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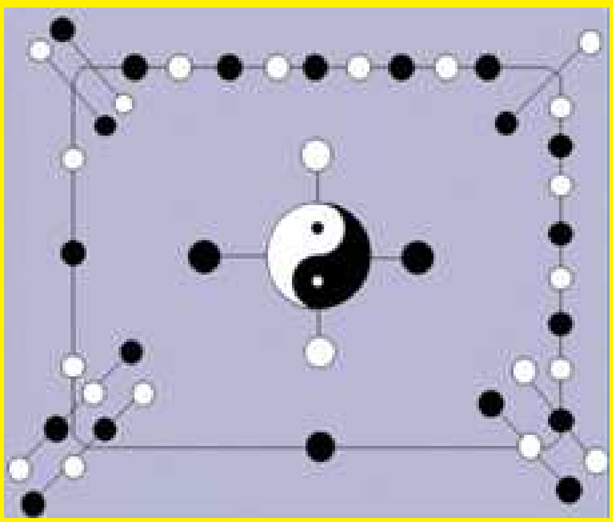

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Few things are impossible in themselves; and it is often for want of will, rather than of means, that man fails to succeed.

By La Rocheforcauld, a French writer.

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Smarandache Curves of Curves lying on Lightlike Cone in $\mathbb{R}^3_1$

Tanju Kahraman[1] and Hasan Hüseyin Uğurlu[2]

1. Celal Bayar University, Department of Mathematics, Manisa-Turkey.
2. Gazi University, Gazi Faculty of Education, Mathematics Teaching Program, Ankara-Turkey.

E-mail: tanju.kahraman@cbu.edu.tr, hugurlu@gazi.edu.tr

Abstract: In this paper, we consider the notion of the Smarandache curves by considering the asymptotic orthonormal frames of curves lying fully on lightlike cone in Minkowski 3-space $\mathbb{R}^3_1$. We give the relationships between Smarandache curves and curves lying on lightlike cone in $\mathbb{R}^3_1$.

Key Words: Minkowski 3-space, Smarandache curves, lightlike cone, curvatures.

AMS(2010): 53A35, 53B30, 53C50

§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the related curves for which there exist corresponding relations between the curves are very interesting and an important problem. The most fascinating examples of such curves are associated curves and special curves. Recently, a new special curve is called Smarandache curve is defined by Turgut and Yılmaz in Minkowski space-time [9]. These curves are called Smarandache curves: If a regular curve in Euclidean 3-space, whose position vector is composed by Frenet vectors on another regular curve, then the curve is called a Smarandache Curve. Then, Ali have studied Smarandache curves in the Euclidean 3-space E^3[1]. Kahraman and Uğurlu have studied dual Smarandache curves of curves lying on unit dual sphere $\tilde{S}^2$ in dual space D^3 [3] and they have studied dual Smarandache curves of curves lying on unit dual hyperbolic sphere $\tilde{H}^2_0$ in D^3_1 [4]. Also, Kahraman, Önder and Uğurlu have studied Blaschke approach to dual Smarandache curves [2]

In this paper, we consider the notion of the Smarandache curves by means of the asymptotic orthonormal frames of curves lying fully on Lightlike cone in Minkowski 3-space $\mathbb{R}^3_1$. We show the relationships between frames and curvatures of Smarandache curves and curves lying on lightlike cone in $\mathbb{R}^3_1$.

§2. Preliminaries

The Minkowski 3-space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ provided with the standart flat metric

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given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of IR_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{x} = \vec{x}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{x}'(s)$ are respectively spacelike, timelike or null (lightlike) [6, 7]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}_1^3$, in the meaning of Lorentz vector product of $\vec{a}$ and $\vec{b}$ is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 -\ a_1 b_2).$$

Denote by $\left\{\vec{T},\ \vec{N},\ \vec{B}\right\}$ the moving Frenet along the curve $x(s)$ in the Minkowski space $\mathbb{R}_1^3$. For an arbitrary spacelike curve $x(s)$ in the space $\mathbb{R}_1^3$, the following Frenet formulae are given ([8]),

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \varepsilon\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = 1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \varepsilon = \pm 1$, $\left\langle \vec{B}, \vec{B} \right\rangle = -\varepsilon$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve $x(s)$ respectively [10]. Here, ε determines the kind of spacelike curve $x(s)$. If $\varepsilon = 1$, then $x(s)$ is a spacelike curve with spacelike first principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $x(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$ [8]. Furthermore, for a timelike curve $x(s)$ in the space $\mathbb{R}_1^3$, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = -1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \left\langle \vec{B}, \vec{B} \right\rangle = 1$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and κ and τ are curvature and torsion of the timelike curve $x(s)$ respectively [10].

Curves lying on lightlike cone are examined using moving asymptotic frame which is denoted by $\{\vec{x},\ \vec{\alpha},\ \vec{y}\}$ along the curve $x(s)$ lying fully on lightlike cone in the Minkowski space $\mathbb{R}_1^3$.

For an arbitrary curve $x(s)$ lying on lightlike cone in $\mathbb{R}_1^3$, the following asymptotic frame formulae are given by

$$\begin{bmatrix} \vec{x}' \\ \vec{\alpha}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -1 \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{bmatrix}$$

where $\langle \vec{x}, \vec{x} \rangle = \langle \vec{y}, \vec{y} \rangle = \langle \vec{x}, \vec{\alpha} \rangle = \langle \vec{y}, \vec{\alpha} \rangle = 0, \quad \langle \vec{x}, \vec{y} \rangle = \langle \vec{\alpha}, \vec{\alpha} \rangle = 1$ and κ is curvature function of curve $\alpha(s)$ [5].

§3. Smarandache Curves of Curves lying on Lightlike Cone in Minkowski 3-Space $\mathbb{R}^3_1$

In this section, we first define the four different type of the Smarandache curves of curves lying fully on lightlike cone in $\mathbb{R}^3_1$. Then, by the aid of asymptotic frame, we give the characterizations between reference curve and its Smarandache curves.

3.1 Smarandache $\vec{x}\vec{\alpha}$-curves of curves lying on lightlike cone in $\mathbb{R}^3_1$

Definition 3.1 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_1 defined by*

$$\vec{\alpha}_1(s) = \vec{x}(s) + \vec{\alpha}(s) \tag{3.1}$$

is called the Smarandache $\vec{x}\vec{\alpha}$-curve of x and α_1 fully lies on Lorentzian sphere S^2_1.

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{\alpha}$-curve α_1 as follows.

Theorem 3.1 *Let $x = x(s)$ be a unit speed regular curve lying on Lightlike cone in $\mathbb{R}^3_1$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{x}\vec{\alpha}$-curve α_1 are given by*

$$\begin{pmatrix} \vec{T}_1 \\ \vec{N}_1 \\ \vec{B}_1 \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{1-2\kappa}} & \frac{1}{\sqrt{1-2\kappa}} & \frac{-1}{\sqrt{1-2\kappa}} \\ \frac{\kappa'+\kappa(-\kappa'-2\kappa+1)}{(1-2\kappa)^2\sqrt{A}} & \frac{\kappa'+2\kappa-4\kappa^2}{(1-2\kappa)^2\sqrt{A}} & \frac{2\kappa-\kappa'-1}{(1-2\kappa)^2\sqrt{A}} \\ \frac{-1}{\sqrt{1-2\kappa}\sqrt{A}} & \frac{\kappa'}{(1-2\kappa)^{3/2}\sqrt{A}} & \frac{\kappa'+\kappa-2\kappa^2}{(1-2\kappa)^{3/2}\sqrt{A}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix} \tag{3.2}$$

where κ is curvature function of $x(s)$ and A is

$$A = \frac{(2\kappa-1)(\kappa')^2 + (8\kappa - 8\kappa^2 - 2)\kappa' - 16\kappa^4 - 24\kappa^3 + 4\kappa^2 - 2\kappa}{(1-2\kappa)^4}.$$

Proof Let the Frenet of Smarandache $\vec{x}\vec{\alpha}$-curve be $\left\{\vec{T}_1, \vec{N}_1, \vec{B}_1\right\}$. Since $\vec{\alpha}_1(s) = \vec{x}(s) + \vec{\alpha}(s)$

and $\vec{T}_1 = \vec{\alpha}_1' / \|\vec{\alpha}_1'\|$, we have

$$\vec{T}_1 = \frac{\kappa}{\sqrt{1-2\kappa}}\vec{x}(s) + \frac{1}{\sqrt{1-2\kappa}}\vec{\alpha}(s) - \frac{1}{\sqrt{1-2\kappa}}\vec{y}(s) \tag{3.3}$$

where

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{1-2\kappa}} \quad \text{and} \quad \kappa < \frac{1}{2}.$$

Since $\vec{N}_1 = \vec{T}_1' \Big/ \left\|\vec{T}_1'\right\|$, we get

$$\vec{N}_1 = \frac{\kappa' - \kappa\kappa' + \kappa - 2\kappa^2}{(1-2\kappa)^2\sqrt{A}}\vec{x}(s) + \frac{\kappa' + 2\kappa - 4\kappa^2}{(1-2\kappa)^2\sqrt{A}}\vec{\alpha}(s) + \frac{2\kappa - \kappa' - 1}{(1-2\kappa)^2\sqrt{A}}\vec{y}(s) \tag{3.4}$$

where

$$A = \frac{(2\kappa-1)(\kappa')^2 + (8\kappa - 8\kappa^2 - 2)\kappa' - 16\kappa^4 - 24\kappa^3 + 4\kappa^2 - 2\kappa}{(1-2\kappa)^4}.$$

Then from $\vec{B}_1 = \vec{T}_1 \times \vec{N}_1$, we have

$$\vec{B}_1 = \frac{-1}{\sqrt{1-2\kappa}\sqrt{A}}\vec{x}(s) + \frac{\kappa'}{(1-2\kappa)^{3/2}\sqrt{A}}\vec{\alpha}(s) + \frac{\kappa' - 2\kappa^2 + \kappa}{(1-2\kappa)^{3/2}\sqrt{A}}\vec{y}(s). \tag{3.5}$$

From (3.3)-(3.5) we have (3.2). □

Theorem 3.2 *The curvature function κ_1 of Smarandache $\vec{x}\vec{\alpha}$-curve α_1 according to curvature function of curve x is given by*

$$\kappa_1 = \frac{\sqrt{A}}{(1-2\kappa)^2}. \tag{3.6}$$

Proof Since $\kappa_1 = \left\|\vec{T}_1'\right\|$. Using the equation (3.3), we get the desired equality (3.6). □

Corollary 3.1 *If curve x is a line. Then Smarandache $\vec{x}\vec{\alpha}$-curve α_1 is line.*

Theorem 3.3 *Torsion τ_1 of Smarandache $\vec{x}\vec{\alpha}$-curve α_1 according to curvature function of curve x is as follows*

$$\begin{aligned}\tau_1 \quad = \quad & \frac{A - A' + 2\kappa A' - \kappa\kappa' A - \kappa' A' - 2\kappa\kappa' A' + 4\kappa^2 A - A\kappa' - 2A\kappa + 2\kappa\kappa'' A}{(1-2\kappa)^3 A^2} \\ & + \frac{\kappa' A - A\kappa\kappa'^2 + \kappa'\kappa'' A - \kappa'^2 A' - 6A\kappa\kappa' - \frac{3A\kappa'^2}{1-2\kappa}}{(1-2\kappa)^4 A^2}\end{aligned}$$

Proof Since $\tau_1 = \left\langle \frac{dB_1}{ds_1}, N_1 \right\rangle$. Using derivation of the equation (3.5), we obtain the wanted equation. □

Corollary 3.2 *If Smarandache $\vec{x}\vec{\alpha}$-curve α_1 is a plane curve. Then, we obtain*

$$\big(A - A' + 2\kappa A' - \kappa\kappa' A - \kappa' A' - 2\kappa\kappa' A' + 4\kappa^2 A - A\kappa' - 2A\kappa + 2\kappa\kappa'' A\big)$$
$$+ (1-2\kappa) + \kappa' A - A\kappa\kappa'^2 + \kappa'\kappa'' A - \kappa'^2 A' - 6A\kappa\kappa' - \frac{3A\kappa'^2}{1-2\kappa} = 0.$$

3.2 Smarandache $\vec{x}\vec{y}$-curves of curves lying on Lightlike cone in $\mathbb{R}_1^3$

In this section, we define the second type of Smarandache curves that is called Smarandache $\vec{x}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{x}\vec{y}$-curve.

Definition 3.2 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_2 defined by*

$$\vec{\alpha}_2(s) = \frac{1}{\sqrt{2}}\left(\vec{x}(s) + \vec{y}(s)\right)$$

is called the Smarandache $\vec{x}\vec{y}$-curve of x and fully lies on Lorentzian sphere S_1^2.

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{y}$-curve α_2 as follows.

Theorem 3.4 *Let $x = x(s)$ be a unit speed regular curve lying on lightlike cone in $\mathbb{R}_1^3$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{x}\vec{y}$-curve α_2 are given by*

$$\begin{pmatrix} \vec{T}_2 \\ \vec{N}_2 \\ \vec{B}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\kappa}{\sqrt{-2\kappa}} & 0 & \frac{-1}{\sqrt{-2\kappa}} \\ \frac{1}{\sqrt{-2\kappa}} & 0 & \frac{-\kappa}{\sqrt{-2\kappa}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix}$$

where κ is curvature function of $x(s)$.

Theorem 3.5 *The curvature function κ_2 of Smarandache $\vec{x}\vec{y}$-curve α_2 according to curvature function of curve x is given*

$$\kappa_2 = \sqrt{-2\kappa}.$$

Corollary 3.3 *Curve x is a line if and only if Smarandache $\vec{x}\vec{y}$-curve α_2 is line.*

Theorem 3.6 *Torsion τ_2 of Smarandache $\vec{x}\vec{y}$-curve α_2 according to curvature function of curve x is as follows*

$$\tau_2 = \frac{-2\sqrt{2}\kappa'}{4\kappa^2 - 4\kappa^3}.$$

Corollary 3.4 *If Smarandache $\vec{x}\vec{y}$-curve α_2 is a plane curve. Then, curvature κ of curve x is constant.*

3.3 Smarandache $\vec{\alpha}\vec{y}$-curves of curves lying on lightlike cone in $\mathbb{R}^3_1$

In this section, we define the third type of Smarandache curves that is called Smarandache $\vec{\alpha}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{\alpha}\vec{y}$-curve.

Definition 3.3 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_3 defined by*

$$\vec{\alpha}_3(s) = \vec{\alpha}(s) + \vec{y}(s)$$

is called the Smarandache $\vec{\alpha}\vec{y}$-curve of x and fully lies on Lorentzian sphere S^2_1.

Now we can give the relationships between x and its Smarandache $\vec{\alpha}\vec{y}$-curve α_3 as follows.

Theorem 3.7 *Let $x = x(s)$ be a unit speed regular curve lying on lightlike cone in $\mathbb{R}^3_1$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{\alpha}\vec{y}$-curve α_3 are given by*

$$\begin{pmatrix} \vec{T}_3 \\ \vec{N}_3 \\ \vec{B}_3 \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{\kappa^2-2\kappa}} & \frac{-\kappa}{\sqrt{\kappa^2-2\kappa}} & \frac{-1}{\sqrt{\kappa^2-2\kappa}} \\ \frac{-\kappa^4+2\kappa^3}{\sqrt{A}} & \frac{-2\kappa^2\kappa'+4\kappa\kappa'+2\kappa^3-4\kappa^2}{\sqrt{A}} & \frac{\kappa\kappa'-\kappa'+\kappa^3-2\kappa^2}{\sqrt{A}} \\ \frac{-3\kappa^2\kappa'+5\kappa\kappa'-\kappa^4+4\kappa^3-4\kappa^2}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} & \frac{\kappa^2\kappa'-\kappa\kappa'}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} & \frac{\kappa^5-4\kappa^4+4\kappa^3+2\kappa^3\kappa'-4\kappa^2\kappa'}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix}$$

where κ is curvature function of $x(s)$ and A is

$$A = (4\kappa^4 - 16\kappa^3 + 16\kappa^2)(\kappa')^2 + \kappa'(-10\kappa^5 + 38\kappa^4 - 36\kappa^3) - 2\kappa^7 + 12\kappa^6 - 24\kappa^5 + 16\kappa^4.$$

Theorem 3.8 *The curvature function κ_3 of Smarandache $\vec{\alpha}\vec{y}$-curve α_3 according to curvature function of curve x is given*

$$\kappa_3 = \frac{\sqrt{(4\kappa^4 - 16\kappa^3 + 16\kappa^2)(\kappa')^2 + \kappa'(-10\kappa^5 + 38\kappa^4 - 36\kappa^3) - 2\kappa^7 + 12\kappa^6 - 24\kappa^5 + 16\kappa^4}}{(\kappa^2 - 2\kappa)^2}.$$

Corollary 3.5 *If curve x is a line. Then, Smarandache $\vec{\alpha}\vec{y}$-curve α_3 is line.*

Theorem 3.9 *Torsion τ_3 of Smarandache $\vec{\alpha}\vec{y}$-curve α_3 according to curvature function of curve x is as follows*

$$\tau_3 = \frac{b_1\left(\kappa\kappa' - \kappa' + \kappa^3 - 2\kappa^2\right)}{A(\kappa^2 - 2\kappa)} + \frac{2b_2\left(\kappa - \kappa'\right)}{A} - \frac{b_3\kappa^2}{A}$$

where b_1, b_2 *and* b_3 *are*

$$\begin{aligned} b_1 &= -6\kappa(\kappa')^2 + 9\kappa^2\kappa' + 5(\kappa')^2 + 5\kappa\kappa'' - 4\kappa^3\kappa' - 8\kappa\kappa' \\ &\quad + (3\kappa^2\kappa' - 5\kappa\kappa' + \kappa^4 - 4\kappa^3 + 4\kappa^2)\left(\frac{A'}{2A} + \frac{\kappa\kappa' - \kappa'}{\kappa^2 - 2\kappa}\right) \\ b_2 &= (\kappa^2 - \kappa)\kappa'' + (2\kappa - 1)(\kappa')^2 - (\kappa^2\kappa' - \kappa\kappa')\left(\frac{A'}{2A} + \frac{\kappa\kappa' - \kappa'}{\kappa^2 - 2\kappa}\right) \\ b_3 &= (5\kappa^2 - 16\kappa + 12)\kappa^2\kappa' + (2\kappa - 8)\kappa(\kappa')^2 + 2\kappa^3\kappa'' \\ &\quad - (2\kappa^3\kappa' - 4\kappa^2\kappa' + \kappa^5 - 4\kappa^4 + 4\kappa^3)\left(\frac{A'}{2A} + \frac{\kappa\kappa' - \kappa'}{\kappa^2 - 2\kappa}\right). \end{aligned}$$

Corollary 3.6 *If Smarandache* $\vec{\alpha}\vec{y}$*-curve* α_3 *is a plane curve. Then, we obtain*

$$b_1\left(\kappa\kappa' - \kappa' + \kappa^3 - 2\kappa^2\right) + (\kappa^2 - 2\kappa)\left(2b_2(\kappa - \kappa') - b_3\kappa^2\right) = 0.$$

3.4 Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curves of curves lying on lightlike cone in $\mathbb{R}^3_1$

In this section, we define the fourth type of Smarandache curves that is called Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve.

Definition 3.4 *Let* $x = x(s)$ *be a unit speed regular curve lying fully on Lightlike cone and* $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ *be its moving asymptotic frame. The curves* α_4 *and* α_5 *defined by*

(i) $\vec{\alpha}_4(s) = \frac{1}{\sqrt{3}}(\vec{x}(s) + \vec{\alpha}(s) + \vec{y}(s))$*;*

(ii) $\vec{\alpha}_5(s) = -\vec{x}(s) + \vec{\alpha}(s) + \vec{y}(s)$

are called the Smarandache $\vec{x}\vec{\alpha}\vec{y}$*-curves of* x *and fully lies on Lorentzian sphere* S^2_1 *and hyperbolic sphere* H^3_0*.*

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 on Lorentzian sphere S^2_1 as follows.

Theorem 3.10 *Let* $x = x(s)$ *be a unit speed regular curve lying on lightlike cone in* $\mathbb{R}^3_1$*. Then the relationships between the asymptotic frame of* x *and Frenet of its Smarandache* $\vec{x}\vec{\alpha}\vec{y}$*-curve* α_4 *are given by*

$$\begin{pmatrix} \vec{T}_4 \\ \vec{N}_4 \\ \vec{B}_4 \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{\kappa^2-4\kappa+1}} & \frac{1-\kappa}{\sqrt{\kappa^2-4\kappa+1}} & \frac{-1}{\sqrt{\kappa^2-4\kappa+1}} \\ \frac{a_1}{\sqrt{2a_1c_1+b_1^2}} & \frac{b_1}{\sqrt{2a_1c_1+b_1^2}} & \frac{c_1}{\sqrt{2a_1c_1+b_1^2}} \\ \frac{c_1(1-\kappa)+b_1}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}} & \frac{c_1\kappa+a_1}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}} & \frac{a_1(\kappa-1)-b_1\kappa}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix}$$

where κ is curvature function of $x(s)$ and a_1, b_1, c_1 are

$$\begin{aligned} a_1 &= \frac{\sqrt{3}(\kappa' - 2\kappa\kappa' - 5\kappa^2 + 5\kappa^3 - \kappa^4 + \kappa)}{(\kappa^2 - 4\kappa + 1)^2} \\ b_1 &= \frac{\sqrt{3}(2\kappa + \kappa' - 8\kappa^2 + 2\kappa\kappa' + 2\kappa^3)}{(\kappa^2 - 4\kappa + 1)^2} \\ c_1 &= \frac{\sqrt{3}(-2\kappa' + \kappa\kappa' + 5\kappa - 5\kappa^2 + \kappa^3 - 1)}{(\kappa^2 - 4\kappa + 1)^2}. \end{aligned}$$

Theorem 3.11 *The curvature function κ_4 of Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 according to curvature function of curve x is given*

$$\kappa_4 = \sqrt{2a_1c_1 + b_1^2}.$$

Corollary 3.7 *If curve x is a line. Then, Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 is line.*

Theorem 3.12 *Torsion τ_4 of Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 according to curvature function of curve x is as follows*

$$\tau_4 = \frac{a_2c_1 + b_1b_2 + a_1c_2}{\sqrt{2a_1c_1 + b_1^2}}$$

where a_2, b_2 and c_2 are

$$\begin{aligned} a_2 &= \frac{\sqrt{3}(c_1' - c_1'\kappa - c_1\kappa' + b_1' + c_1\kappa^2 + a_1\kappa) - \sqrt{3}(c_1 - c_1\kappa + b_1)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}} \\ b_2 &= \frac{\sqrt{3}(c_1'\kappa + c_1\kappa' + a_1' + c_1 - c_1\kappa + b_1 + a_1\kappa - a_1\kappa^2 + b_1\kappa^2) - \sqrt{3}(c_1\kappa + a_1)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}} \\ c_2 &= \frac{\sqrt{3}(a_1'\kappa - a_1' + a_1\kappa' - b_1'\kappa - b_1\kappa' - c_1\kappa - a_1) - \sqrt{3}(a_1\kappa - a_1 - b_1\kappa)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}}. \end{aligned}$$

Corollary 3.8 *If Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 is a plane curve. Then, we obtain*

$$a_2c_1 + b_1b_2 + a_1c_2 = 0.$$

Results of statement (ii) can be given by using the similar ways used for the statement (i).

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On $((r_1, r_2), m, (c_1, c_2))$-Regular Intuitionistic Fuzzy Graphs

N.R.Santhi Maheswari

(Department of Mathematics, G. Venkataswamy Naidu College, Kovilpatti-628502, Tamil Nadu, India)

C.Sekar

(Post Graduate Extension Centre, Nagercoil-629901,Tamil Nadu, India)

E-mail: nrsmaths@yahoo.com, sekar.acas@gmail.com

Abstract: In this paper, $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph and totally $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graphs are introduced. A relation between $((r_1, r_2), m, (c_1, c_2))$-regularity and totally $((r_1, r_2), m, (c_1, c_2))$-regularity on Intuitionistic fuzzy graph is studied. A necessary and sufficient condition under which they are equivalent is provided. Also, $((r_1, r_2), m, (c_1, c_2))$-regularity on some intuitionistic fuzzy graphs whose underlying crisp graphs is a cycle is studied with some specific membership functions.

Key Words: Degree and total degree of a vertex in intuitionistic fuzzy graph, d_m-degree and total d_m-degree of a vertex in intuitionistic fuzzy graph, $(m, (c_1, c_2))$- intuitionistic regular fuzzy graphs, totally $(m, (c_1, c_2))$-intuitionistic regular fuzzy graphs.

AMS(2010): 05C12, 03E72, 05C72

§1. Introduction

In 1965, Lofti A. Zadeh [18] introduced the concept of fuzzy subset of a set as method of representing the phenomena of uncertainty in real life situation. K.T.Attanassov [1]introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. K.T.Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set(and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than one.

Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [1, 2]. Azriel Rosenfeld introduced the concept of fuzzy graphs in 1975 [5]. It has been growing fast and has numerous application in various fields. Bhattacharya [**?**] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Morderson and Peng [9].

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Krassimir T Atanassov [2] introduced the intuitionistic fuzzy graph theory. R.Parvathi and M.G.Karunambigai [8] introduced intuitionistic fuzzy graphs as a special case of Atanassov's IFG and discussed some properties of regular intuitionistic fuzzy graphs [6]. M.G. Karunambigai and R.Parvathi and R.Buvaneswari introduced constant intuitionistic fuzzy graphs [7]. M. Akram, W. Dudek [3] introduced the regular intuitionistic fuzzy graphs. M.Akram and Bijan Davvaz [4] introduced the notion of strong intuitionistic fuzzy graphs and discussed some of their properties.

N.R.Santhi Maheswari and C.Sekar introduced d_2- degree of vertex in fuzzy graphs and introduced $(r, 2, k)$-regular fuzzy graphs and totally $(r, 2, k)$-regular fuzzy graphs [11]. S.Ravi Narayanan and N.R.Santhi Maheswari introduced $((2, (c_1, c_2)$-regular bipolar fuzzy graphs [13]. Also, they introduced d_m-degree, total d_m-degree, of a vertex in fuzzy graphs and introduced an m-neighbourly irregular fuzzy graphs [12, 15], (m, k)-regular fuzzy graphs [14, 15] and (r, m, k)-regular fuzzy graphs [15, 16].

N.R.Santhi Maheswari and C.Sekar introduced d_m- degree of a vertex in intuitionistic fuzzy graphs and introduced $(m, (c_1, c_2))$-regular fuzzy graphs and totally $(m, (c_1, c_2))$-regular fuzzy graphs [17]. These motivates us to introduce $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graphs and totally $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graphs.

§2. Preliminaries

We present some known definitions related to fuzzy graphs and intuitionistic fuzzy graphs for ready reference to go through the work presented in this paper.

Definition 2.1([9]) *A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \rightarrow [0,1]$ is a fuzzy subset of a non empty set V and $\mu : VXV \rightarrow [0, 1]$ is a symmetric fuzzy relation on σ such that for all u, v in V, the relation $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ is satisfied.*

Definition 2.2([12]) *Let $G : (\sigma, \mu)$ be a fuzzy graph. The d_m-degree of a vertex u in G is $d_m(u) = \sum \mu^m(uv)$, where $\mu^m(uv) = sup\{\mu(uu_1) \wedge \mu(u_1u_2) \wedge \ldots, \mu(u_{m-1}v) : u, u_1, u_2, \ldots, u_{m-1}, v$ is the shortest path connecting u and v of length $m\}$. Also, $\mu(uv) = 0$, for uv not in E.*

Definition 2.3([12]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The total d_m-degree of a vertex $u \in V$ is defined as $td_m(u) = \sum \mu^m(uv) + \sigma(u) = d_m(u) + \sigma(u)$.*

Definition 2.4([12]) *If each vertex of G has the same d_m - degree k, then G is said to be an (m, k)-regular fuzzy graph.*

Definition 2.5([12]) *If each vertex of G has the same total d_m - degree k, then G is said to be totally (m, k)-regular fuzzy graph.*

Definition 2.6([15, 16]) *If each vertex of G has the same degree r and has the same d_m-degree k, then G is said to be (r, m, k)-regular fuzzy graph.*

Definition 2.7([15, 16]) *If each vertex of G has the same total degree r and has the same total d_m-degree k, then G is said to be totally (r, m, k)-regular fuzzy.*

Definition 2.8([7]) *An intuitionistic fuzzy graph with underlying set V is defined to be a pair $G = (V, E)$ where*

(1) $V = \{v_1, v_2, v_3, \cdots, v_n\}$ such that $\mu_1 : V \to [0,1]$ and $\gamma_1 : V \to [0,1]$ denote the degree of membership and nonmembership of the element $v_i \in V, (i = 1, 2, 3, \cdots, n)$,such that $0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1$;

(2) $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0,1]$ and $\gamma_2 : V \times V \to [0,1]$ are such that $\mu_2(v_i, v_j) \leq min\{\mu_1(v_i), \mu_1(v_j)\}$ and $\gamma_2(v_i, v_j) \leq max\{\gamma_1(v_i), \gamma_1(v_j)\}$ and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E, (i, j = 1, 2, \cdots, n)$.

Definition 2.9([7]) *If $v_i, v_j \in V \subseteq G$, the μ-strength of connectedness between two vertices v_i and v_j is defined as $\mu_2^{\infty}(v_i, v_j) = sup\{\mu_2^k(v_i, v_j) : k = 1, 2, \cdots, n\}$ and γ-strength of connectedness between two vertices v_i and v_j is defined as $\gamma_2^{\infty}(v_i, v_j) = inf\{\gamma_2^k(v_i, v_j) : k = 1, 2, \cdots, n\}$.*

If u and v are connected by means of paths of length k then $\mu_2^k(u, v)$ is defined as sup $\{\mu_2(u, v_1) \wedge \mu_2(v_1, v_2) \wedge \cdots \wedge \mu_2(v_{k-1}, v)\colon (u, v_1, v_2, \cdots, v_{k-1}, v) \in V\}$ and $\gamma_2^k(u, v)$ is defined as inf $\{\gamma_2(u, v_1) \vee \gamma_2(v_1, v_2) \vee \cdots \vee \gamma_2(v_{k-1}, v)\colon (u, v_1, v_2, \cdots, v_{k-1}, v) \in V\}$.

Definition 2.10([7]) *Let $G = (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. Then the degree of a vertex $v_i \in G$ is defined by $d(v_i) = (d_{\mu_1}(v_i), d_{\gamma_1}(v_i))$, where $d_{\mu_1}(v_i) = \sum \mu_2(v_i, v_j)$ and $d_{\gamma_1}(v_i) = \sum \gamma_2(v_i, v_j)$ for $(v_i, v_j) \in E$ and $\mu_2(v_i, v_j) = 0$ and $\gamma_2(v_i, v_j) = 0$ for $(v_i, v_j) \notin E$.*

Definition 2.11([7]) *Let $G = (V, E)$ be an Intuitionistic fuzzy graph on $G^*(V, E)$. Then the total degree of a vertex $v_i \in G$ is defined by $td(v_i) = (td_{\mu_1}(v_i), td_{\gamma_1}(v_i))$,where $td_{\mu_1}(v_i) = d\mu_1(v_i) + \mu_1(v_i)$ and $td_{\gamma_1}(v_i) = d\gamma_1(v_i) + \gamma_1(v_i)$.*

Definition 2.12([17]) *Let $G = (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. Then the d_m - degree of a vertex $v \in G$ is defined by $d_{(m)}(v) = (d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v))$, where $d_{(m)\mu_1}(v) = \sum \mu_2^{(m)}(u, v)$ where $\mu_2^{(m)}(u, v) = sup\{\mu_2(u, u_1) \wedge \mu_2(u_1, u_2) \wedge \cdots \wedge \mu_2(u_{m-1}, v) : u, u_1, u_2, \cdots, u_{m-1}, v$ is the shortest path connecting u and v of length m)\} and $d_{(m)\gamma_1}(v) = \sum \gamma_2^{(m)}(u, v)$, where $\gamma_2^{(m)}(u, v)$=inf $\{\gamma_2(u, u_1) \vee \gamma_2(u_1, u_2) \vee \cdots \vee \gamma_2(u_{m-1}, v) : u, u_1, u_2, \cdots, u_{m-1}, v$ is the shortest path connecting u and v of length m\}. The minimum d_m-degree of G is $\delta_m(G)$ $= \wedge\{(d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v)) : v \in V\}$.*

The maximum d_m-degree ofG is $\Delta_m(G) = \vee\{(d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v)) : v \in V\}$..

Definition 2.13([17]) *Let G : (V, E) be an intuitionistic fuzzy graph on $G^*(V, E)$. If all the vertices of G have same d_m- degree c_1,c_2, then G is said to be a $(m, (c_1, c_2))$- regular intuitionistic fuzzy graph.*

Definition 2.14([17]) *Let $G = (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. Then the total d_m-degree of a vertex $v \in G$ is defined by $td_{(m)}(v) = (td_{(m)\mu_1}(v), td_{(m)\gamma_1}(v))$, where $td_{(m)\mu_1}(v) = d_{(m)\mu_1}(v) + \mu_1(v)$ and $td_{(m)\gamma_1}(v) = d_{(m)\gamma_1}(v) + \gamma_1(v)$. The minimum td_m-degree of G is $t\delta_m(G) = \wedge\{(td_{(m)\mu_1}(v),\ td_{(m)\gamma_1}(v)) : v \in V\}$. The maximum td_m-degree of G is $t\Delta_m(G) = \vee\{(td_{(m)\mu_1}(v), td_{(m)\gamma_1}(v)) : v \in V\}$.*

Definition 2.15([17]) *Let $G = (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. If each vertex of G has same total d_m - degree c_1,c_2, then G is said to be totally $(m,(c_1,c_2))$ - regular intuitionistic fuzzy graph.*

§3. $((r_1, r_2), m, (c_1, c_2))$- Regular intuitionistic Fuzzy Graphs

Definition 3.1 *Let $G : (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. If $d(v) = (r_1, r_2)$ and $d_{(m)}(v) = (c_1, c_2)$ for all $v \in V$, then G is said to be $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph. That is, if each vertex of G has the same degree (r_1, r_2) and has the same d_m-degree (c_1, c_2), then G is said to be $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph.*

Example 3.2 Consider an intuitionistic fuzzy graph on $G^*(V, E)$, a cycle of length 7.

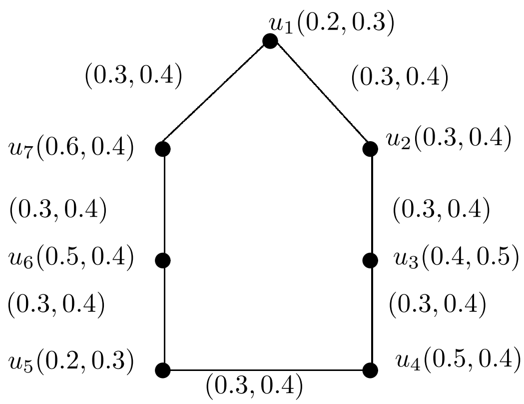

Figure 1

Here, $d_{\mu_1}(u) = 0.6$; $d_{\gamma_1}(u) = 0.8$; $d(u) = (0.6, 0.8)$ for all $u \in V$.
$d_{(3)\mu_1}(u_1) = (0.3 \wedge 0.3 \wedge 0.3) + (0.3 \wedge 0.3 \wedge 0.3) = 0.3 + 0.3 = 0.6$;
$d_{(3)\gamma_1}(u_1) = (0.4 \vee 0.4 \vee 0.4) + (0.4 \vee 0.4 \vee 0.4) = (0.4) + (0.4) = 0.8$;
$d_{(3)}(u_1) = (0.6, 0.8)$; $d_{(3)}(u_2) = (0.6, 0.8)$; $d_{(3)}(u_3) = (0.6, 0.8)$;
$d_{(3)}(u_4) = (0.6, 0.8)$; $d_{(3)}(u_5) = (0.6, 0.8)$; $d_{(3)}(u_6) = (0.6, 0.8)$; $d_{(3)}(u_7) = (0.6, 0.8)$.

Hence G is $((0.6, 0.8), 3, (0.6, 0.8))$-regular intuitionistic fuzzy graph.

Example 3.3 Consider an intuitionistic fuzzy graph on $G^*(V, E)$, a cycle of length 6.

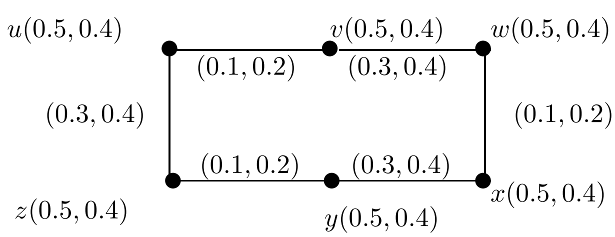

Figure 2

$d_{\mu_1}(u) = 0.4$; $d_{\gamma_1}(u) = 0.6$; $d(u) = (0.4, 0.6)$;

$d_{(3)\mu_1}(u) = \sup\{(0.1 \wedge 0.3 \wedge 0.1), (0.3 \wedge 0.1 \wedge 0.3)\} = \sup\{0.1, 0.1\} = 0.1$;

$d_{(3)\gamma_1}(u) = \inf\{(0.2 \vee 0.4 \vee 0.2), (0.4 \vee 0.2 \vee 0.4)\} = \inf\{0.4, 0.4\} = 0.4$;

$d_{(3)}(u) = (0.1, 0.4)$, $d_{(3)}(u) = (0.1, 0.4)$, for all $u \in V$.

Here, G is $((0.4, 0.6), 3, (0.1, 0.4))$- regular intuitionistic fuzzy graph.

Example 3.4 Non regular intuitionistic fuzzy graphs which is $(m, (c_1, c_2))$-regular intuitionistic fuzzy graph.

