



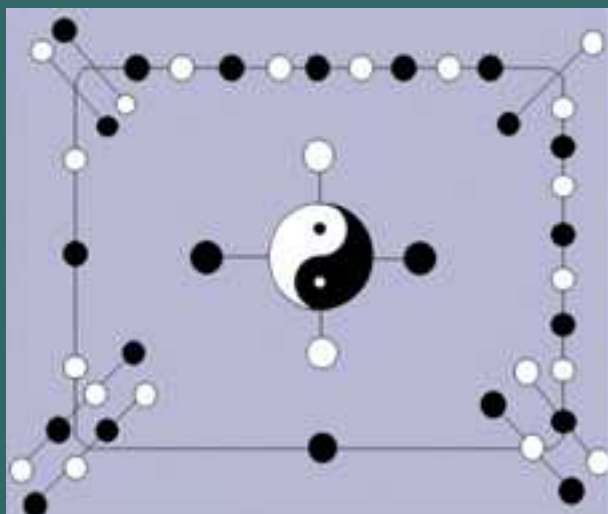
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# MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND  
BEIJING UNIVERSITY OF CIVIL ENGINEERING AND ARCHITECTURE

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**Famous Words:**

*Nature never deceives us, it is always us who deceive ourselves.*

By Jean-Jacques Rousseau, A French philosopher, writer and composer

## Spacelike Curves of Constant Breadth According to Bishop Frame in $E_1^3$

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**Abstract:** In this paper, we study a special case of Smarandache breadth curves, and give some characterizations of the space-like curves of constant breadth according to Bishop frame in Minkowski 3-space.

**Key Words:** Minkowski 3-space, Smarandache breadth curves, curves of constant breadth, Bishop frame.

**AMS(2010):** 53A05, 53B25, 53B30

### §1. Introduction

Curves of constant breadth were introduced by Euler in [3]. Fujivara presented a problem to determine whether there exist space curves of constant breadth or not, and defined the concept of breadth for space curves and also obtained these curves on a surface of constant breadth in [5]. Some geometric properties of curves of constant breadth were given in a plane by [8]. The similar properties were obtained in Euclidean 3-space  $E^3$  in [9]. These kind curves were studied in four dimensional Euclidean space  $E^4$  in [1].

In this paper, we study a special case of Smarandache breadth curves in Minkowski 3-space  $E_1^3$ . A Smarandache curve is a regular curve with 2 breadths or more than 2 breadths in Minkowski 3-space  $E_1^3$ . Also we investigate position vectors of simple closed space-like curves and give some characterizations of curves of constant breadth according to Bishop frame of type-1 in  $E_1^3$ . Thus, we extend this classical topic to the space  $E_1^3$ , which is related to the time-like curves of constant breadth in  $E_1^3$ , see [10] for details. We also use a method which is similar to one in [9].

### §2. Preliminaries

The Minkowski 3-space  $E_1^3$  is an Euclidean 3-space  $E^3$  provided with the standard flat metric

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given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where  $(x_1, x_2, x_3)$  is a rectangular coordinate system of  $E_1^3$ . Since  $\langle , \rangle$  is an indefinite metric recall that a vector  $v \in E_1^3$  can be one of three Lorentzian characters; it can be space-like if  $\langle v, v \rangle > 0$  or  $v = 0$ , time-like if  $\langle v, v \rangle < 0$  and null if  $\langle v, v \rangle = 0$  and  $v \neq 0$ . Similarly, an arbitrary curve  $\varphi = \varphi(s)$  in  $E_1^3$  can locally be space-like, time-like or null (light-like) if all of its velocity vector  $\varphi'$  is respectively space-like, time-like or null (light-like) for every  $s \in J \in \mathbb{R}$ . The pseudo-norm of an arbitrary vector  $a \in E_1^3$  is given by  $\|a\| = \sqrt{|\langle a, a \rangle|}$ . The curve  $\varphi$  is called a unit speed curve if its velocity vector  $\varphi'$  satisfies  $\|\varphi'\| = \mp 1$ . For any vectors  $u, w \in E_1^3$ , they are said to be orthogonal if and only if  $\langle u, w \rangle = 0$ .

Denote by  $\{T, N, B\}$  the moving Frenet frame along curve  $\varphi$  in the space  $E_1^3$ . Let  $\varphi$  be a space-like curve with a space-like binormal in the space  $E_1^3$ , as similar to in [11], the Frenet formulae are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2.1)$$

where  $\kappa$  and  $\tau$  are the first and second curvatures with

$$\begin{aligned} \langle T, T \rangle &= \langle B, B \rangle = 1, \langle N, N \rangle = -1, \\ \langle T, N \rangle &= \langle T, B \rangle = \langle N, B \rangle = 0. \end{aligned}$$

The construction of the Bishop frame is due to L.R.Bishop in [4]. This frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even the space-like curve with a space-like binormal has vanishing second derivative [2]. He used tangent vector and any convenient arbitrary basis for the remainder of the frame. Then, as similar to in [2], the Bishop frame is expressed as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \quad (2.2)$$

and

$$\kappa(s) = \sqrt{|k_1^2 - k_2^2|}, \quad \tau(s) = \frac{d\theta}{ds}, \quad \theta(s) = \tanh^{-1} \frac{k_2}{k_1} \quad (2.3)$$

where  $k_1$  and  $k_2$  are Bishop curvatures.

### §3. Spacelike Curves of Constant Breadth According to Bishop Frame in $E_1^3$

Let  $\vec{\varphi} = \vec{\varphi}(s)$  and  $\vec{\varphi}^* = \vec{\varphi}^*(s)$  be simple closed curves of constant breadth in Minkowski 3-space. These curves will be denoted by  $C$  and  $C^*$ . The normal plane at every point  $P$  on the curve meets the curve in the class  $\Gamma$  having parallel tangents  $\vec{T}$  and  $\vec{T}^*$  in opposite directions

at the opposite points  $\varphi$  and  $\varphi^*$  of the curve as in [5]. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to Bishop frame by the equation

$$\varphi^*(s) = \varphi(s) + m_1 T + m_2 N_1 + m_3 N_2 \quad (3.1)$$

where  $m_i(s)$ ,  $1 \leq i \leq 3$  are arbitrary functions, also  $\varphi$  and  $\varphi^*$  are opposite points. Differentiating (3.1) and considering Bishop equations, we have

$$\begin{aligned} \frac{d\varphi^*}{ds} = \vec{T}^* \frac{ds^*}{ds} = & \left( \frac{dm_1}{ds} + m_2 k_1 + m_3 k_2 + 1 \right) T \\ & + \left( \frac{dm_2}{ds} + m_1 k_1 \right) N_1 + \left( \frac{dm_3}{ds} - m_1 k_2 \right) N_2. \end{aligned} \quad (3.2)$$

Since  $T^* = -T$ , rewriting (3.2), we obtain the following equations

$$\begin{cases} \frac{dm_1}{ds} = -m_2 k_1 - m_3 k_2 - 1 - \frac{ds^*}{ds} \\ \frac{dm_2}{ds} = -m_1 k_1 \\ \frac{dm_3}{ds} = m_1 k_2. \end{cases} \quad (3.3)$$

If we call  $\theta$  as the angle between the tangent of the curve ( $C$ ) at point  $\varphi(s)$  with a given direction and consider  $\frac{d\theta}{ds} = \tau$ , we can rewrite (3.3) as follow;

$$\begin{cases} \frac{dm_1}{d\theta} = -m_2 \frac{k_1}{\tau} - m_3 \frac{k_2}{\tau} - f(\theta) \\ \frac{dm_2}{d\theta} = -m_1 \frac{k_1}{\tau} \\ \frac{dm_3}{d\theta} = m_1 \frac{k_2}{\tau} \end{cases} \quad (3.4)$$

where

$$f(\theta) = \delta + \delta^* \quad \delta = \frac{1}{\tau}, \delta^* = \frac{1}{\tau^*} \quad (3.5)$$

denote the radius of curvature at the points  $\varphi$  and  $\varphi^*$ , respectively. And using the system (3.4), we have the following differential equation with respect to  $m_1$  as

$$\begin{aligned} \frac{d^3 m_1}{d\theta^3} - \left( \frac{\kappa}{\tau} \right)^2 \frac{dm_1}{d\theta} + \left[ \frac{k_2}{\tau} \frac{d}{d\theta} \left( \frac{k_2}{\tau} \right) - \frac{d}{d\theta} \left( \frac{\kappa}{\tau} \right)^2 - \frac{\kappa}{\tau} \frac{d}{d\theta} \left( \frac{k_2}{\tau} \right) \right] m_1 \\ + \left( \int_0^\theta m_1 \frac{k_2}{\tau} d\theta \right) \frac{d^2}{d\theta^2} \left( \frac{k_2}{\tau} \right) - \left( \int_0^\theta m_1 \frac{k_1}{\tau} d\theta \right) \frac{d^2}{d\theta^2} \left( \frac{k_1}{\tau} \right) + \frac{d^2 f}{d\theta^2} = 0. \end{aligned} \quad (3.6)$$

The equation (3.6) is characterization of the point  $\varphi^*$ . If the distance between opposite



points of  $(C)$  and  $(C^*)$  is constant, then we write that

$$\|\varphi^* - \varphi\| = -m_1^2 + m_2^2 + m_3^2 = l^2 \text{ is constant.} \quad (3.7)$$

Hence, from (3.7) we obtain

$$-m_1 \frac{dm_1}{d\theta} + m_2 \frac{dm_2}{d\theta} + m_3 \frac{dm_3}{d\theta} = 0 \quad (3.8)$$

Considering system (3.4), we get

$$m_1 \left[ 2m_3 \frac{k_2}{\tau} + f(\theta) \right] = 0 \quad (3.9)$$

From (3.9), we study the following cases which are depended on the conditions  $2m_3 \frac{k_2}{\tau} + f(\theta) = 0$  or  $m_1 = 0$ .

**Case 1.** If  $2m_3 \frac{k_2}{\tau} + f(\theta) = 0$  then by using (3.4), we obtain

$$\frac{dm_1}{d\theta} - \left( \int_0^\theta m_1 \frac{k_1}{\tau} d\theta \right) \frac{k_1}{\tau} + \frac{f(\theta)}{\tau} = 0. \quad (3.10)$$

Now let us to investigate solution of the equation (3.6) and suppose that  $m_2, m_3$  and  $f(\theta)$  are constants,  $m_1 \neq 0$ , then using (3.4) in (3.6), we have the following differential equation

$$\frac{d^3 m_1}{d\theta^3} - \left( \frac{\kappa}{\tau} \right)^2 \frac{dm_1}{d\theta} - \frac{d}{d\theta} \left( \frac{\kappa}{\tau} \right)^2 m_1 = 0. \quad (3.11)$$

The general solution of (3.11) depends on the character of the ratio  $\frac{\kappa}{\tau}$ . Suppose that  $\varphi$  is not constant breadth. For this reason, we distinguish the following sub-cases.

**Subcase 1.1** Suppose that  $\varphi$  is an inclined curves then the solution of the differential equation (3.11) is

$$m_1 = c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{\frac{\kappa}{\tau}\theta}. \quad (3.12)$$

Therefore, we have  $m_2$  and  $m_3$ , respectively,

$$\begin{aligned} m_2 &= - \int_0^\theta \left( c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{\frac{\kappa}{\tau}\theta} \right) \frac{k_1}{\tau} d\theta \\ m_3 &= \int_0^\theta \left( c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{\frac{\kappa}{\tau}\theta} \right) \frac{k_2}{\tau} d\theta \end{aligned} \quad (3.13)$$

where  $c_1, c_2$  and  $c_3$  are real numbers.

**Subcase 1.2** Suppose that  $\varphi$  is a line. The solution is in the following form

$$m_1 = A_1 \frac{\theta^2}{2} + A_2 \theta + A_3. \quad (3.14)$$

Hence, we have  $m_2$  and  $m_3$  as follows

$$\begin{aligned} m_2 &= -\int_0^\theta (A_1 \frac{\theta^2}{2} + A_2 \theta + A_3) \frac{k_1}{\tau} d\theta \\ m_3 &= \int_0^\theta (A_1 \frac{\theta^2}{2} + A_2 \theta + A_3) \frac{k_2}{\tau} d\theta \end{aligned} \quad (3.15)$$

where  $A_1$ ,  $A_2$  and  $A_3$  are real numbers.

**Case 2.** If  $m_1 = 0$ , then  $m_2 = M_2$  and  $m_3 = M_3$  are constants. Let us suppose that  $m_2 = m_3 = c$  (constant). Thus, the equation (3.4) is obtained as

$$f(\theta) = \frac{-c(k_1 + k_2)}{\tau}.$$

This means that the curve is a circle. Moreover, the equation (3.6) has the form

$$\frac{d^2 f}{d\theta^2} = 0. \quad (3.16)$$

The solution of (3.16) is

$$f(\theta) = l_1 \theta + l_2 \quad (3.17)$$

where  $l_1$  and  $l_2$  are real numbers. Therefore, we write the position vector  $\varphi^*$  as follows

$$\varphi^* = \varphi + M_2 N_1 + M_3 N_2 \quad (3.18)$$

where  $M_2$  and  $M_3$  are real numbers.

Finally, the distance between the opposite points of the curves  $(C)$  and  $(C^*)$  is

$$\|\varphi^* - \varphi\| = M_2^2 + M_3^2 = \text{constant}. \quad (3.19)$$

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## Study Map of Orthotomic of a Circle

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**Abstract:** In this paper, we calculate and discuss the Study map of an spherical orthotomic of a circle which lies on the dual unit sphere in  $D$ -module. In order to do this we use matrix equation of Study mapping. Finally we give some special cases each of which is a geometric result.

**Key Words:** Study mapping,  $D$ -module, orthotomic, spherical orthotomic.

**AMS(2010):** 53A04, 53A25

### §1. Introduction

In linear algebra, dual numbers are defined by W. K. Clifford (1873) by using real numbers. Dual numbers are extended the real numbers by adjoining one new element  $\varepsilon$  with the property  $\varepsilon^2 = 0$ . Dual numbers have the form  $x + \varepsilon y$ , where  $x, y \in R$ . The dual numbers set is two-dimensional commutative unital associative algebra over the real numbers. Its first application was made by E. Study. He used dual numbers and dual vectors in his research on the geometry of lines and kinematics [15]. He devoted special attention to the representation of directed line by dual unit vectors and defined the mapping which is called with his name. There exists one to one correspondence between dual unit points of dual unit sphere and the directed lines of the Euclidean line space  $E^3$ .

Let  $\alpha$  be a regular curve and  $\vec{T}$  be its tangent, and  $u$  be a source. Orthotomic of  $\alpha$  with respect to the source ( $u$ ) is the locus of reflection of  $u$  about the tangents  $\vec{T}$  [7]. Bruce and Giblin studied the unfolding theory to the evolutes and orthotomics of plane and space curves [3], [4] and [5]. Georgiou, Hasanis and Koutroufiotis investigated the orthotomics in Euclidean  $(n + 1)$ -space  $E^{n+1}$  [6]. Alamo and Criado studied the antiorthotomics in Euclidean  $(n+1)$ -space  $E^{n+1}$  [1]. Xiong defined the spherical orthotomic and the spherical antiorthotomic [16]. In this paper we examine the Study Map of the spherical orthotomic of a circle which lies on the dual unit sphere in  $D$ -Module.

### §2. Preliminaries

If  $a$  and  $a^*$  are real numbers and  $\varepsilon^2 = 0$  but  $\varepsilon \notin R$ , a dual number can be written as  $A = a + \varepsilon a^*$ ,

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where  $\varepsilon = (0, 1)$  is the dual unit.

The set  $D = \{A = a + \varepsilon a^* \mid a, a^* \in R\}$  of dual numbers is a commutative ring over the real number field and is denoted by  $D$ . Then the set

$$D^3 = \left\{ \vec{A} = (A_1, A_2, A_3) \mid A_i \in D, 1 \leq i \leq 3 \right\}$$

is a module over the ring  $D$  which is called a  $D$ -Module, under the addition and the scalar multiplication on the set  $D$  ([12]). The elements of  $D^3$  are called dual vectors. Thus a dual vector has the form  $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$ , where  $\vec{a}$  and  $\vec{a}^*$  are real vectors at  $R^3$ . Then, for any vectors  $\vec{A}$  and  $\vec{B}$  in  $D^3$ , the inner product and the vector product of these vectors are defined as

$$\langle \vec{A}, \vec{B} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon (\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle)$$

and

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon (\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b}),$$

respectively. The norm  $\|\vec{A}\|$  of  $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$  is defined as

$$\|\vec{A}\| = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}, \quad \vec{a} \neq 0.$$

The dual vector  $\vec{A}$  with norm 1 is called a dual unit vector. The set of dual unit vectors

$$S^2 = \left\{ \vec{A} = \vec{a} + \varepsilon \vec{a}^* \in D^3 \mid \|\vec{A}\| = 1; \vec{a}, \vec{a}^* \in R^3 \right\}$$

is called the dual unit sphere.

Now, we give the definition of spherical normal, spherical tangent and spherical orthotomic of a spherical curve  $\alpha$ .  $\{\vec{T}, \vec{N}, \vec{B}\}$  be the Frenet frame of  $\alpha$ . The spherical normal of  $\alpha$  is the great circle which passing through  $\alpha(s)$  and normal to  $\alpha$  at  $\alpha(s)$  and is given by

$$\begin{cases} \langle \vec{x}, \vec{x} \rangle = 1 \\ \langle \vec{x}, \vec{T} \rangle = 0 \end{cases}$$

where  $x$  is an arbitrary point of spherical normal. The spherical tangent of  $\alpha$  is the great circle which tangent to  $\alpha$  at  $\alpha(s)$  and is given by

$$\begin{cases} \langle \vec{y}, \vec{y} \rangle = 1 \\ \langle \vec{y}, (\vec{a} \wedge \vec{T}) \rangle = 0 \end{cases}$$

where  $y$  is an arbitrary point of spherical tangent.

Let  $u \in S^2$  be a source. Xiong defined the spherical orthotomic of  $\alpha$  relative to  $u$  to be ([17]) the set of reflections of  $u$  about the planes whose lie on the above great circles for all  $s \in I$  and given by

$$\vec{\tilde{u}} = 2 \langle (\vec{a} - \vec{u}), \vec{v} \rangle \vec{v} + \vec{u}$$

where  $\vec{v} = \frac{\vec{B} - \langle \vec{B}, \vec{\alpha} \rangle \vec{\alpha}}{\|\vec{B} - \langle \vec{B}, \vec{\alpha} \rangle \vec{\alpha}\|}.$

### §3. Study Mapping

**Definition 3.1** *The Study mapping is an one to one mapping between the dual points of a dual unit sphere, in D-Module, and the oriented lines in the Euclidean line space  $E^3$ .*

Let  $K, O$  and  $\{O; \vec{E}_1, \vec{E}_2, \vec{E}_3\}$  denote the unit dual sphere, the center of  $K$  and dual orthonormal system at  $O$ , respectively where

$$\vec{E}_i = \vec{e}_i + \varepsilon \vec{e}_i^*; 1 \leq i \leq 3. \quad (3.1)$$

Let  $S_3$  be the group of all the permutations of the set  $\{1, 2, 3\}$ , then it can be written as

$$\left\{ \begin{array}{l} \vec{E}_{\sigma(1)} = \text{sgn}(\sigma) \vec{E}_{\sigma(2)} \wedge \vec{E}_{\sigma(3)}, \text{sgn}(\sigma) = \pm 1, \\ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} \end{array} \right. \quad (3.2)$$

In the case that the orthonormal system

$$\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

is the system of the line space  $E^3$ . We can write the moment vectors  $\vec{e}_i^*$  as

$$\vec{e}_i^* = \overrightarrow{MO} \wedge \vec{e}_i, 1 \leq i \leq 3. \quad (3.3)$$

Since these moment vectors are the vectors of  $R^3$ , we may write that

$$\vec{e}_i^* = \sum_{j=1}^3 \lambda_{ij} \vec{e}_j, \lambda_{ij} \in R, 1 \leq i \leq 3. \quad (3.4)$$

Hence (3.3) and (3.4) give us

$$\lambda_{ii} = 0, \lambda_{ij} = -\lambda_{ji}, 1 \leq i, j \leq 3$$

and so the scalars  $\lambda_{ij}$  are denoted by  $\lambda_i$ , that is,

$$\lambda_{ij} = \lambda_i.$$

Then (3.4) reduces to

$$\begin{bmatrix} \vec{e}_1^* \\ \vec{e}_2^* \\ \vec{e}_3^* \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & -\lambda_3 \\ -\lambda_1 & 0 & \lambda_2 \\ \lambda_3 & -\lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}. \quad (3.5)$$

Hence the Study mapping

$$K \rightarrow E^3$$

can be given as a mapping from the dual orthonormal system to the real orthonormal system. By using the relations (3.1) and (3.5), we can express Study mapping in the matrix form as follows:

$$\begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1\varepsilon & -\lambda_3\varepsilon \\ -\lambda_1\varepsilon & 1 & \lambda_2\varepsilon \\ \lambda_3\varepsilon & -\lambda_2\varepsilon & 1 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \quad (3.6)$$

which says that Study mapping corresponds with a dual orthogonal matrix. Since we know [14] that the linear mappings are in one to one correspondence with the matrices we may give the following theorem.

**Theorem 3.2** *A Study mapping is a linear isomorphism.*

Since the Euclidean motions in  $E^3$  leave do not change the angle and the distance between two lines, the corresponding mapping in  $D$ -Module leave the inner product invariant.

This is the action of an orthogonal matrix with dual coefficients. Since the center of the dual unit sphere  $K$  must remain fixed the transformation group in  $D$ -module (the image of the Euclidean motions) does not contain any translations. Hence, in order to represents the Euclidean motions in  $D$ -Module we can apply the following theorem [8].

**Theorem 3.3** *The Euclidean motions in  $E^3$  are in one to one correspondence with the dual orthogonal matrices.*

**Definition 3.4** *A ruled surface is a surface that can be swept out by moving a line in space. This line is the generator of surface.*

A differentiable curve

$$t \in R \rightarrow \vec{X}(t) \in K$$

on the dual unit sphere  $K$ , depending on a real parameter  $t$ , represents differentiable family of straight lines of which is ruled surface ([2], [8]). The lines  $\vec{X}(t)$  are the generators of the surface.

Let  $X, Y$  be two different points of  $K$  and  $\Phi$  be the dual angle  $(\vec{OX}, \vec{OY})$ . The dual angle  $\Phi$  has a value  $\varphi + \varepsilon\varphi^*$  which is a dual number, where  $\varphi$  and  $\varphi^*$  are the angle and the minimal distance between the two lines  $\vec{X}$  and  $\vec{Y}$ , respectively. Then we have the following theorem ([8]).

**Theorem 3.5** Let  $\vec{X}, \vec{Y} \in K$ . Then we have

$$\langle \vec{X}, \vec{Y} \rangle = \cos \Phi,$$

where

$$\cos \Phi = \cos \varphi - \varepsilon \varphi^* \sin \varphi. \quad (3.7)$$

The following special cases of Theorem 3.5 are important ([9]):

$$\langle \vec{X}, \vec{Y} \rangle = 0 \implies \varphi = \frac{\pi}{2} \text{ and } \varphi^* = 0; \quad (3.8)$$

meaning that the lines  $\vec{X}$  and  $\vec{Y}$  meet at a right angle.

$$\langle \vec{X}, \vec{Y} \rangle = \text{pure dual} \implies \varphi = \frac{\pi}{2} \text{ and } \varphi^* \neq 0; \quad (3.9)$$

meaning that the lines  $\vec{X}$  and  $\vec{Y}$  are orthogonal skew lines.

$$\langle \vec{X}, \vec{Y} \rangle = \text{pure real} \implies \varphi \neq \frac{\pi}{2} \text{ and } \varphi^* = 0; \quad (3.10)$$

this means that the lines  $\vec{X}$  and  $\vec{Y}$  intersect each other.

$$\langle \vec{X}, \vec{Y} \rangle = \pm 1 \implies \varphi^* = 0 \text{ and } \varphi = 0 (\text{or } \varphi = \pi); \quad (3.11)$$

meaning that the lines  $\vec{X}$  and  $\vec{Y}$  are coincide (their senses are either same or opposite).

#### §4. The Study Map of a Circle

Let  $g$  be the straight line corresponding to the unit dual vector  $\vec{E}_3$ . If we choose the point  $M$  on  $g$  then we have

$$\lambda_2 = \lambda_3 = 0$$

and so the matrix, from (3.6) reduces to

$$\begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \varepsilon & 0 \\ -\lambda_1 \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \quad (4.1)$$

The inverse of this mapping is

$$\begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda_1 \varepsilon & 0 \\ \lambda_1 \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix} \quad (4.2)$$



Let

$$S = \{\vec{X} \mid \langle \vec{X}, \vec{E}_3 \rangle = \cos \Phi = \text{constant}, \vec{X} \in K\}$$

be the circle on the unit dual sphere  $K$ . The spherical orthotomic of the great circle which lies on the plane spanned by  $\{\vec{E}_1, \vec{E}_2\}$  relative to  $S$  is a reflection of  $S$  about this plane and for  $\vec{X} = (X_1, X_2, X_3) \in S$ , it is given by  $\vec{\tilde{X}} = (X_1, X_2, -X_3)$ . Thus the dual vector  $\vec{\tilde{X}}$  can be expressed as

$$\vec{\tilde{X}} = \sin \Phi \cos \Psi \vec{E}_1 + \sin \Phi \sin \Psi \vec{E}_2 - \cos \Phi \vec{E}_3 \quad (4.3)$$

where  $\Phi = \varphi + \varepsilon\varphi^*$  and  $\Psi = \psi + \varepsilon\psi^*$  are the dual angles. Since we have the relations

$$\begin{cases} \vec{\tilde{X}} = \vec{\tilde{x}} + \varepsilon \vec{\tilde{x}}^* \\ \sin \Phi = \sin \varphi + \varepsilon\varphi^* \cos \varphi, \quad \sin \Psi = \sin \psi + \varepsilon\psi^* \cos \psi \\ \cos \Phi = \cos \varphi - \varepsilon\varphi^* \sin \varphi, \quad \cos \Psi = \cos \psi - \varepsilon\psi^* \sin \psi \end{cases}$$

These equations (4.1) and (4.3) give us the vectors  $\vec{\tilde{x}}$  and  $\vec{\tilde{x}}^*$  in the matrix form:

$$\begin{cases} \vec{\tilde{x}} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} \sin \varphi \cos \psi \\ \sin \varphi \sin \psi \\ -\cos \varphi \end{bmatrix} \\ \vec{\tilde{x}}^* = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} \varphi^* \cos \varphi \cos \psi - (\psi^* + \lambda_1) \sin \varphi \sin \psi \\ \varphi^* \cos \varphi \sin \psi + (\psi^* + \lambda_1) \sin \varphi \cos \psi \\ \varphi^* \sin \varphi \end{bmatrix} \end{cases} \quad (4.4)$$

On the other hand, the point  $\vec{X}$  is on the circle with center on the axis  $\vec{E}_3$ . As  $\vec{X}$  is spherical orthotomic of  $\vec{\tilde{X}}$ ,  $\vec{X}$  is on the circle which is reflected about the plane spanned by  $\{\vec{E}_1, \vec{E}_2\}$ . Thus we may write

$$\langle \vec{\tilde{X}}, \vec{E}_3 \rangle = \cos \Phi = \cos \varphi - \varepsilon\varphi^* \sin \varphi = \text{constant} \quad (4.5)$$

which means that  $\varphi = c_1(\text{constant})$  and  $\varphi^* = c_2(\text{constant})$ .

The equation (4.4) and (4.5) let us to write the following relations:

$$\begin{cases} \langle \vec{\tilde{x}}, \vec{\tilde{x}} \rangle = 1 \\ \langle \vec{\tilde{x}}, \vec{\tilde{x}}^* \rangle = 0 \\ \langle \vec{\tilde{x}}, \vec{e}_3 \rangle - \cos \varphi = 0 \\ \langle \vec{\tilde{x}}, \vec{e}_3^* \rangle + \langle \vec{\tilde{x}}^*, \vec{e}_3 \rangle + \varphi^* \sin \varphi = 0 \end{cases} \quad (4.6)$$

The equations (4.6) have only two parameter  $\psi$  and  $\psi^*$  so (4.6) represents a line congruence in  $R^3$ . This congruence is called spherical orthotomic congruence.

Now we may calculate the equations of this spherical orthotomic congruence in Plucker coordinates. Let  $\vec{y}$  be a point of the spherical orthotomic congruence then we have ([12]).

$$\vec{y} = \vec{x}(\psi, \psi^*) \wedge \vec{x}^*(\psi, \psi^*) + v \vec{x}(\psi, \psi^*). \quad (4.7)$$

If the coordinates of  $\vec{y}$  are  $(y_1, y_2, y_3)$  then (4.7) give us

$$\begin{cases} y_1 = \varphi^* \sin \psi + (\psi^* + \lambda_1) \sin \varphi \cos \varphi \cos \psi + v \sin \varphi \cos \psi \\ y_2 = -\varphi^* \cos \psi + (\psi^* + \lambda_1) \sin \varphi \cos \varphi \sin \psi + v \sin \varphi \sin \psi \\ y_3 = (\psi^* + \lambda_1) \sin^2 \varphi - v \cos \varphi \end{cases} \quad (4.8)$$

In this case that  $\varphi \neq \frac{\pi}{2}$  (4.8) give us

$$\frac{y_1^2}{c_2^2} + \frac{y_2^2}{c_2^2} - \frac{[y_3 - (\psi^* + \lambda_1)]^2}{[c_2 \cot c_1]^2} = 1 \quad (4.9)$$

which has two parameters  $\psi^*$  and  $\lambda_1$  so it represents a line congruence with degree two. The lines of this congruence are located so that

- a) The shortest distance of these lines and the line  $g$  is  $\varphi^* = c_2$ ;
- b) The angle of these lines and the line  $g$  is  $\varphi = c_1$ .

Thus, it can be seen that the lines of spherical orthotomic congruence intersect the generators of a cylinder whose radius is  $\varphi^* = \text{constant}$ , and the axis is  $g$ , under the angle  $\varphi = \text{constant}$ .

**Definition 4.1** *If all the lines of a line congruence have a constant angle with a definite line then the congruence is called an inclined congruence.*

According to this definition, (4.9) represents an inclined congruence. Then, we have the following theorem.

**Theorem 4.2** *Let  $S$  be a circle with two parameter on the unit dual sphere  $K$  The Study map of orthotomic of  $S$  is an inclined congruence with degree two.*

In other respect, we know that the shortest distance between the axis  $g$  of the cylinder and the lines of the spherical orthotomic congruence is  $c_2$ . Therefore, this cylinder is the envelope of the lines of the spherical orthotomic congruence. So, we have the following theorem.

**Theorem 4.3** *Let  $K$  be a unit dual sphere and*

$$S = \{ \vec{X} \mid \langle \vec{X}, \vec{G} \rangle = \cos(\varphi + \varepsilon \varphi^*) = \text{constant}, \vec{X} \in K, \vec{G} \in K \}$$

*be a circle on  $K$ . Let  $\zeta$  and  $g$  be the Study maps of spherical orthotomic of  $S$  and  $G$ , respectively. Then the lines of  $\zeta$  has an envelope which is a circular cylinder whose axis is  $g$  and radius is  $c_2$ .*

In the case that  $\psi^* = -\lambda_1$ ,  $\varphi \neq 0$  and  $\varphi^* \neq 0$  (4.9) reduces to

$$\frac{y_1^2}{c_1^2} + \frac{y_2^2}{c_1^2} - \frac{y_3^2}{k^2} = 1, k = c_2 \cot c_1 = \text{constant}, c_1 = \varphi, c_2 = \varphi^* \quad (4.10)$$

which represents an hyperboloid of one sheet.

Since  $\psi^*$  and  $\lambda_1$  are two independent parameters, it can be said that the Study map of spherical orthotomic of  $S$  is, in general, a family of hyperboloids of one sheet with two parameters. Therefore we can give the following theorem.

**Theorem 4.4** *Let  $S$  be a circle on the unit dual sphere  $K$ . Then the Study map of spherical orthotomic of  $S$  is a family of hyperboloid of one sheet with two parameters.*

#### 4.1 The Case that $\varphi^* \neq 0$ and $\varphi = \frac{\pi}{2}$

In this case the lines of the spherical orthotomic congruence (4.9) orthogonally intersect the generators of the cylinder whose axis is  $g$  and the radius is  $\varphi^*$ . Since (4.8) reduces to

$$\begin{cases} y_1 = \varphi^* \sin \psi + v \cos \psi \\ y_2 = -\varphi^* \cos \psi + v \sin \psi \\ y_3 = \psi^* + \lambda_1 \end{cases} \quad (4.11)$$

Then (4.9) becomes

$$\begin{cases} y_1^2 + y_2^2 = c_2^2 + v^2 \\ y_3 = \psi^* + \lambda_1 \end{cases} \quad (4.12)$$

#### 4.2 The Case that $\varphi^* \neq 0$ and $\varphi = 0$ (or $\varphi = \pi$ )

In this case the lines of the spherical orthotomic congruence  $\zeta$  coincide with the generators of the cylinder which is the envelope of the lines of  $\varphi$ . This means noting but the Study map of spherical orthotomic of  $S$  reduces to cylinder whose equations, from (4.8), are

$$\begin{cases} y_1^2 + y_2^2 = c_2^2 \\ y_3 = -v \end{cases}$$

#### 4.3 The Case $\varphi^* = 0$ and $\varphi = 0$ (or $\varphi = \pi$ )

In this case all of the lines of the spherical orthotomic congruence  $\zeta$  are coincided with the line  $g$ . Indeed, in this case, (4.8) reduces to

$$\begin{cases} y_1^2 + y_2^2 = 0 \\ y_3 = -v \end{cases}$$

which represents the line  $g$ .

#### 4.4 The Case $\varphi^* = 0$ and $\varphi \neq 0$

In this case, all of the lines of  $\zeta$  intersect the axis  $g$  under the constant angle  $\varphi$ . So, we can say that the lines of the spherical orthotomic congruence  $\zeta$  are the common lines of two linear line complexes [13]. From (4.8), the equations of  $\zeta$  give us that

$$y_1^2 + y_2^2 - \frac{[y_3 - (\psi^* + \lambda_1)]^2}{[\cot c_1]^2} = 0.$$

#### 4.5 The Case that $\varphi^* = 0$ and $\varphi = \frac{\pi}{2}$

In this case  $S$  is a great circle on  $K$ . Then all of the lines of  $\zeta$  orthogonally intersect the axis  $g$ . This means that the spherical orthotomic inclined congruence reduces to a linear line complex whose axis is  $g$ . Then (4.8) gives us that the equation of  $\zeta$  as

$$\begin{cases} y_1 = v \cos \psi \\ y_2 = v \sin \psi \\ y_3 = \lambda_1 + \psi^* \end{cases}$$

or

$$\begin{cases} y_1^2 + y_2^2 = v^2 \\ y_3 = \lambda_1 + \psi^* \end{cases}$$

**Definition 4.5** *If all the lines of a line congruence orthogonally intersect a constant line then the congruence is called a recticongruence.*

Therefore we can give the following theorem.

**Theorem 4.6** *Let  $S$  be a great circle on  $K$ , that is,*

$$S = \{ \vec{X} \mid \langle \vec{X}, \vec{G} \rangle = 0, \vec{X}, \vec{G} \in K \}.$$

*Then the Study map  $\zeta$  of orthotomic of  $S$  is a recticongruence.*

In the case that  $\lambda_1 = c_3\psi$  and (4.11) reduces to

$$\begin{cases} y_1 = v \cos \psi \\ y_2 = v \sin \psi \\ y_3 = c_3\psi \end{cases}$$

or

$$y_3 = c_3 \arctan \frac{y_2}{y_1}$$

which represents a right helicoid

Since  $\lambda_1$  is a parameter, we can choose it as  $\lambda_1 = c_3\psi$  and so under the corresponding

mapping the image of spherical orthotomic congruence reduce to a right helicoid. Hence we can give the following theorem.

**Theorem 4.8** *It is possible to choose the Study mapping such that the Study maps of spherical orthotomic of dual circles are right helicoids.*

For the spherical orthotomic of great circle which lies on  $Sp \{ \vec{E}_1, \vec{E}_3 \}$ , we have the following theorem.

**Theorem 4.9** *The Study map of spherical orthotomic of  $S$  is given by*

$$\frac{y_1^2}{c_2^2} + \frac{y_3^2}{c_2^2} - \frac{[y_2 - (\psi^* + \lambda_3)]^2}{[c_2 \cot c_1]^2} = 1$$

*which has two parameters, so it represents a line congruence with degree two.*

For a plane spanned by  $\{ \vec{E}_2, \vec{E}_3 \}$ , we obtain the following theorem.

**Theorem 4.10** *The Study map of spherical orthotomic of  $S$  is given by*

$$\frac{y_2^2}{c_2^2} + \frac{y_3^2}{c_2^2} - \frac{[y_1 - (\psi^* + \lambda_2)]^2}{[c_2 \cot c_1]^2} = 1$$

*which has two parameters, so it represents a line congruence with degree two.*

By using the above two theorems, one way modify the study of this paper with choosing the plane spanned by  $\{ \vec{E}_1, \vec{E}_3 \}$  or  $\{ \vec{E}_2, \vec{E}_3 \}$ .

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## Geometry on Non-Solvable Equations

– A Review on Contradictory Systems

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**Abstract:** As we known, an objective thing not moves with one's volition, which implies that all contradictions, particularly, in these semiotic systems for things are artificial. In classical view, a contradictory system is meaningless, contrast to that of geometry on figures of things caught by eyes of human beings. The main objective of sciences is holding the global behavior of things, which needs one knowing both of compatible and contradictory systems on things. Usually, a mathematical system including contradictions is said to be a *Smarandache system*. Beginning from a famous fable, i.e., the 6 blind men with an elephant, this report shows the geometry on contradictory systems, including non-solvable algebraic linear or homogenous equations, non-solvable ordinary differential equations and non-solvable partial differential equations, classify such systems and characterize their global behaviors by combinatorial geometry, particularly, the global stability of non-solvable differential equations. Applications of such systems to other sciences, such as those of gravitational fields, ecologically industrial systems can be also found in this report. All of these discussions show that a non-solvable system is nothing else but a system underlying a topological graph  $G \not\cong K_n$ , or  $\simeq K_n$  without common intersection, contrast to those of solvable systems underlying  $K_n$  being with common non-empty intersections, where  $n$  is the number of equations in this system. However, if we stand on a geometrical viewpoint, they are compatible and both of them are meaningful for human beings.

**Key Words:** Smarandache system, non-solvable system of equations, topological graph,  $G^L$ -solution, global stability, ecologically industrial systems, gravitational field, mathematical combinatorics.

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### §1. Introduction

A *contradiction* is a difference between two statements, beliefs, or ideas about something that can not both be true, exists everywhere and usually with a presentation as argument, debate, disputing,  $\dots$ , etc., even break out a war sometimes. Among them, a widely known contradiction in philosophy happened in a famous fable, i.e., the 6 blind men with an elephant following.

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Fig.1

In this fable, there are 6 blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1. Each of them insisted on his own and not accepted others. They then entered into an endless argument. *All of you are right!* A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.* Thus, the best result on an elephant for these blind men is

$$\begin{aligned} \text{An elephant} = & \{4 \text{ pillars}\} \cup \{1 \text{ rope}\} \cup \{1 \text{ tree branch}\} \\ & \cup \{2 \text{ hand fans}\} \cup \{1 \text{ wall}\} \cup \{1 \text{ solid pipe}\}, \end{aligned}$$

i.e., a *Smarandache multi-space* ([23]-[25]) defined following.

**Definition 1.1**([12]-[13]) *Let  $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$  be  $m$  mathematical systems, different two by two. A Smarandache multi-system  $\tilde{\Sigma}$  is a union  $\bigcup_{i=1}^m \Sigma_i$  with rules  $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$  on  $\tilde{\Sigma}$ , denoted by  $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ .*

Then, what is the philosophical meaning of this fable for one understanding the world? In fact, the situation for one realizing behaviors of things is analogous to the blind men determining what an elephant looks like. Thus, this fable means the limitation or unilateral of one's knowledge, i.e., *science* because of all of those are just correspondent with the sensory cognition of human beings.

Besides, we know that contradiction exists everywhere by this fable, which comes from the limitation of unilateral sensory cognition, i.e., artificial contradiction of human beings, and all scientific conclusions are nothing else but an approximation for things. For example, let  $\mu_1, \mu_2, \dots, \mu_n$  be known and  $\nu_i, i \geq 1$  unknown characters at time  $t$  for a thing  $T$ . Then, the



thing  $T$  should be understood by

$$T = \left( \bigcup_{i=1}^n \{\mu_i\} \right) \cup \left( \bigcup_{k \geq 1} \{\nu_k\} \right)$$

in logic but with an approximation  $T^\circ = \bigcup_{i=1}^n \{\mu_i\}$  for  $T$  by human being at time  $t$ . Even for  $T^\circ$ , these are maybe contradictions in characters  $\mu_1, \mu_2, \dots, \mu_n$  with endless argument between researchers, such as those implied in the fable of 6 blind men with an elephant. Consequently, if one stands still on systems without contradictions, he will never hold the real face of things in the world, particularly, the true essence of geometry for limited of his time.

However, all things are inherently related, not isolated in philosophy, i.e., underlying an invariant topological structure  $G$  ([4],[22]). Thus, one needs to characterize those things on contradictory systems, particularly, by geometry. The main objective of this report is to discuss the geometry on contradictory systems, including non-solvable algebraic equations, non-solvable ordinary or partial differential equations, classify such systems and characterize their global behaviors by combinatorial geometry, particularly, the global stability of non-solvable differential equations. For terminologies and notations not mentioned here, we follow references [11], [13] for topological graphs, [3]-[4] for topology, [12],[23]-[25] for Smarandache multi-spaces and [2],[26] for partial or ordinary differential equations.

## §2. Geometry on Non-Solvable Equations

Loosely speaking, a geometry is mainly concerned with shape, size, position,  $\dots$  etc., i.e., local or global characters of a figure in space. Its mainly objective is to hold the global behavior of things. However, things are always complex, even hybrid with other things. So it is difficult to know its global characters, or true face of a thing sometimes.

Let us beginning with two systems of linear equations in 2 variables:

$$(LES_4^S) \begin{cases} x + 2y = 4 \\ 2x + y = 5 \\ x - 2y = 0 \\ 2x - y = 3 \end{cases} \quad (LES_4^N) \begin{cases} x + 2y = 2 \\ x + 2y = -2 \\ 2x - y = -2 \\ 2x - y = 2 \end{cases}$$

Clearly,  $(LES_4^S)$  is solvable with a solution  $x = 2$  and  $y = 1$ , but  $(LES_4^N)$  is not because  $x + 2y = -2$  is contradictory to  $x + 2y = 2$ , and so that for equations  $2x - y = -2$  and  $2x - y = 2$ . Thus,  $(LES_4^N)$  is a contradiction system, i.e., a Smarandache system defined following.

**Definition 2.1**([11]-[13]) *A rule in a mathematical system  $(\Sigma; \mathcal{R})$  is said to be Smarandachely denied if it behaves in at least two different ways within the same set  $\Sigma$ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways.*

*A Smarandache system  $(\Sigma; \mathcal{R})$  is a mathematical system which has at least one Smarandachely denied rule in  $\mathcal{R}$ .*

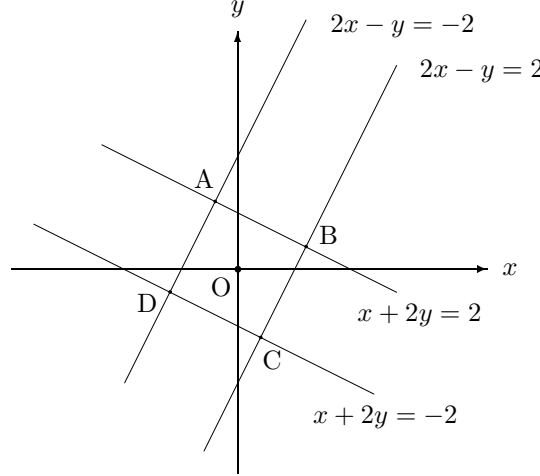
$$S_{f_i} = \{(x_1, x_2, \dots, x_{n+1}) | f_i(x_1, x_2, \dots, x_{n+1}) = 0\} \subset \mathbb{R}^{n+1}$$
[illegible]
$$\bigcap_{i=1}^m S_{f_i} \neq \emptyset \quad \text{or} \quad = \emptyset.$$
$$\mathcal{D}_i = \{(x_1, x_2, \dots, x_{n+1}) | f_k^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0, 1 \leq k \leq m_i\} = \bigcap_{k=1}^{m_i} S_{f_k^{[i]}}.$$
$$\left. \begin{array}{l} f_1^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0 \\ \dots\dots\dots \\ f_{m_i}^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0 \end{array} \right\} 1 \leq i \leq m$$
$$\bigcap_{i=1}^m \mathcal{D}_i \neq \emptyset \quad \text{or} \quad = \emptyset.$$

**Theorem 2.2** *If a geometrical space  $\mathcal{D}$  consists of  $m$  parts  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$ , where,  $\mathcal{D}_i = \{(x_1, x_2, \dots, x_{n+1}) | f_k^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0, 1 \leq k \leq m_i\}$ , then the system  $(ES_m)$  consisting of*

$$\left. \begin{array}{l} f_1^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0 \\ \dots\dots\dots \\ f_{m_i}^{[i]}(x_1, x_2, \dots, x_{n+1}) = 0 \end{array} \right\} 1 \leq i \leq m$$

is non-solvable if  $\bigcap_{i=1}^m \mathcal{D}_i = \emptyset$ .

