leq i \leq n$, we obtain

$$dr(X) = 2r\beta(X) - 2\beta(LX). \tag{2.3}$$

If the scalar curvature r is constant then

$$dr(X) = 0 \quad for \quad all \quad X. \tag{2.4}$$

By virtue of (2.4), (2.3) yields

$$\beta(LX) = r\beta(X), \tag{2.5}$$

i.e.,

$$S(X, \mu) = rg(X, \mu). \tag{2.6}$$

These formula will be used in the sequel.

§3. **Almost Pseudo Ricci Symmetric Spacetime with Viscous Fluid Matter Content**

A viscous fluid spacetime is a connected semi-Riemannian manifold (M^4, g) with signature $(-, +, +, +)$. In general relativity the key role is played by Einstein equation

$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y) \tag{3.1}$$

for all vector fields X, Y, where S is the Ricci tensor of type (0, 2), r is the scalar curvature, λ is the cosmological constant, k is the gravitational constant and T is the energy-momentum tensor of type (0, 2). The matter content of the spacetime is described by the energy-momentum tensor T which is to be determined from physical considerations dealing with distribution of matter and energy. Let us consider the energy-momentum tensor T of a viscous fluid spacetime of the following form [7]

$$T(X, Y) = pg(X, Y) + (\sigma + p)\gamma(X)\gamma(Y) + P(X, Y), \tag{3.2}$$

where σ, p are the energy density and isotropic pressure, respectively, and P denotes the anisotropic pressure tensor of the fluid, μ is the unit timelike vector field, called flow vector field of the fluid associated with the 1-form γ given by $g(X, \mu) = \gamma(X)$ for all X. Then by virtue of (3.2), (3.1) can be written as

$$S(X,Y) = (\frac{r}{2} + kp - \lambda)g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y) + kP(X,Y), \tag{3.3}$$

Case 1. Let us consider $P(X,Y) = tg(X,Y)$, where t is any real number, then equation (3.3) become

$$S(X,Y) = (\frac{r}{2} + k(p+t) - \lambda)g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y), \tag{3.4}$$

which shows that the spacetime under consideration is a $(QE)_4$ with $\varrho = \frac{r}{2} + k(p+t) - \lambda$ and $\zeta = k(\sigma + p)$ as associated scalars; λ as the associated 1-form with generator μ and the anisotropic pressure tensor P which is of the form $P(X,Y) = tg(X,Y)$. Hence we can state

Theorem 3.1 *A viscous fluid spacetime satisfying Einstein's equation with a cosmological constant is a 4-dimensional quasi-Einstein manifold with generator μ as the flow vector field and p as the isotropic pressure tensor when the anisotropic pressure tensor P is of the form $P(X,Y) = tg(X,Y)$.*

In this section we consider a relativistic spacetime as a Lorentzian $A(PRS)_4$ with associate 1-form γ and basic vector field μ whose matter content is the viscous fluid with the velocity vector field μ. Then μ is a timelike unit vector field. Hence $g(\mu,\mu) = -1$ and $g(X,\mu) = \gamma(X)$ for all X. Taking a frame field and contracting (3.5) over X and Y we get

$$r = -k(3p - \sigma) + 4\lambda - 4kt, \tag{3.5}$$

Now putting $Y = \mu$, it follows from (3.4) that

$$S(X,\mu) = (\frac{r}{2} - \lambda - k\sigma + kt)g(X,\mu). \tag{3.6}$$

So, it follows from (3.6) that $\frac{r}{2} - \lambda - k\sigma + kt$ is an eigenvalue of Ricci tensor S and μ is an eigenvector corresponding to the eigenvalue.

Let ρ be another eigenvector os S different from μ. Then ρ must be orthogonal to μ. Hence $g(\mu,\rho) = 0$, i.e., $\gamma(\rho) = 0$.

By putting $Y = \rho$, it follows from (3.4) that

$$S(X,\rho) = (\frac{r}{2} - \lambda + kp + kt)g(X,\rho). \tag{3.7}$$

So from (3.7), $\frac{r}{2} - \lambda + kp + kt$ is another eigenvalue of S corresponding to the eigenvector ρ. Here we see that two eigenvalues are different and for a given eigenvector there is a only one eigenvalue. So, it follows that the Ricci tensor S has only two different eigenvalues, namely $\frac{r}{2} - \lambda - k\sigma + kt$ and $\frac{r}{2} - \lambda - k\sigma + kt$.

Let the multiplicity of $\frac{r}{2} - \lambda - k\sigma + kt$ be m and therefore the multiplicity of $\frac{r}{2} - \lambda - k\sigma + kt$ be $(4 - m)$ because the dimension of the spacetime is 4.

Hence

$$m(\frac{r}{2} - \lambda - k\sigma + kt) + (4 - m)(\frac{r}{2} - \lambda - k\sigma + kt) = r.$$

Then after some calculations and using (3.5), we get

$$(\sigma + p)(m - 1) = 0. \tag{3.8}$$

Since $(\sigma + p) \neq 0$ it follows from (3.8) $m = 1$. Thus the multiplicity of the eigenvalue $\frac{r}{2} - \lambda - k\sigma + kt$ is 1 and the the multiplicity of the eigenvalue $\frac{r}{2} - \lambda - k\sigma + kt$ is 3. Therefore the segre characteristic [6] of S is [(111),1]. Hence we can state

Theorem 3.2 *If in an almost pseudo Ricci symmetric spacetime of basic vector field μ, the matter content is a viscous fluid with μ as the velocity vector field and when the anisotropic pressure tensor P is of the form $P(X, Y) = tg(X, Y)$, then the Ricci tensor of the spacetime is segre characteristic [(111),1].*

Now from (3.6), $S(X, \mu) = (\frac{r}{2} - \lambda - k\sigma + kt)g(X, \mu)$ and on the other hand $S(X, \mu) = rg(X, \mu)$, using this two equations we can write

$$[r - \frac{r}{2} + \lambda + k\sigma - kt]g(X, \mu) = 0. \tag{3.9}$$

Further setting $X = \mu$ in equation (3.9), we get

$$\sigma = \frac{2kt - 2\lambda - r}{2k}. \tag{3.10}$$

Again from (3.5) and using the result of (3.10), we can write

$$p = \frac{2\lambda - r - 2kt}{2k}. \tag{3.11}$$

Here we see that σ and p are constant. Hence we can state the following

Theorem 3.3 *If a viscous fluid $A(PRS)_4$ spacetime obeys Einstein's equation with a cosmological constant and when the anisotropic pressure tensor P is of the form $P(X, Y) = tg(X, Y)$, then energy density and isotropic pressure are constants.*

Let us consider in $A(PRS)_4$ spacetime $p > 0$. Then since $\sigma > 0$, we have from (3.10) and (3.11) that

$$\lambda < kt - \frac{r}{2} \ \text{ and } \ \lambda > \frac{r}{2} + kt.$$

Therefore we can state the following

Theorem 3.4 *If a viscous fluid $A(PRS)_4$ spacetime with positive isotropic pressure obeys Einstein's equation with a cosmological constant λ and when the anisotropic pressure tensor P is of the form $P(X, Y) = tg(X, Y)$, then λ satisfies either $\lambda < kt - \frac{r}{2}$ or $\lambda > \frac{r}{2} + kt$.*

Next we discuss whether a viscous fluid $A(PRS)_4$ spacetime with generator μ as unit timelike flow vector field can admit heat flux or not. Let us consider the energy momentum tensor T is of the following form [10]

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\eta(Y) + \eta(X)\gamma(Y) + P(X,Y),$$

where $\eta(X) = g(X,\rho)$ for all vector field X, ρ being the heat flux vector field, σ, p are energy density and isotropic pressure tensor respectively and P denotes the anisotropic pressure tensor of the fluid. Thus we have $g(\mu,\rho) = 0$ that is $\eta(\mu) = 0$.

Now using this energy momentum tensor in equation (3.1) and then setting $Y = \mu$, we get

$$(\frac{r}{2} + \lambda + k\sigma - kt)g(X,\mu) = -k\eta(X). \tag{3.12}$$

Then putting $X = \mu$ in (3.12), we obtain

$$\frac{r}{2} + \lambda + k\sigma - kt = 0$$

Therefore we can write that from (3.12), $\eta(X) = 0$ for all X, since $k \neq 0$. Thus we have the following

Theorem 3.5 *A viscous fluid $A(PRS)_4$ can not admit heat flux when the anisotropic pressure tensor P is of the form $P(X,Y) = tg(X,Y)$.*

Case 2. Next we consider $P(X,Y) = D(X,Y)$, where $D(X,Y) = D(Y,X)$, $trace(D) = 0$ and $D(X,\mu) = 0$ for all vector field X. Then equation (3.3) become

$$S(X,Y) = (\frac{r}{2} + kp - \lambda)g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y) + kD(X,Y), \tag{3.13}$$

From (3.13) it follows that the spacetime under consideration is a pseudo quasi-Einstein manifold with $\varrho_1 = \frac{r}{2} + kp - \lambda$, $\zeta_1 = k(\sigma + p)$ and $\xi_1 = k$ as associate scalars; γ as the associate 1-form with generator μ and the anisotropic pressure function D as the structure tensor. Hence we can sate the following

Theorem 3.6 *A viscous fluid spacetime satisfying Einstein equation with cosmological constant and when the anisotropic pressure tensor P is of the form $P(X,Y) = D(X,Y)$ is a 4-dimensional connected pseudo quasi-Einstein manifold with generator μ is the flow vector field and the structure tensor D as the anisotropic pressure tensor.*

Next we discuss about the segre characteristic of S, for that after some calculations we get two different eigenvalues of S namely, $\frac{r}{2} - \lambda - k\sigma$ and $\frac{r}{2} - \lambda + kp$. Here we see that two eigenvalues are different and for a given eigenvector there is a only one eigenvalue.

Let the multiplicity of $\frac{r}{2} - \lambda - k\sigma$ be m and therefore the multiplicity of $\frac{r}{2} - \lambda + kp$ be $(4 - m)$ because the dimension of the spacetime is 4.

Hence

$$m(\frac{r}{2} - \lambda - k\sigma) + (4 - m)(\frac{r}{2} - \lambda + kp) = r.$$

Then after some calculations, we get $m = 1$. Thus the multiplicity of the eigenvalue $\frac{r}{2} - \lambda - k\sigma$

is 1 and the the multiplicity of the eigenvalue $\frac{r}{2} - \lambda + kp$ is 3. Therefore the segre characteristic of S is [(111),1]. Hence we can state

Theorem 3.7 *If in an almost pseudo Ricci symmetric spacetime of basic vector field μ, the matter content is a viscous fluid with μ as the velocity vector field and when the anisotropic pressure tensor P is of the form $P(X,Y) = D(X,Y)$, then the Ricci tensor of the spacetime is segre characteristic [(111),1].*

Setting $Y = \mu$ in equation (3.12), we get $S(X,\mu) = (\frac{r}{2} - \lambda - k\sigma)g(X,\mu)$ and on the other hand $S(X,\mu) = rg(X,\mu)$, using this two equations we can write

$$[r - \frac{r}{2} + \lambda + k\sigma]g(X,\mu) = 0. \tag{3.14}$$

Further setting $X = \mu$ in equation (3.14), we get

$$\sigma = \frac{-2\lambda - r}{2k}. \tag{3.15}$$

Taking contraction on (3.13), we get

$$r = 4(\frac{r}{2} - \lambda + kp) - k(\sigma + p).$$

Then using (3.15) in the above equation, we can write

$$p = \frac{2\lambda - r}{2k}. \tag{3.16}$$

Here we see that σ and p are constant. Hence we can state the following

Theorem 3.8 *If a viscous fluid $A(PRS)_4$ spacetime obeys Einstein's equation with a cosmological constant and the anisotropic pressure tensor P is of the form $P(X,Y) = D(X,Y)$, then energy function and isotropic pressure of the fluid are constants.*

Let us consider in $A(PRS)_4$ spacetime $p > 0$. Then since $\sigma > 0$, we have from (3.15) and (3.16) that

$$\lambda < -\frac{r}{2} \text{ and } \lambda > \frac{r}{2}.$$

Therefore we can state the following

Theorem 3.9 *If a viscous fluid $A(PRS)_4$ spacetime with positive isotropic pressure obeys Einstein's equation with a cosmological constant λ and when the anisotropic pressure tensor P is of the form $P(X,Y) = D(X,Y)$, then λ satisfies either $\lambda < -\frac{r}{2}$ or $\lambda > \frac{r}{2}$.*

Next we now discuss whether a viscous fluid $A(PRS)_4$ spacetime with generator μ as unit timelike flow vector field can admit heat flux or not. Let us consider the energy momentum tensor T is of the following form

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\eta(Y) + \eta(X)\gamma(Y) + D(X,Y),$$

where $\eta(X) = g(X, \rho)$ for all vector field X, ρ being the heat flux vector field, σ, p are energy density and isotropic pressure tensor respectively and D denotes the anisotropic pressure tensor of the fluid. Thus we have $g(\mu, \rho) = 0$ that is $\eta(\mu) = 0$.

Now using this energy momentum tensor in equation (3.1) and then setting $Y = \mu$, we get

$$(\frac{r}{2} + \lambda + k\sigma)g(X, \mu) = -k\eta(X). \tag{3.17}$$

Then putting $X = \mu$ in (3.17), we obtain

$$\frac{r}{2} + \lambda + k\sigma = 0$$

Therefore we can write that from (3.16), $\eta(X) = 0$ for all X, since $k \neq 0$. Thus we have the following

Theorem 3.10 *A viscous fluid $A(PRS)_4$ can not admit heat flux when the anisotropic pressure tensor P is of the form $P(X, Y) = D(X, Y)$.*

Case 3. In this case, we consider $P(X, Y) = \gamma(X)\gamma(Y)$. Then equation (3.3) becomes

$$S(X, Y) = (\frac{r}{2} + kp - \lambda)g(X, Y) + k(\sigma + p)\gamma(X)\gamma(Y) + k\gamma(X)\gamma(Y).$$

Therefore,

$$S(X, Y) = (\frac{r}{2} + kp - \lambda)g(X, Y) + k(\sigma + p + 1)\gamma(X)\gamma(Y). \tag{3.18}$$

From (3.18) it follows that the spacetime under consideration is a quasi-Einstein manifold with $\varrho_2 = \frac{r}{2} + kp - \lambda$ and $\zeta_2 = k(\sigma + p + 1)$ as associate scalars; λ as the associated 1-form with generator μ and the anisotropic pressure tensor P which is of the form $P(X, Y) = \gamma(X)\gamma(Y)$. Hence we can state the following

Theorem 3.11 *A viscous fluid spacetime satisfying Einstein's equation with a cosmological constant is a 4-dimensional quasi-Einstein manifold with generator μ as the flow vector field and p as the isotropic pressure tensor when the anisotropic pressure tensor P is of the form $P(X, Y) = \gamma(X)\gamma(Y)$.*

Next we discuss about the segre characteristic of S, for that after some calculations we get two different eigenvalues of S namely, $\frac{r}{2} - \lambda - k\sigma - k$ and $\frac{r}{2} - \lambda + kp$. Here we see that two eigenvalues are different and for a given eigenvector there is a only one eigenvalue.

Let the multiplicity of $\frac{r}{2} - \lambda - k\sigma - k$ be m and therefore the multiplicity of $\frac{r}{2} - \lambda + kp$ be $(4 - m)$ because the dimension of the spacetime is 4.

Hence

$$m(\frac{r}{2} - \lambda - k\sigma - k) + (4 - m)(\frac{r}{2} - \lambda + kp) = r.$$

Then after some calculations, we get $m = 1$. Thus the multiplicity of the eigenvalue $\frac{r}{2} - \lambda - k\sigma - k$ is 1 and the the multiplicity of the eigenvalue $\frac{r}{2} - \lambda + kp$ is 3. Therefore the segre characteristic of S is [(111),1]. Hence we can state

Theorem 3.12 *If in an almost pseudo Ricci symmetric spacetime of basic vector field μ, the matter content is a viscous fluid with μ as the velocity vector field and when the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\gamma(Y)$, then the Ricci tensor of the spacetime is segre characteristic [(111),1].*

Putting $Y = \mu$ in equation (3.18), we get $S(X,\mu) = (\frac{r}{2} - \lambda - k\sigma - k)g(X,\mu)$ and on the other hand $S(X,\mu) = rg(X,\mu)$, using this two equations we can write

$$[r - \frac{r}{2} + \lambda + k\sigma + k]g(X,\mu) = 0. \tag{3.19}$$

Further setting $X = \mu$ in equation (3.19), we get

$$\sigma = \frac{-2\lambda - r - 2k}{2k}. \tag{3.20}$$

Taking contraction on (3.18), we get

$$r = 4(\frac{r}{2} - \lambda + kp) - k(\sigma + p + 1).$$

Then using (3.20) in the above equation, we can write

$$p = \frac{2\lambda - r}{2k}. \tag{3.21}$$

Here we see that σ and p are constant. Hence we can state the following

Theorem 3.13 *If a viscous fluid $A(PRS)_4$ spacetime obeys Einstein's equation with a cosmological constant and the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\gamma(Y)$, then energy function and isotropic pressure of the fluid are constants.*

Let us consider in $A(PRS)_4$ spacetime $p > 0$. Then since $\sigma > 0$, we have from (3.20) and (3.21) that

$$\lambda < -(\frac{r}{2} + k) \ \text{ and } \ \lambda > \frac{r}{2}.$$

Therefore we can state the following

Theorem 3.14 *If a viscous fluid $A(PRS)_4$ spacetime with positive isotropic pressure obeys Einstein's equation with a cosmological constant λ and the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\gamma(Y)$, then λ satisfies either $\lambda < -(\frac{r}{2} + k)$ or $\lambda > \frac{r}{2}$.*

Next we now discuss whether a viscous fluid $A(PRS)_4$ spacetime with generator μ as unit timelike flow vector field can admit heat flux or not. Let us consider the energy momentum tensor T is of the following form

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\eta(Y) + \eta(X)\gamma(Y) + \gamma(X)\gamma(Y),$$

where $\eta(X) = g(X,\rho)$ for all vector field X, ρ being the heat flux vector field, σ, p are energy density and isotropic pressure tensor respectively and P denotes the anisotropic pressure tensor of the fluid which is of the form $P(X,Y) = \gamma(X)\gamma(Y)$. Thus we have $g(\mu,\rho) = 0$ that is $\eta(\mu) = 0$.

Now using this energy momentum tensor in equation (3.1) and then setting $Y = \mu$, we get

$$(\frac{r}{2} + \lambda + k\sigma + k)g(X,\mu) = -k\eta(X). \tag{3.22}$$

Then putting $X = \mu$ in (3.22), we obtain

$$\frac{r}{2} + \lambda + k\sigma + k = 0$$

Therefore we can write that from (3.22), $\eta(X) = 0$ for all X, since $k \neq 0$. Thus we have the following

Theorem 3.15 *A viscous fluid $A(PRS)_4$ can not admit heat flux when the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\gamma(Y)$.*

Case 4. Here we consider $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$, where $\eta(X) = g(X,\rho)$ then equation (3.3) become

$$S(X,Y) = (\frac{r}{2} + kp - \lambda)g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y) + k[\gamma(X)\eta(Y) + \eta(X)\gamma(Y)]. \tag{3.23}$$

From (3.23) it follows that the spacetime under consideration is a generalized quasi-Einstein manifold with $\varrho_3 = \frac{r}{2} + kp - \lambda$, $\zeta_3 = k(\sigma + p)$ and $\xi_2 = k$ as associate scalars; λ as the associated 1-form with generator μ and the anisotropic pressure tensor P which is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$. Hence we can state the following

Theorem 3.16 *A viscous fluid spacetime satisfying Einstein's equation with a cosmological constant is a 4-dimensional generalized quasi-Einstein manifold with generator μ as the flow vector field and p as the isotropic pressure tensor when the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$.*

Next we discuss about the segre characteristic of S, for that after some calculations we does not get two different eigenvalues of S. So, in that case we can't say anything about the segre characteristic of S.

Putting $Y = \mu$ in equation (3.23), we get $S(X,\mu) = (\frac{r}{2} - \lambda - k\sigma)g(X,\mu)$ and on the other hand $S(X,\mu) = rg(X,\mu)$, using this two equations we can write

$$[r - \frac{r}{2} + \lambda + k\sigma]g(X,\mu) = 0. \tag{3.24}$$

Further setting $X = \mu$ in equation (3.24), we get

$$\sigma = \frac{-2\lambda - r}{2k}. \tag{3.25}$$

Taking contraction on (3.23), we get

$$r = 4\lambda - k(3p - \sigma)$$

. Then using (3.25) in the above equation, we can write

$$p = \frac{2\lambda - r}{2k}. \tag{3.26}$$

Here we see that σ and p are constant. Hence we can state the following

Theorem 3.17 *If a viscous fluid $A(PRS)_4$ spacetime obeys Einstein's equation with a cosmological constant and the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$, then energy function and isotropic pressure of the fluid are constants.*

Next we consider in $A(PRS)_4$ spacetime $p > 0$. Then since $\sigma > 0$, we have from (3.25) and (3.26) that

$$\lambda < -\frac{r}{2} \text{ and } \lambda > \frac{r}{2}.$$

Therefore we can state the following

Theorem 3.18 *If a viscous fluid $A(PRS)_4$ spacetime with positive isotropic pressure obeys Einstein's equation with a cosmological constant λ and when the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$, then λ satisfies either $\lambda < -\frac{r}{2}$ or $\lambda > \frac{r}{2}$.*

Next we now discuss whether a viscous fluid $A(PRS)_4$ spacetime with generator μ as unit timelike flow vector field can admit heat flux or not. Let us consider the energy momentum tensor T is of the following form

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\psi(Y) + \psi(X)\gamma(Y) + [\gamma(X)\eta(Y) + \eta(X)\gamma(Y)],$$

where $\psi(X) = g(X, \rho)$ for all vector field X, ρ being the heat flux vector field, σ, p are energy density and isotropic pressure tensor respectively and P denotes the anisotropic pressure tensor of the fluid which is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$. Thus we have $g(\mu, \rho) = 0$ that is $\psi(\mu) = 0$.

Now using this energy momentum tensor in equation (3.1) and then setting $Y = \mu$, we get

$$\left(\frac{r}{2} + \lambda + k\sigma\right) g(X, \mu) = -k(\eta(X) + \psi(X)). \tag{3.27}$$

Then putting $X = \mu$ in (3.27), we obtain

$$\frac{r}{2} + \lambda + k\sigma = 0.$$

So we can write that from (3.27), $\psi(X) = 0$ for all X since $k \neq 0$ and obtain the following

Theorem 3.19 *A viscous fluid $A(PRS)_4$ can not admit heat flux when the anisotropic pressure tensor P is of the form $P(X,Y) = \gamma(X)\eta(Y) + \eta(X)\gamma(Y)$.*

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On $(m,(c_1,c_2))$-Regular Bipolar Fuzzy Graphs

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Abstract: In this paper d_m- degree and total d_m-degree of a vertex in bipolar fuzzy graphs are defined. Also (m,(c_1,c_2))-regularity and totally $(m,(c_1,c_2))$-regularity of bipolar fuzzy graphs are defined. A relation between $(m,(c_1,c_2))$-regularity and totally $(m,(c_1,c_2))$-regularity on bipolar fuzzy graph is studied. $(m,(c_1,c_2))$-regularity on some bipolar fuzzy graphs whose underlying crisp graphs are path on $2m$ vertices, a cycle C_n are studied with some specific membership functions.

Key Words: Degree of a vertex in fuzzy graph, regular fuzzy graph, bipolar fuzzy graph, total degree, totally regular fuzzy graph, d_2- degree of a vertex in bipolar fuzzy graph, total d_2- degree, $(2,(c_1,c_2))$-regular fuzzy graphs, totally $(2,(c_1,c_2))$-regular.

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§1. Introduction

In 1965, Lofti A.Zadeh [16] introduced the concept of fuzzy subset of a set as method of representing the phenomena of uncertainty in real life situation. Azriel Rosenfeld introduced fuzzy graphs in 1975 [12]. It has been growing fast and has numerous application in various fields. Nagoor Gani and Radha [10]introduced regular fuzzy graphs, total degree, totally regular fuzzy graphs. Alison Northup introduced semiregular graphs that we call it as $(2,k)$- regular graphs and discussed some properties of (2,k)-regular graphs. In 1994 W.R.Zhang [15] initiated the concept of bipolar fuzzy sets as generalization of fuzzy sets. Bipolar fuzzy sets are extension of fuzzy sets whose membership value in $[-1,1]$. N.R.Santhi Maheswari and C.Sekar introduced d_2- degree of vertex in graphs and discussed some properties of d_2- degree of a vertex in graphs [13]. S.Ravi Narayanan and N.R.Santhi Maheswari introduced d_2-degree of a vertex in bipolar fuzzy graphs, total d_2-degree of a vertex in bipolar fuzzy graph and discussed some properties of d_2-degree of the vertex in bipolar fuzzy graph [11].

This paper motivates us to introduce d_m- degree in bipolar fuzzy graph. Throughout this paper, the vertices take the membership values (m_1^+,m_1^-) and edges take the membership values (m_2^+,m_2^-) where m_1^+,$m_2^+ \in [0,1]$ and m_1^-,$m_2^- \in [-1,0]$.

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§2. Preliminaries

We present some known definitions related to fuzzy graphs and bipolar fuzzy graphs for ready reference to go through the work presented in this paper.

Definition 2.1([9]) *A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \to [0,1]$ is a fuzzy subset of a non empty set V and $\mu : VXV \to [0, 1]$ is a symmetric fuzzy relation on σ such that for all u, v in V, the relation $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ is satisfied.*

Definition 2.2([2]) *A bipolar fuzzy graph with an underlying set V is defined to be the pair $G = (A, B)$ where $A = (m_1^+, m_1^-)$ is a bipolar fuzzy set on V and $B = (m_2^+, m_2^-)$ is a bipolar fuzzy set on E such that $m_2^+(x, y) \leq \min\{m_1^+(x), m_1^+(y)\}$ and $m_2^-(x, y) \geq \max\{m_1^-(x), m_1^-(y)\}$ for all $(x, y) \in E$. Here A is called a bipolar fuzzy vertex set of V and B is called a bipolar fuzzy edge set of E.*

Definition 2.3([9]) *The strength of connectedness between two vertices u and v is defined as $\mu^\infty(u, v) = sup\{\mu^k(u, v) : k = 1, 2, \cdots\}$ where $\mu^k(u, v) = sup \ \{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \cdots \wedge \mu(u_{k-1}, v)$: $u, u_1, u_2, \cdots, u_{k-1}, v$ is path connecting u and v of length $k\}$.*

Definition 2.4([14]) *The positive degree of a vertex $u \in G$ is $d^+(u) = \sum m_2^+(u, v)$. The negative degree of a vertex $u \in G$ is $d^-(u) = \sum m_2^-(u, v)$. The degree of the vertex u is defined as $d(u) = (d^+(u), d^-(u))$.*

Definition 2.5([14]) *Let $G = (A, B)$ be a bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V. Then G is said to be regular bipolar fuzzy graph if all the vertices of G has same degree (c_1, c_2).*

Definition 2.6([14]) *Let $G = (A, B)$ be a bipolar fuzzy graph. The total degree of a vertex $u \in V$ is denoted by $td(u)$ and defined as $td(u) = (td^+(u), td^-(u))$ where $td^+(u) = \sum m_2^+(u, v) + m_1^+(u)$ and $td^-(u) = \sum m_2^-(u, v) + m_1^-(u)$.*

Definition 2.7([11]) *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. The positive d_2 - degree of a vertex $u \in G$ is defined as $d_2^+(u) = \sum m_2^{(2,+)}(u, v)$, where $m_2^{(2,+)}(u, v) = \sup\{m_2^+(u, u_1) \wedge m_2^+(u_1, v) : u, u_1, v$ is the shortest path connecting u and v of length 2. The negative d_2- degree of a vertex $u \in G$ is defined as $d_2^-(u) = \sum m_2^{(2,-)}(u, v)$ where $m_2^{(2,-)}(u, v) = \inf\{m_2^-(u, u_1) \vee m_2^-(u_1, v) : u, u_1, v$ is the shortest path connecting u and v of length 2. The d_2 - degree of a vertex u is defined as $d_2(u) = (d_2^+(u), d_2^-(u))$. The minimum d_2 - degree of G is $\delta_2(G) = \wedge\{d_2(v) : v \in V\}$. The maximum d_2 - degree of G is $\Delta_2(G) = \vee\{d_2(v) : v \in V\}$*

Definition 2.8([11]) *Let $G : (\sigma, \mu)$ be a bipolar fuzzy graph on $G^* : (V, E)$. If $d_2(v) = (c_1, c_2)$, for all $v \in V$, then G is said to be $(2, (c_1, c_2))$-regular bipolar fuzzy graph.*

Definition 2.9([11]) *Let $G = (A, B)$ be a bipolar fuzzy graph. Then the total d_2- degree of a vertex $u \in V$ is defined as $td_2(u) = (td_2^+(u), td_2^-(u))$ where $td_2^+(u) = d_2^+(u) + m_1^+(u)$ and $td_2^-(u) = d_2^-(u) + m_1^-(u)$. Also it can be defined as $td_2(u) = d_2(u) + A(u)$ where $A(u) =$*

$(m_1^+(u), m_1^-(u))$.

