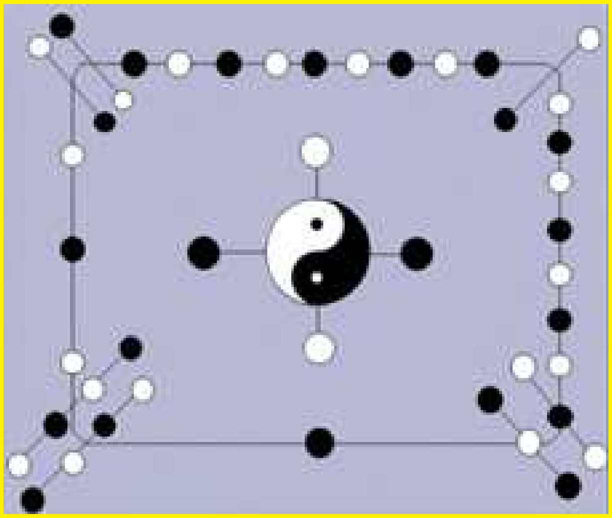
ISBN 978-1-59973-246-6

VOLUME 3, 2013

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

BEIJING UNIVERSITY OF CIVIL ENGINEERING AND ARCHITECTURE

September, 2013

Vol.3, 2013 ISBN 978-1-59973-246-6

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Famous Words:

Do not, for one repulse, give up the purpose that you resolved to effect.

By William Shakespeare, a British dramatist.

Math.Combin.Book Ser. Vol.3(2013), 01-15

Modular Equations for Ramanujan's Cubic Continued Fraction And its Evaluations

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Abstract: In this paper, we establish certain modular equations related to Ramanujan's cubic continued fraction

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+\ldots}, \qquad |q| < 1.$$

and obtain many explicit values of $G(e^{-\pi\sqrt{n}})$, for certain values of n.

Key Words: Ramanujan cubic continued fraction, theta functions, modular equation.

AMS(2010): 33D90, 11A55

§1. Introduction

Let

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+\ldots}, \tag{1.1}$$

denote the Ramanujan's cubic continued fraction for $|q| < 1$. This continued fraction was recorded by Ramanujan in his second letter to Hardy [12]. Chan [11] and Baruah [5] have proved several elegant theorems for $G(q)$. Berndt, Chan and Zhang [8] have proved some general formulas for $G(e^{-\pi\sqrt{n}})$ and $H(e^{-\pi\sqrt{n}})$ where

$$H(q) := -G(-q)$$

and n is any positive rational, in terms of Ramanujan-Weber class invariant G_n and g_n:

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty$$

and

$$g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, q = e^{-\pi\sqrt{n}}.$$

[1]Received June 24, 2013, Accepted July 25, 2013.

For the wonderful introduction to Ramanujan's continued fraction see [3], [6], [11] and for some beautiful subsequent work on Ramanujan's cubic continued fraction [1], [2], [4], [5], [14] and [15].

In this paper, we establish certain general formulae for evaluating $G(q)$. In section 2 of this paper, we setup some preliminaries which are required to prove the general formulae. In section 3, we establish certain modular equations related to $G(q)$ and in the final section, we deduce the above stated general formulae and obtain many explicit values of $G(q)$. We conclude this introduction by recalling an identity for $G(q)$ stated by Ramanujan.

$$1+\frac{1}{G^3(q)}=\frac{\psi^4(q)}{q\psi^4(q^3)} \tag{1.2}$$

where

$$\psi(q):=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.3}$$

The proof of (1.2) follows from Entry 1 (ii) and (iii) of Chapter 20 (6, p.345]).

§2. Some Preliminary Results

As usual, for any complex number a,

$$(a;q)_0:=1$$

and

$$(a;q)_\infty:=\prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n, \qquad |x|<1,$$

with

$$(a)_n:=a(a+1)(a+2)...(a+n-1).$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m:=\frac{z_1}{z_n},$$

the multiplier, where $z_1={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

Theorem 2.1 *Let $G(q)$ be as defined as in (1.1), then*

$$G(q)+G(-q)+2G^2(-q)G^2(q)=0 \tag{2.1}$$

and

$$G^2(q)+2G^2(q^2)G(q)-G(q^2)=0. \tag{2.2}$$

For a proof of Theorem 2.1, see [11].

Theorem 2.2 *Let β and γ be of the third and ninth degrees, respectively, with respect to α. Let $m = z_1/z_3$ and $m' = z_3/z_9$. Then,*

$$(i)\qquad \left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \frac{-3m}{m'} \tag{2.3}$$

and

$$(ii)\qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{2.4}$$

For a proof, see [6], Entry 3 (xii) and $(xiii)$, pp. 352-353.

Theorem 2.3 *Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier connecting α and β and let m' be the multiplier relating γ and δ. Then,*

$$(i)\qquad \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}} \tag{2.5}$$

and

$$(ii)\qquad \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.6}$$

For a proof, see [6], Entry 11 $(viii)$ and (ix), p. 383.

Theorem 2.4 *If β, γ and δ are of degrees 3, 7 and 21 respectively, $m = z_1/z_3$ and $m' = z_7/z_{21}$, then*

$$(i)\qquad \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$
$$-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1+\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\} = mm' \tag{2.7}$$

and

$$(ii)\qquad \left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}$$
$$-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\} = \frac{9}{mm'}. \tag{2.8}$$

For a proof, see [6], Entry 13 (v) and (vi), pp. 400-401.

§3. Modular Equations

Theorem 3.1 *Let*

$$R := \frac{\psi(-q^3)\psi(-q^2)}{q^{3/8}\psi(-q)\psi(-q^6)} \qquad \text{and} \qquad S := \frac{\psi(-q^6)\psi(-q^4)}{q^{3/4}\psi(-q^2)\psi(-q^{12})}$$

then,

$$\left(\sqrt{\frac{R}{S}} + \sqrt{\frac{S}{R}}\right)\left(\sqrt{RS} + \frac{1}{\sqrt{RS}}\right) - 8 = 0. \tag{3.1}$$

Proof From (1.2) and the definition of R and S, it can be seen that

$$B^3(A^3+1)R^4 = A^3(B^3+1) \tag{3.2}$$

and

$$C^3(B^3+1)S^4 = B^3(C^3+1), \tag{3.3}$$

where $A = G(-q), B = G(-q^2)$ and $C = G(-q^4)$.

On changing q to q^2 in (2.1), we have

$$G(q^2) + G(-q^2) + 2G^2(-q^2)G^2(q^2) = 0 \tag{3.4}$$

and also change q to $-q$ in (2.2), we have

$$G^2(-q) + 2G^2(q^2)G(-q) - G(q^2) = 0. \tag{3.5}$$

Eliminating $G(q^2)$ between (3.4) and (3.5) using Maple,

$$2(AB)^4 - 4(AB)^3 + 3(AB)^2 + AB + A^3 + B^3 = 0. \tag{3.6}$$

Now on eliminating A between (3.2) and (3.6) using Maple, we obtain

$$\begin{aligned} &8(BR)^4 - 80(BR)^3 + 63(BR)^2 - 5BR + B^3 - 16B^3R + 72B^3R^2 + 7B^3R^4 \\ &-22B^2R + 2B^2 + 2B^2R^3 - B^2R^4 - 9BR^2 + BR^3 + B + R = 0. \end{aligned} \tag{3.7}$$

Changing q to q^2 in (3.6),

$$2(BC)^4 - 4(BC)^3 + 3(BC)^2 + BC + B^3 + C^3 = 0. \tag{3.8}$$

Eliminating C between (3.3) and (3.8) using Maple,

$$\begin{aligned} &8B^4 + 7B^3 - 16S^3B^3 + 72S^2B^3 - 80SB^3 + S^4B^3 + 2B^2S^4 - B^2 + 2B^2S - 22S^3B^2 \\ &+63B^2S^2 - 9BS^2 + SB - 5BS^3 + BS^4 + S^3 = 0. \end{aligned} \tag{3.9}$$

Finally on eliminating B between (3.7) and (3.9) using Maple, we have

$$L(R,S)M(R,S) = 0,$$

where,

$$L(R,S) = 15S^3R^6 - 1734R^4S^4 + SR + 49S^2R^2 - S^3 - 137S^4R^2 + 8S^4R + 705S^4R^3$$
$$-137S^2R^4 - 8S^2R - 15S^2R^3 + 8SR^4 - 8SR^2 + 16SR^3 + 705S^3R^4 - 15S^3R^2 + 16S^3R - 327S^3R^3$$
$$-120S^3R^5 + 705R^5S^4 + 15S^2R^5 - SR^5 - S^3R^7 - 137R^6S^4 + 8R^7S^4 - 327R^5S^5 + 49R^6S^6$$
$$+8R^4S^7 - R^5S^8 - 15R^5S^6 - 8R^7S^6 - R^8S^5 - 15R^6S^5 + 16R^7S^5 - 8R^6S^7 + 16R^5S^7 + R^7S^7$$
$$-120S^5R^3 + 15S^5R^2 + 705S^5R^4 - 137S^6R^4 + 15S^6R^3 - S^7R^3 - S^5R - R^3 = 0$$

and

$$M(R,S) = R^2S + RS^2 - 8RS + R + S = 0.$$

Using the series expansion of R and S in the above we find that

$$L(R,S) = 223522 + 8q^{-15/2} - 8q^{-57/8} - 2q^{-55/8} - 56q^{-27/4} + 48q^{-13/2} - 24q^{-49/8} + ...$$

and

$$M(R,S) = q^{-15/8} + q^{-3/2} - 8q^{-9/8} + q^{-7/8} + q^{-3/4} + 2q^{-1/2} + ...,$$

where

$$R = \frac{1}{q^{3/8}} + q^{5/8} + 2q^{29/8} + 2q^{21/8} + 2q^{13/8} + ...$$

and

$$S = \frac{1}{q^{3/4}} + q^{5/4} + 2q^{29/4} + 2q^{21/4} + 2q^{13/4} + ...$$

One can see that $q^{-1}L(R,S)$ does not tend to 0 as $q \to 0$ whereas $q^{-1}M(R,S)$ tends to 0 as $q \to 0$. Hence, $q^{-1}M(R,S) = 0$ in some neighborhood of $q = 0$. By analytic continuation $q^{-1}M(R,S) = 0$ in $|q| < 1$. Thus we have

$$M(R,S) = 0.$$

On dividing throughout by RS we have the result. □

Theorem 3.2 *If*

$$R := \frac{\psi^2(-q^3)}{q^{1/2}\psi(-q)\psi(-q^9)} \qquad \textit{and} \qquad S := \frac{\psi^2(-q^6)}{q\psi(-q^2)\psi(-q^{18})},$$

then

$$\left(\frac{R}{S}\right)^4 + \left(\frac{S}{R}\right)^4 + \left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 - \left(RS - \frac{3}{RS}\right)\left\{\left(\frac{R}{S}\right)^3 + \left(\frac{S}{R}\right)^3\right\}$$
$$-3\left(RS - \frac{3}{RS}\right)\left(\frac{R}{S} + \frac{S}{R}\right) - \left\{(RS)^2 + \frac{9}{(RS)^2}\right\} - 6 = 0. \qquad (3.10)$$

Proof Let

$$P := \frac{\psi^2(q^3)}{q^{1/2}\psi(q)\psi(q^9)} \qquad \text{and} \qquad Q := \frac{\psi^2(q^6)}{q\psi(q^2)\psi(q^{18})}.$$

On using Entry 10 $(ii)$and (iii) of Chapter 17 in [6, p.122] in P and Q, we deduce

$$\frac{P}{Q}=\left(\frac{\alpha\gamma}{\beta^2}\right) \qquad \text{and} \qquad \frac{P^2}{Q}=\left(\frac{z_3^2}{z_1z_9}\right)^{1/2}.$$

Employing these in (2.3) and (2.4) it is easy to see that

$$\left\{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right\}^{1/4}=\frac{Q^2(3+P^2)}{P^2(Q^2-P^2)} \quad \text{and} \quad \left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right\}^{1/4}=\frac{P^2(P^2-1)}{Q^2-P^2}.$$

Multiplying these two, we arrive at

$$P^4-4P^2Q^2+Q^4+3Q^2-P^4Q^2=0. \tag{3.11}$$

Changing q to $-q$ in the above,

$$R^4-4R^2Q^2+Q^4+3Q^2-R^4Q^2=0. \tag{3.12}$$

On eliminating Q between (3.11) and (3.12), we have

$$P^4R^4-5P^4-12P^2+16P^2R^2+4P^2R^4-11R^4-8R^6-R^8+12R^2+4P^4R^2$$
$$=(-4P^2-P^4+4R^2+R^4)\sqrt{6R^4-24R^2+8R^6+R^8+9}$$

On squaring the above and then factorizing, we have

$$P^4-2P^2R^2+R^4-P^4R^2-P^2R^4+3P^2+3R^2=0. \tag{3.13}$$

Changing q to q^2 in (3.13), we have

$$Q^4-2Q^2S^2+S^4-Q^4S^2-Q^2S^4+3Q^2+3S^2=0. \tag{3.14}$$

Eliminating Q between (3.12) and (3.14) and then on dividing throughout by $(RS)^4$ and on simplifying, we obtain the required result.

Theorem 3.3 *If*

$$R:=\frac{\psi(-q^3)\psi(-q^5)}{q^{1/4}\psi(-q)\psi(-q^{15})} \qquad \text{and} \qquad S:=\frac{\psi(-q^6)\psi(-q^{10})}{q^{1/2}\psi(-q^2)\psi(-q^{30})},$$

then

$$\left(\frac{R^2}{S^2}+\frac{S^2}{R^2}\right)+\left(\frac{R}{S}+\frac{S}{R}\right)-\left(\sqrt{RS}-\frac{1}{\sqrt{RS}}\right)\left\{\sqrt{\frac{S}{R}}+\sqrt{\frac{R}{S}}+\left(\frac{R}{S}\right)^{3/2}+\left(\frac{S}{R}\right)^{3/2}\right\}$$
$$=RS+\frac{1}{RS}. \tag{3.15}$$

Proof Let

$$P:=\frac{\psi(q^3)\psi(q^5)}{q\psi(q)\psi(q^{15})} \qquad \text{and} \qquad Q:=\frac{\psi(q^6)\psi(q^{10})}{q^2\psi(q^2)\psi(q^{30})},$$

On using Entry 11 (*ii*) and (*iii*) of Chapter 17 in [6, p.122] in P and Q we deduce

$$\frac{P}{Q}=\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} \qquad \text{and} \qquad \frac{P^2}{Q}=\left(\frac{m'}{m}\right)^{1/2}.$$

Employing (2.5) and (2.6) in the above, it is easy to check that

$$\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}=\frac{P(P-1)}{Q-P} \qquad \text{and} \qquad \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}=\frac{Q(P+1)}{P(Q-P)}$$

Multiplying these two, we obtain

$$P^2+Q^2-2PQ-P^2Q+Q=0. \tag{3.16}$$

Changing q to $-q$ in the above

$$R^2+Q^2-2RQ-R^2Q+Q=0. \tag{3.17}$$

Eliminating Q between (3.16) and (3.17), we obtain

$$P^2+R^2+(P+R)(1-PR)=0. \tag{3.18}$$

On Changing q to q^2 in the above

$$Q^2+S^2+(Q+S)(1-QS)=0. \tag{3.19}$$

Finally, on eliminating Q between (3.17) and (3.19) and on dividing through out by $(RS)^2$, we have the result. □

Theorem 3.4 *If*

$$R:=q^2\frac{\psi(-q^3)\psi(-q^{21})}{\psi(-q)\psi(-q^7)} \qquad \textit{and} \qquad S:=q^4\frac{\psi(-q^6)\psi(-q^{42})}{\psi(-q^2)\psi(-q^{14})},$$

then

$$\begin{aligned} &y_8-(4+6x_1)y_7+(24+24x_1+9x_2)y_6-(148+12x_1+36x_2)y_5+(145+252x_1)y_4\\ &-(648+678x_1-36x_2+54x_3)y_3+(2180+360x_1+441x_2-324x_3)y_2-(1016+2016x_1-396x_2-54x_3)y_1\\ &+81x_4-324x_3+1548x_2+1236x_1+5250=0, \end{aligned} \tag{3.20}$$

where

$$x_n=(3RS)^n+\frac{1}{(3RS)^n} \qquad \textit{and} \qquad y_n=\left(\frac{R}{S}\right)^n+\left(\frac{S}{R}\right)^n.$$

Proof Let

$$P:=q^2\frac{\psi(q^3)\psi(q^{21})}{\psi(q)\psi(q^7)} \qquad \text{and} \qquad Q:=q^4\frac{\psi(q^6)\psi(q^{42})}{\psi(q^2)\psi(q^{14})},$$

Using Entry 11 (*ii*) and (*iii*) of Chapter 17 [6, p.122] in P and Q it is easy to deduce

$$\frac{P}{Q}=\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} \qquad \text{and} \qquad \frac{P^2}{Q}=\frac{1}{\sqrt{mm'}}.$$

Employing (2.5) and (2.6) in the above, it is easy to check that

$$\left\{\frac{(P-Q)^2Px}{Q}-PQ-P^2\right\}^2-4P^3Q-Q^2+2PQ-P^2=0$$

and

$$\left\{\frac{(P-Q)^2}{Px}-P-Q\right\}^2-4PQ-9P^2Q^2+18P^3Q-9P^4=0.$$

where

$$x=\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}.$$

Eliminating x between these two we have

$$\begin{aligned}&Q^4+8Q^4P^2-4P^3Q^3-2P^4Q^2-44P^4Q^4+24Q^2P^6-12P^7Q+81P^8Q^4\\&+72P^6Q^4-18Q^2P^8-18Q^6P^4-36Q^5P^5+P^8+Q^8-2Q^6-12P^5Q^3\\&-12P^3Q^5+24Q^6P^2-4Q^5P-36P^7Q^3-12PQ^7=0.\end{aligned}\tag{3.21}$$

On changing q to $-q$ in the above

$$\begin{aligned}&Q^4+8Q^4R^2-4R^3Q^3-2R^4Q^2-44R^4Q^4+24Q^2R^6-12R^7Q+81R^8Q^4\\&+72R^6Q^4-18Q^2R^8-18Q^6R^4-36Q^5R^5+R^8+Q^8-2Q^6-12R^5Q^3\\&-12R^3Q^5+24Q^6R^2-4Q^5R-36R^7Q^3-12RQ^7=0.\end{aligned}\tag{3.22}$$

Now on eliminating Q between (3.21) and (3.22),

$$\begin{aligned}&R^4-2R^6-18P^8R^2+144P^7R^3-450P^6R^4+504P^5R^5-450P^4R^6-12PR^7\\&-12RP^7+78R^2P^6-228R^3P^5+226R^4P^4-228R^5P^3+78R^6P^2-18R^8P^2\\&+P^4-2P^6+P^8+81P^8R^4+R^8+16RP^5-50R^2P^4\\&+56R^3P^3-50R^4P^2+16R^5P+144P^3R^7-4RP^3+6P^2R^2-4PR^3\\&+486R^6P^6-324R^5P^7-324R^7P^5+81R^8P^4=0.\end{aligned}\tag{3.23}$$

On changing q to q^2 in the above

$$\begin{aligned}&Q^4+Q^8-2Q^6+S^4-2S^6+S^8-18Q^8S^2+144Q^7S^3-450Q^6S^4+504Q^5S^5\\&-450Q^4S^6-12QS^7-12SQ^7+78S^2Q^6-228S^3Q^5+226S^4Q^4-228S^5Q^3\\&+78S^6Q^2-18S^8Q^2+81Q^8S^4+16SQ^5-50S^2Q^4+56S^3Q^3\\&-50S^4Q^2+16S^5Q+144Q^3S^7-4SQ^3+6Q^2S^2-4QS^3+486S^6Q^6\\&-324S^5Q^7-324S^7Q^5+81S^8Q^4=0.\end{aligned}\tag{3.24}$$

Finally, on eliminating Q between (3.22) and (3.24), on dividing throughout by $(RS)^8$ and then simplifying we obtain the required result. □

Theorem 3.5 *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}$$

then

$$\left(2(9+(PQ)^4\right)\left(\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right) + 3(PQ)^4 + 27 = 15(PQ)^2\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right). \quad (3.25)$$

Proof Let

$$M_n := \frac{f(-q)}{q^{n/2}f(-q^{3n})}.$$

It is easy to see that

$$P = \frac{M_2^2}{M_1} \qquad \text{and} \qquad Q = \frac{M_{14}^2}{M_7},$$

which implies

$$M_1 = \frac{M_2^2}{P} \qquad \text{and} \qquad M_7 = \frac{M_{14}^2}{Q}. \quad (3.26)$$

From Entry 51 of Chapter 25 [7, p.204], we have

$$(M_1M_2)^2 + \frac{9}{(M_1M_2)^2} = \left(\frac{M_2}{M_1}\right)^6 + \left(\frac{M_1}{M_2}\right)^6. \quad (3.27)$$

Using (3.26) in (3.27), we deduce that

$$M_2^{12} = \frac{P^8(P^4-9)}{P^4-1}. \quad (3.28)$$

On changing q to q^7 in (3.28), we have

$$M_{14}^{12} = \frac{Q^8(Q^4-9)}{Q^4-1}.$$

Thus from the above and (3.28)

$$\left(\frac{M_2}{M_{14}}\right)^{12} = \frac{P^8(P^4-9)(Q^4-1)}{Q^8(P^4-1)(Q^4-9)}. \quad (3.29)$$

From Theorem 3.1(*ii*) of [9], we have

$$LM + \frac{1}{LM} = \left(\frac{L}{M}\right)^3 + \left(\frac{M}{L}\right)^3 + 4\left(\frac{L}{M} + \frac{M}{L}\right), \quad (3.30)$$

where

$$L = \frac{M_1}{M_7} \qquad \text{and} \qquad M = \frac{M_2}{M_{14}}.$$

On using (3.26) in L, we obtain

$$L = \left(\frac{M_2}{M_{14}}\right)^2 \frac{Q}{P} \qquad \text{and} \qquad M = \frac{M_2}{M_{14}}.$$

Employing this in (3.30) and on dividing throughout by $(PQM_2/M_{14})^3$, we have

$$P^6 - 3\left(\frac{M_2}{M_{14}}\right)^6 P^2Q^4 - 3P^4Q^2 - \left(\frac{M_2}{M_{14}}\right)^6 Q^6 = 0. \tag{3.31}$$

Finally, on eliminating M_2/M_{14} between (3.29) and (3.31) and on dividing throughout by $(PQ)^2$, we have the result. □

§4. Evaluations of Ramanujan's Cubic Continued Fraction

Lemma 4.1 *For $q = e^{-\pi\sqrt{n/3}}$, let*

$$A_n := \frac{1}{\sqrt[4]{3}}\frac{\psi(-q)}{\psi(-q^3)}.$$

Then

$$(i)\quad A_nA_{1/n} = 1, \tag{4.1}$$

$$(ii)\quad A_1 = 1, \tag{4.2}$$

$$(iii)\ H(q) = \frac{1}{\sqrt[3]{3A_n^4 + 1}}. \tag{4.3}$$

For a proof see [10].

Lemma 4.2

$$3A_n^2A_{9n}^2 + \frac{3}{A_n^2A_{9n}^2} = 3 + 6\frac{A_{9n}^2}{A_n^2} + \frac{A_{9n}^4}{A_n^4}.$$

For a proof, see [10].

Lemma 4.3

$$3(A_nA_{25n})^2 + \frac{3}{(A_nA_{25n})^2} = \left(\frac{A_{25n}}{A_n}\right)^3 - \left(\frac{A_n}{A_{25n}}\right)^3$$
$$+5\left(\frac{A_{25n}}{A_n}\right)^2 + 5\left(\frac{A_n}{A_{25n}}\right)^2 + 5\left(\frac{A_{25n}}{A_n}\right) - 5\left(\frac{A_n}{A_{25n}}\right),$$

For a proof, see [10].

Theorem 4.1 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\sqrt{\frac{A_{4n}^2}{A_nA_{16n}}} + \sqrt{\frac{A_nA_{16n}}{A_{4n}^2}}\right)\left(\sqrt{\frac{A_{16n}}{A_n}} + \sqrt{\frac{A_n}{A_{16n}}}\right) = 8. \tag{4.4}$$

Proof For proof of (4.4), we use Theorem 3.1 with $R(q) = A_{4n}/A_n$ and $S = A_{16n}/A_{4n}$. □

Theorem 4.2 *We have*

$$A_4 = 2 + \sqrt{3}$$

and

$$A_{1/4} = 2-\sqrt{3}.$$

Proof Put $n=1/4$ in (4.4) and using (4.1) we obtain the result. □

Corollary 4.1 *We have*

$$H(e^{-\pi\sqrt{4/3}}) = \frac{1}{148}(292+168\sqrt{3})^{2/3}(73-42\sqrt{3})$$

and

$$H(e^{-\pi\sqrt{1/12}}) = \frac{1}{148}(292-168\sqrt{3})^{2/3}(73+42\sqrt{3}).$$

Proof On using Theorem 4.2 in (4.3), we have result. □

Theorem 4.3 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^4 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^4 + \left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^2 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^2 - \left(\frac{A_{9n}A_{36n}}{A_nA_{4n}} - 3\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)$$
$$\times\left\{\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^3 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^3\right\} - 3\left(\frac{A_{9n}A_{36n}}{A_nA_{4n}} - 3\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}} + \frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)$$
$$-\left\{\left(\frac{A_{9n}A_{36n}}{A_nA_{4n}}\right)^2 + 9\left(\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)^2\right\} - 6 = 0. \tag{4.5}$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.2. □

Theorem 4.4 *We have*

$$A_6 = \sqrt[4]{6\sqrt{2}-3\sqrt{3}+3\sqrt{6}-6} = A_{1/6}^{-1}$$

and

$$A_{2/3} = \frac{1}{\sqrt{3}}\sqrt[4]{6\sqrt{2}+3\sqrt{3}+3\sqrt{6}+6} = A_{3/2}^{-1}.$$

Proof Setting $n=1/6$ in (4.5) and upon using (4.1), we find that

$$\left(\frac{A_6}{A_{2/3}}\right)^4 + 9\left(\frac{A_{2/3}}{A_6}\right)^4 + 8\left\{\left(\frac{A_6}{A_{2/3}}\right)^2 - 3\left(\frac{A_{2/3}}{A_6}\right)^2\right\} + 2 = 0.$$

Since A_n is real and increasing in n, we have $A_6/A_{2/3} > 1$. Hence

$$\frac{A_6}{A_{2/3}} = \sqrt{3\sqrt{2}-3}. \tag{4.6}$$

Again on setting $n=2/3$ in Lemma 4.2, we have

$$3(A_{2/3}A_6)^2 + \frac{3}{(A_{2/3}A_6)^2} = 3 + 6\frac{A_6^2}{A_{2/3}^2} + \frac{A_6^4}{A_{2/3}^4}.$$

On using (4.6) in this, we obtain

$$A_{2/3}A_6 = \sqrt{2+\sqrt{3}}. \tag{4.7}$$

Finally, on employing (4.6), (4.7) and (4.1) we have the result. □

Corollary 4.2 *We have*

$$H(e^{-\pi\sqrt{2}}) = \frac{1}{(18\sqrt{2}-9\sqrt{3}+9\sqrt{6}-17)^{1/3}}$$

and

$$H(e^{-\pi\sqrt{2/9}}) = \frac{1}{(18\sqrt{2}+9\sqrt{3}+9\sqrt{6}+19)^{1/3}}.$$

Proof On using Theorem 4.4 in (4.3), we have the result. □

Theorem 4.5 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\frac{A_{4n}A_{25n}}{A_nA_{100n}}\right)^2 + \left(\frac{A_nA_{100n}}{A_{4n}A_{25n}}\right)^2 + \left(\frac{A_{4n}A_{25n}}{A_nA_{100n}} + \frac{A_nA_{100n}}{A_{4n}A_{25n}}\right) - \left(\sqrt{\frac{A_{25n}A_{100n}}{A_nA_{4n}}} + \sqrt{\frac{A_nA_{4n}}{A_{25n}A_{100n}}}\right)$$
$$\left(\sqrt{\frac{A_{4n}A_{25n}}{A_nA_{100n}}} + \sqrt{\frac{A_nA_{100n}}{A_{4n}A_{25n}}} + \left(\frac{A_{4n}A_{25n}}{A_nA_{100n}}\right)^{3/2} + \left(\frac{A_nA_{100n}}{A_{4n}A_{25n}}\right)^{3/2}\right) = \frac{A_{25n}A_{100n}}{A_nA_{4n}} + \frac{A_nA_{4n}}{A_{25n}A_{100n}}. \tag{4.8}$$

Proof The proof is similar to Theorem 4.1 by using Theorem 3.3. □

Theorem 4.6 *We have*

$$A_{10} = \sqrt{\frac{2+\sqrt{10}+\sqrt{4\sqrt{10}+10}}{2}}\sqrt[4]{\frac{a-\sqrt{a^2-36}}{6}} = A_{1/10}^{-1}$$

and

$$A_{2/5} = \sqrt{\frac{2+\sqrt{10}-\sqrt{4\sqrt{10}+10}}{2}}\sqrt[4]{\frac{a-\sqrt{a^2-36}}{6}} = A_{5/2}^{-1},$$

where $a = (18+4\sqrt{10})(\sqrt{4\sqrt{10}+10}) + 60 + 20\sqrt{10}$.

Proof Setting $n = 1/10$ in (4.8) and upon using (4.1), we find that

$$x^2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) - 4 = 0,$$

where $x = A_{10}/A_{2/5}$. Since A_n is real and increasing in n, we have $A_{10}/A_{2/5} > 1$. Hence we choose

$$x + \frac{1}{x} = 2 + \sqrt{10}.$$

On solving

$$\frac{A_{10}}{A_{2/5}} = \frac{1}{2}\left(2+\sqrt{10}+\sqrt{4\sqrt{10}+10}\right). \tag{4.9}$$

Put $n = 2/5$ in Lemma 4.3, we have

$$3(A_{2/5}A_{10})^2 + \frac{3}{(A_{2/5}A_{10})^2} = \left(\frac{A_{10}}{A_{2/5}}\right)^3 - \left(\frac{A_{2/5}}{A_{10}}\right)^3$$

$$+5\left\{\left(\frac{A_{10}}{A_{2/5}}\right)^2 + \left(\frac{A_{2/5}}{A_{10}}\right)^2\right\} + 5\left(\frac{A_{10}}{A_{2/5}} - \frac{A_{2/5}}{A_{10}}\right).$$

On employing (4.9) in this, we obtain

$$A_{2/5}A_{10} = \sqrt{\frac{a - \sqrt{a^2 - 36}}{6}}, \tag{4.10}$$

where $a = (18 + 4\sqrt{10})(\sqrt{4\sqrt{10} + 10}) + 60 + 20\sqrt{10}$. On using (4.9) and (4.10) we have the result. □

Theorem 4.7 *If A_n is as defined as in Lemma 4.1, then*

$$\left(2 + 2(A_nA_{49n})^4\right)\left[\left(\frac{A_n}{A_{49n}}\right)^2 - \left(\frac{A_{49n}}{A_n}\right)^2\right] + 3(A_nA_{49n})^4 + 3$$

$$= 5(A_nA_{49n})^2\left[\left(\frac{A_n}{A_{49n}}\right)^2 + \left(\frac{A_{49n}}{A_n}\right)^2\right]. \tag{4.11}$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.5. □

Theorem 4.8 *If A_n is as defined as in Lemma 4.1, then*

$$y_8 - (4 + 6x_1)y_7 + (24 + 24x_1 + 9x_2)y_6 - (148 + 12x_1 + 36x_2)y_5 + (145 + 252x_1)y_4$$

$$-(648+678x_1-36x_2+54x_3)y_3+(2180+360x_1+441x_2-324x_3)y_2-(1016+2016x_1-396x_2-54x_3)y_1$$

$$+81x_4 - 324x_3 + 1548x_2 + 1236x_1 + 5250 = 0, \tag{4.12}$$

where

$$x_m = (3A_nA_{4n}A_{49n}A_{196n})^m + \frac{1}{(3A_nA_{4n}A_{49n}A_{196n})^m}, \qquad m = 1, 2, 3$$

and

$$y_m = \left(\frac{A_{49n}A_{196n}}{A_nA_{4n}}\right)^m + \left(\frac{A_nA_{4n}}{A_{49n}A_{196n}}\right)^m, \qquad m = 1, 2, \cdots, 8$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.4. □

Theorem 4.9 *We have*

$$A_{14} = \frac{1}{\sqrt[4]{34}}\{(a + \sqrt{a^2 - 14})(9 + 10\sqrt{2})\}^{1/4} = A_{1/14}^{-1}$$

and

$$A_{2/7}=\left(\frac{2}{17}\frac{9+10\sqrt{2}}{a+\sqrt{a^2-4}}\right)^{1/4}=A_{7/2}^{-1},$$

where

$$a=\frac{1}{3}(197+18\sqrt{113})^{1/3}+\frac{13}{3(197+18\sqrt{113})^{1/3}}+\frac{2}{3}.$$

Proof On setting $n=1/14$ in (4.12) and upon using (4.1), we find that

$$\begin{aligned}\left(t^8+\frac{1}{t^8}\right) &- 16\left(t^7+\frac{1}{t^7}\right)+90\left(t^6+\frac{1}{t^6}\right)-244\left(t^5+\frac{1}{t^5}\right)+649\left(t^4+\frac{1}{t^4}\right)\\ &- 2040\left(t^3+\frac{1}{t^3}\right)+3134\left(t^2+\frac{1}{t^2}\right)-4148\left(t+\frac{1}{t}\right)+10332=0,\end{aligned}$$

where $t=(A_{2/7}A_{14})^2$. On setting $t+\dfrac{1}{t}=x$, we obtain

$$x^8-16x^7+82x^6-132x^5+129x^4-1044x^3+1332x^2+864x+5184=0.$$

On solving this, we obtain

$$x=6,\quad \frac{1}{3}(197+18\sqrt{113})^{1/3}+\frac{13}{3(197+18\sqrt{113})^{1/3}}+\frac{2}{3}$$

are the double roots and the remaining roots are imaginary. Since A_n is increasing in n, and solving for $(A_{14}/A_{2/7})^2$, it is easy to see that

$$\left(\frac{A_{14}}{A_{2/7}}\right)^2=\frac{a+\sqrt{a^2-4}}{2},$$

where a is as defined earlier. On setting $n=2/7$ in (4.11) and on using the above, we have the result. □

Acknowledgement

The authors are thankful to Dr.K.R.Vasuki, Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore for his valuable suggestions to improve the quality of the paper.

