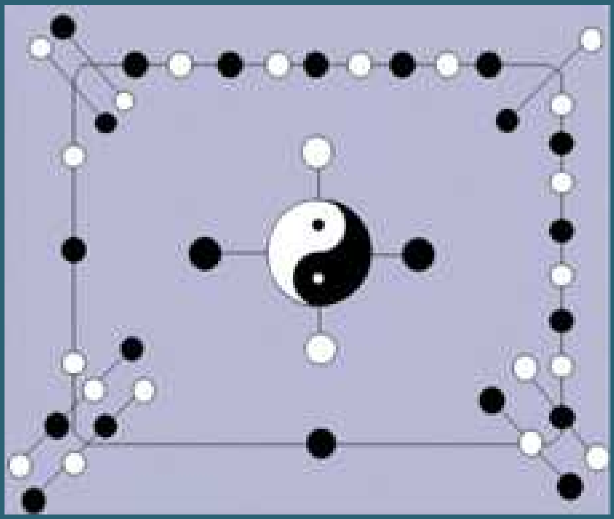
ISBN 978-1-59973-136-0

VOLUME 4, 2010

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES

December, 2010

Vol.4, 2010 ISBN 978-1-59973-136-0

Mathematical Combinatorics

(International Book Series)

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The Madis of Chinese Academy of Sciences

December, 2010

Aims and Scope: The **Mathematical Combinatorics (International Book Series)** (*ISBN 978-1-59973-136-0*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in March, June, September and December in USA, each comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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If you don't learn to think when you are young, you may never learn.

By Thomas Edison, an American inventor.

Math.Combin.Book Ser. Vol.4 (2010), 01-07

On $(\in vq)$-Fuzzy Bigroup

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Abstract: In this paper, we introduce the concept of fuzzy singleton to bigroup, and uses it to define $(\in v\ q)$- fuzzy bigroup and discuss its properties. We investigate whether or not the fuzzy point of a bigroup will belong to or quasi coincident with its fuzzy set if the constituent fuzzy points of the constituting subgroups both belong to or quasi coincident with their respective fuzzy sets, and vise versa. We also prove that a fuzzy bisubset μ is an $(\in vq)$-fuzzy subbigroup of the bigroup G if its constituent fuzzy subsets are $(\in vq)$-fuzzy subgroups of their respective subgroups among others.

Key Words: Bigroups, fuzzy bigroups, fuzzy singleton on bigroup, $(\in vq)$- fuzzy subgroups, $(\in vq)$- fuzzy bigroup

AMS(2010): 03E72, 20D25

§1. Introduction

Fuzzy set was introduced by Zadeh[14] in 1965. Rosenfeld [9] introduced the notion of fuzzy subgroups in 1971. Ming and Ming [8] in 1980 gave a condition for fuzzy subset of a set to be a fuzzy point, and used the idea to introduce and characterize the notions of quasi coincidence of a fuzzy point with a fuzzy set. Bhakat and Das [2,3] used these notions by Ming and Ming to introduce and characterize another class of fuzzy subgroup known as $(\in vq)$- fuzzy subgroups. This concept has been further developed by other researchers. Recent contributions in this direction include those of Yuan et al [12,13].

The notion of bigroup was first introduced by P.L.Maggu [5] in 1994. This idea was extended in 1997 by Vasantha and Meiyappan [10]. These authors gave modifications of some results earlier proved by Maggu. Among these results was the characterization theorems for sub-bigroup. Meiyappan [11] introduced and characterized fuzzy sub-bigroup of a bigroup in 1998.

In this paper, using these mentioned notions and with emphases on the elements that are both in G_1 and G_2 of the bigroup G, we define the notion of $(\in, \in vq)$ - fuzzy sub bigroups as an extension of the notion of $(\in, \in vq)$- fuzzy subgroups and discuss its properties. Apart from this section that introduces the work, section 2 presents the major preliminary results that are useful for the work. In section 3, we define a fuzzy singleton on a bigroup. Using this definition,

[1]Received August 9, 2010. Accepted December 6, 2010.

we investigate whether or not the fuzzy point of a bigroup will belong to or quasi coincident with its fuzzy set if the constituent fuzzy points of the constituting subgroups both belong to or quasi coincident with their respective fuzzy sets, and vise versa. Theorems 3.4 and 3.5 give the results of these findings. In the same section, we define $(\in vq)$- fuzzy subgroup and prove that a fuzzy bisubset μ is an $(\in vq)$- fuzzy sub bigroup of the bigroup G if its constituent fuzzy subsets are $(\in vq)$- fuzzy subgroups of their respective subgroups.

§2. Preliminary Results

Definition 2.1([10,11]) *A set $(G, +, \cdot)$ with two binary operations " $+$ " and " $\cdot$ " is called a bi-group if there exist two proper subsets G_1 and G_2 of G such that*

(i) $G = G_1 \cup G_2$;

(ii) $(G_1, +)$ *is a group;*

(iii) $(G_2, \cdot)$ *is a group.*

Definition 2.2([10]) *A subset $H(\neq 0)$ of a bi-group $(G, +, \cdot)$ is called a sub bi-group of G if H itself is a bi-group under the operations of " $+$ " and " $\cdot$ " defined on G.*

Theorem 2.3([10]) *Let $(G, +, \cdot)$ be a bigroup. If the subset $H \neq 0$ of a bigroup G is a sub bigroup of G, then $(H, +)$ and $(H, \cdot)$ are generally not groups.*

Definition 2.4([14]) *Let G be a non empty set. A mapping $\mu : G \to [0, 1]$ is called a fuzzy subset of G.*

Definition 2.5([14]) *Let μ be a fuzzy set in a set G. Then, the level subset μ_t is defined as: $\mu_t = \{x \in G : \mu(x) \geq t\}$ for $t \in [0, 1]$.*

Definition 2.6([9]) *Let μ be a fuzzy set in a group G. Then, μ is said to be a fuzzy subgroup of G, if the following hold:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $\forall\, x, y \in G$;

(ii) $\mu(x^{-1}) = \mu(x)$ $\forall\, x \in G$.

Definition 2.7([9]) *Let μ be a fuzzy subgroup of G. Then, the level subset μ_t, for $t \in Im\mu$ is a subgroup of G and is called the level subgroup of G.*

Definition 2.8 ([8]) *A fuzzy subset μ of a group G of the form*

$$\mu(y) = \begin{cases} t(\neq 0) & if\ y = x, \\ 0 & if\ y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.9([11]) *Let μ_1 be a fuzzy subset of a set X_1 and μ_2 be a fuzzy subset of a set X_2, then the fuzzy union of the sets μ_1 and μ_2 is defined as a function $\mu_1 \cup \mu_2 : X_1 \cup X_2 \longrightarrow [0, 1]$ given by:*

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} max(\mu_1(x), \mu_2(x)) & if\ x \in X_1 \cap X_2, \\ \mu_1(x) & if\ x \in X_1, x \notin X_2 \\ \mu_2(x) & if\ x \in X_2 \& x \notin X_1 \end{cases}$$

Definition 2.10([11]) *Let $G = (G_1 \cup G_2, +, \cdot)$ be a bigroup. Then $\mu : G \to [0,1]$ is said to be a fuzzy sub-bigroup of the bigroup G if there exist two fuzzy subsets $\mu_1 (of G_1)$ and $\mu_2 (of G_2)$ such that:*

(i) $(\mu_1, +)$ *is a fuzzy subgroup of* $(G_1, +)$*,*

(ii) $(\mu_2, \cdot)$ *is a fuzzy subgroup of* $(G_2, \cdot)$*, and*

(iii) $\mu = \mu_1 \cup \mu_2$.

Definition 2.11([8]) *A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ , written as $x_t \in \mu$ (resp. $x_t\ q\ \mu$) if $\mu(x) \geq t$(resp. $\mu(x) + t > 1$).*
"$x_t \in \mu$ or $x_t\ q\ \mu$" will be denoted by $x_t \in vq\ \mu$).

Definition 2.12([2,3]) *A fuzzy subset μ of G is said to be an ($\in vq$)- fuzzy subgroup of G if for every $x, y \in G$ and $t, r \in (0, 1]$:*

(i) $x_t \in \mu, y_r \in \mu \Rightarrow (xy)_{M(t,r)} \in vq\ \mu$

(ii) $x_t \in \mu \Rightarrow (x^{-1})_t \in vq\ \mu$.

Theorem 2.13([3]) *(i) A necessary and sufficient condition for a fuzzy subset μ of a group G to be an $(\in, \in vq)$-fuzzy subgroup of G is that $\mu(xy^{-1}) \geq M(\mu(x), \mu(y), 0.5)$ for every $x,\ y \in G$.*
(ii). Let μ be a fuzzy subgroup of G. Then $\mu_t = \{x \in G : \mu(x) \geq t\}$ is a fuzzy subgroup of G for every $0 \leq t \leq 0.5$. Conversely, if μ is a fuzzy subset of G such that μ_t is a subgroup of G for every $t \in (0, 0.5]$, then μ is an fuzzy $(\in, \in vq)$-fuzzy subgroup of G.

Definition 2.14([3]) *Let X be a non empty set. The subset $\mu_t = \{x \in X : \mu(x) \geq t\}$ or $\{\mu(x)+ t > 1\} = \{x \in X : x_t \in vq\ \mu\}$ is called ($\in vq$)- level subset of X determined by μ and t.*

Theorem 2.15([3]) *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if μ_t is a subgroup for all $t \in (0, 1]$.*

§3. Main Results

Definition 3.1 *Let $G = G_1 \cup G_2$ be a bi-group. Let $\mu = \mu_1 \cup \mu_2$ be a fuzzy bigroup. A fuzzy subset $\mu = \mu_1 \cup \mu_2$ of the form:*

$$\mu(x) = \begin{cases} M(t, s) \neq 0 & if\ x = y \in G, \\ 0 & if\ x \neq y \end{cases}$$

where $t, s \in [0, 1]$ such that

$$\mu_1(x) = \begin{cases} t \neq 0 & if x = y \in G_1, \\ 0 & if\ x \neq y \end{cases}$$

and

$$\mu_2(x) = \begin{cases} s \neq 0 & if x = y \in G_2, \\ 0 & if\ x \neq y \end{cases}$$

is said to be a fuzzy point of the bi-group G with support x and value $M(t,s)$ and is denoted by $x_{M(t,s)}$.

Theorem 3.2 *Let $x_{M(t,s)}$ be a fuzzy point of the bigroup $G = G_1 \cup G_2$. Then:*
(i) $x_{M(t,s)} = x_t \Leftrightarrow x \in G_1 \cap G_2^c$ *or* $t > s$
(ii) $x_{M(t,s)} = x_s \Leftrightarrow x \in G_1^c \cap G_2$ *or* $t < s$
$\forall\ t, s\ \in [0,1]$, *where x_t, x_s are fuzzy points of the groups G_1 and G_2 respectively.*

Proof (i) Suppose $x_{M(t,s)} = x_t$. Then $M(t,s) = t \Rightarrow t > s$. And $t > s \Rightarrow 0 \leq s < t$. Hence, if $s = 0$ then $x \in G_1 \cap G_2^c$.
Conversely, suppose

$$x \in G_1 \cap G_2^c\ ,\ then\ x \in G_1\ and\ x \notin G_2,$$

which implies that $x_s = 0$. Therefore $x_{M(t,s)} = x_t$. Also, if $t > 0$, $x_{M(t,s)} = x_t$. Hence the proof.
(ii) Similar to that of (i). □

Definition 3.3 *A fuzzy point $x_{M(t,s)}$ of the bigroup $G = G_1 \cup G_2$, is said to belong to (resp. be quasi coincident with) a fuzzy subset $\mu = \mu_1 \cup \mu_2$ of G, written as $x_{M(t,s)} \in \mu$[resp. $x_{M(t,s)} q\mu$] if $\mu(x) \geq M(t,s)$(resp. $\mu(x) + M(t,s) > 1$). $x_{M(t,s)} \in \mu$ or $x_{M(t,s)}\ q\mu$ will be denoted by $x_M(t,s) \in vq\mu$.*

Theorem 3.4 *Let $G = G_1 \cup G_2$ be a bigroup. Let μ_1 and μ_2 be fuzzy subsets of G_1 and G_2 respectively. Suppose that x_t and x_s are fuzzy points of the groups G_1 and G_2 respectively such that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, then $x_{M(t,s)} \in vq\mu$ where $x_{M(t,s)}$ is a fuzzy point of the bigroup G, and $\mu : G \to [0,1]$ is such that $\mu = \mu_1 \cup \mu_2$.*

Proof Suppose that

$$x_t \in vq\mu_1\ and\ x_s \in vq\mu_2,$$

then we have that

$$\mu_1(x) \geq t\ or\ \mu_1(x) + t > 1,$$

and

$$\mu_2(x) \geq s\ or\ \mu_2(x) + s > 1.$$

$$\mu_1(x) \geq t\ and\ \mu_2(x) \geq s \Rightarrow Max[\mu_1(x), \mu_2(x)] \geq M(t,s).$$

This means that

$$(\mu_1 \cup \mu_2)(x) \geq M(t,s)\ since\ x \in G_1 \cap G_2$$

That is

$$\mu(x) \geq M(t,s) \qquad (1)$$

Similarly,

$$\mu_1(x) + t > 1 \; and \; \mu_2(x) + s > 1$$

imply that

$$\mu_1(x) + t \; + \; \mu_2(x) + s > 2$$

$$\Rightarrow 2Max[\mu_1(x), \mu_2(x)] + 2M[t, s] > 2$$

$$\Rightarrow Max[\mu_1(x), \mu_2(x)] + M[t, s] > 1$$

$$\Rightarrow (\mu_1 \cup \mu_2)(x) + M(t, s) > 1$$

$$\Rightarrow \mu(x) + M(t, s) > 1 \qquad (2)$$

Combining (1) and (2), it follows that:

$$\mu(x) \geq M(t, s) \;\; or \;\; \mu(x) + M(t, s) > 1$$

which shows that

$$x_{M(t,s)} \in vq\mu$$

hence the proof. □

Theorem 3.5 *Let $G = G_1 \cup G_2$ be a bigroup. $\mu = \mu_1 \cup \mu_2$ a fuzzy subset of G, where μ_1, μ_2 are fuzzy subsets of G_1 and G_2 respectively. Suppose that $x_{M(t,s)}$ is a fuzzy point of the bigroup G then $x_{M(t,s)} \in vq\mu$ does not imply that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, where x_t and x_s are fuzzy points of the groups G_1 and G_2, respectively.*

Proof Suppose that $x_{M(t,s)} \in vq\mu$, then

$$\mu(x) \geq M(t, s) \;\; or \;\; \mu(x) \; + \; M(t, s) \; > \; 1$$

By definition 2.9, this implies that

$$(\mu_1 \cup \mu_2)(x) \geq M(t, s) \; or \; (\mu_1 \cup \mu_2)(x) \; + \; M(t, s) \; > \; 1$$

$$\Rightarrow Max[\mu_1(x), \mu_2(x)] \geq M(t, s) \; or \; Max[\mu_1(x), \mu_2(x)] \; + \; M(t, s) \; > \; 1$$

Now, suppose that $t \; > \; s$, so that $M(t, s) = t$, we then have that

$$Max[\mu_1(x), \mu_2(x)] \geq t \; or \; (Max[\mu_1(x), \mu_2(x)] \; + \; t) \; > \; 1$$

which means that $x_t \in vqMax[\mu_1, \mu_2]$, and by extended implication, we have that
$x_s \in vqMax[\mu_1, \mu_2]$.
If we assume that $Max[\mu_1, \mu_2] \; = \; \mu_1$, then we have that

$$x_t \in vq\mu_1 \; and \; x_s \in vq\mu_1,$$

and since $0 \leq \; s < t < 1$, we now need to show that at least $x_s \in vq\mu_2$
since by assumption, $\mu_1 > \mu_2$. To this end, let the fuzzy subset μ_2 and the fuzzy singleton x_s

be defined in such a way that $\mu_2 < s < 0.5$, then it becomes a straight forward matter to see that $x_s \bar{\in} \bar{v} q\mu_2$. Even though, $x_{M(t,s)} \in vq\mu$ still holds. And the result follows accordingly. □

Corollary 3.6 *Let $G = G_1 \cup G_2$ be a bigroup. $\mu = \mu_1 \cup \mu_2$ a fuzzy subset of G, where μ_1, μ_2 are fuzzy subsets of G_1 and G_2 respectively. Suppose that $x_{M(t,s)}$ is a fuzzy point of the bigroup G then $x_{M(t,s)} \in vq\mu$ imply that $x_t \in vq\mu_1$ and $x_s \in vq\mu_2$, if and only if*

$$0.5 < min[t, s] \leq min[\mu_1(x), \mu_2(x)] < 1$$

where x_t and x_s are fuzzy points of the groups G_1 and G_2 respectively.

Definition 3.7 *A fuzzy bisubset μ of a bigroup G is said to be an ($\in v\ q$)- fuzzy sub bigroup of G if for every $x,\ y \in G$ and $t_1,\ t_2,\ s_1,\ s_2,\ t,\ s, \in [0, 1]$,*
(i) $x_{M(t_1,t_2)} \in \mu,\ y_{M(s_1,s_2)} \in \mu \Rightarrow (xy)_{M(t,s)} \in vq\mu$
(ii) $x_{M(t_1,t_2)} \in \mu \Rightarrow (x^{-1})_{M(t_1,t_2)} \in vq\mu$
where $t = M(t_1, t_2)$ and $s = M(s_1, s_2)$.

Theorem 3.8 *Let $\mu = \mu_1 \cup \mu_2 :\ G = G_1 \cup G_2 \to [0, 1]$ be a fuzzy subset of G. Suppose that μ_1 is an ($\in vq$)-fuzzy subgroup of G_1 and μ_2 is an ($\in vq$)-fuzzy subgroup of G_2, then μ is an ($\in vq$)-fuzzy subgroup of G.*

Proof Suppose that μ_1 is an ($\in vq$)-fuzzy subgroup of G_1 and μ_2 is an ($\in vq$)-fuzzy subgroup of G_2. Let $x,\ y, \in G_1,\ G_2$ and $t_1,\ t_2,\ s_1,\ s_2 \in [0, 1]$ for which

$$x_{t_1} \in \mu_1\ ,\ y_{s_1} \in \mu_1 \Rightarrow (xy)_{M(t_1,s_1)} \in vq\mu_1$$

and

$$x_{t_2} \in \mu_2\ ,\ y_{s_2} \in \mu_2 \Rightarrow (xy)_{M(t_2,s_2)} \in vq\mu_2.$$

This implies that

$$\mu_1(xy) \geq M(t_1, s_1) \ or\ \mu_1(xy) + M(t_1, s_1) > 1,$$

and

$$\mu_2(xy) \geq M(t_2, s_2) \ or\ \mu_2(xy) + M(t_2, s_2) > 1.$$

$$\Rightarrow \mu_1(xy) + \mu_2(xy) \geq M(t_1, s_1) + M(t_2, s_2)$$

$$or\ \mu_1(xy) + \mu_2(xy) + M(t_1, s_1) + M(t_2, s_2) > 2$$

$$\Rightarrow 2Max[\mu_1(xy), \mu_2(xy)] \geq 2M[t, s]$$

$$or\ 2Max[\mu_1(xy), \mu_2(xy)] + 2M[t, s] > 2$$

$$\Rightarrow Max[\mu_1(xy), \mu_2(xy)] \geq M[t, s]$$

$$or\ Max[\mu_1(xy), \mu_2(xy)] + M[t, s] > 1$$

$$\Rightarrow \mu_1 \cup \mu_2(xy) \geq M[t, s] \ or\ \mu_1 \cup \mu_2(xy) + M[t, s] > 1$$

$$\Rightarrow \mu(xy) \geq M[t,s] \ or \ \mu(xy) + M[t,s] \ > \ 1$$

$$\Rightarrow (xy)_{M(t,s)} \in vq\mu.$$

To conclude the proof, we see that

$$x_{t_1} \in \mu_1 \Rightarrow (x^{-1})_{t_1} \in vq\mu_1, \ and \ x_{t_2} \in \mu_2 \ \Rightarrow (x^{-1})_{t_2} \in vq\mu_2$$

And since it is a straightforward matter to see that

$$(x^{-1})_{t_1} \in vq\mu_1, \ and \ (x^{-1})_{t_2} \in vq\mu_2 \ \Rightarrow \ (x^{-1})_{M(t_1,t_2)} \in vq \ \mu,$$

then, the result follows accordingly. □

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Math.Combin.Book Ser. Vol.4 (2010), 08-12

Connectivity of Smarandachely Line Splitting Graphs

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Abstract: Let $G(V, E)$ be a graph. Let $U \subseteq V(G)$ and $X \subseteq E(G)$. For each vertex $u \in U$, a new vertex u' is taken and the resulting set of vertices is denoted by $V_1(G)$. The *Smarandachely splitting graph* $S^U(G)$ of a graph G is defined as the graph having vertex set $V(G) \bigcup V_1(G)$ with two vertices adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u' of V_1 and the other to a vertex w of G where w is in $N_G(u)$. Particularly, if $U = V(G)$, such a Smarandachely splitting graph $S^{V(G)}(G)$ is abbreviated to *Splitting graph* of G and denoted by $S(G)$. The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i. For each edge $e_i \in X$, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E_1(G)$. The *Smarandachely line splitting graph* $L_s^X(G)$ of a graph G is defined as the graph having vertex set $E(G) \bigcup E_1(G)$ and two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of E_1 and the other to an element e_j of $E(G)$ where e_j is in $N_G(e_i)$. Particularly, if $X = E(G)$, such a Smarandachely line splitting graph $L_S^{V(G)}(G)$ is abbreviated to *Line Splitting Graph* of G and denoted by $L_S(G)$. In this paper, we study the connectivity of line splitting graphs.

Key Words: Line graph, Smarandachely splitting graph, splitting graph, Smarandachely line splitting graph, line splitting graph.

AMS(2010): 05C40

§1. Introduction

By a graph, we mean a finite, undirected graph without loops or multiple edges. Definitions not given here may be found in [2]. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

A vertex-cut in a graph G is a set S of vertices of G such that $G \setminus S$ is disconnected. Similarly, an edge-cut in a graph G is a set X of edges of G such that $G \setminus X$ is disconnected.

The open neighborhood $N(u)$ of a vertex u in $V(G)$ is the set of vertices adjacent to u. $N(u) = \{v/uv \in E(G)\}$.

Let $U \subseteq V(G)$ and $X \subseteq E(G)$. For each vertex $u \in U$, a new vertex u' is taken and the resulting set of vertices is denoted by $V_1(G)$. The *Smarandachely splitting graph* $S^U(G)$ of a

[1]Received September 1, 2010. Accepted December 6, 2010.

graph G is defined as the graph having vertex set $V(G)\bigcup V_1(G)$ with two vertices adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u' of V_1 and the other to a vertex w of G where w is in $N_G(u)$. Particularly, if $U = V(G)$, such a Smarandachely splitting graph $S^{V(G)}(G)$ is abbreviated to *Splitting graph* of G and denoted by $S(G)$. The concept of Splitting Graph was introduced by Sampathkumar and Walikar in [4].

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i. $N(e_i) = \{e_j/e_i, e_j \text{ are adjacent in } E(G)\}$.

For each edge $e_i \in X$, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E_1(G)$. The *Smarandachely line splitting graph* $L_s^X(G)$ of a graph G is defined as the graph having vertex set $E(G)\bigcup E_1(G)$ and two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of E_1 and the other to an element e_j of $E(G)$ where e_j is in $N_G(e_i)$. Particularly, if $X = E(G)$, such a Smarandachely line splitting graph $L_s^{V(G)}(G)$ is abbreviated to *Line Splitting Graph* of G and denoted by $L_s(G)$. The concept of Line splitting graph was introduced by Kulli and Biradar in [3].

We first make the following observations.

Observation 1. The graph G is an induced subgraph of $S(G)$. The line graph $L(G)$ is an induced subgraph of $L_s(G)$.

Observation 2. If $G = L_s(H)$ for some graph H, then $G = S(L(H))$.

The following will be useful in the proof of our results.

Theorem A([1]) *If a graph G is m-edge connected, $m \geq 2$, then its line graph $L(G)$ is m-connected.*

Theorem B([2]) *A graph G is n-connected if and only if every pair of vertices are joined by at least n vertex disjoint paths.*

$2. Main Results

Theorem 1 *Let G be a (p, q) graph. Then $L_s(G)$ is connected if and only if G is a connected graph with $p \geq 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices. Let $V(L_s(G)) = V_1 \bigcup V_2$ where $< V_1 >= L(G)$ and V_2 is the set of all newly introduced vertices, such that $v_1 \to v_2$ is a bijective map from V_1 onto V_2 satisfying $N(v_2) = N(v_1)\bigcap V_1$ for all $v_1 \in V_1$. Let $a, b \in V(L_s(G))$. We consider the following cases.

Case 1. $a, b \in V_1$. Since G is a connected graph with $p \geq 3$, $L(G)$ is a nontrivial connected graph. Since $L(G)$ is an induced subgraph of $L_s(G)$, there exists an $a - b$ path in $L_s(G)$.

Case 2. $a \in V_1$ and $b \in V_2$. Let $v \in V_1$ be such that $N(b) = N(v)\bigcap V_1$. Choose $w \in N(b)$. Since a and $w \in V_1$, as in Case 1, a and w are joined by a path in $L_s(G)$. Hence a and b are connected by a path in $L_s(G)$.

Case 3. $a, b \in V_2$. As in Case 2, there exist w_1 and w_2 in V_1 such that $w_1 \in N(a)$ and

$w_2 \in N(b)$. Consequently, $w_1a, w_2b \in E(L_s(G))$. Also w_1 and w_2 are joined by a path in $L_s(G)$. Hence a and b are connected by a path in $L_s(G)$.

In all the cases, a and b are connected by a path in $L_s(G)$. Thus $L_s(G)$ is connected.

Conversely, if G is disconnected or $G = K_2$, then obviously $L_s(G)$ is disconnected. □

Theorem 2 *For any graph* G, $\kappa(L_s(G)) = min\{2\kappa(L(G)), \delta_e(G) - 2\}$.

Proof By Whitney's result, $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L_s(G))$. Also, $\kappa(L(G)) \leq \lambda(L(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G)$, $\kappa(Ls(G)) \geq \kappa(L(G))$. We have the following cases.

Case 1. If $\kappa(L(G)) = 0$, then obviously $\kappa(L_s(G)) = 0$.

Case 2. If $\kappa(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a cut-vertex e_i.

We consider the following subcases.

Subcase 2.1. $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\kappa(L_s(G)) = \delta(L(G)) = 1$.

Subcase 2.2. $L(G)$ is connected with a cut-vertex e_i. Let e_j be a pendant vertex of $L(G)$ which is adjacent to e_i. Then $e_j^{'}$ is a pendant vertex of $L_s(G)$ and e_i is also a cut-vertex of $L_s(G)$. Hence $\kappa(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 2$, then the removal of a cut-vertex e_i of $L(G)$ and its corresponding vertex $e_i^{'}$ from $L_s(G)$ results in a disconnected graph. Hence $\kappa(Ls(G)) = 2\kappa(L(G))$.

Now suppose $\kappa(L(G)) = n$. Then $L(G)$ has a minimum vertex-cut $\{e_l : 1 \leq l \leq n\}$ whose removal from $L(G)$ results in a disconnected graph. There are two types of vertex-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one vertex-cut contains exactly $2n$ vertices, e_l's and $e_l^{'}$'s of $L_s(G)$ whose removal increases the components of $L_s(G)$ and the other is $\delta(L(G))$-vertex-cut. Thus we have

$$\kappa(L_s(G)) = \begin{cases} 2n, \quad \text{if} \quad n \leq \frac{\delta(L(G))}{2} = \frac{\delta_e(G)-2}{2}; \\ \\ \delta(L(G)) = \delta_e(G) - 2, \quad \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \kappa(L_s(G)) &= min\{2\kappa(L(G)), \delta(L(G))\} \\ &= min\{2\kappa(L(G)), \delta_e(G) - 2\}. \end{aligned}$$ □

Theorem 3 *For any graph* G, $\lambda(L_s(G)) = min\{3\lambda(L(G)), \delta_e(G) - 2\}$.

Proof Since $\delta(L_s(G)) = \delta(L(G))$, by Whitney's result $\kappa(L_s(G)) \leq \lambda(L_s(G)) \leq \delta(L(G))$. Since $L(G)$ is an induced subgraph of $L_s(G), \lambda(L_s(G)) \geq \lambda(L(G))$.

We consider the following cases.

Case 1. If $\lambda(L(G)) = 0$, then obviously $\lambda(L_s(G)) = 0$.

Case 2. If $\lambda(L(G)) = 1$, then $L(G) = K_2$ or it is connected with a bridge $x = e_ie_j$.

We have the following subcases of this case.

Subcase 2.1. $L(G) = K_2$, then $L_s(G) = P_4$. Consequently, $\lambda(L_s(G)) = \delta(L(G)) = 1$.

Subcase 2.2. $L(G)$ is connected with a bridge e_ie_j. If e_i is a pendant vertex, then $L_s(G)$ is connected with the some pendant vertex $e_i^{'}$. There is only one edge incident with $e_i^{'}$ whose removal disconnects it. Thus $\lambda(L_s(G)) = \delta(L(G)) = 1$. If neither e_i nor e_j is a pendant vertex and $\delta(L(G)) = 2$, then $\delta(L_s(G)) = 2$ and let e_k be a vertex of $L_s(G)$ with $deg_{L_s(G)}e_k = 2$. In $L_s(G)$, there are only two edges incident with e_k and the removal of these disconnects $L_s(G)$. So $\lambda(L_s(G)) = \delta(L(G))$. If $\delta(L(G)) \geq 3$, then the removal of edges $e_ie_j, e_i^{'}e_j$ and $e_ie_j^{'}$ from $L_s(G)$ results in a disconnected graph. Hence $\lambda(L_s(G)) = 3\lambda(L(G))$.

Now suppose $\lambda(L(G)) = n$. Then $L(G)$ has a minimum edge-cut $\{e_l = u_lv_l : 1 \leq l \leq n\}$ whose removal from $L(G)$ results in a disconnected graph. As above, there are two types of edge-cuts in $L_s(G)$ depending on the structure of $L(G)$. Among these, one edge-cut contains exactly 3n edges $\{u_lv_l, u_l^{'}v_l, u_lv_l^{'}, 1 \leq l \leq n\}$ whose removal increases the components of $L_s(G)$ and the other is $\delta(L(G))-$edge-cut. Thus we have

$$\lambda(L_s(G)) = \begin{cases} 3n, & \text{if} \quad n \leq \frac{\delta(L(G))}{3} = \frac{\delta_e(G)-2}{3}; \\ \\ \delta(L(G)) = \delta_e(G) - 2, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \lambda(L_s(G)) &= min\{3\lambda(L(G)), \delta(L(G))\} \\ &= min\{3\lambda(L(G)), \delta_e(G) - 2\} \end{aligned}$$

□

Theorem 4 *If a graph G is n-edge connected, $n \geq 2$, then $L_s(G)$ is n-connected.*

Proof Let G be a n-edge connected graph, $n \geq 2$. Then by Theorem A, $L(G)$ is n-connected. We show that there exist n-disjoint paths between any two vertices of $L_s(G)$. Let x and y be two distinct vertices of $L_s(G)$. We consider the following cases.

Case 1. Let $x, y \in E(G)$. Then by Theorem B, x and y are joined by n-disjoint paths in $L(G)$. Since $L(G)$ is an induced subgraph of $L_s(G)$, there exist n-disjoint paths between x and y in $L_s(G)$.

Case 2. Let $x \in E(G)$ and $y \in E_1(G)$. Since $\lambda(G) \leq \delta(G) < 2\delta(G) \leq \delta_e(G)$, there are at least n edges adjacent to x. Let $x_i, i = 1, 2, ..., n$ be edges of G, adjacent to x. Then the vertices $x_i^{'}, i = 1, 2, ..., n$ are adjacent to the vertex x in $L_s(G)$, where $x_i^{'} \in E_1(G), i = 1, 2, ..., n$. It follows from Case 1, that there exist n-disjoint paths from x to x_i, $i = 1, 2, ..., n$ in $L_s(G)$. Since $y \in E_1(G)$, we have $N(y) = N(w) \bigcap E$, for some $w \in E(G)$. Since $|N(w)| \geq n$, let $y_1, y_2, ..., y_n \in E(G)$ such that $y_i \in N(w), i = 1, 2, ..., n$. So $y_i \in N(y), i = 1, 2, ..., n$. Also, since x and $y_i \in E(G)$, i=1,2,...,n, as in Case 1, there exist n-disjoint paths in $L_s(G)$ between x and $y_i, i = 1, 2, ..., n$. Hence x and y are joined by n-disjoint paths in $L_s(G)$.

Case 3. Let $x, y \in E_1(G)$. As in Case 2, $x_i \in N(x), i = 1, 2, ..., n$ and $y_i \in N(y), i = 1, 2, ..., n$ for some $x_i, y_i \in E(G), i = 1, 2, ..., n$. Consequently, x_ix and $y_iy \in E(L_s(G)), i = 1, 2, ..., n$.

Also by Case 1, every pair of x_i and y_i are joined by n-disjoint paths in $L_s(G)$. Hence x and y are joined by n-disjoint paths in $L_s(G)$.

Thus it follows from Theorem B that $L_s(G)$ is n-connected. □

However, the converse of the above Theorem is not true. For example, in Figure 1, $L_s(G_1)$ is 2-connected, whereas G_1 is edge connected.