Definition 2.10([11]) *Let $G = (A, B)$ be a bipolar fuzzy graph. If all the vertices of G have the same total d_2-degree (c_1, c_2) then G is said to be totally $(2,(c_1, c_2))$- regular bipolar fuzzy graph.*

§3. d_m- Degree of Vertex in Bipolar Fuzzy Graphs

Definition 3.1 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. The positive d_m - degree of a vertex $u \in G$ is defined as $d_m^+(u) = \sum m_2^{(m,+)}(u, v)$, where $m_2^{(m,+)}(u, v) = sup\{m_2^+(u, u_1) \wedge m_2^+(u_1, u_2) \wedge \cdots \wedge m_2^+(u_{m-1}, v) : u, u_1, u_2, \cdots, u_{m-1}, v$ is the shortest path connecting u and v of length $m)\}$. The negative d_m- degree of a vertex $u \in G$ is defined as $d_2^-(u) = \sum m_2^{(m,-)}(u, v)$ where $m_2^{(m,-)}(u, v) = inf\{m_2^-(u, u_1) \vee m_2^-(u_1, u_2) \vee \cdots \vee m_2^-(u_{m-1}, v) : u, u_1, u_2, \ldots, u_{m-1}, v$ is the shortest path connecting u and v of length $m)\}$.The d_m - degree of a vertex u is defined as $d_m(u) = (d_m^+(u), d_m^-(u))$.*

The minimum d_m - degree of G is $\delta_m(G) = \wedge\{d_m(v) : v \in V\}$ and the maximum d_m - degree of G is $\Delta_m(G) = \vee\{d_m(v) : v \in V\}$.

Example 3.2 Consider a bipolar fuzzy graph on $G^*(V, E)$

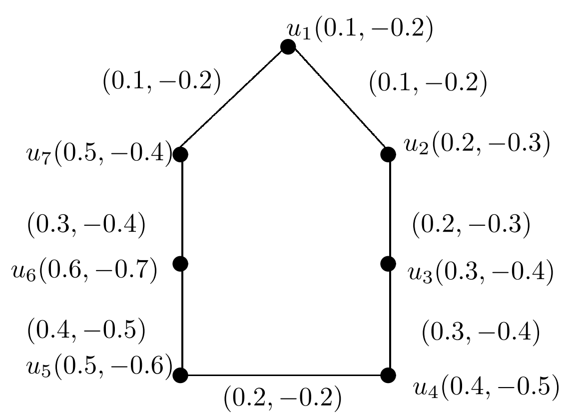

Figure 1

$$d_3^+(u_1) = (0.1 \wedge 0.2 \wedge 0.3) + (0.1 \wedge 0.3 \wedge 0.4) = 0.1 + 0.1 = 0.2$$
$$d_3^-(u_1) = (-0.2 \vee -0.3 \vee -0.2) + (-0.2 \vee -0.4 \vee -0.5)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_1) = (0.2, -0.4)$$

$$d_3^+(u_2) = (0.2 \wedge 0.2 \wedge 0.3) + (0.1 \wedge 0.1 \wedge 0.3) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_2) = (-0.3 \vee -0.2 \vee -0.4) + (-0.2 \vee -0.2 \vee -0.4)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_2) = (0.3, -0.4)$$

$$d_3^+(u_3) = (0.2 \wedge 0.1 \wedge 0.1) + (0.2 \wedge 0.3 \wedge 0.4) = 0.1 + 0.2 = 0.3$$

$$d_3^-(u_3) = (-0.3 \vee -0.2 \vee -0.2) + (-0.2 \vee -0.5 \vee -0.4)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_3) = (0.3, -0.4)$$

$$d_3^+(u_4) = (0.2 \wedge 0.2 \wedge 0.1) + (0.3 \wedge 0.4 \wedge 0.3) = 0.1 + 0.3 = 0.4$$
$$d_3^-(u_4) = (-0.2 \vee -0.3 \vee -0.2) + (-0.4 \vee -0.5 \vee -0.4)$$
$$= (-0.2) + (-0.4) = -0.6$$
$$d_3(u_4) = (0.4, -0.6)$$

$$d_3^+(u_5) = (0.3 \wedge 0.2 \wedge 0.2) + (0.4 \wedge 0.3 \wedge 0.1) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_5) = (-0.4 \vee -0.2 \vee -0.3) + (-0.5 \vee -0.4 \vee -0.2)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_5) = (0.3, -0.4)$$

$$d_3^+(u_6) = (0.4 \wedge 0.3 \wedge 0.2) + (0.3 \wedge 0.1 \wedge 0.1) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_6) = (-0.5 \vee -0.4 \vee -0.2) + (-0.4 \vee -0.2 \vee -0.2)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_6) = (0.3, -0.4)$$

$$d_3^+(u_7) = (0.3 \wedge 0.4 \wedge 0.3) + (0.1 \wedge 0.1 \wedge 0.2) = 0.3 + 0.1 = 0.4$$
$$d_3^-(u_7) = (-0.4 \vee -0.5 \vee -0.4) + (-0.2 \vee -0.2 \vee -0.3)$$
$$= (-0.4) + (-0.2) = -0.6$$
$$d_3(u_7) = (0.4, -0.6)$$

§4. $(m, (c_1, c_2))$-Regular Bipolar Fuzzy Graphs

Definition 4.1 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. If $d_m(v) = (c_1, c_2)$, for all $v \in V$ then G is said to be $(m, (c_1, c_2))$-regular bipolar fuzzy graph.*

Example 4.2 Consider a bipolar fuzzy graph on $G^*(V, E)$

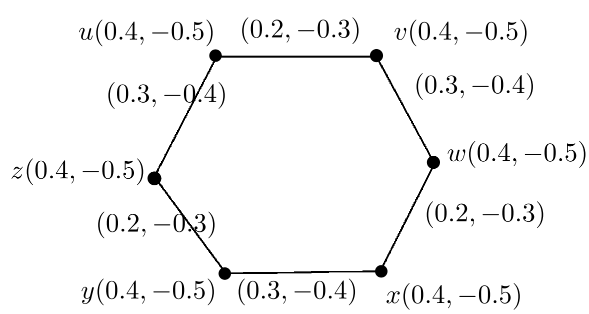

Figure 2

$$d_3^+(u) = sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}$$

$$
\begin{aligned}
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(u) &= inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(u) &= (0.2, -0.3)
\end{aligned}
$$

$$
\begin{aligned}
d_3^+(v) &= sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}\\
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(v) &= inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(v) &= (0.2, -0.3)
\end{aligned}
$$

$$
\begin{aligned}
d_3^+(w) &= sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}\\
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(w) &= inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(w) &= (0.2, -0.3)
\end{aligned}
$$

$$
\begin{aligned}
d_3^+(x) &= sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}\\
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(x) &= inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(x) &= (0.2, -0.3)
\end{aligned}
$$

$$
\begin{aligned}
d_3^+(y) &= sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}\\
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(y) &= inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(y) &= (0.2, -0.3)
\end{aligned}
$$

$$
\begin{aligned}
d_3^+(z) &= sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}\\
&= sup\{0.2, 0.2\} = 0.2\\
d_3^-(z) &= inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}\\
&= inf\{-0.3, -0.3\} = -0.3\\
d_3(z) &= (0.2, -0.3)
\end{aligned}
$$

Here, G is $(3,(0.2,-0.3))$-regular bipolar fuzzy graph.

§5. Totally $(m,(c_1,c_2))$-Regular Bipolar Fuzzy Graphs

Definition 5.1 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. The total d_m-degree of a vertex $u \in V$ is defined as $td_m(u) = d_m(u) + A(u)$.*

The minimum td_m-degree is $t\delta_m(G) = \wedge\{td_m(v) \colon v \in G\}$ and the maximum td_m-degree is $t\Delta_m(G) = \vee\{td_m(v) \colon v \in G\}$

Definition 5.2 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. If each vertex of G has same total d_m - degree, then G is said to be totally (m,(c_1,c_2)) - regular bipolar fuzzy graph.*

Example 5.3 In Figure.2,

$td_3(u) = d_3(u) + A(u) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(v) = d_3(v) + A(v) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(w) = d_3(w) + A(w) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(x) = d_3(x) + A(x) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(y) = d_3(y) + A(y) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(z) = d_3(z) + A(z) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$

Since all the vertices have the same total td_3- degree$(0.6, -0.8)$ This graph is totally $(3, (0.6, -0.8))$-regular bipolar fuzzy graph.

Remark 5.4 *A $(m, (c_1, c_2))$-regular bipolar fuzzy graph need not be totally $(m, (c_1, c_2))$-regular bipolar fuzzy graph*

Example 5.5 Consider a bipolar fuzzy graph on $G^*(V, E)$.

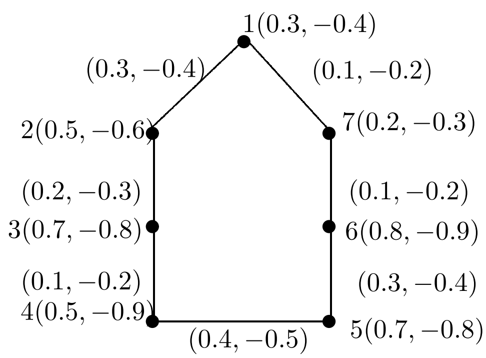

Figure 3

Here, $d_3(v) = (0.2, -0.4)$ for all $v \in V$.Hence G is a $(3, (0.2, -0.4))$-regular bipolar fuzzy graph. But $td_3(1) \neq td_3(7)$.Hence G is not totally $(3, (c_1, c_2))$-regular bipolar fuzzy graph.

Remark 5.6 *A totally $(m, (c_1, c_2))$-regular bipolar fuzzy graph need not be $(m, (c_1, c_2))$-regular bipolar fuzzy graph*

Example 5.7 Consider a bipolar fuzzy graph on $G^*(V, E)$.

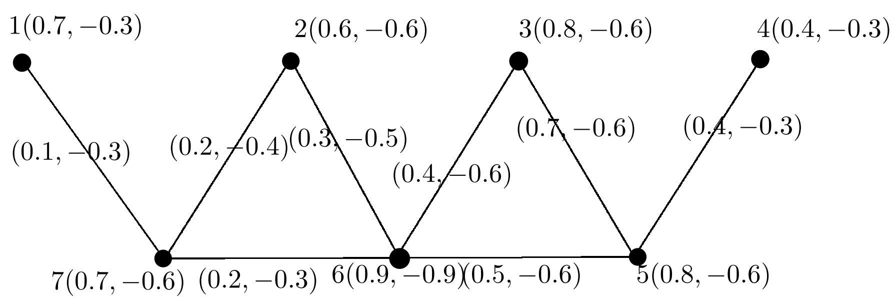

Figure 4

Here, $td_3(v) = (0.9,-0.9)$, for all $v \in V$. Hence G is a totally $(3,(0.9,-0.9))$-regular bipolar fuzzy graph. But $d_3(1) \neq d_3(2)$. Hence G is not $(3,(c_1,c_2))$-regular bipolar fuzzy graph.

Remark 5.8 *A $(m,(c_1,c_2))$-regular bipolar fuzzy graph which is a totally $(m,(c_1,c_2))$-regular bipolar fuzzy graph.*

Example 5.9 Consider a bipolar fuzzy graph on $G^*(V,E)$.

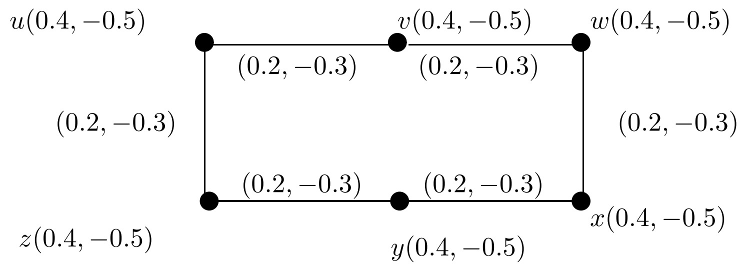

Figure 5

Here, $d_3(v) = (0.2,-0.3)$ and $td_3(v) = (0.6,-0.8)$, for all $v \in V$. Hence G is $(3,(0.2,-0.3))$-regular bipolar fuzzy graph and totally $(3,(0.6,-0.8))$-regular bipolar fuzzy graph.

Theorem 5.10 *Let $G = (A,B)$ be a bipolar fuzzy graph on $G^*(V,E)$. Then $A(u) = (k_1,k_2)$ for all $u \in V$ if and only if the following conditions are equivalent.*

(1) *$G = (A,B)$ is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph;*

(2) *$G = (A,B)$ is totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.*

Proof Suppose $A(u) = (k_1,k_2)$ for all $u \in V$. Assume that G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. Then $d_m(u) = (c_1,c_2)$, for all $u \in V$. So $td_m(u) = d_m(u) + A(u) = (c_1,c_2)+(k_1,k_2) = (c_1+k_1,c_2+k_2)$. Hence G is a totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.. Thus (i) $\Rightarrow$ (ii) is proved.

Now suppose G is totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.

$\Rightarrow td_m(u) = (c_1+k_1,c_2+k_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1+k_1,c_2+k_2)$, for all $u \in V$

$\Rightarrow d_m(u) + (k_1,k_2) = (c_1,c_2) + (k_1,k_2)$, for all $u \in V$

$\Rightarrow d_m(u) = (c_1,c_2)$, for all $u \in V$.

Hence G is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. Thus (i) and (ii) are equivalent.

Conversely assume (i) and (ii) are equivalent. Let G be a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph and totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.

$\Rightarrow td_m(u) = (c_1+k_1,c_2+k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1+k_1,c_2+k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1,c_2) + (k_1,k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow A(u) = (k_1,k_2)$, for all $u \in V$.

Hence $A(u) = (k_1,k_2)$. □

§6. $(m,(c_1,c_2))$- Regularity on Path of $2m$ Vertices with Specific Membership Function

Theorem 6.1 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. If B is constant function then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Proof Suppose that B is constant function, say $B(uv)=(c_1,c_2)$, for all $uv\in E$. Then $d_m(u)=(c_1,c_2)$. Hence G is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. □

Remark 6.2 *The converse of Theorem* 6.1 *need not be true.*

Example 6.3 For example consider $G=(A,B)$ be bipolar fuzzy graph such that $G^*(V,E)$ is path on 6 vertices.

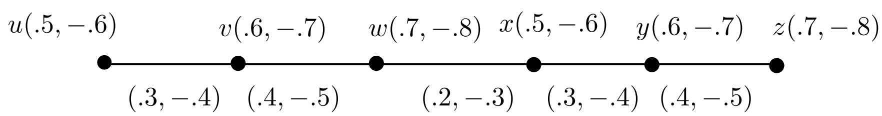

Figure 6

Note that $d_3(u)=(0.2,-0.3)$, for all $u\in V$. So, G is a $(3,(0.2,-0.3))$ regular bipolar fuzzy graph. But B is not constant function.

Theorem 6.4 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. If the alternate edges have same membership values then G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph where $c_1=min\{m_2^{(m,+)}\}$ and $c_2=max\{m_2^{(m,-)}\}$.*

Proof Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. Let $e_1,e_2,\ldots,e_{2m-1}$ be the edges of path G^* in that order. If alternate edges have same membership values then

$$m_2^+(e_i)=\begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even}\end{cases} \quad \text{and} \quad m_2^-(e_i)=\begin{cases} r_1 & \text{if } i \text{ is odd} \\ r_2 & \text{if } i \text{ is even}\end{cases}.$$

For $i=1,2,\cdots,m$,

$d_m^+(v_i)=\{B^+(e_i)\wedge B^+(e_{i+1})\wedge\cdots\wedge B^+(e_{m-2+i})\wedge B^+(e_{m-1+i})\}=min\{k_1,k_2\}$,

$d_m^+(v_i)=c_1$ where $c_1=min\{k_1,k_2\}$.

For $i=1,2,\cdots,m$,

$$d_m^+(v_{m+i})=\{B^+(e_i)\wedge B^+(e_{i+1})\wedge\cdots\wedge B^+(e_{m-2+i})\wedge B^+(e_{m-1+i})\}$$
$$=min\{k_1,k_2\},$$

$d_m^+(v_{m+i})=c_1$ where $c_1=min\{k_1,k_2\}$.

For $i=1,2,\cdots,m$,

$d_m^-(v_i)=\{B^-(e_i)\vee B^-(e_{i+1})\vee\cdots\vee B^-(e_{m-2+i})\vee B^-(e_{m-1+i})\}=max\{r_1,r_2\}$,

$d_m^-(v_i) = c_2$ where $c_2 = max\{r_1, r_2\}$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_{m+i}) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^+(e_{m-1+i})\}$$
$$= max\{r_1, r_2\}.$$

$d_m^-(v_{m+i}) = c_2$, where $c_2 = max\{r_1, r_2\}$.

Since $d_m(v) = (d_m^+(v), d_m^-(v)) = (c_1, c_2)$, for all $v \in V$, we know that G is an $(m,(c_1,c_2))$-regular bipolar fuzzy graph. □

Theorem 6.5 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is path on $2m$ vertices. If the middle edge have positive membership value less than positive membership value of remaining edges and negative membership value greater than negative membership value of remaining edges, then G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph where c_1 and c_2 are membership values of the middle edge.*

Proof Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is path on $2m$ vertices. Let $e_1, e_2, \ldots, e_{2m-1}$ be the edges of path G^* in that order. Let the positive membership values of the edges $e_1, e_2, \cdots, e_{2m-1}$ be $k_1, k_2, \cdots, k_{m-1}, k_m, k_{m+1}, \cdots, k_{2m-1}$ such that $k_m = c_1 \leq k_1, k_2, \cdots, k_{2m-1}$ and the negative values of the edges $e_1, e_2, \cdots, e_{2m-1}$ be then $r_1, r_2, \cdots, r_{m-1}, r_m, r_{m+1}, \ldots, r_{2m-1}$ such that $r_m = c_2 \geq r_1, r_2, \cdots, r_{2m-1}$.

For $i = 1, 2, \cdots, m$,

$$d_m^+(v_i) = \{B^+(e_i) \wedge B^+(e_{i+1}) \wedge \cdots \wedge B^+(e_{m-2+i}) \wedge B^+(e_{m-1+i})\}$$
$$= min\{k_i, k_{i+1}, k_{m-2+i}, k_{m-1+i}\} = k_m,$$

$d_m^+(v_i) = c_1$, where $c_1 = k_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^+(v_{m+i}) = \{B^+(e_i) \wedge B^+(e_{i+1}) \wedge \cdots \wedge B^+(e_{m-2+i}) \wedge B^+(e_{m-1+i})\}$$
$$= min\{k_i, k_{i+1}, k_{m-2+i}, k_{m-1+i}\} = k_m,$$

$d_m^+(v_{m+i}) = c_1$ where $c_1 = k_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_i) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^-(e_{m-1+i})\}$$
$$= \max\{r_i, r_{i+1}, \cdots, r_{m-2+i}, r_{m-1+i}\} = r_m,$$

$d_m^-(v_i) = c_2$ where $c_2 = r_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_{m+i}) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^+(e_{m-1+i})\}$$
$$= \max\{r_i, r_{i+1}, \ldots, r_{m-2+i}, r_{m-1+i}\} = r_m,$$

$d_m^-(v_{m+i}) = c_2$, where $c_2 = r_m$.

Since $d_m(v) = (d_m^+(v), d_m^-(v)) = (c_1, c_2)$, for all $v \in V$ we know that G is an $(m,(c_1,c_2))$-regular bipolar fuzzy graph. □

Remark 6.6 *If A is constant function, then Theorems 6.1, 6.4 and 6.5 hold good for totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

§7. $(m,(c_1,c_2))$ - Regularity on a Cycle with Some Specific Membership Functions

Theorem 7.1 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is the cycle of length $\geq 2m+1$. If m_2^+ and m_2^- are constant functions, then G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. where $(c_1,c_2)=2(m_2^+,m_2^-)$.*

Remark 7.2 *The converse of the Theorem 7.1 need not be true. For example consider $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an odd cycle of length five.*

Example 7.3 See the graph in Figure 7.

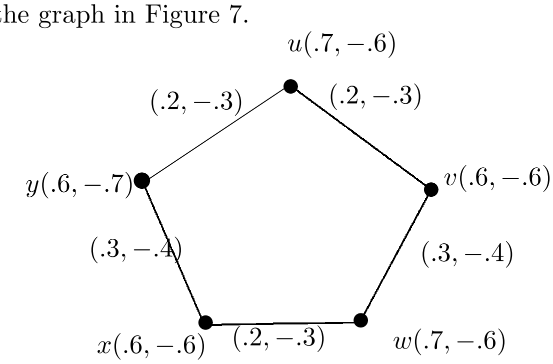

Figure 7

Note that $d_2(u)=(0.4,-0.6)$ for all $u\in V$. So G is a $(2,(0.4,-0.6))$- regular bipolar fuzzy graph. But m_2^+ and m_2^- are not constant functions.

Theorem 7.4 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an even cycle of length $\geq 2m+2$. If the alternate edges have same positive and negative membership values then G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Proof If alternate edges have same positive and negative membership values then

$$m_2^+(e_i)=\begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even}\end{cases} \quad \text{and} \quad m_2^-(e_i)=\begin{cases} k_3 & \text{if } i \text{ is odd} \\ k_4 & \text{if } i \text{ iseven}\end{cases}.$$

Here we have 4 possible cases:

(1) $k_1>k_2$ and $k_3>k_4$,

$d_m^+(v)=min\{k_1,k_2\}+min\{k_1,k_2\}=k_2+k_2=2k_2=c_1$,

$d_m^-(v)=max\{k_3,k_4\}+min\{k_3,k_4\}=k_3+k_3=2k_3=c_2$.

(2) $k_1>k_2$ and $k_3<k_4$,

$d_m^+(v)=min\{k_1,k_2\}+min\{k_1,k_2\}=k_2+k_2=2k_2=c_1$,

$d_m^-(v) = max\{k_3, k_4\} + min\{k_3, k_4\} = k_4 + k_4 = 2k_4 = c_2$.

(3) $k_1 < k_2$ and $k_3 > k_4$,

$d_m^+(v) = min\{k_1, k_2\} + min\{k_1, k_2\} = k_1 + k_1 = 2k_1 = c_1$,

$d_m^-(v) = max\{k_3, k_4\} + min\{k_3, k_4\} = k_3 + k_3 = 2k_3 = c_2$.

(4) $k_1 < k_2$ and $k_3 < k_4$,

$d_m^+(v) = min\{k_1, k_2\} + min\{k_1, k_2\} = k_1 + k_1 = 2k_1 = c_1$,

$d_m^-(v) = max\{k_3, k_4\} + min\{k_3, k_4\} = k_4 + k_4 = 2k_4 = c_2$.

Hence G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph where $d_m(v) = (c_1, c_2)$. □

Remark 7.5 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is an even cycle of length $> 2m+2$. Even if the alternate edges have same positive and same negative membership values, then G need not be a $(m,(c_1,c_2))$- regular bipolar fuzzy graph. Since if $A = (m_1^+, m_1^-)$ is not a constant function, G is not totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Theorem 7.6 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is any cycle of length $> 2m + 1$. Let $k_2 \geq k_1$ and $k_4 \geq k_3$ and*

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ k_2 & if\ i\ is\ even \end{cases} \quad and \quad m_2^-(e_i) = \begin{cases} k_3 & if\ i\ is\ odd \\ k_4 & if\ i\ is\ even \end{cases}.$$

Then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Proof The proof is divided into two cases.

Case 1. Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is an even cycle of length $\leq 2m + 2$. Then by Theorem 7.4, G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Case 2. Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is an odd cycle of length $\leq 2m + 1$.For any $m > 1$, $d_m = (c_1, c_2)$, for all $v \in V$. Hence G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph. □

Remark 7.7 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is any cycle of length $> 2m + 1$. Even if $k_2 \geq k_1$ and $k_4 \geq k_3$,*

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ k_2 & if\ i\ is\ even \end{cases} \quad and \quad m_2^-(e_i) = \begin{cases} k_3 & if\ i\ is\ odd \\ k_4 & if\ i\ is\ even \end{cases}.$$

Then G need not be totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph, since if $A = (m_1^+, m_1^-)$ is not a constant function G is not totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph

Theorem 7.8 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is odd cycle of length*

$> 2m+1$. *Let*

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ membership\ value\ x \geq k_1 & if\ i\ is\ even \end{cases},$$

$$m_2^-(e_i) = \begin{cases} k_2 & if\ i\ is\ odd \\ membership\ value\ y \geq k_2 & if\ i\ is\ even \end{cases},$$

where x is not constant. Then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Proof Let

$$m_2^+(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ \text{membership value } x \geq k_1 & \text{if } i \text{ is even} \end{cases},$$

and

$$m_2^-(e_i) = \begin{cases} k_2 & \text{if } i \text{ is odd} \\ \text{membership value } y \geq k_2 & \text{if } i \text{ is even} \end{cases},$$

where x is not constant. Then

$$\begin{aligned} d_m(v_i) &= (min\{k_1,x\}+max\{k_2,y\})+(min\{k_1,x\}+max\{k_2,y\}) \\ &= (k_1,y)+(k_1,y) = (2k_1,2y), \end{aligned}$$

$d_m(v_i) = (c_1,c_2)$, where $c_1 = 2k_1$,$c_2 = 2y$, for all $v \in V$. Hence G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph. □

Remark 7.9 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is odd cycle of length $> 2m+1$. Even if*

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ membership\ value\ x \geq k_1 & if\ i\ is\ even \end{cases}$$

and

$$m_2^-(e_i) = \begin{cases} k_2 & if\ i\ is\ odd \\ membership\ value\ y \geq k_2 & if\ i\ is\ even \end{cases},$$

where x is not a constant function. Then G need not be totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph, since if $A = (m_1^+, m_1^-)$ is not a constant function G is not totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

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Edge Antimagic Total Labeling of Graphs

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Abstract: The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller as a natural extension of edge-magic labeling defined by Kotzig and Rosa. The present paper deals with a class of subdivided star for all possible values of the parameter $d \in \{0, 1, 2, 3\}$.