Let $G : (V, E)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$, a path on $2m$ vertices. Let all the edges of G have the same membership value (c_1, c_2). Then,

For $i = 1, 2, \cdots, m$,

$$\begin{aligned} d_{(m)\mu_1}(v_i) &= \{\mu(e_i) \wedge \mu(e_{i+1}) \wedge \cdots \wedge \mu(e_{m-2+i}) \wedge \mu(e_{m-1+i})\} \\ &= \{c_1 \wedge c_1 \wedge \cdots \wedge c_1\}, \end{aligned}$$

$$d_{(m)\mu_1}(v_i) = c_1,$$

$$\begin{aligned} d_{(m)\gamma_1}(v_i) &= \{\gamma(e_i) \vee \gamma(e_{i+1}) \vee \cdots \vee \gamma(e_{m-2+i}) \vee \gamma(e_{m-1+i})\} \\ &= \{c_2 \vee c_2 \vee \cdots \vee c_2\} \end{aligned}$$

$$d_{(m)\gamma_1}(v_i) = c_2,$$

$$\begin{aligned} d_{(m)\mu_1}(v_{m+i}) &= \{\mu(e_i) \wedge \mu(e_{i+1}) \wedge \cdots \wedge \mu(e_{m-2+i}) \wedge \mu(e_{m-1+i})\} \\ &= \{c_1 \wedge c_1 \wedge \cdots \wedge c_1\}, \end{aligned}$$

$$d_{(m)\mu_1}(v_{m+i}) = c_1,$$

$$\begin{aligned} d_{(m)\gamma_1}(v_{m+i}) &= \{\gamma(e_i) \vee \gamma(e_{i+1}) \vee \cdots \vee \gamma(e_{m-2+i}) \vee \gamma(e_{m-1+i})\} \\ &= \{c_2 \vee c_2 \vee \cdots \vee c_2\}, \end{aligned}$$

$$d_{(m)\gamma_1}(v_{m+i}) = c_2.$$

Hence G is $(m, (c_1, c_2))$ - regular intuitionistic fuzzy graph.

For $i = 2, 3, \cdots, 2m - 1$,

$d_\mu(v_i) = \mu(e_{i-1}) + \mu(e_i) = c_1 + c_1 = 2c_1$;

$d_\gamma(v_i) = \gamma(e_{i-1}) + \gamma(e_i) = c_2 + c_2 = 2c_2$;

$d(v_i) = (2c_1, 2c_2) = (k_1, k_2)$ where $k_1 = 2c_1$ and $k_2 = 2c_2$;

$d_\mu(v_1) = \mu(e_1) = c_1$ and $d_\gamma(v_1) = \gamma(e_1) = c_2$,

So, $d(v_1) = (c_1, c_2)$, $d_\mu(v_{2m}) = \mu(e_{2m-1}) = c_1$ and $d\gamma(v_{2m}) = \gamma(e_{2m-1}) = c_2$.

So, $d(v_{2m}) = (c_1, c_2)$. Therefore, $d(v_1) \neq d(v_i) \neq d(v_{2m})$ for $i = 2, 3, \cdots, 2m - 1$.

Hence G is non regular intuitionistic fuzzy graph which is $(m, (c_1, c_2))$-regular intuitionistic fuzzy graph.

Example 3.5 Let $G : (V, E)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$, an even cycle of length $\geq 2m + 1$.

Let

$$\mu(e_i) = \begin{cases} k_{1l} & \text{if } i \text{ is odd} \\ \text{membership value x} \geq k_1 & \text{if i is even} \end{cases}$$

and

$$\gamma(e_i) = \begin{cases} k_2 & \text{if } i \text{ is odd} \\ \text{membership value } y \leq k_2 & \text{if } i \text{ is even} \end{cases}$$

where x, y are not constant functions. Then,

$$d_{(m)\mu_1}(v) = min\{k_1, x\} + min\{x, k_1\} = k_1 + k_1 = 2k_1 = c_1, \text{ where } c_1 = 2k_1$$

$$d_{(m)\gamma_1}(v) = \max\{k_2, y\} + max\{y, k_2\} = k_2 + k_2 = 2k_2 = c_2, \text{ where } c_2 = 2k_2.$$

So, $d_{(m)}(v) = (c_1, c_2)$, for all $v \in V$.

Case 1. Let $G : (A, B)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$, an even cycle of length $\leq 2m+2$. Then $d(v_i)$=$(x+c_1, y+c_2)$, for all $i = 1, 2, \cdots, 2m+1$. Hence G is non-regular $(m, (c_1, c_2))$-regular intuitionistic fuzzy graph.

Case 2. Let $G : (A, B)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$, an odd cycle of length $\leq 2m + 1$. Then $d(v_1) = (c_1, c_2) + (c_1, c_2) = (2c_1, 2c_2)$ and $d(v_i) = (x + c_1, y + c_2)$, for all $i = 2, 3, \cdots, 2m$. Hence G is non-regular $(m, (c_1, c_2))$-regular intuitionistic fuzzy graph.

§4. Totally $((r_1, r_2), m, (c_1, c_2))$- Regular Intuitionistic Fuzzy Graphs

Definition 4.1 *Let $G : (A, B)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. If each vertex of G has the same total degree (r_1, r_2) and same total d_m-degree (c_1, c_2) , then G is said to be totally $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph.*

Example 4.2 In Figure 2, $td_{(3)}(v) = d_{(3)}(v) + A(v)$ = (0.1,0.4)+(0.5,0.4)=(0.6,0.8) for all $v \in V$. $td(v) = d(v) + A(v) = (0.4, 0.6) + (0.5, 0.4) = (0.9, 1.0)$ for all $v \in V$ Hence G is ((0.9,1.0),3,(0.6,0.8))- regular intuitionistic fuzzy graph.

Example 4.3 A $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph need not be totally $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph. Consider $G : (A, B)$ be a intuitionistic fuzzy graph on $G^*(V, E)$, a cycle of length 7.

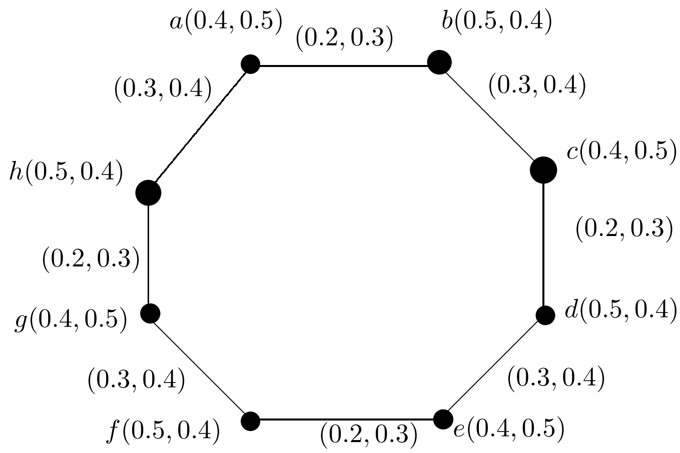

Figure 3

Here $d(v) = (0.5, 0.7)$ for all $v \in V$ and $d_{(3)}(v) = (0.4, 0.8)$, for all $v \in V$. But $td_{(3)}(a) = (0.4, 0.8) + (0.4, 0.5) = (0.8, 1.3)$, $td_{(3)}(b) = (0.4, 0.8) + (0.5, 0.4) = (0.9, 1.2)$. Hence G is ((0.5,0.7), 3 , (0.4,0.8)) - regular intuitionistic fuzzy graph.

But $td_3(a) \neq td_3(b)$. Hence G is not totally $((r_1, r_2), 3, (c_1, c_2))$ - regular intuitionistic fuzzy graph.

Example 4.4 A $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph which is totally $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph.

Consider $G : (A, B)$ be an intuitionistic fuzzy graph on $G^*(V, E)$, a cycle of length 6. For m = 3,

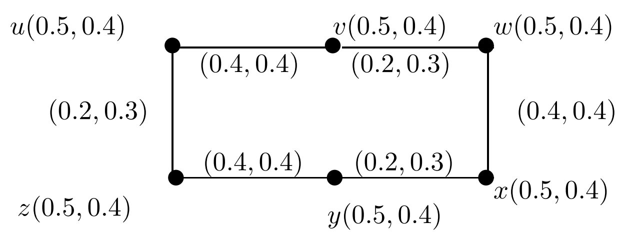

Figure 4

Here, $d(v) = (0.6, 0.8)$ and $d_{(3)}(v) = (0.2,\ 0.3)$, for all $v \in V$. Also, $td(v) = (1.1, 1.4)$ and $td_{(3)}(v) = ((0.7, 0.9)$ for all $v \in V$. Hence G is $((0.6, 0.8), 3, (0.2, 0.3))$ regular intuitionistic fuzzy graph and totally $((1.1, 1.4), 3, (0.7, 0.9))$ - regular intuitionistic fuzzy graph.

Theorem 4.5 *Let $G : (A, B)$ be an intuitionistic fuzzy graph.on $G^*(V, E)$. Then $A(u) = (k_1, k_2)$, for all $u \in V$ if and only if the following are equivalent:*

(i) $G : (V, E)$ is $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph;

(ii) G : (V, E) is totally $((r_1 + k_1, r_2 + k_2), m, (c_1 + k_1, c_2 + k_2))$- regular intuitionistic fuzzy graph.

Proof Suppose $A(u) = (k_1, k_2)$, for all $u \in V$. Assume that G is $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph. Then $d(u) = (r_1, r_2)$ and $d_{(m)}(u) = (c_1, c_2)$, for all $u \in V$.

So, $td(u) = d(u) + A(u)$ and $td_{(m)}(u) = d_{(m)}(u) + A(u) \Rightarrow td(u) = (r_1, r_2) + (k_1, k_2)$ and $td_{(m)}(u) = (c_1, c_2) + (k_1, k_2)$. So, $td(u) = (r_1 + k_1, r_2 + k_2), td_{(m)}(u) = (c_1 + k_1, c_2 + k_2)$. Hence G is totally $((r_1 + k_1, r_2 + k_2), m, (c_1 + k_1, c_2 + k_2))$ - regular intuitionistic fuzzy graph. Thus (i) $\Rightarrow$ (ii) is proved.

Now, suppose G is totally $((r_1 + k_1, r_2 + k_2), m, (c_1 + k_1, c_2 + k_2))$- regular intuitionistic fuzzy graph. Then $td(u) = (r_1 + k_1, r_2 + k_2)$ and $td_{(m)}(u) = (c_1 + k_1, c_2 + k_2)$, for all $u \in V \Rightarrow d(u) + A(u) = (r_1 + k_1, r_2 + k_2)$ and $d_{(m)}(u) + A(u) = (c_1 + k_1, c_2 + k_2)$, for all $u \in V \Rightarrow d(u) + (k_1, k_2) = (r_1, r_2) + (k_1, k_2)$ and $d_{(m)}(u) + (k_1, k_2) = (c_1, c_2) + (k_1, k_2)$, for all $u \in V \Rightarrow$d(u)$= (r_1, r_2)$ and $d_{(m)}(u) = (c_1, c_2)$, for all $u \in V$. Hence G is $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph. Thus (ii) $\Rightarrow$ (i) is proved. Hence (i) and (ii) are equivalent.

Conversely, suppose (i) and (ii) are equivalent. Suppose $A(u)$ is not constant function, then $A(u) \neq A(w)$, for atleast one pair $u, v \in V$. Let G be a $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph and totally $((r_1+k_1, r_2+k_2), m, (c_1+k_1, c_2+k_2))$- regular intuitionistic fuzzy graph. Then $d_{(m)}(u) = d_{(m)}(w) = (c_1, c_2)$ and $d(u) = d(w) = (r_1, r_2)$. Also, $td_{(m)}(u) = d_{(m)}(u) + A(u) = (c_1, c_2) + A(u)$ and $td_{(m)}(w) = d_{(m)}(w) + A(w) = (c_1, c_2) + A(w), td(u) = d(u) + A(u) = (r_1, r_2) + A(u)$ and $td(w) = d(w) + A(w) = (r_1, r_2) + A(w)$. Since $A(u) \neq A(w)$, $(c_1, c_2) + A(u) \neq (c_1, c_2) + A(w)$ and $(r_1, r_2) + A(u) \neq (r_1, r_2) + A(w) \Rightarrow td_{(m)}(u) \neq td_{(m)}(w)$ and $td(u) \neq td(w)$. So, G is not totally $((r_1+k_1, r_2+k_2), m, (c_1+k_1, c_2+k_2))$- regular intuitionistic fuzzy graph. Which is a contradiction.

Now let G be a totally $((r_1 + k_1, r_2 + k_2), m, (c_1 + k_1, c_2 + k_2))$ - regular intuitionistic fuzzy graph. Then $td_{(m)}(u) = td_{(m)}(w)$ and $td(u) = td(w) \Rightarrow d_{(m)}(u) + A(u) = d_{(m)}(w) + A(w)$ and $d(u) + A(u) = d(w) + A(w) \Rightarrow d_{(m)}(u) - d_{(m)}(w) = A(w) - A(u) \neq 0$ and $d(u) - d(w) = A(w) - A(u) \neq 0 \Rightarrow d_{(m)}(u) \neq d_{(m)}(w)$ and $d(u) \neq d(w)$. So, G is not $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph. Which is a contradiction. Hence $A(u) = (k_1, k_2)$, for all $u \in V$. □

Theorem 4.6 *If an intuitionistic fuzzy graph $G : (A, B)$ is both $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph and totally $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph then A is constant function.*

Proof Let G be a $((s_1, s_2), m, (k_1, k_2))$ - regular intuitionistic fuzzy graph and totally $((s_3, s_4), m, (k_3, k_4))$ - regular intuitionistic fuzzy graph. Then, let $d_{(m)}(u) = (k_1, k_2), td_{(m)}(u) = (k_3, k_4)$, $d(u) = (s_1, s_2)$, $td(u) = (s_3, s_4)$ for all $u \in v$. Now, $td_{(m)}(u) = (k_3, k_4)$ and $td(u) = (s_3, s_4)$ for all $u \in v \Rightarrow d_{(m)}(u) + A(u) = (k_3, k_4)$ and $d(u) + A(u) = (s_3, s_4)$ for all $u \in v \Rightarrow (k_1, k_2) + A(u) = (k_3, k_4)$ and $(s_1, s_2) + A(u) = (s_3, s_4)$ for all $u \in v \Rightarrow A(u) = (k_3, k_4) - (k_1, k_2)$ and $A(u) = (s_3, s_4) - (s_1, s_2)$ for all $u \in v \Rightarrow A(u) = (k_3 - k_1, k_4 - k_2)$ and $A(u) = (s_3 - s_1), (s_4 - s_2)$ for all $u \in v$. Hence $A(u)$ is constant function. □

§5. $((r_1, r_2), m, (c_1, c_2))$- Regularity on a Cycle with Some Specific Membership Functions

Theorem 5.1 *For any $m \geq 1$, Let $G : (A, B)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$, a cycle of length $\geq 2m$. If B is constant function then G is $((r_1, r_2), m, (c_1, c_2))$ -*

regular intuitionistic fuzzy graph, where $(k_1, k_2) = 2B(uv)$.

Proof Suppose B is a constant function say $B(uv) = (c_1, c_2)$, for all $uv \in E$. Then $d_\mu(u)$= sup $\{(c_1 \wedge c_1 \wedge \cdots \wedge c_1), (c_1 \wedge c_1 \wedge \cdots \wedge c_1)\} = c_1$ for all $v \in V$. $d_\gamma(u) = \inf\{(c_2 \vee c_2 \vee \cdots \vee c_2), (c_2 \vee c_2 \vee \cdots \vee c_2)\} = c_2$ for all $v \in V$. $d_{(m)}(v) = (c_1, c_2)$ and $d(v) = (c_1, c_2) + (c_1, c_2) = 2(c_1, c_2)$. Hence G is $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph. □

Remark 5.2 The Converse of above theorem need not be true.

Example 5.3 Consider an intuitionistic fuzzy graph on $G^*(V, E)$.

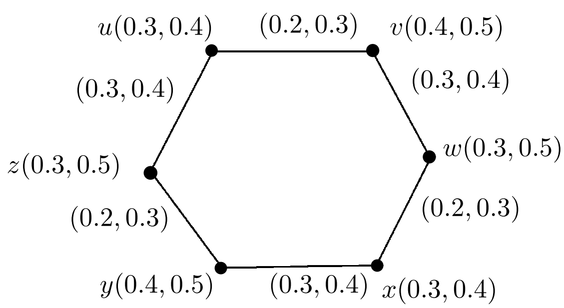

Figure 5

Here, $d(u) = (0.5, 0.7)$ and $d_{(3)}(u) = (0.3, 0.4)$, for all $u \in V$. Hence G is $((0.5, 0.7), 3, (0.3, 0.4))$-regular intuitionistic fuzzy graph. But B is not a constant function.

Theorem 5.4 *For any* $m \geq 1$, *let* $G : (A, B)$ *be an intuitionistic fuzzy graph such that* $G^*(V, E)$, *a cycle of length* $\geq 2m + 1$. *If* B *is constant function, then* G *is* $((r_1, r_2), m, (c_1, c_2))$ *- regular intuitionistic fuzzy graph, where* $(r_1, r_2) = 2B(uv)$ *and* $(c_1, c_2) = 2B(uv)$.

Proof Suppose B is constant function say $B(uv) = (c_1, c_2)$, for all $uv \in E$. Then, $d_{(m)\mu_1}(v)$= $\{c_1 \wedge c_1 \wedge \cdots \wedge c_1\} + \{c_1 \wedge c_1 \wedge \cdots \wedge c_1\} = c_1 + c_1 = 2c_1$, $d_{(m)\gamma_1} = \{c_2 \vee c_2 \vee \cdots \vee c_2\} + \{c_2 \vee c_2 \vee \cdots \vee c_2\} = c_2 + c_2 = 2c_2$, for all $v \in V$. So,$d_{(m)}(v) = 2(c_1, c_2)$, for all $u \in V$. Also, $d(v) = (c_1, c_2) + (c_1, c_2) = 2(c_1, c_2)$ Hence G is $(2(c_1, c_2), m, 2(c_1, c_2))$ - regular intuitionistic fuzzy graph. □

Theorem 5.5 *For any* $m \geq 1$, *let* $G : (A, B)$ *be an intuitionistic fuzzy graph such that* $G^*(V, E)$, *a cycle of length* $\geq 2m + 1$. *If the alternate edges have the same membership values and same non membership values, then* G *is* $((r_1, r_2), m, (c_1, c_2))$ *- regular intuitionistic fuzzy graph.*

Proof If the alternate edges have same membership and same non membership values then, let

$$\mu(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even} \end{cases} \qquad \gamma(e_i) = \begin{cases} k_3 & \text{if } i \text{ is odd} \\ k_4 & \text{if } i \text{ is even} \end{cases}$$

Here, we have 4 possible cases.

Case 1. Suppose $k_1 \leq k_2$ and $k_3 \geq k_4$.

$d_{(m)\mu_1}(v) = min\{k_1, k_2\} + min\{k_1, k_2\} = k_1 + k_1 = 2k_1$;

$d_{(m)\gamma_1}(v) = max\{k_3, k_4\} + max\{k_3, k_4\} = k_3 + k_3 = 2k_3$;

$d_{(m)}(v) = (2k_1, 2k_3)$ and $d(v) = (k_1, k_3) + (k_2, k_4) = (k_1 + k_2, k_3 + k_4)$.

Case 2. Suppose $k_1 \leq k_2$ and $k_3 \leq k_4$.

$d_{(m)\mu_1}(v) = \min\{k_1, k_2\} + min\{k_1, k_2\} = k_1 + k_1 = 2k_1$;

$d_{(m)\gamma_1}(v) = \max\{k_3, k_4\} + max\{k_3, k_4\} = k_4 + k_4 = 2k_4$;

$d_{(m)}(v) = (2k_1, 2k_4)$ and $d(v) = (k_1, k_3) + (k_2, k_4) = (k_1 + k_2, k_3 + k_4)$.

Case 3. Suppose $k_1 \geq k_2$ and $k_3 \leq k_4$.

$d_{(m)\mu_1}(v) = \min\{k_1, k_2\} + \min\{k_1, k_2\} = k_2 + k_2 = 2k_2$;

$d_{(m)\gamma_1}(v) = \max\{k_3, k_4\} + \max\{k_3, k_4\} = k_4 + k_4 = 2k_4$;

$d_{(m)}(v) = (2k_2, 2k_4)$ and $d(v) = (k_1, k_3) + (k_2, k_4) = (k_1 + k_2, k_3 + k_4)$.

Case 4. Suppose $k_1 \geq k_2$ and $k_3 \geq k_4$.

$d_{(m)\mu_1}(v) = min\{k_1, k_2\} + min\{k_1, k_2\} = k_2 + k_2 = 2k_2$;

$d_{(m)\gamma_1}(v) = max\{k_3, k_4\} + max\{k_3, k_4\} = k_3 + k_3 = 2k_3$;

$d_{(m)}(v) = (2k_2, 2k_3)$ and $d(v) = (k_1, k_3) + (k_2, k_4) = (k_1 + k_2, k_3 + k_4)$.

Thus, $d(v) = (r_1, r_2)$ and $d_{(m)}(v) = (c_1, c_2)$ for all $v \in V$. Hence G is $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph. □

Remark 5.6 Even if the alternate edges of an intuitionistic fuzzy graph whose underlying graph is an even cycle of length $\geq 2m + 2$ having same membership values and same non membership values, then G need not be totally $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph, since if A $= (\mu_1, \gamma_1)$ is not a constant function, G is not totally $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph, for any $m \geq 1$.

Theorem 5.7 *For any $m \geq 1$, let $G : (A, B)$ be an intuitionistic fuzzy graph on $G^* : (V, E)$, a cycle of length $\geq 2m + 1$. Let $k_2 \geq k_1$, $k_4 \geq k_3$ and let*

$$\mu(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even} \end{cases} \qquad \gamma(e_i) = \begin{cases} k_3 & \text{if } i \text{ is odd} \\ k_4 & \text{if } i \text{ is even} \end{cases}$$

Then, G is $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph.

Proof We consider cases following.

Case 1. Let $G : (A, B)$ be an intuitionistic fuzzy graph on $G^*(V, E)$, an even cycle of length $\leq 2m + 2$. Then by theorem 5.3, G is $((r_1, r_2), m, (c_1, c_2))$ - regular intuitionistic fuzzy graph.

Case 2. Let $G = (A, B)$ be an intuitionistic fuzzy graph on $G^*(V, E)$ an odd cycle of length $\geq 2m + 1$. For any $m \geq 1$, $d_{(m)}(v) = (2k_1, 2k_4)$, for all $v \in V$. But $d(v_1) = (k_1, k_3) + (k_1, k_3) = 2(k_1, k_3)$ and $d(v_i) = (k_1, k_3) + (k_2, k_4) = ((k_1 + k_2), (k_3 + k_4))$ So, $d(v_i) \neq d(v_1)$ for $i = 2, 3, \ldots, m$

Hence G is not $((r_1, r_2), m, (c_1, c_2))$-regular intuitionistic fuzzy graph. □

Remark 5.8 Let $G : (A, B)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$ is a cycle of length $\geq 2m + 1$. Even if let

$$\mu(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even} \end{cases} \qquad \gamma(e_i) = \begin{cases} k_3 & \text{if } i \text{ is odd} \\ k_4 & \text{if } i \text{ is even} \end{cases}$$

Then G need not be totally $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph, since if $A = (\mu_1, \gamma_1)$ is not a constant function, G is not totally $((r_1, r_2), m, (c_1, c_2))$- regular intuitionistic fuzzy graph.

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Minimum Dominating Color Energy of a Graph

P.Siva Kota Reddy[1], K.N.Prakasha[2] and Gavirangaiah K[3]

1. Department of Mathematics, Siddaganga Institute of Technology, B.H.Road, Tumkur-572 103, India
2. Department of Mathematic, Vidyavardhaka College of Engineering, Mysuru- 570 002, India
3. Department of Mathematics, Government First Grade Collge, Tumkur-562 102, India

E-mail: reddy_math@yahoo.com, prakashamaths@gmail.com, gavirangayya@gmail.com

Abstract: In this paper, we introduce the concept of minimum dominating color energy of a graph, $E_c^D(G)$ and compute the minimum dominating color energy $E_c^D(G)$ of few families of graphs. Further, we establish the bounds for minimum dominating color energy.

Key Words: Minimum dominating set, Smarandachely dominating, minimum dominating color eigenvalues, minimum dominating color energy of a graph.

AMS(2010): 05C50

§1. Introduction

Let G be a graph with n vertices $v_1, v_2, \cdots, v_n$ and m edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

The concept of graph energy originates from chemistry to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph. Here every carbon atom is represented by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The eigenvalues of the molecular graph represent the energy level of the electron in the molecule. An interesting quantity in Hückel theory is the sum of the energies of all the electrons in a molecule, the so called π-electron energy of a molecule.

Prof.Chandrashekara Adiga et al.[5] have defined color energy $E_c(G)$ of a graph G. Rajesh Khanna et al.[2] have defined the minimum dominating energy. Motivated by these two papers, we introduced the concept of minimum dominating color energy $E_c^D(G)$ of a graph G and computed minimum dominating chromatic energies of star graph, complete graph, crown graph, and cocktail party graphs. Upper and lower bounds for $E_c^D(G)$ are also established.

This paper is organized as follows. In section 3, we define minimum dominating color energy of a graph. In section 4,minimum dominating color spectrum and minimum dominating

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color energies are derived for some families of graphs. In section 5 Some properties of minimum dominating color energy of a graph are discussed. In section 6 bounds for minimum dominating color energy of a graph are obtained. section 7 consist some open problems.

§2. Minimum Dominating Energy of a Graph

Let G be a simple graph of order n with vertex set $V = v_1, v_2, v_3, \cdots, v_n$ and edge set E. A subset $D \subseteq V$ is a dominating set if D is a dominating set and every vertex of $V - D$ is adjacent to at least one vertex in D, and generally, for $\forall O \subset V$ with $\langle O \rangle_G$ isomorphic to a special graph, for instance a tree, a Smarandachely dominating set D_S on O of G is such a subset $D_S \subseteq V - O$ that every vertex of $V - D_S - O$ is adjacent to at least one vertex in D_S. Obviously, if $O = \emptyset$, D_S is nothing else but the usual dominating set of graph. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The minimum dominating matrix of G is the $n \times n$ matrix defined by $A_D(G) = (a_{ij})$ ([2]) where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 1 & \text{if } i = j \text{ and } v_i \in D, \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_D(G))$. The minimum dominating eigenvalues of the graph G are the eigenvalues of $A_D(G)$.

Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ The minimum dominating energy of G is defined as

$$E_c^D(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{2}$$

§3. Coloring and Color Energy

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$ ([19]).

Consider the vertex colored graph. Then entries of the matrix $A_c(G)$ are as follows ([5]):

If $c(v_i)$ is the color of v_i, then

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by $f_n(G, \rho) = det(\rho I - A_c(G))$. The color eigenvalues of the graph G are the eigenvalues of $A_c(G)$.

Since $A_c(G)$ is real and symmetric, its eigen values are real numbers and are labelled in non-increasing order $\rho_1 \geq \rho_2 \geq \rho_3 \geq \cdots \geq \rho_n$ The color energy of G is defined as

$$E_c(G) = \sum_{i=1}^{n} |\rho_i|. \tag{3}$$

§4. The Minimum Dominating Color Energy of a Graph

Let G be a simple graph of order n with vertex set $V = v_1, v_2, v_3, \cdots, v_n$ and edge set E. Let D be the minimum dominating set of a graph G. The minimum dominating color matrix of G is the $n \times n$ matrix defined by $A_c^D(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j) \text{ or if } i = j \text{ and } v_i \in D, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non adjacent with } c(v_i) = c(v_j), \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_c^D(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_c^D(G))$. The minimum dominating color eigenvalues of the graph G are the eigenvalues of $A_c^D(G)$.

Since $A_c^D(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 .... \geq \lambda_n$ The minimum dominating color energy of G is defined as

$$E_c^D(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{4}$$

If the color used is minimum then the energy is called minimum dominating chromatic energy and it is denoted by $E_\chi^D(G)$. Note that the trace of $A_c^D(G) = |D|$.

§5. Minimum Dominating Color Energy of Some Standard Graphs

Theorem 5.1 *If K_n is the complete graph with n vertices has $E_\chi^D(G)(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.*

Proof Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum dominating set $= D = \{v_1\}$.

$$A_c^D(K_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}.$$

Its characteristic polynomial is

$$[\lambda+1]^{n-2}[\lambda^2-(n-1)\lambda-1].$$

The minimum dominating color eigenvalues are

$$spec_D(K_n)=\begin{pmatrix} -1 & \frac{n-1+\sqrt{(n^2-2n+5)}}{2} & \frac{n-1-\sqrt{(n^2-2n+5)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

The minimum dominating color energy for complete graph is

$$\begin{aligned} E_\chi^D(K_n) &= |-1|(n-2)+|\frac{(n-1)+\sqrt{(n^2-2n+5)}}{2}| \\ &\quad +|\frac{(n-1)-\sqrt{(n^2-2n+5)}}{2}| \\ &= (n-2)+\sqrt{(n^2-2n+3)}, \end{aligned}$$

i.e.,

$$E_\chi^D(G)(K_n)=(n-2)+\sqrt{(n^2-2n+5)}.$$ □

Definition 5.2 *The crown graph S_n^0 for an integer $n\geq 3$ is the graph with vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ and edge set $\{u_iv_i : 1\leq i,j\leq n, i\neq j\}$. S_n^0 is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.*

Theorem 5.3 *If S_n^0 is a crown graph of order $2n$ then $E_\chi^D(S_n^0)=(2n-3)+\sqrt{(4n^2+4n-7)}$.*

Proof Let S_n^0 be a crown graph of order $2n$ with vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ and minimum dominating set $= D=\{u_1,v_1\}$. Since $\chi(S_n^0)=2$, we have

$$A_\chi(S_n^0)=\begin{pmatrix} 1 & -1 & \cdots & -1 & -1 & 0 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & -1 & -1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 0 & -1 & 1 & 1 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 1 & 1 & -1 & 0 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & -1 & -1 & \cdots\ 0 & -1 & \\ 1 & 1 & \cdots & 1 & 0 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}_{2n\times 2n}.$$

Its characteristic polynomial is

$$\lambda^{n-1}[\lambda-1][\lambda-2]^{n-2}[\lambda^2+(2n-5)\lambda-(6n-8)]$$

and its minimum dominating color eigenvalues are

$$spec_{\chi}^{D}(S_n^0) = \begin{pmatrix} 0 & 1 & 2 & \frac{-(2n-5)+\sqrt{(4n^2+4n-7)}}{2} & \frac{-(2n-5)-\sqrt{(4n^2+4n-7)}}{2} \\ n-1 & 1 & n-2 & 1 & 1 \end{pmatrix}.$$

The minimum dominating color energy of S_n^0 is

$$\begin{aligned} E_{\chi}^{D}(S_n^0) &= |0|(n-1)+|1|(n-1)+|2|+|\frac{-(2n-5)+\sqrt{(4n^2+4n-7)}}{2}| \\ &+|\frac{-(2n-5)-\sqrt{(4n^2+4n-7)}}{2}| \\ &= (2n-3)+\sqrt{4n^2+4n-7}, \end{aligned}$$

i.e.,

$$E_{\chi}^{D}(S_n^0) = (2n-3)+\sqrt{4n^2+4n-7}. \qquad \square$$

.

Theorem 5.4 *If $K_{1,n-1}$ is a star graph of order n, then*

(*i*) $E_{\chi}(K_{1,n-1}) = \sqrt{5}$ *for* $n = 2$;

(*ii*) $E_{\chi}(K_{1,n-1}) = (n-2)+\sqrt{(n^2-2n+3)}$*forn* ≥ 3..

Proof Let $K_{1,n-1}$ be a colored graph on n vertices. Minimum dominating set is $D = \{v_0\}$. Then we have

$$A_{\chi}(K_{1,n-1}) \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & -1 & \cdots & -1 & -1 \\ 1 & -1 & 0 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & 0 & -1 \\ 1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}_{n\times n}.$$

Case 1. The characteristic equation for $n = 2$ is $\lambda^2 - \lambda - 1 = 0$ and the minimum dominating color eigenvalues for $n = 2$ are $= \dfrac{1 \pm \sqrt{5}}{2}$. Whence, $E_{\chi}^{D}(K_{1,n-1}) = \sqrt{5}$.

Case 2. The characteristic equation for $n \geq 3$ is $(\lambda-1)^{n-2}(\lambda^2+(n-3)\lambda-(2n-3)) = 0$ and The minimum dominating color eigenvalues for $n \geq 3$ are

$$\begin{pmatrix} 1 & \frac{(n-3)+\sqrt{(n^2+2n-3)}}{2} & \frac{(n-3)-\sqrt{(n^2+2n-3)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Its minimum dominating color energy is

$$\begin{aligned} E_\chi^D(K_{1,n-1}) &= |1|(n-2) + |\frac{n-3+\sqrt{(n^2+2n-3)}}{2}| \\ &\quad +|\frac{n-3-\sqrt{(n^2+2n-3)}}{2}| \\ &= (n-2)+\sqrt{(n^2-2n+3)}. \end{aligned}$$

Therefore,

$$E_\chi^D(K_{1,n-1}) = (n-2)+\sqrt{(n^2-2n+3)}. \qquad \Box$$

Definition 5.5 *The cocktail party graph, denoted by $K_{n\times 2}$, is graph having vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < j \le n\}$. This graph is also called as complete n-partite graph.*

Theorem 5.6 *If $K_{n\times 2}$ is a cocktail party graph of order $2n$, then $E_\chi^D(K_{n\times 2}) = (4n-5)+\sqrt{(4n^2-4n+9)}$.*

Proof Let $K_{n\times 2}$ be a cocktail party graph of order $2n$ with $V(K_{n\times 2}) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$. The minimum dominating set $= D = \{u_1, v_1\}$. Then,

$$A_\chi^D(K_{n\times 2}) = \begin{pmatrix} 1 & -1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -1 & 0 \end{pmatrix}_{2n\times 2n}.$$

Its characteristic equation is

$$[\lambda+3]^{n-2}[\lambda-1]^{n-1}[\lambda-2][\lambda^2-(2n-5)\lambda-4(n-1)]$$

with the minimum dominating color eigenvalues

$$Espec_\chi^D(K_n\times 2) = \begin{pmatrix} -3 & 1 & 2 & \frac{2n-5+\sqrt{(4n^2-4n+9)}}{2} & \frac{2n-5-\sqrt{(4n^2-4n+9)}}{2} \\ n-2 & n-1 & 1 & 1 & 1 \end{pmatrix}.$$

and the minimum dominating color energy,

$$\begin{aligned} E^D_\chi(K_n \times 2) &= |-3|(n-2)+1(n-1)+|2|+|\frac{2n-5+\sqrt{(4n^2-4n+9)}}{2}| \\ &\quad +|\frac{2n-5-\sqrt{(4n^2-4n+9)}}{2}| \\ &= (4n-5)+\sqrt{(4n^2-4n+9)}. \end{aligned}$$

This completes the proof. □

Definition 5.7 *The friendship graph, denoted by $F_3^{(n)}$, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.*

Theorem 5.8 *If $F_3^{(n)}$ is a friendship graph, then $E^D_\chi(F_3^{(n)}) = (3n-2)+\sqrt{(n^2+6n+1)}$.*

Proof Let $F_3^{(n)}$ be a friendship graph with $V(F_3^{(n)}) = \{v_0, v_1, v_2, \ldots, v_n\}$. The minimum dominating set $= D = \{v_3\}$. Then,

$$A^D_\chi(F_3^{(n)}) == \begin{pmatrix} 0 & 1 & 1 & -1 & \cdots & -1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 1 & -1 & \cdots & 0 & 1 \\ 0 & -1 & 1 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2n+1)\times(2n+1)}.$$

Its characteristic equation is

$$\lambda^{n-1}[\lambda+n][\lambda-2]^{n-1}[\lambda^2+(n-3)\lambda+(2-3n)]$$

with the minimum dominating color eigenvalues

$$Espec^D_\chi(F_3^{(n)}) = \begin{pmatrix} -n & 0 & 2 & \frac{-(n-3)+\sqrt{(n^2+6n+1)}}{2} & \frac{-(n-3)-\sqrt{(n^2+6n+1)}}{2} \\ 1 & n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

and the minimum dominating color energy

$$\begin{aligned} E^D_\chi(F_3^{(n)}) &= |-n|+0+|2|(n-1)+|\frac{-(n-3)+\sqrt{(n^2+6n+1)}}{2}| \\ &\quad +|\frac{-(n-3)-\sqrt{(n^2+6n+1)}}{2}| \\ &= (3n-2)+\sqrt{(n^2+6n+1)}. \end{aligned}$$

This completes the proof. □

§6. Properties of Minimum Dominating Color Energy of a Graph

Theorem 6.1 *Let* $|\lambda I - A_C^D| = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n$ *be the characteristic polynomial of* A_c^D*. Then*

(*i*) $a_0 = 1$;

(*ii*) $a_1 = -|D|$;

(*iii*) $a_2 = (|D|_2) - (m + m_c)$.

where m *is the number of edges and* m_c *is the number of pairs of non-adjacent vertices receiving the same color in* G.

Proof (*i*) It follows from the definition, $P_c(G, \lambda) := det(\lambda I - A_c(G))$, that $a_0 = 1$.

(*ii*) The sum of determinants of all 1×1 principal submatrices of A_c^D is equal to the trace of A_c^D, which $\Rightarrow a_1 = (-1)^1$ trace of $[A_c^D(G)] = -|E|$.

(*iii*) The sum of determinants of all the 2×2 principal submatrices of $[A_c^D]$ is

$$\begin{aligned} a_2 &= (-1)^2 \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= (D|2) - (m + \text{number of pairs of non} - \text{adjacent vertices} \\ &\qquad \text{receiving the same color in G}) \\ &= (D|2) - (m + m_c). \end{aligned}$$

□

Theorem 6.2 *If* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are eigenvalues of* $A_c^D(G)$*, then* $\sum_{i=1}^{n} \lambda_i = |D|$ *and* $\sum_{i=1}^{n} \lambda_i^2 = |D| + 2(m + m_c)$*, where* m_c *is the number of pairs of non-adjacent vertices receiving the same color in* G.

§7. Open Problems

Problem 1. *Determine the class of graphs whose minimum dominating color energy of a graph is equal to number of vertices.*

Problem 2. *Determine the class of graphs whose minimum dominating color energy of a graph equal to usual energy.*

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Cohen-Macaulay of Ideal $I_2(G)$

Abbas Alilou

(Department of Mathematics Azarbaijan Shahid, Madani University Tabriz, Iran)

E-mail: abbasalilou@yahoo.com

Abstract: In this paper, we study the Cohen-Macaulay of ideal $I_2(G)$, where $I_2(G) = \langle xyz \mid x-y-z \text{ is } 2-\text{path in } G\rangle$. Also, we determined the 2-projective dimension R-module, $R/I_2(G)$ denoted by $pd_2(G)$ of some graphs.