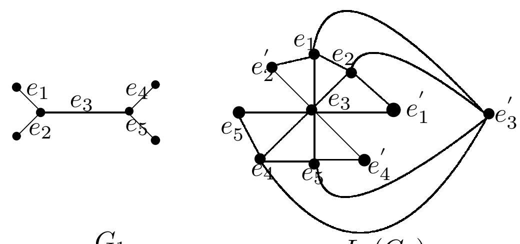

Figure 1

Corollary 5 *If a graph G is n-connected, $n \geq 2$, then $L_s(G)$ is n-connected.*

The converse of above corollary is not true. For instance, In Figure 2, $L_s(G_2)$ is 2-connected, but G_2 is connected.

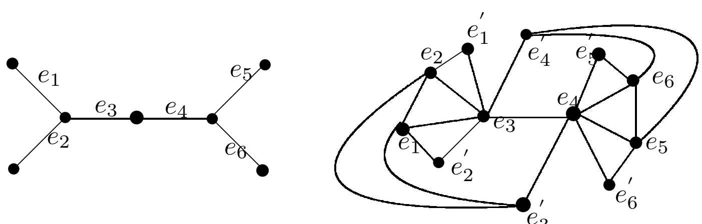

Figure 2

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Math.Combin.Book Ser. Vol.4 (2010), 13-23

Separation for Triple-Harmonic Differential Operator in Hilbert Space

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Abstract: We separate the differential operator A of the form $Au(x) = -\Delta^3 u(x) + V(x)u(x)$ for all $x \in R^n$, in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x)$, where $L(H_1)$ is the space of all bounded operators on an arbitrary Hilbert space H_1, and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator on R^n. Moreover, we study the existence and uniqueness of the solution of the equation $Au(x) = -\Delta^3 u(x) + V(x)u(x) = f(x)$, in the Hilbert space H, where $f(x) \in H$.

Key Words: Separation, Tricomi differential operator, Hilbert space.

AMS(2010): 35K, 46K

§1. Introduction

The concept of separation for differential operators was first introduced by Everitt and Giertz [6, 7]. They have obtained the separation results for the Sturm Liouville differential operator

$$Ay(x) = -y''(x) + V(x)y(x), \quad x \in R, \tag{1}$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $y(x) \in L_2(R)$ and $Ay(x) \in L_2(R)$ imply that both of $y''(x)$ and $V(x)y(x) \in L_2(R)$? More fundamental results of separation of differential operators were obtained by Everitt and Giertz [8,9]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [2], Otelbaev [16], Zettle [21] and Mohamed etal [10-15]. The separation for the differential operators with the matrix potential was first studied by Bergbaev [1]. Brown [3] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second order differential operators have been studied by many authors, see for examples [4,5,11,13,14,15,17,19].

Recently, Zayed [20] has studied the separation for the following biharmonic differential operator

$$Au(x) = \Delta\Delta u(x) + V(x)u(x), \quad x \in R^n, \tag{2}$$

[1]Received October 18, 2010. Accepted December 12, 2010.

in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$ where $\Delta\Delta u(x)$ is the biharmonic differential operator, while $\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$ is the Laplace operator in R^n.

The main objective of the present paper is to study the separation of the differential operator A of the form

$$Au(x) = -\Delta^3 u(x) + V(x)u(x). \tag{3}$$

We construct the coercive estimate for the differential operator (3). The existence and uniqueness Theorem for the solution of the differential equation

$$Au(x) = -\Delta^3 u(x) + V(x)u(x) = f(x) \tag{4}$$

in the Hilbert space $H = L_2(R^n, H_1)$ is also given, where $f(x) \in H$.

§2. Some Notations

In this section we introduce the definitions that will be used in the subsequent section. We consider the Hilbert space H_1 with the norm $\|.\|_1$and the inner product space $\langle u, v\rangle_1$ .We introduce the Hilbert space $H = L_2(R^n, H_1)$ of all functions $u(x), x \in R^n$ equipped with the norm

$$\|u\|^2 = \int_{R^n} \|u(x)\|_1^2 \, dx. \tag{5}$$

The symbol $\langle u, v\rangle$ where $u, v \in H$ denotes the scalar product in H which is defined by

$$\langle u, v\rangle = \int_{R^n} \langle u, v\rangle_1 \, dx. \tag{6}$$

Let $W_2^2(R^n, H_1)$ be the space of all functions $u(x)$, $x \in R$ which have generalized derivatives $D^\alpha u(x)$, $\alpha \leq 2$ such that $u(x)$ and $D^\alpha u(x)$ belong to H. We say that the function $u(x)$ for all $x \in R$ belongs to $W_{2,\ loc}^2(R^n, H_1)$ if for all functions $Q(x) \in C_0^\infty(R^n)$ the functions $Q(x)\, y(x) \in W_{2,\ loc}^2(R^n, H_1)$.

§3. Main Results

Definition 3.1 *The operator A of the form $Au(x) = -\Delta^3 u(x) + V(x)u(x)$, $x \in R^n$ is said to be separated in the Hilbert space H if the following statement holds:*

If $u(x) \in H \cap W_{2,\ loc}^2(R^n, H_1)$ and $Au(x) \in H$ imply that both $-\Delta^3 u(x)$ and $V(x)u(x) \in H$.

Theorem 3.1 *If the following inequalities are holds for all $x \in R^n$,*

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial^3 v}{\partial x_i^3} V^{-1} V u \right\| \leq \sigma_1 \|Vu\|, \tag{7}$$

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial v}{\partial x_i} \frac{\partial^2 u}{\partial x_i^2} \right\| \leq \sigma_2 \|Vu\|, \tag{8}$$

$$\left\| V_0^{-\frac{1}{2}} \frac{\partial u}{\partial x_i} \frac{\partial^2 v}{\partial x_i^2} \right\| \leq \sigma_3 \| V u \| , \tag{9}$$

where σ_1, σ_2 and σ_3 are positive constants satisfy $\sigma_1 + \sigma_2 + \sigma_3 < \frac{2}{n}$, $V_0 = ReV$, then the coercive estimate

$$\| V u \| + \left\| \Delta^3 u \right\| + \left\| \sum_{i=1}^{n} V_0^{\frac{1}{2}} \frac{\partial^3 u}{\partial x_i^3} \right\| \leq N \| V u \| , \tag{10}$$

is valid, where

$$\begin{aligned} N &= 1 + 2 \left[1 - \frac{n}{2\beta} (\sigma_1 + 2\sigma_2 + 3\sigma_3) \right]^{-1} \\ &\quad + \left[1 - \frac{n}{2\beta} (\sigma_1 + 2\sigma_2 + 3\sigma_3) \right]^{-\frac{1}{2}} \left[1 - \frac{n\beta}{2} (\sigma_1 + 2\sigma_2 + 3\sigma_3) \right]^{-\frac{1}{2}} , \end{aligned}$$

is a constant independent on $u(x)$ while β is given by $\frac{n}{2} (\sigma_1 + 2\sigma_2 + 3\sigma_3) < \beta < \frac{2}{n(\sigma_1 + 2\sigma_2 + 3\sigma_3)}$. Then the operator A given by (3) is separated in the Hilbert space H.

Proof From the definition of the inner product in the Hilbert space H, we can obtain $\left\langle \frac{\partial u}{\partial x_i}, v \right\rangle = - \left\langle u, \frac{\partial v}{\partial x_i} \right\rangle, i = 1, 2, 3, ..., n$ for all $u, v \in C_0^\infty (R^n)$.
Hence

$$\langle Au, Vu \rangle = \left\langle -\Delta^3 u + Vu, Vu \right\rangle = \left\langle -\Delta^3 u, Vu \right\rangle + \langle Vu, Vu \rangle$$

Setting $-\Delta^2 u = \Omega$, we have

$$\begin{aligned} \langle Au, Vu \rangle &= \langle \Delta\Omega, Vu \rangle + \langle Vu, Vu \rangle \\ &= \left\langle \sum_{i=1}^{n} \frac{\partial^2 \Omega}{\partial x_i^2}, Vu \right\rangle + \langle Vu, Vu \rangle \\ &= - \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, \frac{\partial (Vu)}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= - \sum_{i=1}^{n} \left\langle \frac{\partial \Omega}{\partial x_i}, V \frac{\partial u}{\partial x_i} + u \frac{\partial V}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, \frac{\partial}{\partial x_i} (V \frac{\partial u}{\partial x_i}) \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, \frac{\partial}{\partial x_i} (u \frac{\partial V}{\partial x_i}) \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, V \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial u}{\partial x_i} \frac{\partial V}{\partial x_i} \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, u \frac{\partial^2 V}{\partial x_i^2} + \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= \sum_{i=1}^{n} \left\langle \Omega, V \frac{\partial^2 u}{\partial x_i^2} \right\rangle + \sum_{i=1}^{n} \left\langle \Omega, u \frac{\partial^2 V}{\partial x_i^2} \right\rangle + 2 \sum_{i=1}^{n} \left\langle \Omega, \frac{\partial u}{\partial x_i} \frac{\partial V}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle . \end{aligned} \tag{11}$$

Setting $\Delta u = W$, we have $\Omega = -\Delta W$ and hence (11) reduces to

$$\begin{aligned}
\langle Au, Vu\rangle &= -\sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, V\frac{\partial^2 u}{\partial x_i^2}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, u\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\
&\quad -2\sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i}\right\rangle + \langle Vu, Vu\rangle \\
&= \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial xi}, \frac{\partial}{\partial x_i}(V\frac{\partial^2 u}{\partial x_i^2})\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_{1i}}, \frac{\partial}{\partial x_i}(u\frac{\partial^2 V}{\partial x_i^2})\right\rangle \\
&\quad +2\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial V}{\partial x_i})\right\rangle + \langle Vu, Vu\rangle \\
&= \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad +3\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial xi}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^i}\right\rangle + 3\sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + \langle Vu, Vu\rangle \\
&= -\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(V\frac{\partial^3 u}{\partial x_i^3})\right\rangle - \sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(u\frac{\partial^3 V}{\partial x_i^3})\right\rangle \\
&\quad -3\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2})\right\rangle - 3\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i})\right\rangle + \langle Vu, Vu\rangle \\
&= -\sum_{i=1}^{n}\left\langle W, \frac{\partial}{\partial x_i}V\frac{\partial^4 u}{\partial x_i^4}\right\rangle - \sum_{i=1}^{n}\left\langle W, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle - 4\sum_{i=1}^{n}\left\langle W, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad -4\sum_{i=1}^{n}\left\langle W, \frac{\partial V}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}\right\rangle - 6\sum_{i=1}^{n}\left\langle W, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu, Vu\rangle . \\
&= -\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, V\frac{\partial^4 u}{\partial x_i^4}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle - 4\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle \\
&\quad -4\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial V}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}\right\rangle - 6\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu, Vu\rangle \qquad (12)
\end{aligned}$$

Let us rewrite (12) as follows:

$$\langle Au, Vu\rangle = -I_1 - I_2 - 4I_3 - 4I_4 - 6I_5 + \langle Vu, Vu\rangle , \qquad (13)$$

where

$$I_1 = \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, V\frac{\partial^4 u}{\partial x_i^4}\right\rangle = -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle , \qquad (14)$$

$$\begin{aligned} I_2 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, u\frac{\partial^4 V}{\partial x_i^4}\right\rangle = \sum_{i=1}^{n}\left\langle u\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^4 V}{\partial x_i^4}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(u\frac{\partial^2 u}{\partial x_i^2}), \frac{\partial^3 V}{\partial x_i^3}\right\rangle = -\sum_{i=1}^{n}\left\langle u\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_i}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial x_i}), \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - 2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle - 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle, \end{aligned} \tag{15}$$

$$I_3 = \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^3 u}{\partial x_i^3}\frac{\partial V}{\partial x_i}\right\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle, \tag{16}$$

$$\begin{aligned} I_4 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial x_i}\frac{\partial^3 V}{\partial x_i^3}\right\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^3 V}{\partial x_i^3}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i^2}), \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial u}{\partial x_i}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 V}{\partial x_i^2}\right\rangle - \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + 2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle, \end{aligned} \tag{17}$$

$$\begin{aligned} I_5 &= \sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 V}{\partial x_i^2}\right\rangle = \sum_{i=1}^{n}\left\langle (\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial^2 V}{\partial x_i^2}\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial}{\partial x_i}(\frac{\partial^2 u}{\partial x_i^2})^2, \frac{\partial V}{\partial x_i}\right\rangle = -2\sum_{i=1}^{n}\left\langle \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial V}{\partial x_i}\right\rangle \\ &= -2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle. \end{aligned} \tag{18}$$

Substitute(14)-(17) into (13) , we get

$$\begin{aligned}\langle Au,Vu\rangle &= \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, V\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, u\frac{\partial^3 V}{\partial x_i^3}\right\rangle + \\ & 2\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle + 3\sum_{i=1}^{n}\left\langle \frac{\partial^3 u}{\partial x_i^3}, \frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu,Vu\rangle .\end{aligned} \tag{19}$$

Equating the real parts of the two sides of (19), we get

$$\begin{aligned}Re\,\langle Au,Vu\rangle &= \sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\rangle + \sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle \\ &+2\sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle \\ &+3\sum_{i=1}^{n} Re\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle + \langle Vu,Vu\rangle .\end{aligned} \tag{20}$$

Since for any complex number Z,we have

$$-|Z| \le ReZ \le |Z|,$$

then by the Cauchy- Schwarz inequality, we obtain

$$Re\,\langle Au,Vu\rangle \le |\langle Au,Vu\rangle| \le \|Au\|\,\|Vu\| . \tag{21}$$

With the help of (21) we have

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\| \\ &\ge -n\sigma_1\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| ,\end{aligned} \tag{22}$$

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\| \\ &\ge -n\sigma_2\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| ,\end{aligned} \tag{23}$$

$$\begin{aligned}Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle &\ge -\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\left\|\sum_{i=1}^{n}V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\| \\ &\ge -n\sigma_3\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|\|Vu\| .\end{aligned} \tag{24}$$

It is easy to see that for any $\beta > 0$ and $y_1, y_2 \in R^n$, the inequality

$$|y_1|\,|y_2| \le \frac{\beta\,|y_1|^2}{2} + \frac{|y_2|^2}{2\beta}. \tag{25}$$

is valid. Applying (25) to (22)-(24), we have

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^3 V}{\partial x_i^3}V^{-1}Vu\right\rangle \geq -\frac{n\beta\sigma_1}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_1}{2\beta}\left\|Vu\right\|^2, \tag{26}$$

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\frac{\partial V}{\partial x_i}\right\rangle \geq -\frac{n\beta\sigma_2}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_2}{2\beta}\left\|Vu\right\|^2, \tag{27}$$

$$Re\sum_{i=1}^{n}\left\langle V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}, V_0^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\frac{\partial^2 V}{\partial x_i^2}\right\rangle \geq -\frac{n\beta\sigma_3}{2}\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 - \frac{n\sigma_3}{2\beta}\left\|Vu\right\|^2, \tag{28}$$

From (20),(21) and (26) - (28), we conclude that

$$\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\|^2 + \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]\left\|Vu\right\|^2 \leq \left\|Vu\right\|\left\|Au\right\| \tag{29}$$

Choosing $\frac{n}{2}(\sigma_1+2\sigma_2+3\sigma_3) < \beta < \frac{2}{n}(\sigma_1+2\sigma_2+3\sigma_3)$, we obtain from (29) that

$$\left\|Vu\right\| \leq \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1}\left\|Au\right\|, \tag{30}$$

$$\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\| \leq \left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\left\|Au\right\|. \tag{31}$$

From (3) and (30) we get

$$\left\|\Delta^3 u\right\| \leq \left\|Vu\right\| + \left\|Au\right\| \leq \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1}\left\|Au\right\|. \tag{32}$$

Consequently, we deduce from (30)-(32) that

$$\left\|Vu\right\| + \left\|\Delta^3 u\right\| + \left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}\right\| \leq N\left\|Au\right\| \tag{33}$$

where

$$\begin{aligned} N &= 1+2\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-1} \\ &\quad + \left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2+3\sigma_3)\right]^{-\frac{1}{2}}, \end{aligned}$$

and this completes the proof. □

Theorem 3.2 *If the operator A given by (3) is separated in the Hilbert space H and if there are positive functions $\psi(x)$ and $t(x) \in C^1(R^n)$ satisfying*

$$\left\|t^{-1}(\frac{\partial t}{\partial x_i})V_0^{-\frac{1}{2}}\right\| \leq 2\sqrt{\sigma_1}, \tag{34}$$

$$\left\|\psi^{-1}(\frac{\partial\psi}{\partial x_i})V_0^{-\frac{1}{2}}\right\| \leq 2\sqrt{\sigma_2}, \tag{35}$$

$$\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}(\frac{\partial u}{\partial x_i})\right\| \leq 2\sqrt{\sigma_3}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|, \tag{36}$$

$$\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}(\frac{\partial\Omega}{\partial x_i})\right\| \leq \sqrt{\sigma_4}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\psi}{\partial x_i}\right\|, \tag{37}$$

where $0 < \sigma_1 + \sigma_2 + \sigma_3 < \frac{\beta}{2n}$, *and* β *is defined in Theorem 1. Then the differential operator* $Au = -\Delta^3 u + Vu = f,\ f \in H$ *, has a unique solution on the Hilbert space* H*.*

Proof First, we prove the differential equation

$$Au = -\Delta^3 u + Vu = 0, \tag{38}$$

as the only zero solution $u(x) = 0$ for all $x \in R^n$. To this end we assume that $t(x)$ and $\psi(x)$ are two positive functions belonging to $C^1(R^n)$.

On setting $\Omega = -\Delta^2 u$, we get

$$\begin{aligned}\langle Vu, t\psi u\rangle &= \langle -\Delta\Omega, t\psi u\rangle = -\left\langle \sum_{i=1}^{n}\frac{\partial^2\Omega}{\partial x_i^2}, t\psi u\right\rangle = \sum_{i=1}^{n}\left\langle\frac{\partial\Omega}{\partial x_i}, \frac{\partial}{\partial x_i}(t\psi u)\right\rangle \\ &= \sum_{i=1}^{n}\left\langle\frac{\partial\Omega}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle\frac{\partial\Omega}{\partial x_i}, tu\frac{\partial\psi}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle\frac{\partial\Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle,\end{aligned} \tag{39}$$

Equating the real parts of both sides of (39), we have

$$\begin{aligned}\langle V_0 u, t\psi u\rangle &= \left\langle t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u, t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\rangle \\ &= \sum_{i=1}^{n} Re\left\langle\frac{\partial\Omega}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle + \sum_{i=1}^{n} Re\left\langle\frac{\partial\Omega}{\partial x_i}, tu\frac{\partial\psi}{\partial x_i}\right\rangle \\ &\quad + \sum_{i=1}^{n} Re\left\langle\frac{\partial\Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle.\end{aligned} \tag{40}$$

By using Cauchy- Schwarz inequality, we obtain

$$\begin{aligned}Re\left\langle\frac{\partial\Omega}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle &= Re\left\langle t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}, t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\right\rangle \\ &\leq \left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial u}{\partial x_i}\right\|,\end{aligned} \tag{41}$$

$$\begin{aligned}Re\left\langle\frac{\partial\Omega}{\partial x_i}, tu\frac{\partial\psi}{\partial x_i}\right\rangle &= Re\left\langle t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}, t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\left[\psi^{-1}\frac{\partial\psi}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\rangle \\ &\leq \left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial\Omega}{\partial x_i}\right\|\left\|t^{\frac{1}{2}}\psi^{\frac{-1}{2}}\left[\psi^{-1}\frac{\partial\psi}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\|,\end{aligned} \tag{42}$$

$$
\begin{aligned}
Re\left\langle \frac{\partial \Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle &= Re\left\langle t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}, t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\left[t^{-1}\frac{\partial t}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\rangle \\
&\leq \left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}\right\|\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\left[t^{-1}\frac{\partial t}{\partial x_i}V_0^{-\frac{1}{2}}\right]V_0^{\frac{1}{2}}u\right\|, \qquad (43)
\end{aligned}
$$

From (25) in (41)- (43), we have

$$
Re\left\langle \frac{\partial \Omega}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle \leq \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_1\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \qquad (44)
$$

$$
Re\left\langle \frac{\partial \Omega}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle \leq \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_2\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \qquad (45)
$$

$$
Re\left\langle \frac{\partial \Omega}{\partial x_i}, u\psi\frac{\partial t}{\partial x_i}\right\rangle \leq \frac{\beta}{2}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}\right\|^2 + \frac{2}{\beta}\sigma_3\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2, \qquad (46)
$$

From (39) and (44)-(46) we have the following inequality:

$$
\begin{aligned}
\left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 &\leq \frac{3\beta}{2}\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \Omega}{\partial x_i}\right\|^2 \\
&\leq \frac{3\beta}{2}\sigma_4\sum_{i=1}^{n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}\frac{\partial \psi}{\partial x_i}\right\|^2 \qquad (47)
\end{aligned}
$$

Choosing $\psi(x) = 1$, for all $x \in R^n$, then if $0 < \frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3) < 1$, we have

$$
0 < \left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|^2 \leq 0, \qquad (48)
$$

and consequently, we have

$$
0 < \left[1-\frac{2n}{\beta}(\sigma_1+\sigma_2+\sigma_3)\right]\int_{R^n}\left\|t^{\frac{1}{2}}\psi^{-\frac{1}{2}}V_0^{\frac{1}{2}}u\right\|_1^2 dx \leq 0. \qquad (49)
$$

The inequality (49) holds only for $u(x) = 0$. This proves that $u(x) = 0$ is the only solution of Eq.(38).

Second, We know that the linear manifold $M = \{f : Au = f, u \in C_0^\infty(R^n)\}$ is dense everywhere in H, so there exist a sequence of functions $\{u_r\} \in C_0^\infty(R^n)$, such that for all $f \in H, \|Au_r - f\| \longrightarrow 0$, as $r \longrightarrow \infty$.

By applying the coercive estimate (33), we find that

$$
\|V(u_p-u_r)\| + \left\|\Delta^3(u_p-u_r)\right\| + \left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3(u_p-u_r)}{\partial x_i^3}\right\| \leq N\left\|A(u_p-u_r)\right\|, \qquad (50)
$$

that $\|V(u_r-w_0)\|, \left\|\Delta^3(u_r-w_1)\right\|$ and $\left\|\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3(u_r-w_2)}{\partial x_i^3}\right\|$ are convergent to zero, as $r \longrightarrow \infty$. This implies that $u_r \longrightarrow V^{-1}w_0 = u$, $\Delta^3 u_r \longrightarrow \Delta^3 u$ and $\sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u_r}{\partial x_i^3} \longrightarrow \sum_{i=1}^{n}V_0^{\frac{1}{2}}\frac{\partial^3 u}{\partial x_i^3}$ as $r \longrightarrow \infty$. Hence for any $f \in H$ there exist $u \in H \cap W_{2,loc}^2(R^n, H_1)$, such that $Au = f$.

Suppose that $\overline{u}$ is another solution of the equation $Au = f$, then $A(u - \overline{u}) = 0$. But $Au = 0$ has only zero solution, then $u = \overline{u}$ and the uniqueness is proved. Hence the proof of theorem 3.2 is completed. □

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Math.Combin.Book Ser. Vol.4 (2010), 24-31

Classification of Differentiable Graph

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Abstract: We will classify the differentiable graph representing the solution of differential equation. Present new types of graphs. Theorems govern these types are introduced. Finally the effect of step size h on the differentiable graph is discussed.

Key Words: Differentiable, graph, numerical methods.

AMS(2010): 08A10, 05C15

§1. Definitions and Background

Definition 1([2]) *A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints*

Definition 2([1,2,4,7,8]) *A loop is an edge whose endpoints are equal. Multiple edges are edges have the same pair of endpoints.*

Definition 3([2,6]) *A simple graph is a graph having no loops or multiple edges . We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with end points u and v.*

Definition 4([2]) *A directed graph or digraph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices. the first vertex of the ordered pair is the tail of the edge, and the second is the head; together, they are the endpoints. We say that an edge is an edge from its tail to its head.*

Definition 5([2]) *A digraph is simple if each ordered pair is the head and tail of at most one edge. In a simple digraph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v. We write $u \to v$ for "there is an edge from u to v".*

Definition 6([7,8]) *A null graph is a graph containing no edges.*

Definition 7([2]) *The order of a graph G, written $n(G)$, is the number of vertices in G. An n-vertex graph is a graph of order n. The size of a graph G, written $e(G)$, is the number of*

[1]Received June 2, 2010. Accepted December 10, 2010.

edges in G *for* $n \in \mathbb{N}$.

Definition 8 *Let* $f\ (x, y)$ *be areal valued function of two variable defined for* $a \leq x \leq b$ *and all real* y, *then*

$$\begin{array}{lll} y' = f\ (x, y), & x \in S = [0, T] \subseteq R & (1) \\ y(x_0) = y_0 & & (2) \end{array} \qquad (1.1)$$

is called initial value problem (I.V.P.), where (1) is called ordinary differential equation (O.D.E) of the first order and equation (2) is called the initial value.

Definition 9([3,6]) *For the problem* (1.1) *where the function* $f\ (x, y)$ *is continuous on the region* $(0 \leq x \leq T, |y| \leq R)$ *and differentiable with respect to* x *such that* $\left|\frac{df}{dy}\right| \leq L, L = const.$*Divide the segment* $[0, T]$ *into* n *equal parts by the points* $x_i = ih, h = \frac{T}{n}$ *is called a step size,* $(i = \overline{0, n})$ *such that* $x_0 = 0 < x_1 < ... < x_{n-1} < x_n = T$ *the approximate numerical solutions for this problem at the mesh points* $x = x_i$ *will be denoted by* y_j.

Definition 10([3]) *Numerical answers to problems generally contain errors. Truncation error occurs as a result of truncating an infinite process to get a finite process.*

Definition 11 *For Riemannian manifolds* M *and* N *(not necessarily of the same dimension), a map* $f : M \to N$ *is said to be a topological folding of* M *into* N *if, for each piecewise geodesic path* $\gamma : I \to M(I = [0, 1] \subseteq R)$, *the induced path* $f \circ \gamma : I \to N$ *is piecewise geodesic. If, in addition,* $f : M \to N$ *preserves lengths of paths, we call* f *an isometric folding of* M *into* N. *Thus an isometric folding is necessarily a topological folding* [9]. *Some applications are introduced in* [5].

§2. Main Results

We will introduce several types of approximate differentiable graph which represent the solution of initial value problems **I.V.P**.

$$\begin{array}{l} y' = f(x, y), \\ y(x_0) = y_0. \end{array} \qquad (2.1)$$

According to the used methods for solving these problems.

Definition 12 *We can study the solution of ordinary differential equation* $y' = f\ (x, y)$ *using differentiable graph which is a smooth graph with vertex set* $\{(x, y(x)) : x, y \in R\}$ *and edge set* $d((x_i, y(x_i)), (x_{i+1}, y(x_{i+1})))$ *where* d *represent the distance function. A differentiable graph is a smooth graph represent the solution of ordinary differential equation* $y' = f\ (x, y), x \in S$ *whose vertices are* $(x, y(x)), \forall x \in S$ *and its edges are the distance between any two consequent points. In this graph the number of vertices is* ∞,*the number of edges is so.*

Since the finite difference methods which solve $(I.V.P.)$ divided into the following:

(i) general multi-step methods (implicit-explicit).

(ii) general single-step methods (implicit-explicit).

So we have the following new types of differentiable graph:

Type [1]: Single-Compound digraph H_{N_1}

Definition 13 *A numerical digraph G_N is a simple differentiable digraph consists of numerical vertices V_N^j which represent the numerical solutions y_j of (I.V.P.), and ordered numerical edge set $E_N = \{e_N^1, e_N^2, ..., e_N^n\}$ where $e_N^{j+1} = |(x_{j+1}, y_{j+1}) - (x_j, y_j)| = \left|v_N^{j+1} - v_N^j\right|, v_N^j$ is the tail of the edge, and $v_N^{j+1}$is the head.*

Definition 14 *A compound graph (digraph) H is a graph (digraph) whose vertex set consists of a set of graphs (digraphs) i.e. $V(G) = \{G_1, G_2, ...\}$ and an edge set of unordered (ordered) pairs of this graphs i.e. $E(G) = \{(G_1, G_2), (G_2, G_3), \cdots\}$.*

Corollary 1 *The compound digraph H of a numerical digraph is numerical digraph H_N.*

Definition 15 *A single-compound digraph H_{N_1} is a compound digraph H_N has one null graph is the tail of digraph.*

Theorem 1 *The single-step methods (implicit) due to a single-compound digraph H_{N_1}.*

Proof The basis of many simple numerical technique for solving the differential equation

$$y' = f(x, y), y(x_0) = y_0, a \leq x \leq b \tag{2.2}$$

is to find some means of expressing the solution at $x + h$ i.e., $y(x + h)$ in terms of $y(x)$.where $(x, y(x))$ represent a vertex in the differentiable graph, $(x + h, y(x + h))$ is the next vertex, the initial value (x_0, y_0) is called the source of graph. An approximate solution can be generated at the discrete points $x_0 + h, x_0 + 2h, ...$representing the vertices of the induced differentiable graph.

All these methods where y_{n+1} is given in terms of y_n alone, $n = 0, 1, 2,...$, are called single step methods. The general linear single step method is given by

$y_{n+1} + \alpha_1 y_n = h[\beta_0 f\ (x_{n+1}, y_{n+1}) + \beta_1 f\ (x_n, y_n)]$ where $\alpha_1, \beta_0, \beta_1$ are constants. If $\beta_0 = 0$ then the method gives y_{n+1} explicitly otherwise it is given implicitly. The trapezium method $y_{n+1} = y_n + \frac{h}{2}[f\ (x_{n+1}, y_{n+1}) + f\ (x_n, y_n)]$ is implicit. In general this equation would be solved by using the iteration method i.e.,

$\{y_{n+1}\}^{r+1} = y_n + \frac{h}{2}[f\ (x_{n+1}, y_{n+1}) + f\ (x_n, \{y_n\}^r)]$, $r = 0, 1, 2, ...$, where $\{y_{n+1}\}^0$ can be obtained from a single -step method and represents a source of numerical digraph G_{N+1}in the vertex V_{n+1} of compound graph H_N. Finally we get A single-compound digraph H_{N_1}. As shown in Figure 1. □

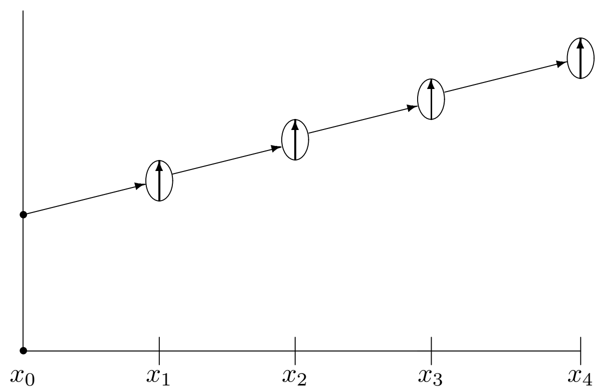

Figure 1. Single Compound Graph H_{N_1}

Definition 16 *A single-compound digraph H_{N_1} is a Compound numerical digraph has a unique null graph which is the source of graph.*

Type [2]: A simple numerical digraph G_N

Definition 17 *A numerical digraph G_N is a simple differentiable digraph consists of numerical vertices V_N^j which represent the numerical solutions y_j of (I.V.P.) and ordered numericaledge set $E_N = \{e_N^1, e_N^2, ..., e_N^n\}$ where $e_N^{j+1} = |(x_{j+1}, y_{j+1}) - (x_j, y_j)| = \left|v_N^{j+1} - v_N^j\right|, v_N^j$ is the tail of the edge, and $v_N^{j+1}$is the head.*

Theorem 2 *The explicit single-step method get a simple numerical digraph G_N.*

Proof The general single step given by

$$y_{n+1} = y_n + h\phi(x_n, y_n, h), x_n = x_0 + nh, y(x_0) = y_0.$$

For example, Euler's method has $\phi(x, y, h) = f(x, y)$,then

$y_{n+1} = y_n + hf(x_n, y_n)$, and for differential equation (2.1) give the following differentiable digraph (Figure 2)

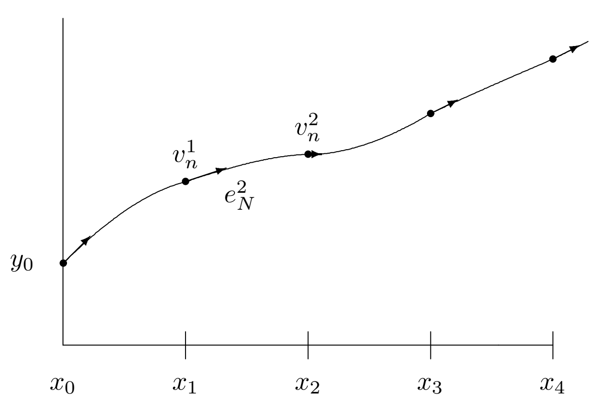

Figure 2: Simple numerical graph

where (x_n, y_n) represent the set of vertices $\left\{v_N^j\right\}, j = 0, 1, ...,$and $|(x_{j+1}, y_{j+1}) - (x_j, y_j)|$ represent the set of edges $\left\{e_N^{j+1}\right\}$.The initial value y_0 represent the source of simple numerical digraph G_N. □

Type [3]: Multi-Compound Digraph H_{N_m}

Definition 18 *A multi-compound digraph H_{N_m} is a compound digraph H_N has m null graphs are the tail of digraph.*

Theorem 3 *The implicit multi-step method give a multi-compound digraph H_{N_m}.*

Proof The general multi-step method is defined to be

$$y_{n+1}+\alpha_1 y_n+...+\alpha_m y_{n-m+1} = h[\beta_0 f_{n+1}+\beta_1 f_n+...+\beta_m f_{n-m+1}], \tag{2.3}$$

where f_p is used to denote $f(x_p, y_p), n = m-1, m-2, ...$. To apply this general method we need m steps which represent m null graphs $G_{N_0}, G_{N_1}, ..., G_{N_{m-1}}$ in a multi-compound digraph H_{N_m} as indicate in the following example. If $\beta_0 = 0$ then the method (2.3) gives y_{n+1} explicitly otherwise it is given implicitly, when $m = 1$ equation (2.3) reduce to the single step method.□

Example 1 Find the differentiable graph of $y' = y^2, y(0) = 1$ using a 3-step method.