Now, *whether is it meaningless for a contradiction system in the world?* Certainly not! As we discussed in the last section, a contradiction is artificial if such a system indeed exists in the world. The objective for human beings is not just finding contradictions, but holds behaviors of such systems. For example, although the system  $(LES_4^N)$  is contradictory, but it really exists, i.e., 4 lines in  $\mathbb{R}^2$ , such as those shown in Fig.2.



**Fig.2**

Generally, let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq)$$

be a linear equation system with

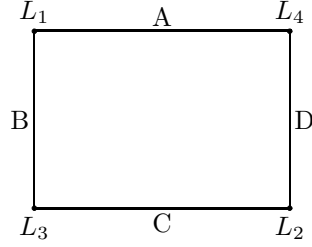
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers  $m, n \geq 1$ . A vertex-edge labeled graph  $G^L[LEq]$  on such a system is defined by:

$V(G^L[LEq]) = \{P_1, P_2, \dots, P_m\}$ , where  $P_i = \{(x_1, x_2, \dots, x_n) | a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i\}$ ,  $E(G^L[LEq]) = \{(P_i, P_j), P_i \cap P_j \neq \emptyset, 1 \leq i, j \leq m\}$  and labeled with  $L : P_i \rightarrow P_i$ ,  $L : (P_i, P_j) \rightarrow P_i \cap P_j$  for integers  $1 \leq i, j \leq m$  with an underlying graph  $\widehat{G}[LEq]$  without labels.

For example, let  $L_1 = \{(x, y) | x + 2y = 2\}$ ,  $L_2 = \{(x, y) | x + 2y = -2\}$ ,  $L_3 = \{(x, y) | 2x - y = 2\}$  and  $L_4 = \{(x, y) | 2x - y = -2\}$  for the system  $(LES_4^N)$ . Clearly,  $L_1 \cap L_2 = \emptyset$ ,  $L_1 \cap L_3 = \{B\}$ ,  $L_1 \cap L_4 = \{A\}$ ,  $L_2 \cap L_3 = \{C\}$ ,  $L_2 \cap L_4 = \{D\}$  and  $L_3 \cap L_4 = \emptyset$ . Then, the system  $(LES_4^N)$  can also appears as a vertex-edge labeled graph  $C_4^l$  in  $\mathbb{R}^2$  with labels vertex labeling

$l(L_i) = L_i$  for integers  $1 \leq i \leq 4$ , edge labeling  $l(L_1, L_3) = B$ ,  $l(L_1, L_4) = A$ ,  $l(L_2, L_3) = C$  and  $l(L_2, L_4) = D$ , such as those shown in Fig.3.



**Fig.3**

We are easily to determine  $\widehat{G}[LEq]$  for systems  $(LEq)$ . For integers  $1 \leq i, j \leq m$ ,  $i \neq j$ , two linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n &= b_j \end{aligned}$$

are called *parallel* if there exists a constant  $c$  such that

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

Otherwise, *non-parallel*. The following result is known in [16].

**Theorem 2.3**([16]) *Let  $(LEq)$  be a linear equation system for integers  $m, n \geq 1$ . Then  $\widehat{G}[LEq] \simeq K_{n_1, n_2, \dots, n_s}$  with  $n_1 + n_2 + \cdots + n_s = m$ , where  $\mathcal{C}_i$  is the parallel family by the property that all equations in a family  $\mathcal{C}_i$  are parallel and there are no other equations parallel to lines in  $\mathcal{C}_i$  for integers  $1 \leq i \leq s$ ,  $n_i = |\mathcal{C}_i|$  for integers  $1 \leq i \leq s$  in  $(LEq)$  and  $(LEq)$  is non-solvable if  $s \geq 2$ .*

Particularly, for linear equation system on 2 variables, let  $H$  be a planar graph with edges straight segments on  $\mathbb{R}^2$ . The *c-line graph*  $L_C(H)$  on  $H$  is defined by

$$\begin{aligned} V(L_C(H)) &= \{\text{straight lines } L = e_1 e_2 \cdots e_l, l \geq 1 \text{ in } H\}; \\ E(L_C(H)) &= \{(L_1, L_2) \mid L_1 = e_1^1 e_2^1 \cdots e_l^1, L_2 = e_1^2 e_2^2 \cdots e_s^2, l, s \geq 1 \\ &\quad \text{and there adjacent edges } e_i^1, e_j^2 \text{ in } H, 1 \leq i \leq l, 1 \leq j \leq s\}. \end{aligned}$$

Then, a simple criterion in [16] following is interesting.

**Theorem 2.4**([16]) *A linear equation system  $(LEq2)$  on 2 variables is non-solvable if and only if  $\widehat{G}[LEq2] \simeq L_C(H)$ , where  $H$  is a planar graph of order  $|H| \geq 2$  on  $\mathbb{R}^2$  with each edge a straight segment*

Generally, a Smarandache multi-system is equivalent to a combinatorial system by following, which implies the *CC Conjecture* for mathematics, i.e., *any mathematics can be recon-*

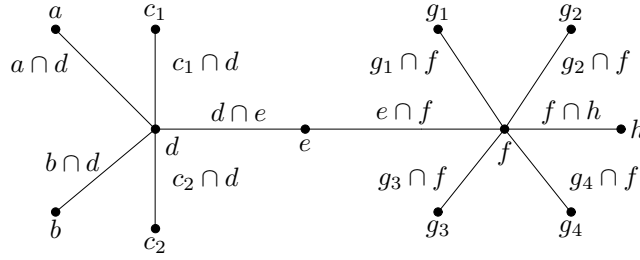
structed from or turned into combinatorization (see [6] for details).

**Definition 2.5**([11]-[13]) For any integer  $m \geq 1$ , let  $(\tilde{\Sigma}; \tilde{\mathcal{R}})$  be a Smarandache multi-system consisting of  $m$  mathematical systems  $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ . An inherited topological structure  $G^L[\tilde{S}]$  of  $(\tilde{\Sigma}; \tilde{\mathcal{R}})$  is a topological vertex-edge labeled graph defined following:

$$\begin{aligned} V(G^L[\tilde{S}]) &= \{\Sigma_1, \Sigma_2, \dots, \Sigma_m\}, \\ E(G^L[\tilde{S}]) &= \{(\Sigma_i, \Sigma_j) \mid \Sigma_i \cap \Sigma_j \neq \emptyset, 1 \leq i \neq j \leq m\} \text{ with labeling} \\ L: \Sigma_i &\rightarrow L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L: (\Sigma_i, \Sigma_j) \rightarrow L(\Sigma_i, \Sigma_j) = \Sigma_i \cap \Sigma_j \end{aligned}$$

for integers  $1 \leq i \neq j \leq m$ .

Therefore, a Smarandache system is equivalent to a combinatorial system, i.e.,  $(\tilde{\Sigma}; \tilde{\mathcal{R}}) \simeq G^L[\tilde{S}]$ , a labeled graph  $\widehat{G}^L[\tilde{S}]$  by this notion. For examples, denoting by  $a = \{\text{tusk}\} b = \{\text{nose}\} c_1, c_2 = \{\text{ear}\} d = \{\text{head}\} e = \{\text{neck}\} f = \{\text{trunk}\} g_1, g_2, g_3, g_4 = \{\text{leg}\} h = \{\text{tail}\}$  for an elephant then a topological structure for an elephant is shown in Fig.4 following.



**Fig.4** Topological structure of an elephant

For geometry, let these mathematical systems  $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$  be geometrical spaces, for instance manifolds  $M_1, M_2, \dots, M_m$  with respective dimensions  $n_1, n_2, \dots, n_m$  in Definition 2.3, we get a geometrical space  $\widetilde{M} = \bigcup_{i=1}^m M_i$  underlying a topological graph  $G^L[\widetilde{M}]$ . Such a geometrical space  $G^L[\widetilde{M}]$  is said to be *combinatorial manifold*, denoted by  $\widetilde{M}(n_1, n_2, \dots, n_m)$ . Particularly, if  $n_i = n, 1 \leq i \leq m$ , then a combinatorial manifold  $\widetilde{M}(n_1, \dots, n_m)$  is nothing else but an  $n$ -manifold underlying  $G^L[\widetilde{M}]$ . However, this presentation of  $G^L$ -systems contributes to manifolds and combinatorial manifolds (See [7]-[15] for details). For example, the fundamental groups of manifolds are characterized in [14]-[15] following.

**Theorem 2.6**([14]) For any locally compact  $n$ -manifold  $M$ , there always exists an inherent graph  $G_{min}^{in}[M]$  of  $M$  such that  $\pi(M) \cong \pi(G_{min}^{in}[M])$ .

Particularly, for an integer  $n \geq 2$  a compact  $n$ -manifold  $M$  is simply-connected if and only if  $G_{min}^{in}[M]$  is a finite tree.

**Theorem 2.7**([15]) Let  $\widetilde{M}$  be a finitely combinatorial manifold. If for  $\forall (M_1, M_2) \in E(G^L[\widetilde{M}])$ ,

$M_1 \cap M_2$  is simply-connected, then

$$\pi_1(\widetilde{M}) \cong \left( \bigoplus_{M \in V(G[\widetilde{M}])} \pi_1(M) \right) \bigoplus \pi_1(G[\widetilde{M}]).$$

Furthermore, it provides one with a listing of manifolds by graphs in [14].

**Theorem 2.8**([14]) *Let  $\mathcal{A}[M] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda \}$  be a atlas of a locally compact  $n$ -manifold  $M$ . Then the labeled graph  $G_{|\Lambda|}^L$  of  $M$  is a topological invariant on  $|\Lambda|$ , i.e., if  $H_{|\Lambda|}^{L_1}$  and  $G_{|\Lambda|}^{L_2}$  are two labeled  $n$ -dimensional graphs of  $M$ , then there exists a self-homeomorphism  $h : M \rightarrow M$  such that  $h : H_{|\Lambda|}^{L_1} \rightarrow G_{|\Lambda|}^{L_2}$  naturally induces an isomorphism of graph.*

For a combinatorial surface consisting of surfaces associated with homogenous polynomials in  $\mathbb{R}^3$ , we can further determine its genus. Let

$$P_1(\overline{x}), P_2(\overline{x}), \dots, P_m(\overline{x}) \quad (ES_m^{n+1})$$

be  $m$  homogeneous polynomials in variables  $x_1, x_2, \dots, x_{n+1}$  with coefficients in  $\mathbb{C}$  and

$$\emptyset \neq S_{P_i} = \{(x_1, x_2, \dots, x_{n+1}) \mid P_i(\overline{x}) = 0\} \subset \mathbb{P}^n \mathbb{C}$$

for integers  $1 \leq i \leq m$ , which are hypersurfaces, particularly, curves if  $n = 2$  passing through the original of  $\mathbb{C}^{n+1}$ .

Similarly, parallel hypersurfaces in  $\mathbb{C}^{n+1}$  are defined following.

**Definition 2.9** *Let  $P(\overline{x}), Q(\overline{x})$  be two complex homogenous polynomials of degree  $d$  in  $n + 1$  variables and  $I(P, Q)$  the set of intersection points of  $P(\overline{x})$  with  $Q(\overline{x})$ . They are said to be parallel, denoted by  $P \parallel Q$  if  $d > 1$  and there are constants  $a, b, \dots, c$  (not all zero) such that for  $\forall \overline{x} \in I(P, Q)$ ,  $ax_1 + bx_2 + \dots + cx_{n+1} = 0$ , i.e., all intersections of  $P(\overline{x})$  with  $Q(\overline{x})$  appear at a hyperplane on  $\mathbb{P}^n \mathbb{C}$ , or  $d = 1$  with all intersections at the infinite  $x_{n+1} = 0$ . Otherwise,  $P(\overline{x})$  are not parallel to  $Q(\overline{x})$ , denoted by  $P \not\parallel Q$ .*

Then, these polynomials in  $(ES_m^{n+1})$  can be classified into families  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$  by this parallel property such that  $P_i \parallel P_j$  if  $P_i, P_j \in \mathcal{C}_k$  for an integer  $1 \leq k \leq l$ , where  $1 \leq i \neq j \leq m$  and it is *maximal* if each  $\mathcal{C}_i$  is maximal for integers  $1 \leq i \leq l$ , i.e., for  $\forall P \in \{P_k(\overline{x}), 1 \leq k \leq m\} \setminus \mathcal{C}_i$ , there is a polynomial  $Q(\overline{x}) \in \mathcal{C}_i$  such that  $P \not\parallel Q$ . The following result is a generalization of Theorem 2.3.

**Theorem 2.10**([19]) *Let  $n \geq 2$  be an integer. For a system  $(ES_m^{n+1})$  of homogenous polynomials with a parallel maximal classification  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$ ,*

$$\widehat{G}[ES_m^{n+1}] \leq K(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l)$$

*and with equality holds if and only if  $P_i \parallel P_j$  and  $P_s \not\parallel P_i$  implies that  $P_s \not\parallel P_j$ , where*

$K(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l)$  denotes a complete  $l$ -partite graphs. Conversely, for any subgraph  $G \leq K(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l)$ , there are systems  $(ES_m^{n+1})$  of homogenous polynomials with a parallel maximal classification  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$  such that

$$G \simeq \widehat{G}[ES_m^{n+1}].$$

Particularly, if all polynomials in  $(ES_m^{n+1})$  be degree 1, i.e., hyperplanes with a parallel maximal classification  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$ , then

$$\widehat{G}[ES_m^{n+1}] = K(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l).$$

The following result is immediately known by definition.

**Theorem 2.11** Let  $(ES_m^{n+1})$  be a  $G^L$ -system consisting of homogenous polynomials  $P(\bar{x}_1), P(\bar{x}_2), \dots, P(\bar{x}_m)$  in  $n+1$  variables with respectively hypersurfaces  $S_{P_i}, 1 \leq i \leq m$ . Then,  $\widetilde{M} = \bigcup_{i=1}^m S_{P_i}$  is an  $n$ -manifold underlying graph  $\widehat{G}[ES_m^{n+1}]$  in  $\mathbb{C}^{n+1}$ .

For  $n = 2$ , we can further determine the genus of surface  $\widetilde{M}$  in  $\mathbb{R}^3$  following.

**Theorem 2.12** ([19]) Let  $\widetilde{S}$  be a combinatorial surface consisting of  $m$  orientable surfaces  $S_1, S_2, \dots, S_m$  underlying a topological graph  $G^L[\widetilde{S}]$  in  $\mathbb{R}^3$ . Then

$$g(\widetilde{S}) = \beta(\widehat{G} \langle \widetilde{S} \rangle) + \sum_{i=1}^m (-1)^{i+1} \sum_{\bigcap_{l=1}^i S_{k_l} \neq \emptyset} \left[ g \left( \bigcap_{l=1}^i S_{k_l} \right) - c \left( \bigcap_{l=1}^i S_{k_l} \right) + 1 \right],$$

where  $g \left( \bigcap_{l=1}^i S_{k_l} \right)$ ,  $c \left( \bigcap_{l=1}^i S_{k_l} \right)$  are respectively the genus and number of path-connected components in surface  $S_{k_1} \cap S_{k_2} \cap \dots \cap S_{k_i}$  and  $\beta(\widehat{G} \langle \widetilde{S} \rangle)$  denotes the Betti number of topological graph  $\widehat{G} \langle \widetilde{S} \rangle$ .

Notice that for a curve  $C$  determined by homogenous polynomial  $P(x, y, z)$  of degree  $d$  in  $\mathbb{P}^2\mathbf{C}$ , there is a compact connected Riemann surface  $S$  by the Noether's result such that

$$h : S - h^{-1}(\text{Sing}(C)) \rightarrow C - \text{Sing}(C)$$

is a homeomorphism with genus

$$g(S) = \frac{1}{2}(d-1)(d-2) - \sum_{p \in \text{Sing}(C)} \delta(p),$$

where  $\delta(p)$  is a positive integer associated with the singular point  $p$  in  $C$ . Furthermore, if  $\text{Sing}(C) = \emptyset$ , i.e.,  $C$  is non-singular then there is a compact connected Riemann surface  $S$  homeomorphism to  $C$  with genus  $\frac{1}{2}(d-1)(d-2)$ . By Theorem 2.12, we obtain the genus of  $\widetilde{S}$

determined by homogenous polynomials following.

**Theorem 2.13**([19]) *Let  $C_1, C_2, \dots, C_m$  be complex curves determined by homogenous polynomials  $P_1(x, y, z), P_2(x, y, z), \dots, P_m(x, y, z)$  without common component, and let*

$$R_{P_i, P_j} = \prod_{k=1}^{\deg(P_i)\deg(P_j)} (c_k^{ij}z - b_k^{ij}y)^{e_k^{ij}}, \quad \omega_{i,j} = \sum_{k=1}^{\deg(P_i)\deg(P_j)} \sum_{e_k^{ij} \neq 0} 1$$

*be the resultant of  $P_i(x, y, z), P_j(x, y, z)$  for  $1 \leq i \neq j \leq m$ . Then there is an orientable surface  $\tilde{S}$  in  $\mathbb{R}^3$  of genus*

$$\begin{aligned} g(\tilde{S}) = & \beta(\widehat{G} \langle \tilde{C} \rangle) + \sum_{i=1}^m \left( \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2} - \sum_{p^i \in \text{Sing}(C_i)} \delta(p^i) \right) \\ & + \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} (-1)^i \sum_{C_{k_1} \cap \dots \cap C_{k_i} \neq \emptyset} \left[ c(C_{k_1} \cap \dots \cap C_{k_i}) - 1 \right] \end{aligned}$$

*with a homeomorphism  $\varphi : \tilde{S} \rightarrow \tilde{C} = \bigcup_{i=1}^m C_i$ . Furthermore, if  $C_1, C_2, \dots, C_m$  are non-singular, then*

$$\begin{aligned} g(\tilde{S}) = & \beta(\widehat{G} \langle \tilde{C} \rangle) + \sum_{i=1}^m \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2} \\ & + \sum_{1 \leq i \neq j \leq m} (\omega_{i,j} - 1) + \sum_{i \geq 3} (-1)^i \sum_{C_{k_1} \cap \dots \cap C_{k_i} \neq \emptyset} \left[ c(C_{k_1} \cap \dots \cap C_{k_i}) - 1 \right], \end{aligned}$$

*where*

$$\delta(p^i) = \frac{1}{2} \left( I_{p^i} \left( P_i, \frac{\partial P_i}{\partial y} \right) - \nu_\phi(p^i) + |\pi^{-1}(p^i)| \right)$$

*is a positive integer with a ramification index  $\nu_\phi(p^i)$  for  $p^i \in \text{Sing}(C_i), 1 \leq i \leq m$ .*

Notice that  $\widehat{G}[ES_m^3] = K_m$ . We then easily get conclusions following.

**Corollary 2.14** *Let  $C_1, C_2, \dots, C_m$  be complex non-singular curves determined by homogenous polynomials  $P_1(x, y, z), P_2(x, y, z), \dots, P_m(x, y, z)$  without common component, any intersection point  $p \in I(P_i, P_j)$  with multiplicity 1 and*

$$\begin{cases} P_i(x, y, z) = 0 \\ P_j(x, y, z) = 0, \quad \forall i, j, k \in \{1, 2, \dots, m\} \\ P_k(x, y, z) = 0 \end{cases}$$

*has zero-solution only. Then the genus of normalization  $\tilde{S}$  of curves  $C_1, C_2, \dots, C_m$  is*

$$g(\tilde{S}) = 1 + \frac{1}{2} \times \sum_{i=1}^m \deg(P_i) (\deg(P_i) - 3) + \sum_{1 \leq i \neq j \leq m} \deg(P_i) \deg(P_j).$$





[illegible]

**Definition 3.1** *A non-solvable  $(PDES_m)$  is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

**Theorem 3.2**([18],[21]) *A Cauchy problem on systems*

$$\left\{ \begin{array}{l} F_1(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0 \\ F_2(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0 \\ \vdots \\ F_m(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0 \end{array} \right.$$

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \dots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \dots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \dots, s_{n-1}), \quad i = 1, 2, \dots, n \end{cases}$$
$$F_k(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0, \quad 1 \leq k \leq m$$
$$F_{k_0}(x_1^0, x_2^0, \dots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \dots, p_n^0) \neq 0$$
$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

**Corollary 3.3** *A Cauchy problem  $(LPDES_m^C)$  on quasilinear, particularly, linear system of*

partial differential equations with initial values  $u|_{x_n=x_n^0} = u_0$  is non-solvable if and only if the system  $(LPDES_m)$  of partial differential equations is algebraically contradictory. Particularly, the Cauchy problem on a quasilinear partial differential equation is always solvable.

Similarly, for integers  $m, n \geq 1$ , let

$$\dot{X} = A_1 X, \dots, \dot{X} = A_k X, \dots, \dot{X} = A_m X \quad (LDES_m^1)$$

be a linear ordinary differential equation system of first order and

$$\begin{cases} x^{(n)} + a_{11}^{[0]} x^{(n-1)} + \dots + a_{1n}^{[0]} x = 0 \\ x^{(n)} + a_{21}^{[0]} x^{(n-1)} + \dots + a_{2n}^{[0]} x = 0 \\ \dots\dots\dots \\ x^{(n)} + a_{m1}^{[0]} x^{(n-1)} + \dots + a_{mn}^{[0]} x = 0 \end{cases} \quad (LDE_m^n)$$

a linear differential equation system of order  $n$  with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \dots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \dots & a_{2n}^{[k]} \\ \dots & \dots & \dots & \dots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \dots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{bmatrix}$$

where each  $a_{ij}^{[k]}$  is a real number for integers  $0 \leq k \leq m, 1 \leq i, j \leq n$ . Then it is known a criterion from [16] following.

**Theorem 3.4**([17]) *A differential equation system  $(LDES_m^1)$  is non-solvable if and only if*

$$(|A_1 - \lambda I_{n \times n}|, |A_2 - \lambda I_{n \times n}|, \dots, |A_m - \lambda I_{n \times n}|) = 1.$$

*Similarly, the differential equation system  $(LDE_m^n)$  is non-solvable if and only if*

$$(P_1(\lambda), P_2(\lambda), \dots, P_m(\lambda)) = 1,$$

where  $P_i(\lambda) = \lambda^n + a_{i1}^{[0]} \lambda^{n-1} + \dots + a_{i(n-1)}^{[0]} \lambda + a_{in}^{[0]}$  for integers  $1 \leq i \leq m$ . Particularly,  $(LDES_1^1)$  and  $(LDE_1^n)$  are always solvable.

According to Theorems 3.3 and 3.4, for systems  $(LPDES_m^C)$ ,  $(LDES_m^1)$  or  $(LDE_m^n)$ , there are equivalent systems  $G^L[LPDES_m^C]$ ,  $G^L[LDES_m^1]$  or  $G^L[LDE_m^n]$  by Definition 2.5, called  $G^L[LPDES_m^C]$ -solution,  $G^L[LDES_m^1]$ -solution or  $G^L[LDE_m^n]$ -solution of systems  $(LPDES_m^C)$ ,  $(LDES_m^1)$  or  $(LDE_m^n)$ , respectively. Then, we know the following conclusion from [17]-[18], [21].

**Theorem 3.5**([17]-[18],[21]) *The Cauchy problem on system  $(PDES_m)$  of partial differential equations of first order with initial values  $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}, 1 \leq i \leq n$  for the  $k$ th equation in*

( $PDES_m$ ),  $1 \leq k \leq m$  such that

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^n p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0,$$

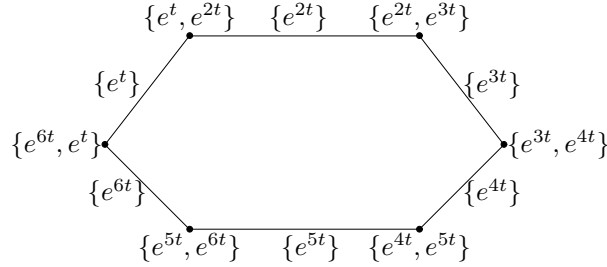
and the linear homogeneous differential equation system ( $LDES_m^1$ ) (or ( $LDE_m^n$ )) both are uniquely  $G^L$ -solvable, i.e.,  $G^L[PDES]$ ,  $G^L[LDES_m^1]$  and  $G^L[LDE_m^n]$  are uniquely determined.

For ordinary differential systems ( $LDES_m^1$ ) or ( $LDE_m^n$ ), we can further replace solution-manifolds  $S^{[k]}$  of the  $k$ th equation in  $G^L[LDES_m^1]$  and  $G^L[LDE_m^n]$  by their solution basis  $\mathcal{B}^{[k]} = \{ \bar{\beta}_i^{[k]}(t)e^{\alpha_i^{[k]}t} \mid 1 \leq i \leq n \}$  or  $\mathcal{C}^{[k]} = \{ t^l e^{\lambda_i^{[k]}t} \mid 1 \leq i \leq s, 1 \leq l \leq k_i \}$  because each solution-manifold of ( $LDES_m^1$ ) (or ( $LDE_m^n$ )) is a linear space.

For example, let a system ( $LDE_m^n$ ) be

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where  $\ddot{x} = \frac{d^2x}{dt^2}$  and  $\dot{x} = \frac{dx}{dt}$ . Then the solution basis of equations (1) – (6) are respectively  $\{e^t, e^{2t}\}$ ,  $\{e^{2t}, e^{3t}\}$ ,  $\{e^{3t}, e^{4t}\}$ ,  $\{e^{4t}, e^{5t}\}$ ,  $\{e^{5t}, e^{6t}\}$ ,  $\{e^{6t}, e^t\}$  with its  $G^L[LDE_m^n]$  shown in Fig.5.



**Fig.5**

Such a labeling can be simplified to labeling by integers for combinatorially classifying systems  $G^L[LDES_m^1]$  and  $G^L[LDE_m^n]$ , i.e., *integral graphs* following.

**Definition 3.6** Let  $G$  be a simple graph. A vertex-edge labeled graph  $\theta : G \rightarrow \mathbb{Z}^+$  is called *integral* if  $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$  for  $\forall uv \in E(G)$ , denoted by  $G^{I_\theta}$ .

For two integral labeled graphs  $G_1^{I_\theta}$  and  $G_2^{I_\tau}$ , they are called *identical* if  $G_1 \stackrel{\varphi}{\simeq} G_2$  and  $\theta(x) = \tau(\varphi(x))$  for any graph isomorphism  $\varphi$  and  $\forall x \in V(G_1) \cup E(G_1)$ , denoted by  $G_1^{I_\theta} = G_2^{I_\tau}$ . Otherwise, *non-identical*.

For example, the graphs shown in Fig.6 are all integral on  $K_4 - e$ , but  $G_1^{I_\theta} = G_2^{I_\tau}$ ,  $G_1^{I_\theta} \neq G_3^{I_\sigma}$ .

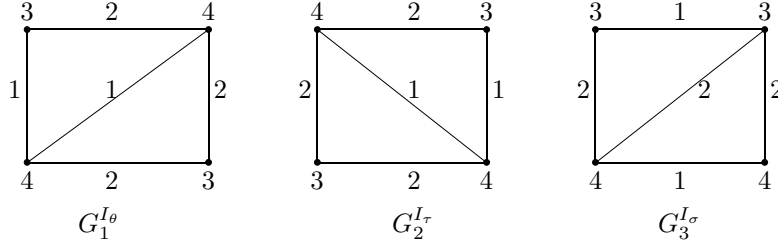


Fig.6

Applying integral graphs, the systems  $(LDES_m^1)$  and  $(LDE_m^n)$  are combinatorially classified in [17] following.

**Theorem 3.7**([17]) *Let  $(LDES_m^1)$ ,  $(LDES_m^1)'$  (or  $(LDE_m^n)$ ,  $(LDE_m^n)'$ ) be two linear homogeneous differential equation systems with integral labeled graphs  $H$ ,  $H'$ . Then  $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$  (or  $(LDE_m^n) \stackrel{\varphi}{\simeq} (LDE_m^n)'$ ) if and only if  $H = H'$ .*

### 3.2 Differential Manifolds on $G^L$ -Systems of Equations

By definition, the union  $\widetilde{M} = \bigcup_{k=1}^m S^{[k]}$  is an  $n$ -manifold. The following result is immediately known.

**Theorem 3.8**([17]-[18],[21]) *For any simply graph  $G$ , there are differentiable solution-manifolds of  $(PDES_m)$ ,  $(LDES_m^1)$ ,  $(LDE_m^n)$  such that  $\widehat{G}[PDES] \simeq G$ ,  $\widehat{G}[LDES_m^1] \simeq G$  and  $\widehat{G}[LDE_m^n] \simeq G$ .*

Notice that a basis on vector field  $T(M)$  of a differentiable  $n$ -manifold  $M$  is

$$\left\{ \frac{\partial}{\partial x_i}, 1 \leq i \leq n \right\}$$

and a vector field  $X$  can be viewed as a first order partial differential operator

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i},$$

where  $a_i$  is  $C^\infty$ -differentiable for all integers  $1 \leq i \leq n$ . Combining Theorems 3.5 and 3.8 enables one to get a result on vector fields following.

**Theorem 3.9**([21]) *For an integer  $m \geq 1$ , let  $U_i, 1 \leq i \leq m$  be open sets in  $\mathbb{R}^n$  underlying a graph defined by  $V(G) = \{U_i | 1 \leq i \leq m\}$ ,  $E(G) = \{(U_i, U_j) | U_i \cap U_j \neq \emptyset, 1 \leq i, j \leq m\}$ . If  $X_i$  is a vector field on  $U_i$  for integers  $1 \leq i \leq m$ , then there always exists a differentiable manifold*

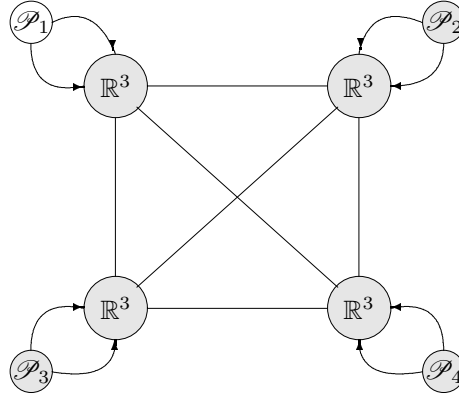
$M \subset \mathbb{R}^n$  with atlas  $\mathcal{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$  underlying graph  $G$  and a function  $u_G \in \Omega^0(M)$  such that  $X_i(u_G) = 0, 1 \leq i \leq m$ .

## §4. Applications

In philosophy, every thing is a  $G^L$ -system with contradictions embedded in our world, which implies that the geometry on non-solvable system of equations is in fact a truthful portraying of things with applications to various fields, particularly, the understanding on gravitational fields and the controlling of industrial systems.

### 4.1 Gravitational Fields

An immediate application of geometry on  $G^L$ -systems of non-solvable equations is that it can provides one with a visualization on things in space of dimension  $\geq 4$  by decomposing the space into subspaces underlying a graph  $G^L$ . For example, a decomposition of a Euclidean space into  $\mathbb{R}^3$  is shown in Fig.7, where  $G^L \simeq K_4$ , a complete graph of order 4 and  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$  are the observations on its subspaces  $\mathbb{R}^3$ . This space model enable one to hold well local behaviors of the spacetime in  $\mathbb{R}^3$  as usual and then determine its global behavior naturally, different from the *string theory* by artificial assuming the dimension of the universe is 11.



**Fig.7**

Notice that  $\mathbb{R}^3$  is in a general position and maybe  $\mathbb{R}^3 \cap \mathbb{R}^3 \neq \mathbb{R}^3$  here. Generally, if  $G^L \simeq K_m$ , we know its dimension following.

**Theorem 4.1** ([9],[13]) *Let  $\mathcal{E}_{K_m}(3)$  be a  $K_m$ -space of  $\underbrace{\mathbb{R}_1^3, \dots, \mathbb{R}_m^3}_m$ . Then its minimum dimension*

$$\dim_{\min} \mathcal{E}_{K_m}(3) = \begin{cases} 3, & \text{if } m = 1, \\ 4, & \text{if } 2 \leq m \leq 4, \\ 5, & \text{if } 5 \leq m \leq 10, \\ 2 + \lceil \sqrt{m} \rceil, & \text{if } m \geq 11 \end{cases}$$

and maximum dimension

$$\dim_{\max} \mathcal{E}_{K_m}(3) = 2m - 1$$

with  $\mathbb{R}_i^3 \cap \mathbb{R}_j^3 = \bigcap_{i=1}^m \mathbb{R}_i^3$  for any integers  $1 \leq i, j \leq m$ .

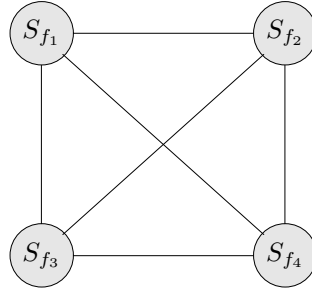
For the gravitational field, by applying the *geometrization of gravitation* in  $\mathbb{R}^3$ , Einstein got his gravitational equations with time ([1])

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi GT^{\mu\nu}$$

where  $R^{\mu\nu} = R_{\alpha}^{\mu\alpha\nu} = g_{\alpha\beta}R^{\alpha\mu\beta\nu}$ ,  $R = g_{\mu\nu}R^{\mu\nu}$  are the respective Ricci tensor, Ricci scalar curvature,  $G = 6.673 \times 10^{-8} \text{cm}^3/\text{gs}^2$ ,  $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} \text{cm}^{-1} \cdot \text{g}^{-1} \cdot \text{s}^2$ , which has a spherically symmetric solution on Riemannian metric, called *Schwarzschild spacetime*

$$ds^2 = f(t) \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1 - \frac{r_s}{r}} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

for  $\lambda = 0$  in vacuum, where  $r_g$  is the *Schwarzschild radius*. Thus, if the dimension of the universe  $\geq 4$ , all these observations are nothing else but a projection of the true faces on our six organs, a pseudo-truth. However, we can characterize its global behavior by  $K_m^L$ -space solutions of  $\mathbb{R}^3$  (See [8]-[10] for details). For example, if  $m = 4$ , there are 4 Einstein's gravitational equations for  $\forall v \in V(K_4^L)$ . We can solving it locally by spherically symmetric solutions in  $\mathbb{R}^3$  and construct a  $K_4^L$ -solution  $S_{f_1}, S_{f_2}, S_{f_3}$  and  $S_{f_4}$ , such as those shown in Fig.8,



**Fig.8**

where, each  $S_{f_i}$  is a geometrical space determined by Schwarzschild spacetime

$$ds^2 = f(t) \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1 - \frac{r_s}{r}} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

for integers  $1 \leq i \leq m$ . Certainly, its global behavior depends on the intersections  $S_{f_i} \cap S_{f_j}, 1 \leq i \neq j \leq 4$ .

## 4.2 Ecologically Industrial Systems

Determining a system, particularly, an industrial system on initial values being stable or not is

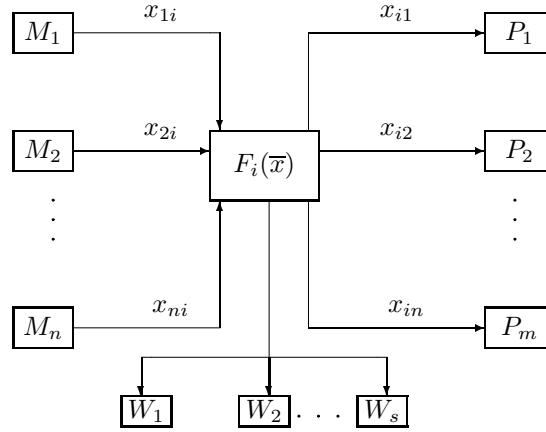
an important problem because it reveals that this system is controllable or not by human beings. Usually, such a system is characterized by a system of differential equations. For example, let

$$\begin{cases} A \rightarrow X \\ 2X + Y \rightarrow 3X \\ B + X \rightarrow Y + D \\ X \rightarrow E \end{cases}$$

be the *Brusselator model* on chemical reaction, where  $A, B, X, Y$  are respectively the concentrations of 4 materials in this reaction. By the chemical dynamics if the initial concentrations for  $A, B$  are chosen sufficiently larger, then  $X$  and  $Y$  can be characterized by differential equations

$$\frac{\partial X}{\partial t} = k_1 \Delta X + A + X^2 Y - (B + 1)X, \quad \frac{\partial Y}{\partial t} = k_2 \Delta Y + BX - X^2 Y.$$

As we known, the stability of a system is determined by its solutions in classical sciences. But if the system of equations is non-solvable, *what is its stability?* It should be noted that non-solvable systems of equations extensively exist in our daily life. For example, an industrial system with raw materials  $M_1, M_2, \dots, M_n$ , products (including by-products)  $P_1, P_2, \dots, P_m$  but  $W_1, W_2, \dots, W_s$  wastes after a produce process, such as those shown in Fig.9 following,



**Fig.9**

which is an opened system and can be transferred to a closed one by letting the environment as an additional cell, called an *ecologically industrial system*. However, such an ecologically industrial system is usually a non-solvable system of equations by the input-output model in economy, see [20] for details.

Certainly, the global stability depends on the local stabilities. Applying the  $G$ -solution of a  $G^L$ -system  $(DES_m^C)$  of differential equations, the global stability is defined following.

**Definition 4.2** Let  $(PDES_m^C)$  be a Cauchy problem on a system of partial differential equations of first order in  $\mathbb{R}^n$ ,  $H \leq G[PDES_m^C]$  a spanning subgraph, and  $u^{[v]}$  the solution of the  $v$ th



equation with initial value  $u_0^{[v]}$ ,  $v \in V(H)$ . It is sum-stable on the subgraph  $H$  if for any number  $\varepsilon > 0$  there exists,  $\delta_v > 0$ ,  $v \in V(H)$  such that each  $G(t)$ -solution with

$$\left| u_0'^{[v]} - u_0^{[v]} \right| < \delta_v, \quad \forall v \in V(H)$$

exists for all  $t \geq 0$  and with the inequality

$$\left| \sum_{v \in V(H)} u'^{[v]} - \sum_{v \in V(H)} u^{[v]} \right| < \varepsilon$$

holds, denoted by  $G[t] \overset{H}{\sim} G[0]$  and  $G[t] \overset{\Sigma}{\sim} G[0]$  if  $H = G[PDES_m^C]$ . Furthermore, if there exists a number  $\beta_v > 0$ ,  $v \in V(H)$  such that every  $G'[t]$ -solution with

$$\left| u_0'^{[v]} - u_0^{[v]} \right| < \beta_v, \quad \forall v \in V(H)$$

satisfies

$$\lim_{t \rightarrow \infty} \left| \sum_{v \in V(H)} u'^{[v]} - \sum_{v \in V(H)} u^{[v]} \right| = 0,$$

then the  $G[t]$ -solution is called asymptotically stable, denoted by  $G[t] \overset{H}{\rightarrow} G[0]$  and  $G[t] \overset{\Sigma}{\rightarrow} G[0]$  if  $H = G[PDES_m^C]$ .

Let  $(PDES_m^C)$  be a system

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= H_i(t, x_1, \dots, x_{n-1}, p_1, \dots, p_{n-1}) \\ u|_{t=t_0} &= u_0^{[i]}(x_1, x_2, \dots, x_{n-1}) \end{aligned} \right\} \quad 1 \leq i \leq m \quad (APDES_m^C)$$

A point  $X_0^{[i]} = (t_0, x_{10}^{[i]}, \dots, x_{(n-1)0}^{[i]})$  with  $H_i(t_0, x_{10}^{[i]}, \dots, x_{(n-1)0}^{[i]}) = 0$  for an integer  $1 \leq i \leq m$  is called an *equilibrium point* of the  $i$ th equation in  $(APDES_m)$ . A result on the sum-stability of  $(APDES_m)$  is known in [18] and [21] following.

**Theorem 4.3**([18],[21]) *Let  $X_0^{[i]}$  be an equilibrium point of the  $i$ th equation in  $(APDES_m)$  for each integer  $1 \leq i \leq m$ . If*

$$\sum_{i=1}^m H_i(X) > 0 \quad \text{and} \quad \sum_{i=1}^m \frac{\partial H_i}{\partial t} \leq 0$$

for  $X \neq \sum_{i=1}^m X_0^{[i]}$ , then the system  $(APDES_m)$  is sum-stability, i.e.,  $G[t] \overset{\Sigma}{\sim} G[0]$ . Furthermore, if

$$\sum_{i=1}^m \frac{\partial H_i}{\partial t} < 0$$

for  $X \neq \sum_{i=1}^m X_0^{[i]}$ , then  $G[t] \overset{\Sigma}{\rightarrow} G[0]$ .

Particularly, if the non-solvable system is a linear homogenous differential equation systems  $(LDES_m^1)$ , we further get a simple criterion on its zero  $G^L$ -solution, i.e., all vertices with 0 labels in [17] following.

**Theorem 4.4**([17]) *The zero  $G$ -solution of linear homogenous differential equation systems  $(LDES_m^1)$  is asymptotically sum-stable on a spanning subgraph  $H \leq G[LDES_m^1]$  if and only if  $\text{Re}\alpha_v < 0$  for each  $\bar{\beta}_v(t)e^{\alpha_v t} \in \mathcal{B}_v$  in  $(LDES^1)$  hold for  $\forall v \in V(H)$ .*

## §5. Conclusions

For human beings, the world is hybrid and filled with contradictions. That is why it is said that *all contradictions are artificial or man-made, not the nature of world* in this paper. In philosophy, a mathematics is nothing else but a set of *symbolic names with relations*. However, as Lao Zi said *name named is not the eternal name, the unnamable is the eternally real and naming is the origin of things* for human beings in his *TAO TEH KING*, a well-known Chinese book. It is difficult to establish such a mathematics join tightly with the world. Even so, for knowing the world, one should develops mathematics well by turning all these mathematical systems with artificial contradictions to a compatible system, i.e., out of the classical run in mathematics but return to their origins. For such an aim, geometry is more applicable, which is an encouraging thing for mathematicians in 21<sup>th</sup> century.

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## On Generalized Quasi-Kenmotsu Manifolds

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**Abstract:** We present a brief analysis on some properties of generalized quasi-Sasakian manifolds, discuss some important properties, particularly, regard the integrability conditions of this kind of manifolds in this paper.

**Key Words:** Riemannian manifold, semi-Riemannian manifold, quasi-Sasakian structure, integrability.

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### §1. Introduction

An interesting topic in the differential geometry is the theory of submanifolds in space endowed with additional structures ([5], [6]). Cr-submanifolds of Kaehler manifolds were studied by A.Bejancu, B.Y.Chen, N.Papaghiuc etc. have studied semi-invariant submanifolds in Sasakian manifolds ([1], [9]). The notion of Kenmotsu manifolds was defined by K.Kenmotsu in 1972 ([10]). N.Papaghiuc have studied semi-invariant submanifolds in a Kenmotsu manifold ([11]). He also studied the geometry of leaves on a semi-invariant  $\xi^\perp$ -submanifolds in a Kenmotsu manifolds ([12]).