Key Words: Cohen-Macaulay, projective dimension, ideal, path.

AMS(2010): 05E15

§1. Introduction

A simple graph is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of G, respectively. A walk is an alternating sequence of vertices and connecting edges. A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length n denotes by P_n. In a graph G, the distance between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting x and y, if such a path exists: otherwise, we set $d(x, y) = \infty$. The diameter of a graph G is $diam(G) = \sup\{d(x, y) : x \text{ and } y \text{ are distinct vertices of } G\}$. Also, a cycle is a path that begins and ends on the same vertex. A cycle with length n denotes by C_n. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. For a positive integer r, a complete r-partite graph is one in which each vertex is joined to every vertex that is not in the some subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a star graph in which the vertex with degree $n-1$ is called the center of the graph. For any graph G, we denote $N[x] = \{y \in V(G) : (x, y) \; is\ an\ edge\ of\ G\}$. Recall that the projective dimension of an R-module M, denoted by $pd(M)$, is the length of the minimal free resolution of M, that is, $pd(M) = max\{I| \; \beta_{i,j}(M) \neq 0 \; for\ some\ j\}$. There is a strong connection between the topology of the simplicial complex and the structure of the free resolution of $K[\Delta]$. Let $\beta_{i,j}(\Delta)$ denotes the N-graded Betti numbers of the Stanley-Reisner ring $K[\Delta]$.

To any finite simple graph G with the vertex set $V(G) = \{x_1, \cdots, x_n\}$ and the edge set $E(G)$, one can attach an ideal in the Polynomial rings $R = K[x_1, \cdots, x_n]$ over the field K, where ideal $l_2(G)$ is called the edge ideal of G such that $l_2(G) =< xyz| \; x - y - z$ is $2 -$

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path in $G >$. Also the edge ring of G, denoted by $K(G)$ is defined to be the quotient ring $K(G) = R/I_2(G)$. Edge ideals and edge rings were first introduced by Villarreal [9] and then they have been studied by many authors in order to examine their algebraic properties according to the combinatorial data of graphs. In this paper, we denote S_n for a star graph with $n+1$ vertices. Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K with the grading induced by $deg(x_i) = d_i$, where d_i is a positive integer. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a finitely generated N-graded module over R, its Hilbert function and Hilbert series are defined by $H(M,i) = l(M_i)$ and $F(M,t) = \sum_{i=0}^{\infty} H(M,i)t^i$ respectively, where $l(M_i)$ denotes the length of M_i as a K-module, in our case $l(M_i) = dim_K(M_i)$.

§2. Cohen-Macaulay of Ideal $I_2(G)$ and $pd_2(G)$ of Some Graph G

Definition 2.1 *Let G be a graph with vertex set V. Then a subset $A \subseteq V$ is a 2-vertex cover for G if for every path xyz of G we have $\{x,y,z\} \cap A \neq \varnothing$. A 2-minimal vertex cover of G is a subset A of V such that A is a 2-vertex cover, and no proper subset of A is a vertex cover for G. The smallest cardinality of a 2-vertex cover of G is called the 2-vertex covering number of G and is denoted by $a_{02}(G)$.*

Example 2.2 Let G be a graph shown in the figure. Then the set $\{x_2, x_4, x_7\}$ is a 2-minimal vertex cover of G and $a_{02}(G) = 3$.

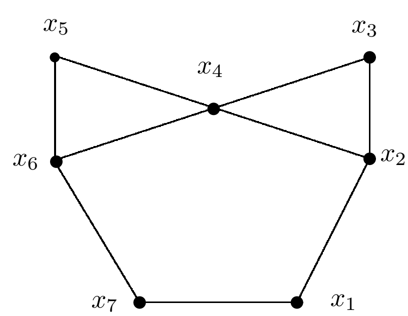

Figure 1

Definition 2.3 *Let G be a graph with vertex set V. A subset $\mathcal{A} \subseteq V$ is a k-independent if for even x of $\mathcal{A}$ we have $deg_{G[S]} \leq k-1$. The maximum possible cardinality of an k-independent set of G, denoted $\beta_{0k}(G)$, is called the k-independence number of G. It is easy see that*

$$\alpha_{02}(G) + \beta_{02}(G) = |V(G)|.$$

Definition 2.4 *Let G be a graph without isolated vertices, Let $\mathcal{S} = K[x_1, \cdots, x_n]$ the polynomial ring on the vertices of G over some fixed field K. The 2-pathes ideal $I_2(G)$ associated to the graph G is the ideal of $\mathcal{S}$ generated by the set of square-free monomials $x_i x_j x_r$ such that $\nu_i \nu_j \nu_r$*

is the path of G, *that is* $I_2(G) =< \{x_i x_j x_r| \ \{\nu_i \nu_j \nu_r\} \in P_2(G)\} >$.

Proposition 2.5 *Let* $\mathcal{S} = K[x_1, \cdots, x_n]$ *be a polynomial ring over a field* K *and* G *a graph with vertices* $\nu_1, \cdots, \nu_n$. *If* P *is an ideal of* R *generated by* $\mathcal{A} = \{x_{i1}, \cdots, x_{ik}\}$ *then* P *is a minimal prime of* $I_2(G)$ *if and only if* $\mathcal{A}$ *is a 2-minimal vertex cover of* G.

Proof It is easy see that $I_2(G) \subseteq P$ if and only if $\mathcal{A}$ is a 2-vertex cover of G. Now, let $\mathcal{A}$ is a 2-minimal vertex cover of G. By Proposition 5.1.3 [9] any minimal prime ideal of $I_2(G)$ is a face ideal thus P is a minimal prime of $I_2(G)$. The converse is clear. □

Corollary 2.6 *If* G *is a graph and* $I_2(G)$ *its 2-path ideal, then*

$$ht\,(I_2(G)) = \alpha_{02}(G).$$

Proof It follows from Proposition 5 and the definition of $\alpha_{02}(G)$. □

Definition 2.7 *A graph* G *is 2-unmixed if all of its 2-minimal vertex covers have the same cardinality.*

Definition 2.8 *A graph* G *with vertex set* $V(G) = \{\nu_1, \nu_2, \cdots, \nu_n\}$ *is 2-cohen-Macaullay over field* K *if the quotient ring* $K[x_1, \cdots, x_n]/I_2(G)$ *is cohen-Macaulay.*

Proposition 2.9 *If* G *is a 2-cohen-Macaulay graph, then* G *is 2-unmixed.*

Proof By Corollary 1.3.6 [9], $I_2(G) = \bigcap_{P \in \min(I_2(G))} P$. Since $R/I_2(G)$ is cohen-Macaullay, all minimal prime ideals of $I_2(G)$ have the same height. Then, by Proposition 5, all 2-minimal vertex cover of G have the same cardinality, as desired. □

Proposition 2.10 *If* G *is a graph and* $G_1, \cdots, G_s$ *are its connected components, then* G *is 2-cohen- Macaulay if and only if for all* i, G_i *is cohen-Macaulay.*

Proof Let $R = K[V(G)]$ and $R_i = L[V(G_i)]$ for all i. Since

$$R/I_2(G) \cong R_1/I_2(G_1) \otimes_K \cdots \otimes_K R_s/I_2(G_s).$$

Hence the results follow from Corollary 2.2.22 [9]. □

Definition 2.11 *For any graph* G *one associates the complementary simplicial complex* $\triangle_2\,(G)$, *which is defined as*

$$\triangle_2\,(G) := \{\mathcal{A} \subset V|\ \mathcal{A}\ is\ 2 - independent\ set\ in\ G\}.$$

This means that the facets of $\Delta_2(G)$ are precisely the maximal 2-independent sets in G, that is the complements in V of the minimal 2-vertex covers. Thus $\Delta_2(G)$ precisely the Stanley-Reisner complex of $I_2(G)$.

It is easy see that $\omega\left(\Delta_2(G)\right)=\{\{x,y,z\}\mid xyz\in P_3(G)\}$. Therefore $I_2(G)=I_{\Delta_2(G)}$, and so G is $2-C-M$ graph if and only if the simplicial complex $\Delta_2(G)$ is cohen-Macaulay.

Now, we can show the following propositiori.

Proposition 2.12 *The following statements hold:*

(a) *For any $n\geq 1$ the complete graph K_n is cohen-Macaulay;*
(b) *The complete bipartite graph $K_{m,n}$ is cohen-Macaulay if and only if $m+n\leq 4$.*

Proof (a) Since $\Delta_2(K_n)=<\{x,y\}\mid x,y\in V(K_n)>$, thus $\Delta_2(K_n)$ is connected l-dimensional simplicial complex, then by Corohary 5.3.7 [9], $\Delta_2(K_n)$ is cohen-Macaula so K_n is cohen-Macaulay.

(b) If $m+n\leq 4$, then $K_{m,n}\cong P_2,P_3,C_4$. It is easy to see that $\Delta_2(K_{m,n})$ is c. So $K_{m,n}$ is cohen-Macaulay.

Conversely, let $K_{m,n}$ is cohen-Macaulay and $m+n\geq 5$. Take $V_1=\{x_1,\cdots,x_n\}$ and $V_2=\{y_1,\cdots,y_m\}$ are the partite sets of $K_{m,n}$. One has

$$\Delta_2(K_{m,n})=<\{x_1,\cdots,x_n\},\{y_1,\cdots,y_m\},\{x_i,y_j\}\mid 1\leq i\leq n,1\leq j\leq m>$$

. Since $m+n\geq 5$, $\Delta_2(K_{m,n})$ is not pure simplicial complex. Then, by 5.3.12 [9] $\Delta_2(K_{m,n})$ is not cohen-Macaulay, a contradiction, as desired. □

Now, we present a result about the Hilbert series of $K[\Delta_2(K_n)]$ and $K[\Delta_2(K_{m,n})]$.

Proposition 2.13 *If $\Delta_2(K_n)$ and $\Delta_2(K_{m,n})$ are the complementary simplicial complexes K_n and $K_{m,n}$ respectively, then*

(a) $F(K[\Delta_2(K_n)],z)=1+nz/(1-z)+n(n-1)/2(1-z)^2$;
(b) $F(K[\Delta_2(K_{n,m})],z)=1/(1-z)^n+1/(1-z)^m+m.nz^2/(1-z)^2-1$.

Proof (a) Since $\Delta_2(K_n)=<\{x,y\}\mid x,y\in V(K_n)>$ hence dime $\Delta_2(K_n)=1$ and $f_{-1}(K_n)=1$, $f_0,(K_n)=n$ and $f_1(K_n)=\binom{n}{2}=n(n-1)/2$. By Corollary 5.4.5 [9]. We have

$$F(K[\Delta_2(K_n)],z)=1+nz/1-z+n(n-1)/2.z^2/2(1-z)^2.$$

(b) Let $\{x_1,\cdots,x_n\}$ and $\{y_1,\cdots,y_m\}$ are the parties sets of $K_{m,n}$. Since

$$\Delta_2(K_{m,n})=<\{x_1,\cdots,x_n\},\{y_1,\cdots,y_m\},\{x_i,y_j\}\mid 1\leq i\leq n,1\leq j\leq m>$$

Then it is easy see that $f_1\left(\Delta_2(K_{m,n})\right)=f_1\left(\Delta(K_{m,n})\right)+mn$ and $f_i\left(\Delta_2(K_{m,n})\right)=f_i\left(\Delta(K_{m,n})\right)$ for all $i\neq 1$. ln the other hand, by 6.6.6[9], $F(K[\Delta_2(K_n)],z)=1/(1-z)^n-1.$, Thus

$$F(K[\Delta_2(K_n)],z)=1/(1-z)^n+1/(1-z)^m+m.nz^2/(1-z)^2-1.$$

This completes the proof. □

Corollary 2.14 $F\left(K\left[\Delta_2(S_n)\right],z\right)=1/(1-z)^n+nz^2/(1-z)^2+z/(1-z)$.

Proof It follows from Proposition 2.13 with assume $m = 1$. □

In this section we mainly present basic properties of 2-shellable graphs.

Lemma 2.15 *Let G be a graph and x be a vertex of degree 1 in G and let $y \in N(x)$ and $G' = G - (\{y\} \cup N(y))$. Then $\Delta_2(G') = lK_{\Delta_2(G)}(\{x, y\})$. Moreover F is a facet of $\Delta_2(G')$ if and only if $F \cup \{x, y\}$ is a facet of $\Delta_2(G)$.*

Proof (a) Let $F \in lK_{\Delta_2(G)}(\{x, y\})$. Then $F \in \Delta_2(G)$, $x, y \notin F$ and $F \cup \{x, y\} \in \Delta_2(G)$. This implies that $(F \cup \{x, y\}) \cap N[y] = \varnothing$ and $F \subseteq (V - \{x, y\}) \cup N[y] = (V - y) \cup N[y] = V(G')$. Thus F is 2-independent in G', it follows that $F \in \Delta_2(G')$. Conversely let $F \in \Delta_2(G')$, then F is 2-independent in G' and $F \cap (x \cup [y]) = \varnothing$. Therefore $F \cup \{x, y\}$ is 2-independent in G and so $F \cup \{x, y\} \in \Delta_2(G)$, $F \cup \{x, y\} = \varnothing$. Thus $F \in lK_{\Delta_2(G)}(\{x, y\})$. Finaly from part one follows that F is a facet of $\Delta_2(G')$ if and only if $F \cup \{x, y\}$ is a facet of $\Delta_2(G)$. □

Definition 2.16 *Fix a field K and set $R = K[x_1, \cdots, x_n]$. If G is a graph with vertex set $V(G) = \{x_1, x_2, \cdots, x_n\}$, we define the projective dimension of G to be the 2-projective dimension R- module $R/I_2(G)$, and we will write $pd_2(G) = pd\,(R/l_2(G))$.*

Proposition 2.17 *If G is a graph and $\{x, y\}$ is a edge of G, then*

$$\begin{aligned} P_2(G) \quad &\leq \quad \max\{P_2\,(G - (N[x] \cup N[y])) + deg(x) + deg(y) \\ &\quad - |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1\}. \end{aligned}$$

Proof Let $N[x] = \{x_1, \cdots, x_\xi\}$ and $N[y] = \{y_1, \cdots, y_r\}$. It is easy to see that

$$I_2(G) : xy = (I_2(G) - (N[x] \cup N[y])\,, x_1, \cdots, x_\xi, y_1, \cdots, y_r)\,.$$

Now, let

$$R' = K\left[V\left(G - \left(N[x] \bigcup N[y]\right)\right)\right].$$

Then

$$depth(R/I_2(G) : xy) = depth(R'/I_2(G - (N[x] \cup N[y]).$$

And so by Auslander-Buchsbaum formula, we have

$$\begin{aligned} pd_2(R/I_2(x) : xy) \quad &= \quad pd_2(G - (N[x] \cup N[y]) + deg(x) + deg(y) - |N[x] \cap N[y]|, \\ pd_2(R/I_2(x), x) \quad &= \quad pd_2(G - x) + 1, \\ pd_2(R/I_2(x), y) \quad &= \quad pd_2(G - y) + 1. \end{aligned}$$

On the other hand by Proposition 2.10, together with the exact sequence

$$0 \longrightarrow R/I_2(G) : xy \longrightarrow R/I_2(G) \longrightarrow R/I_2(G)xy \longrightarrow 0,$$

it follows that

$$\begin{aligned} P_2(G) \quad \leq \quad & \max\left\{P_2(G-(N[x]\cup N[y]))+deg(x)+deg(y)\right.\\ & \left. -\,|N[x]\cap N[y]|, P_2(G-x)+1, P_2(G-y)+1\right\}. \end{aligned}$$ □

Proposition 2.18 *Let G be a graph and $I_2(G)$ is path ideal of G. Then*

$$Bight\,(I_2(G)) \leq pd_2(G).$$

Proof Let P be a minimal vertex cover with maximal cardinality. Then by Proposition 2.5, P is an associated prime of $R/I_2(G)$, so

$$pd_2(G) = pd\,(R/I_2(G)) \geq pd_{R_p}\,(R_p/I_2(G)R_p) = dimR_p = htP.$$ □

Proposition 2.19 *Let K_n denote the complete graph on n vertices and let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices.*

(*a*) $pd_2(K_n) = n-2$;

(*b*) $pd_2(K_{m,n}) = m+n-2$.

Proof (*a*) The proof is by induction on n. If $n = 2$ or 3, then the result easy follows. Let $n \geq 4$ and suppose that for every complete graphs K_n of other less than n the result is true. Since $Bight\,(I_2\,(K_n)) = n-2$ then by Proposition $pd_2(K_n) \geq n-2$. On the other hand by the inductive hypothesis, we have $pd_2(K_{n-1}) = n-3$. So by Proposition 2.17,

$$pd_2(K_n) \leq \max\left\{n-2, n-2\right\}.$$

(*b*) Again we use by induction on $m+n$. If $m+n = 2$ or 3, then it is easy to see that $pd_2(K_{m,n}) = 0$ or 1. Let $m+n \geq 4$ and suppose that for every complete bipartite graph $K_{m,n}$ of order less than $m+n$ the result is true. Since $Bight\,(I_2\,(K_{m,n})) = m+n-2$ then $pd_2(K_{m,n}) \geq m+n-2$. Also, by the inductive hypothesis we have $pd_2(K_{m-1,n}) = m+n-3$ and $pd_2(K_{m,n-1}) = m+n-3$. So by Proposition2.17,

$$pd_2(K_{m,n}) \leq \max\left\{m+n-2, pd_2(K_{m-1,n})+1, pd_2(K_{m,n-1})+1 = m+n-2\right\}.$$

This completes the proof. □

Corollary 2.20 *Let S_n denote the star graph on $n+1$ vertices and $S_{m,n}$ denote the double star, then $pd_2(S_{m,n}) = m+n$.*

Proof It follows from Proposition 2.19 with assume $m = 1$ and it is easy to see that $BightI_2\,(S_{m,n}) = m+n$, and so by Proposition 2.17, it follows that

$$pd_2\,(S_{m,n}) = m+n.$$ □

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Slant Submanifolds of a Conformal (k,μ)-Contact Manifold

Siddesha M.S.

(Department of mathematics, New Horizon College of Engineering, Bangalore, India)

Bagewadi C.S.

(Department of Mathematics, Kuvempu University, Shankaraghatta, Shimoga, Karnataka, India)

E-mail: prof_bagewadi@yahoo.co.in

Abstract: In this paper, we study the geometry of slant submanifolds of conformal (k,μ)-contact manifold when the tensor field Q is parallel. Further, we give a necessary and sufficient condition for a 3-dimensional slant submanifold of a 5-dimensional conformal (k,μ)-contact manifold to be a proper slant submanifold.

Key Words: (k,μ)-contact manifold; conformal (k,μ)-contact manifold; slant submanifold.

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§1. Introduction

Let (M^{2n},J,g) be a Hermitian manifold of even dimension $2n$, where J and g are the complex structure and Hermitian metric respectively. Then (M^{2n},J,g) is a locally conformal Kähler manifold if there is an open cover $\{U_i\}_{i\in I}$ of M^{2n} and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \to R$ such that each local metric $g_i = exp(-f_i)g|U_i$ is Kählerian. Here $g|U_i = \iota_i^* g$ where $\iota_i : U_i \to M^{2n}$ is the inclusion. Also (M^{2n},J,g) is globally conformal Kähler if there is a C^∞ function $f : M^{2n} \to R$ such that the metric $exp(f)g$ is Kählerian [11]. In 1955, Libermann [14] initiated the study of locally conformal Kähler manifolds. The geometrical conditions for locally conformal Kähler manifold have been obtained by Visman [22] and examples of these locally conformal Kähler manifolds were given by Triceri in 1982 [21]. In 2001, Banaru [2] succeeded to classify the sixteen classes of almost Hermitian Kirichenko's tensors. The locally conformal Kähler manifold is one of the sixteen classes of almost Hermitian manifolds. It is known that there is a close relationship between Kähler and contact metric manifolds because Kählerian structures can be made into contact structures by adding a characteristic vector field ξ. The contact structures consists of Sasakian and non-Sasakian cases. In 1972, Kenmotsu introduced a class of contact metric manifolds, called Kenmotsu manifolds, which are not Sasakian [13]. Later in 1995, Blair, Koufogiorgos and Papantoniou [4] introduced the notion of (k,μ)-contact manifold which consists of both Sasakian and non-Sasakian.

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On the other hand, Chen [7] introduced the notion of slant submanifold for an almost Hermitian manifold, as a generalization of both holomorphic and totally real submanifolds. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [8, 9, 10], while slant submanifolds of Kaehler manifold were given by Maeda, Ohnita and Udagawa [17]. The notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by Lotta [15] and he has proved some properties of such immersions. Later, the study of slant submanifolds was enriched by the authors of [6, 12, 16, 18, 19] and many others. Recently, the authors of [1] introduced conformal Sasakian manifold and studied slant submanifolds of the conformal Sasakian manifold. As a generalization to the work of [1] in [20], we defined conformal (k,μ)-contact manifold and studied invariant and anti-invariant submanifolds of it. Our aim in the present paper is to extend the study of slant submanifold to the setting of conformal (k,μ)-contact manifold.

The paper is organized as follows: In section 2, we recall the notion and some results of (k,μ)-contact manifold and their submanifolds, which are used for further study. In section 3, we introduce a conformal (k,μ)-contact manifold and give some properties of submanifolds of it. Section 4 deals with the study of slant submaifolds of (k,μ)-contact manifold. Section 5 is devoted to the study of characterization of three-dimensional slant submanifolds of (k,μ)-contact manifold via covariant derivative of T and T^2, where T is the tangent projection of (k,μ)-contact manifold.

§2. Preliminaries

2.1 (k,μ)-Contact Manifold

Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure $(\tilde{\phi},\tilde{\xi},\tilde{\eta},\tilde{g})$, where $\tilde{\phi},\tilde{\xi},\tilde{\eta}$ are the tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$ respectively, and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$ satisfying

$$\begin{aligned} \tilde{\phi}^2 &= -I+\tilde{\eta}\otimes\tilde{\xi}, \quad \tilde{\eta}(\tilde{\xi})=1, \quad \tilde{\phi}\tilde{\xi}=0, \quad \tilde{\eta}\cdot\tilde{\phi}=0, \\ \tilde{g}(\tilde{\phi}X,\tilde{\phi}Y) &= \tilde{g}(X,Y)-\tilde{\eta}(X)\tilde{\eta}(Y), \quad \tilde{\eta}(X)=\tilde{g}(X,\xi), \end{aligned} \tag{2.1}$$

for all vector fields X,Y on $\tilde{M}$. An almost contact metric structure becomes a contact metric structure if

$$\tilde{g}(X,\tilde{\phi}Y)=d\tilde{\eta}(X,Y).$$

Then the 1-form $\tilde{\eta}$ is contact form and $\tilde{\xi}$ is a characteristic vector field.

We now define a $(1,1)$ tensor field $\tilde{h}$ by $\tilde{h}=\frac{1}{2}\mathfrak{L}_{\tilde{\xi}}\tilde{\phi}$, where $\mathcal{L}$ denotes the Lie differentiation, then $\tilde{h}$ is symmetric and satisfies $\tilde{h}\phi=-\phi\tilde{h}$. Further, a q-dimensional distribution on a manifold $\tilde{M}$ is defined as a mapping D on $\tilde{M}$ which assigns to each point $p\in\tilde{M}$, a q-dimensional subspace D_p of $T_p\tilde{M}$.

The (k,μ)-nullity distribution of a contact metric manifold $\tilde{M}(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned} N(k,\mu): p \rightarrow N_p(k,\mu) = \{Z \in T_p\tilde{M} : \tilde{R}(X,Y)Z &= k[\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y] \\ &\quad +\mu[\tilde{g}(Y,Z)\tilde{h}X - \tilde{g}(X,Z)\tilde{h}Y]\} \end{aligned}$$

for all $X,Y \in T\tilde{M}$. Hence if the characteristic vector field $\tilde{\xi}$ belongs to the (k,μ)-nullity distribution, then we have

$$\tilde{R}(X,Y)\tilde{\xi} = k[\tilde{\eta}(Y)X - \tilde{\eta}(X)Y] + \mu[\tilde{\eta}(Y)\tilde{h}X - \tilde{\eta}(X)\tilde{h}Y]. \tag{2.2}$$

The contact metric manifold satisfying the relation (2.2) is called (k,μ) contact metric manifold [4]. It consists of both k-nullity distribution for $\mu = 0$ and Sasakian for $k = 1$. A (k,μ)-contact metric manifold $\tilde{M}(\phi,\xi,\eta,g)$ satisfies

$$(\tilde{\nabla}_X\tilde{\phi})Y = \tilde{g}(X + \tilde{h}X, Y)\tilde{\xi} - \tilde{\eta}(Y)(X + \tilde{h}X) \tag{2.3}$$

for all $X,Y \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to $\tilde{g}$. From (2.3), we have

$$\tilde{\nabla}_X\tilde{\xi} = -\tilde{\phi}X - \tilde{\phi}\tilde{h}X \tag{2.4}$$

for all $X,Y \in T\tilde{M}$.

2.2 Submanifold

Assume M is a submanifold of a (k,μ)-contact manifold $\tilde{M}$. Let g and ∇ be the induced Riemannian metric and connections of M, respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\tilde{\nabla}_XY = \nabla_XY + \sigma(X,Y), \quad \tilde{\nabla}_XN = -A_NX + \nabla^{\perp}_XN \tag{2.5}$$

for all X,Y on TM and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the normal connection and A is the shape operator of M with respect to the unit normal vector N. The second fundamental form σ and the shape operator A are related by:

$$g(\sigma(X,Y),N) = g(A_NX,Y). \tag{2.6}$$

Let R and $\tilde{R}$ denote the curvature tensor of M and $\tilde{M}$, then, the Gauss and Ricci equations are given by

$$\begin{aligned} \tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)), \\ \tilde{g}(\tilde{R}(X,Y)N_1,N_2) &= g(R^{\perp}(X,Y)N_1,N_2) - g([A_1,A_2]X,Y) \end{aligned}$$

for all $X,Y,Z,W \in TM$, $N_1,N_2 \in T^{\perp}M$ and A_1, A_2 are shape operators corresponding to N_1, N_2 respectively.

For each $x \in M$ and $X \in T_xM$, we decompose ϕX into tangential and normal components as:

$$\phi X = TX + FX, \tag{2.7}$$

where, T is an endomorphism and F is normal valued 1-form on T_xM. Similarly, for any $N \in T_x^{\perp}M$, we decompose ϕV into tangential and normal components as:

$$\phi N = tN + fN, \tag{2.8}$$

where, t is a tangent valued 1-form and f is an endomorphism on $T_x^{\perp}M$.

2.3 Slant Submanifolds of an Almost Contact Metric Manifold

For any $x \in M$ and $X \in T_xM$ such that X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is a constant θ, that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [**?**]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

We mention the following results for later use.

Theorem 2.1([6]) *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi). \tag{2.9}$$

Further more, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary 2.1([6]) *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(TX, TY) = cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{2.10}$$

$$g(FX, FY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{2.11}$$

Lemma 2.1([15]) *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, at each point x of M, $Q|D$ has only one eigenvalue $\lambda_1 = -cos^2\theta$.*

Lemma 2.2([15]) *Let M be a 3-dimensional slant submanifold of an almost contact metric manifold $\tilde{M}$. Suppose that M is not anti invariant. If $p \in M$, then in a neighborhood of p, there exist vector fields e_1, e_2 tangent to M, such that ξ, e_1, e_2 is a local orthonormal frame satisfying*

$$Te_1 = (cos\theta)e_2, \quad Te_2 = -(cos\theta)e_1. \tag{2.12}$$

§3. Conformal (k, μ)-Contact Manifold

A smooth manifold $(\bar{M}^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is called a conformal (k, μ)-contact manifold of a (k, μ)-

contact structure $(\tilde{M}^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ if, there is a positive smooth function $f : \bar{M}^{2n+1} \to R$ such that

$$\tilde{g} = exp(f)\bar{g}, \quad \tilde{\phi} = \bar{\phi}, \quad \tilde{\eta} = (exp(f))^{\frac{1}{2}}\bar{\eta}, \quad \tilde{\xi} = (exp(-f))^{\frac{1}{2}}\bar{\xi}. \tag{3.1}$$

Example 3.1 Let R^{2n+1} be the $(2n+1)$-dimensional Euclidean space spanned by the orthogonal basis $\{\xi, X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n\}$ and the Lie bracket defined as in [?]. Then, the almost contact metric structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined by

$$\begin{aligned}
\bar{\phi}\left(\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + z\frac{\partial}{\partial z}\right)\right) &= \sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{n} Y_i y^i\frac{\partial}{\partial z}, \\
\bar{g} &= exp(-f)\{\bar{\eta}\otimes\bar{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\}\}, \\
\bar{\eta} &= (exp(-f))^{\frac{1}{2}}\left\{\frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)\right\}, \\
\bar{\xi} &= (exp(f))^{\frac{1}{2}}\left\{2\frac{\partial}{\partial z}\right\},
\end{aligned}$$

where $f = \sum_{i=1}^{n}(x^i)^2 + (y^i)^2 + z^2$.

It is easy to reveal that $(R^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is not a (k, μ)-contact manifold, but R^{2n+1} with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined by

$$\begin{aligned}
\tilde{\phi} &= \bar{\phi}, \\
\tilde{g} &= \tilde{\eta}\otimes\tilde{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\}, \\
\tilde{\eta} &= \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \\
\tilde{\xi} &= 2\frac{\partial}{\partial z},
\end{aligned}$$

is a (k, μ)-space form.

Let $\bar{M}$ be a conformal (k, μ)-contact manifold, let $\tilde{\nabla}$ and $\bar{\nabla}$ denote the Riemannian connections of $\bar{M}$ with respect to metrics $\tilde{g}$ and $\bar{g}$, respectively. Using the Koszul formula, we obtain the following relation between the connections $\tilde{\nabla}$ and $\bar{\nabla}$

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - \bar{g}(X, Y)\omega^\sharp\} \tag{3.2}$$

such that $\omega(X) = X(f)$ and $\omega^\sharp = gradf$ is a vector field metrically equivalent to 1-form ω, that is $\bar{g}(\omega^\sharp, X) = \omega(X)$.

Then with a straight forward computation we will have

$$
\begin{aligned}
exp(-f)(\tilde{R}(X,Y,Z,W)) &= \bar{R}(X,Y,Z,W) + \frac{1}{2}\{B(X,Z)\bar{g}(Y,W) - B(Y,Z) \\
&\quad \bar{g}(X,W) + B(Y,W)\bar{g}(X,Z) - B(X,W)\bar{g}(Y,Z)\} \\
&\quad + \frac{1}{4}\|\omega^\sharp\|^2\{\bar{g}(X,Z)\bar{g}(Y,W) - \bar{g}(Y,Z)\bar{g}(X,W)\} \qquad (3.3)
\end{aligned}
$$

for all vector fields X, Y, Z, W on $\bar{M}$, where $B = \bar{\nabla}\omega - \frac{1}{2}\omega \otimes \omega$ and $\bar{R}$, $\tilde{R}$ are the curvature tensors of M related to connections of $\bar{\nabla}$ and $\tilde{\nabla}$, respectively. Furthermore, by the relations, (2.1), (2.3) and (3.2) we get

$$
\begin{aligned}
(\bar{\nabla}_X\bar{\phi})Y &= (exp(f))^{\frac{1}{2}}\{\bar{g}(X + \bar{h}X, Y)\xi - \bar{\eta}(Y)(X + \bar{h}X)\} \\
&\quad -\frac{1}{2}\{\omega(\bar{\phi}Y)X - \omega(Y)\bar{\phi}X + g(X,Y)\bar{\phi}\omega^\sharp - g(X,\bar{\phi}Y)\omega^\sharp\} \qquad (3.4) \\
\bar{\nabla}_X\bar{\xi} &= -(exp(f))^{\frac{1}{2}}\{\bar{\phi}X + \bar{\phi}hX\} + \frac{1}{2}\{\bar{\eta}(X)\omega^\sharp - \omega(\bar{\xi})X\} \qquad (3.5)
\end{aligned}
$$

for all vector fields X, Y on $\bar{M}$. Now assume M is a submanifold of a conformal (k, μ)-contact manifold $\bar{M}$ and ∇, R are the connection, curvature tensor on M, respectively, and g is an induced metric on M.

For all $X, Y \in TM$ and $N \in T^\perp M$, from the Gauss, Weingarten formulas and (3.4), we obtain the following relations:

$$
\begin{aligned}
(\nabla_X T)Y &= A_{FY}X + t\sigma(X,Y) + (exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} \\
&\quad -\frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX + g(X,Y)(\phi\omega^\sharp)^\top - g(X,TY)(\omega^\sharp)^\top\}, \qquad (3.6) \\
(\nabla_X F)Y &= f\sigma(X,Y) - \sigma(X,TY) + \frac{1}{2}\{\omega(Y)FX - g(X,Y)F\omega^\sharp + g(X,TY)\omega^{\sharp\perp}\}, \qquad (3.7) \\
(\nabla_X t)N &= A_{fN}X - PA_NX - \frac{1}{2}\{\omega(\phi N)X - \omega(N)PX + g(X,tN)(\omega^\sharp)^\top\}, \qquad (3.8) \\
(\nabla_X f)N &= -\sigma(X,tN) - FA_NX + \frac{1}{2}\{\omega(N)FX + g(X,tN)(\omega^\sharp)^\perp\}, \qquad (3.9)
\end{aligned}
$$

where, $g = \bar{g}|M$, $\eta = \bar{\eta}|M$, $\xi = \bar{\xi}|M$ and $\phi = \bar{\phi}|M$.

§4 Slant Submanifolds of Conformal (k, μ)-Contact Manifolds

In this section, we prove a characterization theorem for slant submanifolds of a conformal (k, μ)-contact manifold.

Theorem 4.1 *Let M be a slant submanifold of conformal (k, μ)-contact manifold $\bar{M}$ such that $\omega^\sharp \in T^\perp M$ and $\xi \in TM$. Then Q is parallel if and only if one of the following is true:*

(i) M is anti-invariant;
(ii) $dim(M) \geq 3$;
(iii) M is trivial.

Proof Let θ be the slant angle of M in $\bar{M}$, then for any $X, Y \in TM$ and by equation (2.9), we infer

$$T^2Y = QY = cos^2\theta(-Y + \eta(Y)\xi). \tag{4.1}$$

$$\Rightarrow Q(\nabla_X Y) = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi). \tag{4.2}$$

Differentiating (4.1) covariantly with respect to X, we get

$$\nabla_X QY = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi - g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi). \tag{4.3}$$

Subtracting (4.2) from (4.3), we obtain

$$(\nabla_X Q)Y = cos^2\theta[g(\nabla_X Y, \xi)\xi + \eta(Y)\nabla_X\xi]. \tag{4.4}$$

If Q is parallel, then from (4.4) it follows that either $cos(\theta) = 0$ i.e. M is anti-invariant or

$$g(\nabla_X Y, \xi)\xi + \eta(Y)\nabla_X\xi = 0. \tag{4.5}$$

We know $g(\nabla_X\xi, \xi) = 0$, since $g(\nabla_X\xi, \xi) = -g(\xi, \nabla_X\xi)$, which implies $\nabla_X\xi \in D$.

Suppose $\nabla_X\xi \neq 0$, then (4.5) yields $\eta(Y) = 0$ i.e. $Y \in D$. But then (4.5) implies $\nabla_X\xi \in D^{\perp} \oplus < \xi >$, which is absurd.

Hence $\nabla_X\xi = 0$ and therefore either $D = 0$ or we can take at least two linearly independent vectors X and TX to span D. In this case the eigenvalue must be non-zero as $\theta = \frac{\pi}{2}$ has already been taken. Hence $dim(M) \geq 3$. □

Now, we state the the main result of this section.

Theorem 4.2 *Let M be a slant submanifold of conformal (k,μ)-contact manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if*

(1) *The endomorphism $Q|D$ has only one eigen value at each point of M;*

(2) *There exists a function $\lambda : M \to [0, 1]$ such that*

$$\begin{aligned}(\nabla_X Q)Y &= \lambda\{(exp(f))^{\frac{1}{2}}[g(Y, TX + ThX)\xi - \eta(Y)(TX + ThX)] \\ &\quad -\frac{1}{2}\{\omega(\xi)g(X, Y)\xi - \eta(X)\omega(Y)\xi + \omega(\xi)\eta(Y)X - \eta(X)\eta(Y)\omega^{\sharp^T}\}\},\end{aligned} \tag{4.6}$$

for any $X, Y \in TM$. Moreover, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Proof Statement 1 gets from Lemma (2.1). So, it remains to prove statement 2. Let M be a slant submanifold, then by (4.4) we have

$$(\nabla_X Q)Y = cos^2\theta(-g(Y, \nabla_X\xi) + \eta(Y)\nabla_X\xi). \tag{4.7}$$

By putting (3.5) in (4.7), we find (4.6). Conversely, let $\lambda_1(x)$ is the only eigenvalue of $Q|D$ at each point $x \in M$ and $Y \in D$ be a unit eigenvector associated with λ_1, i.e., $QY = \lambda_1 Y$.

Then from statement (2), we have

$$X(\lambda_1)Y + \lambda_1\nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda\{(exp(f))^{\frac{1}{2}}g(X, TY + ThY)\xi - \frac{1}{2}\{\omega(\xi)g(X,Y)\xi - \eta(X)\omega(Y)\xi\}\}, \tag{4.8}$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that $X(\lambda_1) = 0$. Hence λ_1 is constant. So it remains to prove M is slant. For proof one can refer to Theorem (4.3) in [6].

§5. Slant Submanifolds of Dimension Three

Theorem 5.1 *Let M be a 3-dimensional proper slant submanifold of a conformal (k,μ)-contact manifold $\bar{M}$, such that $\xi \in TM$, then*

$$\begin{aligned}(\nabla_X T)Y &= cos^2\theta(exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} + \frac{1}{2}\{\omega(\xi)g(TX,Y)\xi \\ &\quad -\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\}\end{aligned} \tag{5.1}$$

for any $X, Y \in TM$ and θ is the slant angle of M.