Solution 1. by using

$$y_{n+1}-y_n = h[9f_{n+1}+19f_n-5f_{n-1}+f_{n-2}]/24, h = 0.1, \tag{2.4}$$

then $n = 2, 3, ..., \Rightarrow \{y_3\}^{r+1} - y_2 = 0.1[9\{f_3\}^r + 19f_2 - 5f_1 + f_0]/24, r = 0, 1, 2, ...,$ so the iterative vertex $(x_3, \{y_3\}^{r+1})$ depend on the vertex $(x_3, \{y_3\}^0)$ which can be determined from an explicit 3-srep method say

$$y_{n+1} - y_n = h[23f_n - 16f_{n-1} + 5f_{n-2}]l12, \tag{2.5}$$

at $n = 2 \Rightarrow y_3 - y_2 = h[23f(x_2, y_2) - 16f(x_1, y_1) + 5f(x_0, y_0)]l12$, where $V_0 = (x_0, y_0), V_1 = (x_1, y_1), V_2 = (x_2, y_2)$ represent three null graphs $G_{N_0}, G_{N_1}, G_{N_2}$ in the induced compound digraph by predictor method (2.5) we get the vertex $v_{N_3} = (x_3, \{y_3\}^0)$which is the tail of the digraph G_{N_3} in the compound digraph H_{N_3}then correct $\{y_3\}^0$using equation (2.4) until we get the fixed vertex $v_{N_3}^f$. This gives a numerical digraph $G_{N_3} = V_3$ and similarly we get the other vertices (simple digraphs) $V_4 = G_{N_4}, ..., V_l = G_{N_l}, l$ is a + ve integer. Finally we get bounded compound digraph H_{N_3} as shown in Figure 3.

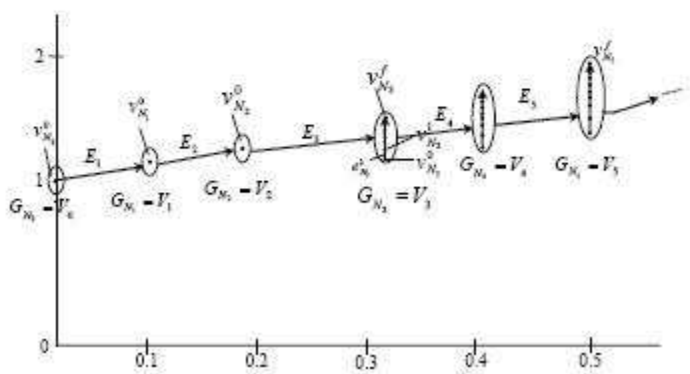

Figure 3: Multi-compound digraph

Definition 19 *A fixed vertex V_N^f is a numerical vertex which all next vertices coincided on it.*

Corollary 2 *The multi-compound digraph H_{N_m} must have null graphs.*

Type [4]: Nonhomogeneous Numerical Digraph G_{N_m}

Definition 20 *A nonhomogeneous graph G is a graph whose vertices divided into multi-groups such that each one has a specific character.*

Definition 21 *A nonhomogeneous numerical digraph G_{N_m} is a numerical digraph whose vertices divided into multi-groups such that each one has a specific character.*

Theorem 4 *The explicit multi -step method give nonhomogeneous numerical digraph G_{N_m}.*

Proof The general explicit multi-step method

$y_{n+1} + \alpha_1 y_n + ... + \alpha_m y_{n-m+1} = h[\beta_1 f_n + \beta_2 f_{n-1} + ... + \beta_m f_{n-m+1}]$, i.e., to determine the vertex (x_{n+1}, y_{n+1}) we need know m vertices begin from (x_0, y_0) up to (x_n, y_n).

for example: The difference method

$y_{n+1} - y_n = h[23f_n - 16f_{n-1} + 5f_{n-2}]/12, n = 2, 3, ...$, is 3-step method, the group of vertices $(x_3, y_3), (x_4, y_4), ...$, are given by this multi-step method whenever the group $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are gotten from single-step method .See Figure 3. □

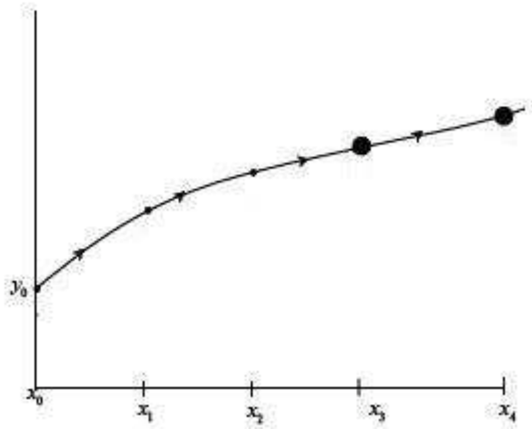

Figure 4: Nonhomogenous Numerical Digraph G_{N_m}

There is an important role to the step size h in the all types of numerical digraphs.

Definition 22 *The initial tight graph (digraph) T is a package of graphs (digraph) which have one source.*

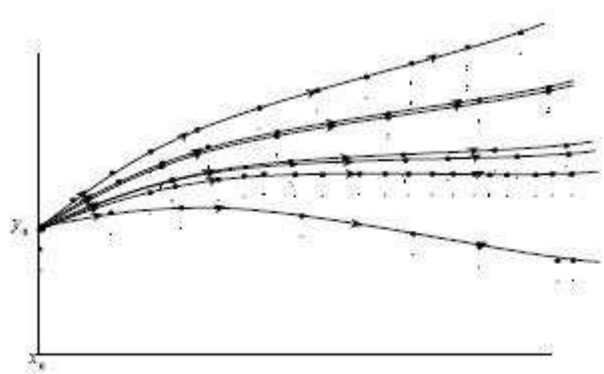

Figure 5: Initial Tight Digraph T

Theorem 5 *As the order of numerical digraph in bounded interval* $\to \infty$ *the consistent digraph is obtained.*

Proof Since the local error of the approximate solution of (I.V.P.)(1.1) depends on the step size h s.t. $\sup|E(h,x)| \leq Mh^k, where\ M, k$ is a positive integers [4], for all sufficiently small h, the order of bounded numerical digraph$\to \infty$, and then the difference method is said to be consistent of order k. □

Theorem 6 *The limit of foldings F_j of initial tight graph give a convergent numerical graph.*

Proof Let $F_i : T \to T$ be a folding map of an initial tight graph T s.t., $F_i(G_N^j) = G_N^m$, where order of $(G_N^j) \leq$ order of (G_N^m), then $\lim_{i\to\infty} F_i$ = The highest order numerical digraph, which is required. As shown in Figure 6. □

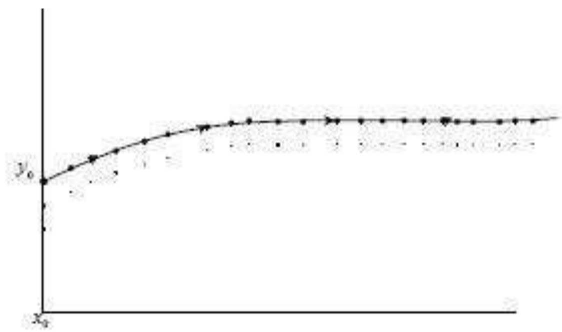

Figure 6: Limit of Foldings F_j

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Math.Combin.Book Ser. Vol.4 (2010), 32-37

On Equitable Coloring of Helm Graph and Gear Graph

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Abstract: The notion of equitable coloring was introduced by Meyer in 1973. In this paper we obtain interesting results regarding the equitable chromatic number $\chi_=$ for the Helm Graph H_n, line graph of Helm graph $L(H_n)$, middle graph of Helm graph $M(H_n)$, total graph of Helm graph $T(H_n)$, Gear graph G_n, line graph of Gear graph $L(G_n)$, middle graph of Gear graph $M(G_n)$, total graph of Gear graph $T(G_n)$.

Key Words: Smarandachely equitable k-colorable graph, Equitable coloring, Helm graph, Gear graph, line graph, middle graph and total graph.

AMS(2010): 05C15, 05C75

§1. Introduction

Let $G = (V, E)$ be a graph. If the vertices of G can be partitioned into m classes $V_1, V_2, ..., V_m$ such that each V_i, $(1 \leq i \leq m)$ are independent and $||V_i| - |V_j|| \leq 1$ holds for every pair $(1 \leq i, j \leq k)$, where $k \leq m$, then the graph G is said to be *Smarandachely equitable k-colorable*. Particularly, if $k = m$, we abbreviated it to *equitably k-colorable*. The smallest integer k for which G is Smarandachely equitable k-colorable or equitable k-colorable is known as the *Smarandachely equitable chromatic number* or *equitable chromatic number* [1,3,7-10] of G and denoted by $\chi_=(G)$, respectively.

This model of graph coloring has many applications. Every time when we have to divide a system with binary conflicting relations into equal or almost equal conflict-free subsystems we can model such situation by means of equitable graph coloring. This subject is widely discussed in literature [3,8-10]. In general, the problem of optimal equitable coloring, in the sense of the number color used, is NP-hard. So we have to look for simplified structure of graphs allowing polynomial-time algorithms. This paper gives such solution for Helm graph and Gear graph families: Helm graph, its line, middle and total graphs; Gear graph, its line, middle and total graphs.

[1]Received August 23, 2010. Accepted December 12, 2010.

§2. Preliminaries

The Helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendent edge at each node of the cycle.

The Gear graph G_n, also known as a bipartite wheel graph, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The line graph [2,5] of G, denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

The middle graph [4] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [2,4,5] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G. Additional graph theory terminology used in this paper can be found in [2,5,9].

§3. Equitable Coloring on Helm Graph

Theorem 3.1 *If $n \geq 4$ the equitable chromatic number of Helm graph H_n,*

$$\chi_{=}(H_n) = \begin{cases} 3 & if\ n\ is\ even, \\ 4 & if\ n\ is\ odd. \end{cases}$$

Proof Let H_n be the Helm graph obtained by attaching a pendant edge at each vertex of the cycle. Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where v_i's are the vertices of cycles taken in cyclic order and u_i's are pendant vertices such that each $v_i u_i$ is a pendant edge and v is a hub of the cycle.

Case i: If n is even.

Case i-a: If $n = 6k - 2$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 2k + 1 \leq i \leq 6k - 2\}$; $V_2 = \{v_{2i-1} : 1 \leq i \leq 3k - 1\} \cup \{u_{2i} : 1 \leq i \leq k\}$; $V_3 = \{v_{2i} : 1 \leq i \leq 3k - 1\} \cup \{u_{2i-1} : 1 \leq i \leq k\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. Also $|V_1| = |V_2| = |V_3| = 4k - 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i, j).

Case i-b: If $n = 6k$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 2k + 2 \leq i \leq 6k\}$; $V_2 = \{v_{2i-1} : 1 \leq i \leq 3k\} \cup \{u_{2i} : 1 \leq i \leq k\}$; $V_3 = \{v_{2i} : 1 \leq i \leq 3k\} \cup \{u_{2i-1} : 1 \leq i \leq k + 1\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. Also $|V_1| = |V_2| = 4k$ and $|V_3| = 4k + 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j).

Case i-c: If $n = 6k+2$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 2k+3 \le i \le 6k+2\}$; $V_2 = \{v_{2i-1} : 1 \le i \le 3k+1\} \cup \{u_{2i} : 1 \le i \le k+1\}$; $V_3 = \{v_{2i} : 1 \le i \le 3k+1\} \cup \{u_{2i-1} : 1 \le i \le k+1\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. $|V_1| = 4k+1$ and $|V_2| = |V_3| = 4k+2$, it holds the inequality $||V_i| - |V_j|| \le 1$ for every pair (i,j).

In all the three subcases of Case i, $\chi_=(H_n) \le 3$. Since $\chi(H_n) \ge 3$, $\chi_=(H_n) \ge \chi(H_n) \ge 3$, $\chi_=(H_n) \ge 3$. Therefore $\chi_=(H_n) = 3$.

Case ii: If n is odd.

Case ii-a: If $n = 6k-3$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 3k+1 \le i \le 5k-1\}$; $V_2 = \{v_{3i-2} : 1 \le i \le 2k-1\} \cup \{u_{3i} : 1 \le i \le k\}$; $V_3 = \{v_{3i-1} : 1 \le i \le 2k-1\} \cup \{u_{3i-2} : 1 \le i \le k\}$; $V_4 = \{v_{3i} : 1 \le i \le 2k-1\} \cup \{u_{3i-1} : 1 \le i \le k\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. Also $|V_1| = 3k-2$ and $|V_2| = |V_3| = |V_4| = 3k-1$, it holds the inequality $||V_i| - |V_j|| \le 1$ for every pair (i,j).

Case ii-b: If $n = 6k-1$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 3k+2 \le i \le 6k-1\}$; $V_2 = \{v_{3i-2} : 1 \le i \le 2k\} \cup \{u_{3i-1} : 1 \le i \le 2k-1\}$; $V_3 = \{v_{3i-1} : 1 \le i \le 2k\} \cup \{u_{3i} : 1 \le i \le 2k-1\}$; $V_4 = \{v_{3i} : 1 \le i \le 2k-1\} \cup \{u_{3i-2} : 1 \le i \le 2k\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. Also $|V_1| = 4k-2$ and $|V_2| = |V_3| = |V_4| = 4k-1$, it holds the inequality $||V_i| - |V_j|| \le 1$ for every pair (i,j).

Case ii-c: If $n = 6k+1$ for some positive integer k, then set the partition of V as below.

$V_1 = \{v\} \cup \{u_i : 3k+2 \le i \le 6k+1\}$;$V_2 = \{v_{3i-2} : 1 \le i \le 2k\} \cup \{u_{3i-1} : 1 \le i \le 2k-1\}$; $V_3 = \{v_{3i-1} : 1 \le i \le 2k\} \cup \{v_n\} \cup \{u_{3i} : 1 \le i \le 2k-1\}$; $V_4 = \{v_{3i} : 1 \le i \le 2k\} \cup \{u_1\} \cup \{u_{3i+1} : 1 \le i \le 2k-1\}$. Clearly V_1, V_2, V_3 are independent sets of $V(H_n)$. $|V_1| = |V_3| = |V_4| = 4k$ and $|V_2| = 4k-1$, it holds the inequality $||V_i| - |V_j|| \le 1$ for every pair (i,j).

In all the three subcases of cases (ii), $\chi_=(H_n) \le 4$. Since $\chi(H_n) \ge 4$, $\chi_=(H_n) \ge \chi(H_n) \ge 4$, $\chi_=(H_n) \ge 4$. Therefore $\chi_=(H_n) = 4$. □

§4. Equitable Coloring on Line graph, Middle Graph and Total Graph of Helm Graph

Theorem 4.1 *If $n \ge 4$ the equitable chromatic number on line graph of Helm graph $L(H_n)$, $\chi_=(L(H_n)) = n$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \le i \le n\} \cup \{e'_i : 1 \le i \le n-1\} \cup \{e'_n\} \cup \{s_i : 1 \le i \le n\}$ where e_i is the edge vv_i $(1 \le i \le n)$, e'_i is the edge v_iv_{i+1} $(1 \le i \le n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \le i \le n)$. By the definition of line graph $V(L(H_n)) = E(H_n) = \{e_i : 1 \le i \le n\} \cup \{e'_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$.

Now, we partition the vertex set of $V(L(H_n))$ as below.

$V_1 = \{e_1, e'_2, s_n\}$; $V_i = \{e_i, e'_{i+1}, s_{i-1} : 2 \leq i \leq n-1\}$; $V_n = \{e_n, e'_1, s_{n-1}\}$. Clearly V_1, V_i, V_n $(2 \leq i \leq n-1)$ are independent sets of $L(H_n)$. Also $|V_1| = |V_i| = |V_n| = 3$ $(2 \leq i \leq n-1)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j). $\chi_=(L(H_n)) \leq n$. Since $e_i (1 \leq i \leq n)$ forms a clique of order n, $\chi(L(H_n)) \geq n$, $\chi_=(L(H_n)) \geq \chi(L(H_n)) \geq n$, $\chi_=(L(H_n)) \geq n$. Therefore $\chi_=(L(H_n)) = n$. □

Theorem 4.2 *If $n \geq 5$ the equitable chromatic number on middle graph of Helm graph $M(H_n)$, $\chi_=(M(H_n)) = n+1$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of middle graph $V(M(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Now, we partition the vertex set of $V(M(H_n))$ as below.

$V_1 = \{e_1, e'_2, u_1, s_n\}$; $V_i = \{v_{i-1}, u_i, e_i, e'_{i+1} : 2 \leq i \leq n-1\} \cup \{s_{i-2} : 3 \leq i \leq n+1\}$; $V_n = \{v_{n-1}, s_{n-2}, e_n, e'_1, \}$; $V_{n+1} = \{v, v_n, s_{n-1}, u_n\}$. Clearly $V_1, V_2, \cdots V_n, V_{n+1}$ are independent sets of $M(H_n)$. Also $|V_1| = |V_2| = |V_n| = |V_{n+1}| = 4$ and $|V_i| = 5$ $(3 \leq i \leq n-1)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j). $\chi_=(M(H_n)) \leq n+1$. Since $ve_i (1 \leq i \leq n)$ forms a clique of order $n+1$, $\chi(M(H_n)) \geq n+1$, $\chi_=(M(H_n)) \geq \chi(M(H_n)) \geq n+1$, $\chi_=(M(H_n)) \geq n+1$. Therefore $\chi_=(M(H_n)) = n+1$. □

Theorem 4.3 *If $n \geq 5$ the equitable chromatic number on total graph of Helm graph $T(H_n)$, $\chi_=(T(H_n)) = n+1$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of total graph $V(T(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Now, we partition the vertex set of $V(T(H_n))$ as below.

$V_1 = \{e_1, e'_2, u_3, v_n\}$; $V_2 = \{e_2, v_2, e'_3, s_n, u_4\}$; $V_i = \{e_i, v_{i-1}, e'_{i+1}, s_{i-2}, u_{i+2} : 3 \leq i \leq n-2\}$; $V_{n-1} = \{e_{n-1}, v_{n-2}, e'_n, s_{n-3}\}$; $V_n = \{e_n, v_{n-1}, e'_1, s_{n-2}\}$; $V_{n+1} = \{v, s_{n-1}, u_1, u_2\}$. Clearly $V_1, V_2, V_i, V_{n-1}, V_n, V_{n+1}$ $(3 \leq i \leq n-2)$ are independent sets of $T(H_n)$. Also $|V_1| = |V_n| = |V_{n+1}| = 4$ and $|V_2| = |V_i| = 5$ $(3 \leq i \leq n-2)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j), $\chi_=(T(H_n)) \leq n+1$. Since $ve_i (1 \leq i \leq n)$ forms a clique of order $n+1$, $\chi(T(H_n)) \geq n+1$, $\chi_=(T(H_n)) \geq \chi(T(H_n)) \geq n+1$, $\chi_=(T(H_n)) \geq n+1$. Therefore $\chi_=(T(H_n)) = n+1$.

§5. Equitable Coloring on Gear Graph, Line Graph, Middle Graph and Total Graph

Theorem 5.1 *If $n \geq 3$ the equitable chromatic number of gear graph G_n, $\chi_=(G_n) = 2$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ where v_i's are the vertices of cycles taken in cyclic order and v is adjacent with $v_{2i-1}(1 \leq i \leq n)$.

Now, we partition the vertex set of $V(G_n)$ as below.

$V_1 = \{v\} \cup \{v_{2i} : 1 \leq i \leq n-1\}$; $V_2 = \{v_{2i-1} : 1 \leq i \leq n\}$. Clearly V_1, V_2 are independent sets of (G_n). Also $|V_1| = n+1$ and $|V_2| = n$, it holds the inequality $||V_i|-|V_j|| \leq 1$ for every pair (i,j). $\chi_=(G_n) \leq 2$. $\chi(G_n) \geq 2$, $\chi_=(G_n) \geq \chi(G_n) \geq 2$, $\chi_=(G_n) \geq 2$. Therefore $\chi_=(G_n) = 2$. □

§6. Equitable Coloring on Line Graph, Middle Graph and Total Graph of Gear Graph

Theorem 6.1 *If $n \geq 3$ the equitable chromatic number on line graph of Gear graph $L(G_n)$, $\chi_=(L(G_n)) = n$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of line graph $V(L(G_n)) = E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

Now, we partition the vertex set of $V(L(H_n))$ as below.

$V_1 = \{e_1, e'_2, e'_{2n-1}\}$; $V_i = \{e_i, e'_{2i}, e'_{2i-3} : 2 \leq i \leq n\}$. Clearly $V_1, V_2, \cdots V_n$ are independent sets of $L(G_n)$. Also $|V_1| = |V_i| = 3$ $(2 \leq i \leq n)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j). $\chi_=(L(G_n)) \leq n$. Since $e_i(1 \leq i \leq n)$ forms a clique of order n, $\chi(L(G_n)) \geq n$, $\chi_=(L(G_n)) \geq \chi(L(G_n)) \geq n$, $\chi_=(L(G_n)) \geq n$. Therefore $\chi_=(L(G_n)) = n$. □

Theorem 6.2 *If $n \geq 5$ the equitable chromatic number on middle graph of Gear graph $M(G_n)$, $\chi_=(M(G_n)) = n+1$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

Now, we partition the vertex set of $V(M(G_n))$ as below.

$V_1 = \{e_1, v_2, e'_{2n-4}, e'_{2n-1}\}$; $V_2 = \{e_2, v_1, v_4, e'_{2n-2}\}$; $V_3 = \{e_3, v_3, e'_1, v_6, e'_{2n}\}$; $V_i = \{e_i, e'_{2i-5}, v_{2i-3}, v_{2i} : 4 \leq i \leq n\} \cup \{e'_{2i-8} : 5 \leq i \leq n\}$; $V_{n+1} = \{v, e'_{2n-6}, v_{2n-1}, e'_{2n-3}\}$. Clearly $V_1, V_2, \cdots V_n, V_{n+1}$ are independent sets of $M(G_n)$. Also $|V_1| = |V_2| = |V_{n+1}| = 4$ and $|V_i| = |V_3| = 5$ $(5 \leq i \leq n)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i,j). $\chi_=(M(G_n)) \leq n+1$. Since $ve_i(1 \leq i \leq n)$ forms a clique of order $n+1$, $\chi(M(G_n)) \geq n+1$, $\chi_=(M(G_n)) \geq \chi(M(G_n)) \geq n+1$, $\chi_=(M(G_n)) \geq n+1$. Therefore $\chi_=(M(G_n)) = n+1$. □

Theorem 6.3 *If $n \geq 5$ the equitable chromatic number on total graph of Gear graph $T(G_n)$, $\chi_=(T(G_n)) = n+1$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of total graph $V(T(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

Now, we partition the vertex set of $V(T(G_n))$ as below.

$V_1 = \{e_1, e'_2, v_4, v_{2n-1}\}$; $V_2 = \{e_2, v_1, e'_4, v_6\}$; $V_i = \{e_i, e'_{2i-5}, v_{2i-3}, e'_{2i}, v_{2i+2} : 3 \leq i \leq n-1\}$;$V_n = \{e_n, e'_{2n-5}, v_{2n-3}, e'_{2n}\}$; $V_{n+1} = \{v, v_2, e'_{2n-1}, e'_{2n-3}\}$.

Clearly $V_1, V_2, V_i, V_n, V_{n+1}$ $(3 \leq i \leq n-1)$ are independent sets of $T(G_n)$. Also $|V_1| = |V_2| = |V_n| = |V_{n+1}| = 4$ and $|V_i| = 5$ $(3 \leq i \leq n-1)$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair (i, j), $\chi_=(T(G_n)) \leq n+1$. Since $ve_i(1 \leq i \leq n)$ forms a clique of order $n+1$, $\chi(T(G_n)) \geq n+1$, $\chi_=(T(G_n)) \geq \chi(T(G_n)) \geq n+1$, $\chi_=(T(G_n)) \geq n+1$. Therefore $\chi_=(T(G_n)) = n+1$. □

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Math.Combin.Book Ser. Vol.4 (2010), 38-45

On the Roman Edge Domination Number of a Graph

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Abstract: For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*. The Roman edge domination number $\gamma_{re}(G)$ of G is the minimum of $f(E) = \sum_{e \in E} f(e)$ over such functions. In this paper we first show that for any connected graph G of $q \geq 3$, $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$ and $\gamma_{re}(G) \leq 4q/5$, where $\gamma_e(G)$ is the edge domination number of G. Also we prove that for any $\gamma_{re}(G)$-function $f = \{E_0, E_1, E_2\}$ of a connected graph G of $q \geq 3$, $|E_0| \geq q/5 + 1$, $|E_1| \leq 4q/5 - 2$ and $|E_2| \leq 2q/5$.

Key Words: Smarandachely Roman s-dominating function, Smarandachely Roman edge s-dominating function.

AMS(2010): 05C69

§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = p$ and $|E| = q$ denote the number of vertices and edges of the graph G, respectively. The open neighborhood $N(e)$ of the edge e is the set of all edges adjacent to e in G. And its closed neighborhood is $N[e] = N(e) \cup \{e\}$. Similarly, the open neighborhood of a set $S \subseteq E$ is the set $N(S) = \bigcup_{e \in S} N(e)$, and its closed neighborhood is $N[S] = N(S) \cup S$.

The degree of an edge $e = uv$ of G is defined by $deg\, e = deg\, u + deg\, v - 2$ and $\delta'(G)$ ($\Delta'(G)$) is the minimum (maximum) degree among the edges of G (the degree of an edge is the number of edges adjacent to it). A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex.

[1]Received August 3, 2010. Accepted December 15, 2010.

Let $e \in S \subseteq E$. Edge h is called a private neighbor of e with respect to S (denoted by h is an S-pn of e) if $h \in N[e] - N[S - \{e\}]$. An S-pn of e is external if it is an edge of $E - S$. The set $pn(e, S) = N[e] - N[S - \{e\}]$ of all S-pn's of e is called the private neighborhood set of e with respect to S. The set S is said to be irredundant if for every $e \in S$, $pn(e, S) \neq \varnothing$. And a set S of edges is called independent if no two edges in S are adjacent.

A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of such a set is called the domination number of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see [5].

Mitchell and Hedetniemi in [6] introduced the notion of edge domination as follows. A set F of edges in a graph G is an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The minimum number of edges in such a set is called the edge domination number of G and is denoted by $\gamma_e(G)$. This concept is also studied in [1].

The concept of Roman dominating function (RDF) was introduced by E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi in [3]. (See also [2,4,8]). A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G.

A Roman edge dominating function (REDF) on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 2$. The weight of a Roman edge dominating function is the value $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G, denoted by $\gamma_{re}(G)$, equals the minimum weight of a Roman edge dominating function on G. A Roman edge dominating function $f : E \to \{0, 1, 2\}$ can be represented by the ordered partition (E_0, E_1, E_2) of E, where $E_i = \{e \in E \mid f(e) = i\}$ and $|E_i| = q_i$ for $i = 0, 1, 2$. This concept is studied in Soner et al. in [9] (see also [7]). A $\gamma - set$, $\gamma_r - set$ and γ_{re}-set, can be defined as a minimum dominating set (MDS), a minimum Roman dominating set (MRDS) and a minimum Roman edge dominating set (MREDS), respectively.

Theorem A. *For a graph G of order p,*

$$\gamma_e(G) \leq \gamma_{re}(G) \leq 2\gamma_e(G).$$

It is clear that if G has at least one edge then $1 \leq \gamma_{re}(G) \leq q$, where q is the number of edges in G. However if a graph is totally disconnected or trivial, we define $\gamma_{re}(G) = 0$. We note that $E(G)$ is the unique maximum REDS of G. Since every edge dominating set in G is a dominating set in the line graph of G and an independent set of edges of G is an independent set of vertices in the line graph of G, the following results can easily be proved from the well-known analogous results for dominating sets of vertices and independent sets.

Proposition 1. *A Roman edge dominating set S is minimal if and only if for each $e \in S$, one of the following two conditions holds.*

(i) $N(e) \cap S = \varnothing$.

(ii) There exists an edge $h \in E - S$, *such that* $N(h) \cap S = \{e\}$.

Proposition 2. *Let* $S = E_1 \cup E_2$ *be a REDS such that* $|E_1| + 2|E_2| = \gamma_{re}(G)$. *Then*

$$|E(G) - S| \leq \sum_{e \in S} deg(e),$$

and the equality holds if and only if S *is independent and for every* $e \in E - S$ *there exists only one edge* $h \in S$ *such that* $N(e) \cap S = \{h\}$.

Proof Since every edge in $E(G) - S$ is adjacent to at least one edge of S, each edge in $E(G) - S$ contributes at least one to the sum of the degrees of the edges of S, hence

$$|E(G) - S| \leq \textstyle\sum_{e \in S} deg(e)$$

Let $|E(G) - S| = \sum_{e \in S} deg(e)$. Suppose S is not independent. Since S is a REDS, every edge in $E - S$ is counted in the sum $\sum_{e \in S} deg(e)$. Hence if e_1 and e_2 have a common point in S, then e_1 is counted in $deg(e_2)$ and vice versa. Then the sum exceeds $|E - S|$ by at least two, contrary to the hypothesis. Hence S must be independent.

Now suppose $N(e) \cap S = \varnothing$ or $|N(e) \cap S| \geq 2$ for $e \in E - S$. Since S is a REDS the former case does not occur. Let e_1 and e_2 belong to $N(e) \cap S$. In this case $\sum_{e \in S} deg(e)$ exceeds $|E(G) - S|$ by at least one since e_1 is counted twice: once in $deg(e_1)$ and once in $deg(e_2)$, a contradiction. Hence equality holds if S is independent and for every $e \in E - S$ there exists only one edge $h \in S$ such that $N(e) \cap S = \{h\}$. Conversely, if S is independent and for every $e \in E - S$ there exists only one edge $h \in S$ such that $N(e) \cap S = \{h\}$, then equality holds. □

Proposition 3. *Let* G *be a graph and* $S = E_1 \cup E_2$ *be a minimum REDS of* G *such that* $|S| = 1$, *then the following condition hold.*

(i) S *is independent.*

(ii) $|E - S| = \sum_{e \in S} deg(e)$.

(iii) $\Delta'(G) = q - 1$.

(iv) $q/(\Delta' + 1) = 1$.

An immediate consequence of the above result is.

Corollary 1 *For any* (p, q) *graph,* $\gamma_{re}(G) = p - q + 1$ *if and only if* G *has* γ_{re} *components each of which is isomorphic to a star.*

Proposition 4. *Let* G *be a graph of* q *edges which contains a edge of degree* $q - 1$, *then* $\gamma_e(G) = 1$ *and* $\gamma_{re}(G) = 2$.

Proposition 5.([9]) *Let* $f = (E_0, E_1, E_2)$ *be any REDF. Then*

(i) $\langle E_1 \rangle$ *has maximum degree one.*

(ii) Each edge of E_0 *is adjacent to at most two edges of* E_1.

(iii) E_2 *is an* γ_e*-set of* $H = G[E_0 \cup E_2]$.

Proposition 6. *Let $f = (E_0, E_1, E_2)$ be any γ_{re}-function. Then*

(i) *No any edge of E_1 is adjacent to any edge of E_2.*

(ii) *Let $H = G[E_0 \cup E_2]$. Then each edge $e \in E_2$ has at least two H-pn's (i.e private neighbors relative to E_2 in the graph H).*

(iii) *If e is isolated in $G[E_2]$ and has precisely one external H-pn, say $h \in E_0$, then $N(h) \cap E_1 = \varnothing$.*

Proof (i) Let $e_1, e_2 \in E$, where e_1 adjacent to e_2, $f(e_1) = 1$ and $f(e_2) = 2$. Form f' by changing $f(e_1)$ to 0. Then f' is a REDF with $f'(E) < f(E)$, a contradiction.

(ii) By Proposition 5(iii), E_2 is an γ_e-set of H and hence is a maximal irredundant set in H. Therefore, each $e \in E_2$ has at least one E_2-pn in H.

Let e be isolated in $G[E_2]$. Then e is a E_2-pn of e. Suppose that e has no external E_2-pn. Then the function produced by changing $f(e)$ from 2 to 1 is an REDF of smaller weight, a contradiction. Hence, e has at least two E_2-pns in H.

Suppose that e is not isolated in $G[E_2]$ and has precisely one E_2-pn (in H), say w. Consider the function produced by changing $f(e)$ to 0 and $f(h)$ to 1. The edge e is still dominated because it has a neighbor in E_2. All of e's neighbors in E_0 are also obtained, since every edge in E_0 has another neighbor in E_2 except for h, which is now in E_1. Therefore, this new function is an REDF of smaller weight, which is a contradiction. Again, we can conclude that e has at least two E_2-pns in H.