### §2. Preliminaries

**Definition 2.1** An  $(2n + 1)$ -dimensional semi-Riemannian manifold  $(\tilde{M}, \tilde{g})$  is said to be an indefinite almost contact manifold if it admits an indefinite almost contact structure  $(\phi, \xi, \eta)$ , where  $\phi$  is a tensor field of type  $(1, 1)$ ,  $\xi$  is a vector field and  $\eta$  is a 1-form, satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad (2.1)$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad (2.2)$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \quad (2.3)$$

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$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \quad (2.4)$$

for all vector fields  $X, Y$  on  $\tilde{M}$  and where  $\epsilon = \tilde{g}(\xi, \xi) = \pm 1$  and  $\tilde{\nabla}$  is the Levi-Civita (L-C) connection for a semi-Riemannian metric  $\tilde{g}$ . Let  $F(\tilde{M})$  be the algebra of the smooth functions on  $\tilde{M}$ .

**Definition 2.2** An almost contact manifold  $\tilde{M}(\phi, \xi, \eta)$  is said to be normal if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \quad X, Y \in \Gamma(T\tilde{M})$$

is the nijenhuis tensor field corresponding to the tensor fields  $\phi$ . The fundamental 2-form  $\Phi$  on  $\tilde{M}$  is defined by

$$\Phi(X, Y) = \tilde{g}(X, \phi Y).$$

In [7]-[8], the authors studied hypersurfaces of an almost contact metric manifold  $\tilde{M}$ . In this paper we define hypersurfaces of an almost contact metric manifold  $\tilde{M}$  whose structure tensor field satisfy the following relation

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\tilde{\nabla}_{\phi^2 X} \xi, Y)\xi - \eta(Y)\tilde{\nabla}_{\phi^2 X} \xi, \quad (2.5)$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of the metric tensor  $\tilde{g}$ . We name this manifold  $\tilde{M}$  equipped with an almost contact metric structure satisfying from (2.5) as generalized Quasi-Kenmotsu manifold, in short G.Q.K.

We define a  $(1, 1)$  tensor field  $F$  by

$$FX = \tilde{\nabla}_X \xi. \quad (2.6)$$

Let us now state the following proposition:

**Proposition 2.1** If  $\tilde{M}$  is a G.Q.K manifold then any integral curve of the structure vector field  $\xi$  is a geodesic i.e.  $\tilde{\nabla}_\xi \xi = 0$ . Again  $d\Phi = 0$  iff  $\xi$  is a Killing vector field.

*Proof* From equation (2.5) putting  $X = Y = \xi$  we can easily prove this assertion.

Next, we derive

$$\begin{aligned} 3d\Phi(X, Y, Z) &= \tilde{g}((\tilde{\nabla}_X \phi)Z, Y) + \tilde{g}((\tilde{\nabla}_Z \phi)Y, X) + \tilde{g}((\tilde{\nabla}_Y \phi)X, Z) \\ &\quad + \eta(X)(\tilde{g}(Y, \tilde{\nabla}_{\phi Z} \xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi)) \\ &\quad + \eta(Y)(\tilde{g}(Z, \tilde{\nabla}_{\phi X} \xi) + \tilde{g}(\phi X, \tilde{\nabla}_Z \xi)) \\ &\quad + \eta(Z)(\tilde{g}(X, \tilde{\nabla}_{\phi Y} \xi) + \tilde{g}(\phi Y, \tilde{\nabla}_X \xi)) = 0. \end{aligned}$$

Therefore, if  $\xi$  is a killing vector field then  $d\Phi = 0$ .

Conversely, Suppose  $d\Phi = 0$ . taking into account  $X = \xi, \eta(Y) = \eta(Z) = 0$ , the last equation implies

$$\tilde{g}(Y, \tilde{\nabla}_{\phi Z} \xi) + \tilde{g}(\phi Z, \tilde{\nabla}_Y \xi) = 0$$

Now substituting  $Z = \phi Z$  and  $Y = Y - \eta(Y)\xi$  we get,

$$\tilde{g}(Y - \eta(Y)\xi, \tilde{\nabla}_{\phi^2 Z} \xi) + \tilde{g}(\phi^2 Z, \tilde{\nabla}_{Y - \eta(Y)\xi} \xi) = 0$$

This implies  $\xi$  is a killing vector field.  $\square$

Let  $\tilde{M}$  be a G.Q.K manifold and considering an  $m$ -dimensional submanifold  $M$ , isometrically immersed in  $\tilde{M}$ . Assuming  $g, \nabla$ , are the induced metric and levi-Civita connvection on  $M$  respectively. Let  $\nabla^\perp$  and  $h$  be the normal connection induced by  $\tilde{\nabla}$  on the normal bundle  $TM^\perp$  and the second fundamental form of  $M$ , respectively.

Therefore, we can decompose the tangent bundle as

$$T\tilde{M} = TM \oplus TM^\perp.$$

The Gauss and Weingarten formulae are characterized by the equations

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.7)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.8)$$

where  $A_N$  is the Weingarten map w.r.t the normal section  $N$  and satisfies

$$g(A_N X, Y) = g(h(X, Y), N) \quad X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp). \quad (2.9)$$

Now we shall give the definition of semi-invariant  $\xi^\perp$ -submanifold. According to Bejancu ([4])  $M$  is a semi-invariant  $\xi^\perp$ -submanifold if there exists two orthogonal distributions,  $D$  and  $D^\perp$  in  $TM$  such that

$$TM = D \oplus D^\perp, \phi D = D, \phi D^\perp \subset TM^\perp, \quad (2.10)$$

where  $\oplus$  denotes the orthogonal sum.

If  $D^\perp = \{0\}$ , then  $M$  is an invariant  $\xi^\perp$ -submanifold. The normal bundle can also be decomposed as

$$TM^\perp = \phi D^\perp \oplus \mu,$$

where  $\phi\mu \subset \mu$ . Hence  $\mu$  contains  $\xi$ .

### §3. Integrability of Distributions on a Semi-Invariant $\xi^\perp$ -Submanifolds

Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a G.Q.K manifold  $\tilde{M}$ . We denote by  $P$  and  $Q$  the projections of  $TM$  on  $D$  and  $D^\perp$  respectively, namely for any  $X \in \Gamma(TM)$ .

$$X = PX + QX, \quad (3.1)$$

Again, for any  $X \in \Gamma(TM)$  and  $N \in \Gamma(TM^\perp)$  we put

$$\phi X = tX + \omega X, \quad (3.2)$$

$$\phi N = BN + CN, \quad (3.3)$$

with  $tX \in \Gamma(D)$ ,  $BN \in \Gamma(TM)$  and  $\omega X, CN \in \Gamma(TM^\perp)$ . Again, for  $X \in \Gamma(TM)$ , the decomposition is

$$FX = \alpha X + \beta X, \alpha X \in \Gamma(D), \beta X \in \Gamma(TM^\perp). \quad (3.4)$$

This section deals with the study of the integrability of both distributions  $D$  and  $D^\perp$ . We have the following proposition:

**Proposition 3.1** *Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a  $G.Q.K$  manifold  $\tilde{M}$ . Then we obtain*

$$(\nabla_X t)Y = A_{\omega Y}X + Bh(X, Y) \quad (3.5a)$$

$$(\nabla_X \omega)Y = Ch(X, Y) - h(X, tY) - g(FX, Y)\xi. \quad (3.5b)$$

*Proof* Notice that

$$\begin{aligned} (\tilde{\nabla}_X \phi)YZ &= -\phi \tilde{\nabla}_X Y + \tilde{\nabla}_X \phi Y \\ &= -\phi(\nabla_X Y + h(X, Y)) + \tilde{\nabla}_X TY + \tilde{\nabla}_X tY + \tilde{\nabla}_X \omega Y. \end{aligned}$$

Using (3.3) and (3.4) and the Gauss and Weingarten formula we get

$$(\tilde{\nabla}_X \phi)Y = (-t\nabla_X Y + \nabla_X tY) + (-\omega\nabla_X Y + \nabla_X^\perp \omega Y) - Bh(X, Y) - Ch(X, Y) + h(X, tY) - A_{\omega Y}X.$$

After some brief calculations we deduce

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X t)Y + (\nabla_X \omega)Y - Bh(X, Y) - Ch(X, Y) + h(X, tY) - A_{\omega Y}X.$$

Again,

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\tilde{\nabla}_{\phi^2 X} \xi, Y)\xi - \eta(Y)\tilde{\nabla}_{\phi^2 X} \xi,$$

Using (2.1) and some steps of calculations, we obtain

$$(\tilde{\nabla}_X \phi)Y = -g(\tilde{\nabla}_X \xi, Y)\xi,$$

as  $(\eta(Y) = 0, g(\xi, Y) = 0$  as  $\xi \perp D, D^\perp$ ). Hence,

$$(\tilde{\nabla}_X \phi)Y = -g(FX, Y)\xi.$$

On comparing the tangential and normal components we shall obtain the results.  $\square$

Taking into the consideration the decomposition of  $TM^\perp$ , we can prove that:

**Proposition 3.2** *Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a G.Q.K manifold  $\tilde{M}$ . Then for any  $N \in \Gamma(TM^\perp)$ , there are*

- (1)  $BN \in D^\perp$ ;
- (2)  $CN \in \mu$ .

*Proof* Let  $N \in \Gamma(TM^\perp)$ ,

$$\phi N = BN + CN,$$

We know  $TM^\perp = \phi D^\perp \oplus \mu$ . Therefore we have

$$BN \in \phi D^\perp \subseteq D^\perp, CN \in \mu. \quad \square$$

**Proposition 3.3** *Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a G.Q.K manifold  $\tilde{M}$ , then*

$$A_{\omega X}Y = A_{\omega Y}X,$$

*for any  $X, Y \in \Gamma(TM^\perp)$ .*

*Proof* From equation (2.9) we have

$$\begin{aligned} g(A_{\omega X}Y, Z) &= g(h(Y, Z), \omega X) = g(\tilde{\nabla}_Z Y, \omega X) \\ &= -g(\omega \tilde{\nabla}_Z Y, X) = g(\omega Y, h(Z, X)) \\ &= g(h(X, Z), \omega Y) = g(A_{\omega Y}X, Z). \end{aligned}$$

Hence the result.  $\square$

**Proposition 3.4** *Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a G.Q.K manifold  $\tilde{M}$ . Then the distribution  $D^\perp$  is integrable.*

*Proof* Let  $Z, X \in \Gamma(D^\perp)$ . Then

$$\begin{aligned} \nabla_Z tX &= (\nabla_Z t)X + t\nabla_Z X \\ \nabla_Z tX &= A_{X\omega}Z + Bh(Z, X) + t\nabla_Z X. \end{aligned}$$

Therefore, (i)

$$t\nabla_Z X = \nabla_Z tX - A_{\omega X}Z - Bh(X, Z)$$

Interchanging  $X$  and  $Z$  we have (ii)

$$t\nabla_X Z = \nabla_X tZ - A_{\omega Z}X - Bh(Z, X).$$

Subtracting equation (ii) from (i) and using Proposition (3.3), we obtain

$$t([Z, X]) = \nabla_Z tX - \nabla_X tZ. \quad \square$$

**Theorem 3.1** *If  $M$  is a semi-invariant  $\xi^\perp$ -submanifold of a G.Q.K manifold  $\tilde{M}$ , then the*



distribution  $D$  is integrable if and only if

$$h(Z, tW) - h(W, tZ) = (\mathbf{L}_\xi \tilde{g})(Z, W)\xi, \quad X, Y \in \Gamma(D).$$

*Proof* From the covariant derivative we have

$$\nabla_Z \omega W = (\tilde{\nabla}_Z \omega W) + \omega \nabla_Z W,$$

$$\nabla_Z \omega W = Ch(Z, W) - h(Z, tW) - g(FZ, W)\xi + \omega(\nabla_Z W + h(Z, W))$$

for  $Z, W \in \Gamma(D)$ . Again using Weingarten formulae we have

$$\nabla_Z \omega W = -A_{\omega W} Z + \nabla_Z^\perp \omega W.$$

Comparing both the equations we get

$$-A_{\omega W} Z + \nabla_Z^\perp \omega W = Ch(Z, W) - h(Z, tW) - g(FZ, W)\xi + \omega(\nabla_Z W + h(Z, W)).$$

On a simplification we obtain

$$\omega \nabla_W Z = \nabla_Z^\perp \omega W - A_{\omega W} Z - Ch(Z, W) + h(W, tZ) - g(FW, Z)\xi.$$

Interchanging  $W$  and  $Z$  in the above equation, we get

$$\omega \nabla_Z W = \nabla_W^\perp \omega Z - A_{\omega Z} W - Ch(W, Z) + h(Z, tW) - g(FZ, W)\xi.$$

Subtracting the above two equations and using Proposition 3.3 we get

$$\omega[Z, W] = h(Z, tW) - h(W, tZ) - g(FZ, W)\xi + g(FW, Z)\xi.$$

We also know that

$$(\mathbf{L}_\xi \tilde{g})(Z, W)\xi = g(FZ, W)\xi - g(FW, Z)\xi.$$

Therefore the distribution  $D$  is integrable if

$$h(Z, tW) - h(W, tZ) = (\mathbf{L}_\xi \tilde{g})(Z, W)\xi. \quad \square$$

**Proposition 3.5** *Let  $M$  be a semi-invariant  $\xi^\perp$ -submanifold of a  $G.Q.K$  manifold  $\tilde{M}$ . Then  $\alpha X = A_\xi X$  and  $\beta X = -\nabla_X^\perp \xi$   $X \in \Gamma(TM)$ .*

*Proof* From Weingarten formulae we get

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi.$$

Again we know from (3.4),

$$\tilde{\nabla}_X \xi = -FX = -\alpha X - \beta X.$$

Comparing these formulae we get  $\alpha X = A_\xi X$  and  $\beta X = -\nabla_X^\perp \xi$  by assuming

$$\{e_i, \phi e_i, e_{2p+j}\}, i = \{1, \dots, p\}, j = \{1, \dots, q\}$$

being an adapted orthonormal local frame on  $M$ , where  $q = \dim D^\perp$  and  $2p = \dim D$ .  $\square$

Similarly, the following theorem is obtained.

**Theorem 3.2** *If  $M$  is a  $\xi^\perp$ -semiinvariant submanifold of a G.Q.K manifold  $\tilde{M}$  one has*

$$\eta(H) = \frac{1}{m} \text{trace}(A_\xi); m = 2p + q.$$

*Proof* From the mean curvature formula

$$H = \frac{1}{m} \sum_{a=1}^s \text{trace}(A_{\xi_a}) \xi_a,$$

where  $\{\xi_1, \dots, \xi_s\}$  is an orthonormal basis in  $TM^\perp$ ,

$$\begin{aligned} \eta(H) &= \frac{1}{m} \sum_{a=1}^s \text{trace}(A_{\xi_a}) \cdot 1, \\ \eta(H) &= \frac{1}{m} \text{trace}(A_\xi). \end{aligned}$$

$\square$

**Corollary 3.1** *If the leaves of the integrable distribution  $D$  are totally geodesic in  $M$  then the structures vector field  $\xi$  is  $D$ -killing, i.e.  $(L_\xi g)(X, Y) = 0$ ,  $X, Y \in \Gamma(D)$ .*

*Proof* We know that

$$\begin{aligned} (L_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(\nabla_X Y, \xi) + g(\xi, \nabla_Y X) = 0, \quad X, Y \in \Gamma(D). \end{aligned}$$

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## On Super $(a, d)$ -Edge-Antimagic Total Labeling of a Class of Trees

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**Abstract:** The concept of labeling has its origin in the works of Stewart (1966), Kotzig and Rosa (1970). Later on Enomoto, Llado, Nakamigawa and Ringel (1998) defined a super  $(a, 0)$ -edge-antimagic total labeling and proposed the conjecture that every tree is a super  $(a, 0)$ -edge-antimagic total graph. In the favour of this conjecture, the present paper deals with different results on antimagicness of a class of trees, which is called subdivided stars.

**Key Words:** Smarandachely super  $(a, d)$ -edge-antimagic total labeling, super  $(a, d)$ -edge-antimagic total labeling, stars and subdivision of stars.

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### §1. Introduction

All graphs in this paper are finite, undirected and simple. For a graph  $G$ ,  $V(G)$  and  $E(G)$  denote the vertex-set and the edge-set, respectively. A  $(v, e)$ -graph  $G$  is a graph such that  $|V(G)| = v$  and  $|E(G)| = e$ . A general reference for graph-theoretic ideas can be seen in [28]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a *total labeling*. Some labelings use the vertex-set only or the edge-set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively.

**Definition 1.1** An  $(s, d)$ -edge-antimagic vertex (abbreviated to  $(s, d)$ -EAV) labeling of a  $(v, e)$ -graph  $G$  is a bijective function  $\lambda : V(G) \rightarrow \{1, 2, \dots, v\}$  such that the set of edge-sums of all edges in  $G$ ,  $\{w(xy) = \lambda(x) + \lambda(y) : xy \in E(G)\}$ , forms an arithmetic progression  $\{s, s + d, s + 2d, \dots, s + (e - 1)d\}$ , where  $s > 0$  and  $d \geq 0$  are two fixed integers.

Furthermore, let  $H \leq G$ . If there is a bijective function  $\lambda : V(H) \rightarrow \{1, 2, \dots, |H|\}$

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such that the set of edge-sums of all edges in  $H$  forms an arithmetic progression  $\{s, s + d, s + 2d, \dots, s + (|E(H)| - 1)d\}$  but for all edges not in  $H$  is a constant, such a labeling is called a Smarandachely  $(s, d)$ -edge-antimagic labeling of  $G$  respect to  $H$ . Clearly, an  $(s, d)$ -EAV labeling of  $G$  is a Smarandachely  $(s, d)$ -EAV labeling of  $G$  respect to  $G$  itself.

**Definition 1.2** A bijection  $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \dots, v + e\}$  is called an  $(a, d)$ -edge-antimagic total  $((a, d)$ -EAT) labeling of a  $(v, e)$ -graph  $G$  if the set of edge-weights  $\{\lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$  forms an arithmetic progression starting from  $a$  and having common difference  $d$ , where  $a > 0$  and  $d \geq 0$  are two chosen integers. A graph that admits an  $(a, d)$ -EAT labeling is called an  $(a, d)$ -EAT graph.

**Definition 1.3** If  $\lambda$  is an  $(a, d)$ -EAT labeling such that  $\lambda(V(G)) = \{1, 2, \dots, v\}$  then  $\lambda$  is called a super  $(a, d)$ -EAT labeling and  $G$  is known as a super  $(a, d)$ -EAT graph.

In Definitions 1.2 and 1.3, if  $d = 0$  then an  $(a, 0)$ -EAT labeling is called an edge-magic total (EMT) labeling and a super  $(a, 0)$ -EAT labeling is called a super edge magic total (SEMT) labeling. Moreover, in general  $a$  is called minimum edge-weight but particularly magic constant when  $d = 0$ . The definition of an  $(a, d)$ -EAT labeling was introduced by Simanjuntak, Bertault and Miller in [23] as a natural extension of *magic valuation* defined by Kotzig and Rosa [17-18]. A super  $(a, d)$ -EAT labeling is a natural extension of the notion of *super edge-magic labeling* defined by Enomoto, Llado, Nakamigawa and Ringel. Moreover, Enomoto et al. [8] proposed the following conjecture.

**Conjecture 1.1** Every tree admits a super  $(a, 0)$ -EAT labeling.

In the favor of this conjecture, many authors have considered a super  $(a, 0)$ -EAT labeling for different particular classes of trees. Lee and Shah [19] verified this conjecture by a computer search for trees with at most 17 vertices. For different values of  $d$ , the results related to a super  $(a, d)$ -EAT labeling can be found for w-trees [13], stars [20], subdivided stars [14, 15, 21, 22, 29, 30], path-like trees [3], caterpillars [17, 18, 25], disjoint union of stars and books [10] and wheels, fans and friendship graphs [24], paths and cycles [23] and complete bipartite graphs [1]. For detail studies of a super  $(a, d)$ -EAT labeling reader can see [2, 4, 5, 7, 9-12].

**Definition 1.4** Let  $n_i \geq 1$ ,  $1 \leq i \leq r$ , and  $r \geq 2$ . A subdivided star  $T(n_1, n_2, \dots, n_r)$  is a tree obtained by inserting  $n_i - 1$  vertices to each of the  $i$ th edge of the star  $K_{1,r}$ . Moreover suppose that  $V(G) = \{c\} \cup \{x_i^{l_i} | 1 \leq i \leq r; 1 \leq l_i \leq n_i\}$  is the vertex-set and  $E(G) = \{cx_i^1 | 1 \leq i \leq r\} \cup \{x_i^{l_i} x_i^{l_i+1} | 1 \leq i \leq r; 1 \leq l_i \leq n_i - 1\}$  is the edge-set of the subdivided star  $G \cong T(n_1, n_2, \dots, n_r)$  then  $v = \sum_{i=1}^r n_i + 1$  and  $e = \sum_{i=1}^r n_i$ .

Lu [29,30] called the subdivided star  $T(n_1, n_2, n_3)$  as a three-path tree and proved that it is a super  $(a, 0)$ -EAT graph if  $n_1$  and  $n_2$  are odd with  $n_3 = n_2 + 1$  or  $n_3 = n_2 + 2$ . Ngurah et al. [21] proved that the subdivided star  $T(n_1, n_2, n_3)$  is also a super  $(a, 0)$ -EAT graph if  $n_3 = n_2 + 3$  or  $n_3 = n_2 + 4$ . Salman et al. [22] found a super  $(a, 0)$ -EAT labeling on the subdivided stars  $T(\underbrace{n, n, n, \dots, n}_{r\text{-times}})$ , where  $n \in \{2, 3\}$ .

Moreover, Javaid et al. [14,15] proved the following results related to a super  $(a, d)$ -EAT labeling on different subclasses of subdivided stars for different values of  $d$ :

- For any odd  $n \geq 3$ ,  $G \cong T(n, n-1, n, n)$  admits a super  $(a, 0)$ -EAT labeling with  $a = 10n + 2$ ;
- For any odd  $n \geq 3$  and  $m \geq 3$ ,  $G \cong T(n, n, m, m)$  admits a super  $(a, 0)$ -EAT labeling with  $a = 6n + 5m + 2$ ;
- For any odd  $n \geq 3$  and  $p \geq 5$ ,  $G \cong T(n, n, n+2, n+2, n_5, \dots, n_p)$  admits a super  $(a, 0)$ -EAT labeling with  $a = 2v + s - 1$ , a super  $(a, 1)$ -EAT labeling with  $a = s + \frac{3}{2}v$  and a super  $(a, 2)$ -EAT labeling with  $a = v + s + 1$  where  $v = |V(G)|$ ,  $s = (2n + 6) + \sum_{m=5}^p [(n+1)2^{m-5} + 1]$  and  $n_r = 1 + (n+1)2^{r-4}$  for  $5 \leq r \leq p$ .

However, the investigation of the different results related to a super  $(a, d)$ -EAT labeling of the subdivided star  $T(n_1, n_2, n_3, \dots, n_r)$  for  $n_1 \neq n_2 \neq n_3, \dots, \neq n_r$  is still open. In this paper, for  $d \in \{0, 1, 2\}$ , we formulate a super  $(a, d)$ -EAT labeling on the subclasses of subdivided stars denoted by  $T(kn, kn, kn, kn, 2kn, n_6, \dots, n_r)$  and  $T(kn, kn, 2n, 2n+2, n_5, \dots, n_r)$  under certain conditions.

## §2. Basic Results

In this section, we present some basic results which will be used frequently in the main results. Ngurah et al. [21] found lower and upper bounds of the minimum edge-weight  $a$  for a subclass of the subdivided stars, which is stated as follows:

**Lemma 2.1** *If  $T(n_1, n_2, n_3)$  is a super  $(a, 0)$ -EAT graph, then  $\frac{1}{2l}(5l^2 + 3l + 6) \leq a \leq \frac{1}{2l}(5l^2 + 11l - 6)$ , where  $l = \sum_{i=1}^3 n_i$ .*

The lower and upper bounds of the minimum edge-weight  $a$  for another subclass of subdivided stars established by Salman et al. [22] are given below:

**Lemma 2.2** *If  $T(\underbrace{n, n, \dots, n}_{n\text{-times}})$  is a super  $(a, 0)$ -EAT graph, then  $\frac{1}{2l}(5l^2 + (9 - 2n)l + n^2 - n) \leq a \leq \frac{1}{2l}(5l^2 + (2n + 5)l + n - n^2)$ , where  $l = n^2$ .*

Moreover, the following lemma presents the lower and upper bound of the minimum edge-weight  $a$  for the most generalized subclass of subdivided stars proved by Javaid and Akhlaq:

**Lemma 2.3**([16]) *If  $T(n_1, n_2, n_3, \dots, n_r)$  has a super  $(a, d)$ -EAT labeling, then  $\frac{1}{2l}(5l^2 + r^2 - 2lr + 9l - r - (l-1)ld) \leq a \leq \frac{1}{2l}(5l^2 - r^2 + 2lr + 5l + r - (l-1)ld)$ , where  $l = \sum_{i=1}^r n_i$  and  $d \in \{0, 1, 2, 3\}$ .*

Bača and Miller [4] state a necessary condition for a graph to be super  $(a, d)$ -EAT, which

provides an upper bound on the parameter  $d$ . Let a  $(v, e)$ -graph  $G$  be a super  $(a, d)$ -EAT. The minimum possible edge-weight is at least  $v + 4$ . The maximum possible edge-weight is no more than  $3v + e - 1$ . Thus  $a + (e - 1)d \leq 3v + e - 1$  or  $d \leq \frac{2v + e - 5}{e - 1}$ . For any subdivided star, where  $v = e + 1$ , it follows that  $d \leq 3$ .

Let us consider the following proposition which we will use frequently in the main results.

**Proposition 2.1**([3]) *If a  $(v, e)$ -graph  $G$  has a  $(s, d)$ -EAV labeling then*

- (1)  $G$  has a super  $(s + v + 1, d + 1)$ -EAT labeling;
- (2)  $G$  has a super  $(s + v + e, d - 1)$ -EAT labeling.

### §3. Super $(a, d)$ -EAT Labeling of Subdivided Stars

**Theorem 3.1** *For any even  $n \geq 2$  and  $r \geq 6$ ,  $G \cong T(n + 3, n + 2, n, n + 1, 2n + 1, n_6, \dots, n_r)$  admits a super  $(a, 0)$ -edge-antimagic total labeling with  $a = 2v + s - 1$  and a super  $(a, 2)$ -edge-antimagic total labeling with  $a = v + s + 1$  where  $v = |V(G)|$ ,  $s = (3n + 7) + \sum_{m=6}^r [2^{m-5}n + 1]$  and  $n_m = 2^{m-4}n + 1$  for  $6 \leq m \leq r$ .*

*Proof* Let us denote the vertices and edges of  $G$ , as follows:

$$V(G) = \{c\} \cup \{x_i^{l_i} | 1 \leq i \leq r; 1 \leq l_i \leq n_i\},$$

$$E(G) = \{cx_i^1 | 1 \leq i \leq r\} \cup \{x_i^{l_i}x_i^{l_i+1} | 1 \leq i \leq r; 1 \leq l_i \leq n_i - 1\}.$$

If  $v = |V(G)|$  and  $e = |E(G)|$ , then

$$v = (6n + 8) + \sum_{m=6}^r [2^{m-6}4n + 1] \text{ and } e = v - 1.$$

Now, we define the labeling  $\lambda : V(G) \rightarrow \{1, 2, \dots, v\}$  as follows:

$$\lambda(c) = (4n + 8) + \sum_{m=6}^r [2^{m-6}2n + 1].$$

For odd  $1 \leq l_i \leq n_i$ , where  $i = 1, 2, 3, 4, 5$  and  $6 \leq i \leq r$ , we define

$$\lambda(u) = \begin{cases} \frac{l_1 + 1}{2}, & \text{for } u = x_1^{l_1}, \\ n + 3 - \frac{l_2 - 1}{2}, & \text{for } u = x_2^{l_2}, \\ (n + 4) + \frac{l_3 - 1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n + 4) - \frac{l_4 - 1}{2}, & \text{for } u = x_4^{l_4}, \\ (3n + 5) - \frac{l_5 - 1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

and

$$\lambda(x_i^{l_i}) = (3n + 5) + \sum_{m=6}^i [2^{m-6}2n + 1] - \frac{l_i - 1}{2},$$

respectively. For even  $1 \leq l_i \leq n_i$ ,  $\alpha = (3n + 5) + \sum_{m=6}^r [2^{m-6}2n + 1]$ ,  $i = 1, 2, 3, 4, 5$  and  $6 \leq i \leq r$ , we define

$$\lambda(u) = \begin{cases} (\alpha + 1) + \frac{l_1 - 2}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha + n + 2) - \frac{l_2 - 2}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha + n + 4) + \frac{l_3 - 2}{2}, & \text{for } u = x_3^{l_3}, \\ (\alpha + 2n + 3) - \frac{l_4 - 2}{2}, & \text{for } u = x_4^{l_4}, \\ (\alpha + 3n + 3) - \frac{l_5 - 2}{2}, & \text{for } u = x_5^{l_5} \end{cases}$$

and

$$\lambda(x_i^{l_i}) = (\alpha + 3n + 3) + \sum_{m=6}^i [2^{m-6}2n] - \frac{l_i - 2}{2},$$

respectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence  $s = \alpha + 2, \alpha + 3, \dots, \alpha + 1 + e$ . Therefore, by Proposition 2.1,  $\lambda$  can be extended to a super  $(a, 0)$ -edge-antimagic total labeling and we obtain the magic constant  $a = v + e + s = 2v + (3n + 6) + \sum_{m=6}^r [2^{m-6}2n + 1]$ .

Similarly by Proposition 2.2,  $\lambda$  can be extended to a super  $(a, 2)$ -edge-antimagic total labeling and we obtain the magic constant  $a = v + 1 + s = v + (3n + 8) + \sum_{m=6}^r [2^{m-6}2n + 1]$ .  $\square$

**Theorem 3.2** For any odd  $n \geq 3$  and  $r \geq 6$ ,  $G \cong T(n + 3, n + 2, n, n + 1, 2n + 1, n_6, \dots, n_r)$  admits a super  $(a, 1)$ -edge-antimagic total labeling with  $a = s + \frac{3v}{2}$  if  $v$  is even, where  $v = |V(G)|$ ,  $s = (3n + 7) + \sum_{m=6}^r [2^{m-5}n + 1]$  and  $n_m = 2^{m-4}n + 1$  for  $6 \leq m \leq r$ .

*Proof* Let us consider the vertices and edges of  $G$ , as defined in Theorem 3.1. Now, we define the labeling  $\lambda : V(G) \rightarrow \{1, 2, \dots, v\}$  as in same theorem. It follows that the edge-weights of all edges of  $G$  constitute an arithmetic sequence  $s = \alpha + 2, \alpha + 3, \dots, \alpha + 1 + e$  with common difference 1, where

$$\alpha = (3n + 5) + \sum_{m=6}^r [2^{m-6}2n + 1].$$

We denote it by  $A = \{a_i; 1 \leq i \leq e\}$ . Now for  $G$  we complete the edge labeling  $\lambda$  for super  $(a, 1)$ -edge-antimagic total labeling with values in the arithmetic sequence  $v + 1, v + 2, \dots, v + e$  with common difference 1. Let us denote it by  $B = \{b_j; 1 \leq j \leq e\}$ . Define  $C = \{a_{2i-1} + b_{e-i+1}; 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1}; 1 \leq j \leq \frac{e+1}{2} - 1\}$ . It is easy to see



that  $C$  constitutes an arithmetic sequence with  $d = 1$  and

$$a = s + \frac{3v}{2} = (12n + 19) + \frac{1}{2} \sum_{m=6}^r [2^{m-3}2n + 5].$$

Since all vertices receive the smallest labels,  $\lambda$  is a super  $(a, 1)$ -edge-antimagic total labeling.  $\square$

**Theorem 3.3** *For any even  $n \geq 2$  and  $r \geq 6$ ,  $G \cong T(n + 2, n, n, n + 1, 2(n + 1), n_6, \dots, n_r)$  admits a super  $(a, 0)$ -edge-antimagic total labeling with  $a = 2v + s - 1$  and a super  $(a, 2)$ -edge-antimagic total labeling with  $a = v + s + 1$  where  $v = |V(G)|$ ,  $s = (3n + 5) + \sum_{m=6}^r [2^{m-5}n + 2]$  and  $n_m = 2^{m-4}n + 2$  for  $6 \leq m \leq r$ .*

*Proof* Let us denote the vertices and edges of  $G$  as follows:

$$V(G) = \{c\} \cup \{x_i^{l_i} | 1 \leq i \leq r; 1 \leq l_i \leq n_i\};$$

$$E(G) = \{cx_i^1 | 1 \leq i \leq r\} \cup \{x_i^{l_i}x_i^{l_i+1} | 1 \leq i \leq r; 1 \leq l_i \leq n_i - 1\}.$$

If  $v = |V(G)|$  and  $e = |E(G)|$ , then

$$v = (6n + 6) + \sum_{m=6}^r [2^{m-6}4(n+)] \text{ and } e = v - 1.$$

Now, we define the labeling  $\lambda : V(G) \rightarrow \{1, 2, \dots, v\}$  as follows:

$$\lambda(c) = (4n + 5) + \sum_{m=6}^r [2^{m-6}2n + 2].$$

For odd  $1 \leq l_i \leq n_i$ , where  $i = 1, 2, 3, 4, 5$  and  $6 \leq i \leq r$ , we define

$$\lambda(u) = \begin{cases} \frac{l_1 + 1}{2}, & \text{for } u = x_1^{l_1}, \\ n + 1 - \frac{l_2 - 1}{2}, & \text{for } u = x_2^{l_2}, \\ (n + 2) - \frac{l_3 + 1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n + 2) - \frac{l_4 - 1}{2}, & \text{for } u = x_4^{l_4}, \\ (3n + 3) - \frac{l_5 - 1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

$$\lambda(x_i^{l_i}) = (3n + 3) + \sum_{m=6}^i [2^{m-6}2n + 2] - \frac{l_i - 1}{2},$$

respectively. For even  $1 \leq l_i \leq n_i$ ,  $\alpha = (3n + 43) + \sum_{m=6}^r [2^{m-6}2n + 2]$ ,  $i = 1, 2, 3, 4, 5$  and

$5 \leq i \leq r$ , we define

$$\lambda(u) = \begin{cases} (\alpha + 1) + \frac{l_1 - 2}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha + n(\alpha + n + 1) - \frac{l_2 - 2}{2}), & \text{for } u = x_2^{l_2}, \\ (\alpha + n + 3) - \frac{l_3 - 2}{2}, & \text{for } u = x_3^{l_3}, \\ (\alpha + 2n + 2) - \frac{l_4 - 2}{2}, & \text{for } u = x_4^{l_4}, \\ (\alpha + 3n + 3) - \frac{l_5 - 2}{2}, & \text{for } u = x_5^{l_5} \end{cases}$$

and

$$\lambda(x_i^{l_i}) = (\alpha + 3n + 3) + \sum_{m=6}^i [2^{m-6} 4(n + 1)] - \frac{l_i - 2}{2},$$

respectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence  $s = \alpha + 2, \alpha + 3, \dots, \alpha + 1 + e$ . Therefore, by Proposition 2.1,  $\lambda$  can be extended to a super  $(a, 0)$ -edge-antimagic total labeling and we obtain the magic constant

$$a = v + e + s = 2v + (3n + 4) + \sum_{m=6}^r [2^{m-6} 2n + 2].$$

Similarly by Proposition 2.2,  $\lambda$  can be extended to a super  $(a, 2)$ -edge-antimagic total labeling and we obtain the magic constant  $a = v + 1 + s = v + (3n + 6) + \sum_{m=6}^r [2^{m-6} 2n + 2]$ .  $\square$

**Theorem 3.4** *For any odd  $n \geq 3$  and  $r \geq 6$ ,  $G \cong T(n + 2, n, n, n + 1, 2(n + 1), n_6, \dots, n_r)$  admits a super  $(a, 1)$ -edge-antimagic total labeling with  $a = s + \frac{3v}{2}$  if  $v$  is even, where  $v = |V(G)|$ ,  $s = (3n + 5) + \sum_{m=6}^r [2^{m-5} n + 2]$  and  $n_m = 2^{m-4} n + 2$  for  $6 \leq m \leq r$ .*

*Proof* Let us consider the vertices and edges of  $G$ , as defined in Theorem 3.3. Now, we define the labeling  $\lambda : V(G) \rightarrow \{1, 2, \dots, v\}$  as in same theorem. It follows that the edge-weights of all edges of  $G$  constitute an arithmetic sequence  $s = \alpha + 2, \alpha + 3, \dots, \alpha + 1 + e$  with common difference 1, where

$$\alpha = (3n + 3) + \sum_{m=6}^r [2^{m-6} 2(n + 1)].$$

We denote it by  $A = \{a_i; 1 \leq i \leq e\}$ . Now for  $G$  we complete the edge labeling  $\lambda$  for super  $(a, 1)$ -edge-antimagic total labeling with values in the arithmetic sequence  $v + 1, v + 2, \dots, v + e$  with common difference 1. Let us denote it by  $B = \{b_j; 1 \leq j \leq e\}$ . Define  $C = \{a_{2i-1} + b_{e-i+1}; 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1}; 1 \leq j \leq \frac{e+1}{2} - 1\}$ . It is easy to see

that  $C$  constitutes an arithmetic sequence with  $d = 1$  and

$$a = s + \frac{3v}{2} = (12n + 14) + \sum_{m=6}^r [2^{m-5}(4n + 3) + 2].$$

Since all vertices receive the smallest labels,  $\lambda$  is a super  $(a, 1)$ -edge-antimagic total labeling.  $\square$

#### §4. Conclusion

In this paper, we have shown that two different subclasses of subdivided stars admit a super  $(a, d)$ -EAT labeling for  $d \in \{0, 1, 2\}$ . However, the problem is still open for the magicness of  $T(n_1, n_2, n_3, \dots, n_r)$ , where  $n_i = n$  and  $1 \leq i \leq r$ .

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## Total Mean Cordial Labeling of Graphs

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**Abstract:** In this paper, we introduce a new type of graph labeling known as total mean cordial labeling. A total mean cordial labeling of a graph  $G = (V, E)$  is a mapping  $f : V(G) \rightarrow \{0, 1, 2\}$  such that  $f(xy) = \left\lceil \frac{f(x) + f(y)}{2} \right\rceil$  where  $x, y \in V(G)$ ,  $xy \in E$ , and the total number of 0, 1 and 2 are balanced. That is  $|ev_f(i) - ev_f(j)| \leq 1$ ,  $i, j \in \{0, 1, 2\}$  where  $ev_f(x)$  denotes the total number of vertices and edges labeled with  $x$  ( $x = 0, 1, 2$ ). If there exists a total mean cordial labeling on a graph  $G$ , we will call  $G$  is Total Mean Cordial. In this paper, we study some classes of graphs and their Total Mean Cordial behaviour.

**Key Words:** Smarandachely total mean cordial labeling, total mean cordial labeling, path, cycle, wheel, complete graph, complete bipartite graph.

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### §1. Introduction

Unless mentioned otherwise, a graph in this paper shall mean a simple finite and undirected. For all terminology and notations in graph theory, we follow Harary [3]. The vertex and edge set of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$  so that the order and size of  $G$  are respectively  $|V(G)|$  and  $|E(G)|$ . Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling plays an important role of various fields of science and few of them are astronomy, coding theory, x-ray crystallography, radar, circuit design, communication network addressing, database management, secret sharing schemes, and models for constraint programming over finite domains [2]. The graph labeling problem was introduced by Rosa and he has introduced graceful labeling of graphs [5] in the year 1967. In 1980, Cahit [1] introduced the cordial labeling of graphs. In 2012, Ponraj et al. [6] introduced the notion of mean cordial labeling. Motivated by these labelings, we introduce a new type of labeling, called total mean cordial labeling. In this paper, we investigate the total mean cordial labeling behaviour of some graphs like path, cycle, wheel, complete graph etc. Let  $x$  be any real number. Then the symbol  $\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$  and  $\lceil x \rceil$  stands for the smallest integer greater than or equal to  $x$ .

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## §2. Main Results

**Definition 2.1** Let  $f$  be a function  $f$  from  $V(G) \rightarrow \{0, 1, 2\}$ . For each edge  $uv$ , assign the label  $\left\lceil \frac{f(u) + f(v)}{2} \right\rceil$ . Then,  $f$  is called a total mean cordial labeling if  $|ev_f(i) - ev_f(j)| \leq 1$  where  $ev_f(x)$  denote the total number of vertices and edges labeled with  $x$  ( $x = 0, 1, 2$ ). A graph with a total mean cordial labeling is called total mean cordial graph.

Furthermore, let  $H \leq G$  be a subgraph of  $G$ . If there is a function  $f$  from  $V(G) \rightarrow \{0, 1, 2\}$  such that  $f|_H$  is a total mean cordial labeling but  $\left\lceil \frac{f(u) + f(v)}{2} \right\rceil$  is a constant for all edges in  $G \setminus H$ , such a labeling and  $G$  are then respectively called Smarandachely total mean cordial labeling and Smarandachely total mean cordial labeling graph respect to  $H$ .

**Theorem 2.2** Any Path  $P_n$  is total mean cordial.

*Proof* Let  $P_n : u_1 u_2 \cdots u_n$  be the path.

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ . Define a map  $f : V(P_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t \\ f(u_{t+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+i}) &= 2 & 1 \leq i \leq t. \end{cases}$$

**Case 2.**  $n \equiv 1 \pmod{3}$ .

Let  $n = 3t + 1$ . Define a function  $f : V(P_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t+1 \\ f(u_{t+1+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+1+i}) &= 2 & 1 \leq i \leq t. \end{cases}$$

**Case 3.**  $n \equiv 2 \pmod{3}$ .

Let  $n = 3t + 2$ . Define a function  $f : V(P_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t+1 \\ f(u_{t+1+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+1+i}) &= 2 & 1 \leq i \leq t \end{cases}$$

and  $f(u_{3t+2}) = 1$ . The following table Table 1 shows that the above vertex labeling  $f$  is a total mean cordial labeling.

Nature of $n$	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod{3}$	$2t - 1$	$2t$	$2t$
$n \equiv 1 \pmod{3}$	$2t + 1$	$2t$	$2t$
$n \equiv 2 \pmod{3}$	$2t + 1$	$2t + 1$	$2t + 1$

**Table 1**

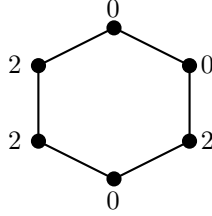
This completes the proof. □

**Theorem 2.3** *The cycle  $C_n$  is total mean cordial if and only if  $n \neq 3$ .*

*Proof* Let  $C_n : u_1 u_2 \dots u_n u_1$  be the cycle. If  $n = 3$ , then we have  $ev_f(0) = ev_f(1) = ev_f(2) = 2$ . But this is an impossible one. Assume  $n > 3$ .

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ ,  $t > 1$ . The labeling given in Figure 1 shows that  $C_6$  is total mean cordial.

**Figure 1**

Take  $t \geq 3$ . Define  $f : V(C_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t \\ f(u_{t+i}) &= 2 & 1 \leq i \leq t \\ f(u_{2t+i}) &= 1 & 1 \leq i \leq t-2. \end{cases}$$

and  $f(u_{3t-1}) = 0$ ,  $f(u_{3t}) = 1$ . In this case  $ev_f(0) = ev_f(1) = ev_f(2) = 2t$ .

**Case 2.**  $n \equiv 1 \pmod{3}$ .

The labeling  $f$  defined in case 2 of Theorem 2.1 is a total mean cordial labeling of here also. In this case,  $ev_f(0) = ev_f(1) = 2t + 1$ ,  $ev_f(2) = 2t$ .

**Case 3.**  $n \equiv 2 \pmod{3}$ .

The labeling  $f$  defined in case 3 of Theorem 2.1 is a total mean cordial labeling. Here,  $ev_f(0) = ev_f(2) = 2t + 1$ ,  $ev_f(1) = 2t + 2$ . □

The following three lemmas 2.4–2.6 are used for investigation of total mean cordial labeling of complete graphs.