Proof Let $X, Y \in TM$ and $p \in M$. Let ξ, e_1, e_2 be the orthonormal frame in a neighborhood U of p given by Lemma (2.2). Put $\xi|U = e_0$ and let α_i^j be the structural 1-forms defined by

$$\nabla_X e_i = \sum_{j=0}^{2} \alpha_i^j e_j. \tag{5.2}$$

In view of orthonormal frame ξ, e_1, e_2, we have

$$Y = \eta(Y)e_0 + g(Y, e_1)e_1 + g(Y, e_2)e_2. \tag{5.3}$$

Thus, we get

$$(\nabla_X T)Y = \eta(Y)(\nabla_X T)e_0 + g(Y, e_1)(\nabla_X T)e_1 + g(Y, e_2)(\nabla_X T)e_2. \tag{5.4}$$

Therefore, for obtaining $(\nabla_X T)Y$, we have to get $(\nabla_X T)e_0$, $(\nabla_X T)e_1$ and $(\nabla_X T)e_2$. By applying (3.5), we get

$$\begin{aligned}(\nabla_X T)e_0 &= \nabla_X(Te_0) - T(\nabla_X e_0) \\ &= (exp(f))^{\frac{1}{2}}(T^2X + T^2hX) + \frac{1}{2}\{\omega(\xi)TX - \eta(X)T\omega^{\sharp^T}\}.\end{aligned} \tag{5.5}$$

Moreover, by using (2.12) we obtain

$$\begin{aligned}(\nabla_X T)e_1 &= \nabla_X(Te_1) - T(\nabla_X e_1)\\ &= \nabla_X((cos\theta)e_2) - T(\alpha_1^0(X)e_0 + \alpha_1^1(X)e_1 + \alpha_1^2(X)e_2)\\ &= (cos\theta)\alpha_2^0(X)e_0. \qquad (5.6)\end{aligned}$$

Similarly, we get

$$(\nabla_X T)e_2 = -(cos\theta)\alpha_1^0(X)e_0. \qquad (5.7)$$

By substituting (5.5)-(5.7) in (5.4), we have

$$\begin{aligned}(\nabla_X T)Y &= (exp(f))^{\frac{1}{2}}\eta(Y)(T^2X + T^2hX) + \frac{1}{2}\{\eta(Y)\omega(\xi)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\}\\ &\quad +cos(\theta)\{g(Y, e_1)\alpha_2^0(X)e_0 - g(Y, e_2)\alpha_1^0(X)e_0\}. \qquad (5.8)\end{aligned}$$

Now, we obtain $\alpha_1^0(X)$ and $\alpha_2^0(X)$ as follows:

$$\begin{aligned}\alpha_1^0(X) &= g(\nabla_X e_1, e_0)\\ &= Xg(e_1, e_0) - g(e_1, \nabla_X e_0)\\ &= -(exp(f))^{\frac{1}{2}}g(e_2, X + hX) + \frac{1}{2}\{\omega(\xi)g(e_1, X) - \eta(X)\omega(e_1)\} \qquad (5.9)\end{aligned}$$

and similarly we get

$$\alpha_2^0(X) = cos\theta g(e_1, X) + cos\theta g(e_1, hX). \qquad (5.10)$$

By using (5.9) and (5.10) in (5.8) and in view of (5.3) and (2.9) we obtain (5.1). □

From, Theorems 4.3 and 5.4, we can state the following:

Corollary 5.1 *Let M be a three dimensional submanifold of a (k, μ)-contact manifold tangent to ξ. Then the following statements are equivalent:*

(1) *M is slant;*

(2) $(\nabla_X T)Y = cos^2\theta(exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} + \frac{1}{2}\{\omega(\xi)g(TX, Y)\xi$

$$-\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\};$$

(3) $(\nabla_X Q)Y = \lambda\{(exp(f))^{\frac{1}{2}}[g(X, TX + ThX)\xi - \eta(Y)(TX + ThX)] - \frac{1}{2}\{\omega(\xi)g(X, Y)\xi$

$$-\eta(X)\omega(Y)\xi + \omega(\xi)\eta(Y)X - \eta(X)\eta(Y)\omega^{\sharp^T}\}\}.$$

The next result characterizes 3-dimensional slant submanifold in terms of the Weingarten map.

Theorem 5.2 *Let M be a 3-dimensional proper slant submanifold of a conformal (k, μ)-contact* *manifold $\bar{M}$, such that $\xi \in TM$. Then, there exists a function $C : M \to [0,1]$ such that*

$$\begin{aligned} A_{FX}Y &= A_{FY}X + C(exp(f))^{\frac{1}{2}}(\eta(X)(Y+hY) - \eta(Y)(X+hX)) + \omega(\xi)g(TX,Y)\xi \\ &+g(X,TY)\omega^\sharp + \frac{1}{2}\{\eta(X)\omega(TY)\xi - \eta(Y)\omega(TX)\xi + \eta(X)\omega(\xi)TY \\ &-\eta(Y)\omega(\xi)TX - \omega(X)TY + \omega(Y)TX + \omega(TX)Y - \omega(TY)X\}, \end{aligned} \tag{5.11}$$

for any $X, Y \in TM$. Moreover in this case, if θ is the slant angle of M then we have $C = sin^2\theta$.

Proof Let $X, Y \in TM$ and M is a slant submanifold. From (3.6) and Theorem 5.1, we have

$$\begin{aligned} t\sigma(X,Y) &= (\lambda-1)(exp(f))^{\frac{1}{2}}\{g(Y, X+hX)\xi - \eta(Y)(X+hX)\} + \frac{1}{2}\{\omega(\xi)g(X,TY)\xi \\ &-\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T} + \omega(TY)X - \omega(Y)TX \\ &+g(X,Y)T\omega^\sharp - g(X,TY)\omega^\sharp\} - A_{FY}X. \end{aligned} \tag{5.12}$$

Now by using the fact that $\sigma(X,Y) = \sigma(Y,X)$, we obtain (5.11). □

Next, we assume that M is a three dimensional proper slant submanifold M of a five-dimensional conformal (k,μ)-contact manifold $\bar{M}$ with slant angle θ. Then for a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (sec\theta)Te_1, \quad e_3 = \xi, \quad e_4 = (csc\theta)Fe_1, \quad e_5 = (csc\theta)Fe_2. \tag{5.13}$$

It is easy to show that $e_1 = -(sec\theta)Te_2$ and by using Corollary 2.1, $\{e_1, e_2, e_3, e_4, e_5\}$ form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. Also we have

$$te_4 = -sin\theta e_1, \quad te_5 = -sin\theta e_2, \quad fe_4 = -cos\theta e_5, \quad fe_5 = -cos\theta e_4. \tag{5.14}$$

If we put $\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r),\ i,j = 1,2,3,\ r = 4,5$, then we have the following result:

Lemma 5.1 *In the above conditions, we have*

$$\begin{aligned} \sigma^4_{12} &= \sigma^5_{11},\ \sigma^4_{22} = \sigma^5_{12}, \\ \sigma^4_{13} &= \sigma^5_{23} = -(exp(f))^{\frac{1}{2}}sin\theta \\ \sigma^4_{32} &= \sigma^4_{33} = \sigma^5_{33} = \sigma^5_{13} = 0. \end{aligned} \tag{5.15}$$

Proof Apply (5.11) by setting $X = e_1$ and $Y = e_2$, we obtain

$$A_{e_4}e_2 = A_{e_5}e_1 + (cot\theta)\{\omega(\xi)\xi - \omega^\sharp + \omega(e_1)e_1 + \omega(e_2)e_2\}.$$

Using (2.6) in the above relation, we get

$$\sigma_{12}^4 = \sigma_{11}^5, \;\; \sigma_{22}^4 = \sigma_{12}^5, \sigma_{23}^4 = \sigma_{13}^5.$$

Further, by taking $X = e_1$ and $Y = e_3$ in (5.11), we have

$$A_{e_4} e_3 = -(exp(f))^{\frac{1}{2}}(sin\theta)(e_1 + he_1). \tag{5.16}$$

After applying (2.6) in (5.16), we obtain

$$\sigma_{13}^4 = -(exp(f))^{\frac{1}{2}}(sin\theta), \;\sigma_{23}^4 = \sigma_{33}^4 = 0.$$

In the similar manner by putting $X = e_2$ and $Y = e_3$, we get

$$\sigma_{23}^5 = -(exp(f))^{\frac{1}{2}}(sin\theta), \;\sigma_{33}^5 = 0. \qquad \square$$

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Operations of n-Wheel Graph via Topological Indices

V. Lokesha and T. Deepika

(Department of studies in Mathematics, Vijayanagara Sri Krishnadevaraya University, Ballari, India)

E-mail: v.lokesha@gmail.com; sastry.deepi@gmail.com

Abstract: In this paper, we discussed the Topological indices viz., Wiener, Harmonic, Geometric-Arithmetic(GA), first and second Zagreb indices of n-wheel graphs with bridges using operator techniques.

Key Words: n-wheel graph, subdivision operator, line graph, complement of n-wheel, wiener index, harmonic, GA, first and second zagreb indices.

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§1. Introduction

For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. A topological index of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [4]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

The Wiener index [17], defined as the sum of all distances between pairs of vertices u and v in a graph G is given by

$$W(G) = \sum_{uv \in E(G)} d(u,v)$$

Another few degree based topological indices are defined as follows:

The Harmonic index according to [13] is given by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$$

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The geometric-arithmetic index of a graph G [3], denoted by $GA(G)$ and is defined by

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u.d_v}}{d_u + d_v}$$

The first and second Zagreb indices [10] is defined as

$$M_1(G) = \sum_{u \in V(G)} d_u^2, \quad M_2(G) = \sum_{uv \in E(G)} d_u.d_v$$

Throughout paper, we have used n-wheel graph with standard operators.

The n-wheel graph is defined as the graph $K_1 + C_n$ where K_1 is the singleton graph and C_n is the cycle graph. The center of the wheel is called the hub and the edges joining the hub and vertices of C_n are called the spokes.

The well known operators are recalled [7, 9, 10].

Adding a additional edge on top most vertex of two or more graphs is defined as bridge operator.

The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting an additional vertex into each edge of G.

The Line graph $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G with two vertices being adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent in G.

Complement graph or inverse of a graph G is a graph G' on the same vertices such that two distinct vertices of G' are adjacent if and only if they are not adjacent in G.

The paper is starting with the preliminaries needed for our study. Section 2, construction of bridge operator in a wheel graph results on different topological indices are discussed. Section 3, complement of wheel graph of constructed graph results are established. Section 4, Subdivision operator of constructed graph results are shown. Final section deals with the line graph of constructed graph results are highlighted.

§2. Distance and Degree-Based Indices of n-Wheel Graph with Bridge Operator

In this section, we constructed a n-wheel graph by attaching bridge at top most vertex of a graphs and established the results on different topological indices.

Here, we denote the edge set of n-wheel graph G_n, then $E_i = \{e = uv \in E(G) | d_u + d_v = i, \forall i = 1, 2, \cdots, n\}$.

Theorem 2.1 *Let G_n be attached wheel graphs of n vertices with bridge operator b_k, $k > 0$, then the harmonic index is*

$$H(G_n, b_k) = (b_k + 1)\left[\frac{n^3 + 8n^2 + 3n - 42}{3n^2 + 15n + 18}\right] + \frac{23b_k + 16}{28}.$$

Proof Consider two wheel graphs with 4 vertices ($n \geq 4$), if a bridge is attached, there are E_6, E_7, E_8 edges for $B(G_1, G_2) = B(G_1, G_2, ; v_1, v_2)$.

(i) The number of edges is $7b_k + 6$

The number of wheel graphs and bridges will increases in a graph G_4 then the harmonic index is:

$$H(G_4, b_k) = \frac{59b_k + 52}{28}.$$

Similarly, if we consider two wheel graphs with 5 vertices, if a bridge is attached, there are E_6, E_7, E_8 edges.

(ii) The number of edges is $9b_k + 8$.

(iii) The number of wheel graphs and bridges will increases in a graph G_5 then the harmonic index is

$$H(G_5, b_k) = \frac{218b_k + 197}{84}.$$

The total number of edges in G_n is $(2n-1)b_k + 2(n-1)$.

Computing for n vertices with b_k bridges of G_n the harmonic index is

$$H(G_n, b_k) = (b_k + 1)\left[\frac{n^3 + 8n^2 + 3n - 42}{3n^2 + 15n + 18}\right] + \frac{23b_k + 16}{28}. \qquad \square$$

Theorem 2.2 *The geometric-arithmetic index of a bridge operator formed by G_n is*

$$GA(G_n, b_k) = (b_k + 1)\left[\frac{\sqrt{3}(n-2)}{\sqrt{n-1}} + \frac{4\sqrt{n-1}}{2n-1} + n - 3\right] + \frac{8\sqrt{3}(b_k + 1)}{7} + 2b_k.$$

Proof If a bridge is formed for two wheel graphs with 4 vertices, having E_6, E_7, E_8 edges and using equation (i), then geometric-arithmetic index is

$$GA(G_4, b_k) = \frac{4(3\sqrt{3} + 7)(b_k + 1)}{7}$$

Similarly, For $n = 5$. Using (ii), then Geometric-Arithmetic index of G_4 is

$$GA(G_5, b_k) = 2(b_k + 1)\frac{4\sqrt{3} + 21}{7} + \frac{4\sqrt{3} + 7}{7} + 5b_k.$$

Computing for n vertices b_k bridges of G_n the geometric-arithmetic index is

$$GA(G_n, b_k) = (b_k + 1)\left[\frac{\sqrt{3}(n-2)}{\sqrt{n-1}} + \frac{4\sqrt{n-1}}{2n-1} + n - 3\right] + \frac{8\sqrt{3}(b_k + 1)}{7} + 2b_k. \qquad \square$$

Theorem 2.3 *The G_n of a bridge operator for wiener index is*

$$W(G_n, b_k) = n^2(b_k + 1) - 3n(b_k + 1) + 3b_k + 2.$$

Proof We adopted the proof technique of Theorem 2.1 and using equations (i) and (ii) in the wiener indices for $n = 4$ and $n = 5$ is

$$W(G_4, b_k) = 7b_k + 6.$$

and

$$W(G_5, b_k) = 13b_k + 12.$$

Computing for the wiener index of graph G_n is

$$W(G_n, b_k) = n^2(b_k + 1) - 3n(b_k + 1) + 3b_k + 2. \qquad \square$$

§3. Complement of a Constructed Graph

In this segment, a complement of a wheel graphs connected with the number of bridges b_k (constructed graph) for $n \geq 5$ with respect to different topological indices are established.

Theorem 3.1 *Let $G_n^{'}$ be a complement of constructed graph then harmonic index is*

$$H(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-3)(n-4)^2}{2} + \frac{2(n-4)}{2n-7}\right] + \frac{b_k}{n-3}.$$

Proof In $G_n^{'}$ having $(n-1)$ vertices with E_2, E_3, E_4 edges. Therefore,

(iv) the total number of edges is $2(b_k + 1)$.

Hence, the harmonic index is

$$H(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-3)(n-4)^2}{2} + \frac{2(n-4)}{2n-7}\right] + \frac{b_k}{n-3}. \qquad \square$$

Theorem 3.2 *The geometric-arithmetic index of $G_n^{'}$ is*

$$GA(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-3)(n-4)}{2} + \frac{2(n-4)\sqrt{(n-4)(n-3)}}{2n-7}\right] + b_k.$$

Proof The proof technique is applied as in Theorem 3.1. Hence using equation (iv) we get the required result.

$$GA(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-3)(n-4)}{2} + \frac{2(n-4)\sqrt{(n-4)(n-3)}}{2n-7}\right] + b_k. \qquad \square$$

Theorem 3.3 *Let $G_n^{'}$ of wiener index is*

$$W(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-1)(n-4)}{2}\right] + b_k.$$

Proof Let $n = 4, 5, 6, 7, \cdots$ having distance b_k, $3b_k + 2$, $6b_k + 5, \cdots$, then the wiener index is

$$W(G_n^{'}, b_k) = (b_k + 1)\left[\frac{(n-1)(n-4)}{2}\right] + b_k. \qquad \square$$

Observation 3.3 If $n = 4$, $H(G_n^{'}, b_k)$=$GA(G_n^{'}, b_k)$=$W(G_n^{'}, b_k)$=b_k.

§4. Subdivision of Constructed Graph on Degree-Based Indices

In this section, the subdivision operator of constructed graph (G_n) are highlighted.

Theorem 4.1 *Let $S(G_n)$ be a subdivision operator of constructed graph then,*

(1) $H[S(G_n, b_k)] = 2(b_k + 1)\left[\frac{3n^2 + 2n - 11}{5(n+1)}\right] + \frac{15b_k + 12}{12}$;

(2) $GA[S(G_n, b_k)] = (b_k + 1)\left[\frac{6\sqrt{6}(n+1)(n-2) + 10(n-1)\sqrt{n+2} + 5\sqrt{2}(n+1)}{5(n+1)}\right] + \frac{b_k}{2}$;

(3) $M_1[S(G_n, b_k)] = (b_k + 1)(n^2 + 7n - 1)$;

(4) $M_2[S(G_n, b_k)] = (b_k + 1)(22n - 25) + 16b_k$.

Proof A subdivision G_n graph having $(3n - 2)$ vertices and $4(n - 1)$ edges among which $2(n-1)$ vertices are of degree 2, $(n-2)$ vertices are of degree 3, n vertices are of degree $(n-1)$ and by attaching bridge there exists $2b_k$ edges and b_k vertices having degree 2. Here also, adopted the similar proof techniques of earlier theorems we obtained the required results. $\square$

§5. Line Graph of Constructed Graph

In this section, the Line graph of bridge graph of n wheel graph related to different topological indices are discussed. The following results are observed.

Theorem 5.1 *Let $L(G_n)$ be the line graph of constructed graph then*

(1) $H[L(G_n, b_k)] = 2(b_k + 1)\left[\frac{(n-3)}{8} + \frac{2(n-2)}{(n+4)} + \frac{(n-1)(n-2)}{4n} + \frac{2}{(n+5)} + \frac{2}{9}\right] + \frac{b_k}{5}$;

(2) $GA[L(G_n, b_k)] = 2(b_k+1)\left[\frac{n-3}{2} + \frac{4\sqrt{n}(n-2)}{(n+4)} + \frac{(n-1)(n-2)}{4} + \frac{2\sqrt{5n}}{(n+5)} + \frac{4\sqrt{5}}{9}\right] + b_k$;

(3) $M_1[L(G_n, b_k)] = (b_k + 1)[n^3 - n^2 + 16n - 7]$;

(4) $M_2[L(G_n, b_k)] = \frac{(b_k + 1)}{2}\left[n^4 - 3n^3 + 18n^2 + 20n - 16\right] + 25b_k$.

Proof Consider $L(G_n)$ be the line graph of wheel graph using bridge graph. In G_n there are two copies of $2(n-1)$ vertices and total number of edges exists $6(n-1)$ and b_k when bridge is attached. Hence the results are proved by adopting same proof technique used in the earlier sections. $\square$

§6. Conclusion

In this paper, degree based and distance based indices for different types of operators on wheel graphs are studied. This type of relationships may be useful to connectivity between graph structures or chemical structures.

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Complexity of Linear and General Cyclic Snake Networks

E. M. Badr

Scientific Computing Department, Faculty of Computers and Informatics

Benha University, Egypt.

E-mail: badrgraph@gmail.com

B. Mohamed

Mathematics Department, Faculty of Science

Menoufia University, Egypt.

Abstract: In this paper we prove that the number of spanning trees of the linear and general cyclic snake networks is the same using the combinatorial approach. We derive the explicit formulas for the subdivided fan network $S(F_n)$ and the subdivided ladder graph $S(L_n)$. Finally, we calculate their spanning trees entropy and compare it between them.

Key Words: Number of spanning trees, Cyclic snakes networks, Entropy

AMS(2010): 05C05, 05C30

§1. Introduction

The complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph G is defined as the total number of distinct connected acyclic spanning subgraphs. There are many techniques to compute this number. Kirchhoff [1] gave the famous matrix tree theorem. In which $\tau(G)$ = any cofactor of $L(G)$, where $L(G)$ is equal to the degree matrix $D(G)$ of G minus the adjacency matrix $A(G)$ of G .

There are other methods for calculating $t(G)$. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_p$ denote the eigenvalues of H matrix of a p point graph. Then it is easily shown that$\mu_p = 0$. In 1974, Kelmans and Chelnokov [2] shown that, $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a *d-regular* graph G can be expressed as $t(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = \lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}$, $n \geqslant 2$. Clark [4] proved that $\tau(K_{p,q}) = p^{q-1}q^{p-1}$, $p, q \geqslant 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively.

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Therefore, many works derive formulas to calculate the complexity for some classes of graphs. Bogdanowicz [5] derived the explicit formula for the fan network if $n \geqslant 1$,

$$\tau(F_n) = \frac{1}{\sqrt{5}}[(\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n].$$

Sedlacek [6] proposed a formula for the number of spanning trees in a ladder graph. The ladder L_n is the Cartesian product of P_2 and P_n. The number of spanning trees in L_n is given by

$$\tau(L_n) = \frac{\sqrt{3}}{6}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$$

for $n \geqslant 1$. A. Modabish and M. El Marraki investigated the number of spanning trees in the star flower planar graph [7]. In [8], E.M. Badr and B.Mohamed derived the explicit formulas for triangular snake ($\Delta_k - snake$),double triangular snake ($2\Delta_k - snake$) and the total graph of path $P_n(T(P_n))$. Badr and Mohamed [9] derived the explicit formulas for the subdivision of ladder, fan, wheel, triangular snake (Δ_k-*snake*), double triangular snake ($2\Delta_k$-*snake*) and the total graph of path $P_n(T(P_n))$.

In this paper we prove that the number of spanning trees of the linear and general cyclic snake networks is the same using the combinatorial approach. We derive the explicit formulas for the subdivided fan network $\mathrm{S}(F_n)$ and the subdivided ladder graph $\mathrm{S}(L_n)$. Finally, we calculate their spanning trees entropy and compare it between them.

§2. Preliminary Notes

The combinatorial method involves the operation of contraction of an edge. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G.e$. Also we denote by $G-e$ the graph obtained from G by deleting the edge e.

Theorem 2.1([10]) *Let G be a planar graph (multiple edges are allowed in here). Then for any edge e,*

$$\tau(G) = \tau(G - e) + \tau(G \cdot e).$$

Remark 2.2 If *G'* is obtained from G by removing all the pendant edges of G, then

$$\tau(G^{'}) = \tau(G).$$

Remark 2.3 If *G'* is obtained from G by removing all the loops of G, then $\tau(G^{'}) = \tau(G)$.

Remark 2.4 If *G'* is obtained from G by removing one or more than one multiple edges of G, then $\tau(G^{'}) < \tau(G)$.

Definition 2.5([11]) *A triangular snake (Δ_k-snake) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path.*

Definition 2.6 *C_n-cyclic snake is a connected graph in which all blocks are C_n and the block-cut-point graph is a path. Furthermore, if the length of its path is exactly k, we call it a kC_n-cyclic snake.*

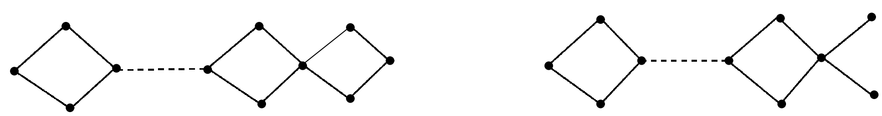
Definition 2.7 *A kC_n -snake is called linear if its block-cut-vertex graph of kC_n -snake has the property that the distance between any two consecutive cut-vertices is $\lfloor \frac{n}{2} \rfloor$.*

§3. Main Results

Theorem 3.1 *The number of spanning trees of the linear kC_4 -snake satisfies the following recursive relation:*

$$\tau(kC_4 - snake) = 4^k$$

Proof Let us consider a graph $kC_4^{/} - snake$ constructed from $kC_4 - snake$ by deleting two edges. See Figure 1



KC_4-Snake $kC_4^{/}$-Snake

Figure 1 Linear kC_4-Snake

We put $kC_4 - snake = \tau(kC_4 - snake)$ and $kC_4^{/} - snake = \tau(kC_4^{/} - snake)$.

It is clear that

$$kC_4 - snake = 3(k-1)C_4 - snake + 4(k-1)C_4^{/} - snake$$

and

$$kC_4 - snake = 2(k-1)C_4 - snake - 4(k-1)C_4^{/} - snake$$

with initial conditions $C_4 - snake = 4$, $C_4^{/} - snake = 1$. Thus, we have

$$\begin{pmatrix} kC_4 - snake \\ kC_4^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_4 - snake \\ (k-1)C_4^{/} - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_4 - snake \\ kC_4^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_4 - snake \\ (k-1)C_4^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_4 - snake \\ C_4^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0, \quad \lambda_1 = -5 \text{ and } \lambda_2 = 4,\ \lambda_1 \neq \lambda_2.$$

Therefore, there is a matrix M invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \frac{1}{9/4} \begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix}.$$

Notice that $A^{n-1} = MB^{n-1}M^{-1}$, where $B^{n-1} = \begin{pmatrix} (-5)^{n-1} & 0 \\ 0 & 4^{n-1} \end{pmatrix}$. We therefore obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-5)^{n-1}}{9} + \frac{2*4^n}{9} & \frac{-4*(-5)^{n-1}}{9} + \frac{4^n}{9} \\ \frac{-2*(-5)^{n-1}}{9} + \frac{2*4^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9} + \frac{4^{n-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.2 *The number of spanning trees of the linear* $kc_6 - snake$*satisfies the following recursive relation* $\tau(\ kc_6 - snake\ ) = 6^k$

Proof Consider a graph $kC_6^{/} - snake$ constructed from $kC_6 - snake$ by deleting two edges. See Figure 2 following.

kC_6-Snake $\qquad$ $kC_6^{/}$-Snake

Figure 2 Linear kC_6-Snake

We put $kC_6 - snake = \tau(kC_6 - snake)$ and $kC_6^/ - snake = \tau(kC_6^/ - snake)$. It is clear that

$$kC_6 - snake = 5((k-1)C_6 - snake) + 6((k-1)C_6^/ - snake)$$

and

$$kC_6^/ - snake = 2((k-1)C_6 - snake) - 6((k-1)C_6^/ - snake)$$

with initial conditions $(C_1 - snake) = 6,\ (C_1^/ - snake) = 1$. Thus, we have

$$\begin{pmatrix} kC_6 - snake \\ kC_6^/ - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^/ - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} 5 & 6 \\ 2 & -6 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_6 - snake \\ kC_6^/ - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^/ - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_6 - snake \\ C_6^/ - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \qquad \lambda_1 = -7 \text{ and } \lambda_2 = 6,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{6} \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{-6}{13} \\ \frac{12}{13} & \frac{6}{13} \end{pmatrix} \quad \Rightarrow \quad A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (6)^{n-1} & 0 \\ 0 & (-7)^{n-1} \end{pmatrix}$. We therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(6)^{n-1}}{13} + \frac{12*(-7)^{n-1}}{13} & \frac{-(6)^n}{13} + \frac{6*(-7)^{n-1}}{13} \\ \frac{-2(6)^{n-1}}{13} + \frac{2*(-7)^{n-1}}{13} & \frac{2*(6)^n}{13} + \frac{(-7)^{n-1}}{13} \end{pmatrix} \frac{1}{2}$$

and hence the result follows. □

Theorem 3.3 *The number of spanning trees of the linear $(kC_n - snake)$satisfies the following recursive relation $\tau(kC_n - snake) = n^k$.*

Proof Consider a graph $kC_n^/ - snake$ constructed from $kc_n - snake$ by deleting two edges. See Figure 3 following.

kC_n-Snake $\qquad$ $kC_n^{/}$-Snake

Figure 3 Linear kC_n-Snake

We put $kC_n - snake = \tau(kC_n - snake)$ and $kC_n^{/} - snake = \tau(kC_n^{/} - snake)$. It is clear that

$$kC_n - snake = 5((k-1)C_n - snake) + 6((k-1)C_n^{/} - snake)$$

and

$$kC_n^{/} - snake = 2((k-1)C_n - snake) - 6((k-1)C_n^{/} - snake)$$

with initial conditions $(C_n - snake) = n$, $C_n^{/} - snake) = 1$. Thus, we have

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix}, \ = \cdots = A^{n-1} \begin{pmatrix} C_n - snake \\ C_n^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \qquad \lambda_1 = -(n+1) \text{ and } \lambda_2 = n,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix} \ \Rightarrow \ M^{-1} = \begin{pmatrix} \frac{1}{(2n+1)} & \frac{-n}{(2n+1)} \\ \frac{2n}{(2n+1)} & \frac{n}{(2n+1)} \end{pmatrix} \ \Rightarrow \ A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (n)^{n-1} & 0 \\ 0 & (-(n+1))^{n-1} \end{pmatrix}$. We therefore obtain

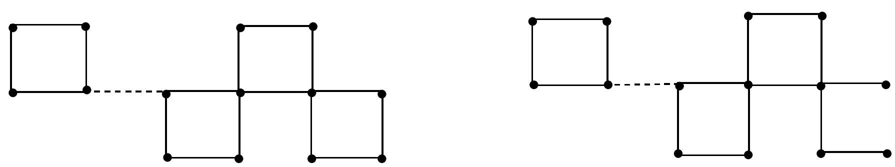
$$A^{n-1} = \begin{pmatrix} \frac{(n)^{n-1}}{(2n+1)} + \frac{2n*(-(n+1))^{n-1}}{(2n+1)} & \frac{-(n)^{n}}{(2n+1)} + \frac{n*(-(n+1))^{n-1}}{(2n+1)} \\ \frac{-2(n)^{n-1}}{(2n+1)} + \frac{2*(-(n+1))^{n-1}}{(2n+1)} & \frac{2*(n)^{n}}{(2n+1)} + \frac{(-(n+1))^{n-1}}{(2n+1)} \end{pmatrix}$$

and hence the result follows. □

Remark 3.4 The number of spanning trees of the subdivision of linear ($kC_n - snake$)satisfies the following recursive relation $\tau(S(kC_n - snake)) = 2n\,\tau((k-1))C_n - snake$, where k is the number of blocks and n is the number of vertices for each block.

Theorem 3.5 *The number of spanning trees of the general $kC_4 - snake$ satisfies the following recursive relation $\tau(\ kC_4 - snake\ ) = 4^k$, where k is the number of blocks.*

Proof Consider a graph $kC_4^{/} - snake$ constructed from $kC_4 - snake$ by deleting two edges. See Figure 4 following.



kC_4-Snake $\qquad$ $kC_4^{/}$-Snake

Figure 4 General kC_4-Snake

We put $kC_4 - snake = \tau(kC_4 - snake)$ and $kC_4^{/} - snake = \tau(kC_4^{/} - snake)$. It is clear that

$$kC_4 - snake = 3(k-1)C_4 - snake + 4(k-1)C_4^{/} - snake$$

and

$$kC_4 - snake = 2(k-1)C_4 - snake - 4(k-1)C_4^{/} - snake$$

with initial conditions $C_4 - snake = 4$, $C_4^{/} - snake = 1$. Thus, we have

$$\begin{pmatrix} kC_4\text{−snake} \\ kC_4^{/}\text{−snake} \end{pmatrix} = A \begin{pmatrix} (k-1)C_4\text{−snake} \\ (k-1)C_4^{/}\text{−snake} \end{pmatrix},$$

where $A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_4\text{−snake} \\ kC_4^{/}\text{−snake} \end{pmatrix} = A \begin{pmatrix} (k-1)C_4\text{−snake} \\ (k-1)C_4^{/}\text{−snake} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_4\text{−snake} \\ C_4^{/}\text{−snake} \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A-\lambda I_2)=\lambda^2+\lambda-20=0, \qquad \lambda_1=-5 \text{ and } \lambda_2=4,\ \lambda_1\neq\lambda_2$$

Then, there is a matrix M invertible such that $A=MBM^{-1}$, where $B=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M=\begin{pmatrix}1 & 1\\ -2 & \frac{1}{4}\end{pmatrix} \Rightarrow M^{-1}=\frac{1}{\frac{9}{4}}\begin{pmatrix}\frac{1}{4} & -1\\ 2 & 1\end{pmatrix} \Rightarrow A^{n-1}=MB^{n-1}M^{-1},$$

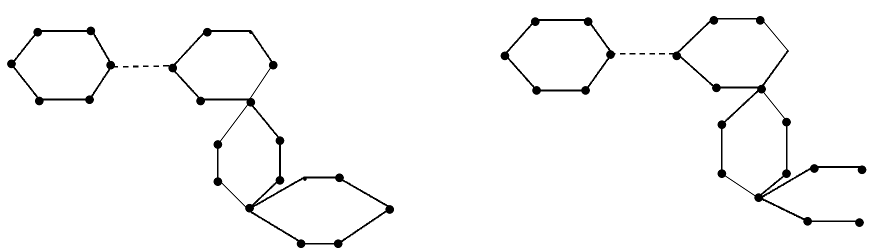
where, $B^{n-1}=\begin{pmatrix}(-5)^{n-1} & 0\\ 0 & (4)^{n-1}\end{pmatrix}$. We therefore obtain

$$\mathrm{A}^{n-1}=\begin{pmatrix}\frac{(-5)^{n-1}}{9}+\frac{2*(4)^{n}}{9} & \frac{-4*(-5)^{n-1}}{9}+\frac{(4)^{n}}{9}\\ \frac{-2*(-5)^{n-1}}{9}+\frac{2*(4)^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9}+\frac{4^{n-1}}{9}\end{pmatrix}$$

and hence the result follows. □

Theorem 3.6 *The number of spanning trees of the general $kC_6-snake$ satisfies the following recursive relation $\tau(\ kC_6-snake\ )=6^k$.*

Proof Consider a graph $kC_6-snake$ constructed from $kC_6^{/}-snake$ by deleting two edges. See Figure 5.



kC_6-Snake $kC_6^{/}$-Snake

Figure 5 General kC_6-Snake

We put $kC_6-snake=\tau(kC_6-snake)$ and $kC_6^{/}-snake=\tau(kC_6^{/}-snake)$. It is clear that

$$kC_6-snake=5((k-1)C_6-snake)+6((k-1)C_6^{/}-snake)$$

and

$$kC_6^{/}-snake=2((k-1)C_6-snake)-6((k-1)C_6^{/}-snake)$$

with initial conditions $(C_1 - snake) = 6(C_1^{/} - snake) = 1$. Thus we have

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix},$$

where $A = \begin{pmatrix} 5 & 6 \\ 2 & -6 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_6 - snake \\ C_6^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \quad \lambda_1 = -7 \text{ and } \lambda_2 = 6,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{6} \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{-6}{13} \\ \frac{12}{13} & \frac{6}{13} \end{pmatrix} \Rightarrow A^{n-1} = MB^{n-1}M^{-1},$$

where, $B^{n-1} = \begin{pmatrix} (6)^{n-1} & 0 \\ 0 & (-7)^{n-1} \end{pmatrix}$. From which, we obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(6)^{n-1}}{13} + \frac{12*(-7)^{n-1}}{13} & \frac{-(6)^{n}}{13} + \frac{6*(-7)^{n-1}}{13} \\ \frac{-2(6)^{n-1}}{13} + \frac{2*(-7)^{n-1}}{13} & \frac{2*(6)^{n}}{13} + \frac{(-7)^{n-1}}{13} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.7 *The number of spanning trees of general $(kC_n - snake)$satisfies the following recursive relation $\tau(kC_n - snake) = n^k$.*

Proof Consider a graph $kC_n - snake$ constructed from $kC_n^{/} - snake$ by deleting two edges. See Figure 6 following.

We put $kC_n - snake = \tau(kC_n - snake)$ and $kC_n^{/} - snake = \tau(kC_n^{/} - snake)$. It is clear that

$$kC_n - snake = 5((k-1)C_n - snake) + 6((k-1)C_n^{/} - snake)$$

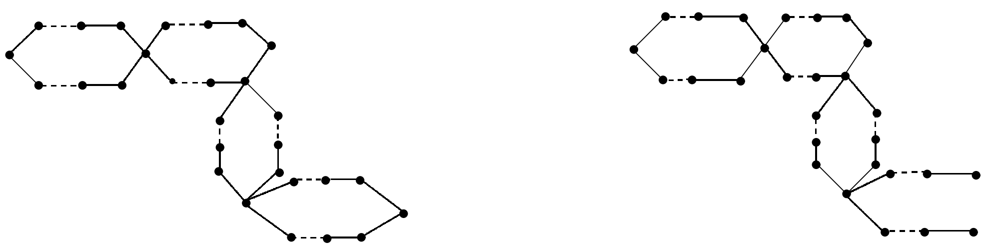
and

$$kC_n^{/} - snake = 2((k-1)C_n - snake) - 6((k-1)C_n^{/} - snake)$$

with initial conditions $(C_n - snake) = n$, $(C_n^{/} - snake) = 1$. Thus we have

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix},$$

where $A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}$,



kC_n-Snake $kC_n^{/}$-Snake

Figure 6 General kC_6-Snake

which implies that

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_n - snake \\ C_n^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \quad \lambda_1 = -(n+1) \text{ and } \lambda_2 = n,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{(2n+1)} & \frac{-n}{(2n+1)} \\ \frac{2n}{(2n+1)} & \frac{n}{(2n+1)} \end{pmatrix} \Rightarrow A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (n)^{n-1} & 0 \\ 0 & (-(n+1))^{n-1} \end{pmatrix}$. From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(n)^{n-1}}{(2n+1)} + \frac{2n*(-(n+1))^{n-1}}{(2n+1)} & \frac{-(n)^{n}}{(2n+1)} + \frac{n*(-(n+1))^{n-1}}{(2n+1)} \\ \frac{-2(n)^{n-1}}{(2n+1)} + \frac{2*(-(n+1))^{n-1}}{(2n+1)} & \frac{2*(n)^{n}}{(2n+1)} + \frac{(-(n+1))^{n-1}}{(2n+1)} \end{pmatrix}$$

and hence the result follows. □

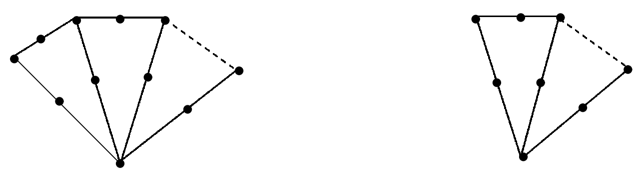
Remark 3.8 The number of spanning trees of the subdivision of general $S(kC_n - snake)$ satisfies the following recursive relation: $\tau(S(kC_n)) = 2n\tau(S(k-1)C_n - snake) = (2n)^k$ where k is the number of blocks.