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Math.Combin.Book Ser. Vol.3(2013), 16-21

Semi-Symmetric Metric Connection on a 3-Dimensional Trans-Sasakian Manifold

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Abstract: The object of the present paper is to study the nature of curvature tensor, Ricci tensor, scalar curvature and Weyl conformal curvature tensors with respect to a semi-symmetric metric connection on a 3-dimensional trans-Sasakian manifold.We have given an example regarding it.

Key Words: α -Sasakian manifold, β-Kenmotsu manifold, cosymplectic manifold, Levi-Civita connection, semi-symmetric connection, Weyl conformal curvature tensor.

AMS(2010): 53C25

§1. Introduction

The notion of locally φ-symmetric Sasakian manifold was introduced by T. Takahashi [14] in 1977. Also J.A. Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [13], α-Sasakian [11], Kenmotsu [11], β-Kenmotsu [11] and cosymplectic [11] manifolds, which was called trans-Sasakian manifold [12]. After him many authors [4],[5],[10],[12] have studied various type of properties in trans-Sasakian manifold.

In this paper we have obtained the curvature tensor and also the first Bianchi identity with respect to a semi-symmetric connection on a 3-dimensional trans-Sasakian manifold. We also find out the condition of Ricci tensor to be symmetric under this connection. We have shown that the Riemannian Weyl conformal curvature tensor is equal to the Weyl conformal curvature tensor with respect to semi-symmetric connection and also equal to the curvature tensor with respect to semi-symmetric connection when the Ricci tensor under this connection vanishes.

[1]Received April 12, 2013, Accepted August 2, 2013.

§2. Preliminaries

Let M^n be an n-dimensional (n is odd) almost contact C^∞ manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric.

Then the manifold satisfies the following relations ([3]):

(2.1) $\phi^2(X) = -X + \eta(X)\xi, \quad \eta \circ \phi = 0;$

(2.2) $\eta(X) = g(X, \xi), \quad \eta(\xi) = 1;$

(2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$

Now an almost contact manifold is called trans-Sasakian manifold if it satisfies ([13]):

(2.4) $(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X].$

From (2.4) it follows

(2.5) $(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)], \forall\ X, Y \in \chi(M)$

where $\alpha, \beta \in F(M)$ and ∇ be the Levi-Civita connection on M^n.

A linear connection $\bar{\nabla}$ on M^n is said to be semi-symmetric [1] if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

(2.6) $\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y,$

where π is a 1-form on M^n with U as associated vector field, i.e,

(2.7) $\pi(X) = g(X, U)$

for any differentiable vector field X on M^n.

A semi-symmetric connection $\bar{\nabla}$ is called semi-symmetric metric connection [2] if it further satisfies

(2.8) $\bar{\nabla} g = 0.$

In [2] Sharfuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold by identifying the 1-form π of [1] with the contact 1-form η i.e., by setting

(2.9) $T(X, Y) = \eta(Y)X - \eta(X)Y.$

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) has been obtained by K.Yano [9], which is given by

(2.10) $\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)U.$

Further, a relation between the curvature tensor R and $\bar{R}$ of type $(1,3)$ of the connections ∇ and $\bar{\nabla}$ respectively are given by [7],[8],[9]

(2.11) $\bar{R}(X, Y)Z = R(X, Y)Z + \hat{\alpha}(X, Z)Y - \hat{\alpha}(Y, Z)X - g(Y, Z)LX + g(X, Z)LY,$

where,

(2.12) $\hat{\alpha}(Y, Z) = g(LY, Z) = (\nabla_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(U)g(Y, Z).$

The Weyl conformal curvature tensor of type $(1,3)$ of the manifold is defined by

(2.13) $C(X, Y)Z = R(X, Y)Z + \lambda(Y, Z)X - \lambda(X, Z)Y + g(Y, Z)QX - g(X, Z)QY,$

where,

(2.14) $\lambda(Y,Z) = g(QY,Z) = -\frac{1}{n-2}S(Y,Z) + \frac{r}{2(n-1)(n-2)}g(Y,Z)$,

where S and r denote respectively the $(0,2)$ Ricci tensor and scalar curvature of the manifold.

We shall use these results in the next sections for a 3-dimensional trans-Sasakian manifold with semi-symmetric metric connection.

§3. Curvature tensors with Respect to the Semi-Symmetric Metric Connection On a 3-Dimensional Trans-Sasakian Manifold

From (2.5), (2.9) and (2.12) we have

(3.1) $\hat{\alpha}(Y,Z) = -\alpha g(\phi Y,Z) - (\beta+1)\eta(Y)\eta(Z) + (\beta+\frac{1}{2})g(Y,Z)$.

Using (2.12), we get from (3.1)

(3.2) $LY = -\alpha\phi Y - (\beta+1)\eta(Y)\xi + (\beta+\frac{1}{2})Y$.

Now using (3.1) and (3.2), we get from (2.11) after some calculations

$$\begin{aligned}(3.3)\ \bar{R}(X,Y)Z \ = \ & R(X,Y)Z - \alpha[g(\phi X,Z)Y - g(\phi Y,Z)X] \\ & -\alpha[g(X,Z)\phi Y - g(Y,Z)\phi X] + (2\beta+1)[g(X,Z)Y - g(Y,Z)X] \\ & -(\beta+1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ & -(\beta+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi.\end{aligned}$$

Thus we can state

Theorem 3.1 *The curvature tensor with respect to $\bar{\nabla}$ on a 3-dimensional trans-Sasakian manifold is of the form* (3.3).

From (3.3) it is seen that

(3.4) $\bar{R}(Y,X)Z = -\bar{R}(X,Y)Z$.

We now define a tensor $\bar{R}'$ of type $(0,4)$ by

(3.5) $\bar{R}'(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$.

From (3.4) and (3.5) it follows that

(3.6) $\bar{R}'(Y,X,Z,V) = -\bar{R}'(X,Y,Z,V)$.

Combining (3.6) and (3.4) we can see that

(3.7) $\bar{R}'(X,Y,Z,V) = \bar{R}'(Y,X,V,Z)$.

Again from (3.3) exchanging X,Y,Z cyclically and adding them, we get

(3.8) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 2\alpha[g(\phi X,Y)Z + g(\phi Y,Z)X + g(\phi Z,X)Y]$.

This is the first Bianchi identity with respect to $\bar{\nabla}$. Thus we state

Theorem 3.2 *The first Bianchi identity with respect to $\bar{\nabla}$ on a 3-dimensional trans-Sasakian manifold is of the form* (3.8).

Let $\bar{S}$ and S denote respectively the Ricci tensor of the manifold with respect to $\bar{\nabla}$ and ∇. From (3.3) we get by contracting X,

(3.11) $\bar{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y, Z) - (3\beta+1)g(Y,Z) + (\beta+1)\eta(Y)\eta(Z)$.

In (3.11) we put $Y = Z = e_i, 1 \leq i \leq 3$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. Then summing over i, we get

(3.12) $\bar{r} = r - 2(4\beta+1)$.

From (3.11), we get

(3.13) $\bar{S}(Y,Z) - \bar{S}(Z,Y) = \alpha(g(\phi Y,Z) - g(\phi Z,Y)) = 2\alpha g(\phi Y,Z)$.

But $g(\phi Y, Z)$ is not identically zero. So $\bar{S}(Y,Z)$ is not symmetric. Thus we state

Theorem 3.3 *The Ricci tensor of a* 3*-dimensional trans-Sasakian manifold with respect to the semi-symmetric metric connection is not symmetric.*

The Weyl conformal curvature tensor of type $(1,3)$ of the 3-dimensional trans-sasakian manifold with respect to the semi-symmtric metric connection $\bar{\nabla}$ is defined by

(3.14) $\bar{C}(X,Y)Z = \bar{R}(X,Y)Z + \bar{\lambda}(Y,Z)X - \bar{\lambda}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y$,

where,

(3.15) $\bar{\lambda}(Y,Z) = g(\bar{Q}Y,Z) = -\frac{1}{2}\bar{S}(Y,Z) + \frac{\bar{r}}{4}g(Y,Z)$.

Putting the values of $\bar{S}$ and $\bar{r}$ from (3.11) and (3.12) respectively in (3.15) we get

(3.16) $\bar{\lambda}(Y,Z) = g(\bar{Q}Y,Z) = \lambda(Y,Z) - \alpha g(\bar{Y},Z) + \frac{2\beta+1}{2}g(Y,Z) - (\beta+1)\eta(Y)\eta(Z)$.

and,

(3.17) $\bar{Q}Y = QY - \alpha\bar{Y} + \frac{2\beta+1}{2}Y - (\beta+1)\eta(Y)\xi$.

Using (3.3),(3.16) and (3.17), we get from (3.14) after a brief calculations

(3.18) $\bar{C}(X,Y)Z = C(X,Y)Z$.

Thus we can state

Theorem 3.4 *The Weyl conformal curvature tensors of the* 3*-dimensional trans-sasakian manifold with respect to the Levi-Civita connection and the semi-symmetric metric connection are equal.*

If in particular $\bar{S} = 0$, then $\bar{r} = 0$, so from (3.15) we get

(3.19) $\bar{\lambda}(Y,Z) = 0$.

From (3.19) and (3.14) we get

(3.20) $\bar{C}(X,Y)Z = \bar{R}(X,Y)Z$.

From (3.18) and (3.20) we have

(3.21) $C(X,Y)Z = \bar{R}(X,Y)Z.$

Corollary 3.5 *If the Ricci tensor of a 3-dimensional trans-Sasakian manifold with respect to the semi-symmetric metric connection vanishes, the Weyl conformal curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to the semi-symmetric metric connection.*

§4. Example of a 3-Dimensional Trans-Sasakian Manifold Admitting A Semi-Symmetric Metric Connection

Let the 3-dim. C^∞ real manifold $M = \{(x,y,z) : (x,y,z) \in R^3, z \neq 0\}$ with the basis $\{e_1, e_2, e_3\}$, where $e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z}$.

We consider the Riemannian metric g defined by

$$g(e_i, e_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j. \end{cases}$$

Now we define a $(1,1)$ tensor field ϕ by $\phi(e_1) = -e_2,\ \phi(e_2) = e_1$ and $\phi(e_3) = 0$, and choose the vector field $\xi = e_3$ and define a 1-form η by $\eta(X) = g(X, e_3), \forall\ X \in \chi(M)$. Then $\eta(e_1) = \eta(e_2) = 0$ and $\eta(e_3) = 1$.

From the above construction we can easily show that

$$\begin{aligned} &\phi^2(X) = -X + \eta(X)\xi, \quad \eta \circ \phi = 0 \\ ,\qquad &\eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \end{aligned}$$

Thus M is a 3-dim. almost contact C^∞ manifold with the almost contact structure (ϕ, ξ, η, g).

We also obtain $[e_1, e_2] = 0, [e_2, e_3] = -e_2$ and $[e_1, e_3] = -e_1$. By Koszul's formula we get

$$\begin{aligned} &\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \\ &\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_3} e_2 = 0, \\ &\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

Then it can be shown that M is a trans-Sasakian manifold of type $(0, -1)$.

Now we define a linear connection $\bar{\nabla}$ such that

$$\bar{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \eta(e_j) e_i - g(e_i, e_j) e_3, \forall\ i, j = 1, 2, 3.$$

Then we get

$$\begin{aligned} &\bar{\nabla}_{e_1} e_1 = 0, \quad \bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_3} e_1 = 0, \\ &\bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_2} e_2 = 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \\ &\bar{\nabla}_{e_1} e_3 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0. \end{aligned}$$

If $\bar{T}$ is the torsion tensor of the connection $\bar{\nabla}$, then we have

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y \text{ and } (\bar{\nabla}_X g)(Y,Z) = 0,$$

which implies that $\bar{\nabla}$ is a semi-symmetric metric connection on M.

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Math.Combin.Book Ser. Vol.3(2013), 22-34

On Mean Graphs

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Abstract: Let $G(V,E)$ be a graph with p vertices and q edges. For every assignment $f : V(G) \to \{0,1,2,3,\ldots,q\}$, an induced edge labeling $f^* : E(G) \to \{1,2,3,\ldots,q\}$ is defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity} \\ \dfrac{f(u)+f(v)+1}{2} & \text{otherwise} \end{cases}$$

for every edge $uv \in E(G)$. If $f^*(E) = \{1,2,\ldots,q\}$, then we say that f is a mean labeling of G. If a graph G admits a mean labeling, then G is called a mean graph. In this paper, we prove that the graphs double sided step ladder graph $2S(T_m)$, Jelly fish graph $J(m,n)$ for $|m-n| \leq 2$, $P_n(+)N_m$, $(P_2 \cup kK_1) + N_2$ for $k \geq 1$, the triangular belt graph $TB(\alpha)$, $TBL(n,\alpha,k,\beta)$, the edge mC_n- snake, $m \geq 1, n \geq 3$ and $S_t(B(m)_{(n)})$ are mean graphs. Also we prove that the graph obtained by identifying an edge of two cycles C_m and C_n is a mean graph for $m,n \geq 3$.

Key Words: Smarandachely edge 2-labeling, mean graph, mean labeling, Jelly fish graph, triangular belt graph.

AMS(2010): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. $K_{1,m}$ is called a star and it is denoted by S_m. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. $B_{m,m}$ is often denoted by $B(m)$. The join of two graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G with each vertex of H by means of an edge and it is denoted by $G + H$. The edge mC_n- snake is a graph obtained from m copies of C_n by identifying the edge $v_{k+1}v_{k+2}$ in each copy of C_n, n is either $2k+1$ or $2k$ with the edge v_1v_2 in the successive

[1]Received April 11, 2013, Accepted August 5, 2013.

copy of C_n. The graph $P_n \times P_2$ is called a ladder. Let P_{2n} be a path of length $2n-1$ with $2n$ vertices $(1,1),(1,2),\ldots,(1,2n)$ with $2n-1$ edges $e_1,e_2,\ldots,e_{2n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i, for $i=1,2,\ldots,n$, we erect a ladder with $i+1$ steps including the edge e_i and on each edge e_i, for $i=n+1,n+2,\ldots,2n-1$, we erect a ladder with $2n+1-i$ steps including the edge e_i. The resultant graph is called double sided step ladder graph and is denoted by $2S(T_m)$, where $m=2n$ denotes the number of vertices in the base.

A vertex labeling of G is an assignment $f : V(G) \to \{0,1,2,\ldots,q\}$. For a vertex labeling f, the induced edge labeling f^* is defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity} \\ \dfrac{f(u)+f(v)+1}{2} & \text{otherwise} \end{cases}$$

A vertex labeling f is called a mean labeling of G if its induced edge labeling $f^* : E(G) \to \{1,2,\ldots,q\}$ is a bijection, that is, $f^*(E) = \{1,2,\ldots,q\}$. If a graph G has a mean labeling, then we say that G is a mean graph. It is clear that a mean labeling is a Smarandachely edge 2-labeling of G.

A mean labeling of the Petersen graph is shown in Figure 1.

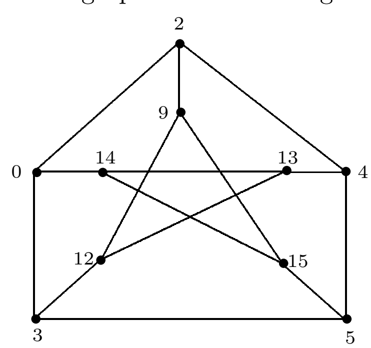

Figure 1

The concept of mean labeling was introduced and studied by S.Somasundaram and R.Ponraj [4]. Some new families of mean graphs are studied by S.K.Vaidya et al. [6], [7]. Further some more results on mean graphs are discussed in [2], [3], [5].

In this paper, we establish the meanness of the graphs double sided step ladder graph $2S(T_m)$, Jelly fish graph $J(m,n)$ for $|m-n| \leq 2$, $P_n(+)N_m$, $(P_2 \cup kK_1) + N_2$ for $k \geq 1$, the triangular belt graph $TB(\alpha)$, $TBL(n,\alpha,k,\beta)$, the edge mC_n-snake $m \geq 1, n \geq 3$ and $S_t(B(m)_{(n)})$. Also we prove that the graph obtained by identifying an edge of two cycles C_m and C_n is a mean graph for $m,n \geq 3$.

§2. Mean Graphs

Theorem 2.1 *The double sided step ladder graph $2S(T_m)$ is a mean graph where $m=2n$ denotes the number of vertices in the base.*

Proof Let P_{2n} be a path of length $2n-1$ with $2n$ vertices $(1,1),(1,2),\cdots,(1,2n)$ with $2n-1$ edges, $e_1,e_2,\cdots,e_{2n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge e_i, for $i=1,2,\cdots,n$, we erect a ladder with $i+1$ steps including the edge e_i and on each edge e_i, for $i=n+1,n+2,\cdots,2n-1$, we erect a ladder with $2n+1-i$ steps including the edge e_i.

The double sided step ladder graph $2S(T_m)$ has vertices denoted by $(1,1),(1,2),\ldots,(1,2n)$, $(2,1),(2,2),\cdots,(2,2n),(3,2),(3,3),\cdots,(3,2n-1),(4,3),(4,4),\cdots,(4,2n-2),\cdots,(n+1,n),(n+1,n+1)$. In the ordered pair (i,j), i denotes the row (counted from bottom to top) and j denotes the column (from left to right) in which the vertex occurs. Define $f: V(2S(T_m)) \to \{0,1,2,\ldots,q\}$ as follows:

$$f(i,j) = (n+1-i)(2n-2i+3)+j-1, \quad 1 \le j \le 2n, i=1,2$$
$$f(i,j) = (n+1-i)(2n-2i+3)+j+1-i, \quad i-1 \le j \le 2n+2-i, 3 \le i \le n+1.$$

Then, f is a mean labeling for the double sided step ladder graph $2S(T_m)$. Thus $2S(T_m)$ is a mean graph. □

For example, a mean labeling of $2S(T_{10})$ is shown in Figure 2.

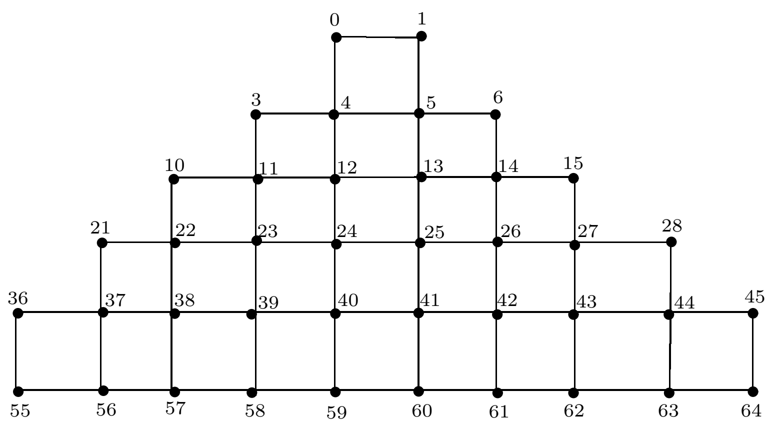

Figure 2

For integers $m,n \ge 0$ we consider the graph $J(m,n)$ with vertex set $V(J(m,n)) = \{u,v,x,y\} \cup \{x_1,x_2,c\ldots,x_m\} \cup \{y_1,y_2,\cdots,y_n\}$ and edge set $E(J(m,n)) = \{(u,x),(u,v),(u,y),(v,x),(v,y)\} \cup \{(x_i,x): i=1,2,\cdots,m\} \cup \{(y_i,y): i=1,2,\cdots,n\}$. We will refer to $J(m,n)$ as a Jelly fish graph.

Theorem 2.2 *A Jelly fish graph $J(m,n)$ is a mean graph for $m,n \ge 0$ and $|m-n| \le 2$.*

Proof The proof is divided into cases following.

Case 1 $m=n$.

Define a labeling $f : V(J(m, n)) \to \{0, 1, 2, \ldots, q = m + n + 5\}$ as follows:

$$f(u) = 2,\ f(y) = 0,$$
$$f(v) = m + n + 4,\ f(x) = m + n + 5,$$
$$f(x_i) = 4 + 2(i - 1), \quad 1 \le i \le m$$
$$f(y_{n+1-i}) = 3 + 2(i - 1), \quad 1 \le i \le n$$

Then f provides a mean labeling.

Case 2 $m = n + 1$ or $n + 2$

Define $f : V(J(m, n)) \to \{0, 1, 2, \ldots, q = m + n + 5\}$ as follows:

$$f(u) = 2,\ f(v) = 2n + 4,\ f(y) = 0,$$
$$f(x) = \begin{cases} m + n + 5 & \text{if } m = n + 1 \\ m + n + 4 & \text{if } m = n + 2 \end{cases}$$
$$f(x_i) = \begin{cases} 4 + 2(i - 1), & 1 \le i \le n \\ 2n + 5 + 2(i - (n + 1)), & n + 1 \le i \le m \end{cases}$$
$$f(y_{n+1-i}) = 3 + 2(i - 1), \quad 1 \le i \le n.$$

Then f gives a mean labeling. Thus $J(m, n)$ is a mean graph for $m, n \ge 0$ and $|m - n| \le 2$. □

For example, a mean labeling of $J(6, 6)$ and $J(9, 7)$ are shown in Figure 3.

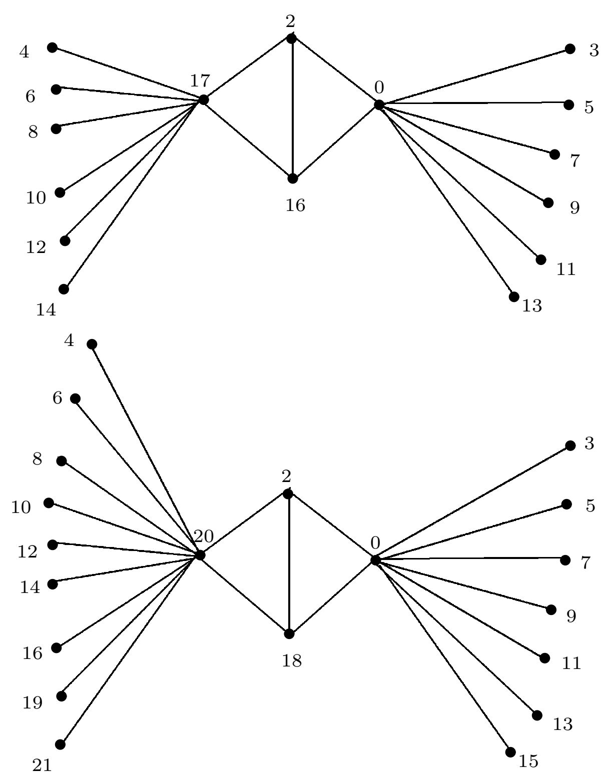

Figure 3

Let $P_n(+)N_m$ be the graph with $p = n + m$ and $q = 2m + n - 1$. $V(P_n(+)N_m) = \{v_1, v_2, \cdots, v_n, y_1, y_2, \cdots, y_m\}$, where $V(P_n) = \{v_1, v_2, \cdots, v_n\}$, $V(N_m) = \{y_1, y_2, \cdots, y_m\}$ and

$$E(P_n(+)N_m) = E(P_n) \bigcup \left\{ \begin{array}{l} (v_1, y_1), (v_1, y_2), \cdots, (v_1, y_m), \\ (v_n, y_1), (v_n, y_2), \cdots, (v_n, y_m). \end{array} \right\}$$

Theorem 2.3 *$P_n(+)N_m$ is a mean graph for all $n, m \geq 1$.*

Proof Let us define $f: V(P_n(+)N_m) \to \{1, 2, 3, \cdots, 2m + n - 1\}$ as follows:

$$\begin{aligned} f(y_i) &= 2i - 1,\ 1 \leq i \leq m, \\ f(v_1) &= 0, \\ f(v_i) &= 2m + 1 + 2(i - 2), \quad 2 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil \\ f(v_{n+1-i}) &= 2m + 2 + 2(i - 1), \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

Then, f gives a mean labeling. Thus $P_n(+)N_m$ is a mean graph for $n, m \geq 1$. □

For example, a mean labeling of $P_8(+)N_5$ and $P_7(+)N_6$ are shown in Figure 4.

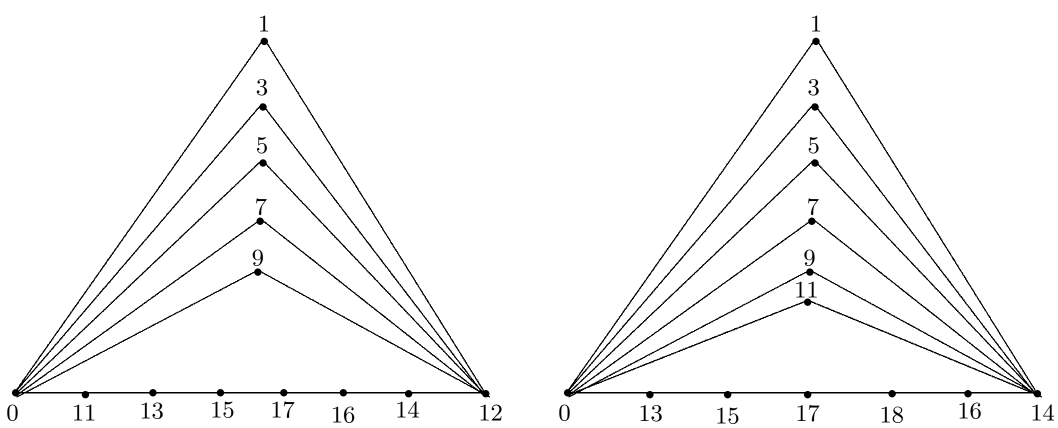

Figure 4

Theorem 2.4 *For $k \geq 1$, the planar graph $(P_2 \cup kK_1) + N_2$ is a mean graph.*

Proof Let the vertex set of $P_2 \cup kK_1$ be $\{z_1, z_2, x_1, x_2, \cdots, x_k\}$ and $V(N_2) = \{y_1, y_2\}$. We have $q = 2k + 5$. Define a labeling $f: V((P_2 \cup kK_1) + N_2) \to \{1, 2, \cdots, 2k + 5\}$ by

$$\begin{aligned} &f(y_1) = 0,\ f(y_2) = 2k + 5,\ f(z_1) = 2 \\ &f(z_2) = 2k + 4 \\ &f(x_i) = 4 + 2(i - 1), \quad 1 \leq i \leq k \end{aligned}$$

Then, f is a mean labeling and hence $(P_2 \cup kK_1) + N_2$ is a mean graph for $k \geq 1$. □

For example, a mean labeling of $(P_2 \cup 5K_1) + N_2$ is shown in Figure 5.

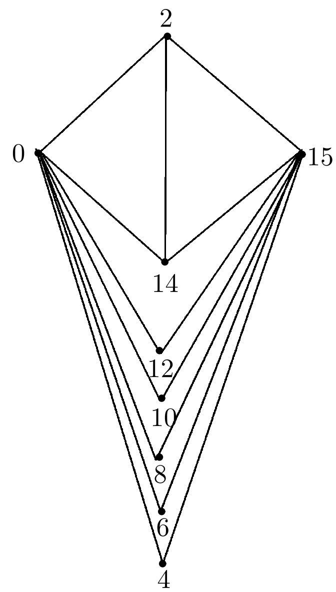

Figure 5

Let $S = \{\uparrow, \downarrow\}$ be the symbol representing, the position of the block as given in Figure 6.

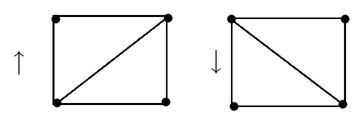

Figure 6

Let α be a sequence of n symbols of S, $\alpha \in S^n$. We will construct a graph by tiling n blocks side by side with their positions indicated by α. We will denote the resulting graph by $TB(\alpha)$ and refer to it as a triangular belt.

For example, the triangular belts corresponding to sequences $\alpha_1 = \{\downarrow\uparrow\uparrow\}$, $\alpha_2 = \{\downarrow\downarrow\uparrow\downarrow\}$ respectively are shown in Figure 7.

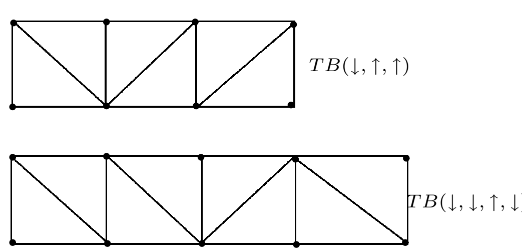

Figure 7

Theorem 2.5 *A triangular belt $TB(\alpha)$ is a mean graph for any α in S^n with the first and last block are being $\downarrow$ for all $n \geq 1$.*

Proof Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the top vertices of the belt and $v_1, v_2, \ldots, v_n, v_{n+1}$ be the bottom vertices of the belt. The graph $TB(\alpha)$ has $2n+2$ vertices and $4n+1$ edges. Define $f : V(TB(\alpha)) \rightarrow \{0, 1, 2, \ldots, q = 4n+1\}$ as follows :

$$f(u_i) = 4i, \quad 1 \leq i \leq n$$
$$f(u_{n+1}) = 4n+1$$
$$f(v_1) = 0$$
$$f(v_i) = 2 + 4(i-2), \quad 2 \leq i \leq n$$

Then f gives a mean labeling. Thus $TB(\alpha)$ is a mean graph for all $n \geq 1$. □

For example, a mean labeling of $TB(\alpha)$, $TB(\beta)$ and $TB(\gamma)$ are shown in Figure 8.

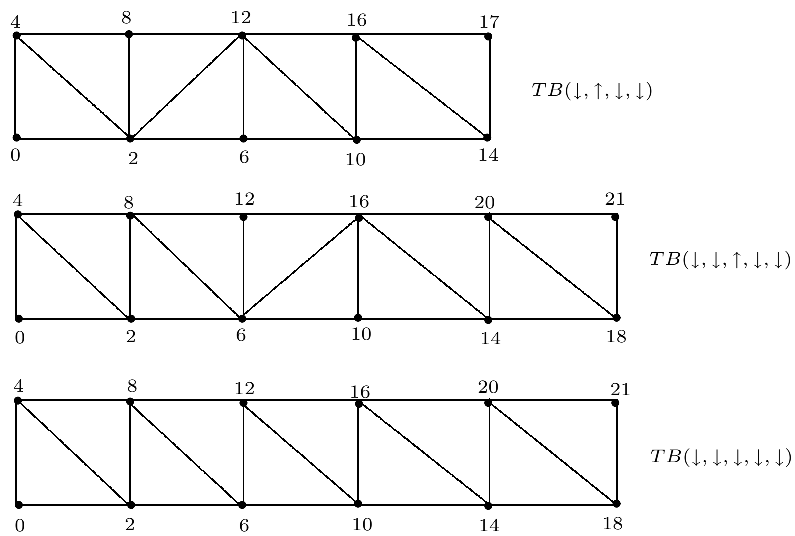

Figure 8

Corollary 2.6 *The graph P_n^2 is a mean graph.*

Proof The graph P_n^2 is isomorphic to $TB(\downarrow, \downarrow, \downarrow, \ldots, \downarrow)$ or $TB(\uparrow, \uparrow, \uparrow, \ldots, \uparrow)$. Hence the result follows from Theorem 2.5. □

We now consider a class of planar graphs that are formed by amalgamation of triangular belts. For each $n \geq 1$ and α in S^n n blocks with the first and last block are $\downarrow$ we take the triangular belt $TB(\alpha)$ and the triangular belt $TB(\beta)$, β in S^k where $k > 0$.

