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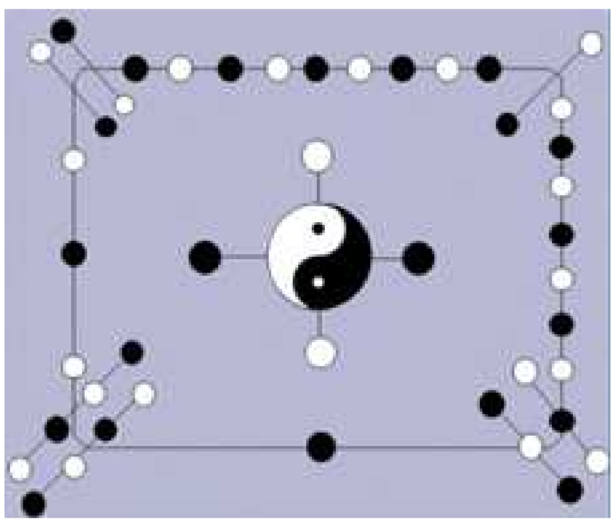
MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



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Think for yourself. What everyone else is doing may not be the right thing.

By Aesop, an ancient Greek fable writer.

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Combinatorial Field - An Introduction

Dedicated to Prof. Feng Tian on his 70th Birthday

Linfan Mao

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Abstract: A *combinatorial field* $\mathscr{W}_G$ is a multi-field underlying a graph G, established on a smoothly combinatorial manifold. This paper first presents a quick glance to its mathematical basis with motivation, such as those of *why the WORLD is combinatorial?* and *what is a topological or differentiable combinatorial manifold?* After then, we explain how to construct principal fiber bundles on combinatorial manifolds by the voltage assignment technique, and how to establish differential theory, for example, connections on combinatorial manifolds. We also show applications of combinatorial fields to other sciences in this paper.

Key Words: Combinatorial field, Smarandache multi-space, combinatorial manifold, WORLD, principal fiber bundle, gauge field.

AMS(2000): 51M15, 53B15, 53B40, 57N16, 83C05, 83F05.

§1. Why is the WORLD a Combinatorial One?

The multiplicity of the WORLD results in modern sciences overlap and hybrid, also implies its combinatorial structure. To see more clear, we present two meaningful proverbs following.

Proverb 1. *Ames Room*

An Ames room is a distorted room constructed so that from the front it appears to be an ordinary cubic-shaped room, with a back wall and two side walls parallel to each other and perpendicular to the horizontally level floor and ceiling. As a result of the optical illusion, a person standing in one corner appears to the observer to be a giant, while a person standing in the other corner appears to be a dwarf. The illusion is convincing enough that a person walking back and forth from the left corner to the right corner appears to grow or shrink. For details, see Fig.1.1 below.

[1]Reported at Nanjing Normal University, 2009.
[2]Received July 6, 2009. Accepted Aug.8, 2009.

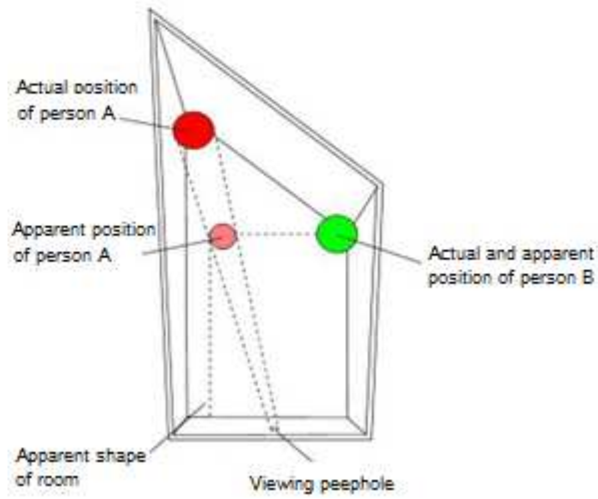


Fig.1.1

This proverb means that it is not all right by our visual sense for the multiplicity of world.

Proverb 2. *Blind men with an elephant*

In this proverb, there are six blind men were be asked to determine what an elephant looked like by feeling different parts of the elephant's body, seeing Fig.1.2 following. The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They then entered into an endless argument and each of them insisted his view right.

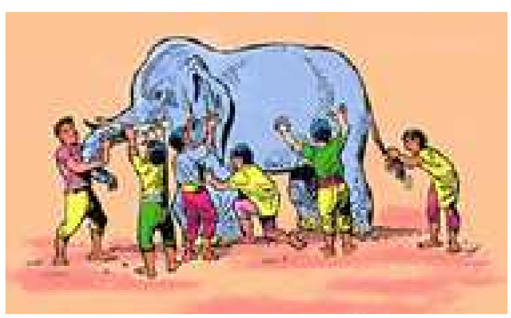


Fig.1.2

All of you are right! A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*. Then

What is the meaning of Proverbs 1 *and* 2 *for understanding the structure of WORLD*?

The situation for one realizing behaviors of the WORLD is analogous to the observer in Ames room or these blind men in the second proverb. In fact, we can distinguish the WORLD by known or unknown parts simply, such as those shown in Fig.1.3.

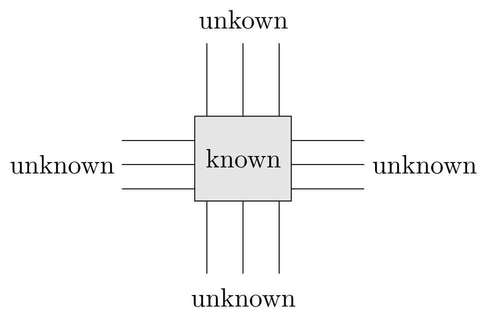


Fig.1.3

The laterality of human beings implies that one can only determines lateral feature of the WORLD by our technology. Whence, the WORLD should be the union of all characters determined by human beings, i.e., a Smarandache multi-space underlying a combinatorial structure in logic. Then *what can we say about the unknown part of the WORLD*? *Is it out order*? No! It must be in order for any thing having its own right for existing. Therefore, these is an underlying combinatorial structure in the WORLD by the *combinatorial notion*, shown in Fig.1.4.

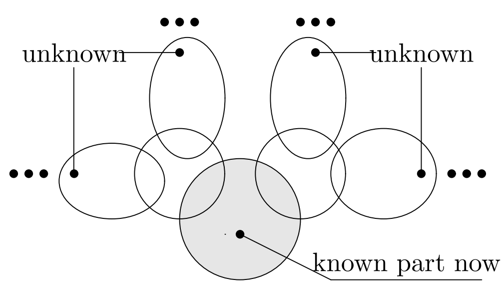


Fig.1.4

In fact, this combinatorial notion for the WORLD can be applied for all sciences. I presented this combinatorial notion in Chapter 5 of [8], then formally as the *CC conjecture for mathematics* in [11], which was reported at *the 2nd Conference on Combinatorics and Graph Theory of China* in 2006.

Combinatorial Conjecture *A mathematical science can be reconstructed from or made by combinatorialization.*

This conjecture opens an entirely way for advancing the modern sciences. It indeed means a deeply *combinatorial notion* on mathematical objects following for researchers.

(i) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(*ii*) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(*iii*) One can make one combination of different branches in mathematics and find new results after then.

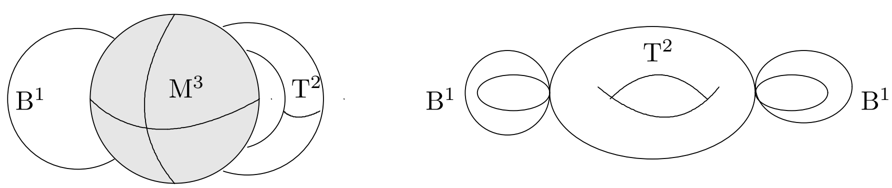
(*iv*) One can understand our WORLD by this combinatorial notion, establish combinatorial models for it and then find its behavior, and so on.

This combinatorial notion enables ones to establish a combinatorial model for the WORLD and develop modern sciences combinatorially. Whence, a science can not be ended if its combinatorialization has not completed yet.

§2. Topological Combinatorial Manifold

Now *how can we characterize these unknown parts in Fig.*1.4 *by mathematics*? Certainly, these unknown parts can be also considered to be fields. Today, we have known a best tool for understanding the known field, i.e., a topological or differentiable manifold in geometry ([1], [2]). So it is more natural to think each unknown part is itself a manifold. That is the motivation of combinatorial manifolds.

Loosely speaking, a combinatorial manifold is a combination of finite manifolds, such as those shown in Fig.2.1.



(a) (b)

Fig.2.1

In where (a) represents a combination of a 3-manifold, a torus and 1-manifold, and (b) a torus with 4 bouquets of 1-manifolds.

2.1 Euclidean Fan-Space

A *combinatorial Euclidean space* is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ underlying a connected graph G defined by

$$V(G) = \{\mathbf{R}^{n_1}, \mathbf{R}^{n_2}, \cdots, \mathbf{R}^{n_m}\},$$

$$E(G) = \{\ (\mathbf{R}^{n_i}, \mathbf{R}^{n_j})\ |\ \mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} \neq \emptyset, 1 \leq i, j \leq m\},$$

denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$, which enables us to view an Euclidean space $\mathbf{R}^n$ for $n \geq 4$. Whence it can be used for models of spacetime in physics.

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is the combinatorial Euclidean space $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers $i, j,\ 1 \leq i \neq j \leq m$,

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$

A combinatorial fan-space is in fact a *p-brane* with $p = \dim \bigcap\limits_{k=1}^{m} \mathbf{R}^{n_k}$ in *String Theory* ([21], [22]), seeing Fig.2.2 for details.

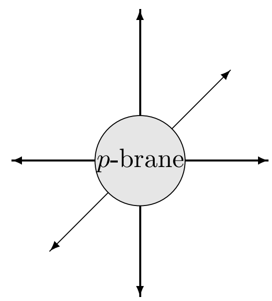


Fig.2.2

For $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$,

$$[\overline{x}] = \left[\begin{array}{cccccccc} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{array}\right].$$

Let $\mathscr{M}_{n\times s}$ denote all $n \times s$ matrixes for integers $n, s \geq 1$. We introduce the *inner product* $\langle (A), (B) \rangle$ for $(A), (B) \in \mathscr{M}_{n\times s}$ by

$$\langle (A), (B) \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

Then we easily know that $\mathscr{M}_{n\times s}$ forms an Euclidean space under such product.

2.2 Topological Combinatorial Manifold

For a given integer sequence $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, a *combinatorial manifold* $\widetilde{M}$ is a *Hausdorff space* such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\},$$

$$\bigcup_{p\in\widetilde{M}}\{n_1(p),n_2(p),\cdots,n_{s(p)}(p)\}=\{n_1,n_2,\cdots,n_m\},$$

denoted by $\widetilde{M}(n_1,n_2,\cdots,n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}}=\{(U_p,\varphi_p)|p\in\widetilde{M}(n_1,n_2,\cdots,n_m))\}$$

an atlas on $\widetilde{M}(n_1,n_2,\cdots,n_m)$.

A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. Certainly, a finitely combinatorial manifold is indeed a combinatorial manifold. Examples of combinatorial manifolds can be seen in Fig.2.1.

For characterizing topological properties of combinatorial manifolds, we need to introduced the vertex-edge labeled graph. A *vertex-edge labeled graph* $G([1,k],[1,l])$ is a connected graph $G=(V,E)$ with two mappings

$$\tau_1:V\rightarrow\{1,2,\cdots,k\},\quad \tau_2:E\rightarrow\{1,2,\cdots,l\}$$

for integers $k,l\geq 1$. For example, two vertex-edge labeled graphs on K_4 are shown in Fig.2.3.

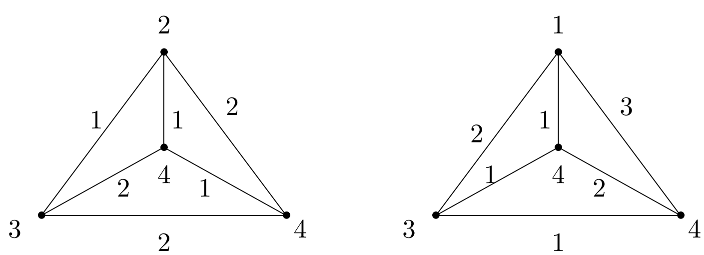


Fig.2.3

Let $\widetilde{M}(n_1,n_2,\cdots,n_m)$ be a finitely combinatorial manifold and $d,d\geq 1$ an integer. We construct a vertex-edge labeled graph $G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)]$ by

$$V(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=V_1\bigcup V_2,$$

where $V_1=\{n_i-\text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1,\cdots,n_m)|1\leq i\leq m\}$ and $V_2=\{$isolated intersection points $O_{M^{n_i},M^{n_j}}$ of M^{n_i},M^{n_j} in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ for $1\leq i,j\leq m\}$. Label n_i for each n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=E_1\bigcup E_2,$$

where $E_1=\{(M^{n_i},M^{n_j})$ labeled with $\dim(M^{n_i}\bigcap M^{n_j})\mid\dim(M^{n_i}\bigcap M^{n_j})\geq d,1\leq i,j\leq m\}$ and $E_2=\{(O_{M^{n_i},M^{n_j}},M^{n_i}),(O_{M^{n_i},M^{n_j}},M^{n_j})$ labeled with $0|M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i},M^{n_j}}$ for $1\leq i,j\leq m\}$.

Now denote by $\mathcal{H}(n_1,n_2,\cdots,n_m)$ all finitely combinatorial manifolds $\widetilde{M}(n_1,n_2,\cdots,n_m)$ and $\mathcal{G}[0,n_m]$ all vertex-edge labeled graphs G^L with $\theta_L:V(G^L)\cup E(G^L)\rightarrow\{0,1,\cdots,n_m\}$ with conditions following hold.

(1)Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2)For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0, n_m], [0, n_m])$ following.

Theorem 2.1 *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ defines a vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$. Conversely, every vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with a $1-1$ mapping $\theta : G([0, n_m]) \to \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m]))$.*

2.4 Fundamental d-Group

For two points p, q in a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of d-dimensional open balls with two conditions following hold.

(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ for any integer $i, 1 \leq i \leq s$ and $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \bigcap B_{i+1}) \geq d$ for $\forall i, 1 \leq i \leq s - 1$.

Then points p, q are called d-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p, q)$. If each pair p, q of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called d-pathwise connected and say its connectivity$\geq d$.

Choose a graph with vertex set being manifolds labeled by its dimension and two manifold adjacent with a label of the dimension of the intersection if there is a d-path in this combinatorial manifold. Such graph is denoted by G^d. For example, these correspondent labeled graphs gotten from finitely combinatorial manifolds in Fig.2.1 are shown in Fig.2.4, in where $d = 1$ for (a) and (b), $d = 2$ for (c) and (d).

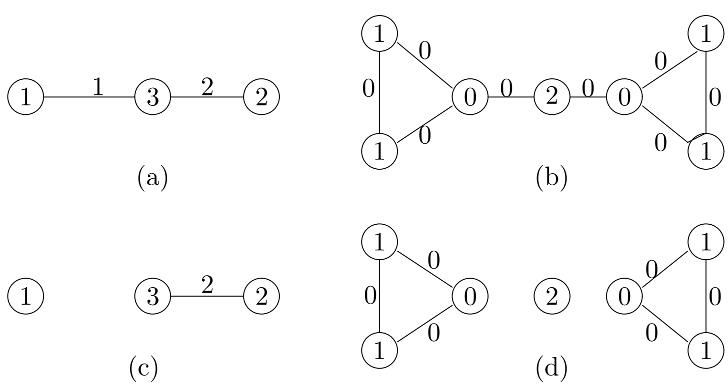


Fig.2.4

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold of d-arcwise connectedness for an integer $d, 1 \leq d \leq n_1$ and $\forall x_0 \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a *fundamental d-group* at the point

x_0, denoted by $\pi^d(\widetilde{M}(n_1,n_2,\cdots,n_m),x_0)$ is defined to be a group generated by all homotopic classes of closed d-pathes based at x_0. If $d=1$, then it is obvious that $\pi^d(\widetilde{M}(n_1,n_2,\cdots,n_m),x_0)$ is the common fundamental group of $\widetilde{M}(n_1,n_2,\cdots,n_m)$ at the point x_0 ([18]). For some special graphs, their fundamental d-groups can be immediately gotten, for example, the d-dimensional graphs following.

A combinatorial Euclidean space $\mathscr{E}_G(\overbrace{d,d,\cdots,d}^{m})$ of $\mathbf{R}^d$ underlying a combinatorial structure $G, |G|=m$ is called a *d-dimensional graph*, denoted by $\widetilde{M}^d[G]$ if

(1) $\widetilde{M}^d[G]\setminus V(\widetilde{M}^d[G])$ is a disjoint union of a finite number of open subsets $e_1,e_2,\cdots,e_m$, each of which is homeomorphic to an open ball B^d;

(2) the boundary $\overline{e}_i-e_i$ of e_i consists of one or two vertices B^d, and each pair $(\overline{e}_i,e_i)$ is homeomorphic to the pair $(\overline{B}^d,S^{d-1})$.

Then we get the next result by definition.

Theorem 2.2 $\pi^d(\widetilde{M}^d[G],x_0)\cong\pi_1(G,x_0)$, $x_0\in G$.

Generally, we know the following result for fundamental d-groups of combinatorial manifolds ([13], [17]).

Theorem 2.3 *Let $\widetilde{M}(n_1,n_2,\cdots,n_m)$ be a d-connected finitely combinatorial manifold for an integer d, $1\leq d\leq n_1$. If $\forall(M_1,M_2)\in E(G^L[\widetilde{M}(n_1,n_2,\cdots,n_m)])$, $M_1\cap M_2$ is simply connected, then*

(1) *for $\forall x_0\in G^d$, $M\in V(G^L[\widetilde{M}(n_1,n_2,\cdots,n_m)])$ and $x_{0M}\in M$,*

$$\pi^d(\widetilde{M}(n_1,n_2,\cdots,n_m),x_0)\cong(\bigoplus_{M\in V(G^d)}\pi^d(M,x_{M0}))\bigoplus\pi(G^d,x_0),$$

where $G^d=G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)]$ in which each edge (M_1,M_2) passing through a given point $x_{M_1M_2}\in M_1\cap M_2$, $\pi^d(M,x_{M0}),\pi(G^d,x_0)$ denote the fundamental d-groups of a manifold M and the graph G^d, respectively and

(2) *for $\forall x,y\in\widetilde{M}(n_1,n_2,\cdots,n_m)$,*

$$\pi^d(\widetilde{M}(n_1,n_2,\cdots,n_m),x)\cong\pi^d(\widetilde{M}(n_1,n_2,\cdots,n_m),y).$$

2.5 Homology Group

For a subspace A of a topological space S and an inclusion mapping $i:A\hookrightarrow S$, it is readily verified that the induced homomorphism $i_\sharp:C_p(A)\to C_p(S)$ is a monomorphism. Let $C_p(S,A)$ denote the quotient group $C_p(S)/C_p(A)$. Similarly, we define the *p-cycle group* and *p-boundary group* of (S,A) by ([19])

$$Z_p(S,A)=\mathrm{Ker}\partial_p=\{\ u\in C_p(S,A)\mid\partial_p(u)=0\ \},$$

$$B_p(S,A)=\mathrm{Im}\partial_{p+1}=\partial_{p+1}(C_{p+1}(S,A)),$$

for any integer $p \geq 0$.It follows that $B_p(S,A) \subset Z_p(S,A)$ and the *pth relative homology group* $H_p(S,A)$ is defined to be

$$H_p(S,A) = Z_p(S,A)/B_p(S,A).$$

We know the following result.

Theorem 2.4 *Let $\widetilde{M}$ be a combinatorial manifold, $\widetilde{M}^d(G) \prec \widetilde{M}$ a d-dimensional graph with $E(\widetilde{M}^d(G)) = \{e_1, e_2, \cdots, e_m\}$ such that*

$$\widetilde{M} \setminus \widetilde{M}^d[G] = \bigcup_{i=2}^{k} \bigcup_{j=1}^{l_i} B_{i_j}.$$

Then the inclusion $(e_l, \dot{e}_l) \hookrightarrow (\widetilde{M}, \widetilde{M}^d(G))$ induces a monomorphism $H_p(e_l, \dot{e}_l) \to H_p(\widetilde{M}, \widetilde{M}^d(G))$ for $l = 1, 2\cdots, m$ and

$$H_p(\widetilde{M}, \widetilde{M}^d(G)) \cong \begin{cases} \underbrace{\mathbf{Z} \oplus \cdots \mathbf{Z}}_{m}, & if\ p = d, \\ 0, & if\ p \neq d. \end{cases}$$

§3. **Differentiable Combinatorial Manifolds**

3.1 Definition

For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorial C^h-differential manifold $(\widetilde{M}(n_1, \cdots, n_m); \widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$, $\widetilde{M}(n_1, \cdots, n_m) = \bigcup\limits_{i \in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

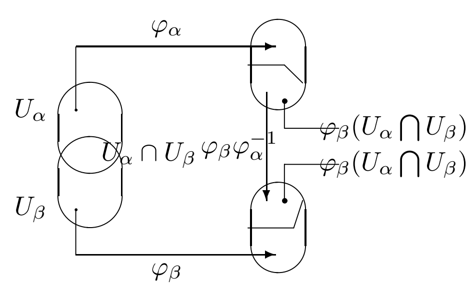


Fig.3.1

(2) For $\forall \alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\beta) \quad \text{and} \quad \varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are C^h-mappings, such as those shown in Fig.3.1.

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U;\varphi)$ is a local chart of $\widetilde{M}(n_1,n_2,\cdots,n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U;\varphi)\in\widetilde{\mathcal{A}}$.

Denote by $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{\mathcal{A}})$ a combinatorial differential manifold. A finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is said to be *smooth* if it is endowed with a C^∞-differential structure. For the existence of combinatorial differential manifolds, we know the following result ([13],[17]).

Theorem 3.1 *Let $\widetilde{M}(n_1,\cdots,n_m)$ be a finitely combinatorial manifold and $d, 1\leq d\leq n_1$ an integer. If for $\forall M\in V(G^d[\widetilde{M}(n_1,\cdots,n_m)])$ is C^h-differential and*

$$\forall(M_1,M_2)\in E(G^d[\widetilde{M}(n_1,\cdots,n_m)])$$

there exist atlas

$$\mathcal{A}_1=\{(V_x;\varphi_x)|\forall x\in M_1\}\quad \mathcal{A}_2=\{(W_y;\psi_y)|\forall y\in M_2\}$$

such that $\varphi_x|_{V_x\bigcap W_y}=\psi_y|_{V_x\bigcap W_y}$ for $\forall x\in M_1, y\in M_2$, then there is a differential structures

$$\widetilde{\mathcal{A}}=\{(U_p;[\varpi_p])|\forall p\in\widetilde{M}(n_1,\cdots,n_m)\}$$

such that $(\widetilde{M}(n_1,\cdots,n_m);\widetilde{\mathcal{A}})$ is a combinatorial C^h-differential manifold.

3.2 Local Properties of Combinatorial Manifolds

Let $\widetilde{M}_1(n_1,\cdots,n_m)$, $\widetilde{M}_2(k_1,\cdots,k_l)$ be smoothly combinatorial manifolds and

$$f:\widetilde{M}_1(n_1,\cdots,n_m)\to\widetilde{M}_2(k_1,\cdots,k_l)$$

be a mapping, $p\in\widetilde{M}_1(n_1,n_2,\cdots,n_m)$. If there are local charts $(U_p;[\varpi_p])$ of p on $\widetilde{M}_1(n_1,n_2,\cdots,n_m)$ and $(V_{f(p)};[\omega_{f(p)}])$ of $f(p)$ with $f(U_p)\subset V_{f(p)}$ such that the composition mapping

$$\widetilde{f}=[\omega_{f(p)}]\circ f\circ[\varpi_p]^{-1}:[\varpi_p](U_p)\to[\omega_{f(p)}](V_{f(p)})$$

is a C^h-mapping, then f is called a C^h-mapping at the point p. If f is C^h at any point p of $\widetilde{M}_1(n_1,\cdots,n_m)$, then f is called a C^h-mapping. Denote by $\mathscr{X}_p$ all these C^∞-functions at a point $p\in\widetilde{M}(n_1,\cdots,n_m)$.

Now let $(\widetilde{M}(n_1,\cdots,n_m),\widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p\in\widetilde{M}(n_1,\cdots,n_m)$. A tangent vector $\overline{v}$ at p is a mapping $\overline{v}:\mathscr{X}_p\to\mathbf{R}$ with conditions following hold.