Theorem 3.9 *The number of spanning trees of the subdivided fan graph satisfies the following recurrence relation*

$$\tau(S(F_n)) = \frac{1}{2\sqrt{5}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n],$$

where $\tau($ S(F_1) $)$ =1 *and* $\tau($ S(F_2) $)$ =6.

Proof Consider a graph $S(F_n)$ constructed from $S(F_n^{/})$ by deleting two edges. See Figure 7 following.



$S(F_n)$ $S(F_n^{/})$

Figure 7 Subdivided Fan Graph

We put $S(F_n) = \tau(S(F_n))$ and $S(F_n^{/}) = \tau(S(F_n^{/}))$, It is clear that

$$S(F_n) = 32S(F_{n-2}) - 24S(F_{n-3}^{/}),$$

where $S(F_n^{/})$ is the number of odd block and

$$S(F_n^{/}) = 6S(F_{n-1}) - 4S(F_{n-2}^{/}),$$

where $S(F_n)$ is the number of even block with initial conditions $S(F_1) = 6$, $S(F_1^{/}) = 1$ and

$$\begin{pmatrix} S(F_n) \\ S(F_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(F_{n-1}) \\ S(F_{n-1}^{/}) \end{pmatrix},$$

where, $A = \begin{pmatrix} 6 & -4 \\ 32 & -24 \end{pmatrix}$, which implies that

$$\begin{pmatrix} S(F_n) \\ S(F_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(F_{n-1}) \\ S(F_{n-1}^{/}) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(F_1) \\ S(F_1^{/}) \end{pmatrix},$$

$$\lambda_1 = \frac{1061}{1250} \quad \text{and} \quad \lambda_2 = \frac{23561}{1250}, \quad \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MBM^{-1}$ where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 1.2878 & 6.2121 \end{pmatrix} \; ; \; \mathrm{M}^{-1} = \begin{pmatrix} 1.2615 & -0.2031 \\ -0.2615 & 0.2031 \end{pmatrix} \; ; \; \mathrm{A}^{n-1} = MB^{n-1}M^{-1} \; ,$$

$$B^{n-1} = \begin{pmatrix} (0.8488)^{n-1} & 0 \\ 0 & (-18.8488)^{n-1} \end{pmatrix}.$$

From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} 1.2615(0.8488)^{n-1} - 0.2615(-18.8488)^{n-1} & -0.2031(0.8488)^{n-1} + 0.2031(-18.8488)^{n-1} \\ 1.6246(0.8488)^{n-1} - 1.6245(-18.8488)^{n-1} & -0.2616(0.8488)^{n-1} + 1.2617(-18.8488)^{n-1} \end{pmatrix}$$

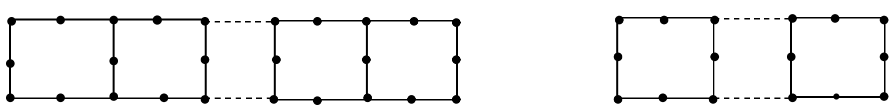
and hence the result follows. □

Theorem 3.10 *The number of spanning trees of the subdivided ladder graph satisfies the following recurrence relation*

$$\tau(S(L_n)) = \frac{2^{n-2}}{\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$$

for any $n \geqslant 1$, *where* $\tau(\ S(L_1)\ ) = 1$ *and* $\tau(\ S(L_2)\ ) = 8$.

Proof Consider a graph $S(F_n)$ constructed from $S(F_n^{/})$ by deleting two edges. See Figure 8 following.



$S(L_n)$ $S(L_n^{/})$

Figure 8 Subdivided Ladder Graphs $S(L_n)$ and $S(L_n^{/})$

We put $S(L_n) \ = \tau(S(L_n)\ )$ and $S(L_n^{/}) \ = \tau(S(L_n^{/})\ )$, It is clear that

$$S(L_n) = 8S(L_{n-1}^{/}) \ - \ 4S(L_{n-2}) \ ,$$

where $S(L_n)$ is the number of even block,

$$S(L_n^{/}) = 60S(L_{n-2}^{/}) \ - \ 32S(L_{n-3})$$

with $S(L_n^{/})$ the number of its odd block with initial conditions $S(L_1) = 8$, $S(L_1^{/}) = 1$. Thus, we have

$$\begin{pmatrix} S(L_n) \\ S(L_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(L_{n-1}) \\ S(L_{n-1}^{/}) \end{pmatrix},$$

where $A = \begin{pmatrix} 8 & -4 \\ 60 & -32 \end{pmatrix}$, which implies that

$$\begin{pmatrix} S(L_n) \\ S(L_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(L_{n-1}) \\ S(L_{n-1}^{/}) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(L_1) \\ S(L_1^{/}) \end{pmatrix},$$

$$\lambda_1 = 0.49 \quad \text{and} \quad \lambda_2 = -24.49\,, \quad \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MBM^{-1}$ where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 1.8775 & 8.1225 \end{pmatrix}\ ;\ \mathrm{M}^{-1} = \begin{pmatrix} 1.3006 & -0.1601 \\ -0.3006 & 0.1601 \end{pmatrix}\ ;\ \mathrm{A}^{n-1} = MB^{n-1}M^{-1}\,,$$

with $B^{n-1} = \begin{pmatrix} (0.49)^{n-1} & 0 \\ 0 & (-24.49)^{n-1} \end{pmatrix}$. From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} 1.3006(0.49)^{n-1} - 0.3006(-24.49)^{n-1} & -0.1601(0.49)^{n-1} + 0.1601(-24.49)^{n-1} \\ 2.4419(0.49)^{n-1} - 2.4416(-24.49)^{n-1} & -0.3022(0.49)^{n-1} + 1.3004(-24.49)^{n-1} \end{pmatrix}$$

and hence the result follows. □

§4. **Spanning Tree Entropy**

The entropy of spanning trees of a network or the asymptotic complexity is a quantitative measure of the number of spanning trees and it characterizes the network structure. We use this entropy to quantify the robustness of networks. The most robust network is the network that has the highest entropy. We can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined as in [15, 16] as:

$$Z(G) = \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|},$$

$$Z(KC_4\text{−snake}) = \lim_{n\to\infty} \frac{\ln 4^k}{3k+1} = 0.4621,$$

$$Z(KC_6\text{−snake}) = \lim_{n\to\infty} \frac{\ln 6^k}{5k+1} = 0.3584$$

$$Z(KC_n - snake) = \lim_{k\to\infty} \frac{\ln n^k}{(n-1)k+1} = \frac{\ln(n)}{n-1},$$

$$Z(S(F_n)) = \lim_{n\to\infty} \frac{\ln(\frac{1}{2\sqrt{5}} * (3+\sqrt{5})^n - (3-\sqrt{5})^n)}{3n+1} = \ln(\sqrt[3]{3+\sqrt{5}}) = 0.5513$$

$$Z(S(L_n)) = \lim_{n\to\infty} \frac{\ln(\frac{2^{n-2}}{\sqrt{3}} * (2+\sqrt{3})^n - (2-\sqrt{3})^n)}{5n-2} = \ln(\sqrt[5]{2+\sqrt{3}}) + \frac{\ln(2)}{5} = 0.4020$$

§5. Conclusion

In this paper, we described how to propose the combinatorial approach to facilitate the calculation of the number of spanning trees in linear and general cyclic snake networks. In particular, we derived the explicit formulas for the linear $kc_4 - snake$, linear $kc_6 - snake$ and linear $kc_n - snake$. Finally, we derived explicit formulas for the general $kc_4 - snake$, general $kc_6 - snake$ and general $kc_n - snake$.

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Strong Domination Number of Some Cycle Related Graphs

Samir K. Vaidya

(Department of Mathematics, Saurashtra University, Rajkot - 360 005, Gujarat, India)

Raksha N. Mehta

(Atmiya Institute of Technology and Science ,Rajkot - 360 005, Gujarat, India)

E-mail: samirkvaidya@yahoo.co.in, rakshaselarka@gmail.com

Abstract: Let $G = (V(G), E(G))$ be a graph and $u, v \in V(G)$. If $uv \in E(G)$ and $deg(u) \geq deg(v)$, then we say that u strongly dominates v or v weakly dominates u. A subset D of $V(G)$ is called a strong dominating set of G if every vertex $v \in V(G) - D$ is strongly dominated by some $u \in D$. The smallest cardinality of strong dominating set is called a strong domination number. In this paper we explore the concept of strong domination number and investigate strong domination number of some cycle related graphs.

Key Words: Dominating strong set, Smarandachely strong dominating set, strong domination number, $d-$balanced graph.

AMS(2010): 05C69, 05C76.

§1. Introduction

In this paper we consider finite, undirected, connected and simple graph G. The vertex set and edge set of the graph G is denoted by $V(G)$ and $E(G)$ respectively. For any graph theoretic terminology and notations we rely upon Chartrand and Lesniak [2]. We denote the degree of a vertex v in a graph G by $deg(v)$. The maximum and minimum degree of the graph G is denoted by $\triangle(G)$ and $\delta(G)$ respectively.

A subset $D \subseteq V(G)$ is independent if no two vertices in D are adjacent. A set $D \subseteq V(G)$ of vertices in the graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset $D^{'} \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set of the graph G. A detailed bibliography on the concept of domination can be found in Hedetniemi and Laskar [7] as well as Cockayne and Hedetniemi [3]. A dominating set $D \subseteq V(G)$ is called an independent dominating set if it is also an independent set. The minimum cardinality of an independent dominating set in G is called the independent domination number $i(G)$ of the graph G. For the better understanding of domination and its related concepts we refer to Haynes et al [6].

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We will give some definitions which are useful for the present work.

Definition 1.1([10]) *For graph G and $uv \in E(G)$, we say u strongly dominates v (v weakly dominates u) if $deg(u) \geq deg(v)$.*

Definition 1.2([10]) *A subset D is a strong(weak) dominating set $sd-set(wd-set)$ if every vertex $v \in V(G)-D$ is strongly(weakly) dominated by some u in D. The strong(weak) domination number $\gamma_{st}(G)(\gamma_w(G))$ is the minimum cardinality of a $sd-set(wd-set)$.*

Generally, for a subset $O \subset V(G)$ with $\langle O\rangle_G$ isomorphic to a special graph, for instance a tree, a subset D_S of $V(G)$ is a Smarandachely strong(weak) dominating set of G on O if every vertex $v \in V(G)-D-O$ is strongly(weakly) dominated by some vertex in D_S. Clearly, if $O=\emptyset$, D_S is nothing else but the strong dominating set of G.

The concepts of strong and weak domination were introduced by Sampathkumar and Pushpa Latha [10]. In the same paper they have defined the following concepts.

Definition 1.3 *The independent strong(weak) domination number $i_{st}(G)$ $(i_w(G))$ of the graph G is the minimum cardinality of a strongly(weakly) dominating set which is independent set.*

Definition 1.4 *Let $G=(V(G),E(G))$ be a graph and $D \subset V(G)$. Then D is s-full (w-full) if every $u \in D$ strongly (weakly) dominates some $v \in V(G)-D$.*

Definition 1.5 *A graph G is domination balanced (d-balanced) if there exists an sd-set D_1 and a wd-set D_2 such that $D_1 \cap D_2=\phi$.*

Several results on the concepts of strong and weak domination have also been explored by Domke et al [4]. The bounds on strong domination number and the influence of special vertices on strong domination is discussed by Rautenbach [8,9] while Hattingh and Henning have investigated bounds on strong domination number of connected graphs in [5]. For regular graphs $\gamma_{st}=\gamma_w=\gamma$ as reported by Swaminathan and Thangaraju in [11]. Therefore we consider the graph G which is not regular.

§2. Main Results

We begin with propositions which are useful for further results.

Proposition 2.1([10]) *For a graph G of order n, $\gamma \leq \gamma_{st} \leq n-\triangle(G)$.*

Proposition 2.2([1]) *For a nontrivial path P_n,*

$$\gamma_{st}(P_n)=\lceil \tfrac{n}{3}\rceil \text{ and } \gamma_w(P_n)=\begin{cases}\left\lceil \dfrac{n}{3}\right\rceil & if\ n\ \equiv\ 1(mod\ 3)\\ \left\lceil \dfrac{n}{3}\right\rceil+1 & otherwise\end{cases}$$

Proposition 2.3([1]) *For cycle C_n, $\gamma_{st}(C_n)=\gamma_w(C_n)=\gamma(C_n)=\lceil \frac{n}{3}\rceil$.*

Proposition 2.4([11]) *For any non regular graph G, $\gamma_{st}(G)+\triangle(G)=n$ and $\gamma_w(G)+\delta(G)=n$*

if and only if

(1) *for every vertex u of degree δ, $V(G) - N[u]$ is an independent set and every vertex in $N(u)$ is adjacent to every vertex in $V(G) - N(u)$.*

(2) *for every vertex v of degree $\triangle$, $V(G)-N[v]$ is independent, each vertex in $V(G)-N(v)$ is of degree $\geq \delta+1$ and no vertex of $N(v)$ strongly dominates two or more vertices of $V(G)-N[v]$.*

Proposition 2.5([10]) *For a graph G, the following statements are equivalent.*

(1) *G is d-balanced;*

(3) *There exists an sd-set D which is s-full;*

(3) *There exists an wd-set D which is w-full.*

Theorem 2.6 *Let G be the graph of order n. If there exists a vertex u_1 with $deg(u_1) = \triangle$ and $deg(u_i) = m$, where $2 \leq i \leq n$ then, $\gamma_{st}(G) = \gamma(G)$.*

Proof Let G be the graph of order n and let u_1 be the vertex with $deg(u_1) = \triangle(G)$. The set $V(G) - N(u_1)$ contains the vertices of degree m. It is clear that the graph G contains two types of vertices: a vertex of degree $\triangle$ and remaining vertices of degree m. The vertex $u_1 \in \gamma_{st}$ - set as it is of maximum degree.

To prove the result we consider following two cases.

Case 1. $N[u_1] = V(G)$.

If $N[u_1] = V(G)$ implies that $\gamma(G) = 1$. Hence $deg(D) > deg(V(G) - D)$. Therefore $u_1 \in D$ strongly dominates $V(G) - D$. Thus $\gamma_{st}(G) = \gamma(G) = 1$.

Case 2. $N[u_1] \neq V(G)$.

Let us partition the vertex set $V(G)$ into V_1 and V_2. Now to construct a dominating set or a strong dominating set of minimum cardinality the vertex u_1 must belong to every strong dominating set. So let $N[u_1] \in V_1$ and remaining $n - \triangle - 1$ vertices are in V_2. Now the vertices in V_2 are of degree m. Thus the vertices V_2 forms a regular graph. For regular graphs $\gamma(G) = \gamma_{st}(G) = \gamma_w(G)$. Let k be the domination number of vertex set V_2. Therefore $\gamma(G) = \gamma_{st}(G) = \gamma_{st}(V_1) + \gamma_{st}(V_2) = 1 + k$.

In any case, if G contains a vertex of degree $\triangle(G)$ and remaining vertices of same degree m then $\gamma_{st}(G) = \gamma(G)$. □

Corollary 2.7 $\gamma(K_{1,n}) = \gamma_{st}(K_{1,n}) = i_{st}(K_{1,n}) = 1$.

Corollary 2.8 $\gamma(W_n) = \gamma_{st}(W_n) = i_{st}(W_n) = 1$.

Definition 2.9 *One point union $C_n^{(k)}$ of k copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two cycles $C_n^{(i)}$ and $C_n^{(j)}$ $(i \neq j)$ are edge disjoint and do not have any vertex in common except v.*

Corollary 2.10 $\gamma_{st}(C_n^{(k)}) = \gamma(C_n^{(k)}) = 1 + k\lceil \frac{n-3}{3} \rceil$, *for $n \geq 3$.*

Proof Let $v_1^p, v_2^p, \ldots v_n^p$ be the vertices of p^{th} copy of cycle C_n for $1 \leq p \leq k$, $k \in \mathbb{N}$ and

v be the common vertex in graph C_n^k such that $v = v_1^1 = v_1^2 = v_1^3 = \cdots = v_1^p$. Consequently $|V(C_n^k)| = kn - k + 1$.

The $deg(v) = 2k$ which is of maximum degree, then it must be in every dominating set D and the vertex v will dominate $2k+1$ vertices.

Now to dominate the remaining k disconnected copies of path each of length $n-3$ we require minimum $k\left\lceil \frac{n-3}{3} \right\rceil$ vertices.

This implies that $\gamma(C_n^k) \geq 1 + k\left\lceil \frac{n-3}{3} \right\rceil$. Let us partition the vertex set $V(C_n^k)$ into $V_1(C_n^k)$ and $V_2(C_n^k)$ such that $V(C_n^k) = V_1(C_n^k) \bigcup V_2(C_n^k)$ depending on the degree of vertices. Let $V_1(C_n^k)$ contain $N[v]$ which forms a star graph $K_{1,2k}$. Thus, from above Corollary 2.7 $\gamma(K_{1,n}) = 1$. Let $V_2(C_n^k)$ contain the remaining vertices, that is, $|V_2(C_n^k)| = |V(C_n^k)|$ - $|V_1(C_n^k)| = kn-k+1$ $-(2k+1) = kn - 3k$ in k copies. Thus, in one copy there are $n-3$ vertices which forms a path of order $n-3$. Therefore, from above Proposition 2.2, $\gamma(P_{n-3}) = \left\lceil \frac{n-3}{3} \right\rceil$. For k copies of path the domination number is $\gamma[k(P_{n-3})] = k\left\lceil \frac{n-3}{3} \right\rceil$. Hence, $\gamma(C_n^k) = \gamma(K_{1,n}) + \gamma[k(P_{n-3})]$ $= 1 + k\left\lceil \frac{n-3}{3} \right\rceil$, for $n \geq 3$. Therefore D is a dominating set of minimum cardinality. Thus, D is also the strong dominating set of minimum cardinality. Therefore,

$$\gamma_{st}(C_n^{(k)}) = \gamma(C_n^{(k)}) = 1 + k\lceil \frac{n-3}{3} \rceil,$$

for $n \geq 3$. □

Definition 2.11 *Duplication of a vertex v_i by a new edge $e' = u'v'$ in a graph G results into a graph G' such that $N(u') = \{v_i, v'\}$ and $N(v') = \{v_i, u'\}$.*

Theorem 2.12 *If G' is the graph obtained by duplication of each vertex of graph G by a new edge then $\gamma_{st}(G') = \gamma(G') = n$.*

Proof Let $V(G)$ be the set of vertices and $E(G)$ be the set of edges for the graph G. Let us denote vertices of graph G by $u_1, u_2, u_3 \cdots, u_n$. Hence $|V(G)| = n$ and $|E(G)| = m$. Each vertex of G is duplicated by a new edge. Let us denote these new added vertices by $v_1, v_2, v_3 \cdots, v_n$ and $w_1, w_2, w_3 \cdots,\ w_n$ respectively. Hence, the obtained graph G' contains $3n$ vertices and $3n + m$ edges. Thus the degree of u_i $(1 \leq i \leq n)$ will increase by two and the degree of v_i and w_i $(1 \leq i \leq n)$ is two. The graph G' contains n vertex disjoint cycles of order 3. By Proposition 2.3, $\gamma_{st}(C_3) = 1$. Thus minimum n vertices are essential to strongly dominate n vertex disjoint cycles. Hence, $\gamma_{st}(G') \geq n$. Since u_i are the vertices of maximum degree, they must be in every strong dominating set. We claim that it is enough to take u_i in strong dominating set as the vertices v_i and w_i are adjacent to a common vertex u_i. Thus, $D = \{u_1, u_2, u_3 \cdots, u_n\}$ is the only strong dominating set with minimum cardinality. Hence,

$$\gamma_{st}(G') = \gamma(G') = n.$$ □

Theorem 2.13 *If G' is the graph obtained by duplication of each vertex of graph G by a new edge then G' is $d-$ balanced.*

Proof As argued in Theorem 2.12, $D = \{u_1, u_2, u_3 \cdots, u_n\}$ is the only strong dominating set. Hence it is the strong dominating set with minimum cardinality. The vertices u_i $(1 \leq i \leq n)$ strongly dominates v_i and w_i in $V(G') - D$ where $(1 \leq i \leq n)$. Thus, D is $s-$full. Hence from Proposition 2.5 G' is $d-$ balanced. □

Definition 2.14 *The switching of a vertex v of G means removing all the edges incident to v and adding edges joining to every vertex which is not adjacent to v in G. We denote the resultant graph by $\widetilde{G}$.*

Theorem 2.15 *If $\widetilde{C_n}$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n, $(n > 3)$ then,*

$$\gamma_{st}(\widetilde{C_n}) = \begin{cases} 1 & if\ n = 4 \\ 2 & if\ n = 5 \\ 3 & if\ n \geq 6 \end{cases}$$

Proof Let $v_1, v_2, v_3 \ldots, v_n$ be the vertices of the cycle C_n. Without loss of generality we switch the vertex v_1 of C_n. We consider following cases to prove the theorem.

Case 1. $n = 4$.

The graph $\widetilde{C_4}$ is obtained by switching of vertex v_1 in cycle C_4 which is same as $K_{1,3}$. Hence $D = \{v_3\}$ is the only strong dominating set as discussed in Corollary 2.7. It is the only strong dominating set with minimum cardinality. Therefore the strong domination number $\gamma_{st}(\widetilde{C_4}) = 1$.

Case 2. $n = 5$.

The graph $\widetilde{C_5}$ obtained by switching of vertex v_1 in cycle C_5. The degree $deg(v_1) = 2$, $deg(v_2) = deg(v_5) = 1$ while $deg(v_3) = deg(v_4) = 3$. The vertex v_3 strongly dominates v_1, v_2 and v_4 along with itself. It is enough to take the vertex v_4 in the strong dominating set to strongly dominate the vertex v_5. Thus $D = \{v_3, v_4\}$ is the only strong dominating set with minimum cardinality. Hence arbitrary switching of a vertex of cycle C_5 results into $\gamma_{st}(\widetilde{C_5}) = 2$.

Case 3. $n \geq 6$.

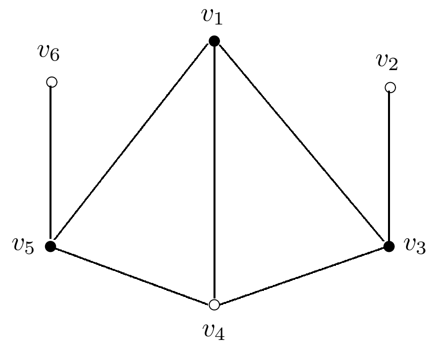
Let $\widetilde{C_n}$ be the graph obtained by switching of vertex v_1 in cycle C_n. The degree $deg(v_1) = n - 3$ while $deg(v_2) = deg(v_n) = 1$ and remaining $n - 3$ vertices are of degree three. Thus, $|V(\widetilde{C_n})| = n$. By the Proposition 2.1 $\gamma_{st}(\widetilde{C_n}) \leq n - \triangle(\widetilde{C_n}) = n - (n-3)$, implying $\gamma_s(\widetilde{C_n}) \leq 3$.

The degree $deg(v_1) = n-3$ which is of maximum degree, that is, v_1 must be in every strong dominating set and v_1 will strongly dominate $n-2$ vertices except the pendant vertices v_2 and v_n. Hence either these pendant vertices must be in every strong dominating set or the supporting vertices v_{n-1} and v_3. Thus, $D_1 = \{v_1, v_2, v_n\}$ or $D_2 = \{v_1, v_3, v_{n-1}\}$ or $D_3 = \{v_1, v_2, v_{n-1}\}$ or $D_4 = \{v_1, v_n, v_3\}$ are strong dominating sets with minimum cardinality. Therefore,

$$\gamma_{st}(\widetilde{C_n}) = 3.$$

for $n \geq 6$. □

Illustration 2.16 In Figure 2.1, the solid vertices are the elements of strong dominating sets of $\widetilde{C_6}$ as shown below.



$$\gamma(\widetilde{C_6}) = \gamma_{st}(\widetilde{C_6}) = 3$$

Figure 2.1

Corollary 2.17 *$\gamma_{st}(\widetilde{C_n}) = \gamma(\widetilde{C_n})$ for $n > 3$.*

Proof We continue with the terminology and notations used in Theorem 2.15 and consider the following cases to prove the corollary.

Case 1. $n = 4$.

As shown in Theorem 2.15, $D = \{v_3\}$ is the only strong dominating set with minimum cardinality which is also the dominating set of minimum cardinality. As discussed in Corollary 2.7, $\gamma_{st}(\widetilde{C_4}) = \gamma(\widetilde{C_4})$.

Case 2. $n = 5$.

As shown in Theorem 2.15, $D = \{v_3, v_4\}$ is the only strong dominating set with minimum cardinality which is also the dominating set of minimum cardinality. Hence $\gamma_{st}(\widetilde{C_5}) = \gamma(\widetilde{C_5})$.

Case 3. $n \geq 6$.

As shown in Theorem 2.15 we have obtained four possible strong dominating sets. The strong dominating sets $D_1 = \{v_1, v_2, v_n\}$ or $D_2 = \{v_1, v_3, v_{n-1}\}$ or $D_3 = \{v_1, v_2, v_{n-1}\}$ or $D_4 = \{v_1, v_n, v_3\}$ are strong dominating sets with minimum cardinality which are also the dominating set of minimum cardinality. Thus, $\gamma_{st}(\widetilde{C_n}) = \gamma(\widetilde{C_n})$, for $n \geq 6$. □

Theorem 2.18 *If $\widetilde{C_n}$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n then,$\widetilde{C_n}$ $(n > 3)$ is $d-$ balanced.*

Proof We continue with the terminology and notations used in Theorem 2.15 and consider the following cases to prove the corollary.

Case 1. $n = 4$.

As discussed in Theorem 2.15 the set $D = \{v_3\}$ is the strong dominating with minimum cardinality. The set D is $s-$ full since the vertex v_3 strongly dominates remaining three vertices in $V(\widetilde{C_4}) - D$. Hence from Proposition 2.5 $\widetilde{C_4}$ is $d-$ balanced.

Case 2. $n = 5$.

As shown in Theorem 2.15, the set $D = \{v_3, v_4\}$ is a strong dominating set with minimum cardinality. The set $D = \{v_3, v_4\}$ is $s-$ full since v_i $(i = 3, 4)$ strongly dominates v_2, v_4 and v_5 in $V(\widetilde{C_5}) - D$. Hence from Proposition 2.5 $\widetilde{C_5}$ is $d-$ balanced.

Case 3. $n \geq 6$.

In Theorem 2.15 we have obtained the strong dominating set $D_2 = \{v_1, v_3, v_{n-1}\}$ of minimum cardinality. The set D_2 is $s-$ full as v_1, v_3 and v_{n-1} strongly dominates remaining vertices in $V(\widetilde{C_n}) - D_2$. Thus from Proposition 2.5 $\widetilde{C_n}$ $(n \geq 6)$ is $d-$ balanced. □

Definition 2.19 *The book B_m is a graph $S_m \times P_2$ where $S_m = K_{1,m}$.*

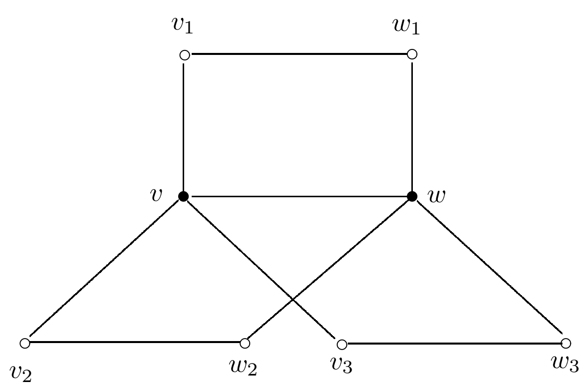
Theorem 2.20 $\gamma_{st}(B_m) = 2$ *for* $m \geq 3$.

Proof Let S_m be the graph with vertices $u, u_1, u_2, u_3 \cdots, u_m$ where u is the vertex of degree m and $u_1, u_2, u_3 \cdots, u_m$ are pendant vertices. Let P_2 be the path with vertices a_1 and a_2. We consider $v = (u, a_1), v_1 = (u_1, a_1), v_2 = (u_2, a_1) \cdots, v_m = (u_m, a_1)$ and $w = (u, a_2), w_1 = (u_1, a_2), w_2 = (u_2, a_2) \cdots, w_m = (u_m, a_2)$. Hence $|V(B_m)| = 2m + 2$.

In B_m there is no vertex with degree $2m + 1$, implying that $\gamma(B_m) > 1$. The $deg(v) = deg(w) = m + 1$ are the vertices of maximum degree. Let us partition the vertex set $V(B_m)$ into V_1 and V_2 such that $V(B_m) = V_1 \bigcup V_2$. Let $N[v] \in V_1$ and $N[w] \in V_2$. Then in both the partitions a star graph $K_{1,m}$ is formed. Thus from above Corollary 2.7, $\gamma(K_{1,m}) = \gamma_{st}(K_{1,m}) = 1$. Thus, it is enough to take v and w in strong dominating set as it strongly dominates $2m + 2$ vertices. Therefore $D = \{v, w\}$ is the strong dominating set with minimum cardinality. Hence,

$$\gamma_{st}(B_m) = 2 \quad \text{if} \quad m \geq 3. \qquad \square$$

Illustration 2.21 In Figure 2.2, the solid vertices are the elements of strong dominating set of B_3 as shown below.



$\gamma(B_3) = \gamma_{st}(B_3) = 2$

Figure 2.2

Corollary 2.22 $\gamma_{st}(B_m) = \gamma(B_m)$ *for* $m \geq 3$.

Proof As shown in Theorem 2.20 we have obtained the strong dominating set $D = \{v, w\}$. The set D also forms the dominating set of minimum cardinality. Thus, $\gamma_{st}(B_m) = \gamma(B_m)$, for $n \geq 3$. □

Theorem 2.23 *The book graph B_m is $d-$balanced.*

Proof In Theorem 2.20 we have obtained the strong dominating set $D = \{v, w\}$ of minimum cardinality. The vertex v strongly dominates $v_1, v_2 \cdots, v_m$ while the vertex w strongly dominate $w_1, w_2 \cdots, w_m$ in $V(B_m) - D$ respectively. Hence D is $s-$full set. Hence from Proposition 2.5 the book graph B_m is $d-$balanced. □

§3. Concluding Remarks

The strong domination in graph is a variant of domination. The strong domination number of various graphs are known. We have investigated the strong domination number of some graphs obtained from C_n by means of some graph operations. This work can be applied to rearrange the existing security network in the case of high alert situation and to beef up the surveillance.

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Minimum Equitable Dominating Randic Energy of a Graph

P.Siva Kota Reddy

(Department of Mathematics, Siddaganga Institute of Technology, B.H.Road, Tumkur-572103, India)

K. N. Prakasha

(Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru- 570 002, India)

Gavirangaiah K

(Department of Mathematics, Government First Grade Collge, Tumkur-562 102, India)

E-mail: reddy_math@yahoo.com, prakashamaths@gmail.com, gavirangayya@gmail.com

Abstract: In this paper, we introduce the minimum equitable dominating Randic energy of a graph and computed the minimum dominating Randic energy of graph. Also, established the upper and lower bounds for the minimum equitable dominating Randic energy of a graph.

Key Words: Minimum equitable dominating set, Smarandachely equitable dominating set, minimum equitable dominating Randic eigenvalues, minimum equitable dominating Randic energy.

AMS(2010): 05C50.

§1. Introduction

Let G be a simple, finite, undirected graph, The energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of graph see [5, 6].

The Randic matrix $R(G) = (R_{ij})_{n\times n}$ is given by [1-3].

$$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 0 & otherwise \end{cases}$$

We can see lower and upper bounds on Randic energy in [1,2]. Some sharp upper bounds for Randic energy of graphs were obtain in [3].

§2. The Minimum Equitable Dominating Randic Energy of Graph

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ and edge set E. A subset U of $V(G)$ is an equitable dominating set, if for every $v \in V(G) - U$ there exists a

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vertex $u \in U$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, and a Smarandachely equitable dominating set is its contrary, i.e., $|deg(u) - deg(v)| \geq 1$ for such an edge uv, where $deg(x)$ denotes the degree of vertex x in $V(G)$. Any equitable dominating set with minimum cardinality is called a minimum equitable dominating set. Let E be a minimum equitable dominating set of a graph G. The minimum equitable dominating Randic matrix $R^E(G) = (R^E_{ij})_{n\times n}$ is given by

$$R^E_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 1 & \text{if } i = j \text{ and } v_i \in E, \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of $R^E(G)$ is denoted by $\phi^E_R(G, \lambda) = det(\lambda I - R^E(G))$. Since the minimum equitable dominating Randic Matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 > \lambda_2 > \cdots \lambda_n$. The minimum equitable dominating Randic Energy is given by

$$RE_E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

Definition 2.1 *The spectrum of a graph G is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots \lambda_r$, with their multiplicities $m_1, m_2, \ldots, m_r$, and we write it as*

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

This paper is organized as follows. In the Section 3, we get some basic properties of minimum equitable dominating Randic energy of a graph. In the Section 4, minimum equitable dominating Randic energy of some standard graphs are obtained.

§3. Some Basic Properties of Minimum Equitable Dominating Randic Energy of a Graph

Let us consider

$$P = \sum_{i<j} \frac{1}{d_i d_j},$$

where $d_i d_j$ is the product of degrees of two vertices which are adjacent.

Proposition 3.1 *The first three coefficients of $\phi^E_R(G, \lambda)$ are given as follows:*

(*i*) $a_0 = 1$;

(*ii*) $a_1 = -|E|$;

(*iii*) $a_2 = |E|C_2 - P$.

Proof (*i*) From the definition $\Phi_R^E(G,\lambda) = det[\lambda I - R^E(G)]$, we get $a_0 = 1$.

(*ii*) The sum of determinants of all 1×1 principal submatrices of $R^E(G)$ is equal to the trace of $R^E(G) \Rightarrow a_1 = (-1)^1$ trace of $[R^E(G)] = -|E|$.

(*iii*)

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= |E|C_2 - P.\end{aligned}$$

□

Proposition 3.2 *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the minimum equitable dominating Randic eigenvalues of $R^E(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = |E| + 2P.$$

Proof We know that

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\ &= 2\sum_{i<j}(a_{ij})^2 + |E| \\ &= |E| + 2P.\end{aligned}$$

□

Theorem 3.3 *Let G be a graph with n vertices and Then*

$$RE^E(G) \le \sqrt{n(|E| + 2[P])}$$

where

$$P = \sum_{i<j} \frac{1}{d_i d_j}$$

for which $d_i d_j$ is the product of degrees of two vertices which are adjacent.

Proof Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $R^E(G)$. Now by Cauchy - Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i =| \lambda_i |$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

Thus,

$$[RE^E]^2 \leq n(|E| + 2P),$$

which implies that

$$[RE^E] \leq \sqrt{n(|E| + 2P)},$$

i.e., the upper bound. □

Theorem 3.4 *Let G be a graph with n vertices. If $R=$ det $R^E(G)$, then*

$$RE^E(G) \geq \sqrt{(|E| + 2P) + n(n-1)R^{\frac{2}{n}}}.$$

Proof By definition,

$$\begin{aligned}\left(RE^E(G)\right)^2 &= \left(\sum_{i=1}^{n} | \lambda_i |\right)^2 \\ &= \sum_{i=1}^{n} | \lambda_i | \sum_{j=1}^{n} | \lambda_j | \\ &= \left(\sum_{i=1}^{n} | \lambda_i |^2\right) + \sum_{i \neq j} | \lambda_i || \lambda_j | .\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} | \lambda_i || \lambda_j | \geq \left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}[RE^E(G)]^2 &\geq \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)\left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)\left(\prod_{i=1}^{n} | \lambda_i |^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)R^{\frac{2}{n}} \\ &= (|E| + 2P) + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Thus,

$$RE^E(G) \geq \sqrt{(|E| + 2P) + n(n-1)R^{\frac{2}{n}}}. \qquad \square$$

§4. Minimum Equitable Dominating Randic Energy of Some Standard Graphs

Theorem 4.1 *The minimum equitable dominating Randic energy of a complete graph K_n is $RE^E(K_n) = \frac{3n-5}{n-1}$.*

Proof Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum equitable dominating set $= E = \{v_1\}$. The minimum equitable dominating Randic matrix is

$$R^E(K_n) = \begin{bmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \end{bmatrix}.$$

The characteristic equation is

$$\left(\lambda + \frac{1}{n-1}\right)^{n-2} \left(\lambda^2 - \frac{2n-3}{n-1}\lambda + \frac{n-3}{n-1}\right) = 0$$

and the spectrum is $Spec_R^E(K_n) = \begin{pmatrix} \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{-1}{n-1} \\ 1 & 1 & n-2 \end{pmatrix}$.

Therefore, $RE^E(K_n) = \dfrac{3n-5}{n-1}$. $\square$

Theorem 4.2 *The minimum equitable dominating Randic energy of star graph $K_{1,n-1}$ is*

$$RE^E(K_{1,n-1}) = \sqrt{5}.$$

Proof Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. Here v_0 be the center. The minimum equitable dominating set $= E = V(G)$. The minimum equitable dominating Randic matrix is

$$R^E(K_{1,n-1}) = \begin{bmatrix} 1 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \ldots & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{n-1}} & 1 & 0 & \ldots & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 1 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

The characteristic equation is

$$\lambda(\lambda-1)^{n-2}[\lambda-2] = 0$$

spectrum is $Spec_R^E(K_{1,n-1}) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & n-2 & 1 \end{pmatrix}$.