We rotate $TB(\beta)$ by 90 degrees counter clockwise and amalgamate the last block with the first block of $TB(\alpha)$ by sharing an edge. The resulting graph is denoted by $TBL(n, \alpha, k, \beta)$, which has $2(nk+1)$ vertices, $3(n+k)+1$ edges with

$$V(TBL(n, \alpha, k, \beta)) = \{u_{1,1}, u_{1,2}, \cdots, u_{1,n+1}, u_{2,1}, u_{2,2}, \cdots, u_{2,n+1}, v_{3,1}, v_{3,2}, \cdots, v_{3,k-1}, v_{4,1}, v_{4,2}, \cdots, v_{4,k-1}\}.$$

Theorem 2.7 *The graph $TBL(n,\alpha,k,\beta)$ is a mean graph for all α in S^n with the first and last block are $\downarrow$ and β in S^k for all $k > 0$.*

Proof Define $f : V(TBL(n,\alpha,k,\beta)) \to \{0,1,2,\ldots,3(n+k)+1\}$ as follows:

$$f(u_{1,i}) = 4k + 4i, \quad 1 \le i \le n$$
$$f(u_{1,n+1}) = 4(n+k) + 1$$
$$f(u_{2,1}) = 4k$$
$$f(u_{2,i}) = 4k + 2 + 4(i-2), \quad 2 \le i \le n+1$$
$$f(v_{3,i}) = 4i - 4, \quad 1 \le i \le k$$
$$f(v_{4,i}) = 4i - 2, \quad 1 \le i \le k$$

Then f provides a mean labeling and hence $TBL(n,\alpha,k,\beta)$ is a mean graph. □

For example, a mean labeling of $TBL(4,\downarrow,\uparrow,\uparrow,\downarrow,2,\uparrow,\uparrow)$ and $TBL(5,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,3,\uparrow,\downarrow,\uparrow)$ is shown in Figure 9.

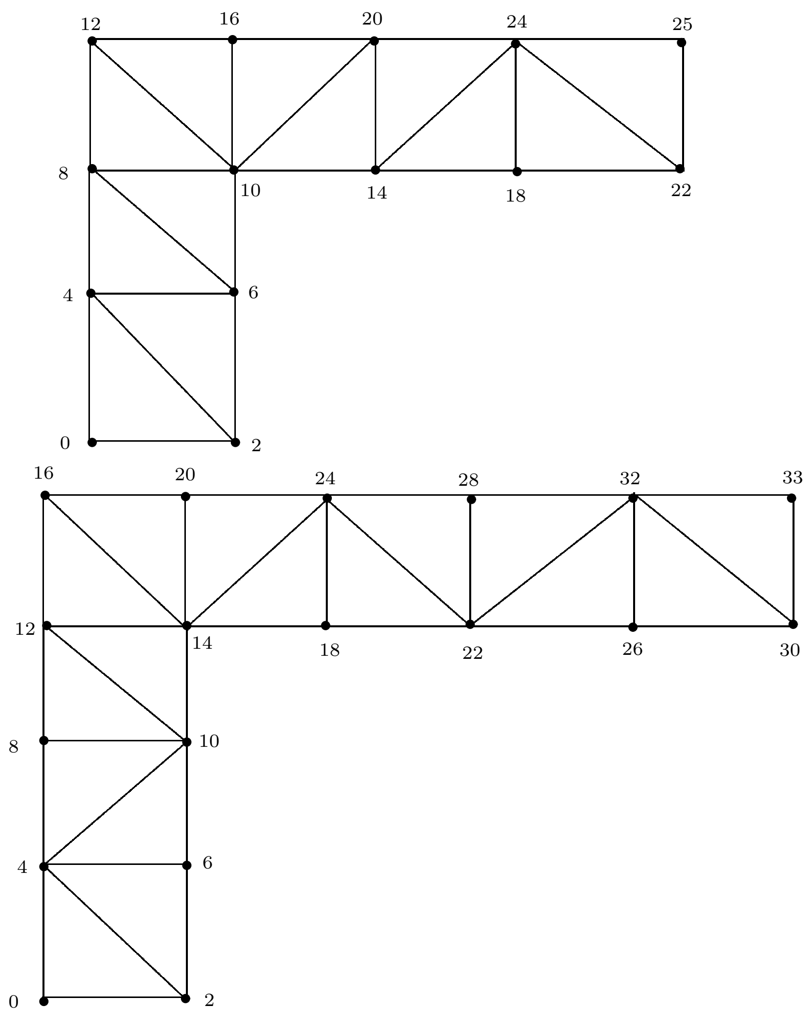

Figure 9

Theorem 2.8 *The graph edge mC_n-snake, $m \geq 1, n \geq 3$ has a mean labeling.*

Proof Let $v_{1_j}, v_{2_j}, \ldots, v_{n_j}$ be the vertices and $e_{1_j}, e_{2_j}, \ldots, e_{n_j}$ be the edges of edge mC_n-snake for $1 \leq j \leq m$.

Case 1 n is odd

Let $n = 2k+1$ for some $k \in Z^+$. Define a vertex labeling f of edge mC_n-snake as follows:

$$\begin{aligned} f(v_{1_1}) &= 0,\ f(v_{2_1}) = 1 \\ f(v_{i_1}) &= 2i-2, \quad 3 \leq i \leq k+1 \\ f(v_{(k+1+i)_1}) &= n - 2(i-1), \quad 1 \leq i \leq k \\ f(v_{1_2}) &= f(v_{(k+2)_1}),\ f(v_{2_2}) = f(v_{(k+1)_1}), \\ f(v_{i_2}) &= n + 4 + 2(i-3), \quad 3 \leq i \leq k+1 \\ f(v_{(k+1+i)_2}) &= 2n - 2 - 2(i-1), \quad 1 \leq i \leq k-1 \\ f(v_{n_2}) &= n+2 \\ f(v_{i_j}) &= f(v_{i_{j-2}}) + 2n - 2, \quad 3 \leq j \leq m, \quad 1 \leq i \leq n. \end{aligned}$$

Then f gives a mean labeling.

Case 2 n is even

Let $n = 2k$ for some $k \in Z^+$. Define a labeling f of edge mC_n-snake as follows:

$$\begin{aligned} f(v_{1_1}) &= 0,\ f(v_{2_1}) = 1, \\ f(v_{i_1}) &= 2i-2, \quad 3 \leq i \leq k+1 \\ f(v_{(k+1+i)_1}) &= n - 1 - 2(i-1), \quad 1 \leq i \leq k-1 \\ f(v_{i_j}) &= f(v_{i_{j-1}}) + n - 1, \quad 2 \leq j \leq m, \quad 1 \leq i \leq n \end{aligned}$$

Then f is a mean labeling. Thus the graph edge mC_n-snake is a mean graph for $m \geq 1$ and $n \geq 3$. □

For example, a mean labeling of edge $4C_7$-snake and $5C_6$-snake are shown in Figure 10.

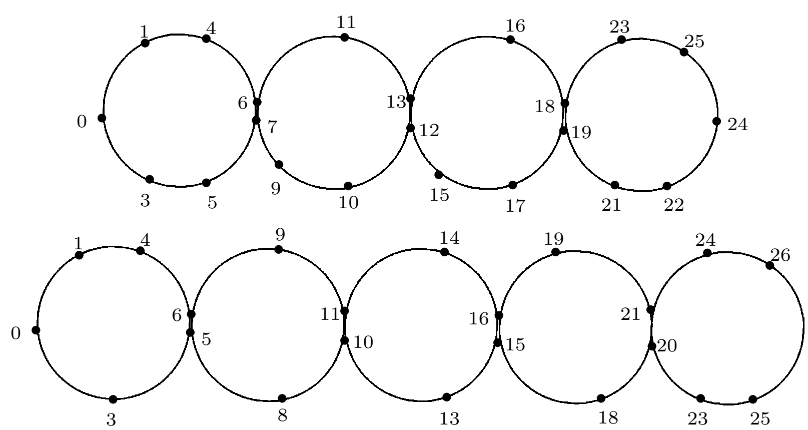

Figure 10

Theorem 2.9 *Let G' be a graph obtained by identifying an edge of two cycles C_m and C_n. Then G' is a mean graph for $m, n \geq 3$.*

Proof Let us assume that $m \leq n$.

Case 1 m is odd and n is odd

Let $m = 2k+1$, $k \geq 1$ and $n = 2l+1$, $l \geq 1$. The G' has $m+n-2$ vertices and $m+n-1$ edges. We denote the vertices of G' as follows:

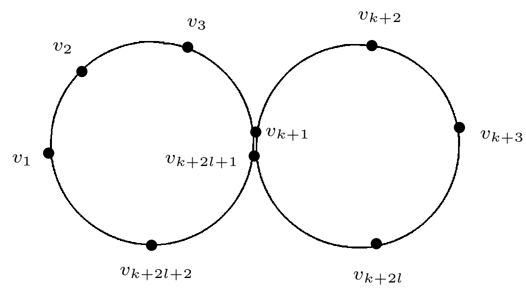

Figure 11

Define $f : V(G') \rightarrow \{0, 1, 2, 3, \ldots, q = m+n-1\}$ as follows:

$$f(v_1) = 0,\ f(v_i) = 2i-1, \quad 2 \leq i \leq k+1$$
$$f(v_i) = m+3+2(i-k-2), \quad k+2 \leq i \leq k+l$$
$$f(v_i) = m+n-1-2(i-k-l-1), \quad k+l+1 \leq i \leq k+2l$$
$$f(v_i) = m-1-2(i-k-2l-1), \quad k+2l+1 \leq i \leq 2k+2l$$

Then f is a mean labeling.

Case 2 m is odd and n is even

Let $m = 2k+1$, $k \geq 1$ and $n = 2l$, $l \geq 2$. Define $f : V(G') \rightarrow \{0, 1, 2, 3, \ldots, q = m+n-1\}$ as follows:

$$f(v_1) = 0,\ f(v_i) = 2i-1, \quad 2 \leq i \leq k+1$$
$$f(v_i) = m+3+2(i-k-2), \quad k+2 \leq i \leq k+l$$
$$f(v_i) = m+n-2-2(i-k-l-1), \quad k+l+1 \leq i \leq k+2l-1$$
$$f(v_i) = m-1-2(i-k-2l), \quad k+2l \leq i \leq 2k+2l-1$$

Then, f gives a mean labeling.

Case 3 m and n are even

Let $m = 2k$, $k \geq 2$ and $n = 2l$, $l \geq 2$. Define f on the vertex set of G' as follows:

$$f(v_1) = 0,\ f(v_i) = 2i-2, \quad 2 \leq i \leq k+1$$
$$f(v_i) = m+3+2(i-k-2), \quad k+2 \leq i \leq k+l$$
$$f(v_i) = m+n-2-2(i-k-l-1), \quad k+l+1 \leq i \leq k+2l-1$$
$$f(v_i) = m-1-2(i-k-2l), \quad k+2l \leq i \leq 2k+2l-2$$

Then, f is a mean labeling. Thus G' is a mean graph. □

For example, a mean labeling of the graph G' obtained by identifying an edge of C_7 and C_{10} are shown in Figure 12.

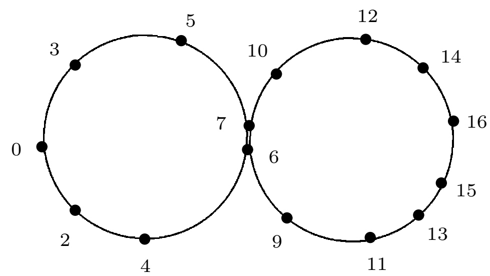

Figure 12

Theorem 2.10 *Let $\{u_iv_iw_iu_i : 1 \leq i \leq n\}$ be a collection of n disjoint triangles. Let G be the graph obtained by joining w_i to $u_{i+1}, 1 \leq i \leq n-1$ and joining u_i to u_{i+1} and $v_{i+1}, 1 \leq i \leq n-1$. Then G is a mean graph.*

Proof The graph G has $3n$ vertices and $6n-3$ edges respectively. We denote the vertices of G as in Figure 13.

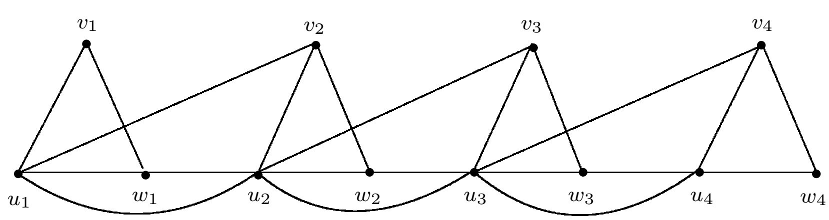

Figure 13

Define $f : V(G) \to \{0, 1, 2, \ldots, 6n-3\}$ as follows:

$$f(u_i) = 6i-4, 1 \leq i \leq n$$
$$f(v_i) = 6i-6, 1 \leq i \leq n$$
$$f(w_i) = 6i-3, 1 \leq i \leq n.$$

Then f gives a mean labeling and hence G is a mean graph. □

For example, a mean labeling of G when $n = 6$ is shown Figure 14.

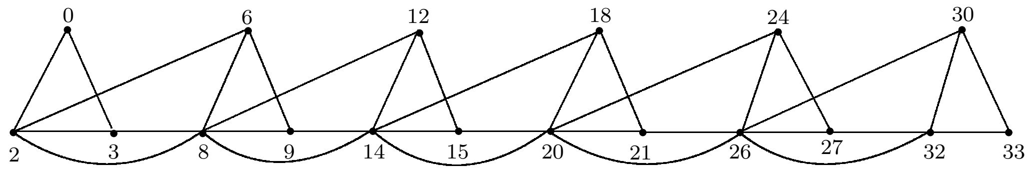

Figure 14

The graph obtained by attaching m pendant vertices to each vertex of a path of length $2n-1$ is denoted by $B(m)_{(n)}$. Dividing each edge of $B(m)_{(n)}$ by t number of vertices, the resultant graph is denoted by $S_t(B(m)_{(n)})$.

Theorem 2.11 *The $S_t(B(m)_{(n)})$ is a mean graph for all $m, n, t \geq 1$.*

Proof Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the path of length $2n-1$ and $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ be the pendant vertices attached at $v_i, 1 \leq i \leq 2n$ in the graph $B(m)_{(n)}$. Each edge $v_iv_{i+1}, 1 \leq i \leq 2n-1$, is subdivided by t vertices $x_{i,1}, x_{i,2}, \ldots, x_{i,t}$ and each pendant edge $v_iu_{i,j}$, $1 \leq i \leq 2n, 1 \leq j \leq m$ is subdivided by t vertices $y_{i,j,1}, y_{i,j,2}, \ldots, y_{i,j,t}$.

The vertices and their labels of $S_t(B(m)_{(1)})$ are shown in Figure 15.

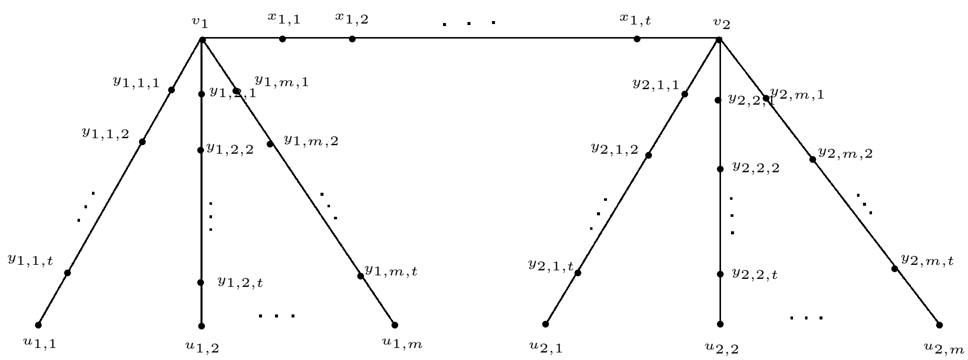

Figure 15

Define $f : V(S_t(B(m)_{(n)})) \to \{0, 1, 2, \ldots, (t+1)(2mn+2n-1)\}$ as follows:

$$f(v_i) = \begin{cases} (t+1)(m+1)(i-1) & \text{if } i \text{ is odd and } 1 \leq i \leq 2n-1 \\ (t+1)[(m+1)i-1] & \text{if } i \text{ is even and } 1 \leq i \leq 2n-1 \end{cases}$$

$$f(x_{i,k}) = \begin{cases} (t+1)[(m+1)i+m-1]+k & \text{if } i \text{ is odd, } 1 \leq i \leq 2n-1 \text{ and } 1 \leq k \leq t \\ (t+1)[(m+1)i-1]+k & \text{if } i \text{ is even, } 1 \leq i \leq 2n-1 \text{ and } 1 \leq k \leq t \end{cases}$$

$$f(y_{i,j,k}) = \begin{cases} (t+1)(m+1)(i-1) & \text{if } i \text{ is odd,} \\ +(2t+2)(j-1)+k, & 1 \leq i \leq 2n, 1 \leq j \leq m \text{ and } 1 \leq k \leq t \\ (t+1)[(m+1)(i-2)+1] & \text{if } i \text{ is even,} \\ +(2t+2)(j-1)+k, & 1 \leq i \leq 2n, 1 \leq j \leq m \text{ and } 1 \leq k \leq t \end{cases}$$

$$\text{and } f(u_{i,j}) = \begin{cases} (t+1)[(m+1)(i-1)+1] & \text{if } i \text{ is odd,} \\ +(2t+2)(j-1), & 1 \leq i \leq 2n \text{ and } 1 \leq j \leq m \\ (t+1)[(m+1)(i-2)+2] & \text{if } i \text{ is even,} \\ +(2t+2)(j-1), & 1 \leq i \leq 2n \text{ and } 1 \leq j \leq m. \end{cases}$$

Then, f is a mean labeling. Thus $S_t(B(m)_{(n)})$ is a mean graph. □

For example, a mean labeling of $S_3(B(4)_{(2)})$ is shown in Figure 16.

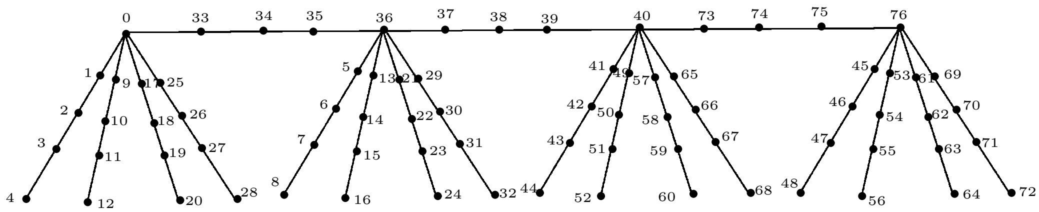

Figure 16

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Math.Combin.Book Ser. Vol.3(2013), 35-43

Special Kinds of Colorable Complements in Graphs

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Abstract: Let $G = (V, E)$ be a graph and $C = \{C_1, C_2, \cdots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. Then the graph G is a k-colorable complement graph G^C_k(with respect to C) if for all C_i and $C_j, i \neq j$, remove the edges between C_i and C_j, and add the edges which are not in G between C_i and C_j. Similarly, the $k(i)$- colorable complement graph $G^C_{k(i)}$ of a graph G is obtained by removing the edges in $\langle C_i \rangle$ and $\langle C_j \rangle$ and adding the missing edges in them. This paper aims at the study of Special kinds of colorable complements of a graph and its relationship with other graph theoretic parameters are explored.

Key Words: Graph, complement, k-complement, k(i)-complement, colorable complement.

AMS(2010): 05C15, 05C70

§1. Introduction

All the graphs considered here are finite, undirected and connected with no loops and multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges at a graph G, respectively. For the open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V/uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood is $N[v] = N(v) \bigcup \{v\}$. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X. If $deg(v)$ is the degree of vertex v and usually, $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree. The complement G_c of a graph G defined to be graph which has V as its sets of vertices and two vertices are adjacent in G_c if and only if they are not adjacent in G. Further, a graph G is said to be self-complementary (s.c), if $G \cong G_c$. For notation and graph theory terminology we generally follow [3], and [5].

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of V. Then k-complement G^P_k and $k(i)$-complement $G^P_{k(i)}$(with respect to P) are defined as follows: For all V_i and $V_j, i \neq j$, remove the edges between V_i andV_j, and add the edges which are not in G between V_i and V_j. The graph G^P_k thus obtained is called the k-complement of a graph G with respect to P. Similarly, the $k(i)$-complement of $G^P_{k(i)}$ of a graph G is obtained by removing the edges in $\langle V_l \rangle$ and $\langle V_j \rangle$ and adding the missing edges in them for $l \neq j$. This concept was first

[1]Received June 13, 2013, Accepted August 10, 2013.

introduced by Sampathkumar et al. [9] and [10]. For more detail on complement graphs, we refer [1], [2], [4], [8], [11] and [12].

A graph is said to be k-vertex colorable (or k-colorable) if it is possible to assign one color from a set of k colors to each vertex such that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An k-coloring of a graph G uses k colors: it there by partitions V into k color classes. The chromatic number $\chi(G)$ is defined as the minimum k for which G has an k-coloring. Hence, graph G is a k-colorable if and only if $\chi(G) \leq k$, [7].

We make use of the following results in sequel [6].

Theorem 1.1 *For any non-trivial graph G,*

$$\sum_{x_i \in V} deg(x_i) = 2m.$$

Theorem 1.2(Konig's [5]) *In a bipartite graph G, $\alpha_1(G) = \beta_0(G)$. Consequently, if a graph G has no vertex of degree 0, then $\alpha_0(G) = \beta_1(G)$.*

§2. k-Colorable Complement

Let $G = (V, E)$ be a graph. If there exists a k-coloring of a graph G if and only if $V(G)$ can be partitioned into k subsets $C_1, C_2, \cdots, C_k$ such that no two vertices in color classes of $C_i, i = 1, 2, \cdots, k$, are adjacent. Then, we have the following definitions.

Definition 2.1 *The k-colorable complement graph G_k^C (with respect to C) of a graph G is obtained by for every C_i and C_j, $i \neq j$, remove the edges between C_i and C_j in G, and add the edges which are not in a graph G.*

Definition 2.2 *The graph G is k-self colorable complement graph, if $G \cong G_k^C$.*

Definition 2.3 *The graph G is k-co-self colorable complement graph, if $G_c \cong G_k^C$.*

Lemma 2.1 *Let G be a k-colorable graph. Then in any k-coloring of G, the subgraph induced by the union of any two color classes is connected.*

Proof If possible, let C_1 and C_2 be two color classes of vertex set $V(G)$ such that the subgraph induced by $C_1 \cup C_2$ is disconnected. Let G_1 be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no vertex of G_1 is adjacent to a vertex in $V(G) - V(G_1)$, which is assign the color either C_1 or C_2. Thus interchanging the colors of the vertices in G_1 and retaining the original colors for all other vertices, we gets a different k-coloring of a graph G, which is a contradiction. □

Theorem 2.1 *Let G be a (n, m)-graph. If for every C_l and C_j, $l \neq j$, and each vertex of C_l is adjacent to each vertex of C_j, then $m(G_k^C) = \varnothing$.*

Proof If for every C_l and C_j, $l \neq j$ in a (n, m)- graph with $\langle C_k \rangle$ is totally disconnected, where C_k is the partition of color classes of vertex set $V(G)$, then by the definition of k-colorable complement, $m(G_k^C) = \varnothing$ follows. Conversely, suppose the given condition is not satisfied, then there exist at least two vertices u and v such that $u \in C_l$ is not adjacent to vertex $u \in C_j$ with $l \neq j$. Thus by above lemma, this implies that $m(G_k^C) \geq 1$, which is a contradiction. □

A graph that can be decomposed into two partite sets but not fewer is bipartite; three sets but not fewer, tripartite; k sets but not fewer, k-partite; and an unknown number of sets, multipartite. An 1-partite graph is the same as an independent set, or an empty graph. A 2-partite graph is the same as a bipartite graph. A graph that can be decomposed into k partite sets is also said to be k-colorable. That is $\chi(K_n) = n$, but the chromatic number of complete k- partite graph $\chi(K_{r_1,r_2,r_3,\cdots,r_k}) = k < n$ for $r_i > 2$, where $i = 1, 2, \cdots, k$. By virtue of the facts, we have following corollaries.

Corollary 2.1 *Let G be a complete graph K_n; $n \geq 1$ vertices and $m = \dfrac{n(n-1)}{2}$ edges with $\chi(K_n) = n$. Then $m(G_n^C) = \varnothing$.*

Corollary 2.2 *Let G be a complete bipartite graph K_{r_1,r_2}; $1 \leq r_1 \leq r_2$, with $\chi(K_{r_1,r_2}) = 2$ for $n = (r_1 + r_2)$- vertices and m =$(r_1.r_2)$ edges. Then $m(G_2^C) = \varnothing$.*

Theorem 2.2 *Let G be a path P_n with $\chi(P_n) = 2$; $n \geq 2$ vertices. Then*

$$m(G_2^C) = \begin{cases} \frac{1}{4}(n-2)^2 & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)(n-3) & \text{if } n \text{ is odd.} \end{cases}$$

Proof Let G be a path P_n with $\chi(P_n) = 2$; $n \geq 2$ vertices, and $C = \{C_1, C_2\}$ be a partition of colorable class of vertex set of P_n. We have the following cases.

Case 1 If $\{u_1, u_2, \cdots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \cdots, v_{t-1}, v_t\} \in C_2$ with $v_1 - v_t$ is path of even length. Then $u_1, u_2, \cdots, u_{t-1}$ are adjacent $(t-2)$-vertices, that is $deg(u_i) = (t-2)$ if $1 \leq i \leq t-1$. Similarly, $v_1, v_2, \cdots, v_t$ are adjacent to $(t-2)$- vertices that is $deg(u_i) = (t-2)$ if $2 \leq i \leq t-1$, and v_1 and u_t are adjacent to $(t-1)$- vertices in G_2^C. Thus, $2(t-1) + (n-2)(t-2) = 2m(G_2^C)$. By Theorem 1.1, with the fact that $n = 2t$ and $m(G) = n-1$. Hence $m(G_2^C) = \dfrac{1}{4}(n-2)^2$.

Case 2 If $\{u_1, u_2, \cdots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \cdots, v_t, v_{t+1}\} \in C_2$ with $v_1 - v_{t+1}$ is path of even length. Then $u_1, u_2, \cdots, u_t$ are adjacent $(t-1)$-vertices, $v_2, v_3, \cdots, v_t$ are adjacent to $(t-2)$- vertices and, v_1 and u_{t-1} are adjacent to $(t-1)$ - vertices in G_2^C. Thus, $t(t-1) + (t-1)(t-2) + 2(t-1) = 2m(G_2^C)$. By theorem 1.1, with the fact that $n = 2t+1$ and $m(G) = n-1$. Hence $m(G_2^C) = \dfrac{1}{4}(n-1)(n-3)$. □

Theorem 2.3 *Let G be a cycle C_n; $n \geq 3$ vertices. Then*

(i) $m(G_2^C) = \dfrac{(n-4)n}{4}$, *if $\chi(C_n) = 2$ and n is even.*

(ii) $m(G_3^C) = \dfrac{(n+1)(n-3)}{4}$, *if $\chi(C_n) = 3$ and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle C_n.*

Proof The proof follows from Theorem 2.2, with even cycle of C_n and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle C_n. □

Theorem 2.4 *Let G be a Wheel W_n; $n \geq 4$ vertices and $m = 2(n-1)$ edges. Then*

(i) $m(G_4^C) = \dfrac{(n-4)n}{4}$, *if $\chi(C_n) = 4$ and n is even.*

(ii) $m(G_3^C) = \dfrac{(n+1)(n-3)}{4}$, *if $\chi(W_n) = 3$ and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle C_{n-1} of W_n.*

Proof By Theorem 2.3 and $m(K_1) = 0$ due to the fact of $W_n = K_1 + C_{n-1}$, the result follows. □

Theorem 2.5 *Let T be a nontrivial tree with $\chi(T) = 2$. Then*

$$m(G_2^C) = (r_1.r_2) - n(T) + 1.$$

Proof Let $C = \{C_1, C_2\}$ be a partition of colorable class of a tree T with $n \geq 2$ vertices and $m(T) = n(T) - 1$. If every vertex in C_1 is adjacent to every vertex in C_2, that is K_{r_1,r_2} with $m(K_{r_1,r_2}) = r_1.r_2$. By definition of G_k^C with $\chi(T) = 2$, we have $m(G_2^C) = m(K_{r_1,r_2}) - m(T)$. Thus the results follows. □

Theorem 2.6 *For any non trivial graph G is k - self colorable complement if and only if $G \cong P_7$ or $2K_2$.*

Proof By definition of k-self colorable complement. It is clear that both G and G_2^C are isomorphic to P_7 or $2K_2$ with $\chi(P_7) = \chi(2K_2) = 2$. On the other hand, suppose G is k-self colorable complement, when G is not isomorphic with P_7 or $2K_2$. Then there exist at least two adjacent vertices u and v in G such that $u \in C_1$ and $v \in C_2$ are in disjoint color classes of $C = \{C_1, C_2\}$ with $\chi(P_7) = \chi(2K_2) = 2$. This implies that, u and v are not adjacent in G_2^C or they are in one color classes in G_1^C, that is totally disconnected graph. Thus the graph G and its colorable complements G_k^C are not isomorphic to each other, which is a contradiction. Hence the results follows. □

Theorem 2.7 *Let G be a k-self colorable complement graph. Then G has a vertex of degree at least* $\dfrac{n(\chi(G)-1)}{2\chi(G)}$.

Proof Let G be a (n,m)- graph with $G \cong G_k^C$ and $C = \{C_1, C_2, \cdots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. Suppose, if $\chi(G) = k$ and $V(G)$ is partitioned into k independent sets $C_1, C_2, \cdots, C_k$. Thus, $n = |V(G)| = |C_1, C_2, \cdots, C_k| = \sum_{i=1}^{k} |V(G)| \leq k\beta(G)$, where $\beta(G)$ is the independence number of a graph G. There fore $\chi(G) = k = n/\beta(G)$. Also, suppose $v \in C_i$, where C_i is a colorable set in C with at most $n/\chi(G)$. Then the sum of the degree of v in G and G_k^C is greater than $\dfrac{n(\chi(G)-1)}{\chi(G)}$. This implies that the degree of v is at least $\dfrac{1}{2}(n - \dfrac{n}{\chi(G)})$. Hence the result follows. □

Theorem 2.8 *Let G be a k-self colorable complement graph. Then*

$$\frac{(k-1)(2n-k)}{4} \leq m(G) \leq \frac{2n(n-k)+k(k-1)}{4}.$$

Proof Let G be a k-self colorable complement graph and $C = \{C_1, C_2, \cdots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. If $|C_t| = n_t$ for $1 \leq t \leq k$, then the total number of edges between C_l and C_j in C, $l \neq j$, in both the graph G and its colorable complement graph G_k^C is $\sum\limits_{l \neq j} n_l n_j$. Since the graph G is k-self colorable complement graph G_k^C, half of these edges are not there in G. Hence $m(G) \leq \begin{pmatrix} n \\ 2 \end{pmatrix} - \sum\limits_{l \neq j} n_l n_j$. Clearly, $\sum\limits_{l \neq j} n_l n_j$ is minimum, when $n_t = 1$ for $k-1$ of the indices. Thus, we have

$$m(G) \leq \begin{pmatrix} n \\ 2 \end{pmatrix} - \frac{1}{2}[\begin{pmatrix} k-1 \\ 2 \end{pmatrix} + (k-1)(n-k+1)].$$

Hence the upper bound follow. To establish the lower bound, the graph G being k-self colorable complement has at least $\sum\limits_{l \neq j} n_l n_j$- edges. So, $\frac{1}{2}[\begin{pmatrix} k-1 \\ 2 \end{pmatrix} + (k-1)(n-k+1)] \leq m(G)$ and the result follows. □

Theorem 2.9 *For any non trivial graph G is k - co - self colorable complement if and only if $G \cong K_n$.*

Proof On contrary, suppose given condition is not satisfied, then there exists at least three vertices u, v and w such that v is adjacent to both u and w, and u is not adjacent to w. This implies that an edge $e = uw \in G_c$ and induced subgraph $\langle u, v, w \rangle$ in G_2^C is totally disconnected. Thus $E(G_2^C) \subset E(G_c)$, which is a contradiction to the fact of $G_c \cong G_n^C$ with $\chi(K_n) = n$. Converse is obvious. □

§3. $k(i)$-Colorable Complement

Let $G = (V, E)$ be a graph and $C = \{C_1, C_2, \cdots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. Then, we have the following definitions.