(1) $\forall g,h\in\mathscr{X}_p,\forall\lambda\in\mathbf{R}$, $\overline{v}(h+\lambda h)=\overline{v}(g)+\lambda\overline{v}(h)$;

(2) $\forall g,h\in\mathscr{X}_p,\overline{v}(gh)=\overline{v}(g)h(p)+g(p)\overline{v}(h)$.

Let $\gamma:(-\epsilon,\epsilon)\to\widetilde{M}$ be a smooth curve on $\widetilde{M}$ and $p=\gamma(0)$. Then for $\forall f\in\mathscr{X}_p$, we usually define a mapping $\overline{v}:\mathscr{X}_p\to\mathbf{R}$ by

$$\overline{v}(f)=\frac{df(\gamma(t))}{dt}|_{t=0}.$$

We can easily verify such mappings $\overline{v}$ are tangent vectors at p.

Denote all tangent vectors at $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ by $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and define addition+and scalar multiplication·for $\forall \overline{u}, \overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\lambda \in \mathbf{R}$ and $f \in \mathscr{X}_p$ by

$$(\overline{u} + \overline{v})(f) = \overline{u}(f) + \overline{v}(f), \quad (\lambda\overline{u})(f) = \lambda \cdot \overline{u}(f).$$

Then it can be shown immediately that $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is a vector space under these two operations+and·. Let

$$\mathscr{X}(\widetilde{M}(n_1, n_2, \cdots, n_m)) = \bigcup_{p \in \widetilde{M}} T_p\widetilde{M}(n_1, n_2, \cdots, n_m).$$

A *vector field* on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is a mapping $X : \widetilde{M} \to \mathscr{X}(\widetilde{M}(n_1, n_2, \cdots, n_m))$, i.e., chosen a vector at each point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then the dimension and basis of the tangent space $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ are determined in the next result.

Theorem 3.2 *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix

$$[\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} =$$

$$\left[\begin{array}{ccccccccc} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{array}\right]$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely there is a smoothly functional matrix $[v_{ij}]_{s(p) \times n_{s(p)}}$ such that for any tangent vector $\overline{v}$ at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = \left\langle [v_{ij}]_{s(p) \times n_{s(p)}}, [\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} \right\rangle,$$

where $\langle [a_{ij}]_{k \times l}, [b_{ts}]_{k \times l} \rangle = \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} b_{ij}$, the inner product on matrixes.

For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$, the dual space $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called a *co-tangent vector space* at p. Let $f \in \mathscr{X}_p, d \in T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$. Then the action of d on f, called a *differential operator* $d : \mathscr{X}_p \to \mathbf{R}$, is defined by

$$df \;=\; \overline{v}(f).$$

We know the following result.

Theorem 3.3 *For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ with a basis matrix* $[d\overline{x}]_{s(p)\times n_{s(p)}} =$

$$\left[\begin{array}{cccccccc} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}} \end{array}\right]$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely for any co-tangent vector d at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$, there is a smoothly functional matrix $[u_{ij}]_{s(p)\times s(p)}$ such that,

$$d = \left\langle [u_{ij}]_{s(p)\times n_{s(p)}}, [d\overline{x}]_{s(p)\times n_{s(p)}} \right\rangle.$$

3.3 Tensor Field

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A *tensor of type* (r, s) at the point p on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function τ,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$. Denoted by $T_s^r(p, \widetilde{M})$ all tensors of type (r, s) at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$. We know its structure as follows.

Theorem 3.4 *Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, \cdots, n_m)$. Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p)))^{r+s}.$$

3.4 Curvature Tensor

A *connection on tensors* of a smoothly combinatorial manifold $\widetilde{M}$ is a mapping $\widetilde{D} : \mathscr{X}(\widetilde{M}) \times T_s^r\widetilde{M} \to T_s^r\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X, \tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T_s^r(\widetilde{M})$,$\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;
(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;
(3) for any contraction C on $T_s^r(\widetilde{M})$,

$$\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau).$$

A *combinatorial connection space* is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ on its tensors. Let $(\widetilde{M}, \widetilde{D})$ be a combinatorial connection space. For $\forall X, Y \in \mathscr{X}(\widetilde{M})$, a *combinatorial curvature operator* $\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is defined by

$$\widetilde{\mathcal{R}}(X,Y)Z = \widetilde{D}_X\widetilde{D}_Y Z - \widetilde{D}_Y\widetilde{D}_X Z - \widetilde{D}_{[X,Y]}Z$$

for $\forall Z \in \mathscr{X}(\widetilde{M})$.

Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup\limits_{p\in\widetilde{M}} T_2^0(p, \widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a *combinatorial Riemannian manifold*, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_Z Y)$$

then $\widetilde{M}$ is called a *combinatorial Riemannian geometry*, denoted by $(\widetilde{M}, g, \widetilde{D})$. In this case, calculation shows that ([14])

$$\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda}$$

with

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}\Big(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}\Big) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

§4. Principal Fiber Bundles

In classical differential geometry, a principal fiber bundle ([3]) is an application of covering space to smoothly manifolds. Topologically, a covering space ([18]) S' of S consisting of a space S' with a continuous mapping $\pi : S' \to S$ such that each point $x \in S$ has an arcwise connected neighborhood U_x and each arcwise connected component of $\pi^{-1}(U_x)$ is mapped homeomorphically onto U_x by π, such as those shown in Fig.4.1.

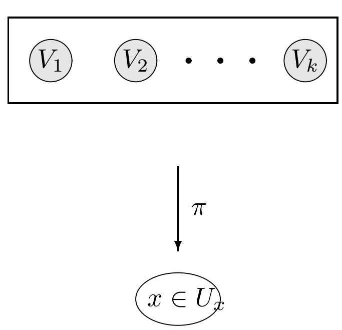


Fig.4.1

where $V_i = \pi^{-1}(U_x)$ for integers $1 \leq i \leq k$.

A *principal fiber bundle* ([3]) consists of a manifold P action by a Lie group $\mathscr{G}$, which is a manifold with group operation $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(g, h) \to g \circ h$ being C^∞ mapping, a projection $\pi : P \to M$, a base pseudo-manifold M, denoted by $(P, M, \mathscr{G})$, seeing Fig.4.2 such that conditions (1), (2) and (3) following hold.

(1) there is a right freely action of $\mathscr{G}$ on P,, i.e., for $\forall g \in \mathscr{G}$, there is a diffeomorphism $R_g : P \to P$ with $R_g(p) = pg$ for $\forall p \in P$ such that $p(g_1 g_2) = (pg_1)g_2$ for $\forall p \in P$, $\forall g_1, g_2 \in \mathscr{G}$ and $pe = p$ for some $p \in P$, $e \in \mathscr{G}$ if and only if e is the identity element of $\mathscr{G}$.

(2) the map $\pi : P \to M$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in \mathscr{G}\}$.

(3) for $\forall x \in M$ there is an open set U with $x \in U$ and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times \mathscr{G}$ of the form $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to \mathscr{G}$ has the property $s_U(pg) = s_U(p)g$ for $\forall g \in \mathscr{G}, p \in \pi^{-1}(U)$.

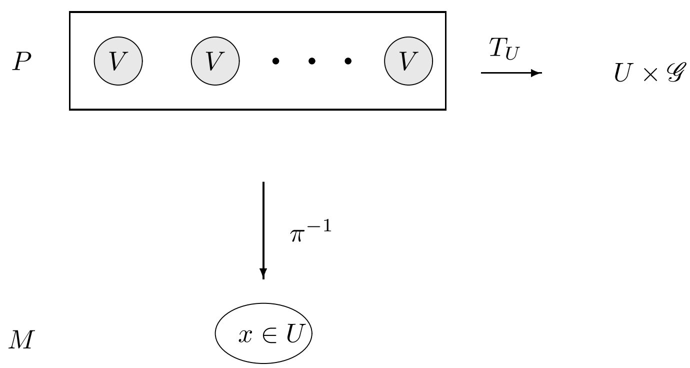


Fig.4.2

where $V = \pi^{-1}(U)$. Now *can we establish principal fiber bundles on smoothly combinatorial manifolds*? This question can be formally presented as follows:

Question *For a family of k principal fiber bundles $P_1(M_1, \mathscr{G}_1)$, $P_2(M_2, \mathscr{G}_2)$,$\cdots$, $P_k(M_k, \mathscr{G}_k)$ over manifolds M_1, M_2, $\cdots$, M_k, how can we construct principal fiber bundles on a smoothly combinatorial manifold consisting of $M_1, M_2, \cdots, M_k$ underlying a connected graph G?*

The answer is YES! For this object, we need some techniques in combinatorics.

4.1 Voltage Graph with Its Lifting

Let G be a connected graph and $(\Gamma; \circ)$ a group. For each edge $e \in E(G), e = uv$, an *orientation* on e is an orientation on e from u to v, denoted by $e = (u, v)$, called *plus orientation* and its *minus orientation*, from v to u, denoted by $e^{-1} = (v, u)$. For a given graph G with plus and minus orientation on its edges, a *voltage assignment* on G is a mapping α from the plus-edges of G into a group Γ satisfying $\alpha(e^{-1}) = \alpha^{-1}(e), e \in E(G)$. These elements $\alpha(e), e \in E(G)$ are called voltages, and (G, α) a *voltage graph* over the group $(\Gamma; \circ)$.

For a voltage graph (G,α), its lifting (See [6], [9] for details) $G^\alpha = (V(G^\alpha), E(G^\alpha); I(G^\alpha))$ is defined by

$$V(G^\alpha) = V(G) \times \Gamma, \ (u,a) \in V(G) \times \Gamma \text{ abbreviated to } u_a;$$

$$E(G^\alpha) = \{(u_a, v_{a\circ b}) | e^+ = (u,v) \in E(G), \alpha(e^+) = b\}.$$

For example, let $G = K_3$ and $\Gamma = Z_2$. Then the voltage graph (K_3,α) with $\alpha : K_3 \to Z_2$ and its lifting are shown in Fig.4.3.

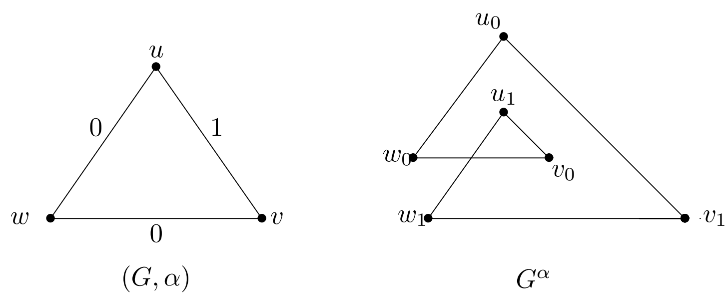


Fig.4.3

Similarly, let G^L be a connected vertex-edge labeled graph with $\theta_L : V(G) \cup E(G) \to L$ of a label set and Γ a finite group. A *voltage labeled graph* on a vertex-edge labeled graph G^L is a 2-tuple $(G^L;\alpha)$ with a voltage assignments $\alpha : E(G^L) \to \Gamma$ such that

$$\alpha(u,v) = \alpha^{-1}(v,u), \quad \forall (u,v) \in E(G^L).$$

Similar to voltage graphs, the importance of voltage labeled graphs lies in their *labeled lifting* G^{L_α} defined by

$$V(G^{L_\alpha}) = V(G^L) \times \Gamma, \ (u,g) \in V(G^L) \times \Gamma \text{ abbreviated to } u_g;$$

$$E(G^L_\alpha) = \{ \ (u_g, v_{g\circ h}) \mid \text{for } \ \forall (u,v) \in E(G^L) \text{ with } \alpha(u,v) = h \ \}$$

with labels $\Theta_L : G^{L_\alpha} \to L$ following:

$$\Theta_L(u_g) = \theta_L(u), \ \text{ and } \ \Theta_L(u_g, v_{g\circ h}) = \theta_L(u,v)$$

for $u,v \in V(G^L)$, $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$ and $g,h \in \Gamma$.

For a voltage labeled graph (G^L,α) with its lifting G^L_α, a *natural projection* $\pi : G^{L_\alpha} \to G^L$ is defined by $\pi(u_g) = u$ and $\pi(u_g, v_{g\circ h}) = (u,v)$ for $\forall u,v \in V(G^L)$ and $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$. Whence, (G^{L_α}, π) is a covering space of the labeled graph G^L. In this covering, we can find

$$\pi^{-1}(u) = \{ \ u_g \mid \forall g \in \Gamma\}$$

for a vertex $u \in V(G^L)$ and

$$\pi^{-1}(u,v) = \{ \ (u_g, v_{g\circ h}) \mid \forall g \in \Gamma \ \}$$

for an edge $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$. Such sets $\pi^{-1}(u)$, π^{-1}(u,v) are called *fibres* over the vertex $u \in V(G^L)$ or edge $(u,v) \in E(G^L)$, denoted by fib_u or $\mathrm{fib}_{(u,v)}$, respectively. A voltage labeled graph with its labeled lifting are shown in Fig.4.4, in where, $G^L = C_3^L$ and $\Gamma = Z_2$.

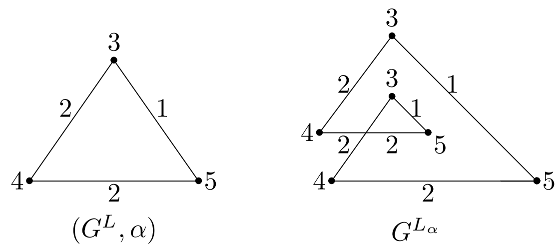


Fig.4.4

A mapping $g : G^L \to G^L$ is *acting on a labeled graph* G^L with a labeling $\theta_L : G^L \to L$ if $g\theta_L(x) = \theta_L g(x)$ for $\forall x \in V(G^L) \cup E(G^L)$, and a group Γ is acting on a labeled graph G^L if each $g \in \Gamma$ is acting on G^L. Clearly, if Γ is acting on a labeled graph G^L, then $\Gamma \leq \mathrm{Aut}G$.

Now let A be a group of automorphisms of G^L. A voltage labeled graph (G^L, α) is called *locally A-invariant* at a vertex $u \in V(G^L)$ if for $\forall f \in A$ and $W \in \pi_1(G^L, u)$, we have

$$\alpha(W) = identity \ \Rightarrow \ \alpha(f(W)) = identity$$

and *locally f-invariant* for an automorphism $f \in \mathrm{Aut}G^L$ if it is locally invariant with respect to the group $\langle f \rangle$ in $\mathrm{Aut}G^L$. Then we know a criterion for lifting automorphisms of voltage labeled graphs.

Theorem 4.1 *Let (G^L, α) be a voltage labeled graph with $\alpha : E(G^L) \to \Gamma$ and $f \in \mathrm{Aut}G^L$. Then f lifts to an automorphism of G^{L_α} if and only if (G^L, α) is locally f-invariant.*

4.2 Combinatorial Principal Fiber Bundles

For construction principal fiber bundles on smoothly combinatorial manifolds, we need to introduce the conception of Lie multi-group. A *Lie multi-group* $\mathscr{L}_G$ is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a multi-group $(\widetilde{\mathscr{A}(\mathscr{L}_G)}; \mathscr{O}(\mathscr{L}_G))$, where $\widetilde{\mathscr{A}(\mathscr{L}_G)} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O}(\mathscr{L}_G) = \bigcup\limits_{i=1}^{m} \{\circ_i\}$ such that

(i) $(\mathscr{H}_i; \circ_i)$ is a group for each integer i, $1 \leq i \leq m$;

(ii) $G^L[\widetilde{M}] = G$;

(iii) the mapping $(a,b) \to a \circ_i b^{-1}$ is C^∞-differentiable for any integer i, $1 \leq i \leq m$ and $\forall a, b \in \mathscr{H}_i$.

Notice that if $m = 1$, then a Lie multi-group $\mathscr{L}_G$ is nothing but just the Lie group ([24]) in classical differential geometry.

Now let $\widetilde{P}$, $\widetilde{M}$ be a differentiably combinatorial manifolds and $\mathscr{L}_G$ a Lie multi-group $(\widetilde{\mathscr{A}(\mathscr{L}_G)}; \mathscr{O}(\mathscr{L}_G))$ with

$$\widetilde{P}=\bigcup_{i=1}^{m}P_i,\ \widetilde{M}=\bigcup_{i=1}^{s}M_i,\ \widetilde{\mathscr{A}}(\mathscr{L}_G)=\bigcup_{i=1}^{m}\mathscr{H}_{\circ_i},\mathscr{O}(\mathscr{L}_G)=\bigcup_{i=1}^{m}\{\circ_i\}.$$

Then a *differentiable principal fiber bundle over* $\widetilde{M}$ *with group* $\mathscr{L}_G$ consists of a differentiably combinatorial manifold $\widetilde{P}$, an action of $\mathscr{L}_G$ on $\widetilde{P}$, denoted by $\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ satisfying following conditions PFB1-PFB3:

PFB1. For any integer $i,\ 1\leq i\leq m$, $\mathscr{H}_{\circ_i}$ acts differentiably on P_i to the right without fixed point, i.e.,

$$(x,g)\in P_i\times\mathscr{H}_{\circ_i}\rightarrow x\circ_i g\in P_i \text{ and } x\circ_i g=x \text{ implies that } g=1_{\circ_i};$$

PFB2. For any integer $i,\ 1\leq i\leq m$, $M_{\circ_i}$ is the quotient space of a covering manifold $P\in\Pi^{-1}(M_{\circ_i})$ by the equivalence relation R induced by $\mathscr{H}_{\circ_i}$:

$$R_i=\{(x,y)\in P_{\circ_i}\times P_{\circ_i}|\exists g\in\mathscr{H}_{\circ_i}\Rightarrow x\circ_i g=y\},$$

written by $M_{\circ_i}=P_{\circ_i}/\mathscr{H}_{\circ_i}$, i.e., an orbit space of $P_{\circ_i}$ under the action of $\mathscr{H}_{\circ_i}$. These is a canonical projection $\Pi:\widetilde{P}\rightarrow\widetilde{M}$ such that $\Pi_i=\Pi|_{P_{\circ_i}}:P_{\circ_i}\rightarrow M_{\circ_i}$ is differentiable and each fiber $\Pi_i^{-1}(x)=\{p\circ_i g|g\in\mathscr{H}_{\circ_i},\Pi_i(p)=x\}$ is a closed submanifold of $P_{\circ_i}$ and coincides with an equivalence class of R_i;

PFB3. For any integer $i,\ 1\leq i\leq m$, $P\in\Pi^{-1}(M_{\circ_i})$ is locally trivial over $M_{\circ_i}$, i.e., any $x\in M_{\circ_i}$ has a neighborhood U_x and a diffeomorphism $T:\Pi^{-1}(U_x)\rightarrow U_x\times\mathscr{L}_G$ with

$$T|_{\Pi_i^{-1}(U_x)}=T_i^x:\Pi_i^{-1}(U_x)\rightarrow U_x\times\mathscr{H}_{\circ_i};\ x\rightarrow\ T_i^x(x)=(\Pi_i(x),\epsilon(x)),$$

called a local trivialization (abbreviated to LT) such that $\epsilon(x\circ_i g)=\epsilon(x)\circ_i g$ for $\forall g\in\mathscr{H}_{\circ_i}$, $\epsilon(x)\in\mathscr{H}_{\circ_i}$.

Certainly, if $m=1$, then $\widetilde{P}(\widetilde{M},\mathscr{L}_G)=P(M,\mathscr{H})$ is just the common principal fiber bundle over a manifold M.

4.3 Construction by Voltage Assignment

Now we show how to construct principal fiber bundles over a combinatorial manifold $\widetilde{M}$.

Construction 4.1 *For a family of principal fiber bundles over manifolds* $M_1,M_2,\cdots,\ M_l$*, such as those shown in Fig.*4.5*,*

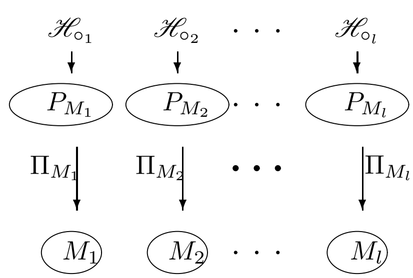


Fig.4.5

where $\mathscr{H}_{\circ_i}$ is a Lie group acting on P_{M_i} for $1 \leq i \leq l$ satisfying conditions PFB1-PFB3, let $\widetilde{M}$ be a differentiably combinatorial manifold consisting of M_i, $1 \leq i \leq l$ and $(G^L[\widetilde{M}], \alpha)$ a voltage graph with a voltage assignment $\alpha : G^L[\widetilde{M}] \to \mathfrak{G}$ over a finite group $\mathfrak{G}$, which naturally induced a projection $\pi : G^L[\widetilde{P}] \to G^L[\widetilde{M}]$. For $\forall M \in V(G^L[\widetilde{M}])$, if $\pi(P_M) = M$, place P_M on each lifting vertex M^{L_α} in the fiber $\pi^{-1}(M)$ of $G^{L_\alpha}[\widetilde{M}]$, such as those shown in Fig.4.6.

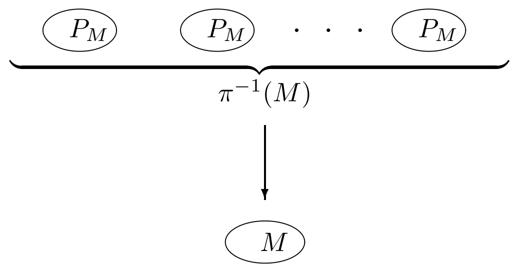


Fig.4.6

Let $\Pi = \pi \Pi_M \pi^{-1}$ for $\forall M \in V(G^L[\widetilde{M}])$. Then $\widetilde{P} = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} P_M$ is a smoothly combinatorial manifold and $\mathscr{L}_G = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} \mathscr{H}_M$ a Lie multi-group by definition. Such a constructed combinatorial fiber bundle is denoted by $\widetilde{P}^{L_\alpha}(\widetilde{M}, \mathscr{L}_G)$.

For example, let $\mathfrak{G} = Z_2$ and $G^L[\widetilde{M}] = C_3$. A voltage assignment $\alpha : G^L[\widetilde{M}] \to Z_2$ and its induced combinatorial fiber bundle are shown in Fig.4.7.

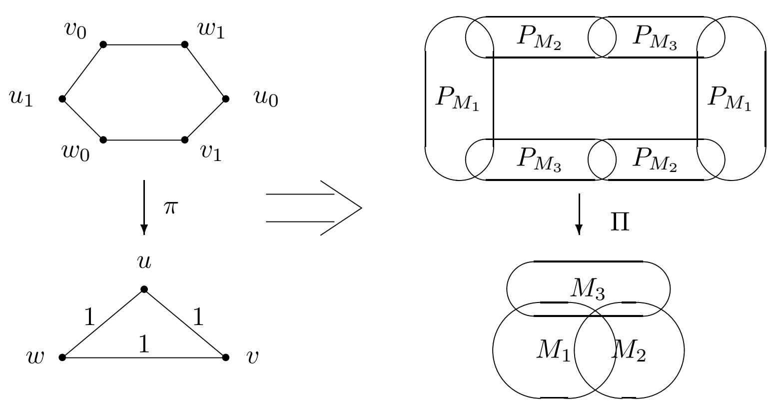


Fig.4.7

Then we know the existence result following.

Theorem 4.2 *A combinatorial fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is a principal fiber bundle if and only if for $\forall (M', M'') \in E(G^L[\widetilde{M}])$ and $(P_{M'}, P_{M''}) = (M', M'')^{L_\alpha} \in E(G^L[\widetilde{P}])$, $\Pi_{M'}|_{P_{M'} \cap P_{M''}} = \Pi_{M''}|_{P_{M'} \cap P_{M''}}$.*

We assume $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ satisfying conditions in Theorem 4.2, i.e., it is indeed a principal fiber bundle over $\widetilde{M}$. An *automorphism of* $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is a diffeomorphism $\omega : \widetilde{P} \to \widetilde{P}$ such that $\omega(p \circ_i g) = \omega(p) \circ_i g$ for $g \in \mathscr{H}_{\circ_i}$ and

$$p \in \bigcup_{P \in \pi^{-1}(M_i)} P, \quad \text{where } 1 \leq i \leq l.$$

Theorem 4.3 *Let $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ be a principal fiber bundle. Then*

$$\mathrm{Aut}\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G) \geq \langle \mathfrak{L} \rangle ,$$

where $\mathfrak{L} = \{ \widehat{h}\omega_i \mid \widehat{h} : P_{M_i} \to P_{M_i}$ is $1_{P_{M_i}}$ determined by $h((M_i)_g) = (M_i)_{g \circ_i h}$ for $h \in \mathfrak{G}$ and $g_i \in \mathrm{Aut}P_{M_i}(M_i, \mathscr{H}_{\circ_i}),\ 1 \leq i \leq l\}$.