(iii) Suppose the contrary. Define a new function f' with $f'(e) = 0$, $f'(e') = 0$ for $e' \in N(h) \cap E_1$, $f'(h) = 2$, and $f'(x) = f(x)$ for all other edges x. $f'(E) = f(E) - |N(h) \cap E_1| < f(E)$, contradicting the minimality of f. □

Proposition 7. *Let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of an isolate-free graph G, such that $|E_2| = q_2$ is a maximum. Then*

(i) *E_1 is independent.*

(ii) *The set E_0 dominates the set E_1.*

(iii) *Each edge of E_0 is adjacent to at most one edge of E_1.*

(iv) *Let $e \in G[E_2]$ have exactly two external H-pn's e_1 and e_2 in E_0. Then there do not exist edges $h_1, h_2 \in E_1$ such that (h_1, e_1, e, e_2, h_2) is the edge sequence of a path P_6.*

Proof (i) By Proposition 5(i), $G[E_1]$ consists of disjoint K_2's and P_3's. If there exists a P_3, then we can change the function values of its edges to 0 and 2. The resulting function $g = (W_0, W_1, W_2)$ is a γ_{re}-function with $|W_2| > |E_2|$, which is a contradiction. Therefore, E_1 is an independent set.

(ii) By (i) and Proposition 6(i), no edge $e \in E_1$ is adjacent to an edge in $E_1 \cup E_2$. Since G is isolate-free, e is adjacent to some edge in E_0. Hence the set E_0 dominates the set E_1.

(iii) Let $e \in E_0$ and $B = N(e) \cap E_1$, where $|B| = 2$. Note that $|B| \leq 2$, by Proposition 5(ii). Let

$$W_0 = (E_0 \cup B) - \{e\},$$
$$W_1 = E_1 - B,$$

$W_2 = E_2 \cup \{e\}$.

We know that E_2 dominates E_0, so that $g = (W_0, W_1, W_2)$ is an REDF.

$g(E) = |W_1| + 2|W_2| = |E_1| - B + 2|E_2| - 2 = f(E)$. Hence, g is a γ_{re}-function with $|W_2| > |E_2|$, which is a contradiction.

iv) Suppose the contrary. Form a new function by changing the function values of (h_1, e_1, e, e_2, h_2) from $(1, 0, 2, 0, 1)$ to $(0, 2, 0, 0, 2)$. Then the new function is a γ_{re}-function with bigger value of q_2, which is a contradiction. □

§2. Graph for Which $\gamma_{re}(G) = 2\gamma_e(G)$

From Theorem A we know that for any graph G, $\gamma_{re}(G) \leq 2\gamma_e(G)$. We will say that a graph G is a Roman edge graph if $\gamma_{re}(G) = 2\gamma_e(G)$.

Proposition 8. *A graph G is Roman edge graph if and only if it has a γ_{re}-function $f = (E_0, E_1, E_2)$ with $q_1 = |E_1| = 0$.*

Proof Let G be a Roman edge graph and let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of G. Proposition 5(iii) we know that E_2 dominates E_0, and $E_1 \cup E_2$ dominates E, and hence

$$\gamma_e(G) \leq |E_1 \cup E_2| = |E_1| + |E_2| \leq |E_1| + 2|E_2| = \gamma_{re}(G).$$

But since G is Roman edge, we know that

$$2\gamma_e(G) = 2|E_1| + 2|E_2| = \gamma_{re}(G) = |E_1| + 2|E_2|.$$

Hence, $q_1 = |E_1| = 0$.

Conversely, let $f = (E_0, E_1, E_2)$ be a γ_{re}-function of G with $q_1 = |E_1| = 0$. Then, $\gamma_{re}(G) = 2|E_2|$, and since by definition $E_1 \cup E_2$ dominates E, it follows that E_2 is a dominating set of G. But by Proposition 5(iii), we know that E_2 is a γ_e-set of $G[E_0 \cup E_2]$, i.e. $\gamma_e(G) = |E_2|$ and $\gamma_{re}(G) = 2\gamma_e(G)$, i.e. G is a Roman edge graph. □

§3. Bound on the Sum $\gamma_{re}(G) + \gamma_e(G)/2$

For q-edge graphs, always $\gamma_{re}(G) \leq q$, with equality when G is isomorphic with mK_2 or mP_3. In this section we prove that $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$ and $\gamma_{re}(G) \leq 4q/5$ when G is a connected q-edge graph.

Theorem 9. *For any connected graph G of $q \geq 3$,*

(*i*) $\gamma_{re}(G) + \gamma_e(G)/2 \leq q$.
(*ii*) $\gamma_{re}(G) \leq 4q/5$.

Proof Let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(G)$-function such that $|E_2|$ is maximum. It is proved in Proposition 6(i) that for such a function no edge of E_1 is adjacent to any edge of E_2 and every edge e of E_2 has at least two E_2-private neighbors, one of them can be e itself if it is isolated in

E_2 (true for every $\gamma_{re}(G)$-function). The set E_1 is independent and every edge of E_0 has at most one neighbor in E_1. Moreover we add the condition the number $\mu(f)$ of edges of E_2 with only one neighbor in E_0 is minimum. Suppose that $N_{E_0}(e) = \{h\}$ for some $e \in E_2$. Then partition $E_0' = (E_0\backslash\{h\}) \cup \{e\} \cup N_{E_1}(h)$, $E_1' = E_1\backslash N_{E_1}(h)$ and $E_2' = (E_2\backslash\{e\}) \cup \{h\}$ is a Roman edge dominating function f' such that $w(f') = w(f) - 1$ if $N_{E_1}(h) \neq \varnothing$, or $w(f') = w(f)$, $|E_2'| = |E_2|$ but $\mu(f') < \mu(f)$ if $N_{E_1}(h) = \varnothing$ since then, G being connected $q \geq 3$, h is not isolated in E_0. Therefore every edge of E_2 has at least two neighbors in E_0. Let A be a largest subset of E_2 such that for each $e \in A$ there exists a subset A_e of $N_{E_0}(e)$ such that the set A_e is disjoint, $|A_e| \geq 2$ and sets $\cup_{e\in A} A_e = \cup_{e\in A} N_{E_0}(e)$. Note that A_e contains all the external E_2-private neighbors of e. $A' = E_2\backslash A$.

Case 1 $A' = \varnothing$.

In this case $|E_0| \geq 2|E_2|$ and $|E_1| \leq |E_0|$ since every edge of E_0 has at most one neighbor in E_1. Since E_0 is an edge dominating set of G and $|E_0|/2 \geq |E_2|$ we have

(i) $\gamma_{re}(G) + \gamma_e(G)/2 \leq |E_1| + 2|E_2| + |E_0|/2 \leq |E_0| + |E_1| + |E_2| = q$.

(ii) $5\gamma_{re}(G) = 5|E_1| + 10|E_2| = 4q - 4|E_0| + |E_1| + 6|E_2| = 4q - 3(|E_0| - 2|E_2|) - (|E_0| - |E_1|) \leq 4q$. Hence $\gamma_{re}(G) \leq 4q/5$.

Case 2 $A' \neq \varnothing$.

Let $B = \cup_{e\in A} A_e$ and $B' = E_0\backslash B$. Every edge ε in A' has exactly one E_2-private neighbor ε' in E_0 and $N_{B'}(\varepsilon) = \{\varepsilon'\}$ for otherwise ε could be added to A. This shows that $|A'| = |B'|$. Moreover since $|N_{E_0}(\varepsilon)| \geq 2$, each edge $\varepsilon \in A'$ has at least one neighbor in B. Let $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$ and let ε_A be the edge of A such that $\varepsilon_B \in A_{\varepsilon_A}$. The edge ε_A is well defined since the sets A_e with $e \in A$ form a partition of B.

Claim 1 $|A_{\varepsilon_A}| = 2$ *for each* $\varepsilon \in A'$ *and each* $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$.

Proof of Claim 1 If $|A_{\varepsilon_A}| > 2$, then by putting $A'_{\varepsilon_A} = A_{\varepsilon_A}\backslash\{\varepsilon_B\}$ and $A_\varepsilon = \{\varepsilon', \varepsilon_B\}$ we can see that $A_1 = A \cup \{\varepsilon\}$ contradicts the choice of A. Hence $|A_{\varepsilon_A}| = 2$, ε_A has a unique external E_2-private neighbor ε'_A and $A_{\varepsilon_A} = \{\varepsilon_B, \varepsilon'_A\}$. Note that the edges ε_A and ε are isolated in E_2 since they must have a second E_2-private neighbor.

Claim 2 *If* $\varepsilon, y \in A'$ *then* $\varepsilon_B \neq y_B$ *and* $A_{\varepsilon_A} \neq A_{y_A}$.

Proof of Claim 2 Let ε' and y' be respectively the unique external E_2-private neighbors of ε and y. Suppose that $\varepsilon_B = y_B$, and thus $\varepsilon_A = y_A$. The function $g : E(G) \to \{0, 1, 2\}$ defined by $g(\varepsilon_B) = 2$, $g(\varepsilon) = g(y) = g(\varepsilon_A) = 0$, $g(\varepsilon'_A) = g(y') = g(\varepsilon') = 1$ and $g(e) = f(e)$ otherwise, is a REDF of G of weight less than $\gamma_{re}(G)$, a contradiction. Hence $\varepsilon_B \neq y_B$. Since $A_{\varepsilon_A} \supseteq \{\varepsilon_B, \varepsilon'_A\}$ and $|A_{\varepsilon_A}| = 2$, the edge y_B is not in A_{ε_A}. Therefore $A_{\varepsilon_A} \neq A_{y_A}$.

Let $A'' = \{\varepsilon_A \mid \varepsilon \in A' \text{ and } \varepsilon_B \in B \cap N_{E_0}(\varepsilon)\}$ and $B'' = \cup_{e\in A''} A_e$. By Claims 1 and 2,

$$|B''| + 2|A''| \quad \text{and} \quad |A''| \geq |A'|.$$

Let $A''' = E_2\backslash(A' \cup A'')$ and $B''' = \cup_{e\in A'''} A_e = E_0\backslash(B' \cup B'')$. By the definition of the sets A_e,

$$|B'''| \geq |2A'''|.$$

Claim 3 *If $\varepsilon \in A'$ and $\varepsilon_B \in B \cap N_{E_0}(\varepsilon)$, then $\varepsilon', \varepsilon_B$ and ε'_A have no neighbor in E_1. Hence B''' dominates E_1.*

Proof of Claim 3 Let h be a edge of E_1. If h has a neighbor in $B' \cup B''$, Let $g : E(G) \to \{0, 1, 2\}$ be defined by $g(\varepsilon'_A) = 2$, $g(h) = g(\varepsilon_A) = 0$, $g(e) = f(e)$ otherwise if h is adjacent to ε'_A, $g(\varepsilon') = 2$, $g(h) = g(\varepsilon) = 0$, $g(e) = f(e)$ otherwise if h is adjacent to ε',

$g(\varepsilon_B) = 2$, $g(h) = g(\varepsilon_A) = g(\varepsilon) = 0$, $g(\varepsilon'_A) = g(\varepsilon') = 1$, $g(e) = f(e)$ otherwise if h is adjacent to ε_B. In each case, g is a REDF of weight less than $\gamma_{re}(G)$, a contradiction. Therefore $N(h) \subseteq B'''$.

We are now ready to establish the two parts of the Theorem.

(i) By Claim 3, $B''' \cup A' \cup A''$ is an edge dominating set of G. Therefore, since $|A'| = |B'|$ and $|B'''| \geq |2A'''|$ we have,

$$\gamma_e(G) \leq |B'''| + |A'| + |A''| \leq |B'''| + |B''| \leq (2|B'''| - 2|A'''|) + (2|B''| - 2|A''|) + (2|B'| - 2|A'|).$$

Hence $\gamma_e(G) \leq 2|E_0| - 2|E_2|$ and $\gamma_{re}(G) + \gamma_e(G)/2 \leq (|E_1| + 2|E_2|) + (|E_0| - |E_2|) = q$.

(ii) By Claim 3 and since each edge of E_1 has at most one neighbor in E_0 and $|E_1| \leq |B'''|$. Using this inequality and since $|A'| = |B'|$ and $|B'''| \geq |2A'''|$ we get

$$\begin{aligned} 5\gamma_{re}(G) &= 5|E_1| + 10|E_2| = 4q - 4|E_0| + |E_1| + 6|E_2| \leq 4q - 4|B'| - 4|B''| - 4|B'''| \\ &\quad + |B'''| + 6|A'| + 6|A''| + 6|A'''| \leq 4q + 2(|A'| - |A''|) + 3(2|A'''| - |B'''|) \leq 4q. \end{aligned}$$

Hence $\gamma_{re}(G) \leq 4q/5$. □

Corollary 10 *Let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(G) -$ function of a connected graph G. If $k|E_2| \leq |E_0|$ such that $k \geq 4$, then $\gamma_{re}(G) \leq (k-1)q/k$.*

§4. Bounds on $|E_0|$, $|E_1|$ and $|E_2|$ for a $\gamma_{re}(G)$-Function (E_0, E_1, E_2)

Theorem 11. *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re}(G) -$ function of a connected graph G of $q \geq 3$. Then*

(1) $1 \leq |E_2| \leq 2q/5$;
(2) $0 \leq |E_1| \leq 4q/5 - 2$;
(3) $q/5 + 1 \leq |E_0| \leq q - 1$.

Proof By Theorem 9, $|E_1| + 2|E_2| \leq 4q/5$.

(1) If $E_2 = \varnothing$, then $E_1 = q$ and $E_0 = \varnothing$. The REDF $(0, q, 0)$ is not minimum since $|E_1| + 2|E_2| > 4q/5$. Hence $|E_2| \geq 1$. On the other hand, $|E_2| \leq 2q/5 - |E_1|/2 \leq 2q/5$.

(2) Since $|E_2| \geq 1$, then $|E_1| \leq 4q/5 - 2|E_2| \leq 4q/5 - 2$.

(3) The upper bound comes from $|E_0| \leq q - |E_2| \leq q - 1$. For the lower bound, adding on side by side $2|E_0| + 2|E_1| + 2|E_2| = 2q, -|E_1| - 2|E_2| \geq -4q/5$ and $-|E_1| \geq -4q/5 + 2$ gives $2|E_0| \geq 2q/5 + 2$. Therefor, $|E_0| \geq q/5 + 1$. □

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Math.Combin.Book Ser. Vol.4 (2010), 46-52

The Upper Monophonic Number of a Graph

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Abstract: For a connected graph $G = (V, E)$, a Smarandachely k-monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a path with less or equal k chords joining some pair of vertices in M. The Smarandachely k-monophonic number $m_S^k(G)$ of G is the minimum order of its Smarandachely k-monophonic sets. Particularly, a Smarandachely 0-monophonic path, a Smarandachely 0-monophonic number is abbreviated to a *monophonic path*, *monophonic number* $m(G)$ of G respectively. Any monophonic set of order $m(G)$ is a minimum monophonic set of G. A monophonic set M in a connected graph G is called a minimal monophonic set if no proper subset of M is a monophonic set of G. The upper monophonic number $m^+(G)$ of G is the maximum cardinality of a minimal monophonic set of G. Connected graphs of order p with upper monophonic number p and $p-1$ are characterized. It is shown that for every two integers a and b such that $2 \le a \le b$, there exists a connected graph G with $m(G) = a$ and $m^+(G) = b$.

Key Words: Smarandachely k-monophonic path, Smarandachely k-monophonic number, monophonic path, monophonic number.

AMS(2010): 05C12

§1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [1]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. A vertex x is said to *lie on* a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diam* G of G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any

[1]Received August 6, 2010. Accepted December 16, 2010.

geodetic set of cardinality $g(G)$ is a *minimum geodetic set* of G. The geodetic number of a graph is introduced in [2] and further studied in [3]. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. A vertex v is an *extreme vertex* of a graph G if $< N(v) >$ is complete. A *chord* of a path $u_0, u_1, u_2, \ldots, u_h$ is an edge $u_i u_j$, with $j \geq i + 2$. An $u - v$ path is called a *monophonic path* if it is a chordless path. A Smarandachely k-monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a path with less or equal k chords joining some pair of vertices in M. The Smarandachely k-monophonic number $m_S^k(G)$ of G is the minimum order of its Smarandachely k-monophonic sets. Particularly, a Smarandachely 0-monophonic path, a Smarandachely 0-monophonic number is abbrevated to *monophonic path*, *monophonic number* $m(G)$ of G respectively. Thus, a *monophonic set* of G is a set $M \subseteq V$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* or simply a $m - set$ of G. It is easily observed that no cut vertex of G belongs to any minimum monophonic set of G. The monophonic number of a graph is studied in [4, 5, 6]. For the graph G given in Figure 1.1, $S_1 = \{v_2, v_4, v_5\}$, $S_2 = \{v_2, v_4, v_6\}$ are the only minimum geodetic sets of G so that $g(G) = 3$. Also, $M_1 = \{v_2, v_4\}, M_2 = \{v_4, v_6\}, M_3 = \{v_2, v_5\}$ are are the only minimum monophonic sets of G so that $m(G) = 2$.

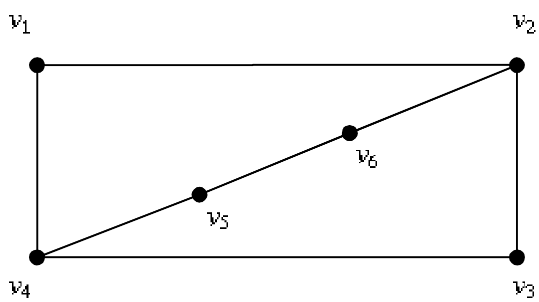

Figure 1: G

§2. The Upper Monophonic Number of a Graph

Definition 2.1 *A monophonic set M in a connected graph G is called a* minimal monophonic set *if no proper subset of M is a monophonic set of G. The upper monophonic number $m^+(G)$ of G is the maximum cardinality of a minimal monophonic set of G.*

Example 2.2 For the graph G given in Figure 1.1, $M_4 = \{v_1, v_3, v_5\}$ and $M_5 = \{v_1, v_3, v_6\}$ are minimal monophonic sets of G so that $m^+(G) \geq 3$. It is easily verified that no four element subsets or five element subsets of $V(G)$ is a minimal monophonic set of G and so $m^+(G) = 3$.

Remark 2.3 Every minimum monophonic set of G is a minimal monophonic set of G and the converse is not true. For the graph G given in Figure 1.1, $M_4 = \{v_1, v_3, v_5\}$ is a minimal monophonic set but not a minimum monophonic set of G.

Theorem 2.4 *Each extreme vertex of G belongs to every monophonic set of G.*

Proof Let M be a monophonic set of G and v be an extreme vertex of G. Let $\{v_1, v_2, \ldots, v_k\}$ be the neighbors of v in G. Suppose that $v \notin M$. Then v lies on a monophonic path $P : x = x_1, x_2, \ldots, v_i, v, v_j, \ldots, x_m = y$, where $x, y \in M$. Since v_iv_j is a chord of P and so P is not a monophonic path, which is a contradiction. Hence it follows that $v \in M$. □

Theorem 2.5 *Let G be a connected graph with cut-vertices and S be a monophonic set of G. If v is a cut -vertex of G, then every component of $G - v$ contains an element of S.*

Proof Suppose that there is a component G_1 of $G - v$ such that G_1 contains no vertex of S. By Theorem 2.4, G_1 does not contain any end-vertex of G. Thus G_1 contains at least one vertex, say u. Since S is a monophonic set, there exists vertices $x, y \in S$ such that u lies on the $x - y$ monophonic path $P : x = u_0, u_1, u_2, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be a $x - u$ sub path of P and P_2 be a $u - y$ subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of $G - v$ contains an element of S. □

Theorem 2.6 *For any connected graph G, no cut-vertex of G belongs to any minimal monophonic set of G.*

Proof Let M be a minimal monophonic set of G and $v \in M$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - v$. By Theorem 2.5, each component $G_i (1 \leq i \leq r)$ contains an element of M. We claim that $M_1 = M - \{v\}$ is also a monophonic set of G. Let x be a vertex of G. Since M is a monophonic set, x lies on a monophonic path P joining a pair of vertices u and v of M. Assume without loss of generality that $u \in G_1$. Since v is adjacent to at least one vertex of each $G_i (1 \leq i \leq r)$, assume that v is adjacent to z in G_k, $k \neq 1$. Since M is a monophonic set, z lies on a monophonic path Q joining v and a vertex w of M such that w must necessarily belongs to G_k. Thus $w \neq v$. Now, since v is a cut vertex of G, $P \cup Q$ is a path joining u and w in M and thus the vertex x lies on this monophonic path joining two vertices u and w of M_1. Thus we have proved that every vertex that lies on a monophonic path joining a pair of vertices u and v of M also lies on a monophonic path joining two vertices of M_1. Hence it follows that every vertex of G lies on a monophonic path joining two vertices of M_1, which shows that M_1 is a monophonic set of G. Since $M_1 \subsetneq M$, this contradicts the fact that M is a minimal monophonic set of G. Hence $v \notin M$ so that no cut vertex of G belongs to any minimal monophonic set of G. □

Corollary 2.7 *For any non-trivial tree T, the monophonic number $m^+(T) = m(T) = k$, where k is number of end vertices of T.*

Proof This follows from Theorems 2.4 and 2.6. □

Corollary 2.8 *For the complete graph $K_p(p \geq 2)$, $m^+(K_p) = m(K_p) = p$.*

Proof Since every vertex of the complete graph, $K_p(p \geq 2)$ is an extreme vertex, the vertex set of K_p is the unique monophonic set of K_p. Thus $m^+(K_p) = m(K_p) = p$. □

Theorem 2.9 *For a cycle $G = C_p(p \geq 4)$, $m^+(G) = 2 = m(G)$.*

Proof Let x, y be two independent vertices of G. Then $M = \{x, y\}$ is a monophonic set of G so that $m(G) = 2$. We show that $m^+(G) = 2$. Suppose that $m^+(G) > 2$. Then there exists a minimal monophonic set M_1 such that $|M_1| \geq 3$. Now it is clear that $M \subsetneq M_1$, which is a contradiction to M_1 a minimal monophonic set of G. Therefore, $m^+(G) = 2$. □

Theorem 2.10 *For a connected graph G, $2 \leq m(G) \leq m^+(G) \leq p$.*

Proof Any monophonic set needs at least two vertices and so $m(G) \geq 2$. Since every minimal monophonic set is a monophonic set, $m(G) \leq m^+(G)$. Also, since $V(G)$ is a monophonic set of G, it is clear that $m^+(G) \leq p$. Thus $2 \leq m(G) \leq m^+(G) \leq p$. □

The following Theorem is proved in [3].

Theorem A *Let G be a connected graph with diameter d .Then $g(G) \leq p - d + 1$.*

Theorem 2.11 *Let G be a connected graph with diameter d .Then $m(G) \leq p - d + 1$.*

Proof Since every geodetic set of G is a monophonic set of G, the assertion follows from Theorem 2.10 and Theorem A. □

Theorem 2.12 *For a non-complete connected graph G, $m(G) \leq p - k(G)$, where $k(G)$ is vertex connectivity of G.*

Proof Since G is non complete, it is clear that $1 \leq k(G) \leq p - 2$. Let $U = \{u_1, u_2, \ldots, u_k\}$ be a minimum cutset of vertices of$G$. Let $G_1, G_2, \ldots, G_r(r \geq 2)$ be the components of $G - U$ and let $M = V(G) - U$. Then every vertex $u_i(1 \leq i \leq k)$ is adjacent to at least one vertex of $G_j(1 \leq j \leq r)$. Then it follows that the vertex u_i lies on the monophonic path x, u_i, y, where $x, y \in M$ so that M is a monophonic set. Thus $m(G) \leq p - k(G)$. □

The following Theorems 2.13 and 2.15 characterize graphs for which $m^+(G) = p$ and $m^+(G) = p - 1$ respectively.

Theorem 2.13 *For a connected graph G of order p, the following are equivalent:*

(*i*) $m^+(G) = p$;
(*ii*) $m(G) = p$;
(*iii*) $G = K_p$.

Proof $(i) \Rightarrow (ii)$. Let $m^+(G) = p$. Then $M = V(G)$ is the unique minimal monophonic set of G. Since no proper subset of M is a monophonic set, it is clear that M is the unique minimum monophonic set of G and so $m(G) = p$. $(ii) \Rightarrow (iii)$. Let $m(G) = p$. If $G \neq K_p$, then by Theorem 2.11, $m(G) \leq p-1$, which is a contradiction. Therefore $G = K_p$. $(ii) \Rightarrow (iii)$. Let $G = K_p$. Then by Corollary 2.8, $m^+(G) = p$. □

Theorem 2.14 *Let G be a non complete connected graph without cut vertices. Then $m^+(G) \leq p-2$.*

Proof Suppose that $m^+(G) \geq p-1$. Then by Theorem 2.13, $m^+(G) = p-1$. Let v be a vertex of G and let $M = V(G) - \{v\}$ be a minimal monophonic set of G. By Theorem 2.4, v is not an extreme vertex of G. Then there exists $x, y \in N(v)$ such that $xy \notin E(G)$. Since v is not a cut vertex of G, $< G - v >$ is connected. Let $x, x_1, x_2, \ldots, x_n, y$ be a monophonic path in $< G - v >$. Then $M_1 = M - \{x_1, x_2, \ldots, x_n\}$ is a monophonic set of G. Since $M_1 \subsetneq M$, M_1 is not a minimal monophonic set of G, which is a contradiction. Therefore $m^+(G) \leq p-2$. □

Theorem 2.15 *For a connected graph G of order p, the following are equivalent:*

(i) $m^+(G) = p-1$;
(ii) $m(G) = p-1$;
(iii) $G = K_1 + \bigcup m_j K_j$, $\sum m_j \geq 2$.

Proof $(i) \Rightarrow (ii)$. Let $m^+(G) = p-1$. Then it follows from Theorem 2.13 that G is non-complete. Hence by Theorem 2.14, G contains a cut vertex, say v. Since $m^+(G) = p-1$, hence it follows from Theorem 2.6 that $M = V - \{v\}$ is the unique minimal monophonic set of G. We claim that $m(G) = p-1$. Suppose that $m(G) < p-1$.Then there exists a minimum monophonic set M_1 such that $|M_1| < p-1$. It is clear that $v \notin M_1$. Then it follows that $M_1 \subsetneq M$, which is a contradiction. Therefore $m(G) = p-1$. $(ii) \Rightarrow (iii)$. Let $m(G) = p-1$. Then by Theorem 2.11, $d \leq 2$. If $d = 1$, then $G = K_p$, which is a contradiction. Therefore $d = 2$. If G has no cut vertex, then by Theorem 2.12, $m(G) \leq p-2$, which is a contradiction. Therefore G has a unique cut-vertex, say v. Suppose that $G \neq K_1 + \bigcup m_j K_j$. Then there exists a component, say G_1 of $G - v$ such that $< G_1 >$ is non complete. Hence $|V(G_1)| \geq 3$. Therefore $< G_1 >$ contains a chordless path P of length at least two. Let y be an internal vertex of the path P and let $M = V(G) - \{v, y\}$. Then M is a monophonic set of G so that $m(G) \leq p-2$, which is a contradiction. Thus $G = K_1 + \bigcup m_j K_j$. $(iii) \Rightarrow (i)$). Let $G = K_1 + \bigcup m_j K_j$. Then by Theorems 2.4 and 2.6, $m^+(G) = p-1$. □

In the view of Theorem 2.10, we have the following realization result.

Theorem 2.16 *For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $m(G) = a$ and $m^+(G) = b$.*

Proof Let G be a graph given in Figure 2.1 obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding the new vertices $v_1, v_2, \ldots, v_{b-a+1}$ and $w_1, w_2, \ldots, w_{a-1}$ and joining each v_i $(1 \leq i \leq b-a+1)$ to each $v_j (1 \leq j \leq b-a+1)$, $i \neq j$, and also joining each w_i $(1 \leq i \leq a-1)$ with u_1 and u_2. First we show that $m(G) = a$. Let M be a monophonic set of G and let $W = \{w_1, w_2, \ldots, w_{a-1}\}$. By Theorem 2.4, $W \subseteq M$. It is easily seen that W is not a monophonic set of G. However, $W \cup \{u_3\}$ is a monophonic set of G and so $m(G) = a$. Next we show that $m^+(G) = b$. Let $M_1 = W \cup \{v_1, v_2, \ldots, v_{b-a+1}\}$. Then M_1 is a monophonic

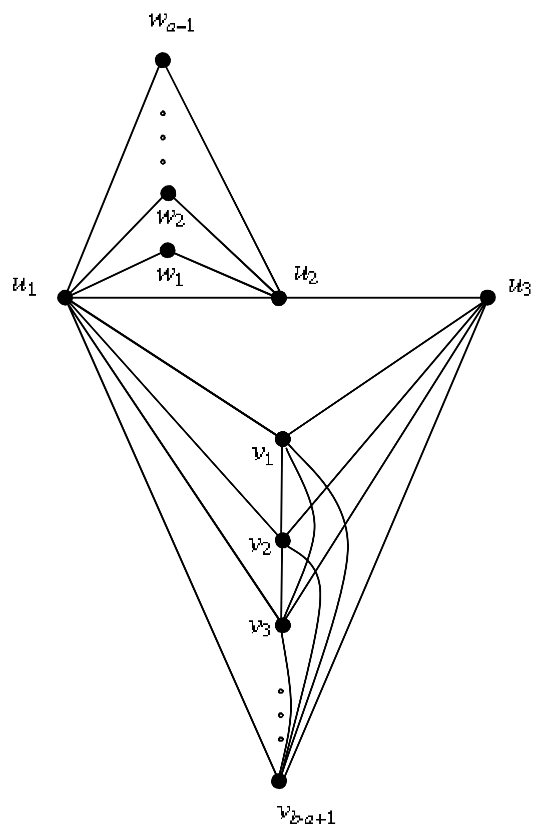

Figure 2: G

set of G. If M_1 is not a minimal monophonic set of G, then there is a proper subset T of M_1 such that T is a monophonic set of G. Then there exists $v \in M_1$ such that $v \notin T$. By Theorem 2.4, $v \neq w_i$ $(1 \leq i \leq a-1)$. Therefore $v = v_i$ for some i $(1 \leq i \leq b-a+1)$. Since $v_iv_j(1 \leq i, j \leq b-a+1)$, $i \neq j$ is a chord, v_i does not lie on a monophonic path joining some vertices of T and so T is not a monophonic set of G, which is a contradiction. Thus M_1 is a minimal monophonic set of G and so $m^+(G) \geq b$. Let T' be a minimal monophonic set of G with $|T'| \geq b+1$. By Theorem2.4, $W \subseteq T'$. Since $W \cup \{u_3\}$ is a monophonic set of G, $u_3 \notin T'$. Since M_1 is a monophonic set of G, there exists at least one v_i such that $v_i \notin T'$. Without loss of generality let us assume that $v_1 \notin T'$. Since $|T'| \geq b+1$, then u_1, u_2 must belong to T'. Now it is clear that v_1 does not lie on a monophonic path joining a pair of vertices of T', it follows that T' is not a monophonic set of G, which is a contradiction. Therefore $m^+(G) = b$. □

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Math.Combin.Book Ser. Vol.4 (2010), 53-61

Some Results on Pair Sum Labeling of Graphs

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Abstract: Let G be a (p, q) graph. An injective map $f : V(G) \to \{\pm 1, \pm 2, \cdots, \pm\ p\}$ is called a pair sum labeling if the induced edge function, $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm\ k_1, \pm\ k_2, \cdots, \pm\ k_{\frac{q}{2}}\}$ or $\{\pm\ k_1, \pm\ k_2, \ldots, \pm\ k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$ according as q is even or odd. Here we study about the pair sum labeling of some standard graphs.

Key Words: Path, cycle, star, ladder, quadrilateral snake, Smarandachely pair sum V-labeling.

AMS(2010): 05C78

§1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. p and q denote respectively the number of vertices and edges of G. The Union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If P_n denotes a path on n vertices, the graph $L_n = P_2 \times P_n$ is called a ladder. Let G be a graph. A 1-1 mapping $f : V(G) \to \{\pm\ 1, \pm\ 2, \ldots, \pm\ |V|\}$ is said to be a Smarandachely pair sum V-labeling if the induced edge function, $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ for $uv \in E(G)$ is also one-one and $f_e(E(G))$ is either of the form $\{\pm\ k_1, \pm\ k_2, \ldots \pm\ k_{(|E|2}\}$ if $|E| \equiv 0(mod\ 2)$ or $\{\pm\ k_1, \pm\ k_2, \ \ \pm\ \ k_{\frac{|E|-1)}{2}}\} \cup \{k_{\frac{(|E|+1)}{2}}\}$ if $|E| \equiv 1(mod\ 2)$. Particularly we abbreviate a Smarandachely pair sum V-labeling to a pair sum labeling and define a graph with a pair sum labeling to be a pair sum graph. The notion of pair sum labeling has been introduced in [4]. In [4] we investigate the pair sum labeling behavior of complete graph, cycle, path, bistar etc. Here we study pair sum labeling of union of some standard graphs and we find the maximum size of a pair sum graph. Terms not defined here are used in the sense of Gary Chartrand [2] and Harary [3].

[1]Received August 23, 2010. Accepted December 18, 2010.

§2. Pair Sum Labeling

Definition 2.1 *Let G be a (p,q) graph. A one - one map $f : V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e : E(G) \rightarrow Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.*

Notation 2.2 Let G be a pair sum graph with pair sum labeling f. We denote $M = Max\{f(u) : u \in V(G)\}$ and $m = Min\{f(u) : u \in V(G)\}$.