Key Words: Smarandachely super (a, d)-edge-antimagic total labeling, super (a, d)-EAT labeling, subdivision of stars.

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§1. Introduction

All graphs in this paper are finite, undirected and simple. For a graph G we denote the vertex-set and edge-set by $V(G)$ and $E(G)$, respectively. A (v, e)-graph G is a graph such that $v = |V(G)|$ and $e = |E(G)|$. A general reference for graph-theoretic ideas can be seen [21]. There are many types of labeling, for example magic, antimagic, gracegraceful, odd gracefu, cordial, radio, sum and mean labeling. This paper deals with different results on super (a, d)-edge-antimagic total $((a, d) - EAT)$ labelings for a subclass of a subdivided star trees. The more details on antimagic total labeling can be seen in [5,9]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [12, 13] on what they called magic valuations of graphs. The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [18] as a natural extension of edge-magic labeling defined by Kotzig and Rosa.

Conjecture 1.1([6]) *Every tree admits a super edge-magic total labeling.*

For supporting this conjecture, many authors have considered super edge-magic total labeling for many particular classes of trees for example [1, 2, 3, 4, 5, 7, 8, 10, 11, 16, 17, 18, 19]. Lee and Shah [14] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still as an open problem.

A star is a particular type of tree graph and many authors have proved the magicness for subdivided stars. Ngurah et. al. [15] proved that $T(m, n, k)$ is also super edge-magic if $k = n + 3$ or $n + 4$. In [20], Salman et. al. found the super edge-magic total labeling of a

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subdivision of a star S_n^m for $m = 1, 2$. Javaid et. al. [11] proved super edge-magic total labeling on subdivided star $K_{1,4}$ and w-trees.

However, super (a, d)-edge-antimagic total labeling of $G \cong T(n_1, n_2, n_3, \cdots, n_r)$ for different $\{n_i : 1 \le i \le r\}$ is still open.

Definition 1.1 *A graph G is called (a, d)-edge-antimagic total $((a, d) - EAT)$ if there exist integers $a > 0$, $d \ge 0$ and a bijection*

$$\lambda : V(G) \cup E(G) \to \{1, 2, 3, \cdots, v + e\}$$

such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic sequence starting from a with the common difference d, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ for every $xy \in E(G)$. W is called the set of edge-weights of the graph G.

Furthermore, let $H \le G$. If there is a bijective function $\lambda : V(H) \to \{1, 2, \cdots, |H|\}$ such that the set of edge-sums of all edges in H forms an arithmetic progression $\{a, a + d, a + 2d, \cdots, a + (|E(H)| - 1)d\}$ but for all edges not in H is a constant, such a labeling is called a Smarandachely (a, d)-edge-antimagic labeling of G respect to H. Clearly, an (a, d)-EAV labeling of G is a Smarandachely (s, d)-EAV labeling of G respect to G itself.

Definition 1.2 *A (a, d)-edge-antimagic total labeling λ is called super (a, d)-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, 3, \cdots, v\}$.*

Lemma 1.1([3]) *If f is a super edge-magic total labeling of G with the magic constant c, then the function $f_1 : V(G) \cup E(G) \to \{1, 2, 3, \cdots, v + e\}$ defined by*

$$f_1(x) = \begin{cases} v + 1 - f(x), & \text{for } x \in V(G), \\ 2v + e + 1 - f(x), & \text{for } x \in E(G). \end{cases}$$

is also a super edge-magic total labeling of G with the magic constant $c_1 = 4v + e + 3 - c$.

Definition 1.3 *For $n_i \ge 1$ and $r \ge 3$, let $G \cong T(n_1, n_2, n_3, \cdots, n_p)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the i-th edge of the star $K_{1,p}$, where $1 \le i \le p$.*

§2. Main Results

We consider the following proposition which we will use frequently in the main results.

Proposition 2.1([4]) *If a (v, e)-graph G has a (s, d)-EAV labeling then*

(1) *G has a super $(s + v + 1, d + 1)$-EAT labeling;*
(2) *G has a super $(s + v + e, d - 1)$-EAT labeling.*

Theorem 2.1 *For all positive integers n, $G \cong T(n_1, n_2, n_3, n_4)$ admits super $(a, 0)$-EAT labeling with $a = v + e + s$ and admits super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $n_1 = n, n_2 = n + 1, n_3 = 2n + 1, n_4 = 4n + 2, s = 4n + 5$.*

Proof Suppose that the vertex-set and edge-set of G are as follows:

$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \le i \le 4\ ;\ 1 \le l_i \le n_i\}$,
$E(G) = \{c\ x_i^1 \mid 1 \le i \le 4\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \le i \le 4\ ;\ 1 \le l_i \le n_i - 1\}$.
If $v = |V(G)|$ and $e = |E(G)|$ then $v = 8n + 5$ and $e = v - 1$.
We define the labeling $\eta : V(G) \to \{1, 2, \cdots, |V(G)|\}$ as follows:

$\eta(c) = 4(n + 1)$. For l_i odd $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4$, we define

$$\eta(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+3) - \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ 4(n+1) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4}. \end{cases}$$

For l_i even $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4$, we define

$$\eta(u) = \begin{cases} 4(n+1) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ 5(n+1) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (6n+5) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (8n+6) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}. \end{cases}$$

The set of all edge-sums $\{\eta(x) + \eta(y) : xy \in E(G)\} = \{4n + 4 + j : 1 \le j \le e\}$ form an arithmatic sequence starting with minimum edge-sum $4(n + 1)$. Therefore, by Proposition 2.1, η can be extended to a super $(a, 0)$-EAT labeling and we obtain the magic constant $a = v + e + s = 20n + 14$. Similarly, by Proposition 2.1, η can be extended to a super $(a, 2)$-EAT labeling and we obtain the minimum edge weight is $a = v + 1 + s = 12n + 11$. □

Theorem 2.2 *For all positive integers n, $G \cong T(n_1, n_2, n_3, n_4)$ admits super $(a, 1)$-EAT labeling with $a = 2v + s - 1$ and admits super $(a, 3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$, $s = 3$, $n_1 = n$, $n_2 = n + 1$, $n_3 = 2n + 1$, $n_4 = 4n + 2$.*

Proof Suppose that the $V(G)$ and $E(G)$ are defined as in the proof of the Theorem 2.1. Let us consider $v = |V(G)|$ and $e = |E(G)|$ then $v = (8n + 5)$ and $e = 4(2n + 1)$. We define the vertex labeling $\eta : V(G) \to \{1, 2, \cdots, V(G)\}$ as follows:

$\eta(c) = 2$. For $1 \le l_i \le n_i$ with $i = 1, 2, 3, 4$, we define

$$\eta(u) = \begin{cases} l_1, & \text{for } u = x_1^{l_1} \text{ and } l_1 \equiv 1 (\text{mod} 2), \\ 2 + l_1, & \text{for } u = x_1^{l_1} \text{ and } l_1 \equiv 0 (\text{mod} 2), \\ (2n+1) - (l_2 - 1), & \text{for } u = x_2^{l_2} \text{ and } l_2 \equiv 1 (\text{mod} 2), \\ (2n+2) - (l_2 - 2), & \text{for } u = x_2^{l_2} \text{ and } l_2 \equiv 0 (\text{mod} 2), \\ 4(n+1) - l_3, & \text{for } u = x_3^{l_3}, \\ (8n+6) - l_4, & \text{for } u = x_4^{l_4}, \end{cases}$$

The set of edge-sums $\{\eta(x)+\eta(y) : xy \in E(G)\}$ generated by the above scheme forms a integer sequence $3, 3+2, \cdots, 3+2(e-1)$ with difference difference 2. Therefore, by Proposition 2.1, η can be extended to a super $(a,1)$-EAT labeling and the minimum edge weight is $a = v+e+s = 2(v+1)$ Similarly, by Proposition 2.1, η can be extended to a super $(a,3)$-EAT labeling and the minimum edge weight is $a = v+1+s = v+4$. □

Theorem 2.3 *For all positive integers n, $G \cong T(n_1, n_2, n_3, n_4, n_5)$ admits super $(a,0)$-EAT labeling with $a = v+e+s$ and admits super $(a,2)$-EAT labeling with $a = v+s+1$, where $n_1 = n$, $n_2 = n+1$, $n_3 = 2n+1$, $n_4 = 4n+2$,$n_5 = 8n+4$ and $s = 8(n+1)$.*

Proof Suppose that the vertex-set and edge-set of the graph G are as follows:

$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \le i \le 5\ ;\ 1 \le l_i \le n_i\}$,
$E(G) = \{c\ x_i^1 \mid 1 \le i \le 5\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \le i \le 5\ ;\ 1 \le l_i \le n_i - 1\}$.

If $v = |V(G)|$ and $e = |E(G)|$ then $v = 16n+9$ and $e = 8(2n+1)$. We define the labeling $\eta : V(G) \to \{1, 2, \cdots, |V(G)|\}$ as follows:

$\eta(c) = 8n+6$. For l_i odd $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4, 5$, we define

$$\eta(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+3) - \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ 4(n+1) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4}, \\ (8n+6) - \frac{l_5+1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

For l_i even $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4, 5$, we define

$$\eta(u) = \begin{cases} (8n+6) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (9n+7) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (10n+6) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (12n+8) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4} \\ (16n+9) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

The set of all edge-sums $\{\eta(x)+\eta(y) : xy \in E(G)\}$=$\{8n+6+j : 1 \le j \le e\}$ form an arithmatic sequence starting with minimum edge-sum $(8n+7)$. Therefore, by Proposition 2.1, η can be extended to a super $(a,0)$-EAT labeling and we obtain the magic constant $a = v+e+s = 40n+25$. Similarly, by Proposition 2.1, η can be extended to a super $(a,2)$-EAT labeling and the minimum edge weight is $a = v+1+s = 24n+18$. □

Theorem 2.4 *For all positive integers n , $G \cong T(n_1, n_2, n_3, n_4, n_5)$ admits super $(a,1)$-EAT labeling with $a = 2v+s-1$ and admits super $(a,3)$-EAT labeling with $a = v+s+1$, where $v = |V(G)|$, $s = 3$,$n_1 = n$, $n_2 = n+1$, $n_3 = 2n+1$, $n_4 = 4n+2$.*

Proof Suppose that the $V(G)$ and $E(G)$ are defined as in the proof of the Theorem 2.3.

Let us consider $v = |V(G)|$ and $e = |E(G)|$ then $v = 16n + 9$ and $e = 8(2n + 1)$. We define the vertex labeling $\eta : V(G) \to \{1, 2, \cdots, V(G)\}$ as follows:

$\eta(c) = 2$. For $1 \leq l_i \leq n_i$, where $i = 1, 2, 3, 4, 5$, we define

$$\eta(u) = \begin{cases} l_1, & \text{for } u = x_1^{l_1} \text{ and } l_1 \equiv 1(\text{mod}2), \\ 2 + l_1, & \text{for } u = x_1^{l_1} \text{ and } l_1 \equiv 0(\text{mod}2), \\ (2n+1) - (l_2 - 1), & \text{for } u = x_2^{l_2} \text{ and } l_2 \equiv 1(\text{mod}2), \\ 2(n+1) - (l_2 - 2), & \text{for } u = x_2^{l_2} \text{ and } l_2 \equiv 0(\text{mod}2), \\ 4(n+1) - l_3, & \text{for } u = x_3^{l_3}, \\ 2(4n+3) - l_4, & \text{for } u = x_4^{l_4}, \\ 2(8n+5) - l_5, & \text{for } u = x_5^{l_5}. \end{cases}$$

The set of edge-sums $\{\eta(x) + \eta(y) : xy \in E(G)\}$ generated by the above scheme forms a integer sequence $3, 3+2, \cdots, 3+2(e-1)$ with difference difference 2. Therefore, by Proposition 2.1, η can be extended to a super $(a, 1)$-EAT labeling and the minimum edge weight is $a = v + e + s = 2(v + 1)$. Similarly, by Proposition 2.1, η can be extended to a super $(a, 3)$-EAT labeling and the minimum edge weight is $a = v + 1 + s = v + 4$.

Theorem 2.5 *For all positive integers n, $G \cong T(n_1, n_2, n_3, n_4, n_5, n_6, ..., n_p)$ admits super $(a, 0)$-EAT labeling with $a = v + e + s$ and admits super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $n_1 = n, n_2 = n + 1, n_3 = 2n + 1, n_4 = 4n + 2, n_5 = 8n + 4$,*

$$s = 8(n+1) + \sum_{m=6}^{p} [2^{m-3}(2n+1)] \quad and n_m = 2^{m-3}(2n+1)$$

for $6 \leq m \leq p$.

Proof Suppose that the vertex-set and edge-set of G are as follows:

$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \leq i \leq p \; ; \; 1 \leq l_i \leq n_i\}$,

$E(G) = \{c \; x_i^1 \mid 1 \leq i \leq p\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \leq i \leq p \; ; \; 1 \leq l_i \leq n_i - 1\}$.

If $v = |V(G)|$ and $e = |E(G)|$ then

$$v = (16n+9) + \sum_{m=6}^{p} [2^{m-2}(2n+1)] \quad \text{and} \quad e = (16n+8) + \sum_{m=6}^{p} [2^{m-2}(2n+1)].$$

We define the labeling $\eta : V(G) \to \{1, 2, \cdots, |V(G)|\}$ as follows:

$$\eta(c) = (8n+6) + \sum_{m=6}^{p} [2^{m-3}(2n+1)].$$

For l_i odd $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4, 5$ and $6 \le m \le p$ we define

$$\eta(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (2n+3) - \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ 4(n+1) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4}. \\ (8n+6) - \frac{l_5+1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

and

$$\eta(x_i^{l_i}) = (8n+6) + \sum_{m=6}^{i} [2^{m-3}(2n+1)] - \frac{l_i+1}{2}$$

respectively. For l_i even $1 \le l_i \le n_i$, where $i = 1, 2, 3, 4, 5$ and $\eta(c) = \beta$ we define

$$\eta(u) = \begin{cases} \beta + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (\beta+n) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (\beta+2n-1) - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (\beta+4n+1) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (\beta+8n+3) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

and

$$\eta(x_i^{l_i}) = (\beta+8n+3) + \sum_{m=6}^{i} [2^{m-3}(2n+1)] - \frac{l_i}{2}$$

respectively. The set of all edge-sums $\{\eta(x)+\eta(y) : xy \in E(G)\}$=$\{\beta + j : 1 \le j \le e\}$ form an arithmatic sequence starting with minimum edge-sum β. Therefore, by Proposition 2.1, η can be extended to a super $(a, 0)$-EAT labeling and we obtain the magic constant

$$a = v + e + s = (2v + 8n + 7) + \sum_{m=6}^{p} [2^{m-3}(2n+1)]$$

Similarly, by Proposition 2.1, η can be extended to a super $(a, 2)$-EAT labeling and obtain the minimum edge weight is

$$a = v + 1 + s = (v + 8n + 9) + \sum_{m=6}^{p} [2^{m-3}(2n+1)]. \qquad \square$$

Theorem 2.6 *For all positive integers n, $G \cong T(n_1, n_2, n_3, n_4, n_5, n_6, \cdots, n_p)$ admits super $(a, 1)$-EAT labeling with $a = 2v + s - 1$ and admits super $(a, 3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$, $s = 3$, $n_1 = n$, $n_2 = n + 1$, $n_3 = 2n + 1$, $n_4 = 4n + 2$, $n_5 = 8n + 4$,*

$$s = 8(n+1) + \sum_{m=6}^{p} [2^{m-3}(2n+1)] \quad \textit{and} \quad n_m = 2^{m-3}(2n+1)$$

for $6 \leq m \leq p$.

Proof Suppose that the $V(G)$ and $E(G)$ are defined as in the proof of the Theorem 2.5. Let us consider $v = |V(G)|$ and $e = |E(G)|$ then

$$v = (16n+9) + \sum_{m=6}^{p}[2^{m-2}(2n+1)] \quad \text{and} \quad e = 8(2n+1) + \sum_{m=6}^{p}[2^{m-2}(2n+1)].$$

We define the vertex labeling $\eta : V(G) \to \{1, 2, \cdots, V(G)\}$ as follows:

$\eta(c) = 2$. For $1 \leq l_i \leq n_i$, where $i = 1, 2, 3, 4, 5$ and $6 \leq m \leq p$, we define

$$\eta(u) = \begin{cases} l_1, & \text{for } \ u = x_1^{l_1} \text{ and } l_1 \equiv 1(\text{mod}2), \\ (2+l_1), & \text{for } \ u = x_1^{l_1} \text{ and } l_1 \equiv 0(\text{mod}2), \\ (2n+1)-(l_2-1), & \text{for } \ u = x_2^{l_2} \text{ and } l_2 \equiv 1(\text{mod}2), \\ 2(n+1)-(l_2-2), & \text{for } \ u = x_2^{l_2} \text{ and } l_2 \equiv 0(\text{mod}2), \\ 4(n+1)-l_3, & \text{for } \ u = x_3^{l_3}, \\ (8n+6)-l_4, & \text{for } \ u = x_4^{l_4}, \\ (16n+10)-l_5, & \text{for } \ u = x_5^{l_5}. \end{cases}$$

and

$$\eta(x_i^{l_i}) = (16n+10) + \sum_{m=6}^{i}[2^{m-3}(2n+1)] - l_i$$

respectively. The set of edge-sums $\{\eta(x) + \eta(y) : xy \in E(G)\}$ generated by the above scheme forms a integer sequence $3, 3+2, \cdots, 3+2(e-1)$ with difference difference 2. Therefore, by Proposition 2.1, η can be extended to a super $(a, 1)$-EAT labeling and obtain the minimum edge weight is $a = v + e + s = 2(v+1)$ Similarly, by Proposition 2.1, η can be extended to a super $(a, 3)$-EAT labeling and obtain the minimum edge weight is $a = v + 1 + s = v + 4$. □

§3. Conclusion

In this paper, we show that a subclass of a tree, namely a subdivision of a star tree denoted by $G \cong T(n_1, n_2, n_3, ..., n_p)$ admits super (a, d)-EAT labeling for all positive integers n and all possible values of the parameter d. However, for different values of the minimum edge-weight a and n_i, problem is still open.

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Galilean Bobillier Formula for One-Parameter Planar Motions

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Abstract: In this present paper, Galilean Euler-Savary formula for the radius of curvature of the trajectory of a point in the moving Galilean plane (or called Isotropic plane) during one-parameter planar motion is taken into consideration. Galilean Bobillier formula is obtained by using the geometrical interpretation of the Galilean Euler-Savary formula. Moreover, a direct way is chosen to obtain Bobillier formula without using the Euler-Savary formula in the Galilean plane. As a consequence, the Galilean Euler-Savary will appear as a specific case of Bobillier formula given in the Galilean plane.

Key Words: Galilean plane, Euler-Savary formula, Bobillier formula.

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§1. Introduction

The study of kinematic analysis and synthesis to describe a motion and to design a mechanism for a desired range of motion, respectively, were examined by many researchers [1]-[9].

In 1959, H. R. Müller defined one-parameter planar motion in the Euclidean plane E^2, studied the moving coordinate system and Euler-Savary formula which gives the relationship between the curvature of trajectory curves, during one-parameter planar motions, [8]. Then A. A. Ergin, by considering the Lorentzian plane L^2, instead of the Euclidean plane E^2, introduced the one-parameter planar motion in L^2 and gave the relations between both the velocities and accelerations and also defined the moving coordinate system [10]-[11]. Euler-Savary formula is studied in Lorentzian plane for the one-parameter Lorentzian motions by using two different ways: In 2002, I. Aytun studied the this formula for the one-parameter Lorentzian motions with using the Müller's Method [12]. In 2003, T. Ikawa gave this formula on Minkowski plane by taking a new aspect without using the Müller's Method [13]. Ikawa gives relation between curvature of roulette and curvatures of these base curve and rolling curve, [13]. Euler-Savary formula is a well documented and an admitted formula in the literature and it takes place in a lot of studies of engineering and mathematics, [14]-[20].

In 1983, the kinematics in the isotropic plane is studied by O. Röschel. In [21], the fundamental properties of the point-paths are investigated, a formula analog to the well-known

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formula of Euler-Savary is developed and special motions: an isotropic elliptic motion and an isotropic four-bar-motion are studied. Besides, in 1985, the motions $\sum / \sum_0$ in the isotropic plane is studied in [22]. Given C^2 -curve k in the moving frame $\sum$. Röschel found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondence between the isotropic curvatures A and A_0 of k and k_0. Then third-order properties of the point-paths are investigated.

Moreover, M. Akar and S. Yüce, [23], introduced the one-parameter motions in the Galilean plane $\mathbb{G}^2$ (or called Isotropic) and gave same concepts analog with [8] or [10]. They analyzed the relationships between the absolute, relative and sliding velocities of one-parameter Galilean planar motion as well as the related pole lines. Also in [24], one Galilean plane moving relative to two other Galilean planes, one moving and the other fixed, was taken into consideration and the relation between the absolute, relative and sliding velocities of this motion and pole points were obtained. Also, a canonical relative system for one-parameter Galilean planar motion was defined. Furthermore, Euler-Savary formula was obtained with the aim of this canonical relative system by using Müller's method in [24]. On the other hand, Euler-Savary formula with using the Ikawa's method is examined in [25].

In 1988, M. Fayet introduced a new formula relative to the curvatures in an one planar motion Euclidean planar motion and called it *Bobillier formula* which may obtained by using Euler-Savary formula and without using Euler-Savary formula [26, 27]. In addition to this, Bobillier formula gave a new analytically aspect to graphically viewpoint of Bobillier construction which was studied by [15]-[20], [26]-[29]. Bobillier formula was established also with concerning second order properties of one-parameter planar motion in the complex plane in [30] and with regarding Lorentzian planar motion in [31].

In this respect, we bring a new breath of Bobillier formula in the Galilean plane in this study. We introduce Bobillier formula with two ways: by using Galilean Euler-Savary equation with respect to one-parameter Galilean motion and a direct way towards to it.

§2. Preliminaries

The study of mechanics of rectilinear motions reduces to a geometry of two dimensional space. The geometry is invariant under transformation stated by I. M. Yaglom

$$\begin{aligned} x' &= \quad\quad x + a \\ y' &= y + vx + b \end{aligned}$$

which is called *Galilean transformation for rectilinear motions* [32]. This geometry is called *two dimensional Galilean geometry* is represented by $\mathbb{G}^2$. Yaglom also expressed three dimensional Galilean geometry which is obtained by plane-parallel motions, is denoted by $\mathbb{G}^3$.

Galilean geometry is a geometry of the Galilean Relativity or shortly a non-Euclidean geometry. It is a "bridge" from Euclidean geometry to Special Relativity. Two and three dimensional Galilean geometry were worked in detail in the literature and the further information about the Galilean geometry can be found in [32]-[34]. Also, many studies are conducted in the Galilean plane (or Isotropic plane) and Galilean space, [33]-[34].

The basic notation about Galilean plane geometry can be given as below:

The distance between points $A(x_1, x_2)$ and $B(y_1, y_2)$ in $\mathbb{G}^2$ is defined by as follows:

$$d(A,B) = \begin{cases} |x_{1-}y_1|, & x_1 \neq y_1 \\ |x_{2-}y_2|, & x_1 = y_1. \end{cases}$$

In this paper, we will denote the inner product of two vectors in the sense of Galilean by notation $\langle , \rangle_{\mathbb{G}}$. Moreover, we will define the Galilean cross product as below:

$$(a \times_{\mathbb{G}} b) = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} & if \;\; a_1 \neq 0 \;\text{ or }\; \mathrm{b}_1 \neq 0 \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} & if \;\; a_1 = \mathrm{b}_1 = 0 \end{cases}$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$. On the other hand, Galilean circle is defined by

$$S_{\mathbb{G}}(m, r) = \{x - m \in \mathbb{G}^2 : \langle x - m, x - m \rangle_{\mathbb{G}} = r^2\}.$$

So the unit Galilean circle is $x = \pm 1$. Hence, we have

$$\cos_g \alpha = 1, \;\; \sin_g \alpha = \alpha$$

for all α. Also, by using another circle definition in Euclidean geometry, we get a set

$$ax^2 + 2b_1 x + 2b_2 y + c = 0$$

which are (Euclidean) parabolas. This set is called a *Galilean cycle* and denoted by Z.

§3. One-Parameter Planar Motion in the Galilean Plane $\mathbb{G}$

Let $\mathbb{G}'$ and $\mathbb{G}$ be fixed and moving Galilean planes with the perpendicular coordinate systems $\{O'; \mathbf{g_1'}, \mathbf{g_2'}\}$ and $\{O; \mathbf{g_1}, \mathbf{g_2}\}$, respectively. If we take M_1, M_2 and M_3 are points linked to moving Galilean plane $\mathbb{G}$ then there are the conjugate points M_1', M_2' and M_3' of these points which are the curvature centers of the trajectory drawn M_1, M_2 and M_3 in the fixed Galilean plane $\mathbb{G}'$.

The normals of this trajectory pass from an instantaneous center of rotation that is denoted by P and called as *pole point*.

Since there exist pole points in every moment t, during the one-parameter planar motion

$\mathbb{G}/\mathbb{G}'$, any pole point P is situated varied position on the planes $\mathbb{G}$ and $\mathbb{G}'$. The position of pole point P on the moving plane $\mathbb{G}$ is usually a curve called *moving pole curve* and denoted by (P). Also the position of this pole point P on the fixed plane $\mathbb{G}'$ is usually a curve called *fixed pole curve* denoted by (P') [23].

The axis $\mathbf{x}$ is the common tangent and the axis $\mathbf{y}$ is the common normal to pole curves (P) and (P') at P, see Figure 1.

If θ is the rotation angle of motion of the Galilean plane $\mathbb{G}$ with respect to $\mathbb{G}'$ at each t moment, then each point M makes a rotation motion with $\dot{\theta}$ angular velocity at the instantaneous center P.