A principal fiber bundle $\widetilde{P}(\widetilde{M}, \mathscr{L}_G)$ is called to be *normal* if for $\forall u, v \in \widetilde{P}$, there exists an $\omega \in \mathrm{Aut}\widetilde{P}(\widetilde{M}, \mathscr{L}_G)$ such that $\omega(u) = v$. We get the necessary and sufficient conditions of normally principal fiber bundles $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ following.

Theorem 4.4 *$\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is normal if and only if $P_{M_i}(M_i, \mathscr{H}_{\circ_i})$ is normal, $(\mathscr{H}_{\circ_i}; \circ_i) = (\mathscr{H}; \circ)$ for $1 \leq i \leq l$ and $G^{L_\alpha}[\widetilde{M}]$ is transitive by diffeomorphic automorphisms in $\mathrm{Aut}G^{L_\alpha}[\widetilde{M}]$.*

4.4 Connection on Principal Fiber Bundles over Combinatorial Manifolds

A *local connection* on a principal fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is a linear mapping ${}^i\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P})$ for an integer i, $1 \leq i \leq l$ and $u \in \Pi_i^{-1}(x) = {}^iF_x$, $x \in M_i$, enjoys with properties following:

- (i) $(d\Pi_i){}^i\Gamma_u$ = identity mapping on $T_x(\widetilde{M})$;
- (ii) ${}^i\Gamma_{{}^iR_g \circ_i u} = d\,{}^iR_g \circ_i {}^i\Gamma_u$, where ${}^iR_g$ is the right translation on P_{M_i};
- (iii) the mapping $u \to {}^i\Gamma_u$ is C^∞.

Similarly, a *global connection* on a principal fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ is a linear mapping $\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P})$ for a $u \in \Pi^{-1}(x) = F_x$, $x \in \widetilde{M}$ with conditions following hold:

- (i) $(d\Pi)\Gamma_u$ = identity mapping on $T_x(\widetilde{M})$;
- (ii) $\Gamma_{R_g \circ u} = dR_g \circ \Gamma_u$ for $\forall g \in \mathscr{L}_G$, $\forall \circ \in \mathscr{O}(\mathscr{L}_G)$, where R_g is the right translation on $\widetilde{P}$;
- (iii) the mapping $u \to \Gamma_u$ is C^∞.

Local or global connections on combinatorial principal fiber bundles are characterized by results following.

Theorem 4.5 *For an integer i, $1 \leq i \leq l$, a local connection ${}^i\Gamma$ in $\widetilde{P}$ is an assignment ${}^iH : u \to {}^iH_u \subset T_u(\widetilde{P})$, of a subspace ${}^iH_u$ of $T_u(\widetilde{P})$ to each $u \in {}^iF_x$ with*

- (i) *$T_u(\widetilde{P}) = {}^iH_u \oplus {}^iV_u$, $u \in {}^iF_x$;*
- (ii) *$(d\,{}^iR_g)\,{}^iH_u = {}^iH_{u \circ_i g}$ for $\forall u \in {}^iF_x$ and $\forall g \in \mathscr{H}_{\circ_i}$;*
- (iii) *${}^iH$ is a C^∞-distribution on $\widetilde{P}$.*

Theorem 4.6 *A global connection Γ in $\widetilde{P}$ is an assignment $H : u \to H_u \subset T_u(\widetilde{P})$, of a subspace H_u of $T_u(\widetilde{P})$ to each $u \in F_x$ with*

- (i) *$T_u(\widetilde{P}) = H_u \oplus V_u$, $u \in F_x$;*
- (ii) *$(dR_g)H_u = H_{u \circ g}$ for $\forall u \in F_x$, $\forall g \in \mathscr{L}_G$ and $\circ \in \mathscr{O}(\mathscr{L}_G)$;*
- (iii) *H is a C^∞-distribution on $\widetilde{P}$.*

Theorem 4.7 *Let* ${}^i\Gamma$ *be a local connections on* $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$ *for* $1 \leq i \leq l$. *Then a global connection on* $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$ *exists if and only if* $(\mathscr{H}_{\circ_i};\circ_i) = (\mathscr{H};\circ)$, *i.e.,* $\mathscr{L}_G$ *is a group and* ${}^i\Gamma|_{M_i\cap M_j} = {}^j\Gamma|_{M_i\cap M_j}$ *for* $(M_i, M_j) \in E(G^L[\widetilde{M}])$, $1 \leq i, j \leq l$.

A *curvature form* of a local or global connection is a $\mathfrak{Y}(\mathscr{H}_{\circ_i},\circ_i)$ or $\mathfrak{Y}(\mathscr{L}_G)$-valued 2-form

$$ {}^i\Omega = (d\ {}^i\omega)h, \quad \text{or} \quad \Omega = (d\omega)h, $$

where $(d\ {}^i\omega)h(X,Y) = d\ {}^i\omega(hX,hY)$, $\quad (d\omega)h(X,Y) = d\omega(hX,hY)$ for $X,Y \in \mathscr{X}(P_{M_i})$ or $X,Y \in \mathscr{X}(\widetilde{P})$. Notice that a 1-form $\omega h(X_1,X_2) = 0$ if and only if ${}^ih(X_1) = 0$ or ${}^ih(X_12) = 0$. We generalize classical structural equations and Bianchi's identity on principal fiber bundles following.

Theorem 4.8(E.Cartan) *Let* ${}^i\omega$, $1 \leq i \leq l$ *and* ω *be local or global connection forms on a principal fiber bundle* $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$. *Then*

$$ (d\ {}^i\omega)(X,Y) = -[\ {}^i\omega(X),{}^i\omega(Y)] +\ {}^i\Omega(X,Y) $$

and

$$ d\omega(X,Y) = -[\omega(X),\omega(Y)] + \Omega(X,Y) $$

for vector fields $X,Y \in \mathscr{X}(P_{M_i})$ *or* $\mathscr{X}(\widetilde{P})$.

Theorem 4.9(Bianchi) *Let* ${}^i\omega$, $1 \leq i \leq l$ *and* ω *be local or global connection forms on a principal fiber bundle* $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$. *Then*

$$ (d\ {}^i\Omega)h = 0, \quad \textit{and} \quad (d\Omega)h = 0. $$

§5. Applications

A *gauge field* is such a mathematical model with local or global symmetries under a group, a finite-dimensional Lie group in most cases action on its gauge basis at an individual point in space and time, together with a set of techniques for making physical predictions consistent with the symmetries of the model, which is a generalization of Einstein's principle of covariance to that of internal field characterized by the following ([3],[23],[24]).

Gauge Invariant Principle *A gauge field equation, particularly, the Lagrange density of a gauge field is invariant under gauge transformations on this field.*

We wish to find gauge fields on combinatorial manifolds, and then to characterize WORLD by combinatorics. A *globally* or *locally combinatorial gauge field* is a combinatorial field $\widetilde{M}$ under a gauge transformation $\tau_{\widetilde{M}} : \widetilde{M} \to \widetilde{M}$ independent or dependent on the field variable $\overline{x}$. If a combinatorial gauge field $\widetilde{M}$ is consisting of gauge fields $M_1, M_2, \cdots, M_m$, we can easily find that $\widetilde{M}$ is a globally combinatorial gauge field only if each gauge field is global.

Let M_i, $1 \leq i \leq m$ be gauge fields with a basis B_{M_i} and $\tau_i : B_{M_i} \to B_{M_i}$ a gauge transformation, i.e., $\mathscr{L}_{M_i}(B_{M_i}^{\tau_i}) = \mathscr{L}_{M_i}(B_{M_i})$. A gauge transformation $\tau_{\widetilde{M}} : \bigcup\limits_{i=1}^{m} B_{M_i} \to \bigcup\limits_{i=1}^{m} B_{M_i}$ is such a transformation on the gauge multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$ and Lagrange density $\mathscr{L}_{\widetilde{M}}$ with $\tau_{\widetilde{M}}|_{M_i} = \tau_i, \mathscr{L}_{\widetilde{M}}|_{M_i} = \mathscr{L}_{M_i}$ for integers $1 \leq i \leq m$ such that

$$\mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i})^{\tau_{\widetilde{M}}} = \mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i}).$$

A multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$ is a *combinatorial gauge basis* if for any automorphism $g \in \mathrm{Aut}G^L[\widetilde{M}]$,

$$\mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i})^{\tau_{\widetilde{M}} \circ g} = \mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i}),$$

where $\tau_{\widetilde{M}} \circ g$ means $\tau_{\widetilde{M}}$ composting with an automorphism g, a bijection on gauge multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$. Whence, a combinatorial field consisting of gauge fields $M_1, M_2, \cdots, M_m$ is a combinatorial gauge field if $M_1^\alpha = M_2^\alpha$ for $\forall M_1^\alpha, M_2^\alpha \in \Omega_\alpha$, where Ω_α, $1 \leq \alpha \leq s$ are orbits of $M_1, M_2, \cdots, M_m$ under the action of $\mathrm{Aut}G^L[\widetilde{M}]$. Therefore, combining existent gauge fields underlying a connected graph G in space enables us to find more combinatorial gauge fields. For example, combinatorial gravitational fields $\widetilde{M}(t)$ determined by tensor equations

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2} g_{(\mu\nu)(\sigma\tau)} R = -8\pi G \mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

in a combinatorial Riemannian manifold $(\widetilde{M}, g, \widetilde{D})$ with $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Now let $\overset{1}{\omega}$ be the local connection 1-form, $\overset{2}{\Omega} = \widetilde{d} \overset{1}{\omega}$ the curvature 2-form of a local connection on $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ and $\Lambda : \widetilde{M} \to \widetilde{P}$, $\Pi \circ \Lambda = \mathrm{id}_{\widetilde{M}}$ be a local cross section of $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$. Consider

$$\widetilde{A} = \Lambda^* \overset{1}{\omega} = \sum_{\mu\nu} A_{\mu\nu} dx^{\mu\nu},$$

$$\widetilde{F} = \Lambda^* \overset{2}{\Omega} = \sum F_{(\mu\nu)(\kappa\lambda)} dx^{\mu\nu} \wedge dx^{\kappa\lambda}, \qquad \widetilde{d}\, \widetilde{F} = 0,$$

called the *combinatorial gauge potential* and *combinatorial field strength*, respectively. Let $\gamma : \widetilde{M} \to \mathbf{R}$ and $\Lambda' : \widetilde{M} \to \widetilde{P}$, $\Lambda'(\overline{x}) = e^{i\gamma(\overline{x})}\Lambda(\overline{x})$. If $\widetilde{A}' = \Lambda'^* \overset{1}{\omega}$, then we have

$$\overset{1}{\omega}{}' (X) = g^{-1} \overset{1}{\omega} (X') g + g^{-1} dg, \ g \in \mathscr{L}_G,$$

for $dg \in T_g(\mathscr{L}_G)$, $X = \widetilde{d} R_g X'$ by properties of local connections on combinatorial principal fiber bundles discussed in Section 4.4, which finally yields equations following

$$\widetilde{A}' = \widetilde{A} + \widetilde{d}\, \widetilde{A}, \qquad \widetilde{d}\, \widetilde{F}' = \widetilde{d}\, \widetilde{F},$$

i.e., the gauge transformation law on field. This equation enables one to obtain the local form of $\widetilde{F}$ as they contributions to Maxwell or Yang-Mills fields in classical gauge fields theory.

Certainly, combinatorial fields can be applied to any many-body system in natural or social science, such as those in mechanics, cosmology, physical structure, economics, $\cdots$, etc..

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A Spacetime Geodesics of the Schwarzschild Space and Its Deformation Retract

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Abstract: A Smarandache multi-space is such a union space $\bigcup_{i=1}^{n} S_i$ of $S_1, S_2, \cdots, S_n$ for an integer $n \geq 1$. In this paper, we deduce the spacetime geodesic of Schwarzschild space, i.e., a Smarandachely Schwarzschild space with $n = 1$ by using Lagrangian equations. The deformation retract of this space will be presented. The relation between folding and deformation retract of this space will be achieved.

Keywords: Schwarzschild space, spacetime, folding, Smarandache multi-space, deformation retract.

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§1. Introduction

The Dirichlet problem for boundaries which are S^1-bundles over some compact manifolds. In general relativity such boundaries often arise in gravitational thermodynamics. The classic example is that of the trivial bundle $\sum \equiv S^1 \times S^2$. Manifolds with complete Ricci-flat metrics [1] admitting such boundaries are known; they are the Euclidean's Schwarzschild metric and the flat metric with periodic identification. The Schwarzschild metric result by taking the limit k $\to$ 0 and L $\to$ 0 while keeping r+ fixed:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{1}$$

Here $t \in [0,\infty)$ and replaces the ψ coordinate in the previous two examples. The metric has a bolt singularity at r = 2m which can be made regular by identifying the coordinate t with a period of 8πm. The radial coordinate r has the range [2m,∞) and constant r slices of the regular metric have the trivial product topology of $S^1 \times S^2$. The four-metric therefore has the topology of $R^1 \times S^2$. For a boundary $\sum \equiv S^1 \times S^2$, the pair ($\alpha$, β) constitutes the canonical boundary data with the interpretation that α represents the radius of a spherical cavity immersed in a thermal bath with temperature T = $\frac{1}{2\pi\beta}$. It is known that for such canonical boundary data, apart from the obvious infilling flat-space solution with proper identification, there are in general two black hole solutions distinguished by their masses which become degenerate at a certain

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value of the squashing, i.e., the ratio of the two radii $\frac{\beta}{\alpha}$. This can be seen in the following way. First rewrite the Schwarzschild metric (1) in the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (2)$$

where t $= 8\pi\tau$ such that τ has unit period. With this definition one can simply read off the proper length – alternatively the radius – of the S^1fibre and that of the S^2base. They are

$$\alpha^2 = r^2 \qquad (3)$$

and

$$\beta^2 = 16m^2 \left(1 - \frac{2m}{r}\right) \qquad (4)$$

It is easy to see that for a given $(\alpha,\, \beta)$, r is uniquely determined whereas m is given by the positive solutions of the following equation:

$$m^3 - \frac{1}{2}\alpha m^2 + \frac{1}{32}\alpha\beta^2 = 0 \qquad (5)$$

By solving this equation for m, the two Schwarzschild infilling geometries are found3. There are in general two positive roots of Eq.(5) provided $\frac{\beta^2}{\alpha^2} \leq \frac{16}{27}$. When the equality holds the two solutions become degenerate and beyond this value of squashing they turn complex. The remaining root of Eq.(5) is always negative. Therefore the two infilling solutions appear and disappear in pairs as the boundary data is varied [1,12].

Next let us recall the concept of a metric in four-dimensions. Considering only flat space, we have

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \qquad (6)$$

Now we see that $ds^2 > 0, ds^2 = 0$ and $ds^2 < 0$ correspond to space-like, null and time-like geodesics, we note that massless particles, such as the photon, move on null geodesics. That can be interpreted as saying that in 4-dimensional space, the photon does not move and that for a photon, and time does not pass. Particularly intriguing is the mathematical possibility of a negative metric. Now it is extremely interesting that there is a geometry in which two separated points may still have a zero distance analogous to the corresponding to a null geodesic [7].

§2. **Definitions and Background**

(i) Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \to M$, the induced path $f \circ \gamma : J \to N$ is a piecewise geodesic and of the same length as γ [13]. If f does not preserve the lengths, it is called topological folding. Many types of foldings are discussed in [3,4,5,6,8,9]. Some applications are discussed in [2,10].

(ii) A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that([11])

(a) X is open

(b) $r(a) = a, \forall\ a \in A$.

(iii) A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times I \to X$ such that([11])

$f(x, 0) = x, \forall\ x \in X,$

$f(x, 1) = r(x), \forall\ x \in X,$

$f(a, t) = a, \forall\ a \in A, t \in [0, 1]$.

§3. Main Results

In this paper we discuss the deformation retract of the Schwarzschild space with metric:

$$ds^2 = \left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

Then, the coordinate of Schwarzschild space are given by:

$x_1 = \pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}$

$x_2 = \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}$

$x_3 = \pm\sqrt{c_3 + r^2\theta^2}$

$x_4 = \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}$

where c_1, c_2, c_3 and c_4 are the constant of integration. Now, by using the Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial x_i'}\right) - \frac{\partial T}{\partial x_i} = 0, \quad i = 1, 2, 3, 4,$$

find a geodesic which is a subspace of Schwarzschild space.

Since

$$T = \frac{1}{2}\overline{ds}^2$$

$$T = \frac{1}{2}\left\{\left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

Then, the Lagrangian equations are

$$\frac{d}{ds}\left(\left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 t'\right) = 0 \qquad (7)$$

$$\frac{d}{ds}\left(\left(1-\frac{2m}{r}\right)^{-1}r'\right)-$$
$$\left(128\,\pi^2 m^3 dt^2+\frac{2m}{(2m-r)^2}dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right)=0 \tag{8}$$
$$\frac{d}{ds}\left(r^2\theta'\right)-r^2\sin 2\theta\,d\phi^2=0 \tag{9}$$
$$\frac{d}{ds}\left(r^2\sin^2\theta\ \phi'\right)=0 \tag{10}$$

From equation (7), we obtain

$$\left(1-\frac{2m}{r}\right)64\,\pi^2 m^2 t'=\delta=\text{constant},$$

if $\delta=0$, we have two cases:

(i) $t'=0$, or $t=cons\tan t=\beta$, if $\beta=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1}$

$x_2=\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}$

$x_3=\pm\sqrt{c_3+r^2\theta^2}$

$x_4=\pm\sqrt{c_4+r^2\sin^2\theta\phi^2}$

This is the geodesic hyper spacetime S_1 of the Schwarzchild space$S$, i.e. $dS^2\succ 0$. This is a retraction.

(ii) If $m=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1}$

$x_2=\pm\sqrt{c_2+r^2+8\pi^2\ln(r)}$

$x_3=\pm\sqrt{c_3+r^2\theta^2}$

$x_4=\pm\sqrt{c_4+r^2\sin^2\theta\phi^2}$

This is the geodesic hyper spacetime S_2 of the Schwarzchild space$S$, i.e. $dS^2\succ 0$. This is a retraction.

From equation (8), we obtain

$$r^2\sin^2\theta\ \phi'=\alpha=\text{constant},$$

if $\alpha=0$, we have two cases:

(a) If $\phi'=0$, or $\phi=cons\tan t=\zeta$, if $\zeta=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}$

$x_2=\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}$

$x_3 = \pm\sqrt{c_3 + r^2\theta^2}$

$x_4 = \pm\sqrt{c_4}$

This is the geodesic hyper spacetime S_3 of the Schwarzchild space$S$, i.e. $dS^2 \succ 0$. This is a retraction.

(b) If $\theta = 0$, , we obtain the following coordinates:

$x_1 = \pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}$

$x_2 = \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}$

$x_3 = \pm\sqrt{c_3}$

$x_4 = \pm\sqrt{c_4}$

If $c_3 = c_4 = 0$, then $x_1^2 + x_2^2 + x_3^2 - x_4^2 \succ 0$, which is the great circle S_4in the Schwarzchild spacetime geodesic. These geodesic is a retraction in Schwarzchild space.

Now, we are in a postion to formulate the following theorem.

Theorem 1 *The retraction of Schwarzchild space are spacetime geodesic.*

The deformation retract of the Schwarzchild space is defined by:$\varphi : S \times I \to S$,where S is the Schwarzchild space and I is the closed interval [0, 1]. The retraction of Schwarzchild space$S$ is given by:$R : S \to S_1, S_2, S_3, S_4$.

Then, the deformation retracts of the Schwarzchild space$S$ into a hyper spacetime geodesic $S_1 \subset S$ is given by:

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}, \\ &\pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \pm\sqrt{c_3 + r^2\theta^2}, \\ &\pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\} + t\ \{\pm\sqrt{c_1}, \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \\ &\pm\sqrt{c_3 + r^2\theta^2}, \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\},\end{aligned}$$

where $\varphi(m, 0) = S$ and $\varphi(m, 1) = S_1$.

The deformation retracts of the Schwarzchild space S into a hyper spacetime geodesic $S_2 \subset S$ is given by:

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}, \\ &\pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \pm\sqrt{c_3 + r^2\theta^2}\ \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\} \\ &+t\ \{\pm\sqrt{c_1}, \pm\sqrt{c_2 + r^2 + 8\pi^2 \ln(r)}, \pm\sqrt{c_3 + r^2\theta^2}\ , \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\},\end{aligned}$$

The deformation retracts of the Schwarzchild space S into a hyper spacetime geodesic $S_3 \subset S$ is given by

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4}\}\end{aligned}$$

The deformation retracts of the Schwarzchild space S into a spacetime geodesic $S_4 \subset S$ is given by

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3}, \pm\sqrt{c_4}\}\end{aligned}$$

Now, we are going to discuss the folding f of the Schwarzchild space$S$. Let $f : S \to S$, where $f(x_1, x_2, x_3, x_4) = (|x_1|, x_2, x_3, x_4)$.An isometric folding of the Schwarzchild space$S$ into itself may be defined by

$$\begin{aligned}&f : \{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space$S$ into the folded hyper spacetime geodesic $S_1 \subset S$ is

$$\begin{aligned}&\varphi_f : \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \times I \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \\ &+t\ \{|\pm\sqrt{c_1}|\ , \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \\ &\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_2 \subset S$ is

$$\begin{aligned}&\varphi_f : \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \times I \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm \\ &\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \\ &+t\ \{|\pm\sqrt{c_1}|\ , \pm\sqrt{c_2+r^2+8\pi^2\ln(r)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_3 \subset S$ is

$$\begin{aligned}&\varphi_f : \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \times I \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$
\begin{aligned}
\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\
&\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\ \pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\
&+t\ \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\
&\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4}\}
\end{aligned}
$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_4 \subset S$ is

$$
\begin{aligned}
&\varphi_f:\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\
&\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\times I\to\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\
&c_2+r^2+4mr+8\pi^2\ln(r-2m),\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\
&\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\ \pm\sqrt{c_3+r^2\theta^2},\\
&\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}+t\ \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|\\
&\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\pm\sqrt{c_3},\pm\sqrt{c_4}\}
\end{aligned}
$$

Then the following theorem has been proved.

Theorem 2 *Under the defined folding, the deformation retract of the folded Schwarzchild space into the folded hyper spacetime geodesic is different from the deformation retract of Schwarzchild space into hyper spacetime geodesic.*

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Degree Equitable Sets in a Graph

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called a *Smarandachely degree equitable k-set* for any integer k, $0 \leq k \leq \Delta(G)$ if the degrees of any two vertices in S differ by at most k. It is obvious that $S = V(G)$ if $k = \Delta(G)$. A Smarandachely degree equitable 1-set is usually called a *degree equitable set*. The degree equitable number $D_e(G)$, the lower degree equitable number $d_e(G)$, the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ are defined by

$D_e(G) = \max\{|S| : S$ is a degree equitable set in $G\}$,
$d_e(G) = \min\{|S| : S$ is a maximal degree equitable set in $G\}$,
$D_{ie}(G) = \max\{|S| : S$ is an independent and degree equitable set in $G\}$ and
$d_{ie}(G) = \min\{|S| : S$ is a maximal independent and degree equitable set in $G\}$.

In this paper we initiate a study of these four parameters on Smarandachely degree equitable 1-sets.

Key Words: Smarandachely degree equitable k-set, degree equitable set, degree equitable number, lower, degree equitable number, independent degree equitable number, lower independent degree equitable number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1]. For any graph G, the set $D(G)$ of all distinct degrees of the vertices of G is called the degree set of G. In this paper we introduce four graph theoretic parameters which just depend on the basic concept of vertex degrees. We need the following definitions and theorems.