**Lemma 2.4** *There are infinitely many values of  $n$  for which  $12n^2 + 12n + 9$  is not a perfect square.*

*Proof* Suppose  $12n^2 + 12n + 9$  is a square,  $\alpha^2$ , say. Then  $3/\alpha$ . So  $\alpha = 3\beta$ . This implies  $12n^2 + 12n + 9 = 9\beta^2$ . Hence we obtain  $4n^2 + 4n + 3 = 3\beta^2$ . On rewriting, we have  $(2n + 1)^2 - 3\beta^2 = -2$ . Substituting  $2n + 1 = U$ ,  $\beta = V$ , we get the Pell's equation  $U^2 - 3V^2 = -2$ . The fundamental solutions of the equations  $U^2 - 3V^2 = -2$  and  $A^2 - 3B^2 = 1$  are  $1 + \sqrt{3}$  and  $2 + \sqrt{3}$ , respectively. Therefore, all the integral solutions  $u_k + \sqrt{3}v_k$  of the equation  $U^2 - 3V^2 = -2$  are given by  $(1 + \sqrt{3})(2 + \sqrt{3})^k$ , where  $k = 0, \pm 1, \pm 2, \dots$ . Applying the result of Mohanty and Ramasamy [4] on Pell's equation, it is seen that the solutions  $u_k + \sqrt{3}v_k$  of the equation  $U^2 - 3V^2 = -2$  are proved by the recurrence relationships  $u_0 = -1$ ,  $u_1 = 1$ ,  $u_{k+2} = 4u_{k+1} - u_k$  and  $v_0 = 1$ ,  $v_1 = 1$ ,  $v_{k+2} = 4v_{k+1} - v_k$ . Hence the square values of  $12n^2 + 12n + 9$  are given by the sequence  $\{n_k\}$  where  $n_1 = 0$ ,  $n_2 = 2$ ,  $n_{k+2} = 4n_{k+1} - n_k + 1$ . It follows that such of those integers of the form  $12m^2 + 12m + 9$  which are not in the sequence  $\{n_k\}$  are not perfect squares.  $\square$

**Lemma 2.5** *There are infinitely many values of  $n$  for which  $12n^2 + 12n - 15$  is not a perfect square.*

*Proof* As in Lemma 2.4 the square values of  $12n^2 + 12n - 15$  are given by the sequence  $\{n_k\}$  where  $n_1 = 1$ ,  $n_2 = 4$ ,  $n_{k+2} = 4n_{k+1} - n_k + 1$ . It follows that such of those integers of the form  $12m^2 + 12m - 15$  which are not in the sequence  $\{n_k\}$  are not perfect squares.  $\square$

**Lemma 2.6** *There are infinitely many values of  $n$  for which  $12n^2 + 12n + 57$  is not a perfect square.*

*Proof* As in Lemma 2.4 the square values of  $12n^2 + 12n + 57$  are given by the sequence  $\{n_k\}$  where  $n_1 = 1$ ,  $n_2 = 7$ ,  $n_{k+2} = 4n_{k+1} - n_k + 1$ . It follows that such of those integers of the form  $12m^2 + 12m - 15$  which are not in the sequence  $\{n_k\}$  are not perfect squares.  $\square$

**Theorem 2.7** *If  $n \equiv 0, 2 \pmod{3}$  and  $12n^2 + 12n + 9$  is not a perfect square then the complete graph  $K_n$  is not total mean cordial.*

*Proof* Suppose  $f$  is a total mean cordial labeling of  $K_n$ . Clearly  $|V(K_n)| + |E(K_n)| = \frac{n(n+1)}{2}$ . If  $n \equiv 0, 2 \pmod{3}$  then 3 divides  $\frac{n(n+1)}{2}$ . Clearly  $ev_f(0) = m + \binom{m}{2}$  where  $m \in \mathbb{N}$ . Then

$$\begin{aligned} \frac{m(m+1)}{2} &= \frac{n(n+1)}{6} \\ \implies m &= \frac{-3 \pm \sqrt{12n^2 + 12n + 9}}{2}, \end{aligned}$$

a contradiction since  $12n^2 + 12n + 9$  is not a perfect square.  $\square$

**Theorem 2.8** *If  $n \equiv 1 \pmod{3}$ ,  $12n^2 + 12n - 15$  and  $12n^2 + 12n + 57$  are not perfect squares then the complete graph  $K_n$  is not total mean cordial.*

*Proof* Suppose there exists a total mean cordial labeling of  $K_n$ , say  $f$ . It is clear that  $ev_f(0) = \frac{n^2 + n - 2}{6}$  or  $ev_f(0) = \frac{n^2 + n + 4}{6}$ .



**Case 1.**  $ev_f(0) = \frac{n^2 + n - 2}{6} = m$ .

Suppose  $k$  zeros are used in the vertices. Then  $k + \binom{k}{2} = m$  where  $k \in \mathbb{N}$ .

$$\begin{aligned} \implies k(k+1) &= \frac{n^2 + n - 2}{3} \\ \implies 3k^2 + 3k - (n^2 + n - 2) &= 0 \\ \implies k &= \frac{-3 \pm \sqrt{12n^2 + 12n - 15}}{6}. \end{aligned}$$

a contradiction since  $12n^2 + 12n - 15$  is not a perfect square.

**Case 2.**  $ev_f(0) = \frac{n^2 + n + 4}{6} = m$ .

Suppose  $k$  zeros are used in the vertices. Then  $k + \binom{k}{2} = m$  where  $k \in \mathbb{N}$ .

$$\begin{aligned} \implies k(k+1) &= \frac{n^2 + n + 4}{3} \\ \implies 3k^2 + 3k - (n^2 + n + 4) &= 0 \\ \implies k &= \frac{-3 \pm \sqrt{12n^2 + 12n + 57}}{6}. \end{aligned}$$

a contradiction since  $12n^2 + 12n + 57$  is not a perfect square.  $\square$

**Theorem 2.9** *The complete graph  $K_n$  is not total mean cordial for infinitely many values of  $n$ .*

*Proof* Proof follow from Lemmas 2.4 – 2.6 and Theorems 2.7 – 2.8.  $\square$

**Theorem 2.10** *The star  $K_{1,n}$  is total mean cordial.*

*Proof* Let  $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$  and  $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$ . Define a map  $f : V(K_{1,n}) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$ ,

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ f(u_{\lfloor \frac{n}{3} \rfloor + i}) &= 2 & 1 \leq i \leq \lceil \frac{2n}{3} \rceil \end{cases}$$

The Table 2 shows that  $f$  is a total mean cordial labeling.

Values of $n$	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod{3}$	$\frac{2n+3}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod{3}$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$
$n \equiv 2 \pmod{3}$	$\frac{2n-1}{3}$	$\frac{2n+2}{3}$	$\frac{2n+2}{3}$

**Table 2**

This completes the proof.  $\square$

The join of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1 + G_2$  with

$$V(G_1 + G_2) = V(G_1) \cup V(G_2),$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

**Theorem 2.11** *The wheel  $W_n = C_n + K_1$  is total mean cordial if and only if  $n \neq 4$ .*

*Proof* Let  $C_n : u_1u_2 \dots u_nu_1$  be the cycle. Let  $V(W_n) = V(C_n) \cup \{u\}$  and  $E(W_n) = E(C_n) \cup \{uu_i : 1 \leq i \leq n\}$ . Here  $|V(W_n)| = n + 1$  and  $|E(W_n)| = 2n$ .

**Case 1.**  $n \equiv 0 \pmod{6}$ .

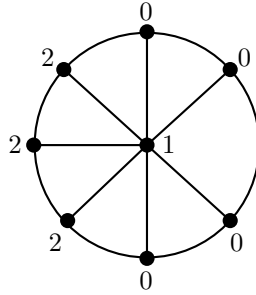
Let  $n = 6k$  where  $k \in \mathbb{N}$ . Define a map  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$ ,

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq 2k \\ f(u_{5k+i}) &= 1 & 1 \leq i \leq k \\ f(u_{2k+i}) &= 2 & 1 \leq i \leq 3k. \end{cases}$$

In this case,  $ev_f(0) = ev_f(2) = 6k$ ,  $ev_f(1) = 6k + 1$ .

**Case 2.**  $n \equiv 1 \pmod{6}$ .

Let  $n = 6k - 5$  where  $k \in \mathbb{N}$  and  $k > 1$ . Suppose  $k = 2$  then the Figure 2 shows that  $W_7$  is total mean cordial.



**Figure 2**

Assume  $k > 2$ . Define a function  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$  and

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq 2k - 2 \text{ \& } i = 5k - 3 \\ 1 & \text{if } 5k - 2 \leq i \leq 6k - 5 \\ 2 & \text{if } 2k - 1 \leq i \leq 5k - 4. \end{cases}$$

It is clear that  $ev_f(0) = 6k - 4$ ,  $ev_f(1) = ev_f(2) = 6k - 5$ .

**Case 3.**  $n \equiv 2 \pmod{6}$ .

Let  $n = 6k - 4$  where  $k \in \mathbb{N}$  and  $k > 1$ . Define  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$  and

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq 2k - 1 \\ 1 & \text{if } 5k - 2 \leq i \leq 6k - 4 \\ 2 & \text{if } 2k \leq i \leq 5k - 3. \end{cases}$$

Note that  $ev_f(0) = 6k - 3$ ,  $ev_f(1) = ev_f(2) = 6k - 4$ .

**Case 4.**  $n \equiv 3 \pmod{6}$ .

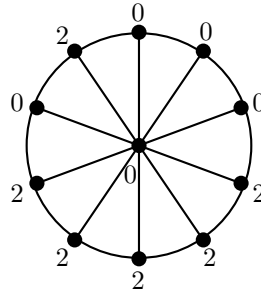
Let  $n = 6k - 3$  where  $k \in \mathbb{N}$ . Define a function  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$ ,

$$\begin{cases} f(u_i) & = 0 & 1 \leq i \leq 2k - 1 \\ f(u_{5k-2+i}) & = 1 & 1 \leq i \leq k - 1 \\ f(u_{2k-1+i}) & = 2 & 1 \leq i \leq 3k - 1. \end{cases}$$

In this case  $ev_f(0) = ev_f(1) = 6k - 3$ ,  $ev_f(2) = 6k - 2$ .

**Case 5.**  $n \equiv 4 \pmod{6}$ .

When  $n = 4$  it is easy to verify that the total mean cordiality condition is not satisfied. Let  $n = 6k - 2$  where  $k \in \mathbb{N}$  and  $k > 1$ . From Figure 3, it is clear that  $ev_f(0) = 11$ ,  $ev_f(1) = ev_f(2) = 10$  and hence  $W_{10}$  is total mean cordial.



**Figure 3**

Let  $k \geq 3$ . Define a function  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$ ,  $f(u_{6k-3}) = 0$ ,  $f(u_{6k-2}) = 1$  and

$$\begin{cases} f(u_i) & = 0 & 1 \leq i \leq 2k - 1 \\ f(u_{5k-2+i}) & = 1 & 1 \leq i \leq k - 2 \\ f(u_{2k-1+i}) & = 2 & 1 \leq i \leq 3k - 1. \end{cases}$$

In this case  $ev_f(0) = 6k - 1$ ,  $ev_f(1) = ev_f(2) = 6k - 2$ .

**Case 6.**  $n \equiv 5 \pmod{6}$ .

Let  $n = 6k - 1$  where  $k \in \mathbb{N}$ . For  $k = 1$  the Figure 4 shows that  $W_5$  is total mean cordial.

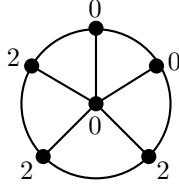


Figure 4

Assume  $k \geq 2$ . Define a function  $f : V(W_n) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$  and

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq 2k \\ 1 & \text{if } 5k + 1 \leq i \leq 6k - 1 \text{ \& } i = 2k + 1 \\ 2 & \text{if } 2k + 2 \leq i \leq 5k. \end{cases}$$

It is clear that  $ev_f(0) = 6k$ ,  $ev_f(1) = ev_f(2) = 6k - 1$ . □

**Theorem 2.12**  $K_2 + mK_1$  is total mean cordial if and only if  $m$  is even.

*Proof* Clearly  $|V(K_2 + mK_1)| = 3m + 3$ . Let  $V(K_2 + mK_1) = \{u, v, u_i : 1 \leq i \leq m\}$  and  $E(K_2 + mK_1) = \{uv, uu_i, vu_i : 1 \leq i \leq m\}$ .

**Case 1.**  $m$  is even.

Let  $m = 2t$ . Define  $f : V(K_2 + mK_1) \rightarrow \{0, 1, 2\}$  by  $f(u) = 0$ ,  $f(v) = 2$

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t \\ f(u_{t+i}) &= 2 & 1 \leq i \leq t. \end{cases}$$

Then  $ev_f(0) = ev_f(1) = ev_f(2) = 2t + 1$  and hence  $f$  is a total mean cordial labeling.

**Case 2.**  $m$  is odd.

Let  $m = 2t + 1$ . Suppose  $f$  is a total mean cordial labeling, then  $ev_f(0) = ev_f(1) = ev_f(2) = 2t + 2$ .

**Subcase 1.**  $f(u) = 0$  and  $f(v) = 0$ .

Then  $ev_f(2) \leq 2t + 1$ , a contradiction.

**Subcase 2.**  $f(u) = 0$  and  $f(v) \neq 0$ .

Since the vertex  $u$  has label 0, we have only  $2t + 1$  zeros. While counting the total number of zeros each vertices  $u_i$  along with the edges  $uu_i$  contributes 2 zeros. This implies  $ev_f(0)$  is an odd number, a contradiction.

**Subcase 3.**  $f(u) \neq 0$  and  $f(v) \neq 0$ .

Then  $ev_f(0) \leq 2t + 1$ , a contradiction. □

The corona of  $G$  with  $H$ ,  $G \odot H$  is the graph obtained by taking one copy of  $G$  and  $p$  copies

of  $H$  and joining the  $i^{th}$  vertex of  $G$  with an edge to every vertex in the  $i^{th}$  copy of  $H$ .  $C_n \odot K_1$  is called the crown,  $P_n \odot K_1$  is called the comb and  $P_n \odot 2K_1$  is called the double comb.

**Theorem 2.13** *The comb  $P_n \odot K_1$  admits a total mean cordial labeling.*

*Proof* Let  $P_n : u_1 u_2 \dots u_n$  be the path. Let  $V(P_n \odot K_1) = \{V(P_n) \cup \{v_i : 1 \leq i \leq n\}$  and  $E(P_n \odot K_1) = E(P_n) \cup \{u_i v_i : 1 \leq i \leq n\}$ . Note that  $|V(P_n \odot K_1)| + |E(P_n \odot K_1)| = 4n - 1$ .

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ . Define a map  $f : V(P_n \odot K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq 2t \\ f(u_{2t+i}) &= 1 & 1 \leq i \leq t \\ f(v_i) &= 2 & 1 \leq i \leq 3t. \end{cases}$$

**Case 2.**  $n \equiv 1 \pmod{3}$ .

Let  $n = 3t + 1$ . Define a function  $f : V(P_n \odot K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq 2t + 1 \\ f(u_{2t+1+i}) &= 1 & 1 \leq i \leq t \\ f(v_i) &= 2 & 1 \leq i \leq 3t + 1. \end{cases}$$

**Case 3.**  $n \equiv 2 \pmod{3}$ .

Let  $n = 3t + 2$ . Define a function  $f : V(P_n \odot K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq 2t + 2 \\ f(u_{2t+2+i}) &= 1 & 1 \leq i \leq t \\ f(v_i) &= 2 & 1 \leq i \leq 3t + 2. \end{cases}$$

From Table 3 it is easy that the labeling  $f$  is a total mean cordial labeling.

Nature of $n$	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod{3}$	$4t - 1$	$4t$	$4t$
$n \equiv 1 \pmod{3}$	$4t + 1$	$4t + 1$	$4t + 1$
$n \equiv 2 \pmod{3}$	$4t + 3$	$4t + 2$	$4t + 2$

**Table 3**

This completes the proof. □

**Theorem 2.14** *The double comb  $P_n \odot 2K_1$  is total mean cordial.*

*Proof* Let  $P_n : u_1 u_2 \dots u_n$  be the path. Let  $V(P_n \odot 2K_1) = \{V(P_n) \cup \{v_i, w_i : 1 \leq$

$i \leq n\}$  and  $E(P_n \odot 2K_1) = E(P_n) \cup \{u_i v_i, u_i w_i : 1 \leq i \leq n\}$ . Note that  $|V(P_n \odot 2K_1)| + |E(P_n \odot 2K_1)| = 6n - 1$ .

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ . Define a map  $f : V(P_n \odot 2K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= f(v_i) &= f(w_i) &= 0 & 1 \leq i \leq t \\ f(u_{t+i}) &= f(v_{t+i}) &= f(w_{t+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+i}) &= f(v_{2t+i}) &= f(w_{2t+i}) &= 2 & 1 \leq i \leq t \end{cases}$$

**Case 2.**  $n \equiv 1 \pmod{3}$ .

Let  $n = 3t + 1$ . Define a function  $f : V(P_n \odot 2K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t+1 \\ f(u_{t+1+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+1+i}) &= 2 & 1 \leq i \leq t \\ f(v_i) &= 0 & 1 \leq i \leq t \\ f(v_{t+i}) &= 1 & 1 \leq i \leq t+1 \\ f(v_{2t+1+i}) &= 2 & 1 \leq i \leq t \\ f(w_i) &= 0 & 1 \leq i \leq t \\ f(w_{t+i}) &= 1 & 1 \leq i \leq t \\ f(w_{2t+i}) &= 2 & 1 \leq i \leq t+1 \end{cases}$$

**Case 3.**  $n \equiv 2 \pmod{3}$ .

Let  $n = 3t + 2$ . Define a function  $f : V(P_n \odot 2K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t+1 \\ f(u_{t+1+i}) &= 1 & 1 \leq i \leq t+1 \\ f(u_{2t+2+i}) &= 2 & 1 \leq i \leq t \\ f(v_i) &= 0 & 1 \leq i \leq t+1 \\ f(v_{t+1+i}) &= 1 & 1 \leq i \leq t \\ f(v_{2t+1+i}) &= 2 & 1 \leq i \leq t+1 \\ f(w_i) &= 0 & 1 \leq i \leq t \\ f(w_{t+i}) &= 1 & 1 \leq i \leq t+1 \\ f(w_{2t+1+i}) &= 2 & 1 \leq i \leq t+1 \end{cases}$$

The Table 4 shows that the labeling  $f$  is a total mean cordial labeling.

Nature of $n$	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod{3}$	$6t - 1$	$6t$	$6t$
$n \equiv 1 \pmod{3}$	$6t + 1$	$6t + 2$	$6t + 2$
$n \equiv 2 \pmod{3}$	$6t + 3$	$6t + 4$	$6t + 4$

**Table 4**

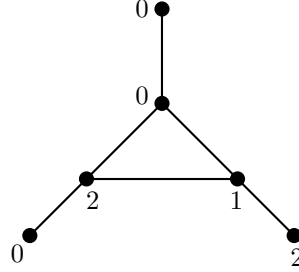
This completes the proof.  $\square$

**Theorem 2.15** *The crown  $C_n \odot K_1$  is total mean cordial.*

*Proof* Let  $C_n : u_1 u_2 \dots u_n u_1$  be the cycle. Let  $V(C_n \odot K_1) = \{V(C_n) \cup \{v_i : 1 \leq i \leq n\}\}$  and  $E(C_n \odot K_1) = E(C_n) \cup \{u_i v_i : 1 \leq i \leq n\}$ . Note that  $|V(C_n \odot K_1)| + |E(C_n \odot K_1)| = 4n$ .

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ . For  $t = 1$  we refer Figure 5.

**Figure 5**

Let  $t > 1$ . Define a map  $f : V(C_n \odot K_1) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) & = f(v_i) & = 0 & 1 \leq i \leq t \\ f(u_{t+i}) & = f(v_{t+i}) & = 1 & 1 \leq i \leq t-1 \\ f(u_{2t-1+i}) & = f(v_{2t-1+i}) & = 2 & 1 \leq i \leq t-1 \end{cases}$$

and  $f(u_{3t-1}) = 2, f(u_{3t}) = 1, f(v_{3t-1}) = 1, f(v_{3t}) = 0$ . Here  $ev_f(0) = ev_f(1) = ev_f(2) = 4t$ .

**Case 2.**  $n \equiv 1 \pmod{3}$ .

The labeling  $f$  defined in case 2 of Theorem 2.13 is a total mean cordial labeling. Here,  $ev_f(0) = 4t + 1, ev_f(1) = 4t + 2, ev_f(2) = 4t + 1$ .

**Case 3.**  $n \equiv 2 \pmod{3}$ .

The labeling  $f$  defined in case 3 of Theorem 2.13 is a total mean cordial labeling. Here,  $ev_f(0) = ev_f(1) = 4t + 3, ev_f(2) = 4t + 2$ .  $\square$

The triangular snake  $T_n$  is obtained from the path  $P_n$  by replacing every edge of the path by a triangle.

**Theorem 2.16** *The triangular snake  $T_n$  is total mean cordial if and only if  $n > 2$ .*

*Proof* Let  $P_n : u_1 u_2 \dots u_n$  be the path and  $V(T_n) = V(P_n) \cup \{v_i : 1 \leq i \leq n-1\}$ . Let  $E(T_n) = E(P_n) \cup \{u_i v_i, v_i u_{i+1} : 1 \leq i \leq n-1\}$ . If  $n = 2$ ,  $T_2 \cong C_3$ , by Theorem 2.3,  $T_2$  is not total mean cordial. Let  $n \geq 3$ . Here  $|V(T_n)| + |E(T_n)| = 5n - 4$ .

**Case 1.**  $n \equiv 0 \pmod{3}$ .

Let  $n = 3t$ . For  $T_3$ , the vertex labeling in Figure 6 is a total mean cordial labeling.

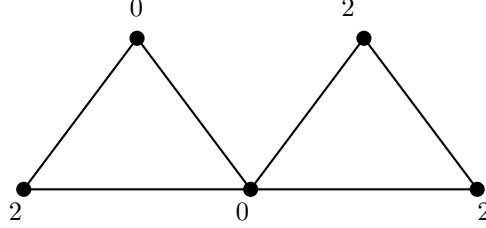


Figure 6

Let  $t \geq 2$ . Define a map  $f : V(T_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t \\ f(u_{t+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+i}) &= 2 & 1 \leq i \leq t-1 \\ f(v_i) &= 0 & 1 \leq i \leq t \\ f(v_{t+i}) &= 1 & 1 \leq i \leq t-1 \\ f(v_{2t-1+i}) &= 2 & 1 \leq i \leq t \end{cases}$$

and  $f(u_{3t}) = 1$ .

**Case 2.**  $n \equiv 1 \pmod{3}$ .

Let  $n = 3t + 1$ . Define  $f : V(T_n) \rightarrow \{0, 1, 2\}$  by

$$\begin{cases} f(u_i) &= 0 & 1 \leq i \leq t+1 \\ f(u_{t+1+i}) &= 1 & 1 \leq i \leq t \\ f(u_{2t+1+i}) &= 2 & 1 \leq i \leq t \\ f(v_i) &= 0 & 1 \leq i \leq t \\ f(v_{t+i}) &= 1 & 1 \leq i \leq t \\ f(v_{2t+i}) &= 2 & 1 \leq i \leq t \end{cases}$$

**Case 3.**  $n \equiv 2 \pmod{3}$ .



Let  $n = 3t + 2$ . Define  $f : V(T_n) \rightarrow \{0, 1, 2\}$  by

$$\left\{ \begin{array}{ll} f(u_i) & = 0 \quad 1 \leq i \leq t+1 \\ f(u_{t+1+i}) & = 2 \quad 1 \leq i \leq t \\ f(u_{2t+1+i}) & = 1 \quad 1 \leq i \leq t \\ f(v_i) & = 0 \quad 1 \leq i \leq t \\ f(v_{t+i}) & = 2 \quad 1 \leq i \leq t+1 \\ f(v_{2t+1+i}) & = 1 \quad 1 \leq i \leq t \end{array} \right.$$

and  $f(u_{3t+2}) = 0$ . The Table 5 shows that  $T_n$  is total mean cordial.

Nature of $n$	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod{3}$	$5t - 2$	$5t - 1$	$5t - 1$
$n \equiv 1 \pmod{3}$	$5t + 1$	$5t$	$5t$
$n \equiv 2 \pmod{3}$	$5t + 2$	$5t + 2$	$5t + 2$

**Table 5**

This completes the proof. □

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## Smarandache's Conjecture on Consecutive Primes

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**Abstract:** Let  $p$  and  $q$  two consecutive prime numbers, where  $q > p$ . Smarandache's conjecture states that the nonlinear equation  $q^x - p^x = 1$  has solutions  $> 0.5$  for any  $p$  and  $q$  consecutive prime numbers. This article describes the conditions that must be fulfilled for Smarandache's conjecture to be true.

**Key Words:** Smarandache conjecture, Smarandache constant, prime, gap of consecutive prime.

**AMS(2010):** 11A41

### §1. Introduction

We note  $\mathbb{P}_{\geq k} = \{p \mid p \text{ prime number, } p \geq k\}$  and two consecutive prime numbers  $p_n, p_{n+1} \in \mathbb{P}_{\geq 2}$ .

**Smarandache Conjecture** *The equation*

$$p_{n+1}^x - p_n^x = 1, \quad (1.1)$$

*has solutions  $> 0.5$ , for any  $n \in \mathbb{N}^*$  ([18], [25]).*

Smarandache's constant([18], [29]) is  $c_S \approx 0.567148130202539 \dots$ , the solution for the equation

$$127^x - 113^x = 1.$$

**Smarandache Constant Conjecture** *The constant  $c_S$  is the smallest solution of equation (1.1) for any  $n \in \mathbb{N}^*$ .*

The function that counts the the prime numbers  $p$ ,  $p \leq x$ , was denoted by Edmund Landau in 1909, by  $\pi$  ([10], [27]). The notation was adopted in this article.

We present some conjectures and theorems regarding the distribution of prime numbers.

**Legendre Conjecture**([8], [20]) *For any  $n \in \mathbb{N}^*$  there is a prime number  $p$  such that*

$$n^2 < p < (n+1)^2.$$

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The smallest primes between  $n^2$  and  $(n+1)^2$  for  $n = 1, 2, \dots$ , are 2, 5, 11, 17, 29, 37, 53, 67, 83,  $\dots$ , [24, A007491].

The largest primes between  $n^2$  and  $(n+1)^2$  for  $n = 1, 2, \dots$ , are 3, 7, 13, 23, 31, 47, 61, 79, 97,  $\dots$ , [24, A053001].

The numbers of primes between  $n^2$  and  $(n+1)^2$  for  $n = 1, 2, \dots$  are given by 2, 2, 2, 3, 2, 4, 3, 4,  $\dots$ , [24, A014085].

**Bertrand Theorem** *For any integer  $n$ ,  $n > 3$ , there is a prime  $p$  such that  $n < p < 2(n-1)$ .*

Bertrand formulated this theorem in 1845. This assumption was proven for the first time by Chebyshev in 1850. Ramanujan in 1919 ([19]), and Erdős in 1932 ([5]), published two simple proofs for this theorem.

Bertrand's theorem stated that: *for any  $n \in \mathbb{N}^*$  there is a prime  $p$ , such that  $n < p < 2n$ .* In 1930, Hoheisel, proved that there is  $\theta \in (0, 1)$  ([9]), such that

$$\pi(x + x^\theta) - \pi(x) \approx \frac{x^\theta}{\ln(x)} . \quad (1.2)$$

Finding the smallest interval that contains at least one prime number  $p$ , was a very hot topic. Among the most recent results belong to Andy Loo whom in 2011 ([11]) proved any for  $n \in \mathbb{N}^*$  there is a prime  $p$  such that  $3n < p < 4n$ . Even ore so, we can state that, if Riemann's hypothesis

$$\pi(x) = \int_2^x \frac{du}{\ln(u)} + O(\sqrt{x} \ln(x)) , \quad (1.3)$$

stands, then in (1.2) we can consider  $\theta = 0.5 + \varepsilon$ , according to Maier ([12]).

**Brocard Conjecture**([17,26]) *For any  $n \in \mathbb{N}^*$  the inequality*

$$\pi(p_{n+1}^2) - \pi(p_n^2) \geq 4$$

*holds.*

Legendre's conjecture stated that between  $p_n^2$  and  $a^2$ , where  $a \in (p_n, p_{n+1})$ , there are at least two primes and that between  $a^2$  and  $p_{n+1}^2$  there are also at least two prime numbers. Namely, is Legendre's conjecture stands, then there are at least four prime numbers between  $p_n^2$  and  $p_{n+1}^2$ .

Concluding, if Legendre's conjecture stands then Brocard's conjecture is also true.

**Andrica Conjecture**([1],[13],[17]) *For any  $n \in \mathbb{N}^*$  the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 , \quad (1.4)$$

*stands.*

The relation (1.4) is equivalent to the inequality

$$\sqrt{p_n + g_n} < \sqrt{p_n} + 1 , \quad (1.5)$$

where we denote by  $g_n = p_{n+1} - p_n$  the gap between  $p_{n+1}$  and  $p_n$ . Squaring (1.5) we obtain the equivalent relation

$$g_n < 2\sqrt{p_n} + 1 . \quad (1.6)$$

Therefore Andrica's conjecture equivalent form is: *for any  $n \in \mathbb{N}^*$  the inequality (1.6) is true.*

In 2014 Paz ([17]) proved that if Legendre's conjecture stands then Andrica's conjecture is also fulfilled. Smarandache's conjecture is a generalization of Andrica's conjecture ([25]).

**Cramér Conjecture** ([4, 7, 21, 23]) *For any  $n \in \mathbb{N}^*$*

$$g_n = O(\ln(p_n)^2) , \quad (1.7)$$

where  $g_n = p_{n+1} - p_n$ , namely

$$\limsup_{n \rightarrow \infty} \frac{g_n}{\ln(p_n)^2} = 1 .$$

Cramér proved that

$$g_n = O(\sqrt{p_n} \ln(p_n)) ,$$

a much weaker relation (1.7), by assuming Riemann hypothesis (1.3) to be true.

Westzynthius proved in 1931 that the gaps  $g_n$  grow faster than the prime numbers logarithmic curve ([30]), namely

$$\limsup_{n \rightarrow \infty} \frac{g_n}{\ln(p_n)} = \infty .$$

**Cramér-Granville Conjecture** *For any  $n \in \mathbb{N}^*$*

$$g_n < R \cdot \ln(p_n)^2 , \quad (1.8)$$

*stands for  $R > 1$ , where  $g_n = p_{n+1} - p_n$ .*

Using Maier's theorem, Granville proved that Cramér's inequality (1.8) does not accurately describe the prime numbers distribution. Granville proposed that  $R = 2e^{-\gamma} \approx 1.123 \dots$  considering the small prime numbers ([6, 13]) (a prime number is considered small if  $p < 10^6$ , [3]).

Nicely studied the validity of Cramér-Granville's conjecture, by computing the ratio

$$R = \frac{\ln(p_n)}{\sqrt{g_n}} ,$$

using large gaps. He noted that for this kind of gaps  $R \approx 1.13 \dots$ . Since  $1/R^2 < 1$ , using the ratio  $R$  we can not produce a proof for Cramér-Granville's conjecture ([14]).

**Oppermann Conjecture** ([16],[17]) *For any  $n \in \mathbb{N}^*$ , the intervals*

$$[n^2 - n + 1, n^2 - 1] \quad \text{and} \quad [n^2 + 1, n^2 + n]$$

*contain at least one prime number  $p$ .*

**Firoozbakht Conjecture** For any  $n \in \mathbb{N}^*$  we have the inequality

$${}^{n+1}\sqrt{p_{n+1}} < \sqrt[n]{p_n} \quad (1.9)$$

or its equivalent form

$$p_{n+1} < p_n^{1+\frac{1}{n}}.$$

If Firoozbakht's conjecture stands, then for any  $n > 4$  we the inequality

$$g_n < \ln(p_n)^2 - \ln(p_n), \quad (1.10)$$

is true, where  $g_n = p_{n+1} - p_n$ . In 1982 Firoozbakht verified the inequality (1.10) using maximal gaps up to  $4.444 \times 10^{12}$  ([22]), namely close to the 48th position in Table 1.

Currently the table was completed up to the position 75 ([15, 24]).

**Paz Conjecture** ([17]) *If Legendre's conjecture stands then:*

- (1) *The interval  $[n, n + 2\lfloor\sqrt{n}\rfloor + 1]$  contains at least one prime number  $p$  for any  $n \in \mathbb{N}^*$ ;*
- (2) *The interval  $[n - \lfloor\sqrt{n}\rfloor + 1, n]$  or  $[n, n + \lfloor\sqrt{n}\rfloor - 1]$  contains at least one prime number  $p$ , for any  $n \in \mathbb{N}^*$ ,  $n > 1$ .*

**Remark 1.1** According to Case (1) and (2), if Legendre's conjecture holds, then Andrica's conjecture is also true ([17]).

**Conjecture Wolf** *Furthermore, the bounds presented below suggest yet another growth rate, namely, that of the square of the so-called Lambert W function. These growth rates differ by very slowly growing factors (like  $\ln(\ln(p_n))$ ). Much more data is needed to verify empirically which one is closer to the true growth rate.*

Let  $P(g)$  be the least prime such that  $P(g) + g$  is the smallest prime larger than  $P(g)$ . The values of  $P(g)$  are bounded, for our empirical data, by the functions

$$P_{min}(g) = 0.12 \cdot \sqrt{g} \cdot e^{\sqrt{g}},$$

$$P_{max}(g) = 30.83 \cdot \sqrt{g} \cdot e^{\sqrt{g}}.$$

For large  $g$ , there bounds are in accord with a conjecture of Marek Wolf ([15, 31, 32]).

## §2. Proof of Smarandache Conjecture

In this article we intend to prove that there are no equations of type (1.1), in respect to  $x$  with solutions  $\leq 0.5$  for any  $n \in \mathbb{N}^*$ .

Let  $f : [0, 1] \rightarrow \mathbb{R}$ ,

$$f(x) = (p + g)^x - p^x - 1, \quad (2.1)$$

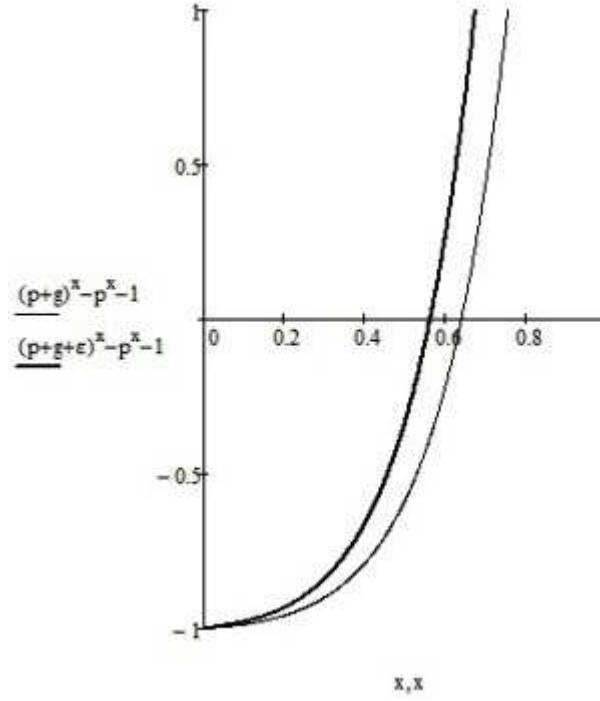
where  $p \in \mathbb{P}_{\geq 3}$ ,  $g \in \mathbb{N}^*$  and  $g$  the gap between  $p$  and the consecutive prime number  $p + g$ . Thus

the equation

$$(p + g)^x - p^x = 1 . \quad (2.2)$$

is equivalent to equation (1.1).

Since for any  $p \in \mathbb{P}_{\geq 3}$  we have  $g \geq 2$  (if *Goldbach's conjecture* is true, then  $g = 2 \cdot \mathbb{N}^{*1}$ ).



**Figure 1** The functions (2.1) and  $(p + g + \varepsilon)^x - p^x - 1$  for  $p = 89$ ,  $g = 8$  and  $\varepsilon = 5$

**Theorem 2.1** *The function  $f$  given by (2.1) is strictly increasing and convex over its domain.*

*Proof* If we compute the first and second derivative of function  $f$ , namely

$$f'(x) = \ln(p + g)(p + g)^x - \ln(p)p^x$$

and

$$f''(x) = \ln(p + g)^2(p + g)^x - \ln(p)^2 p^x .$$

it follows that  $f'(x) > 0$  and  $f''(x) > 0$  over  $[0, 1]$ , thus function  $f$  is strictly increasing and convex over its domain.  $\square$

**Corollary 2.2** *Since  $f(0) = -1 < 0$  and  $f(1) = g - 1 > 0$  because  $g \geq 2$  if  $p \in \mathbb{P}_{\geq 3}$  and, also since function  $f$  is strictly monotonically increasing function it follows that equation (2.2) has a unique solution over the interval  $(0, 1)$ .*

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<sup>1</sup> $2 \cdot \mathbb{N}^*$  is the set of all even natural numbers

**Theorem 2.3** For any  $g$  that verifies the condition  $2 \leq g < 2\sqrt{p} + 1$ , function  $f(0.5) < 0$ .

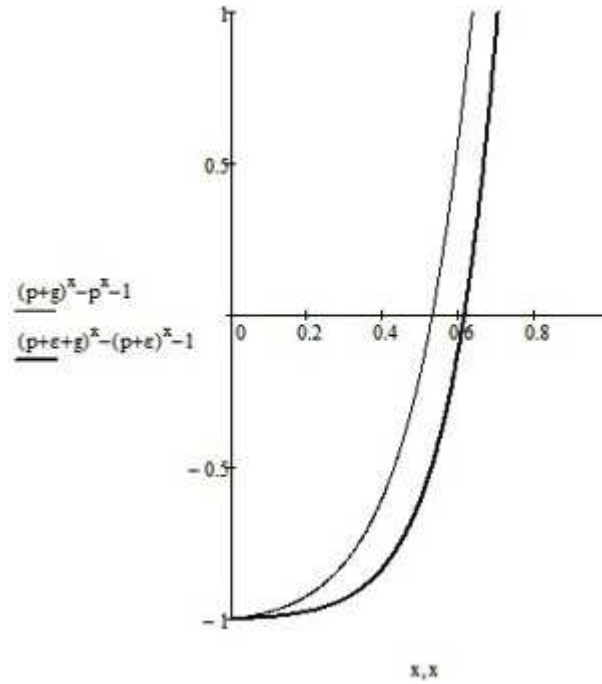
*Proof* The inequality  $\sqrt{p+g} - \sqrt{p} - 1 < 0$  in respect to  $g$  had the solution  $-p \leq g < 2\sqrt{p} + 1$ . Considering the give condition it follows that for a given  $g$  that fulfills  $2 \leq g < 2\sqrt{p} + 1$  we have  $f(0.5) < 0$  for any  $p \in \mathbb{P}_{\geq 3}$ .  $\square$

**Remark 2.4** The condition  $g < 2\sqrt{p} + 1$  represent Andrica's conjecture (1.6).

**Theorem 2.5** Let  $p \in \mathbb{P}_{\geq 3}$  and  $g \in \mathbb{N}^*$ , then the equation (2.2) has a greater solution  $s$  then  $s_\varepsilon$ , the solution for the equation  $(p+g+\varepsilon)^x - p^x - 1 = 0$ , for any  $\varepsilon > 0$ .

*Proof* Let  $\varepsilon > 0$ , then  $p+g+\varepsilon > p+g$ . It follows that  $(p+g+\varepsilon)^x - p^x - 1 > (p+g)^x - p^x - 1$ , for any  $x \in [0, 1]$ . Let  $s$  be the solution to equation (2.2), then there is  $\delta > 0$ , that depends on  $\varepsilon$ , such that  $(p+g+\varepsilon)^{s-\delta} - p^{s-\delta} - 1 = 0$ . Therefore  $s$ , the solution for equation (2.2), is greater that the solution  $s_\varepsilon = s - \delta$  for the equation  $(p+g+\varepsilon)^x - p^x - 1 = 0$ , see Figure 1.  $\square$

**Theorem 2.6** Let  $p \in \mathbb{P}_{\geq 3}$  and  $g \in \mathbb{N}^*$ , then  $s < s_\varepsilon$ , where  $s$  is the equation solution (2.2) and  $s_\varepsilon$  is the equation solution  $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 = 0$ , for any  $\varepsilon > 0$ .



**Figure 2** The functions (2.1) and  $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1$  for  $p = 113, \varepsilon = 408, g = 14$

*Proof* Let  $\varepsilon > 0$ , Then  $p+\varepsilon+g > p+g$ , from which it follows that  $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 < (p+g)^x - p^x - 1$ , for any  $x \in [0, 1]$  (see Figure 2). Let  $s$  the equation solution (2.2), then there  $\delta > 0$ , which depends on  $\varepsilon$ , so  $(p+\varepsilon+g)^{s+\delta} - (p+\varepsilon)^{s+\delta} - 1 = 0$ . Therefore the solution  $s$ , of the equation (2.2), is lower than the solution  $s_\varepsilon = s + \delta$  of the equation  $(p+\varepsilon+g)^x - (p+\varepsilon)^x - 1 = 0$ , see Figure 2.  $\square$

**Remark 2.7** Let  $p_n$  and  $p_{n+1}$  two prime numbers in Table maximal gaps corresponding the maximum gap  $g_n$ . The Theorem 2.6 allows us to say that all solutions of the equation  $(q + \gamma)^x - q^x = 1$ , where  $q \in \{p_n, \dots, p_{n+1} - 2\}$  and  $\gamma < g_n$  solutions are smaller that the solution of the equation  $p_{n+1}^x - p_n^x = 1$ , see Figure 2.

Let:

- (1)  $g_A(p) = 2\sqrt{p} + 1$ , Andrica's gap function ;
- (2)  $g_{CG}(p) = 2 \cdot e^{-\gamma} \cdot \ln(p)^2$ , Cramér-Grandville's gap function ;
- (3)  $g_F(p) = g_1(p) = \ln(p)^2 - \ln(p)$ , Firoozbakht's gap function;
- (4)  $g_c(p) = \ln(p)^2 - c \cdot \ln(p)$ , where  $c = 4(2 \ln(2) - 1) \approx 1.545 \dots$ ,
- (5)  $g_b(p) = \ln(p)^2 - b \cdot \ln(p)$ , where  $b = 6(2 \ln(2) - 1) \approx 2.318 \dots$ .

**Theorem 2.8** The inequality  $g_A(p) > g_\alpha(p)$  is true for:

- (1)  $\alpha = 1$  and  $p \in \mathbb{P}_{\geq 3} \setminus \{7, 11, \dots, 41\}$ ;
- (2)  $\alpha = c = 4(2 \ln(2) - 1)$  and  $p \in \mathbb{P}_{\geq 3}$ ;
- (3)  $\alpha = b = 6(2 \ln(2) - 1)$  and  $p \in \mathbb{P}_{\geq 3}$  and the function  $g_A$  increases at a higher rate than function  $g_b$ .

*Proof* Let the function

$$d_\alpha(p) = g_A(p) - g_\alpha(p) = 1 + 2\sqrt{p} + \alpha \cdot \ln(p) - \ln(p)^2$$

The derivative of function  $d_\alpha$  is

$$d'_\alpha(p) = \frac{\alpha - 2 \ln(p) + \sqrt{p}}{p}.$$

The analytical solutions for function  $d'_1$  are  $5.099 \dots$  and  $41.816 \dots$ . At the same time,  $d'_1(p) < 0$  for  $\{7, 11, \dots, 41\}$  and  $d'_1(p) > 0$  for  $p \in \mathbb{P}_{\geq 3} \setminus \{7, 11, \dots, 41\}$ , meaning that the function  $d_1$  is strictly increasing only over  $p \in \mathbb{P}_{\geq 3} \setminus \{7, 11, \dots, 41\}$  (see Figure 3).