Let $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ be unit vectors, then these unit vectors can be given as

$$\mathbf{X_1} = \frac{\mathbf{PM_1}}{\|\mathbf{PM_1}\|_{\mathbb{G}}}, \mathbf{X_2} = \frac{\mathbf{PM_2}}{\|\mathbf{PM_2}\|_{\mathbb{G}}}, \mathbf{X_3} = \frac{\mathbf{PM_3}}{\|\mathbf{PM_3}\|_{\mathbb{G}}}. \tag{3.1}$$

If the abscissae of the points M_1 and M_1' on the axis $(P, \mathbf{X_1})$ are ρ_1 and ρ_1' respectively, then it can be written that

$$\langle \mathbf{PM_1}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1, \text{ and } \langle \mathbf{PM_1'}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1'. \tag{3.2}$$

Similarly,

$$\langle \mathbf{PM_2}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2, \text{ and } \langle \mathbf{PM_2'}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2',$$

$$\langle \mathbf{PM_3}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3, \text{ and } \langle \mathbf{PM_3'}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3'.$$

§4. Inflection Points, Inflection Cycle and Euler-Savary Formula in Galilean Plane $\mathbb{G}$

Let M be an arbitrary point on moving Galilean plane $\mathbb{G}$ and M' be its conjugate point on fixed plane $\mathbb{G}'$. Let the coordinates of points M and M' be (m_1, m_2) and (m_1', m_2') in the canonical relative system, respectively. The vectors $\mathbf{PM}$ and $\mathbf{PM'}$ have same direction which passes the pole point P. So we can write

$$m_1' = \lambda m_1, \;\; m_2' = \lambda m_2,$$

where λ is an unknown ratio. From the definition of Euler-Savary equation in Galilean plane [24], we get the relation between the points M and M' such as

$$m_1' = \frac{m_1 m_2}{m_2 - m_1^2 \frac{d\theta}{ds}}, \qquad m_2' = \frac{m_2 m_2}{m_2 - m_1^2 \frac{d\theta}{ds}}.$$

From the fact that, an inflection point may be defined to be a point whose trajectory momentarily has an infinite radius of curvature [14, 15], we get the inflection cycle such that

$$m_2 = m_1^2 \frac{d\theta}{ds}.$$

Let the inflection points linked to the points M_1, M_2 and M_3, by referring to Figure 1 be

M_1^*, M_2^* and M_3^*, respectively. The locus of such points is a cycle in the moving Galilean plane $\mathbb{G}$ called as an *inflection cycle*. The abscissae of the inflection points can be written as below:

$$\langle \mathbf{PM_1^*}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1^*, \quad \langle \mathbf{PM_2^*}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2^*, \quad \langle \mathbf{PM_3^*}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3^*. \tag{4.1}$$

Let the diameter of the inflection cycle be h. Then there is a relationship between h and ρ_1^* as follows:

$$h \sin_g \theta_1 = \rho_1^*, \tag{4.2}$$

where θ_1 is angle of the motion $\mathbb{G}/\mathbb{G}'$.

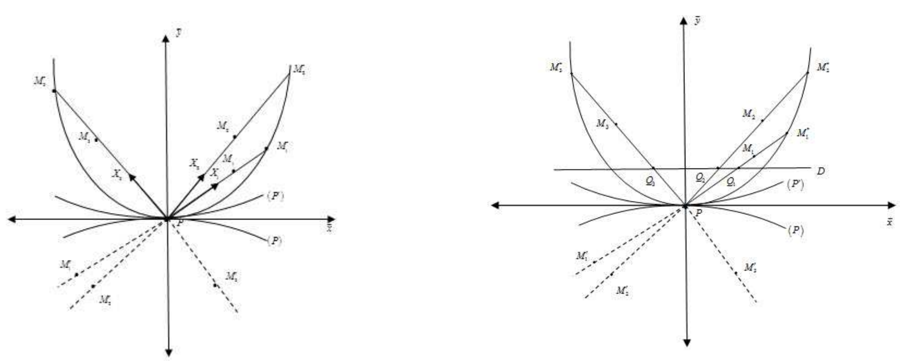
During one-parameter planar motion $\mathbb{G}/\mathbb{G}'$, the point M_1 in the moving Galilean plane $\mathbb{G}$ draws a trajectory with instantaneous curvature center M_1' in the fixed Galilean plane $\mathbb{G}'$. In reverse motion, the point M_1' in $\mathbb{G}'$ draws a trajectory in $\mathbb{G}$, being the curvature center at the point M_1, (see Figure 1). This interconnection between the points M_1 and M_1' is given by Euler-Savary formula

$$\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1} \right) \sin_g \theta_1 = \frac{1}{R_1'} - \frac{1}{R_1}, \tag{4.3}$$

where R_1' and R_1 are the abscissae on $(O, \vec{y})$ of the curvature centers of pole curves (P') and (P), respectively [24]. From the equations (4.2) and (4.3) it is seen that

$$\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1} \right) \sin_g \theta_1 = \frac{1}{R_1'} - \frac{1}{R_1} = \frac{ds}{d\theta_1} = \frac{1}{h}$$

in which $\frac{1}{h} = \frac{1}{R_1} - \frac{1}{R_1'}$ (first form) or $\frac{1}{h} = \pm \frac{\omega}{V}$ (second form) where ω is the angular velocity of the motion of the plane $\mathbb{G}$ with respect to $\mathbb{G}'$ and V is the common velocity of P on (P') and (P).



$\mathbf{PM_i}, \mathbf{PM_i^*}, 1 \leq i \leq 3$, vectors $\qquad$ $\mathbf{Q_i}, 1 \leq i \leq 3$, points

Figure 2

§5. Galilean Bobillier Formula Obtained by Galilean Euler-Savary Formula

Let consider the points Q_j are defined by $\mathbf{PQ_j} = \frac{1}{\rho_j^*} \mathbf{X_j}$ where $\frac{1}{\rho_j^*} = \frac{1}{\rho_j} - \frac{1}{\rho_j'}$ for $1 \leqslant j \leqslant 3$. Then the points Q_1, Q_2, and Q_3 are the images of points M_1^*, M_2^*, and M_3^* of the inflection cycle which belong to $(P,\mathbf{X_1})$, $(P,\mathbf{X_2})$ and $(P,\mathbf{X_3})$, respectively (see Figure 2). Therefore the following equations can be written as follows:

$$\langle \mathbf{PQ_1},\mathbf{X_1}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_1^*},\ \ \langle \mathbf{PQ_2},\mathbf{X_2}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_2^*},\ \ \langle \mathbf{PQ_3},\mathbf{X_3}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_3^*},\ \ \langle \mathbf{PQ},\mathbf{x}\rangle_{\mathbb{G}}=\tfrac{1}{h}. \tag{5.1}$$

From the definition of Euler-Savary formula in Galilean plane [24] and the equation (5.1)

$$\mathbf{PQ_1}\sin_g\theta_1=\frac{1}{\rho_1^*}\mathbf{X_1}\sin_g\theta_1\ =\frac{1}{h}\mathbf{X_1},$$

$$\mathbf{PQ_2}\sin_g\theta_2=\frac{1}{\rho_2^*}\mathbf{X_2}\sin_g\theta_2\ =\frac{1}{h}\mathbf{X_2},$$

and

$$\mathbf{PQ_3}\sin_g\theta_3=\frac{1}{\rho_3^*}\mathbf{X_3}\sin_g\theta_3\ =\frac{1}{h}\mathbf{X_3},$$

are obtained. By taking into account the last three equations, we have

$$\langle \mathbf{PQ}_1,\mathbf{X_1}\rangle_{\mathbb{G}}\sin_g\theta_1=\langle \mathbf{PQ_2},\mathbf{X_2}\rangle_{\mathbb{G}}\ \sin_g\theta_2=\langle \mathbf{PQ_3},\mathbf{X_3}\rangle_{\mathbb{G}}\ \sin_g\theta_3=\frac{1}{h}.$$

Thus, the set of the points Q is a straight line which is denoted by D parallel to real axis $\mathbf{x}$. Thus the line $\mathbf{x}$ is an image of the inflection cycle by this inversion, see Figure 1. From the fact that the vectors $\mathbf{PQ_1}-\mathbf{PQ_2}$ and $\mathbf{PQ_2}-\mathbf{PQ_3}$ are linearly dependent, the following equation can be written:

$$(\mathbf{PQ_1}\times\mathbf{PQ_2})-(\mathbf{PQ_2}\times\mathbf{PQ_2})-(\mathbf{PQ_1}\times\mathbf{PQ_3})+(\mathbf{PQ_2}\times\mathbf{PQ_3})=\mathbf{0}.$$

Since $\mathbf{PQ_1}=\frac{1}{\rho_1^*}\mathbf{X}_1$, $\mathbf{PQ_2}=\frac{1}{\rho_2^*}\mathbf{X}_2$, $\mathbf{PQ_3}=\frac{1}{\rho_3^*}\mathbf{X}_3$ and $\rho_1^*\rho_2^*\rho_3^*$ never vanishes, we get

$$\rho_1^*\left(\mathbf{X}_2\times\mathbf{X}_3\right)+\rho_2^*\left(\mathbf{X}_3\times\mathbf{X}_1\right)+\rho_3^*\left(\mathbf{X}_1\times\mathbf{X}_2\right)=\mathbf{0}$$

and for the sake of brevity, if we take

$$\theta_3-\theta_2=\theta_{23},\theta_1-\theta_3=\theta_{31},\theta_2-\theta_1=\theta_{12}, \tag{5.2}$$

then we find

$$\rho_1^*\sin_g\theta_{23}+\rho_2^*\sin_g\theta_{31}+\rho_3^*\sin_g\theta_{12}=0\ \ , \tag{5.3}$$

where $\frac{1}{\rho_j^*}=\frac{1}{\rho_j}-\frac{1}{\rho_j'}$ for $1\leqslant j\leqslant 3$.

This is Bobillier formula for one-parameter planar motion in Galilean plane $\mathbb{G}$ analog with Bobillier formula given in Euclidean plane [27], complex plane [30] and Lorentzian plane [31]. With using the Galilean trigonometric properties we can write

$$\rho_1^*\theta_{23}+\rho_2^*\theta_{31}+\rho_3^*\theta_{12}=0\ \ . \tag{5.4}$$

The equation (5.4) is called the *Galilean Bobillier formula* during the one-parameter planar motions $\mathbb{G}/\mathbb{G}'$.

§6. Galilean Bobillier Formula Deduced from a Direct Way in the Galilean Plane $\mathbb{G}$

In this section, we will introduce Galilean Bobillier formula from a direct way. Let us examine the trajectory velocity and trajectory acceleration of the points in moving Galilean plane $\mathbb{G}$. Suppose that $\mathbf{V}'(M_1)$ and $\mathbf{J}'(M_1)$ are absolute velocity and absolute acceleration vector of the point M_1, respectively. Let denote the angular velocity of planar motion $\mathbb{G}/\mathbb{G}'$ by ω, then $\omega = \frac{\Delta\theta}{\Delta t}$ where θ is the rotation angle. By taking an orthogonal vector to the Galilean planes $\mathbb{G}$ and $\mathbb{G}'$ as $\mathbf{z}$, the angular velocity vector can be defined by $\omega = \omega\mathbf{z}$. Moreover, the sliding velocity vector of the point M_1 is

$$\mathbf{V}(M_1) = \omega \times \mathbf{PM_1} = \omega \left\|\mathbf{PM_1}\right\|_{\mathbb{G}} \sin_g \theta. \tag{6.1}$$

The relation between velocities during one-parameter planar motion in Galilean plane is

$$\mathbf{V}'(M_1) = \mathbf{V}'(P) + \mathbf{V}(M_1),$$

where $\mathbf{V}'(M_1), \mathbf{V}'(P)$ and $\mathbf{V}(M_1)$ denote the absolute, sliding and relative velocity vectors of $\mathbb{G}/\mathbb{G}'$, respectively [23]. With using the equation (6.1), we have

$$\mathbf{V}'(M_1) = \mathbf{V}'(P) + (\omega \times \mathbf{PM_1}) \ . \tag{6.2}$$

By differentiating the equation (6.2) with respect to time t , we obtain

$$\mathbf{J}'(M_1) = \mathbf{J}'(P) + \left(\dot{\omega}\mathbf{z} \times \mathbf{PM_1}\right) + \omega^2\mathbf{PM}_1, \tag{6.3}$$

where $\mathbf{J}'(P)$ is acceleration vector of the point on $\mathbb{G}'$ that coincides instantaneously with P. Here the first term is the trajectorywise invariant acceleration component, the second term is tangential acceleration component and the third term is centripental acceleration component. With considering this explanation for the inflection points whose acceleration normal is zero, then the absolute velocity vector and acceleration vectors of the point M_1^* on the inflection cycle becomes linearly dependent, so

$$\mathbf{V}'(M_1^*) \times \mathbf{J}'(M_1^*) = \mathbf{0}$$

can be written. If we substitute the method of the (6.2) and (6.3) into the last equation, the equation rewritten as follows:

$$(\mathbf{V}'(P) + (\omega\mathbf{z} \times \mathbf{PM_1^*})) \times \left(\mathbf{J}'(P) + \left(\dot{\omega}\mathbf{z} \times \mathbf{PM_1^*}\right) + \omega^2\mathbf{PM}_1^*\right) = \mathbf{0}.$$

From $\mathbf{V}'(P) = \mathbf{0}$ and the equation (4.1) $\|\mathbf{PM_1^*}\|_{\mathbb{G}} = \rho_1^*$, we obtain

$$\langle \mathbf{PM_1}^*, \mathbf{J}'(P)\rangle_{\mathbb{G}}\, \mathbf{z} - \omega^2 \left\|\mathbf{PM}_1^*\right\|_{\mathbb{G}}^2 \mathbf{z} = \mathbf{0}.$$

With simplifying calculations and using $\mathbf{PM_j}^* = \left\|\mathbf{PM_j}^*\right\|_{\mathbb{G}} \mathbf{X}_j$ for $1 \leq j \leq 3$, we obtain the equations as follows,

$$\rho_1^* = \frac{\langle \mathbf{X}_1, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2} \tag{6.4}$$

$$\rho_2^* = \frac{\langle \mathbf{X}_2, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2}, \tag{6.5}$$

$$\rho_3^* = \frac{\langle \mathbf{X}_3, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2}, \tag{6.6}$$

for the points M_1^*, M_2^* and M_3^*, respectively. It is easily seen from the equations (6.4), (6.5) and (6.6) that ρ_1^*, ρ_2^*, and ρ_3^* are the orthogonal projections of the same vector $\frac{\mathbf{J}'(P)}{\omega^2}$ onto the vectors $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$, respectively. The relationship between these unit vectors is indicated with the equation

$$\lambda_1 \mathbf{X}_1 + \lambda_2 \mathbf{X}_2 + \lambda_3 \mathbf{X}_3 = \mathbf{0},$$

where λ_1, λ_2 and λ_3 are quantities. By successive Galilean cross products with $\mathbf{X}_1$ and $\mathbf{X}_2$ the quantities λ_1, λ_2 and λ_3 are obtained and using the specification (5.2), then the previous linear combination becomes

$$\sin_g \theta_{12} \mathbf{X}_3 + \sin_g \theta_{23} \mathbf{X}_1 + \sin_g \theta_{31} \mathbf{X}_2 = \mathbf{0}.$$

The product of the previous equation with the vector $\frac{\mathbf{J}'(I)}{\omega^2}$ is given as below:

$$\sin_g \theta_{12} \frac{\langle \mathbf{X}_3, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2} + \sin_g \theta_{23} \frac{\langle \mathbf{X}_1, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2} + \sin_g \theta_{31} \frac{\langle \mathbf{X}_2, \mathbf{J}'(P)\rangle_{\mathbb{G}}}{\omega^2} = \mathbf{0}.$$

Finally, if we substitute the equations (6.4), (6.5) and (6.6) into the last equation, we obtain the equation (5.4) which was called *Bobillier formula*. It can be noticed that, the direct way gives us the Bobillier formula without using the Euler-Savary formula. Therefore the following theorem can be given.

Theorem 6.1 *During the one-parameter planar motion $\mathbb{G}/\mathbb{G}'$, the relation between the distances of inflection points of points in the moving plane $\mathbb{G}$ and the pole point is given by the equation (5.4) which is called Bobillier formula.*

Let us analyze a particular case of Theorem 6.1. If a point K linked to moving plane $\mathbb{G}$ is coincident with instantaneous pole center P, then $\mathbf{V}'(K) = \mathbf{0}$ and similarly $\mathbf{J}'(K) = \mathbf{0}$. From this place, the vector $\mathbf{X}_2$ is equal to $\mathbf{x}$ which is the normal to the path of K at P. Hence, in the equation (6.5) ρ_2^* is equal to zero. Thus we can express the following corollary.

Corollary 6.2 *Let a point K linked to moving plane $\mathbb{G}$ be coincident with instantaneous pole center P. In that case Bobillier formula in the Galilean plane becomes*

$$\rho_1^* + \rho_3^* \theta_{12} = 0.$$

In conclusion, the corollary simply a particular case of Bobillier formula in the Galilean plane $\mathbb{G}$.

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Generalized h-Kropina Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a generalized $h-$Kropina change of Finsler metric becomes a projective change. The condition under which a generalized $h-$Kropina change of Finsler metric of Douglas space leads to a Douglas space have also been found.

Key Words: Finsler metric, Kropina change, generalized Kropina change, $h-$vector, projective change, Douglas space.

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§1. Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$.

In the paper [3], Shukla, Pandey and Mishra have investigated the necessary and sufficient conditions under which a generalized Kropina change of Finsler metric becomes a projective change. They have also obtained the condition under which such change of metric of a Douglas space give rise to a Douglas space.

The generalized Kropina change of Finsler metric is given by

$$\overline{L}(x,y) = \frac{L^{m+1}}{\beta^m}, \qquad \text{where} \quad \beta = b_i(x)\, y^i \tag{1.1}$$

and m is a constant not equal to -1, 0.

In the present paper we have considered the transformation (1.1), in which $b_i(x)$ in β has been replaced by $h-$vector $b_i(x, y)$ so that $\frac{\partial b_i}{\partial y^j}$ is proportional to the angular metric tensor h_{ij}.

Let

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{1.2}$$

where ρ is any scalar function of x, y and $h_{ij} = g_{ij} - l_i\, l_j$. It has been shown by Shukla, Pandey and Joshi in [2] that

$$\dot{\partial}_k \rho = -\frac{\rho}{L}\, l_k, \quad \text{for} \quad n > 2, \tag{1.3}$$

where $\dot{\partial}_k \equiv \frac{\partial}{\partial y^k}$.

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We shall use the equation (1.3) without quoting it in the present paper.

Let $\beta = b_i(x,y)\, y^i$ be defined on the manifold M^n. Then $L \to \frac{L^{m+1}}{\beta^m}$ is called generalized $h-$Kropina change of Finsler metric. If we write $\overline{L} = \frac{L^{m+1}}{\beta^m}$ and $\overline{F}^n = (M^n, \overline{L})$ then the Finsler space $\overline{F}^n$ is said to be obtained from F^n by a generalized $h-$Kropina change.

If $m = 1$, then generalized $h-$Kropina change reduces to $h-$Kropina change of Finsler metric. The quantities corresponding to $\overline{F}^n$ will be denoted by putting bar over those quantities.

The fundamental quantities of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} \qquad \text{and} \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \qquad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \qquad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then

$$L_i = l_i, \qquad L^{-1}\, h_{ij} = L_{ij}$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds}\right) = 0,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \qquad F = \frac{L^2}{2}$$

where g^{ij} are the inverse of g_{ij}.

Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of Finsler space $F^n = (M^n, L)$ is given by [5]

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

The Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, G^i_{jk})$ is constructed from L with the help of following axioms [5]:

1. Cartan connection $C\Gamma$ is $v-$metrical;
2. Cartan connection $C\Gamma$ is $h-$metrical;
3. The $(v)v$ torsion tensor field S of Cartan connection vanishes;
4. The $h(h)$ torsion tensor field T of Cartan connection vanishes;
5. The deflection tensor field D of Cartan connection vanishes.

The $h-$ and $v-$covariant derivatives with respect to Cartan connection are denoted by $_{|k}$ and $|_k$ respectively. It is clear that the $h-$covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the $h-$covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}.$$

§2. Difference Tensor $\mathbf{D}^i$

The generalized $h-$Kropina change of Finsler metric L is given by

$$\overline{L} = \frac{L^{m+1}}{\beta^m}, \quad \text{where} \quad \beta(x,y) = b_i(x,y)\,y^i \quad \text{and} \quad m \neq -1, 0. \tag{2.1}$$

We may put

$$\overline{G}^i = G^i + D^i. \tag{2.2}$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equation $L_{ij|k} = 0$, [4] i.e.

$$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0. \tag{2.3}$$

Since $\dot{\partial}_i \beta = b_i$, from (2.1), we have

$$\text{(a)} \qquad \overline{L}_i = (m+1)\frac{L^m}{\beta^m}\,L_i - m\frac{L^{m+1}}{\beta^{m+1}}\,b_i; \tag{2.4}$$

$$\text{(b)} \qquad \overline{L}_{ij} = \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2]L_{ij} + m(m+1)\frac{L^{m-1}}{\beta^m}L_iL_j$$
$$-m(m+1)\frac{L^m}{\beta^{m+1}}(L_ib_j + L_jb_i) + m(m+1)\frac{L^{m+1}}{\beta^{m+2}}b_ib_j;$$

$$\text{(c)} \qquad \partial_j\overline{L}_i = m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_i - Lb_i)\partial_j L + m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i - \beta L_i)\partial_j\beta$$
$$+(m+1)\frac{L^m}{\beta^m}\partial_j L_i - m\frac{L^{m+1}}{\beta^{m+1}}\partial_j b_i;$$

$$\text{(d)} \qquad \partial_k\overline{L}_{ij} = \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2]\partial_k L_{ij} + \big[\frac{mL^{m-1}}{\beta^{m+1}}((m+1)\beta$$
$$-\rho(m+2)L^2)L_{ij} + m(m^2-1)\frac{L^{m-2}}{\beta^m}L_iL_j - m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}(L_ib_j$$
$$+L_jb_i) + m(m+1)^2\frac{L^m}{\beta^{m+2}}b_ib_j\big]\,\partial_k L + \big\{(\rho L^2 - \beta)m(m+1)\frac{L^m}{\beta^{m+2}}L_{ij}$$
$$-m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}L_iL_j + m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_ib_j + L_jb_i)$$
$$-m(m+1)(m+2)\frac{L^{m-1}}{\beta^{m+3}}b_ib_j\big\}\partial_k\beta + m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_j - Lb_j)\,\partial_k L_i$$
$$+m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_i - Lb_i)\partial_k L_j + m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_j - \beta L_j)\partial_k b_i$$
$$+m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i - \beta L_i)\partial_k b_j - m\frac{L^{m+2}}{\beta^{m+1}}L_{ij}\partial_k\rho$$

and

$$\text{(e)} \qquad \overline{L}_{ijk} = \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2]L_{ijk} + m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta - \rho L^2)(L_iL_{jk}+$$

$$+L_jL_{ik}+L_kL_{ij})+m(m+1)\frac{L^m}{\beta^{m+2}}(\rho L^2-\beta)(b_iL_{jk}+b_jL_{ik}+b_kL_{ij})$$

$$-m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}(L_iL_jb_k+L_iL_kb_j+L_jL_kb_i)$$

$$+m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_ib_jb_k+L_jb_kb_i+L_kb_ib_j)$$

$$+m(m^2-1)\frac{L^{m-2}}{\beta^m}L_iL_jL_k-m(m+1)(m+2)\frac{L^{m+1}}{\beta^{m+3}}b_ib_jb_k.$$

Since $\overline{L}_{ij|k}=0$ in $\overline{F}^n$, after using (2.2), we have

$$\partial_k\overline{L}_{ij}-\overline{L}_{ijr}\overline{G}^r_k-\overline{L}_{jr}F^r_{ik}-\overline{L}_{ir}F^r_{jk}=0,$$

where $\overline{F}^i_{jk}=F^i_{jk}+{}^cD^i_{jk}$ [1].

Substituting in the above equation the values of $\partial_k\overline{L}_{ij}$, $\overline{L}_{ir}$ and $\overline{L}_{ijr}$ from (2.4) and using (2.3) and then contracting the resulting equation with y^k, we get

$$\begin{aligned}2\overline{L}_{ijr}D^r+\overline{L}_{jr}D^r_i+\overline{L}_{ir}D^r_j-Lw(Lb_j-\beta L_j)(r_{i0}+s_{i0})-Lw(Lb_i-\\ -\beta L_i)(r_{j0}+s_{j0})-\big\{m(m+1)\frac{L^m}{\beta^{m+2}}(\rho L^2-\beta)L_{ij}-m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}\\ L_iL_j+m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_ib_j+L_jb_i)-m(m+1)(m+2)\frac{L^{m+1}}{\beta^{m+3}}b_ib_j\big\}r_{00}\\ +m\frac{L^{m+2}}{\beta^{m+1}}\rho_0L_{ij}+2\rho m\frac{L^{m+1}}{\beta^{m+1}}L_rL_{ij}G^r=0.\end{aligned} \tag{2.5}$$

where '0' stands for the contraction with y^k viz. $r_{j0}=r_{jk}y^k$, $r_{00}=r_{ij}y^iy^j$ and we have use the fact that $D^i_{jk}y^k={}^cD^i_{jk}y^k=D^i_j$ [4].

Next, we deal with $\overline{L}_{i|j}=0$, that is $\partial_j\overline{L}_i-\overline{L}_{ir}\overline{G}^r_j-\overline{L}_r\overline{F}^r_{ij}=0$, then we have

$$\partial_j\overline{L}_i-\overline{L}_{ir}(G^r_j+D^r_j)-\overline{L}_r(F^r_{ij}+{}^cD^r_{ij})=0. \tag{2.6}$$

Putting the values of $\partial_j\overline{L}_i$, $\overline{L}_{ir}$ and $\overline{L}_r$ from (2.4) in (2.6) and using equation $L_{i|j}=\partial_jL_i-L_{ir}G^r_j-L_rF^r_{ij}=0$, and rearranging the terms, we get

$$-m\frac{L^{m+1}}{\beta^{m+1}}b_{i|j}=\overline{L}_{ir}D^r_j+\overline{L}_r\,{}^cD^r_{ij}+m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_i-Lb_i)(r_{0j}+s_{0j}),$$

which after using $2r_{ij}=b_{i|j}+b_{j|i}$ and $2s_{ij}=b_{i|j}-b_{j|i}$, we get

$$\begin{aligned}-2m\frac{L^{m+1}}{\beta^{m+1}}r_{ij}=&\overline{L}_{ir}D^r_j+\overline{L}_{jr}D^r_i+2\overline{L}_r\,{}^cD^r_{ij}+m(m+1)\frac{L^m}{\beta^{m+2}}\times\\ &\times(\beta L_i-Lb_i)(r_{0j}+s_{0j})+m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_j-Lb_j)(r_{i0}+s_{i0})\end{aligned} \tag{2.7}$$

and

$$-2m\frac{L^{m+1}}{\beta^{m+1}}s_{ij}=\overline{L}_{ir}D^r_j-\overline{L}_{jr}D^r_i+m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_i-Lb_i)\times \times(r_{0j}+s_{0j})-m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_j-Lb_j)(r_{i0}+s_{i0}). \tag{2.8}$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i, we get

$$-2\overline{L}_{jr}D^r+m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_j-\beta L_j)\,r_{00}-2m\frac{L^{m+1}}{\beta^{m+1}}r_{0j}=2\overline{L}_rD^r_j. \tag{2.9}$$

Contracting (2.9) with y^j, we get

$$\left[(m+1)\frac{L^m}{\beta^m}\,L_r-m\frac{L^{m+1}}{\beta^{m+1}}\,b_r\right]\,D^r=-\frac{1}{2}m\frac{L^{m+1}}{\beta^{m+1}}r_{00}. \tag{2.10}$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with y^j, we have

$$\left[\frac{L^m}{\beta^{m+1}}\{(m+1)\beta-\rho mL^2\}L_{ir}+m(m+1)\frac{L^{m-1}}{\beta^m}L_iL_r -m(m+1)\frac{L^m}{\beta^{m+1}}(L_ib_r+L_rb_i)+m(m+1)\frac{L^{m+1}}{\beta^{m+2}}b_ib_r\right]D^r =-m\frac{L^{m+1}}{\beta^{m+1}}s_{i0}+\frac{1}{2}m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i-\beta L_i)r_{00}. \tag{2.11}$$

In view of $LL_{ir}=g_{ir}-L_iL_r$, the equation (2.11) may be written as

$$\left[(m+1)\frac{L^{m-1}}{\beta^m}-\rho m\frac{L^{m+2}}{\beta^{m+1}}\right]g_{ir}D^r+\left[\frac{L^{m-1}}{\beta^{m+1}}\{(m^2-1)\beta+\rho mL^2\}L_i -m(m+1)\frac{L^m}{\beta^{m+1}}b_i\right]L_rD^r+m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i-\beta L_i)\,b_rD^r =-m\frac{L^{m+1}}{\beta^{m+1}}s_{i0}+\frac{1}{2}m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i-\beta L_i)r_{00}. \tag{2.12}$$

Contracting (2.12) with $b^i(=g^{ij}b_j)$, we get

$$-2\frac{L^m}{\beta^{m+1}}[m(m+1)L^2\triangle+(m+1)\beta^2-\rho mL^2\beta]L_rD^r +2\frac{L^{m+1}}{\beta^{m+2}}[m(m+1)L^2\triangle+(m+1)\beta^2-\rho mL^2\beta]b_rD^r =\frac{L^{m+3}}{\beta^{m+2}}[-2m\beta s_0+m(m+1)\triangle r_{00}], \tag{2.13}$$

where $\triangle=b^2-\frac{\beta^2}{L^2}$ and $s_0=s_{r0}b^r$.