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Definition 1.1 *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. The corona $G_1 \circ G_2$ is defined to be the graph obtained by taking n_1 copies of G_2 and joining the i^{th} vertex of G_1 to all the vertices of the i^{th} copy of G_2.*

Definition 1.2 *A set S of vertices is said to be an independent set if no two vertices in S are adjacent. The maximum number of vertices in an independent set of a graph G is called the independence number of G and is denoted by $\beta_0(G)$.*

Definition 1.3 *A dominating set S of a graph G is called an independent dominating set of G if S is independent in G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.*

Definition 1.4 *Let $\mathcal{F}$ be a family of nonempty subsets of a set S. The intersection graph $\Omega(\mathcal{F})$ is the graph whose vertex set is $\mathcal{F}$ and two distinct elements $A, B \in \mathcal{F}$ are adjacent in $\Omega(\mathcal{F})$ if $A \cap B \neq \emptyset$.*

Definition 1.5 *A graph G is called a block graph if each block of G is a complete subgraph.*

Definition 1.6 *A split graph is a graph $G = (V, E)$ whose vertices can be partitioned into two sets V' and V'', where the vertices in V' form a complete graph and the vertices in V'' are independent.*

Definition 1.7 *A clique in G is a complete subgraph of G. The maximum order of a clique in G is called the clique number of G and is denoted by $\omega(G)$ or simply ω.*

Theorem 1.8([1], Page 59) *Let T be a non-trivial tree with $\Delta(T) = k$ and let n_i be the number of vertices of degree i in T, $1 \leq i \leq k$. Then $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2$.*

Theorem 1.9([1], Page 130) *Let G be a maximal planar graph of order $n \geq 4$ and let n_i denote the number of vertices of degree i in G, $3 \leq i \leq k = \Delta(G)$. Then $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (k-6)n_k + 12$.*

Theorem 1.10([2]) *Given a graph G and a positive integer $k \leq |V|$, the problem of determining whether G contains an independent set of cardinality at least k is NP-complete even when G is restricted to cubic planar graphs.*

§2. Degree Equitable Sets

In social network theory one studies the relationships that exist on the members of a group. The people in such a group are called actors, relationships among the actors is usually defined in terms of a dichotomous property. A social network graph is a graph in which the vertices represent the actors and an edge between the two actors indicates the property under consideration holds between the corresponding actors. In the social network graph the degree of a vertex v gives a measure of influence the corresponding actor has within the group. Hence identifying the maximum number of actors who have almost equal influence within the group is a significant problem. This motivates the following definition of degree equitable sets.

Definition 2.1 *Let $G = (V, E)$ be a graph. A subset S of V is called a degree equitable set if the degrees of any two vertices in S differ by at most one. The maximum cardinality of a degree equitable set in G is called the degree equitable number of G and is denoted by $D_e(G)$. The minimum cardinality of a maximal degree equitable set in G is called the lower degree equitable number of G and is denoted by $d_e(G)$.*

Observation 2.2 If S is a degree equitable set in G, then any subset of S is degree equitable, so that degree equitableness is a hereditary property. Hence a degree equitable set S is maximal if and only if S is 1-maximal, or equivalently $S \cup \{v\}$ is not a degree equitable set for all $v \in V - S$. Thus a degree equitable set S is maximal if and only if for every $v \in V - S$, there exists $u \in S$ such that $|deg\ u - deg\ v| \geq 2$.

Example 2.3

1. For the complete bipartite graph $K_{r,s}$, we have

$$D_e(K_{r,s}) = \begin{cases} \max\{r, s\} & \text{if } |r - s| \geq 2 \\ r + s & \text{otherwise.} \end{cases}$$

$$d_e(K_{r,s}) = \begin{cases} \min\{r, s\} & \text{if } |r - s| \geq 2 \\ r + s & \text{otherwise.} \end{cases}$$

2. For the wheel W_n on n-vertices, we have

$$D_e(W_n) = \begin{cases} n & \text{if } n = 4 \text{ or } 5 \\ n - 1 & \text{otherwise.} \end{cases}$$

$$d_e(W_n) = \begin{cases} n & \text{if } n = 4 \text{ or } 5 \\ 1 & \text{otherwise.} \end{cases}$$

3. If G is any connected graph, then for the corona $H = G \circ K_1$, $|S_1(H)| \geq |V(G)| = \frac{|V(H)|}{2}$ and hence $D_e(H) = |S_1(H)|$.

Observation 2.4 If G_1 and G_2 are two graphs with same degree sequence, then $D_e(G_1) = D_e(G_2)$ and $d_e(G_1) = d_e(G_2)$. Further a subset S of V is degree equitable in G if and only if it is degree equitable in the complement $\overline{G}$ and hence $D_e(G) = D_e(\overline{G})$ and $d_e(G) = d_e(\overline{G})$.

Observation 2.5 Clearly $1 \leq d_e(G) \leq D_e(G) \leq n$ and $D_e(G) = d_e(G) = n$ if and only if either $D(G) = \{k\}$ or $D(G) = \{k, k+1\}$ for some non-negative integer k. Also $D_e(G) = 1$ if and only if $G = K_1$ and $d_e(G) = 1$ if and only if there exists a vertex $u \in V(G)$ such that $deg\ u = k$ and $|deg\ u - deg\ v| \geq 2$ for all $v \in V - \{u\}$.

Observation 2.6 For any integer i with $\delta \leq i \leq \Delta - 1$, let $S_i = \{v \in V : deg\ v = i \text{ or } i + 1\}$. Clearly a nonempty subset A of V is a maximal degree equitable set if and only if $A = S_i$ for some i. Hence $D_e(G) = \max\{|S_i| : \delta \leq i \leq \Delta - 1\}$ and $d_e(G) = \min\{|S_i| : \delta \leq i \leq \Delta - 1$ and $S_i \neq \emptyset\}$. Since the degrees of the vertices of G and the sets S_i, $\delta \leq i \leq \Delta - 1$, can be determined in linear time, it follows that $D_e(G)$ and $d_e(G)$ can be computed in linear time.

Observation 2.7 Let n and k be positive integers with $k \leq n$. Then there exists a graph G of order n with $d_e(G) = k$. If $k < \frac{n}{2}$, we take G to be the graph obtained from the path $P = (v_1, v_2, \ldots, v_k)$ and the complete graph K_{n-k} by joining v_1 to a vertex of K_{n-k}. If $k \geq \frac{n}{2}$, we take G to be the graph obtained from the cycle C_k by attaching exactly one leaf at $n - k$ vertices of C_k.

Theorem 2.8 *Let G be a non-trivial graph on n vertices. Then $2 \leq D_e(G) \leq n$ and $D_e(G) = 2$ if and only if $G = K_2$ or $\overline{K_2}$.*

Proof The inequalities are trivial.

Suppose $D_e(G) = 2$. Let $D(G) = \{d_1, d_2, \ldots, d_k\}$, where $d_1 < d_2 < \ldots < d_k$. Clearly $k \leq n - 1$ and there exist at most two vertices with degree $d_i, 1 \leq i \leq k$. Let $d_{i_1} \in D(G)$ be such that exactly two vertices have degree d_{i_1}. Since $D_e(G) = 2$, it follows that $d_{i_1} - 1, d_{i_1} + 1 \notin D(G)$ if $i_1 < k$ and $d_k - 1 \notin D(G)$ if $i_1 = k$. Hence by Pigeonhole principle, there exists $d_{i_2} \in D(G) - \{d_{i_1}\}$ such that exactly two vertices of G have degree d_{i_2}. Continuing this process we get for each $d_i \in D(G)$, there exist exactly two vertices with degree d_i and $|d_i - d_j| \geq 2$ if $i \neq j$. Hence the degree sequence of G is given by $\Pi_1 = (1, 1, 3, 3, 5, 5, \ldots, n-1, n-1)$ or $\Pi_2 = (0, 0, 2, 2, 4, 4, \ldots, n-2, n-2)$. Hence it follows that $n = 2$ and $G = K_2$ or $\overline{K_2}$. □

Theorem 2.9 *If a and b are positive integers with $a \leq b$, then there exists a graph G with $d_e(G) = a$ and $D_e(G) = b$, except when $a = 1$ and $b = 2$.*

Proof If $a = b$, then for any regular graph G of order a, we have $d_e(G) = D_e(G) = a$. Hence we assume that $a < b$. If $b \geq a + 2$, then for the graph G consisting of a copy of K_a and a copy of K_b along with a unique edge joining a vertex of K_a to a vertex of K_b, we have $d_e(G) = a$ and $D_e(G) = b$. If $b = a + 1$ and $a > 3$, then for the graph G consisting of the cycle C_a and the complete graph K_b with an edge joining a vertex of C_a to a vertex of K_b, we have $d_e(G) = a$ and $D_e(G) = b$. For the graphs G_1 and G_2 given in Fig.1, we have $d_e(G_1) = 3$ and $D_e(G_1) = 4$ and $d_e(G_2) = 2, D_e(G_2) = 3$. Also it follows from Theorem 2.8 that there is no graph G with $d_e(G) = 1$ and $D_e(G) = 2$. □

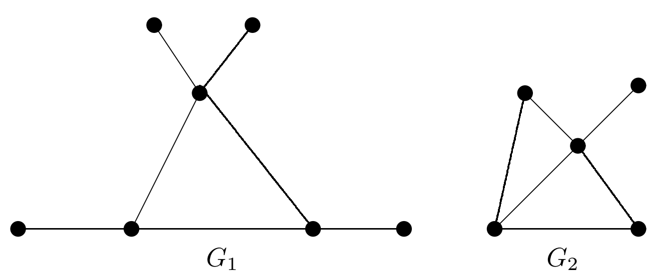

Fig.1

Proposition 2.10 *For a tree T, $D_e(T) = |S_1(T)| = |\{v \in V : deg\ v = 1$ or $2\}|$.*

Proof Let n_i denote the number of vertices of degree i in T where $1 \leq i \leq \Delta$. Clearly $|S_i(T)|$

$= n_i + n_{i+1}$, where $1 \le i \le \Delta - 1$. By Theorem 1.8, $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2$. Hence $|S_1(T)| \ge |S_i(T)| + 2$, for all i, $2 \le i \le \Delta - 1$, so that $D_e(T) = |S_1(T)|$. □

Proposition 2.11 *Let G be a maximal planar graph with $\delta(G) = 5$. Then $D_e(G) = |S_5(G)|$.*

Proof It follows from Theorem 1.9 that $n_5 = n_7 + 2n_8 + 3n_9 + \cdots + (\Delta - 6)n_\Delta + 12$ and hence $D_e(G) = |S_5(G)|$. □

Proposition 2.12 *For any unicyclic graph G with cycle C, $D_e(G) = |S_1(G)|$.*

Proof If $G= C$, then $D_e(G) = |V(G)| = |S_1(G)|$. Suppose $G \ne C$. Let $e = uv$ be any edge of C and let $T = G - e$. It follows from Proposition 2.10 that, $D_e(T) = |S_1(T)|$ and $|S_1(T)| \ge |S_i(T)| + 2$, for all $i = 2, 3, \ldots, \Delta - 1$.

Clearly, $|S_i(T)| - 2 \le |S_i(G)| \le |S_i(T)| + 2$. If $|S_1(T)| = |S_1(G)|$, then $|S_1(G)| = |S_1(T)| \ge |S_i(T)| + 2 \ge |S_i(G)|$, for all $i = 2, 3, \ldots, \Delta - 1$. Suppose $|S_1(G)| \ne |S_1(T)|$. Then the vertices u and v have degree either 2 or 3 and at least one of the vertices have degree 3 in G. Let $deg\ u = k_1$ and $deg\ v = k_2$.

Case 1. $k_1 = 3$ and $k_2 = 2$.

Then $|S_1(G)| = |S_1(T)| - 1$, $|S_2(G)| = |S_2(T)| + 1$, $|S_3(G)| = |S_3(T)| + 1$ and $|S_i(G)| = |S_i(T)|$, for all $i \ge 4$. Hence $|S_1(G)| = |S_1(T)| - 1 \ge |S_i(T)| + 2 - 1 \ge |S_i(T)| + 1 \ge |S_i(G)|$, for all $i = 2, 3, \ldots, \Delta - 1$.

Case 2. $k_1 = k_2 = 3$.

Then $|S_1(G)| = |S_1(T)| - 2, |S_2(G)| = |S_2(T)|, |S_3(G)| = |S_3(T)| + 2$ and $|S_i(G)| = |S_i(T)|$, for all $i \ge 4$. We claim that $|S_1(G)| \ge |S_i(G)|$ for all $i = 2, 3, \ldots, \Delta - 1$. Since $|S_1(G)| = |S_1(T)| - 2$, $|S_1(T)| \ge |S_i(T)| + 2$, for all $i = 2, 3, \ldots, \Delta - 1$ and $|S_i(G)| = |S_i(T)|$ for all $i \ne 3$, it follows that $|S_1(G)| \ge |S_i(G)|$ if $i \ne 3$. We now prove that $|S_1(G)| \ge |S_3(G)|$. Let n_i denote the number of vertices of degree i in G, $1 \le i \le \Delta$. Since G is unicyclic, $n_1 + 2n_2 + 3n_3 + \cdots + \Delta n_\Delta = 2n$. Also $n_1 + n_2 + \cdots + n_\Delta = n$. Hence it follows that $n_1 = n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta$. Since $|S_3(G)| = n_3 + n_4$ it follows that $n_1 > |S_3(G)|$ and hence $|S_1(G)| > |S_3(G)|$. Thus $|S_1(G)| \ge |S_i(G)|$ for all $i = 2, 3, \ldots, \Delta - 1$ and hence $D_e(G) = |S_1(G)|$. □

The study of the effect of the removal of a vertex or an edge on any graph theoretic parameter has interesting applications in the context of a network since the removal of a vertex can be interpreted as a faulty component in the network, and the removal of an edge can be interpreted as the failure of a link joining two elements of the network.

We now proceed to investigate the effect of the removal of a vertex on $D_e(G)$.

Observation 2.13 On the removal of a vertex, $D_e(G)$ may increase arbitrarily or decrease arbitrarily or remain unaltered. For the complete bipartite graph $G = K_{r,r+2}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+2}\}$, $D_e(G) = r + 2$ and

$$D_e(G - v) = \begin{cases} 2r + 1 & \text{if } v \in Y \\ r + 2 & \text{if } v \in X. \end{cases}$$

Also for the graph $G = K_{r,r+1}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+1}\}$, $D_e(G) = 2r+1$ and $D_e(G-v) = r+1$ for all $v \in X$.

Hence the vertex set of G can be partitioned into three sets (not necessarily nonempty) as follows.

$V^0 = \{v \in V : D_e(G) = D_e(G-v)\}$,

$V^+ = \{v \in V : D_e(G) < D_e(G-v)\}$ and

$V^- = \{v \in V : D_e(G) > D_e(G-v)\}$.

Example 2.14

1. For any regular graph G, we have $V = V^-$ and $V^0 = V^+ = \emptyset$.

2. There exist graphs for which all the sets V^0, V^+ and V^- are nonempty. For the graph G given in Fig.2, $D_e(G) = 6$, $V^0 = \{6, 5, 3, 2, 1\}$, $V^+ = \{4\}$ and $V^- = \{8, 9, 7\}$.

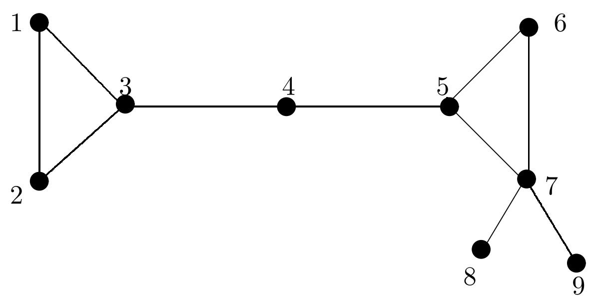

Fig.2 G

We now proceed to determine the sets V^0, V^+ and V^- for trees and unicyclic graphs. We need the following lemma.

Lemma 2.15 *Let G be a disconnected graph in which every component is either a tree or a unicyclic graph. Then $D_e(G) = \max\{|S_0(G)|, |S_1(G)|\}$.*

Proof Let d_0, d_1 and d_2 denote respectively the number of vertices of degree zero, one and two in G. Then $|S_0(G)| = d_0 + d_1$ and $|S_1(G)| = d_1 + d_2$. Hence $|S_0(G)| \geq |S_1(G)|$ if $d_0 \geq d_2$ and $|S_0(G)| < |S_1(G)|$ if $d_0 < d_2$. Also it follows from Proposition 2.10 and Proposition 2.12 that $|S_1(G)| \geq |S_i(G)|$ for all $i \geq 2$. Hence $D_e(G) = \max\{|S_0(G)|, |S_1(G)|\}$. □

Theorem 2.16 *Let G be a tree or a unicyclic graph and let $v \in V(G)$. Let $N(v) = \{w_1, w_2, \ldots, w_k\}$. Let k_1, k_2 and k_3 denote respectively the number of vertices in $N(v)$ with degrees 1,2 and 3 respectively. Let m_2 denote the number of vertices of degree 2 in G.*

(a) If deg $v = 1$, then $v \in V^0$ if and only if deg $w_1 = 3$ and $v \in V^-$ otherwise.

(b) If deg $v = 2$, then $v \in V^+$ if deg $w_1 =$ deg $w_2 = 3$, $v \in V^0$ if deg $w_1 = 2$ and deg $w_2 = 3$ or deg $w_1 = 3$ and deg $w_2 \geq 4$ and in all other cases $v \in V^-$.

(c) If deg $v \geq 3$, then $v \in V^-$ if $m_2 > k_2$ and $k_1 > k_3$, $v \in V^+$ if $k_3 > k_1$ and in all other cases $v \in V^0$.

Proof We prove the theorem for a tree T. The proof for unicyclic graphs is similar.

a) Suppose $deg\ v = 1$. Then $T_1 = T - v$ is also a tree. Further,

$$|S_1(T_1)| = \begin{cases} |S_1(T)| & \text{if } deg\ w_1 = 3 \\ |S_1(T)| - 1 & \text{otherwise} \end{cases}$$

Hence it follows from Proposition 2.10 that $v \in V^0$ if and only if $deg\ w_1 = 3$ and $v \in V^-$ otherwise.

b) Let $deg\ v = 2$. Then $F = T - v$ is a forest with two components T_1 and T_2.

If $deg\ w_1 = deg\ w_2 = 3$, then $|S_1(F)| = |S_1(T)| + 1$. Also by Lemma 2.15, $D_e(F) = |S_1(F)| > |S_1(T)| = D_e(T)$. Hence $v \in V^+$.

If $deg\ w_1 = 2$ and $deg\ w_2 = 3$ or $deg\ w_1 = 3$ and $deg\ w_2 \geq 4$, then $|S_1(F)| = |S_1(T)|$. Hence $D_e(F) = |S_1(F)| = |S_1(T)| = D_e(T)$, so that $v \in V^0$.

If $deg\ w_1 = deg\ w_2 = 1$, then $T = K_{1,2}$ and hence $D_e(F) = 2$ and $D_e(T) = 3$. If $deg\ w_1 = 1$ and $deg\ w_2 = 2$, then $|S_0(F)| = k_1 < |S_1(T)|$ and $|S_1(F)| = |S_1(T)| - 2$. Hence $D_e(F) = \max\{|S_0(F)|, |S_1(F)|\} < D_e(T)$. If $deg\ w_1 = deg\ w_2 = 2$, then $D_e(F) = |S_1(F)| = |S_1(T)| - 1$. If $deg\ w_1 = 2$ and $deg\ w_2 \geq 4$ or if $deg\ w_1 \geq 4$ and $deg\ w_2 \geq 4$, then $D_e(F) = |S_1(F)| = |S_1(T)| - 1 = D_e(T) - 1$. Hence in all cases $D_e(F) < D_e(T)$, so that $v \in V^-$.

c) Let $deg\ v \geq 3$.

In this case F is a forest with k components, where $k = deg\ v$. Then $|S_0(F)| = |S_1(T)| - m_2 + k_2$ and $|S_1(F)| = |S_1(T)| - k_1 + k_3$. Since $m_2 \geq k_2$, we have $|S_0(F)| \leq |S_1(T)|$. Now, if $m_2 > k_2$ and $k_1 > k_3$ then $|S_0(F)| < |S_1(T)|$ and $|S_1(F)| < |S_1(T)|$. Hence $D_e(F) < D_e(T)$ so that $v \in V^-$. If $k_1 < k_3$, then $|S_1(F)| > |S_1(T)|$. Hence $D_e(F) = \max\{|S_0(F)|, |S_1(F)|\} = |S_1(F)| > |S_1(T)| > D_e(T)$, so that $v \in V^+$. If $m_2 = k_2$ and $k_1 > k_3$, then $|S_0(F)| = |S_1(T)|$ and $|S_1(F)| < |S_1(T)|$. If $k_1 = k_3$ then $|S_1(F)| = |S_1(T)|$. Thus in both cases, $D_e(F) = |S_1(T)| = D_e(T)$ and hence $v \in V^0$. □

We now proceed to investigate the effect of the removal of an edge on $D_e(G)$. Let $e = uv \in E(G)$ and let $H = G - e$. Since $d_H(u) = d_G(u) - 1$, $d_H(v) = d_G(v) - 1$ and $d_H(w) = d_G(w)$, for all $w \in V - \{u, v\}$, it follows that $D_e(G) - 2 \leq D_e(G - e) \leq D_e(G) + 2$. Hence the edge set of G can be partitioned into five subsets as follows.

$E^{-2} = \{e \in E : D_e(G) = D_e(G - e) + 2\}$,

$E^{-1} = \{e \in E : D_e(G) = D_e(G - e) + 1\}$,

$E^0 = \{e \in E : D_e(G) = D_e(G - e)\}$,

$E^1 = \{e \in E : D_e(G) = D_e(G - e) - 1\}$ and

$E^2 = \{e \in E : D_e(G) = D_e(G - e) - 2\}$.

The following examples illustrate that all five types of edges can exist.

Example 2.17

1. For the graph G_1 given in Fig.3, $D_e(G_1) = 5, D_e(G_1 - e_1) = 4, D_e(G_1 - e_2) = 3$ and $D_e(G_1 - e_3) = 5$. Hence $e_3 \in E^0, e_1 \in E^{-1}$ and $e_2 \in E^{-2}$.

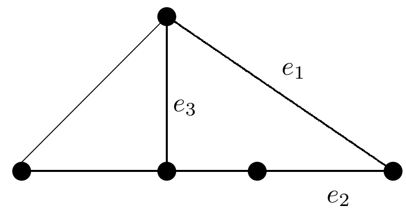

Fig.3 G_1

2. For the graph G_2 given in Fig.4, $D_e(G_2) = 4$ and $D_e(G_2 - e) = 6$, so that $e \in E^2$.

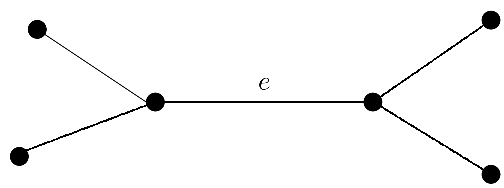

Fig.4 G_2

3. For the graph G_3 given in Fig.5, $D_e(G_3) = 6$ and $D_e(G_3 - e) = 7$, so that $e \in E^1$.

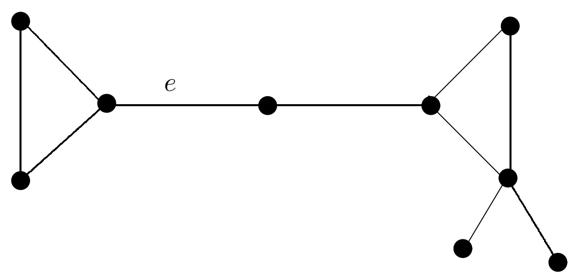

Fig.5 G_3

Theorem 2.18 *Let $T \neq K_2$ be a tree and let $e = uv$ be an edge of T.*

(a) If either u or v is a leaf, then $e \in E^0$ if T has no vertex of degree 2 and $e \in E^{-1}$ otherwise.

(b) If deg $u \geq 4$ and deg $v \geq 4$, then $e \in E^0$.

(c) If deg $u \geq 4$ and deg $v = 2$, then $e \in E^0$.

(d) If deg $u \geq 4$ and deg $v = 3$, then $e \in E^1$.

(e) If deg $u =$ deg $v = 3$, then $e \in E^2$.

(f) If deg $u = 3$ and deg $v = 2$, then $e \in E^1$.

(g) If deg $u =$ deg $v = 2$, then $e \in E^0$.

Proof Let $T_1 = T - uv$. Clearly T_1 is a forest with exactly two components. Suppose v is a leaf. If $deg\ u = 3$ and T has no vertex of degree 2, then $S_0(T_1) \subseteq S_1(T)$ and $|S_1(T_1)| = |S_1(T)|$. Hence $D_e(T_1) = D_e(T)$, so that $e \in E^0$. If T has a vertex of degree 2, then $S_0(T_1) \subsetneq S_1(T)$ and $|S_1(T_1)| = |S_1(T)| - 1$, so that $D_e(T_1) = D_e(T) - 1$ and $e \in E^{-1}$.