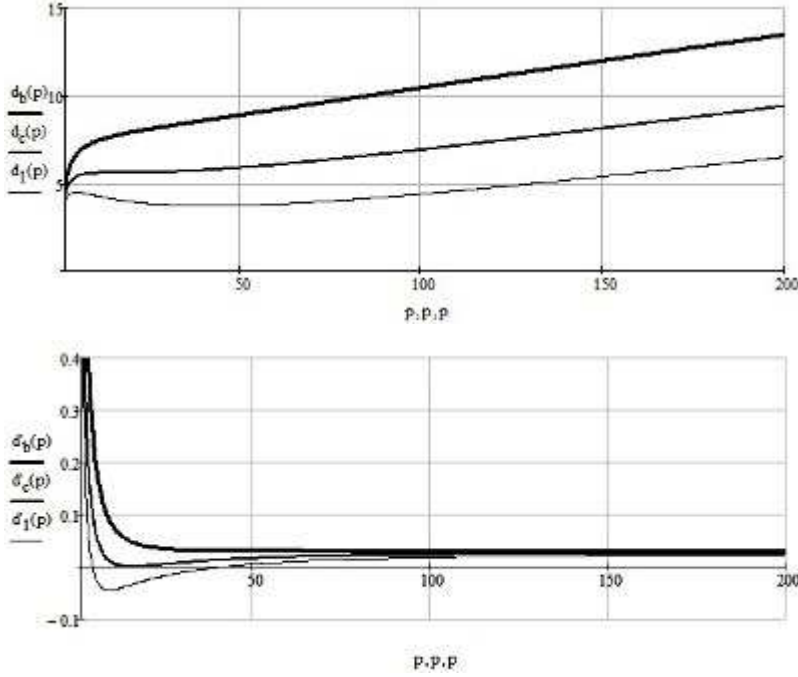
For  $\alpha = c = 4(2 \ln(2) - 1) \approx 1.5451774444795623 \dots$ ,  $d'_c(p) > 0$  for any  $p \in \mathbb{P}_{\geq 3}$ , ( $d'_c$  is nulled for  $p = 16$ , but  $16 \notin \mathbb{P}_{\geq 3}$ ), then function  $d_c$  is strictly increasing for  $p \in \mathbb{P}_{\geq 3}$  (see Figure c). Because function  $d_c$  is strictly increasing and  $d_c(3) = \ln(3)(8 \ln(2) - 4 - \ln(3)) + 2\sqrt{3} + 1 \approx 4.954 \dots$ , it follows that  $d_c(p) > 0$  for any  $p \in \mathbb{P}_{\geq 3}$ .

In  $\alpha = b = 6(2 \ln(2) - 1) \approx 2.3177661667193434 \dots$ , function  $d_b$  is increasing fastest for any  $p \in \mathbb{P}_{\geq 3}$  (because  $d'_b(p) > d'_\alpha(p)$  for any  $p \in \mathbb{P}_{\geq 3}$  and  $\alpha \geq 0$ ,  $\alpha \neq b$ ). Since  $d'_b(p) > 0$  for any  $p \in \mathbb{P}_{\geq 3}$  and because

$$d_b(3) = \ln(3)(12 \ln(2) - 6 - \ln(3)) + 2\sqrt{3} + 1 \approx 5.803479047342222 \dots$$

It follows that  $d_b(p) > 0$  for any  $p \in \mathbb{P}_{\geq 3}$  (see Figure 3). □





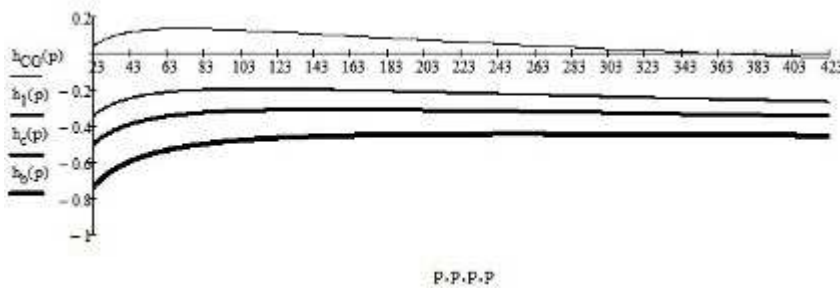
**Figure 3**  $d_\alpha$  and  $d'_\alpha$  functions

**Remark 2.9** In order to determine the value of  $c$ , we solve the equation  $d'_\alpha(p) = 0$  in respect to  $\alpha$ . The solution  $\alpha$  in respect to  $p$  is  $\alpha(p) = 2 \ln(p) - \sqrt{p}$ . We determine  $p$ , the solution of  $\alpha'(p) = \frac{4 - \sqrt{p}}{2p}$ . Then it follows that  $c = \alpha(16) = 4(2 \ln(2) - 1)$ .

**Remark 2.10** In order to find the value for  $b$ , we solve the equation  $d''_\alpha(p) = 0$  in respect to  $\alpha$ . The solution  $\alpha$  in respect to  $p$  is  $\alpha(p) = 2 \ln(p) - \frac{\sqrt{p}}{2} - 2$ . We determine  $p$ , the solution of  $\alpha'(p) = \frac{8 - \sqrt{p}}{4p}$ . It follows that  $b = \alpha(8) = 6(2 \ln(2) - 1)$ .

Since function  $d_b$  manifests the fastest growth rate we can state that the function  $g_A$  increases more rapidly then function  $g_b$ .

$$\text{Let } h(p, g) = f(0.5) = \sqrt{p+g} - \sqrt{p} - 1.$$



**Figure 4** Functions  $h_b$ ,  $h_c$ ,  $h_F$  and  $h_{CG}$

**Theorem 2.11** *For*

$$h_{CG}(p) = h(p, g_{CG}(p)) = \sqrt{p + 2e^{-\gamma} \ln(p)^2} - \sqrt{p} - 1$$

$h_{CG}(p) < 0$  for  $p \in \{3, 5, 7, 11, 13, 17\} \cup \{359, 367, \dots\}$  and

$$\lim_{p \rightarrow \infty} h_{CG}(p) = -1 .$$

*Proof* The theorem can be proven by direct computation, as observed in the graph from Figure 4.  $\square$

**Theorem 2.12** *The function*

$$h_F(p) = h_1(p) = h(p, g_F(p)) = \sqrt{p + \ln(p)^2 - \ln(p)} - \sqrt{p} - 1$$

reaches its maximal value for  $p = 111.152 \dots$  and  $h_F(109) = -0.201205 \dots$  while  $h_F(113) = -0.201199 \dots$  and

$$\lim_{p \rightarrow \infty} h_F(p) = -1 .$$

*Proof* Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4.  $\square$

**Theorem 2.13** *The function*

$$h_c(p) = h(p, g_c(p)) = \sqrt{p + \ln(p)^2 - c \ln(p)} - \sqrt{p} - 1$$

reaches its maximal value for  $p = 152.134 \dots$  and  $h_c(151) = -0.3105 \dots$  while  $h_c(157) = -0.3105 \dots$  and

$$\lim_{p \rightarrow \infty} h_c(p) = -1 .$$

*Proof* Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4.  $\square$

**Theorem 2.14** *The function*

$$h_B(p) = h(p, g_B(p)) = \sqrt{\ln(p)^2 - b \ln(p) + p} - \sqrt{p} - 1$$

reaches its maximal value for  $p = 253.375 \dots$  and  $h_B(251) = -0.45017 \dots$  while  $h_B(257) = -0.45018 \dots$  and

$$\lim_{p \rightarrow \infty} h_B(p) = -1 .$$

*Proof* Again, the theorem can be proven by direct calculation as one can observe from the graph in Figure 4.  $\square$

Table 1: Maximal gaps [24, 14, 15]

#	$n$	$p_n$	$g_n$
1	1	2	1
2	2	3	2
3	4	7	4
4	9	23	6
5	24	89	8
6	30	113	14
7	99	523	18
8	154	887	20
9	189	1129	22
10	217	1327	34
11	1183	9551	36
12	1831	15683	44
13	2225	19609	52
14	3385	31397	72
15	14357	155921	86
16	30802	360653	96
17	31545	370261	112
18	40933	492113	114
19	103520	1349533	118
20	104071	1357201	132
21	149689	2010733	148
22	325852	4652353	154
23	1094421	17051707	180
24	1319945	20831323	210
25	2850174	47326693	220
26	6957876	122164747	222
27	10539432	189695659	234
28	10655462	191912783	248
29	20684332	387096133	250
30	23163298	436273009	282
31	64955634	1294268491	288

#	$n$	$p_n$	$g_n$
32	72507380	1453168141	292
33	112228683	2300942549	320
34	182837804	3842610773	336
35	203615628	4302407359	354
36	486570087	10726904659	382
37	910774004	20678048297	384
38	981765347	22367084959	394
39	1094330259	25056082087	456
40	1820471368	42652618343	464
41	5217031687	127976334671	468
42	7322882472	182226896239	474
43	9583057667	241160624143	486
44	11723859927	297501075799	490
45	11945986786	303371455241	500
46	11992433550	304599508537	514
47	16202238656	416608695821	516
48	17883926781	461690510011	532
49	23541455083	614487453523	534
50	28106444830	738832927927	540
51	50070452577	1346294310749	582
52	52302956123	1408695493609	588
53	72178455400	1968188556461	602
54	94906079600	2614941710599	652
55	251265078335	7177162611713	674
56	473258870471	13829048559701	716
57	662221289043	19581334192423	766
58	1411461642343	42842283925351	778
59	2921439731020	90874329411493	804
60	5394763455325	171231342420521	806
61	6822667965940	218209405436543	906
62	35315870460455	1189459969825483	916
63	49573167413483	1686994940955803	924
64	49749629143526	1693182318746371	1132

#	$n$	$p_n$	$g_n$
65	1175661926421598	43841547845541059	1184
66	1475067052906945	55350776431903243	1198
67	2133658100875638	80873624627234849	1220
68	5253374014230870	203986478517455989	1224
69	5605544222945291	218034721194214273	1248
70	7784313111002702	305405826521087869	1272
71	8952449214971382	352521223451364323	1328
72	10160960128667332	401429925999153707	1356
73	10570355884548334	418032645936712127	1370
74	20004097201301079	804212830686677669	1442
75	34952141021660495	1425172824437699411	1476

We denote by  $a_n = \lfloor g_A(p_n) \rfloor$  (Andrica's conjecture), by  $cg_n = \lfloor g_{CG}(p_n) \rfloor$  (Cramér-Grandville's conjecture) by  $f_n = \lfloor g_F(p_n) \rfloor$  (Firoozbakht's conjecture), by  $c_n = \lfloor g_c(p_n) \rfloor$  and  $b_n = \lfloor g_b(p_n) \rfloor$ .

The columns of Table 2 represent the values of the maximal gaps  $a_n$ ,  $cg_n$ ,  $f_n$ ,  $c_n$ ,  $b_n$  and  $g_n$ , [14, 2, 28, 15]. Note the Cramér-Grandville's conjecture as well as Firoozbakht's conjecture are confirmed when  $n \geq 9$  (for  $p_9 = 23$ , the forth row in the table of *maximal gaps*).

Table 2: Approximative values of maximal gaps

#	$a_n$	$cg_n$	$f_n$	$c_n$	$b_n$	$g_n$
1	3	0	-1	-1	-2	1
2	4	1	0	-1	-2	2
3	6	4	1	0	-1	4
4	10	11	6	4	2	6
5	19	22	15	13	9	8
6	22	25	17	15	11	14
7	46	43	32	29	24	18
8	60	51	39	35	30	20
9	68	55	42	38	33	22
10	73	58	44	40	35	34
11	196	94	74	69	62	36

#	$a_n$	$cg_n$	$f_n$	$c_n$	$b_n$	$g_n$
12	251	104	83	78	70	44
13	281	109	87	82	74	52
14	355	120	96	91	83	72
15	790	160	131	123	115	86
16	1202	183	150	143	134	96
17	1217	184	151	144	134	112
18	1404	192	158	151	141	114
19	2324	223	185	177	166	118
20	2330	223	185	177	166	132
21	2837	236	196	188	177	148
22	4314	264	220	211	200	154
23	8259	311	260	251	238	180
24	9129	318	267	257	244	210
25	13759	350	294	285	271	220
26	22106	389	328	317	303	222
27	27547	407	344	333	319	234
28	27707	408	344	334	319	248
29	39350	439	371	360	345	250
30	41775	444	375	365	349	282
31	71952	494	419	407	391	288
32	76241	499	423	412	396	292
33	95937	521	443	431	414	320
34	123978	546	464	452	435	336
35	131186	552	469	457	440	354
36	207142	598	510	497	479	382
37	287598	633	540	527	509	384
38	299113	637	544	531	512	394
39	316583	643	549	536	517	456
40	413051	672	574	561	542	464
41	715476	734	628	614	594	468
42	853761	754	646	632	612	474
43	982163	771	660	646	626	486
44	1090874	783	671	657	636	490

#	$a_n$	$cg_n$	$f_n$	$c_n$	$b_n$	$g_n$
45	1101584	784	672	658	637	500
46	1103811	785	672	658	637	514
47	1290905	803	689	674	653	516
48	1358957	810	694	679	659	532
49	1567786	827	709	694	673	534
50	1719108	838	719	704	683	540
51	2320599	875	752	736	715	582
52	2373770	878	754	739	717	588
53	2805843	899	773	757	735	602
54	3234157	918	788	773	751	652
55	5358046	983	846	830	807	674
56	7437486	1028	885	868	845	716
57	8850161	1051	906	889	865	766
58	13090804	1106	953	936	912	778
59	19065606	1159	1000	983	958	804
60	26171079	1206	1041	1023	998	806
61	29543826	1224	1057	1039	1013	906
62	68977097	1353	1170	1151	1124	916
63	82146088	1380	1194	1175	1148	924
64	82296594	1380	1194	1175	1148	1132
65	418767467	1648	1430	1409	1379	1184
66	470534915	1668	1447	1426	1396	1198
67	568765768	1701	1476	1455	1425	1220
68	903297246	1783	1548	1526	1496	1224
69	933883765	1789	1553	1532	1501	1248
70	1105270694	1820	1580	1558	1527	1272
71	1187469955	1833	1592	1570	1538	1328
72	1267169959	1844	1602	1580	1549	1356
73	1293108884	1848	1605	1583	1552	1370
74	1793558286	1908	1658	1636	1604	1442
75	2387612050	1962	1705	1682	1650	1476

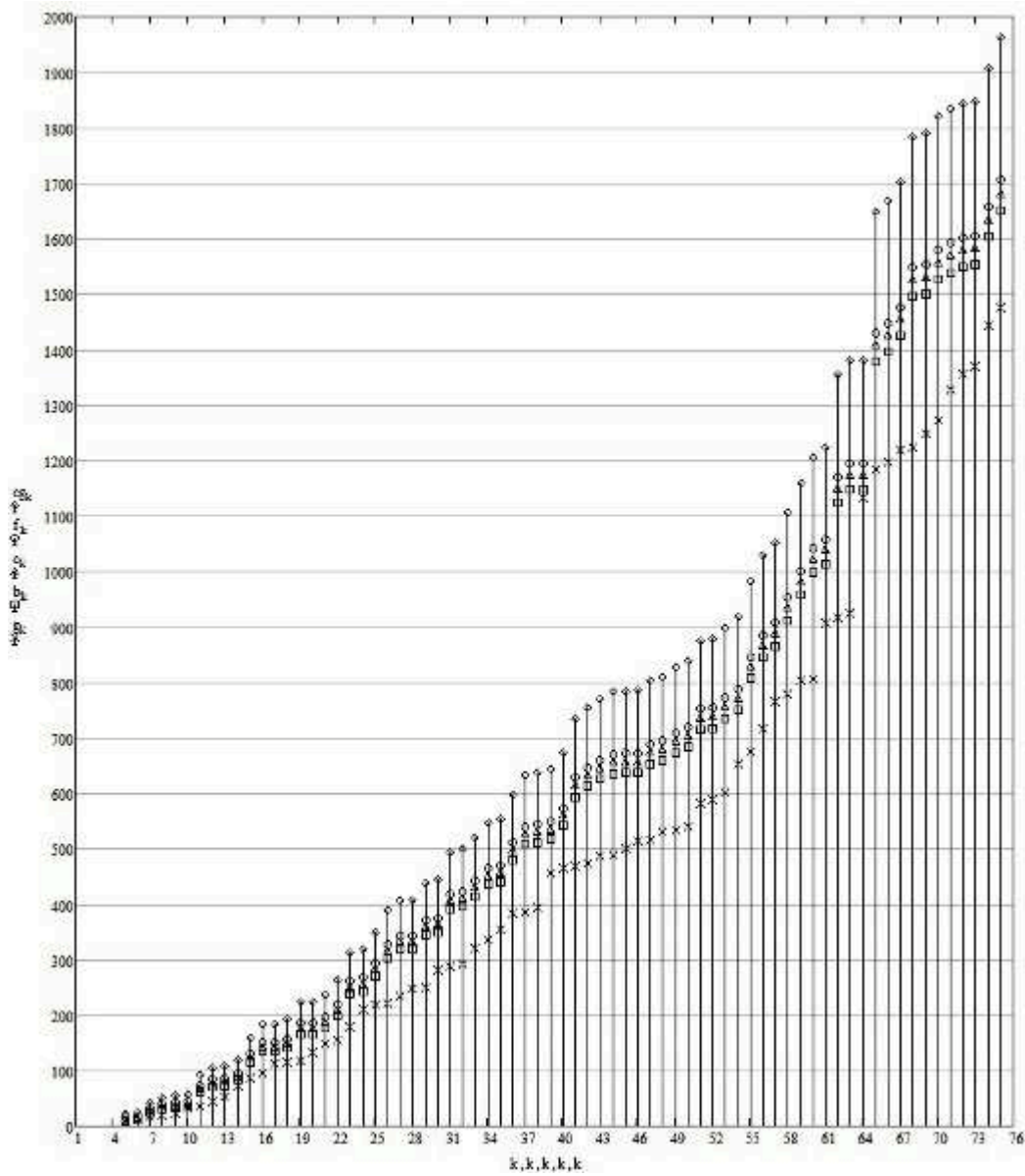
Table 2, the graphs in 5 and 6 stand proof that

$$g_n = p_{n+1} - p_n < \ln(p_n)^2 - c \cdot \ln(p_n) , \quad (2.3)$$

for  $p \in \{89, 113, \dots, 1425172824437699411\}$ . By Theorem 2.6 we can say that inequality (2.3) is true for any  $p \in \mathbb{P}_{\geq 89} \setminus \mathbb{P}_{\geq 1425172824437699413}$ .

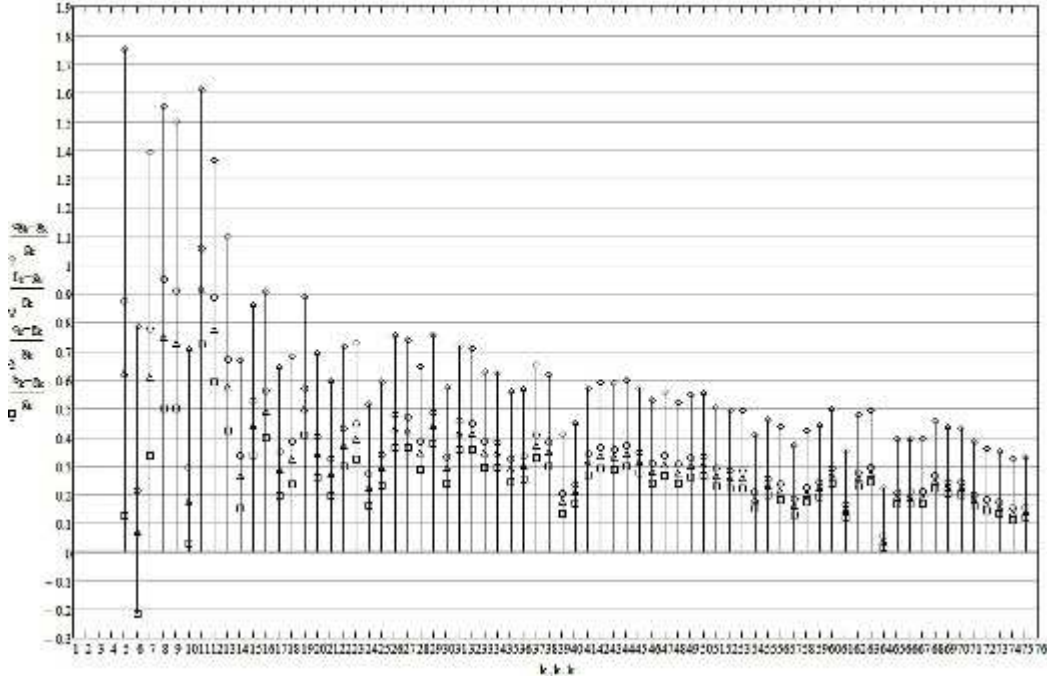
This valid statements in respect to the inequality (2.3) allows us to consider the following hypothesis.

**Conjecture 2.1** *The relation (2.3) is true for any  $p \in \mathbb{P}_{\geq 29}$ .*



**Figure 5** Maximal gaps graph





**Figure 6** Relative errors of  $cg$ ,  $f$ ,  $c$  and  $b$  in respect to  $g$

Let  $g_\alpha : \mathbb{P}_{\geq 3} \rightarrow \mathbb{R}_+$ ,

$$g_\alpha(p) = \ln(p)^2 - \alpha \cdot \ln(p)$$

and  $h_\alpha : \mathbb{P}_{\geq 3} \times [0, 1] \rightarrow \mathbb{R}$ , with  $p$  fixed,

$$h_\alpha(p, x) = (p + g_\alpha(p))^x - p^x - 1$$

that, according to Theorem 2.1, is strictly increasing and convex over its domain, and according to the Corollary 2.2 has a unique solution over the interval  $[0, 1]$ .

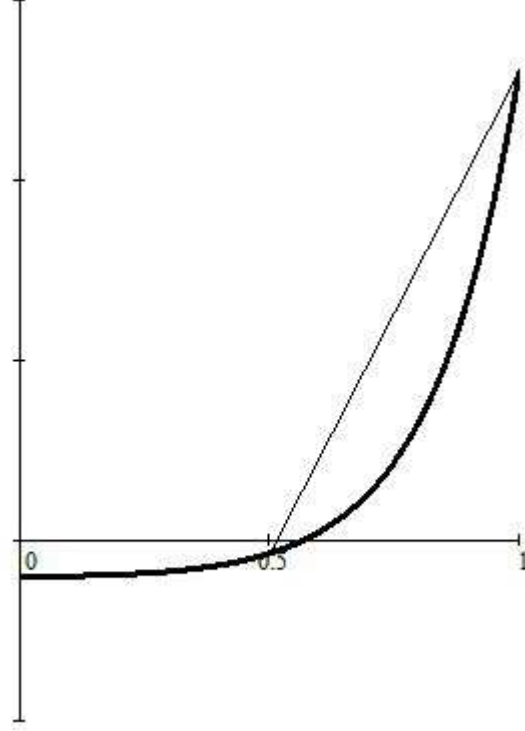
We solve the following equation, equivalent to (2.2)

$$h_c(p, x) = (p + \ln(p)^2 - c \ln(p))^x - p^x - 1 = 0, \quad (2.4)$$

in respect to  $x$ , for any  $p \in \mathbb{P}_{\geq 29}$ . In accordance to Theorem 2.5 the solution for equation (2.2) is greater then the solution to equation (2.4). Therefore if we prove that the solutions to equation (2.4) are greater then 0.5 then, even more so, the solutions to (2.2) are greater then 0.5 .

For equation  $h_\alpha(p, x) = 0$  we consider the secant method, with the initial iterations  $x_0$  and  $x_1$  (see Figure 7). The iteration  $x_2$  is given by

$$x_2 = \frac{x_1 \cdot h_\alpha(p, x_0) - x_0 \cdot h_\alpha(p, x_1)}{h_\alpha(p, x_1) - h_\alpha(p, x_0)}. \quad (2.5)$$



**Figure 7** Function  $f$  and the secant method

If Andrica's conjecture,  $\sqrt{p+g} - \sqrt{p} - 1 < 0$  for any  $p \in \mathbb{P}_{\geq 3}$ ,  $g \in \mathbb{N}^*$  and  $p > g \geq 2$ , is true, then  $h_\alpha(p, 0.5) < 0$  (according to Remark 1.1 if Legendre's conjecture is true then Andrica's conjecture is also true), and  $h_\alpha(p, 1) > 0$ . Since function  $h_\alpha(p, \cdot)$  is strictly increasing and convex, iteration  $x_2$  **approximates the solution to the equation**  $h_\alpha(p, x) = 0$ , (in respect to  $x$ ). Some simple calculation show that  $a$  the solution  $x_2$  in respect to  $h_\alpha$ ,  $p$ ,  $x_0$  and  $x_1$  is:

$$a(p, h_\alpha, x_0, x_1) = \frac{x_1 \cdot h_\alpha(p, x_0) - x_0 h_\alpha(p, x_1)}{h_\alpha(p, x_1) - h_\alpha(p, x_0)}. \quad (2.6)$$

Let  $a_\alpha(p) = a(p, h_\alpha, 0.5, 1)$ , then

$$a_\alpha(p) = \frac{1}{2} + \frac{1 + \sqrt{p} - \sqrt{\ln(p)^2 - \alpha \ln(p) + p}}{2(\ln(p)^2 - \alpha \ln(p) + \sqrt{p} - \sqrt{\ln(p)^2 - \alpha \ln(p) + p})}. \quad (2.7)$$

**Theorem 2.15** *The function  $a_c(p)$ , that approximates the solution to equation (2.4) has values in the open interval  $(0.5, 1)$  for any  $p \in \mathbb{P}_{\geq 29}$ .*

*Proof* According to Theorem 2.8 for  $\alpha = c = 4(2 \ln(2) - 1)$  we have  $\ln(p)^2 - c \cdot \ln(p) < 2\sqrt{p} + 1$  for any  $p \in \mathbb{P}_{\geq 29}$ .

We can rewrite function  $a_c$  as

$$a_c(p) = \frac{1}{2} + \frac{1 + \sqrt{p} - \sqrt{p+c}}{2(c + \sqrt{p} - \sqrt{p+c})}.$$

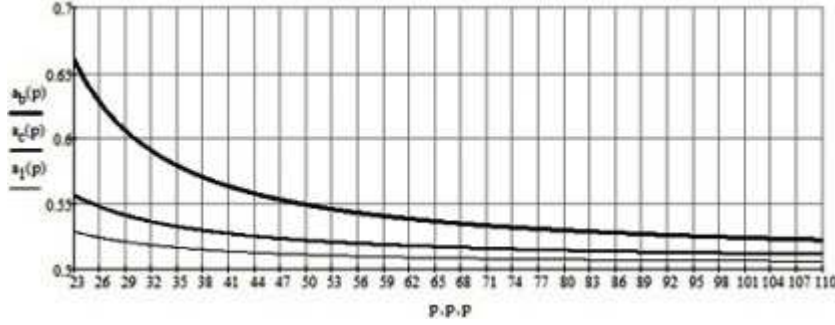
which leads to

$$\frac{1 + \sqrt{p} - \sqrt{p+c}}{2(c + \sqrt{p} - \sqrt{p+c})} > 0 ,$$

it follows that  $a_c(p) > \frac{1}{2}$  for  $p \in \mathbb{P}_{\geq 3}$  (see Figure 8) and we have

$$\lim_{p \rightarrow \infty} a_c(p) = \frac{1}{2} .$$

□



**Figure 8** The graphs for functions  $a_b$ ,  $a_c$  and  $a_1$

For  $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$  and the respective gaps we solve the following equations (2.2).

$$\left\{ \begin{array}{ll} (2+1)^x - 2^x = 1, & s = 1 \\ (3+2)^x - 3^x = 1, & s = 0.7271597432435757 \dots \\ (5+2)^x - 5^x = 1, & s = 0.7632032096058607 \dots \\ (7+4)^x - 7^x = 1, & s = 0.5996694211239202 \dots \\ (11+2)^x - 11^x = 1, & s = 0.8071623463868518 \dots \\ (13+4)^x - 13^x = 1, & s = 0.6478551304201904 \dots \\ (17+2)^x - 17^x = 1, & s = 0.8262031187421179 \dots \\ (19+4)^x - 19^x = 1, & s = 0.6740197879899883 \dots \\ (23+6)^x - 23^x = 1, & s = 0.6042842019286720 \dots \end{array} \right. \quad (2.8)$$

**Corollary 2.9** *We proved that the approximative solutions of equation (2.4) are  $> 0.5$  for any  $n \geq 10$ , then the solutions of equation (2.2) are  $> 0.5$  for any  $n \geq 10$ . If we consider the exceptional cases (2.8) we can state that the equation (1.1) has solutions in  $s \in (0.5, 1]$  for any  $n \in \mathbb{N}^*$ .*

### §3. Smarandache Constant

We order the solutions to equation (2.2) in Table 1 using the maximal gaps.

Table 3: Equation (2.2) solutions

$p$	$g$	solution for (2.2)
113	14	0.5671481305206224...
1327	34	0.5849080865740931...
7	4	0.5996694211239202...
23	6	0.6042842019286720...
523	18	0.6165497314215637...
1129	22	0.6271418980644412...
887	20	0.6278476315319166...
31397	72	0.6314206007048127...
89	8	0.6397424613256825...
19609	52	0.6446915279533268...
15683	44	0.6525193297681189...
9551	36	0.6551846556887808...
155921	86	0.6619804741301879...
370261	112	0.6639444999972240...
492113	114	0.6692774164975257...
360653	96	0.6741127001176469...
1357201	132	0.6813839139412406...
2010733	148	0.6820613370357171...
1349533	118	0.6884662952427394...
4652353	154	0.6955672852207547...
20831323	210	0.7035651178160084...
17051707	180	0.7088121412466053...
47326693	220	0.7138744163020114...
122164747	222	0.7269826061830018...
3	2	0.7271597432435757...
191912783	248	0.7275969819805509...
189695659	234	0.7302859105830866...
436273009	282	0.7320752818323865...
387096133	250	0.7362578381533295...
1294268491	288	0.7441766589716590...
1453168141	292	0.7448821415605216...

$p$	$g$	solution for (2.2)
2300942549	320	0.7460035467176455...
4302407359	354	0.7484690049408947...
3842610773	336	0.7494840618593505...
10726904659	382	0.7547601234459729...
25056082087	456	0.7559861641728429...
42652618343	464	0.7603441937898209...
22367084959	394	0.7606955951728551...
20678048297	384	0.7609716068556747...
127976334671	468	0.7698203623795380...
182226896239	474	0.7723403816143177...
304599508537	514	0.7736363009251175...
241160624143	486	0.7737508697071668...
303371455241	500	0.7745991865337681...
297501075799	490	0.7751693424982924...
461690510011	532	0.7757580339651479...
416608695821	516	0.7760253389165942...
614487453523	534	0.7778809828805762...
1408695493609	588	0.7808871027951452...
1346294310749	582	0.7808983645683428...
2614941710599	652	0.7819658004744228...
1968188556461	602	0.7825687226257725...
7177162611713	674	0.7880214782837229...
13829048559701	716	0.7905146362137986...
19581334192423	766	0.7906829063252424...
42842283925351	778	0.7952277512573828...
90874329411493	804	0.7988558653770882...
218209405436543	906	0.8005126614171458...
171231342420521	806	0.8025304565279002...
1693182318746371	1132	0.8056470803187964...
1189459969825483	916	0.8096231085041140...
1686994940955803	924	0.8112057874892308...
43841547845541060	1184	0.8205327998695296...
55350776431903240	1198	0.8212591131062218...

$p$	$g$	solution for (2.2)
80873624627234850	1220	0.8224041089823987...
218034721194214270	1248	0.8258811322716928...
352521223451364350	1328	0.8264955008480679...
1425172824437699300	1476	0.8267652954810718...
305405826521087900	1272	0.8270541728027422...
203986478517456000	1224	0.8271121951019150...
418032645936712100	1370	0.8272229385637846...
401429925999153700	1356	0.8272389079572986...
804212830686677600	1442	0.8288714147741382...
2	1	1

#### §4 Conclusions

Therefore, if Legendre's conjecture is true then Andrica's conjecture is also true according to Paz [17]. Andrica's conjecture validated the following sequence of inequalities  $a_n > cg_n > f_n > c_n > b_n > g_n$  for any  $n$  natural number,  $5 \leq n \leq 75$ , in Tables 2. The inequalities  $c_n < g_n$  for any natural  $n$ ,  $5 \leq n \leq 75$ , from Table 2 allows us to state Conjecture 2.1.

If Legendre's conjecture and Conjecture 2.1 are true, then Smarandache's conjecture is true.

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## The Neighborhood Pseudochromatic Number of a Graph

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**Abstract:** A *pseudocoloring* of  $G$  is a coloring of  $G$  in which adjacent vertices can receive the same color. The *neighborhood pseudochromatic number* of a non-trivial connected graph  $G$ , denoted  $\psi_{nhd}(G)$ , is the maximum number of colors used in a pseudocoloring of  $G$  such that every vertex has at least two vertices in its closed neighborhood receiving the same color. In this paper, we obtain  $\psi_{nhd}(G)$  of some standard graphs and characterize all graphs for which  $\psi_{nhd}(G)$  is 1, 2,  $n - 1$  or  $n$ .

**Key Words:** Coloring, pseudocoloring, neighborhood, domination.

**AMS(2010):** 05C15

### §1. Introduction

Historically, the coloring terminology comes from the map-coloring problem which involves coloring of the countries in a map in such a way that no two adjacent countries are colored with the same color. The committee scheduling problem is another problem which can be rephrased as a vertex coloring problem. As such, the concept of graph coloring motivates varieties of graph labelings with an addition of various conditions and has a wide range of applications - channel assignment in wireless communications, traffic phasing, fleet maintenance and task assignment to name a few. More applications of graph coloring can be found in [2,17]. A detailed discussion on graph coloring and some of its variations can be seen in [1,5-8,16,18].

Throughout this paper, we consider a graph  $G$  which is simple, finite and undirected. A vertex  $k$ -coloring of  $G$  is a surjection  $c : V(G) \rightarrow \{1, 2, \dots, k\}$ . A vertex  $k$ -coloring  $c$  of a graph  $G$  is said to be a *proper  $k$ -coloring* if vertices of  $G$  receive different colors whenever they are adjacent in  $G$ . Thus for a proper  $k$ -coloring, we have  $c(u) \neq c(v)$  whenever  $uv \in E(G)$ . The minimum  $k$  for which there is a proper  $k$ -coloring of  $G$  is called the chromatic number of  $G$ , denoted  $\chi(G)$ . It can be seen that a proper  $k$ -coloring of  $G$  is simply a vertex partition of  $V(G)$  into  $k$  independent subsets called color classes. For any vertex  $v \in V(G)$ ,  $N[v] = N(v) \cup \{v\}$

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where  $N(v)$  is the set of all the vertices in  $V(G)$  which are adjacent to  $v$ . As discussed in [14], a dominating set  $S$  of a graph  $G(V, E)$  is a subset of  $V$  such that every vertex in  $V$  is either an element of  $S$  or is adjacent to an element of  $S$ . The minimum cardinality of a dominating set of a graph  $G$  is called its dominating number, denoted  $\gamma(G)$ . Further, a dominating set of  $G$  with minimum cardinality is called a  $\gamma$ -set of  $G$ .

As introduced in 1967 by Harary et al. [10, 11], a *complete  $k$ -coloring* of a graph  $G$  is a proper  $k$ -coloring of  $G$  such that, for any pair of colors, there is at least one edge of  $G$  whose end vertices are colored with this pair of colors. The greatest  $k$  for which  $G$  admits a complete  $k$ -coloring is the *achromatic number*  $\alpha(G)$ . In 1969, while working on the famous Nordaus - Gaddum inequality [16], R. P. Gupta [9] introduced a new coloring parameter, called the *pseudochromatic number*, which generalizes the achromatic number. A pseudo  $k$ -coloring of  $G$  is a  $k$ -coloring in which adjacent vertices may receive same color. A *pseudocomplete  $k$ -coloring* of a graph  $G$  is a pseudo  $k$ -coloring such that, for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors. The *pseudochromatic number*  $\psi(G)$  is the greatest  $k$  for which  $G$  admits a pseudocomplete  $k$ -coloring. This parameter was later studied by V. N. Bhawe [3], E. Sampath Kumar [19] and V. Yegnanarayanan [20]. Motivated by the above studies, we introduce here a new graph invariant and study some of its properties in this paper. We use standard notations, the terms not defined here may be found in [4, 12, 14, 15].

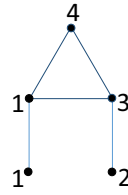
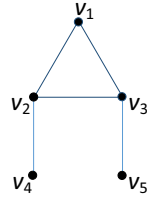


Figure 1. The graph  $G$       Figure 2. A pseudo 4-coloring of  $G$

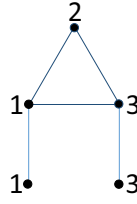
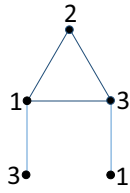


Figure 3. A complete 3-coloring of  $G$       Figure 4. A pseudocomplete 3-coloring of  $G$

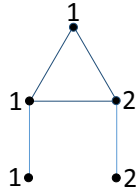


Figure 5. A neighborhood pseudo 2-coloring of  $G$

**Definition 1.1** A neighborhood pseudo  $k$ -coloring of a connected graph  $G(V, E)$  is a pseudo  $k$ -coloring  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  of  $G$  such that for every  $v \in V$ ,  $c|_{N[v]}$  is not an injection.

In other words, a connected graph  $G = (V, E)$  is said to have a neighborhood pseudo  $k$ -coloring if there exists a pseudo  $k$ -coloring  $c$  of  $G$  such that  $\forall v \in V(G)$ ,  $\exists u, w \in N[v]$  with  $c(u) = c(w)$ .

**Definition 1.2** The maximum  $k$  for which  $G$  admits a neighborhood pseudo  $k$ -coloring is called the neighborhood pseudochromatic number of  $G$ , denoted  $\psi_{nhd}(G)$ . Further, a coloring  $c$  for which  $k$  is maximum is called a maximal neighborhood pseudocoloring of  $G$ .

The Figures 1-5 show a graph  $G$  and its various colorings. The above Definition 1.1 can be extended to disconnected graphs as follows.

**Definition 1.3** If  $G$  is a disconnected graph with  $k$  components  $H_1, H_2, \dots, H_k$ , then

$$\psi_{nhd}(G) = \sum_{i=1}^k \psi_{nhd}(H_i).$$

**Observation 1.4** For any graph  $G$  of order  $n$ ,  $1 \leq \psi_{nhd}(G) \leq n$ . In particular, if  $G$  is connected, then  $1 \leq \psi_{nhd}(G) \leq n - 1$ .

**Observation 1.5** If  $H$  is any connected subgraph of a graph  $G$ , then  $\psi_{nhd}(H) \leq \psi_{nhd}(G)$ .

## §2. Preliminary Results

In this section, we study the neighborhood pseudochromatic number of standard graphs. We also obtain certain bounds on the neighborhood pseudochromatic number of a graph. We end the section with a few characterizations. We first state the following theorem whose proof is immediate.

**Theorem 2.1** If  $n$  is an integer and  $n_i \in \mathbb{Z}^+$  for each  $i = 1, 2, \dots$ ,

- (1)  $\psi_{nhd}(\overline{K}_n) = n$  for  $n \geq 1$ ;
- (2)  $\psi_{nhd}(K_n) = n - 1$  for  $n \geq 2$ ;
- (3)  $\psi_{nhd}(P_n) = \lfloor \frac{n}{2} \rfloor$  for  $n \geq 2$ ;
- (4)  $\psi_{nhd}(C_n) = \begin{cases} 2 & \text{for } n = 3 \\ \lfloor \frac{n}{2} \rfloor & \text{for } n > 3 \end{cases}$
- (5)  $\psi_{nhd}(K_{1,n}) = 1$  for  $n \geq 1$ ;
- (6)  $\psi_{nhd}(K_{n_1, n_2, \dots, n_k}) = \sum_{i=1}^k n_i - 2$  where each  $n_i \geq 2$ .

**Corollary 2.2** For any graph  $G$  having  $k$  components,  $\psi_{nhd}(G) \geq k$ .

**Corollary 2.3** For a connected graph  $G$  with diameter  $d$ ,  $\psi_{nhd}(G) \geq \lfloor \frac{d}{2} \rfloor$ .

**Corollary 2.4** If  $G$  is a graph with  $k$  non-trivial components  $H_1, H_2, \dots, H_k$  and  $\omega(H_i)$  is the clique number of  $H_i$ , then

$$\psi_{nhd}(G) \geq \sum_{i=1}^k \omega(H_i) - k.$$

**Corollary 2.5**  $\psi_{nhd}(G) \leq n - k$  for a connected graph  $G$  of order  $n \geq 3$  with  $k$  pendant vertices.

**Lemma 2.6** For a connected graph  $G$ ,  $\psi_{nhd}(G) \geq 2$  if and only if  $G$  has a subgraph isomorphic to  $C_3$  or  $P_4$ .

*Proof* If  $G$  contains  $C_3$  or  $P_4$ , from Observation 1.5 and Theorem 2.1,  $\psi_{nhd}(G) \geq 2$ . Conversely, let  $G$  be a connected graph with  $\psi_{nhd}(G) \geq 2$ . If possible, suppose that  $G$  has neither a  $C_3$  or nor a  $P_4$  as its subgraph, then  $G$  is isomorphic to  $K_{1,n}$ . But then,  $\psi_{nhd}(G) = 1$ , a contradiction by Theorem 2.1.  $\square$

As a consequence of Lemma 2.6, we have a consequence following.

**Corollary 2.7** A non-trivial graph  $G$  is a star if and only if  $\psi_{nhd}(G) = 1$ .

**Theorem 2.8** A graph  $G$  of order  $n$  is totally disconnected if and only if  $\psi_{nhd}(G) = n$ .

*Proof* If a graph  $G$  of order  $n$  is totally disconnected, then by Theorem 2.1,  $\psi_{nhd}(G) = n$ . Conversely, if  $G$  is not totally disconnected, then  $G$  has an edge, say,  $e$ . Now for an end vertex of  $e$ , at least one color should repeat in  $G$ , so  $\psi_{nhd}(G) < |V| = n$ . Hence the theorem.  $\square$

### §3. Characterization of a Graph $G$ with $\psi_{nhd}(G) = n - 1$

**Theorem 3.1** For a connected graph  $G$  of order  $n$ ,  $\psi_{nhd}(G) = n - 1$  if and only if  $G \cong G_1 + P_2$  for some graph  $G_1$  of order  $n - 2$ .

*Proof* Let  $G_1$  be any graph on  $n - 2$  vertices and  $G = G_1 + P_2$ . By Observation 1.4,  $\psi_{nhd}(G) \leq n - 1$ . Now to prove the reverse inequality, let  $V(G) = \{v_1, v_2, \dots, v_{n-2}, v_{n-1}, v_n\}$  with  $v_1, v_2$  being the vertices of  $P_2$ . Define a coloring  $c : V(G) \rightarrow \{1, 2, \dots, n - 1\}$  as follows:

$$c(v_i) = \begin{cases} 1 & \text{if } i = 1, 2 \\ i - 1 & \text{otherwise} \end{cases}$$

It can be easily seen that  $c$  is a neighborhood pseudo  $k$ -coloring of  $G$  with  $k = n - 1$  implies that  $\psi_{nhd}(G) \geq n - 1$ . Hence  $\psi_{nhd}(G) = n - 1$ .

Conversely, let  $G = (V, E)$  be a connected graph of order  $n$  with  $\psi_{nhd}(G) = n - 1$ . Thus there exists a neighborhood pseudo  $k$ -coloring, say  $c$  with  $k = n - 1$  colors. This implies that all vertices but two in  $V$  receive different colors under  $c$ . Without loss of generality, let the only two vertices receiving the same color be  $v_1$  and  $v_2$  and other  $n - 2$  vertices of  $G$  be  $v_3, v_4, \dots, v_n$ .

Now, for each  $i$ ,  $3 \leq i \leq n$ , we have known that  $c|_{N[v_i]}$  is an injection unless both  $v_1$  and  $v_2$  are in  $N[v_i]$ . Thus each  $v_i$  is adjacent to both  $v_1$  and  $v_2$  in  $G$ . Further, if  $v_1$  is not adjacent to  $v_2$ , then, as  $c$  assigns  $n - 1$  colors to the graph  $G - \{v_2\}$ , we get that  $c|_{N[v_1]}$  is an injection from  $V(G)$  onto  $\{1, 2, \dots, n - 1\}$ , which is a contradiction to the fact that  $c$  is a neighborhood pseudo  $n - 1$  coloring of  $G$ . Thus,  $G \cong P_2 + G_1$  for some graph  $G_1$  on  $n - 2$  vertices.  $\square$

#### §4. A Bound in Terms of the Domination Number

In this section, we establish a bound on the neighborhood pseudochromatic number of a graph in terms of its domination number. Using this result, we give a characterization of graphs  $G$  with  $\psi_{nhd}(G) = 2$ .