Definition 3.1 *The $k(i)$ - colorable complement graph $G_{k(i)}^C$ (with respect to C) of a graph G is obtained by removing the edges in $\langle C_l \rangle$ and $\langle C_j \rangle$ and adding the missing edges in them for $l \neq j$.*

Definition 3.2 *The graph G is $k(i)$-self colorable complement graph, if $G \cong G_{k(i)}^C$.*

Definition 3.3 *The graph G is $k(i)$-co-self colorable complement graph, if $G_c \cong G_{k(i)}^C$.*

Theorem 3.1 *For any graph G, $m(G_{k(i)}^C) = \dfrac{n(n-1)}{2}$ if and only if the graph G is isomorphic with complete n- partite graph $K_{r_1, r_2, r_3, \cdots, r_n}$ or $(K_n)_c$.*

Proof To prove the necessity, we use the mathematical induction. Let G be a graph with $n=1$ vertex. Then $\chi(G)=1$ and $m(G^C_{1(i)})=\varnothing$. Hence the result follows. Suppose the graph G with $n>1$ vertices. Then the following cases are arises.

Case 1 If the graph G is totally disconnected, that is $(K_n)_c$, complement of a complete graph K_n, then G has a only one color class C_1 with $\chi((K_n)_c)=1$. By the definition of $G^C_{1(i)}$, the induced subgraph of $\langle C_1\rangle$ is complete, which form a $\frac{n(n-1)}{2}$- edges.

Case 2. If the graph G is complete n- partite graph $K_{r_1,r_2,r_3,\ldots,r_n}$, then for every two color classes C_l and C_j for $l\neq j$, and each vertex C_l adjacent to each vertex of C_j in complete n-partite graph $K_{r_1,r_2,r_3,\ldots,r_n}$ with $m(K_{r_1,r_2,r_3,\ldots,r_n})=r_1r_2r_3\ldots r_n$. By the definition of $G^C_{n(i)}$ with $G=K_{r_1,r_2,r_3,\ldots,r_n}$, we have

$$m(G^C_{n(i)})=\left(\begin{array}{c} r_1\\ 2\end{array}\right)+\left(\begin{array}{c} r_2\\ 2\end{array}\right)+\ldots+\left(\begin{array}{c} r_n\\ 2\end{array}\right)+r_1r_2r_3\ldots r_n,$$

where $\left(\begin{array}{c} r_t\\ 2\end{array}\right)$ is the maximum number edges of induced subgraph $\langle C_t\rangle$ if $t=1,2,\ldots,n$, which are complete. This forms $\frac{n(n-1)}{2}$- edges.

Conversely, suppose the graph G is not isomorphic to complete n- partite graph $K_{r_1,r_2,r_3,\ldots,r_n}$ or $(K_n)_c$. Then there exist at least three vertices $\{a,b,c\}$ such that at least two adjacent vertices a and b are not adjacent to isolated vertex c. By the definition of $G^C_{k(i)}$ with $\chi(G)=k\geq 2$, which form a path $(a-b-c)$ or $(b-a-c)$ of length 2, which is not a complete, a contradiction. This proves the sufficiency. □

Theorem 3.2 *Let G be a path P_n with $\chi(P_n)=2$ and $n\geq 2$ vertices. Then*

$$m(G^C_{2(i)})=\begin{cases}\frac{1}{4}[n^2+2n-4]^2 & \textit{if } n \textit{ is even}\\ \frac{1}{4}(n-1)(n+3) & \textit{if } n \textit{ is odd}\end{cases}$$

Proof Let G be a path P_n with $\chi(P_n)=2$; $n\geq 2$ vertices, and $C=\{C_1,C_2\}$ be a partition of colorable class of vertex set of P_n. We have the following cases.

Case 1 Let $C=\{C_1,C_2\}$ be a partition of colorable class of P_n. If $\{u_1,u_2,\cdots,u_{t-1},u_t\}\in C_1$ and $\{v_1,v_2,\cdots,v_{t-1},v_t\}\in C_2$ with v_1-u_t is path of even length. Then $\langle C_1\rangle$ and $\langle C_2\rangle$ are complete in $G^C_{2(i)}$ and also v_1-u_t path have $(n-1)$ - edges in both the graph G and its $k(i)$-colorable complement graph $G^C_{2(i)}$. Thus, $m(G)+t(t-1)=(n-1)+n(n-2)/4=m(G^C_{2(i)})$ and this implies $m(G^C_{2(i)})=\frac{1}{4}[n^2+2n-4]^2$.

Case 2 Let $C=\{C_1,C_2\}$ be a partition of colorable class of P_n. If $\{u_1,u_2,\cdots,u_{t-1},u_t\}\in C_1$ and $\{v_1,v_2,\cdots,v_{t-1},v_t\}\in C_2$ with v_1-u_{t+1} is path of odd length. Then $\langle C_1\rangle$ and $\langle C_2\rangle$ are complete in $G^C_{2(i)}$ and also v_1-u_{t+1} path have $(n-1)$ - edges in both the graph G and its $2(i)$-colorable complement graph $G^C_{2(i)}$. Thus, $m(G)+t(t-1)/2+t(t+1)/2=(n-1)[1+(n-3)/8+(n+1)/8]=m(G^C_{2(i)})$ and this implies $m(G^C_{2(i)})=\frac{1}{4}(n-1)(n+3)$. □

Theorem 3.3 *Let G be a cycle C_n; $n \geq 3$ vertices. Then*

(*i*) $m(G^C_{2(i)}) = \frac{1}{4}[n(n+2)]$, *if* $\chi(C_n) = 2$ *and* n *is even.*

(*ii*) $m(G^C_{3(i)}) = \frac{1}{4}(n^2+3)$, *if* $\chi(C_n) = 3$ *and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle* C_n.

Proof The proof follows from Theorem 3.2, with even cycle of C_n and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle C_n. □

Theorem 3.4 *Let T be a nontrivial tree with $\chi(T) = 2$. If $C = \{C_1, C_2\}$ be a partition of colorable class of a tree T, then*

$$m(G^C_{2(i)}) = \frac{1}{2}[r^2 + s^2 + n - 2],$$

where $|C_1| = r$ *and* $|C_2| = s$.

Proof Let $C = \{C_1, C_2\}$ be a partition of colorable class of a tree T with $\chi(T) = 2$ and $m(T) = n(T) - 1 = r + s + 1$. Then by definition of $G^C_{k(i)}$, we have $\langle C_1\rangle$ and $\langle C_2\rangle$ are complete.

There fore, $m(C_1) = \begin{pmatrix} r \\ 2 \end{pmatrix}$ and $m(C_2) = \begin{pmatrix} s \\ 2 \end{pmatrix}$.

Thus, we have

$$m(G^C_{2(i)}) = \begin{pmatrix} r \\ 2 \end{pmatrix} + \begin{pmatrix} s \\ 2 \end{pmatrix} + m(T) = \frac{1}{2}[r(r+1) + s(s+1) - 2].$$

Hence the result follows. □

Theorem 3.5 *For any non trivial graph G is $k(i)$ - self colorable complement if and only if G is isomorphic with K_n.*

Proof Let $G = K_n$ be a complete graph with $\chi(G) = n$. Then by the definition of $G^C_{k(i)}$, the induced subgraph $\langle C_t\rangle$ for $t = 1, 2, \ldots, n$ are connected and $|C_t| = 1$ for $t = 1, 2, \ldots, n$. Thus $G^C_{n(i)} \cong K_n$ and the result follows. Conversely, suppose given condition is not satisfied, then there exists at least two non adjacent vertices u and v in a graph G such that $\chi(G) = 1$ and $m(G) = \varnothing$. By the definition of $G^C_{k(i)}$, we have $\chi(G^C_{1(i)}) = 2$ with an induced subgraph $\langle u, v\rangle$ in $G^C_{1(i)}$ is connected. Thus $m(G) < m(G^C_{1(i)})$, which is a contradiction to the fact of $G \cong G^C_{k(i)}$. □

§4. $\{G, G^p_k, G^p_{k(i)}\}$ - Realizability

Here, we show the $G, G^p_k, G^p_{k(i)}$ - Realizability for some graph theoretic parameter.

Let G be a graph. Then $S \subseteq V(G)$ is a separating set if $G - S$ has more than one component. The connectivity $\kappa(G)$ of G is the minimum size of $S \subseteq V(G)$ such that $G - S$ is disconnected or a single vertex. For any $k \leq \kappa(G)$, we say that G is k-connected. Then, we have

Theorem 4.1 *Let G be a graph with $C = \{C_1, C_2\}$ be a partition of colorable class of a vertex set V. If $\langle C_1\rangle$ and $\langle C_2\rangle$ are $(t-1)$ -colorable with Max.$\{\chi(G_k^C), \chi(G_{k_i}^C)\} \geq t$, then Min.$\{k(G), k(G_k^C), k(G_{k_i}^C)\}$ has at least $(t-1)$ -edges.*

Theorem 4.2 *Let G be a (n,m)- graph. Then*

(*i*) $\chi(G_k^C) = 1$ *if and only if G is isomorphic with K_n or $(K_n)_c$ or $K_{r_1,r_2,r_3,\cdots,r_k}$.*
(*ii*) $\chi(G_{k(i)}^C) = n$ *if and only if G is isomorphic with K_n or $(K_n)_c$ or $K_{r_1,r_2,r_3,\cdots,r_k}$.*

Proof By the definition of G_k^C and Theorem 2.1, (i) follows. Also by the definition of $G_{k(i)}^C$ and Theorem 3.1, (ii) follows. □

A set M of vertices in a graph G is independent if no two vertices of M are adjacent. The number of vertices in a maximum independent set of G is denoted by $\beta(G)$. Opposite to an independent set of vertices in a graph is a clique. A clique in a graph G is a complete subgraph of G. The order of the largest clique in a graph G and its clique number, which is denoted by $\omega(G)$. In fact $\beta(G) = k$ if and only if $\omega(\overline{G}) = k$. Then, we have

Theorem 4.3 *Let G be a nontrivial (n,m)- graph. Then*

(*i*) $\beta(G_{k(i)}^C) \leq \beta(G) \leq \beta(G_k^C)$.
(*ii*) $\omega(G_k^C) \leq \omega(G) \leq \omega(G_{k_i}^C)$.

Acknowledgement

Thanks are due to Prof. E.Sampathkumar for his help and valuable suggestions in the preparation of this paper.

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Math.Combin.Book Ser. Vol.3(2013), 44-49

Vertex Graceful Labeling-Some Path Related Graphs

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Abstract: In this article, we show that an algorithm for VG of a caterpillar and proved that $A(m_j, n)$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$ and $m_1 < m_2$, $(m_j, n) \cup P_3$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$, $m_1 < m_2$ and $C_n \cup C_{n+1}$ is vertex graceful if and only if $n \geq 4$.

Key Words: Vertex graceful graphs, vertex graceful labeling, caterpillar, actinia graphs, Smarandachely vertex m-labeling.

AMS(2010): 05C78

§1. Introduction

A graph G with p vertices and q edges is said to be vertex graceful if a labeling $f : V(G) \rightarrow \{1, 2, 3 \cdots p\}$ exists in such a way that the induced labeling $f^+ : E(G) \rightarrow Z_q$ defined by $f^+((u, v)) = f(u) + f(v)(mod\ q)$ is a bisection. The concept of vertex graceful (VG) was introduced by Lee, Pan and Tsai in 2005. Generally, if replacing q by an integer m and $f^S : E(G) \rightarrow Z_m$ also is a bijection, such a labeling is called a *Smarandachely vertex m-labeling*. Thus a vertex graceful labeling is in fact a Smarandachely vertex q-labeling.

All graphs in this paper are finite simple graphs with no loops or multiple edges. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G. The cardinality of the vertex set is called the order of G. The cardinality of the edge set is called the size of G. A graph with p vertices and q edges is called a (p, q) graph.

§2. Main Results

Algorithm 2.1

1. Let $v_1, v_2 \cdots v_n$ be the vertices of a path in the caterpillar. (refer Figure 1).
2. Let v_{ij} be the vertices, which are adjacent to v_i for $1 \leq i \leq n$ and for any j.
3. Draw the caterpillar as a bipartite graph in two partite sets denoted as Left (L) which

[1]Received April 10, 2013, Accepted August 15, 2013.

contains $v_1, v_{2j}, v_3, v_{4j}, \cdots$ and for any j and Right (R) which contains $v_{1j}, v_2, v_{3j}, v_4, \cdots$ and for any j. (refer Figure 2).

4. Let the number of vertices in L be x.
5. Number the vertices in L starting from top down to bottom consecutively as $1, 2, \cdots, x$.
6. Number the vertices in R starting from top down to bottom consecutively as $(x + 1), \cdots, q$. Note that these numbers are the vertex labels.
7. Compute the edge labels by adding them modulo q.
8. The resulting labeling is vertex graceful labeling.

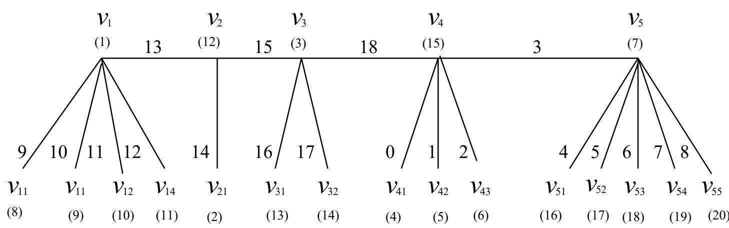

Figure 1: A caterpillar

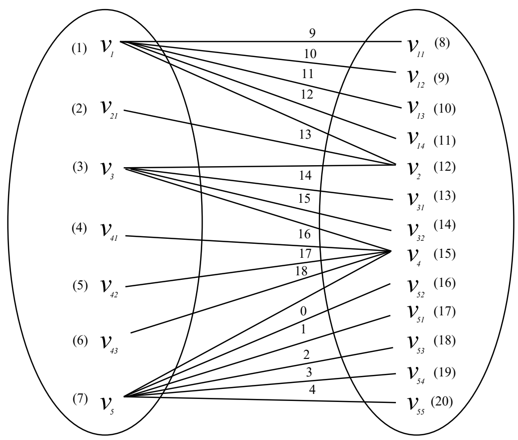

Figure 2: A caterpillar as bipartite graph

Definition 2.2 *The graph $A(m, n)$ obtained by attaching m pendent edges to the vertices of the cycle C_n is called Actinia graph.*

Theorem 2.3 *A graph $A(m_j, n), m_j$ is monotonically increasing with difference one, $2 \le j \le n$ is vertex graceful, $1 \le m_2 \le 3$ when n is odd.*

Proof Let the graph $G = A(m_j, n), m_j$ be monotonically increasing with difference one, $2 \le j \le n, n$ be odd with $p = n + m_n(\frac{m_n+1}{2}) - m_1(\frac{m_1+1}{2}), m_1 = m_2 - 1$ vertices and $q = p$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Let $v_{ij} (j = 1, 2, 3, \cdots, n)$ denote the vertices which are adjacent to v_i. By definition of vertex graceful labeling, the required vertices labeling are

$$v_i = \begin{cases} \dfrac{(i-1)}{2}\left(m_2+\frac{(i+1)}{2}\right)+1, 1 \leq i \leq n,\ i \text{ is odd,} \\ (m_2+1)\frac{(n+1)}{2}+\left(\frac{n-1}{2}\right)^2+\frac{(i-2)}{2}\left(m_2+\frac{i}{2}\right)+\frac{i}{2}, 1 \leq i \leq n,\ i \text{ iseven.} \end{cases}$$

$$v_{ij} = \begin{cases} \dfrac{(n-1)}{2}\left(m_2+\frac{(n+1)}{2}\right)+\frac{i-1}{2}\left(m_2+\frac{i-3}{2}\right)+\frac{i+1}{2}+j, 1 \leq j \leq m_2+i-1, i \text{ is odd;} \\ \dfrac{(i-2)}{2}\left(m_2+\frac{i-2}{2}\right)+\frac{i}{2}+j, 1 \leq i \leq m_2+i-1, i \text{ is even.} \end{cases}$$

The corresponding edge set labels are as follows:

Let $A = \{e_i = v_iv_{i+1}/1 \leq i \leq n-1 \cup e_n = v_nv_1\}$, where

$$e_i = \left[\frac{(m_2+1)(n+1)}{2}+\left(\frac{n-1}{2}\right)^2+m_2(i-1)+\frac{i(i+1)}{2}+1\right] (\text{mod } q)$$

for $1 \leq i \leq n$. $B = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2}\left(m_2+\frac{(n+1)}{2}\right)+(i-1)\left(m_2+\frac{i-1}{2}\right)+\frac{(i+1)}{2}+j+1\right] (\text{mod } q)$$

for $1 \leq i \leq n$ and i is odd, $j = 1, 2, \cdots, m_2+i-1$. $C = \{e_{ij} = v_iv_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[(m_2+1)\frac{(n+1)}{2}+\left(\frac{n-1}{2}\right)^2+\frac{i-2}{2}(2m_2+i-1)+i+j\right] (\text{mod } q)$$

for $1 \leq i \leq n$ and i is even, $j = 1, 2, \cdots, m_2+i-1$.

Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n)$ is vertex graceful for n is odd, and $m \geq 1$. □

Theorem 2.4 *A graph $A(m_j, n) \cup P_3, m_j$ be monotonically increasing, $2 \leq j \leq n$ is vertex graceful, $1 \leq m_2 \leq 3, n$ is odd.*

Proof Let the graph $G = A(m_j, n) \cup P_3, m_j$ be monotonically increasing ,$2 \leq j \leq n$, n is odd with $p = n+3+m_n\frac{(m_n+1)}{2}-m_1\frac{(m_1+1)}{2}, m_1 < m_2$ vertices and $q = p-1$ edges. Let$v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Let $v_{ij}(j = 1, 2, 3, \cdots, n)$ denote the vertices which are adjacent to v_i. Let u_1, u_2, u_3 be the vertices of the path P_3. By definition of vertex graceful labeling, the required vertices labeling are

$$v_i = \begin{cases} \frac{i-1}{2}\left(m_2+\frac{i+1}{2}\right)+1; 1 \leq i \leq n, i \text{ is odd;} \\ (m_2+1)\frac{(n+1)}{2}+\left(\frac{n-1}{2}\right)^2+\frac{(i-2)}{2}\left(m_2+\frac{i}{2}\right)+\frac{i}{2}+2; 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

$$v_{ij} = \begin{cases} \frac{n-1}{2}\left(m_2+\frac{n+1}{2}\right)+\frac{i-1}{2}\left(m_2+\frac{i-3}{2}\right)+\frac{i+1}{2}+j+2; 1 \leq i \leq n, i \text{ is odd,} \\ \dfrac{i-2}{2}\left(m_2+\frac{i-2}{2}\right)+\frac{i}{2}+j+2; 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

$u_i = \frac{n-1}{2}\left(m_2+\frac{n+1}{2}\right)+\frac{i+1}{2} for\ \ i = 1, 3$ and $u_2 = p$.

The corresponding edge labels are as follows:

Let $A = \{e_i = v_iv_{i+1}/1 \leq i \leq n-1 \cup e_n = v_nv_1\}$, where

$$e_i = \left[\frac{(m_2+1)(n+1)}{2}+\left(\frac{n-1}{2}\right)^2+m_2(i-1)+\frac{i(i+1)}{2}+3\right] (\text{mod } q)$$

for $1 \leq i \leq n$. $B = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2}\left(m_2 + \frac{(n+1)}{2}\right) + (i-1)\left(m_2 + \frac{i-1}{2}\right) + \frac{(i+1)}{2} + j + 3\right] (\text{mod } q)$$

for $1 \leq i \leq n$ and i is odd, $j = 1, 2, \cdots, m_2 + i - 1$. $C = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[(m_2+1)\frac{(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + \frac{i-2}{2}(2m_2 + i - 1) + i + j + 2\right] (\text{mod } q)$$

for$1 \leq i \leq n$ and i is even, $j = 1, 2, \cdots, m_2 + i - 1$. $D = \{e_i = u_i u_{i+1}$ for $i = 1, 2\}$, where

$$e_i = \left[\frac{n-1}{2}(m_2 + \frac{n+1}{2} + i + 1\right] (\text{mod } q)$$

for $i = 1, 2$. Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n) \cup P_3$ is vertex graceful for n is odd. □

Definition 2.5 *A regular lobster is defined by each vertex in a path is adjacent to the path* P_2.

Theorem 2.6 *A regular lobster is vertex graceful.*

Proof Let G be a 1- regular lobster with $3n$ vertices and $q = 3n - 1$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of a path P_n . Let v_i be the vertices, which are adjacent to v_{i1}^i and v_{i1}^i adjacent to v_{i2}^i for $1 \leq i \leq n$ and n is even .The theorem is proved by two cases. By definition of Vertex graceful labeling, the required vertices labeling are

Case 1 n is even

$$v_i = \begin{cases} \frac{3i-1}{2}; 1 \leq i \leq n, i \text{ is odd,} \\ \dfrac{3(n+i)}{2}; 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

$$v_{i1} = \begin{cases} \dfrac{3(n+i)-1}{2} / \ 1 \leq i \leq n, i \text{ is odd} \\ \dfrac{3i-2)}{2} + 3 / \ 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

$$v_{i2} = \begin{cases} \frac{3(i-1)}{2} + 2; 1 \leq i \leq n, i \text{ is odd,} \\ \dfrac{3(n+i)}{2} - 1; 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

The corresponding edge labels are as follows:

Let $A = \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}$, where $e_i = \left(\dfrac{3(n+2i)}{2} + 1\right) (\text{mod } q)$ for $1 \leq i \leq n-1$, $B = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)}{2} - 1\right) (\text{mod } q)$ for $1 \leq i \leq n$ and i is odd, $C = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)}{2}\right) (\text{mod } q)$ for $1 \leq i \leq n$ and is even, $D = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)}{2}\right) (\text{mod } q)$ for $1 \leq i \leq n$ and i is odd, $E = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)}{2} - 1\right) (\text{mod } q)$ for $1 \leq i \leq n$ and is even.

Case 2 n is odd

$$v_i = \begin{cases} \dfrac{3i-1}{2};\ 1 \le i \le n, i \text{ is odd}, \\ \dfrac{3(n+i)+1}{2}; 1 \le i \le n, i \text{ is even}, \end{cases}$$

$$v_{i1} = \begin{cases} \frac{3(n+i)}{2}; 1 \le i \le n, i \text{ is odd}, \\ \dfrac{3(i-2)}{2} + 3; 1 \le i \le n, i \text{ is even}, \end{cases}$$

$$v_{i2} = \begin{cases} \frac{3(i-1)}{2} + 2; 1 \le i \le n, i \text{ is odd}, \\ \dfrac{3(n+i-1)}{2} + 1; 1 \le i \le n, i \text{ is even}. \end{cases}$$

The corresponding edge labels are determined by $A = \{e_i = v_i v_{i+1}/1 \le i \le n-1\}$, where $e_i = \left(\dfrac{3(n+2i+1)}{2}\right)$ (mod q) for $1 \le i \le n-1$, $B = \{e_{i1} = v_i v_{i1}/1 \le i \le n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)-1}{2}\right)$ (mod q) for $1 \le i \le n$ and i is odd, $C = \{e_{i1} = v_i v_{i1}/1 \le i \le n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)+1}{2}\right)$ (mod q) for $1 \le i \le n$ and is even, $D = \{e_{i2} = v_{i1} v_{i2}/1 \le i \le n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)+1}{2}\right)$ (mod q) for $1 \le i \le n$ and i is odd, $E = \{e_{i2} = v_i 1 v_{i2}/1 \le i \le n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)-1}{2}\right)$ (mod q) for $1 \le i \le n$ and is even. Hence the induced edge labels of G are q distinct edges. Therefore, the graph G is vertex graceful. □

Theorem 2.7 *$C_n \cup C_{n+1}$ is vertex graceful if and only if $n \ge 4$.*

Proof Let $G = C_n \cup C_{n+1}$ with $p = 2n+1$ vertices and $q = 2n+1$ edges. Suppose that the vertices of the cycle C_n run consecutively $u_1, u_2, \cdots, u_n$ with u_n joined to u_1 and that the vertices of the cycle C_{n+1} run consecutively $v_1, v_2, \cdots, v_{n+1}$ with v_{n+1} joined to v_1.

By definition of vertex graceful labeling

(a) $u_1 = 1, u_n = 2, u_i = 2i$ for $i = 2, 3, \cdots, \lfloor (n+1)/2 \rfloor, u_j = 2(n-j)+3$ for $j = \lfloor (n+3)/2 \rfloor, \cdots, n-1$.

(b) $v_1 = 2, v_2 = 2n-1$ and

(i) $v_{3s+t} = 2n - 4t - 6s + 7, t = 0, 1, 2, s = 1, 2, \cdots, \lfloor (n+1-3t)/6 \rfloor$ if $s = \lfloor \dfrac{n+1-3t}{6} \rfloor < 1$ then no s.

(ii) Write $\alpha(0) = 0, \alpha(1) = 4, \alpha(2) = 2, \beta(0) = 0, \beta(1) = 3 = \beta(2)$
$v_{n+1-3s-t} = 2n - 6s - \alpha(t), t = 0, 1, 2, s = 0, 1, \cdots, \lfloor \dfrac{n-5-\beta(t)}{6} \rfloor$. If $s = \lfloor \dfrac{n-5-\beta(t)}{6} \rfloor < 0$ then no s value exists.

(iii) We consider as that v_i to $f(i)$; and suppose that $n - 2 = \theta\ mod(3), 0 \le \theta \le 2$. There are $2 + \theta$ vertices as yet unlabeled. These middle vertices are labeled according to congruence class of modulo 6.

Congruence class	
n = 0 (mod 6)	f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3, f((n + 6)/2) = n + 4
n = 1 (mod 6)	f((n + 1)/2) = n + 2, f((n + 3)/2) = n + 3, f((n + 5)/2) = n + 4, f((n + 7)/2) = n+5
n = 2 (mod 6)	f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3
n = 2 (mod 6)	f((n + 1)/2) = n+4, f((n+3)/2) = n+3, f((n+5)/2) = n+2
n = 4 (mod 6)	f((n + 2)/2) = n+5, f((n+3)/2) = n+4, f((n+4)/2) = n+3, f((n+5)/2) = n+2
n = 4 (mod 6)	f((n + 3)/2) = n+3, f((n+5)/2) = n+2

To check that f is vertex graceful is very tedious. But we can give basic idea. The C_n cycle has edges with labels $\{2k+2/k = 4, 5, \cdots, n-1\} \cup \{0, 3, 5, 7\}$. In this case all the labeling of the edges of the cycle C_{n+1} run consecutively v_1v_2 as follows:

$1, (2n-1, 2n-3), (2n-11, 2n-13, 2n-15), \cdots, (2n+1-12k, 2n-1-12k, 2n-3-12k), \cdots,$ middle labels,$\cdots, (2n+3-12k, (2n+5-12k, (2n+7-12k), \cdots, (2n-21, 2n-19, 2n-17), (2n-9, 2n-7, 2n-5), 2$. The middle labels depend on the congruence class modulo and are best summarized in the following table. If n is small the terms in brackets alone occur.

Congruence class	
$n = 0(\text{mod}6)$	$\cdots(11, 9), 6, 4, 7, (13, 15, 17)\cdots$
$n = 1(\text{mod}6)$	$\cdots(13, 11), 6, 4, 7, (13, 15, 17)\cdots$
$n = 2(\text{mod}6)$	$\cdots(11), 6, 4, 7, (9)\cdots$
$n = 2(\text{mod}6)$	$\cdots(13), 7, 4, 6, (9, 11)\cdots$
$n = 4(\text{mod}6)$	$\cdots(15, 9), 6, 4, 7(11, 13)\cdots$
$n = 4(\text{mod}6)$	$\cdots(9), 7, 6, 4(11, 13, 15)\cdots$

Thus, all these edge labelings are distinct. □

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Math.Combin.Book Ser. Vol.3(2013), 50-55

Total Semirelib Graph

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Abstract: In this paper, the concept of Total semirelib graph of a planar graph is introduced. We present a characterization of those graphs whose total semirelib graphs are planar, outer planar, Eulerian, hamiltonian with crossing number one.

Key Words: Blocks, edge degree, inner vertex number, line graph, regions Smarandachely semirelib M-graph.

AMS(2010): 10C75, 10C10

§1. Introduction

The concept of block edge cut vertex graph was introduced by Venkanagouda M Goudar [4]. For the graph G(p,q), if $B = u_1, u_2, \cdots, u_r : r \geq 2$ is a block of G, then we say that the vertex u_i and the block B are incident with each other. If two blocks B_1 and B_2 are incident with a common cutvertex, then they are adjacent blocks.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected, planar and without loops or multiple edges.

The semirelib graph of a planar graph G is introduced by Venkanagouda M Goudar and Manjunath Prasad K B [5] denoted by $R_s(G)$ is the graph whose vertex set is the union of set of edges, set of blocks and set of regions of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent, the corresponding edges lies on the blocks and the corresponding edges lies on the region. Now we define the total semirelib graph.

Let M be a maximal planar graph of a graph G. A *Smarandachely semirelib M-graph* $T_s^M(G)$ of M is the graph whose vertex set is the union of set of edges, set of blocks and set of regions of M in which two vertices are adjacent if and only if the corresponding edges of M are adjacent, the corresponding edges lies on the blocks, the corresponding edges lies on the region, the corresponding blocks are adjacent and the graph $G\setminus M$. Particularly, if G is a planar graph, such a $T_s^M(G)$ is called the *total semirelib graph* of G denoted, denoted by $T_s(G)$.

The *edge degree* of an edge uv is the sum of the degree of the vertices of u and v. For the planar graph G, the inner vertex number i(G) of a graph G is the minimum number of vertices

[1]Received March 8, 2013, Accepted August 18, 2013.

not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if i(G)=1 as was given by Kulli [4].

§2. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1([1]) *If G is a (p,q) graph whose vertices have degree* d_i *then the line graph L(G) has q vertices and* q_L *edges, where* $q_L = -q + \frac{1}{2}\sum d_i^2$ *edges.*

Theorem 2.2([1]) *The line graph L(G) of a graph is planar if and only if G is planar,* $\Delta(G) \leq 4$ *and if* $degv = 4$*, for a vertex v of G, then v is a cutvertex.*

Theorem 2.3([2]) *A graph is planar if and only if it has no subgraph homeomorphic to* K_5 *or* $K_{3,3}$.

Theorem 2.4([3]) *A graph is outerplanar if and only if it has no subgraph homeomorphic to* K_4 *or* $K_{2,3}$.

§3. Main Results

We start with few preliminary results.

Lemma 3.1 *For any planar graph G,* $L(G) \subseteq R_s(G) \subseteq T_s(G)$.

Lemma 3.2 *For any graph with block degree* n_i*, the block graph has* $\begin{pmatrix} n_i \\ 2 \end{pmatrix}$ *edges.*

Definition 3.3 *For the graph G the block degree of a cutvertex* v_i *is the number of blocks incident to the cutvertex* v_i *and is denoted by* n_i.

In the following theorem we obtain the number of vertices and edges of a Total semirelib graph of a graph.

Theorem 3.4 *For any planar graph G, the total semirelib graph* $T_s(G)$ *whose vertices have degree* d_i*, has* $q + r + b$ *vertices and* $\frac{1}{2}\sum d_i^2 + \sum q_j$ *edges where r and b be the number of regions and blocks respectively.*

Proof By the definition of $T_s(G)$, the number of vertices is the union of edges, regions and blocks of G. Hence $T_s(G)$ has (q + r + b) vertices. Further by the Theorem 2.1, number of edges in $L(G)$ is $q_L = -q + \frac{1}{2}\sum d_i^2$. Thus the number of edges in $T_s(G)$ is the sum of the number of edges in $L(G)$, the number of edges bounded by the regions which is q, the number of edges lies on the blocks is $\sum q_j$ and the number the sum of the block degree of cutvertices which is $\sum\binom{n_i}{2}$ by the Lemma 3.2. Hence

$$E[T_s(G)] = -q + \frac{1}{2}\sum d_i^2 + q + \sum q_j + \sum\binom{n_i}{2} = \frac{1}{2}\sum d_i^2 + \sum q_j + \sum\binom{n_i}{2}. \qquad \square$$

Theorem 3.5 *For any edge in a plane graph G with edge degree e_i is n, the degree of the corresponding vertex in $T_s(G)$ is i). n if e_i is incident to a cutvertex and ii). n+1 if e_i is not incident to a cutvertex.*

Proof Suppose an edge $e_i \in E(G)$ have degree n. By the definition of total semirelib graph, the corresponding vertex in $T_s(G)$ has n-1. Since edge lies on a block, we have the degree of the vertex is $n - 1 + 1 = n$. Further, if $e_i \neq b_i \in E(G)$ then by the definition of total semirelib graph, $\forall e_i \in E(G)$, e_i is adjacent to all vertices e_j of $T_s(G)$ which are adjacent edges of e_i of G. Also the block vertex of $T_s(G)$ is adjacent to e_i. Clearly degree of e_i is $n + 1$. $\square$

Theorem 3.6 *For any planar graph G with n blocks which are K_2 then $T_s(G)$ contains n pendent vertices.*

Theorem 3.7 *For any graph G, $T_s(G)$ is nonseparable.*

Proof Let $e_1, e_2, \cdots, e_n \in E(G)$, $b_1 = e_1, b_2 = e_2, \cdots, b_n = e_n$ be the blocks and $r_1, r_2, \cdots, r_k$ be the regions of G. By the definition of line graph L(G), $e_1, e_2, \cdots, e_n$ form a subgraph without isolated vertex. By the definition of $T_s(G)$, the region vertices are adjacent to these vertices to form a graph without isolated vertex. Since there are n blocks which are K_2, we have each $b_1 = e_1, b_2 = e_2, \cdots, b_n = e_n$ are adjacent to $e_1, e_2, \cdots, e_n$. Hence semirelib graph $R_s(G)$ contains n pendent vertices. By the definition of total semirelib graph, the block vertices are also adjacent. Hence $T_s(G)$ is nonseparable. $\square$

In the following theorem we obtain the condition for the planarity on total semirelib graph of a graph.