Observation 2.3

(a) *If G is an even size pair sum graph then $G - e$ is also a pair sum graph for every edge e.*

(b) *Let G be an odd size pair sum graph with - $f_e(e) \notin f_e(E)$. Then $G - e$ is a pair sum graph.*

Proof These results follow from Definition 2.1. □

Observation 2.4 *Let G be a pair sum graph with even size and let f be a pair sum labeling of G with $f(u) = M$. Then the graph G^* with $V(G^*) = V(G) \cup \{v\}$ and $E(G^*) = E(G) \cup \{uv\}$ is also a pair sum graph.*

Proof Define $f^* : V(G^*) \rightarrow \{\pm 1, \pm 2, , \pm (p+1)\}$ by $f^*(w) = f(w)$ for all $w \in V(G)$ and $f^*(v) = p + 1$. Then $f_e(E(G^*)) = f_e(E(G)) \cup \{M + p + 1\}$. Hence f is a pair sum labeling. □

S.M. Lee and W. Wei define super vertex-graceful labeling of a graph [1].

Definition 2.5 *A (p,q) graph is said to be super vertex-graceful if there is a bijection f from V to $\{0, \pm 1, \pm 2, \ldots, \pm \frac{p-1)}{2}\}$ when p is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm \frac{p}{2}\}$ when p is even such that the induced edge labeling f^+ defined by $f^+(uv) = f(u) + f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \ldots, \pm \frac{q-1}{2}\}$ when q is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm \frac{q}{2}\}$ when q is even.*

Observation 2.6 *Let G be an even order and even size graph. If G is super vertex graceful then G is a pair sum graph.*

Remark. *K_4 is a pair sum graph but not super vertex graceful graph.*

Theorem 2.7 *If G is a (p,q) pair sum graph then $q \leq 4p - 2$.*

Proof Let f be a pair sum labeling of G. Obviously $-2p+1 \leq f_e(uv) \leq 2p-1, f_e(uv) \neq 0$ for all uv. This forces $q \leq 4p - 2$. □

We know that $K_{1,n}$ and $K_{2,n}$ are pair sum graph [4]. Now we have

Corollary 2.8 *If $m, n \geq 8$, then $K_{m,n}$ is not a pair sum graph.*

Proof This result follows from the inequality $(m-4)(n-4) \leq 14$ and the condition $m \geq 8, n \geq 8$. □

§3. Pair Sum Labeling of Union of Graphs

Theorem 3.1 *$K_{1,n} \cup K_{1,m}$ is a pair sum graph.*

Proof Let $u, u_1, u_2, \ldots u_n$ be the vertices of $K_{1,n}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let $v, v_1, v_2, \ldots, v_m$ be the vertices of $K_{1,m}$ and $E(K_{1,m}) = \{vv_i : 1 \leq i \leq n\}$.

Case 1 $m = n$.

Define

$$\begin{aligned} &f(u) = 1, \\ &f(u_i) = i+1, && 1 \leq i \leq m \\ &f(v) = -1, \\ &f(v_i) = -(i+1), && 1 \leq i \leq m \end{aligned}$$

Case 2 $m > n$.

Define

$$\begin{aligned} &f(u) = 1, \\ &f(u_i) = i+1, && 1 \leq i \leq n, \\ &f(v) = -1, \\ &f(v_i) = -(i+1), && 1 \leq i \leq n, \\ &f(v_{n+2i-1}) = -(n+1+i), && 1 \leq i \leq \frac{m-n}{2} \text{ if } m-n \text{ is even or} \\ & && 1 \leq i \leq \frac{m-n-1}{2} \text{ if } m-n \text{ is odd,} \\ &f(v_{n+2i}) = n+i+3, && 1 \leq i \leq \frac{m-n}{2} \text{ if } m-n \text{ is even or} \\ & && 1 \leq i \leq \frac{m-n-1}{2} \text{ if } m-n \text{ is odd.} \end{aligned}$$

Then clearly f is a pair sum labeling. □

Theorem 3.2 *$P_m \cup K_{1,n}$ is a pair sum graph.*

Proof Let $u_1, u_2, \ldots u_m$ be the path P_m. Let $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$.

Case 1 $m = n$.

Define

$$f(u) = 1, \qquad 1 \le i \le m,$$
$$f(v) = -1,$$
$$f(v_i) = -2i, \qquad 1 \le i \le m,$$

Case 2 $n > m$.

Define

$$f(u_i) = i, \qquad 1 \le i \le m,$$
$$f(v) = -1,$$
$$f(v_i) = -2i, \qquad 1 \le i \le m-1,$$
$$f(v_{m+2i-1}) = 2m + i, \qquad 1 \le i \le \frac{n-m+1}{2} \text{ if } n-m \text{ is odd or } 1 \le i \le \frac{n-m}{2} \text{ if } n-m \text{ is even,}$$
$$f(v_{m+2i-2}) = -(2m + i - 2), \qquad 1 \le i \le \frac{n-m+1}{2} \text{ if } n-m \text{ is odd or } 1 \le i \le \frac{n-m}{2} + 1 \text{ if } n-m \text{ is even.}$$

Then f is a pair sum labeling. □

Theorem 3.3 *If $m = n$, then $C_m \cup C_n$ is a pair sum graph.*

Proof Let $u_1u_2, \ldots u_nu_1$ be the first copy of the cycle in $C_n \cup C_n$ and $v_1v_2 \ldots v_nv_1$ be the second copy of the cycle in $C_n \cup C_n$.

Case 1 $m = n = 4k$.

Define

$$f(u_i) = i, \qquad 1 \le i \le 2k-1,$$
$$f(u_{2k}) = 2k+1,$$
$$f(u_{2k+i}) = -i, \qquad 1 \le i \le 2k-1,$$
$$f(u_n) = -2k-1,$$
$$f(v_i) = 2k + 2i, \qquad 1 \le i \le 2k,$$
$$f(v_{2k+i}) = -2k - 2i, \qquad 1 \le i \le 2k.$$

Case 2 $m = n = 4k+2$.

Define

$$f(u_i) = i, \qquad 1 \le i \le 2k+1,$$
$$f(u_{2k+1+i}) = -i, \qquad 1 \le i \le 2k+1,$$
$$f(v_i) = 2k + 2i, \qquad 1 \le i \le 2k+1,$$
$$f(v_{2k+1+i}) = -2k - 2i, \qquad 1 \le i \le 2k+1.$$

Case 3 $m = n = 2k + 1$.

Assigning $-i$ to u_i and i to v_i, we get a pair sum labeling. □

Remark. *mG denotes the union of m copies of G.*

Theorem 3.4 *If $n \leq 4$, then mK_n is a pair sum graph.*

Proof If $n = 1$, the result is obvious.

Case 1 $n = 2$.

Assign the label i and $i + 1$ to the vertices of i^{th} copy of K_2 for all odd i. For even values of i, label the vertices of the i^{th} copy of K_2 by $-i + 1$ and $-i$.

Case 2 $n = 3$.

Subcase 1 m is even.

Label the vertices of first $\frac{m}{2}$ copies by $3i - 2, 3i - 1, 3i (1 \leq i \leq m/2)$. Remaining $\frac{m}{2}$ copies are labeled by $-3i + 2, -3i + 1, -3i$.

Subcase 2 m is odd.

Label the vertices of first $(m - 1)$ copies as in Subcase (a). In the last copy label the vertices by $\frac{3(m-1)}{2} + 1, \frac{-3(m-1)}{2} - 2, \frac{3(m-1)}{2} + 3$ respectively.

Case 3 $n = 4$.

Subcase 1 m is even.

Label the vertices of first $\frac{m}{2}$ copies by $4i - 3, 4i - 2, 4i - 1, 4i$ $(1 \leq i \leq \frac{m}{2})$. Remaining $\frac{m}{2}$ copies are labeled by $-4i + 3, -4i + 2, -4i + 1, -4i$.

Subcase 2 m is odd.

Label the vertices of first $(m - 1)$ copies as in Sub case (a). In the last copy label the vertices by $-2m, 2m + 1, 2m + 2$ and $-2m - 3$ respectively. □

Theorem 3.5 *If $n \geq 9$, then mK_n is not a pair sum graph.*

Proof Suppose mK_n is a pair sum graph. By Theorem 2.7, we know that $\frac{mn(n-1)}{2} \leq 4mn - 2$, i.e., $mn(n-1) \leq 8mn - 4$. That is $8mn - mn^2 + mn - 4 \geq 0$. Whence, $9mn(9-n) - 4 \geq 0$, a contradiction. □

§4. Pair Sum Labeling on Standard Graphs

Theorem 4.1 *Any ladder L_n is a pair sum graph.*

Proof Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n - 1\}$.

Case 1 n is odd.

Let $n = 2m+1$. Define $f : V(L_n) \to \{\pm 1, \pm 2, \ldots, \pm (4m+2)\}$ by

$$
\begin{aligned}
&f(u_i) = -4(m+1) + 2i, && 1 \le i \le m,\\
&f(u_{m+1}) = -(2m+1), &&\\
&f(u_{m+1+i}) = 2m + 2i + 2, && 1 \le i \le m,\\
&f(v_i) = -4m - 3 + 2i, && 1 \le i \le m,\\
&f(v_{m+1}) = 2m + 2 &&\\
&f(v_{m+1+i}) = 2m + 2i + 1, && 1 \le i \le m.
\end{aligned}
$$

Case 2 n is even.

Let $n = 2m$. Define $f : V(L_n) \to \{\pm 1, \pm 2, \ldots, \pm (4m+2)\}$ by

$$
\begin{aligned}
&f(u_{m+1-i}) = -2i, && 1 \le i \le m,\\
&f(u_{m+i}) = 2i - 1, && 1 \le i \le m,\\
&f(u_{m+i}) = 2i, && 1 \le i \le m,\\
&f(u_{m+1-i}) = -(2i-1), && 1 \le i \le m.
\end{aligned}
$$

Then obviously f is a pair sum labeling. □

Notation 4.2 We denote the vertices and edges of the Quadrilateral Snake Q_n as follows:

$$V(Q_n) = \{u_i, v_j, w_j : 1 \le i \le n+1, 1 \le j \le n\}$$
$$E(Q_n) = \{u_i v_i, v_i w_i, u_i u_{i+1}, u_{i+1} w_i : 1 \le i \le n\}.$$

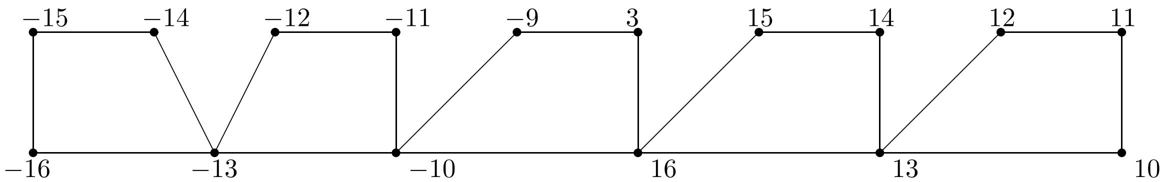
Theorem 4.3 *The quadrilateral snake Q_n is a pair sum graph if n is odd.*

Proof Let $n = 2m+1$. Define $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm (6m+4)\}$ by

$$
\begin{aligned}
&f(u_i) = -3n + 3i - 4, && 1 \le i \le m+1,\\
&f(u_{m+i}) = 3n - 3i + 4, && 1 \le i \le m+1,\\
&f(v_i) = -3n + 3i - 3, && 1 \le i \le m+1,\\
&f(v_{m+1+i}) = 3n - 3i + 3, && 1 \le i \le m,\\
&f(w_i) = -3n + 3i - 2, && 1 \le i \le m,\\
&f(w_{m+1}) = 3, &&\\
&f(w_{m+i+1}) = 3n - 3i + 2, && 1 \le i \le m,
\end{aligned}
$$

Then f is a pair sum labeling. □

Example 4.4 A pair sum labeling of Q_5 is shown in the following figure.



Notation 4.5 We denote the vertices and edges of the triangular snake T_n as follows:

$$V(T_n) = \{u_i, v_j : 1 \leq i \leq n+1, 1 \leq j \leq n\},$$
$$E(T_n) = \{u_i u_{i+1}, u_i v_j, v_i v_{j+1} : 1 \leq i \leq n, 1 \leq j \leq n-1\}.$$

Theorem 4.6 *Any triangular snake T_n is a pair sum graph.*

Proof The proof is divided into three cases following.

Case 1 $n = 4m-1$.

Define

$$\begin{aligned} &f(u_i) = 2i-1, && 1 \leq i \leq 2m, \\ &f(u_{2m+i}) = -2i+1, && 1 \leq i \leq 2m, \\ &f(v_i) = 2i, && 1 \leq i \leq 2m-1, \\ &f(v_{2m}) = -8m+3, && \\ &f(v_{2m+i}) = -2i, && 1 \leq i \leq 2m-1. \end{aligned}$$

Case 2 $n = 4m+1$.

Define

$$\begin{aligned} &f(u_i) = -8m-3+2(i-1), && 1 \leq i \leq 2m+1, \\ &f(u_{2m+1+i}) = 8m+3-2(i-1), && 1 \leq i \leq 2m+1, \\ &f(v_i) = -2+2(i-1), && 1 \leq i \leq 2m, \\ &f(v_{2m+1}) = 3, && \\ &f(v_{2m+i+1}) = 8m+2-2(i-1), && 1 \leq i \leq 2m. \end{aligned}$$

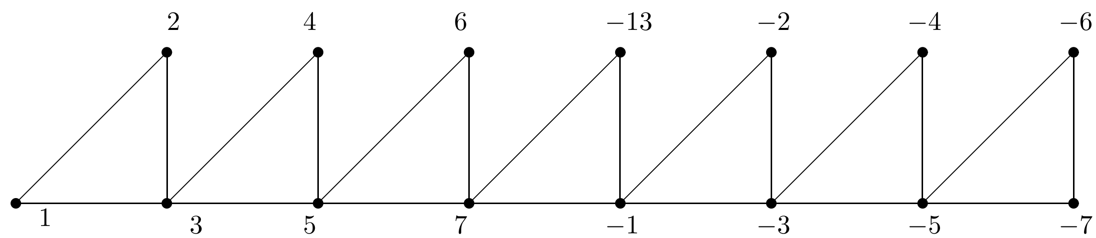
Case 3 $n = 2m$.

Define

$$\begin{aligned}
&f(u_{m+1}) = 1, \\
&f(u_{m+1+i}) = 2i, && 1 \le i \le m, \\
&f(u_{m+1-i}) = -2i, && 1 \le i \le m, \\
&f(v_m) = 3, \\
&f(v_{m+1}) = -5, \\
&f(v_{m+1+i}) = 5 + 2i, && 1 \le i \le m-1, \\
&f(v_{m-i}) = -(5 + 2i), && 1 \le i \le m-1.
\end{aligned}$$

Clearly f is a pair sum labeling. □

Example 4.7 A pair sum labeling of T_7 is shown in the following figure.



Theorem 4.8 *The crown $C_n \odot K_1$ is a pair sum graph.*

Proof Let C_n be the cycle given by $u_1u_2, \ldots, u_nu_1$ and let $v_1, v_2, \ldots, v_n$ be the pendent vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

Case 1 n is even.

Subcase (a) $n = 4m$.

Define

$$\begin{aligned}
&f(u_i) = 2i - 1, && 1 \le i \le 2m, \\
&f(u_{2m+i}) = -2i + 1, && 1 \le i \le 2m, \\
&f(v_i) = 4m + (2i - 1), && 1 \le i \le 2m, \\
&f(v_{2m+i}) = -(4m + 2i - 1), && 1 \le i \le 2m, .
\end{aligned}$$

Subcase (b) $n = 4m + 2$.

Define

$$f(u_i) = i, \qquad 1 \le i \le 2m+1,$$
$$f(u_{2m+1+i}) = -i, \qquad 1 \le i \le 2m+1,$$
$$f(v_i) = 4m+i, \qquad 1 \le i \le 2m+1,$$
$$f(v_{2m+1+i}) = -(4m+i), \qquad 1 \le i \le 2m+1.$$

obviously f is a pair sum labeling.

Case 2 $n = 2m+1$.

Define

$$f(u_1) = m-1,$$
$$f(u_i) = 2m+2i+1, \qquad 2 \le i \le m+1,$$
$$f(u_{m+1+i}) = -(2m+2i+1), \qquad 1 \le i \le m,$$
$$f(v_1) = -3m+3,$$
$$f(v_i) = f(u_i)+1, \qquad 2 \le i \le m+1,$$
$$f(v_{m+1+i}) = f(u_{m+1+i})-1, \qquad 1 \le i \le m.$$

Clearly f is a pair sum labeling. □

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Weierstrass Formula for Minimal Surface in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

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Abstract: In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.

Key Words: Weierstrass representation, Kenmotsu manifold, minimal surface.

AMS(2010): 53C41, 53A10

§1. Introduction

The study of minimal surfaces played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds.

Minimal surface, such as soap film, has zero curvature at every point. It has attracted the attention for both mathematicians and natural scientists for different reasons. Mathematicians are interested in studying minimal surfaces that have certain properties, such as completedness and finite total curvature, while scientists are more inclined to periodic minimal surfaces observed in crystals or biosystems such as lipid bilayers.

Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [8], statistical physics [14], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3, 4] have

[1]Received September 8, 2010. Accepted December 18, 2010.

made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [12] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of 2+1 dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterized by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.

§2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [2]

$$\phi\xi=0,\quad \eta(\xi)=1,\quad \eta(\phi X)=0, \tag{2.1}$$
$$\phi^2(X)=-X+\eta(X)\xi, \tag{2.2}$$
$$g(X,\xi)=\eta(X), \tag{2.3}$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X\phi)Y=-\eta(Y)\phi(X)-g(X,\phi Y)\xi,\qquad X,Y\in\chi(M), \tag{2.5}$$

$$\nabla_X\xi=X-\eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [2].

In Kenmotsu manifolds the following relations hold [2]:

$$(\nabla_X\eta)Y=g(\phi X,\phi Y), \tag{2.7}$$
$$\eta(R(X,Y)Z)=\eta(Y)g(X,Z)-\eta(X)g(Y,Z), \tag{2.8}$$
$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{2.9}$$
$$R(\xi,X)Y=\eta(Y)X-g(X,Y)\xi, \tag{2.10}$$
$$R(\xi,X)\xi=X-\eta(X)\xi, \tag{2.11}$$
$$S(\phi X,\phi Y)=S(X,Y)+2n\eta(X)\eta(Y), \tag{2.12}$$
$$S(X,\xi)=-2n\eta(X), \tag{2.13}$$
$$(\nabla_X R)(X,Y)\xi=g(Z,X)Y-g(Z,Y)X-R(X,Y)Z, \tag{2.14}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. In a Riemannian manifold we also have

$$g\left(R\left(W,X\right)Y,Z\right)+g\left(R\left(W,X\right)Z,Y\right)=0, \tag{2.15}$$

for every vector fields X,Y,Z.

§3. Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

Definition 3.1 *The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies*

$$\left(\nabla_X S\right)\left(\phi Y,\phi Z\right)=0.$$

The notion of Ricci $\eta-$parallelity for Sasakian manifolds was introduced by M. Kon [13].

We consider the three-dimensional manifold

$$\mathbb{K}=\left\{(x,y,z)\in\mathbb{R}^3:(x,y,z)\neq(0,0,0)\right\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1=x^3\frac{\partial}{\partial x^1},\quad \mathbf{e}_2=x^3\frac{\partial}{\partial x^2},\quad \mathbf{e}_3=-x^3\frac{\partial}{\partial x^3} \tag{3.1}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g\left(\mathbf{e}_1,\mathbf{e}_1\right)&=g\left(\mathbf{e}_2,\mathbf{e}_2\right)=g\left(\mathbf{e}_3,\mathbf{e}_3\right)=1, \\ g\left(\mathbf{e}_1,\mathbf{e}_2\right)&=g\left(\mathbf{e}_2,\mathbf{e}_3\right)=g\left(\mathbf{e}_1,\mathbf{e}_3\right)=0. \end{aligned} \tag{3.2}$$

The characterizing properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{3.3}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{ for any }Z\in\chi(M).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{3.4}$$

$$\phi^2(Z)=-Z+\ \eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g\left(\phi Z,\phi W\right)=g\left(Z,W\right)-\eta(Z)\eta(W), \tag{3.6}$$

for any $Z,W\in\chi(M)$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on $\mathbb{M}$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= 0,\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= 0,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= 0,\ \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0,\ \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{3.7}$$

§4. Minimal Surfaces in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

In this section, we obtain an integral representation formula for minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor.

We will denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also, we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{4.1}$$

For $X \in \chi(\mathbb{K})$, denote by $\mathsf{ad}(X)^*$ the adjoint operator of $\mathsf{ad}(X)$, i.e., it satisfies the equation

$$g([X,Y], Z) = g(Y, \mathsf{ad}(X)^*(Z)), \tag{4.2}$$

for any $Y, Z \in \chi(\mathbb{K})$. Let U be the symmetric bilinear operator on $\chi(\mathbb{M})$ defined by

$$U(X,Y) := \frac{1}{2}\{\mathsf{ad}(X)^*(Y) + \mathsf{ad}(Y)^*(X)\}. \tag{4.3}$$

Lemma 4.1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the orthonormal basis for an orthonormal basis for $\chi(\mathbb{K})$ defined in (3.1). Then,*

$$\begin{aligned} U(\mathbf{e}_1,\mathbf{e}_1) &= \mathbf{e}_3,\ U(\mathbf{e}_1,\mathbf{e}_3) = -\frac{1}{2}\mathbf{e}_1, \\ U(\mathbf{e}_2,\mathbf{e}_2) &= \mathbf{e}_3,\ U(\mathbf{e}_2,\mathbf{e}_3) = -\frac{1}{2}\mathbf{e}_2, \\ U(\mathbf{e}_1,\mathbf{e}_2) &= U(\mathbf{e}_3,\mathbf{e}_3) = 0. \end{aligned} \tag{4.4}$$

Proof Using (4.2) and (4.3), we have

$$2g(U(X,Y), Z) = g([X,Z], Y) + g([Y,Z], X).$$

Thus, direct computations lead to the table of U above. Lemma 4.1 is proved. □

Lemma 4.2(see [10]) *Let D be a simply connected domain. A smooth map $\varphi : D \longrightarrow \mathbb{K}$ is harmonic if and only if*

$$\left(\varphi^{-1}\varphi_u\right)_u + \left(\varphi^{-1}\varphi_v\right)_v - \mathsf{ad}\left(\varphi^{-1}\varphi_u\right)^*\left(\varphi^{-1}\varphi_u\right) - \mathsf{ad}\left(\varphi^{-1}\varphi_v\right)^*\left(\varphi^{-1}\varphi_v\right) = 0 \tag{4.5}$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, $\bar{z}$, the harmonic map equation (4.5) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \tag{4.6}$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then, (4.6) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U\left(A, \bar{A}\right). \tag{4.7}$$

The Maurer–Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = \left[A, \bar{A}\right]. \tag{4.8}$$

(4.7) and (4.8) can be combined to a single equation

$$A_{\bar{z}} = U\left(A, \bar{A}\right) + \frac{1}{2}\left[A, \bar{A}\right]. \tag{4.9}$$

(4.9) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Let $D(z, \bar{z})$ be a simply connected domain and $\varphi : D \longrightarrow \mathbb{K}$ a smooth map. If we write $\varphi(z) = \left(x^1(z), x^2(z), x^3(z)\right)$, then by direct calculation

$$A = \left(x^3\right)^{-1}\left(x_z^1\mathbf{e}_1 + x_z^2\mathbf{e}_2 + x_z^3\mathbf{e}_3\right). \tag{4.10}$$

It follows from the harmonic map equation (4.7) that

Theorem 4.3 *$\varphi : D \longrightarrow \mathbb{K}$ is harmonic if and only if the following equations hold:*

$$x_z^1 x_{\bar{z}}^3 + x_z^3 x_{\bar{z}}^1 = \left(x^3\right)^{-1} x_{\bar{z}}^3 x_z^1 - 2x_{z\bar{z}}^1 + \left(x^3\right)^{-1} x_z^3 x_{\bar{z}}^1, \tag{4.11}$$

$$x_z^2 x_{\bar{z}}^3 + x_z^3 x_{\bar{z}}^2 = \left(x^3\right)^{-1} x_{\bar{z}}^3 x_z^2 - 2x_{z\bar{z}}^2 + \left(x^3\right)^{-1} x_z^3 x_{\bar{z}}^2, \tag{4.12}$$

$$2\left(x_z^1 x_{\bar{z}}^1 + x_z^2 x_{\bar{z}}^2\right) = \left(x^3\right)^{-1} x_{\bar{z}}^3 x_z^3 - 2x_{z\bar{z}}^3 + \left(x^3\right)^{-1} x_z^3 x_{\bar{z}}^3. \tag{4.13}$$

Proof From (4.10), we have

$$\bar{A} = \left(x^3\right)^{-1}\left(x_{\bar{z}}^1\mathbf{e}_1 + x_{\bar{z}}^2\mathbf{e}_2 + x_{\bar{z}}^3\mathbf{e}_3\right) \tag{4.14}$$

Using (4.10) and (4.14), we obtain

$$\begin{aligned} U\left(A, \bar{A}\right) &= \left(x^3\right)^{-1}[-\left(\frac{1}{2}x_z^1 x_{\bar{z}}^3 + \frac{1}{2}x_z^3 x_{\bar{z}}^1\right)\mathbf{e}_1 - \left(\frac{1}{2}x_z^2 x_{\bar{z}}^3 + \frac{1}{2}x_z^3 x_{\bar{z}}^2\right)\mathbf{e}_2 \\ &\quad + \left(x_z^1 x_{\bar{z}}^1 + x_z^2 x_{\bar{z}}^2\right)\mathbf{e}_3]. \end{aligned} \tag{4.15}$$

On the other hand, we have

$$A_{\bar{z}} = -\left(x^3\right)^{-2} x^3_{\bar{z}} \left(x^1_z \mathbf{e}_1 + x^2_z \mathbf{e}_2 + x^3_z \mathbf{e}_3\right) + \left(x^3\right)^{-1} \left(x^1_{z\bar{z}} \mathbf{e}_1 + x^2_{z\bar{z}} \mathbf{e}_2 + x^3_{z\bar{z}} \mathbf{e}_3\right), \tag{4.16}$$

$$\bar{A}_z = -\left(x^3\right)^{-2} x^3_z \left(x^1_{\bar{z}} \mathbf{e}_1 + x^2_{\bar{z}} \mathbf{e}_2 + x^3_{\bar{z}} \mathbf{e}_3\right) + \left(x^3\right)^{-1} \left(x^1_{z\bar{z}} \mathbf{e}_1 + x^2_{z\bar{z}} \mathbf{e}_2 + x^3_{z\bar{z}} \mathbf{e}_3\right). \tag{4.17}$$

By direct computation, we obtain

$$\begin{aligned} A_{\bar{z}} \;\; = \;\; & \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^1_z + \left(x^3\right)^{-1} x^1_{z\bar{z}}\right) \mathbf{e}_1 + \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^2_z + \left(x^3\right)^{-1} x^2_{z\bar{z}}\right) \mathbf{e}_2 \\ & + \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^3_z + \left(x^3\right)^{-1} x^3_{z\bar{z}}\right) \mathbf{e}_3, \end{aligned} \tag{4.18}$$

$$\begin{aligned} \bar{A}_z \;\; = \;\; & \left(-\left(x^3\right)^{-2} x^3_z x^1_{\bar{z}} + \left(x^3\right)^{-1} x^1_{z\bar{z}}\right) \mathbf{e}_1 + \left(-\left(x^3\right)^{-2} x^3_z x^2_{\bar{z}} + \left(x^3\right)^{-1} x^2_{z\bar{z}}\right) \mathbf{e}_2 \\ & + \left(-\left(x^3\right)^{-2} x^3_z x^3_{\bar{z}} + \left(x^3\right)^{-1} x^3_{z\bar{z}}\right) \mathbf{e}_3. \end{aligned} \tag{4.19}$$

Hence, using (4.7) we obtain (4.11)-(4.13). This completes the proof of the Theorem. □

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z} dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}, \tag{4.20}$$

with respect to the conformal structure of D. Let

$$\wp^1 = \left(x^3\right)^{-1} x^1_z dz, \quad \wp^2 = \left(x^3\right)^{-1} x^2_z dz, \;\; \wp^3 = \left(x^3\right)^{-1} x^3_z dz. \tag{4.21}$$

Theorem 4.4 *The triplet* $\{\wp^1, \wp^2, \wp^3\}$ *of* $(1,0)$*-forms satisfies the following differential system:*

$$\bar{\partial}\wp^1 = -x^3 \left(\wp^1 \wedge \overline{\wp^3} + \wp^3 \wedge \overline{\wp^1}\right), \tag{4.22}$$

$$\bar{\partial}\wp^2 = -x^3 \left(\wp^2 \wedge \overline{\wp^3} + \wp^3 \wedge \overline{\wp^2}\right), \tag{4.23}$$

$$\bar{\partial}\wp^3 = 2x^3 \left(\wp^1 \wedge \overline{\wp^1} + \wp^2 \wedge \overline{\wp^2}\right). \tag{4.24}$$

Proof From (4.11)-(4.13), we have (4.22)-(4.24). Thus proof is complete. □

Theorem 4.5 *Let* $\{\wp^1, \wp^2, \wp^3\}$ *be a solution to (4.22)-(4.24) on a simply connected coordinate region D. Then*

$$\varphi\left(z, \bar{z}\right) = 2Re \int_{z_0}^{z} \left(x^3 \wp^1, x^3 \wp^2, x^3 \wp^3\right) \tag{4.25}$$

is a harmonic map into $\mathbb{K}$.

Conversely, any harmonic map of D into $\mathbb{K}$ *can be represented in this form.*

Proof By theorem 4.3, we see that $\varphi\left(z, \bar{z}\right)$ is a harmonic curve if and only if $\varphi\left(z, \bar{z}\right)$ satisfy (4.11)-(4.13). From (4.21), we have

$$\begin{aligned} x^1(z,\bar{z}) &= 2Re\int_{z_0}^{z} x^3\wp^1,\ x^2(z,\bar{z}) = 2Re\int_{z_0}^{z} x^3\wp^2, \\ x^3(z,\bar{z}) &= 2Re\int_{z_0}^{z} x^3\wp^3, \end{aligned}$$

which proves the theorem. □

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Math.Combin.Book Ser. Vol.4 (2010), 70-78

On the Basis Number of the Wreath Product of Wheels with Stars

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Abstract: A cycle basis $\mathcal{B}$ of G is called a *Smarandachely* (k,d)*-fold* for integers $, d, k,\ d-k \geq 0$ if each edge of G occurs in at least k and at most d of the cycles in $\mathcal{B}$. Particularly, a Smarandachely $(0,d)$-fold basis is abbreviated to a d-fold basis. The basis number of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. In this work, the basis number for the wreath product of wheels with stars is investigated.

Key Words: Wreath product, Smarandachely (k,d)-fold basis, basis number.

AMS(2010): 05C38, 05C75

§1. Introduction

For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X \backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \varnothing$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number,* $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a *Smarandachely* (k,d)*-fold* for integers $, d, k,\ d-k \geq 0$ if each edge of G occurs in at least k and at most d of the cycles in $\mathcal{B}$. Particularly, a Smarandachely $(0,d)$-fold basis is abbreviated to a d-fold basis. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The first important use of the basis number goes bask to 1937 when MacLane proved the following result (see [17]):

Theorem 1.1 (MacLane) *The graph G is planar if and only if $b(G) \leq 2$.*

Later on, Schmeichel [17] proved the existence of graphs that have arbitrary large basis number. In fact he proved that for any integer r there exists a graph with basis greater than

[1]Received September 8, 2010. Accepted December 20, 2010.

or equal to r. Also, he proved that for $n \geq 5$, $b(K_n) = 3$ where K_n is the complete graph of n vertices. There after, Banks and Schmeichel [8] proved that $b(Q_n) = 4$ where Q_n is the n-cube. For the completeness, it should be mentioned that a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$. The following result will be used frequently in the sequel [15]:

Lemma 1.2 (Jaradat, et al.) *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and that $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \varnothing$). Then $A \cup B$ is linearly independent.*

From 1982 more attention has given to address the problem of finding the basis number in graph products. In the literature there are a lot of graph products. In fact, there are more than 256 different kind of products, we mention out of these product the most common ones, The Cartesian, the direct, the strong the lexicographic, semi-composite and the wreath product. The first four of the above products were extensively studied by many authors, we refer the reader to the following articles and references cited there in: [2], [4], [5], [6], [7], [9], [10], [11], [13], [14], [15] and [16]. In contrast to the first four products, a very little is known about the basis number of the wreath products, ρ, of graphs. Schmeichel [18] proved that $b(P_2 \rho N_m) \leq 4$. Ali [1] proved that $b(K_n \rho N_m) \leq 9$. Al-Qeyyam and Jaradat [3] proved that $b(S_n \rho P_m), b(S_n \rho S_m) \leq 4$. In this paper, we investigate the basis number of the wreath product of wheel graphs with stars.

For completeness we give the definition of the following products: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The Cartesian product $G \Box H$ has the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$. (2) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$. (3) The wreath product $G \rho H$ has the vertex set $V(G \rho H) = V(G) \times V(H)$ and the edge set $E(G \rho H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$.