Now, suppose $deg\ u \geq 2$ and $deg\ v \geq 2$, so that $|S_0(T_1)| \leq |S_1(T)|$. Now if (b) or (c) holds then $|S_1(T_1)| = |S_1(T)|$. If (d) or (f) holds then $|S_1(T_1)| = |S_1(T)| + 1$ and if (e) holds, $|S_1(T_1)| = |S_1(T)| + 2$. Hence the result follows. □

We now consider the effect of removal of a vertex or an edge on the lower degree equitable number $d_e(G)$.

Observation 2.19 On the removal of a vertex, $d_e(G)$ may increase arbitrarily or decrease arbitrarily or remain unaltered. For the complete bipartite graph $G = K_{r,r+2}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+2}\}$, $d_e(G) = r$ and

$$d_e(G - v) = \begin{cases} 2r + 1 & \text{if } v \in Y \\ r - 1 & \text{if } v \in X. \end{cases}$$

This shows that $d_e(G)$ may increase arbitrarily on vertex removal.

Also for the bistar $G = B(n_1, n_2)$ with $|n_1 - n_2| = 1$, $d_e(G) = 2$ and $d_e(G - v) = 2$, where v is any leaf of G, so that $d_e(G)$ remains unaltered.

The following example shows that $d_e(G)$ may decrease arbitrarily on vertex removal. Let G_1 be a 4-regular graph on n_1 vertices and let G_2 be a 6-regular graph on n_2 vertices where $n_1 < n_2$. Let G_3 be a $n_1 + 1$-regular graph on n_3 vertices where $n_3 > n_1 + n_2$. Let G be the graph obtained from G_1, G_2 and G_3 as follows.

Add a new vertex v and join it to all vertices of G_1. Remove two disjoint edges x_1y_1 and x_2y_2 from G_3 and remove an edge x_3y_3 from G_2 and add the edges vx_1, vy_1, x_3x_2 and y_3y_2. Clearly $D(G) = (5, 6, n_1 + 1, n_1 + 2)$. Also $|S_5| = n_1 + n_2$ and $|S_{n_1+1}| = n_3$. Since $n_3 > n_1 + n_2$ it follows that $d_e(G) = n_1 + n_2$. Now, $D(G - \{v\}) = \{4, 6, n_1, n_1 + 1\}$. Also $|S_4| = n_1, |S_6| = n_2$ and $|S_{n_1}| = n_3$. Hence $d_e(G - v) = n_1$.

Theorem 2.20 *Given a positive integer k, there exist graphs G_1 and G_2 such that $d_e(G_1) - d_e(G_1 - e) = k$ and $d_e(G_2 - e) - d_e(G_2) = k$.*

Proof Let $G_1 = P_{k+3} = (v_1, v_2, \ldots, v_{k+3})$. Then $d_e(G_1) = k + 3$ and $d_e(G_1 - v_1v_2) = 3$ and hence $d_e(G_1) - d_e(G_1 - e) = k$. Let H be the complete bipartite graph, $K_{k+4,k+8}$ with bipartition $X = \{x_1, x_2, \ldots, x_{k+4}\}$ and $Y = \{y_1, y_2, \ldots, y_{k+8}\}$. Let G_2 be the graph obtained from H by adding the edges y_1y_2, y_2y_3, y_3y_4. Then $d_e(G_2) = 4$, $d_e(G_2 - y_2y_3) = k + 4$ and hence $d_e(G_2 - e) - d_e(G_2) = k$. □

Hence for $d_e(G)$ the vertex set $V(G)$ and the edge set $E(G)$ can be partitioned into subsets V_0, V_+, V_- and E_0, E_+, E_- as follows.

$V_- = \{v \in V : d_e(G) > d_e(G - v)\}$,

$V_0 = \{v \in V : d_e(G) = d_e(G - v)\}$,

$V_+ = \{v \in V : d_e(G) < d_e(G - v)\}$,

$E_- = \{e \in E : d_e(G) > d_e(G - e)\}$,

$E_0 = \{e \in E : d_e(G) = d_e(G - e)\}$ and

$E_+ = \{e \in E : d_e(G) < d_e(G - e)\}$.

Example 2.21

1. For the complete graph K_n, we have $V = V_-$.
2. For the corona of the cycle $C_n \circ K_1$, we have $V = V_0$.
3. For the graph $G = K_{1,3}$, we have $V = V_+$ and $E = E_+$.
4. For any regular graph G we have $E = E_0$.
5. For the graph G given in Fig.6, we have $d_e(G) = 12$ and $d_e(G-e) = 10$ for every $e \in E(G)$ and hence $E = E_-$.

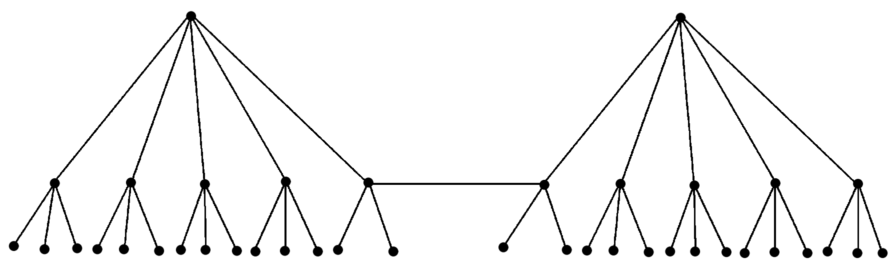

Fig.6 G

The following are some interesting problems for further investigation.

Problem 2.22

1. *Characterize graphs for which* $V = V_-$.
2. *Characterize graphs for which* $V = V_0$.
3. *Characterize graphs for which* $V = V_+$.
4. *Characterize graphs for which* $E = E_-$.
5. *Characterize graphs for which* $E = E_0$.
6. *Characterize graphs for which* $E = E_+$.
7. *Characterize vertices and edges in different classes.*

§3. Independent Degree Equitable Sets

In this section we consider subsets which are both degree equitable and independent. We introduce the concepts of independent degree equitable number and the lower independent degree equitable number, and present some basic results on these parameters.

Definition 3.1 *The independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ of a graph G are defined by $D_{ie}(G) = \max\{|S| : S \subseteq V$ and S is an independent and degree equitable set in $G\}$ and $d_{ie}(G) = \min\{|S| : S$ is a maximal independent and degree equitable set in $G\}$.*

Example 3.2

1. For the complete bipartite graph $K_{m,n}$, we have $D_{ie}(K_{m,n}) = \max\{m, n\}$ and $d_{ie}(G) = \min\{m, n\}$.
2. For the wheel W_n on n-vertices, we have $D_{ie}(W_n) = \beta_0(C_{n-1})$ and $d_{ie}(W_n) = 1$.

Observation 3.3 Let $H_i = \langle S_i \rangle$. Then $D_{ie}(G) = max\{\beta_o(H_i) : \delta \leq i \leq \Delta - 1\}$ and $d_{ie}(G) = min\{i(H_i) : \delta \leq i \leq \Delta - 1\}$.

Observation 3.4 For any graph G, we have $d_{ie}(G) \leq D_{ie}(G) \leq \beta_0(G)$. Also, since $D_{ie}(K_{a,b}) = \max\{a, b\}$ and $d_{ie}(K_{a,b}) = \min\{a, b\}$, the difference between the parameters $D_{ie}(G)$ and $d_{ie}(G)$ can be made arbitrarily large.

Observation 3.5 For any regular graph, we have $D_{ie}(G) = \beta_0(G)$. By Theorem 1.10 the computation of $\beta_0(G)$ is NP-complete even for cubic planar graphs. Hence it follows that the computation of $D_{ie}(G)$ is NP-complete.

Observation 3.6 The difference between $\beta_0(G)$ and $D_{ie}(G)$ can also be made arbitrarily large. If $G_i = K_{2i+1}$, $i = 1, 2, \ldots, k+1$ and G is the graph obtained from $G_1, G_2, \ldots, G_{k+1}$ by joining a vertex of G_i to a vertex of G_{i+1},where $1 \leq i \leq k$, then $D_{ie}(G) = 1$ and $\beta_0(G) = k+1$. Hence $\beta_0(G) - D_{ie}(G) = k$.

Observation 3.7 For any connected graph G, $D_{ie}(G) = n - 1$ if and only if $G \cong K_{1,n-1}$.

Observation 3.8 Let G be a graph with $\beta_0(G) = n-2$. If A is any β_0-set in G, then $deg\ v = 1$ or 2 for all $v \in A$ and hence $D_{ie}(G) = \beta_0(G) = n - 2$.

Proposition 3.9 *For any connected graph G, $d_{ie}(G) = 1$ if and only if either $\Delta = n - 1$ or for any two nonadjacent vertices $u, v \in V(G)$, $|deg\ u - deg\ v| \geq 2$.*

Proof Suppose $d_{ie}(G) = 1$ and $\Delta(G) < n-1$. Let u and v be any two nonadjacent vertices in G. Since $d_{ie}(G) = 1$, $\{u, v\}$ is not a degree equitable set and hence $|deg\ u - deg\ v| \geq 2$. The converse is obvious. □

Proposition 3.10 *For any connected graph G, $D_{ie}(G) = 1$ if and only if $G \cong K_n$ or for any two nonadjacent vertices $u, v \in V(G)$, $|deg\ u - deg\ v| \geq 2$.*

Proof Suppose $D_{ie}(G) = 1$. If $G \neq K_n$, let u and v be any two nonadjacent vertices in G. Since $D_{ie}(G) = 1$, $\{u, v\}$ is not a degree equitable set and hence $|deg\ u - deg\ v| \geq 2$. The converse is obvious. □

Observation 3.11 Every independent set of a graph G is degree equitable if and only if for any two nonadjacent vertices u and v, $|deg\ u - deg\ v| \leq 1$.

Theorem 3.12 *In a tree T every independent set is degree equitable if and only if T is a star or a path.*

Proof Let T be a tree. Suppose every independent set in T is degree equitable. If all the vertices of T are of degree 1 or 2 then T is a path. If there exists a vertex v with $deg\ v > 2$, then all the leaves of T are adjacent to v and hence T is a star. The converse is obvious. □

Theorem 3.13 *Let G be a unicyclic graph with cycle C. Then every independent set of G is degree equitable if and only if $G = C$ or the graph obtained from the cycle C_3 by attaching at least one leaf at a vertex or the graph obtained from a cycle $C_k, k \geq 4$ by attaching exactly one leaf at a vertex.*

Proof Let G be a unicyclic graph with cycle C. Suppose every independent set in G is degree equitable. If all the vertices of G are of degree 2, then $G = C$. Suppose there exists a vertex v on C with $deg\ v > 2$. Then $\delta = 1$. If there exists a leaf w which is not adjacent to v, then $\{w, v\}$ is an independent set in G and is not degree equitable, which is a contradiction. Thus every leaf of G is adjacent to v and hence all the vertices of C other than v are of degree 2. Also if the length of the cycle C is at least 4 and $deg\ v \geq 4$, then $\{v, x\}$ where x is any vertex on C which is not adjacent to v is an independent set which is not degree equitable. Hence G is isomorphic to one of the graphs given in the theorem. The converse is obvious. □

We now consider the effect of removal of a vertex or an edge on the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$.

Observation 3.14

1. For the complete graph K_n, $D_{ie}(K_n) = d_{ie}(K_n) = 1$ and $D_{ie}(K_n - v) = d_{ie}(K_n - v) = 1$ for all $v \in V(K_n)$.

2. For the wheel $G = W_n$ on n vertices, we have $d_{ie}(G) = 1$. Further if v is the central vertex of G, then $G - v$ is the cycle C_{n-1} and hence $d_{ie}(G - v) = \left\lfloor \frac{n-1}{2} \right\rfloor$. This shows that the lower independent degree equitable number may increase arbitrarily on vertex removal.

3. For the graph G obtained from a copy of K_5 and a copy K_6 by joining a vertex u of K_5 with a vertex v of K_6, we have $d_{ie}(G) = D_{ie}(G) = 2$. Also $d_{ie}(G - w) = D_{ie}(G - w) = 1$ for any $w \in V(K_5) - \{u\}$.

4. For the graph G obtained from a copy of K_5 and a copy of K_7 by joining a vertex u of K_5 with a vertex v of K_7, we have $d_{ie}(G) = D_{ie}(G) = 1$. Also $D_{ie}(G - w) = d_{ie}(G - w) = 2$ for any $w \in V(K_7) - \{v\}$.

Thus the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ may increase or decrease or remain same on removal of a vertex. Hence the vertex set $V(G)$ can be partitioned into subsets as follows.

$V^{(-)} = \{v \in V : D_{ie}(G) > D_{ie}(G - v)\}$,

$V^{(0)} = \{v \in V : D_{ie}(G) = D_{ie}(G - v)\}$,

$V^{(+)} = \{v \in V : D_{ie}(G) < D_{ie}(G - v)\}$,
$V_{(-)} = \{v \in V : d_{ie}(G) > d_{ie}(G - v)\}$,
$V_{(0)} = \{v \in V : d_{ie}(G) = d_{ie}(G - v)\}$ and
$V_{(+)} = \{v \in V : d_{ie}(G) < d_{ie}(G - v)\}$.

The following theorem shows that on removal of an edge, $D_{ie}(G)$ can decrease by at most 1 and increase by at most 2.

Theorem 3.15 *Let G be a graph. Let $e = uv \in E(G)$. Then $D_{ie}(G) - 1 \leq D_{ie}(G - e) \leq D_{ie}(G) + 2$.*

Proof Let S be an independent degree equitable set in G with $|S| = D_{ie}(G)$. Then at most one of the vertices u, v belong to S. If $u \notin S$ and $v \notin S$, then S is an independent degree equitable set in $G - e$ and if $u \in S$, $v \notin S$, then $S - \{u\}$ is an independent degree equitable set in $G - e$. Hence $D_{ie}(G - e) \geq D_{ie}(G) - 1$.

Now, let S be an independent degree equitable set in $G - e$ with $|S| = D_{ie}(G - e)$. If both u and v are in S, then $S - \{u, v\}$ is an independent degree equitable set in G. If $u \in S$ and $v \notin S$, then $S - \{u\}$ is an independent degree equitable set in G. If both u and v are not in S, then S is an independent degree equitable set in G. Hence $D_{ie}(G) \geq D_{ie}(G - e) - 2$. □

Observation 3.16

1. For the complete graph K_n, $n \geq 3$, we have $D_{ie}(G) = d_{ie}(K_n) = 1$ and $D_{ie}(K_n - e) = 2$, $d_{ie}(K_n - e) = 1$ for any edge $e \in E(K_n)$.

2. For the path $P_n = (v_1, v_2, \ldots, v_n)$ we have $D_{ie}(P_n) = d_{ie}(P_n) = \lceil \frac{n}{2} \rceil$. Also $d_{ie}(P_n - v_1v_2) = 3$ and

$$D_{ie}(P_n - v_1v_2) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \\ \\ \lceil \frac{n}{2} \rceil + 1 & \text{if } n \text{ is even.} \end{cases}$$

3. For the corona $G = K_3 \circ K_1$, we have $d_{ie}(G) = 1$ and $d_{ie}(G - e) = 2$ for any edge $e \in E(K_3)$.

4. For the graph G given in Fig.7, $D_{ie}(G) = 6$ and $D_{ie}(G - e) = 5$.

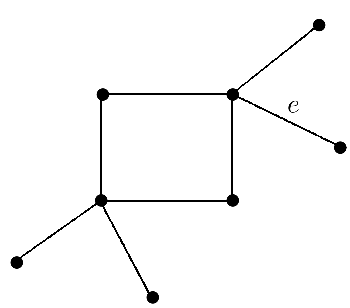

Fig.7 G

5. Let H be a split graph with split partition X, Y such that X is independent, $\langle Y \rangle$ is complete, $|Y| \geq |X| + 3$ and $D(X) = \{|Y| - 4, |Y| - 3\}$ and at least two vertices in X have degree $|Y| - 3$. Let $G = H + uv$ where $u, v \in X$ and $deg\ u = deg\ v = |Y| - 3$. Then $D_{ie}(G) = |X| - 2$ and $D_{ie}(G - uv) = D_{ie}(H) = |X|$.

Thus the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ may increase or decrease or remain same on removal of an edge. Hence for $D_{ie}(G)$ the edge set $E(G)$ can be partitioned into 4 subsets as follows.

$E^{(-1)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) - 1\}$,

$E^{(0)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G)\}$,

$E^{(1)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) + 1\}$ and

$E^{(2)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) + 2\}$.

Hence for $d_{ie}(G)$ the edge set $E(G)$ can be partitioned into 3 subsets as follows.

$E_{(-)} = \{e \in E : d_{ie}(G) > d_{ie}(G - e)\}$,

$E_{(0)} = \{e \in E : d_{ie}(G) = d_{ie}(G - e)\}$ and

$E_{(+)} = \{e \in E : d_{ie}(G) < d_{ie}(G - e)\}$.

Example 3.17

1. *For the complete graph K_n where $n \geq 3$, we have $V = V_{(0)} = V^{(0)}$ and $E = E_{(0)}$.*
2. *For the complete graph K_2, we have $E = E_{(+)}$.*
3. *For any odd cycle C_{2n+1} where $n \geq 2$, we have $E = E^{(1)}$.*
4. *For any even cycle C_{2n}, we have $E = E^{(0)}$.*

Problem 3.18

1. *Characterize graphs for which $V = V^{(0)}$.*
2. *Characterize graphs for which $V = V_{(0)}$.*
3. *Characterize graphs for which $E = E^{(0)}$.*
4. *Characterize graphs for which $E = E_{(0)}$.*
5. *Characterize graphs for which $E = E^{(1)}$.*
6. *Characterize graphs for which $E = E_{(+)}$.*
7. *Characterize vertices and edges in different classes.*

§4. Degree Equitable Graphs

Given a graph $G = (V, E)$, we define another graph G^{de} using the concept of degree equitableness and present some basic results.

Definition 4.1 *Let $G = (V, E)$ be a graph. The degree equitable graph of G, denoted by G^{de} is defined as follows.*

$V(G^{de}) = V(G)$ and two vertices u and v are adjacent in G^{de} if and only if $|deg\ u - deg\ v| \leq 1$.

Observation 4.2 For any maximal degree equitable set S_i in G, the induced subgraph $\langle S_i \rangle$ of G^{de} is a clique in G^{de} and hence it follows that the clique number $\omega(G^{de})$ is equal to the degree equitable number $D_e(G)$.

Theorem 4.3 *Let G be any graph. Then the number of edges in G^{de} is given by $\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i \cap S_{i+1}|}{2}$.*

Proof Each $\langle S_i \rangle$ is complete in G^{de} and hence the subgraph $\langle S_i \rangle$ has $\binom{|S_i|}{2}$ edges. Also the edges in the subgraph $\langle S_{i+1} \cap S_i \rangle$ are counted twice in $\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2}$. Hence the number of edges in $G^{de} = \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i \cap S_{i+1}|}{2}$. □

Theorem 4.4 *Let G be any graph. Then the following are equivalent.*

(i) G^{de} is connected.

(ii) $D(G) = \{\delta, \delta+1, \ldots, \Delta\}$.

(iii) The intersection graph H of the set of all maximal degree equitable sets of G is a path.

Proof Suppose G^{de} is connected. If there exists an integer i such that $i, i+2 \in D(G)$ and $i+1 \notin D(G)$, then $S_i \cap S_{i+1} = \emptyset$ and no edge in G^{de} joins a vertex of S_i and a vertex of S_{i+1}. Now, $V_1 = S_\delta \cup S_{\delta+1} \cup \cdots \cup S_i$ and $V_2 = S_{i+1} \cup \cdots \cup S_{\Delta-1}$ forms a partition of V and no edge of G^{de} joins a vertex of V_1 and a vertex of V_2. Hence G^{de} is disconnected, which is a contradiction. Hence $D(G) = \{\delta, \delta+1, \ldots, \Delta\}$, so that (i) implies (ii).

Now, if $D(G) = \{\delta, \delta+1, \ldots, \Delta\}$, then $S_i \cap S_{i+1} \neq \emptyset$ and $S_i \cap S_j = \emptyset$ if $|i - j| \geq 2$. Hence H is a path, so that (ii) implies (iii).

Now, suppose H is a path. Then $S_i \cap S_{i+1} \neq \emptyset$ and since $\langle S_i \rangle$ is a complete graph in G^{de}, it follows that G^{de} is connected. Thus (iii) implies (i). □

Theorem 4.5 *Let G be a connected graph. Then G^{de} is a connected block graph if and only if $|S_i \cap S_{i+1}| = 1$ for every i, $\delta \leq i \leq \Delta - 1$.*

proof The induced subgraph $\langle S_i \rangle$ of G^{de} is complete and each $\langle S_i \rangle$ is a block in G^{de} if and only if $|S_i \cap S_{i+1}| = 1$. □

Definition 4.6 *A graph H is called a degree equitable graph if there exists a graph G such that H is isomorphic to G^{de}.*

Example 4.7 Any complete graph K_n is a degree equitable graph, since $K_n = G^{de}$ for any regular graph G.

Theorem 4.8 *Any triangle free graph H is not a degree equitable graph.*

Proof Suppose $H = G^{de}$ for some graph G. Then $D_e(G) = 2$. Hence it follows from Theorem 2.8 that $G = K_2$ or $\overline{K_2}$, which is a contradiction. Hence any triangle free graph is not a degree equitable graph. □

Problem 4.9 *Characterize degree equitable graphs.*

Conclusion and Scope. In this paper we have introduced the concept of degree equitable sets. The concept of degree equitableness can be combined with any other graph theoretic property concerning subsets of V. For example one can consider concepts such as degree equitable dominating sets or degree equitable connected sets and study the existence of such sets in graphs.

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Smarandachely k-Constrained Number of Paths and Cycles

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Abstract: A *Smarandachely k-constrained labeling* of a graph $G(V, E)$ is a bijective mapping $f : V \cup E \to \{1, 2, .., |V| + |E|\}$ with the additional conditions that $|f(u) - f(v)| \geq k$ whenever $uv \in E$, $|f(u)-f(uv)| \geq k$ and $|f(uv)-f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a *Smarandachely k-constrained total graph*, abbreviated as $k - CTG$. The minimum number of isolated vertices required for a given graph G to make the resultant graph a $k - CTG$ is called the *k-constrained number* of the graph G and is denoted by $t_k(G)$. In this paper we settle the open problems 3.4 and 3.6 in [4] by showing that $t_k(P_n) = 0$, if $k \leq k_0$; $2(k - k_0)$, if $k > k_0$ and $2n \equiv 1$ or 2 (mod 3); $2(k - k_0) - 1$ if $k > k_0$; $2n \equiv 0$(mod 3) and $t_k(C_n) = 0$, if $k \leq k_0$; $2(k - k_0)$, if $k > k_0$ and $2n \equiv 0$ (mod 3); $3(k - k_0)$ if $k > k_0$ and $2n \equiv 1$ or 2 (mod 3), where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$.

Key Words: Smarandachely k-constrained labeling, Smarandachely k-constrained total graph, k-constrained number, minimal k-constrained total labeling.

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§1. Introduction

All the graphs considered in this paper are simple, finite and undirected. For standard terminology and notations we refer [1], [3]. There are several types of graph labelings studied by various authors. We refer [2] for the entire survey on graph labeling. In [4], one such labeling called Smarandachely labeling is introduced. Let $G = (V, E)$ be a graph. A bijective mapping $f : V \cup E \to \{1, 2, ..., |V| + |E|\}$ is called a *Smarandachely* $k - constrained$ *labeling* of G if it satisfies the following conditions for every $u, v, w \in V$ and $k \geq 2$;

1. $|f(u) - f(v)| \geq k$
2. $|f(u) - f(uv)| \geq k$,

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3. $|f(uv) - f(vw)| \geq k$

whenever $uv, vw \in E$ and $u \neq w$.

A graph G which admits a such labeling is called a *Smarandachely k-constrained total graph*, abbreviated as $k - CTG$. The minimum number of isolated vertices to be included for a graph G to make the resultant graph is a $k - CTG$ is called *k-constrained number of the graph* G and is denoted by $t_k(G)$, the corresponding labeling is called a *minimal k-constrained total labeling* of G.

We recall the following open problems from [4], for immediate reference.