Theorem 3.8 *For any planar graph G, the $T_s(G)$ is planar if and only if G is a tree such that $\Delta(G) \leq 3$.*

Proof Suppose $R_s(G)$ is planar. Assume that $\exists v_i \in G$ such that $deg v_i \geq 4$. Suppose $deg v_i = 4$ and e_1, e_2, e_3, e_4 are the edges incident to v_i. By the definition of line graph, e_1, e_2, e_3, e_4 form K_4 as an induced subgraph. In $T_s(G)$, the region vertex r_i is adjacent with all vertices of $L(G)$ to form K_5 as an induced subgraph. Further the corresponding block vertices $b_1, b_2, b_3, \cdots, b_{n-1}$ of of blocks $B_1, B_2, B_3, \cdots, B_n$ in G are adjacent to vertices of K_4 and the corresponding blocks are adjacent. Clearly $T_s(G)$ forms graph homeomorphic to K_5. By the Theorem 2.3, it is non planar, a contradiction.

Conversely, Suppose $deg v \leq 3$ and let e_1, e_2, e_3 be the edges of G incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph. By the definition of $T_s(G)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as a subgraph. Further, by the Lemma 3.2, the blocks $b_1, b_2, b_3, \cdots, b_n$ of T with n vertices such that $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1}$ becomes $p-1$ pendant vertices.By the definition of $T_s(G)$, these block vertices are adjacent. Hence $T_s(G)$ is planar. $\square$

In the following theorem we obtain the condition for the outer planarity on total semirelib graph of a graph.

Theorem 3.9 *For any planar graph G, $T_s(G)$ is outer planar if and only if G is a path P_3.*

Proof Suppose $T_s(G)$ is outer planar. Assume that G is a tree with at least one vertex v such that $degv = 3$. Let e_1, e_2, e_3 be the edges of G incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph.In $T_s(G)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as induced subgraph. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1}$ becomes n-1 pendant vertices in $R_s(G)$. By the definition of $T_s(G)$, $i[R_s(G) \geq 1]$, which is non-outer planar ,a contradiction.

Conversely, Suppose G is a path P_3. Let $e_1, e_2 \in E(G)$. By the definition of line graph $L[P_3](G) = P_2$. Further by definition of $T_s(G), b_1 = e_1, b_2 = e_2$ forms and the vertices of line graph form C_4 . Further the region vertex r_1 is adjacent to all the vertices of $T_s(G)$ which is outer planar. □

In the following theorem we obtain the condition for the minimally non outer planar on total semirelib graph of a graph.

Theorem 3.10 *For any planar graph G, $T_s(G)$ is minimally non-outer planar if and only if G is P_4.*

Proof Suppose $T_s(G)$ is minimally non-outer planar. Assume that $G \neq P_4$.Consider the following cases.

Case 1 Assume that $G = K_{1,n}$ for $n \geq 3$. Then there exist at least one vertex of degree at least 3. Suppose $degv = 3$ for any $v \in G$. By the definition of line graph, $L[K_{1,3}] = K_3$. By the definition of $T_s(G)$,these vertices are adjacent to a region vertex r_1, which form K_4. Further the block vertices form K_3 and it has e_1, e_2, e_3 as its internal vertices. Clearly, T_s is not minimally non-outer planar, a contradiction.

Case 2 Suppose $G \neq K_{1,n}$. By the Theorem 3.9, $T_s(G)$ is non-outer planar,a contradiction.

Case 3 Assume that $G = P_n$,for $n \geq 5$. Suppose $n = 5$. By the definition of line graph,$L[P_5](G) = P_4$ and e_2, e_3 are the internal vertices of $L(G)$. By the definition of T_s, the region vertex r_1 is adjacent to all vertices of $L(G)$ to form connected graph. Further the block vertices are adjacent to all vertices of $L(G)$. Clearly the vertices e_2, e_3 becomes the internal vertices of P_s. Clearly $i[T_s] = 2$, which is not minimally nonouterplanar, a contradiction.

Conversely, suppose $G = P_4$ and let $e_1, e_2, e_3 \in E(G)$. By the definition of line graph, $L[P_4] = P_3$. Let r_1 be the region vertex in $T_s(G)$ such that r_1 is adjacent to all vertices of $L(G)$. Further the blocks b_i are adjacent to the vertices e_j for $i = j$. Clearly $i[T_s(G)] = 1$. Hence G is minimally non-outer planar. □

In the following theorem we obtain the condition for the non Eulerian on total semirelib graph of a graph.

Theorem 3.11 *For any planar graph G, $T_s(G)$ is always non Eulerian.*

Proof We consider the following cases.

Case 1 Assume that G is a tree. In a tree each edge is a block and hence $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(G)$ and $\forall b_{n-1} \in V[T_s(G)]$. In $T_s(G)$, the degree of a block vertex b_i is always even, but the pendent edges of G becomes the odd degree vertex in $T_s(G)$, which is non Eulerian.

Case 2 Assume that G is K_2 -free graph. We have the following subcases of Case 2.

Subcase 1 Suppose G itself is a block with even number of edges. Clearly each edge of G is of even degree. By the definition of $T_s(G)$, both the region vertices and blocks have even degree. By the Theorem 2.3, $e_i = b_i \in V[T_s(G)]$ is of odd degree, which is non Eulerian. Further if G is a block with odd number of edges, then by the Theorem 3.3, each $e_i = b_i \in V[T_s(G)]$ is of even degree. Also the block vertex and region vertex b_i, r_i are adjacent to these vertices. Clearly degree of b_i and r_i is odd, which is non Eulerian.

Subcase 2 Suppose G is a graph such that it contains at least one cutvertex. If each edge is even degree then by the sub case 1, it is non Eulerian. Assume that G contains at least one edge with odd edge degree. Clearly for any $e_j \in E(G)$, degree of $e_j \in V[T_s(G)]$ is odd, which is non Eulerian. Hence for any graph G $T_s(G)$ is always non Eulerian. □

In the following theorem we obtain the condition for the hamiltonian on total semirelib graph of a graph.

Theorem 3.12 *For any graph G, $T_s(G)$ is always hamiltonian.*

Proof Suppose G is any graph. We have the following cases.

Case 1 Consider a graph G is a tree. In a tree, each edge is a block and hence $b_1 = e_1, b_2 = e_2, \cdots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(G)$ and $\forall b_{n-1} \in V[T_s(G)]$. Since a tree T contains only ne region r_1 which is adjacent to all vertices $e_1, e_2, \cdots, e_{n-1}$ of $T_s(G)$. Also the block vertices are adjacent to each vertex e_i which corresponds to the edge of G and it is a block in G. Clearly $r_1, e_1, b_1, b_2, e_2, e_3, b_3, \cdots, r_1$ form a hamiltonian cycle. Hence$T_s(G)$ is hamiltonian graph.

Case 2 Suppose G is not a tree. Let $e_1, e_2, \cdots, e_{n-1} \in E(G)$, $b_1, b_2, \cdots, b_i$ be the blocks and $r_1, r_2, \cdots, r_k$ be the regions of G such that $e_1, e_2, \cdots, e_l \in V(b_1)$, $e_{l+1}, e_{l+2}, \cdots, e_m \in V(b_2), \cdots$, $e_{m+1}, e_{m+2}, \cdots, e_{n-1} \in V(b_i)$. By the Theorem 3.3, $V[T_s(G)] = e_1, e_2, \cdots, e_{n-1} \cup b_1, b_2, \cdots, b_i \cup r_1, r_2, \cdots, r_k$. By theorem 3.7, $T_s(G)$ is non separable. By the definition, $b_1 e_1, e_2, \cdots, e_{l-1} r_1 b_2 \cdots r_2 e_m b_3 \cdots e_{k+1}, e_{k+2}, \cdots, e_{n-1} b_k r_k e_l b_1$ form a cycle which contains all the vertices of $T_s(G)$. Hence $T_s(G)$ is hamiltonian. □

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Math.Combin.Book Ser. Vol.3(2013), 56-68

On Some Characterization of Ruled Surface of a Closed Spacelike Curve with Spacelike Binormal in Dual Lorentzian Space

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Abstract: In this paper, we investigate the relations between the pitch, the angle of pitch and drall of parallel ruled surface of a closed spacelike curve with spacelike binormal in dual Lorentzian space.

Key Words: Spacelike dual curve, ruled surface, Lorentzian space, dual numbers.

AMS(2010): 53A04, 45F10

§1. Introduction

Dual numbers were introduced by W.K. Clifford [5] as a tool for his geometrical investigations. After him,e E.Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [7]. The pitches and the angles of the pitches of the closed ruled surfaces corresponding to the one parameter dual unit spherical curves and oriented lines in $\mathbb{R}^3$ were calculated respectively by Hacısalihoğlu [10] and Gürsoy [8]. Definitions of the parallel ruled surface were presented by Wilhelm Blaschke [6]. The integral invariants of the paralel ruled surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ corresponding to the unit dual spherical parallel curves were calculated by Senyurt [14]. The integral invariants of ruled surface of a timelike curve in dual Lorentzian space were calculated by Bektaş and Şenyurt [2]. The integral invariants of ruled surface of a closed spacelike curve with timelike binormal in dual Lorentzian space were calculated by Bektaş and Şenyurt [3].

The set $D = \{\hat{\lambda} = \lambda + \varepsilon\lambda^* | \lambda, \lambda^* \in \mathbb{R}, \varepsilon^2 = 0\}$ is called *dual numbers* set, see [5].On this set, product and addition operations are respectively

$$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$$

and

$$(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = \lambda\beta + \varepsilon(\lambda\beta^* + \lambda^*\beta).$$

[1]Received May 15, 2013, Accepted August 22, 2013.

The elements of the set $D^3 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \,\middle|\, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}^3 \right\}$ are called *dual vectors*. On this set addition and scalar product operations are respectively

$$\begin{array}{l} \oplus : D^3 \times D^3 \to D^3 \\ \left(\overrightarrow{A}, \overrightarrow{B}\right) \to \overrightarrow{A} \oplus \overrightarrow{B} = \overrightarrow{a} + \overrightarrow{b} + \varepsilon\left(\overrightarrow{a}^* + \overrightarrow{b}^*\right), \end{array}$$

$$\begin{array}{l} \odot : D \times D^3 \to D^3 \\ \left(\lambda, \overrightarrow{A}\right) \to \lambda \odot \overrightarrow{A} = (\lambda + \varepsilon\lambda^*) \odot (\overrightarrow{a} + \varepsilon \overrightarrow{a}^*) = \lambda \overrightarrow{a} + \varepsilon (\lambda \overrightarrow{a}^* + \lambda^* \overrightarrow{a}) \end{array}$$

The set $\left(D^3, \oplus\right)$ is a module over the ring $(D, +, \cdot)$, called the $D-$ *Modul*.

The Lorentzian inner product of dual vectors $\overrightarrow{A}$, $\overrightarrow{B} \in D^3$ is defined by

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon\left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle\right)$$

where $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle$ is the following Lorentzian inner product of vectors $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$,i.e.,

$$\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The set D^3 equipped with the Lorentzian inner product $\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle$ is called 3-dimensional dual Lorentzian space and is denoted in what follows by $D_1^3 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \,\middle|\, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}_1^3 \right\}$ [17].

A dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in D_1^3$ is called dual space-like vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle > 0$ or $\overrightarrow{A} = 0$, a dual time-like vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle < 0$, a dual null (light-like) vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle = 0$ for $\overrightarrow{A} \neq 0$. For $\overrightarrow{A} \neq 0$, the *norm* $\left\| \overrightarrow{A} \right\|$ of $\overrightarrow{A}$ is defined by

$$\left\| \overrightarrow{A} \right\| = \sqrt{\left| \left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle \right|} = \| \overrightarrow{a} \| + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{\| \overrightarrow{a} \|}, \quad \| \overrightarrow{a} \| \neq 0.$$

The dual Lorentzian cross-product of $\overrightarrow{A}$, $\overrightarrow{B} \in D^3$ is defined as

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \times \overrightarrow{b} + \varepsilon\left(\overrightarrow{a} \times \overrightarrow{b}^* + \overrightarrow{a}^* \times \overrightarrow{b}\right)$$

where $\overrightarrow{a} \times \overrightarrow{b}$ is the cross-product [14] of $\overrightarrow{a}$, $\overrightarrow{b} \in \mathbb{R}^3$ given by

$$\overrightarrow{a} \times \overrightarrow{b} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1).$$

Theorem 1.1(E. Study) *The oriented lines in $\mathbb{R}^3$ are in one to one correspondence with the points of the dual unit sphere $\| \overrightarrow{A} \| = (1, 0)$ where $\overrightarrow{A} \neq (\overrightarrow{0}, \overrightarrow{a}) \in$ D-Modul , see [9].*

The dual number $\Phi = \varphi + \varepsilon\varphi^*$ is called dual angle between the unit dual vectors $\overrightarrow{A}$ ve $\overrightarrow{B}$ and keep in mind that

$$\begin{aligned} \sin(\varphi + \varepsilon\varphi^*) &= \sin\varphi + \varepsilon\varphi^* \cos\varphi\,, \\ \cos(\varphi + \varepsilon\varphi^*) &= \cos\varphi - \varepsilon\varphi^* \sin\varphi. \end{aligned}$$

§2. Characterization of Ruled Surface of a Closed Spacelike Curve with Spacelike Binormal in Dual Lorentzian Space (D_1^3)

Let $U : I \to D_1^3,\ t \to \overrightarrow{U}(t) = \overrightarrow{U}_1(t),\ \left\|\overrightarrow{U}(t)\right\| = 1$ be a differentiable spacelike curve with spacelike binormal in the dual unit sphere. Denote by $(\overrightarrow{U})$ the closed ruled generated by this curve.

Let $\left\{\overrightarrow{U_1}, \overrightarrow{U_2}, \overrightarrow{U_3}\right\}$ be the Frenet frame of the curve $\overrightarrow{U} = \overrightarrow{U}_1$ with

$$\overrightarrow{U}_1 = \overrightarrow{U} \quad , \quad \overrightarrow{U}_2 = \overrightarrow{U}' \Big/ \left\|\overrightarrow{U}\right\| \quad , \quad \overrightarrow{U}_3 = \overrightarrow{U}_1 \times \overrightarrow{U}_2$$

Definition 2.1 *The closed ruled surface $(\overrightarrow{U})$ corresponding to the dual spacelike curve $\overrightarrow{U}(t)$ which makes the fixed dual angle $\Phi = \varphi + \varepsilon\varphi^*$ with $\overrightarrow{U}(t)$ determines*

$$\overrightarrow{V} = \cos\Phi\overrightarrow{U_1} + \sin\Phi\overrightarrow{U_3} \tag{2.1}$$

The surface $(\overrightarrow{V})$ corresponding to the dual spacelike vector $\overrightarrow{V}$ is called the *parallel ruled surface* of surface $(\overrightarrow{U})$ in the dual Lorentzian space D_1^3.

Now, take $\overrightarrow{U}(t)$ as a closed spacelike curve with curvature $\kappa = k_1 + \varepsilon k_1^*$ and torsion $\tau = k_2 + \varepsilon k_2^*$. Recall that in the Frenet frames associated to the curve $\overrightarrow{U_1}$ and $\overrightarrow{U_3}$ are spacelike vectors and $\overrightarrow{U_2}$ is timelike vector and we have

$$\overrightarrow{U_1} \times \overrightarrow{U_2} = -\overrightarrow{U_3} \quad , \quad \overrightarrow{U_2} \times \overrightarrow{U_3} = -\overrightarrow{U_1} \quad , \quad \overrightarrow{U_3} \times \overrightarrow{U_1} = \overrightarrow{U_2}. \tag{2.2}$$

Under these conditions, the Frenet formulas are given by ([18])

$$\overrightarrow{U_1}' = \kappa\overrightarrow{U_2} \quad , \quad \overrightarrow{U_2}' = \kappa\overrightarrow{U_1} + \tau\overrightarrow{U_3} \quad , \quad \overrightarrow{U_3}' = \tau\overrightarrow{U_2}. \tag{2.3}$$

The Frenet instantaneous rotation vector (also called instantaneous Darboux vector) for the spacelike curve is given by ([16])

$$\overrightarrow{\Psi} = -\tau\overrightarrow{U_1} + \kappa\overrightarrow{U_3}, \tag{2.4}$$

Let be $\overrightarrow{V}_1 = \overrightarrow{V}$. Differentiating of the vector $\overrightarrow{V_1}$ with respect the parameter t and using the Eq.(2.3) we get

$$\overrightarrow{V_1}' = (\kappa\cos\Phi + \tau\sin\Phi)\overrightarrow{U_2} \tag{2.5}$$

and the norm of that vector denoted by P is

$$P = \kappa\cos\Phi + \tau\sin\Phi. \tag{2.6}$$

Then, substituting the values of (2.5) and (2.6) into Frenet equations gives

$$\overrightarrow{V_2} = \overrightarrow{U_2} \tag{2.7}$$

For the vector $\overrightarrow{V_3}$,we have

$$\overrightarrow{V_3} = \sin\Phi\overrightarrow{U_1} - \cos\Phi\overrightarrow{U_3} \tag{2.8}$$

If Eq.(2.1), (2.7) and (2.8) are written matrix form, we have

$$\begin{bmatrix} \overrightarrow{V_1} \\ \overrightarrow{V_2} \\ \overrightarrow{V_3} \end{bmatrix} = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & -\cos\Phi \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{U_1} \\ \overrightarrow{U_2} \\ \overrightarrow{U_3} \end{bmatrix}$$

or

$$\begin{bmatrix} \overrightarrow{U_1} \\ \overrightarrow{U_2} \\ \overrightarrow{U_3} \end{bmatrix} = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & -\cos\Phi \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{V_1} \\ \overrightarrow{V_2} \\ \overrightarrow{V_3} \end{bmatrix}$$

The real and dual parts of $\overrightarrow{U_1}, \overrightarrow{U_2}, \overrightarrow{U_3}$ are

$$\begin{cases} \overrightarrow{u}_1 = \cos\varphi \overrightarrow{v}_1 + \sin\varphi \overrightarrow{v}_3 \\ \overrightarrow{u}_2 = \overrightarrow{v}_2 \\ \overrightarrow{u}_3 = \sin\varphi \overrightarrow{v}_1 - \cos\varphi \overrightarrow{v}_3 \\ \overrightarrow{u}_1^* = \cos\varphi \overrightarrow{v}_1^* + \sin\varphi \overrightarrow{v}_3^* - \varphi^*(\sin\varphi \overrightarrow{v}_1 - \cos\varphi \overrightarrow{v}_3) \\ \overrightarrow{u}_2^* = \overrightarrow{v}_2^* \\ \overrightarrow{u}_3^* = \sin\varphi \overrightarrow{v}_1^* - \cos\varphi \overrightarrow{v}_3^* + \varphi^*(\cos\varphi \overrightarrow{v}_1 - \sin\varphi \overrightarrow{v}_3) \end{cases} \tag{2.9}$$

Let $P = p + \varepsilon p^*$ be the curvature and $Q = q + \varepsilon q^*$ the torsion of curve $\overrightarrow{V}(t)$. Then, the following relating holds between the vectors

$$\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3} \text{ and } \overrightarrow{V_1}', \overrightarrow{V_2}', \overrightarrow{V_3}' \quad [18]$$

$$\begin{cases} \overrightarrow{V}_1{}' = P\overrightarrow{V}_2, \quad \overrightarrow{V}_2{}' = P\overrightarrow{V}_1 + Q\overrightarrow{V}_3, \quad \overrightarrow{V}_3{}' = Q\overrightarrow{V}_2 \\ P = \sqrt{<\overrightarrow{V}_1{}', \overrightarrow{V}_1{}'>}, \quad Q = \frac{det(\overrightarrow{V}_1, \overrightarrow{V}_1{}', \overrightarrow{V}_1{}'')}{<\overrightarrow{V}_1{}', \overrightarrow{V}_1{}'>}. \end{cases} \tag{2.10}$$

If Eq.(2.10) is separated into its real and dual parts, we get

$$\begin{cases} \overrightarrow{v}_1{}' = p\overrightarrow{v}_2, \quad \overrightarrow{v}_2{}' = p\overrightarrow{v}_1 + q\overrightarrow{v}_3, \quad \overrightarrow{v}_3{}' = q\overrightarrow{v}_2 \\ \overrightarrow{v}'_1{}^* = p\overrightarrow{v}_2^* + p^*\overrightarrow{v}_2, \\ \overrightarrow{v}'_2{}^* = p\overrightarrow{v}_1^* + p^*\overrightarrow{v}_1 + q^*\overrightarrow{v}_3 + q\overrightarrow{v}_3{}^*, \\ \overrightarrow{v}'_3{}^* = q\overrightarrow{v}_2^* + q^*\overrightarrow{v}_2 \end{cases} \tag{2.11}$$

Now, we are ready to calculate the value of Q as function of κ and τ. Differentiating Eq. (2.5) with respect to the curve parameter t we get

$$\begin{aligned} \overrightarrow{V}''_1 =& (+\kappa^2\cos\Phi + \kappa\tau\sin\Phi)\overrightarrow{U}_1 + \\ &+ (\kappa\cos\Phi + \tau\sin\Phi)'\overrightarrow{U}_2 + (\kappa\tau\cos\Phi + \tau^2\sin\Phi)\overrightarrow{U}_3 \end{aligned} \tag{2.12}$$

Using Eqs.(2.1), (2.5) and (2.12) into Eq.(2.10), we get

$$Q = -\kappa\sin\Phi + \tau\cos\Phi \tag{2.13}$$

and separating Eq.(2.6) and Eq.(2.13) into its dual and real parts gives

$$\begin{cases} p = k_1 \cos\varphi + k_2 \sin\varphi \\ p^* = k_1^* \cos\varphi + k_2^* \sin\varphi - \varphi^*(k_1 \sin\varphi - k_2 \cos\varphi) \\ q = -k_1 \sin\varphi + k_2 \cos\varphi \\ q^* = -k_1^* \sin\varphi + k_2^* \cos\varphi - \varphi^*(k_1 \cos\varphi + k_2 \sin\varphi) \end{cases} \tag{2.14}$$

In its dual unit spherical motion the dual orthonormal system $\left\{\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3}\right\}$ at any t makes a dual rotation motion around the instantaneous dual Darboux vector. This vector is determined by the following equation ([16]).

$$\overrightarrow{\overrightarrow{\Psi}} = -Q\overrightarrow{V_1} + P\overrightarrow{V_3}. \tag{2.15}$$

For the Steiner vector of the motion, we can write

$$\overrightarrow{\overrightarrow{D}} = \oint \overrightarrow{\overrightarrow{\Psi}} \tag{2.16}$$

or

$$\overrightarrow{\overrightarrow{D}} = -\overrightarrow{V_1} \oint Qdt + \overrightarrow{V_3} \oint Pdt \tag{2.17}$$

Using the values of the vectors $\overrightarrow{U_1}$ and $\overrightarrow{U_3}$ into Eq.(2.4), gives

$$\overrightarrow{\Psi} = -\tau(cos\Phi\overrightarrow{V}_1 + sin\Phi\overrightarrow{V}_3) + \kappa(sin\Phi\overrightarrow{V}_1 - cos\Phi\overrightarrow{V}_3),$$

$$\overrightarrow{\Psi} = -Q\overrightarrow{V}_1 - P\overrightarrow{V}_3 \tag{2.18}$$

Because of the equations $\overrightarrow{D} = \oint \overrightarrow{\Psi}$ for the dual Steiner vector of the motion, we may write

$$\overrightarrow{D} = -\overrightarrow{V_1} \oint Qdt - \overrightarrow{V_3} \oint Pdt \tag{2.19}$$

The real and dual parts of $\overrightarrow{D}$ are

$$\begin{cases} \overrightarrow{d} = -\overrightarrow{v}_1 \oint qdt - \overrightarrow{v}_3 \oint pdt, \\ \overrightarrow{d}^* = -\overrightarrow{v}_1 \oint q^*dt - \overrightarrow{v}_1^* \oint qdt - \overrightarrow{v}_3 \oint p^*dt - \overrightarrow{v}_3^* \oint pdt \end{cases} \tag{2.20}$$

$$\overrightarrow{D} = -\overrightarrow{U_1} \oint \tau dt + \overrightarrow{U_3} \oint \kappa dt \tag{2.21}$$

Eq.(2.21) can be written type of the dual and real part as follow

$$\begin{cases} \overrightarrow{d} = -\overrightarrow{u}_1 \oint k_2dt + \overrightarrow{u}_3 \oint k_1dt, \\ \overrightarrow{d}^* = -\overrightarrow{u}_1^* \oint k_2dt - \overrightarrow{u}_1 \oint k_2^*dt + \overrightarrow{u}_3^* \oint k_1dt + \overrightarrow{u}_3 \oint k_1^*dt \end{cases} \tag{2.22}$$

If the equation (2.3) is separated into the dual and real part, we can obtain

$$\begin{cases} \overrightarrow{u}_1' = k_1\overrightarrow{u}_2, \quad \overrightarrow{u}_2' = k_1\overrightarrow{u}_1 + k_2\overrightarrow{u}_3, \quad \overrightarrow{u}_3' = k_2\overrightarrow{u}_2 \\ \overrightarrow{u}_1'^* = k_1^*\overrightarrow{u}_2 + k_1\overrightarrow{u}_2^*, \\ \overrightarrow{u}_2'^* = k_1^*\overrightarrow{u}_1 + k_2^*\overrightarrow{u}_3 + k_1\overrightarrow{u}_1^* + k_2\overrightarrow{u}_3^* \\ \overrightarrow{u}_3'^* = k_2^*\overrightarrow{u}_2 + k_2\overrightarrow{u}_2^* \end{cases} \tag{2.23}$$

Now, let us calculate the integral invariants of the respective closed ruled surfaces. The pitch of the first closed surface (U_1) is obtained as

$$L_{u_1} = \left\langle \overrightarrow{d}, \overrightarrow{u_1}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{u_1} \right\rangle,$$

$$L_{u_1} = -\oint k_2^* dt. \tag{2.24}$$

The dual angle of the pitch of the closed surface U_1 is

$$\Lambda_{U_1} = -\left\langle \overrightarrow{D}, \overrightarrow{U_1} \right\rangle.$$

and from Eq.(2.21) we obtain

$$\Lambda_{U_1} = \oint \tau dt. \tag{2.25}$$

The real and dual of U_1 are

$$\lambda_{u_1} = \oint k_2 dt \quad , \quad L_{u_1} = -\oint k_2^* dt \tag{2.26}$$

The *drall* of the closed surface (U_1) is

$$P_{U_1} = \frac{\langle d\overrightarrow{u_1}, d\overrightarrow{u_1}^* \rangle}{\langle d\overrightarrow{u_1}, d\overrightarrow{u_1} \rangle}$$

Using the values $d\overrightarrow{u_1}$ and $d\overrightarrow{u_1}^*$ given by Eq.(2.23), we get

$$P_{U_1} = \frac{k_1^*}{k_1} \tag{2.27}$$

The pitch of the closed surface (U_2) is given by

$$L_{u_2} = 0. \tag{2.28}$$

The dual angle of the pitch of the closed surface (U_2) is

$$\Lambda_{U_2} = -\left\langle \overrightarrow{D}, \overrightarrow{U_2} \right\rangle,$$

$$\Lambda_{U_2} = 0. \tag{2.29}$$

The drall of the closed surface (U_2), we may write

$$P_{U_2} = \frac{\langle d\overrightarrow{u_2}, d\overrightarrow{u_2}^* \rangle}{\langle d\overrightarrow{u_2}, d\overrightarrow{u_2} \rangle}$$

Using the values $d\overrightarrow{u_2}$ and $d\overrightarrow{u_2}^*$ given by Eq.(2.23), we get

$$P_{U_2} = \frac{k_1 k_1^* + k_2 k_2^*}{k_1^2 + k_2^2} \tag{2.30}$$

The pitch of the closed surface (U_3) is

$$L_{u_3} = \left\langle \overrightarrow{d}, \overrightarrow{u_3}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{u_3} \right\rangle,$$

$$L_{u_3} = \oint k_1^* dt \tag{2.31}$$

The dual angle of the pitch of the closed surface (U_3) is

$$\Lambda_{U_3} = -\left\langle \overrightarrow{D}, \overrightarrow{U_3} \right\rangle$$

which gives using (2.21)

$$\Lambda_{U_3} = -\oint \kappa dt \tag{2.32}$$

The real and dual parts of Λ_{U_3} are

$$\lambda_{u_3} = -\oint k_1 dt \quad , \quad L_{u_3} = \oint k_1^* dt \tag{2.33}$$

The drall of the closed surface (U_3) is

$$P_{U_3} = \frac{\langle d\overrightarrow{u_3}, d\overrightarrow{u_3}^* \rangle}{\langle d\overrightarrow{u_3}, d\overrightarrow{u_3} \rangle}$$

Using the values of $d\overrightarrow{u_3}$ and $d\overrightarrow{u_3}^*$ given in Eq.(2.23) gives

$$P_{U_3} = \frac{k_2^*}{k_2}. \tag{2.34}$$

Let $\Omega(t) = \omega(t) + \varepsilon\omega^*(t)$ be the Lorentzian timelike angle between the instantaneous dual Pfaffion vector $\overrightarrow{\Psi}$ and the vector $\overrightarrow{U_3}$. In this case dual Pfaffion vector $\overrightarrow{\Psi}$ is spacelike vector and so,

$$\kappa = \left\| \overrightarrow{\Psi} \right\| \cos\Omega \quad , \quad \tau = \left\| \overrightarrow{\Psi} \right\| \sin\Omega$$

then $\overrightarrow{C} = \overrightarrow{c} + \varepsilon \overrightarrow{c}^*$, the unit vector in the $\overrightarrow{\Psi}$ direction is

$$\overrightarrow{C} = -\sin\Omega\overrightarrow{U_1} + \cos\Omega\overrightarrow{U_3} \tag{2.35}$$

and the real and dual parts of $\overrightarrow{C}$ are

$$\begin{cases} \overrightarrow{c} = -\sin\omega\overrightarrow{u_1} + \cos\omega\overrightarrow{u_3} \\ \overrightarrow{c}^* = -\sin\omega\overrightarrow{u_1}^* + \cos\omega\overrightarrow{u_3}^* - \omega^*\cos\omega\overrightarrow{u_1} - \omega^*\sin\omega\overrightarrow{u_3} \end{cases} \tag{2.36}$$

The pitch of the closed surface $(\overrightarrow{C})$ generated by $\overrightarrow{C}$ is given by

$$L_C = < \overrightarrow{d}, \overrightarrow{c}^* > + < \overrightarrow{d}^*, \overrightarrow{c} >$$

$$L_C = \cos\omega \oint k_1^* dt + \sin\omega \oint k_2^* dt - \omega^*(\sin\omega \oint k_1 dt - \cos\omega \oint k_2 dt) \tag{2.37}$$

If we use Eq.(2.26) and Eq.(2.33) into Eq.(2.37) we get

$$L_C = -\sin\omega L_{u_1} + \cos\omega L_{u_3} + \omega^*(\cos\omega\lambda_{u_1} + \sin\omega\lambda_{u_3}) \tag{2.38}$$