**Lemma 4.1** *Every connected graph  $G(V, E)$  has a  $\gamma$ -set  $S$  satisfying the property that for every  $v \in S$ , there exists a vertex  $u \in V - S$  such that  $N(u) \cap S = \{v\}$ .*

*Proof* Consider any  $\gamma$ -set  $S$  of a connected graph  $G$ . We construct a  $\gamma$ -set with the required property as follows. Firstly, we obtain a  $\gamma$ -set of  $G$  with the property that  $\deg_G(v) \geq 2$  whenever  $v \in S$ . Let  $S_1$  be the set of all pendant vertices of  $G$  in  $S$ . If  $S_1 = \emptyset$ , then  $S$  itself is the required set. Otherwise, consider the set  $S_2 = (S - S_1) \cup_{v \in S_1} N(v)$ . It is easily seen that  $S_2$  is a dominating set. Also,  $|S| = |S_2|$  since each vertex of degree 1 in  $S_1$  is replaced by a unique vertex in  $V - S$ . Otherwise, at least two vertices in  $S_1$ , say  $u$  and  $v$ , are replaced by a unique vertex in  $V - S$ , say  $w$ , in which case  $S' = (S - \{u, v\}) \cup \{w\}$  is a dominating set of  $G$  with  $|S'| < |S|$ , a contradiction to the fact that  $S$  is a  $\gamma$ -set. Further,  $\deg(v) \geq 2$  for all  $v \in S_2$  failing which  $G$  will not remain connected. In this case,  $S_2$  is the required set. Now, we replace  $S$  by  $S_2$  and proceed further.

If for all  $v \in S$ , there exists  $u \in V - S$  such that  $N(u) \cap S = \{v\}$ , then we are done with the proof. If not, let  $D = \{v \in S : N(u) \cap S - \{v\} \neq \emptyset, u \in N(v)\}$ . Then, for each vertex  $v$  in  $D$ , every vertex  $u \in N(v)$  is dominated by some vertex  $w \in S - \{v\}$ . We now claim that  $w$  is adjacent to another vertex  $x \neq u \in V - S$ . Otherwise,  $(S - \{v, w\}) \cup \{u\}$  is a dominating set having lesser elements than in  $S$ , again a contradiction. Now, replace  $S$  by  $(S - \{v\}) \cup \{u\}$ . Repeating this procedure for every vertex in  $D$  will provide a  $\gamma$ -set  $S$  of  $G$  with the property that for all  $v \in S$ , there exists  $u \in V - S$  such that  $N(u) \cap S = \{v\}$ .  $\square$

**Theorem 4.2** *For any graph  $G$ ,  $\psi_{nhd}(G) \geq \gamma(G)$ .*

*Proof* Let  $G = (V, E)$  be a connected graph with  $V = \{v_1, v_2, \dots, v_n\}$  with  $\gamma(G) = k$ . By Lemma 4.1,  $G$  has a  $\gamma$ -set, say  $S$ , satisfying the property that for all  $v \in S$ , there exists  $u \in V - S$  such that  $N(u) \cap S = \{v\}$ . Without loss in generality, we take  $S = \{v_1, v_2, \dots, v_k\}$ ,  $S_1$  as the set of all those vertices in  $V - S$  which are adjacent to exactly one vertex in  $S$  and  $S_2$  as the set of all the remaining vertices in  $V - S$  so that  $V = S \cup S_1 \cup S_2$ .

We define a coloring  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  as follows:

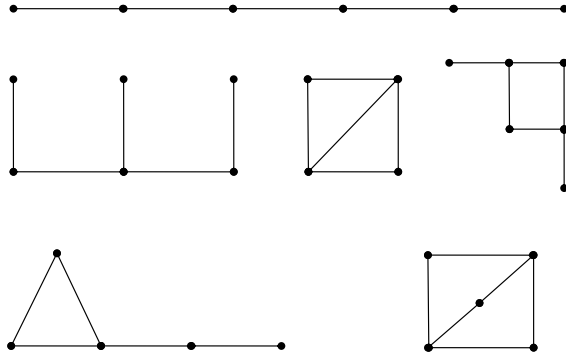
$$c(v_i) = \begin{cases} i & \text{if } v_i \in S \\ j & \text{if } v_i \in S_1 \text{ where } j \text{ is the index of the vertex in } S \text{ adjacent to } v_i \\ k & \text{otherwise where } k \text{ is the index of any vertex in } S \text{ adjacent to } v_i \end{cases}$$

Then for every vertex  $v_i \in S$ , there exists a vertex, say  $v_j$  in  $V - S$  with  $c(v_i) = c(v_j)$  and vice-versa. This ensures that  $c$  is a neighborhood pseudocoloring of  $G$ . Hence  $\psi_{nhd}(G) \geq k = \gamma(G)$ . The result obtained for connected graphs can be easily extended to disconnected graphs.  $\square$

### §5. Characterization of a Graph $G$ with $\psi_{nhd}(G) = 2$

Using the results in Section 4, we give a characterization of a graph  $G$  with pseudochromatic number 2 through the following observation in this section.

**Observation 5.1** The following are the six forbidden subgraphs in any non-trivial connected graph  $G$  with  $\psi_{nhd}(G) \leq 2$ , i. e., a non-trivial connected graph  $G$  has  $\psi_{nhd}(G) \geq 3$  if  $G$  has a subgraph isomorphic to one of the six graphs in Figure 6.



**Figure 6.** Forbidden subgraphs in a non-trivial connected graph with  $\psi_{nhd}(G) \leq 2$

*Proof* The result follows directly from Observation 1.5 and the fact that the neighborhood pseudochromatic number of each of the graphs in Figure 6 is 3.  $\square$

**Theorem 5.2** For a non-trivial connected graph  $G$ ,  $\psi_{nhd}(G) = 2$  if and only if  $G$  is isomorphic to one of the three graphs  $G_1, G_2$  or  $G_3$  or is a member of one of the graph families  $G_4, G_5, G_6, G_7$  or  $G_8$  in Figure 7.

*Proof* Let  $G$  be a non-trivial connected graph. Suppose  $G$  is isomorphic to one of the three graphs  $G_1, G_2$  or  $G_3$  or is a member of one of the graph families  $G_4, G_5, G_6, G_7$  or  $G_8$  in Figure 7. Then it is easy to observe that  $\psi_{nhd}(G) = 2$ .

Conversely, suppose  $\psi_{nhd}(G) = 2$ . By Theorem 4.2,  $\gamma(G) \leq \psi_{nhd}(G) = 2$  implies that

either  $\gamma(G) = 1$  or  $\gamma(G) = 2$ .

If  $\gamma(G) = 1$ , then  $G$  is a star  $K_{1,n}$ ,  $n \geq 1$  or is isomorphic to  $G_1$  or a member of the family  $G_4$  in Figure 7 or has a subgraph isomorphic to  $H_3$  in Figure 6. Similarly, if  $\gamma(G) = 2$ , then  $G$  is one of the graphs  $G_2$  or  $G_3$  or is a member of the family  $G_5, G_6, G_7$  or  $G_8$  in Figure 7 or has a subgraph isomorphic to one of the graphs in Figure 6.

However, since  $\psi_{nhd}(G) = 2$ , by Observation 5.1,  $G$  cannot have a subgraph isomorphic to any of the graphs in Figure 6. Thus, the only possibility is that  $G$  is isomorphic to one of the three graphs  $G_1, G_2$  or  $G_3$  or is a member of one of the graph families  $G_4, G_5, G_6, G_7$  or  $G_8$  in Figure 7.  $\square$

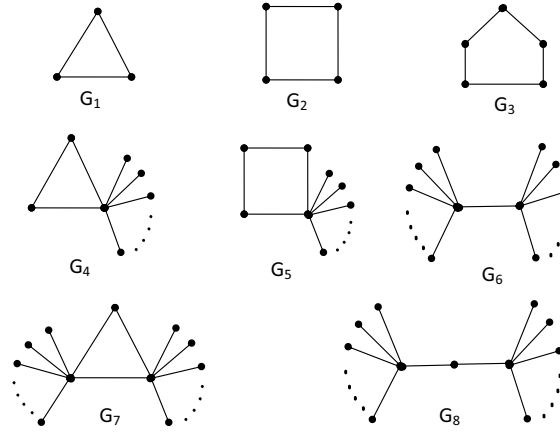


Figure 7. Graphs or graph families with  $\psi_{nhd}(G) = 2$

## §6. Conclusion

In this paper, we have obtained the neighborhood pseudochromatic number of some standard graphs. We have established some trivial lower bounds on this number. Improving on these lower bounds remains an interesting open problem.

We have also characterized graphs  $G$  for which  $\psi_{nhd}(G) = 1, 2, n - 1$  or  $n$ . However, the problem of characterizing graphs for which  $\psi_{nhd}(G) = 3$  still remains open.

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## Signed Domatic Number of Directed Circulant Graphs

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**Abstract:** A function  $f : V \rightarrow \{-1, 1\}$  is a signed dominating function (SDF) of a directed graph  $D$  ([4]) if for every vertex  $v \in V$ ,

$$f(N^-[v]) = \sum_{u \in N^-[v]} f(u) \geq 1.$$

In this paper, we introduce the concept of signed efficient dominating function (SEDF) for directed graphs. A SDF of a directed graph  $D$  is said to be SEDF if for every vertex  $v \in V$ ,  $f(N^-[v]) = 1$  when  $|N^-[v]|$  is odd and  $f(N^-[v]) = 2$  when  $|N^-[v]|$  is even. We study the signed domatic number  $d_S(D)$  of directed graphs. Actually, we give a lower bound for signed domination number  $\gamma_S(G)$  and an upper bound for  $d_S(G)$ . Also we characterize some classes of directed circulant graphs for which  $d_S(D) = \delta^-(D) + 1$ . Further, we find a necessary and sufficient condition for the existence of SEDF in circulant graphs in terms of covering projection.

**Key Words:** Signed graphs, signed domination, signed efficient domination, covering projection.

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### §1. Introduction

Let  $D$  be a simple finite digraph with vertex set  $V(D) = V$  and arc set  $E(D) = E$ . For every vertex  $v \in V$ , in-neighbors of  $v$  and out-neighbors of  $v$  are defined by  $N^-[v] = N_D^-[v] = \{u \in V : (u, v) \in E\}$  and  $N^+[v] = N_D^+[v] = \{u \in V : (v, u) \in E\}$  respectively. For a vertex  $v \in V$ ,  $d_D^+(v) = d^+(v) = |N^+(v)|$  and  $d_D^-(v) = d^-(v) = |N^-(v)|$  respectively denote the outdegree and indegree of the vertex  $v$ . The minimum and maximum indegree of  $D$  are denoted by  $\delta^-(D)$  and  $\Delta^-(D)$  respectively. Similarly the minimum and maximum outdegree of  $D$  are denoted by  $\delta^+(D)$  and  $\Delta^+(D)$  respectively.

In [2], J.E. Dunbar et al. introduced the concept of signed domination number of an undirected graph. In 2005, Bohdan Zelinka [1] extended the concept of signed domination in directed graphs.

A function  $f : V \rightarrow \{-1, 1\}$  is a signed dominating function (SDF) of a directed graph  $D$

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([4]) if for every vertex  $v \in V$ ,

$$f(N^-[v]) = \sum_{u \in N^-[v]} f(u) \geq 1.$$

The signed domination number, denoted by  $\gamma_S(D)$ , is the minimum weight of a signed dominating function of  $D$  [4]. In this paper, we introduce the concept of signed efficient dominating function (SEDF) for directed graphs. A SDF of a directed graph  $D$  is said to be SEDF if for every vertex  $v \in V$ ,  $f(N^-[v]) = 1$  when  $|N^-[v]|$  is odd and  $f(N^-[v]) = 2$  when  $|N^-[v]|$  is even.

A set  $\{f_1, f_2, \dots, f_d\}$  of signed dominating functions on a graph (directed graph)  $G$  with the property that

$$\sum_{i=1}^d f_i(x) \leq 1$$

for each vertex  $x \in V(G)$ , is called a signed dominating family on  $G$ . The maximum number of functions in a signed dominating family on  $G$  is the signed domatic number of  $G$ , denoted by  $d_S(G)$ .

The signed domatic number of undirected and simple graphs was introduced by Volkmann and Zelinka [6]. They determined the signed domatic number of complete graphs and complete bipartite graphs. Further, they obtained some bounds for domatic number. They proved the following results.

**Theorem 1.1([6])** *Let  $G$  be a graph of order  $n(G)$  with signed domination number  $\gamma_S(G)$  and signed domatic number  $d_S(G)$ . Then  $\gamma_S(G) \cdot d_S(G) \leq n(G)$ .*

**Theorem 1.1([6])** *Let  $G$  be a graph with minimum degree  $\delta(G)$ , then  $1 \leq d_S(G) \leq \delta(G) + 1$ .*

In this paper, we study some of the properties of signed domination number and signed domatic number of directed graphs. Also, we study the signed domination number and signed domatic number of directed circulant graphs  $Cir(n, A)$ . Further, we obtain a necessary and sufficient condition for the existence of SEDF in  $Cir(n, A)$  in terms of covering projection.

## §2. Signed Domatic Number of Directed Graphs

In this section, we study the signed domatic number of directed graphs. Actually, we give a lower bound for  $\gamma_S(G)$  and an upper bound for  $d_S(G)$ .

**Theorem 2.1** *Let  $D$  be a directed graph of order  $n$  with signed domination number  $\gamma_S(D)$  and signed domatic number  $d_S(D)$ . Then  $\gamma_S(D) d_S(D) \leq n$ .*

*Proof* Let  $d = d_S(D)$  and  $\{f_1, f_2, \dots, f_d\}$  be a corresponding signed dominating family

on  $D$ . Then

$$\begin{aligned} d\gamma_S(D) &= \sum_{i=1}^d \gamma_S(D) \leq \sum_{i=1}^d \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^d f_i(v) \leq \sum_{v \in V(D)} 1 = n. \end{aligned} \quad \square$$

In [4], H. Karami et al. proved the following result.

**Theorem 2.2**([4]) *Let  $D$  be a digraph of order  $n$  in which  $d^+(x) = d^-(x) = k$  for each  $x \in V$ , where  $k$  is a nonnegative integer. Then  $\gamma_S(D) \geq \frac{n}{k+1}$ .*

In the view of Theorems 2.1 and 2.2, we have the following corollary.

**Corollary 2.3** *Let  $D$  be a digraph of order  $n$  in which  $d^+(x) = d^-(x) = k$  for each  $x \in V$ , where  $k$  is a nonnegative integer. Then  $d_S(D) \leq k+1$ .*

The next result is a more general form of the above corollary.

**Theorem 2.4** *Let  $D$  be a directed graph with minimum in degree  $\delta^-(D)$ , then  $1 \leq d_S(D) \leq \delta^-(D) + 1$ .*

*Proof* Note that the function  $f : V(D) \rightarrow \{+1, -1\}$ , defined by  $f(v) = +1$  for all  $v \in V(D)$ , is a SDF and  $\{f\}$  is a signed domatic family on  $D$ . Hence  $d_S(D) \geq 1$ . Let  $d = d_S(D)$  and  $\{f_1, f_2, \dots, f_d\}$  be a corresponding signed dominating family of  $D$ . Let  $v \in V$  be a vertex of minimum degree  $\delta^-(D)$ .

Then,

$$\begin{aligned} d &= \sum_{i=1}^d 1 \leq \sum_{i=1}^d \sum_{x \in N^-[v]} f_i(x) \\ &= \sum_{x \in N^-[v]} \sum_{i=1}^d f_i(x) \leq \sum_{x \in N^-[v]} 1 = \delta^-(D) + 1. \end{aligned} \quad \square$$

**Theorem 2.5**([6]) *The signed domination number is an odd integer.*

**Remark 2.6** The signed domination number of a directed graph may not be an odd integer. For example, for the directed circulant graph  $Cir(10, \{1, 2, 3, 4\})$ , the signed domination number is 2.

**Theorem 2.7** *Let  $D$  be a directed graph such that  $d^+(x) = d^-(x) = 2g$  for each  $x \in V$  and let  $u \in V(D)$ . If  $d = d_S(D) = 2g + 1$  and  $\{f_1, f_2, \dots, f_d\}$  is a signed domatic family of  $D$ , then*

$$\sum_{i=1}^d f_i(u) = 1 \quad \text{and} \quad \sum_{x \in N^-[u]} f_i(x) = 1$$

for each  $u \in V(D)$  and each  $1 \leq i \leq 2g + 1$ .

*Proof* Since  $\sum_{i=1}^d f_i(u) \leq 1$ , this sum has at least  $g$  summands which have the value  $-1$ . Since  $\sum_{x \in N^-[u]} f_i(x) \geq 1$  for each  $1 \leq i \leq 2g + 1$ , this sum has at least  $g + 1$  summands which have the value  $1$ .

Also the sum

$$\sum_{x \in N^-[u]} \sum_{i=1}^d f_i(x) = \sum_{i=1}^d \sum_{x \in N^-[u]} f_i(x)$$

has at least  $dg$  summands of value  $-1$  and at least  $d(g + 1)$  summands of value  $1$ . Since the sum

$$\sum_{x \in N^-[u]} \sum_{i=1}^d f_i(x) = \sum_{i=1}^d \sum_{x \in N^-[u]} f_i(x)$$

contains exactly  $d(2g + 1)$  summands, it is easy to observe that  $\sum_{i=1}^d f_i(u)$  have exactly  $g$  summands of value  $-1$  and  $\sum_{x \in N^-[u]} f_i(x)$  has exactly  $g + 1$  summands of value  $1$  for each  $1 \leq i \leq r + 1$ .

Hence we must have

$$\sum_{i=1}^d f_i(u) = 1 \quad \text{and} \quad \sum_{x \in N^-[u]} f_i(x) = 1$$

for each  $u \in V(D)$  and for each  $1 \leq i \leq 2g + 1$ .  $\square$

### §3. Signed Domatic Number and SEDF in Directed Circulant Graphs

Let  $\Gamma$  be a finite group and  $e$  be the identity element of  $\Gamma$ . A generating set of  $\Gamma$  is a subset  $A$  such that every element of  $\Gamma$  can be written as a product of finitely many elements of  $A$ . Assume that  $e \notin A$  and  $a \in A$  implies  $a^{-1} \in A$ . Then the corresponding *Cayley graph* is a graph  $G = (V, E)$ , where  $V(G) = \Gamma$  and  $E(G) = \{(x, y)_a \mid x, y \in V(G), y = xa \text{ for some } a \in A\}$ , denoted by  $\text{Cay}(\Gamma, A)$ . It may be noted that  $G$  is connected regular graph degree of degree  $|A|$ . A Cayley graph constructed out of a finite cyclic group  $(\mathbb{Z}_n, \oplus_n)$  is called a circulant graph and it is denoted by  $\text{Cir}(n, A)$ , where  $A$  is a generating set of  $\mathbb{Z}_n$ . When we leave the condition that  $a \in A$  implies  $a^{-1} \in A$ , then we get directed circulant graphs. In a directed circulant graph  $\text{Cir}(n, A)$ , for every vertex  $v$ ,  $|N^-[v]| = |N^+[v]| = |A| + 1$ .

Throughout this section,  $n(\geq 3)$  is a positive integer,  $\Gamma = (\mathbb{Z}_n, \oplus_n)$ , where  $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$  and  $D = \text{Cir}(n, A)$ , where  $A = \{1, 2, \dots, r\}$  and  $1 \leq r \leq n - 1$ . From here, the operation  $\oplus_n$  stands for modulo  $n$  addition in  $\mathbb{Z}_n$ . In this section, we characterize the circulant graphs for which  $d_S(D) = \delta^-(D) + 1$ . Also we find a necessary and sufficient condition for the existence of SEDF in  $\text{Cir}(n, A)$  in terms of covering projection.

**Theorem 3.1** *Let  $n \geq 3$  and  $1 \leq r \leq n - 1$  ( $r$  is even) be integers and  $D = \text{Cir}(n, \{1, 2, \dots, r\})$*

be a directed circulant graph. Then  $d_S(D) = r + 1$  if, and only if,  $r + 1$  divides  $n$ .

*Proof* Assume that  $d_S(D) = r + 1$  and  $\{f_1, f_2, \dots, f_{r+1}\}$  is a signed domatic family on  $D$ . Since  $d^+(v) = d^-(v) = r$ , for all  $v \in V(D)$ , by Theorems 2.1 and 2.2, we have  $\gamma_S(D) = \frac{n}{r+1}$ .

Suppose  $n$  is not a multiple of  $r + 1$ . Then  $n = k(r + 1) + i$  for some  $1 \leq i \leq r$ . Let  $t = \gcd(i, r + 1)$ . Then there exist relatively prime integers  $p$  and  $q$  such that  $r + 1 = qt$  and  $i = pt$ . Let  $a$  and  $b$  be the smallest integers such that  $a(r + 1) = bn$ . Then  $\gcd(a, b) = 1$ ; otherwise  $a$  and  $b$  will not be the smallest.

Now  $aqt = a(r + 1) = b(k(r + 1) + i) = b(kqt + pt) = bt(kq + p)$ . That is  $aq = b(kp + q)$ . Note that  $\gcd(a, b) = \gcd(p, q) = 1$ . Hence  $a = kp + q$  and  $b = q$ . Thus the subgroup  $\langle r + 1 \rangle$  of the finite cyclic group  $\mathbb{Z}_n$ , generated by  $r + 1$ , must have  $kp + q$  elements. But  $t = \frac{r+1}{q} = \frac{n}{kp+q}$ . Thus the subgroup  $\langle t \rangle$  of  $\mathbb{Z}_n$ , generated by the element  $t$ , also have  $kp + q$  elements and hence  $\langle t \rangle = \langle r + 1 \rangle$ . Since  $d_S(D) = r + 1$  and  $\{f_1, f_2, \dots, f_{r+1}\}$  is a signed domatic family of  $D$ , by Theorem 2.7, we have

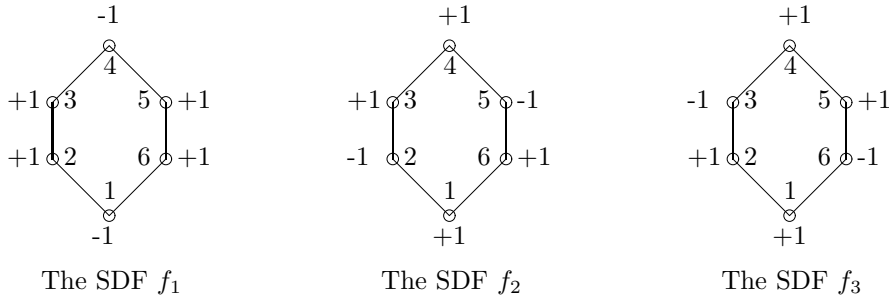
$$\sum_{i=1}^d f_i(u) = 1 \quad \text{and} \quad \sum_{x \in N^-[u]} f_i(x) = 1$$

for each  $u \in V(D)$  and each  $1 \leq i \leq r + 1$ .

From the above fact and since  $|N^-[v]| = r + 1$  for all  $v \in V(D)$ , it follows that if  $f(a) = +1$ , then  $f(a \oplus_n (r + 1)) = +1$  and if  $f(a) = -1$ , then  $f(a \oplus_n (r + 1)) = -1$ . Thus all the elements of the subgroup  $\langle t \rangle$  have the same sign and hence all the elements in each of the co-set of  $\langle t \rangle$  have the same sign. By Lagrange's theorem on subgroups,  $\mathbb{Z}_n$  can be written as the union of co-sets of  $\langle t \rangle = \langle r + 1 \rangle$ . This means that  $\gamma_S(D)$  must be a multiple of the number of elements of  $\langle t \rangle$ , that is a multiple of  $\frac{n}{t}$  (since  $n$  is a multiple of  $t$ ). Since  $t < r + 1$ , it follows that  $\frac{n}{r+1} < \frac{n}{t} \leq \gamma_S(D)$ , a contradiction to  $\gamma_S(D) = \frac{n}{r+1}$ .

Conversely suppose  $r + 1$  divides  $n$ . By theorem 2.4,  $d_S(D) \leq r + 1$ . For each  $1 \leq i \leq r + 1$ , define  $f_i(i) = f_i(i \oplus_{r+1} 1) = \dots = f_i(i \oplus_{r+1} (g - 1)) = -1$  and  $f_i(i \oplus_{r+1} g) = \dots = f_i(i \oplus_{r+1} 2g) = +1$ , where  $g = \frac{r}{2}$ , and for the remaining vertices,  $f_i(v) = f_i(v \bmod (r + 1))$  for  $v \in \{r + 2, r + 3, \dots, n\}$ .

Notice that  $\{f_1, f_2, \dots, f_{r+1}\}$  are SDFs on  $D$  with the property that  $\sum_{i=1}^{r+1} f_i(x) \leq 1$  for each vertex  $x \in V(D)$ . Hence  $d_S(D) \geq r + 1$ .  $\square$



**Fig.1**

**Example 3.2** Let  $n = 6$  and  $r = 2$ . Then  $n$  is a multiple of  $r + 1$ , and  $r + 1 = 3$  SDFs  $f_1$ ,  $f_2$  and  $f_3$  (as discussed in the above theorem) of  $D = \text{Cir}(6, \{1, 2\})$  are as given in Fig.1 following, where  $V(D) = \{1, 2, 3, 4, 5, 6\}$ .

**Theorem 3.3** Let  $n \geq 3$  be an integer and  $1 \leq r \leq n - 1$  be an integer. Let  $D = \text{Cir}(n, \{1, 2, \dots, r\})$  be a directed circulant graph. If  $n$  is a multiple of  $r+1$ , then  $\gamma_S(D) = \frac{n}{r+1}$ .

*Proof* Assume that  $n$  is a multiple of  $r + 1$ . By Theorem 2.2, we have  $\gamma_S(D) \geq \frac{n}{r+1}$ . It remains to show that there exists a SDF  $f$  with the property that  $f(D) = \frac{n}{r+1}$ .

Define a function  $f$  on  $V(D)$  by  $f(1) = f(2) = \dots = f(g) = -1$  and  $f(g+1) = f(g+2) = \dots = f(2g+1) = +1$ , where  $g = \frac{n}{r+1}$ ; and for the remaining vertices,  $f(v) = f(v \bmod(r+1))$  for  $v \in \{r+2, r+3, \dots, n\}$ .

It is clear that  $f$  is a SDF and

$$f(D) = (g+1) \left( \frac{n}{r+1} \right) - (g) \left( \frac{n}{r+1} \right) = \frac{n}{r+1}. \quad \square$$

A graph  $\tilde{G}$  is called a covering graph of  $G$  with covering projection  $f : \tilde{G} \rightarrow G$  if there is a surjection  $f : V(\tilde{G}) \rightarrow V(G)$  such that  $f|_{N(\tilde{v})} : N(\tilde{v}) \rightarrow N(v)$  is a bijection for any vertex  $v \in V(G)$  with  $\tilde{v} \in f^{-1}(v)$  ([5]).

In 2001, J.Lee has studied the domination parameters through covering projections ([5]). In this paper, we introduce the concept of covering projection for directed graphs and we study the SDF through covering projections.

A directed graph  $D$  is called a covering graph of another directed graph  $H$  with covering projection  $f : D \rightarrow H$  if there is a surjection  $f : V(D) \rightarrow V(H)$  such that  $f|_{N^+(u)} : N^+(u) \rightarrow N^+(v)$  and  $f|_{N^-(u)} : N^-(u) \rightarrow N^-(v)$  are bijections for any vertex  $v \in V(H)$  with  $u \in f^{-1}(v)$ .

**Lemma 3.4** Let  $f : D \rightarrow H$  be a covering projection from a directed graph  $D$  on to another directed graph  $H$ . If  $H$  has a SEDF, then so is  $D$ .

*Proof* Let  $f : D \rightarrow H$  be a covering projection from a directed graph  $D$  on to another directed graph  $H$ . Assume that  $H$  has a SEDF  $h : V(H) \rightarrow \{1, -1\}$ .

Define a function  $g : V(D) \rightarrow \{1, -1\}$  defined by  $g(u) = h(f(u))$  for all  $u \in V(D)$ . Since  $h$  is a function from  $V(H)$  to  $\{1, -1\}$  and  $f : V(D) \rightarrow V(H)$ ,  $g$  is well defined. We prove that for the graph  $D$ ,  $g$  is a SEDF.

Firstly, we prove  $g(N^-[u]) = 1$  when  $u \in V(D)$  and  $|N^-[u]|$  is odd. In fact, let  $u \in V(D)$  and assume that  $|N^-[u]|$  is odd. Since  $f$  is a covering projection,  $|N^-(u)|$  and  $|N^-(f(u))|$  are equal. Also  $f|_{N^-(u)} : N^-(u) \rightarrow N^-(f(u))$  is a bijection. Also for each vertex  $x \in N^-[u]$ , we have  $g(x) = h(f(x))$ . Since  $h(N^-[f(u)]) = 1$ , we have  $g(N^-[u]) = 1$ . Similarly, we can prove that  $g(N^-[u]) = -1$  when  $u \in V(D)$  and  $|N^-[u]|$  is even. Hence  $g$  is a SEDF on  $D$ .  $\square$

**Theorem 3.5** Let  $D = \text{Cir}(n, \{1, 2, \dots, r\})$ ,  $r = 2g$  and  $\gamma_S(D) = \frac{n}{r+1}$ . Then  $D$  has a SEDF if and only if, there exists a covering projection from  $D$  onto the graph  $H = \text{Cir}(r+1, \{1, 2, \dots, r\})$ .

*Proof* Suppose  $D$  has a SEDF  $f$ . Then  $\sum_{x \in N^-[u]} f(x) = 1$  for all  $u \in v(D)$ . Thus we can have  $f(a \oplus_n r + 1) = \pm 1$  when ever  $f(a) = \pm 1$ . Thus the elements of the subgroup  $\langle r + 1 \rangle$ , generated by  $r + 1$  have the same sign.

Suppose  $n$  is not a multiple of  $r + 1$ , then  $n = i(r + 1) + j$  for some  $1 \leq j \leq r$ . Let  $t = \gcd(r + 1, j)$ . Then by Theorem 3.1, we have  $\gamma_S(D) > \frac{n}{r+1}$ , a contradiction. Hence  $n$  must be a multiple of  $r + 1$ .

In this case, define  $F : D \rightarrow H = Cir(r + 1, \{1, 2, \dots, r\})$ , defined by  $F(x) = x \pmod{r+1}$ . Note that,  $|N^-[x]| = |N^+[x]| = |N^-[y]| = |N^+[y]| = r + 1$  for all  $x \in V(D)$  and  $y \in V(H)$ . We prove that the function  $F$  is a covering projection.

Let  $x \in V(D)$ . Then  $F(x) = x \pmod{r+1} = i$  for some  $i \in V(H)$  with  $1 \leq i \leq r + 1$ . Note that by the definition of  $D$  and  $H$ ,  $N^+(x) = \{x \oplus_n 1, x \oplus_n 2, \dots, x \oplus_n r\}$  and  $N^+(i) = \{i \oplus_{r+1} 1, i \oplus_{r+1} 2, \dots, i \oplus_{r+1} r\}$ .

Also, for each  $1 \leq g \leq r + 1$ , we have  $F(x \oplus_n g) = (x \oplus_n g) \pmod{r+1} = (x \oplus_{r+1} g) \pmod{r+1}$  (since  $n$  is a multiple of  $r + 1$ ).

Thus  $F(x \oplus_n g) = (i \oplus_{r+1} g) \pmod{r+1}$  (since  $x \pmod{r+1} = i$ ). Thus  $F|_{N^+(x)} : N^+(x) \rightarrow N^+(F(x))$  is a bijection. Similarly, we can prove that  $F|_{N^-(x)} : N^-(x) \rightarrow N^-(F(x))$  is also a bijection and hence  $F$  is a covering projection from  $D$  onto  $H$ .

Conversely, suppose there exists a covering projection  $F$  from  $D$  onto the graph  $H = Cir(r + 1, \{1, 2, \dots, r\})$ . Define  $h : V(H) \rightarrow \{+1, -1\}$  defined by  $h(x) = -1$  when  $1 \leq x \leq g$  and  $h(x) = +1$  when  $g + 1 \leq x \leq 2g + 1$ . Then  $h$  is a SEDF of  $H$  and hence by Lemma 3.4,  $G$  has a SEDF.  $\square$

**Theorem 3.6** Let  $D = Cir(n, \{1, 2, \dots, r\})$ ,  $r$  be an odd integer and  $\gamma_S(D) = \frac{n}{r+1}$ . Then  $D$  has a SEDF if and only if, there exists a covering projection from  $D$  onto the graph  $H = Cir(r + 1, \{1, 2, \dots, r\})$ .

*Proof* Suppose  $D$  has a SEDF  $f$ . Let  $H = Cir(r + 1, \{1, 2, \dots, r\})$ . Note that,  $|N^-[x]| = |N^+[x]| = |N^-[y]| = |N^+[y]| = r + 1 = 2g$  (say), an even integer, for all  $x \in V(D)$  and  $y \in V(H)$ . Thus

$$\sum_{x \in N^-[u]} f(x) = 2$$

for all  $u \in v(D)$ . Thus we can have  $f(a \oplus_n r + 1) = \pm 1$  when ever  $f(a) = \pm 1$ . Thus the elements of the subgroup  $\langle r + 1 \rangle$ , generated by  $r + 1$  have the same sign.

Suppose  $n$  is not a multiple of  $r + 1$ , As in the proof of Theorem 3.5, we can get a contradiction. Also the function  $F$  defined in Theorem 3.5 is a covering projection from  $D$  onto  $H$ .

Conversely suppose there exists a covering projection  $F$  from  $D$  onto the graph  $H = Cir(r + 1, \{1, 2, \dots, r\})$ . Define  $h : V(H) \rightarrow \{+1, -1\}$  defined by  $h(x) = -1$  when  $1 \leq x \leq g - 1$  and  $h(x) = +1$  when  $g \leq x \leq 2g$ . Note that  $h$  is a SEDF of  $H$  and

$$\sum_{x \in N^-[u]} f(x) = 2$$

for all  $u \in v(G)$ . Thus by Lemma 3.4,  $G$  has a SEDF.  $\square$

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## Neighborhood Total 2-Domination in Graphs

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**Abstract:** Let  $G = (V, E)$  be a graph without isolated vertices. A set  $S \subseteq V$  is called the neighborhood total 2-dominating set (nt2d-set) of a graph  $G$  if every vertex in  $V - S$  is adjacent to at least two vertices in  $S$  and the induced subgraph  $\langle N(S) \rangle$  has no isolated vertices. The minimum cardinality of a nt2d-set of  $G$  is called the neighborhood total 2-domination number of  $G$  and is denoted by  $\gamma_{2nt}(G)$ . In this paper we initiate a study of this parameter.

**Key Words:** Neighborhood total domination, neighborhood total 2-domination.

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### §1. Introduction

The graph  $G = (V, E)$  we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of  $G$  are denoted by  $n$  and  $m$  respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3] and Haynes et.al [5-6].

Let  $v \in V$ . The open neighborhood and closed neighborhood of  $v$  are denoted by  $N(v)$  and  $N[v] = N(v) \cup \{v\}$  respectively. If  $S \subseteq V$  then  $N(S) = \bigcup_{v \in S} N(v)$  and  $N[S] = N(S) \cup S$ .

If  $S \subseteq V$  and  $u \in S$  then the private neighbor set of  $u$  with respect to  $S$  is defined by  $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$ . The chromatic number  $\chi(G)$  of a graph  $G$  is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour.  $H(m_1, m_2, \dots, m_n)$  denotes the graph obtained from the graph  $H$  by attaching  $m_i$  edges to the vertex  $v_i \in V(H)$ ,  $1 \leq i \leq n$ .  $H(P_{m_1}, P_{m_2}, \dots, P_{m_n})$  is the graph obtained from the graph  $H$  by attaching the end vertex of  $P_{m_i}$  to the vertex  $v_i$  in  $H$ ,  $1 \leq i \leq n$ .

A subset  $S$  of  $V$  is called a dominating set if every vertex in  $V - S$  is adjacent to at least one vertex in  $S$ . The minimum cardinality of a dominating set is called the domination number of  $G$  and is denoted by  $\gamma(G)$ . Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al., Fink and Jacobson [4] introduced the concept of  $k$ -domination in graphs. A dominating set  $S$  of  $G$  is called a  $k$ -dominating set if every vertex in  $V - S$  is adjacent to at least  $k$  vertices in  $S$ . The minimum cardinality of a  $k$ -dominating set is called  $k$ -domination number of  $G$  and is denoted by  $\gamma_k(G)$ . F. Harary and T.W. Haynes [4] introduced the concept of double

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domination in graphs. A dominating set  $S$  of  $G$  is called a double dominating set if every vertex in  $V - S$  is adjacent to at least two vertices in  $S$  and every vertex in  $S$  is adjacent to at least one vertex in  $S$ . The minimum cardinality of a double dominating set is called double domination number of  $G$  and is denoted by  $dd(G)$ . S. Arumugam and C. Sivagnanam [1], [2] introduced the concept of neighborhood connected domination and neighborhood total domination in graphs. A dominating set  $S$  of a connected graph  $G$  is called a neighborhood connected dominating set (ncd-set) if the induced subgraph  $\langle N(S) \rangle$  is connected. The minimum cardinality of a ncd-set of  $G$  is called the neighborhood connected domination number of  $G$  and is denoted by  $\gamma_{nc}(G)$ . A dominating set  $S$  of a graph  $G$  without isolate vertices is called the neighborhood total dominating set (ntd-set) if the induced subgraph  $\langle N(S) \rangle$  has no isolated vertices. The minimum cardinality of a ntd-set of  $G$  is called the neighborhood total domination number of  $G$  and is denoted by  $\gamma_{nt}(G)$ . Sivagnanam et.al [8] studied the concept of neighborhood connected 2-domination in graphs. A set  $S \subseteq V$  is called a neighborhood connected 2-dominating set (nc2d-set) of a connected graph  $G$  if every vertex in  $V - S$  is adjacent to at least two vertices in  $S$  and the induced subgraph  $\langle N(S) \rangle$  is connected. The minimum cardinality of a nc2d-set of  $G$  is called the neighborhood connected 2-domination number of  $G$  and is denoted by  $\gamma_{2nc}(G)$ . In this paper we introduce the concept of neighborhood total 2-domination and initiate a study of the corresponding parameter.

Through out this paper we assume the graph  $G$  has no isolated vertices.

## §2. Neighborhood Total 2-Dominating Sets

**Definition 2.1** A set  $S \subseteq V$  is called the neighborhood total 2-dominating set (nt2d-set) of a graph  $G$  if every vertex in  $V - S$  is adjacent to at least two vertices in  $S$  and the induced subgraph  $\langle N(S) \rangle$  has no isolated vertices. The minimum cardinality of a nt2d-set of  $G$  is called the neighborhood total 2-domination number of  $G$  and is denoted by  $\gamma_{2nt}(G)$ .

**Remark 2.2** (i) Clearly  $\gamma_{2nt}(G) \geq \gamma_{nt}(G) \geq \gamma(G)$ ,  $\gamma_{2nt}(G) \leq \gamma_{2nc}(G)$  and  $\gamma_{2nt}(G) \geq \gamma_2(G)$ .

(ii) A graph  $G$  has  $\gamma_{2nt}(G) = 2$  if and only if there exist two vertices  $u, v \in V$  such that (a)  $\deg u = \deg v = n - 1$  or (b)  $\deg u = \deg v = n - 2$ ,  $uv \notin E(G)$  with  $G - \{u, v\}$  has no isolated vertices. Thus  $\gamma_{2nt}(G) = 2$  if and only if  $G$  is isomorphic to either  $H + K_2$  for some graph  $H$  or  $H + \overline{K_2}$  for some graph  $H$  with  $\delta(H) \geq 1$ .

### Examples A

- (1)  $\gamma_{2nt}(K_n) = 2, n \geq 2$ ;
- (2)  $\gamma_{2nt}(K_{1,n-1}) = n$ ;
- (3) Let  $K_{r,s}$  be a complete bipartite graph and not a star then

$$\gamma_{2nt}(K_{r,s}) = \begin{cases} 3 & r \text{ or } s = 2 \\ 4 & r, s \geq 3 \end{cases}$$

$$(4) \gamma_{2nt}(W_n) = \left\lceil \frac{n}{3} \right\rceil + 1.$$

**Theorem 2.3** For any non trivial path  $P_n$ ,

$$\gamma_{2nt}(P_n) = \begin{cases} \left\lceil \frac{3n}{5} \right\rceil + 1 & \text{if } n \equiv 0, 3 \pmod{5} \\ \left\lceil \frac{3n}{5} \right\rceil & \text{otherwise} \end{cases}$$

*Proof* Let  $P_n = (v_1, v_2, \dots, v_n)$  and  $n = 5k + r$ , where  $0 \leq r \leq 4$ ,  $S = \{v_i \in V : i = 5j + 1, 5j + 3, 5j + 4, 0 \leq j \leq k\}$  and

$$S_1 = \begin{cases} S & \text{if } n \equiv 1, 4 \pmod{5} \\ S \cup \{v_n\} & \text{if } n \equiv 0, 2 \pmod{5} \\ S \cup \{v_{n-1}\} & \text{if } n \equiv 3 \pmod{5} \end{cases}$$

Clearly  $S_1$  is a nt2d-set of  $P_n$  and hence

$$\gamma_{2nt}(P_n) \leq \begin{cases} \left\lceil \frac{3n}{5} \right\rceil + 1 & \text{if } n \equiv 0, 3 \pmod{5} \\ \left\lceil \frac{3n}{5} \right\rceil & \text{otherwise} \end{cases}$$

Let  $S$  be any  $\gamma_{2nt}$ -set of  $P_n$ . Since any 2-dominating set  $D$  of order either  $\left\lceil \frac{3n}{5} \right\rceil, n \equiv 0, 3 \pmod{5}$  or  $\left\lceil \frac{3n}{5} \right\rceil - 1, n \equiv 1, 2, 4 \pmod{5}$ ,  $N(D)$  contains isolated vertices, we have

$$|S| \geq \begin{cases} \left\lceil \frac{3n}{5} \right\rceil + 1 & \text{if } n \equiv 0, 3 \pmod{5} \\ \left\lceil \frac{3n}{5} \right\rceil & \text{otherwise} \end{cases}$$

Hence,

$$\gamma_{2nt}(P_n) = \begin{cases} \left\lceil \frac{3n}{5} \right\rceil + 1 & \text{if } n \equiv 0, 3 \pmod{5} \\ \left\lceil \frac{3n}{5} \right\rceil & \text{otherwise.} \end{cases} \quad \square$$

**Theorem 2.4** For the cycle  $C_n$  on  $n$  vertices  $\gamma_{2nt}(C_n) = \left\lceil \frac{3n}{5} \right\rceil$ .

*Proof* Let  $C_n = (v_1, v_2, \dots, v_n, v_1)$ ,  $n = 5k + r$ , where  $0 \leq r \leq 4$ ,  $S = \{v_i : i = 5j + 1, 5j + 3, 5j + 4, 0 \leq j \leq k\}$  and

$$S_1 = \begin{cases} S \cup \{v_n\} & \text{if } n \equiv 2 \pmod{5} \\ S & \text{otherwise} \end{cases}$$

Clearly  $S_1$  is a nt2d-set of  $C_n$  and hence  $\gamma_{2nt}(C_n) \leq \left\lceil \frac{3n}{5} \right\rceil$ . Now, let  $S$  be any  $\gamma_{2nt}$ -set of  $C_n$ . Since any 2-dominating set  $D$  of order  $\left\lceil \frac{3n}{5} \right\rceil - 1$ ,  $N(D)$  contains isolated vertices, we have

$|S| \geq \lceil \frac{3n}{5} \rceil$ . Hence,

$$\gamma_{2nt}(C_n) = \lceil \frac{3n}{5} \rceil. \quad \square$$

We now proceed to obtain a characterization of minimal nt2d-sets.