In the rest of this paper, we let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of the wheel W_n(the star S_n), with $d_{W_n}(u_1) = n - 1$ ($d_{S_n}(u_1) = n - 1$) and $\{v_1, v_2, \ldots, v_m\}$ be the vertex set S_m with $d_{S_m}(v_1) = m - 1$. Also, $C_{n-1} = u_2u_3 \ldots u_nu_2$ and N_{m-1} is the null graph with vertex set $\{v_2, v_3, \ldots, v_m\}$. Wherever they appear a, b, c, d and l stand for vertices. Also, $f_B(e)$ stands for the number of elements of B containing the edge e, and $E(B) = \cup_{C \in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

§2. The Basis Number of $W_n \rho S_m$

Throughout this work, we set the following sets of cycles:

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_i)(a, v_{j+1})(b, v_{i+1})(a, v_j) \mid 2 \leq i, j \leq m - 1\},$$

$$\mathcal{E}_{cab} = \left\{\mathcal{E}_{cab}^{(j)} = (c, v_2)(a, v_j)(b, v_m)(a, v_{j+1})(c, v_2) \mid 2 \leq j \leq m - 1\right\},$$

$$\mathcal{G}_{ab} = \left\{\mathcal{G}_{ab}^{(j)} = (a, v_1)(a, v_j)(b, v_2)(a, v_{j+1})(a, v_1) \mid 2 \leq j \leq m-1\right\},$$

$$\mathcal{W}_{cab} = \{(c, v_1)(c, v_2)(a, v_2)(b, v_m)(b, v_1)(a, v_1)(c, v_1)\},$$

and

$$\mathcal{S}_{ab} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\}.$$

Note that $\mathcal{H}_{ab}$ is the Schemeichel's 4-fold basis of $\mathcal{C}(ab\rho N_{m-1})$ (see Theorem 2.4 in [18]). Moreover, (1) if $e = (a, v_2)(b, v_m)$ or $e = (a, v_m)(b, v_2)$ or $e = (a, v_2)(b, v_2)$ or $e = (a, v_m)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$; (2) if $e = (a, v_2)(b, v_i)$ or $(a, v_j)(b, v_2)$ or $(a, v_m)(b, v_i)$ or $(a, v_j)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) \leq 2$; and (3) if $e \in E(ab\rho N_{m-1})$ and it is not of the above forms, then $f_{\mathcal{H}_{ab}}(e) \leq 4$.

Lemma 2.1 *Every linear combination of cycles of $\mathcal{E}_{cab}$ contains at least one edge of the form $(b, v_i)(a, v_m)$ where $2 \leq i \leq m$ and at least one edge of the form $(c, v_2)(b, v_i)$ where $2 \leq i \leq m$.*

Proof Note that $E(\mathcal{E}_{cab}) \subseteq \{(b, v_m)(a, v_j), (c, v_2)(a, v_j) | j = 2, 3, \ldots, m\}$. Since each of $\{(b, v_m)(a, v_j) | j = 2, 3, \ldots, m\}$ and $\{(c, v_2)(a, v_j) | j = 2, 3, \ldots, m\}$ forms an edge set of a star and since any linear combination of cycles is a cycle or an edge disjoint union of cycles, any linear combination of cycles of $\mathcal{E}_{cab}$ must contains at least one edge of $\{(b, v_m)(a, v_j) | j = 2, 3, \ldots, m\}$ and at least one edge of $\{(c, v_2)(a, v_j) | j = 2, 3, \ldots, m\}$. □

Using the same argument as in Lemma 2.1, we get the following result.

Lemma 2.2 *Every linear combination of cycles of $\mathcal{G}_{ab}$ contains at least one edge of the form $(a, v_1)(a, v_i)$ where $2 \leq i \leq m$ and at least one of the form $(b, v_2)(a, v_i)$ where $2 \leq i \leq m$.* □

We now set the following cycles:

$$\mathcal{U}_{lab}^{(j)} = (l, v_j)(a, v_j)(b, v_j)(l, v_j), j = 1, 2, \ldots, m$$
$$\mathcal{U}_{ab} = (a, v_1)(a, v_2)(b, v_m)(b, v_1)(a, v_1).$$

Let

$$\mathcal{O}_{labc} = \mathcal{H}_{bc} \cup \mathcal{G}_{cb} \cup \{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{E}_{cba}$$

Lemma 2.3 *$\mathcal{O}_{labc}$ is linearly independent.*

Proof Note that $\mathcal{H}_{bc}$ is isomorphic to the Schemeichel's 4-fold basis of $bc\rho N_{m-1}$. Thus, $\mathcal{H}_{bc}$ is a linearly independent set. By Lemma 3.2.3 of [3], each of $\mathcal{G}_{cb}$, and $\mathcal{E}_{cba}$ is linearly independent. The cycle $\mathcal{U}_{bc}$ contains the edge $(b, v_2)(c, v_m)$ which does not occur in $\mathcal{U}_{lbc}^{(1)}$, thus $\{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\}$ is linearly independent. It is easy to see that any linear combination of cycles of $\{\mathcal{U}_{lbc}^{(1)}, \mathcal{U}_{bc}\}$ contains either $(b, v_1)(c, v_1)$ or $(l, v_1)(b, v_1)$ and non of them occurs in any cycle of $\mathcal{H}_{bc}$, thus $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc}$ is linearly independent. By Lemma 2.1, any linear combination of cycles of $\mathcal{E}_{cba}$ contains an edge of the form $(b, v_i)(a, v_m)$ for $2 \leq i \leq m$ which does not occur in any cycle of $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc}$. Thus $\{\mathcal{U}_{l,bc}^{(1)}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc} \cup \mathcal{E}_{cba}$ is linearly independent. Finally, by Lemma 2.2, any linear combination of cycles of $\mathcal{G}_{cb}$ contains an edge of the form $(c, v_1)(c, v_i)$

for $2 \leq i < m$ which appears in no cycle of $\{\mathcal{U}^{(1)}_{lbc}, \mathcal{U}_{bc}\} \cup \mathcal{H}_{bc} \cup \mathcal{E}_{cba}$. Thus $\mathcal{O}_{labc}$ is linearly independent. □

The coming result follows from being that

$$E(\mathcal{O}_{labc}) - \{(b, v_i)(c, v_j) | 2 \leq i, j \leq m\} \cup \{(b, v_1)(c, v_1)\}$$

forms and edge set of a tree and the fact that any linear combination of an independent set of cycles is a cycle or an edge disjoint union of cycles.

Lemma 2.4 *Any linear combination of cycles of $\mathcal{O}_{labc}$ must contain an edge of $\{(b, v_i)(c, v_j)|$*

$2 \leq i, j \leq m\} \cup \{(b, v_1)(c, v_1)\}$. □

Now, we let

$$\mathcal{O}^*_{lbc} = \mathcal{H}_{bc} \cup \mathcal{G}_{cb} \cup \{\mathcal{U}^{(1)}_{lbc}, \mathcal{U}_{bc}\} \text{ and } \mathcal{O}^{\circledast}_{labcd} = \mathcal{O}_{labc} \cup \mathcal{E}_{dcb}.$$

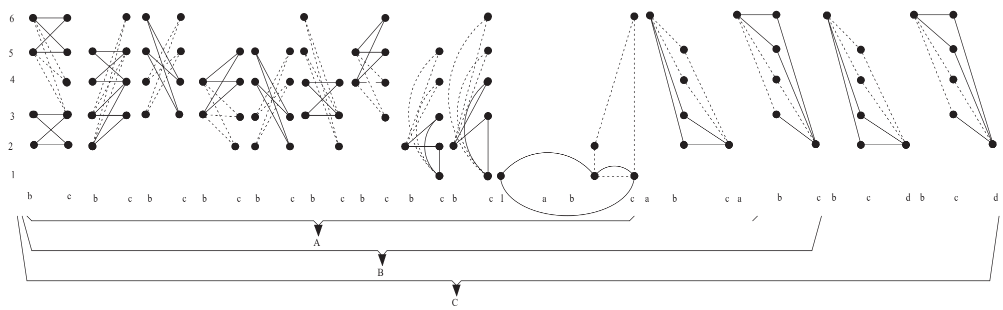

Figure 1: A represents the cycles of $\mathcal{O}^*_{lbc}$, B represents the cycles of $\mathcal{O}_{labc}$ and C represents the cycles of $\mathcal{O}^{\circledast}_{labcd}$ for $m = 6$.

Remark 2.5 Let $e \in E(\mathcal{O}^{\circledast}_{labcd})$. From the definitions of $\mathcal{O}^*_{lab}, \mathcal{O}_{labc}$ and $\mathcal{O}^{\circledast}_{labcd}$ and by the aid of Figure 1 below, one can easily see the following:

(1) If $e = (l, v_1)(b, v_1)$ or $(l, v_1)(c, v_1)$or $(b, v_1)(b, v_2)$ or $(c, v_1)(c, v_2)$ or $(c, v_1)(c, v_2)$, then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 1$. (2) If $e = (c, v_1)(c, v_j)$ such that $3 \leq j \leq m$ or $e = (b, v_1)(c, v_1)$, then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (3) If $e = (a, v_m)(b, v_j)$ such that $3 \leq j \leq m-1$, then $f_{\mathcal{O}^*_{lab}}(e) = 0$ and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (4) If $e = (c, v_j)(d, v_2)$ such that $3 \leq j \leq m-1$, then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = 0$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (5) If $e = (a, v_m)(b, v_m)$ or $(a, v_m)(b, v_2)$ or $(c, v_m)(d, v_2)$, then $f_{\mathcal{O}^*_{lab}}(e) = 0$and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 1$. (6) If $e = (b, v_2)(c, v_2)$, then $f_{\mathcal{O}^*_{lab}}(e) = 2$ and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 3$. (7) If $e = (b, v_m)(c, v_2)$, then $f_{\mathcal{O}^*_{lab}}(e) = 1$,$f_{\mathcal{O}_{labc}}(e) = 2$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 2$. (8) If $e = (b, v_j)(c, v_2)$ such that $3 \leq j \leq m-1$, then $f_{\mathcal{O}^*_{lab}}(e) = 2$and $f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 4$. (9) If $e = (b, v_m)(c, v_j)$ such that $3 \leq j \leq m-1$, then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = 2$ and $f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 4$. (10) If $e = (b, v_2)(c, v_m)$, then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) = 3$. (10) If $e = (b, v_j)(c, v_k)$ such that $2 \leq j, k \leq m$ and e is not as in (1)-(10), then $f_{\mathcal{O}^*_{lab}}(e) = f_{\mathcal{O}_{labc}}(e) = f_{\mathcal{O}^{\circledast}_{labcd}}(e) \leq 4$.

The automorphism group of S_m is isomorphic to the symmetric group on the set $\{v_2, v_3, \dots, v_m\}$ with $\alpha(v_1) = v_1$ for any $\alpha \in Aut(G)$. Therefore, for any two vertices v_i, v_j $(2 \le i, j \le m)$, there is $\alpha \in Aut(G)$ such that $\alpha(v_i) = v_j$. Hence, $W_n\rho S_m$ is decomposable into $S_n\rho S_m \cup C_{n-1}[N_{m-1}] \cup \{(u_j, v_1)(u_{j+1}, v_1)|2 \le j < n\} \cup \{(u_n, v_1)(u_2, v_1)\}$. Thus

$$|E(W_n\rho S_m)| = |E(S_n\rho S_m)| + (n-1)(m-1)^2 + (n-1) = |E(S_n\rho S_m)| + (n-1)(m^2 - 2m + 2).$$

Hence,

$$\dim \mathcal{C}(W_n\rho S_m) = \dim \mathcal{C}(S_n\rho S_m) + (n-1)(m^2 - 2m + 2).$$

By Theorem 3.2.5 of [3], we have

$$\dim \mathcal{C}(S_n\rho S_m) = (n-1)(m^2 - 2m + 1). \tag{1}$$

Therefore,

$$\dim \mathcal{C}(W_n\rho S_m) = (n-1)(2m^2 - 4m + 3). \tag{2}$$

Lemma 2.6 *The set $\mathcal{O} = \mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}) \cup \mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$ is linearly independent subset of $\mathcal{C}(W_n\rho S_m)$.*

Proof By Lemmas 2.3, $\mathcal{O}^*_{u_1u_2u_3}$ is linearly independent. Note that, $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3} = \mathcal{O}_{u_1,u_{n-1}u_nu_2} \cup \mathcal{E}_{u_1,u_3u_2u_n}$. By Lemma 2.1, any linear combination of cycles of $\mathcal{E}_{u_3u_2u_n}$ contains an edge of $\{(u_3, v_2)(u_2, v_j)|2 \le j \le m\}$ which is not in any cycle of $\mathcal{O}_{u_1u_{n-1}u_nu_2}$. Thus, $\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}$ is linearly independent. We now use mathematical induction on n to show that $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ is linearly independent. If $n = 4$, then $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} = \mathcal{O}_{u_1u_2u_3u_4}$. And so, the result is followed from Lemma 2.3. Assume that n is greater than 3 and it is true for less than n. Note that $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} = \cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}} \cup \mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$. By Lemma 2.3 and the inductive step, each of $\cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ and $\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$ is linearly independent. Since any linear combination of cycles of $\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}$ contains an edge of $\{(u_n, v_i)(u_{n-1}, v_j)|2 \le i, j \le m\} \cup \{(u_n, v_1)(u_{n-1}, v_1)\}$ (Lemma 2.4) which does not occur in any cycle of $\cup_{i=3}^{n-2} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$, $\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}$ is linearly independent. Also, since $E(\mathcal{O}^*_{u_1u_2u_3}) \cap E(\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}) = \{(u_3, v_1)(u_1, v_1), (u_3, v_1)(u_3, v_2)\} \cup \{(u_2, v_m)(u_3, v_i) \mid 2 \le i \le m\}$ which is an edge set of a tree, $\mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}})$ is linearly independent by Lemma 1.2. Similarly, $E(\mathcal{O}^*_{u_1u_2u_3} \cup (\cup_{i=3}^{n-1} \mathcal{O}_{u_1u_{i-1}u_iu_{i+1}})) \cap E(\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}) = \{(u_1, v_1)(u_2, v_1), (u_2, v_1)(u_2, v_2), (u_1, v_1)(u_n, v_1), (u_n, v_1)(u_n, v_2)\} \cup \{(u_{n-1}, v_m)(u_n, v_i), (u_3, v_2)(u_2, v_i) \mid 2 \le i \le m\}$ which is an edge set of a tree. Thus, $\mathcal{O}$ is linearly independent by Lemma 1.2. □

Now, let

$$\mathcal{L}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab} \text{ and } \mathcal{Y}_{cab} = \mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca} \cup \mathcal{W}_{cab}.$$

Theorem 2.7 *For any wheel W_n of order $n \ge 4$ and star S_m* of order $m \ge 3$,

$$3 \le b(W_n\rho S_m) \le 4.$$

Proof To prove the first inequality, we assume that $W_n\rho S_m$ has a 2-fold basis for $n \ge 4$ and $m \ge 3$, say $\mathcal{B}$. Since the girth of $W_n\rho S_m$ is 3, we have $4|\mathcal{B}| \le 3|E(W_n\rho S_m)|$. Hence,

$3(n-1)(2m^2-4m+3) \leq 2[(n-1)(2m^2-4m+3)+nm]$, which implies that $n(2m^2-6m+3)-2m^2+4m-3 \leq 0$. But $n \geq 3$, thus, $4(2m^2-6m+3)-2m^2+4m-3 \leq 0$, that is $m \leq \frac{20}{6}-\frac{9}{m}$. Therefore, $m < 4$.

To prove the second inequality, it is enough to exhibit a 4-fold basis. Define $\mathcal{B}(W_n\rho S_m) = B(S_n\rho S_m) \cup \mathcal{O}$ where $B(S_n\rho S_m) = (\cup_{i=2}^{n-1}\mathcal{Y}_{u_{i+1}u_1u_i}) \cup \mathcal{L}_{u_1u_2}$ is the cycle basis of $S_n\rho S_m$ (Theorem 3.2.5 of [3]). By Lemma 2.6 $\mathcal{O}$ is linearly independent. Since

$$E(\mathcal{B}(S_n\rho S_m)) \cap E(\mathcal{O}) = E((N_{n-1}\square S_m) \cup (S_n\square v_1)) \tag{3}$$

which is an edge set of a tree, $\mathcal{B}(W_n\rho S_m)$ is linearly independent by Lemma 1.2**.** Note that,

$$|\mathcal{H}_{ab}| = (m-2)^2 \text{ and } |\mathcal{G}_{ba}| = |\mathcal{E}_{cba}| = (m-2). \tag{4}$$

Thus by (4),

$$\begin{aligned} |\mathcal{O}^*_{u_1u_2u_3}| &= |\mathcal{O}^*_{lab}| = |\mathcal{H}_{ab}| + |\mathcal{G}_{ba}| + 2 \\ &= (m-2)^2 + (m-2) + 2 \\ &= m^2 - 3m + 4, \end{aligned} \tag{5}$$

and so

$$\begin{aligned} |\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}| &= |\mathcal{O}_{labc}| = |\mathcal{O}^*_{lab}| + |\mathcal{E}_{cba}| \\ &= m^2 - 3m + 4 + (m-2) \\ &= m^2 - 2m + 2, \end{aligned} \tag{6}$$

$$\begin{aligned} |\mathcal{O}^{\circledast}_{u_{n-1}u_nu_2u_3}| &= |\mathcal{O}^{\circledast}{}_{labcd}| = |\mathcal{O}_{labc}| + |\mathcal{E}_{dcb}| \\ &= m^2 - 2m + 2 + (m-2) \\ &= m^2 - m. \end{aligned} \tag{7}$$

Hence (5), (6) and (7), imply

$$\begin{aligned} |\mathcal{O}| &= |\mathcal{O}^*_{u_1u_2u_3}| + \sum_{i=3}^{n-1} |\mathcal{O}_{u_{i-1}u_iu_{i+1}}| + |\mathcal{O}^{\circledast}_{u_{n-1}u_nu_2u_3}| \\ &= m^2 - 3m + 4 + (n-3)(m^2 - 2m + 2) + m^2 - m \\ &= (n-1)(m^2 - 2m + 2). \end{aligned} \tag{8}$$

Thus (1), (2) and (8), give

$$\begin{aligned} |\mathcal{B}(W_n\rho S_m)| &= |\mathcal{B}(S_n\rho S_m)| + |\mathcal{O}| \\ &= (n-1)(m^2 - 2m + 1) + (n-1)(m^2 - 2m + 2). \\ &= (n-1)(2m^2 - 4m + 3) \\ &= \dim \mathcal{C}(W_n\rho S_m) \end{aligned}$$

Therefore, $\mathcal{B}(W_n\rho S_m)$ forms a basis for $\mathcal{C}(W_n\rho S_m)$. To this end, we show that $f_{\mathcal{B}(W_n\rho S_m)}(e) \leq 4$ for each edge $e \in E(W_n\rho S_m)$. To do that we count the number of cycles of $\mathcal{B}(W_n\rho S_m)$ that contain the edge e. To this end, and according to (3) we split our work into two cases.

Case 1. $e \in E(W_n \rho S_m) - E((N_{n-1} \square S_m) \cup (S_n \square v_1))$.

Then we have the following:

Subcase 1.1. $e \in E(u_2 u_3 \rho S_m) - (E(S_n \rho S_m) \cup E(C_{n-1} \square v_1))$. Then e occurs only in cycles of $\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3} \cup \mathcal{O}^{*}_{u_1 u_2 u_3} \cup \mathcal{O}_{u_1 u_2 u_3 u_4}$. By the help of Remark 2.5, we have the following: (1) If $e = (u_3, v_2)(u_2, v_j)$ such that $2 \leq j \leq m-1$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) + f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) \leq 2+2$. (2) If $e = (u_3, v_2)(u_2, v_m)$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) + f_{\mathcal{O}_{u_1 u_2 u_3 u_4}}(e) + f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) \leq 1+1+2$. (3) If $e = (u_2, v_m)(u_3, v_j)$ such that $3 \leq j \leq m$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) + f_{\mathcal{O}_{u_1 u_2 u_3 u_4}}(e) \leq 1+1+2$. (4) If $e = (u_2, v_j)(u_3, v_k)$ such that $2 \leq j, k \leq m$ and e is not as in in (1) or (2) or (3), then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) \leq 4$.

Subcase 1.2. $e \in E(u_i u_j \rho S_m) - (E(S_n \rho S_m) \cup E(C_{n-1} \square v_1))$ such that $3 \leq j \leq n-2$. Then e occurs only in cycles of $\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}} \cup \mathcal{O}_{u_1 u_i u_{i+1} u_{i+2}}$. By the help of Remark 3.5, we have the following: (1) If $e = (u_i, v_m)(u_{i+1}, v_j)$ such that $2 \leq j \leq m$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) + f_{\mathcal{O}_{u_1 u_i u_{i+1} u_{i+2}}}(e) \leq 2+2$. (2) If $e = (u_i, v_j)(u_{i+1}, v_k)$ such that $2 \leq j, k \leq m$ and e is not as in (1), then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) \leq 4$.

Subcase 1.3. $e \in E(u_{n-1} u_n \rho S_m) - (E(S_n \rho S_m) \cup E(C_{n-1} \square v_1))$. Then e occurs only in cycles of $\mathcal{O}_{u_1 u_{n-2} u_{n-1} u_n} \cup \mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}$. By the help of Remark 3.5, we have the following: (1) If $e = (u_{n-1}, v_m)(u_n, v_j)$ such that $2 \leq j \leq m$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) + f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) \leq 2+2$. (2) If $e = (u_i, v_j)(u_{i+1}, v_k)$ such that $2 \leq j, k \leq m$ and e is not as in (1), then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) \leq 4$.

Subcase 1.4. $e \in E(u_n u_2 \rho S_m) - (E(S_n \rho S_m) \cup E(C_{n-1} \square v_1))$. Then e occur only in cycles of $\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}$. By Remark 2.5, $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) \leq 4$.

Subcase 1.5. $e \in C_{n-1} \square v_1$. By Remark 2.5, we have the following: (1) If $e = (u_2, v_1)(u_3, v_1)$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) = 2$. (2) If $e = (u_i, v_1)(u_{i+1}, v_1)$ such that $3 \leq i \leq n-1$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) = 2$. (3) If $e = (u_2, v_1)(u_n, v_1)$, then $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) = 2$.

Subcase 1.6. $e \in (S_n \rho S_m) - E((N_{|V(S_n - \{u_1\})|} \square S_m) \cup (S_n \square v_1))$. Then e occurs only in cycles of $\mathcal{B}(S_n \rho S_m)$. Thus by Theorem 3.2.5 of [3], $f_{\mathcal{B}(W_n \rho S_m)}(e) \leq f_{\mathcal{B}(S_n \rho S_m)} \leq 4$.

Case 2. $e \in E((N_{|V(S_n - \{u_1\})|} \square S_m) \cup (S_n \square v_1))$.

Then by the aid of Remark 2.5 and Theorem 3.2.5 of [3] we have the following.

Subcase 2.1. $e \in \cup_{i=4}^{n} (u_i \square S_m)$.

Then e occurs only in cycles of $\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}} \cup \mathcal{B}(S_n \rho S_m)$. Thus, $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}_{u_1 u_{i-1} u_i u_{i+1}}}(e) + f_{\mathcal{B}(S_n \rho S_m)} \leq 2+2$.

Subcase 2.2. $e \in (u_3 \square S_m)$. Then e occurs only in cycles of $\mathcal{O}^{*}_{u_1 u_2 u_3} \cup \mathcal{B}(S_n \rho S_m)$. Thus, $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{*}_{u_1 u_2 u_3}}(e) + f_{\mathcal{B}(S_n \rho S_m)} \leq 2+2$.

Subcase 2.3. $e \in (u_2 \square S_m)$. Then e occurs only in cycles of $\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3} \cup \mathcal{B}(S_n \rho S_m)$. Thus, $f_{\mathcal{B}(W_n \rho S_m)}(e) = f_{\mathcal{O}^{\circledast}_{u_1 u_{n-1} u_n u_2 u_3}}(e) + f_{\mathcal{B}(S_n \rho S_m)} \leq 2+2$.

Subcase 2.4. $e \in S_n \square v_1$. Then we have the following: (1) If $e = (u_1, v_1)(u_2, v_1)$, then

$f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}^*_{u_1u_2u_3}}(e) + f_{\mathcal{O}_{u_1u_2u_3u_4}} + f_{\mathcal{B}(S_n\rho S_m)}$. (2) If $e = (u_1, v_1)(u_3, v_1)$, then $f_{\mathcal{O}^*_{u_1u_2u_3}}(e) + f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+2$. (3) If $e = (u_1, v_1)(u_i, v_1)$ such that $3 \leq i \leq n-1$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{i-1}u_iu_{i+1}}}(e) + f_{\mathcal{O}_{u_1u_iu_{i+1}u_{i+2}}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+2$. (4) If $e = (u_1, v_1)(u_n, v_1)$, then $f_{\mathcal{B}(W_n\rho S_m)}(e) = f_{\mathcal{O}_{u_1u_{n-2}u_{n-1}u_n}}(e) + f_{\mathcal{O}^{\circledast}_{u_1u_{n-1}u_nu_2u_3}}(e) + f_{\mathcal{B}(S_n\rho S_m)} \leq 1+1+1$. □

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Math.Combin.Book Ser. Vol.4 (2010), 79-85

Entire Semitotal-Point Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph. Then the semitotal-point graph is denoted by $T_2(G)$. Let the vertices and edges of G be the elements of G. An entire dominating set X of a graph $T_2(G)$ is an entire semitotal-point (ESP) dominating set if every element not in X is either adjacent or incident to at least one element in X. An entire ESP domination number $\varepsilon_{tp}(G)$ of G is the minimum cardinality of an ESP dominating set of G. In this paper many bounds on $\varepsilon_{tp}(G)$ are obtained in terms of elements of G. Also its relationship with other domination parameters are investigated.

Key Words: Smarandachely total graph, domination, entire domination, entire semitotal-point domination numbers.

AMS(2010): 05C69

§1. Introduction

The graphs considered here are finite, nontrivial, undirected, connected, without loops or multiple edges. The vertices and edges of a graph G are called the elements of G. In a graph G if $deg(v) = 1$, then v is called a pendant vertex of G. A dominating set X of a graph G is called an entire dominating set of G, if every element not in X is either adjacent or incident to at least one element in X. The entire domination number $\varepsilon(G)$ of G is the minimum cardinality of an entire dominating set of G. For an early survey on entire domination number,refer[5].For undefined terms or notations in this paper may be found in Harary [3].

Let D be a minimum dominating set in $G = (V, E)$. If $V - D$ contains a dominating set D' of G then D' is called an inverse dominating set with respect to D. *The inverse domination number* $\gamma^{-1}(G)$ of G is the cardinality of a smallest inverse dominating set of G.

Let S be the set of elements of a graph G and X be the minimum entire dominating set of G. If $S - X$ contains an entire dominating set say X' then X' is called inverse entire dominating set of G with respect to X. The *inverse entire domination number* $\varepsilon^{-1}(G)$ of G is the minimum number of elements in an inverse entire dominating set of G.

A dominating set D of a graph G, is a *split dominating set*, if the induced subgraph $\langle V - D\rangle$

[1]Supported by the University Grants Commission, New Delhi,India-No.F.4-3/2006(BSR)/7-101/2007(BSR), dated: September 2009 and No.F.FIP/11^{th} Plan/KAKA047TF02, dated: October 8, 2009.

[2]Received August 18, 2010. Accepted December 22, 2010.

is disconnected. The *split domination number* $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set [6].

A dominating set D of a graph G, is a *nonsplit dominating set*, if the induced subgraph $\langle V-D\rangle$ is connected. The *nonsplit domination number* $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set [7].

A dominating set D of a graph G, is a *connected dominating set*, if the induced subgraph $\langle D\rangle$ is connected. The *connected domination number* $\gamma_c(G)$ of G, is the minimum cardinality of a connected dominating set [10].

A set D of vertices of a graph G is a *maximal dominating set* of G, if D is a dominating set and $V-D$ is not a dominating set. The *maximal domination number* $\gamma_m(G)$ of G, is the minimum cardinality of a maximal dominating set [8].
Let $G=(V,E)$ be a graph and $W=W_1\cup W_2$ with $W_1\subset V$ and $W_2\subset E$. The *Smarandachely total graph* $T_{W_1,W_2}(G)$ is defined to be a graph $G^*=(V^*,E^*)$, where $V^*=V\cup E$ and two vertices are adjacent in G^* if and only if they are adjacent in or incident W in W. Particularly, if $W_1=V$ and $W_2=E$, such a Smarandachely total graph $T_{V,E}(G)$ is called the *semitotal-point graph*, denoted by $T_2(G)$.

For any graph $G=(V,E)$, the semitotal-point graph $T_2(G)$ is the graph whose vertex set is the union of vertices and edges in which two vertices are adjacent if and only if they are adjacent vertices of G or one is a vertex and other is an edge of G incident with it [9].

An entire dominating set X of a graph $T_2(G)$ is an entire semitotal-point(ESP) dominating set if every element not in X is either adjacent or incident to at least one element in X. An ESP domination number $\varepsilon_{tp}(G)$ of G is the minimum cardinality of an ESP dominating set of G.

In this paper, we have initiated a study on an entire domination on semitotal-point graphs and expressed the results in terms of elements of G and other different domination parameters of G.

The following figure shows the formation of $\varepsilon(G)$ and $\varepsilon_{tp}(G)$.

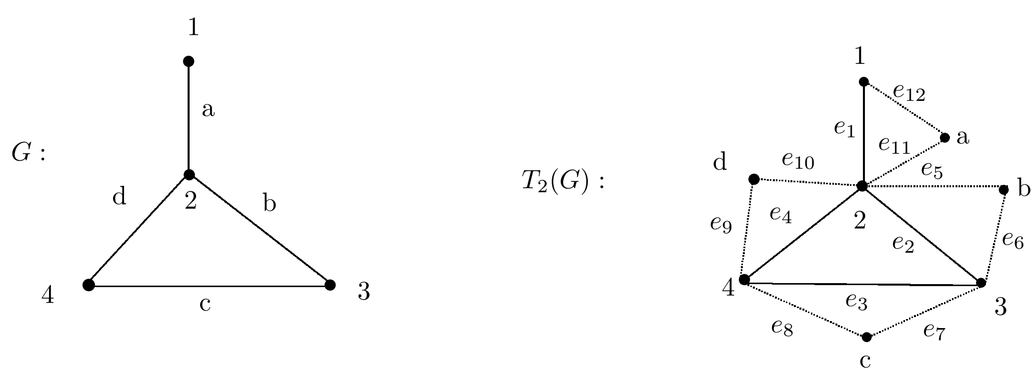

Figure 1

In Figure 1, $V(G)=p=4$ and $E(G)=q=4$. In $T_2(G)$, $V(T_2(G))=p+q$ and $E(T_2(G))=3q$. $X=\{2,c\}$ or $\{4,a\}$ etc. $X'=\{2,e_{12},4,3\}$ etc. $\varepsilon(G)=|X|=2$, $\varepsilon_{tp}(G)=|X'|=4=p$.

§2. Preliminary Results

We need the following theorems for our further results.

Theorem A([5]) *For any connected graph G of order p, $\varepsilon(G) \leq \lceil \frac{p}{2} \rceil$ and the equality is holds for $G = K_p$.*

Theorem B([4]) *For any connected graph G of order p, $\gamma(G) \geq \lceil \frac{p}{\Delta(G)+1} \rceil$, where $\Delta(G)$ is the maximum degree of G.*

Theorem C([2]) *For any tree T, $\gamma_{tp}(T) \geq \gamma(T)$.*

Theorem D([6]) *For any graph G with an pendant vertex, $\gamma(G) = \gamma_s(G)$.*

Theorem E([10]) *If T is a tree with $p \geq 3$ vertices, then $\gamma_c(T) = p - e$, where e is the number of pendant vertices in a tree.*

Theorem F([1]) *Let G be a (p, q) graph with edge domination number $\gamma'(G)$. Then $\gamma'(G) \leq \lfloor \frac{p}{2} \rfloor$.*

Proposition 1 *For any graph G, $\gamma(G) \leq \varepsilon(G)$.*

Proposition 2 *If $G = K_p$; $p \geq 2$ vertices. then $\varepsilon_{tp}(K_p) = p$.*

Observation. In this paper, $\gamma(T_2(G))$ and $\gamma_{tp}(G)$ both denote the domination number of the semitotal-point graph $T_2(G)$.

§3. Main Results

Theorem 3.1 *For any graph G of order p, $\varepsilon_{tp}(G) = p$.*

Proof Let G be a (p, q) graph. We consider the following cases.

Case 1 When $q \leq p$. Let $X = \{v_1, v_2. \cdots, v_k\}$ be the set of vertices in $T_2(G)$ and by definition, $V(T_2(G)) = p + q$. By Theorem A,

$$\varepsilon_{tp}(G) \leq \lceil \frac{p+q}{2} \rceil \leq p.$$

Let $F = \{v_1, v_2, \cdots, v_m\}$ be the maximum independent set of $T_2(G)$. Since every maximum independent set is a minimal dominating set, therefore $\gamma(T_2(G)) \leq \frac{p+q}{2}$. Let $\{e_1, e_2, \cdots, e_q\}$ be the set of edge vertices in $T_2(G)$. Let $F' = \{e'_1, e'_2, \cdots, e'_n\}$ be the edge subset of $E(T_2(G))$ such that no edge in F' is incident with a vertex in F. By Theorem F, $|F'| = \lfloor \frac{p+q}{2} \rfloor$. Clearly $F \cup F'$ is an entire dominating set of $T_2(G)$.

Therefore, $\varepsilon_{tp}(G) \leq |F \cup F'| = \frac{p+q}{2} + \lfloor \frac{p+q}{2} \rfloor \geq p$.

Thus from the above two results we get $\varepsilon_{tp}(G) = p$.