The dual angle of the pitch of that closed ruled surface $(\overrightarrow{C})$, we have

$$\Lambda_C = -\left\langle \overrightarrow{D}, \overrightarrow{C} \right\rangle$$

and from Eq.(2.21) and (2.35) it follows that

$$\Lambda_{U_3} = -<\overrightarrow{U_1}\oint \tau dt + \overrightarrow{U_3}\oint \kappa dt, -\sin\Omega\overrightarrow{U_1} + \cos\Omega\overrightarrow{U_3}>,$$

$$\Lambda_C = -\sin\Omega\oint \tau dt - \cos\Omega\oint \kappa dt \tag{2.39}$$

Using Eg.(2.25) and (2.32) gives

$$\Lambda_C = -\sin\Omega\Lambda_{U_1} + \cos\Omega\Lambda_{U_3} \tag{2.40}$$

The drall of the closed surface $(\overrightarrow{C})$ is

$$P_C = \frac{\langle d\overrightarrow{c}, d\overrightarrow{c}^*\rangle}{\langle d\overrightarrow{c}, d\overrightarrow{c}\rangle}$$

$$P_C = \frac{\omega^{'}\omega^{*'} - (k_2\cos\omega - k_1\sin\omega)\left[(k_2^* - k_1\omega^*)\cos\omega - (k_2\omega^* + k_1^*)\sin\omega\right]}{\omega'^2 - (k_2\cos\omega - k_1\sin\omega)^2} \tag{2.41}$$

Now, let us calculate the integral invariants of the respective closed ruled surfaces. The pitch of the closed (V_1) surface is given by

$$L_{V_1} = \left\langle \overrightarrow{d}, \overrightarrow{v_1}^*\right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_1}\right\rangle,$$

$$L_{V_1} = -\oint q^* dt. \tag{2.42}$$

Substituting by the value q^* into Eq.(2.42)

$$L_{V_1} = \sin\varphi\oint k_1^* dt - \cos\varphi\oint k_2^* dt + \varphi^*(\cos\varphi\oint k_1 dt + \sin\varphi\oint k_2 dt) \tag{2.43}$$

or

$$L_{V_1} = \cos\varphi L_{u_1} + \sin\varphi L_{u_3} + \varphi^*\left(\sin\varphi\lambda_{u_1} - \cos\varphi\lambda_{u_3}\right). \tag{2.44}$$

The dual angle of the pitch of the closed ruled surface (V_1) , we have

$$\Lambda_{V_1} = -\left\langle \overrightarrow{D}, \overrightarrow{V_1}\right\rangle$$

and using Eq.(2.19) we obtain

$$\Lambda_{V_1} = -<-\overrightarrow{V_1}\oint Qdt - \overrightarrow{V_3}\oint Pdt, \overrightarrow{V_1}>,$$

$$\Lambda_{V_1} = \oint Qdt. \tag{2.45}$$

Using Eq.(2.13) into the last equation, we get

$$\Lambda_{V_1} = -\sin\Phi\oint \kappa dt + \cos\Phi\oint \tau dt$$

or

$$\Lambda_{V_1} = \cos\Phi\Lambda_{U_1} + \sin\Phi\Lambda_{U_3} \tag{2.46}$$

Separating Eq.(2.46) into its real and dual parts gives

$$\begin{cases} \lambda_{V_1} = \cos\varphi\lambda_{u_1} + \sin\varphi\lambda_{u_3} \\ L_{V_1} = \cos\varphi L_{u_1} + \sin\varphi L_{u_3} + \varphi^*\left(\sin\varphi\lambda_{u_1} - \cos\varphi\lambda_{u_3}\right) \end{cases} \tag{2.47}$$

The drall of the closed surface (V_1) is

$$P_{V_1} = \frac{\langle d\overrightarrow{v_1}, d\overrightarrow{v_1}^*\rangle}{\langle d\overrightarrow{v_1}, d\overrightarrow{v_1}\rangle}$$

which gives using the values of $d\overrightarrow{v_1}$ and $d\overrightarrow{v_1}^*$ in Eq.(2.11)

$$P_{V_1} = \frac{p^*}{p} \tag{2.48}$$

and using the values of p and p^* given by Eq.(2.14) gives

$$P_{V_1} = \frac{k_1^*\cos\varphi + k_2^*\sin\varphi}{k_1\cos\varphi + k_2\sin\varphi} - \varphi^*\frac{k_1\sin\varphi - k_2\cos\varphi}{k_1\cos\varphi + k_2\sin\varphi} \tag{2.49}$$

Theorem 2.1 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_1) are*

$$1-)L_{V_1} = -\oint q^* dt \qquad 2-)\Lambda_{V_1} = \oint Q dt \qquad 3-)P_{V_1} = \frac{p^*}{p}$$

Corollary 2.1 *Let (V_1) be the parallel surface of the surface (U_1). The pitch and the dual of the pitch of the ruled surface (V_1) related to the invariants of the surface (U_1) are written as follow*

$1^-)$ $L_{V_1} = \cos\varphi L_{u_1} + \sin\varphi L_{u_3} + \varphi^*\left(\sin\varphi\lambda_{u_1} - \cos\varphi\lambda_{u_3}\right)$;
$2^-)$ $\Lambda_{V_1} = \cos\Phi\Lambda_{U_1} + \sin\Phi\Lambda_{U_3}$

The pitch of the closed surface (V_2) is given by

$$L_{V_2} = \left\langle \overrightarrow{d}, \overrightarrow{v_2}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_2} \right\rangle$$

$$L_{V_2} = 0 \tag{2.50}$$

The dual angle of the pitch of the closed ruled surface (V_2) is

$$\Lambda_{V_2} = -\left\langle \overrightarrow{D}, \overrightarrow{V_2} \right\rangle$$

Using Eq.(2.19) we get

$$\Lambda_{V_2} = 0 \tag{2.51}$$

The drall of the closed surface (V_2) is

$$P_{V_2} = \frac{\langle d\overrightarrow{v_2}, d\overrightarrow{v_2}^*\rangle}{\langle d\overrightarrow{v_2}, d\overrightarrow{v_2}\rangle}$$

Using the values of $d\overrightarrow{v_2}$ and $d\overrightarrow{v_2}^*$ given by Eq.(2.11) gives

$$P_{V_2} = \frac{pp^* + qq^*}{p^2 + q^2} \tag{2.52}$$

and with the values of p, p^*, q and q^* given by Eq.(2.14) we get

$$P_{V_2} = \frac{k_1 k_1^* + k_2 k_2^*}{k_1^2 + k_2^2} \tag{2.53}$$

Theorem 2.2 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_2) are*

$$1-) L_{V_2} = 0 \qquad 2-) \Lambda_{V_2} = 0 \qquad 3-) P_{V_2} = \frac{pp^* + qq^*}{p^2 + q^2}$$

The pitch of the closed surface (V_3) is given by

$$L_{V_3} = \left\langle \overrightarrow{d}, \overrightarrow{v_3}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_3} \right\rangle,$$

$$L_{V_3} = -\oint p^* dt \tag{2.54}$$

and using Eq.(2.54)

$$L_{V_3} = -\cos\varphi \oint k_1^* dt - \sin\varphi \oint k_2^* dt + \varphi^* (\sin\varphi \oint k_1 dt - \cos\varphi \oint k_2 dt) \tag{2.55}$$

or

$$L_{V_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* \left(\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3}\right) \tag{2.56}$$

The dual angle of the pitch of the closed ruled surface (V_3) is

$$\Lambda_{V_3} = -\left\langle \overrightarrow{D}, \overrightarrow{V_3} \right\rangle$$

Due to Eq.(2.19) we have

$$\Lambda_{V_3} = -< \overrightarrow{V_1} \oint Qdt - \overrightarrow{V_3} \oint Pdt, \overrightarrow{V_3} >,$$

$$\Lambda_{V_3} = \oint Pdt. \tag{2.57}$$

and using Eq.(2.6) into the last equation gives

$$\Lambda_{V_3} = \cos\Phi \oint \kappa dt + \sin\Phi \oint \tau dt$$

or

$$\Lambda_{V_3} = \sin\Phi \Lambda_{U_1} - \cos\Phi \Lambda_{U_3} \tag{2.58}$$

Separating Eq.(2.58) into its real and dual parts give

$$\begin{cases} \lambda_{v_3} = \sin\varphi \lambda_{u_1} - \cos\varphi \lambda_{u_3} \\ L_{v_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* \left(\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3}\right) \end{cases} \tag{2.59}$$

The drall of the closed surface (V_3) is

$$P_{V_3} = \frac{\langle d\overrightarrow{v_3}, d\overrightarrow{v_3}^* \rangle}{\langle d\overrightarrow{v_3}, d\overrightarrow{v_3} \rangle}$$

Using the values of $d\overrightarrow{v_3}$ and $d\overrightarrow{v_3}^*$ given by Eq.(2.11) gives

$$P_{V_3} = \frac{q^*}{q} \tag{2.60}$$

and using the values of q and q^* given by Eq.(2.14) into the last equations, we get

$$P_{V_3} = \frac{-k_1^* \sin\varphi - k_2^* \cos\varphi}{-k_1 \sin\varphi + k_2 \cos\varphi} - \varphi^* \left(\frac{k_1 \cos\varphi + k_2 \sin\varphi}{-k_1 \sin\varphi + k_2 \cos\varphi} \right) \tag{2.61}$$

Theorem 2.3 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_3) are*

$$1-) L_{V_3} = -\oint p^* dt \qquad 2-) \Lambda_{V_3} = \oint P dt \qquad 3-) P_{V_3} = \frac{q^*}{q}.$$

Corollary 2.2 *Let (V_1) be the parallel surface of the surface (U_1). The pitch and the dual of the pitch of the ruled surface (V_3) related to the invariants of the surface (U_1) are written as follow*

$1^-)$ $L_{V_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* (\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3})$;
$2^-)$ $\Lambda_{V_3} = \sin\Phi \Lambda_{U_1} - \cos\Phi \Lambda_{U_3}$.

Let $\Theta(t) = \theta(t) + \varepsilon\theta^*(t)$ be the Lorentzian timelike angle between the instantaneous dual Pfaffion vector $\overrightarrow{\overline{\Psi}}$ and the vector $\overrightarrow{V_3}$.

In this case dual Pfaffion vector $\overrightarrow{\overline{\Psi}}$ is spacelike vector,

$$P = \left\| \overrightarrow{\overline{\Psi}} \right\| \cos\Theta, \qquad Q = \left\| \overrightarrow{\overline{\Psi}} \right\| \sin\Theta$$

The unit vector $\overrightarrow{\overline{C}} = \overrightarrow{\overline{c}} + \varepsilon \overrightarrow{\overline{c}}^*$, in the $\overrightarrow{\overline{\Psi}}$ direction is

$$\overrightarrow{\overline{C}} = -\sin\Theta \overrightarrow{V_1} + \cos\Theta \overrightarrow{V_3} \tag{2.62}$$

Using the values of the vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_3}$ given by Eq.(2.9) into Eq.(2.62), we get

$$\overrightarrow{\overline{C}} = -\sin\Theta \left(\cos\Phi \overrightarrow{U_1} + \sin\Phi \overrightarrow{U_3} \right) + \cos\Theta \left(\sin\Phi \overrightarrow{U_1} - \cos\Phi \overrightarrow{U_3} \right)$$

$$\overrightarrow{\overline{C}} = \sin(\Theta - \Phi) \overrightarrow{U_1} - \cos(\Theta - \Phi) \overrightarrow{U_3} \tag{2.63}$$

The real and dual parts of $\overrightarrow{\overline{C}}$ are

$$\begin{cases} \overrightarrow{\overline{c}} = -\sin\theta \overrightarrow{v_1} + \cos\theta \overrightarrow{v_3} \\ \overrightarrow{\overline{c}}^* = -\sin\theta \overrightarrow{v_1}^* + \cos\theta \overrightarrow{v_3}^* - \theta^* \cos\theta \overrightarrow{v_1} - \theta^* \sin\theta \overrightarrow{v_3} \end{cases} \tag{2.64}$$

The pitch of the closed surface $(\overrightarrow{\overline{C}})$ is given by

$$L_{\overline{C}} = \left\langle \overrightarrow{d}, \overrightarrow{c}^{\,*} \right\rangle + \left\langle \overrightarrow{d}^{\,*}, \overrightarrow{c} \right\rangle$$

and using the values of $\overrightarrow{d}$ and $\overrightarrow{d}^{\,*}$ given by Eq.(2.22) into the last equation we get

$$L_{\overline{C}} = -\cos\theta \oint p^* dt + \sin\theta \oint q^* dt + \theta^* \left(\cos\theta \oint q dt + \sin\theta \oint p dt \right) \tag{2.65}$$

or

$$L_{\overline{C}} = -\sin\theta L_{V_1} + \cos\theta L_{V_3} + \theta^* \left(\cos\theta \lambda_{V_1} + \sin\theta \lambda_{V_3} \right) \tag{2.66}$$

Finally if we use Eq.(2.47) and Eq.(2.59) into Eq.(2.66), we get

$$\begin{aligned} L_{\overline{C}} = & \sin(\varphi - \theta) L_{U_1} - \cos(\varphi - \theta) L_{U_3} + \\ & (\varphi^* - \theta^*) \left(\cos(\varphi - \theta) \lambda_{U_1} \right) + \sin(\varphi - \theta) \lambda_{U_3} \end{aligned} \tag{2.67}$$

The dual angle of the pitch of the closed ruled surface $(\overrightarrow{\overline{C}})$, we may write

$$\Lambda_{\overline{C}} = -\left\langle \overrightarrow{D}, \overrightarrow{\overline{C}} \right\rangle$$

and using Eq.(2.21) and Eq.(2.62) we get

$$\Lambda_{\overline{C}} = - < -\overrightarrow{V_1} \oint Q dt - \overrightarrow{V_3} \oint P dt, -\sin\Theta V_1 + \cos\Theta V_3 >,$$

$$\Lambda_{\overline{C}} = -\sin\Theta \oint Q dt + \cos\Theta \oint P dt \tag{2.68}$$

If we use the Eqs.(2.45) and (2.57) into the last equation, we get

$$\Lambda_{\overline{C}} = -\sin\Theta \Lambda_{V_1} + \cos\Theta \Lambda_{V_3} \tag{2.69}$$

If we use Eq.(2.46), we get

$$\Lambda_{\overline{C}} = -\sin(\Theta - \Phi)\Lambda_{U_1} - \cos(\Theta - \Phi)\Lambda_{U_3} \tag{2.70}$$

The drall of the closed surface $(\overrightarrow{\overline{C}})$, we may write

$$P_{\overline{C}} = \frac{\left\langle d\overrightarrow{c}, d\overrightarrow{c}^{\,*} \right\rangle}{\left\langle d\overrightarrow{c}, d\overrightarrow{c} \right\rangle}$$

$$P_{\overline{C}} = \frac{\theta' \theta^{*'} - (q\cos\theta - p\sin\theta)\left[(q^* - p\theta^*)\cos\theta - (q\theta^* + p^*)\sin\theta \right]}{\theta'^2 - (q\cos\theta - p\sin\theta)^2} \tag{2.71}$$

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Math.Combin.Book Ser. Vol.3(2013), 69-77

Some Prime Labeling Results of H-Class Graphs

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Abstract: Prime labeling originated with Entringer and was introduced by Tout, Dabboucy and Howalla [3]. A Graph $G(V,E)$ is said to have a *prime labeling* if its vertices are labeled with distinct integers $1,2,3,\cdots,|V(G)|$ such that for each edge xy the labels assigned to x and y are relatively prime. A graph admits a prime labeling is called a *prime graph*. We investigate the prime labeling of some H-class graphs.

Key Words: labeling, prime labeling, prime graph, H-class graph

AMS(2010): 05C78

§1. Introduction

A simple graph $G(V,E)$ is said to have a prime labeling (or called prime) if its vertices are labeled with distinct integers $1,2,3,\ldots,|V(G)|$, such that for each edge $xy \in E(G)$, the labels assigned to x and y are relatively prime [1].

We begin with listing a few definitions/notations that are used.

(1) A graph $G=(V,E)$ is said to have order $|V|$ and size $|E|$.

(2) A vertex $v \in V(G)$ of degree 1 is called pendant vertex.

(3) P_n is a path of length n.

(4) The H-graph is defined as the union of two paths of length n together with an edge joining the mid points of them. That is, it is obtained from two copies of P_n with vertices $v_1,v_2,\ldots,v_n$ and $u_1,u_2,\ldots,u_n$ by joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ by means of an edge if n is odd and the vertices $v_{(n/2)+1}$ and $u_{n/2}$ if n is even [4].

(5) The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 [1].

§2. Prime Labeling of H-Class Graphs

Theorem 2.1 *The H-graph of a path of length n is prime.*

Proof Let $G=(V,E)$ be a H-graph of a path of length n. It is obtained from two copies

[1]Received April 15, 2013, Accepted August 24, 2013.

of paths of length n. It has $2n$ vertices and $2n-1$ edges.

$$V(G) = \{u_i, v_i/1 \leq i \leq n\}$$
$$E(G) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_{\lceil (n/2) \rceil}v_{\lceil (n/2) \rceil}\}$$

Define $f : V(G) \to \{1, 2, \ldots, 2n\}$ by

$$f(v_{(n+1)/2}) = 1 \text{ if } n \text{ is odd}$$
$$f(v_{n/2}) = 1 \text{ if } n \text{ is even}$$
$$f(u_i) = i+1, 1 \leq i \leq n$$
$$f(v_i) = n+i+1, 1 \leq i \leq (n/2)-1, \text{ when } i \neq (n/2) \text{ if } n \text{ is even}$$
$$f(v_i) = n+i, (n/2)+1 \leq i \leq n, \text{ if } n \text{ is even}$$
$$f(v_i) = n+i+1, 1 \leq i \leq (n-1)/2, \text{ when } i \neq (n+1)/2, \text{ if } n \text{ is odd}$$
$$f(v_i) = n+i, (n+3)/2 \leq i \leq n, \text{ when } i \neq (n+1)/2, \text{ if } n \text{ is odd}$$

Clearly, it is easy to check that $GCD\ (f(u), f(v)) = 1$, for every edge $uv \in E(G)$. Therefore, the H-graph of a path of length n admits prime labeling. □

Example 2.2 The prime labeling for H-graph with $n = 14, 16$ are shown in Fig.1 and 2.

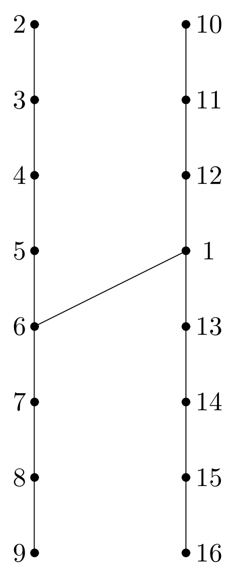

Fig.1 $n \equiv 0(\text{mod}2)$

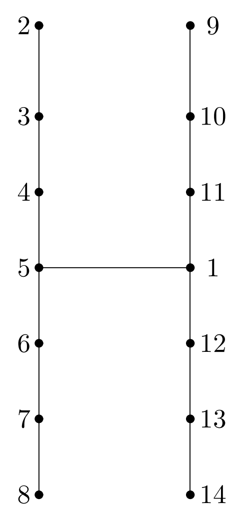

Fig.2 $n \equiv 1(\text{mod}2)$

Theorem 2.3 *The graph $G \odot K_1$ is a prime.*

Proof $G \odot K_1$ is obtained from H-graph by attaching pendant vertices to each of the vertices. The graph has $4n$ vertices and $4n-1$ edges, where $n = |G|$.

$$V(G \odot K_1) = \{u_i, v_i / 1 \leq i \leq 2n\}$$
$$E(G \odot K_1) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i u_{n+i}, v_i v_{n+i} / 1 \leq i \leq n\}$$
$$\cup \{u_{(n+1)/2} v_{(n+1)/2} \ n \text{ is odd or } u_{(n/2)+1} v_{n/2}, n \text{ is even.}\}$$

Define $f : V(G \odot K_1) \to \{1, 2, \ldots, 4n\}$ by

$$f(u_i) = 2i+1, 1 \leq i \leq n \text{ and } \begin{cases} i \neq (n+1)/2 & \text{if } n \text{ is odd} \\ i \neq n/2 & \text{if } n \text{ is even} \end{cases}$$

$$f(u_{(n+1)/2}) = 1 \ n \text{ odd}$$
$$f(u_{(n/2)+1}) = 1 \ n \text{ even}$$
$$f(u_{n+i}) = 2i, \ 1 \leq i \leq n$$
$$f(v_i) = 2n + 2i - 1, 1 \leq i \leq n$$
$$f(v_{n+i}) = 2n + 2i, 1 \leq i \leq n$$
$$GCD(f(u_i), f(u_{i+1})) = GCD(2i+1, 2i+3) = 1, 1 \leq i \leq (n-3)/2, n \text{ odd}$$
$$GCD(f(u_i), f(u_{i+1})) = GCD(2i+1, 2i+3) = 1, (n+3)/2 \leq i \leq n-1, n \text{ odd}$$
$$GCD(f(u_i), f(u_{i+1})) = GCD(2i+1, 2i+3) = 1, 1 \leq i \leq (n/2)-1, n \text{ even}$$
$$GCD(f(u_i), f(u_{i+1})) = GCD(2i+1, 2i+3) = 1, (n/2)+2 \leq i \leq n-1, n \text{ even}$$
$$GCD(f(v_i), f(v_{i+1})) = GCD(2n+2i-1, 2n+2i+1) = 1, 1 \leq i \leq n$$
$$GCD(f(v_i), f(v_{n+i})) = GCD(2n+2i-1, 2n+2i) = 1, 1 \leq i \leq n.$$

In this case it can be easily verified that $GCD(f(u), f(v)) = 1$ for remaining edges $uv \in E(G \odot K_1)$. Therefore, $G \odot K_1$ admits prime labeling. □

Example 2.4 The prime labeling for $G \odot K_1$ and $G \odot K_1$ are shown in Fig.3 and 4.

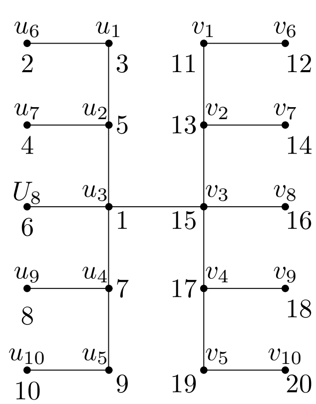

Fig.3 $G \odot K_1$

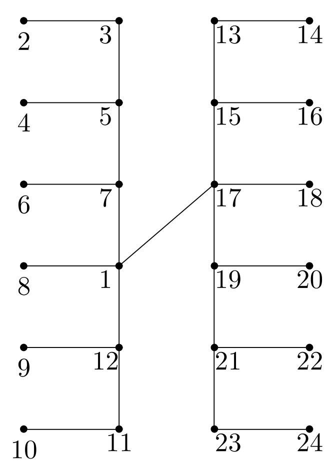

Fig.4 $G \odot K_1$

Theorem 2.5 *The graph $G \odot S_2$ is prime.*

Proof The graph $G \odot S_2$ has $6n$ vertices and $6n-1$ edges, where $n = |G|$.

$$V(G \odot S_2) = \{u_i, v_i / 1 \le i \le n\} \cup \{u_i^{(1)}, u_i^{(2)}, v_i^{(1)}, v_i^{(2)} / 1 \le i \le n\}$$
$$E(G \odot S_2) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \le i \le n-1\} \cup \{u_i u_i^{(1)}, u_i u_i^{(2)}, v_i v_i^{(1)}, v_i v_i^{(2)} / 1 \le i \le n\}$$
$$\cup \{u_{n+1/2} v_{n+1/2} \ n \text{ is odd or } u_{n/2+1} v_{n/2} \ n \text{ is even.}$$

Define $f : V \to \{1, 2, \ldots, 6n\}$ by
$f(u_{n+1/2}) = 1, f(u^{(1)}_{(n+1)/2}) = 6n-1, f(u^{(2)}_{(n+1)/2}) = 6n.$

Case 1 Suppose $n \equiv 1(mod\ 2)$.

Subcase 1.1 $n \equiv 1(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1)+3, 1 \le i \le (n-1)/4$$
$$f(u_{2i-1}) = f(u_{(n-1)/2}) + 6 + 6[i - ((n-1)/4)+2],$$
$$((n-1)/4)+2 \le i \le ((n-1)/2)+1$$
$$f(u_{2i}) = 6(i-1)+5, 1 \le i \le (n-1)/4$$
$$f(u_{2i}) = f(u_{(n-1)/2}) + 4 + 6[i - ((n-1)/4)+1],$$
$$((n-1)/4)+1 \le i \le (n-1)/2$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n-1)/2}) + 7 + 6[i - ((n-1)/4)+2],$$
$$((n-1)/4)+2 \le i \le (n-1)/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(2)}_{2i-1}) = f(u_{(n-1)/2}) + 8 + 6[i - ((n-1)/4)+2],$$
$$((n-1)/4)+2 \le i \le ((n-1)/2)+1$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(1)}_{2i}) = f(u_{(n-1)/2}) + 3 + 6[i - ((n-1)/4+1],$$
$$((n-1)/4)+1 \le i \le (n-1)/2$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(2)}_{2i}) = f(u_{(n-1)/2}) + 5 + 6[i - ((n-1)/4)+1],$$
$$((n-1)/4)+1 \le i \le (n-1)/2$$
$$f(v_1) = 3n, f(v_2) = 3n+2$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \le i \le (n+1)/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \le i \le (n-1)/2$$
$$f(v^{(1)}_{2i-1}) = f(v_1) - 1 + 6(i-1), 1 \le i \le (n+1)/2$$

$$f(v^{(2)}_{2i-1}) = f(v_1) + 1 + 6(i-1), 1 \le i \le (n+1)/2$$
$$f(v^{(1)}_{2i}) = f(v_2) + 1 + 6(i-1), 1 \le i \le (n-1)/2$$
$$f(v^{(2)}_{2i}) = f(v_2) + 2 + 6(i-1), 1 \le i \le (n-1)/2$$
$$GCD(f(u_{2i-1}), f(u_{2i})) = GCD(6i-3, 6i-1) = 1, 1 \le i \le (n-1)/4$$
$$GCD(f(u_{2i}), f(u_{2i+1})) = GCD(6i-1, 6i+3) = 1, 1 \le i \le ((n-1)/4) - 1$$
$$GCD(f(u_{2i-1}), f(u_{2i})) = GCD(6i-7, 6i-3) = 1, \quad ((n-1)/4) + 2 \le i \le (n-1)/2$$
$$GCD(f(u_{2i}), f(u_{2i+1})) = GCD(6i-3, 6i-1) = 1, \quad ((n-1)/4) + 1 \le i \le (n-1)/2$$
$$GCD(f(u_{2i-1}), f(u^{(2)}_{2i-1})) = GCD(6i-7, 6i-5) = 1, \quad ((n-1)/4) + 2 \le i \le ((n-1)/2) + 1$$
$$GCD(f(u_{2i}), f(u^{(2)}_{2i})) = GCD(6i-1, 6i+1) = 1, 1 \le i \le (n-1)/4$$
$$GCD(f(u_{2i}), f(u^{(2)}_{2i})) = GCD(6i-3, 6i-2) = 1, \quad ((n-1)/4) + 1 \le i \le (n-1)/2$$
$$GCD(f(v_1), f(v_2)) = GCD(3n, 3n+2) = 1$$
$$GCD(f(v_{2i-1}), f(v_{2i})) = GCD(3n+6i-6, 3n+6i-4) = 1, \quad 2 \le i \le (n-1)/2$$
$$GCD(f(v_{2i}), f(v_{2i+1})) = GCD(3n+6i-4, 3n+6i) = 1, \quad 1 \le i \le (n-1)/2$$
$$GCD(f(v_{2i}), f(v^{(2)}_{2i})) = GCD(3n+6i-4, 3n+6i-2) = 1, \quad 1 \le i \le (n-1)/2.$$

Subcase 1.2 $n \equiv 3(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \le i \le (n+1)/4$$
$$f(u_{2i-1}) = f(u_{(n-1)/2}) + 2 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4 + 1 \le i \le (n+1)/2$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \le i \le ((n+1)/4) - 1$$
$$f(u_{2i}) = f(u_{(n-1)/2}) + 6 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4) + 1 \le i \le ((n+1)/2) - 1$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \le i \le (n+1)/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n-1)/2}) + 3 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4) + 1 \le i \le (n+1)/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \le i \le (n+1)/4$$

$$f(u^{(2)}_{2i-1}) = f(u_{(n-1)/2}) + 4 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4) + 1 \leq i \leq (n+1)/2$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq ((n+1)/4) - 1$$
$$f(u^{(1)}_{2i}) = f(u_{(n-1)/2}) + 5 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4) + 1 \leq i \leq ((n+1)/2) - 1$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq ((n+1)/4) - 1$$
$$f(u^{(2)}_{2i}) = f(u_{(n-1)/2}) + 7 + 6[i - ((n+1)/4) + 1], \quad ((n+1)/4) + 1 \leq i \leq ((n+1)/2) - 1$$
$$f(v_1) = 3n, f(v_2) = 3n + 2$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq (n+1)/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq ((n+1)/2) - 1$$
$$f(v^{(1)}_{2i-1}) = f(v_1) - 1 + 6(i-1), 1 \leq i \leq (n+1)/2$$
$$f(v^{(2)}_{2i-1}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq (n+1)/2$$
$$f(v^{(1)}_{2i}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq ((n+1)/2) - 1$$
$$f(v^{(2)}_{2i}) = f(v_2) + 2 + 6(i-1), 1 \leq i \leq ((n+1)/2) - 1.$$

As in the above case it can be verified that $GCD(f(u), f(v)) = 1$ for every edge $uv \in E(G \odot S_2)$.

Case 2. $n \equiv 0(mod\ 2)$

$$f(u_{(n/2)+1}) = 1, f(u^{(1)}_{(n/2)+1}) = 6n - 1, f(u^{(2)}_{(n/2)+1}) = 6n.$$

Subcase 2.1 $n \equiv 0(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \leq i \leq n/4$$
$$f(u_{2i-1}) = f(u_{(n/2)}) + 6 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \leq i \leq n/4$$
$$f(u_{2i}) = f(u_{(n/2)}) + 4 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n/2)}) + 7 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(2)}_{2i-1}) = f(u_{(n/2)}) + 8 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(1)}_{2i}) = f(u_{(n/2)}) + 3 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(2)}_{2i}) = f(u_{(n/2)}) + 5 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$

$$f(v_1) = 3n-1, f(v_2) = 3(n+1)$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v^{(1)}_{2i-1}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v^{(2)}_{2i-1}) = f(v_1) + 2 + 6 + (i-1), 1 \leq i \leq n/2$$
$$f(v^{(1)}_{2i}) = f(v_2) - 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v^{(2)}_{2i}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq n/2$$

Clearly $GCD(f(u), f(v)) = 1$ for every edge $uv \in E(G \odot S_2)$.

Subcase 2.2 $n \equiv 2(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \leq i \leq (n+2)/4$$
$$f(u_{2i-1}) = f(u_{(n/2)}) + 2 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \leq i \leq (n-2)/4$$
$$f(u_{2i}) = f(u_{(n/2)}) + 6 + 6]i - ((n-2)/4) + 2], ((n-2)/4) + 2 \leq i \leq n/2$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \leq i \leq (n+2)/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n/2)}) + 3 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \leq i \leq (n+2)/4$$
$$f(u^{(2)}_{2i-1}) = f(u_{(n/2)}) + 4 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq (n-2)/4$$
$$f(u^{(1)}_{2i}) = f(u_{(n/2)}) + 5 + 6[i - ((n-2)/4) + 2], ((n-2) + 2 \leq i \leq n/2$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq (n-2)/4$$
$$f(u^{(2)}_{2i}) = f(u_{(n/2)}) + 7 + 6[i - ((n-2)/4) + 2], ((n-2)/4) + 2 \leq i \leq n/2$$
$$f(v_1) = 3n-1, f(v_2) = 3(n+1)$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v^{(1)}_{2i-1}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v^{(2)}_{2i-1}) = f(v_1) + 2 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v^{(1)}_{2i}) = f(v_2) - 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v^{(2)}_{2i}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq n/2.$$

In this case also it is easy to check that $GCD(f(u), f(v)) = 1$. Therefore $G \odot S_2$ admits prime labeling. □

Example 2.6 The prime labelings for $G \odot S_2$ with $n \equiv 1(\text{mod}4)$, $n \equiv 2(\text{mod}4)$, $n \equiv 3(\text{mod}4)$, $n \equiv 0(\text{mod}4)$ are respectively shown in Fig.5-8 following.