Therefore, $RE^E(K_{1,n-1}) = n$. □

Theorem 4.3 *The minimum equitable dominating Randic energy of Crown graph S_n^0 is*

$$RE^E(S_n^0) = \frac{(4n-7)+\sqrt{4n^2-8n+5}}{n-1}.$$

Proof Let S_n^0 be a crown graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and minimum dominating set $= E = \{u_1, v_1\}$. The minimum equitable dominating Randic matrix is

$$R^E(S_n^0) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 1 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\left(\lambda+\frac{1}{n-1}\right)^{n-2}\left(\lambda-\frac{1}{n-1}\right)^{n-2}\left(\lambda^2-\frac{1}{n-1}\lambda-1\right)\left(\lambda^2-\frac{2n-3}{n-1}\lambda+\frac{n-3}{n-1}\right)=0$$

spectrum is $Spec_R^E(S_n^0)$

$$=\begin{pmatrix} \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} & \frac{1+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{1}{n-1} & \frac{-1}{n-1} & \frac{1-\sqrt{4n^2-8n+5}}{2(n-1)} \\ 1 & 1 & 1 & n-2 & n-2 & 1 \end{pmatrix}.$$

Therefore, $RE^E(S_n^0)=\dfrac{(4n-7)+\sqrt{4n^2-8n+5}}{n-1}$. □

Theorem 4.4 *The minimum equitable dominating Randic energy of complete bipartite graph $K_{n,n}$ of order $2n$ with vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ is*

$$RE^E(K_{n,n})=\frac{2\sqrt{n-1}}{\sqrt{n}}+2.$$

Proof Let $K_{n,n}$ be the complete bipartite graph of order $2n$ with vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$. The minimum equitable dominating set $= E = \{u_1,v_1\}$ with a minimum equitable dominating Randic matrix

$$R^E(K_{n,n})=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 1 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^{2n-4}(\lambda^2-\frac{n-1}{n})[\lambda^2-2\lambda+\frac{n-1}{n}]=0$$

Hence, spectrum is

$$Spec_R^E(K_{n,n})=\begin{pmatrix} 1+\sqrt{\frac{1}{n}} & \frac{\sqrt{n-1}}{\sqrt{n}} & 1-\sqrt{\frac{1}{n}} & 0 & -\frac{\sqrt{n-1}}{\sqrt{n}} \\ 1 & 1 & 1 & 2n-4 & 1 \end{pmatrix}.$$

Therefore, $RE^E(K_{n,n})=\dfrac{2\sqrt{n-1}}{\sqrt{n}}+2$. □

Theorem 4.5 *The minimum equitable dominating Randic energy of cocktail party graph $K_{n\times 2}$ is*

$$RE^E(K_{n\times 2}) = \frac{4n-6}{n-1}.$$

Proof Let $K_{n\times 2}$ be a Cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The minimum equitable dominating set $= E = \{u_1, v_1\}$ with a minimum equitable dominating Randic matrix

$$R^E(K_{n\times 2}) = \begin{bmatrix} 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^{n-1}\left(\lambda + \frac{1}{n-1}\right)^{n-2}(\lambda-1)[\lambda^2 - \frac{2n-3}{n-1}\lambda + \frac{n-3}{n-1}] = 0$$

Hence, spectrum is

$$Spec_R^E(K_{n\times 2}) = \begin{pmatrix} \frac{2n-3+\sqrt{4n-3}}{2(n-1)} & 1 & \frac{2n-3-\sqrt{4n-3}}{2(n-1)} & 0 & \frac{-1}{n-1} \\ 1 & 1 & 1 & n-1 & n-2 \end{pmatrix}.$$

Therefore, $RE^E(K_{n\times 2}) = \frac{4n-6}{n-1}$. □

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Cordiality in the Context of Duplication in Web and Armed Helm

U M Prajapati

(St. Xavier's College, Ahmedabad-380009, Gujarat, India)

R M Gajjar

(Department of Mathematics, School of Sciences, Gujarat University, Ahmedabad-380009, Gujarat, India)

E-mail: udayan64@yahoo.com, roopalgajjar@gmail.com

Abstract: Let $G = (V(G), E(G))$ be a graph and let $f : V(G) \to \{0, 1\}$ be a mapping from the set of vertices to {0,1} and for each edge $uv \in E$ assign the label $|f(u) - f(v)|$. If the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labled with 0 and the number of edges labeled with 1 differ by at most 1, then f is called a cordial labeling. We discuss cordial labeling of graphs obtained from duplication of certain graph elements in web and armed helm.

Key Words: Graph labeling, cordial labeling, cordial graph, Smarandachely cordial labeling, Smarandachely cordial graph.

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§1. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For all other terminology we follow West [1]. We will give the brief summary of definitions which are useful for the present work.

Definition 1.1 *The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s).*

A detailed survey of various graph labeling is explained in Gallian [3].

Definition 1.2 *For a graph $G = (V(G), E(G))$, a mapping $f : V(G) \to \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$.*

Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3 *Duplication of a vertex v of a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.*

Definition 1.4 *Duplication of an edge $e = uv$ of a graph G produces a new graph G' by adding*

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an edge $e' = u'v'$ *such that* $N(u') = N(u) \cup \{v'\} - \{v\}$ *and* $N(v') = N(v) \cup \{u'\} - \{u\}$.

Definition 1.5 *The wheel* W_n*, is join of the graphs* C_n *and* K_1*. i.e* $W_n = C_n + K_1$*. Here vertices corresponding to* C_n *are called rim vertices and* C_n *is called rim of* W_n *while, the vertex corresponding to* K_1 *is called the apex vertex, edges joining the apex vertex and a rim vertex is called spoke.*

Definition 1.6([3]) *The helm* H_n*, is the graph obtained from the wheel* W_n *by adding a pendant edge at each rim vertex.*

Definition 1.7([3]) *The web* Wb_n*, is the graph obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle, here vertices corresponding to this outer cycle are called outer rim vertices and vertices corresponding to wheel except the apex vertex are called inner rim vertices.*

We define one new graph family as follows:

Definition 1.8 *An armed helm is a graph in which path* P_2 *is attached at each vertex of wheel* W_n *by an edge. It is denoted by* AH_n *where* n *is the number of vertices in cycle* C_n*.*

Definition 1.9 *A binary vertex labeling* f *of a graph* G *is called a cordial labeling if* $|v_f(1) - v_f(0)| \leq 1$ *and* $|e_f(1) - e_f(0)| \leq 1$*, and a binary vertex labeling* f *of a graph* G *is called a Smarandachely cordial labeling if* $|v_f(1) - v_f(0)| \geq 1$ *or* $|e_f(1) - e_f(0)| \geq 1$.

A graph G *is said to be cordial if it admits cordial labeling, and Smarandachely cordial if it admits Smarandachely cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2] in which he proved that the wheel W_n is cordial if and only if $n \not\equiv 3(mod 4)$. Vaidya and Dani [4] proved that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Prajapati and Gajjar [5] proved that complement of wheel graph and complement of cycle graph are cordial if $n \not\equiv 4(mod 8)$ or $n \not\equiv 7(mod 8)$. Prajapati and Gajjar [6] proved that cordial labeling in the context of duplication of cycle graph and path graph.

§2. Main Results

Theorem 2.1 *The graph obtained by duplicating all the vertices of the web* Wb_n *is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices in Wb_n. Let $t', u'_1, u'_2, \cdots, u'_n, v'_1, v'_2, \cdots, v'_n, w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\} \cup \{u_i, v_i, w_i, u'_i, v'_i, w'_i/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv'_i, u_iv'_i, u'_iv_i, tu'_i, v_iw'_i, t'u_i$ $/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1, v'_nv_1, v_nv'_1, u'_nu_1, u_nu'_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v'_iv_{i+1}, v_iv'_{i+1}, u'_iu_{i+1}, u_iu'_{i+1}$ $/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 6n + 2$ and $|E(G)| = 15$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t'; \\ 1 & \text{if } x = w'_i, i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\ldots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } \ e \in \{u_iu'_{i+1}, u'_iu_{i+1}, v_iv'_{i+1}, v'_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{t'u_i, tu'_i, u'_iv_i, u_iv'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } \ e = v_iw_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e = v_iw_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1, v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{15n}{2}$ and $e_f(0) = \dfrac{15n}{2}$.

Case 2 n is odd.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t'; \\ 1 & \text{if } x = w'_i, i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } x = w_i, i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } x \in \{u_i, v'_i\}, i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } \ e \in \{u_iu'_{i+1}, u'_iu_{i+1}, v_iv'_{i+1}, v'_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{t'u_i, tu'_i, u'_iv_i, u_iv'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } \ e = v_iw_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } \ e = v_iw_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1, v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{15n-1}{2}$ and $e_f(0) = \dfrac{15n+1}{2}$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.2 *The graph obtained by duplicating all the pendent vertices of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the pendent vertices in Wb_n. Let $w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, w'_i/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, v_iw'_i/1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 4n+1$ and $|E(G)| = 6n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, w'_i\},\ i \in \{1,2,\ldots,n-1,n\}. \end{cases}$$

Thus $v_f(1) = 2n$ and $v_f(0) = 2n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e = v_iw'_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and

$|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.3 *The graph obtained by duplicating the outer rim vertices and the apex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1\}$. Let G be the graph obtained by duplicating the outer rim vertices and the apex in Wb_n. Let $t', v_1', v_2', \cdots, v_n'$ be the new vertices of G by duplicating $t, v_1, v_2, \cdots, v_n$ respectively. Then $V(G) = \{t, t'\} \bigcup \{u_i, v_i, w_i, v_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', t'u_i/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}'/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1, v_n'v_1, v_nv_1'\}$. Therefore $|V(G)| = 4n+2$ and $|E(G)| = 10n$. Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{u_i, w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x \in \{v_i, v_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 2n + 1$ and $v_f(0) = 2n + 1$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}'\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e = t'u_i,\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1.v_n'v_1, v_nv_1'\}. \end{cases}$$

Thus $e_f(1) = 5n$ and $e_f(0) = 5n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.4 *The graph obtained by duplicating all the vertices except the apex vertex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the apex vertex in Wb_n. Let $u_1', u_2', \cdots, u_n', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \bigcup \{u_i, v_i, w_i, u_i', v_i', w_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', u_i'v_i, v_iw_i', tu_i'/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1, v_n'v_1, v_nv_1', u_n'u_1, u_nu_1'\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}', u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 6n + 1$ and $|E(G)| = 14n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i, u_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, w_i', v_i'\},\ i \in \{1,2,\ldots,n-1,n\}. \end{cases}$$

Thus $v_f(1) = 3n$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', u_i'v_i, tu_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e = v_iw_i',\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}', u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1.v_n'v_1, v_nv_1', u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = 7n$ and $e_f(0) = 7n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.5 *The graph obtained by duplicating all the inner rim vertices and the apex vertex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the inner rim vertices and the apex vertex in Wb_n. Let $t', u_1', u_2', \cdots, u_n'$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n$ respectively. Then $V(G) = \{t, t'\} \cup \{u_i, v_i, w_i, u_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, u_i'v_i, tu_i', t'u_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \cup \{u_nu_1, v_nv_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 4n+2$ and $|E(G)| = 10n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, u_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 2n+1$ and $v_f(0) = 2n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } \ e \in \{u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{tu_i', t'u_i, u_i'v_i\},\ i \in \{1,2,\ldots,n-1,n\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = 5n$ and $e_f(0) = 5n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.6 *The graph obtained by duplicating all the edges other than spoke edges of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{j_i = tu_i, l_i = u_iv_i, o_i = v_iw_i/1 \leq i \leq n\} \bigcup \{k_i = u_iu_{i+1}, m_i = v_iv_{i+1}/1 \leq i \leq n-1\} \bigcup \{k_n = u_nu_1, m_n = v_nv_1\}$. Let G be the graph obtained by duplicating all the edges other than spoke edges in Wb_n. For each $i \in 1, 2, \cdots, n$, let $k'_i = a_ib_i, l'_i = c_id_i, m'_i = e_if_i$ and $o'_i = g_ih_i$ be the new edges of G by duplicating k_i, l_i, m_i and o_i respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i/1 \leq i \leq n\}$ and $E(G) = \{b_iu_{i+2}, v_if_{i+2}/1 \leq i \leq n-2\} \bigcup \{tu_i, u_iv_i, a_ib_i, e_if_i, c_id_i, g_ih_i, tc_i, tb_i, g_iu_i, ta_i, a_iv_i, e_iu_i, d_iw_i, v_iw_i, e_iw_i/1 \leq i \leq n\} \bigcup \{b_iv_{i+1}, u_ia_{i+1}, f_iu_{i+1}, v_ie_{i+1}, u_iu_{i+1}, v_iv_{i+1}, d_iv_{i+1}, v_id_{i+1}, c_iu_{i+1}, u_ic_{i+1}, g_iv_{i+1}, v_ig_{i+1}, f_iw_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1, d_nv_1, v_nd_1, c_nu_1, u_nc_1, g_nv_1, v_ng_1, b_{n-1}u_1, b_nu_2, v_{n-1}f_1, f_nv_2, b_nv_1, u_na_1, f_nu_1, v_ne_1, f_nw_1\}$. Therefore $|V(G)| = 11n + 1$ and $|E(G)| = 30n$. Using parity of n, we have the following cases:

Case 1 n is even.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, a_i, d_i, f_i, g_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x \in \{u_i, b_i, c_i, e_i, h_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = w_i, \ i \in \{1, 3, \ldots, n-3, n-1\}; \\ 0 & \text{if } x = w_i, \ i \in \{2, 4, \ldots, n-2, n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{11n}{2}$ and $v_f(0) = \dfrac{11n}{2} + 1$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, a_ib_i, e_if_i, c_id_i, g_ih_i, tc_i, tb_i, g_iu_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{ta_i, a_iv_i, e_iu_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}, d_iv_{i+1}, v_id_{i+1}, c_iu_{i+1}, u_ic_{i+1}, g_iv_{i+1}, v_ig_{i+1}\}, 1 \leq i \leq n-1; \\ 1 & \text{if } \ e \in \{b_iv_{i+1}, u_ia_{i+1}, f_iu_{i+1}, v_ie_{i+1}\}, \ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{b_iu_{i+2}, v_if_{i+2}\}, \ i \in \{1, 2, \ldots, n-3, n-2\}; \\ 1 & \text{if } \ e \in \{d_iw_i, v_iw_i\}, \ i \in \{1, 3, \ldots, n-3, n-1\} \end{cases}$$

and

$$f^*(e)=\begin{cases}0 & \text{if } \ e\in\{d_iw_i, v_iw_i\},\ i\in\{2,4,\ldots,n-2,n\};\\ 0 & \text{if } \ e\in\{e_iw_i, f_iw_{i+1}\},\ i\in\{1,3,\ldots,n-3,n-1\};\\ 1 & \text{if } \ e=e_iw_i,\ i\in\{2,4,\ldots,n-2,n\};\\ 1 & \text{if } \ e=f_iw_{i+1},\ i\in\{2,4,\ldots,n-4,n-2\};\\ 0 & \text{if } \ e\in\{u_nu_1, v_nv_1, d_nv_1, v_nd_1, c_nu_1, u_nc_1, g_nv_1, v_ng_1, b_{n-1}u_1, b_nu_2, v_{n-1}f_1, f_nv_2\};\\ 1 & \text{if } \ e\in\{b_nv_1, u_na_1, f_nu_1, v_ne_1, f_nw_1\}.\end{cases}$$

Thus $e_f(1)=15n$ and $e_f(0)=15n$.

Case 2 n is odd.

Define a vertex labeling $f:V(G)\to\{0,1\}$ as follows:

$$f(x)=\begin{cases}0 & \text{if } x=t;\\ 0 & \text{if } x\in\{v_i,a_i,d_i,f_i,g_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } x\in\{u_i,b_i,c_i,e_i,h_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } x=w_i,\ i\in\{1,3,\ldots,n-2,n\};\\ 0 & \text{if } x=w_i,\ i\in\{2,4,\ldots,n-3,n-1\}.\end{cases}$$

Thus $v_f(1)=\dfrac{11n+1}{2}$ and $v_f(0)=\dfrac{11n+1}{2}$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } \ e\in\{tu_i, u_iv_i, a_ib_i, e_if_i, c_id_i, g_ih_i, tc_i, tb_i, g_iu_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } \ e\in\{ta_i, a_iv_i, e_iu_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } \ e\in\{u_iu_{i+1}, v_iv_{i+1}, d_iv_{i+1}, v_id_{i+1}, c_iu_{i+1}, u_ic_{i+1}, g_iv_{i+1}, v_ig_{i+1}\}, 1\le i\le n-1;\\ 1 & \text{if } \ e\in\{b_iv_{i+1}, u_ia_{i+1}, f_iu_{i+1}, v_ie_{i+1}\},\ i\in\{1,2,\ldots,n-2,n-1\};\\ 0 & \text{if } \ e\in\{b_iu_{i+2}, v_if_{i+2}\},\ i\in\{1,2,\ldots,n-3,n-2\};\\ 1 & \text{if } \ e\in\{d_iw_i, v_iw_i\},\ i\in\{1,3,\ldots,n-2,n\};\\ 0 & \text{if } \ e\in\{d_iw_i, v_iw_i\},\ i\in\{2,4,\ldots,n-3,n-1\};\\ 0 & \text{if } \ e=e_iw_i,\ i\in\{1,3,\ldots,n-2,n\};\\ 1 & \text{if } \ e\in\{e_iw_i, f_iw_{i+1}\},\ i\in\{2,4,\ldots,n-3,n-1\};\\ 0 & \text{if } \ e=f_iw_{i+1},\ i\in\{1,3,\ldots,n-4,n-2\};\\ 0 & \text{if } \ e\in\{u_nu_1, v_nv_1, d_nv_1, v_nd_1, c_nu_1, u_nc_1, g_nv_1, v_ng_1, b_{n-1}u_1, b_nu_2, v_{n-1}f_1, f_nv_2, f_nw_1\};\\ 1 & \text{if } \ e\in\{b_nv_1, u_na_1, f_nu_1, v_ne_1\}.\end{cases}$$

Thus $e_f(1)=15n$ and $e_f(0)=15n$.

Therefore f satisfies the conditions $|v_f(1)-v_f(0)|\le 1$ and $|e_f(1)-e_f(0)|\le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.7 *The graph obtained by duplicating all the vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\}\bigcup\{u_i, v_i, w_i/1 \le i \le n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \le i \le n\}\bigcup\{u_iu_{i+1};\ 1 \le i \le n-1\}\bigcup\{u_nu_1\}$. Let G be the graph obtained by duplicating all the vertices in AH_n. Let $t', u'_1, u'_2, \cdots, u'_n, v'_1, v'_2, \cdots, v'_n, w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\}\bigcup\{u_i, v_i, w_i, u'_i, v'_i, w'_i/1 \le i \le n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w'_iv_i, w_iv'_i, u_iv'_i, t'u_i, u'_iv_i, tu'_i;\ 1 \le i \le n\}\bigcup\{u_iu_{i+1}, u'_iu_{i+1}, u_iu'_{i+1};\ 1 \le i \le n-1\}\bigcup\{u_nu_1, u'_nu_1, u_nu'_1\}$. Therefore $|V(G)| = 6n+2$ and $|E(G)| = 12n$. Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, u'_i, w'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x \in \{v_i, w_i, v'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, u_iv'_i, v_iw'_i, u'_iv_i, tu'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{v_iw_i, t'u_i, w_iv'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, u'_iu_{i+1}, u_iu'_{i+1}\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, u'_nu_1, u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = 6n$ and $e_f(0) = 6n$. Therefore, f satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.8 *The graph obtained by duplicating all the vertices other than the rim vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\}\bigcup\{u_i, v_i, w_i/1 \le i \le n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \le i \le n\} \cup \{u_nu_1\}\bigcup\{u_iu_{i+1}/1 \le i \le n-1\}$. Let G be the graph obtained by duplicating all the vertices other than the rim vertices in AH_n. Let $t', v'_1, v'_2, \cdots, v'_n, w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $t, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\}\bigcup\{u_i, v_i, w_i, v'_i, w'_i/1 \le i \le n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w'_iv_i, w_iv'_i, u_iv'_i, t'u_i/1 \le i \le n\}\bigcup\{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_nu_1\}$. Therefore $|V(G)| = 5n+2$ and $|E(G)| = 8n$. Using parity of n, we have the following cases:

Case 1 n is even.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, w'_i\},\ i \in \{1, 2, \ldots, n-1, n\} \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } x \in \{v_i, w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = t'; \\ 1 & \text{if } x = v_i',\ i \in \{1, 3, \ldots, n-3, n-1\}; \\ 0 & \text{if } x = v_i',\ i \in \{2, 4, \ldots, n-2, n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{5n+2}{2}$ and $v_f(0) = \dfrac{5n+2}{2}$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$.
Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_i v_i, v_i w_i', t_i' u_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } e = u_i v_i',\ i \in \{1, 3, \ldots, n-3, n-1\}; \\ 1 & \text{if } e = w_i v_i',\ i \in \{2, 4, \ldots, n-2, n\}; \\ 0 & \text{if } e \in \{t u_i, v_i w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } e = u_i u_{i+1},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } e = u_i v_i',\ i \in \{2, 4, \ldots, n-2, n\}; \\ 0 & \text{if } e = w_i v_i',\ i \in \{1, 3, \ldots, n-3, n-1\}; \\ 0 & \text{if } e \in \{u_n u_1\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$.

Case 2 n is odd.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t'; \\ 1 & \text{if } x \in \{v_i, w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = v_i',\ i \in \{1, 3, \ldots, n-2, n\}; \\ 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, w_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x = v_i',\ i \in \{2, 4, \ldots, n-3, n-1\}. \end{cases}$$

Thus $v_f(1) = \dfrac{5n+3}{2}$ and $v_f(0) = \dfrac{5n+1}{2}$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_i v_i, v_i w_i', t_i' u_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } e = u_i v_i',\ i \in \{1, 3, \ldots, n-2, n\}; \\ 1 & \text{if } e = w_i v_i',\ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 0 & \text{if } e \in \{t u_i, v_i w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } e = u_i u_{i+1},\ i \in \{1, 2, \ldots, n-2, n-1\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } \ e = u_iv_i', \ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e = w_iv_i', \ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } \ e \in \{u_nu_1\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.9 *The graph obtained by duplicating all the rim vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i//1 \leq i \leq n\} \bigcup \{u_nu_1\} \bigcup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the rim vertices in AH_n. Let $u_1', u_2', \cdots, u_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n$ respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, u_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, u_i'v_i, tu_i'/1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \cup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 4n+1$ and $|E(G)| = 8n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, u_i'\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x \in \{v_i, w_i\}, \ i \in \{1,2,\ldots,n-1,n\}. \end{cases}$$

Thus $v_f(1) = 2n+1$ and $v_f(0) = 2n$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, tu_i', u_i'v_i\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e = v_iw_i, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, u_iu_{i+1}', u_i'u_{i+1}\}, \ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$. Therefore, f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.10 *The graph obtained by duplicating all the vertices except the apex vertex of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1\} \bigcup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the apex vertex in AH_n. Let $u_1', u_2', \cdots, u_n', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \bigcup \{u_i, v_i, w_i, u_i', v_i', w_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_i'v_i, w_iv_i', u_iv_i', u_i'v_i, tu_i'/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \bigcup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore

$|V(G)| = 6n+1$ and $|E(G)| = 11n$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 1 & \text{if } x \in \{v_i, u_i'\},\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{2,4,\dots,n-2,n\}; \\ 1 & \text{if } x = w_i',\ i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } x \in \{u_i, v_i'\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } x = w_i, i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } x = w_i', i \in \{2,4,\dots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } \ e = v_iw_i,\ i \in \{1,3,\dots,n-3,n-1\}; \\ 1 & \text{if } \ e \in \{w_iv_i', v_iw_i'\},\ i \in \{2,4,\dots,n-2,n\}; \\ 1 & \text{if } \ e \in \{u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } \ e = v_iw_i,\ i \in \{2,4,\dots,n-2,n\}; \\ 0 & \text{if } \ e = u_iu_{i+1},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } \ e = u_iv_i',\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } \ e \in \{w_iv_i', v_iw_i'\},\ i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } \ e \in \{u_i'v_i, tu_i'\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } \ e = u_nu_1; \\ 1 & \text{if } \ e \in \{u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = \dfrac{11n}{2}$ and $e_f(0) = \dfrac{11n}{2}$.

Case 2. n is odd.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 1 & \text{if } x = w_i,\ i \in \{2,4,\dots,n-3,n-1\}; \\ 1 & \text{if } x \in \{u_i', v_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } x = w_i',\ i \in \{1,3,\dots,n-2,n\} \end{cases}$$

and

$$f(x)=\begin{cases}0 & \text{if } x\in\{u_i,v_i'\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } x=w_i,\ i\in\{1,3,\ldots,n-2,n\};\\ 0 & \text{if } x=w_i',\ i\in\{2,4,\ldots,n-3,n-1\}.\end{cases}$$

Thus $v_f(1)=3n+1$ and $v_f(0)=3n$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e\in\{tu_i,u_iv_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } e=v_iw_i,\ i\in\{1,3,\ldots,n-2,n\};\\ 1 & \text{if } e\in\{w_iv_i',v_iw_i'\},\ i\in\{2,4,\ldots,n-3,n-1\};\\ 1 & \text{if } e\in\{u_i'u_{i+1},u_iu_{i+1}'\},\ i\in\{1,2,\ldots,n-2,n-1\};\\ 0 & \text{if } e=v_iw_i,\ i\in\{2,4,\ldots,n-3,n-1\};\\ 0 & \text{if } e=u_iu_{i+1},\ i\in\{1,2,\ldots,n-2,n-1\};\\ 0 & \text{if } e=u_iv_i',\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } e\in\{w_iv_i',v_iw_i'\},\ i\in\{1,3,\ldots,n-2,n\};\\ 0 & \text{if } e\in\{u_i'v_i,tu_i'\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } e=u_nu_1;\\ 1 & \text{if } e\in\{u_n'u_1,u_nu_1'\}.\end{cases}$$

Thus $e_f(1)=\dfrac{11n-1}{2}$ and $e_f(0)=\dfrac{11n+1}{2}$.

From both the cases we can conclude $|v_f(1)-v_f(0)|\le 1$ and $|e_f(1)-e_f(0)|\le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.11 *The graph obtained by duplicating all the edges other than spoke edges of of the armed helm AH_n is cordial.*

Proof Let $V(AH_n)=\{t\}\bigcup\{u_i,v_i,w_i/1\le i\le n\}$ and $E(AH_n)=\{j_i=tu_i,l_i=u_iv_i,m_i=u_iw_i/1\le i\le n\}\bigcup\{k_i=u_iu_{i+1}/1\le i\le n-1\}\bigcup\{k_n=u_nu_1\}$. Let G be the graph obtained by duplicating all the edges other than spoke edges in AH_n. For each $i\in 1,2,\cdots,n$, let $k_i'=a_ib_i, l_i'=c_id_i$ and $m_i'=e_if_i$ be the new edges of G by duplicating $k_i,l_i$and m_i respectively. Then $V(G)=\{t\}\bigcup\{u_i,v_i,w_i,a_i,b_i,c_i,d_i,e_i,f_i/1\le i\le n\}$ and $E(G)=\{b_iu_{i+2}/1\le i\le n-2\}\bigcup\{tu_i,u_iv_i,v_iw_i,c_id_i,a_ib_i,a_iv_i,ta_i,tb_i,tc_i,e_if_i,e_iu_i,d_iw_i/1\le i\le n\}\bigcup\{u_iu_{i+1},b_iv_{i+1},u_ia_{i+1},c_iu_{i+1},u_ic_{i+1}/1\le i\le n-1\}\bigcup\{u_nu_1,b_nv_1,u_na_1,c_nu_1,u_nc_1,b_{n-1}u_1, b_nu_2\}$. Therefore $|V(G)|=9n+1$ and $|E(G)|=18n$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, v_i, c_i, d_i\},\ i \in \{1, 2, \dots, n-1, n\} \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } x \in \{w_i, a_i, b_i, f_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } x = e_i,\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 1 & \text{if } x = e_i,\ i \in \{2, 4, \dots, n-2, n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{9n}{2} + 1$ and $v_f(0) = \dfrac{9n}{2}$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, v_iw_i, a_iv_i, tc_i, d_iw_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iv_i, , ta_i, tb_i, a_ib_i, c_id_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 1 & \text{if } \ e \in \{b_iv_{i+1}, u_ia_{i+1}\},\ i \in \{1, 2, \dots, n-2, n-1\}; \\ 1 & \text{if } \ e = b_iu_{i+2},\ i \in \{1, 2, \dots, n-3, n-2\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, c_iu_{i+1}, u_ic_{i+1}\},\ i \in \{1, 2, \dots, n-2, n-1\}; \\ 0 & \text{if } \ e = u_ie_i,\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 1 & \text{if } \ e = u_ie_i,\ i \in \{2, 4, \dots, n-2, n\}; \\ 1 & \text{if } \ e = e_if_i,\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 0 & \text{if } \ e = e_if_i,\ i \in \{2, 4, \dots, n-2, n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, c_nu_1, u_nc_1\}; \\ 1 & \text{if } \ e \in \{b_nv_1, u_na_1, b_n - 1u_1, b_nu_2\}. \end{cases}$$

Thus $e_f(1) = 9n$ and $e_f(0) = 9n$.

Case 2. n is odd.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, v_i, c_i, d_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 1 & \text{if } x \in \{w_i, a_i, b_i, f_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } x = e_i,\ i \in \{1, 3, \dots, n-2, n\}; \\ 1 & \text{if } x = e_i,\ i \in \{2, 4, \dots, n-3, n-1\}. \end{cases}$$

Thus $v_f(1) = \dfrac{9n+1}{2}$ and $v_f(0) = \dfrac{9n+1}{2}$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$

is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{tu_i, v_iw_i, a_iv_i, tc_i, d_iw_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } e \in \{u_iv_i, , ta_i, tb_i, a_ib_i, c_id_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } e \in \{b_iv_{i+1}, u_ia_{i+1}\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 1 & \text{if } e = b_iu_{i+2},\ i \in \{1, 2, \ldots, n-3, n-2\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } e \in \{u_iu_{i+1}, c_iu_{i+1}, u_ic_{i+1}\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } e = u_ie_i,\ i \in \{1, 3, \ldots, n-2, n\}; \\ 1 & \text{if } e = u_ie_i,\ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 1 & \text{if } e = e_if_i,\ i \in \{1, 3, \ldots, n-2, n\}; \\ 0 & \text{if } e = e_if_i,\ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 0 & \text{if } e \in \{u_nu_1, c_nu_1, u_nc_1\}; \\ 1 & \text{if } e \in \{b_nv_1, u_na_1, b_{n-1}u_1, b_nu_2\}. \end{cases}$$

Thus $e_f(1) = 9n$ and $e_f(0) = 9n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

§3. Conclusion

we have derived eleven new results by investigating cordial labeling in the context of duplication in Web and Armed Helm. More exploration is possible for other graph families and in the context of different graph labeling problems.

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A Study on
Equitable Triple Connected Domination Number of a Graph

M. Subramanian

Department of Mathematics

Anna University Regional Campus - Tirunelveli, Tirunelveli - 627 007, India

T. Subramanian

Department of Mathematics, University Voc College of Engineering

Anna University:Thoothukudi Campus, Thoothukudi- 628008, India

E-mail: ms_akce@yahoo.com, tsubramanian001@gmail.com

Abstract: A graph G is said to be *triple connected* if any three vertices lie on a path in G. A dominating set S of a connected graph G is said to be a *triple connected dominating set* of G if the induced subgraph $\langle S\rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the *triple connected domination number* and is denoted by γ_{tc}. A triple connected dominating set S of V in G is said to be an *equitable triple connected dominating set* if for every vertex u in $V-S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v)-deg(u)|\leq 1$.The minimum cardinality taken over all equitable triple connected dominating sets is called the *equitable triple connected domination number* and is denoted by γ_{etc}. In this paper we initiate a study on this parameter. In addition, we discuss the related problem of finding the stability of γ_{etc} upon edge addition on some classes of graphs.

Key Words: Connected domination, triple connected domination, equitable triple connected dominating set, equitable triple connected domination number, Smarandachely equitable dominating set.

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§1. Introduction

By a *graph*, we mean a finite, simple, connected and undirected graph $G(V,E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G is connected and has p vertices and q edges. For graph theoretic terminology, we refer to Harary [1].

Definition 1.1([2]) *A subset S of V in G is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G.*

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Definition 1.2([6]) *A dominating set S of V in G is said to be an equitable dominating set if for every vertex u in $V - S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v) - deg(u)| \leq 1$, and Smarandachely equitable dominating set if $|deg(v) - deg(u)| \geq 1$ for all such an edge. The minimum cardinality taken over all equitable dominating sets in G is the equitable domination number of G and is denoted by γ_e.*

Definition 1.3([2]) *A dominating set S of V in G is said to be a connected dominating set of G if the induced sub graph $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets in G is the connected domination number of G and is denoted by γ_c.*

Definition 1.4([3]) *A connected dominating set S of V in G is said to be an equitable connected dominating set if for every vertex u in $V - S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v) - deg(u)| \leq 1$. The minimum cardinality taken over all equitable connected dominating sets in G is the equitable connected domination number of G and is denoted by γ_{ec}.*

The concept of triple connected graphs has been introduced by Paulraj Joseph et. al. [5] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be *triple connected* if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. But the star $K_{1,p-1}, p \geq 4$ is not a triple connected graph.

Definition 1.5([4]) *A dominating set S of a connected graph G is said to be a triple connected dominating set of G if the induced sub graph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all connected dominating sets is the triple connected domination number and is denoted by γ_{tc}.*

In this paper, we extend the concept of triple connected domination to an equitable triple connected domination and study its properties.

Notation 1.6 Let G be a connected graph on m vertices $v_1, v_2, \ldots, v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \cdots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Notation 1.7 We have $C_p(nP_k, 0, 0, \cdots, 0) \cong C_p(0, nP_k, 0, \cdots, 0) \cong \cdots \cong C_p(nP_k)$.

Example 1.8 The graph G_1 in Figure 1 is isomorphic to $C_3(2P_2)$.

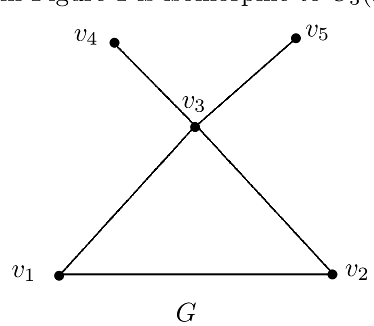


Figure 1 The graph $C_3(2P_2)$.

Proposition 1.9 *For any connected graph G with p vertices, $1 \leq \gamma_e(G) \leq p$.*

Proposition 1.10 *For any connected graph G with p vertices, $1 \leq \gamma_{ec}(G) \leq p$.*

§2. Equitable Triple Connected Domination Number of a Graph

In this section, we define the concept of equitable triple connected domination number of a graph.

Definition 2.1 *A subset S of V of a nontrivial graph G is said to be an equitable triple connected dominating set, if $\langle S \rangle$ is triple connected and for every vertex u in $V - S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v) - deg(u)| \leq 1$. The minimum cardinality taken over all equitable triple connected dominating sets is called an equitable triple connected domination number of G and is denoted by $\gamma_{etc}(G)$. Any equitable triple connected dominating set with γ_{etc} vertices is called a γ_{etc}-set of G.*

Example 2.2 For the graph G_1 in Figure 2, $S = \{v_2, v_3, v_5, v_6\}$ forms a γ_{etc}-set of G_1. Hence $\gamma_{etc}(G_1) = 4$.

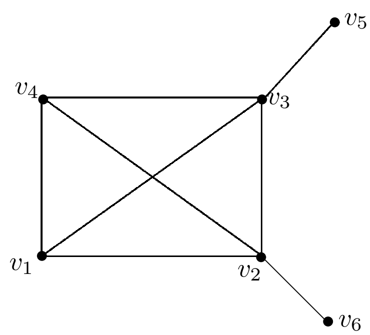


Figure 2 Graph with $\gamma_{etc} = 4$.

Remark 2.3 Any equitable triple connected dominating set is obviously equitable connected dominating set and any equitable connected dominating set is also an equitable dominating set and finally and any equitable dominating set is a dominating set. So it is permissible for the equitable triple connected dominating set S can have less than three vertices. If S has 1 (or 2) vertex (vertices) then S can be viewed as an equitable dominating set (or connected equitable dominating set).

Throughout this paper, we consider only connected graphs for which equitable triple connected dominating set exists.

Definition 2.4 *A bistar, denoted by $B(m, n)$ is the graph obtained by joining the centers of the stars $K_{1,m}$ and $K_{1,n}$. The center of a star $K_{1,p-1}$ with $p > 2$ vertices is its unique vertex of maximum degree.*

Definition 2.5 *A helm graph, denoted by H_n is the graph obtained from the wheel W_n by attaching a pendant vertex to each vertex in the outer cycle of W_n (the number of vertices of H_n is, $p = 2n - 1$).*

Definition 2.6 *The friendship graph F_n is the graph constructed by identifying n copies of the cycle C_3 at a common vertex.*

Remark 2.7 It is to be noted that not every graph has a triple connected dominating set likewise not all graphs have an equitable triple connected dominating set. For example, the star graph $K_{1,3}$ does not have an equitable triple connected dominating set.

§3. Preliminary Results

We now proceed to determine the equitable triple connected domination number for some standard graphs.

(1) For any path of order $p, \gamma_{etc}(P_p) = \begin{cases} p & \text{if } p = 1 \\ p-1 & \text{if } p = 2 \\ p-2 & \text{if } p \geq 3. \end{cases}$

(2) For any cycle of order $p, \gamma_{etc}(C_p) = p - 2$.

(3) For any complete bipartite graph other than star of order $p = m + n$,

$$\gamma_{etc}(K_{m,n}) = \begin{cases} 2 & \text{if } |m-n| \leq 1 \text{ and } m, n \neq 1 \\ p & \text{if } |m-n| \geq 2 \text{ and } m, n \neq 1. \end{cases}$$

(4) For any complete graph of order $p, \gamma_{etc}(K_p) = 1$.

(5) For any wheel of order $p, \gamma_{etc}(W_p) = \begin{cases} 1 & \text{if } p = 4, 5 \\ 3 & \text{if } p = 6 \\ p-4 & \text{if } p \geq 7 \end{cases}$

Equitable triple connected dominating set does not exist for the following special graphs:

(6) For any star $K_{1,p-1}$ other than $K_{1,2}$.

(7) Helm graph H_n.

(8) Bistar $B(m,n)$.

Consider any star $K_{1,p-1}$ of order $p > 3$. Let v be its center and $v_1, v_2, \ldots, v_{p-1}$ be the pendant vertices which are adjacent to v. Since every minimum equitable dominating set S must contain all the pendant vertices $v_1, v_2, \cdots, v_{p-1}$ and we have $\langle S \rangle$ is not triple connected if $p - 1 > 2$. Hence $\gamma_{etc}(K_{1,p-1})$ does not exist if $p > 3$. Similarly we can prove all the other results.