Case 2 When $q > p$, then G must be either K_p or $K_p - x_i$, where x_i; $i \geq 1$ denotes the edges of K_p. Suppose $G = K_p$, then by Proposition 2, the result follows. If $G \neq K_p$ and $q > p$, then by Case 1, $\varepsilon_{tp}(G) = p$. Hence the result. □

Theorem 3.2 *For any graph G, $\varepsilon(G) < \varepsilon_{tp}(G)$.*

Proof Let D and D' be minimal entire dominating sets of G and $T_2(G)$ respectively. By Theorem 3.1, $\varepsilon_{tp}(G) = p$ and by Theorem A, we have $\varepsilon(G) \leq \lceil \frac{p}{2} \rceil$. Hence from these two results we get the required result. □

Theorem 3.3 *For any (p, q) graph G with maximum degree Δ,*

$$\frac{p+q}{2\Delta(G)+1} \leq \varepsilon_{tp}(G).$$

Proof By Theorem B, we have $\gamma(G) \geq \lceil \frac{p}{\Delta(G)+1} \rceil$. Therefore, $\gamma_{tp}(G) \geq \lceil \frac{p+q}{2\Delta(G)+1} \rceil$. Since $\gamma_{tp}(G) \leq \varepsilon_{tp}(G)$, therefore $\varepsilon_{tp}(G) \geq \lceil \frac{p+q}{2\Delta(G)+1} \rceil$. □

Corollary 3.3.1 *For any (p, q) graph G,*

$$\frac{p}{\Delta(G)+1} < \varepsilon_{tp}(G).$$

In the following theorem we obtain the relation between $\varepsilon_{tp}(G)$ and $\gamma_s(G)$.

Theorem 3.4 *For any graph G with a pendant vertex, $\gamma_s(G) \leq \varepsilon_{tp}(G)$ and the equality holds for $G = K_p$ with $p \geq 2$ vertices.*

Proof For any graph G, we have $\gamma(G) \leq \varepsilon(G)$ and also by Theorem 3.2, $\gamma(G) \leq \varepsilon(G) < \varepsilon_{tp}(G)$. Therefore by from Theorem D, we get $\gamma_s(G) < \varepsilon_{tp}(G)$.

Since $\gamma_s(K_p) = p$, therefore by Theorem 3.1, the equality follows. □

The next result relates $\varepsilon_{tp}(G)$ with $\beta_0(G)$.

Theorem 3.5 *For any graph $G \neq K_p$ and tree T with $p \geq 3$ vertices, $2\beta_0 \leq \varepsilon_{tp}(G)$, where β_0 is the vertex independence number of G.*

Proof Let B be the independent set of G which is also a dominating set of G. Then $\gamma(G) \leq |B| \leq \frac{p}{2}$. Now $2\beta_0(G) = 2|B| \leq 2\frac{p}{2} = p$. Also by using Theorem 3.1, we get the required result. □

Theorem 3.6 *For any graph G with $\Delta(G) < p - 1$,*

$$\frac{\gamma(G) + \beta_0(G)}{2} < \varepsilon_{tp}(G) \leq \gamma(G) + \beta_0(G) + 1.$$

Proof By Theorem 3.1, we have $\varepsilon_{tp}(G) = p$. Also for any graph G without isolated vertices we have $\gamma(G) \leq \frac{p}{2}$ and $\beta_0(G) \leq \frac{p}{2}$. Therefore from these the lower bound is attained. The upper bound is obvious. □

In the following result we establish the relation between $\varepsilon_{tp}(G)$ and diameter of G.

Theorem 3.7 *For any graph G,*

$$\frac{diam(G)+1}{3} < \varepsilon_{tp}(G),$$

where $diam(G)$ is the diameter of G.

Proof In [4], we have $\lceil \frac{diam(G)+1}{3} \rceil \leq \gamma(G)$ and by Proposition 1 and Theorem 3.2, the result follows. □

Next we obtain the relation between $\varepsilon_{tp}(G)$, $\alpha_0(G)$ and $\beta_0(G)$.

Theorem 3.8 *For any graph G, $\varepsilon_{tp}(G) = \alpha_0(G) + \beta_0(G)$.*

Proof The result follows from Theorem 3.1, also by the fact that $\alpha_0(G) + \beta_0(G) = p$. □

The immediate consequence of the above theorem is the following result.

Theorem 3.9 *For any graph G, $\varepsilon_{tp}(G) = \alpha_1(G) + \beta_1(G)$, where α_1 and β_1 denote the edge covering number and edge independence number of G.*

Theorem 3.10 *If a graph G and its complement $\overline{G}$ are connected, then, $\varepsilon_{tp}(\overline{G}) = p$.*

Proof Let G with its complement $\overline{G}$ be connected. Then the proof follows by that of Theorem 3.1. □

The next result gives the relation between $\gamma_c(T)$ and $\varepsilon_{tp}(T)$.

Theorem 3.11 *For any non trivial tree T, $\varepsilon_{tp}(T) = \gamma_c(T) + e$, where e is the number of pendant vertices in T.*

Proof Let G be any non trivial tree T. Then by Theorem E, $\gamma_c(T) = p - e$. Substituting for $\gamma_c(T)$ in the required result, the result follows. □

The next result gives the relation between $\gamma_m(G)$ and $\varepsilon_{tp}(G)$.

Theorem 3.12 *For any connected graph G of order ≥ 2, $\gamma_m(G) \leq \varepsilon_{tp}(G)$. Furthermore, the equality holds for $G = K_p$.*

Proof Let G be a (p, q) graph G which is not K_p. Let D be a maximal dominating set of G. Then $V - D$ contains at least one vertex v_i which does not form a dominating set of G. Hence $|V - D| < |V(G)|$. Thus by Theorem 3.1, $\gamma_m(G) < \varepsilon_{tp}(G)$.

For the equality, suppose $G = K_p$, then $\gamma_m(K_p) = p$. Hence by Theorem 3.1, we get $\gamma_m(G) = \varepsilon_{tp}(G)$. □

Theorem 3.13 *For any graph G, $\varepsilon^{-1}(G) < \varepsilon_{tp}(G)$.*

Proof Since every inverse entire dominating set is an entire dominating set, therefore $\varepsilon(G) \leq \varepsilon^{-1}(G)$. By Proposition 1 and Theorem 3.2, we have $\varepsilon^{-1}(G) < \varepsilon_{tp}(G)$. □

In the next theorem we establish the relation between $\gamma_{ns}(G)$ and $\varepsilon_{tp}(G)$.

Theorem 3.14 *Let D be a γ_{ns}- set of a connected graph G. If no two vertices in $V - D$ are* *adjacent to a common vertex in D, then $\gamma_{ns}(G)+\xi(T)\le\varepsilon_{tp}(G)$, where $\xi(T)$ is the maximum number of pendant vertices in any spanning tree T of G.*

Proof Let G be a graph such that no two vertices in $V-D$ are adjacent to a common vertex in D. By Theorem 3.1, we have $\varepsilon_{tp}(G)=p$.

Let D be a γ_{ns}-set of G. Since for any two vertices $u,v\in V-D$, there exist no vertices $u_1,v_1\in D$ such that u_1 is adjacent to u but not v and v_1 is adjacent to v but not to u_1. This implies that there exist a spanning tree T of $\langle V-D\rangle$ in which each vertex of $V-D$ is adjacent to a vertex of D. This shows that $\xi(T)\ge|V-D|$.

Thus from above two results we get that $\gamma_{ns}(G)+\xi(T)\le\varepsilon_{tp}(G)$. □

The next theorem gives the relation between $\varepsilon_{tp}(G)$ and $\gamma^{-1}(G)$.

Theorem 3.15 *Let T be a tree with every nonpendant vertex adjacent to at least one pendant vertex. Then $\gamma(T)+\gamma^{-1}(T)=\varepsilon_{tp}(T)$.*

Proof Let T be a tree with every nonpendant vertex is adjacent to at least one pendant vertex. If every nonpendant vertex is adjacent to at least two pendant vertices, then the set of all nonpendant vertices is a minimum dominating set and the set of all pendant vertices is a minimum inverse dominating dominating set. Suppose there are nonpendant vertices which are adjacent to exactly one pendant vertex. Let D and D' denote the minimum dominating and inverse dominating sets respectively. Let u be a nonpendant vertex adjacent to exactly one pendant vertex v. If $u\in D$ then $v\in D'$ and if $u\in D'$ then $v\in D$. Therefore $|D|+|D'|=p$. Also by Theorem 3.1, $\varepsilon_{tp}(T)=p$. Thus $\gamma(T)+\gamma^{-1}(T)=\varepsilon_{tp}(T)$. □

Next, we establish Nordhaus-Gaddum type results.

Theorem 3.16 *For any (p,q) graph G,*

(i) $\varepsilon_{tp}(G)+\varepsilon_{tp}(\overline{G})\le 2p$;
(ii) $\varepsilon_{tp}(G)\varepsilon_{tp}(\overline{G})\le p^2$.

Furthermore, the equality holds for $G=C_5$ or P_4.

Proof From Theorems 3.1 and 3.10, this result follows. □

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Math.Combin.Book Ser. Vol.4 (2010), 86-93

On k-Equivalence Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called an *equivalence set* if every component of the induced subgraph $\langle S\rangle$ is complete. If further at least one component of $\langle V - S\rangle$ is not complete, then S is called a Smarandachely equivalence set. Let k be any nonnegative integer. An equivalence set $S \subseteq V$ is called a *k-equivalence set* if $\Delta(\langle S\rangle) \leq k$. A k-equivalence set which dominates G is called a *k-equivalence dominating set* of G. In this paper we introduce some parameters using the just defined notion and discuss their relations with other graph theoretic parameters.

Key Words: Domination, irredundance, Smarandachely equivalence set, k-equivalence set, k-equivalence domination, k-equivalence irredundance.

AMS(2010): 05C69

In this paper we consider only finite undirected simple graphs. For graph theoretic terminology we rely on [5]. Throughout this article, let G be a graph with vertex set V and edge set E.

One of the dominant areas in graph theory is the study of domination and related notions such as independence, irredundance, covering and matching. (In this connection see [9-10].)

Let $v \in V$. The open neighbourhood of v denoted by $N(v)$ and the closed neighbourhood of v denoted by $N[v]$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is said to be an *independent set* if no two vertices in S are adjacent. A subset S of V is called a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The cardinality of a minimum dominating set is called the *domination number* and it is denoted by $\gamma(G)$.

There are many variations of domination in graphs. In the book by Haynes et al. [9] it is proposed that a type of domination is "fundamental" if every connected nontrivial graph has a dominating set of this type and this type of dominating set S is defined in terms of some "natural" property of the subgraph induced by S. Examples include total domination, independent domination, connected domination and paired domination. In this paper we introduce

[1]Received September 21, 2010. Accepted December 24, 2010.

the concept of k-equivalence domination, which is a fundamental concept in the above sense.

An *equivalence graph* is a vertex disjoint union of complete graphs. An *equivalence covering* of a G is a family of equivalence subgraphs of G such that every edge of G is an edge of at least one member of the family. The *equivalence covering number* of G is the cardinality of a minimum equivalence covering of G. The equivalence covering number was first studied in [6]. Interesting bounds for the equivalence covering number in terms of maximal degree of the complement were obtained in [2]. The computation of the equivalence covering number of split graphs was considered in [4].

An important concept which uses equivalence graph is subcoloring studied in [1,8,11]. A *subcoloring* of G is a partition of its vertex set into subsets $X_1, X_2, \ldots, X_k$, where for each $i \leq k$ the induced subgraph $\langle X_i \rangle$ is an equivalence graph. The order of a minimum subcoloring is called the *subchromatic number* of G. The notion of subchromatic number is a natural generalization of the well studied chromatic number since for any independent set S, the induced subgraph $\langle S \rangle$ is trivially an equivalence graph.

The concept of equivalence graph also arises naturally in the study of domination in claw-free graphs, as shown by the following theorem proved in [7].

Theorem 1([7]) *Any minimal dominating set of a $K_{1,3}$-free graph is a collection of disjoint complete subgraphs.*

Motivated by these observations, we have introduced the concept of equivalence set and equivalence domination number in [3].

Definition 2 *A subset S of V is called an equivalence set if every component of the induced subgraph $\langle S \rangle$ is complete. A dominating set of G which is also an equivalence set is called an equivalence dominating set of G. The equivalence domination number $\gamma_e(G)$ is defined to be the cardinality of a minimum equivalence dominating set of G. An equivalence set S is called a Smarandachely equivalence set if at least one component of $\langle V - S \rangle$ is not complete.*

In this paper we introduce the concept of k-equivalence set and several parameters using this concept and investigate their relation with the six basic parameters of the domination chain. (For details see [9, §3.5].)

Definition 3 *Let k be any nonnegative integer. A subset S of V is called a k-equivalence set if every component of the induced subgraph $\langle S \rangle$ is complete—i.e., if S is an equivalence set of G—and $\Delta(\langle S \rangle) \leq k$.*

The concept of k-equivalence set is a natural generalization of the concept of independence, since every independent set is obviously 0-equivalence set. Also every $(k-1)$-equivalence set is a k-equivalence set and k-equivalence is a hereditary property. Hence a k-equivalence set S is a maximal k-equivalence set if and only if $S \cup \{v\}$ is not a k-equivalence set for all $v \in V - S$. Thus a k-equivalence set $S \subseteq V$ is maximal if and only if for every $v \in V - S$, there exists a clique C in $\langle S \rangle$ such that v is adjacent to a vertex in C and v is not adjacent to a vertex in C or there exist two cliques C_1 and C_2 in $\langle S \rangle$ such that v is adjacent to a vertex in C_1 and to a vertex in C_2 or there exists a clique C in $\langle S\rangle$ such that $|C| = k+1$ and v is adjacent to all vertices in C.

Definition 4 *The k-equivalence number $\beta_e^k(G)$ and the lower k-equivalence number $i_e^k(G)$ are defined as follows.*

$$\beta_e^k(G) = \max\{|S| : S \text{ is a maximal } k\text{-equivalence set of } G\} \text{ and}$$
$$i_e^k(G) = \min\{|S| : S \text{ is a maximal } k\text{-equivalence set of } G\}.$$

Clearly $i_e^k(G) \leq \beta_e^k(G)$ and $\beta_0(G) \leq \beta_e^k(G)$.

Definition 5 *A dominating set S of V which is also a k-equivalence set is called a k-equivalence dominating set of G. The k-equivalence domination number $\gamma_e^k(G)$ and the upper k-equivalence domination number $\Gamma_e^k(G)$ are defined by*

$$\gamma_e^k(G) = \min\{|S| : S \text{ is a minimal } k\text{-equivalence dominating set of } G\} \text{ and}$$
$$\Gamma_e^k(G) = \max\{|S| : S \text{ is a minimal } k\text{-equivalence dominating set of } G\}.$$

Since every maximal k-equivalence set is a dominating set of G and every maximal independent set is a minimal k-equivalence dominating set, the parameters $\gamma_e^k(G)$ and $\Gamma_e^k(G)$ fit into the domination chain, thus leading to the following extended domination chain: $ir(G) \leq \gamma(G) \leq \gamma_e^k(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_e^k(G) \leq \Gamma(G) \leq IR(G)$.

Definition 6 *An irredundant set which is also a k-equivalence set is called a k-equivalence irredundant set. The k-equivalence irredundance number $ir_e^k(G)$ and the upper k-equivalence irredundance number $IR_e^k(G)$ are defined by*

$$ir_e^k(G) = \min\{|I| : I \text{ is a maximal } k\text{-equivalence irredundant set of } G\} \text{ and}$$
$$IR_e^k(G) = \max\{|I| : I \text{ is a maximal } k\text{-equivalence irredundant set of } G\}.$$

Remark 7 Let S be a minimal k-equivalence dominating set of G. Since S is a minimal dominating set, it is a maximal irredundant set. Thus S is a maximal k-equivalence irredundant set of G. Thus we have the following: Any minimal k-equivalence dominating set is a maximal k-equivalence irredundant set.

For any G, we have $ir_e^k(G) \leq \gamma_e^k(G) \leq \Gamma_e^k(G) \leq IR_e^k(G)$ and $ir_e^k(G) \leq \gamma_e^k(G) \leq i_e^k(G) \leq \beta_e^k(G)$.

Lemma 8 *If D is a minimal k-equivalence dominating set of G, then D is both a minimal dominating set and a minimal $(k+1)$-equivalence dominating set of G.*

Proof Assume that D is a minimal k-equivalence dominating set of G, and let $x \in D$. Then $D - \{x\}$ is not a k-equivalence dominating set and $\Delta(\langle D - \{x\}\rangle) \leq \Delta(\langle D\rangle) \leq k$. Therefore D is a minimal dominating set of G and D is a $(k+1)$-equivalence set. □

Corollary 9 *For every nonnegative integer k, $\gamma_e^{k+1}(G) \leq \gamma_e^k(G)$ and $\Gamma_e^k(G) \leq \Gamma_e^{k+1}(G)$.*

The proof of the next result is similar to that of Theorem 3.2 in [3].

Theorem 10 *For any graph G, $\gamma(G) \leq 2ir_e^k(G)$.*

Proof Let $I = \{x_1, x_2, \ldots, x_k\}$ be an ir_e^k-set of G. Let y_i be a private neighbor of x_i with respect to I and let $A = I \cup \{y_1, y_2, \ldots, y_k\}$. If there exists a vertex x in $V - A$ such that $N(x) \cap (V - A) = \emptyset$, then $B = I \cup \{x\}$ is an k-equivalence set of G and x is an isolated vertex in $\langle B \rangle$. Further for each i, y_i is a private neighbor of x_i with respect to B; therefore B is a k-equivalence irredundant set— a contradiction. Whence A is a dominating set of G; therefore $\gamma(G) \leq 2ir_e^k(G)$. □

Let k be any integer ≥ 2. Consider the graphs H_1, H_2 displayed in Figure 1 and Figure 2 respectively. Obviously $ir_e^k(H_1) = 4$ and $\gamma(H_1) = 5$. Since $\{a, b, c\}$ is a maximal equivalence irredundant set in H_2, $ir_e^k(H_2) = 3$. Since $\{a, e, f, g\}$ is a maximal irredundant set in H_2, $ir(H_2) = 4$. Now for the graph $H_3 = P_3 \circ 2K_1$, we have $ir(H_3) = 3$, $ir_e^k(H_3) \geq 4$ and $\gamma(H_3) = 3$. From these information, it is clear that the parameters ir and ir_e^k and the parameters γ and ir_e^k are not comparable. It is not difficult to show that the just mentioned statement holds when $k \leq 1$.

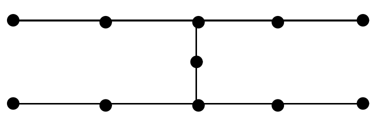

Figure 1

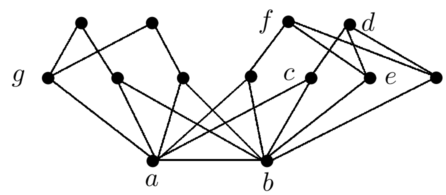

Figure 2

For the complete bipartite graph $H_4 = K_{2,r}, r \geq 3$, we have $i_e^k(H_4) = 2$ and $\beta_0(H_4) = \Gamma_e^k(H_4) = \Gamma(H_4) = IR_e^k(H_4) = \beta_e^k(H_4) = r$. Also $i_e^k(K_n) = \beta_e^k(K_n) = k + 1 \leq n$ whereas $i(K_n) = \beta_0(K_n) = \Gamma_e^k(K_n) = \Gamma(K_n) = IR_e^k(K_n) = IR(K_n) = 1$. Further $i(K_n \circ 2K_1) = 2n - 1$ and $i_e^k(K_n \circ 2K_1) = 2n - (k+1)$. Hence i_e is not comparable with any of $IR, IR_e^k, \Gamma, \Gamma_e^k, i(G)$ and β_0. For the graph H_5 obtained from $K_{4,4,4} \circ K_1$, by adding edges in such a way that the subgraph induced by the set of all pendant vertices of the latter is a cycle, we have $\Gamma(H_5) = IR(H_5) = 12$ and $\beta_e^k(H_5) < 12$. Thus β_e^k is not comparable with IR and Γ.

Let H_6 be the graph obtained from the path $P_6 := (a_1, a_2, a_3, a_4, a_5, a_6)$ and the complete graph K_6 with $V(K_6) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ by adding the edges $a_1b_1, a_2b_2, a_4b_4, a_5b_5$ and a_6b_6. At least one vertex of $V(K_6)$ belongs to every dominating set of H_6 whence $\Gamma(H_6) = 4$. Since $\{b_1, b_2, b_4, b_5, b_6\}$ is an equivalence irredundant set of H_6, $IR_e^k(H_6) > \Gamma(H_6)$, when $k \geq 4$. It is not difficult to show that the just mentioned statement holds when $k \leq 3$. Also for the graph $H_7 = C_5 \Box K_2$, we have $\Gamma(H_7) = 5$ and $IR_e^k(H_7) = 4$. Thus Γ and IR_e^k are not comparable.

The following Hasse diagram summarizes the relationship between the various parameters for the graph G.

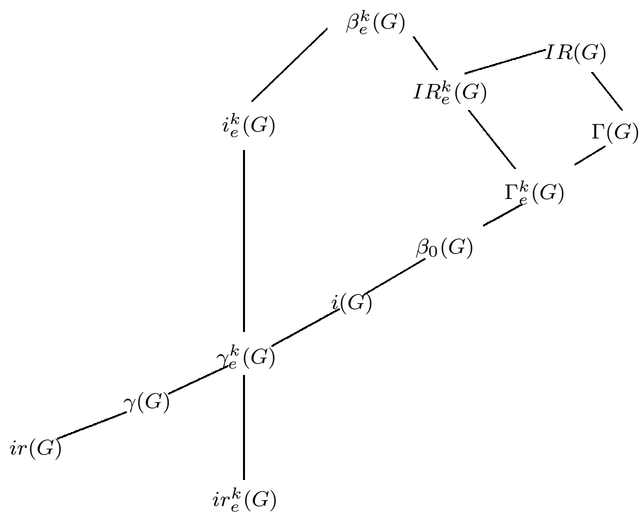

Figure 3. Relationship between parameters

Remark 11 It is easy to show that $\gamma_e^k(G) \leq i(G) \leq |V(G)| - \Delta(G)$.

Proposition 12 If G is connected, then

$$\gamma_e^k(G) \leq n - \left\lfloor \frac{2(\text{diam}(G)+1)}{3} \right\rfloor.$$

Proof Consider an arbitrary induced path P of length diam(G) in a connected graph G. Every interior vertex in diametrical path dominates at least 3 vertices in G and also there exists maximal k-equivalence set in $\langle V - P \rangle$. Therefore

$$\gamma_e^k(G) \leq n - (\text{diam}(G)+1) + \left\lfloor \frac{\text{diam}(G)+1}{3} \right\rfloor = n - \left\lfloor \frac{2}{3}(\text{diam}(G)+1) \right\rfloor.$$

Also this bound is sharp when $G \cong P_n$, where $n \equiv 2 \pmod 3$. □

Theorem 13 *If* $\Delta(G) \geq 3$ *and* k *is an integer such that* $0 \leq k \leq \Delta - 3$, *then* $\gamma_e^k(G) \leq (\Delta - k - 1)\gamma_e(G) - (k+1)(\Delta - k - 2)$.

Proof Let D be a γ_e-set of G. If D is k-equivalence set, then $\gamma_e^k(G) = \gamma_e(G)$. Assume $\Delta(\langle D \rangle) \geq k+1$. Let $x \in D$ such that $\deg_{\langle D \rangle}(x) \geq k+1$ and let $Q = N(x) \cap (V - D)$. Let P be the set of all private neighbors of x with respect to D. Clearly $P \neq \emptyset$. Let R be a minimum k-equivalence dominating set of $\langle P \rangle$ and let $D' = (D - \{x\}) \cup R$. Now $|R| \leq |Q| \leq \Delta - (k+1)$. It follows that the set D' is an equivalence dominating set of G and $\langle D' \rangle$ has fewer vertices of degree at least $k+1$ than $\langle D \rangle$. Let E be a minimal equivalence dominating set of G such that $E \subseteq D'$. Then

$$|E| \leq |D'| = |D| - 1 + |R| = \gamma_e(G) - 1 + |R| \leq \gamma_e(G) + \Delta - k - 2.$$

Continue to repeat the above process until no more vertices of degree larger than k exist in the resultant set. (Note that the number of such repetitions is at most $|D|-(k+1)$.) Hence

$$\begin{aligned}\gamma_e^k(G) &\le |D| \le \gamma_e(G) + (|D|-(k+1))(\Delta-k-2)\\ &= (\Delta-k-1)\gamma_e(G) - (k+1)(\Delta-k-2).\end{aligned}$$

The above bound is attained when $G = K_3 \circ 2K_1$. Here $\gamma_e(G) = \gamma_e^2(G) = 3, \gamma_e^1(G) = 4$. □

Theorem 14 *If k is an integer such that $0 \le k \le \omega - 3$, then $\gamma_e^k(G) \le \left(\frac{\omega-k}{2}\right)\gamma_e^{k+1}(G)$.*

Proof Let D be a γ_e^{k+1}-set of G. If D is k-equivalence set, then $\gamma_e^k(G) = \gamma_e^{k+1}(G)$. Let k be any nonnegative integer not more than $\omega - 3$. Suppose D is not a k-equivalence set. Let X be a subset of D such that for all x, $\deg_{\langle D\rangle}(x) = k+1$ and let Y be a minimum independent set of $\langle X\rangle$. Since every vertex of $X - Y$ has at least one of its $(k+1)$ neighbors in Y, $D - Y$ is a k-equivalence set. Note that there are $|Y|(k+1)$ edges between Y and $D - Y$. Since D is $(k+1)$-equivalence set, $|Y|(k+1) \le |D-Y|(k+1)$. Thus $|Y| \le \frac{1}{2}|D|$.

Let P be the set of all private neighbors of Y with respect to D and R be a minimum k-equivalence dominating set of $\langle P\rangle$. Then R dominates P and $D - Y$ dominates $V - P$. Therefore $R \cup (D-Y)$ is a k-equivalence dominating set and there are no edges between $D-Y$ and R. Since $|R| \le |P| \le |Y|(\omega-k-1)$, we obtain

$$\begin{aligned}\gamma_e^k(G) &\le |D| - |Y| + |R| \le |D| - |Y| + |Y|(\omega-k-1) = |D| + |Y|(\omega-k-2)\\ &\le |D| + \frac{|D|}{2}(\omega-k-2) = \left(\frac{\omega-k}{2}\right)\gamma_e^{k+1}(G).\end{aligned}$$

□

Theorem 15 *If $\gamma_e^k(G) \ge 2$, then $m \le \left\lfloor \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2)\right\rfloor$, where n and m are respectively, the order and the size of the graph G.*

Proof We prove this result by induction on number of vertices. We can assume that $n > 2$ for otherwise the proof is obvious; we can also assume that the result holds for any graph whose order is less than n. If $\gamma_e^k(G) = 2$, then also the conclusion holds. So assume that $\gamma_e^k(G) \ge 3$. Let $v \in V(G)$ with $\deg(v) = \Delta(G)$. Then by Remark 11, $|N(v)| = \Delta(G) \le n - \gamma_e^k(G)$; i.e., $\Delta(G) = n - \gamma_e^k(G) - r$ where $0 \le r \le n - \gamma_e^k(G)$. Let $S = V - N[v]$. Then $|S| = \gamma_e^k(G) + r - 1$. If $u \in N(v)$, then $(S - N(u)) \cup \{u, v\}$ is a dominating set of G and $\gamma_e^k(G) \le |S - N(u)| + 2$. Thus $\gamma_e^k(G) \le \gamma_e^k(G) + r - 1 - |S \cap N(u)| + 2$ and so $|S \cap N(u)| \le r + 1$ for all $u \in N(v)$. Hence the number of edges between $N(v)$ and S, say ℓ_1, is at most $\Delta(G)(r+1)$.

Further, if D is a γ_e^k-set of $\langle S\rangle$, then $D \cup \{v\}$ is a k-equivalence dominating set of G. Hence $\gamma_e^k(G) \le |D \cup \{v\}|$, implying that $\gamma_e^k(\langle S\rangle) \ge \gamma_e^k(G) - 1 \ge 2$. Let ℓ_2 be the size of $\langle S\rangle$. By the inductive hypothesis,

$$\begin{aligned}\ell_2 &\le \left\lfloor \frac{1}{2}(|S| - \gamma_e^k(\langle S\rangle))(|S| - \gamma_e^k(\langle S\rangle) + 2)\right\rfloor\\ &\le \left\lfloor \frac{1}{2}(\gamma_e^k(G) + r - 1 - \gamma_e^k(G) + 1)(\gamma_e^k(G) + r - 1 - \gamma_e^k(G) + 1 + 2)\right\rfloor\\ &= \frac{1}{2}r(r+2).\end{aligned}$$

Let $\ell_3 = |E\langle N[v]\rangle|$. Note that for each u in $N(v)$ there are at most $r+1$ vertices in S which are adjacent to u. Therefore,

$$\begin{aligned}|E| &= \ell_1 + \ell_2 + \ell_3 \\ &\leq \Delta(G)\cdot(r+1) + \frac{1}{2}r\cdot(r+2) + \Delta(G) + \frac{1}{2}\Delta(G)(\Delta(G)-r-2) \\ &= \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2) - \frac{1}{2}\Delta(G)(n-\gamma_e^k(G)-\Delta(G)) \\ &\leq \frac{1}{2}(n-\gamma_e^k(G))(n-\gamma_e^k(G)+2).\end{aligned}$$

□

Concluding Remarks

We have proved that the decision problem corresponding to the parameters γ_e and Γ_e are NP-complete in [3]. Therefore the computations of γ_e^k and Γ_e^k are also NP-complete. The problem of designing efficient algorithms for computing the parameters in connection with a notion of k-equivalence for special classes of graphs is an interesting direction for further research. In particular one can attempt the design of such algorithms for families of graphs with bounded tree-width.

Acknowledgement

We are thankful to the National Board for Higher Mathematics, Mumbai, for its support through the project 48/5/2008/R&D-II/561, awarded to the first author. We are also thankful to G. R. Vijayakumar for giving suggestions to improve the presentation of the paper.

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Math.Combin.Book Ser. Vol.4 (2010), 94-99

On Near Mean Graphs

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Abstract: Let $G = (V, E)$ be a graph with p vertices and q edges and let $f : V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be an injection. The graph G is said to have a *near mean labeling* if for each edge, there exist an induced injective map $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

We extend this notion to *Smarandachely near m-mean labeling* (as in [9]) if for each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely m-labeling f^* is defined by

$$f^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

A graph that admits a Smarandachely near mean m-labeling is called *Smarandachely near m-mean graph*. The graph that admits a near mean labeling is called a near mean graph (NMG). In this paper, we proved that the graphs $P_n, C_n, K_{2,n}$ are near mean graphs and $K_n(n > 4)$ and $K_{1,n}(n > 4)$ are not near mean graphs.

Key Words: Labeling, near mean labeling, near mean graph, Smarandachely near m-labeling, Smarandachely near m-mean graph.

AMS(2010): 05C78

§1. Introduction

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by $V(G)$ and $E(G)$ respectively. Let $f : V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be an injection. The graph G is said to have a *near mean labeling* if for each edge, there exist an induced injective map $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

We extend this notion to *Smarandachely near m-mean labeling* (as in [9]) if for each edge $e = uv$

[1]Received August 2, 2010. Accepted December 25, 2010.

and an integer $m \geq 2$, the induced Smarandachely m-labeling f^* is defined by

$$f^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

A graph that admits a Smarandachely near mean m-labeling is called *Smarandachely near m-mean graph*. A path P_n is a graph of length $n-1 \cdot K_n$ and C_n are complete graph and cycle with n vertices respectively. Terms and notations not used here are as in [2].

§2. Preliminaries

The mean labeling was introduced in [3]. Let G be a (p,q) graph. In [4], we proved that the graphs Book B_n, Ladder L_n, Grid $P_n \times P_n$, Prism $P_m \times C_3$ and $L_n \odot K_1$ are near mean graphs. In [5], we proved that *Join* of graphs, $K_2 + mK_1, K_n^1 + 2K_2$, $S_m + K_1P_n + 2K_1$ and double fan are near mean graphs. In [6], we proved Family of trees, Bi-star, Sub-division Bi-star $P_m \ominus 2K_1, P_m \ominus 3K_1$, $P_m \ominus K_{1,4}$ and $P_m \ominus K_{1,3}$ are near mean graphs. In [7], special class of graphs triangular snake, quadrilateral snake, C_n^+, $S_{m,3}, S_{m,4}$, and parachutes are proved as near mean graphs. In [8], we proved the graphs armed and double armed crown of C_3 and C_4 are near mean graphs. In this paper we proved that the graphs $P_n, C_n, K_{2,n}$ are near mean graphs and $K_n(n>4)$ and $K_{1,n}(n>4)$ are not near mean graphs.

§3 Near Mean Graphs

Theorem 3.1 *The path P_n is a near mean graph.*

Proof Let P_n be a path of n vertices with $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(P_n) = \{(u_iu_{i+1})/i = 1, 2, \ldots, n-1\}$. Define $f : V(P_n) \to \{0, 1, 2, \ldots, n-1, n+1\}$ by

$$f(u_i) = i-1, 1 \leq i \leq n$$
$$f(u_n) = n+1.$$

Clearly, f is injective. It can be verified that the induced edge labeling given by $f^*(u_iu_{i+1}) = i(1 \leq i \leq n)$ are distinct. Hence, P_n is a near mean graph. □

Example 3.2 A near mean labeling of P_4 is shown in Figure 1.