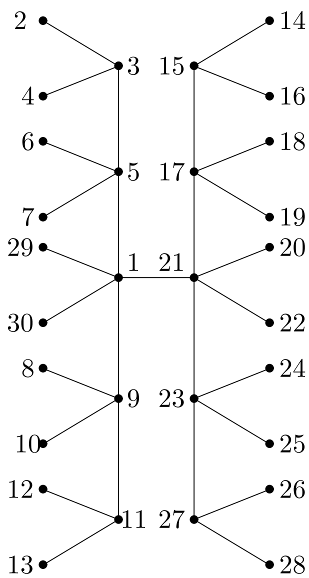

Fig.5 $G \odot S_2$ $n \equiv 1(\text{mod}4)$

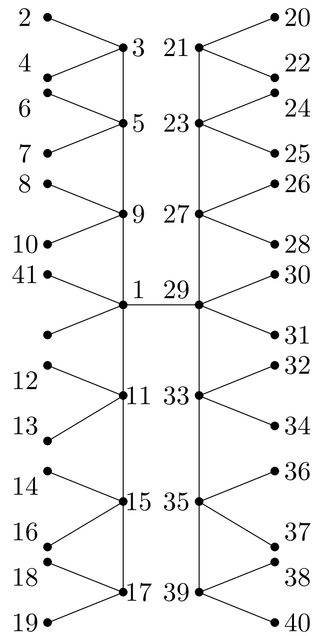

Fig.6 $G \odot S_2$ $n \equiv 3(\text{mod}4)$

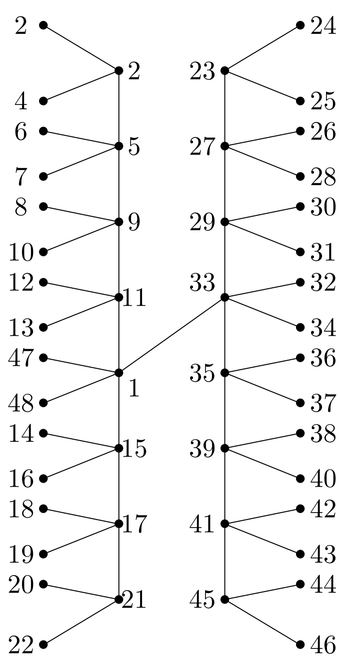

Fig.7 $G \odot S_2$ $n \equiv 0(\text{mod}4)$

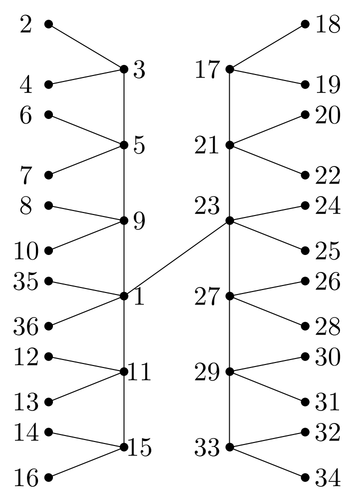

Fig.8 $G \odot S_2$ $n \equiv 2(\text{mod}4)$

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On Mean Cordial Graphs

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Abstract: Let f be a function from the vertex set $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ respectively are denote the number of vertices and edges labeled with x $(x = 0,1,2)$. A graph with a mean cordial labeling is called a mean cordial graph. In this paper we investigate mean cordial labeling behavior of union of some graphs, square of paths, subdivision of comb and double comb and some more standard graphs.

Key Words: Path, star, complete graph, comb.

AMS(2010): 05C78

§1. Introduction

All graphs in this paper are finite, undirected and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The corona of G with H, $G \odot H$ is the graph obtained by taking one copy of G and p copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H. The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The triangular snake T_n is obtained from the path P_{n+1} by replacing each edge of the path by the triangle C_3. mG denotes the m copies of the graph G. The square G^2 of a graph G has the vertex set $V(G^2) = V(G)$, with u,v adjacent in G^2 whenever $d(u,v) \leq 2$ in G. The powers G^3, $G^4 \ldots$ of G are similarly defined. Ponraj et al. defined the mean cordial labeling of a graph in [4]. Mean cordial labeling behavior of path, cycle, star, complete graph, wheel, comb etc have been investigated in [4]. Here we investigate the mean cordial labeling behavior of some standard graphs. The symbol $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms and definitions are not defined here are used in the sense of Harary [3].

[1]Received January 18, 2013, Accepted August 27, 2013.

§2. Mean Cordial Labeling

Definition 2.1 *Let f be a function from $V(G)$ to $\{0,1,2\}$. For each edge uv of G assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i)-v_f(j)| \leq 1$ and $|e_f(i)-e_f(j)| \leq 1$, $i,j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x $(x=0,1,2)$ respectively. A graph with a mean cordial labeling is called a mean cordial graph.*

Theorem 2.2 *If $m \equiv 0 \pmod 3$ then mG is mean cordial for all m.*

Proof Let $m=3t$. Assign the label 0 to all the vertices of first t copies of the graph G. Then assign 1 to the vertices of next t copies of G. Finally assign 2 to remaining vertices of mG. Therefore $v_f(0)=v_f(1)=v_f(2)=pt$, $e_f(0)=e_f(1)=e_f(2)=qt$. □

Theorem 2.3 *If G is mean cordial, then mG, $m \equiv 1 \pmod 3$ is also mean cordial.*

Proof $(m-1)G$ is mean cordial by theorem 2.2. Let g be a mean cordial labeling of $(m-1)G$. Using the mean cordial labeling g of $(m-1)G$ and the mean cordial labeling of G, we get a mean cordial labeling of mG. □

Theorem 2.4 *$P_m \cup P_n$ is mean cordial.*

Proof Let $u_1u_2\ldots u_m$ and $v_1v_2\ldots v_n$ be the paths P_m and P_n respectively. Clearly $P_m \cup P_n$ has $m+n$ vertices and $m+n-2$ edges. Assume $m \geq n$.

Case 1 $m+n \equiv 0 \pmod 3$

Let $m+n=3t$. Define

$$\begin{aligned} f(u_i) &= 2, \quad 1 \leq i \leq t, \\ f(u_{t+i}) &= 1, \quad 1 \leq i \leq m-t, \\ f(v_i) &= 1, \quad 1 \leq i \leq n-t, \\ f(v_{n-t+i}) &= 0, \quad 1 \leq i \leq t. \end{aligned}$$

Clearly $v_f(0)=v_f(1)=v_f(2)=t$ and $e_f(0)=e_f(1)=t-1$, $e_f(2)=t$. Therefore f is a mean cordial labeling.

Case 2 $m+n \equiv 1 \pmod 3$

Similar to Case 1.

Case 3 $m+n \equiv 2 \pmod 3$

Let $m+n=3t+2$. Define

$$\begin{aligned} f(u_i) &= 2, \quad 1 \le i \le t, \\ f(u_{t+i}) &= 1, \quad 1 \le i \le m-t, \\ f(v_i) &= 1, \quad 1 \le i \le n-t-1, \\ f(v_{n-t+i}) &= 0, \quad 1 \le i \le t+1. \end{aligned}$$

Clearly $v_f(0) = v_f(1) = t+1$, $v_f(2) = t$ and $e_f(0) = t-1$, $e_f(1) = e_f(2) = t$. Therefore $P_m \cup P_n$ is mean cordial. □

Theorem 2.5 *$C_n \cup P_m$ is mean cordial if $m \ge n$.*

Proof Let C_n be the cycle $u_1u_2\ldots u_mu_1$ and P_m be the path $v_1v_2\ldots v_n$ respectively. Clearly $C_n \cup P_m$ has $m+n$ vertices and $m+n-1$ edges. Assume $m \ge n$.

Case 1 $m+n \equiv 0 \pmod 3$

Let $m+n=3t$. Define

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le t, \\ f(u_{t+i}) &= 1, \quad 1 \le i \le n-,t \\ f(v_i) &= 1, \quad 1 \le i \le m-t, \\ f(v_{m-t+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

Clearly $e_f(0) = t-1$, $e_f(1) = e_f(2) = t$.

Case 2 $m+n \equiv 1 \pmod 3$

Let $m+n=3t+1$. Define

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le t+1, \\ f(u_{t+1+i}) &= 1, \quad 1 \le i \le n-t-1, \\ f(v_i) &= 1, \quad 1 \le i \le m-t, \\ f(v_{m-t+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

Clearly $e_f(0) = e_f(1) = t-1$, $e_f(2) = t$.

Case 3 $m+n \equiv 2 \pmod 3$

Let $m+n=3t+2$. Define

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le t+1, \\ f(u_{t+1+i}) &= 1, \quad 1 \le i \le n-t-1, \\ f(v_i) &= 1, \quad 1 \le i \le m-t-1, \\ f(v_{m-t-1+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

Clearly $e_f(0) = e_f(1) = t$, $e_f(2) = t+1$. Hence $C_n \cup P_m$ is mean cordial. □

Theorem 2.6 *$K_{1,n} \cup P_m$ is mean cordial.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \le i \le n\}$ and $E(K_{1,n}) = \{uu_i : 1 \le i \le n\}$. Let P_m be the path $v_1v_2 \ldots v_m$ respectively. Clearly $K_{1,n} \cup P_m$ has $m+n+1$ vertices and $m+n-1$ edges.

Case 1 $m+n \equiv 0 \pmod 3$

Let $m+n = 3t$. Define $f(u) = 1$

$$\begin{aligned} f(u_i) &= 1, & 1 \le i \le t-1, \\ f(u_{t-1+i}) &= 2, & 1 \le i \le n-t+1, \\ f(v_i) &= 2, & 1 \le i \le m-t-1, \\ f(v_{m-t-1+i}) &= 0, & 1 \le i \le t+1. \end{aligned}$$

Clearly $e_f(0) = e_f(1) = t$, $e_f(2) = t-1$.

Case 2 $m+n \equiv 1 \pmod 3$

Similar to Case 1.

Case 3 $m+n \equiv 2 \pmod 3$

Let $m+n = 3t+2$. Assign the labels to the vertices as in case 1 and then $e_f(0) = e_f(2) = t$, $e_f(1) = t+1$. Hence $K_{1,n} \cup P_m$ is mean cordial. □

Example 2.7 A mean cordial labeling of $K_{1,8} \cup P_6$ is given in Figure 1.

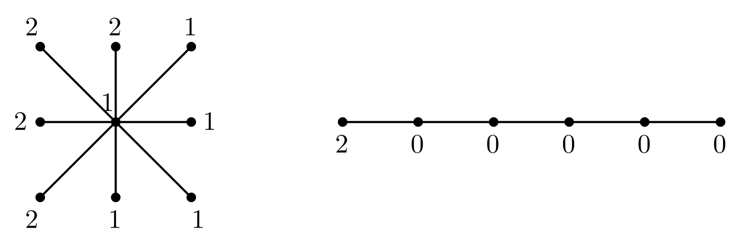

Figure 1

Theorem 2.8 *$S(P_n \odot K_1)$ is mean cordial where $S(G)$ and $G \odot H$ respectively denotes the subdivision of G and corona of G with H.*

Proof Let P_n be the path $u_1u_2 \ldots u_n$ and v_i be the pendant vertices adjacent to u_i. Let the edges u_iu_{i+1}, u_iv_i be subdivided by the vertices z_i and w_i respectively.

Case 1 $n \equiv 0 \pmod 3$

Let $n = 3t$. Define $f(u_i) = f(v_i) = f(w_i) = 2$, $f(u_{t+i}) = f(v_{t+i}) = f(w_{t+i}) = 1$, $f(u_{2t+i}) = f(v_{2t+i}) = f(w_{2t+i}) = 0$, $1 \le i \le t$.

$$\begin{aligned} f(z_i) &= 2, & 1 \le i \le t, \\ f(z_{t+i}) &= 1, & 1 \le i \le t-1, \\ f(z_{2t-1+i}) &= 0, & 1 \le i \le t. \end{aligned}$$

Here $v_f(0) = v_f(2) = 4t$, $v_f(1) = 4t-1$ and $e_f(0) = e_f(1) = 4t-1$, $e_f(2) = 4t$. Hence $S(P_n \odot K_1)$ is mean cordial graph.

Case 2 $n \equiv 1 \pmod 3$

Label the vertices z_i, u_i, v_i $(1 \le i \le n-1)$, w_i $(1 \le i \le n-2)$ as in Case 1. Then assign the labels $0,1,1,2$ to the vertices z_n, u_n, w_{n-1}, v_n respectively. Hence $v_f(0) = v_f(1) = v_f(2) = 4t+1$, $e_f(0) = 4t$, $e_f(1) = e_f(2) = 4t+1$. Hence $S(P_n \odot K_1)$ is mean cordial.

Case 3 $n \equiv 2 \pmod 3$

Label the vertices z_i, u_i, v_i $(1 \le i \le n-2)$, w_i $(1 \le i \le n-3)$ as in case 1. Assign the labels $0,1,2,2,1,1,0,0$ to the vertices u_{n-1}, u_n, v_{n-1}, v_n, w_{n-2}, w_{n-1}, z_{n-1}, z_n respectively. Here $v_f(1) = v_f(2) = 4t+2$, $v_f(0) = 4t+3$, $e_f(0) = e_f(1) = e_f(2) = 4t+2$. Hence $S(P_n \odot K_1)$ is mean cordial. □

Example 2.9 Mean cordial labeling of $S(P_4 \odot K_1)$ is given in Figure 2.

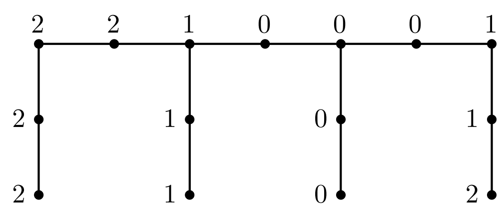

Figure 2

Theorem 2.10 *$(P_n \odot 2K_1)$ is mean cordial.*

Proof Let P_n be the path $u_1u_2\ldots u_n$ and v_i and w_i be the pendant vertices adjacent to u_i $(1 \le i \le n)$. Let the edges u_iu_{i+1}, u_iv_i, u_iw_i be subdivided by the vertices x_i and y_i, z_i respectively.

Case 1 $n \equiv 0 \pmod 3$

Let $n = 3t$. Define $f(u_i) = f(v_i) = f(w_i) = f(y_i) = f(z_i) = 2$, $f(u_{t+i}) = f(v_{t+i}) = f(w_{t+i}) = f(y_{t+i}) = f(z_{t+i}) = 1$, $f(u_{2t+i}) = f(v_{2t+i}) = f(w_{2t+i}) = f(y_{2t+i}) = f(z_{2t+i}) = 0$, $1 \le i \le t$.

$$\begin{aligned} f(x_i) &= 2 \quad 1 \le i \le t \\ f(x_{t+i}) &= 1 \quad 1 \le i \le t-1 \\ f(x_{2t-1+i}) &= 0 \quad 1 \le i \le t. \end{aligned}$$

Here $v_f(0) = v_f(2) = 6t$, $v_f(1) = 6t-1$, $e_f(0) = e_f(1) = 6t-1$, $e_f(2) = 6t$. Hence $S(P_n \odot 2K_1)$ is mean cordial.

Case 2 $n \equiv 1 \pmod 3$

Label the vertices u_i, v_i, w_i, y_i and z_i $(1 \le i \le n-1)$, x_i $(1 \le i \le n-2)$ as in case 1. Assign the labels $0,2,1,0,2,1$ to the vertices u_n, v_n, w_n, x_{n-1}, y_n and z_n respectively. Hence

$v_f(0) = v_f(2) = 6t+2$, $v_f(1) = 6t+1$, $e_f(0) = e_f(2) = 6t+1$, $e_f(1) = 6t+2$. Hence $S(P_n \odot 2K_1)$ is mean cordial.

Case 3 $n \equiv 2 \pmod 3$

Label the vertices u_i, v_i, w_i, y_i and z_i $(1 \le i \le n-2)$, x_i $(1 \le i \le n-3)$ as in case 1. Assign the labels $0,0,2,2,2,2,0,0,1,1,1,1$ to the vertices u_{n-1}, u_n, v_{n-1}, v_n, w_{n-1}, w_n, x_{n-2}, x_{n-1}, y_{n-1}, y_n z_{n-1} and z_n respectively. Hence $v_f(0) = v_f(2) = 6t+4$, $v_f(1) = 6t+3$, $e_f(0) = e_f(1) = 6t+3$, $e_f(2) = 6t+4$. Hence $S(P_n \odot 2K_1)$ is mean cordial. □

Theorem 2.11 *P_n^2 is mean cordial iff $n \equiv 1 \pmod 3$ and $n \ge 7$.*

Proof Let P_n be the path $u_1u_2\ldots u_n$. Clearly P_n^2 $(n \le 6)$ are not mean cordial. Assume $n \ge 7$. Clearly the order and size of P_n^2 are n and $2n-3$ respectively.

Case 1 $n \equiv 0 \pmod 3$

Let $n = 3t$. In this case $e_f(0) = (t-1)+(t-2) \le 2t-3$. which is a contradiction to the size of P_n^2.

Case 2 $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Define

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le t+1, \\ f(u_{t+1+i}) &= 1, \quad 1 \le i \le t, \\ f(u_{2t+1+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

Here $v_f(0) = t+1$, $v_f(1) = v_f(2) = t$, $e_f(0) = 2t-1$, $e_f(1) = e_f(2) = 2t$. Therefore P_n^2 is mean cordial.

Case 3 $n \equiv 2 \pmod 3$

Let $n = 3t+2$. Here $e_f(0) \le 2t-1$, a contradiction to the size of P_n^2. Therefore P_n^2 is not mean cordial. □

Example 2.12 A mean cordial labeling of P_{10}^2 is given in Figure 3.

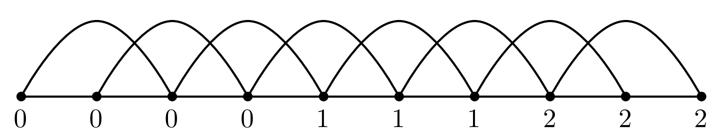

Figure 3

Theorem 2.13 *The triangular snake T_n $(n > 1)$ is mean cordial iff $n \equiv 0 \pmod 3$.*

Proof Let $V(T_n) = \{u_i, v_j : 1 \le i \le n+1,\ 1 \le j \le n\}$ and $E(T_n) = \{u_iu_{i+1} : 1 \le i \le n\} \cup \{u_iv_i, v_iu_{i+1} : 1 \le i \le n\}$.

Case 1 $n \equiv 0 \pmod 3$

Let $n = 3t$. Define

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le t+1, \\ f(u_{t+1+i}) &= 1, \quad 1 \le i \le t, \\ f(u_{2t+1+i}) &= 2, \quad 1 \le i \le t, \\ f(v_i) &= 0, \quad 1 \le i \le m-t-1, \\ f(v_{t+i}) &= 1, \quad 1 \le i \le t, \\ f(v_{2t+i}) &= 2, \quad 1 \le i \le t. \end{aligned}$$

Here $v_f(0) = t+1$, $v_f(1) = v_f(2) = t$, $e_f(0) = e_f(1) = e_f(2) = 3t$. Therefore triangular snake T_n is mean cordial.

Case 2 $n \equiv 1 \pmod 3$

Let $n = 3t+1$. Here $v_f(0) = 2t+1$. But $e_f(0) \le 3t$, a contradiction.

Case 3 $n \equiv 2 \pmod 3$

Let $n = 3t+2$. In this case $v_f(0) = 2t+1$ or $2t+2$. But $e_f(0) \le 3t+1$, a contradiction.□

§3. Conclusion

In this paper we have studied the mean cordial behavior of $P_m \cup P_n$, $C_n \cup P_m$, $S(P_n \odot K_1)$, $S(P_n \odot 2K_1)$, P_n^2, T_n. Mean cordial labeling behavior of join and product of given two graphs are the open problems for future research.

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More on p^* Graceful Graphs

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Abstract: A p^* graceful labeling of a graph G is an assignment f_p of labels to the vertices of G, that induces for each edge uv, a label $f_p^* = |f_p(u) - f_p(v)|$ so that the resulting edge labels are distinct pentagonal numbers. In this paper, we investigate the p^* graceful nature of some graphs based on some graph theoretic operations.

Key Words: Pentagonal numbers, p^*-graceful graphs, comb graph, twig graph, banana trees

AMS(2010): 05C78

§1. Introduction

Unless otherwise mentioned, a graph in this paper means a simple graph without isolated vertices. For all the terminology and notations in graph theory, we follow [1] and [2] and for the definition regarding p^* graceful graphs, we follow [4].

A labeling f of a graph G is one-one mapping from the vertex set of G into the set of integers. Consider a graph G with q edges. Let $f_p : V(G) \rightarrow \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p^*(uv) = |f_p(u) - f_p(v)|$. If f_p^* is a sequence of distinct consecutive pentagonal numbers, then the function f_p is said to be p^* graceful labeling and the graph which admits the p^* graceful labeling is called p^* graceful graph. Here $\omega^p(q) = \dfrac{q(3q-1)}{2}$ is the q^{th} pentagonal number.

In [4], we proved that the paths, star graphs, comb graphs and twig graphs are p^* graceful. In this paper, we are having some generalizations on p^* graceful graphs.

Theorem 1.1 *$S(n, 1, n)$ is p^* graceful.*

Proof Let $G = S(n, 1, n)$. Let u_1, u_2, u_3 be the vertices of P_3 and $u_{1i}, u_{21}, u_{3i}, i = 1, 2, \cdots n$ be the pendant vertices attached with the vertices of P_3. Define $f_p : V(G) \rightarrow \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p(u_1) = 0$

$f_p(u_{1i}) = \omega^p(i),\ i = 1, 2, \cdots, n;$
$f_p(u_2) = \omega^p(q),\ f_p(u_{21}) = f_p(u_2) - \omega^p(q-1);$
$f_p(u_3) = f_p(u_2) - \omega^p(q-2),\ f_p(u_{3i}) = f_p(u_3) + \omega^p(q-2-i),\quad i = 1, 2, \ldots, n.$

Then we can easily verify that f_p generates f_p^* as required. Hence the result. □

[1]Received May 13, 2013, Accepted August 30, 2013.

Theorem 1.2 *The union of two p^* graceful trees is p^* graceful.*

Proof Let G_1 and G_2 be two p^* graceful trees. Let n_1 be the number of edges of G_1 and n_2 be the number of edges of G_2 such that $n_1 + n_2 = q$, the number of edges of $G_1 \cup G_2$. The p^* graceful labeling of $G_1 \cup G_2$ can be obtained as by assigning the vertices in the first copy of $G_1 \cup G_2$ i.e, G_1 in such a way as to get the edge labels $\{\omega^p(q), \ldots, \omega^p(q-(n_1-1))\}$ and then by assigning the first vertex of G_2 by $\omega^p(q-(n_1-1))-1$. The remaining vertices of G_2 are labeled so as to get $\{\omega^p(q-n_1), \cdots, \omega^p(1)\}$ as edge labels. □

Corollary 1.1 *The union of n, p^* graceful graphs is p^* graceful.*

Definition 1.1 *Let S_n be a star with n pendant vertices. Take m isomorphic copies of S_n. Let u_i and u_{ij}, $j = 1, 2, \cdots, n$ for $i = 1, 2, \cdots, m$ be the vertices of the i^{th} copy of S_n. Join u_1 to $u_{1+i,1}$ for $i = 1, 2, \cdots, m-1$. The resultant graph is denoted by S_n^m. Note that S_n^m has $mn + n$ vertices and $m(n+1)-1$ edges.*

Theorem 1.3 *The graph S_n^m exhibits p^* gracefulness.*

Proof Let the vertex set of S_n^m be $\{u_i u_{ij} / i = 1, 2, \cdots, m, j = 1, 2, \ldots n\}$. Define $f_p : V(S_n^m) \to \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p(u_1) = \omega^p(q), f_p(u_{11}) = 0$;

$f_p(u_{1i}) = f_p(u_1) - \omega^p(q-(i-1)),\ i = 2, 3, \cdots, n;$
$f_p(u_{k1}) = |f_p(u_1) - \omega^p(q-(k-1)n-(k-2))|,\ k = 2, 3, \cdots, m;$
$f_p(u_k) = |f_p(u_{k1}) - \omega^p(q-(k-1)n-(k-2)-1)|,\ k = 2, 3, \cdots, m;$
$f_p(u_{ki}) = |f_p(u_k) - \omega^p(q-(k-1)n-(k-2)-i)|,\ i = 2, 3, \cdots, n.$

If the vertex labeling is less than the corresponding $\omega^p(n)$, instead of subtraction, addition may be done. Clearly f_p defined in this manner generates f_p^* as required. □

For example, the p^* graceful labeling of ${S_4}^5$ is shown in Figure 1.

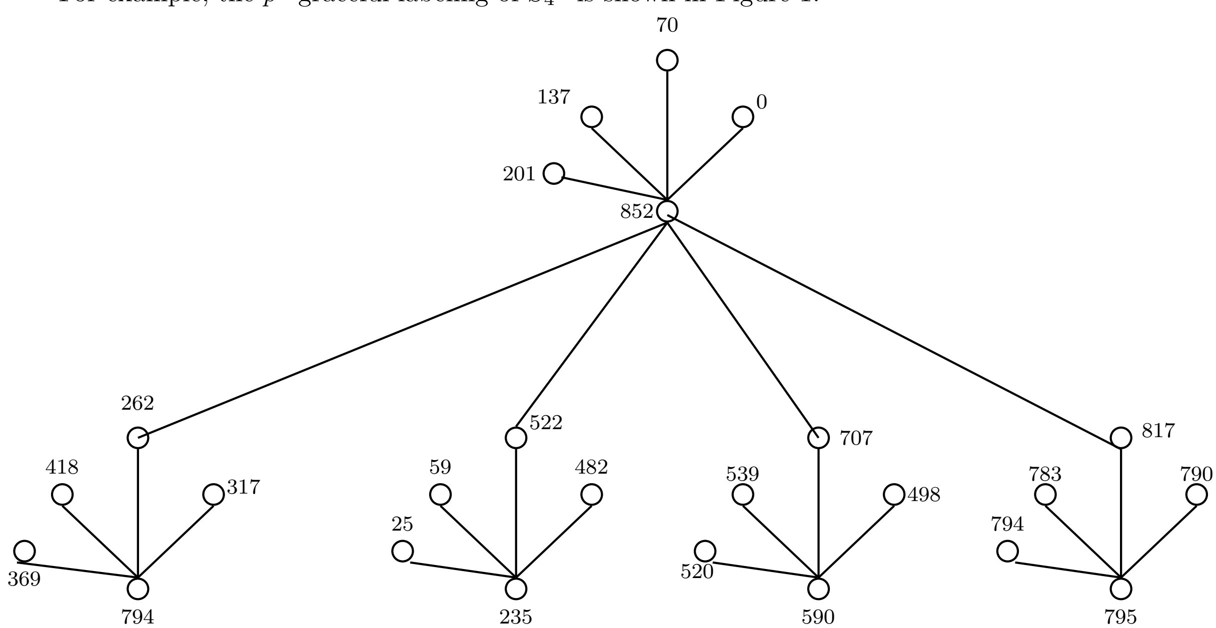

Figure 1

§2. On Cycles and Related Graphs

Theorem 2.1 *Cycles are p^* graceful graphs for some $n \geqslant 6$.*

Proof Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle.

Case 1 $n \equiv 0 (\mathrm{mod}\ 4)$

Let $n = 4k$ for some k. Define $f_p : V(C_n) \to \{0, 1, \cdots, \omega^p(q)\}$ as follows.

$f_p(u_1) = 0,\ f_p(u_2) = \omega^p(q)$;
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 3),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2$;
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 4$ and $f_p(u_q) = \omega^p(q - 1)$.

As we reach $u_{\lfloor \frac{n}{2} \rfloor - 1}$ and $u_{q - \lfloor \frac{n}{2} \rfloor + 3}$, a stage may be reached when the vertex label is big enough to accommodate two or more consecutive $\omega^p(i)$. Hence or otherwise we can complete the proof in Case 1, by allotting all pentagonal numbers from $\omega^p(1)$ to $\omega^p(q)$. For example, p^* graceful labeling of C_{16} is shown in Figure 2.

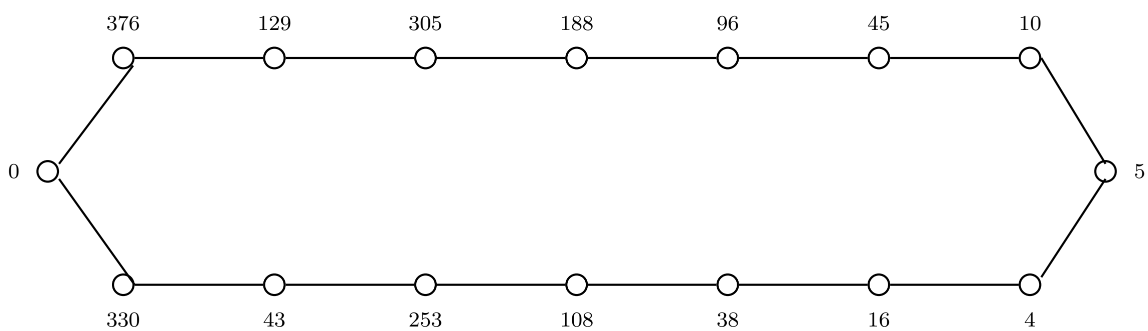

Figure 2

Case 2 $n \equiv 2 (\mathrm{mod}\ 4)$

Let $n = 4k + 2$ for some k. Define $f_p : V(C_n) \to \{0, 1, \cdots, \omega^p(q)\}$ such that

$f_p(u_1) = 0, \quad f_p(u_2) = \omega^p(q)$;
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 4),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2$;
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i - 1),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 4$ and $f_p(u_q) = \omega^p(q - 1)$.

As discussed in the earlier case, after the above defined stages we may make suitable increments or decrements depending upon the size of vertex labels, to get the remaining $\omega^p(i)$. As an example consider the labeling of C_{14} in Figure 3.

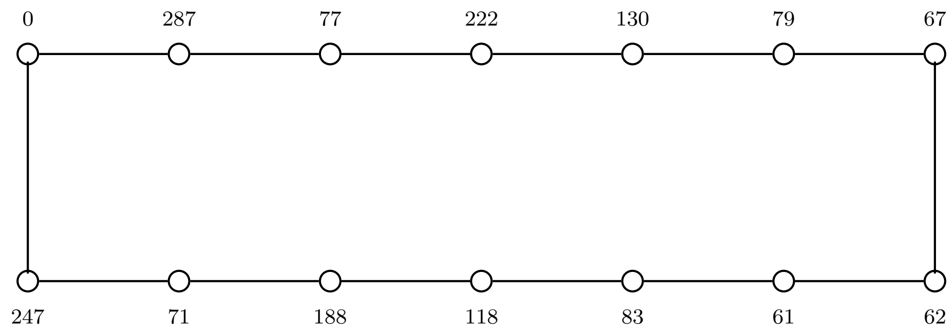

Figure 3

Case 3 $n \equiv 3 (\text{mod } 4)$

Let $n = 4k - 1$ for some k. Here we define f_p on $V(C_n)$ as follows:

$f_p(u_1) = 0,\ f_p(u_2) = \omega^p(q)$;
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 4),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 1$;
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i - 1),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 3$ and $f_p(u_q) = \omega^p(q-1)$.

As we reach the vertex at $\lfloor \frac{n}{2} \rfloor$ ie, $u_{\lfloor \frac{n}{2} \rfloor}$ and the vertex $u_{q - \lfloor \frac{n}{2} \rfloor + 2}$ a stage will be reached where the vertex labels is big enough to accommodate two or more consecutive $\omega^p(i)$. Hence or otherwise we can complete the labeling in the required manner. For example, consider the p^* graceful labeling of C_{15} in Figure 4.

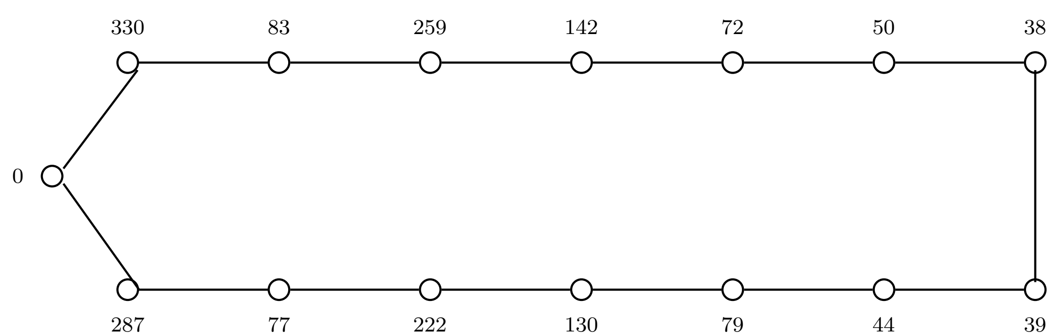

Figure 4

Definition 2.1 *The armed crown is a graph obtained from cycle C_n by attaching a path P_m at each vertex of C_n and is denoted by $C_n \Theta P_m$.*

Definition 2.2 *Biarmed crown $C_n \Theta 2P_n$ is a graph obtained from C_n by identifying the pendant vertices of two vertex disjoint paths of same length $m-1$ at each vertex of the cycle.*

Corollary 2.1 *The armed crown $C_n \Theta P_m$ and bi-armed crown $C_n \Theta 2P_m$ are p^* graceful for some n and m.*

§3. p^* Gracefulness of Some Duplicate Graphs

Definition 3.1 *Let G be a graph with $V(G)$ as vertex set. Let V' be the set of vertices $|V'| = |V|$ where each $a \in V$ is associated with a unique $a' \in V'$. The duplicate graph of G, denoted by $D(G)$ has the vertex set $V \cup V'$ and $E(D(G))$ defined as,*

$$E(D(G)) = \{ab' \text{ and } a'b : ab \in E(G)\} \text{ (see [2])}$$

For example, $D(C_3) = C_6$.