Lemma 3.1 *If $\gamma_e(G) = 1$, then $\gamma_{etc}(G) = 1$.*

Lemma 3.2 *If $\gamma_{ec}(G) = 1$ (or 2 or 3), then $\gamma_{etc}(G) = 1$ (or 2 or 3).*

Lemma 3.3 *If $\gamma_{ec}(G) = 4$, then $\gamma_{etc}(G)$ need not be equal to 4.*

For $C_3(2P_2), \gamma_{ec}(C_3(2P_2)) = 4$, but $\gamma_{etc}(C_3(2P_2))$-set does not exist.

Theorem 3.4 *For any connected graph G with p vertices, we have $1 \le \gamma_{etc}(G) \le p$ and the bounds are sharp.*

Proof The lower bound follows from Definition 2.1 and the upper bound is obvious. For K_4 the lower bound is attained and for $K_{2,4}$ the upper bound is attained. □

Observation 3.5 For any connected graph G of order 1, $\gamma_{etc}(G) = p$ if and only if G is isomorphic to K_1.

Lemma 3.6 *There exists no connected graph G with $2 \le p \le 4$ vertices such that $\gamma_{etc}(G) = p$.*

Proof The proof is divided into cases following.

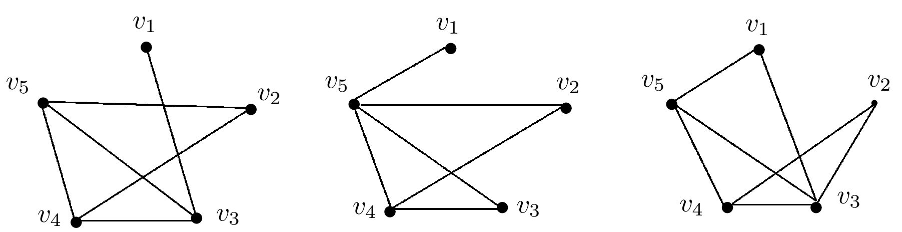
Case 1. The only connected graph with of order 2 is K_2 and for $K_2, \gamma_{etc}(K_2) = 1 = p - 1$ ([1]).

Case 2. There are only two connected graphs with three vertices which are P_3 or K_3 and for $G \cong P_3, K_3, \gamma_{etc}(G) = 1 = p - 2$ ([1]).

Case 3. The various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2), C_4, K_4 - \{e\}, K_4$. If G is isomorphic to $P_4, C_4, C_3(P_2)$, $\gamma_{etc}(G) = 2 = p - 2$. If G is isomorphic to $K_4, K_4 - \{e\}, \gamma_{etc}(G) = 1 = p - 3$. If $G \cong K_{1,3}, \gamma_{etc}(G)$ does not exist. □

Theorem 3.7 *Let G be a connected graph with $p = 5$ vertices, then $\gamma_{etc}(G) = p$ if and only if G is isomorphic to $\overline{C_3 \cup 2K_1}$.*

Proof ([1]) For the various possibilities of connected graphs on five vertices are: $K_{1,4}$, $P_3(0, P_2, P_2), P_5$, $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, $C_3(P_3)$, $C_4(P_2)$, C_5, F_2, $\overline{P}_5$, $K_{2,3}, K_4(P_2), \overline{C_3 \bigcup 2K_1}$, $\overline{P_2 \bigcup P_3}, \overline{P_3 \bigcup 2K_1}, W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3.



G_1 G_2 G_3

Figure 3 Graphs on 5 vertices.

If $G \cong K_5, W_5, K_5 - \{e\}$, then $\gamma_{etc}(G) = 1 = p - 4$. If $G \cong K_4(P_2)$, $C_3(P_3), K_{2,3}, \overline{P}_5$, $\overline{P_3 \cup 2K_1}$, $\overline{P_2 \cup P_3}, G_3$, then $\gamma_{etc}(G) = 2 = p - 3$. If $G \cong P_5, C_5, F_2$, $C_4(P_2), G_1, G_2$, then $\gamma_{etc}(G) = 3 = p - 2$. If $G \cong C_3(P_2, P_2, 0)$, then $\gamma_{etc}(G) = 4 = p - 1$. If $G \cong \overline{C_3 \cup 2K_1}$, then $\gamma_{etc}(G) = 5 = p$. If $G \cong K_{1,4}, C_3(2P_2)$, $P_3(0, P_2, P_2)$, then $\gamma_{etc}(G)$ does not exist. □

Theorem 3.8 *Let G be a connected graph with $p = 6$ vertices, then $\gamma_{etc}(G) = p$ if and only if*

G is isomorphic to $K_{2,4}$ *or any one of the graphs:* G_1, G_2, G_3 *in Figure 4.*

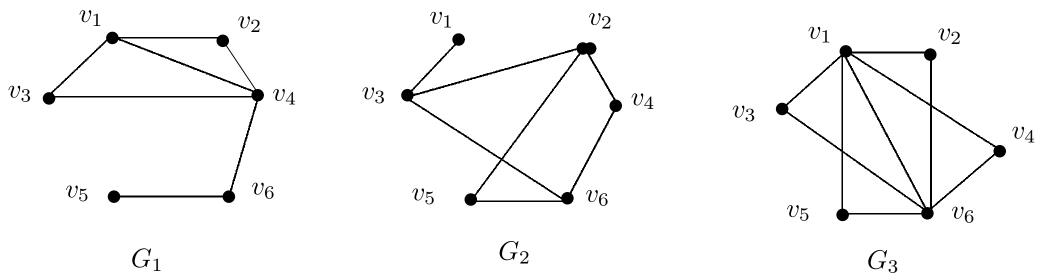


Figure 4 Graphs on 6 vertices with $\gamma_{etc}(G) = 6$.

Proof Let G be a connected graph with $p = 6$ vertices, and let $\gamma_{etc}(G) = 6$ ([1]). Among all of the connected graphs on 6 vertices, it can be easily verified that $G \cong K_{2,4}$ or any one of the graphs G_1, G_2, G_3 in Figure 4.The converse part is obvious. □

Lemma 3.9 *Let G be a connected graph of order 2 such that* $\gamma(G) = \gamma_{etc}(G)$*. Then* $G \cong K_2$.

Lemma 3.10 *Let G be a connected graph of order 3 such that* $\gamma(G) = \gamma_{etc}(G)$*. Then* $G \cong K_3, P_3$.

Lemma 3.11 *Let G be a connected graph of order 4 such that* $\gamma(G) = \gamma_{etc}(G)$*. Then G is isomorphic to one of the following graphs:* $P_4, C_4, K_4, K_4 - \{e\}$.

Proof For the various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2)$, $C_4, K_4 - \{e\}, K_4$. If $G \cong P_4, C_4, \gamma(G) = \gamma_{etc}(G) = 2$. If $G \cong K_4, K_4 - \{e\}, \gamma(G) = \gamma_{etc}(G) = 1$. If $G \cong C_3(P_2), K_{1,3}, \gamma(C_3(P_2)) = 1$ but $\gamma_{etc}(C_3(P_2)) = 2$ and $\gamma(K_{1,3}) = 1$ but $\gamma_{etc}(K_{1,3})$ does not exist. Hence the lemma. □

Theorem 3.12 *Let G be a connected graph on order 5 such that* $\gamma(G) = \gamma_{etc}(G)$*. Then G is isomorphic to one of the following graphs:* $C_3(P_3), \overline{P}_5,\ K_{2,3}, \overline{P_2 \cup P_3}, W_5, K_5 - \{e\}, K_5$.

Proof For the various possibilities of connected graphs on five vertices are: $K_{1,4}, P_3(0, P_2, P_2)$, $P_5, C_3(2P_2), C_3(P_2, P_2, 0), C_3(P_3), C_4(P_2), C_5, F_2, \overline{P}_5, K_{2,3}, K_4(P_2),\ \overline{C_3 \cup 2K_1},\ \overline{P_2 \cup P_3}, \overline{P_3 \cup 2K_1}$, $W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3. Among all the above possibilities it can be easily verified that $\gamma(G) = \gamma_{etc}(G)$ only if $G \cong C_3(P_3), \overline{P}_5, K_{2,3}$, $\overline{P_2 \cup P_3},\ W_5, K_5 - \{e\}, K_5$. □

Theorem 3.13 *Let G be a connected graph of order 6 such that* $\gamma(G) = \gamma_{etc}(G)$*. Then G is isomorphic to* $K_{3,3}, K_6 - \{e\}, K_6$*, or any one of the graphs:* $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}$, G_{11}, G_{12} *in Figure* 5.

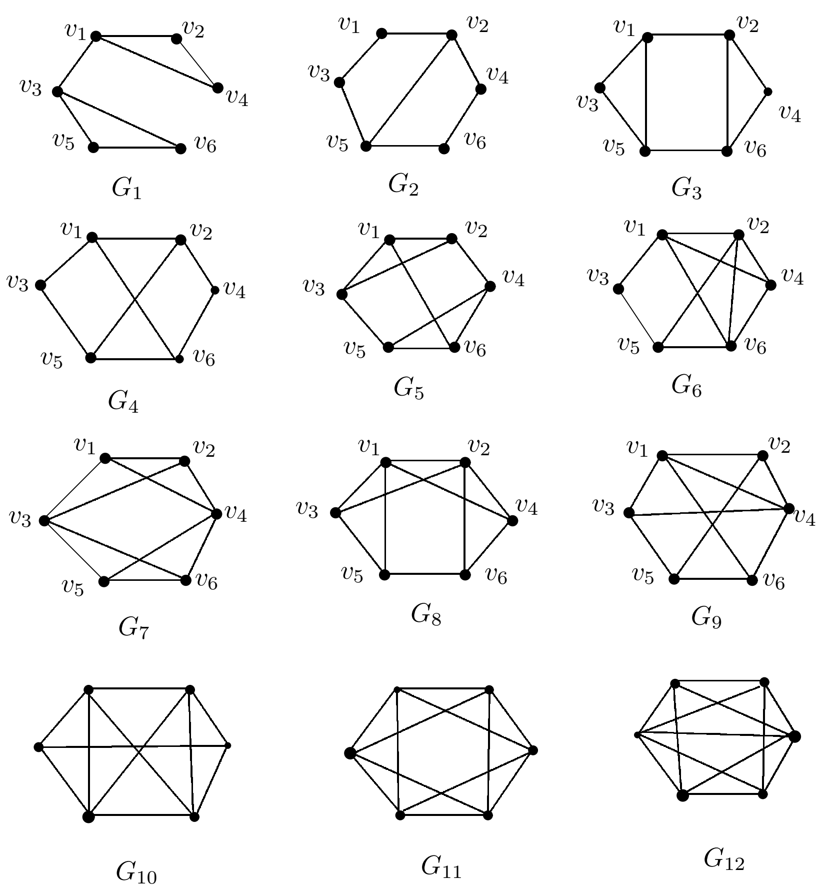


Figure 5 Graphs on 6 vertices such that $\gamma(G) = \gamma_{etc}(G)$.

Proof Let G be a connected graph of order 6 such that $\gamma(G) = \gamma_{etc}(G)$. It is straight forward to observe that $\gamma(G) = \gamma_{etc}(G)$ only if $G \cong K_{3,3}, K_6 - \{e\}, K_6$ or any one of the graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{12}$ in Figure 5. □

Observation 3.14 For any connected graph $G, \gamma_e(G) \leq \gamma_{ec}(G) \leq \gamma_{etc}(G)$ and the bounds can be strict as well as sharp for all possible cases.

(1) For the complete graph $K_5, \gamma_e(K_5) = \gamma_{ec}(K_5) = \gamma_{etc}(K_5) = 1$.

(2) For $K_4(P_3), \gamma_e(K_4(P_3)) = 2 < \gamma_{ec}(K_4(P_3)) = \gamma_{etc}(K_4(P_3)) = 3$.

(3) For the graph G_1 in Figure 6, $\gamma_e(G_1) = 3 < \gamma_{ec}(G_1) = 4 < \gamma_{etc}(G_1) = 5$.

(4) For the graph G_2 in Figure 6, $\gamma_e(G_2) = \gamma_{ec}(G_2) = 4 < \gamma_{etc}(G_2) = 5$.

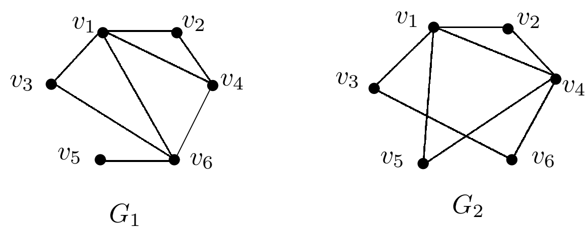


Figure 6

Lemma 3.15 *Let G be a connected graph of order 1 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_1$.*

Lemma 3.16 *Let G be a connected graph of order 2 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_2$.*

Lemma 3.17 *Let G be a connected graph of order 3 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_3, P_3$.*

Lemma 3.18 *Let G be a connected graph of order 4 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $P_4, C_4, K_4, C_3(P_2), K_4 - \{e\}$.*

Proof The various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2)$, $C_4, K_4 - \{e\}, K_4$. If $G \cong P_4, C_4, C_3(P_2), \gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 2$. If $G \cong K_4, K_4 - \{e\}$, $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 1$. And if $G \cong K_{1,3}, \gamma_e(G) = \gamma_{ec}(G) = 4$ and $\gamma_{etc}(G)$ does not exist. Hence the lemma. □

Theorem 3.19 *Let G be a connected graph of order 5 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $C_3(P_3), C_4(P_2), \overline{P}_5, K_{2,3}, F_2, K_4(P_2)$, $\overline{P_2 \cup P_3}, \overline{P_3 \cup 2K_1}, W_5, K_5 - \{e\}, K_5$ or the graphs: G_1 to G_3 in Figure* 3.

Proof For the various possibilities of connected graphs on five vertices are: $K_{1,4}, P_3(0, P_2, P_2)$, $P_5, C_3(2P_2), C_3(P_2, P_2, 0), C_3(P_3), C_4(P_2), C_5, F_2, \overline{P}_5, K_{2,3}, K_4(P_2), \overline{C_3 \cup 2K_1}, \overline{P_2 \cup P_3}, \overline{P_3 \cup 2K_1}$, $W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3. If $G \cong K_5, W_5, K_5 - \{e\}$, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 1$. If $G \cong K_4(P_2), C_3(P_3), K_{2,3}, \overline{P}_5, \overline{P_3 \cup 2K_1}$, $\overline{P_2 \cup P_3}, G_3$, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 2$. If $G \cong F_2$ or $C_4(P_2)$ then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 3$. If $G \cong G_1$ or G_2 then $\gamma_e(G) = 2$, but $\gamma_{ec}(G) = \gamma_{etc}(G) = 3$. If $G \cong C_3(P_2, P_2, 0)$, then $\gamma_e(G) = 3$, but $\gamma_{ec}(G) = \gamma_{etc}(G) = 4$. If $G \cong \overline{C_3 \cup 2K_1}$, then$\gamma_e(G) = \gamma_{ec}(G) = 4$, but $\gamma_{etc}(G) = 5$. If $G \cong K_{1,4}$, then $\gamma_e(G) = \gamma_{ec}(G) = 5$, but $\gamma_{etc}(G)$ does not exist. If $G \cong P_3(0, P_2, P_2)$, then $\gamma_e(G) = 3, \gamma_{ec}(G) = 4$ and $\gamma_{etc}(G)$ does not exist. If $G \cong C_3(2P_2)$ then $\gamma_e(G) = \gamma_{ec}(G) = 4$, but $\gamma_{etc}(G)$ does not exist. If $G \cong P_5, C_5$, then $\gamma_e(G) = 2$,but $\gamma_{ec}(G) = \gamma_{etc}(G) = 3$. □

Theorem 3.20 *Let G be a connected graph of order 6 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_{2,4}$.*

Proof Let G be a connected graph of order 6 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Among all of the connected graphs on 6 vertices, it can be easily seen that $K_{2,4}$ is the only graph with $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 6$. □

Theorem 3.21 *If G is a connected graph of order $p = 2n$ for some positive integer $n \geq 2$ such that its vertex set and edge set are $V(G) = \{v_i : 1 \leq i \leq p\}$ and $E(G) = \{v_iv_{i+1} : 1 \leq i \leq p-1\} \cup \{v_iv_j :$ for $i = 1$ to $\frac{p}{2}$ and $j = \left(\frac{P}{2}+1\right)$ to $p\}$ respectively, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = n-1$.*

Example 3.22 For $p = 6(= 2n)$, By Theorem 3.21, the graph constructed is shown in Figure 7. Clearly any two adjacent vertices from the set $\{v_2, v_3, v_4, v_5\}$ forms a minimum equitable triple connected dominating set. Hence $\gamma_{etc}(G) = 2 = n-1$.

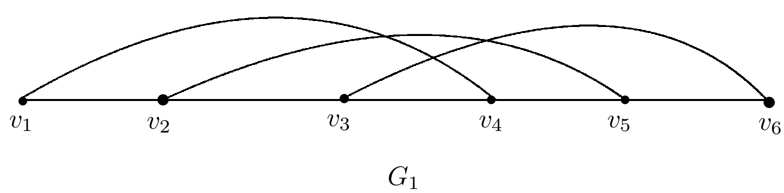


Figure 7 Graph illustrating the Theorem 3.21

Proposition 3.23 *Let G be a triple connected graph on order p. If its vertex set $V(G)$ can be partitioned into k sets $\{S_1, S_2, \cdots, S_k\}$ such that $S_1 = \{v : deg(v) = m_1\}$, $S_2 = \{v : deg(v) = m_2 \geq m_1 + 2\}$, $S_3 = \{v : deg(v) = m_3 \geq m_2 + 2\}, \cdots, S_k = \{v : deg(v) = m_k \geq m_{k-1} + 2\}$ where m_i's are increasing sequence of positive integers and $\langle S_i \rangle$ is $\overline{K_n}$ for some positive integer n, for $1 \leq i \leq k$, then $\gamma_e(G) = \gamma_{ce}(G) = \gamma_{etc}(G) = p$.*

Remark 3.24 The converse of Proposition 3.23 need not be true. Let G be a triple connected graph given in Figure 8. Clearly $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = p$, but there is no such partition of $V(G)$ as stated in Proposition 3.23. Since $V(G)$ can be partitioned in to $S_1 = \{v_{11}, v_{12}\}$ of degree $1, S_2 = \{v_3, v_4, v_5, v_7, v_8, v_9\}$ of degree $2, S_3 = \{v_2\}$ of degree $3, S_4 = \{v_6, v_{10}\}$ of degree 4 and $S_5 = \{v_1\}$ of degree 7 such that $\langle S_i \rangle$ is totally disconnected, for $1 \leq i \leq 5$.

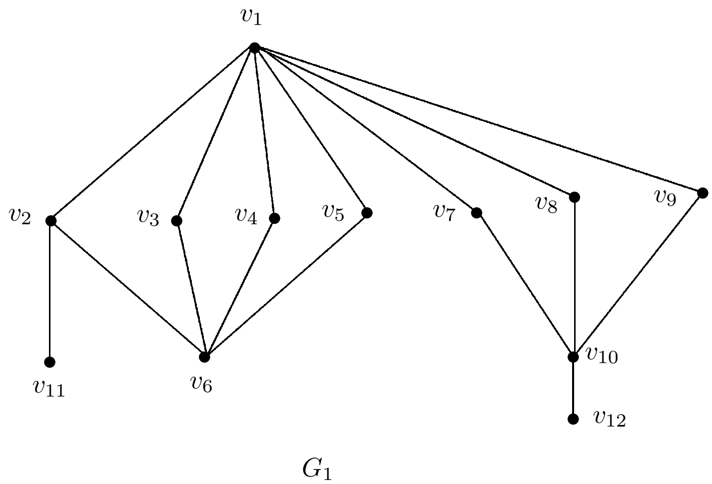


Figure 8 Counter example for Proposition 3.23

Lemma 3.25 *Let T be any tree, $\gamma_{etc}(T) = p$ if and only if $T \cong K_1$.*

Proof Let $T \cong K_1$, then clearly $\gamma_{etc}(T) = p$. Conversely, let T be a tree such that $\gamma_{etc}(T) = p$. Now $\langle T \rangle$ is triple connected, it follows that $T \cong P_p$ ([5] since a tree T is triple connected if and only if $T \cong P_p; p \geq 3$) and given that $\gamma_{etc}(T) = p$, we have $T \cong K_1$. □

Lemma 3.25 *Let T be any tree, $\gamma_{etc}(T) = p - 1$ if and only if $T \cong K_2$.*

Proof Let $T \cong K_2$, then clearly $\gamma_{etc}(T) = p - 1$. Conversely, let T be a tree such that $\gamma_{etc}(T) = p - 1$. Let v_p be the vertex not in $\gamma_{etc}(T)$-set. Suppose $deg(v_p) > 1$, then we can find a cycle in T, which is a contradiction. Hence $deg(v_p) = 1$. Since v_p is a pendant vertex we have $T - \{v_p\}$ is also a tree. Then $\langle T - \{v_p\} \rangle$ is triple connected, which follows that $T - \{v_p\} \cong P_{p-1}$ (from [5]) and hence $T \cong P_p$ and given that $\gamma_{etc}(T) = p - 1$, we have $T \cong K_2$. □

Theorem 3.27 *Let T be any tree on $p > 2$ vertices. Then either $\gamma_{etc}(T) = p - 2$ if $T \cong P_p$ or γ_{etc}-set does not exist.*

Proof The proof is divided into cases following.

Case 1. If T contains two pendant vertices. Then $T \cong P_p$ for which $\gamma_{etc}(T) = p - 2$, where $p > 2$.

Case 2. If T contains more than two pendant vertices.

Since any equitable triple connected dominating set must contain all the pendant vertices or its supports and also T is connected and acyclic it follows that γ_{etc}-set does not exist. □

§4. Equitable Triple Connected Domination Edge Addition Stable Graphs

In this section, we consider the problem of finding the stability of γ_{etc} upon edge addition of some classes of graphs such as cycles and complete bipartite graphs.

Definition 4.1 *A connected graph G is said to be an equitable triple connected domination edge addition stable (γ_{etc}-stable), if both G and $G+e$ have the same equitable triple connected domination number, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) = \gamma_{etc}(G+e)$.*

Definition 4.2 *A connected graph G is said to be an equitable triple connected domination edge addition positive critical (γ^+_{etc}- critical), if $G+e$ has greater equitable triple connected domination number than G, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) < \gamma_{etc}(G+e)$.*

Definition 4.3 *A connected graph G is said to be an equitable triple connected domination edge addition negative critical (γ^-_{etc}-critical), if G has greater equitable triple connected domination number than $G+e$, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) > \gamma_{etc}(G+e)$.*

Theorem 4.4 *The cycle $C_p(p>3)$, is γ^-_{etc}-critical.*

Proof Let $C_p = v_1v_2\cdots v_pv_1$ be any cycle of length $p, p>3$. Now $S=\{v_2, v_3, \cdots, v_{p-1}\}$ is the minimum equitable triple connected dominating set, we have $\gamma_{etc}(C_p) = p-2$. Consider C_p+e, where $e = v_iv_j$.

Case 1. Let C_p+e contain $C_3 = v_1v_2v_3v_1$, where $e = v_iv_j = v_3v_1$. Since $S = \{v_3, v_4, \cdots, v_{p-1}\}$ forms a minimum equitable triple connected dominating set, we have $\gamma_{etc}(C_p+e) = p-3$.

Case 2. Let C_p+e does not contain C_3. Let $e = v_iv_j$. Now $S = V(C_p+e)-\{v_{i+1}, v_{i+2}, v_{j+1}, v_{j+2}\}$, where $v_{i+2} = N(v_{i+1})-v_i$ and $v_{j+2} = N(v_{j+1})-v_j$ forms a minimum equitable triple connected dominating set. Hence $\gamma_{etc}(C_p+e) = p-4$.

In both cases $\gamma_{etc}(C_p+e) < \gamma_{etc}(C_p)$. Hence $C_p(p>3)$, is γ^-_{etc}-critical. □

Lemma 4.5 *The complete bipartite graph $K_{1,2}$ is γ_{etc}-stable.*

Lemma 4.6 *The complete bipartite graph $K_{2,2}$ is γ^-_{etc}-critical.*

Lemma 4.7 *The complete bipartite graph $K_{n,n}, (n>2, p=2n)$ is γ_{etc}-stable.*

Proof Let $K_{n,n}, (n>2)$ be a complete bipartite graph and its vertex partition is given by $V = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, \cdots, u_n\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. Now $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n,n}) = 2$. If we add any edge to $K_{n,n}$ there is no change in the equitable triple connected domination number. Hence $K_{n,n}, (n>2)$ is γ_{etc}-stable. □

Lemma 4.8 *If an edge e is added between the vertices of V_1. Then the complete bipartite graph $K_{3,2}$, is γ_{etc}-stable, where $V(K_{3,2}) = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2\}$.*

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 2$. If we add an edge e is added between the vertices of V_1 we see that there is no change in the equitable triple connected domination number. □

Lemma 4.9 *If an edge e is added between the vertices of V_2. Then the complete bipartite graph $K_{3,2}$ is γ^+_{etc}-critical, where $V(K_{3,2}) = V_1 \bigcup V_2$ such that $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2\}$.*

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 2$. By adding an edge between the vertices of V_2, we see that $S = \{u_1, u_2, u_3, v_1, v_2\}$ is a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 5$. □

Lemma 4.10 *If an edge e is added between the vertices of* V_1. *Then the complete bipartite graph* $K_{n+1,n}, (n > 2)$ *is* γ_{etc}*-stable, where* $V(K_{n+1,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 2$. If we add an edge e is added between the vertices of V_1 we see that there is no change in the equitable triple connected domination number. □

Lemma 4.11 *If an edge e is added between the vertices of* V_2. *Then the complete bipartite graph* $K_{n+1,n}, (n > 2)$ *is* γ^+_{etc}*-critical, where* $V(K_{n+1,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 2$. By adding an edge e between the vertices of V_2 say $e = v_1v_2$, we see that $S = \{v_1, u_1, v_i\}$ for $i \neq 2$ is a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 3$. □

Lemma 4.12 *If an edge e is added between the vertices of* V_1. *Then the complete bipartite graph* $K_{n+2,n}, (n > 1)$ *is* γ^-_{etc}*-critical, where* $V(K_{n+2,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p$. By adding an edge e between the vertices of V_1 say $e = u_1u_2$, we see that $S = \{u_3, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p - 2$. □

Lemma 4.13 *If an edge e is added between the vertices of* V_2. *Then the complete bipartite graph* $K_{n+2,n}, (n > 1)$ *is* γ_{etc}*-stable, where* $V(K_{n+2,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p$. By adding an edge e between the vertices of V_2 say $e = v_1v_2$, we see that there is no change in the equitable triple connected domination number. □

Theorem 4.14 *The complete bipartite graph* $K_{m,n}, (m - n > 2$ *and* $m + n = p)$ *is* γ_{etc}*-stable.*

Proof Let $K_{m,n}, (m - n > 2$ and $m + n = p)$ be a complete bipartite graph and its vertex partition is given by $V = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. Now $S = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{m,n}) = p$. If we add any edge to $K_{m,n}$ there is no change in the equitable triple connected domination number. □

§5. Conclusion

We conclude this paper by giving the following interesting open problems for further study:

(1) Characterize connected graphs of order p for which $\gamma_{etc} = p$.

(2) For which graphs, $\gamma_e = \gamma_{ec} = \gamma_{etc} = p$.

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Path Related n-Cap Cordial Graphs

A. Nellai Murugan

(Department of Mathematics, V.O.Chidambaram College, Tamil Nadu, India)

P. Iyadurai Selvaraj

(Department of Computer Science, V.O.Chidambaram College, Tamil Nadu, India)

E-mail: anellai.vocc@gmail.com, iyaduraiselvaraj@gmail.com

Abstract: Let G = (V,E) be a graph with p vertices and q edges. A *n-cap* $\overline{(\bigwedge)}$ cordial labeling of a graph G with vertex set V is a bijection from V to {0,1} such that if each edge uv is assigned the label

$$f(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 1 \\ 1, & \text{otherwise.} \end{cases}$$

with the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. The graph that admits a $\overline{\bigwedge}$ cordial labeling is called a $\overline{\bigwedge}$ cordial graph (nCCG). In this paper, we proved that Path P_n, Comb $(P_n \odot K_1)$, $P_m \odot 2K_1$ and Fan $(F_n = P_n + K_1)$ are $\overline{\bigwedge}$ cordial graphs.

Key Words: $\overline{\bigwedge}$ cordial labeling, Smarandachely cordial labeling, $\overline{\bigwedge}$ cordial labeling graph.

AMS(2010): 05C78.

§1. Introduction

A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G which is called edges. Each pair $e = \{uv\}$ of vertices in E is called an edge or a line of G. In this paper, we proved that Path P_n, Comb $(P_n \odot_1)$, $P_m \odot 2K_1$ and Fan $(F_n = P_n + K_1)$ are $\overline{\bigwedge}$ cordial graphs.

§2. Preliminaries

Let $G = (V, E)$ be a graph with p vertices and q edges. A *n-cap* $(\overline{\bigwedge})$ cordial labeling of a graph G with vertex set V is a bijection from V to $\{0, 1\}$ such that if each edge uv is assigned the

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label

$$f(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 1 \\ 1, & \text{otherwise.} \end{cases}$$

with the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1, and it is said to be a Smarandachely cordial labeling if the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at least 1 and the number of edges labeled with 0 or the number of edges labeled with 1 differ by at least 1.

The graph that admits a $\overline{\bigwedge}$ cordial labeling is called a $\overline{\bigwedge}$ cordial graph. We proved that Path P_n, Comb $(P_n \odot K_1)$, $P_m \odot 2K_1$ and Fan $(F_n = P_n + K_1)$ are $\overline{\bigwedge}$ cordial graphs.

Definition 2.1 *A path is a graph with sequence of vertices $u_1, u_2, \cdots, u_n$ such that successive vertices are joined with an edge, denoted by P_n, which is a path of length $n-1$.*

A closed path of length n is cycle C_n.

Definition 2.2 *A comb is a graph obtained from a path P_n by joining a pendent vertex to each vertices of P_n, it is denoted by $P_n \odot K_1$*

Definition 2.3 *A graph obtained from a path P_m by joining two pendent vertices at each vertices of P_m is denoted by $P_m \odot 2K_1$*

Definition 2.4 *A fan is a graph obtained from a path P_n by joining each vertices of P_n to a pendent vertex, it is denoted by $F_n = P_n + K_1$*

§3. Main Results

Theorem 3.1 *A path P_n is a $\overline{\bigwedge}$ cordial graph*

Proof Let $V(P_n) = \{u_i : 1 \le i \le n\}$ and $E(P_n) = \{u_i u_{i+1} : 1 \le i \le n-1\}$ Define $f : V(P_n) \to \{0,1\}$ with the vertex labeling determined following.

Case 1. n is odd.

Define

$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2}, \\ 1, & \frac{n+1}{2} \le i \le n. \end{cases}$$

The induced edge labeling are

$$f^*(u_i u_{i+1}) = \begin{cases} 1, & 1 \le i \le \frac{n}{2}, \\ 0, & \frac{n}{2} \le i \le n. \end{cases}$$

Here $V_0(f)+1 = V_1(f)$ and $e_0(f) = e_1(f)$. Clearly, it satisfies the condition $|V_0(f) - V_1(f)| \le 1$ and $|e_0(f) - e_1(f)| \le 1$.

Case 2. n is even.

Define

$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2}, \\ 1, & \frac{n}{2}+1 \le i \le n. \end{cases}$$

The induced edge labeling are

$$f^*(u_i u_{i+1}) = \begin{cases} 1, & 1 \le i \le \frac{n}{2}, \\ 0, & \frac{n}{2}+1 \le i \le n. \end{cases}$$

Here $V_0(f) = V_1(f)$ and $e_0(f)+1 = e_1(f)$ which satisfies the condition $|V_0(f) - V_1(f)| \le 1$ and $|e_0(f) - e_1(f)| \le 1$. Hence, a path P_n is a $\overline{\wedge}$ cordial graph. □

For example, P_5 and P_6 are $\overline{\wedge}$ cordial graph shown in the Figure 1.

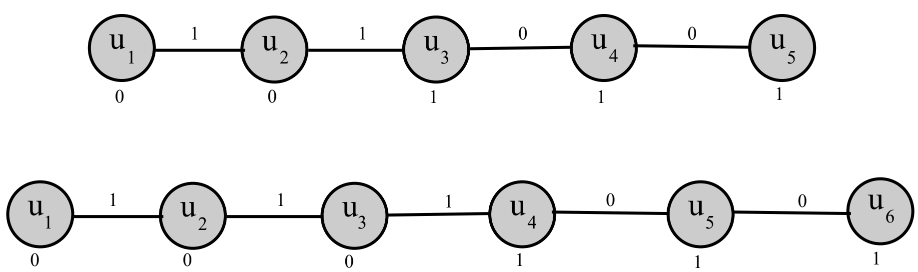

Figure 1

Theorem 3.2 *A comb $P_n \odot K_1$ is a $\overline{\wedge}$ cordial graph*

Proof Let G be a comb $P_n \odot K_1$ and let $V(G) = \{(u_i, v_i) : 1 \le i \le n\}$ and $E(G) = \{[(u_i u_{i+1}) : 1 \le i \le n-1] \bigcup [(u_i v_i) : 1 \le i \le n]\}$. Define $f : V(G) \to \{0, 1\}$ with a vertex labeling

$$\begin{aligned} f(u_i) &= 1,\ 1 \le i \le n, \\ f(v_i) &= 0,\ 1 \le i \le n. \end{aligned}$$

The induced edge labeling are

$$\begin{aligned} f^*(u_i u_{i+1}) &= 1,\ 1 \le i < n, \\ f^*(u_i v_i) &= 0,\ 1 \le i \le n. \end{aligned}$$

Here $V_0(f) = V_1(f)$ and $e_0(f) = e_1(f)+1$ which satisfies the condition $|V_0(f) - V_1(f)| \le 1$ and $|e_0(f) - e_1(f)| \le 1$. Hence, a comb $P_n \odot K_1$ is a $\overline{\wedge}$ cordial graph. □

For example, $P_5 \odot K_1$ is a $\overline{\wedge}$ cordial graph shown in Figure 2.

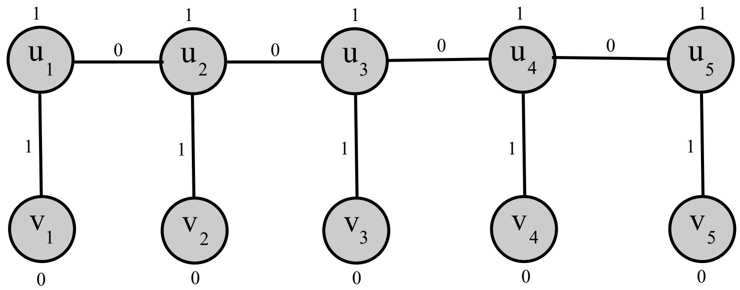

Figure 2

Theorem 3.3 *A graph $P_m \odot 2K_1$ is a $\overline{\bigwedge}$ cordial graph.*

Proof Let G be a $P_m \odot 2K_1$ with $V(G) = \{u_i, v_{1i}, v_{2i}, 1 \leq i \leq n\}$ and $E(G) = \{[(u_i u_{i+1}) : 1 \leq i < n] \bigcup [(u_i v_{1i}) : 1 \leq i \leq n] \bigcup [(u_i v_{2i}) : 1 \leq i \leq n]\}$. Define $f : V(C_n) \to \{0, 1\}$ by a vertex labeling $f(u_i) = \{1, 1 \leq i \leq n\}$, $f(v_{1i}) = \{0, 1 \leq i \leq n\}$ and if n is even,

$$f(v_{2i}) = \begin{cases} 1, & 1 \leq i \leq \frac{n}{2}, \\ 0, & \frac{n}{2} + 1 \leq i \leq n, \end{cases}$$

if n is odd

$$f(v_{2i}) = \begin{cases} 1, & 1 \leq i \leq \frac{n+1}{2}, \\ 0, & \frac{n+1}{2} + 1 \leq i \leq n. \end{cases}$$

The induced edge labeling are

$$\begin{aligned} f^*(u_i u_{i+1}) &= \{0, 1 \leq i \leq n\}, \\ f^*(u_i v_{1i}) &= \{1, 1 \leq i \leq n\} \end{aligned}$$

and if n is even

$$f^*(u_i v_{2i}) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2}, \\ 1, & \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Here $V_0(f) = V_1(f)$ and $e_0(f) + 1 = e_1(f)$ which satisfies the condition $|V_0(f) - V_1(f)| \leq 1$ and $|e_0(f) - e_1(f)| \leq 1$, and if n is odd

$$f^*(u_i v_{2i}) = \begin{cases} 0, & 1 \leq i \leq \frac{n+1}{2}, \\ 1, & \frac{n+1}{2} \leq i \leq n. \end{cases}$$

Here $V_0(f) + 1 = V_1(f)$ and $e_0(f) = e_1(f)$ which satisfies the condition $|V_0(f) - V_1(f)| \leq 1$ and $|e_0(f) - e_1(f)| \leq 1$. Hence, $P_m \odot 2K_1$ is a $\overline{\bigwedge}$ cordial graph. □

For example, $P_5 \odot 2K_1$ is a $\overline{\bigwedge}$ cordial graph shown in the Figures 3.