Problem 1.1 *For any integers $n, k \geq 3$, determine the value of $t_k(P_n)$.*

Problem 1.2 *For any integers $n, k \geq 3$, determine the value of $t_k(C_n)$.*

§2. k-Constrained Number of a Path

Let $V(P_n) = \{v_1, v_2, . . . , v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. Designate the vertex v_i of P_n as $2i-1$ and the edge $v_j v_{j+1}$ as $2j$, for each $i, 1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Lemma 2.1 *Let $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ and $S_l = \{3l-2, 3l-1, 3l\}$ for $1 \leq l \leq k_0$. Let f be a minimal k-constrained total labeling of P_n. Then for each $i, 1 \leq i \leq k_0$, there exist a $l, 1 \leq l \leq k_0$ and a $x \in S_l$ such that $f(x) = i$.*

Proof For $1 \leq l \leq k_0$, let S_l=$\{l_1, l_2, l_3\}$, where $l_1 = 3l-2, l_2 = 3l-1, l_3 = 3l$. Let $S = \{1, 2, 3, ..., k_0\}$ and f be a minimal k-constrained total labeling of $P_n, 2n \equiv 0 \pmod 3$ and $k > k_0$, then by the definition of f it follows that $|f(S_i) \cap S| \leq 1$, for each $i, 1 \leq i \leq k_0 + 1$, otherwise if $f(l_i), f(l_j) \in S$ for $1 \leq i, j \leq 3, i \neq j$, then $|f(l_i) - f(l_j)| < k_0 < k$, a contradiction. Further, if $f(l_j) \neq i$ for any l, j with $1 \leq l \leq k_0, 1 \leq j \leq 3$ for some $i \in S$, then i should be assigned to an isolated vertex. So, span of f will increase, hence f can not be minimal. □

Lemma 2.2 *Let $S_l = \{3l-2, 3l-1, 3l\}$ and f be a minimal k-constrained total labeling of P_n. Let $f(x) = s_1$ and $f(y) = s_2$ for some $x \in S_l$ and $y \in S_{l+1}$ for some $l, 1 \leq l < m \leq k_0$ and $1 \leq s_1, s_2 \leq k_0$, where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then $y = x + 3$.*

Proof Let x_1, x_2, x_3 be the elements of S_l and x_4, x_5, x_6 be that of S_{l+1} (i.e. if x_1 is a vertex of P_n then x_3, x_5 are vertices and x_2 is an edge x_1x_3 ; x_4 is an edge x_3x_5 and x_6 is incident with x_5 or if x_1 is an edge, then x_1 is incident with x_2; x_2, x_4, x_6 are vertices and x_3 is an edge x_2x_4, x_5 is an edge x_4x_6).

Let f be a minimal k-constrained total labeling of P_n and $S_1, S_2, ..., S_{k_0}$ be the sets as defined in the Lemma 2.1. Let S_α be the set of first k_0 consecutive positive integers required for labeling of exactly one element of S_l for each $l, 1 \leq l \leq k_0$ as in Lemma 2.1. Then each set $S_l, 1 \leq l \leq k_0$ contains exactly two unassigned elements. Again by Lemma 2.1 exactly one of these unassigned element can be assigned by the set S_β containing next possible k_0 consecutive positive integers not in S_α. After labeling the elements of the set $S_l, 1 \leq l \leq k_0$ by the labels in

$S_\alpha \cup S_\beta$, each S_l contains exactly one element unassigned. Thus these elements can be assigned as per Lemma 2.1 again by the set S_γ having next possible k_0 consecutive positive integers not in $S_\alpha \cup S_\beta$.

Let us now consider two consecutive sets S_l, S_{l+1} (Two sets S_i and S_j are said to be consecutive if they are disjoint and there exists $x \in S_i$ and $y \in S_j$ such that xy is an edge). Let $\alpha_1, \alpha_2 \in S_\alpha, x_i \in S_l$ and $x_j \in S_{l+1}$ such that $f(x_i) = \alpha_1$ and $f(x_j) = \alpha_2$ (such α_1, α_2, x_i and x_j exist by Lemma 2.1). Then, as f is a minimal k-constrained total labeling of P_n, it follows that $|j - i| > 2$ implies $j \geq i + 3$. Now we claim that $j = i + 3$. We note that if $i = 3$, then the claim is obvious. If $i \neq 3$, then we have the following cases.

Case 1 $i = 1$

If $j \neq 4$ then

Subcase 1 $j = 5$

By Lemma 2.1, there exists $\beta_1, \beta_2 \in S_\beta$ and $x_r \in S_l, x_s \in S_{l+1}$ such that $f(x_r) = \beta_1$ and $f(x_s) = \beta_2$. Now $f(x_1) = \alpha_1$, $f(x_5) = \alpha_2$ implies $r = 2$ or $r = 3$ (i.e. $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$).

Subsubcase 1 $r = 2$ (i.e. $f(x_2) = \beta_1$)

In this case, $f(x_6) = \beta_2$ (since $f(x_i) = \beta_1$ and $f(x_j) = \beta_2$ implies $|j - i| > 2$) and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is inadmissible as x_3 and x_4 are incident to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subsubcase 2 $r = 3$ (i.e. $f(x_3) = \beta_1$)

Again in this case, $f(x_6) = \beta_2$. So $f(x_2) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is contradiction as x_2 and x_4 are adjacent to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subcase 2 $j = 6$

Now $f(x_1) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$.

Subsubcase 1 $f(x_2) = \beta_1$

In this case, $f(x_5) = \beta_2$ and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$, which is a contradiction as x_3 and x_4 are incident to each other.

Subsubcase 2 $f(x_3) = \beta_1$

In this case, $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$ none of them is possible.

Thus we conclude in Case 1 that if $i = 1$, then $j = 4$, so $j = i + 3$.

Case 2 $i = 2$

In this case we have $j \geq i + 3$, so $j \geq 5$. If $j \neq 5$ then $j = 6$. Now $f(x_2) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_1) = \beta_1$ or or $f(x_3) = \beta_1$.

`Subcase 1:` $f(x_1) = \beta_1$

But then $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$.

Subsubcase 1 $f(x_4) = \beta_2$

In this case, $f(x_4) = \beta_2$ and by Lemma 2.1 $f(x_3) = \gamma_1, f(x_5) = \gamma_2$, which is a contradiction as x_3 and x_5 are adjacent to each other.

Subsubcase 2 $f(x_5) = \beta_2$

In this case, $f(x_5) = \beta_2$ and by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$, which is not possible as x_3 and x_4 are incident to each other.

Subcase 2 $f(x_3) = \beta_1$

In this case, $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$ none of them is possible.

Thus in this case 2, we conclude that if $i = 2$, then $j = 5$, so $j = i + 3$.

Thus, we conclude that the labels in S_α preserves the position in S_l. The similar argument can be extended for the sets S_β and S_γ also. □

Remark 2.3 *Let* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ *and* l *be an integer such that* $1 \le l \le k_0$. *Let* f *be a minimal* k*-constrained total labeling of a path* P_n *and* $S_\alpha = \{\alpha, \alpha+1, \alpha+2, ..., \alpha+k_0-1\}$. *Let* $S_l = \{3l-2, 3l-1, 3l\}$ *and* $f(x) = \alpha + i$ *for some* $x \in S_l$ *Then* $f(y) = \alpha + i + k$ *implies* $y \in S_l$.

Proof After assigning the integers 1 to k_0 one each for exactly one element of S_l, for each $l, 1 \le l \le k_0$, an unassigned element in the set containing the element labeled by 1 can be labeled by $k+1$. But no unassigned element of any other set can be labeled by $k+1$. Thus, if the label $k+1$ is not assigned to an element of the set whose one of the element is labeled by 1, then it should be excluded for the labeling of the elements of P_n and hence the number of isolated vertices required to make P_n a k-constrained graph will increase. Therefore, every minimal k-constrained total labeling should include label $k+1$ for an element of the set whose one of the element is labeled by 1. After including $k+1$, by continuing the same argument for $k+2, k+3, \cdots, k+k_0$ one by one we can conclude that the label $k+i$ (and then $2k+i$) can be labeled only for the element of the set whose one of the element is labeled by i. □

Remark 2.4 *If* $1 \in f(S_1)$, *then from the above Lemmas* 2.1, 2.2 *and Remark* 2.3, *it is clear that* $l, l+k, l+2k \in f(S_l)$ *for every* $l, 1 \le l \le k_0$, *where* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$.

Lemma 2.5 *Let* $S_i = \{3i-2, 3i-1, 3i\}$ *and* f *be a minimal* k*-constrained total labeling of* P_n *such that* $f(x) = s$ *for some* $x \in S_i$ *for some* $i, 1 \le i \le k_0$, *where* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. *Then* $f(y) = s+1$ *implies* $y \in S_{l+1}$ *or* $y \in S_{l-1}$ *and hence by Lemma* 2.2 *we have* $|x-y| = 3$.

Proof Suppose the contrary that $y \in S_j$ for some j where $|j-i| > 1$ and $1 \le j \le k_0$. Without loss of generality, we now assume that $j > i+1$ (otherwise relabel the set S_m as S_{k_0-m} for each $l, 1 \le m \le k_0$). Now by repeated application of Lemma 2.1 we get the sequence of consecutive sets $S_i, S_{i+1}, S_{i+2}, ..., S_j$ and the sequence of elements $s = s_0, s_1 = s+1, \ldots, s_{j-i} = s+1$ where $s_t \in S_{i+t}$ for each $t, 0 \le t \le j$. As $j > i+1$, this sequence of elements (labels) is neither an increasing nor a decreasing sequence. So, there exists a positive integer l such that $s_{l-1} < s_l$ and $s_{l+1} < s_l$. Also, Remark 2.4 $s_{l+k}, s_{l+2k} \in f(S_{i+l})$, $s_{l+1+k}, s_{l+1+2k} \in f(S_{i+l+1})$ and $s_{l-1+k}, s_{l-1+2k} \in f(S_{i+l-1})$. Let $l_1 = 3(i+l)-2, l_2 = 3(i+l)-1, l_3 = 3(i+l)$. We now discuss the following 3! cases.

Case 1 $f(l_1) = s_l, f(l_2) = s_l + k, f(l_3) = s_l + 2k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}, f(l_2-3)=s_{l-1}+k, f(l_3-3)=s_{l-1}+2k$ and $f(l_1+3)=s_{l+1}, f(l_2+3)=s_{l+1}+k, f(l_3+3)=s_{l+1}+2k$. So, $|f(l_1-2)-f(l_1)|\geq k \Rightarrow |s_{l-1}+k-s_l|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 2 $f(l_1)=s_l, f(l_2)=s_l+2k, f(l_3)=s_l+k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}, f(l_2-3)=s_{l-1}+2k, f(l_3-3)=s_{l-1}+k$ and $f(l_1+3)=s_{l+1}, f(l_2+3)=s_{l+1}+2k, f(l_3+3)=s_{l+1}+k$. So, $|f(l_1-1)-f(l_1)|\geq k \Rightarrow |s_{l-1}+k-s_l|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 3 $f(l_1)=s_l+k, f(l_2)=s_l, f(l_3)=s_l+2k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+k, f(l_2-3)=s_{l-1}, f(l_3-3)=s_{l-1}+2k$ and $f(l_1+3)=s_{l+1}+k, f(l_2+3)=s_{l+1}, f(l_3+3)=s_{l+1}+2k$. So, $|f(l_1-1)-f(l_1)|\geq k \Rightarrow |(s_{l-1}+2k)-(s_l+k)|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 4 $f(l_1)=s_l+2k, f(l_2)=s_l, f(l_3)=s_l+k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+2k, f(l_2-3)=s_{l-1}, f(l_3-3)=s_{l-1}+k$ and $f(l_1+3)=s_{l+1}+2k, f(l_2+3)=s_{l+1}, f(l_3+3)=s_{l+1}+k$. So, $|f(l_1-1)-f(l_2)|\geq k \Rightarrow |(s_{l-1}+k)-s_l)|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 5 $f(l_1)=s_l+k, f(l_2)=s_l+2k, f(l_3)=s_l$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+k, f(l_2-3)=s_{l-1}+2k, f(l_3-3)=s_{l-1}$ and $f(l_1+3)=s_{l+1}+k, f(l_2+3)=s_{l+1}+2k, f(l_3+3)=s_{l+1}$. So, $|f(l_3+1)-f(l_3)|\geq k \Rightarrow |(s_{l+1}+k)-s_l)|\geq k \Rightarrow |k-(s_l-s_{l+1})|\geq k \Rightarrow s_l-s_{l+1}\leq 0 \Rightarrow s_l\leq s_{l+1}$, a contradiction.

Case 6 $f(l_1)=s_l+2k, f(l_2)=s_l+k, f(l_3)=s_l$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+2k, f(l_2-3)=s_{l-1}+k, f(l_3-3)=s_{l-1}$ and $f(l_1+3)=s_{l+1}+2k, f(l_2+3)=s_{l+1}+k, f(l_3+3)=s_{l+1}$. So, $|f(l_3+1)-f(l_2)|\geq k \Rightarrow |(s_{l+1}+2k)-(s_l+k))|\geq k \Rightarrow |k-(s_l-s_{l+1})|\geq k \Rightarrow s_l-s_{l+1}\leq 0 \Rightarrow s_l\leq s_{l+1}$, a contradiction. □

Lemma 2.6 *Let P_n be a path on n vertices and $k_0=\lfloor\frac{2n-1}{3}\rfloor$. Then $t_k(P_n)\geq 2(k-k_0)-1$ whenever $2n\equiv 0(mod 3)$ and $k>k_0$.*

Proof For $1\leq l\leq k_0$, let S_l=$\{l_1,l_2,l_3\}$, where $l_1=3l-2, l_2=3l-1, l_3=3l$. Let $S_{k_0+1}=\{2n-2,2n-1\}$ and $T=\{1,2,3,...,k_0\}$. Let f be a minimal k-constrained total labeling of $P_n, 2n\equiv 0$ (mod 3) and $k>k_0$, then by Lemma 2.1, we have $|f(S_i)\cap T|=1$ for each i (i.e. exactly one element of S_i mapped to distinct element of T for each $i, 1\leq i\leq k_0$) and $f(l_j)=m\in T$ for some $j, 1\leq j\leq 3$, then for other element l_i of $S_l, i\neq j$, we have $|f(l_i)-f(l_j)|\geq k$ implies $f(l_i)\geq k+m$. Thus f excludes the elements of the set $T_1=\{k_0+1,k_0+2,...,k\}$ for the next assignments of the elements of $S_l, l\neq k_0+1$.

Let $f(l_i)=t$ for some $t\in T$, where $l_i\in S_l$. Then for the minimum span$f$, by Remark 2.3 $f(l_j)=k+t$ for $i\neq j$ and $l_j\in S_l$.

Again by Lemma 2.3, we get $|f(S_i)\cap T'|=1$, for each $i, 1\leq i\leq k_0$, where $T'=\{k+1,k+$

$2, ..., k+k_0\}$. Further, if f assigns each element of S to exactly one element of $S_l, 1 \le l \le k_0$, for the next assignments, f should leaves all the elements of the set $T_2 = \{k+k_0+1, k+k_0+2, ..., 2k\}$. The above arguments show that while assigning the labels for the elements of P_n not in S_{k_0+1}, f leaves at least $2(k-k_0)$ elements which are in the set $T_1 \cup T_2$.

In view of Lemma 2.2, there are only two possibilities for the assignments of elements of S_{k_0+1} depending upon whether f assigns an element of T_1 to an element of S_{k_0+1} or not.

Let us now consider the first case. Let $x \in S_{k_0+1}$ such that $f(x) = t$ for some $t \in T_1$.

Claim $x = 2n-1$

If not, $f(2n-2) = t$, but then $f(2n-3) \notin T \cup T_1$ and $f(2n-4) \notin T \cup T_1$. Then by Lemma 2.2 $f(2n-5) \in T \cup T_1$ and by Lemma 2.5 $f(2n-5) = t-1$. Then again as above $f(2n-8) = t-2$. Continuing this argument, we conclude that $f(1) = 1$ and $f(4) = 2$. But then, by above argument, we get $f(x) = k+1$ and $f(x+3) = k+2$ for some $x \in S_1$ and $x \in \{2,3\}$. So, $|f(x) - f(4)| = |k+1-2| \ngeq k$ and $|4-x| \le 2$, a contradiction. Hence the claim.

By the above claim we get $f(2n-1) \in T_1$. We now suppose that $f(2n-2) \notin T_2$(note that $f(2n-2) \notin T \cup T_1$), then by above argument for the minimality of f we have $f(2n-2) = k+k_0+1$ and hence $f(1) = k+1$ and $f(2) = 1$. So, by Lemma 2.5, $f(4) = k+2$ and $f(5) = 2$. So, $f(3) \neq 2k+1$ (Since $|f(3) - f(4)| = |2k+1-(k+2)| \ngeq k$, which is inadmissible). This shows that f includes either at most one element of $T_1 \cup T_2$ to label the elements of S_{k_0+1} or leaves one more element namely $2k+1$ to label the elements of P_n (Since the label $2k+1$ is possible only for the element in S_1. Thus f leaves at least $2(k-k_0)-1$ elements.

If the second case follows then the result is immediate because f leaves $(k-k_0)$ elements in the first round of assignment and uses exactly one element of T_2 in the second round. □

Remark 2.7 *In the above Lemma* 2.6 *if* $2n \not\equiv 0(mod\ 3)$ *, then* $t_k(P_n) \ge 2(k-k_0)$.

Proof If the hypothesis hold, then $S_{k_0+1} = \emptyset$ or $S_{k_0+1} = \{2n-1\}$. In the first case, if $S_{k_0+1} = \emptyset$, then by the proof of the Lemma we see that any minimal k-constrained total labeling f should leave exactly $2(k-k_0)$ integers for the labeling of the elements of the path P_n. In the second case when $S_{k_0+1} = 2n-1$, by Lemma 2.5 $f(2n-1) = k_0+1$ (we can assume that $f(1) \in f(S_1)$ because only other possibility by Lemma 2.5 is that the labeling of elements of P_n is in the reverse order, in such a case relabel the sets S_l as S_{k_0-l}). But then, again by Lemma 2.2 and Lemma 2.5 it forces to take $f(1) = 1$ and $f(4) = 2$ hence by Remark 2.4, $f(x) = k+1$ only if $x = 2$ or $x = 3$. In either of the cases $|f(4) - f(x)| \ngeq k$, a contradiction. Hence neither k_0+1 nor $k+1$ can be assigned. Further, if k_0+1 is not assigned, then in the similar way we can argue that either $k+k_0+1$ or $2k+1$ can not be assigned while assigning the second elements of each of the sets $S_l, 1 \le l \le k_0$. Thus, in both the cases f should leave at least $2(k-k_0)$ integers for the assignment of P_n, whenever $2n \not\equiv 0$ (mod 3). □

Theorem 2.8 *Let* P_n *be a path on* n *vertices and* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. *Then*

$$t_k(P_n) = \begin{cases} 0 \quad if \quad k \le k_0, \\ 2(k-k_0)-1 \quad if \quad k > k_0 \quad and \quad 2n \equiv 0(mod\ 3), \\ 2(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2(mod\ 3). \end{cases}$$

Proof If $k \le k_0$, then the result follows by Theorem 3.3 of [4]. Consider the case $k > k_0$.

Case i $2n \equiv 0 \pmod 3$

By Lemma 2.6 we have $t_k(P_n) \ge 2(k-k_0)-1$. Now, the function $f : V(P_n) \cup E(P_n) \cup \overline{K}_{2(k-k_0)-1} \to \{1,2,\ldots,2(n+k-k_0)-2\}$ defined by $f(1) = 2k+1, f(2) = k+1, f(3) = 1$ and $f(i) = f(i-3)+1$ for all $i, 4 \le i \le 2n-3, f(2n-2) = 2k+1+k_0$, $f(2n-1) = k+1+k_0$ and the vertices of $\overline{K}_{2(k-k_0)-1}$ to the remaining, is a Smarandachely k-constrained labeling of the graph $P_n \cup \overline{K}_{2(k-k_0)-1}$. Hence $t_k(P_n) \le 2(k-k_0)-1$.

Case ii $2n \not\equiv 0 \pmod 3$

By Remark 2.7 we have $t_k(P_n) \ge 2(k-k_0)$. On the other hand, the function $f : V(P_n) \cup E(P_n) \cup \overline{K}_{2(k-k_0)} \to \{1,2,\ldots,2(n+k-k_0)-1\}$ defined by $f(1) = 2k+1, f(2) = k+1, f(3) = 1$, $f(i) = f(i-3)+1$ for all $i, 4 \le i \le 2n-1$ and the vertices of $\overline{K}_{2(k-k_0)}$ to the remaining, is a Smarandachely k-constrained labeling of the graph $P_n \cup \overline{K}_{2(k-k_0)}$. Hence $t_k(P_n) \le 2(k-k_0)$.□

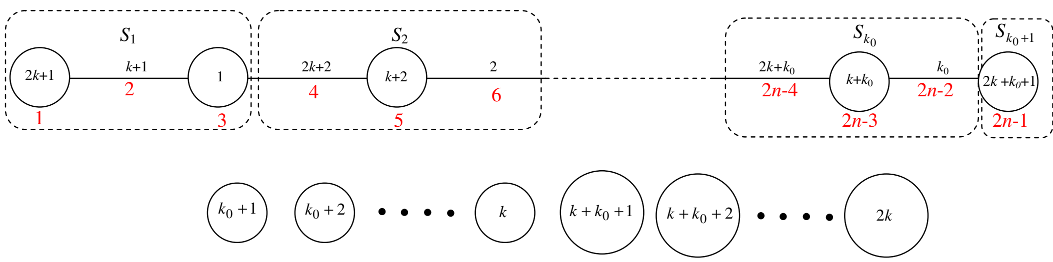

Figure 1: A k-constrained total labeling of the path $P_n \cup \overline{K}_{2(k-k_0)}$, where $2n \equiv 2 \pmod 3$.

§3. k-Constrained Number of a Cycle

Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{v_n v_1\}$. Due to the symmetry in C_n, without loss of generality, we assume that the integer 1 is labeled to the vertex v_1 of C_n. Define $S_\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$, for all $\alpha \in Z^+, 1 \le \alpha \le k_0$, where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ and $\alpha_1 = v_{\frac{3\alpha-1}{2}}, \alpha_2 = v_{\frac{3\alpha-1}{2}} v_{\frac{3\alpha+1}{2}}, \alpha_3 = v_{\frac{3\alpha+1}{2}}$ for all odd α and if α is even, then and $\alpha_1 = v_{\frac{3\alpha}{2}-1} v_{\frac{3\alpha}{2}}, \alpha_2 = v_{\frac{3\alpha}{2}}, \alpha_3 = v_{\frac{3\alpha}{2}} v_{\frac{3\alpha}{2}+1}$.

Case 1 $2n \equiv 0 \pmod 3$

In this case set of elements (edges and vertices) of C_n is $S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1}$, where $S_{k_0+1} = \{v_{n-1}v_n, v_n, v_n v_1\}$.

We now assume the contrary that $t_k(C_n) < 2(k-k_0)$. Then there exists a minimal k-constrained labeling f such that span f is less that $k_0 + 2k + 3$ (since span f = number of vertices + edges + $t_k(C_n) < 3(k_0+1) + 2(k-k_0)$). Now our proof is based on the following observations.

Observation 3.1 Let L_1 be the set of first possible consecutive integers (labels) that can be assigned for the elements of C_n. Then exactly one element of each set $S_\alpha, 1 \le \alpha \le k_0+1$, can receive one distinct label in L_1 and for the minimum span all the labels in L_1 to be assigned. Thus $|L_1| = k_0 + 1$.

Observation 3.2 The labels in L_1 can be assigned only for the elements of S_α in identical places (i.e. $\alpha_i \in S_\alpha$ receives $f(\alpha_i) \in L_1$ and $\beta_j \in S_\beta$ receives $f(\beta_j) \in L_1$ if and only if $i = j$ for all α, β). In fact, since $\alpha_1 = 1$, when $\alpha = 1$, we get $f(\beta_1) \in L_1$, where $\beta = k_0 + 1$, hence $f(\gamma_1) \in L_1$, where $\gamma = k_0$, and so on $\cdots$.

Observation 3.3 The observation 3.2 holds for next labelings for the remaining unlabeled elements also.