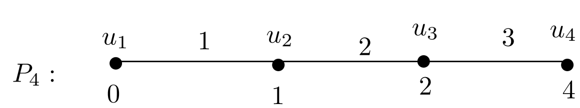


Figure 1: P_4

Theorem 3.3 $K_n, (n>4)$ *is not a near mean graph.*

Proof Let $f : V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$. To get the edge label 1 we must have either 0 and 1 as vertex labels or 0 and 2 as vertex labels.

In either case 0 must be label of some vertex. In the same way to get edge label q, we must have either $q-1$ and $q+1$ as vertex labels or $q-2$ and $q+1$ as vertex labels. Let u be a vertex whose veretx label 0.

Case i To get the edge label q. Assign vertex labels $q-1$ and $q+1$ to the vertices w and x and respectively.

Subcase a. Let v be a vertex whose vertex label be 2, then the edges vw and ux get the same label.

Subcase b. Let v be a vertex whose vertex label be 1.

Then the edges vw and ux get the same label when q is odd. Similarly, when q is even, the edges uw and vw get the same label as well the edges ux and vx get the same label.

Case ii. To get the edge label q assign the vertex label $q-2$ and $q+1$ to the vertices w and x respectively.

Subcase a. Let v be the vertex whose vertex label be 1.

As $n > 4$, to get edge label 2, there should be a vertex whose vertex label is either 3 or 4. Let it be z (say). When vertex label of z is 3, the edges ux and wz have the same label also the edges uz and vz get the same edge label. When the vertex label of z is 4, the edges vx and wz have the same label.

Subcase b. Let v be a vertex whose vertex label 2.

As $n > 4$, to get edge label 2, there should be a vertex, say z whose vertex label is either 3 or 4. When vertex label of z is 3, the edges ux and wz get the same label. Suppose the vertex label of z is 4.

If q is even then the edges ux and wz have the same label. If q is odd then the edges vw and ux have the same label. Hence $K_n (n \geq 5)$ is not a near mean graph. □

Remark 3.4 *K_2, K_3 and K_4 are near mean graphs.*

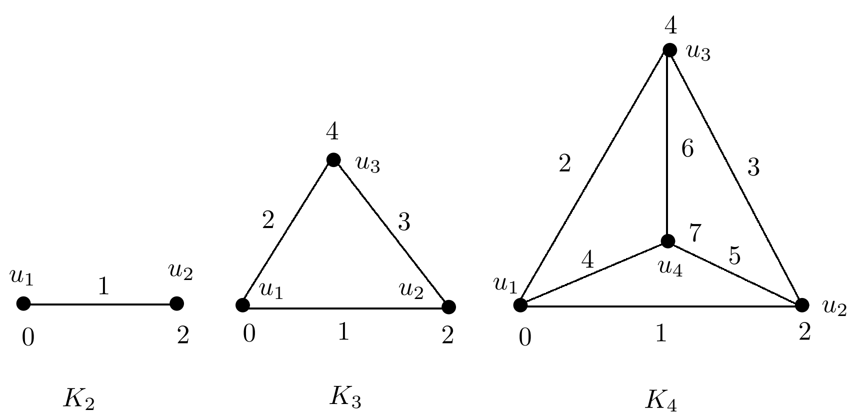


Figure 2: K_2, K_3, K_4

Theorem 3.5 *A cycle C_n is a near mean graph for any integer $n \geq 1$.*

Proof Let $V(C_n) = (u_1, u_2, u_3, \ldots, u_n, u_1\}$ and $E(C_n) = \{[(u_i u_{i+1}) : 1 \leq i \leq n-1] \cup (u_1 u_n)\}$.

Case i Let n be even, say $n = 2m$.

Define $f : V(C_n) \to \{0, 1, 2, \ldots, 2m, 2m+2\}$ by

$$f(u_i) = i - 1, 1 \leq i \leq m.$$
$$f(u_{m+j}) = m + j, 1 \leq i < m.$$
$$f(u_n) = 2m + 1.$$

Clearly f is injective. The set of edge labels of C_n is $\{1, 2, \ldots, q\}$.

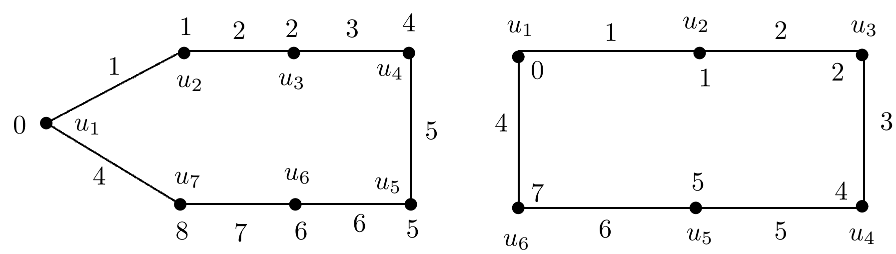
Case ii. Let n be odd, say $n = 2m + 1$.

Define $f : V(C_n) \to \{0, 1, 2, \ldots, 2m-1, 2m+1\}$ by

$$f(u_i) = i - 1, 1 \leq i \leq m$$
$$f(u_{m+j}) = m + j, 1 \leq j \leq m.$$
$$f(u_{2m+1}) = 2m + 2.$$

Clearly f is injective. The set of edge labels of C_n is $\{1, 2, \ldots, q\}$. □

Example 3.6 A near mean labeling of C_6 and C_7 is shown in Figure 3.



C_7-odd C_6-even

Figure 3: C_6, C_7

Theorem 3.7 *$K_{1,n} (n > 4)$ is not a near mean graph.*

Proof Let $V(K_{1,n}) = \{u, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{(uv_i) : 1 \leq i \leq n\}$. To get the edge label 1, either 0 and 1 (or) 0 and 2 are assigned to u and v_i for some i. In either case 0 must be label of some vertex.

Suppose if $f(u) = 0$, then we can not find an edge label q. Suppose if $f(v_1) = 0$, then either $f(u) = 1$ or $f(u) = 2$.

Case i. Let $f(u) = 1$.

To get edge label q, we need the following possibilities either $q-1$ and $q+1$ or $q-2$ and $q+1$. If $f(u) = 1$, it is possible only when q is either 2 or 3. But $q > 4$, so it is not possible to get edge value q.

Case ii. Let $f(u) = 2$.

As in Case i, if $f(u) = 2$ and if one of the edge value is q, then the value of q is either 3 or 4. From both the cases it is not possible to get the edge value q, when $q > 4$.

Hence, $K_{1,n}(n > 5)$ is not a near mean graph. □

Remark 3.8 *$K_{1,n}, n \leq 4$ is a near mean graph. For example, one such a near mean labeling is shown in Figure 4.*

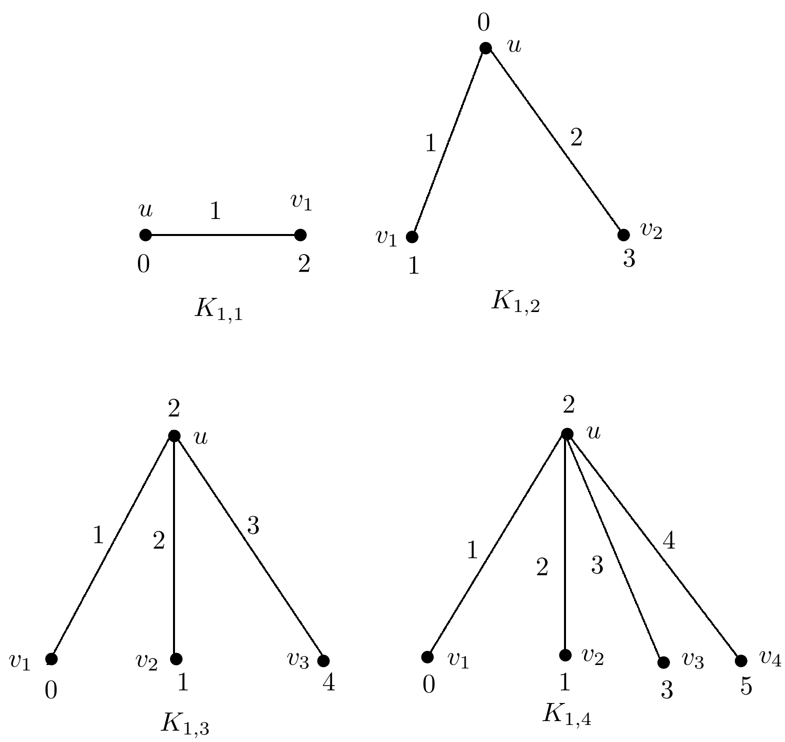


Figure 4: $K_{1,n}, n \leq 4$

Theorem 3.9 *$K_{2,n}$ admits near mean graph.*

Proof Let (V_1, V_2) be the bipartition of $V(K_{2,n})$ with $V_1 = \{u_1u_2\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. $E(K_{2,n}) = \{(u_1v_i), (u_2v_i) : 1 \leq i \leq n\}$.

Define an injective map $f : V(K_{2,n}) \to \{0, 1, 2, \ldots, 2n-1, 2n+1\}$ by

$$f(u_1) = 1$$
$$f(u_2) = 2n + 1$$
$$f(v_i) = 2(i-1), 1 \leq i \leq n.$$

Then, it can be verified $f^*(u_1v_i) = i, 1 \leq i \leq n$, $f^*(u_2v_i) = n+i, 1 \leq i \leq n$ and the edge values are distinct. Hence, $K_{2,n}$ is a near mean graph. □

Example 3.10 A near mean labeling of $K_{2,4}$ is shown in Figure 5.

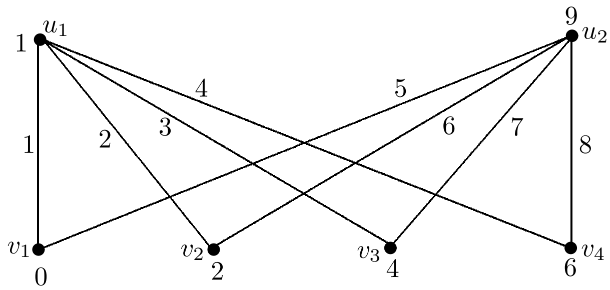


Figure 5: $K_{2,4}$

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Math.Combin.Book Ser. Vol.4 (2010), 100-107

On Pathos Lict Subdivision of a Tree

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Abstract: Let G be a graph and $E_1 \subset E(G)$. A Smarandachely E_1-lict graph $n^{E_1}(G)$ of a graph G is the graph whose point set is the union of the set of lines in E_1 and the set of cutpoints of G in which two points are adjacent if and only if the corresponding lines of G are adjacent or the corresponding members of G are incident.Here the lines and cutpoints of G are member of G. Particularly, if $E_1 = E(G)$, a Smarandachely $E(G)$-lict graph $n^{E(G)}(G)$ is abbreviated to *lict graph of* G and denoted by $n(G)$. In this paper, the concept of pathos lict sub-division graph $P_n[S(T)]$ is introduced. Its study is concentrated only on trees. We present a characterization of those graphs, whose lict sub-division graph is planar, outerplanar, maximal outerplanar and minimally nonouterplanar. Further, we also establish the characterization for $P_n[S(T)]$ to be eulerian and hamiltonian.

Key Words: pathos, path number, Smarandachely lict graph, lict graph, pathos lict sub-division graphs, Smarandache path k-cover, pathos point.

AMS(2010): 05C10, 05C99

§1. Introduction

The concept of *pathos* of a graph G was introduced by Harary [1] as a collection of minimum number of line disjoint open paths whose union is G. The path number of a graph G is the number of paths in a pathos. Stanton [7] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs. The *subdivision of a graph* G is obtained by inserting a point of degree 2 in each line of G and is denoted by $S(G)$. The path number of a subdivision of a tree $S(T)$ is equal to K, where $2K$ is the number of odd degree point of $S(T)$. Also, the end points of each path of any pathos of $S(T)$ are odd points. The *lict graph* $n(G)$ of a graph G is the graph whose point set is the union of the set of lines and the set of cutpoints of G in which two points are adjacent if and only if the corresponding lines of G are adjacent or the corresponding members of G are incident.Here the lines and cutpoints of G are member of G.

For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most

[1]Received July 18, 2010. Accepted December 25, 2010.

k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G. See [8].

By a graph we mean a finite, undirected graph without loops or multiple lines. We refer to the terminology of [1]. The *pathos lict subdivision of a tree* T is denoted as $P_n[S(T)]$ and is defined as the graph, whose point set is the union of set of lines, set of paths of pathos and set of cutpoints of $S(T)$ in which two points are adjacent if and only if the corresponding lines of $S(T)$ are adjacent and the line lies on the corresponding path P_i of pathos and the lines are incident to the cutpoints. Since the system of path of pathos for a $S(T)$ is not unique, the corresponding pathos lict subdivision graph is also not unique. The pathos lict subdivision graph is defined for a tree having at least one cutpoint.

In Figure 1, a tree T and its subdivision graph $S(T)$, and their pathos lict subdivision graphs $P_n[S(T)]$ are shown.

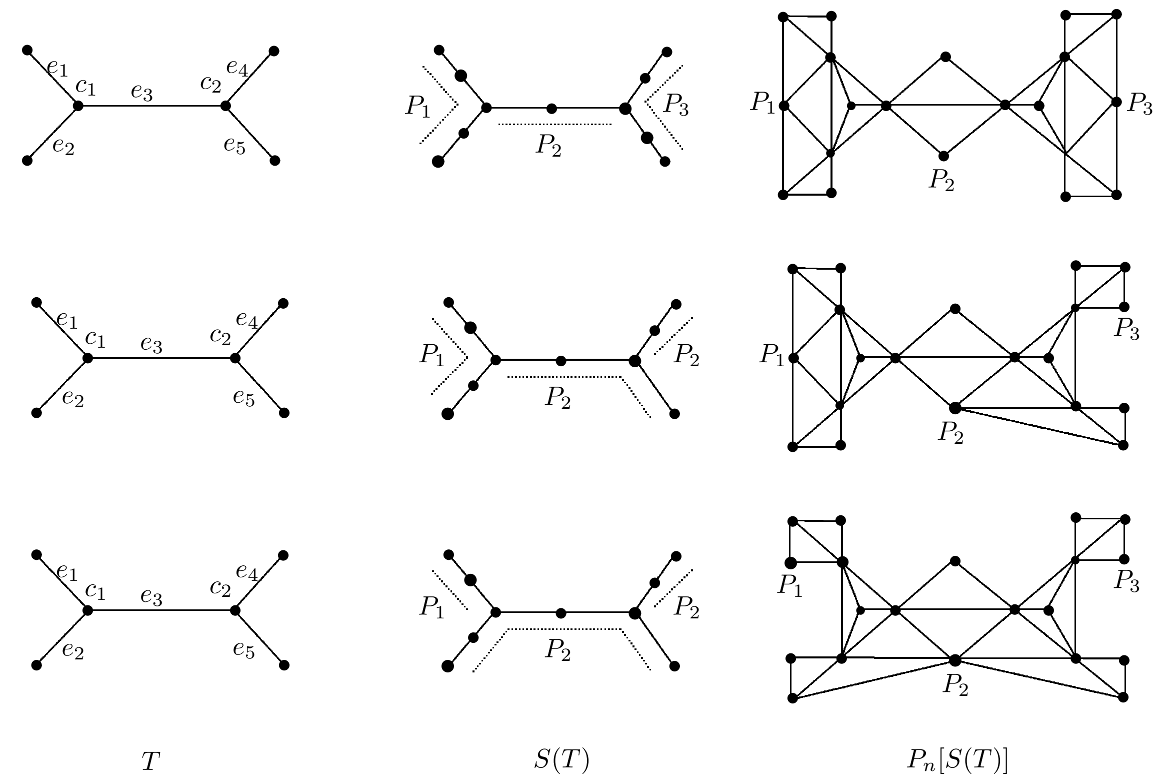

Figure 1

The *line degree* of a line uv in $S(T)$ is the sum of the degrees of u and v. The pathos length is the number of lines which lies on a particular path P_i of pathos of $S(T)$. A pendant pathos is a path P_i of pathos having unit length which corresponds to a pendant line in $S(T)$. A pathos point is a point in $P_n[S(T)]$ corresponding to a path of pathos of $S(T)$. If G is planar graph,the *innerpoint number* $i(G)$ of a graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of the plane. A graph is said to be minimally nonouterplanar if $i(G) = 1$ was given by [4].

We need the following for immediate use.

Remark 1.1 *For any tree T, $n[S(T)]$ is a subgraph of $P_n[S(T)]$.*

Remark 1.2 *For any tree T, $T \subseteq S(T)$.*

Remark 1.3 *If the line degree of a nonpendant line in $S(T)$ is odd(even), the correspondig point in $P_n[S(T)]$ is of even(odd) degree.*

Remark 1.4 *The pendant line in $S(T)$ is always odd degree and the corresponding point in $P_n[S(T)]$ is of odd degree.*

Remark 1.5 *For any tree T with C cutpoints, the number of cutpoints in $n[S(T)]$ is equal to sum of the lines incident to C in T.*

Remark 1.6 *For any tree T, the number of blocks in $n[S(T)]$ is equal to the sum of the cutpoints and lines of T.*

Remark 1.7 *$n[S(T)]$ is connected if and only if T is connected.*

Theorem 1.1([5]) *If G is a non trivial connected (p, q) graph whose points have degree d_i and l_i be the number of lines to which cutpoint C_i belongs in G, then lict graph $n(G)$ has $q + \sum C_i$ points and $-q + \sum[\frac{d_i^2}{2} + l_i]$ lines.*

Theorem 1.2([5]) *The lict graph $n(G)$ of a graph G is planar if and only if G is planar and the degree of each point is atmost 3.*

Theorem 1.3([2]) *Every maximal outerplanar graph G with p points has $2p - 3$ lines.*

Theorem 1.4([6]) *A graph is a nonempty path if and only if it is a connected graph with $p \geq 2$ points and $\sum d_i^2 - 4p + 6 = 0$.*

Theorem 1.5([2]) *A graph G is eulerian if and only if every point of G is of even degree.*

§2. Pathos Lict Subdivision Graph

In the following Theorem we obtain the number of points and lines of $P_n[S(T)]$.

Theorem 2.1 *For any (p, q) graph T, whose points have degree d_i and cutpoints C have degree C_j, then the pathos lict sub-division graph $P_n[S(T)]$ has $(3q + C + P_i)$ points and $\frac{1}{2}\sum d_i^2 + 4q + \sum C_j$ lines.*

Proof By Theorem 1.1, $n(T)$ has $q + \sum c$ points by subdivision of T $n(S(T))$contains $2q + q + \sum c$ points and by Remark 1.1, $P_nS(T)$ will contain $3q + \sum c + P_i$ points, where P_i is the path number. By the definition of $n(T)$, it follows that $L(T)$ is a subgraph of $n(T)$. Also, subgraphs of $L(T)$ are line-disjoint subgraphs of $n[S(T)]$ whose union is $L(T)$ and the cutpoints c of T having degree C_j are also the members of $n[s(T)]$. Hence this implies that $n[s(T)]$ contains $-q + \frac{1}{2}\sum d_i^2 + \sum c_j$ lines. Apart from these lines every subdivision of T generates a line and a cutpoint c of degree 2. This creates $q + 2q$ lines in $n[s(T)]$. Thus $n[S(T)]$ has $\frac{1}{2}\sum d_i^2 + \sum c_j + 2q$ lines. Further, the pathos contribute $2q$ lines to $P_nS(T)$. Hence $P_n[S(T)]$ contains $\frac{1}{2}\sum d_i^2 + \sum c_j + 4q$lines. □

Corollary 2.1 *For any (p, q) graph T, the number of regions in $P_n[S(T)]$ is $2(p + q) - 3$.*

§3. Planar Pathos Lict Sub-division Graph

In this section we obtain the condition for planarity of pathos.

Theorem 3.1 *$P_n[S(T)]$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_n[S(T)]$ is planar. Assume $\Delta(T) \leq 4$. Let v be a point of degree 4 in T. By Remark 1.1, $n(S(T))$ is a subgraph of $P_n[S(T)]$ and by Theorem 1.2, $P_n[S(T)]$ is non-planar. Clearly, $P_n[S(T)]$ is non-planar, a contradiction.

Conversely, suppose $\Delta(T) \leq 3$. By Theorem 1.2, $n[S(T)]$ is planar. Further each block of $n[S(T)]$ is either K_3 or K_4. The pathos point is adjacent to atmost two vertices of each block of $n[S(T)]$. This gives a planar $P_n[S(T)]$. □

We next give a characterization of trees whose pathos lict subdivision of trees are outerplanar and maximal outerplanar.

Theorem 3.2 *The pathos lict sub-division graph $P_n[S(T)]$ of a tree T is outerplanar if and only if $\Delta(T) \leq 2$.*

Proof Suppose $P_n[S(T)]$ is outerplanar. Assume T has a point v of degree 3. The lines incident to v and the cut-point v form $\langle K_4\rangle$ as a subgraph in $n[S(T)]$. Hence $P_n[S(T)]$ is non-outerplanar, a contradiction.

Conversely, suppose T is a path P_m of length $m \geq 1$, by definition each block of $n[S(T)]$ is K_3 and $n[S(T)]$ has $2m - 1$ blocks. Also, $S(T)$ has exactly one path of pathos and the pathos point is adjacent to atmost two points of each block of $n[S(T)]$. The pathos point together with each block form $2m - 1$ number of $\langle K_4 - x\rangle$ subgraphs in $P_n[S(T)]$. Hence $P_n[S(T)]$ is outerplanar. □

Theorem 3.3 *The pathos lict sub-division graph $P_n[S(T)]$ of a tree T is maximal outerplanar if and only if.*

Proof Suppose $P_n[S(T)]$ is maximal outerplanar. Then $P_n[S(T)]$ is connected. Hence by Remark 1.7, T is connected. Suppose $P_n[S(T)]$ is $K_4 - x$, then clearly, T is K_2. Let T be any connected tree with $p > 2$ points, q lines and having path number k and C cut-points. Then clearly, $P_n[S(T)]$ has $3q+k+C$ points and $\frac{1}{2}\sum d_i^2 + 4q + \sum C_j$ lines. Since $P_n[S(T)]$ is maximal outerplanar, by Theorem 1.3, it has $[2(3q+k+C)-3]$ lines. Hence

$$\begin{aligned}\frac{1}{2}\sum d_i^2+4q+\sum C_j &= [2(3q+k+C)-3]\\ &= [2(3(p-1)+k+C)-3]\\ &= 6p-6+2k+2C-3\\ &= 6p+2k+2C-9.\end{aligned}$$

But for $k=1$,

$$\sum d_i^2+8q+2\sum C_j = 12p+4C-18+4,$$
$$\sum d_i^2+2\sum C_j = 4p+4C-6,$$
$$\sum d_i^2+2\sum C_j-4p-4C+6=0.$$

Since every cut-point is of degree two in a path, we have,

$$\sum C_j = 2C.$$

Therefore

$$\sum d_i^2+6-4p = 4C-2x2C=0.$$

Hence $\sum d_i^2+6-4p=0$. By Theorem 1.4, it follows that T is a non-empty path.

Conversely, Suppose T is a non-empty path. We now prove that $P_n[S(T)]$ is maximal outerplanar by induction on the number of points (≥ 2). Suppose T is K_2. Then $P_n[S(T)] = K_4-x$. Hence it is maximal outerplanar. As the inductive hypothesis, let the pathos lict subdivision of a non-empty path P with n points be maximal outerplanar. We now show that $P_n[S(T)]$ of a path P with $n+1$ points is maximal outerplanar. First we prove that it is outerplanar. Let the point and line sequence of the path P' be $v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_n, e_n, v_{n+1}$. P', $S(P')$ and $P_n[S(P')]$ are shown in Figure 2. Without loss of generality, $P'-v_{n+1}=P$. By inductive hypothesis $P_n[S(P)]$ is maximal outerplanar. Now the point v_{n+1} is one point more in $P_n[S(P')]$ than in $P_n[S(P)]$. Also there are only eight lines (e'_{n-1}, e_n), (e'_{n-1}, e_{n-1}), (e_{n-1}, e_n), (e_n, R), (e_n, e'_n), (e_n, C'_n), (C'_n, e'_n), (e'_n, R) more in $P_n[S(P')]$. Clearly, the induced subgraph on the points e'_{n-1}, C_{n-1}, e_n, e'_n, C'_n, R is not K_4. Hence $P_n[S(P')]$ is outerplanar. We now prove $P_n[S(P')]$ is maximal outerplanar. Since $P_n[S(P)]$ is maximal outerplanar, it has $2(3q+C+1)-3$ lines. The outerplanar graph $P_n[S(P')]$ has $2(3q+C+1)-3+8$ lines $=2[3(q+1)+(C+)+1]-3$ lines. By Theorem 1.3, $P_n[S(P')]$ is maximal outerplanar. □

Theorem 3.4 *For any tree T, $P_n[S(T)]$ is minimally nonouterplanar if and only if $\Delta(T)\leq 3$ and T has a unique point of degree 3.*

Proof Suppose $P_n[S(T)]$ is minimally non-outerplanar. Assume $\Delta(T)>3$. By Theorem 3.1, $P_n[S(T)]$ is nonplanar, a contradiction. Hence $\Delta(T)\leq 3$.

Assume $\Delta(T)<3$. By Theorem 3.2, $P_n[S(T)]$ is outerplanar, a contradiction. Thus $\Delta(T)=3$.

Assume there exist two points of degree 3 in T. Then $n[S(T)]$ has at least two blocks as K_4. Any pathos point of $S(T)$ is adjacent to atmost two points of each block in $n[S(T)]$ which gives $i(P_n[S(T)])>1$, a contradiction. Hence T has exactly one point point of degree 3.

Conversely, suppose every point of T has degree ≤ 3 and has a unique point of degree 3, then $n[S(T)]$ has exactly one block as K_4 and remaining blocks are K_3's. Each pathos point is adjacent to atmost two points of each block. Hence $i(P_n[S(T)]) = 1$. □

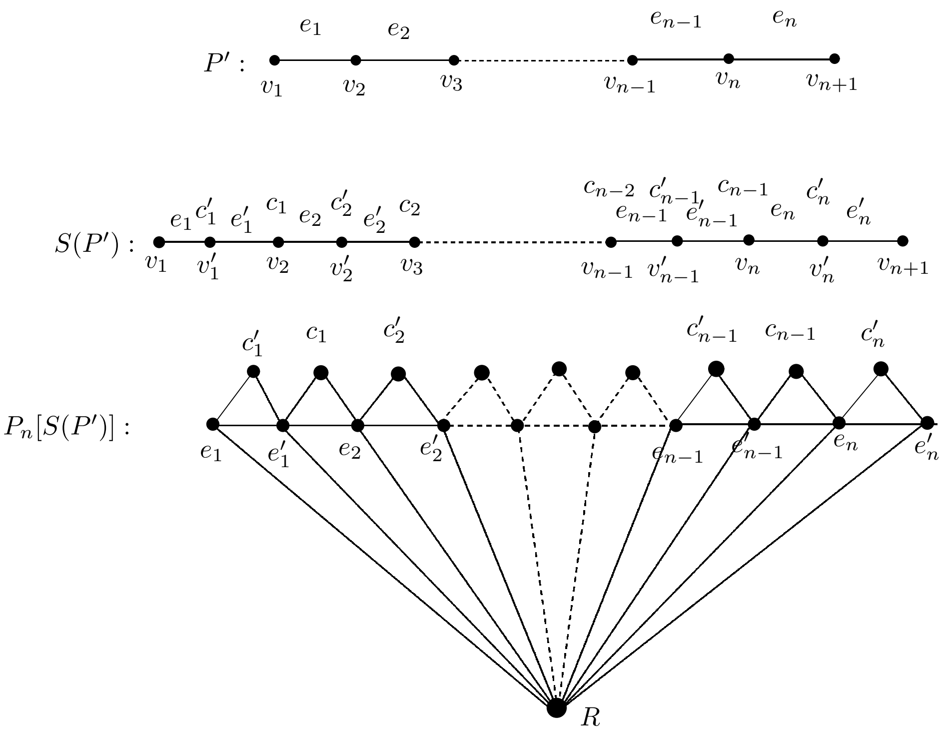

Figure 2

§4. Traversability in Pathos Lict Subdivision of a Tree

In this section, we characterize the trees whose $P_n[S(T)]$ is eulerian and hamiltonian.

Theorem 4.1 *For any non-trivial tree T, the pathos lict subdivision of a tree is non-eulerian.*

Proof Let T be a non-trivial tree. Remark 1.4 implies $P_n[S(T)]$ always contains a point of odd degree. Hence by Theorem 1.5, the result follows. □

Theorem 4.2 *The pathos lict subdivision $P_n[S(T)]$ of a tree T is hamiltonian if and only if every cut-point of T is even of degree.*

Proof If $T = P_2$, then $P_n[S(T)]$ is $K_4 - x$. If T is a tree with $p \geq 3$ points. Suppose $P_n[S(T)]$ is hamiltonian. Assume that T has at least one cut-point v of odd degree m. Then $G = K_{1,m}$ is a subgraph of T. Clearly, $n(S(K_{1,m})) = K_{m+1}$, together with each point of K_m incident to a line of K_3. In number of path of pathos of $S(T)$ there exist at least one path of pathos P_i such that it begins with the cut-point v of $S(T)$. In $P_n[S(T)]$ each pathos point is adjacent to exactly two points of K_m. Further the pathos beginning with the cut-point v of $S(T)$ is adjacent to exactly one point of K_m in $n(S(T))$. Hence this creates a cut-point in $P_n[S(T)]$, a contradiction.

Conversely, suppose every cut-point of T is even. Then every path of pathos starts and ends at pendant points of T.

We consider the following cases.

Case 1 If T has only cut-points of degree two. Clearly, T is a path. Further $S(T)$ is also a path with $p+q$ points and has exactly one path of pathos. Let $T = P_l$, $v_1, v_2, \cdots, v_l$ is a path. Now $S(T)$:$v_1, v_1', v_2, v_2', \cdots, v_{l-1}', v_l$ for all $v_i \in V[S(P_l)]$ such that $v_i v_i' = e_i$, $v_i' v_{i+1} = e_i'$ are consecutive lines and for all $e_i, e_i' \in E[S(P_n)]$. Further $V[n(S(T))]$=$\{e_1, e_1', e_2, e_2', \cdots, e_i, e_i'\} \cup \{C_1', C_1, C_2', C_2, \cdots, C_i'\}$ where,$(C_1', C_1, C_2', C_2, \cdots, C_i')$ are cut-points of $S(T)$. Since each block is a triangle in $n(S(T))$ and each block consist of points as $B_1 = (e_1 C_1' e_1')$, $B_2 = (e_2 C_2' e_2')$, $\cdots$, $B_m = (e_i C_i' e_i')$. In $P_n[S(T)]$, the pathos point w is adjacent to $e_1, e_1', e_2, e_2', \cdots, e_i, e_i'$. Hence, $P_n[S(T)]$= $e_1, e_1', e_2, e_2', \cdots, e_i, e_i' \cup (C_1', C_1, C_2', C_2, \cdots, C_i') \cup w$ form a cycle as $w e_1 C_1' e_1' C_1 e_2 C_2' e_2' \cdots e_i' w$ containing all the points of $P_n[S(T)]$.Hence $P_n[S(T)]$ is hamiltonian.

Case 2 If T has all cut-points of even degree and is not a path.

we consider the following subcases of this case.

Subcase 2.1. If T has exactly one cut-point v of even degree m, $v = \Delta(T)$ and is $K_{1,m}$. Clearly, $S(K_{1,m}) = F$, such that $E(F)$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\}$. Now $n(F)$ contains point set as $\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{v, C_1', C_2', C_3', \cdots, C_q'\}$. For $S[K_{1,m}]$, it has $\frac{m}{2}$ paths of pathos with pathos point as $P_1, P_2, \cdots, P_{\frac{m}{2}}$. By definition of $P_n[S(T)]$, each pathos point is adjacent to exactly two points of $n(S(T))$. Also, $V[P_n[S(T)]]$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{v, C_1', C_2', C_3', \cdots, C_q'\} \cup \{P_1, P_2, \cdots, P_{\frac{m}{2}}\}$. Then there exist a cycle containing all the points of $P_n[S(T)]$ as $P_1, e_1', C_1', e_1, v, e_2, C_2', e_2', P_2, \cdots, P_{\frac{m}{2}}, e_{q-1}', C_{q-1}', e_{q-1}, e_q, C_q', e_q', P_1$.

Subcase 2.2. Assume T has more than one cut-point of even degree. Then in $n(S(T))$ each block is complete and every cut-point lies on exactly two blocks of $n(S(T))$. Let $V[n(S(T))]$=$\{e_1, e_1', e_2, e_2', \cdots, e_q, e_q'\} \cup \{C_1, C_2, \cdots, C_i\} \cup \{C_1', C_2', C_3', \cdots, C_q'\} \cup \{P_1, P_2, \cdots, P_j\}$. But each P_j is adjacent to exactly two point of the block B_j except $\{C_1, C_2, \cdots, C_i\} \cup \{C_1', C_2', C_3', \cdots, C_q'\}$ and all these points together form a hamiltonian cycle of the type, $\{P_1, e_1', C_1', e_1, v, e_2, C_2', e_2', P_2, \cdots, P_r, e_k', C_k', e_k, e_{k+1}, C_{k+1}', e_{k+1}', P_{r+1}, \cdots, P_j, e_{q-1}', C_{q-1}', e_{q-1}, e_q, C_q', e_q', P_1\}$.

Hence $P_n[S(T)]$ is hamiltonian. □

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If winter comes, can spring be far behind?

By P.B.Shelley, a British poet.

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Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

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Mathematical Combinatorics (*International Book Series*) **December, 2010**

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ISBN 9781599731360

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