The equations (2.10) and (2.13) constitute the system of algebraic equations in L_rD^r and b_rD^r whose solution is given by

$$L_rD^r=\frac{-mL[\{(m+1)\beta-\rho mL^2\}r_{00}+2mL^2s_0]}{2[m(m+1)L^2\triangle+\beta\{(m+1)\beta-\rho mL^2\}} \tag{2.14}$$

and

$$b_r D^r = -\frac{-2m(m+1)L^2\beta s_0 + [m(m+1)(\triangle L^2 - \beta^2) + \rho m^2 L^2 \beta] r_{00}}{2[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]}. \tag{2.15}$$

Contracting (2.12) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (2.14) and (2.15) respectively, we get

$$D^i = \frac{m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]\times}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{2.16}$$

$$\frac{[(m+1)L^2 b^i - \{2(m+1)\beta - \rho m L^2\}\, y^i]}{} - \frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

where $l^i = \dfrac{y^i}{L}$.

Proposition 2.1 *The difference tensor $D^i = \overline{G}^i - G^i$ of generalized $h-$Kropina change of Finsler metric is given by (2.16).*

Remark *The difference tensor for $h-$Kropina change of Finsler metric is obtained by putting $m = 1$ in equation (2.16).*

§3. Conditions for Projective Change

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. It is well known that the change $L \to \overline{L}$ is projective if $\overline{G}^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor [6].

Thus from (2.2) it follows that $L \to \overline{L}$ is projective iff $D^i = Py^i$. Now we consider that the generalized $h-$Kropina change $L \to \overline{L} = \frac{L^{m+1}}{\beta^m}$ is projective. Then from equation (2.16), we have

$$Py^i = \frac{m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]\times}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{3.1}$$

$$\frac{[(m+1)L^2 b^i - \{2(m+1)\beta - \rho m L^2\}\, y^i]}{} - \frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

Contracting (3.1) with $y_i (= g_{ij} y^j)$ and using the fact that $s^i_0 y_i = 0$ and $y_i y^i = L^2$, we get

$$P = \frac{-m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]}{2[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]}. \tag{3.2}$$

Putting the value of P from (3.2) in (3.1), we get

$$\frac{m(m+1)[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0](\beta y^i - L^2 b^i)}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{3.3}$$

$$= -\frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

Transecting (3.3) by b_i, we get

$$r_{00} = \frac{2\beta\, s_0}{(m+1)\,\triangle}, \quad \text{where} \quad \triangle = b^2 - \frac{\beta^2}{L^2} \neq 0. \tag{3.4}$$

Putting the value of r_{00} from (3.4) in (3.2), we get

$$P = -\frac{ms_0}{(m+1)\,\triangle}. \tag{3.5}$$

Eliminating P and r_{00} from (3.5), (3.4) and (2.16), we get

$$s^i_0 = \left[b^i - \frac{\beta}{L^2} y^i\right] \frac{s_0}{\triangle}. \tag{3.6}$$

The equations (3.4) and (3.6) give the necessary conditions under which a generalized $h-$Kropina change becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting these conditions in (2.16), we get

$$D^i = -\frac{ms_0}{(m+1)\triangle}\, y^i \;\; \text{i.e.} \;\; D^i = Py^i, \quad \text{where} \quad P = -\frac{ms_0}{(m+1)\triangle}.$$

Thus $\overline{F}^n$ is projective to F^n.

Theorem 3.1 *The generalized $h-$Kropina change of a Finsler space is a projective change iff equations (3.4) and (3.6) hold the projective factor P is given by equation (3.5).*

If $m = 1$, then the equations (3.4) and (3.6) are reduced to the equations

$$r_{00} = \frac{\beta s_0}{\triangle}, \tag{3.7}$$

and

$$s^i_0 = \left[b^i - \frac{\beta}{L^2} y^i\right] \frac{s_0}{\triangle} \tag{3.8}$$

respectively and the projective factor is given by $P = -\frac{s_0}{2\triangle}$. Thus, we have

Corollary 3.1 *The $h-$Kropina change of Finsler metric is projective iff the conditions (3.7) and (3.8) hold.*

§4. Douglas Space

The Finsler space F^n is called a Douglas space if and only if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [7]. We shall write $hp(r)$ to denote a homogeneous polynomial in y^i of degree r. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$B^{ij} = \frac{m(m+1)L^2[\{(m+1)\beta - \rho mL^2\}r_{00} + 2mL^2 s_0](b^i y^j - b^j y^i)}{2\{m+1)\beta - \rho mL^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho mL^2\}]} - \frac{mL^2}{(m+1)\beta - \rho mL^2}(s_0^i y^j - s_0^j y^i). \tag{4.1}$$

From (4.1), we find if a Douglas space is transformed to a Douglas space by generalized $h-$Kropina change of Finsler metric, then B^{ij} must be $hp(3)$ and if B^{ij} is $hp(3)$ then generalized $h-$Kropina change transforms a Douglas space into a space of the same kind.

Theorem 4.1 *The generalized $h-$Kropina change of Douglas space leads to a Douglas space iff B^{ij} given by (4.1) is $hp(3)$.*

If $m = 1$, then the equation (4.1) becomes

$$B^{ij} = \frac{L^2[(2\beta - \rho L^2)r_{00} + 2L^2 s_0](b^i y^j - b^j y^i)}{(2\beta - \rho L^2)[2L^2\triangle + \beta(2\beta - \rho L^2)]} - \frac{L^2}{(2\beta - \rho L^2)}(s_0^i y^j - s_0^j y^i). \tag{4.2}$$

Thus, we have

Corollary 4.1 *The $h-$Kropina change of Douglas space leads to a Douglas space iff B^{ij} given by (4.2) is $hp(3)$.*

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Relation Between
Sum-Connectivity Index and Average Distance of Trees

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Abstract: The sum-connectivity index of a simple graph G is defined in mathematical chemistry as

$$R^{+}(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u+d_v}}$$

where $E(G)$ is the edge set of G and d_u is the degree of vertex u in G. We report relation between Sum connectivity index and Average distance $ad(T)$ of tree T.

Key Words: Sum connectivity index, average distance, tree.

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§1. Introduction and Preliminaries

Let G be a simple graph, the vertex-set and edge-set of which are represented by $V(G)$ and $E(G)$ respectively. The connectivity index introduced in 1975 by Milan Randic [9], who has shown this index to reflect molecular branching. The *Randic index* was defined as follows:

$$\chi(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u d_v}}.$$

It is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies [4], [9]. Mathematical properties of this descriptor as summarized in [2] and its generalizations/variants [8] have also been studied extensively. We also call the $R(G)$ index as the product-connectivity index of G.

Motivated by Randics definition of the product-connectivity index, the sumconnectivity

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index was recently proposed in [9]. The *sum-connectivity index* of the graph G is defined as

$$R^{+}(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u+d_v}}.$$

It is interesting to note that various kinds of variants and extension of the product connectivity index have been reported, but there is not a single one on the additive version of the connectivity index before the sum-connectivity index was proposed [17].

The product-connectivity index and the sum-connectivity index are highly inter-correlated quantities. For example, the correlation coefficient between the product connectivity index and the sum-connectivity index for 137 alkane-trees [6] is 0.9996.

The Wiener index is a distance-based topological index defined as the sum of distances between all pairs of vertices in a graph and is denoted by $W(G)$. It was the first topological index in chemistry [15], introduced in 1945. Since then, the Wiener index has been used to explain various chemical and physical properties of molecules. However the Wiener index can be cubic in the number of vertices of a graph, hence for the sake of simplicity we use the average distance instead [1],[5], [10],[12], [14],[16.

$$ad(G)=\frac{W(G)}{\binom{n}{2}}.$$

In order to state the results of this paper let us introduce some notation. By $distG(u,v)$ we denote the length of the shortest path between vertices u and v in a connected graph G. The length $max_{u,v}\ distG(u,v)$ is a diameter of a graph G, which we denote by $diam(G)$. In particular for trees the diameter is the length of the longest simple path. A star is a tree where at most one vertex is of degree greater than one. Note that a tree with at most three vertices is a star. Also notice that stars are precisely the trees with diameter at most two. The thorn graph of the graph G, with parameters $p_1,p_2,\cdots,p_n$, is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, $i=1,2,\cdots,n$. ([3])

In [7] author defined special class of thorn tree named as as *Double comet*. It is defined as $\tau_{n,p}$ which denotes a set of all trees with exactly n nodes and p pendent vertices. For $a,b\geq 1$, $n\geq a+b+2$ Double Comet $DC(n,a,b)$, is a tree composed of a path containing $n-a-b$ vertices with a pendent vertices attached to one of the ends of the path and b pendent vertices attached to the other end of the path. Thus, $DC(n,a,b)$ has n vertices and $a+b$ leaves, i.e.,$DC(n,a,b)\in\tau_{n,a+b}$.

In this paper we present a relation between sum-connectivity index and average distance of a trees. This paper is motivated from [7].

§2. Relation Between Sum-Connectivty Index and Average Distance of a Tree

The main result of this paper is the following theorem.

Theorem 2.1 *For any tree T with n vertices and p leaves the following inequality holds:*

$$R^+(T) \geq ad(T) + min(0, \sqrt{p} - 2), when\ p = 2.$$
$$R^+(T) \geq ad(T) + max(0, \frac{p-1}{\sqrt{p+1}} - 2), when\ p \geq 3.$$

The inequality is sharp for stars, if we consider the limit when n goes to infinity.

Proof Let $\tau_{n,p}$ be the set of all trees with exactly n nodes and p pendent vertices.

First we prove a theorem for stars and path in the following lemma.

Lemma 2.1 *Let $\tau_{n,p}$ be a tree such that $p \leq 2$ (a path) or $p = n - 1$ (a star). Then*

$$R^+(T) \geq ad(T) + min(0, \sqrt{p} - 2), when\ p = 2,$$
$$R^+(T) \geq ad(T) + max(0, \frac{p-1}{\sqrt{p+1}} - 2), when\ p \geq 3.$$

Proof If $n \leq 2$ then the inequality trivially holds, hence we assume $n \geq 3$.

If T is a star then by calculation we obtain $ad(T) = 2 - \frac{2}{n}$ and $R^+(T) = \frac{(n-1)}{\sqrt{n}}$.

If T is a path then by calculation we obtain $ad(T) = \frac{(n+1)}{3}$ and $R^+(T) = \frac{2}{\sqrt{3}} + \frac{(n-3)}{2}$.

In both cases the lemma holds. □

Now we assume that $T \in \tau_{n,p}$ and $3 \leq p \leq n - 2$, for this we make use of the following lemma [18], [19].

Lemma 2.2([19]) *Let T be a tree with n vertices and p pendant vertices, where $3 \leq p \leq n - 2$. Then*

$$R^+(T) \geq \frac{n-p-2}{2} + \frac{1}{\sqrt{p+2}} + \frac{p-1}{\sqrt{p+1}} + \frac{1}{\sqrt{3}}.$$

From this lemma we obtain

Corollary 2.1 *Let T be a tree with n vertices and p pendant vertices, where $3 \leq p \leq n - 2$.*

$$R^+(T) \geq \frac{n-p-2}{2} + \frac{p-1}{\sqrt{p+1}} + 1.025.$$

Since each tree can be transformed into a double comet without decreasing the average distance and simultaneously preserving the number of vertices and leaves(lemma 2.4 in [7]) and in Corollary 2.1 we only use n and p disregarding the actual structure of a tree it is enough to show that for double comets we have

$$\frac{p-1}{\sqrt{p+1}} + \frac{n-p-2}{2} + 1.025 \geq ad(T) + max(0, \frac{p-1}{\sqrt{p+1}} - 2).$$

Lemma 2.3 *Let $\tau_{n,p}$ be a double comet $DC(n,a,b)$ for $a,b \geq 1$, where $3 \leq p \leq n-2$. Then*

$$\frac{p-1}{\sqrt{p+1}} + \frac{n-p-2}{2} + 1.025 \geq ad(T) + max(0, \frac{p-1}{\sqrt{p+1}} - 2).$$

Proof To prove this lemma first we have to find average distance of double comet $ad(G) = \frac{W(G)}{\binom{n}{2}}$, i.e to find the wiener index of the double comet.

In order to compute $W(DC(n,a,b))$ we distinguish four types of pairs of vertices of

TYPE 1. Distance between inner vertices;

TYPE 2. Distance between leaves and inner vertices;

TYPE 3. Distance between the leaves from a to b;

TYPE 4. Distance between leaves with the distance 2.

Let the contributions of all such vertex pairs to $DC(n,a,b)$ be denoted by F_1 , F_2 , F_3 and F_4, respectively. Then,

$$W(DC(n,a,b)) = F_1 + F_2 + F_3 + F_4. \quad (2.1)$$

There are $n-(a+b)$ vertices of Type 1. So

$$F_1 = \binom{n-p+1}{3}.$$

There are $p_i = a, b$ vertices of Type 2. Whence

$$F_2 = \frac{(a+b)(n-p)(n-p+1)}{2}.$$

There are ab vertex pair of Type 3, each of them separated by distance $(n-p+1)$. Thus

$$F_3 = ab(n-p+1).$$

There are $\binom{p_i}{2}$, $p_i = a, b$ vertex pairs of Type 4, each of them at distance 2. Consequently

$$F_4 = 2\binom{a}{2} + 2\binom{b}{2}.$$

Substituting the above relations back into Eq.(2.1) we arrive at the wiener index of the Double comet. i.e

$$W(DC(n,a,b)) = \binom{n-p+1}{3} + \frac{(a+b)(n-p)(n-p+1)}{2} + ab(n-p+1) + 2\binom{a}{2} + 2\binom{b}{2}.$$

Therefore

$$\begin{aligned}\binom{n}{2}ad(T) &= W(T)\\ &= \binom{n-p+1}{3} + \frac{(a+b)(n-p)(n-p+1)}{2} + ab(n-p+1) + 2\binom{a}{2} + 2\binom{b}{2}.\end{aligned}$$

Using following inequalities:

$$2ab + 2\binom{a}{2} + 2\binom{b}{2} \leq (a+b)^2 = p^2,$$
$$ab \leq \frac{p^2}{4},$$
$$\binom{n-p+1}{3} \leq \frac{(n-p)^3}{6},$$

we obtain

$$\binom{n}{2} ad(T) \leq \frac{(n-p)^3}{6} + \frac{p(n-p)(n-p+1)}{2} + \frac{p^2(n-p+1)}{4} + p^2$$
$$= \frac{(n-p)^3}{6} + \frac{p(n-p)^2}{2} + \frac{p(n-p)}{2} + \frac{p^2(n-p)}{4} + \frac{5p^2}{4}.$$

For the sake of simplicity we put $x = p$, $y = n - p$.

$$\binom{n}{2} ad(T) \leq \frac{y^3}{6} + \frac{xy^2}{2} + \frac{xy}{2} + \frac{x^2y}{4} + \frac{5x^2}{4}.$$

In order to prove the lemma it is enough to show the following inequality:

$$(\binom{n}{2})(\frac{n-p-2}{2} + \frac{p-1}{\sqrt{p+1}} + 1.025 - max(0, \frac{p-1}{\sqrt{p+1}} - 2) - ad(T)) \geq 0.$$

Now multiply the expression by 4 , put x, y instead of $p, n-p$ and use the previously obtained inequality for $\binom{n}{2} ad(T)$

$$4(\binom{n}{2})(\frac{n-p-2}{2} + \frac{p-1}{\sqrt{p+1}} + 1.025 - max(0, \frac{p-1}{\sqrt{p+1}} - 2) - ad(T))$$
$$= 2(x+y)(x+y-1)(\frac{y-2}{2} + min(\frac{x-1}{\sqrt{x+1}} + 1.025, 3.025)) - 4ad(T)\binom{n}{2}$$
$$\geq 2(x^2 + y^2 + 2xy - x - y)(min(\frac{x-1}{\sqrt{x+1}} + 1.025, 3.025)) - 7xy - 3y^2 - 7x^2 + 2x$$
$$+ 2y + \frac{y^3}{3}.$$

We consider two cases. Either $p = x = 3$ or $p = x \geq 4$. If $x = 3$ from the above expression we obtain

$$2(y^2 + 5y + 6)(min(\frac{2}{\sqrt{4}} + 1.025, 3.025)) - 19y - 3y^2 + \frac{y^3}{3} - 57$$
$$\geq 4(y^2 + 5y + 6) + \frac{y^3}{3} - 3y^2 - 19y - 57$$
$$= \frac{y^3}{3} + y^2 + y + 9.$$

Using the assumption $p \leq n-2$ which is equivalent to $y \geq 2$, we obtain

$$\frac{y^3}{3} + y^2 + y + 9 \geq \frac{53}{3} \geq 0.$$

Now we assume $x \geq 4$. Hence $min(\frac{x-1}{\sqrt{x+1}} + 1.025, 3.025) \geq 2.4$. Since $x, y \geq 2$, then $xy \geq x + y$, so we have

$$\begin{aligned}&(2x^2 + 2y^2 + 4xy - 2x - 2y)2.4 - 7xy - 3y^2 - 7x^2 + 2x + 2y + \frac{y^3}{3}\\&= \frac{y^3}{3} - 2.2x^2 - 1.2y^2 + 2.6xy - (2.8x + 6y) \geq 0.\end{aligned}$$

This the proof of the lemma. □

In Lemma 2.1 we proved Theorem 2.1 for paths and stars whereas by Lemmas 2.2 and 2.3 together with Corollary 2.2 we obtain the inequality for all other trees, thus completing the proof of Theorem 2.1. □

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Line Cut Vertex Digraphs of Digraphs

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Abstract: In this paper we define the digraph valued function (digraph operator) namely, the line cut vertex digraph $L_c(D)$ of a digraph D. The problem of reconstructing a digraph from its line cut vertex digraph is presented. Also, outer planarity and maximal outer planarity properties of these digraphs are discussed.

Key Words: Line digraph, cut vertex, complete bipartite subdigraph, line cut vertex digraph, Smarandachely line cut vertex digraph.

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§1. Introduction

Notations and definitions not introduced here can be found in [2,3]. For a simple graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and cut vertex set $C(G) = \{C_1, C_2, \cdots, C_r\}$, detailed by V.R.Kulli et al.[4] gave the following definition. The *line cut vertex graph* of G, written $n(G)$, is the graph whose vertice are the edges $E(G)$ and cut vertices $C(G)$ of G, with two vertices of $n(G)$ adjacent whenever the corresponding edges of G are adjacent or the corresponding members of G are incident, and where the edges and cut vertices of G are called its *members*. Generally, a *Smarandachely line cut vertex digraph* $n_S(G)$ for $S \subset E(G) \bigcup C(G)$ is such a graph with vertices $E(G) \bigcup C(G)$ and members are adjacent if and only if they are adjacent or incident in $\langle S \rangle_G$. Clearly, if $S = E(G) \bigcup C(G)$, then the Smarandachely line cut vertex digraph $n_S(G)$ of G is nothing else but $n(G)$.

Recently, there has been an extension of this topic to trees [5]. In this paper, we extend the definition of the line cut vertex graph of a graph to a directed graph. M.Aigner [1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \cdots, v_n$ and m arcs and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'} = m$ and $m^{'} = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$. Furthermore, the in-degree, respectively out-degree of a vertex $v^{'} = (v_i, v_j)$ in $L(D)$ are $d^-(v^{'}) = d^-(v_i)$ and $d^+(v^{'}) = d^+(v_j)$. Also, a digraph D is

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said to be a *line digraph* if it is isomorphic to the line digraph of a certain digraph H [6].

We need some concepts and notations on directed graphs. A *directed graph*(or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u, v) or uv of D, the first vertex u is its *tail*, and the second vertex v is its *head*. A digraph without any arcs is called *totally disconnected*. An out-star (an in-star) in D is a star in the underlying undirected graph of D such that all arcs are directed out of (into) the center. The out-star and in-star of order k is denoted by S_k^+ and S_k^-, respectively. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v. The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$. A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0, d^-(v) = 0$ or $d^+(v) = 0, d^-(v) > 0$.

A *cut set* of D is defined as a minimal set of vertices whose removal increases the number of connected components of D. A cut set of size one is called a *cut vertex*. A *tournament* is a nontrivial complete asymmetric digraph. A tournament of order n is denoted by T_n. A *semi-star digraph*, denoted by $D_{1,n}, n \geq 2$, is a directed graph with $(n + 1)$ vertices and n arcs; n vertices have total degree exactly one, and one has n.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions hold good for planar digraphs. A *planar drawing* of a digraph D is a drawing of D in which no two distinct arcs intersect. A digraph is said to be *planar* if it admits a planar drawing. If D is a planar digraph, then the *inner vertex number* $i(D)$ of D is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of D in the plane. A digraph D is *outerplanar* if $i(D) = 0$. An outerplanar digraph is said to be *maximal outerplanar* if no arc can be added to it without losing its outer planarity.

§2. Line Cut Vertex Digraph of a Digraph

For a connected digraph D, the *line cut vertex digraph* $Q = L_c(D)$ has vertex set $V(Q) = A(D) \cup C(D)$, where $C(D)$ is the cut vertex set of D. The arc set

$$A(Q) = \begin{cases} ab : a, b \in A(D), \text{the head of } a \text{ coincides with the tail of } b, \\ Cd : C \in C(D), d \in A(D), \text{the tail of } d \text{ is } C, \\ dC : C \in C(D), d \in A(D), \text{the head of } d \text{ is } C. \end{cases}$$

§3. Decomposition and Reconstruction

A digraph is a *complete bipartite digraph* if its vertex set can be partitioned into two sets A, B in such a way that every arc has its initial vertex in A and its terminal vertex in B and any two vertices $a \in A$ and $b \in B$ are joined by an arc. An arc (u, v) of D is said to be an *end arc* if u is the transmitter and v is the receiver.

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and cut vertex set $C(D) = \{C_1, C_2, \cdots, C_r\}$. We consider the following three cases.

Case 1. Let v be a vertex of D with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(D)$ into mutually arc disjoint complete bipartite subdigraphs.

Case 2. Let C_j be a cut vertex of D with $d^-(C_j) = \alpha'$. Then α' arcs coming into C_j give rise to a complete bipartite subdigraph with α' tails and a single head(i.e., C_j) and α' arcs joining each tail with C_j.

Case 3. Let C_j be a cut vertex of D with $d^+(C_j) = \beta'$. Then β' arcs going out from C_j give rise to a complete bipartite subdigraph with a single tail (i.e., C_j) and β' heads and β' arcs joining C_j with each head.

Hence by all the above cases, $Q = L_c(D)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = A(D) \cup C(D)$ and arc sets (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of D, respectively; (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j' \times C_k$ such that $Z_j' \times C_k = \phi$ for $j \neq k$; (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where Z_j' and Z_j are the sets of in-coming and out-going arcs at C_j of D, respectively.

Conversely, let H' be a digraph of the type described above. Let us denote each of the complete bipartite subdigraphs obtained by Case 1 by $T_1, T_2, \ldots, T_l$. The vertex set of H is $V(H) = \{t_0, t_1, \ldots, t_l, t_{l+1}\}$. The arcs of H are obtained by the following procedure (see [3]).

For each vertex $v \in L(D)$, we draw an arc a_v to H as follows.

Step 1: If $d^+_{L(D)}(v) > 0$, and $d^-_{L(D)}(v) = 0$, then $a_v = (t_0, t_i)$ is an arc, where i is the base(or index) of T_i such that $v \in X_i$;

Step 2: If $d^+_{L(D)}(v) = 0$, and $d^-_{L(D)}(v) > 0$, then $a_v = (t_j, t_{l+1})$ is an arc, where j is the base of T_j such that $v \in Y_j$;

Step 3: If $d^+_{L(D)}(v) > 0$, and $d^-_{L(D)}(v) > 0$, then $a_v = (t_i, t_j)$ is an arc, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$. Finally, if $L(D)$ has an isolated vertex, then $a_v = (t_0, t_{l+1})$. Note that this method always constructs H with only one vertex of in-degree zero and one vertex of out-degree zero.

We now mark the cut vertices of H as follows. From Case 2 and Case 3, we observe that for every cut vertex C, there exists at most two complete bipartite subdigraphs, one containing C as the tail, and other as head. Let it be C_j' and C_j'', $1 \leq j \leq r$ such that C_j' contains C as the head and C_j'' contains C as the tail. Now, if the tails of C_j' and the heads of C_j'' are the heads and tails of a single T_i, $1 \leq i \leq l$, then the vertex t_i is a cut vertex in H, where i is the index of T_i. Furthermore, a vertex of an end arc in H whose total degree at least two is a cut vertex. The digraph H thus constructed apparently has H' as line cut vertex digraph. Therefore we have,

Theorem 3.1 *A digraph Q is the line cut vertex digraph of a certain digraph D if and only if $V(Q) = A(D) \cup C(D)$ such that the arc set $A(Q)$ equals: (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are* *the sets of in-coming and out-going arcs at v_i of D, respectively, (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = \phi$ for $j \neq k$, (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where $Z_j^{'}$ and Z_j are the sets of in-coming and out-going arcs at C_j of D, respectively.*

Proposition 3.2 *Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and cut vertex set $C(D) = \{C_1, C_2, \ldots, C_r\}$. Then the order and size of $L_c(D)$ are*

$$m + \sum_{j=1}^{r} C_j \quad \text{and} \quad \sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\},$$

respectively, where m is the size of D.

Proof Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $C(D) = \{C_1, C_2, \ldots, C_r\}$. Then the order of $L_c(D)$ equals the sum of size and cut vertices of D. Thus, $V(L_c(D)) = m + \sum_{j=1}^{r} C_j$. Now, the size of $L_c(D)$ equals the sum of size of $L(D)$ and the total degree of cut vertices of D. Hence

$$A(L_c(D)) = \sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\}. \qquad \square$$

Proposition 3.3 *Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. Then $L(D) \simeq L_c(D)$ if and only if D is a block.*

Proof Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. Suppose that D is a block. Clearly, D does not have any cut vertex. Thus, the order of $L_c(D)$ is exactly m, where m is the size of D. But, $V(L(D)) = m$. Hence $L(D) \simeq L_c(D)$.