Observation 3.4 Since the smallest label in L_1 is 1, by observation 3.1, it follows that the largest label in L_1 is $k_0 + 1$ and next minimum possible integer(label) in the set L_2, consisting of consecutive integers used for the labeling of elements unassigned by the set L_1, is $k + 2$ (we observe that $k + i$, for $k_0 - k + 1 < i < 1$ can not be used for the labeling of any element in the set $S_\alpha, 1 \leq \alpha \leq k_0 + 1$ (since an element of each of S_α has already received a label x in L_1, $1 \leq x \leq k_0 + 1$ and $(k + i) - (x) = k + (i - x) < k$. Also if $k + 1$ is assigned, then $k + 1$ is assigned only to 2^{nd} or 3^{rd} element (viz α_2 or α_3, where $\alpha = 1$) of S_1, but then difference of labels of first element of S_2 labeled by an integer in L_1 (which is greater than 1) with $k + 1$ differs by at most by $k - 1$).

Observation 3.5 By observation 3.4 it follows that the minimum integer label in L_2 is $k + 2$, so the maximum integer label is $k + k_0 + 2$.

Observation 3.6 Let L_3 be the set of next consecutive integers which can be used for the labeling of the elements not assigned by $L_1 \cup L_2$. Then, as span is less than $k_0 + 2k + 3$, the maximum label in L_3 is at most $k_0 + 2k + 2$ and hence the minimum is at most $2k + 2$.

We now suppose that $f(\alpha_i) \in L_3$ and $f(\alpha_i) = \min L_3$, for some $\alpha, 1 \leq \alpha \leq k_0+1$. Then, as $f(\alpha_i) = \min L_3$, $f(\alpha_i) = 2k + j$ for some $j \leq 2$. Further, as $f(\alpha_i) \notin L_2$, we have $k_0 + 2 - k \leq j$. Combining these two we get $k_0 + 2 - k \leq j \leq 2$.

Subcase 1 $i = 2$

In this case $f(\alpha_2) \in L_3$ and already $f(\alpha_1) \in L_1$, so $f(\alpha_3) \in L_2$ and hence $f(\beta_3) \in L_2$ (by Observation 3.2), where $\beta = \alpha - 1$ (or $\beta = k_0 + 1$ if $\alpha = 1$). Thus, $f(\beta_3) = k + l$ for some $l, 2 \leq l \leq k + 2 + k_0$

Now $|f(\alpha_2) - f(\beta_3)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k$ implies that $j - l \geq 0$ hence $j \geq l$. But $j \leq 2 \leq l$ implies $j = l = 2$. Therefore, $f(\alpha_2) = 2k + 2$ and $f(\beta_3) = k + l = k + 2=$ $\min L_2$

In this case $f(\alpha_3) \in L_2$ implies that $f(\alpha_3) = k+m$, for some $m > 2$. So, $|f(\alpha_2)-f(\alpha_3)| =$ $|(2k + 2) - (k + m)| = |k + (2 - m)| < k$ as $m > 2$, which is a contradiction.

Subcase 2 $i = 3$

In this case $f(\alpha_3) \in L_3$ and already $f(\alpha_1) \in L_1$, so $f(\alpha_2) \in L_2$ and hence $f(\beta_2) \in L_2$ (by Observation 3.2), where $\beta = \alpha - 1$ (or $\beta = 1$ if $\alpha = k_0 + 1$). Thus, $f(\beta_2) = k + l$ for some $l, 2 \leq l \leq k + 2 + k_0$.

Now $|f(\alpha_3) - f(\beta_2)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k$ implies that $j - l \geq 0$ hence

$j \geq l$. But $j \leq 2 \leq l$ implies $j = l = 2$. Therefore, $f(\alpha_3) = 2k + 2$ and $f(\beta_2) = k + l = k + 2 =$ min L_2.

In this case $f(\alpha_2) \in L_2$ implies that $f(\alpha_2) = k + m$, for some $m > 2$. So, $|f(\alpha_3) - f(\alpha_2)| = |(2k + 2) - (k + m)| = |k + (2 - m)| < k$ as $m > 2$, which is a contradiction.

Hence in either of the cases we get $t_k(C_n) \geq 2(k - k_0)$.

Case 2 $2n \not\equiv 0 \pmod 3$

Let f be a minimal k-constrained total labeling of C_n. Let L_1, L_2, L_3 be the sets as defined as in Observations 3.1, 3.4 and 3.6 above. Let L_4 be the set of possible consecutive integers used for labeling the elements of C_n which are not assigned by the set $L_1 \cup L_2 \cup L_3$.

We first take the case $2n \equiv 1 \pmod 3$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span f is less than $3k + 1$.

Observation 3.7 Since minimum label in L_1 is 1 and f is a minimal k-constrained labeling, we have $f(x) \geq k + 1$ for all x such that $f(x) \in L_2$.

We have $f(\alpha_1) = 1$ for $\alpha = 1$. Let β be the smallest index such that $f(\beta_1) \in L_1$ and $f(\gamma_1) \notin L_1$, where $\gamma = \beta + 1$ (such index β exists because $f(\alpha_1) = 1$ for $\alpha = 1$ and γ exists because $2n \not\equiv 0 \pmod 3$, the elements labeled by L_1 differ by it position by exactly multiples of 3 apart on either sides of the element labeled by 1). Now consider the set $S = \{\beta_2, \beta_3, \gamma_1\}$. None of the elements of S can be labeled by any the label in L_1 and no two of them receive the label for a single set L_i, for any $i, 2 \leq i \leq 4$. Let s_1, s_2, s_3 be the elements of S arranged accordingly $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$.

Since span $f \leq 3k$, we have $f(s_3) \leq 3k$, so $f(s_2) \leq 2k$ and hence $f(s_1) \leq k$, which is a contradiction (follows by Observation 3.7). Hence for any minimal k-constrained labeling f we get $t_k(C_n) \geq 3(k - k_0)$ whenever $2n \equiv 1 \pmod 3$.

We now take the case $2n \equiv 2 \pmod 3$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span f is less than or equal to $3k + 1$. The element of C_n is the set $S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1}$, where $S_{k_0+1} = \{v_n, v_n v_1\}$. We now claim that the label of the first element namely α_1 of the set S_α is in the set L_1 for all $\alpha, 1 \leq \alpha \leq k_0$ if and only if $k_0 > 2$.

Suppose that α is the least positive index such that $f(\alpha_1) \notin L_1$ and $1 < \alpha \leq k_0$. Then for all β such that $1 \leq \beta < \alpha, f(\beta_1) \in L_1$. Let $\beta = \alpha - 1$. Consider the set $S = \{\beta_2, \beta_3, \alpha_1\}$. Let s_1, s_2, s_3 be the rearrangements of the elements in the set S such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$ respectively.
Since $f(s_3) \in L_4$ and span f is less than or equal to $3k + 1$ it follows that $f(s_3) \leq 3k + 1$ and hence $f(s_2) \leq 2k + 1, f(s_1) \leq k + 1$. But, the least element in L_1 is 1 implies that the least element in L_2 is greater than or equal to $k + 1$, so $f(s_1) \geq k + 1$. Therefore, $f(s_1) = k + 1$, so that $f(s_2) = 2k + 1$ and $f(s_3) = 3k + 1$. There are two possible cases depending on $s_3 \in S_\alpha$ or not. Before considering these cases we make the the following observations.

Observation 3.8 Since $f(\alpha_1) \in L_4$, we find $f(\alpha_1) = 3k + 1$ for any $\alpha > 1$. Suppose for any δ, $\delta > \alpha$, if $f(\delta_1) \in L_1$, then for any γ, $\gamma > \delta$, we find $f(\gamma_1) \in L_1$. In fact, for $\gamma > \delta$, if $f(\gamma_1) \notin L_1$ and $f(\eta_1) \in L_1$ for $\eta = \gamma - 1$, then sequence s_1, s_2, s_3 of the elements of the set $S = \{\eta_2, \eta_3, \gamma_1\}$

taken accordingly as $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$ as above, we get $f(s_3) \leq 3k$ (since $3k+1$ is already assigned). Therefore, $f(s_2) \leq 2k$ and hence $f(s_1) \leq k$, which is imposable (since $f(s_1 \notin L_1)$.

This shows that if $f(\delta_1) \in L_1$, where $\delta = \alpha + 1$, we arrive at the situation that $f(\eta_1) \in L_1$, where $\eta = k_0$.

Now taking the set $\{\eta_2, \eta_3, v_n\}$ and rearranging these elements as s_1, s_2, s_3 such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$, we get $f(s_1) \leq k$ which is again a contradiction.

Observation 3.9 Observation 3.8 shows that $f(\delta_1) \notin L_1$ for any $\delta, \alpha < \delta \leq k_0$.

Observation 3.10 Starting from the vertex v_1, consider the sets $\acute{S}_1 = \{v_1, v_1v_n, v_n\}$, $\acute{S}_2 = S_{k_0}$, $\acute{S}_3 = S_{k_0-1}$, . . . , $\acute{S}_{k_0-\delta+2} = S_\delta$. By taking these sets, we arrive at the conclusion, as in Observation 3.8, that $f(\delta_3) \in L_1$ for every $\delta > \alpha$.

We now continue the main proof for the first case $s_3 \in S_\alpha$. In this case $s_3 = \alpha_1$, therefore $s_1 \in S_\beta$. But $f(s_3) \in L_4$ implies that $f(s_3) \leq 3k+1$, so $f(s_2) \leq 2k+1$ and hence $f(s_1) \leq k+1$. On the other hand $f(\beta_1) \in L_1$ implies that $f(\beta_2)$ or $f(\beta_3)$ is greater than or equal to $k+1$ (since $\min L_1 = 1$), that is, $f(s_1) \geq k+1$. Thus, $f(s_1) = k+1$. This yields $f(\beta_1) = 1$, so $\beta = 1$ and $\alpha = 2$. Also $f(s_2) = 2k+1$ and $f(s_3) = 3k+1$.

Let us now suppose that $\alpha < k_0$. Then there exists an index δ such that $\delta = \alpha + 1 \leq k_0$.

If $f(\beta_2) = 2k+1, f(\beta_3) = k+1$, then $f(\alpha_2) \geq 2k+1$ (since $f(\beta_3) = k+1$) and $f(\alpha_2) \leq 2k+1$ (since $f(\alpha_1) = 3k+1$). So, $f(\alpha_2) = 2k+1$ and hence $f(\alpha_2) = f(\beta_2)$ which is not possible (since $\alpha \neq \beta$).

If $f(\beta_2) = k+1, f(\beta_3) = 2k+1$, then $f(\alpha_2) \leq k+1$ implies $f(\alpha_2) \in L_1$ (since $f(\alpha_2) \neq k+1 = f(\beta_2)$). Further by Observation 3.10, we have $f(\delta_3) \in L_1$. Consider the set $\{\alpha_3, \delta_1, \delta_2\}$ (we note that none of the elements of this set is labeled by the set L_1) and let s_1, s_2, s_3 be the elements of this set taken in order such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$. Since $3k+1$ is already assigned we get $f(s_3) \leq 3k$ and hence as above $f(s_1) \leq k$, which is a contradiction to the fact $f(s_1) \notin L_1$.

We now continue the main proof for the second case $s_3 \notin S_\alpha$. In this case $s_3 \in S_\beta$. Now by assumption we have $f(\alpha_3) \in L_1$ and $k+1$ is already labeled for an element of $S_\beta = S_1$, therefore, $f(\alpha_1) = 2k+1$. Now by Observation 3.10, $f(\delta_3) \in L_1$, where $\delta = \alpha + 1$. If $f(\alpha_2) \in L_1$, then by taking the set $\{\alpha_3, \delta_1, \delta_2\}$ and arranging as above we can show that one of these elements must be labeled by an element of the set L_4 and hence that label should be at most $3k$, so the smallest label of the element of the set is less than or equal k, a contradiction to the fact that the smallest label is not in L_1. Thus, $f(\alpha_2) \notin L_1$.

If $f(\beta_3) = 3k+1$, then $f(\alpha_2) \in L_2$, and hence $f(\alpha_2) \geq k+2$, which is not possible because $f(\alpha_1) = 2k+1$. Therefore, $f(\beta_2) = 3k+1$ and $f(\beta_3) = k+1$. But then, only possibility is that $f(\alpha_2) \in L_4$ implies that $f(\alpha_2) \leq 3k$, which is impossible because $f(\alpha_1) = 2k+1$. Hence the claim.

By the above claim we have either first element of all the sets $S_1, S_2, \ldots S_{k_0}$ are labeled by the elements of the set L_1 or the graph is the cycle C_4. For the graph C_4, it is easy to observe that no three consecutive integers can be used for the labeling and hence each of the sets L_1, L_2, L_3 and L_4 should have at most two elements. Thus, span $f \geq 3k+2$. The equality holds by the following Figure 2.

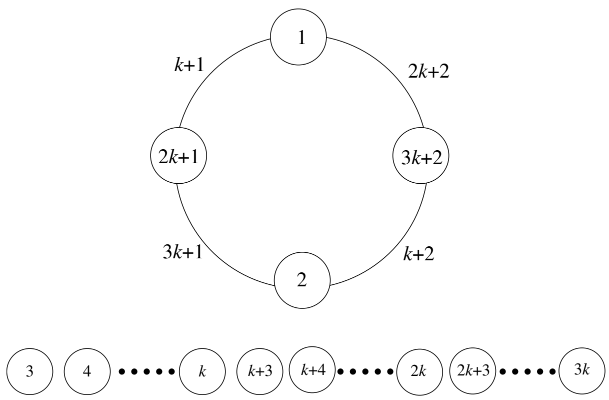

Figure 2: A k-constrained total labeling of the graph $C_4 \cup \overline{K}_{3k-6}$

If the graph is not C_4, then consider the set $T = \{v_{n_1}, v_{n-1}v_n, v_n, v_nv_1\}$. Since $f(v_{n-2}v_{n-1}) \in L_1$ (follows by Observation 3.10) and $f(v_1) = 1 \in L_1$ (follows by the assumption) none of the elements of the set T is labeled by the set L_1 and exactly two elements namely v_{n-1} and v_nv_1 are labeled by same set.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_2, then either $f(v_{n-1}v_n)$ and $f(v_n)$ is in L_4. Suppose $f(v_{n-1}v_n)$ (similarly $f(v_n) \in L_4$), then $f(v_n) \in L_3$ ($f(v_{n-1}v_n) \in L_3$), so $f(v_{n-1}v_n) \leq 3k+1$ and hence $f(v_n) \leq 2k+1$. Therefore both $f(v_{n-1})$ and $f(v_nv_1)$ must be less than or equal to $k+1$, which is not possible because minimum of L_2 is $k+1$.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_3, then $f(v_n) \in L_4$ (or $f(v_{n-1}v_n) \in L_4$) so $f(v_nv_1) \leq 2k+1$ and $f(v_{n-1}) \leq 2k+1$ (since $f(v_n) \leq 3k+1$). Therefore, at least one of $f(v_nv_1)$ or $f(v_{n-1})$ is less than or equal to $2k$, which yields that $f(v_{n-1}v_n) \leq k$ ($f(v_n) \leq k$). Thus, either $f(v_{n-1}v_n)$ or $f(v_n)$ are in L_1, a contradiction.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_4, then at least one of them must be less than $3k+1$. Hence either $f(v_n)$ or $f(v_{n-1}v_n)$ is less than or equal to k (as above), which is again a contradiction.

Thus, we conclude

Lemma 3.11 *Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then*

$$t_k(C_n) \geq \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0) \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 \ (mod\ 3), \\ 3(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2(mod\ 3). \end{cases}$$

Now to prove the reverse inequality, designate the vertex v_i of C_n as $2i-1$ and the edge v_jv_{j+1} as $2j$, v_nv_1 as $2n$. For each $i, 1 \leq i \leq n$ and $1 \leq j \leq n-1$ and for the case $2n \equiv 0$ (mod 3), define a function $f : V(C_n) \cup E(C_n) \cup V(\overline{K}_{2(k-k_0)}) \to \{1, 2, 3, \ldots, 2k+k_0+3\}$ by $f(1) = 1, f(2) = k+2, f(3) = 2k+3$, $f(i) = f(i-3)+1$, for $4 \leq i \leq 2n$ and the vertices of $\overline{K}_{2(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{2(k-k_0)}$. Hence $t_k(C_n) \leq 2(k-k_0)$.

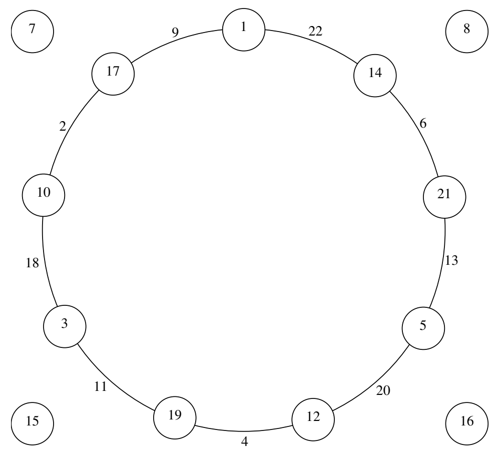

Figure 3: A 7-constrained total labeling of the graph $C_9 \cup \overline{K}_{2(k-k_0)}$

For the case $2n \equiv 1 \pmod 3$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \rightarrow \{1,2,3,\ldots,3k+1\}$ by $f(1) = 1, f(2) = 2k+2, f(3) = k+2$, $f(i) = f(i-3)+1$ for $4 \leq i \leq 2n-4$, $f(2n-3) = k_0$, $f(2n-2) = 3k+1$, $f(2n-1) = 2k+1$, $f(2n) = k+1$ and the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \leq 3(k-k_0)$.

For the case $2n \equiv 2 \pmod 3$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \rightarrow \{1,2,3,\ldots,3k+2\}$ by $f(1) = 1, f(2) = k+2, f(3) = 2k+3$, $f(i) = f(i-3)+1$, for $4 \leq i \leq 2n-6$, $f(2n-5) = 3k+1$, $f(2n-4) = k_0$, $f(2n-3) = 2k+1$, $f(2n-2) = 3k+2$, $f(2n-1) = k+1$, $f(2n) = 2k+2$ the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \leq 3(k-k_0)$.

Hence, in view of Lemma 3.11, we get

Theorem 3.12 *Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then*

$$t_k(C_n) = \begin{cases} 0 \;\; if \;\; k \leq k_0, \\ 2(k-k_0) \;\; if \;\; k > k_0 \;\; and \;\; 2n \equiv 0 \; (mod\ 3), \\ 3(k-k_0) \;\; if \; k > k_0 \;\; and \;\; 2n \equiv 1 \; or \; 2 (mod\ 3). \end{cases}$$

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On Functions Preserving Convergence of Series in Fuzzy n-Normed Spaces

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Abstract: The purpose of this paper is to introduce finite convergence sequences and functions preserving convergence of series in fuzzy n-normed spaces.

Keywords: Pseudo-Euclidean space, Smarandache space, fuzzy n-normed spaces, n-seminorm; function preserving convergence

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§1. Introduction

A *Pseudo-Euclidean space* is a particular Smarandache space defined on a Euclidean space $\mathbf{R}^n$ such that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p > 0$, then p is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean. In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [9,12,14,15]. In [9], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1,2,4,5,6,11,13,18]. In [16], A. Narayanan and S. Vijayabalaji have extended the n-normed linear space to fuzzy n-normed linear space and in [20] the authors have studied the completeness of fuzzy n-normed spaces.

The main purpose of this paper is to study the results concerning infinite series (see, [3,17,19,21]) in fuzzy n-normed spaces. In section 2, we quote some basic definitions of fuzzy n-normed spaces. In section 3, we consider the absolutely convergent series in fuzzy n- normed spaces and obtain some results on it. In section 4, we study the property of finite convergence sequences in fuzzy n-normed spaces. In the last section we introduce and study the concept of

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function preserving convergence of series in fuzzy n-norm spaces and obtain some results.

§2. Preliminaries

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [16].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(S2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ for any real c;

(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2.2 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if:*

(F1) For all $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$;

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(F3) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(F4) For all $t > 0$ and $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min \{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The pair (X, N) will be called a *fuzzy $n-$normed space.*

Theorem 2.1 *Let $\mathcal{A}$ be the family of all finite and nonempty subsets of fuzzy n-normed space (X, N) and $A \in \mathcal{A}$. Then the system of neighborhoods*

$$\mathcal{B} = \{B(t, r, A) \colon t > 0,\ 0 < r < 1,\ A \in \mathcal{A}\}$$

where $B(t,r,A) = \{x \in X \colon N(a_1, \cdots, a_{n-1}, x, t) > 1-r,\ a_1, \cdots, a_{n-1} \in A\}$ *is a base of the null vector* θ*, for a linear topology on* X*, named* N*-topology generated by the fuzzy* n*-norm* N*.*

Proof We omit the proof since it is similar to the proof of Theorem 3.6 in [8]. □

Definition 2.3 *A sequence* $\{x_k\}$ *in a fuzzy* $n-$ *normed space* (X, N) *is said to converge to* x *if given* $r > 0,\ t > 0,\ 0 < r < 1,$ *there exists an integer* $n_0 \in \mathbf{N}$ *such that* $N(x_1, x_2, \ldots, x_{n-1}, x_k - x, t) > 1 - r$ *for all* $k \geq n_0$*.*

Definition 2.4 *A sequence* $\{x_k\}$ *in a fuzzy* $n-$ *normed space* (X, N) *is said to be Cauchy sequence if given* $\epsilon > 0,\ t > 0,\ 0 < \epsilon < 1,$ *there exists an integer* $n_0 \in \mathbf{N}$ *such that* $N(x_1, x_2, \ldots, x_{n-1}, x_m - x_k, t) > 1 - \epsilon$ *for all* $m,\ k \geq n_0$*.*

Theorem 2.1([13]) *Let* N *be a fuzzy* $n-$ *norm on* X*. Define for* $x_1, x_2, \ldots, x_n \in X$ *and* $\alpha \in (0,1)$

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}.$$

Then the following statements hold.

(A_1) *for every* $\alpha \in (0,1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an* $n-$ *seminorm on* X;

(A_2) *If* $0 < \alpha < \beta < 1$ *and* $x_1, x_2, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\beta.$$

Example 2.3[10, Example 2.3] Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Then define $N(x_1, x_2, \ldots, x_n, t) = 0$ if $t \leq 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

§3. Absolutely Convergent Series in Fuzzy $n-$Normed Spaces

In this section we introduce the notion of the absolutely convergent series in a fuzzy n-normed space (X, N) and give some results on it.

Definition 3.1 *The series* $\sum\limits_{k=1}^{\infty} x_k$ *is called absolutely convergent in* (X, N) *if*

$$\sum_{k=1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha < \infty$$

for all $a_1, ..., a_{n-1} \in X$ *and all* $\alpha \in (0,1)$*.*

Using the definition of $\|...\|_\alpha$ the following lemma shows that we can express this condition directly in terms of N.

Lemma 3.1 *The series* $\sum_{k=1}^{\infty} x_k$ *is absolutely convergent in* (X,N) *if, for every* $a_1,...,\ a_{n-1} \in X$ *and every* $\alpha \in (0,1)$ *there are* $t_k \geq 0$ *such that* $\sum_{k=1}^{\infty} t_k < \infty$ *and* $N(a_1,\dots,a_{n-1},x_k,t_k) \geq \alpha$ *for all* k.

proof Let $\sum_{k=1}^{\infty} x_k$ be absolutely convergent in (X,N). Then

$$\sum_{k=1}^{\infty} \|a_1,...,\ a_{n-1},\ x_k\|_\alpha < \infty$$

for every $a_1,...,\ a_{n-1} \in X$ and every $\alpha \in (0,1)$. Let $a_1,...,\ a_{n-1} \in X$ and $\alpha \in (0,1)$. For every k there is $t_k \geq 0$ such that $N(a_1,\dots,a_{n-1},x_k,t_k) \geq \alpha$ and

$$t_k < \|a_1,...,\ a_{n-1},\ x_k\|_\alpha + \frac{1}{2^k}.$$

Then

$$\sum_{k=1}^{\infty} t_k < \sum_{k=1}^{\infty} \|a_1,...,\ a_{n-1},\ x_k\|_\alpha + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

The other direction is even easier to show. □

Definition 3.2 *A fuzzy* $n-$*normed space* (X,N) *is said to be sequentially complete if every Cauchy sequence in it is convergent.*

Lemma 3.2 *Let* (X,N) *be sequentially complete, then every absolutely convergent series* $\sum_{k=1}^{\infty} x_k$ *converges and*

$$\left\|a_1,...,\ a_{n-1},\ \sum_{k=1}^{\infty} x_k\right\|_\alpha \leq \sum_{k=1}^{\infty} \|a_1,...,\ a_{n-1},\ x_k\|_\alpha$$

for every $a_1,...,\ a_{n-1} \in X$ *and every* $\alpha \in (0,1)$.