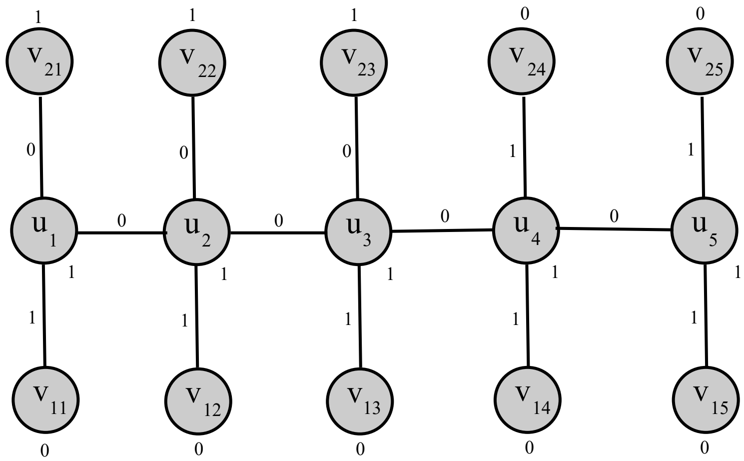

Figure 3

Theorem 3.4 *A fan $F_n = P_n + K_1$ is a $\overline{\bigwedge}$ cordial graph if n is even.*

Proof Let G be a fan $F_n = P_n + K_1$ and n is even with $V(G) = \{u, v_i : 1 \leqslant i \leqslant n\}$ and $E(G) = \{(u, v_i) : 1 \leqslant i \leqslant n\}$. Define $f : V(G) \to \{0, 1\}$ with a vertex labeling $f(u) = \{1\}$ and

$$f(v_i) = \begin{cases} 1, & 1 \leq i \leq \frac{n}{2}, \\ 0, & \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

The induced edge labeling are

$$f^*(uv_i) = \begin{array}{l} 0,\ 1 \leq i \leq \frac{n}{2}, \\ 1,\ \frac{n}{2} + 1 \leq i \leq n, \end{array} \qquad \text{and} \qquad f^*(v_i v_{i+1}) = \begin{array}{l} 0,\ 1 \leq i \leq \frac{n}{2}, \\ 1,\ \frac{n}{2} \leq i \leq n. \end{array}$$

Here $V_0(f) + 1 = V_1(f)$ and $e_0(f) + 1 = e_1(f)$ which satisfies the conditions $|V_0(f) - V_1(f)| \leq 1$ and $|e_0(f) - e_1(f)| \leq 1$. Hence, a fan $F_n = P_n + K_1$ is a $\overline{\bigwedge}$ cordial graph if n is even. □

For example, a fan $F_6 = P_6 + K_1$ is $\overline{\bigwedge}$ cordial shown in Figure 4.

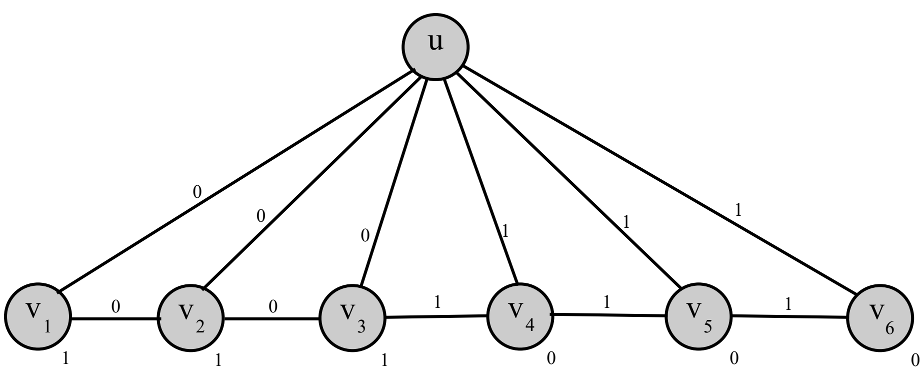

Figure 4

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Some New Families of 4-Prime Cordial Graphs

R.Ponraj

(Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India)

Rajpal Singh and R.Kala

(Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, India)

E-mail: ponrajmaths@gmail.com, rajpalsinh@outlook.com, karthipyi91@yahoo.co.in

Abstract: Let G be a (p,q) graph. Let $f : V(G) \to \{1,2,\ldots,k\}$ be a function. For each edge uv, assign the label $\gcd(f(u),f(v))$. f is called k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i,j \in \{1,2,\ldots,k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with admits a k-prime cordial labeling is called a k-prime cordial graph. In this paper we investigate 4-prime cordial labeling behavior of shadow graph of a path, cycle, star, degree splitting graph of a bistar, jelly fish, splitting graph of a path and star.

Key Words: Cordial labeling, Smarandachely cordial labeling, cycle, star, bistar, splitting graph.

AMS(2010): 05C78.

§1. Introduction

In this paper graphs are finite, simple and undirected. Let G be a (p,q) graph where p is the number of vertices of G and q is the number of edge of G. In 1987, Cahit introduced the concept of cordial labeling of graphs [1]. Sundaram, Ponraj, Somasundaram [5] have been introduced the notion of prime cordial labeling and discussed the prime cordial labeling behavior of various graphs. Recently Ponraj et al. [7], introduced k-prime cordial labeling of graphs. A 2-prime cordial labeling is a product cordial labeling [6]. In [8, 9] Ponraj et al. studied the 4-prime cordial labeling behavior of complete graph, book, flower, mC_n, wheel, gear, double cone, helm, closed helm, butterfly graph, and friendship graph and some more graphs. Ponraj and Rajpal singh have studied about the 4-prime cordiality of union of two bipartite graphs, union of trees, durer graph, tietze graph, planar grid $P_m \times P_n$, subdivision of wheels and subdivision of helms, lotus inside a circle, sunflower graph and they have obtained some 4-prime cordial graphs from 4-prime cordial graphs [10, 11, 12]. Let x be any real number. In this paper we have studied about the 4-prime cordiality of shadow graph of a path, cycle, star, degree splitting graph of a

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bistar, jelly fish, splitting graph of a path and star. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here follow from Harary [3] and Gallian [2].

§2. k-Prime Cordial Labeling

Let G be a (p,q) graph. Let $f : V(G) \to \{1,2,\cdots,k\}$ be a map. For each edge uv, assign the label $\gcd(f(u),f(v))$. f is called k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i,j \in \{1,2,\cdots,k\}$ and $|e_f(0) - e_f(1)| \leq 1$, and conversely, if $|v_f(i) - v_f(j)| \geq 1$, $i,j \in \{1,2,\cdots,k\}$ or $|e_f(0) - e_f(1)| \geq 1$, it is called a Smarandachely cordial labeling, where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is called a k-prime cordial graph.

First we investigate the 4-prime cordiality of shadow graph of a path, cycle and star. A shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 2.1 *$D_2(P_n)$ is 4-prime cordial if and only if $n \neq 2$.*

Proof It is easy to see that $D_2(P_2)$ is not 4-prime cordial. Consider $n > 2$. Let $V(D_2(P_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(D_2(P_n)) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_{i+1}, v_iu_{i+1} : 1 \leq i \leq n-1\}$. In a shadow graph of a path, $D_2(P_n)$, there are $2n$ vertices and $4n-4$ edges.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_{2t}$ then assign 2 to the vertices $v_1, v_2, \cdots, v_{2t}$. For the vertices v_{2t+1}, v_{2t+2}, we assign $3,1$ respectively. Put the label 1 to the vertices $v_{2t+3}, v_{2t+5}, \cdots, v_{4t-1}$. Now we assign the label 3 to the vertices $v_{2t+4}, v_{2t+6}, \cdots, v_{4t-2}$. Then assign the label 1 to the vertex v_{4t}. Next we consider the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{4t}$. Put $3,3$ to the vertices u_{2t+1}, u_{2t+2}. Then fix the number 1 to the vertices $u_{2t+3}, u_{2t+5}, \cdots, u_{4t-1}$. Finally assign the label 3 to the vertices $u_{2t+4}, u_{2t+6}, \cdots, u_{4t}$.

Case 2. $n \equiv 1 \pmod 4$.

Take $n = 4t+1$. Assign the label 4 to the vertices $u_1, u_2, \cdots, u_{2t+1}$. Then assign the label 3 to the vertices $u_{2t+2}, u_{2t+4}, \cdots, u_{4t}$ and put the number 1 to the vertices $u_{2t+3}, u_{2t+5}, \cdots, u_{4t+1}$. Next we turn to the vertices $v_1, v_2, \cdots, v_{2t+1}$. Assign the label 2 to the vertices $v_1, v_2, \cdots, v_{2t+1}$. The remaining vertices v_i $(2t+2 \leq i \leq 4t+1)$ are labeled as in u_i $(2t+2 \leq i \leq 4t+1)$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t+2$. Assign the labels to the vertices u_i, v_i $(1 \leq i \leq 2t+1)$ as in case 2. Now we consider the vertices $u_{2t+2}, u_{2t+3}, \cdots, u_{4t+2}$. Assign the labels $3,1$ to the vertices u_{2t+2}, u_{2t+3} respectively. Then assign the label 1 to the vertices $u_{2t+4}, u_{2t+6}, \cdots, u_{4t+2}$. Put the integer 3 to the vertices $u_{2t+5}, u_{2t+7}, \cdots, u_{4t+1}$. Now we turn to the vertices $v_{2t+2}, v_{2t+3}, \cdots, v_{4t+2}$. Put the labels $3,3,1$ to the vertices $v_{2t+2}, v_{2t+3}, v_{2t+4}$ respectively. The remaining vertices

$v_i\,(2t+5 \leq i \leq 4t+2)$ are labeled as in $u_i\,(2t+5 \leq i \leq 4t+2)$.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t+3$. Assign the label 2 to the vertices $u_i\,(1 \leq i \leq 2t+2)$. Then put the number 3 to the vertices $u_{2t+3}, u_{2t+5}, \cdots, u_{4t+1}$. Then assign 1 to the vertices $u_{2t+4}, u_{2t+6}, \cdots, u_{4t+2}$ and u_{4t+3}. Now we turn to the vertices $v_1, v_2, \cdots, v_{4t+3}$. Assign the label 4 to the vertices $v_i\,(1 \leq i \leq 2t+2)$. The remaining vertices $v_i\,(2t+3 \leq i \leq 4t+3)$ are labeled as in $u_i\,(2t+3 \leq i \leq 4t+3)$. Then relabel the vertex v_{4t+3} by 3.

The vertex and edge conditions of the above labeling is given in Table 1.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$2n-2$	$2n-2$
$n \equiv 1 \pmod 2$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$2n-2$	$2n-2$

Table 1

It follows that $D_2(P_n)$ is a 4-prime cordial graph for $n \neq 2$. □

Theorem 2.2 *$D_2(C_n)$ is 4-prime cordial if and only if $n \geq 7$.*

Proof Let $V(D_2(C_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(D_2(C_n)) = \{u_iu_{i+1},\ v_iv_{i+1},\ u_iv_{i+1},\ v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1,\ v_nu_1,\ u_nu_1,\ v_nv_1\}$. Clearly $D_2(C_n)$ consists of $2n$ vertices and $4n$ edges. We consider the following cases.

Case 1. $n \equiv 0 \pmod 4$.

One can easily check that $D_2(C_4)$ can not have a 4-prime cordial labeling. Define a vertex labeling f from the vertices of $D_2(C_n)$ to the set of first four consecutive positive integers as given below.

$$\begin{array}{lclcll} f(v_{2i}) &=& f(u_{2i-1}) &=& 2, & 1 \leq i \leq \frac{n}{4} \\ f(v_{2i+1}) &=& f(u_{2i}) &=& 4, & 1 \leq i \leq \frac{n}{4} \\ f(v_{\frac{n}{2}+2+2i}) &=& f(u_{\frac{n}{2}+2+2i}) &=& 1, & 1 \leq i \leq \frac{n-4}{4} \\ f(v_{\frac{n}{2}+3+2i}) &=& f(u_{\frac{n}{2}+3+2i}) &=& 3, & 1 \leq i \leq \frac{n-8}{4} \end{array}$$

$f(u_{\frac{n}{2}+1}) = f(u_{\frac{n}{2}+2}) = f(u_{\frac{n}{2}+3}) = f(v_{\frac{n}{2}+2}) = 3$ and $f(v_1) = f(v_{\frac{n}{2}+3}) = 1$.

Case 2. $n \equiv 1 \pmod 4$.

It is easy to verify that $D_2(C_5)$ is not a prime graph. Now we construct a map $f : V(D_2(C_n)) \rightarrow \{1, 2, 3, 4\}$ as follows:

$$\begin{array}{lclcll} & & f(u_{2i-1}) &=& 2, & 1 \leq i \leq \frac{n+3}{4} \\ & & f(u_{2i}) &=& 4, & 1 \leq i \leq \frac{n+3}{4} \\ & & f(v_{2i}) &=& 2, & 1 \leq i \leq \frac{n-1}{4} \\ & & f(v_{2i+1}) &=& 4, & 1 \leq i \leq \frac{n-1}{4} \\ f(v_{\frac{n+5}{2}+2i}) &=& f(u_{\frac{n+5}{2}+2i}) &=& 1, & 1 \leq i \leq \frac{n-5}{4} \\ f(v_{\frac{n+7}{2}+2i}) &=& f(u_{\frac{n+7}{2}+2i}) &=& 3, & 1 \leq i \leq \frac{n-9}{4} \end{array}$$

$f(v_1) = f(v_{\frac{n+5}{2}}) = f(u_{\frac{n+5}{2}}) = f(u_{\frac{n+7}{2}}) = 3$ and $f(v_{\frac{n+7}{2}}) = f(v_{\frac{n+3}{2}}) = 1$.

Case 3. $n \equiv 2 \pmod 4$.

Obviously $D_2(C_6)$ does not permit a 4-prime cordial labeling. For $n \neq 6$, we define a function f from $V(D_2(C_n))$ to the set $\{1, 2, 3, 4\}$ by

$$\begin{array}{rcrcll} & & f(u_{2i-1}) & = & 2, & 1 \le i \le \frac{n+2}{4} \\ & & f(u_{2i}) & = & 4, & 1 \le i \le \frac{n+2}{4} \\ & & f(v_{2i}) & = & 2, & 1 \le i \le \frac{n-2}{4} \\ & & f(v_{2i+1}) & = & 4, & 1 \le i \le \frac{n-2}{4} \\ f(v_{\frac{n+6}{2}+2i}) & = & f(u_{\frac{n+6}{2}+2i}) & = & 1, & 1 \le i \le \frac{n-6}{4} \\ f(v_{\frac{n+8}{2}+2i}) & = & f(u_{\frac{n+8}{2}+2i}) & = & 3, & 1 \le i \le \frac{n-8}{4} \end{array}$$

and

$$f(v_1) = f(v_{\frac{n+6}{2}}) = f(u_{\frac{n+4}{2}}) = f(u_{\frac{n+6}{2}}) = f(u_{\frac{n+8}{2}}) = 3,$$
$$f(v_{\frac{n+2}{2}}) = f(v_{\frac{n+4}{2}}) = f(v_{\frac{n+8}{2}}) = 1.$$

Case 4. $n \equiv 3 \pmod 4$.

Clearly $D_2(C_3)$ is not a 4-prime cordial graph. Let $n \neq 3$. Define a map $f : V(D_2(C_n)) \to \{1, 2, 3, 4\}$ by $f(v_1) = 1$,

$$\begin{array}{rcrcll} f(v_{2i}) & = & f(u_{2i-1}) & = & 2, & 1 \le i \le \frac{n+1}{4} \\ f(v_{2i+1}) & = & f(u_{2i}) & = & 4, & 1 \le i \le \frac{n+1}{4} \\ f(v_{\frac{n+3}{2}+2i}) & = & f(u_{\frac{n+3}{2}+2i}) & = & 1, & 1 \le i \le \frac{n-3}{4} \\ f(v_{\frac{n+5}{2}+2i}) & = & f(u_{\frac{n+5}{2}+2i}) & = & 3, & 1 \le i \le \frac{n-5}{4} \end{array}$$

and $f(u_{\frac{n+3}{2}}) = f(u_{\frac{n+5}{2}}) = f(v_{\frac{n+5}{2}}) = 3$. The Table 2 gives the vertex and edge condition of f.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$2n$	$2n$
$n \equiv 1, 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$2n$	$2n$

Table 2

It follows that $D_2(C_n)$ is 4-prime cordial iff $n \ge 7$. □

Example 2.1 A 4-prime cordial labeling of $D_2(C_9)$ is given in Figure 1.

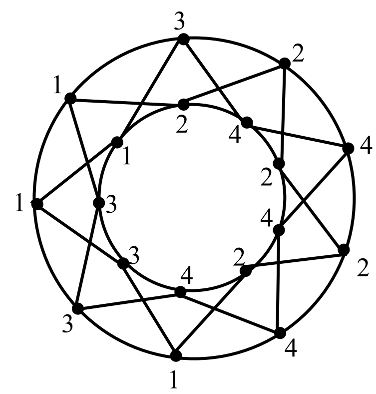

Figure 1

Theorem 2.3 *$D_2(K_{1,n})$ is 4-prime cordial if and only if $n \equiv 0 \pmod 2$.*

Proof It is clear that $D_2(K_{1,n})$ has $2n+2$ vertices and $4n$ edges. Let $V(D_2(K_{1,n})) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(D_2(K_{1,n})) = \{uu_i, vv_i, vu_i, uv_i : 1 \le i \le n\}$.

Case 1. $n \equiv 0 \pmod 2$.

Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}+1}$. Then assign 4 to the vertices $u_{\frac{n}{2}+2}, \cdots, u_n, u, v$. Now we move to the vertices v_i where $1 \le i \le n$. Assign the label 3 to the vertices v_i $(1 \le i \le \frac{n}{2})$ then the remaining vertices are labeled with 1. In this case $v_f(1) = v_f(3) = \frac{n}{2}$, $v_f(2) = v_f(4) = \frac{n}{2} + 1$ and $e_f(0) = e_f(1) = 2n$.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2t+1$. Suppose there exists a 4-prime cordial labeling g, then $v_g(1) = v_g(2) = v_g(3) = v_g(4) = t+1$.

Subcase 2a. $g(u) = g(v) = 1$.

Here $e_g(0) = 0$, a contradiction.

Subcase 2b. $g(u) = g(v) = 2$.

In this case $e_g(0) \le (t-1) + (t-1) + (t+1) + (t+1) = 4t$, a contradiction.

Subcase 2c. $g(u) = g(v) = 3$.

Then $e_g(0) \le (t-1) + (t-1) = 2t-2$, a contradiction.

Subcase 2d. $g(u) = g(v) = 4$.

Similar to Subcase 2b.

Subcase 2e. $g(u) = 2$, $g(v) = 4$ or $g(v) = 2$, $g(u) = 4$.

Here $e_g(0) \le t + t + t + t = 4t$, a contradiction.

Subcase 2f. $g(u) = 2$, $g(v) = 3$ or $g(v) = 2$, $g(u) = 3$.

Here $e_g(0) \le (t+1) + t + t = 3t+1$, a contradiction.

Subcase 2g. $g(u) = 4$, $g(v) = 3$ or $g(v) = 4$, $g(u) = 3$.

Similar to Subcase 2f.

Subcase 2h. $g(u) = 2$, $g(v) = 1$ or $g(v) = 2$, $g(u) = 1$.

Similar to Subcase 2f.

Subcase 2i. $g(u) = 4$, $g(v) = 1$ or $g(v) = 4$, $g(u) = 1$.

Similar to Subcase 2h.

Subcase 2j. $g(u) = 3$, $g(v) = 1$ or $g(v) = 3$, $g(u) = 1$.

In this case $e_g(0) \leq t$, a contradiction.

Hence, if $n \equiv 1 \pmod 2$, $D_2(K_{1,n})$ is not a 4-prime cordial graph. □

The next investigation is about 4-prime cordial labeling behavior of splitting graph of a path, star. For a graph G, the splitting graph of G, $S'(G)$, is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v. Note that if G is a (p, q) graph then $S'(G)$ is a $(2p, 3q)$ graph.

Theorem 2.4 *$S'(P_n)$ is 4-prime cordial for all n.*

Proof Let $V(S'(P_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(S'(P_n)) = \{u_iu_{i+1}, u_iv_{i+1}, v_iu_{i+1} : 1 \leq i \leq n-1\}$. Clearly $S'(P_n)$ has $2n$ vertices and $3n-3$ edges. Figure 2 shows that $S'(P_2)$, $S'(P_3)$ are 4-prime cordial.

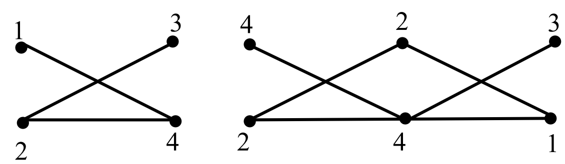

Figure 2

For $n > 3$, we consider the following cases.

Case 1. $n \equiv 0 \pmod 4$.

We define a function f from the vertices of $S'(P_n)$ to the set $\{1, 2, 3, 4\}$ by

$$\begin{array}{lllll} f(v_{2i}) & = & f(u_{2i-1}) & = & 2, \quad 1 \leq i \leq \frac{n}{4} \\ f(v_{2i+1}) & = & f(u_{2i}) & = & 4, \quad 1 \leq i \leq \frac{n}{4} \\ f(v_{\frac{n+2}{2}+2i}) & = & f(u_{\frac{n+2}{2}+2i}) & = & 1, \quad 1 \leq i \leq \frac{n-4}{4} \\ f(v_{\frac{n+4}{2}+2i}) & = & f(u_{\frac{n+4}{2}+2i}) & = & 3, \quad 1 \leq i \leq \frac{n-4}{4} \end{array}$$

and $f(u_{\frac{n+2}{2}}) = f(u_{\frac{n+4}{2}}) = 3$, $f(v_1) = f(v_{\frac{n+4}{2}}) = 1$.

In this case $v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{n}{2}$, and $e_f(0) = \frac{3n-4}{2}$, $e_f(1) = \frac{3n-2}{2}$.

Case 2. $n \equiv 1 \pmod 4$.

We define a map $f : V(S'(P_n)) \to \{1, 2, 3, 4\}$ by

$$\begin{array}{lllll} & & f(u_{2i-1}) & = & 2, \quad 1 \le i \le \frac{n+3}{4} \\ & & f(u_{2i}) & = & 4, \quad 1 \le i \le \frac{n-1}{4} \\ & & f(v_{2i-1}) & = & 4, \quad 1 \le i \le \frac{n+3}{4} \\ & & f(v_{2i}) & = & 2, \quad 1 \le i \le \frac{n-1}{4} \\ f(v_{\frac{n-1}{2}+2i}) & = & f(u_{\frac{n-1}{2}+2i}) & = & 3, \quad 1 \le i \le \frac{n-1}{4} \\ f(v_{\frac{n+1}{2}+2i}) & = & f(u_{\frac{n+1}{2}+2i}) & = & 1, \quad 1 \le i \le \frac{n-1}{4} \end{array}$$

Here $v_f(1) = v_f(3) = \frac{n-1}{2}$, $v_f(2) = v_f(4) = \frac{n+1}{2}$, and $e_f(0) = e_f(1) = \frac{3n-3}{2}$.

Case 3. $n \equiv 2 \pmod 4$.

Define a vertex labeling $f : V(S'(P_n)) \to \{1, 2, 3, 4\}$ by $f(v_1) = 3$, $f(v_{\frac{n}{2}+1}) = 1$,

$$\begin{array}{lllll} & & f(u_{2i-1}) & = & 2, \quad 1 \le i \le \frac{n+2}{4} \\ & & f(u_{2i}) & = & 4, \quad 1 \le i \le \frac{n+2}{4} \\ & & f(v_{2i}) & = & 2, \quad 1 \le i \le \frac{n-2}{4} \\ & & f(v_{2i+1}) & = & 4, \quad 1 \le i \le \frac{n-2}{4} \\ f(v_{\frac{n}{2}+2i}) & = & f(u_{\frac{n}{2}+2i}) & = & 3, \quad 1 \le i \le \frac{n-2}{4} \\ f(v_{\frac{n+2}{2}+2i}) & = & f(u_{\frac{n+2}{2}+2i}) & = & 1, \quad 1 \le i \le \frac{n-2}{4} \end{array}$$

Here $v_f(1) = v_f(2) = v_f(3) = v_f(4) = \frac{n}{2}$, and $e_f(0) = \frac{3n-4}{2}$, $e_f(1) = \frac{3n-2}{2}$.

Case 4. $n \equiv 3 \pmod 4$.

Construct a vertex labeling f from the vertices of $S'(P_n)$ to the set $\{1, 2, 3, 4\}$ by $f(u_n) = 1$, $f(v_n) = 3$,

$$\begin{array}{lllll} f(v_{2i}) & = & f(u_{2i-1}) & = & 2, \quad 1 \le i \le \frac{n+1}{4} \\ f(v_{2i-1}) & = & f(u_{2i}) & = & 4, \quad 1 \le i \le \frac{n+1}{4} \\ f(v_{\frac{n-1}{2}+2i}) & = & f(u_{\frac{n-1}{2}+2i}) & = & 3, \quad 1 \le i \le \frac{n-3}{4} \\ f(v_{\frac{n+1}{2}+2i}) & = & f(u_{\frac{n+1}{2}+2i}) & = & 1, \quad 1 \le i \le \frac{n-3}{4} \end{array}$$

In this case $v_f(1) = v_f(3) = \frac{n-1}{2}$, $v_f(2) = v_f(4) = \frac{n+1}{2}$, and $e_f(0) = e_f(1) = \frac{3n-3}{2}$.

Hence $S'(P_n)$ is 4-prime cordial for all n. □

Theorem 2.5 *$S'(K_{1,n})$ is 4-prime cordial for all n.*

Proof Let $V(S'(K_{1,n})) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(S'(K_{1,n})) = \{uu_i, vu_i, uv_i : 1 \le i \le n\}$. Clearly $S'(K_{1,n})$ has $2n+2$ vertices and $3n$ edges. The Figure 3 shows that $S'(K_{1,2})$ is a 4-prime cordial graph.

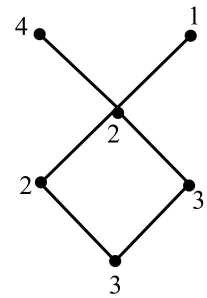

Figure 3

Now for $n > 2$, we define a map $f : V(S'(K_{1,n})) \to \{1, 2, 3, 4\}$ by $f(u) = 2$, $f(v) = 3$, $f(u_n) = 1$,

$$\begin{array}{lcl} f(u_i) & = & 2, \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ f(u_{\lfloor \frac{n}{2} \rfloor + i}) & = & 3, \quad 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor \\ f(v_i) & = & 4, \quad 1 \le i \le \left\lceil \frac{n+1}{2} \right\rceil \\ f(v_{\lceil \frac{n+1}{2} \rceil + i}) & = & 1, \quad 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor \end{array}$$

The Table 3 shows that f is a 4-prime cordial labeling of $S'(K_{1,n})$. □

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \bmod 2$	$\frac{n}{2}$	$\frac{n}{2} + 1$	$\frac{n}{2}$	$\frac{n}{2} + 1$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \bmod 2$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 3

Next we investigate the 4-prime cordial behavior of degree splitting graph of a star. Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V - \bigcup_{i=1}^{t} S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2 \cdots, w_t$ and joining w_i to each vertex of S_i $(1 \le i \le t)$.

Theorem 2.6 *$DS(B_{n,n})$ is 4-prime cordial if* $n \equiv 1, 3 \pmod 4$.

Proof Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \le i \le n\}$. Let $V(DS(B_{n,n})) = V(B_{n,n}) \cup \{w_1, w_2\}$ and $E(DS(B_{n,n})) = E(B_{n,n}) \cup \{w_1u_i, w_1v_i, w_2u, w_2v : 1 \le i \le n\}$. Clearly $DS(B_{n,n})$ has $2n + 4$ vertices and $4n + 3$ edges.

Case 1. $n \equiv 1 \pmod 4$.

Let $n = 4t + 1$. Assign the label 3 to the vertices $v_1, v_2, \cdots, v_{2t+1}$ and 1 to the vertices $v_{2t+2}, v_{2t+3}, \cdots, v_{4t+1}$. Next assign the label 4 to the vertices $u_1, u_2, \cdots, u_{2t+2}$ and 2 to the vertices $u_{2t+3}, u_{2t+4}, \cdots, u_{4t+1}$. Finally, assign the labels $1, 2, 2$ and 2 to the vertices w_2, u, v and w_1 respectively.

Case 2. $n \equiv 3 \pmod 4$.

As in case 1 assign the labels to the vertices u_i, v_i, u, v, w_1 and w_2 $(1 \le i \le n - 2)$. Next assign the labels $1, 3, 2$ and 4 respectively to the vertices v_{n-1}, v_n, u_{n-1} and u_n. The Table 4 establishes that this vertex labeling f is a 4-prime cordial labeling. □

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$4t+1$	$2t+1$	$2t+2$	$2t+1$	$2t+2$	$8t+3$	$8t+4$
$4t+3$	$2t+2$	$2t+3$	$2t+2$	$2t+3$	$8t+7$	$8t+8$

Table 4

The final investigation is about 4-prime cordiality of jelly fish graph.

Theorem 2.7 *The Jelly fish $J(n, n)$ is 4-prime cordial.*

Proof Let $V(J(n, n)) = \{u, v, u_i, v_i, w_1, w_2 : 1 \leq i \leq n\}$ and $E(J(n, n))$ =$\{uu_i,\ vu_i,\ uw_1,$ $w_1v,\ vw_2,\ uw_2,\ w_1w_2 : 1 \leq i \leq n\}$. Note that $J(n, n)$ has $2n + 4$ vertices and $2n + 5$ edges.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$. Assign the label 1 to the vertices $u_1, u_2, \cdots, u_{2t+1}$. Next assign the label 3 to the vertices $u_{2t+2}, u_{2t+3}, \cdots, u_{4t}$. We now move to the other side pendent vertices. Assign the label 3 to the vertices u_1, u_2. Next assign the label 2 to the vertices $u_3, u_4, \ldots, u_{2t+3}$. Then assign the label 4 to the remaining pendent vertices. Finally assign the label 4 to the vertices u, v, w_1, w_2.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$. In this case, assign the label 1 to the vertices $v_1, v_2, \cdots, v_{2t+1}$ and 3 to the vertices $v_{2t+1}, v_{2t+3}, \cdots, v_{4t+1}$. Next assign the label 2 to the vertices $u_1, u_2, \cdots, u_{2t+2}$, and 3 to the vertices u_{2t+3} and u_{2t+4}. Next assign the label 4 to the remaining pendent vertices $u_{2t+5}, u_{2t+6}, \cdots, u_{4t+1}$. Finally assign the label 4 to the vertices u, v, w_1, w_2.

Case 3. $n \equiv 2 \pmod 4$.

As in Case 2, assign the label to the vertices $u_i, v_i (1 \leq i \leq n-1), u, v, w_1, w_2$. Next assign the labels $1, 4$ respectively to the vertices u_n and v_n.

Case 4. $n \equiv 3 \pmod 4$.

Assign the labels to the vertices $u, v, w_1, w_2, u_i, v_i (1 \leq i \leq n-1)$ as in case 3. Finally assign the labels $2, 1$ respectively to the vertices u_n, v_n. The Table 5establishes that this vertex labeling f is obviously a 4-prime cordial labeling. □

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$4t$	$2t+1$	$2t+1$	$2t+1$	$2t+1$	$4t+3$	$4t+2$
$4t+1$	$2t+1$	$2t+2$	$2t+2$	$2t+1$	$4t+4$	$4t+3$
$4t+2$	$2t+2$	$2t+2$	$2t+2$	$2t+2$	$4t+5$	$4t+4$
$4t+3$	$2t+3$	$2t+3$	$2t+2$	$2t+2$	$4t+6$	$4t+5$

Table 5

Corollary 2.1 *The Jelly fish $J(m, n)$ where $m \geq n$ is 4-prime cordial.*

Proof Let $m = n + r$, $r \geq 0$. Use of the labeling f given in theorem **??** assign the label to the vertices u, v, w_1, w_2, u_i, v_i $(1 \leq i \leq n)$.

Case 1. $r \equiv 0 \pmod 4$.

Let $r = 4k$, $k \in N$. Assign the label 2 to the vertices $u_{n+1}, u_{n+2}, \cdots, u_{n+k}$ and to the vertices $u_{n+k+1}, u_{n+k+2}, \cdots, u_{n+2k}$. Then assign the label 1 to the vertices $u_{n+2k+1}, u_{n+2k+2}, \cdots, u_{n+3k}$ and 3 to the vertices u_{n+3k+1}, u_{n+3k+2}, $\ldots$, u_{n+4k}. Clearly this vertex labeling is a 4-prime cordial labeling.

Case 2. $r \equiv 1 \pmod 4$.

Let $r = 4k + 1$, $k \in N$. Assign the labels to the vertices u_{n+i} $(1 \leq i \leq r - 1)$ as in case 1. If $n \equiv 0, 1, 2 \pmod 4$, then assign the label 1 to the vertex u_r; otherwise assign the label 4 to the vertex u_r.

Case 3. $r \equiv 2 \pmod 4$.

Let $r = 4k + 2$, $k \in N$. As in Case 2 assign the labels to the vertices u_{n+i} $(1 \leq i \leq r - 1)$. Then assign the label 4 to the vertex u_r.

Case 4. $r \equiv 3 \pmod 4$.

Let $r = 4k + 3$, $k \in N$. In this case assign the label 3 to the last vertex and assign the label to the vertices u_{n+i} $(1 \leq i \leq r - 1)$ as in Case 3. □

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Linfan Mao PhD Won the Albert Nelson Marquis Lifetime Achievement Award

W.Barbara

(AMCA, 36 South 18th Ave, Suite A, Brighton, Co 80601, United States)

The president of *Academy of Mathematical Combinatorics with Applications* (AMCA), Linfan Mao PhD won the Albert Nelson Marquis Lifetime Achievement Award and has been endorsed by Marquis Who's Who as a leader in the fields of mathematics and engineering, which was acknowledged by the notification of Marquis Who's Who to Dr.Mao in June 26, 2017 and then the released in September 30, 2017.

Marquis Who's Who is the world's premier publisher of biographical profiles since 1899 when A.N.Marquis printed the First Edition of Who's Who in America, which has chronicled the lives of the most accomplished individuals and innovators from every significant field of endeavor, including politics, business, medicine, law, education, art, religion and entertainment. Today, Marquis Who's Who remains an essential biographical source for thousands of researchers, journalists, librarians and executive search firms around the world.

Dr.Mao was born in December 31, 1962, a worker's family of China. After graduating from Wanyuan school, a middle school in the southwestern mountainous area of China, Dr.Mao began working as a scaffold erector in *China Construction Second Engineering Bureau, First Company* in 1981, while in the pursuit of his doctorate degree. He was then appointed to serve as a technician, technical adviser, director of construction management department, and then finally, the general engineer in construction project, respectively. He obtained an undergraduate diploma in applied mathematics and Bachelor of Science of *Peking University* in 1995. After then, he attended postgraduate courses in graph theory, combinational mathematics, and other areas. Dr.Mao completed a PhD with a doctoral dissertation "*A Census of Maps on Surface with Given Underlying Graph*" under the supervisor of Prof.Yanpei Liu at *Northern Jiaotong University* in 2002, and conducted postdoctoral research on automorphism groups of maps and surfaces at the *Chinese Academy of Mathematics and System*

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Science, finished his postdoctoral report "*On Automorphism Groups of Maps, Surfaces and Smarandache Geometries*" with co-advisor Prof.Feng Tian from 2003 to 2005.

In his postdoctoral report, Dr.Mao pointed out that the motivation for developing mathematics for understanding the reality of things is a combinatorial notion, i.e., *mathematical combinatorics* on Smarandache multispaces, i.e., establishing an envelope mathematical theory by combining different branches of classical mathematics into a union one such that the classical branch is its special or local case, or determining the combinatorial structure of classical mathematics and then extending classical mathematics under a given combinatorial structure, characterizing and finding its invariants today, which is in fact the global mathematics for hold on the behavior of complex systems such as those of interaction system, biological system or the adaptable system. Generally, a thing is complex and hybrid with other things but the understanding of human beings is limitation, which results in the difficult to hold on the true face of things in the world. However, there always exist universal connection between things. By this philosophical principle, Dr.Mao has found a natural road from combinatorics to topology, topology to geometry, and then from geometry to theoretical physics and other sciences, i.e., his combinatorial notion, or *Mathematical Combinatorics*.

Dr.Mao's combinatorial notion on things in the world was praised by many mathematicians in the world. For example, Prof.L.Lovasz, the chairman of *International Mathematical Union* (IMU) appraise it "*an interesting paper*", and said "*I agree that combinatorics, or rather the interface of combinatorics with classical mathematics, is a major theme today and in the near future*" in 2007, and Prof.F.Smarandache of University of New Mexico presented a paper *Mathematics for Everything with Combinatorics on Nature – A Report on the Promoter Dr.Linfan Mao* in 2016.

As a corresponding member of the Chinese Academy of Mathematics and Systems, Dr.Mao is a mathematician, also a consultant with nearly 35 years of experience and research in applied mathematics and engineering. His main interests is mainly on mathematical combinatorics and Smarandache multi-spaces with applications to sciences, research fields including combinatorics, graph theory, algebra, topology, geometry,

differential equations, complex network, biological mathematics, theoretical physics, parallel universe, purchasing and circular economy. Now, he has published 9 books and more than 80 research papers on mathematics and engineering management for the guidance of young teachers and post-graduate students.

Dr.Mao's work on Florentin Smarandache's notion, particularly, the Smarandache multispaces applies mathematics to the understanding of natural phenomena. For example, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, *Combinatorial Geometry with Applications to Field Theory* and *Smarandache Multi-Space Theory*, 3 books on mathematics with applications in 2011, *The Foundation of Bidding Theory* and *The Provisions of Clauses of the Law on Tendering and Bidding of P.R.China with Cases Analysis* in 2013, and famous papers, such as those of *Combinatorial Speculation and Combinatorial Conjecture for Mathematics*, *Mathematics on Non-mathematics* on *International J.Mathematical Combinatorics* respectively in 2007 and 2014, and *Mathematics with Natural Reality – Action Flows* on *Bulletin of Calcutta Mathematical Society* in 2015, an established journal of more than 100 years.

Dr.Mao currently serves also as the vice president of the China Academy of Urban Governance, the chief advisor of China Purchasing Association, the editor-in-chief of the *International Journal of Mathematical Combinatorics*, and the editor of Mathematical Combinatorics, an international book series since 2008 and also an honorary member of the Neutrosophic Science International Associations since 2015. He was included in *Who's Who in Science and Engineering* and *Who's Who in the World* beginning in 2006.

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The Marquis Who's Who announced in September 30, 2017: "*an accomplished listee, Dr.Mao celebrates many years' experience in his professional network, and has been noted for achievements, leadership qualities, and the credentials and successes he has accrued in his field*" and also "*In recognition of outstanding contributions to his profession and the Marquis Who's Who community, Linfan Mao, PhD, has been featured on the Albert Nelson Marquis Lifetime Achievement* ".

I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

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