Conversely, suppose that $L(D) \simeq L_c(D)$. Assume that D is not a block. Then there exists at least one cut vertex in D. The order of $L_c(D)$ equals the sum of size and cut vertices of D. But, $V(L(D)) = m$. Thus, the order of $L(D)$ is less than the order of $L_c(D)$. Clearly, $L(D) \neq L_c(D)$, a contradiction. This completes the proof. $\square$

Theorem 3.4 *The line cut vertex digraph $L_c(D)$ of a digraph D is outerplanar if,*

(a) D is a directed path $\vec{P_n}$ on $n \geq 3$ vertices;
(b) D is an in-star(or an out-star) of order $k, k \geq 3$;
(c) D is the semi-star digraph $D_{1,3}$.

Proof We consider the cases following:

Case 1. Suppose that D is a directed path $\vec{P_n}$ on $n \geq 3$ vertices. Then $L_c(D)$ is a connected digraph in which every block is T_3. Clearly, $L_c(D)$ is outerplanar.

Case 2. Suppose that D is an in-star(or an out-star) of order $k, k \geq 3$. Then $L(D)$ is totally disconnected of order $(k-1)$. The number of cut vertex of D is exactly one. Then $L_c(D)$ is an in-star(or an out-star) of order k. Thus, $L_c(D)$ is outerplanar.

Case 3. Suppose that D is the semi-star digraph $D_{1,3}$. Clearly, D contains exactly one cut vertex C. By definition, $L(D)$ is either $D_{1,2}$ or totally disconnected of order three. We consider the following two subcases of Case 3.

Subcase 1 If $L(D)$ is $D_{1,2}$, then $L_c(D)$ is $T_4 - e$, which is outerplanar.

Subcase 2 If $L(D)$ is totally disconnected, then $L_c(D)$ is $D_{1,3}$, which is also outerplanar. This completes the proof. □

Theorem 3.5 *The line cut vertex digraph $L_c(D)$ of a digraph D is maximal outerplanar if and only if D is the directed path $\vec{P_3}$.*

Proof We prove this by the method of contradiction. Suppose $L_c(D)$ is maximal outerplanar. Assume that D is the directed path $\vec{P_4}$. By Proposition 3.2, the order and size of $L_c(D)$ are $n = 5$ and $m = 6$, respectively. But, $m = 6 < 7 = 2n - 3$. Since every maximal outerplanar digraph with n vertices contains $2n-3$ arcs, $L_c(D)$ is not maximal outerplanar, a contradiction.

Conversely, suppose that D is $\vec{P_3}$. Then $L_c(D)$ is T_3, which is maximal outerplanar. This completes the proof. □

§4. Open Problems

We present the following open problems:

(1) Characterize the digraphs whose line cut vertex digraphs are planar, minimally non-outerplanar, and have crossing number one.

(2) One can naturally extend this concept to directed trees. ([See [5]).

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Further Results on Super Geometric Mean Graphs

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Abstract: Let G be a graph and $f : V(G) \rightarrow \{1,2,3,\cdots,p+q\}$ be an injection. For each edge uv, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1,2,3,\cdots,p+q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph. In this paper, we have discussed the super geometric meanness of the graphs $P_n \cup C_m, T_n \cup C_m, mC_n$, the complete graph $K_n, [P_n; S_m]$, subdivision of $P_n \odot K_1, TW(P_n)$, middle graph of a path, triangular ladder, $C_n \odot K_1$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

Key Words: Labeling, super geometric mean labeling, super geometric mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling, we refer [4].

A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. The union of m copies of a graph G is denoted by mG. A complete graph K_n is a graph on n vertices in which every pair of distinct vertices are joined by an edge. A star graph S_n is a complete bipartite graph $K_{1,n}$. Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots, v_{m+1}^{(i)}$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the i^{th} copy of the star graph $S_m, 1 \leq i \leq n$ and the path P_n respectively. The graph $[P_n; S_m]$ is obtained from n copies of S_m and the path P_n by joining u_i with the central vertex $v_1^{(i)}$ of the i^{th} copy of S_m by means of an edge, for $1 \leq i \leq n$. For a graph G, the graph $S(G)$ is obtained by subdividing each edge of G by a vertex. A twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices to each internal vertices of the path.

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)$ and e is incident with $v\}$.

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A ladder L_n is a graph $P_2 \times P_n$ with $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}$. A triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder $L_n \cong P_2 \times P_n$ by adding the edges u_iv_{i+1} for $1 \leq i \leq n-1$. $G \odot K_1$ is the graph obtained from G by attaching a new pendant vertex at each vertex of G.

Duplication of a vertex v_k of a graph G produces a new graph G' by adding a vertex $v'_k$with $N(v_k) = N(v'_k)$. Duplication of an edge $e = uv$ of a graph G by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

In [6], S.K. Vaidya et al. discussed the harmonic mean labeling of duplication of a vertex and edge of a cycle. In [7], R. Vasuki et al. discussed the super mean labeling of some standard graphs. A. Durai Baskar et al. [1,2] discussed the geometric mean labeling some standard graphs. Motivated by these works, the concept of super geometric mean labeling was introduced and studied in [3].

A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph.

The graph shown in Figure 1 is a super geometric mean graph.

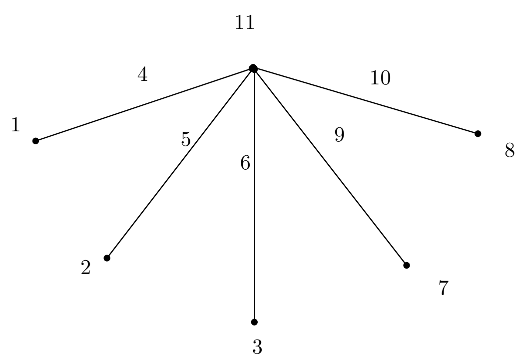

Figure 1

In this paper, we have established the super geometric meanness of the graphs $P_n \cup C_m$ for $n \geq 1$ and $m \geq 3, T_n \cup C_m$ for $n \geq 4$ and $m \geq 3, mC_n$, the complete graph $K_n, [P_n; S_m]$ for $n \geq 1$ and $m \leq 2$, subdivision of $P_n \odot K_1, TW(P_n)$ for $n \geq 3$, middle graph of a path, triangular ladder, $C_n \odot K_1$ for $n \geq 3$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

§2. Main Results

Theorem 2.1 *$P_n \cup C_m$ is a super geometric mean graph, for $n \geq 1$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively.

Case 1. $m \geq 4$.

We define $f : V(P_n \cup C_m) \cup E(P_n \cup C_m) \to \{1, 2, 3, \ldots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \le i \le \lfloor \frac{m}{2} \rfloor \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is odd} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is even} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 5 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \le i \le m \end{cases}$$

and $f(v_i) = 2m + 2i - 1$ for $1 \le i \le n$. The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 4i - 2 & 1 \le i \le \lfloor \frac{m}{2} \rfloor \\ 2m - 1 & i = \lfloor \frac{m}{2} \rfloor + 1 \\ 2m - 2 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 5 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 3 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \le i \le m - 1, \end{cases}$$

$f^*(u_1 u_m) = 3$ and $f^*(v_i v_{i+1}) = 2m + 2i$ for $1 \le i \le n - 1$.

Case 2. $m = 3$.

We define $f : V(P_n \cup C_3) \cup E(P_n \cup C_3) \to \{1, 2, 3, \cdots, 2n + 5\}$ as follows $f(u_1) = 1, f(u_2) = 4, f(u_3) = 6$ and $f(v_i) = 5 + 2i$ for $1 \le i \le n$. The induced edge labeling is as follows:

$f^*(u_1 u_2) = 2$, $f^*(u_2 u_3) = 5$, $f^*(u_3 u_1) = 3$ and $f^*(v_i v_{i+1}) = 6 + 2i$ for $1 \le i \le n - 1$.

Hence, f is a super geometric mean labeling of $P_n \cup C_m$. Thus the graph $P_n \cup C_m$ is a super geometric mean graph for $n \ge 1$ and $m \ge 3$. □

The super geometric mean labeling of $P_5 \cup C_6$ and $P_4 \cup C_3$ are shown in Figure 2.

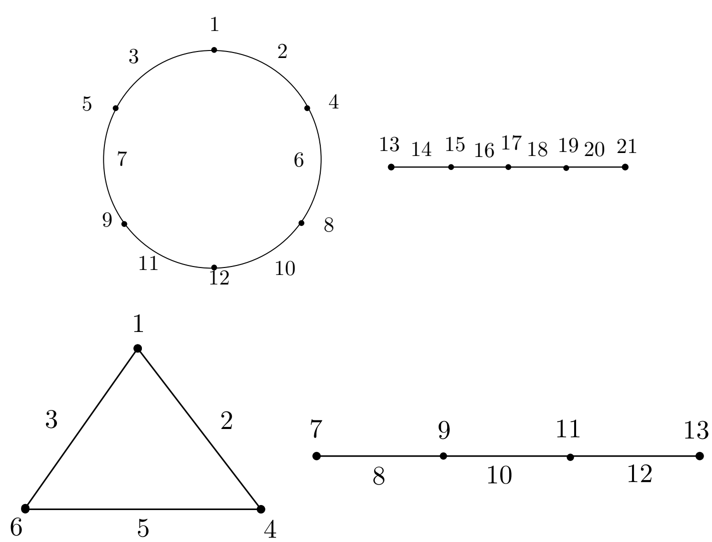

Figure 2

Theorem 2.2 *For a T-graph $T_n, T_n \cup C_m$ is a super geometric mean graph, for $n \geq 4$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ be the vertices of the cycle C_m and $v_1, v_2, \cdots, v_{n-1}$ be the vertices of the path P_{n-1} and let v_n be the pendant vertex identified with v_{n-2} in T_n.

Case 1. $m \geq 4$.

We define $f : V(T_n \cup C_m) \cup E(T_n \cup C_m) \to \{1, 2, 3, \cdots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is odd} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is even} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 5 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \leq i \leq m, \end{cases}$$

$f(v_i) = 2m + 2i - 1$ for $1 \leq i \leq n - 3$, $f(v_{n-2}) = 2m + 2n - 3$, $f(v_{n-1}) = 2m + 2n - 6$ and $f(v_n) = 2m + 2n - 1$.

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 4i - 2 & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 2m - 1 & i = \lfloor \frac{m}{2} \rfloor + 1 \\ 2m - 2 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 5 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 3 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \leq i \leq m - 1, \end{cases}$$

$$f^*(u_1 u_m) = 3, f^*(v_i v_{i+1}) = 2m + 2i \text{ for } 1 \leq i \leq n - 4,$$
$$f^*(v_{n-3} v_{n-2}) = 2m + 2n - 5, f^*(v_{n-2} v_{n-1}) = 2m + 2n - 4 \text{ and}$$
$$f^*(v_{n-2} v_n) = 2m + 2n - 2.$$

Case 2. $m = 3$.

We define $f : V(T_n \cup C_3) \cup E(T_n \cup C_3) \to \{1, 2, 3, \cdots, 2n + 5\}$ as follows:

$f(u_1) = 1, f(u_2) = 4, f(u_3) = 6, f(v_i) = 5 + 2i$ for $1 \leq i \leq n - 3$, $f(v_{n-2}) = 2n + 3$, $f(v_{n-1}) = 2n$ and $f(v_n) = 2n + 5$.

The induced edge labeling as follows:

$$f^*(u_1u_2) = 2, f^*(u_2u_3) = 5, f^*(u_3u_1) = 3,$$
$$f^*(v_iv_{i+1}) = 6 + 2i \text{ for } 1 \le i \le n - 4,$$
$$f^*(v_{n-3}v_{n-2}) = 2n + 1, f^*(v_{n-2}v_{n-1}) = 2n + 2 \text{ and}$$
$$f^*(v_{n-2}v_n) = 2n + 4.$$

Hence, f is a super geometric mean labeling of $T_n \cup C_m$. Thus the graph $T_n \cup C_m$ is a super geometric mean graph for $n \ge 4$ and $m \ge 3$. □

The super geometric mean labeling of $T_6 \cup C_7$ and $T_7 \cup C_3$ are shown in Figure 3.

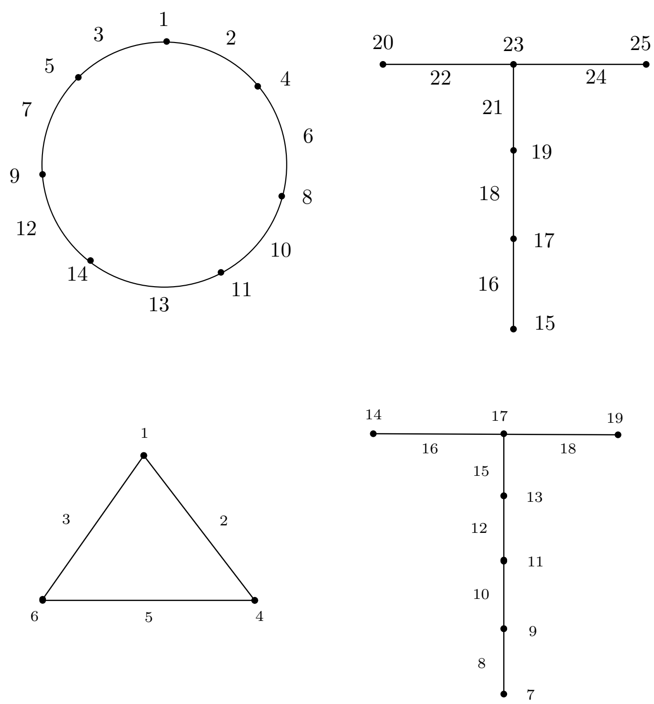

Figure 3

Theorem 2.3 *mC_n is a super geometric mean graph, for any m and n.*

Proof Let $\left\{v_j^{(i)} : 1 \le j \le n\right\}$ be the vertices of the i^{th} copy of the cycle C_n, $1 \le i \le m$.

Case 1. $n \ge 5$.

We define $f : V(mC_n) \cup E(mC_n) \to \{1, 2, 3, \cdots, 2mn\}$ as follows:

$$f\left(v_j^{(1)}\right)=\begin{cases}1 & j=1\\ 4j-4 & 2\leq j\leq\left\lfloor\frac{n}{2}\right\rfloor\\ 2n-3 & j=\left\lfloor\frac{n}{2}\right\rfloor+1 \text{ and } n \text{ is odd}\\ 2n & j=\left\lfloor\frac{n}{2}\right\rfloor+1 \text{ and } n \text{ is even}\\ 2n & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is odd}\\ 2n-3 & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is even}\\ 4n+5-4i & \left\lfloor\frac{n}{2}\right\rfloor+3\leq i\leq n,\end{cases}$$

$$f\left(v_j^{(2)}\right)=\begin{cases}2n+1 & j=1\\ 2n+4j-5 & 2\leq j\leq\left\lfloor\frac{n}{2}\right\rfloor\\ 4n-3 & j=\left\lfloor\frac{n}{2}\right\rfloor+1 \text{ and } n \text{ is odd}\\ 4n & j=\left\lfloor\frac{n}{2}\right\rfloor+1 \text{ and } n \text{ is even}\\ 4n & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is odd}\\ 4n-3 & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is even}\\ 6n+6-4j & \left\lfloor\frac{n}{2}\right\rfloor+3\leq j\leq n\end{cases}$$

and $f\left(v_j^{(i)}\right)=2n+f\left(v_j^{(i-1)}\right)$ for $3\leq i\leq m$ and $1\leq j\leq n$.

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)}v_{j+1}^{(1)}\right)=\begin{cases}4j-2 & 1\leq j\leq\left\lfloor\frac{n}{2}\right\rfloor\\ 2n-1 & j=\left\lfloor\frac{n}{2}\right\rfloor+1\\ 2n-2 & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is odd}\\ 2n-5 & j=\left\lfloor\frac{n}{2}\right\rfloor+2 \text{ and } n \text{ is even}\\ 4n+3-4j & \left\lfloor\frac{n}{2}\right\rfloor+3\leq j\leq n-1,\end{cases}$$

$$f^*\left(v_1^{(1)}v_n^{(1)}\right)=3,$$

$$f^*\left(v_j^{(2)}v_{j+1}^{(2)}\right)=\begin{cases}2n+2 & j=1\\ 2n+4j-3 & 2\leq j\leq\left\lfloor\frac{n}{2}\right\rfloor\text{-1}\\ 4n-5 & j=\left\lfloor\frac{n}{2}\right\rfloor \text{ and } n \text{ is odd}\\ 4n-2 & j=\left\lfloor\frac{n}{2}\right\rfloor \text{ and } n \text{ is even}\\ 4n-1 & j=\left\lfloor\frac{n}{2}\right\rfloor+1\\ 6n+4-4j & \left\lfloor\frac{n}{2}\right\rfloor+2\leq j\leq n-1,\end{cases}$$

$$f^*\left(v_1^{(2)}v_n^{(2)}\right)=2n+4,$$

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right)=2n+f^*\left(v_j^{(i-1)}v_{j+1}^{(i-1)}\right) \text{ for } 3\leq i\leq m \text{ and } 1\leq j\leq n-1$$

and $$f^*\left(v_1^{(i)}v_n^{(i)}\right)=2n+f^*\left(v_1^{(i-1)}v_n^{(i-1)}\right) \text{ for } 3\leq i\leq m.$$

Case 2. $n = 4$.

We define $f : V(mC_4) \cup E(mC_4) \rightarrow \{1, 2, 3, \cdots, 8m\}$ as follows:

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 5j - 6 & 2 \leq j \leq 3 \\ 5 & j = 4 \end{cases} \text{ and } f\left(v_j^{(2)}\right) = \begin{cases} 8 & j = 1 \\ 6j & 2 \leq j \leq 3 \\ 13 & j = 4. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j - 2 & 1 \leq j \leq 2 \\ 7 & j = 3 \end{cases}, f^*\left(v_1^{(1)} v_4^{(1)}\right) = 3$$

$$f^*\left(v_j^{(2)} v_{j+1}^{(2)}\right) = \begin{cases} 5j + 5 & 1 \leq j \leq 2 \\ 16 & j = 3 \end{cases} \text{ and } f^*\left(v_1^{(2)} v_4^{(2)}\right) = 11.$$

Subcase 2.1 m is odd and $m \geq 3$.

$$f\left(v_j^{(3)}\right) = \begin{cases} 14 & j = 1 \\ 2j + 16 & 2 \leq j \leq 4, \end{cases}$$

$$f\left(v_j^{(4)}\right) = \begin{cases} 2j + 23 & 1 \leq j \leq 3 \\ 34 & j = 4, \end{cases}$$

$$f\left(v_j^{(5)}\right) = \begin{cases} 31 & j = 1 \\ 3j + 29 & 2 \leq j \leq 3 \\ 40 & j = 4 \end{cases} \text{ and}$$

$$f\left(v_j^{(i)}\right) = f\left(v_j^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 4.$$

The induced edge labeling is as follows

$$f^*\left(v_j^{(3)} v_{j+1}^{(3)}\right) = \begin{cases} 17 & j = 1 \\ 2j + 17 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_i^{(3)} v_4^{(3)}\right) = 19,$$

$$f^*\left(v_j^{(4)} v_{j+1}^{(4)}\right) = \begin{cases} 26 & j = 1 \\ 4j + 20 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(4)} v_4^{(4)}\right) = 30,$$

$$f^*\left(v_j^{(5)} v_{j+1}^{(5)}\right) = \begin{cases} 33 & j = 1 \\ 2j + 33 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(5)} v_4^{(5)}\right) = 36,$$

$$f^*\left(v_j^{(i)} v_{j+1}^{(i)}\right) = f^*\left(v_j^{(i-2)} v_{j+1}^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 3$$

and

$$f^*\left(v_1^{(i)} v_4^{(i)}\right) = f^*\left(v_1^{(i-2)} v_4^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m.$$

Subcase 2.2 m is even and $m \geq 4$.

$$f\left(v_j^{(3)}\right) = \begin{cases} 14 & j=1 \\ 3j+13 & 2 \leq j \leq 3 \\ 26 & j=4 \end{cases}, \quad f\left(v_j^{(4)}\right) = \begin{cases} 23 & j=1 \\ 3j+21 & 2 \leq j \leq 3 \\ 32 & j=4, \end{cases}$$

$$f\left(v_j^{(5)}\right) = \begin{cases} 2j+31 & 1 \leq j \leq 3 \\ 42 & j=4 \end{cases} \quad \text{and}$$

$$f\left(v_j^{(i)}\right) = f\left(v_j^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 4.$$

The induced edge labeling is as follows

$$f^*\left(v_j^{(3)}v_{j+1}^{(3)}\right) = \begin{cases} 17 & j=1 \\ 3j+15 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(3)}v_4^{(3)}\right) = 20,$$

$$f^*\left(v_j^{(4)}v_{j+1}^{(4)}\right) = \begin{cases} 25 & j=1 \\ 2j+25 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(4)}v_4^{(4)}\right) = 28,$$

$$f^*\left(v_j^{(5)}v_{j+1}^{(5)}\right) = \begin{cases} 34 & j=1 \\ 4j+28 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(5)}v_4^{(5)}\right) = 38,$$

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right) = f^*\left(v_j^{(i-2)}v_{j+1}^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 3$$

$$\text{and } f^*\left(v_1^{(i)}v_4^{(i)}\right) = f^*\left(v_1^{(i-2)}v_4^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m.$$

Case 3. $n = 3$.

We define $f : V(mC_3) \cup E(mC_3) \to \{1, 2, 3, \cdots, 6m\}$ as follows:

$$f\left(v_1^{(i)}\right) = 6i-5 \text{ for } 1 \leq i \leq m, \; f\left(v_2^{(i)}\right) = \begin{cases} 4 & i=1 \\ 6i-3 & 2 \leq i \leq m \end{cases}$$

and $f\left(v_3^{(i)}\right) = 6i$ for $1 \leq i \leq m$. The induced edge labeling is as follows:

$$f^*\left(v_1^{(i)}v_2^{(i)}\right) = 6i-4 \text{ for } 1 \leq i \leq m, \; f^*\left(v_2^{(i)}v_3^{(i)}\right) = 6i-1 \text{ for } 1 \leq i \leq m \text{ and}$$

$$f^*\left(v_3^{(i)}v_1^{(i)}\right) = \begin{cases} 3 & i=1 \\ 6i-2 & 2 \leq i \leq m \end{cases}.$$

Hence, f is super geometric mean labeling of mC_n. Thus the graph mC_n is a super geometric mean graph for any m and n. □

The super geometric mean labeling of $4C_6, 7C_4$ and $5C_3$ are shown in Figure 4.

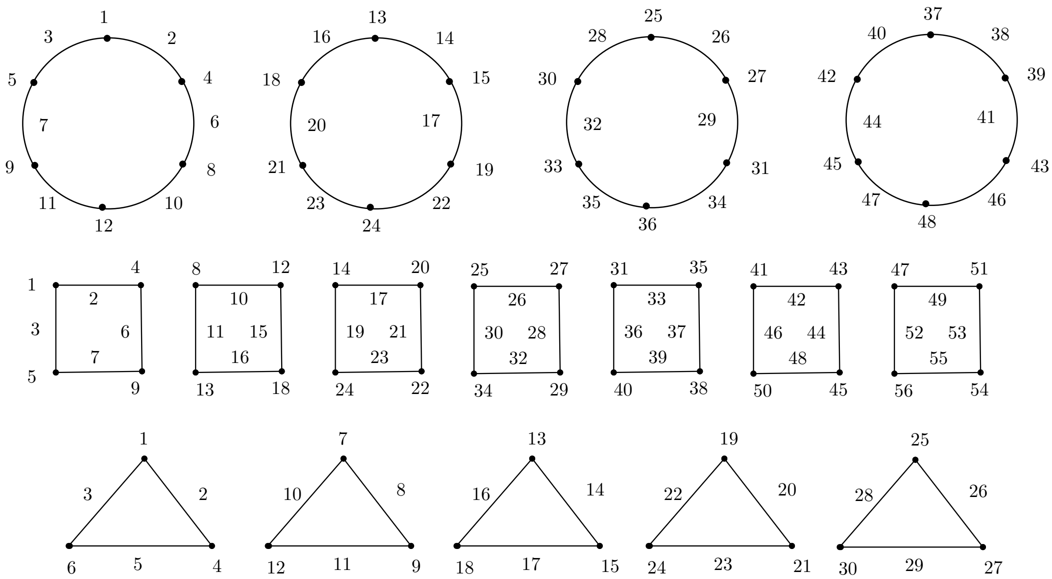

Figure 4

Corollary 2.4 *$mC_n \cup P_k$ is a super geometric mean graph for any m, n and k.*

Proof By the above Theorems 2.1 and 2.3 the results follows. □

Theorem 2.5 *K_n is a super geometric mean graph if and only if $n \leq 3$.*

Proof Based on the definition of super geometric mean labeling, 1 and $p+q$ should be the vertex labels.

For all $p \geq 5$, the edge having the end vertices whose labels are 1 and $p+q$ is less than or equal to $p-1$. So we cannot have distinct edge labels for the edges incident with a vertex whose vertex label is 1.

When $p = 4, 1, p+q = 10$ and $p+q-2 = 8$ are to be the vertex labels whose induced edge labels are $3, 4$ and 9. So we cannot label for the 4^{th} vertex in which the edge label is 2. Also 2 cannot be the vertex label. □

The super geometric mean labeling of K_1, K_2 and K_3 are shown in Figure 5.

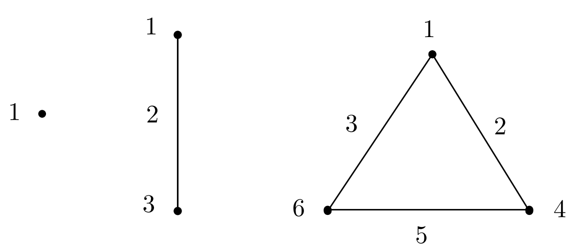

Figure 5

Theorem 2.6 *$[P_n; S_m]$ is a super geometric mean graph, for $n \geq 1$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, \cdots, v_m^{(i)}$ be the pendant vertices at each vertex u_i of the path P_n, for $1 \le i \le n$.

Case 1. $m = 1$.

We define $f : V([P_n; S_1]) \cup E([P_n; S_1]) \to \{1, 2, 3, \cdots, 6n-1\}$ as follows:

$$f(u_i) = \begin{cases} 5 & i = 1 \\ 6i-5 & 2 \le i \le n \end{cases}, f\left(v_1^{(i)}\right) = 6i-3 \text{ for } 1 \le i \le n,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 1 & i = 1 \\ 6i & 2 \le i \le n-1 \end{cases} \text{ and } f\left(v_2^{(n)}\right) = 6n-1.$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 6 & i = 1 \\ 6i-2 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(u_iv_1^{(i)}\right) = \begin{cases} 4 & i = 1 \\ 6i-4 & 2 \le i \le n, \end{cases}$$

$$f^*\left(v_1^{(i)}v_2^{(i)}\right) = \begin{cases} 2 & i = 1 \\ 6i-1 & 2 \le i \le n-1 \end{cases} \text{ and } f^*\left(v_1^{(n)}v_2^{(n)}\right) = 6n-2.$$

Case 2. $m = 2$.