**Lemma 2.5** *A superset of a nt2d-set is a nt2d-set.*

*Proof* Let  $S$  be a nt2d-set of a graph  $G$  and let  $S_1 = S \cup \{v\}$ , where  $v \in V - S$ . Clearly  $v \in N(S)$  and  $S_1$  is a 2-dominating set of  $G$ . Suppose there exists an isolated vertex  $y$  in  $\langle N(S_1) \rangle$ . Then  $N(y) \subseteq S - N(S)$  and hence  $y$  is an isolated vertex in  $\langle N(S) \rangle$ , which is a contradiction. Hence  $\langle N(S_1) \rangle$  has no isolated vertices and  $S_1$  is a nt2d-set.  $\square$

**Theorem 2.6** *A nt2d-set  $S$  of a graph  $G$  is a minimal nt2d-set if and only if for every  $u \in S$ , one of the following holds.*

- (1)  $|N(u) \cap S| \leq 1$ ;
- (2) *there exists a vertex  $v \in (V - S) \cap N(u)$  such that  $|N(v) \cap S| = 2$ ;*
- (3) *there exists a vertex  $x \in N(S - \{u\})$  such that  $N(x) \cap N(S - \{u\}) = \phi$ .*

*Proof* Let  $S$  be a minimal nt2d-set and let  $u \in S$ . Let  $S_1 = S - \{u\}$ . Then  $S_1$  is not a nt2d-set. This gives either  $S_1$  is not a 2-dominating set or  $\langle N(S_1) \rangle$  has an isolated vertex. If  $S_1$  is not a 2-dominating set then there exists a vertex  $v \in V - S_1$  such that  $|N(v) \cap S_1| \leq 1$ . If  $v = u$  then  $|N(u) \cap (S - \{u\})| \leq 1$  which gives  $|N(u) \cap S| \leq 1$ . Suppose  $v \neq u$ . If  $|N(v) \cap S_1| < 1$  then  $|N(v) \cap S| \leq 1$  and hence  $S$  is not a 2-dominating set which is a contradiction. Hence  $|N(v) \cap S_1| = 1$ . Thus  $v \in N(u)$ . So  $v \in (V - S) \cap N(u)$  such that  $|N(v) \cap S| = 2$ . If  $S_1$  is a 2-dominating set and if  $x \in N(S_1)$  is an isolated vertex in  $\langle N(S_1) \rangle$  then  $N(x) \cap N(S_1) = \phi$ . Thus  $N(x) \cap N(S - \{u\}) = \phi$ . Conversely, if  $S$  is a nt2d-set of  $G$  satisfying the conditions of the theorem, then  $S$  is 1-minimal nt2d-set and hence the result follows from Lemma 2.5.  $\square$

**Remark 2.7** Any nt2d-set contains all the pendant vertices of the graph.

**Remark 2.8** Since any nt2d-set of a spanning subgraph  $H$  of a graph  $G$  is a nt2d-set of  $G$ , we have  $\gamma_{2nt}(G) \leq \gamma_{2nt}(H)$ .

**Remark 2.9** If  $G$  is a disconnected graph with  $k$  components  $G_1, G_2, \dots, G_k$  then  $\gamma_{2nt}(G) = \gamma_{2nt}(G_1) + \gamma_{2nt}(G_2) + \dots + \gamma_{2nt}(G_k)$ .

**Theorem 2.10** *Let  $G$  be a connected graph on  $n \geq 2$  vertices. Then  $\gamma_{2nt}(G) \leq n$  and equality holds if and only if  $G$  is a star.*

*Proof* The inequality is obvious. Let  $G$  be a connected graph on  $n$  vertices and  $\gamma_{2nt}(G) = n$ . If  $n = 2$  then nothing to prove. Let us assume  $n \geq 3$ . Suppose  $G$  contains a cycle  $C$ . Let  $x \in V(C)$ . Then  $V(G) - x$  is a nt2d-set of  $G$ , which is a contradiction. Hence  $G$  is a tree.

Let  $u$  be a vertex such that  $\deg u = \Delta$ . Let  $v$  be a vertex such that  $d(u, v) \geq 2$ . Let  $(u, x_1, x_2, \dots, x_k, v)$ ,  $k \geq 1$  be the shortest  $u - v$  path. Then  $S_1 = V - \{x_k\}$  is a nt2d-set of  $G$  which is a contradiction. Hence  $d(u, v) = 1$  for all  $v \in V(G)$ . Thus  $G$  is a star. The converse is

obvious. □

**Corollary 2.11** *Let  $G$  be a disconnected graph with  $\gamma_{2nt}(G) = n$ . Then  $G$  is a galaxy.*

**Theorem 2.12** *Let  $T$  be a tree with  $n \geq 3$  vertices. Then  $\gamma_{2nt}(T) = n - 1$  if and only if  $T$  is a bistar  $B(n - 3, 1)$  or a tree obtained from a bistar by subdividing the edge of maximum degree once.*

*Proof* Let  $u$  be a support with maximum degree. Suppose there exists a vertex  $v \in V(T)$  such that  $d(u, v) \geq 4$ . Let  $(u, x_1, x_2, \dots, x_k, v)$ ,  $k \geq 3$  be the shortest  $u - v$  path then  $S_1 = V - \{u, x_k\}$  is a nt2d-set of  $T$  which is a contradiction. Hence  $d(u, v) \leq 3$  for all  $v \in V(T)$ .

**Case 1.**  $d(u, v) = 3$  for some  $v \in V(T)$ .

Suppose there exists an vertex  $w \in V(T)$ ,  $w \neq v$  such that  $d(u, v) = d(u, w) = 3$ . Let  $P_1$  be the  $u - v$  path and  $P_2$  be the  $u - w$  path. Let  $P_1 = (u, v_1, v_2, v)$  and  $P_2 = (u, w_1, w_2, w)$ . If  $v_1 \neq w_1$  then  $V - \{v_1, w_1\}$  is a nt2d-set of  $T$  which is a contradiction. If  $v_1 = w_1$  and  $v_2 \neq w_2$  then  $V - \{v_2, w_2\}$  is a nt2d-set of  $T$  which is a contradiction. Hence all the pendant vertices  $w$  such that  $d(u, w) = 3$  are adjacent to the same support. Let it be  $x$ . Let  $P = (u, v_1, x)$  be the unique  $u - x$  path in  $T$ . Let  $y \in N(u) - \{v_1\}$ . If  $\deg y \geq 2$  then  $V - \{x, y\}$  is a nt2d-set of  $T$  which is a contradiction. Hence  $T$  is a tree obtained from a bistar by subdividing the edge of maximum degree once.

**Case 2.**  $d(u, v) \leq 2$  for all  $v \in V(T)$ .

If  $d(u, v) = 1$  for all  $v \in V(T)$  and  $v \neq u$  then  $T$  is a star, which is a contradiction. Hence  $d(u, v) = 2$  for some  $v \in V(T)$ . Suppose there exist two vertices  $v$  and  $w$  such that  $d(u, v) = d(u, w) = 2$ . Let  $P_1$  be the  $u - v$  path and  $P_2$  be the  $u - w$  path. Let  $P_1 = (u, v_1, v)$  and  $P_2 = (u, w_1, w)$ . If  $v_1 \neq w_1$  then  $V - \{v_1, w_1\}$  is a nt2d-set of  $T$  which is a contradiction. If  $v_1 = w_1$  then  $V - \{u, v_1\}$  is a nt2d-set of  $T$  which is a contradiction. Hence exactly one vertex  $v \in V$  such that  $d(u, v) = 2$ . Hence  $T$  is isomorphic to  $B(n - 3, 1)$ . The converse is obvious. □

**Theorem 2.13** *Let  $G$  be an unicyclic graph. Then  $\gamma_{2nt}(G) = n - 1$  if and only if  $G$  is isomorphic to  $C_3$  or  $C_4$  or  $K_3(n_1, 0, 0)$ ,  $n_1 \geq 1$ .*

*Proof* Let  $G$  be an unicyclic graph with cycle  $C = (v_1, v_2, \dots, v_r, v_1)$ . If  $G = C$  then by theorem 2.4,  $G = C_3$  or  $C_4$ . Suppose  $G \neq C$ . Let  $A$  be the set of all pendant vertices in  $G$ . Clearly  $A$  is a subset of any  $\gamma_{2nt}$ -set of  $G$ .

**Claim 1.** Vertices of  $C$  of degree more than two or non adjacent.

Let  $v_i$  and  $v_j$  be the vertices of degree more than two in  $C$ . If  $v_i$  and  $v_j$  are adjacent then  $V - \{v_i, v_j\}$  is a nt2d-set of  $G$  which is a contradiction. Hence vertices of  $C$  of degree more than two or non adjacent.

**Claim 2.**  $d(C, w) = 1$  for all  $w \in A$ .

Suppose  $d(C, w) \geq 2$  for some  $w \in A$ . Let  $(v_1, w_1, w_2, \dots, w_k, w)$  be the unique  $v_1 - w$  path in  $G$ ,  $k \geq 1$ . Then  $V - \{w_1, w_2\}$  is a nt2d-set of  $G$  which is a contradiction. Hence  $d(C, w) = 1$

for all  $w \in A$ .

**Claim 3.**  $r = 3$ .

Suppose  $r \geq 5$ . Let  $v_1 \in V(C)$  such that  $\deg v_1 \geq 3$ . Then  $V - \{v_1, v_3\}$  is a nt2d-set of  $G$  which is a contradiction. If  $r = 4$  then  $V - \{v_2, v_4\}$  is a nt2d-set of  $G$  which is a contradiction. Hence  $r = 3$  and  $G$  is isomorphic to  $K_3(n_1, 0, 0)$ ,  $n_1 \geq 1$ . The converse is obvious.  $\square$

**Problem 2.14** Characterize the class of graphs for which  $\gamma_{2nt}(G) = n - 1$ .

**Theorem 2.15** Let  $G$  be a graph with  $\delta(G) \geq 2$  then  $\gamma_{2nt}(G) \leq 2\beta_1(G)$ .

*Proof* Let  $G$  be a graph with  $\delta(G) \geq 2$  and  $M$  be a maximum set of independent edges in  $G$ . Let  $S$  be the vertices in the set of edges of  $M$ . Since  $V - S$  is an independent set, each  $v \in V - S$  must have at least two neighbors in  $S$ . Also since  $S$  contains no isolated vertices,  $\langle N(S) \rangle = G$  and hence  $\langle N(S) \rangle$  contains no isolated vertices. Hence  $S$  is a nt2d-set of  $G$ . Thus  $\gamma_{2nt}(G) \leq 2\beta_1(G)$ .  $\square$

**Problem 2.16** Characterize the class of graphs for which  $\gamma_{2nt}(G) = 2\beta_1(G)$ .

**Notation 2.17** The graph  $G^*$  is a graph with the vertex set can be partition into two sets  $V_1$  and  $V_2$  satisfying the following conditions:

- (1)  $\langle V_1 \rangle = K_2 \cup \overline{K_s}$ ;
- (2)  $\langle V_2 \rangle$  is totally disconnected;
- (3)  $\deg w = 2$  for all  $w \in V_2$ ;
- (4)  $\langle V_2 \cup \{u, v\} \rangle$ , where  $u, v \in V_1$  with  $\deg_{\langle V_1 \rangle} u = \deg_{\langle V_1 \rangle} v = 1$ , has no isolated vertices.

**Theorem 2.18** For any graph  $G$ ,  $\gamma_{2nt}(G) \geq \frac{2n+1-m}{2}$  and the equality holds if and only if  $G$  is isomorphic to  $B(2, 2)$  or  $K_3(1, 1, 0)$  or  $K_4 - e$  or  $K_2 + \overline{K_{n-2}}$  or  $G^*$ .

*Proof* Let  $S$  be a  $\gamma_{2nt}$ -set of  $G$ . Then each vertex of  $V - S$  is adjacent to at least two vertices in  $S$  and since  $\langle N(S) \rangle$  has no isolated vertices either  $V - S$  or  $S$  contains at least one edge. Hence the number of edges  $m \geq 2|V - S| + 1 = 2n - 2\gamma_{2nt} + 1$ . Then  $\gamma_{2nt} \geq \frac{2n+1-m}{2}$ .

Let  $G$  be a graph with  $\gamma_{2nt}(G) = \frac{2n+1-m}{2}$  and let  $S$  be the  $\gamma_{2nt}$ -set of  $G$ . Suppose  $|E(\langle S \rangle \cup \langle V - S \rangle)| \geq 2$ . Then  $m \geq 2(|V - S|) + 2$  and hence  $\gamma_{2nt}(G) \geq \frac{2n+2-m}{2}$  which is a contradiction. Hence either  $|E(\langle S \rangle)| = 1$  or  $|E(\langle V - S \rangle)| = 1$ . Suppose  $|E(\langle V - S \rangle)| = 1$  then  $|E(\langle S \rangle)| = 0$  and hence  $V - S = N(S)$ . Since  $\langle N(S) \rangle$  has no isolated vertices,  $V - S = K_2$ . Let  $V - S = \{u, v\}$ . If  $\deg u \geq 3$  or  $\deg v \geq 3$  then  $m \geq 2(V - S) + 2$ . Hence  $\gamma_{2nt} \geq \frac{2n+2-m}{2}$  which is a contradiction. Hence  $\deg u = 2$  and  $\deg v = 2$ . Then  $|S| \leq 4$ . If  $|S| = 4$  then  $G$  is isomorphic to  $B(2, 2)$ . If  $|S| = 3$  then  $G$  is isomorphic to  $K_3(1, 1, 0)$ . If  $|S| = 2$  then  $G$  is isomorphic to  $K_4 - e$ .

Suppose  $|E(\langle S \rangle)| = 1$  then  $|E(\langle V - S \rangle)| = 0$ . Let  $|S| = 2$ . Since every vertex in  $V - S$  is adjacent to both the vertices in  $S$  we have  $G$  is isomorphic to  $K_2 + \overline{K_{n-2}}$ . If  $|S| \geq 3$  then  $G$  is isomorphic to  $G^*$ . The converse is obvious.  $\square$

**Corollary 2.19** For a tree  $T$ ,  $\gamma_{2nt}(T) \geq \frac{n+2}{2}$ .

**Problem 2.20** Characterize the class of trees for which  $\gamma_{2nt}(T) = \frac{n+2}{2}$ .

**Theorem 2.21** For any graph  $G$ ,  $\gamma_{2nt}(G) \geq \frac{2n}{\Delta+2}$

*Proof* Let  $S$  be a minimum nt2d-set and let  $k$  be the number of edges between  $S$  and  $V - S$ . Since the degree of each vertex in  $S$  is at most  $\Delta$ ,  $k \leq \Delta\gamma_{2nt}$ . But since each vertex in  $V - S$  is adjacent to at least 2 vertices in  $S$ ,  $k \geq 2(n - \gamma_{2nt})$  combining these two inequalities produce  $\gamma_{2nt}(G) \geq \frac{2n}{\Delta+2}$ .  $\square$

**Problem 2.22** Characterize the class of graphs for which  $\gamma_{2nt}(G) = \frac{2n}{\Delta+2}$ .

### §3. Neighborhood Total 2-Domination Numbers and Chromatic Numbers

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J. Paulraj Joseph and S. Arumugam [7] proved that  $\gamma(G) + \chi(G) \leq n + 1$ . They also characterized the class of graphs for which the upper bound is attained. In the following theorems we find an upper bound for the sum of the neighborhood total 2-domination number and chromatic number of a graph, also we characterized the corresponding extremal graphs.

We define the following graphs:

- (1)  $G_1$  is the graph obtained from  $K_4 - e$  by attaching a pendant vertex to any one of the vertices of degree two by an edge.
- (2)  $G_2$  is the graph obtained from  $K_4 - e$  by attaching a pendant vertex to any one of the vertices of degree three by an edge.
- (3)  $G_3$  is the graph obtained from  $(K_4 - e) \cup K_1$  by joining a vertex of degree three, vertex of degree two to the vertex of degree zero by an edge.
- (4)  $G_4$  is the graph obtained from  $C_5 + e$  by adding an edge between two non adjacent vertices of degree two.
- (5)  $G_5$  is the graph obtained from  $K_4$  by subdividing an edge once.
- (6)  $G_6$  is the graph obtained from  $C_5 + e$  by adding an edge between two non adjacent vertices with one has degree three and another has degree two.
- (7)  $G_7 = K_5 - Y_1$  where  $Y_1$  is a maximum matching in  $K_5$ .

**Theorem 3.1** For any connected graph  $G$ ,  $\gamma_{2nt}(G) + \chi(G) \leq 2n$  and equality holds if and only if  $G$  is isomorphic to  $K_2$ .

*Proof* The inequality is obvious. Now we assume that  $\gamma_{2nt}(G) + \chi(G) = 2n$ . This implies  $\gamma_{2nt}(G) = n$  and  $\chi(G) = n$ . Then  $G$  is a complete graph and a star. Hence  $G$  is isomorphic to  $K_2$ . The converse is obvious.  $\square$

**Theorem 3.2** Let  $G$  be a connected graph. Then  $\gamma_{2nt}(G) + \chi(G) = 2n - 1$  if and only if  $G$  is isomorphic to  $K_3$  or  $P_3$ .

*Proof* Let us assume  $\gamma_{2nt}(G) + \chi(G) = 2n - 1$ . This is possible only if (i)  $\gamma_{2nt}(G) = n$

and  $\chi(G) = n - 1$  or (ii)  $\gamma_{2nt}(G) = n - 1$  and  $\chi(G) = n$ . Let  $\gamma_{2nt}(G) = n$  and  $\chi(G) = n - 1$ . Then  $G$  is a star and hence  $n = 3$ . Thus  $G$  is isomorphic to  $P_3$ . Suppose (ii) holds. Then  $G$  is a complete graph with  $\gamma_{2nt}(G) = n - 1$ . Then  $n = 3$  and hence  $G$  is isomorphic to  $K_3$ . The converse is obvious.  $\square$

**Theorem 3.3** *For any connected graph  $G$ ,  $\gamma_{2nt}(G) + \chi(G) = 2n - 2$  if and only if  $G$  is isomorphic to  $K_4$  or  $K_{1,3}$  or  $K_3(1, 0, 0)$ .*

*Proof* Let us assume  $\gamma_{2nt}(G) + \chi(G) = 2n - 2$ . This is possible only if  $\gamma_{2nt}(G) = n$  and  $\chi(G) = n - 2$  or  $\gamma_{2nt}(G) = n - 1$  and  $\chi(G) = n - 1$  or  $\gamma_{2nt}(G) = n - 2$  and  $\chi(G) = n$ .

Let  $\gamma_{2nt}(G) = n$  and  $\chi(G) = n - 2$ . Since  $\gamma_{2nt}(G) = n$  we have  $G$  is a star with  $\chi(G) = n - 2$ . Hence  $n = 4$ . Thus  $G$  isomorphic to  $K_{1,3}$ .

Suppose  $\gamma_{2nt}(G) = n - 1$  and  $\chi(G) = n - 1$ . Since  $\chi(G) = n - 1$ ,  $G$  contains a complete subgraph  $K$  on  $n - 1$  vertices. Let  $V(K) = \{v_1, v_2, \dots, v_{n-1}\}$  and  $V(G) - V(K) = \{v_n\}$ . Then  $v_n$  is adjacent to  $v_i$  for some vertex  $v_i \in V(K)$ . If  $\deg(v_n) = 1$  and  $n \geq 4$  then  $\{v_i, v_j, v_n\}$ ,  $i \neq j$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 4$  and  $K = K_3$ . Thus  $G$  is isomorphic to  $K_3(1, 0, 0)$ . If  $\deg v_n = 1$  and  $n = 3$  then  $G$  is isomorphic to  $P_3$  which is a contradiction to  $\gamma_{2nt} = n - 1$ . If  $\deg(v_n) > 1$  then  $\gamma_{2nt} = 2$ . Then  $n = 3$  which gives  $G$  is isomorphic to  $K_3$  which is a contradiction to  $\chi(G) = n - 1$ .

Suppose  $\gamma_{2nt}(G) = n - 2$  and  $\chi(G) = n$ . Since  $\chi(G) = n$ ,  $G$  is isomorphic to  $K_n$ . But  $\gamma_{2nt}(K_n) = 2$  we have  $n = 4$ . Hence  $G$  is isomorphic to  $K_4$ . The converse is obvious.  $\square$

**Theorem 3.4** *Let  $G$  be a connected graph. Then  $\gamma_{2nt}(G) + \chi(G) = 2n - 3$  if and only if  $G$  is isomorphic to  $C_4$  or  $K_{1,4}$  or  $P_4$  or  $K_5$  or  $K_3(2, 0, 0)$  or  $K_4(1, 0, 0, 0)$  or  $K_4 - e$ .*

*Proof* Let  $\gamma_{2nt}(G) + \chi(G) = 2n - 3$ . This is possible only if (i)  $\gamma_{2nt}(G) = n$ ,  $\chi(G) = n - 3$  or (ii)  $\gamma_{2nt}(G) = n - 1$ ,  $\chi(G) = n - 2$  or (iii)  $\gamma_{2nt}(G) = n - 2$ ,  $\chi(G) = n - 1$  or (iv)  $\gamma_{2nt}(G) = n - 3$ ,  $\chi(G) = n$ .

Suppose (i) holds. Then  $G$  is a star with  $\chi(G) = n - 3$ . Then  $n = 5$ . Hence  $G$  is isomorphic to  $K_{1,4}$ . Suppose (ii) holds. Since  $\chi(G) = n - 2$ ,  $G$  is either  $C_5 + K_{n-5}$  or  $G$  contains a complete subgraph  $K$  on  $n - 2$  vertices. If  $G = C_5 + K_{n-5}$  then  $\gamma_{2nt}(G) + \chi(G) \neq 2n - 3$ . Thus  $G$  contains a complete subgraph  $K$  on  $n - 2$  vertices. Let  $X = V(G) - V(K) = \{v_1, v_2\}$  and  $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ .

**Case 1.**  $\langle X \rangle = K_2$ .

Since  $G$  is connected, without loss of generality we assume  $v_1$  is adjacent to  $v_3$ . If  $|N(v_1) \cap N(v_2)| \geq 2$  then  $\gamma_{2nt}(G) = 2$  and hence  $n = 3$  which is a contradiction. So  $|N(v_1) \cap N(v_2)| \leq 1$ . Then  $\{v_2, v_3, v_4\}$  is a  $\gamma_{2nt}$ -set of  $G$  and hence  $n = 4$ . If  $|N(v_1) \cap N(v_2)| = 1$  then  $G$  is either  $K_4 - e$  or  $K_3(1, 0, 0)$ . For these graphs  $\chi(G) = 3$  which is a contradiction. If  $N(v_1) \cap N(v_2) = \emptyset$ . Then  $G$  is isomorphic to  $P_4$  or  $C_4$  or  $K_3(1, 0, 0)$ . Since  $\chi[K_3(1, 0, 0)] = 3$ , we have  $G$  is isomorphic to  $P_4$  or  $C_4$ .

**Case 2.**  $\langle X \rangle = \overline{K_2}$ .

Since  $G$  is connected  $v_1$  and  $v_2$  are adjacent to at least one vertex in  $K$ . If  $\deg v_1 =$



$\deg v_2 = 1$  and  $N(v_1) \cap N(v_2) \neq \phi$  then  $|V(K)| \neq 1$ . So  $|V(K)| \geq 2$ . If  $|V(K)| = 2$  then  $G$  is isomorphic to  $K_{1,3}$  which is a contradiction. Hence  $|V(K)| \geq 3$ . Then  $\{v_1, v_2, v_3, v_4\}$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 5$ . Thus  $G$  is isomorphic to  $K_3(2, 0, 0)$ . If  $\deg v_1 = \deg v_2 = 1$  and  $N(v_1) \cap N(v_2) = \phi$  then  $|V(K)| \geq 2$ . If  $|V(K)| = 2$  then  $G$  is isomorphic to  $P_4$ . If  $|V(K)| \geq 3$  then  $\{v_1, v_2, v_3, v_4\}$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 5$ . Thus  $G$  is isomorphic to  $K_3(1, 1, 0)$ . But  $\gamma_{2nt}[K_3(1, 1, 0)] = 3$  which is a contradiction. Suppose  $\deg v_1 \geq 2$  and  $|N(v_1) \cap N(v_2)| \leq 1$  then  $\{v_2, v_3, v_4\}$  where  $v_3, v_4 \in N(v_1)$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 4$ . Then  $G$  is isomorphic to  $K_3(1, 0, 0)$ . For this graph  $\gamma_{2nt}(G) = 3$  and  $\chi(G) = 3$  which is a contradiction. If  $\deg v_1 \geq 2$  and  $|N(v_1) \cap N(v_2)| \geq 2$  then  $\{v_3, v_4\}$  where  $v_3, v_4 \in N(v_1) \cap N(v_2)$  is a  $\gamma_{2nt}$ -set of  $G$ . Then  $n = 3$  which gives a contradiction.

Suppose (3) holds. Since  $\chi(G) = n - 1$ ,  $G$  contains a clique  $K$  on  $n - 1$  vertices. Let  $X = V(G) - V(K) = \{v_1\}$ . If  $\deg v_1 \geq 2$  then  $\gamma_{2nt}(G) = 2$ . Hence  $n = 4$ . Thus  $G$  is isomorphic to  $K_4 - e$ . If  $\deg v_1 = 1$  then  $|V(K)| \geq 3$  and hence  $\{v_1, v_2, v_3\}$  where  $v_2 \in N(v_1)$  is a  $\gamma_{2nt}$ -set of  $G$  and hence  $n = 5$ . Thus  $G$  is isomorphic to  $K_4(1, 0, 0, 0)$ .

Suppose (iv) holds. Since  $\chi(G) = n$ ,  $G$  is a complete graph. Then  $\gamma_{2nt}(G) = 2$  and hence  $n = 5$ . Therefore  $G$  is isomorphic to  $K_5$ . The converse is obvious.  $\square$

**Theorem 3.5** *Let  $G$  be a connected graph. Then  $\gamma_{2nt}(G) + \chi(G) = 2n - 4$  if and only if  $G$  is isomorphic to one of the following graphs  $P_5$ ,  $K_6$ ,  $C_5$ ,  $K_{1,5}$ ,  $B(2, 1)$ ,  $K_5(1, 0, 0, 0, 0)$ ,  $K_4(2, 0, 0, 0)$ ,  $K_4(1, 1, 0, 0)$ ,  $K_3(1, 1, 0)$ ,  $K_3(3, 0, 0)$ ,  $C_5 + e$ ,  $2K_2 + K_1$  and  $K_5 - Y$ , where  $Y$  is the set of edges incident to a vertex with  $|Y| = 1$  or  $2$ ,  $G_i$ ,  $1 \leq i \leq 7$ .*

*Proof* Let  $\gamma_{2nt}(G) + \chi(G) = 2n - 4$ . This is possible only if

- (1)  $\gamma_{2nt}(G) = n$ ,  $\chi(G) = n - 4$ , or
- (2)  $\gamma_{2nt}(G) = n - 1$ ,  $\chi(G) = n - 3$ , or
- (3)  $\gamma_{2nt}(G) = n - 2$ ,  $\chi(G) = n - 2$ , or
- (4)  $\gamma_{2nt}(G) = n - 3$ ,  $\chi(G) = n - 1$ , or
- (5)  $\gamma_{2nt}(G) = n - 4$ ,  $\chi(G) = n$ .

**Case 1.**  $\gamma_{2nt}(G) = n$ ,  $\chi(G) = n - 4$ .

Then  $G$  is a star and hence  $G$  is isomorphic to  $K_{1,5}$ .

**Case 2.**  $\gamma_{2nt}(G) = n - 1$ ,  $\chi(G) = n - 3$ .

Since  $\chi(G) = n - 3$ ,  $G$  contains a clique  $K$  on  $n - 3$  vertices. Let  $X = V(G) - V(K) = \{v_1, v_2, v_3\}$  and let  $V(G) = \{v_1, v_2, \dots, v_n\}$ .

**Subcase 2.1**  $\langle X \rangle = \overline{K_3}$ .

Let all  $v_1, v_2$  and  $v_3$  be pendant vertices and  $|V(K)| = 1$  then  $G$  is a star which is a contradiction. So we assume that  $v_1, v_2$  and  $v_3$  be the pendant vertices and  $|V(K)| = 2$ . If all  $v_1, v_2$  and  $v_3$  are adjacent to same vertices in  $K$  then  $G$  is isomorphic to  $K_{1,4}$  which is a contradiction. If  $N(v_1) \cap N(v_2) = \{v_4\}$  and  $N(v_3) = \{v_5\}$  then  $G$  is isomorphic to  $B(2, 1)$ . Let  $v_1, v_2$  and  $v_3$  be the pendant vertices and  $|V(K)| = 3$ . If all  $v_1, v_2$  and  $v_3$  are adjacent to same vertices in  $K$  then  $G$  is isomorphic to  $K_3(3, 0, 0)$ . If  $N(v_1) \cap N(v_i) = \phi, i \neq 1$  then  $\gamma_{2nt} \neq n - 1$

which is a contradiction. If  $|V(K)| \geq 4$  then  $\gamma_{2nt} \leq n - 2$  which is a contradiction.

Suppose  $\deg v_1 \geq 2$  then  $\{v_2, v_3, v_4, v_5\}$  where  $v_4, v_5 \in N(v_1)$  is a nt2d-set then  $\gamma_{2nt}(G) \leq 4$ . Hence  $n \leq 5$ . Since  $\deg v_1 \geq 2$ ,  $n = 5$ . Then  $G$  contains  $K_3$  and hence  $\chi(G) \neq n - 3$  which is a contradiction.

**Subcase 2.2**  $\langle X \rangle = K_2 \cup K_1$ .

Let  $v_1 v_2 \in E(G)$  and  $\deg v_3 = 1$ . Suppose  $\deg v_2 = 1$  and  $\deg v_1 = 2$  and  $N(v_1) \cap N(v_3) = \{v_4\}$ . If  $\deg v_4 = 2$  then  $G$  is isomorphic to  $P_4$  and  $\gamma_{2nt}(G) + \chi(G) \neq 2n - 4$  which is a contradiction. If  $\deg v_4 \geq 3$  then  $\{v_2, v_3, v_4, v_5\}$  is  $\gamma_{2nt}$ -set of  $G$ . Therefore  $n = 5$ . Then  $G$  is isomorphic to a bistar  $B(2, 1)$ . If  $N(v_1) \cap N(v_3) = \emptyset$  then  $K$  contains at least 2 vertices. If  $|V(K)| \geq 3$  then  $\gamma_{2nt}(G) = 4$  and hence  $n = 5$  which is a contradiction. So  $|V(K)| = 2$  and hence  $G$  is isomorphic to  $P_5$ .

Suppose  $\deg v_3 = 1$ ,  $\deg v_2 = 1$  and  $\deg v_1 \geq 3$ . Then  $\gamma_{2nt}(G) \leq 4$  and hence  $n = 5$ . This gives  $|V(K)| = 2$ . Then  $G$  is isomorphic to  $K_3(1, 1, 0)$  and hence  $\gamma_{2nt}(G) = 3$  which is a contradiction.

Suppose  $\deg v_1 \geq 3$ ,  $\deg v_2 \geq 2$  and  $\deg v_3 = 1$ . Then  $\gamma_{2nt}(G) \leq 4$  and hence  $n = 5$ . Then  $G$  is isomorphic to the either  $K_4(1, 0, 0, 0)$  or a graph obtained from  $K_4 - e$  by attaching a pendant vertex to one of the vertices of degree 2. For this graphs  $\chi(G) \neq n - 3$  which is a contradiction. If  $\deg v_1 \geq 3$ ,  $\deg v_2 \geq 3$  and  $\deg v_3 \geq 2$  then  $\gamma_{2nt}(G) \leq 4$  and hence  $n = 5$ . Then  $G$  is isomorphic to the graph which is obtained from  $K_4 \cup K_1$  by including two edges between a vertex of degree zero and any two vertices of degree three. For this graph  $\chi(G) = 4$  which is a contradiction.

**Subcase 2.3.**  $\langle X \rangle = P_3$ .

Let  $\langle X \rangle = (v_1, v_2, v_3)$ . Since  $G$  is connected at least one vertex of  $\langle X \rangle$  is adjacent to  $K$ . Let  $N(v_1) \cap V(K) \neq \emptyset$  and  $N(v_i) \cap V(K) = \emptyset$  for  $i = 2, 3$ . Let  $|N(v_1) \cap V(K)| = 1$  then  $\{v_1, v_3, v_4, v_5\}$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 5$ . Therefore  $G$  is isomorphic to  $P_5$ . If  $|N(v_1) \cap V(K)| \geq 2$  then  $\gamma_{2nt}(G) \leq 4$ . Hence  $n = 5$ . Then  $G$  is isomorphic to  $K_3(P_3, P_1, P_1)$ . For this graph  $\gamma_{2nt}(G) = 3$  which is a contradiction.

Suppose  $N(v_2) \cap V(K) \neq \emptyset$  and  $N(v_i) \cap V(K) = \emptyset$  for  $i = 1, 3$ . If  $|N(v_2) \cap V(K)| = 1$  then  $G$  is isomorphic to  $B(2, 1)$ . If  $|N(v_2) \cap V(K)| \geq 2$  then  $\gamma_{2nt}(G) = 4$  and hence  $n = 5$ . Hence  $G$  is isomorphic to  $K_3(2, 0, 0)$ . For this graph  $\chi(G) = 3 \neq n - 3$  which is a contradiction. If  $|N(v_1) \cap V(K)| \geq 2$ ,  $|N(v_2) \cap V(K)| = 1$  and  $N(v_3) \cap V(K) = \emptyset$  then  $G$  is a graph obtained from  $K_4 - e$  by attaching a pendant vertex to one of the vertices of degree 2. For this graph  $\chi(G) \neq n - 3$  which is a contradiction.

If  $|N(v_1) \cap V(K)| \geq 2$  and  $|N(v_2) \cap V(K)| \geq 2$  and  $N(v_3) \cap V(K) = \emptyset$  then  $G$  is isomorphic to  $K_4(1, 0, 0, 0)$ . For this graph  $\chi(G) = 4$  which is a contradiction. If  $|N(v_i) \cap V(K)| \geq 1$  for all  $i = 1, 2, 3$  then  $\gamma_{2nt}(G) \leq 4$ . Hence  $n = 5$ . Then  $G$  is isomorphic to any of the following graphs:

(i) the graph obtained from  $K_4 - e$  by attaching a pendant vertex to any one of the vertices of degree 3;

(ii) the graph obtained from  $K_4 - e$  by subdividing the edge with the end vertices having

degree 3 once;

(iii)  $C_5 + e$ .

For these graphs either  $\gamma_{2nt}(G) \neq n - 1$  or  $\chi(G) \neq n - 3$  which is a contradiction.

**Subcase 2.4**  $\langle X \rangle = K_3$ .

Then any two vertices from  $X$  and two vertices from  $V - X$  form a nt2d-set and hence  $\gamma_{2nt}(G) \leq 4$ . Then  $n \leq 5$ . For these graphs  $\chi(G) \geq 3$  which is a contradiction.

**Case 3.**  $\gamma_{2nt}(G) = n - 2$  and  $\chi(G) = n - 2$ .

Since  $\chi(G) = n - 2$ ,  $G$  is either  $C_5 + K_{n-5}$  or  $G$  contains a clique  $K$  on  $n - 2$  vertices. If  $G = C_5 + K_{n-5}$  and  $n \geq 6$  then  $\gamma_{2nt}(G) + \chi(G) \neq 2n - 4$  which is a contradiction. Hence  $n = 5$ . Thus  $G = C_5$ . Let  $G$  contains a clique  $K$  on  $n - 2$  vertices. Let  $X = V(G) - V(K) = \{v_1, v_2\}$ .

**Subcase 3.1**  $\langle X \rangle = \overline{K_2}$ .

Since  $G$  is connected  $v_1$  and  $v_2$  are adjacent to at least one vertex in  $K$ . If  $\deg v_1 = \deg v_2 = 1$  and  $N(v_1) \cap N(v_2) \neq \phi$  then  $V(K) = 4$ . Hence  $G$  is isomorphic to  $K_4(2, 0, 0, 0)$ . If  $\deg v_1 = \deg v_2 = 1$  and  $N(v_1) \cap N(v_2) = \phi$  then  $G$  is isomorphic to  $K_3(1, 1, 0)$  or  $K_4(1, 1, 0, 0)$ .

Suppose  $\deg v_1 \geq 2$  and  $|N(v_1) \cap N(v_2)| \leq 1$  then  $\{v_2, v_3, v_4\}$  where  $v_3, v_4 \in N(v_1)$  is a  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 5$ . Then  $G$  is isomorphic to  $G_1$  or  $G_2$  or  $G_3$ . If  $\deg v_1 \geq 2$  and  $|N(v_1) \cap N(v_2)| \geq 2$  then  $\gamma_{2nt}(G) = 2$ . Hence  $n = 4$  with  $\chi(G) = 3$  which is a contradiction.

**Subcase 3.2**  $\langle X \rangle = K_2$ .

Since  $G$  is connected, without loss of generality we assume  $v_1$  is adjacent to  $v_3$ . If  $|N(v_1) \cap N(v_2)| \geq 2$  then  $\gamma_{2nt}(G) = 2$  and hence  $n = 4$ . Thus  $G$  is  $K_4$  which is a contradiction. So  $|N(v_1) \cap N(v_2)| \leq 1$ . Then  $\{v_2, v_3, v_4\}$  is a  $\gamma_{2nt}$ -set of  $G$  and hence  $n = 5$ . If  $\deg v_2 = 1$  then  $G$  is isomorphic to  $K_3(P_3, P_1, P_1)$  or the graph obtained from  $K_4 - e$  by attaching a pendant vertex to any one of the vertices of degree 2. If  $\deg(v_2) \geq 2$  then  $G$  is isomorphic to  $C_5 + e$  or  $2K_2 + K_1$  or  $G_4$  or  $G_5$  or  $G_6$  or  $G_7$ .

**Case 4.**  $\gamma_{2nt}(G) = n - 3$  and  $\chi(G) = n - 1$ .

Then  $G$  contains a clique  $K$  on  $n - 1$  vertices. Let  $X = V(G) - V(K) = \{v_1\}$ . If  $\deg v_1 \geq 2$  then  $\gamma_{2nt}(G) = 2$ . Hence  $n = 5$ . Thus  $G$  is isomorphic to  $K_5 - Y$  where  $Y$  is the set of edges incident to a vertex with  $|Y| = 1$  or  $2$ . If  $\deg v_1 = 1$  then  $\{v_1, v_2, v_3\}$  be the  $\gamma_{2nt}$ -set of  $G$ . Hence  $n = 6$ . Thus  $G$  is isomorphic to  $K_5(1, 0, 0, 0, 0)$ .

**Case 5.**  $\gamma_{2nt}(G) = n - 4$  and  $\chi(G) = n$

Then  $G$  is a complete graph. Hence  $n = 6$ . Therefore  $G$  is isomorphic to  $K_6$ . The converse is obvious.  $\square$

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## Smarandache Lattice and Pseudo Complement

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**Abstract:** In this paper, we introduce Smarandache - 2-algebraic structure of lattice, namely *Smarandache lattices*. A Smarandache 2-algebraic structure on a set  $N$  means a weak algebraic structure  $A_0$  on  $N$  such that there exists a proper subset  $M$  of  $N$  which is embedded with a stronger algebraic structure  $A_1$ , where a stronger algebraic structure means such a structure which satisfies more axioms, by proper subset one can understand a subset different from the empty set, by the unit element if any, and from the whole set. We obtain some of its characterization through pseudo complemented.

**Key Words:** Lattice, Boolean algebra, Smarandache lattice and Pseudo complemented lattice.

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### §1. Introduction

New notions are introduced in algebra to study more about the congruence in number theory by Florentin Smarandache [1]. By  $\langle$ proper subset $\rangle$  of a set  $A$ , we consider a set  $P$  included in  $A$ , different from  $A$ , also different from the empty set and from the unit element in  $A$  - if any they rank the algebraic structures using an order relationship.

The algebraic structures  $S_1 \ll S_2$  if both of them are defined on the same set; all  $S_1$  laws are also  $S_2$  laws; all axioms of  $S_1$  law are accomplished by the corresponding  $S_2$  law;  $S_2$  law strictly accomplishes more axioms than  $S_1$  laws, or in other words  $S_2$  laws has more laws than  $S_1$ . For example, a semi-group  $\ll$  monoid  $\ll$  group  $\ll$  ring  $\ll$  field, or a semi group  $\ll$  commutative semi group, ring  $\ll$  unitary ring,  $\dots$  etc. They define a general special structure to be a structure  $SM$  on a set  $A$ , different from a structure  $SN$ , such that a proper subset of  $A$  is an  $SN$  structure, where  $SM \ll SN$ .

### §2. Preliminaries

**Definition 2.1** Let  $P$  be a lattice with  $0$  and  $x \in P$ . We say  $x^*$  is a pseudo complemented of

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$x$  iff  $x^* \in P$  and  $x \wedge x^* = 0$ , and for every  $y \in P$ , if  $x \wedge y = 0$  then  $y \leq x^*$ .

**Definition 2.2** Let  $P$  be a pseudo complemented lattice.  $N_P = \{ x^* : x \in P \}$  is the set of complements in  $P$ .  $N_P = \{ N_P \leq N, \neg N, 0_N, 1_N, N, N \}$ , where

- (1)  $\leq_N$  is defined by: for every  $x, y \in N_P$ ,  $x \leq_N y$  iff  $x \leq_P y$ ;
- (2)  $\neg N$  is defined by: for every  $x \in N_P$ ,  $\neg N(x) = x^*$ ;
- (3)  $\wedge_N$  is defined by: for every  $x, y \in N_P$ ,  $x \wedge_N y = x \wedge_P y$ ;
- (4)  $\vee_N$  is defined by: for every  $x, y \in N_P$ ,  $x \vee_N y = (x^* \wedge_P y^*)^*$ ;
- (5)  $1_N = 0_{P^*}$ ,  $0_N = 0_P$ .

**Definition 2.3** Let  $P$  be a lattice with  $0$ . Define  $I_P$  to be the set of all ideals in  $P$ , i.e.,  $I_P = \langle I_P, \leq_1, \wedge_1, \vee_1, 0_1, 1_1 \rangle$ , where

$$\leq = \subseteq, i \wedge_1 j = I \cap J, i \vee_1 j = (I \cup J], 0_1 = 0_A, 1_1 = A.$$

**Definition 2.4** If  $P$  is a distributive lattice with  $0$ ,  $I_P$  is a complete pseudo complemented lattice, let  $P$  be a lattice with  $0$  and  $NI_P$ , the set of normal ideals in  $P$ , is given by  $NI_P = \{ I^* : I_P : II_P \}$ . Alternatively,  $NI_P = \{ II_P : I = I^{**} \}$ . Thus  $NI_P = \{ NI_P, \subseteq, \cap, \cup, \wedge_{NI}, \vee_{NI} \}$ , which is the set of pseudo complements in  $I_P$ .

**Definition 2.5** A Pseudo complemented distributive lattice  $P$  is called a stone lattice if, for all  $a \in P$ , it satisfies the property  $a^* \vee a^{**} = 1$ .

**Definition 2.6** Let  $P$  be a pseudo complemented distributive lattice. Then for any filter  $F$  of  $P$ , define the set  $\delta(F)$  by  $\delta(F) = \{ a^* \in P / a^* \in F \}$ .

**Definition 2.7** Let  $P$  be a pseudo complemented distributive lattice. An ideal  $I$  of  $P$  is called a  $\delta$ -ideal if  $I = \delta(F)$  for some filter  $F$  of  $P$ .

Now we have introduced a definition by [4]:

**Definition 2.8** A lattice  $S$  is said to be a Smarandache lattice if there exist a proper subset  $L$  of  $S$ , which is a Boolean algebra with respect to the same induced operations of  $S$ .

### §3. Characterizations

**Theorem 3.1** Let  $S$  be a lattice. If there exist a proper subset  $N_P$  of  $S$  defined in Definition 2.2, then  $S$  is a Smarandache lattice.

*Proof* By hypothesis, let  $S$  be a lattice and whose proper subset  $N_P = \{ x^* : x \in P \}$  is the set of all pseudo complements in  $P$ . By definition, if  $P$  is a pseudo complement lattice, then  $N_P = \{ x^* : x \in P \}$  is the set of complements in  $P$ , i.e.,  $N_P = \{ N_P, \leq_N, \neg_N, 0_N, 1_N, \wedge_N, \vee_N \}$ , where

- (1)  $\leq_N$  is defined for every  $x, y \in N_P$ ,  $x \leq_N y$  iff  $x \leq_P b$ ;
- (2)  $\neg N$  is defined for every  $x \in N_P$ ,  $\neg N(x) = x*$ ;
- (3)  $\wedge_N$  is defined for every  $x, y \in N_P$ ,  $x \wedge_N y = x \wedge_P y$ ;
- (4)  $\vee_N$  is defined for every  $x, y \in N_P$ ,  $x \vee_N y = (x * \wedge_P y*)*$ ;
- (5)  $1_N = 0_{P*}$ ,  $0_N = 0_P$ .

It is enough to prove that  $N_P$  is a Boolean algebra.

(1) For every  $x, y \in N_P$ ,  $x \wedge_N y \in N_P$  and  $\wedge_N$  is meet under  $\leq_N$ . If  $x, y \in N_P$ , then  $x = x**$  and  $y = y**$ .