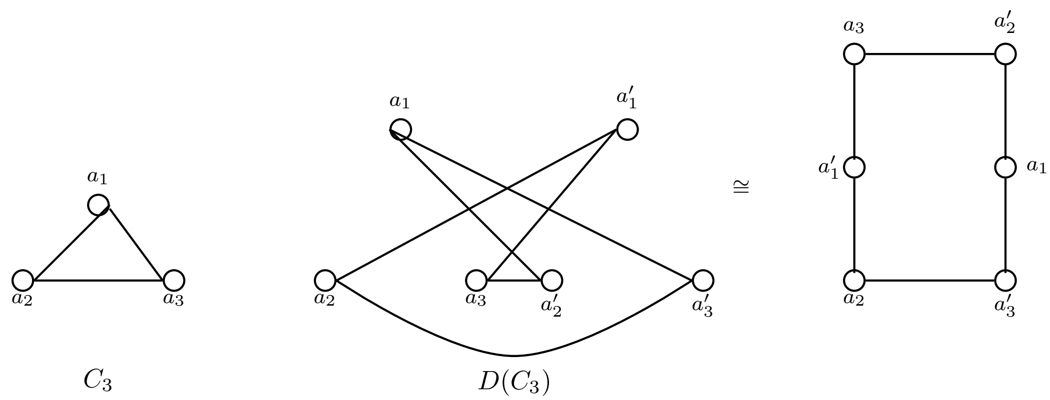

Figure 5

Theorem 3.1 *The duplicate graph of a path is p^* graceful.*

Proof Let P_n be a path.

$$D(P_n) = P_n \cup P_n$$

By Theorem 1.2, $D(P_n)$ is p^* graceful. □

Theorem 3.2 *The duplicate graph of a star S_n is p^* graceful.*

Proof Let $S_n = K_{1,n}$ be a star.

$$D(S_n) = S_n \cup S_n$$

By Theorem 1.2, $D(S_n)$ is p^* graceful. □

Theorem 3.3 *The duplicate graph of H graph admits p^* graceful labeling.*

Proof Let G be an H-graph on $2n$ vertices. $D(G) = G \cup G$. Again by the same theorem mentioned above, we have the result. □

Theorem 3.4 *The duplicate graph $C_3 \hat{o} K_{1,n}$ $n \geqslant 5$ admits p^* graceful labeling.*

Proof $D(C_3 \hat{o} K_{1,n}) = C_6 \hat{o} 2K_{1,n}$. Let $u_i, i = 1, 2, \ldots, 6$ be the vertices of C_6 and u_{1i} and u_{4i}; $i = 1, 2, \ldots, n$ be the pendant vertices attached with u_1 and u_4 respectively.

Consider the mapping f_p on the vertices of $G = C_6\hat{o}2K_{1,n}$ as $f_p : V(G) \to \{0, 1, \ldots, \omega^p(q)\}$ such that

$$f_p(u_1) = 0, \qquad f_p(u_2) = \omega^p(6);$$
$$f_p(u_3) = 29, \qquad f_p(u_4) = 24, \qquad f_p(u_5) = 23, \qquad f_p(u_6) = 35;$$
$$f_p(u_{1i}) = \omega^p(6 + n + i), \qquad i = 1, 2, \ldots, n;$$
$$f_p(u_{4i}) = f_p(u_4) + \omega^p(7 + i - 1), \qquad i = 1, 2, \cdots, n.$$

Obviously f_p defined as above give rise to f_p^* as required. Hence the result. □

In general $D(C_m\hat{o}K_{1,n})$ is p^* graceful for some m.

Remark 3.1 *$D(C_{2n}) = C_{2n} \cup C_{2n}$ for all n is not p^* graceful.*

But $D(C_{2n+1}) = C_{2(2n+1)}$ is p^* graceful, if C_{2n+1} is so.

Conjecture *All trees are p^* graceful.*

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Math.Combin.Book Ser. Vol.3(2013), 91-95

Symmetric Hamilton Cycle Decompositions of Complete Graphs Plus a 1-Factor

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Abstract: Let $n \geq 2$ be an integer. The complete graph K_n with 1-factor I added has a decomposition into Hamilton cycles if and only if n is even. We show that $K_n + I$ has a decomposition into Hamilton cycles which are symmetric with respect to the 1-factor I added. We also show that the complete bipartite graph $K_{n,n}$ plus a 1-factor has a symmetric Hamilton decomposition, where n is odd.

Key Words: Complete graphs, complete bipartite graph, 1-factor, Hamilton cycle decomposition.

AMS(2010):

§1. Introduction

By a decomposition of a nonempty graph G is meant a family of subgraphs $G_1, G_2, \cdots, G_k$ of G such that their edge set form a partition of the edge set of G. Any member of the family is called a part (of the decomposition). This decomposition is usually denoted by $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$.

Let $n \geq 2$ be an integer. The complete graph K_n has many Hamilton cycles and since its vertices have degree $n-1$, K_n has a decomposition into Hamilton cycles if and only if n is odd. Suppose that $n = 2m+1$. The familiar Hamilton cycle decomposition of K_n referred to as the Walecki decomposition in [1] is a symmetric decomposition in that each Hamilton cycle H in the decomposition is symmetric in the following sense. Let the vertices of K_n be labeled as $0, 1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}$. Then each H is invariant under the involution $i \to \bar{i}$, where $\bar{\bar{i}} = i$; the vertex 0 is a fixed point of this involution. A symmetric Hamilton cycle decomposition of K_n different from Walecki's is constructed in [1].

Let G be a graph, then G[2] is a graph whereby each vertex x is replaced by a pair of two independent vertices $x, \bar{x}$ and each edge xy is replaced by four edges $xy, x\bar{y}, \bar{x}y, \bar{x}\bar{y}$.

Now suppose that n is even. Adding the edges of a 1-factor I to K_n results in a graph

[1]Received December 10, 2012, Accepted September 2, 2013.

K_n+I each of whose vertices has even degree n. The graph K_n+I does have a decomposition into Hamilton cycles (see [3]). The complete solution to the problem of decomposing K_n+I into cycles of given uniform length is given in [3].

The degrees of vertices of the complete bipartite graph $K_{n,n}$ equal n, and $K_{n,n}$ has a decomposition into Hamilton cycles if and only if n is even. If n is odd, adding a 1-factor I to $K_{n,n}$ results in a graph $K_{n,n}+I$ with all vertices of even degree $n+1$ and $K_{n,n}+I$ also has a decomposition into Hamilton cycles.

Let $n=2m$ be an even integer with $m\geq 1$. Consider the complete bipartite graph $K_{n,n}$ with vertex bipartition into sets $\{1,2,\cdots,n\}$ and $\{\bar{1},\bar{2},\cdots,\bar{n}\}$. By a symmetric Hamilton cycle in $K_{n,n}$, we mean a Hamilton cycle such that $i\bar{j}$ is an edge if and only if $\bar{i}j$ is an edge. Thus a Hamilton cycle in $K_{n,n}$ is symmetric if and only if it is invariant under the involution $i\to\bar{i}$.

A symmetric hamilton cycle decomposition of $K_{n,n}$ is a partition of the edges of $K_{n,n}$ into m symmetric Hamilton cycles. Now let $n=2m+1$ be an odd integer with $m\geq 1$, and consider the 1-factor $I=\{\{1,\bar{n}\},\{2,\overline{n-1}\},\cdots,\{n,\bar{1}\}\}$ of $K_{n,n}+I$. A symmetric Hamilton cycle decomposition of $K_{n,n}+I$ is a partition of the edges of $K_{n,n}+I$ into $m+1$ symmetric Hamilton cycles.

Let $m>1$ be even, consider the vertex set of the complete graph K_{2m} to be $\{1,2,\cdots,m\}\cup\{\bar{1},\bar{2},\cdots,\bar{m}\}$, where $I=\{1\bar{1},2\bar{2},\cdots,m\bar{m}\}$ is a 1-factor of K_{2m}.

The edges of $K_{2m}+I$ are naturally partitioned into edges of K_m on $\{1,2,\cdots,m\}$, the edges of $K_{m,m}+I$, and the edges of K_m on $\{\bar{1},\bar{2},\cdots,\bar{m}\}$. We denote the complete graph on $\{\bar{1},\bar{2},\cdots,\bar{m}\}$ by $\bar{K_m}$. We abuse terminology and write this edge partition as:

$$K_{2m}+I=K_m\cup(K_{m,m}+I)\cup\bar{K_m}$$

By a symmetric Hamilton cycle of $K_{2m}+I$ we mean a Hamilton cycle such that

(1) ij is an edge in (K_m) if and only if $(\overline{ij})$ is an edge in $\bar{K_m}$ and

(2) $i\bar{j}$ is an edge in $(K_{m,m}+I)$ if and only if $j\bar{i}$ is an edge in $(K_{m,m}+I)$.

Thus a Hamilton cycle of $K_{2m}+I$ is symmetric if and only if it is invariant under the fixed point free involution ϕ of $K_{2m}+I$, where $\phi(a)=\bar{a}$ for all a in $\{1,2,\cdots,m\}\cup\{\bar{1},\bar{2},\cdots,\bar{m}\}$ and $\bar{\bar{a}}=a$. A symmetric Hamilton cycle decomposition of $K_{2m}+I$ is a decomposition of $K_{2m}+I$ into m symmetric Hamilton cycles. Thus ϕ is a nontrivial automorphism of $K_{2m}+I$, which acts trivially on the cycles in a symmetric Hamilton cycle decomposition of $K_{2m}+I$.

A double cover of K_{2m} by Hamilton cycles is a collection $C_1,C_2,\cdots,C_{2m-1}$ of $2m-1$ Hamilton cycles such that each edge of K_{2m} occurs as an edge of exactly two of these Hamilton cycles. Note that the sum of the number edges in these Hamilton cycles equals

$$(2m-1)2m=2\left(\begin{array}{c}2m\\2\end{array}\right),$$

twice the number of edges of K_{2m}, and this also equals half the number of edges of $K_{4m}-I$.

We use K_n+I to denote the multigraph obtained by adding the edges of a 1-factor I to K_n, thus duplicating $\frac{n}{2}$ edges.

Let k be a positive integer and $L \subseteq \{1, 2, \cdots, \lfloor \frac{k}{2} \rfloor\}$. A circulant graph $X = X(k; L)$ is a graph with vertex set $V(X) = \{u_1, u_2, \cdots, u_k\}$ and edge set $E(X)$, where $E(X) = \{u_i u_{i+l} : i \in Z_k, l \in L - \{\frac{k}{2}\}\} \cup \{u_i u_{i+k} : i \in \{1, 2, \cdots, \frac{k}{2}\}\}$ if $\frac{k}{2} \in L$, and $E(X) = \{u_i u_{i+l} : i \in Z_k, l \in L\}$ otherwise. An edge $u_i u_{i+l}$, where $l \in L$ is said to be of length l and L is called the edge length set of the circulant X.

Notice that K_n is isomorphic to the circulant $X(n; \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\})$. If n is even, $K_n - I$ is isomorphic to $X(n; \{1, 2, \cdots, \frac{n}{2} - 1\})$ and $K_n + I$ is isomorphic to $X(n; \{1, 2, \cdots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2}\})$.

Let $X = X(k; L)$ be a circulant graph with vertex set $\{u_1, u_2, \cdots, u_k\}$. By the rotation ρ we mean the cyclic permutation $\{u_1, u_2, \cdots, u_k\}$.

If $P = x_0 x_1 \cdots x_p$ is a path, $\overleftarrow{P}$ denotes the path $x_p x_{p-1} \cdots x_1 x_0$, the reverse of P.

§2. Proof of the Result

In order that $K_n + I$ have a symmetric Hamilton cycle decomposition, it is necessary that n be even.

Theorem 2.1 *Let $m \geq 2$ be an integer. There is a symmetric Hamilton cycle decomposition of $K_{2m} + I$.*

Proof View the graph $K_{2m} + I$ as the circulant graph $X(2m; \{1, 2, \cdots, m-1, m, m\})$ with vertex set $\{x_1, x_2, \cdots, x_{2m}\}$. Let P be the zig-zag $(m-1)$ path

$$P = x_1 x_{+1} x_{-1} x_{+2} x_{-2} \cdots x_A$$

where $A = 1 - 2 + 3 - \cdots + (-1)^m (m-1)$. Thus P has edge length set $L_p = \{1, 2, \cdots, m-1\}$. It is easy to see that

$$C = P \cup \rho^m(\overleftarrow{P}) x_1$$

is an $2m$-cycle and $\{\rho^i(C) : i = 0, 1, \cdots, m-1\}$ is a Hamilton cycle decomposition of $K_{2m} + I$.

Next relabel the vertices of the graph $K_{2m} + I$ by defining a function f as follows: $f : x_i \to x_i$ for $1 \leq i \leq m$ and $f : x_i \to \bar{x}_{i-m}$ for $m \leq i \leq 2m$. Relabeling of the vertices of each Hamilton cycle C_{2m} with the new labels gives symmetric Hamilton cycle. Hence $K_{2m} + I$ can be decomposed into symmetric Hamilton cycle. □

Lemma 2.2 *Let $m \geq 2$ be an integer, and let C be a symmetric Hamilton cycle of $K_{2m} + I$. Then*

(1) *If x is any vertex of $K_{2m} + I$, the distance between x and $\bar{x}$ in C is odd;*

(2) *C is of the form $x_1, x_2, \cdots, x_m, \bar{x}_m, \bar{x}_{m-1}, \cdots, \bar{x}_2, \bar{x}_1 x_1$ where $x_i \in \{1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}\}$;*

(3) *The number of edges $x_i \bar{x}_i$ in each symmetric Hamilton cycle is $2, 1 \leq i \leq m$.*

Proof Let x be a vertex of $K_{2m} + I$ and let the distance between x and $\bar{x}$ in C be k. Then there is a path $x = x_1, \cdots, x_{\frac{k+1}{2}}, \bar{x}_{\frac{k+1}{2}}, \cdots, \bar{x}_2 \bar{x}_1 = \bar{x}$ in C. Since for each $x_i, i \in N$ we have

$k, \dfrac{k+1}{2} \in N$. Suppose k is even, then $\dfrac{k+1}{2} \notin N$.Therefore k is odd which proves (1).

Assertion (2) is now an immediate consequence. Since the cycle C is given as in (2), we have edges $\{x_1\bar{x}_1\}$ and $\{x_m\bar{x}_m\}$ which proves (3). □

Theorem 2.3 *Let m be an even integer, then the graph $K_m + I[2]$ has a symmetric Hamilton cycle decomposition.*

Proof From the definition of the graph $K_m + I[2]$, each vertex x in $K_m + I$ is replaced by a pair of two independent vertices $x, \bar{x}$ and each edge xy is replaced by four edges $xy, x\bar{y}, \bar{x}y, \bar{x}\bar{y}$. Also note that if the graph H decomposes the graph G, then $H[2]$ decomposes $G[2]$.

By [3], cycle C_m decomposes $K_m + I$, then we have

$$K_m + I[2] = C_m[2] \oplus C_m[2] \oplus \cdots \oplus C_m[2]$$

Now label the vertices of each graph $C_m[2]$ as $x_i\bar{x}_i$, where $i = 1, 2, \cdots, m$. By [2], each graph $C_m[2]$ decomposes into symmetric Hamilton cycle C_{2m}. Therefore $K_m + I[2]$ decomposes into symmetric Hamilton cycles. □

Theorem 2.4 *Let $m \geq 4$ be an even integer. From a symmetric Hamilton cycle decomposition of $K_m + I[2]$ we can construct a double cover of $K_m + I$ by Hamilton cycles.*

Proof By Theorem 2.3, a symmetric Hamilton cycle of $K_m + I[2]$ is of the form $x_1, x_2, \cdots, x_m$, $\bar{x}_m, \bar{x}_{m-1}, \cdots, \bar{x}_2, \bar{x}_1 x_1$ where $x_i \in \{1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}\}$. Thus $x_1x_2...x_m$ is a path of length $m-1$ in $K_m + I[2]$ and $\bar{x}_m\bar{x}_{m-1}\cdots\bar{x}_2\bar{x}_1$ is its mirror image. Let

$$b_i = \begin{cases} x_i & \text{if } x_i \in \{1, 2, \cdots, m\} \\ \bar{x_i} & \text{if } x_i \in \{\bar{1}, \bar{2}, \cdots, \bar{m}\} \end{cases}$$

Then $b_1, b_2, \cdots, b_m, b_1$ is a Hamilton cycle in $K_m + I$, the projection of C on $K_m + I$. Now assume we have a symmetric Hamilton cycle decomposition of $K_m + I[2]$. Then for each edge x_ix_j in $K_m + I$, there are distinct symmetric Hamilton cycles C and C' in our decomposition such that x_ix_j and $\bar{x}_i\bar{x}_j$ are edges of C and $x_i\bar{x}_j$ and $\bar{x}_ix_j$ are edges of C'. Hence from a symmetric Hamilton cycle decomposition of $K_m + I[2]$, we get a double cover of $K_m + I$ from the projections of each symmetric Hamilton cycle. □

Theorem 2.5 *Let $m \geq 4$ be even integer. Then $K_{2m} + I$ has a double cover by Hamilton cycles.*

Proof There is a Hamilton cycle C in $K_{2m} + I$, and there exists disjoint 1-factor I_1 and I_2 whose union is the set of edges of C. The vertices of the graphs $K_{2m} + I_1$ and $K_{2m} - I_2$ have degrees equal to the even number. The graphs $K_{2m} + I_1$ and $K_{2m} - I_2$ have decompositions into Hamilton cycles $C_1, C_2, \cdots, C_m$ and $D_1, D_2, \cdots, D_{m-1}$ respectively. Then $C, C_1, C_2, \cdots, C_m$, $D_1, D_2, . \cdots, D_{m-1}$ is a double cover of $K_{2m} + I$ by Hamilton cycles . □

Theorem 2.6 *For each integer $m \geq 1$, there exist a symmetric Hamilton cycle decomposition of $K_{2m+1,2m+1} + I$.*

Proof Let $n = 2m + 1$, we consider the complete bipartite graph $K_{n,n}$ with vertex bipartition $\{1, 2, 3, \cdots, n\}$ and $\{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. Let I be $\{\{1, \bar{n}\}, \{2, \overline{n-1}\}, \{3, \overline{n-2}\}, \cdots, \{n, \bar{1}\}\}$ in $K_{n,n} + I$.

Let the sum of edge $a\bar{b}$ be $a + b \mod n$. Let S_k be the set of edges whose sum is k. Let i be an integer with $1 \leq i \leq m + 1$. Consider the union $S_{2i-1} \cup S_{2i}$, $2i$ is calculated modulo n. observe that this collection of edges yields the following symmetric Hamilton cycle of $K_{n,n} + I$;

$$n, \overline{2i-1}, 1, \overline{2i-2}, 2, \overline{2i-3}, 3, \cdots, \overline{2i}, n$$

For each i, let H_i equal $S_{2i-1} \cup S_{2i}$. Then $H_1, H_2, \cdots, H_{m+1}$ is a symmetric Hamilton cycle decomposition of $K_{n,n} + I$. □

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Math.Combin.Book Ser. Vol.3(2013), 96-103

Ratio by Using Coefficients of Fibonacci Sequence

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Abstract: In this paper, ratio by using coefficients of Fibonacci sequence has been discussed in detail. The Fibonacci series is made from $F_{n+2} = F_n + F_{n+1}$. New sequences from the formula $F_{n+2} = aF_n + bF_{n+1}$ by using a and b, where a and b are consecutive coefficients of Fibonacci sequence are formed. These all new sequences have their own ratios. When find the ratio of these ratios, it always becomes 1.6, which is known as golden ratio in Fibonacci series.

Key Words: Fibonacci series, Fibonacci in nature, golden ratio, ratios of new sequences, ratio of all new ratios.

AMS(2010): 05C78

§1. Introduction

The Fibonacci numbers were first discovered by a man named Leonardo Pisano. He was known by his nickname, Fibonacci. The Fibonacci sequence is a sequence in which each term is the sum of the 2 numbers preceding it. The first 10 Fibonacci numbers are: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$ and 89. These numbers are obviously recursive. Leonardo Pisano Bogollo, (c.1170 - c.1250) known as Leonardo of Pisa, Fibonacci was an Italian mathematician (Anderson, Frazier, & Popendorf, 1999). He is considered as the most talented mathematician of the middle ages (Eves, 1990). Fibonacci was first introduced to the number system we currently use with symbols from 0 to 9 along with the Fibonacci sequence by Indian merchants when he was in northern Africa (Anderson, Frazier, & Popendorf, 1999). He then introduced the Fibonacci sequence and the number system we currently use to the western Europe In his book Liber Abaci in 1202 (Singh, Acharya Hemachandra and the (so called) Fibonacci Numbers, 1986) (Singh, The so-called Fibonacci numbers in ancient and medieval India, 1985). Fibonacci was died around 1240 in Italy. He played an important role in reviving ancient mathematics and made significant contributions of his own. Fibonacci numbers are important to perform a run-time analysis of Euclid's algorithm to Find the greatest common divisor (GCD) of two integers. A pair of two consecutive Fibonacci numbers makes a worst case input for this algorithm (Knuth, Art of Computer Programming, Volume 1: Fundamental Algorithms, 1997). Fibonacci numbers

[1]Received May 31, 2013, Accepted September 5, 2013.

have their application in the Polyphone version of the Merge Sort algorithm. This algorithm divides an unsorted list in two Lists such that the length of lists corresponds to two sequential Fibonacci numbers.

If we take the ratio of two successive numbers in Fibonacci series, $(1, 1, 2, 3, 5, 8, 13,\ cdots)$ we find

$$1/1 = 1,\ 2/1 = 2,\ 3/2 = 1.5,\ 5/3 = 1.666...; 8/5 = 1.6; 13/8 = 1.625.$$

Greeks called the golden ratio and has the value 1.61803. It has some interesting properties, for instance, to square it, you just add 1. To take its reciprocal, you just subtract 1. This means all its powers are just whole multiples of itself plus another whole integer (and guess what these whole integers are? Yes! The Fibonacci numbers again!) Fibonacci numbers are a big factor in Math.

1.1 **Fibonacci Credited Two Things**

1. Introducing the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe. (Can you imagine us trying to multiply numbers using Roman numerals?)
2. Developing a sequence of numbers (later called the Fibonacci sequence) in which the first two numbers are one, then they are added to get 2, 2 is added to the prior number of 1 to get 3, 3 is added to the prior number of 2 to get 5, 5 is added to the prior number of 3 to get 8, etc. Hence, the sequence begins as $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$, etc Allows users to distribute parallelized workloads to a shared pool of resources to automatically find and use the best available resource. The ability to have pieces of work run in parallel on different nodes in the grid allows the over all job to complete much more quickly than if all the pieces were run in sequence.

1.2 **List of Fibonacci Numbers**

The first 21 Fibonacci numbers F_n for $n = 0, 1, 2, \cdots, ..., 20$ are respectively

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765.$$

The Fibonacci sequence can be also extended to negative index n using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1}.$$

This yields the sequence of *negafibonacci* numbers satisfying

$$F_{-n} = (-1)^{n+1} F_n.$$

Thus the bidirectional sequence is

F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
−21	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21

§2. Fibonacci Sequence in Nature

2.1 Sunflower

The Fibonacci numbers have also been observed in the family tree of honeybees. The Fibonacci sequence is a pattern of numbers starting with 0 and 1 and adding each number in sequence to the next $\cdots$, $0+1=1, 1+1=2$ so the first few numbers are $0, 1, 1, 2, 3, 5, 8, \cdots$ and so on and so on infinitely.

Fig.1.1 Sunflower head displaying florets in spirals of 34 and 55 around the outside

One of the most common experiments dealing with the Fibonacci sequence is his experiment with rabbits. Fibonacci put one male and one female rabbit in a field. Fibonacci supposed that the rabbits lived infinitely and every month a new pair of one male and one female was produced. Fibonacci asked how many would be formed in a year. Following the Fibonacci sequence perfectly the rabbit's reproduction was determined 144 rabbits. Though unrealistic, the rabbit sequence allows people to attach a highly evolved series of complex numbers to an everyday, logical, comprehendible thought.

Fibonacci can be found in nature not only in the famous rabbit experiment, but also in beautiful flowers. On the head of a sunflower and the seeds are packed in a certain way so that they follow the pattern of the Fibonacci sequence. This spiral prevents the seed of the sunflower from crowding themselves out, thus helping them with survival. The petals of flowers and other plants may also be related to the Fibonacci sequence in the way that they create new petals.

2.2 Petals on Flowers

Probably most of us have never taken the time to examine very carefully the number or arrangement of petals on a flower. If we were to do so, we would find that the number of petals on a flower that still has all of its petals intact and has not lost any, for many flowers is a Fibonacci number:

(1) 3 petals: lily, iris;
(2) 5 petals: buttercup, wild rose, larkspur, columbine (aquilegia);
(3) 8 petals: delphiniums;
(4) 13 petals: ragwort, corn marigold, cineraria;
(5) 21 petals: aster, black-eyed susan, chicory;
(6) 34 petals: plantain, pyrethrum;

(7) 55, 89 petals: michaelmas daisies, the asteraceae family.

2.3 Fibonacci Numbers in Vegetables and Fruits

Romanesque Brocolli/Cauliflower (or Romanesco) looks and tastes like a cross between brocolli and cauliflower. Each floret is peaked and is an identical but smaller version of the whole thing and this makes the spirals easy to see.

Fig.1.2 Brocolli/Cauliflower

2.4 Human Hand

Every human has two hands, each one of these has five fingers, each finger has three parts which are separated by two knuckles. All of these numbers fit into the sequence. However keep in mind, this could simply be coincidence.

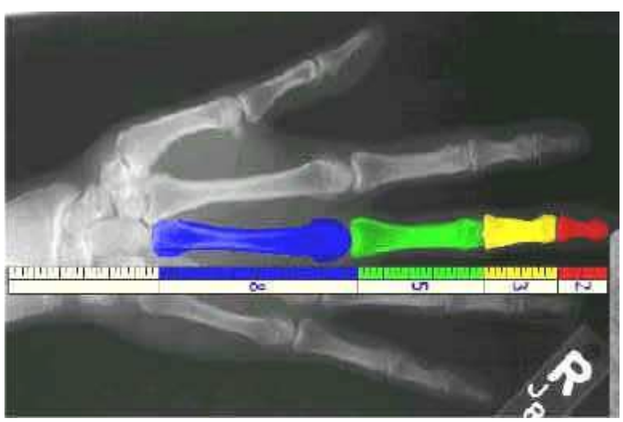

Fig.1.3 Human hand

Subject: The Fibonacci series is a sequence of numbers first created by Leonardo Fibonacci in 1202. The first two numbers of the series are 1 and 1 and each subsequent number is sum of the previous two. Fibonacci numbers are used in computer algorithms. The Fibonacci sequence first appears in the book Liber Abaci by Leonardo of Pisa known as Fibonacci. Fibonacci considers the growth of an idealized rabbit population, assuming that a newly born pair of rabbits, one male, one female and do the study on it. The Fibonacci series become $1, 1, 2, 3, 5, 8, 13, 21 \cdots$.

§3. Ratio by Using Coefficients of Fibonacci Sequence

3.1 Ratio By Using $1, 2$ as Coefficients

Apply the formula by using the next two coefficients of Fibonacci series i.e. 1 for F_{n+1} and 2 for F_n. So the series that becomes from this formula is $F_{n+2} = 2F_n + F_{n+1}$, $F_1 = 1$, $F_2 = 1$, $F_3 = 3$, $5, 11, 21, 43, 85, 171, 341, 683, 1365, \cdots$. K

From this sequence, find the ratio by dividing two consecutive numbers.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1$$
$$\frac{F_3}{F_2} = \frac{3}{1} = 3$$
$$\frac{F_4}{F_3} = \frac{5}{3} = 1.66$$
$$\frac{F_5}{F_4} = \frac{11}{5} = 2.2$$
$$\frac{F_6}{F_5} = \frac{21}{11} = 1.9$$
$$\frac{F_7}{F_6} = \frac{43}{21} = 2.0$$
$$\frac{F_8}{F_7} = \frac{85}{43} = 1.9$$
$$\frac{F_9}{F_8} = \frac{171}{85} = 2.0$$

From here the conclusion is that the ratio (in integer) of this series is 2.

3.2 Ratio by Using 2, 3 as Coefficients

The series that becomes by using $2, 3$ as coefficients is $F_{n+2} = 3F_n + 2F_{n+1}$, i.e., $F_1 = 1, F_2 = 1, F_3 = 5$, $13, 41, 121, 365, 1093, 3281, 9841, \cdots$.

From this sequence, find the ratio by dividing two consecutive numbers.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1$$
$$\frac{F_3}{F_2} = \frac{5}{1} = 5$$
$$\frac{F_4}{F_3} = \frac{13}{5} = 2.6$$
$$\frac{F_5}{F_4} = \frac{41}{13} = 3.15$$

$$\frac{F_6}{F_5} = \frac{121}{41} = 3.15$$
$$\frac{F_7}{F_6} = \frac{365}{121} = 3.01$$
$$\frac{F_8}{F_7} = \frac{1093}{365} = 2.99$$
$$\frac{F_9}{F_8} = \frac{3281}{1093} = 3.01$$

From here the conclusion is that the ratio (in integer) of this series is 3.

3.3 Ratio by Using 3, 5 as Coefficients

The series that becomes by using 3, 5 as coefficients is $F_{n+2} = 5F_n + 3F_{n+1}$, i.e., $F_1 = 1, F_2 = 1, F_3 = 8,\ 29, 127, 526, 2213, 9269, 38872, 162961, \cdots$.

From this sequence, find the ratio by dividing two consecutive numbers.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1$$
$$\frac{F_3}{F_2} = \frac{8}{1} = 8$$
$$\frac{F_4}{F_3} = \frac{29}{8} = 3.6$$
$$\frac{F_5}{F_4} = \frac{127}{29} = 4.3$$
$$\frac{F_6}{F_5} = \frac{526}{127} = 4.14$$
$$\frac{F_7}{F_6} = \frac{2213}{526} = 4.20$$
$$\frac{F_8}{F_7} = \frac{9269}{2213} = 4.18$$
$$\frac{F_9}{F_8} = \frac{38872}{9269} = 4.19$$

From here the conclusion is that the ratio (in integer) of this series is 4.

3.4 Ratio by Using 5, 8 as Coefficients

The series that becomes by using 5, 8 as coefficients is $F_{n+2} = 8F_n + 5F_{n+1}$, i.e., $F_1 = 1, F_2 = 1, F_3 = 13,\ 73, 469, 2929, 18397, 115417, 724229, \cdots$.

From this sequence , find the ratio by dividing two consecutive numbers.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1$$
$$\frac{F_3}{F_2} = \frac{13}{1} = 13$$
$$\frac{F_4}{F_3} = \frac{73}{13} = 5.6$$

$$\frac{F_5}{F_4} = \frac{469}{73} = 6.4$$
$$\frac{F_6}{F_5} = \frac{2929}{469} = 6.24$$
$$\frac{F_7}{F_6} = \frac{18397}{2929} = 6.28$$
$$\frac{F_8}{F_7} = \frac{115417}{18397} = 6.27$$
$$\frac{F_9}{F_8} = \frac{724229}{115417} = 6.27$$

From here the conclusion is that the ratio (in integer) of this series is 6.

Continuing in this way, find that the ratio of

$F_{n+2} = 13F_n + 8F_{n+1}$ is 9 (in integer);
$F_{n+2} = 21F_n + 13F_{n+1}$ is 14 (in integer);
$F_{n+2} = 34F_n + 21F_{n+1}$ is 22 (in integer);
....

§4. Conclusion

Therefore the sequence becomes from all the ratios by using the consecutive numbers as the coefficients of Fibonacci sequence is:

$$2, 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, 234, 378, \cdots$$

Now find the ratio that on dividing consecutive integers, of this sequence is:

$$3/2 = 1.5, 4/3 = 1.33, 6/4 = 1.5, 14/9 = 1.6, 22/14 = 1.6, 35/22 = 1.6$$

and $$56/35 = 1.6, 90/1.6, 145/90 = 1.6, 234/145 = 1.6 \cdots$$

It always become 1.6, yes it is again the golden ratio of Fibonacci sequence. So the conclusion is that the ratio of these ratios is always become golden ratio in Fibonacci series.

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I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

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September 2013

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An International Journal on Mathematical Combinatorics

ISBN 9781599732466

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