We define $f : V([P_n; S_2]) \cup E([P_n; S_2]) \to \{1, 2, 3, \cdots, 8n-1\}$ as follows:

$$f(u_i) = \begin{cases} 2i+5 & 1 \le i \le 2 \\ 8i-8 & 3 \le i \le n, \end{cases}$$

$$f\left(v_1^{(i)}\right) = \begin{cases} 5 & i = 1 \\ 8i-5 & 2 \le i \le n-1, \end{cases}$$

$$f\left(v_1^{(n)}\right) = 8n-3,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 1 & i = 1 \\ 8i-1 & 2 \le i \le n-1, \end{cases} f\left(v_2^{(n)}\right) = 8n-6,$$

$$f\left(v_3^{(i)}\right) = \begin{cases} 2 & i = 1 \\ 8i+1 & 2 \le i \le n-1 \end{cases} \text{ and } f\left(v_3^{(n)}\right) = 8n-1.$$

The induced edge labeling is as follows

$$f^*(u_iu_{i+1}) = \begin{cases} 8 & i=1 \\ 8i-4 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(u_iv_1^{(i)}\right) = \begin{cases} 6 & i=1 \\ 8i-6 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(u_nv_1^{(n)}\right) = 8n-5, f^*\left(v_1^{(i)}v_2^{(i)}\right) = \begin{cases} 3 & i=1 \\ 8i-3 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(v_1^{(n)}v_2^{(n)}\right) = 8n-4 \text{ and } f^*\left(v_1^{(i)}v_3^{(i)}\right) = \begin{cases} 4 & i=1 \\ 8i-2 & 2 \le i \le n. \end{cases}$$

Hence, f is a super geometric mean labeling of $[P_n; S_m]$. Thus the graph $[P_n; S_m]$ is a super geometric mean graph, for $n \ge 1$ and $m \le 2$. □

The super geometric mean labeling of $[P_6; S_1]$ and $[P_5; S_2]$ are shown in Figure 6.

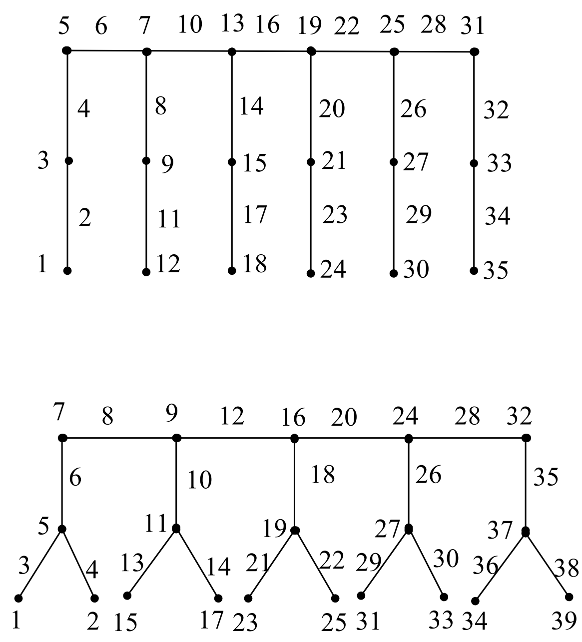

Figure 6

Theorem 2.7 *$S(P_n \odot K_1)$ is a super geometric mean graph, for $n \ge 1$.*

Proof Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$. Let x_i be the vertex which divides the edge

u_iv_i, for $1 \le i \le n$ and y_i be the vertex which divides the edge u_iv_{i+1}, for $1 \le i \le n-1$. Then

$$V\left(S(P_n \odot K_1)\right) = \{u_i, v_i, x_i, y_j : 1 \le i \le n, 1 \le j \le n-1\}$$
$$E\left(S(P_n \odot K_1)\right) = \{u_ix_i, v_ix_i : 1 \le i \le n\} \cup \{u_iy_i, y_iu_{i+1} : 1 \le j \le n-1\}$$

We define $f : V\left(S(P_n \odot K_1)\right) \bigcup E\left(S(P_n \odot K_1)\right) \to \{1, 2, 3, \cdots, 8n-3\}$ as follows:

$$f(u_i) = \begin{cases} 5 & i = 1 \\ 8i-7 & 2 \le i \le n \end{cases}, f(y_i) = 8i-1 \text{ for } 1 \le i \le n-1,$$

$$f(x_i) = 8i-5 \text{ for } 1 \le i \le n, f(v_i) = \begin{cases} 1 & i = 1 \\ 8i-2 & 2 \le i \le n-1 \end{cases} \text{ and}$$

$$f(v_n) = 8n-3.$$

The induced edge labeling is as follows

$$f^*(u_iy_i) = \begin{cases} 6 & i = 1 \\ 8i-4 & 2 \le i \le n-1 \end{cases}, f^*(y_iu_{i+1}) = 8i \text{ for } 1 \le i \le n-1,$$

$$f^*(u_ix_i) = \begin{cases} 4 & i = 1 \\ 8i-6 & 2 \le i \le n \end{cases}, f^*(x_iv_i) = \begin{cases} 2 & i = 1 \\ 8i-3 & 2 \le i \le n-1 \end{cases},$$

and $f^*(x_nv_n) = 8n-4$.

Hence, f is a super geometric mean labeling of $S(P_n \odot K_1)$. Thus the graph $S(P_n \odot K_1)$ is a super geometric mean graph, for $n \ge 1$. □

A super geometric mean labeling of $S(P_4 \odot K_1)$ is shown in Figure 7.

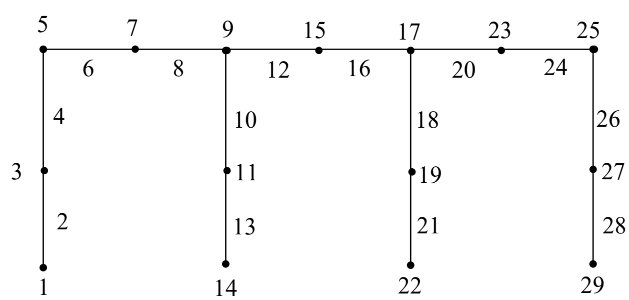

Figure 7

Theorem 2.8 *$TW(P_n)$ is a super geometric mean graph, for $n \ge 3$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}$ be the pendant vertices at each vertex u_i of the path P_n, for $2 \leq i \leq n-1$. Then

$$V\left(TW(P_n)\right) = V(P_n) \cup \left\{v_1^{(i)}, v_2^{(i)} : 2 \leq i \leq n-1\right\} \text{ and}$$
$$E\left(TW(P_n)\right) = E(P_n) \cup \left\{u_i v_1^{(i)}, u_i v_2^{(i)} : 2 \leq i \leq n-1\right\}.$$

We define $f : V(TW(P_n)) \cup E(TW(P_n)) \to \{1, 2, 3, \ldots, 6n-9\}$ as follows

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 6i-7 & 2 \leq i \leq n-2, \end{cases}$$
$$f(u_{n-1}) = 6n-11, f(u_n) = 6n-9,$$
$$f\left(v_1^{(i)}\right) = \begin{cases} 2 & i = 2 \\ 6i-9 & 3 \leq i \leq n-2 \end{cases}, f\left(v_1^{(n-1)}\right) = 6n-16,$$
$$f\left(v_2^{(i)}\right) = 6i-5 \text{ for } 2 \leq i \leq n-2 \textit{ and } f\left(v_2^{(n-1)}\right) = 6n-14.$$

The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 3 & i = 1 \\ 6i-4 & 2 \leq i \leq n-3 \end{cases}, f^*(u_{n-2}u_{n-1}) = 6n-15,$$
$$f^*(u_{n-1}u_n) = 6n-10,$$
$$f^*\left(u_i v_1^{(i)}\right) = 6i-8 \text{ for } 2 \leq i \leq n-2, f^*\left(u_{n-1} v_1^{(n-1)}\right) = 6n-13 \text{ and}$$
$$f^*\left(u_i v_2^{(i)}\right) = 6i-6 \text{ for } 2 \leq i \leq n-1.$$

Hence, f is a super geometric mean labeling of $TW(P_n)$. Thus the graph $TW(P_n)$ is a super geometric mean graph, for $n \geq 3$. □

A super geometric mean labeling of $TW(P_8)$ is shown in Figure 8.

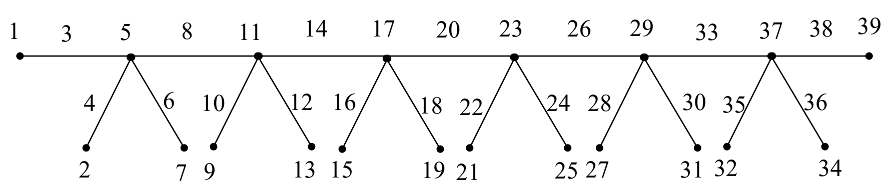

Figure 8

Theorem 2.9 *$M(P_n)$ is a super geometric mean graph, for $n \geq 4$.*

Proof Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then

$$V\left(M(P_n)\right) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\} \text{ and}$$
$$E\left(M(P_n)\right) = \{v_ie_i, e_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{e_ie_{i+1} : 1 \leq i \leq n-2\}.$$

We define $f : V(M(P_n)) \cup E(M(P_n)) \to \{1, 2, 3, \cdots, 5n-5\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 2i+1 & 2 \leq i \leq 3 \\ 5i-5 & 4 \leq i \leq n \end{cases} \text{ and } f(e_i) = \begin{cases} 8i-5 & 1 \leq i \leq 2 \\ 5i-2 & 3 \leq i \leq n-1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(e_ie_{i+1}) = \begin{cases} 6i & 1 \leq i \leq 2 \\ 5i+1 & 3 \leq i \leq n-2 \end{cases}, f^*(e_iv_i) = \begin{cases} 2 & i = 1 \\ 2i+4 & 2 \leq i \leq 3 \\ 5i-3 & 4 \leq i \leq n-1 \end{cases}$$

and $f^*(e_iv_{i+1}) = 5i-1$ for $1 \leq i \leq n-1$.

Hence, f is a super geometric mean labeling of $M(P_n)$. Thus the graph $M(P_n)$ is a super geometric mean graph for $n \geq 4$. □

A super geometric mean labeling of $M(P_8)$ is shown in Figure 9.

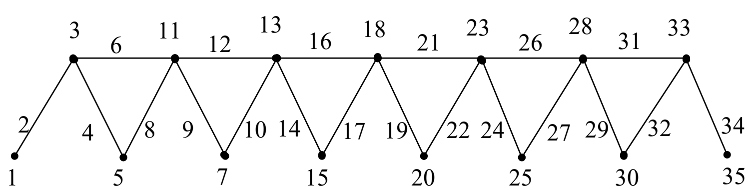

Figure 9

Theorem 2.10 *TL_n is a super geometric mean graph, for $n \geq 3$.*

Proof Let the vertex set of TL_n be $\{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and the edge set of TL_n be $\{u_iu_{i+1}, u_iv_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Then TL_n has $2n$ vertices and $4n-3$ edges. We define $f : V(TL_n) \cup E(TL_n) \to \{1, 2, 3, \cdots, 6n-3\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 6i-6 & 2 \leq i \leq n \end{cases}, f(u_i) = 6i-2 \text{ for } 1 \leq i \leq n-1$$

and $f(u_n) = 6n-3$. The induced edge labeling is as follows:

$$f^*(v_iv_{i+1}) = 6i-3 \text{ for } 1 \leq i \leq n-1, f^*(u_iu_{i+1}) = 6i+1 \text{ for } 1 \leq i \leq n-1,$$
$$f^*(u_iv_i) = 6i-4 \text{ for } 1 \leq i \leq n \text{ and } f^*(u_iv_{i+1}) = 6i-1 \text{ for } 1 \leq i \leq n-1.$$

Hence, f is a super geometric mean labeling of TL_n. Thus the graph TL_n is a super geometric mean graph for $n \geq 3$. □

A super geometric mean labeling of TL_7 are shown in Figure 10.

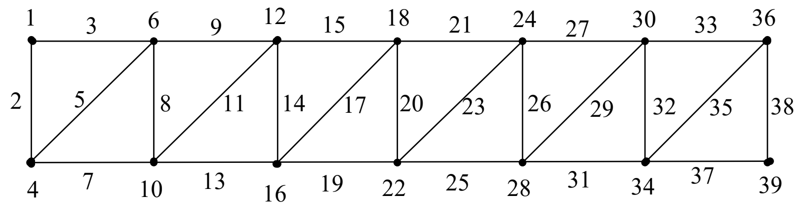

Figure 10

Theorem 2.11 *$C_n \odot K_1$ is a super geometric mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and $u_1, u_2, \cdots, u_n$ be the pendant vertices of the cycle C_n.

Case 1. $n \geq 7$.

We define $f : V(C_n \odot K_1) \cup E(C_n \odot K_1) \to \{1, 2, 3, \cdots, 4n\}$ as follows:

$$f(v_i) = \begin{cases} 3 & i = 1 \\ 8i - 11 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 8n + 12 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \end{cases}$$

$$f(u_i) = \begin{cases} 7i - 6 & 1 \leq i \leq 3 \\ 8i - 9 & 4 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 4n - 5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 8n + 10 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 4 & i = 1 \\ 8i - 7 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4n - 11 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\ 4n - 6 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\ 4n - 3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 8n + 8 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n - 1, \end{cases}$$

$f^*(v_1 v_n) = 6$ and

$$f^*(u_iv_i) = \begin{cases} 5i-3 & 1 \le i \le 2 \\ 8i-10 & 3 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 4n-6 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n-1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 8n+11-8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n \end{cases}$$

Case 2. $n = 3, 4, 5, 6$.

In this case, the super geometric mean labelings are given in Figure 11.

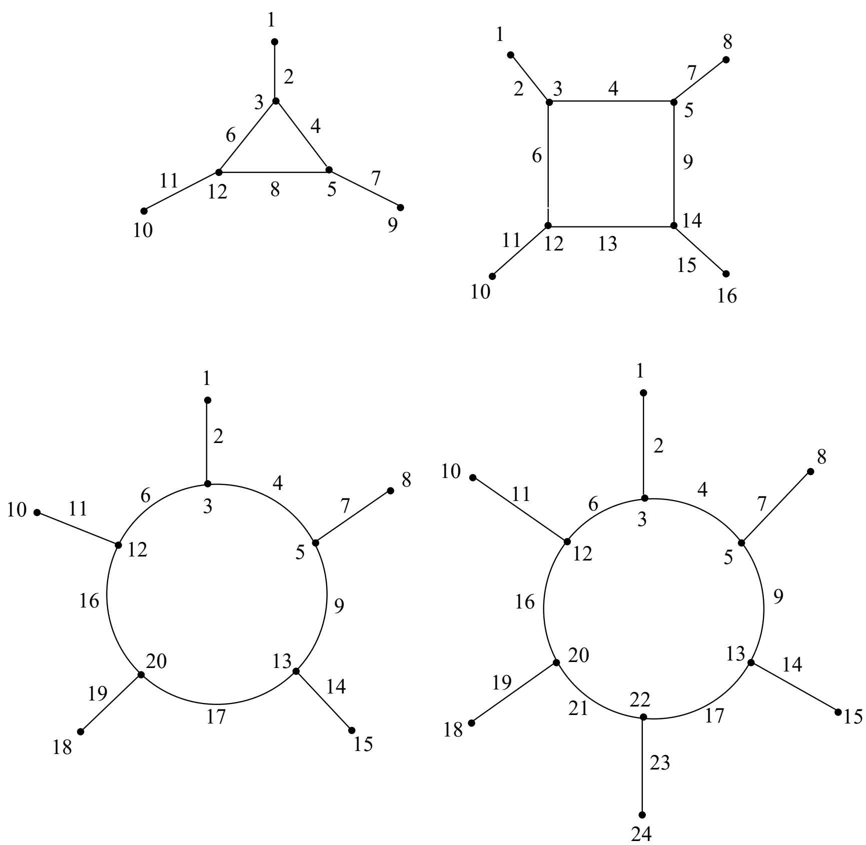

Figure 11

Hence, f is a super geometric mean labeling of $C_n \odot K_1$. Thus the graph $C_n \odot K_1$ is a super geometric mean graph. □

A super geometric mean labeling of $C_9 \odot K_1$ is shown in Figure 12.

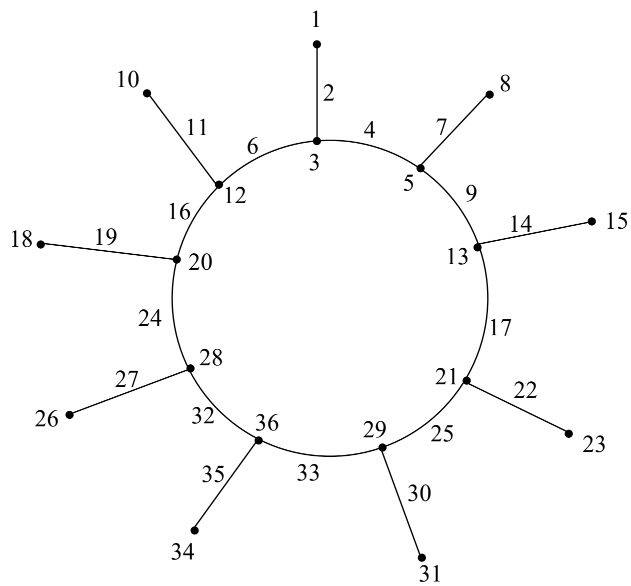

Figure 12

Theorem 2.12 *The graph obtained by duplication of an arbitrary vertex in cycle C_n is a super geometric mean graph, for $n \geq 4$.*

Proof Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the cycle C_n, for $n \geq 4$. Without loss of generality we duplicate the vertex $v = v_1$ and its duplicated vertex is v_1'. Then the resultant graph G will have $n+1$ vertices and $n+2$ edges.

We define $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 2n+3\}$ as follows:

$$f(v_1') = 1,$$

$$f(v_i) = \begin{cases} 8 - 2i & 1 \leq i \leq 2 \\ 4i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 2n & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n-1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n+7-4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n-1 \end{cases}$$

and $$f(v_n) = 9.$$

The induced edge labeling is as follows

$$f^*(v_1'v_2) = 2, \quad f^*(v_1'v_n) = 3,$$

$$f^*(v_iv_{i+1}) = \begin{cases} 2i+3 & 1 \le i \le 2 \\ 4i+2 & 3 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \\ 2n+1 & i = \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is odd} \\ 2n+2 & i = \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is even} \\ 2n+2 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 2n+1 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 4n+5-4i & \lfloor \frac{n}{2} \rfloor + 2 \le i \le n-2, \end{cases}$$

$$f^*(v_{n-1}v_n) = 10 \text{ and } f^*(v_1v_n) = 8.$$

Hence, f is a super geometric mean labeling of G. Thus the graph obtained by duplication of an arbitrary vertex in the cycle C_n is a super geometric mean graph, for $n \ge 4$. □

The graph obtained by duplication of a vertex in C_9 and its super geometric mean labeling is shown in Figure 13.

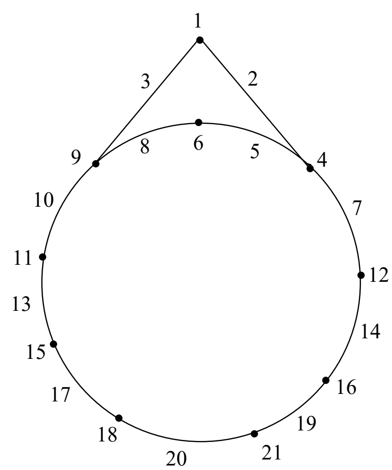

Figure 13

Theorem 2.13 *The graph obtained by duplication of an arbitrary edge in cycle C_n is a super geometric mean graph, for $n \ge 3$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Without loss of generality we duplication an edge $e = v_1v_2$ and its duplicated edge is $e' = v_1'v_2'$. Then the resultant graph G will have $n+2$ vertices and $n+3$ edges.

Case 1. $n \ge 6$.

We define $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, 2n+5\}$ as follows:

$f(v_1') = 1, f(v_2') = 3$ and

$$f(v'_i) = \begin{cases} 9 & i = 1 \\ 5i-5 & 2 \le i \le 3 \\ 4i-2 & 4 \le i \le \lfloor \frac{n}{2} \rfloor + 1 \\ 2n+5 & i = \lfloor \frac{n}{2} \rfloor + 2 \\ 2n+2 & i = \lfloor \frac{n}{2} \rfloor + 3 \text{ and } n \text{ is odd} \\ 2n+1 & i = \lfloor \frac{n}{2} \rfloor + 3 \text{ and } n \text{ is even} \\ 4n+13-4i & \lfloor \frac{n}{2} \rfloor + 4 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v'_1v'_2) = 2, f^*(v'_1v'_n) = 4, f^*(v'_2v'_3) = 6,$$

$$f^*(v_iv_{i+1}) = \begin{cases} i+6 & 1 \le i \le 2 \\ 4i & 3 \le i \le \lfloor \frac{n}{2} \rfloor \\ 2n+3 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 2n+4 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 2n+4 & i = \lfloor \frac{n}{2} \rfloor + 2 \text{ and } n \text{ is odd} \\ 2n+3 & i = \lfloor \frac{n}{2} \rfloor + 2 \text{ and } n \text{ is even} \\ 4n+11-4i & \lfloor \frac{n}{2} \rfloor + 3 \le i \le n-1 \end{cases}$$

and $$f^*(v_1v_n) = 11.$$

Case 2. $n = 3, 4, 5$.

In this case, the super geometric mean labelings are given in Figure 14.

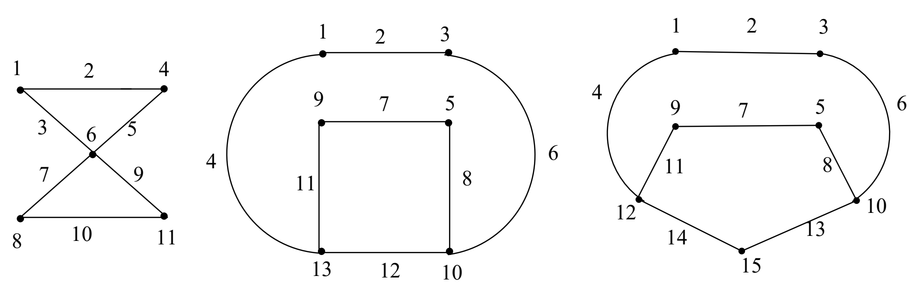

Figure 14

Hence, f is a super geometric mean labeling of G. Thus the graph obtained by duplication of an arbitrary edge in cycle C_n is a super geometric mean graph, for $n \ge 3$. □

The graph obtained by duplication of an edge in C_8 and its super geometric mean labeling is shown in Figure 15.

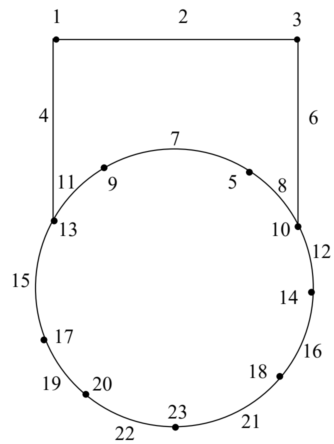

Figure 15

A triangular grid $T_n(G)$ with n vertices in each side are constructed as follows: The vertices of $T_n(G)$ are $\left\{v_i^{(j)} : 1 \leq j \leq n,\ \ 1 \leq i \leq n+1-j\right\}$ and the edges are $\left\{v_i^{(j)} v_{i+1}^{(j)} : 1 \leq j \leq n-1, \right.$ $\left. 1 \leq i \leq n-j\right\} \bigcup \left\{v_i^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq i \leq n-j\right\} \bigcup \left\{v_{i+1}^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq \right.$ $\left. i \leq n-j\right\}$. The triangular grid graph $T_6(G)$ is shown in Figure 16.

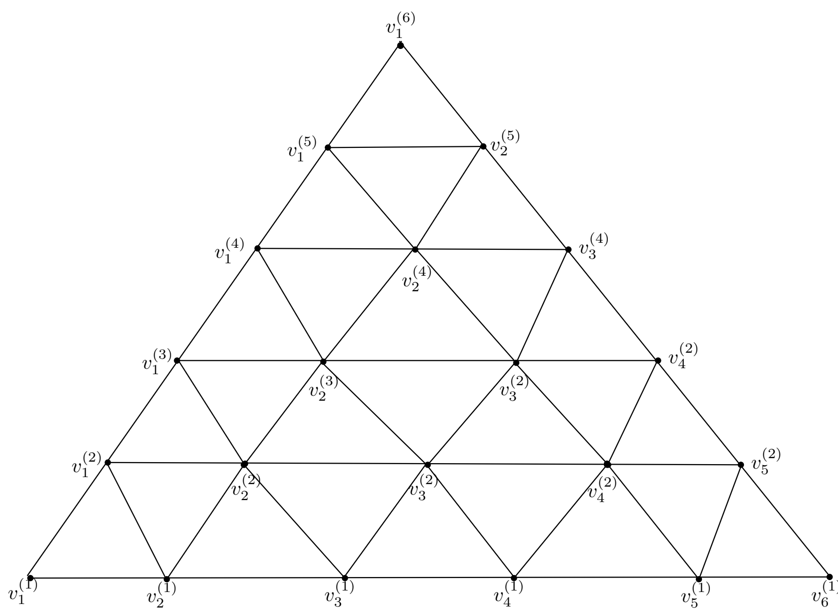

Figure 16

Theorem 2.14 *The triangular grid graph $T_n(G)$ is a super geometric mean graph.*

Proof Let $\left\{v_i^{(j)} : 1 \leq j \leq n, 1 \leq i \leq n+1-j\right\}$ be the vertex set of $T_n(G)$. Then the edge set of $T_n(G)$ are $\left\{v_i^{(j)} v_{i+1}^{(j)} : 1 \leq j \leq n-1, 1 \leq i \leq n-j\right\} \bigcup \left\{v_i^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq i \leq \right.$ $\left. n-j\right\} \bigcup \left\{v_{i+1}^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, \right.$
$\left. 1 \leq i \leq n-j\right\}$.

We define $f : V(T_n(G)) \cup E(T_n(G)) \to \{1, 2, 3, \cdots, n(2n-1)\}$ as follows

$$f\left(v_i^{(1)}\right) = i(2i-1) \text{ for } 1 \le i \le n \text{ and}$$
$$f\left(v_i^{(j)}\right) = f\left(v_{i+1}^{(j-1)}\right) - 2 \text{ for } 2 \le j \le n \text{ and } 1 \le i \le n+1-j.$$

The induced edge labeling is as follows

$$f^*\left(v_i^{(1)}v_{i+1}^{(1)}\right) = i(2i+1) \text{ for } 1 \le i \le n-1,$$
$$f^*\left(v_i^{(j)}v_{i+1}^{(j)}\right) = f^*\left(v_{i+1}^{(j-1)}v_{i+2}^{(j-1)}\right) - 2 \text{ for } 2 \le j \le n-1 \text{ and } 1 \le i \le n-j,$$
$$f^*\left(v_i^{(1)}v_i^{(2)}\right) = (i+1)(2i-1) \text{ for } 1 \le i \le n-1,$$
$$f^*\left(v_i^{(j)}v_i^{(j+1)}\right) = f^*\left(v_{i+1}^{(j-1)}v_{i+1}^{(j)}\right) - 2 \text{ for } 2 \le j \le n-1 \text{ and } 1 \le i \le n-j,$$
$$f^*\left(v_{i+1}^{(1)}v_i^{(2)}\right) = i(2i-1) + 4i \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f^*\left(v_{i+1}^{(j)}v_i^{(j+1)}\right) = f^*\left(v_{i+2}^{(j-1)}v_{i+1}^{(j)}\right) - 2 \text{ for } 2 \le j \le n-1 \text{ and } 1 \le i \le n-j.$$

Hence, f is a super geometric mean labeling of $T_n(G)$. Thus the graph $T_n(G)$ is a super geometric mean graph. □

A super geometric mean labeling of $T_7(G)$ is shown in Figure 17.