Since  $x \wedge_P y \leq_P x$ , by result if  $x \leq_N y$  then  $y* \leq_N x*$ ,  $x* \leq_P (x \wedge_P y)*$ , and with by result if  $x \leq_N y$  then  $y* \leq_N x*$ ,  $(x \wedge_P y)** \leq_P x$ . Similarly,  $(x \wedge_P y)** \leq_P y$ . Hence  $(x \wedge_P y) \leq_P (x \wedge_P y)**$ .

By result,  $x \leq_N x**$ ,  $(x \wedge_P y) \leq_P (x \wedge_P y)**$ . Hence  $(x \wedge_P y) \in N_P$ ,  $(x \wedge_N y) \in N_P$ . If  $a \in N_P$  and  $a \leq_N x$  and  $a \leq_N y$ , then  $a \leq_P x$  and  $a \leq_P y$ ,  $a \leq_P (x \wedge_P y)$ . Hence  $a \leq_N (x \wedge_N y)$ . So, indeed  $\wedge_N$  is meet in  $\leq_N$ .

(2) For every  $x, y \in N_P$ ,  $x \vee_N y \in N_P$  and  $\vee_N$  is join under  $\leq_N$ . Let  $x, y \in N_P$ . Then  $x*, y* \in N_P$ . By (1),  $(x * \wedge_P y*) \in N_P$ . Hence  $(x * \wedge_P y*)* \in N_P$ , and hence  $(x \vee_N y) \in N_P$ ,  $(x * \wedge_P y*) \leq_P x*$ . By result  $x \leq_N x**$ ,  $x** \leq_P (x * \wedge_P y*)*$  and  $N_P = \{x \in P; x = x**\}$ ,  $x \leq_P (x * \wedge_P y*)*$ .

Similarly,  $y \leq_P (x * \wedge_P y*)*$ . If  $a \in N_P$  and  $x \leq_N a$  and  $y \leq_N a$ , then  $x \leq_P a$  and  $y \leq_P a$ , then by result if  $x \leq_N y$ , then  $y* \leq_N x*$ ,  $a* \leq_P x*$  and  $a* \leq_P y*$ . Hence  $a* \leq_P (x * \wedge_P y*)$ . By result if  $x \leq_N y$  then  $y* \leq_N x*$ ,  $(x * \wedge_P y*)* \leq_P a**$ . Thus, by result  $N_P = \{x \in P : x = x**\}$ ,  $(x * \wedge_P y*)* \leq_P x$  and  $x \vee_N y \leq_N a$ . So, indeed  $\vee_N$  is join in  $\leq_N$ .

(3)  $0_N, 1_N \in N_P$  and  $0_N, 1_N$  are the bounds of  $N_P$ . Obviously  $1_N \in N_P$  since  $1_N = 0_{P*}$  and for every  $a \in N_P$ ,  $a \wedge_P 0_P = 0_P$ . For every  $a \in N_P$ ,  $a \leq_P 0_{P*}$ . Hence  $a \leq_N 1_N$ ,  $0_{P*}, 0_{P**} \in N_P$  and  $0_P * \wedge_P 0_{P**} \in N_P$ . But of course,  $0_P * \wedge_P 0_{P**} = 0_P$ . Thus,  $0_P \in N_P$ ,  $0_N \in N_P$ . Obviously, for every  $a \in N_P$ ,  $0_P \leq_P a$ . Hence for every  $a \in N_P$ ,  $0_N \leq_N a$ . So  $N_P$  is bounded lattice.

(4) For every  $a \in N_P$ ,  $\neg N(a) \in N_P$  and for every  $a \in N_P$ ,  $a \wedge_N \neg N(a) = 0_N$ , and for every  $a \in N_P$ ,  $a \vee_N \neg N(a) = 1_N$ .

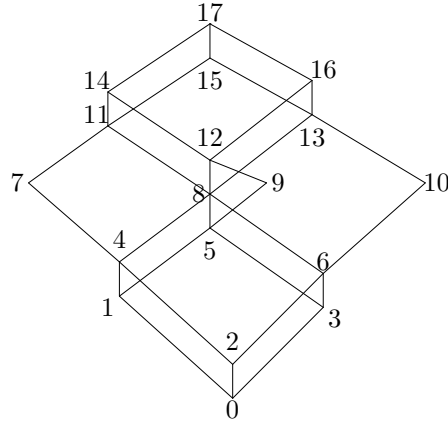
Let  $a \in N_P$ . Obviously,  $\neg N(a) \in N_P$ ,  $a \vee_N \neg N(a) = a \vee_N a* = ((a * \wedge_P b**))* = (a * \wedge_P a)* = 0_{P*} = 1_N$ ,  $a \wedge_N \neg N(a) = a \wedge_P a* = 0_P = 0_N$ . So  $N_P$  is a bounded complemented lattice.

(5) Since  $x \leq_N (x \vee_N (y \wedge_N z))$ ,  $(x \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$ . Also  $(y \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$ . Obviously, if  $a \leq_N b$ , then  $a \wedge_N b* = 0_N$ . Since  $b \wedge b* = 0_N$ , so  $(x \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$  and  $(y \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$ ,  $x \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$ ,  $y \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$ .

By definition of pseudo complement:  $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N x*$ ,  $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N y*$ , Hence,  $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N x * \wedge_N y*$ . Once again, If  $a \leq_N b$ , then  $a \wedge_N b* = 0_N$ . Thus,  $z \wedge_N (x \vee_N (y \wedge_N z)) * \wedge (x * \wedge_N y*)* = 0_N$ ,  $z \wedge_N (x * \wedge_N y*)* \leq_N (x \vee_N (y \wedge_N z))**$ . Now, by definition of  $\wedge_N$ :  $z \wedge_N (x * \vee_N y*)* = z \wedge_N (x \vee_N y)$  and by  $N_P = \{x \in P : x = x**\}$ :  $(x \vee_N (y \wedge_N z))** = x \vee_N (y \wedge_N z)$ . Hence,  $z \wedge_N (x \vee_N y) \leq_N x \vee_N (y \wedge_N z)$ .

Hence, indeed  $N_P$  is a Boolean Algebra. Therefore by definition,  $S$  is a Smarandache lattice.  $\square$

For example, a distributive lattice  $D_3$  is shown in Fig.1,



**Fig.1**

where  $D_3$  is pseudo complemented because

$$0* = 17,$$

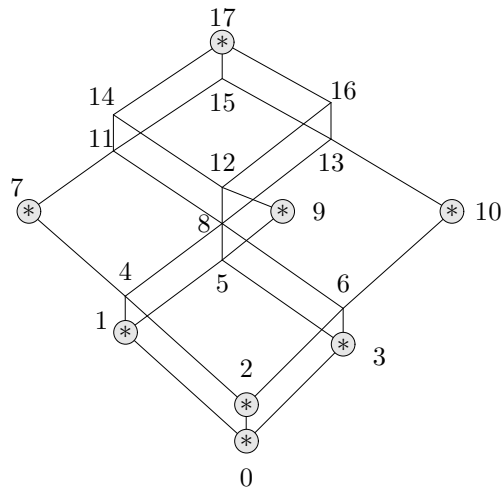
$$8* = 11* = 12* = 13* = 14* = 15* = 16* = 17* = 0,$$

$$1* = 10, 6* = 10* = 1,$$

$$2* = 9, 5* = 9* = 2,$$

$$3* = 7, 4* = 7* = 3$$

and its correspondent Smarandache lattice is shown in Fig.2.



**Fig.2**



**Theorem 3.2** *Let  $S$  be a distributive lattice with 0. If there exist a proper subset  $NI_P$  of  $S$ , defined Definition 2.4. Then  $S$  is a Smarandache lattice.*

*Proof* By hypothesis, let  $S$  be a distributive lattice with 0 and whose proper subset  $NI_P = \{I^* \in I_P, I \in I_P\}$  is the set of normal ideals in  $P$ . We claim that  $NI_P$  is Boolean algebra since  $NI_P = \{I^* \in I_P : I \in I_P\}$  is the set of normal ideals in  $P$ .

Alternatively,  $NI_P = \{I \in I_P : I = I^{**}\}$ . Let  $I \in I_P$ . Take  $I^* = \{y \in P : \text{forevery } i \in I : y \wedge i = 0\}$ ,  $I^* \in I_P$ . Namely, if  $a \in I^*$  then for every  $i \in I : a \wedge i = 0$ . Let  $b \leq a$ . Then, obviously, for every  $i \in I, b \wedge i = 0$ . Thus  $b \in I^*$ . If  $a, b \in I^*$ , then for every  $i \in I, a \wedge i = 0$ , and for every  $i \in I, b \wedge i = 0$ .

Hence for every  $i \in I, (a \wedge i) \vee (b \wedge i) = 0$ . By distributive, for every  $i \in I, i \wedge (a \vee b) = 0$ , i.e.,  $a \vee b \in I^*$ . Thus  $I^* \in I_P, I \cap I^* = I \cap \{y \in P, \text{forevery } i \in I, y \wedge i = 0\} = \{0\}$ .

Let  $I \cap J = \{0\}$  and  $j \in J$ . Suppose that for some  $i \in I, i \wedge j \neq 0$ . Then  $i \wedge j \in I \cap J$ . Because  $I$  and  $J$  are ideals, so  $I \cap J \neq \{0\}$ . Hence, for every  $i \in I, j \wedge i = 0$ , and  $j \subseteq I$ .

Consequently,  $I^*$  is a pseudo complement of  $I$  and  $I_P$  is a pseudo complemented. Therefore  $I_P$  is a Boolean algebra. Thus  $NI_P$  is the set of all pseudo complements lattice in  $I_P$ .

Notice that we have proved that pseudo complemented form a Boolean algebra in Theorem 3.1. Whence,  $NI_P$  is a Boolean algebra. By definition,  $S$  is a Smarandache lattice.  $\square$

**Theorem 3.3** *Let  $S$  be a lattice. If there exist a pseudo complemented distributive lattice  $P$ ,  $X^*(P)$  is a sub-lattice of the lattice  $I^\delta(P)$  of all  $\delta$ -ideals of  $P$ , which is the proper subset of  $S$ . Then  $S$  is a Smarandache lattice.*

*Proof* By hypothesis, let  $S$  be a lattice and there exist a pseudo complemented distributive lattice  $P$ ,  $X^*(P)$  is a sub-lattice of the lattice  $I^\delta(P)$  of all  $\delta$ -ideals of  $P$ , which is the proper subset of  $S$ .

Let  $(a^*), (b^*) \in X^*(P)$  for some  $a, b \in P$ . Clearly,  $(a^*) \cap (b^*) \in X^*(P)$ . Again,  $(a^*) \cup (b^*) = \delta([a]) \cup \delta([b]) = \delta([a] \cup [b]) = \delta([a \cap b]) = ((a \cap b)^*) \in X^*(P)$ . Hence  $X^*(P)$  is a sub-lattice of  $I^\delta(P)$  and it is a distributive lattice. Clearly  $(0^{**})$  and  $(0^*)$  are the least and greatest elements of  $X^*(P)$ .

Now for any  $a \in P$ ,  $(a^*) \cap (a^{**}) = (0)$  and  $(a^*) \cup (b^{**}) = \delta([a]) \cup \delta([a^*]) = \delta([a]) \cup ([a^*]) = \delta([a \cap a^*]) = \delta([0]) = \delta(P) = P$ . Hence  $(a^{**})$  is the complement of  $(a^*)$  in  $X^*(P)$ .

Therefore  $\{X^*(P), \cup, \cap\}$  is a bounded distributive lattice in which every element is complemented.

Thus  $X^*(P)$  is also a Boolean algebra, which implies that  $S$  is a Smarandache lattice.  $\square$

**Theorem 3.4** *Let  $S$  be a lattice and  $P$  a pseudo complemented distributive lattice. If  $S$  is a Smarandache lattice. Then the following conditions are equivalent:*

- (1)  $P$  is a Boolean algebra;
- (2) every element of  $P$  is closed;
- (3) every principal ideal is a  $\delta$ -ideal;
- (4) for any ideal  $I$ ,  $a \in I$  implies  $a^{**} \in I$ ;

- (5) for any proper ideal  $I$ ,  $I \cap D(P) = \phi$ ;
- (6) for any prime ideal  $A$ ,  $A \cap D(P) = \phi$ ;
- (7) every prime ideal is a minimal prime ideal;
- (8) every prime ideal is a  $\delta$ -ideal;
- (9) for any  $a, b \in P$ ,  $a* = b*$  implies  $a = b$ ;
- (10)  $D(P)$  is a singleton set.

*Proof* Since  $S$  is a Smarandache lattice. By definition and previous theorem, we observe that there exists a proper subset  $P$  of  $S$  such that which is a Boolean algebra. Therefore,  $P$  is a Boolean algebra.

(1)  $\implies$  (2) Assume that  $P$  is a Boolean algebra. Then clearly,  $P$  has a unique dense element, precisely the greatest element. Let  $a \in P$ . Then  $a * \wedge a = 0 = a * \wedge a **$ . Also  $a * \vee a$ ,  $a * \vee a ** \in D(P)$ . Hence  $a * \vee a = a * \vee a **$ . By the cancellation property of  $P$ , we get  $a = a **$ . Therefore every element of  $P$  is closed.

(2)  $\implies$  (3) Let  $I$  be a principal ideal of  $P$ . Then  $I = (a]$  for some  $a \in P$ . By condition (2),  $a = a **$ . Now,  $(a] = (a * *) = \delta([a*])$ . So  $(a]$  is a  $\delta$ -ideal.

(3)  $\implies$  (4) Notice that  $I$  be a proper ideal of  $P$ . Let  $a \in I$ . Then there must be  $(a] = \delta(F)$  for some filter  $F$  of  $P$ . Hence, we get that  $a *** = a * \in F$ . Therefore  $a *** \in \delta(F) = (a] \subseteq I$ .

(4)  $\implies$  (5) Let  $I$  be a proper ideal of  $P$ . Suppose  $a \in I \cap D(P)$ . Then  $a *** \in P$  and  $a * = 0$ . Therefore  $1 = 0 * = a *** \in P$ , a contradiction.

(5)  $\implies$  (6) Let  $I$  be a proper ideal of  $P$ ,  $I \cap D(P) = \phi$ . Then  $P$  is a prime ideal of  $P$ ,  $A \cap D(P) = \phi$ .

(6)  $\implies$  (7) Let  $A$  be a prime ideal of  $P$  such that  $A \cap D(P) = \phi$  and  $a \in A$ . Clearly  $a \wedge a * = 0$  and  $a \vee a * \in D(P)$ . So  $a \vee a * \notin A$ , i.e.,  $a * \notin A$ . Therefore  $A$  is a minimal prime ideal of  $P$ .

(7)  $\implies$  (8) Let  $A$  be a minimal prime ideal of  $P$ . It is clear that  $P \setminus A$  is a filter of  $P$ . Let  $a \in A$ . Since  $A$  is minimal, there exists  $b \notin A$  such that  $a \wedge b = 0$ . Hence  $a * \wedge b = b$  and  $a * \notin A$ . Whence,  $a * \in (P \setminus A)$ , which yields  $a \in \delta(P \setminus A)$ . Conversely, let  $a \in \delta(P \setminus A)$ . Then we get  $a * \notin A$ . Thus, we have  $a \in A$  and  $P = \delta(P \setminus A)$ . Therefore  $A$  is  $\delta$ -ideal of  $P$ .

(8)  $\implies$  (9) Assume that every prime ideal of  $P$  is a  $\delta$ -ideal. Let  $a, b \in P$  be chosen that  $a * = b*$ . Suppose  $a \neq b$ . Then there exists a prime ideal  $A$  of  $P$  such that  $a \in A$  and  $b \notin A$ . By hypothesis,  $A$  is a  $\delta$ -ideal of  $P$ . Hence  $A = \delta(F)$  for some filter  $F$  of  $P$ . Consequently,  $a \in A = \delta(F)$ , We get  $b * = a * \in F$ . Thus,  $b \in \delta(F) = A$ , a contradiction. Therefore  $a = b$ .

(9)  $\implies$  (10) Suppose  $x, y$  be two elements of  $D(P)$ . Then  $x * = 0 = y*$ , which implies that  $x = y$ . Therefore  $D(P)$  is a singleton set.

(10)  $\implies$  (1) Assume that  $D(P) = \{d\}$  is singleton set. Let  $a \in P$ . We always have  $a \vee a * \in D(P)$ . Whence,  $a \wedge a * = 0$  and  $a \vee a * = d$ . This true for all  $a \in P$ . Also  $0 \leq a \leq a \vee a * = d$ .

Therefore  $P$  is a bounded distributive lattice, in which every element is complemented, Hence the above conditions are equivalent.  $\square$

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## On Soft Mixed Neutrosophic N-Algebraic Structures

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**Abstract:** Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft mixed neutrosophic N-algebraic with the discussion of some of their characteristics. We also introduced soft mixed dual neutrosophic N-algebraic structures, soft weak mixed neutrosophic N-algebraic structures, soft Lagrange mixed neutrosophic N-algebraic structures, soft weak Lagrange mixed neutrosophic and soft Lagrange free mixed neutrosophic N-algebraic structures. the so called soft strong neutrosophic loop which is of pure neutrosophic character. We also introduced some of new notions and some basic properties of this newly born soft mixed neutrosophic N-structures related to neutrosophic theory.

**Key Words:** Neutrosophic mixed N-algebraic structure, soft set, soft neutrosophic mixed neutrosophic N-algebraic structure.

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### §1. Introduction

Smarandache proposed the concept of neutrosophy in 1980, which is basically a new branch of philosophy that actually deals the origion, nature, and scope of neutralities. He also introduced the neutrosophic logic due to neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset  $T$ , the percentage of indeterminacy in a subset  $I$  and the percentage of falsity in a subset  $F$ . Basically, a neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set etc. Neutrosophic logic is used to overcome the problems of imperciseness, indeterminate and inconsistentness of the data. The theoey of neutrosophy is also applicable in the field of algebra. For example, Kandasamy and Smarandache introduced neutrosophic fields, neutrosophic rings, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups and neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemi-

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groups, and neutrosophic N-semigroups, neutrosophic loops, neutrosophic biloops, and neutrosophic N-loops, and so on. Mumtaz ali et.al. introduced neutrosophic LA-semigroups and also give their generalization.

Molodtsov introduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in [5, 6, 7, 8, 10]. Some properties and algebra may be found in [11]. Feng et.al. introduced soft semirings in [9].

In this paper we introduced soft mixed neutrosophic N-algebraic structures. The organization of this paper is follows. In section one we put the basic concepts about mixed neutrosophic N-algebraic structures and soft sets with some of their operations. In the next sections we introduce soft mixed neutrosophic N-algebraic structures with the construction of some their related theory. At the end we concluded the paper.

## §2. Basic Concepts

### 2.1 Mixed Neutrosophic N-Algebraic Structures

**Definition 2.1** Let  $\{\langle M \cup I \rangle = (M_1 \cup M_2 \cup \dots \cup M_N, *_1, *_2, \dots, *_N)\}$  such that  $N \geq 5$ . Then  $\langle M \cup I \rangle$  is called a mixed neutrosophic N-algebraic structure if

- (1)  $\langle M \cup I \rangle = M_1 \cup M_2 \cup \dots \cup M_N$ , where each  $M_i$  is a proper subset of  $\langle M \cup I \rangle$  for all  $i$ ;
- (2) some of  $(M_i, *_i)$  are neutrosophic groups;
- (3) some of  $(M_j, *_j)$  are neutrosophic loops;
- (4) some of  $(M_k, *_k)$  are neutrosophic groupoids;
- (5) some of  $(M_r, *_r)$  are neutrosophic semigroups.
- (6) the rest of  $(M_t, *_t)$  can be loops or groups or semigroups or groupoids. ('or' not used in the mutually exclusive sense).

**Definition 2.2** Let  $\{\langle D \cup I \rangle = (D_1 \cup D_2 \cup \dots \cup D_N, *_1, *_2, \dots, *_N)\}$ . Then  $\langle D \cup I \rangle$  is called a mixed dual neutrosophic N-algebraic structure if

- (1)  $\langle D \cup I \rangle = D_1 \cup D_2 \cup \dots \cup D_N$ , where each  $D_i$  is a proper subset of  $\langle D \cup I \rangle$  for all  $i$ ;
- (2) some of  $(D_i, *_i)$  are neutrosophic groups;
- (3) some of  $(D_j, *_j)$  are neutrosophic loops;
- (4) some of  $(D_k, *_k)$  are neutrosophic groupoids;
- (5) some of  $(D_r, *_r)$  are neutrosophic semigroups;
- (6) the rest of  $(D_t, *_t)$  can be loops or groups or semigroups or groupoids. ('or' not used in the mutually exclusive sense).

**Definition 2.3** Let  $\{\langle W \cup I \rangle = (W_1 \cup W_2 \cup \dots \cup W_N, *_1, *_2, \dots, *_N)\}$ . Then  $\langle W \cup I \rangle$  is called a weak mixed neutrosophic N-algebraic structure if

- (1)  $\langle W \cup I \rangle = W_1 \cup W_2 \cup \dots \cup W_N$ , where each  $W_i$  is a proper subset of  $\langle W \cup I \rangle$  for all  $i$ ;
- (2) some of  $(W_i, *_i)$  are neutrosophic groups or neutrosophic loops;
- (3) some of  $(W_k, *_k)$  are neutrosophic groupoids or neutrosophic semigroups;
- (4) the Rest of  $(W_t, *_t)$  can be loops or groups or semigroups or groupoids. i.e in the collection, all the algebraic neutrosophic structures may not be present.

At most 3-algebraic neutrosophic structures are present and at least 2-algebraic neutrosophic structures are present. Rest being non-neutrosophic algebraic structures.

**Definition 2.4** Let  $\{\langle V \cup I \rangle = (V_1 \cup V_2 \cup \dots \cup V_N, *_1, *_2, \dots, *_N)\}$ . Then  $\langle V \cup I \rangle$  is called a weak mixed dual neutrosophic N-algebraic structure if

- (1)  $\langle V \cup I \rangle = V_1 \cup V_2 \cup \dots \cup V_N$ , where each  $W_i$  is a proper subset of  $\langle V \cup I \rangle$  for all  $i$ ;
- (2) some of  $(V_i, *_i)$  are neutrosophic groups or neutrosophic loops;
- (3) some of  $(V_k, *_k)$  are neutrosophic groupoids or neutrosophic semigroups;
- (4) the rest of  $(V_t, *_t)$  can be loops or groups or semigroups or groupoids.

**Definition 2.5** Let  $\{\langle M \cup I \rangle = (M_1 \cup M_2 \cup \dots \cup M_N, *_1, *_2, \dots, *_N)\}$  be a neutrosophic N-algebraic structure. A proper subset  $\{\langle P \cup I \rangle = (P_1 \cup P_2 \cup \dots \cup P_N, *_1, *_2, \dots, *_N)\}$  is called a mixed neutrosophic sub N-structure if  $\langle P \cup I \rangle$  is a mixed neutrosophic N-structure under the operation of  $\langle M \cup I \rangle$ .

**Definition 2.6** Let  $\{\langle W \cup I \rangle = (W_1 \cup W_2 \cup \dots \cup W_N, *_1, *_2, \dots, *_N)\}$  be a mixed neutrosophic N-algebraic structure. We call a finite non-empty subset  $P$  of  $\langle W \cup I \rangle$  to be a weak mixed deficit neutrosophic sub N-algebraic structure if  $\{\langle P \cup I \rangle = (P_1 \cup P_2 \cup \dots \cup P_t, *_1, *_2, \dots, *_t)\}$ ,  $1 < t < N$  with  $P_i = P \cap L_k$ ,  $1 \leq i \leq t$ , and  $1 \leq k \leq N$  and some  $P'_i$ 's are neutrosophic groups or neutrosophic loops, some  $P'_j$ 's are neutrosophic groupoids or neutrosophic semigroups and the rest of  $P'_k$ 's are groups or loops or groupoids or semigroups.

**Definition 2.7** Let  $\{\langle M \cup I \rangle = (M_1 \cup M_2 \cup \dots \cup M_N, *_1, *_2, \dots, *_N)\}$  be a mixed neutrosophic N-algebraic structure of finite order. A proper mixed neutrosophic sub N-structure  $P$  of  $\langle M \cup I \rangle$  is called Lagrange mixed neutrosophic sub N-structure if  $o(P)/o\langle M \cup I \rangle$ .

If every mixed neutrosophic sub N-structure of  $\langle M \cup I \rangle$  is a Lagrange mixed neutrosophic sub N-structures. Then  $\langle M \cup I \rangle$  is said to be a Lagrange mixed neutrosophic N-structure.

If some mixed neutrosophic sub N-structure of  $\langle M \cup I \rangle$  are Lagrange mixed neutrosophic sub N-structures. Then  $\langle M \cup I \rangle$  is said to be a weak Lagrange mixed neutrosophic N-structure.

If every mixed neutrosophic sub N-structure of  $\langle M \cup I \rangle$  is not a Lagrange mixed neutrosophic sub N-structures. Then  $\langle M \cup I \rangle$  is said to be a Lagrange free mixed neutrosophic N-structure.

## 2.2 Soft Sets

Throughout this subsection  $U$  refers to an initial universe,  $E$  is a set of parameters,  $P(U)$  is the power set of  $U$ , and  $A \subset E$ . Molodtsov [12] defined the soft set in the following manner.

**Definition 2.8** A pair  $(F, A)$  is called a soft set over  $U$  where  $F$  is a mapping given by  $F : A \longrightarrow P(U)$ .

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . For  $e \in A$ ,  $F(a)$  may be considered as the set of  $a$ -elements of the soft set  $(F, A)$ , or as the set of  $e$ -approximate elements of the soft set.

**Definition 2.9** For two soft sets  $(F, A)$  and  $(H, B)$  over  $U$ ,  $(F, A)$  is called a soft subset of  $(H, B)$  if

- (1)  $A \subseteq B$  and
- (2)  $F(a) \subseteq G(a)$ , for all  $a \in A$ .

This relationship is denoted by  $(F, A) \widetilde{\subset} (H, B)$ . Similarly  $(F, A)$  is called a soft superset of  $(H, B)$  if  $(H, B)$  is a soft subset of  $(F, A)$  which is denoted by  $(F, A) \widetilde{\supset} (H, B)$ .

**Definition 2.10** Two soft sets  $(F, A)$  and  $(H, B)$  over  $U$  are called soft equal if  $(F, A)$  is a soft subset of  $(H, B)$  and  $(H, B)$  is a soft subset of  $(F, A)$ .

**Definition 2.11**  $(F, A)$  over  $U$  is called an absolute soft set if  $F(a) = U$  for all  $a \in A$  and we denote it by  $\mathcal{F}_U$ .

**Definition 2.12** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$  such that  $A \cap B \neq \phi$ . Then their restricted intersection is denoted by  $(F, A) \cap_R (G, B) = (H, C)$  where  $(H, C)$  is defined as  $H(c) = F(a) \cap G(a)$  for all  $a \in C = A \cap B$ .

**Definition 2.13** The extended intersection of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $a \in C$ ,  $H(a)$  is defined as

$$H(a) = \begin{cases} F(a) & \text{if } a \in A - B \\ G(a) & \text{if } a \in B - A \\ F(a) \cap G(a) & \text{if } a \in A \cap B. \end{cases}$$

We write  $(F, A) \cap_\varepsilon (G, B) = (H, C)$ .

**Definition 2.14** The restricted union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $a \in C$ ,  $H(a)$  is defined as the soft set  $(H, C) = (F, A) \cup_R (G, B)$  where  $C = A \cap B$  and  $H(a) = F(a) \cup G(a)$  for all  $a \in C$ .

**Definition 2.15** The extended union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$  is the soft set  $(H, C)$ , where  $C = A \cup B$ , and for all  $a \in C$ ,  $H(a)$  is defined as

$$H(a) = \begin{cases} F(a) & \text{if } a \in A - B \\ G(a) & \text{if } a \in B - A \\ F(a) \cup G(a) & \text{if } a \in A \cap B. \end{cases}$$

We write  $(F, A) \cup_\varepsilon (G, B) = (H, C)$ .

### §3. Soft Mixed Neutrosophic N-Algebraic Structures

**Definition 3.1** Let  $\langle M \cup I \rangle$  be a mixed neutrosophic N-algebraic structure and let  $(F, A)$  be a soft set over  $\langle M \cup I \rangle$ . Then  $(F, A)$  is called a soft mixed neutrosophic N-algebraic structure if and only if  $F(a)$  is a mixed neutrosophic sub N-algebraic structure of  $\langle M \cup I \rangle$  for all  $a \in A$ .

**Example 3.1** Let  $\{\langle M \cup I \rangle = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5, *_1, *_2, \dots, *_5\}$  be a mixed neutrosophic 5-structure, where

$M_1 = \langle \mathbb{Z}_3 \cup I \rangle$ , a neutrosophic group under multiplication mod3,

$M_2 = \langle \mathbb{Z}_6 \cup I \rangle$ , a neutrosophic semigroup under multiplication mod6,

$M_3 = \{0, 1, 2, 3, I, 2I, 3I\}$ , a neutrosophic groupoid under multiplication mod4,

$M_4 = S_3$ , and

$M_5 = \{\mathbb{Z}_{10}\}$ , a semigroup under multiplication mod10}.

Let  $A = \{a_1, a_2, a_3\} \subset E$  be a set of parameters and let  $(F, A)$  be a soft set over  $\langle M \cup I \rangle$ , where

$$\begin{aligned} F(a_1) &= \{1, I\} \cup \{0, 3, 3I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 2, 4, 6, 8\}, \\ F(a_2) &= \{2, I\} \cup \{0, 2, 4, 2I, 4I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 5\}, \\ F(a_3) &= \{1, 2\} \cup \{0, 3\} \cup \{0, 2\} \cup A_3 \cup \{0, 2, 4, 6, 8\}. \end{aligned}$$

Clearly  $(F, A)$  is a soft mixed neutrosophic 5-algebraic structure over  $\langle M \cup I \rangle$ .

**Proposition 3.1** Let  $(F, A)$  and  $(H, A)$  be two soft mixed neutrosophic N-algebraic structures over  $\langle M \cup I \rangle$ . Then their intersection is again a soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ .

*Proof* The proof is straightforward.  $\square$

**Proposition 3.2** Let  $(F, A)$  and  $(H, B)$  be two soft mixed neutrosophic N-algebraic structures over  $\langle M \cup I \rangle$ . If  $A \cap B = \phi$ , then  $(F, A) \cap (H, B)$  is a soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ .

*Proof* The proof is straightforward.  $\square$

**Proposition 3.3** Let  $(F, A)$  and  $(H, B)$  be two soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ . Then

- (1) their extended intersection is a soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ ;
- (2) their restricted intersection is a soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ ;
- (3) their AND operation is a soft mixed neutrosophic N-algebraic structure over  $\langle M \cup I \rangle$ .

**Remark 3.1** Let  $(F, A)$  and  $(H, B)$  be two soft mixed neutrosophic N-algebraic structure over



$\langle M \cup I \rangle$ . Then

(1) their restricted union may not be a soft mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

(2) their extended union may not be a soft mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

(3) their  $OR$  operation may not be a soft mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

To establish the above remark, see the following example.

**Example 3.2** Let  $\{\langle M \cup I \rangle = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5, *_1, *_2, \dots, *_5\}$  be a mixed neutrosophic 5-structure, where

$M_1 = \langle \mathbb{Z}_3 \cup I \rangle$ , a neutrosophic group under multiplication mod3,

$M_2 = \langle \mathbb{Z}_6 \cup I \rangle$ , a neutrosophic semigroup under multiplication mod6,

$M_3 = \{0, 1, 2, 3, I, 2I, 3I\}$ , a neutrosophic groupoid under multiplication mod4,

$M_4 = S_3$ , and

$M_5 = \{\mathbb{Z}_{10}\}$ , a semigroup under multiplication mod10.

Let  $A = \{a_1, a_2, a_3\} \subset E$  be a set of parameters and let  $(F, A)$  be a soft set over  $\langle M \cup I \rangle$ , where

$$\begin{aligned} F(a_1) &= \{1, I\} \cup \{0, 3, 3I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 2, 4, 6, 8\}, \\ F(a_2) &= \{2, I\} \cup \{0, 2, 4, 2I, 4I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 5\}, \\ F(a_3) &= \{1, 2\} \cup \{0, 3\} \cup \{0, 2\} \cup A_3 \cup \{0, 2, 4, 6, 8\}. \end{aligned}$$

Let  $B = \{a_1, a_4\}$  be a another set of parameters and let  $(H, B)$  be a another soft mixed neutrosophic 5-algebraic structure over  $\langle M \cup I \rangle$ , where

$$\begin{aligned} H(a_1) &= \{1, I\} \cup \{0, 3I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 2, 4, 6, 8\}, \\ H(a_4) &= \{1, 2\} \cup \{0, 3I\} \cup \{0, 2I\} \cup A_3 \cup \{0, 5\}. \end{aligned}$$

Let  $C = A \cap B = \{a_1\}$ . The restricted union  $(F, A) \cup_R (H, B) = (K, C)$ , where

$$K(a_1) = F(a_1) \cup H(a_1) = \{1, I, 2\} \cup \{0, 3I\} \cup \{0, 2, 2I\} \cup A_3 \cup \{0, 2, 4, 5, 6, 8\}$$

Thus clearly  $\{1, 2, I\}$  and  $\{0, 2, 4, 5, 6, 8\}$  in  $H(a_1)$  are not subgroups. This shows that  $(K, C)$  is not a soft mixed neutrosophic 5-algebraic structure over  $\langle M \cup I \rangle$ . Similarly one can easily show 2 and 3 by the help of examples.

**Definition 3.2** Let  $\langle D \cup I \rangle$  be a mixed dual neutrosophic  $N$ -algebraic structure and let  $(F, A)$  soft set over  $\langle D \cup I \rangle$ . Then  $(F, A)$  is called a soft mixed dual neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is a mixed dual neutrosophic sub  $N$ -algebraic structure  $\langle D \cup I \rangle$  of for all  $a \in A$ .

**Example 3.3** Let  $\langle D \cup I \rangle = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5, *, *_1, *_2, \dots, *_5$  be a mixed dual neutrosophic 5-algebraic structure, where

$$\begin{aligned} D_1 &= L_7(4), D_2 = S_4, D_3 = \{\mathbb{Z}_{10}, \text{ a semigroup under multiplication modulo } 10\}, \\ D_4 &= \{0, 1, 2, 3, \text{ a groupoid under multiplication modulo } 4\}, \\ D_5 &= \langle L_7(4) \cup I \rangle. \end{aligned}$$

Let  $A = \{a_1, a_2\}$  be a set of parameters and let  $(F, A)$  be a soft set over  $\langle D \cup I \rangle$ , where

$$\begin{aligned} F(a_1) &= \{e, 2\} \cup A_4 \cup \{0, 2, 4, 6, 8\} \cup \{0, 2\} \cup \{e, eI, 2, 2I\}, \\ F(a_2) &= \{e, 3\} \cup S_4 \cup \{0, 5\} \cup \{0, 2\} \cup \{e, eI, 3, 3I\} \end{aligned}$$

Clearly  $(F, A)$  is a soft mixed dual neutrosophic -structure over  $\langle D \cup I \rangle$ .

**Theorem 3.1** *If  $\langle D \cup I \rangle$  is a mixed dual neutrosophic N-algebraic structure. Then  $(F, A)$  over  $\langle D \cup I \rangle$  is also a soft mixed dual neutrosophic N-algebraic structure.*

**Proposition 3.4** *Let  $(F, A)$  and  $(H, B)$  be two soft mixed dual neutrosophic N-algebraic structures over  $\langle D \cup I \rangle$ . Then their intersection is again a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ .*

*Proof* The proof is straightforward.  $\square$

**Proposition 3.5** *Let  $(F, A)$  and  $(H, B)$  be two soft mixed dual neutrosophic N-algebraic structures over  $\langle D \cup I \rangle$ . If  $A \cap B = \phi$ , then  $(F, A) \cap (H, B)$  is a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ .*

*Proof* The proof is straightforward.  $\square$

**Proposition 3.6** *Let  $(F, A)$  and  $(H, B)$  be two soft mixed dual neutrosophic N-algebraic structures over  $\langle D \cup I \rangle$ . Then*

- (1) *their extended intersection is a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ ;*
- (2) *their restricted intersection is a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ ;*
- (3) *their AND operation is a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ .*

**Remark 3.2** Let  $(F, A)$  and  $(H, B)$  be two soft mixed Dual neutrosophic N-algebraic structures over  $\langle D \cup I \rangle$ . Then

- (1) *their restricted union may not be a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ ;*
- (2) *their extended union may not be a soft mixed dual neutrosophic N-algebraic structure over  $\langle D \cup I \rangle$ ;*

(3) their *OR* operation may not be a soft mixed dual neutrosophic  $N$ -algebraic structure over  $\langle D \cup I \rangle$ .

One can easily establish the above remarks by the help of examples.

**Definition 3.3** Let  $\langle W \cup I \rangle$  be a weak mixed neutrosophic  $N$ -algebraic structure and let  $(F, A)$  soft set over  $\langle W \cup I \rangle$ . Then  $(F, A)$  is called a soft weak mixed neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is a weak mixed neutrosophic sub  $N$ -structure of  $\langle W \cup I \rangle$  for all  $a \in A$ .

**Theorem 3.2** If  $\langle W \cup I \rangle$  is a weak mixed neutrosophic  $N$ -algebraic structure. Then  $(F, A)$  over  $\langle W \cup I \rangle$  is also a soft weak mixed neutrosophic  $N$ -algebraic structure.

The restricted intersection, extended intersection and the *AND* operation of two soft weak mixed neutrosophic  $N$ -algebraic structures is again soft weak mixed neutrosophic  $N$ -algebraic structures.

The restricted union, extended union and the *OR* operation of two soft weak mixed neutrosophic  $N$ -algebraic structures may not be soft weak mixed neutrosophic  $N$ -algebraic structures.

**Definition 3.4** Let  $\langle V \cup I \rangle$  be a weak mixed dual neutrosophic  $N$ -algebraic structure and let  $(F, A)$  soft set over  $\langle V \cup I \rangle$ . Then  $(F, A)$  is called a soft weak mixed dual neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is a weak mixed dual neutrosophic sub  $N$ -structure of  $\langle V \cup I \rangle$  for all  $a \in A$ .

**Theorem 3.3** If  $\langle V \cup I \rangle$  is a weak mixed dual neutrosophic  $N$ -algebraic structure. Then  $(F, A)$  over  $\langle V \cup I \rangle$  is also a soft weak mixed dual neutrosophic  $N$ -algebraic structure.

The restricted intersection, extended intersection and the *AND* operation of two soft weak mixed dual neutrosophic  $N$ -algebraic structures is again a soft weak mixed dual neutrosophic  $N$ -algebraic structures.

The restricted union, extended union and the *OR* operation of two soft weak mixed dual neutrosophic  $N$ -algebraic structures may not be soft weak mixed dual neutrosophic  $N$ -algebraic structures.

**Definition 3.5** Let  $(F, A)$  and  $(H, B)$  be two soft mixed neutrosophic  $N$ -algebraic structures over  $\langle M \cup I \rangle$ . Then  $(H, B)$  is called soft mixed neutrosophic sub  $N$ -algebraic structure of  $(F, A)$ , if

- (1)  $B \subseteq A$ ;
- (2)  $H(a)$  is a mixed neutrosophic sub  $N$ -structure of  $F(a)$  for all  $a \in A$ .

It is important to note that a soft mixed neutrosophic  $N$ -algebraic structure can have soft weak mixed neutrosophic sub  $N$ -algebraic structure. But a soft weak mixed neutrosophic sub  $N$ -structure cannot in general have a soft mixed neutrosophic  $N$ -structure.

**Definition 3.6** Let  $\langle V \cup I \rangle$  be a weak mixed neutrosophic  $N$ -algebraic structure and let  $(F, A)$  be a soft set over  $\langle V \cup I \rangle$ . Then  $(F, A)$  is called a soft weak mixed deficit neutrosophic  $N$ -

algebraic structure if and only if  $F(a)$  is a weak mixed deficit neutrosophic sub  $N$ -structure of  $\langle V \cup I \rangle$  for all  $a \in A$ .

**Proposition 3.7** Let  $(F, A)$  and  $(H, B)$  be two soft weak mixed deficit neutrosophic  $N$ -algebraic structures over  $\langle V \cup I \rangle$ . Then

- (1) their extended intersection is a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ ;
- (2) their restricted intersection is a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ ;
- (3) their  $AND$  operation is a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ .

**Remark 3.3** Let  $(F, A)$  and  $(H, B)$  be two soft weak mixed deficit neutrosophic  $N$ -algebraic structures over  $\langle V \cup I \rangle$ . Then

- (1) their restricted union may not be a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ ;
- (2) their extended union may not be a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ ;
- (3) their  $OR$  operation may not be a soft weak mixed deficit neutrosophic  $N$ -algebraic structure over  $\langle V \cup I \rangle$ .

One can easily establish the above remarks by the help of examples.

**Definition 3.7** Let  $\langle M \cup I \rangle$  be a mixed neutrosophic  $N$ -algebraic structure and let  $(F, A)$  soft set over  $\langle M \cup I \rangle$ . Then  $(F, A)$  is called a soft Lagrange mixed neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is a Lagrange mixed neutrosophic sub  $N$ -structure of  $\langle M \cup I \rangle$  for all  $a \in A$ .

**Theorem 3.4** If  $\langle M \cup I \rangle$  is a Lagrange mixed neutrosophic  $N$ -algebraic structure. Then  $(F, A)$  over  $\langle M \cup I \rangle$  is also a soft Lagrange mixed neutrosophic  $N$ -algebraic structure.

**Remark 3.4** Let  $(F, A)$  and  $(H, B)$  be two soft Lagrange mixed neutrosophic  $N$ -algebraic structures over  $\langle M \cup I \rangle$ . Then

- (1) their restricted union may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (2) their extended union may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (3) their  $AND$  operation may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (4) their extended intersection may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (5) their restricted intersection may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

(6) their *OR* operation may not be a soft Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

One can easily establish the above remarks by the help of examples.

Now on similar lines, we can define soft Lagrange weak deficit mixed neutrosophic  $N$ -algebraic structures.

**Definition 3.8** Let  $\langle M \cup I \rangle$  be a mixed neutrosophic  $N$ -algebraic structure and let  $(F, A)$  be a soft set over  $\langle M \cup I \rangle$ . Then  $(F, A)$  is called a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is not a Lagrange mixed neutrosophic sub  $N$ -structure of  $\langle M \cup I \rangle$  for some  $a \in A$ .

**Remark 3.5** Let  $(F, A)$  and  $(H, B)$  be two soft weak Lagrange mixed neutrosophic  $N$ -algebraic structures over  $\langle M \cup I \rangle$ . Then

- (1) their restricted union may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (2) their extended union may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (3) their *AND* operation may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (4) their extended intersection may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (5) their restricted intersection may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (6) their *OR* operation may not be a soft weak Lagrange mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

One can easily establish the above remarks by the help of examples. Similarly we can define soft weak Lagrange weak deficit mixed neutrosophic  $N$ -algebraic structures.

**Definition 3.9** Let  $\langle M \cup I \rangle$  be a mixed neutrosophic  $N$ -algebraic structure and let  $(F, A)$  be a soft set over  $\langle M \cup I \rangle$ . Then  $(F, A)$  is called a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure if and only if  $F(a)$  is not a Lagrange mixed neutrosophic sub  $N$ -structure of  $\langle M \cup I \rangle$  for all  $a \in A$ .

**Theorem 3.5** If  $\langle M \cup I \rangle$  is a Lagrange free mixed neutrosophic  $N$ -algebraic structure. Then  $(F, A)$  over  $\langle M \cup I \rangle$  is also a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure.

**Remark 3.6** Let  $(F, A)$  and  $(H, B)$  be two soft Lagrange free mixed neutrosophic  $N$ -algebraic structures over  $\langle M \cup I \rangle$ . Then

- (1) their restricted union may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;
- (2) their extended union may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;

(3) their  $AND$  operation may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;

(4) their extended intersection may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;

(5) their restricted intersection may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ ;

(6) their  $OR$  operation may not be a soft Lagrange free mixed neutrosophic  $N$ -algebraic structure over  $\langle M \cup I \rangle$ .

One can easily establish the above remarks by the help of examples. Similarly we can define soft Lagrange free weak deficit mixed neutrosophic  $N$ -algebraic structures.

#### §4. Conclusion

This paper is an extension of soft sets to mixed neutrosophic  $N$ -algebraic structures. Their related properties and results are explained with illustrative examples.

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December 2014

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