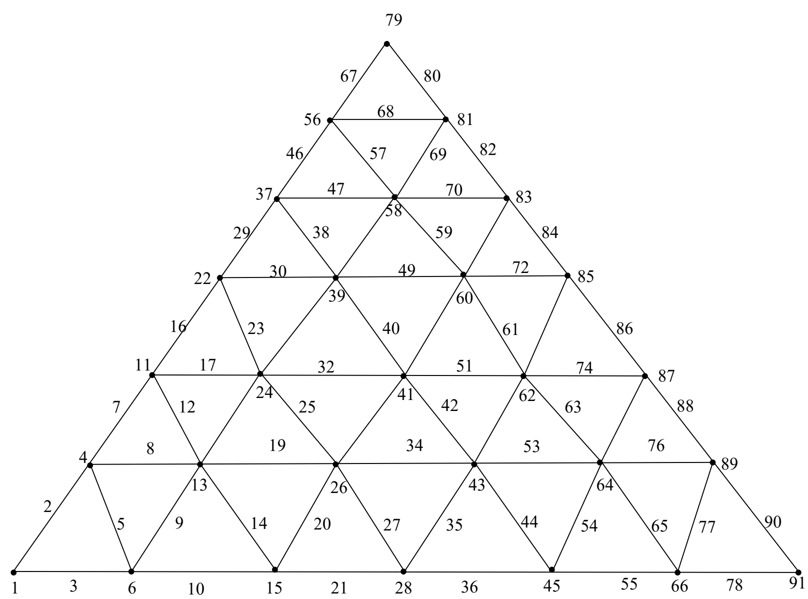

Figure 17

The graph $G'(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ by identifying the j^{th} cycle and $(j+1)^{th}$ cycle by the edges $v^{(j)}_{\frac{p_j+1}{2}} v^{(j)}_{\frac{p_j+3}{2}}$ and $v_1^{(j+1)}v_{p_{j+1}}^{(j+1)}$, for $1 \le j \le n-1$.

Theorem 2.15 *The graph $G'(p_1, p_2, \ldots, p_n)$ is a super geometric mean graph all p_j's are odd or all p_j's are even with $p_j \neq 4$ for $1 \le j \le n$.*

Proof Let $\left\{v_i^{(j)} : 1 \leq j \leq n \text{ and } 1 \leq i \leq p_j\right\}$ be the vertices of the n number of cycles with $p_j \neq 4$. For $1 \leq j \leq n-1$, the j^{th} cycle and $(j+1)^{th}$ cycle by the edges $v^{(j)}_{\frac{p_j+1}{2}} v^{(j)}_{\frac{p_j+3}{2}}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$. We define $f : V(G') \cup E(G') \to \left\{1, 2, 3, \cdots, \sum\limits_{i=1}^{n} 2p_i - 3n + 3\right\}$ as follows.

Case 1. p_j is odd.

When $p_1 = 5$, define

$f\left(v_1^{(1)}\right) = 3$, $f\left(v_2^{(1)}\right) = 1$, $f\left(v_3^{(1)}\right) = 10$, $f\left(v_4^{(1)}\right) = 8$ and $f\left(v_5^{(1)}\right) = 6$.

The induced edge labeling is as follows:

$f^*\left(v_1^{(1)} v_2^{(1)}\right) = 2$, $f^*\left(v_2^{(1)} v_3^{(1)}\right) = 4$, $f^*\left(v_3^{(1)} v_4^{(1)}\right) = 9$,

$f^*\left(v_4^{(1)} v_5^{(1)}\right) = 7$ and $f^*\left(v_1^{(1)} v_5^{(1)}\right) = 5$.

When $p_1 \geq 7$, define

$$f\left(v_i^{(1)}\right) = \begin{cases} 4i - 3 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4i - 2 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4i - 9 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1 + 4 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq i \leq p_1 \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(1)} v_{i+1}^{(1)}\right) = \begin{cases} 4i - 1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor - 1 \\ 4i & i = \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4i - 3 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4p_1 + 2 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq i \leq p_1 - 1 \end{cases}$$

and $$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 2.$$

For $2 \leq j \leq n$, define

$$f\left(v_i^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 5 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1 \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 2 & i = 1 \\ \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 3 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 2 \end{cases}$$

and $$f^*\left(v_{p_j-1}^{(j)} v_{p_j}^{(j)}\right) = \sum_{k=1}^{j-1} 2p_k - 3(j-2) + 1.$$

Case 2. p_j is even.

When $p_1 = 6$, define

$f\left(v_1^{(1)}\right) = 6, f\left(v_2^{(1)}\right) = 8, f\left(v_3^{(1)}\right) = 10, f\left(v_4^{(1)}\right) = 12, f\left(v_5^{(1)}\right) = 1$ and $f\left(v_6^{(1)}\right) = 3.$

The induced edge labeling is as follows:

$f^*\left(v_1^{(1)}v_2^{(1)}\right) = 7, f^*\left(v_2^{(1)}v_3^{(1)}\right) = 9, f^*\left(v_3^{(1)}v_4^{(1)}\right) = 11,$

$f^*\left(v_4^{(1)}v_5^{(1)}\right) = 4, f^*\left(v_5^{(1)}v_6^{(1)}\right) = 2$ and $f^*\left(v_1^{(1)}v_6^{(1)}\right) = 5.$

When $p_1 \geq 8$, define

$$f\left(v_i^{(1)}\right) = \begin{cases} 4i-3 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1+4-4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(1)}v_{i+1}^{(1)}\right) = \begin{cases} 4i-1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1+2-4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1 - 1 \end{cases}$$

and $f^*\left(v_1^{(1)}v_{p_1}^{(1)}\right) = 2.$

Subcase 2.1 $2 \leq j \leq n$ and j is odd.

$$\text{Let } f\left(v_i^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 5 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 6 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1. \end{cases}$$

The induced edge labeling is as follows

$$f^*\left(v_i^{(j)}v_{i+1}^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 2 & i = 1 \\ \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 3 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 4 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 5 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 2 \end{cases}$$

and $$f^*\left(v_{p_j-1}^{(j)}v_{p_j}^{(j)}\right) = \sum_{k=1}^{j-1} 2p_k - 3(j-2) + 1.$$

Subcase 2.2 $2 \leq j \leq n$ and j is even.

$$\text{Let } f\left(v_i^{(j)}\right)=\begin{cases}\sum\limits_{k=1}^{j-1}2p_k-3(j-2)+4i-4 & 2\le i\le\left\lfloor\frac{p_j}{2}\right\rfloor-1\\ \sum\limits_{k=1}^{j-1}2p_k+2p_j-3(j-2)-3 & i=\left\lfloor\frac{p_j}{2}\right\rfloor\\ \sum\limits_{k=1}^{j-1}2p_k+2p_j-3(j-2)-6 & i=\left\lfloor\frac{p_j}{2}\right\rfloor+1\\ \sum\limits_{k=1}^{j-1}2p_k+4p_j-3(j-2)-4i-1 & \left\lfloor\frac{p_j}{2}\right\rfloor+2\le i\le p_j-1.\end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(j)}v_{i+1}^{(j)}\right)=\begin{cases}\sum\limits_{k=1}^{j-1}2p_k-3(j-2)+1 & i=1\\ \sum\limits_{k=1}^{j-1}2p_k-3(j-2)+4i-2 & 2\le i\le\left\lfloor\frac{p_j}{2}\right\rfloor-2\\ \sum\limits_{k=1}^{j-1}2p_k+2p_j-3(j-2)-5 & i=\left\lfloor\frac{p_j}{2}\right\rfloor-1\\ \sum\limits_{k=1}^{j-1}2p_k+2p_j-3(j-2)-4 & i=\left\lfloor\frac{p_j}{2}\right\rfloor\\ \sum\limits_{k=1}^{j-1}2p_k+4p_j-3(j-2)-4i-3 & \left\lfloor\frac{p_j}{2}\right\rfloor+1\le i\le p_j-2\end{cases}$$

and $f^*\left(v_{p_j-1}^{(j)}v_{p_j}^{(j)}\right)=\sum\limits_{k=1}^{j-1}2p_k-3(j-2)+2$.

Hence, f is a super geometric mean labeling of $G'(p_1,p_2,\ldots,p_n)$. Thus it is a super geometric mean graph with $p_j\neq 4$ for $1\le j\le n$. □

A super geometric mean labeling of $G'(7,13,11,5)$ and $G'(8,10,12,8)$ are shown in Figure 18.

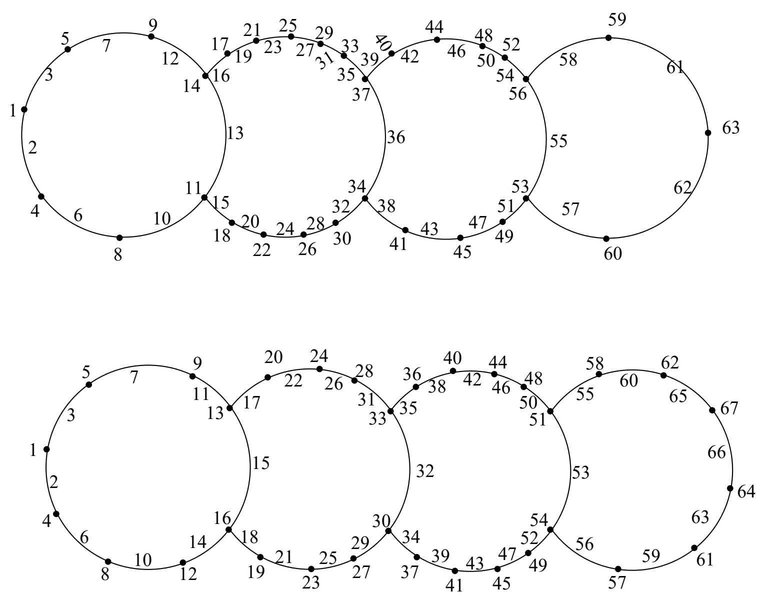

Figure 18

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Resolving Connected Domination in Graphs

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Abstract: For an ordered subset $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph $G = (V, E)$, the (metric) representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. The set W is a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a minimum resolving set and the cardinality of it is a dimension of G, denoted by $dim(G)$. In this paper, we introduce resolving connected domination number $\gamma_{rc}(G)$ of graphs. We investigate the relationship between resolving connected domination number, connected domination number, resolving domination number and dimension of a graph G. Bounds for $\gamma_{rc}(G)$ are determined. Exact values of $\gamma_{rc}(G)$ for some standard graphs are found.

Key Words: Resolving dominating set, resolving connected dominating set, resolving connected domination number, dimension of a graph.

AMS(2010): 05C69, 05C12.

§1. Introduction

In this paper, we consider the connected simple graph $G = (V, E)$, that finite, have no loops, multiple and directed edges, and there is a path between any pair of its vertices. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The distance $d(u, v)$ between two vertices u and v of a graph G is the minimum length of the paths connecting them (i.e., the number of edges between them). A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $S \subseteq V(G)$, the subgraph $\langle S \rangle$ of G is called the subgraph induced by S if $E(\langle S \rangle) = \{uv \in E(G) | u, v \in S\}$. We refer to [3], for graph theory notation and terminology not described here.

A set D of vertices in a graph G is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. The concept of connected domination number was introduced by E. Sampathkumar and H. Walikar [7]. A dominating set D of a graph G is connected dominating set if a subgraph induced by D is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set in G . for more details in domination theory of graphs we refer to [5].

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Let G be a connected graph of order n and let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of vertices of G. For a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)).$$

where $d(v, u)$ represents the distance between the vertices v and u, is called the representation of v with respect to W. The set W is a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for every pair u, v of vertices of G. A resolving set of minimum cardinality is called a minimum resolving set or a basis of a graph G and the cardinality of a basis of G is its dimension and denoted by $dim(G)$. The concepts of resolving set and minimum resolving set have previously appeared in the literature in [9] and later in [10],, Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number of G. Slater described the usefulness of these ideas when working with U. S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [4], investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted.

C. Robert and et al. in [6], introduced the concept of resolving domination in graphs. A set D of vertices of a graph G that is both resolving and dominating is a resolving dominating set. The minimum cardinality of a resolving dominating set is called the resolving domination number $\gamma_r(G)$. Motivated by this paper, we introduce the concept of resolving connected domination number of graphs. We investigate the relationship between resolving connected domination number, connected domination number, resolving domination number and dimension of graphs. Exact values of $\gamma_{rc}(G)$ for some standard graphs are computed. Bounds for $\gamma_{rc}(G)$ of a graph are found.

Before we are starting in the main results of resolving connected domination, we consider the following useful results on dimension and resolving domination numbers of graphs.

Theorem 1.1([1, 4]) *Let G be a connected graph of order $n > 2$. Then*

(*a*) *$dim(G) = 1$ if and only if $G = P_n$;*

(*b*) *$dim(G) = n - 1$ if and only if $G = K_n$;*

(*c*) *For $n > 4$, $dim(G) = n - 2$ if and only if $G = K_{r,s}$, $(r, s > 1)$, $G = K_r + K_s$, $(r > 1; s > 2)$, or $G = K_r + (K_1 \cup K_s)$, $(r, s > 1)$.*

Theorem 1.1([8, 4]) (*a*) *For a cycle C_n , $n \geq 3$, $dim(C_n) = 2$;*

(*b*) *For $n \geq 3$, let $W_{1,n}$ be the wheel graph on $n + 1$ vertices. Then*

$$dim(W_{1,n}) = \begin{cases} 3, & \text{if } n = 3 \text{ or } n = 6; \\ \lfloor \frac{2n+2}{5} \rfloor, & \text{otherwise.} \end{cases}$$

The following definitions are stated in [1, 4].

Definition 1.3 *Fix a graph G. A vertex $v \in V(G)$ is called a major vertex if $d(v) \geq 3$. An* *end-vertex u is called a terminal vertex of a major vertex v if $d(u,v) < d(u,w)$ for every other major vertex w in G. The terminal degree of a major vertex v is the number of terminal vertices of v. A major vertex v is an exterior major vertex if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G, and let $ext(G)$ the number of exterior major vertices of G.*

Theorem 1.4([1]) *If T is a tree that is not a path, then $dim(T) = \sigma(T) - ext(T)$.*

Corollary 1.5([5]) *If T is a tree of order $n > 3$, then $\gamma_c(T) = n - l(T)$. Where $l(T)$ denote the number of end-vertex of T.*

Lemma 1.6([6]) *Let u and v be vertices of a connected graph G. If either*
(1) u and v are not adjacent and $N(u) = N(v)$. or
(2) u and v are adjacent and $N[u] = N[v]$, then every resolving set of G contains at least one of u and v.

Proposition 1.7([6]) *If G is a connected graph of order $n > 2$ and diameter d, then $\gamma_r \geq f(n,d)$, where*

$$f(n,d) = min\{k + \sum_{i=1}^{k} \binom{k}{i} (d-1)^{k-i}\}.$$

§2. **Main Results**

A connected graph G ordinarily contains many dominating sets. Indeed, every superset of a dominating set is also a dominating set. The same statement is true for a connected dominating sets, also for resolving sets. In this paper we study those connected dominating sets that are resolving sets as well. Such sets will be called resolving connected dominating sets. Thus a resolving connected dominating set D of vertices of G not only dominates all the vertices of G but has the added feature that the subgraph $\langle D \rangle$ induced by it is connected, also distinct vertices of G have distinct representations with respect to D. The cardinality of a minimum resolving connected dominating set is called the resolving connected domination number of G and is denoted by $\gamma_{rc}(G)$. A resolving dominating and resolving connected dominating sets of cardinality $\gamma_r(G)$ and $\gamma_{rc}(G)$, is called a $\gamma_r(G)$-set and $\gamma_{rc}(G)$-set, for G, respectively. To illustrate these concepts, consider the following graph G in Figure 1.

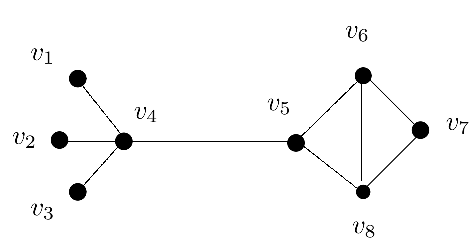

Figure 1. A graph with $\gamma = 2$, $\gamma_c = 3$, $\gamma_r = 4$, $\gamma_{rc} = 5$ and $dim = 3$.

By Lemma 1.6, every resolving set of G contains at least two vertices from set $W = \{v_1, v_2, v_3\}$. Since no 2-element subset of W is a resolving set, it follows that $dim(G) \geq 3$. On the other hand, the set $\{v_1, v_2, v_6\}$ is a resolving set for G, implying that $dim(G) = 3$. The set $\{v_4, v_6\}$ is a γ-set of G and so $\gamma(G) = 2$, the set $\{v_4, v_5, v_6\}$ is a γ_c-set of G so $\gamma_c(G) = 3$, the set $\{v_1, v_2, v_4, v_6\}$ is a γ_r-set of G so $\gamma_r(G) = 4$ and the set $\{v_1, v_2, v_4, v_5, v_6\}$ is a γ_{rc}-set of G and so $\gamma_{rc}(G) = 5$.

2.1 Exact Values of Resolving Connected Domination of Some Standard Graphs

In this section, we present The exact values of resolving connected domination numbers of some well-known classes of graphs as following:

Proposition 2.1

(1) $\gamma_{rc}(K_n) = \gamma_{rc}(K_{1,n}) = n - 1$, *for* $n \geq 2$*;*

(2) $\gamma_{rc}(P_n) = n - 2$, *for* $n \geq 4$*;*

(3) $\gamma_{rc}(C_n) = n - 2$, *for* $n \geq 3$*;*

(4) $\gamma_{rc}(K_{r,s}) = r + s - 2$, *for* $r, s \geq 2$*;*

(5) *For integers* $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ *with* $n_1 + n_2 + \cdots + n_k = n$ *and* $k \geq 2$, $\gamma_{rc}(K_{n_1,n2,\cdots,nk}) = n - k$.

Theorem 2.2 *For a wheel graph* $W_{1,n}$ *of order* $n \geq 7$

$$\gamma_{rc}(W_{1,n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1.$$

Proof In $W_{1,n} = K_1 + C_n$, $n \geq 7$, let $V(W_{1,n}) = \{v_0, v_1, v_2, \cdots, v_n\}$, where v_0 is a central vertex and $v_1, v_2, \cdots, v_n$ are vertices of C_n. Let R be a minimum resolving set of $W_{1,n}$. Since $d(v_0, v_i) = 1$ for all i with $1 \leq i \leq n$ it follows that v_0 does not belong to any minimum resolving set of $W_{1,n}$. Hence, $v_0 \notin R$. In other hand, the set $\{v_0\}$ is a γ-set of $W_{1,n}$ and it is a connected set so the set $\{v_0\}$ is also a γ_c-set of $W_{1,n}$. Thus, the set $D = \{v_0\} \cup R$ is a γ_r-set of $W_{1,n}$. Since the subgraph $\langle D \rangle$ is connected it follows that the set D is a γ_{rc}-set of $W_{1,n}$. Therefore, by this and Theorem 1.2 we get

$$|D| = |R \cup \{v_0\}| = |R| + |\{v_0\}| = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1.$$

And this completes the proof. □

Theorem 2.3 *Let* T *be a tree of order* $n \geq 4$*, that is not a path. If every major vertex of* T *adjacent to its terminal vertex, then*

$$\gamma_{rc}(T) = n - ext(T).$$

Proof Let W, S and D be a resolving set, a connected dominating set and a resolving connected dominating set of a tree T, respectively, with minimum cardinality. Since every superset of a resolving set is a resolving set and every superset of a connected dominating set is a dominating set it follows that $S \cup W$ is a resolving connected dominating set of T. Thus, $D \subseteq (S \cup W)$. it follows that

$$|D| \leq |S \cup W| \leq |S| + |W| \tag{1}$$

Conversely, Since a resolving connected dominating set is both a resolving set and a connected dominating set it follows that $W \subseteq D$ and $S \subseteq D$. Hence, $(W \cup S) \subseteq D$. Therefore, $|S| + |W| - |S \cap W| \leq |D|$. Now, let $L(T)$ be the set of all end-vertex of T. Then from Definition 1.3 we get $|L(T)| = l(T) = \sigma(T)$. From Theorem 1.4 and lemma 1.6 and since every major vertex of T adjacent to it terminal vertex, we conclude that a connected domination set S dose not containing any resolving set. Then $W \subset L(T)$. Since $L(T) \subseteq V(T) - S$ it follows that $S \cap W = \phi$. Hence,

$$|S \cup W| \leq |D|. \tag{2}$$

From equations (1) and (2) we have $|D| = |S| + |W|$. Therefore, by Theorem 1.4 and Corollary 1.5 we get

$$\begin{aligned}\gamma_{rc}(T) = |D| = |S| + |W| &= \gamma_c(T) + dim(T) \\ &= n - \sigma(T) + \sigma(T) - ext(T) = n - ext(T).\end{aligned}$$

□

Corollary 2.4 *Let T be a tree of order $n \geq 4$, that is not a path. If every major vertex of T adjacent to its terminal vertex, then*

$$\gamma_{rc}(T) = \gamma_c(T) + dim(T).$$

2.2 Bounds on Resolving Connected Domination Number

In this section we investigate with some bounds on resolving connected domination number of graphs.

Theorem 2.5 *For any connected graph of order $n \geq 2$, $\gamma_{rc}(G) \leq n - 1$. The bound is sharp, K_n and $K_{1,n}$ attainting this bound.*

Corollary 2.6 *For any connected graph G of order n and size m, $\gamma_{rc}(G) \leq m$.*

Theorem 2.7 *For any tree T of order $n \geq 4$, that is not a star,*

$$\gamma_{rc}(T) \leq n - 2.$$

Proof Let T be a tree of order $n \geq 4$, that is not a star, on contrary we suppose that $\gamma_{rc}(T) \geq n - 1$. if $T = P_n$ then by proposition 1.1 $\gamma_{rc}(T) = n - 2$, contradiction. Now, if $T \neq P_n$, then T has at least one vertex (say v) with $d(v) \geq 3$. Then v is a major vertex of T which is an exterior vertex. Consider the following cases.

Case 1. T has only one a vertex v as a major vertex. Since T not a star, it follows that there exists a vertex $u \in V(T)$ such that $d(v,u) \geq 2$. Without loss the generality, and for simplicity we consider T is a broom graph (see Figure 2).

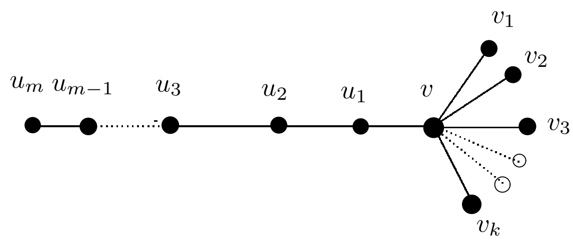

Figure 2. A broom graph

The set $S = V(T) - \{v_1, v_2, \cdots, v_k, u_m\}$ is a γ_c-set of T and a set $W = \{v_1, v_2, \cdots, v_{k-1}, u_1\}$ is a resolving set of T. Hence, a set $D = S \cup W$ is a γ_{rc}- set of T with minimum cardinality. Therefore, $\gamma_{rc}(T) = n - 2 \leq n - 1$, contradiction to hypothesis.

Case 2. T has at least two an exterior vertices, then $\gamma_{rc}(T) \leq n - ext(T) \leq n-1$, contradiction. Therefore, the theorem is true. □

Proposition 2.8 *For every connected graph, necessarily,*

$$dim(G) \leq \gamma_r(G) \leq \gamma_{rc}(G),$$
$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{rc}(G)$$

and

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_{rc}(G).$$

Theorem 2.9 *Let G be a connected graph of order and size $n,\ m$, respectively. Then $\gamma_{rc}(G) = m$ if and only if $G = K_{1,n}$.*

Proof If $G = K_{1,n}$, then $\gamma_{rc}(G) = n - 1 = m$.

Conversely, suppose that $\gamma_{rc}(G) = m$. Then by Theorem 2.5, $m \leq n - 1$. Since G is a connected it follows that $m = n-1$. Hence G must be a tree. If $n \leq 3$, it is clear that G is a star and the theorem is holding. Otherwise if $n \geq 4$, by Theorem 2.7 $\gamma_{rc}(T) \leq n - 2 < n - 1 = m$, contradiction. Therefore, must G be a star. □

Theorem 2.10 *Let G be a connected graph of order $n \geq 2$ such that the complement $\bar{G}$ of its is a connected. Then*

$$\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) \leq \frac{n^2 - n}{2}.$$

The equality is holding if and only if $G = k_{1,2}$.

Proof Let m and $m^{'}$ be the size of G and $\bar{G}$, respectively. By Corollary 2.6, we have

$$\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) \leq m + m^{'} = \frac{n^2 - n}{2}.$$

To prove the second part of theorem, let $G = K_{1,2}$. Then $\gamma_{rc}(G) = 2$ and $\gamma_{rc}(\bar{G}) = \gamma_{rc}(K_2) = 1$. Hence, $\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) = 3 = \frac{9-3}{2}$.
Conversely, if $\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) = \frac{n^2-n}{2}$, we should have $\gamma_{rc}(G) = m$ and $\gamma_{rc}(\bar{G}) = m'$. but this imply by Theorem 2.8, that G is a star. which requires that $n = 3$. So that $G \cong k_{1,2}$. This completes the proof. □

There are only finitely many connected graphs having a fixed resolving connected domination number. To verify this, we first, motivation by the lower bound of a resolving domination number in Proposition 1.7, establish a lower bound for a resolving connected domination number of a graph.

Theorem 2.11 *Let G be a connected graph of order $n \geq 2$ and diameter d. Then*

$$\gamma_{rc}(G) \geq f(n, d).$$

From Theorem 2.11 we have the following result.

Corollary 2.12 *Let G be a connected graph of order $n \geq 2$, diameter d and resolving connected domination number k. Then*

$$n \leq k + \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (d-1)^{k-i}.$$

Theorem 2.13 *For every positive integer k, there are only finitely many connected graphs G with resolving connected domination number k.*

Proof Let G be a connected graph of order $n \geq 2$ with $\gamma_{rc}(G) = k$. Since $\gamma_c(G) \leq \gamma_{rc}(G) = k$ it follows that the diameter of G is at most $k+1$. By Corollary 2.12 we get

$$n \leq k + \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} k^{k-i}.$$

Hence n is finite, and the result is follows. □

It is an immediate observation that the only nontrivial graph having resolving connected domination number 1 is K_2. It is clear form the previous theorem , the order of any connected graph G with resolving connected domination number 2 is at most 5. By Theorem 2.13, the order of any connected graph G with resolving connected domination number 3 is at most 40. In fact, we can improve upon this statement.

Theorem 2.14 *The order of every connected graph of order n with resolving connected domination number 3 is at most 12.*

Proof Let G be a connected graph with $\gamma_{rc}(G) = 3$ and let $D = \{v_1, v_2, v_3\}$ be a γ_{rc}-set for G. Since every vertex in $V(G) - D$ is adjacent to at least one vertex of D and has distance at most 3 from the other, the representations $(v|D)$ of a vertex v in $V(G) - D$ with respect to D is 3-vector, every coordinate of which is a positive integer not exceeding 3, at least one coordinate of which is 1. The only possible representations $(v|D)$ for every $v \in V(G) - D$ are $(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,2,3), (2,1,1), (2,1,2), (2,2,1)$ and $(3,2,1)$. Then the order of G at most 12. □

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To most men, experience is like the stern light of a ship which illuminates only the track it has passed.

By Samuel Tylor Coleridge, a British poet.

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[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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