Proof Let $\sum_{k=1}^{\infty} x_k$ be an infinite series such that $\sum_{k=1}^{\infty} \|a_1,...,\ a_{n-1},\ x_k\|_\alpha < \infty$ for every $a_1,...,\ a_{n-1} \in X$ and every $\alpha \in (0,1)$. Let $y_n = \sum_{k=1}^{n} x_k$ be a partial sum of the series. Let $a_1,...,\ a_{n-1} \in X$, $\alpha \in (0,1)$ and $\epsilon > 0$. There is N such that $\sum_{k=N+1}^{\infty} \|a_1,...,\ a_{n-1},\ x_k\|_\alpha < \epsilon$.

Then, for $n > m \geq N$,

$$\begin{aligned} \left| \|a_1, ..., \ a_{n-1}, \ y_n\|_\alpha - \|a_1, ..., \ a_{n-1}, \ y_m\|_\alpha \right| &\leq \|a_1, ..., \ a_{n-1}, \ y_n - y_m\|_\alpha \\ &\leq \sum_{k=m+1}^{n} \|a_1, ..., \ a_{n-1}, \ x_k\|_\alpha \\ &\leq \sum_{k=N+1}^{\infty} \|a_1, ..., \ a_{n-1}, \ x_k\|_\alpha \\ &< \epsilon. \end{aligned}$$

This is shows that $\{y_n\}$ is a Cauchy sequence in (X, N) . But since (X, N) is sequentially complete, the sequence $\{y_n\}$ converges and so the series $\sum\limits_{k=1}^{\infty} x_k$ converges. □

Definition 3.3 *Let I be any denumerable set . We say that the family $(x_\alpha)_{\alpha \in I}$ of elements in a complete fuzzy n−normed space (X, N) is absolutely summable, if for a bijection Ψ of* ***N(t****he set of all natural numbers*) *onto I the series $\sum\limits_{n=1}^{\infty} x_{\Psi(n)}$ is absolutely convergent.*

The following result may not be surprising but the proof requires some care.

Theorem 3.1 *Let $(x_\alpha)_{\alpha \in I}$ be an absolutely summable family of elements in a sequentially complete fuzzy n− normed space (X, N). Let (B_n) be an infinite sequence of a non-empty subset of A, such that $A = \bigcup\limits_n B_n$, $B_i \cap \ B_j = \varnothing$ for $i \neq j$, then if $z_n = \sum\limits_{\alpha \in B_n} x_\alpha$, the series $\sum\limits_{n=0}^{\infty} z_n$ is absolutely convergent and $\sum\limits_{n=0}^{\infty} z_n = \sum\limits_{\alpha \in I} x_\alpha$.*

Proof It is easy to see that this is true for finite disjoint unions $I = \bigcup\limits_{n=1}^{N} B_n$. Now consider the disjoint unions $I = \bigcup\limits_{n=1}^{\infty} B_n$. By Lemma 3.2

$$\begin{aligned} \sum_{n=1}^{\infty} \|a_1, ..., \ a_{n-1}, \ z_n\|_\alpha &\leq \sum_{n=1}^{\infty} \sum_{i \in B_n} \|a_1, ..., \ a_{n-1}, \ x_i\|_\alpha \\ &= \sum_{i \in I} \|a_1, ..., \ a_{n-1}, \ x_i\|_\alpha < \infty \end{aligned}$$

for every $a_1, ..., \ a_{n-1} \in X$, and every $\alpha \in (0, 1)$. Therefore, $\sum\limits_{n=0}^{\infty} z_n$ is absolutely convergent. Let $y = \sum\limits_{i \in I} x_i, \ z = \sum\limits_{n=1}^{\infty} z_n$. Let $\epsilon > 0$, $a_1, ..., \ a_{n-1} \in X$ and $\alpha \in (0, 1)$. There is a finite set $J \subset I$ such that

$$\sum_{i \notin J} \|a_1, ..., \ a_{n-1}, \ x_i\|_\alpha < \frac{\epsilon}{2}.$$

Choose N large enough such that $B = \bigcup\limits_{n=1}^{N} B_n \supset J$ and

$$\left\| a_1, ..., \ a_{n-1}, \ z - \sum_{n=1}^{N} z_n \right\|_\alpha < \frac{\epsilon}{2}.$$

Then

$$\left\|a_1, ..., \ a_{n-1}, \ y - \sum_{i \in B} x_i \right\|_\alpha < \frac{\epsilon}{2}.$$

By the first part of the proof

$$\sum_{n=1}^{N} z_n = \sum_{i \in B} x.$$

Therefore, $\|a_1, ..., \ a_{n-1}, \ y - z\|_\alpha < \epsilon$. This is true for all ϵ so $\|a_1, ..., \ a_{n-1}, \ y - z\|_\alpha = 0$. This is true for all $a_1, ..., \ a_{n-1} \in X$, $\alpha \in (0, \ 1)$ and $(X, \ N)$ is Hausdorff see [8, Theorem 3.1]. Hence $y = z$. □

Definition 3.4 *Let (X^*, N) be the dual of fuzzy n-normed space (X, N). A linear functional $f : X^* \to K$ where K is a scalar field of X is said to be bounded linear operator if there exists a $\lambda > 0$ such that*

$$\|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha \le \lambda \|a_1, \cdots, a_{n-1}, x_k\|_\alpha,$$

for all $a_1, \cdots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

Definition 3.5 *The series $\sum_{k=1}^{\infty} x_k$ is said to be weakly absolutely convergent in (X, N) if*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha < \infty$$

for all $f \in X^*$, all $a_1, \cdots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

Theorem 3.2 *Let the series $\sum_{k=1}^{\infty} x_k$ be weakly absolutely convergence in (X, N). Then there exists a constant $\lambda > 0$ such that*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha \le \lambda \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha$$

Proof Let $\{e_r\}_{r=1}^{\infty}$ be a standard basis of the space (X, N). Define continuous operators $S_r : X^* \to X$ where $r \in \mathbb{Z}$ by the formula $S_r(f) = \sum_{k=1}^{r} f(x_k) e_k$, we have

$$\|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha = \sum_{k=1}^{r} \|a_1, \cdots, a_{n-1}, f(x_k) e_k\|_\alpha.$$

Since for any fixed $f \in X^*$, the numbers $\|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha$ are bounded by $\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha$, by Banach-Steinhaus theorem, we have

$$\sup_r \|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha = \lambda < \infty.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha &= \sup_r \|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha \\ &\le \lambda \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha. \end{aligned}$$

§4. Finite Convergent Sequences in Fuzzy $n-$Normed Spaces

In this section our principal goal is to show that every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property in every metrizable fuzzy $n-$normed space (X, N).

Definition 4.1 *A sequence $\{x_k\}$ in a fuzzy $n-$normed space (X, N) is said to have finite convergent property if*

$$\sum_{j=1}^{\infty} \|a_1, ..., \ a_{n-1}, \ x_j - x_{j-1}\|_{\alpha} < \infty$$

for all $a_1, ..., \ a_{n-1} \in X$ and all $\alpha \in (0,1)$.

Definition 4.2 *A fuzzy $n-$ normed space (X, N) is said to be metrizable, if there is a metric d which generates the topology of the space.*

Theorem 4.1 *Let (X, N) be a metrizable fuzzy $n-$normed space, then every sequence having finite convergent propperty is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property.*

proof Since X is metrizable, there is a sequence $\left\{\|a_{1,r}, ..., \ a_{n-1,r}, \ x\|_{\alpha_r}\right\}$ for all $a_{1,r}, ..., \ a_{n-1,r} \in X$ and all $\alpha_r \in (0,1)$ generating the topology of X. We choose an increasing sequence $\{m_{k,1}\}$ such that

$$\sum_{k=1}^{\infty} \left\|a_{1,1}, ..., \ a_{n-1,1}, \ x_{m_{k+1,1}} - x_{m_{k,1}}\right\|_{\alpha_1} < \infty$$

where $a_{1,1}, ..., \ a_{n-1,1} \in X$ and $\alpha_1 \in (0,1)$. Then we choose a subsequence $m_{k,2}$ of $m_{k,1}$ such that

$$\sum_{k=1}^{\infty} \left\|a_{1,2}, ..., \ a_{n-1,2}, \ x_{m_{k+1,2}} - x_{m_{k,2}}\right\|_{\alpha_2} < \infty$$

where $a_{1,2}, ..., \ a_{n-1,2} \in X$ and $\alpha_2 \in (0,1)$. Continuing in this way we construct recursively sequences $m_{k,r}$ such that $m_{k,r+1}$ is a subsequence of $m_{k,r}$ and such that

$$\sum_{k=1}^{\infty} \left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{k+1,r}} - x_{m_{k,r}}\right\|_{\alpha_r} < \infty$$

for all $a_{1,r}, ..., \ a_{n-1,r} \in X$ and all $\alpha_r \in (0,1)$. Now consider the diagonal sequence $m_k = m_{k,k}$. Let $r \in \mathbb{N}$. The sequence $\{m_k\}_{k=r}^{\infty}$ is a subsequence of $\{m_{k,r}\}_{k=r}^{\infty}$. Let $k \geq r$. There are pairs of integers (u, v), $u < v$ such that $m_k = m_{u,r}$ and $m_{k+1} = m_{v,r}$. Then by the triangle inequality

$$\left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{k+1}} - x_{m_k}\right\|_{\alpha_r} \leq \sum_{i=u}^{v-1} \left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{i+1,r}} - x_{m_{i,r}}\right\|_{\alpha_r}$$

and therefore,

$$\sum_{k=r}^{\infty} \left\|a_1, ..., \ a_{n-1}, \ x_{m_{k+1}} - x_{m_k}\right\|_{\alpha} \leq \sum_{j=r}^{\infty} \left\|a_1, ..., \ a_{n-1}, \ x_{m_{j+1,r}} - x_{m_{j,r}}\right\|_{\alpha}$$

for all $a_1, ..., \ a_{n-1} \in X$ and all $\alpha \in (0,1)$. The statement of the theorem follows. □

The above theorem shows that many Cauchy sequence has a subsequence which has finite convergent. Therefore, it is natural to ask for an example of Cauchy sequence has a subsequence which has not finite convergent property.

Example 4.2 We consider the set S consisting of all convergent real sequences. Let X be the space of all functions $f : S \to \mathbb{R}$ equipped with the topology of pointwise convergence. This topology is generated by

$$\|f_{1,s}, ..., \ f_{n-1,s}, \ f\|_{\alpha_s} = |f(s)|,$$

for all $f_{1,s}, ..., \ f_{n-1,s}, f \in X$ and all $\alpha_s \in (0,1)$, where $s \in S$. Then consider the sequence $f_n \in X$ defined by $f_n(s) = s_n$ where $s = (s_n) \in S$. The sequence f_n is a Cauchy sequence in X but there is no subsequence f_{n_k} such that

$$\sum_{k=1}^{\infty} \left\|f_{1,s}, ..., \ f_{n-1,s}, \ f_{n_{k+1}} - f_{n_k}\right\|_{\alpha_s} < \infty$$

for all $s \in S$. We see this as follows. If $n_1 < n_2 < n_3 < ...$ is a sequence then define $s_n = (-1)^k \ \frac{1}{k}$ for $n_k \leq n < n_{k+1}$. Then $s = (s_n) \in S$ but

$$\sum_{k=1}^{\infty} \left\|f_{1,s}, ..., \ f_{n-1,s}, \ f_{n_{k+1}} - f_{n_k}\right\|_{\alpha_s} = \sum_{k=1}^{\infty} \left|s_{n_{k+1}} - s_{n_k}\right| \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

§5. Functions Preserving Convergence of Series in Fuzzy $n-$Normed Spaces

In this section we shall introduce the functions $f : X \to X$ that preserve convergence of series in fuzzy $n-$normed spaces. Our work is an extension of functions $f : \mathbb{R} \to \mathbb{R}$ that preserve convergence of series studied in [19] and [3].

We read in Cauchy's condition in (X, N) as follows: the series $\sum_{k=1}^{\infty} x_k$ converges if and only if for every $\epsilon > 0$ there is an N so that for all $n \geq m \geq N$,

$$\|a_1 \cdots, a_{n-1}, \sum_{k=m}^{n} x_k\| < \epsilon,$$

where $a_1 \cdots, a_{n-1} \in X$.

Definition 5.1 *A function $f : X \times X \to X$ is said to be additive in fuzzy n-normed space (X, N) if*

$$\|a_1, \cdots, a_{n-1}, f(x,y)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(x)\|_\alpha + \|a_1, \cdots, a_{n-1}, f(y)\|_\alpha,$$

for each $x, y \in X$, $a_1, \cdots, a_{n-1} \in X$ and for all $\alpha \in (0,1)$.

Definition 5.2 *A function $f : X \to X$ is convergence preserving (abbreviated CP) in (X, N) if for every convergent series $\sum_{k=1}^{\infty} x_k$, the series $\sum_{k=1}^{\infty} f(x_k)$ is also convergent, i.e., for every $a_1, \cdots, a_{n-1} \in X$,*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha < \infty$$

whenever $\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, x_k\|_\alpha < \infty$.

Theorem 5.1 *Let* (X, N) *be a fuzzy* n*-normed space and* $f : X \to X$ *be an additive and continuous function in the neighborhood* $B(t, r, A)$. *Then the function* f *is* CP *of infinite series in* (X, N).

Proof Assume that f is additive and continuous in $B(\alpha, \delta, A) = \{x \in X : \|a_1, \cdots, a_{n-1}, x\|_\alpha < \delta\}$, where $a_1, \cdots, a_{n-1} \in A$ and $\delta > 0$. From additivity of f in $B(\alpha, \delta, A)$ implies that $f(0) = 0$. Let $\sum_{k=1}^{\infty} x_k$ be a absolute convergent series and $x_k \in X$ $(k = 1, 2, 3, \cdots)$. We show that $\sum_{k=1}^{\infty} f(x_k)$ is also absolute convergent.

By Cauchy condition for convergence of series, there exists a $k \in \mathbb{N}$ such that for every $p \in \mathbb{N}$

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} x_j\|_\alpha < \frac{\delta}{2}.$$

From this we have

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} x_j\|_\alpha < \frac{\delta}{2}.$$

By the additivity of f in $B(\alpha, \delta, A)$, we get

$$\|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha < \frac{\delta}{2}.$$

Now, let $y_p = \sum_{j=k+1}^{k+p} x_j$ for $p = 1, 2, 3, \cdots$ and $y = \sum_{j=k+1}^{\infty} x_j$ belong to the neighborhood $B(\alpha, \delta, A)$. The function f is continuous in $B(\alpha, \delta, A)$, i.e., $f(y_p) \to f(y)$ because $y_p \to y$ for $p \to \infty$. Hence

$$\lim_{p\to\infty} \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{\infty} x_j)\|_\alpha.$$

This implies

$$\lim_{p\to\infty} \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} f(x_j)\|_\alpha$$

and this guarantee the convergence of the series $\sum_{j=k+1}^{\infty} f(x_j)$ and therefore the series $\sum_{j=1}^{\infty} f(x_j)$ must also be convergent in X, i.e., the function f is CP infinite series in (X, N). □

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Achromatic Coloring on Double Star Graph Families

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Abstract: The purpose of this article is to find the achromatic number, i.e., Smarandachely achromatic 1-coloring for the central graph, middle graph, total graph and line graph of double star graph $K_{1,n,n}$ denoted by $C(K_{1,n,n})$, $M(K_{1,n,n})$, $T(K_{1,n,n})$ and $L(K_{1,n,n})$ respectively.

Keywords: Smarandachely achromatic k-coloring, Smarandachely achromatic number, central graph, middle graph, total graph, line graph and achromatic coloring.

AMS(2000): 05C15

§1. Preliminaries

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called central graph [10] of G denoted by $C(G)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [4] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [1,5] of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices

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x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The line graph [1,5] of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n + 1$ vertices and $2n$ edges.

For given graph G, an integer $k \geq 1$, a *Smarandachely achromatic k-coloring* of G is a proper vertex coloring of G in which every pair of colors appears on at least k pairs of adjacent vertices. The *Smarandachely achromatic number* of G denoted $\chi_c^S(G)$, is the greatest number of colors in a Smarandachely achromatic k-coloring of G. Certainly, $\chi_c^S(G) \geq k$. Now if $k = 1$, i.e., a Smarandachely achromatic 1-coloring and $\chi_c^S(G)$ are usually abbreviated to *achromatic coloring* [2,3,6,7,8,9,11] and $\chi_c(G)$.

The achromatic number was introduced by Harary, Hedetniemi and Prins [6]. They considered homomorphisms from a graph G onto a complete graph K_n. A homomorphism from a graph G to a graph G' is a function $\phi : V(G) \to V(G')$ such that whenever u and v are adjacent in G, $u\phi$ and $v\phi$ are adjacent in G'. They show that, for every (complete) n-coloring τ of a graph G there exists a (complete) homomorphism ϕ of G onto K_n and conversely. They noted that the smallest n for which such a complete homomorphism exists is just the chromatic number $\chi = \chi(G)$ of G. They considered the largest n for which such a homomorphism exists. This was later named as the achromatic number $\psi(G)$ by Harary and Hedetniemi [6]. In the first paper [6] it is shown that there is a complete homomorphism from G onto K_n if and if only $\chi(G) \leq n \leq \psi(G)$.

§2. Achromatic Coloring on central graph of double star graph

Algorithm 2.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for the vertices in $C(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{u_i\}$;
$C(u_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = n + 1$;
$V_3 = \{e_i\}$;
$C(e_i) = i$;
$V_4 = \{v_i\}$;
$C(v_i) = n + 1 + i$;
}

$V_4 = \{v\};$
$C(v) = n + 1;$
$V = V_1 \cup V_2 \cup V_3 \cup V_4;$
end

Theorem 2.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[C(K_{1,n,n})] = 2n + 1.$$

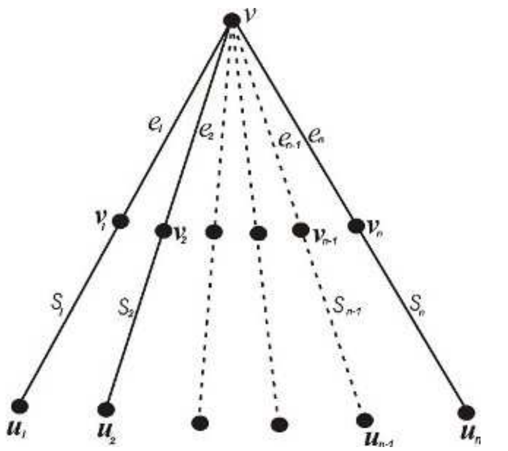

Fig.1

Double star graph $K_{1,n,n}$

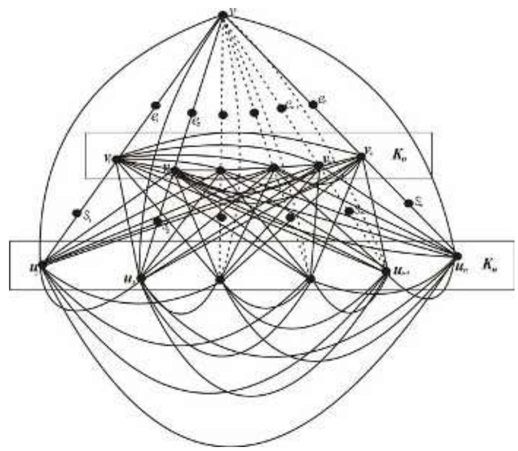

Fig.2

Central graph of double star graph $C(K_{1,n,n})$

Proof Let v,v_1,v_2,$\cdots$,v_n and u_1 ,u_2,..,u_n be the vertices in $K_{1,n,n}$, the vertex v be adjacent to $v_i(1 \le i \le n)$. The vertices $v_i(1 \le i \le n)$ be adjacent to $u_i(1 \le i \le n)$. Let the edge vv_i and uu_i $(1 \le i \le n)$ be subdivided by the vertices $e_i(1 \le i \le n)$ and $s_i(1 \le i \le n)$ in $C(K_{1,n,n})$. Clearly $V[C(K_{1,n,n})] = \{v\} \cup \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\} \cup \{e_i/1 \le i \le n\} \cup \{s_i/1 \le i \le n\}$. The vertices $v_i(1 \le i \le n)$ induce a clique of order n (say K_n) and the vertices $v, u_i(1 \le i \le n)$ induce a clique of order $n+1$ (say K_{n+1}) in $C(K_{1,n,n})$ respectively. Now consider the vertex set $V[C(K_{1,n,n})]$ and the color classes $C_1 = \{c_1, c_2, c_3, .., c_n\}$ and $C_2 = \{c_1, c_2, c_3, ..., c_n, c_{n+1}\}$, assign a proper coloring to $C(K_{1,n,n})$ by Algorithm 2.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 2, 3..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..n$. The edges joining the vertices (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..n-1$, then the edges joining the vertex pair (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will stand for the color pair (c_i, c_j). Now this coloring will accommodate all the pairs of the color class. Thus we have $\chi_c[C(K_{1,n,n})] \geq 2n+1$. The number of edges of

$$C[K_{1,n,n}] = \left\{5n + 2\frac{n(n-1)}{2} + n(n-1)\right\} = 4n + 2n^2 < \binom{2n+1}{2}.$$

Therefore, $\chi_c[C(K_{1,n,n})] \leq 2n+1$. Hence $\chi_c[C(K_{1,n,n})] = 2n+1$. □

Example 2.3

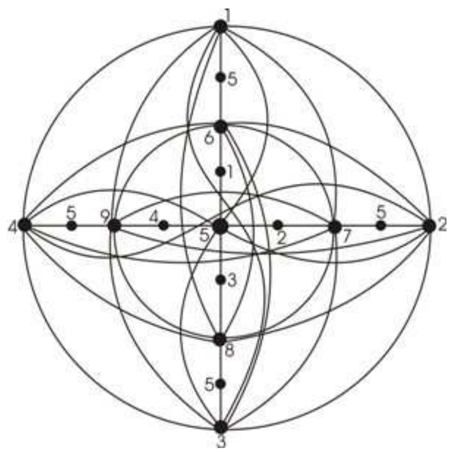

Fig.3
Central graph of $C(K_{1,4,4})$
$\chi_c[C(K_{1,4,4})] = 9$

§3. Achromatic coloring on middle graph of double star graph

Algorithm 3.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $M(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n+1$;
for $i = 1$ to n

{
$V_3 = \{v_i\};$
$C(v_i) = n + 2;$
}
for $i = 2$ to n
{
$V_4 = \{s_i\};$
$C(s_i) = n + 3;$
}
$C(s_1) = n + 1;$
for $i = 1$ to $n - 1$
{
$V_5 = \{u_i\};$
$C(u_i) = 1;$
}
$C(u_n) = C(v);$
$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5;$
end

Theorem 3.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[M(K_{1,n,n})] = n + 3.$$

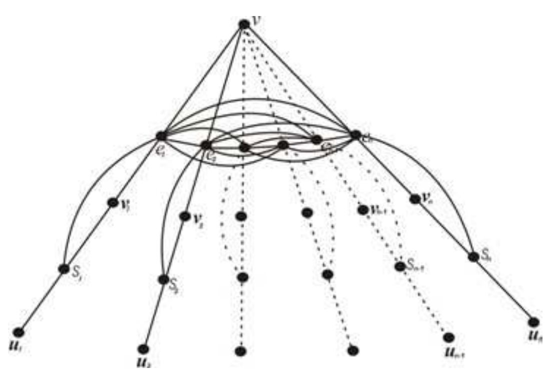

Fig.4

Middle graph of double star graph $M(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$. By definition of middle graph, each edge vv_i and $v_iu_i(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices u_i and s_i in $M(K_{1,n,n})$ and the vertices v,e_1,e_2,...,e_n induce a clique of order $n + 1$(say K_{n+1})

in $M(K_{1,n,n})$. i.e.,$V[M(K_{1,n,n})] = \{v\} \cup \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\} \cup \{e_i/1 \le i \le n\} \cup \{s_i/1 \le i \le n\}$. Now consider the vertex set $V[M(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, .., c_n, c_{n+1}, c_{n+2}, c_{n+3}\}$, assign a proper coloring to $M(K_{1,n,n})$ by Algorithm 3.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j)

Step 1

If $i = 1$, $j = 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..., n-1$, then the edges joining the vertex pair (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will stand for the color pair (c_i, c_j). Now this coloring will accommodate all the pairs of the color class.

Thus we have $\chi_c[M(K_{1,n,n})] \ge n + 3$. The number of edges in $M[K_{1,n,n}] = \dfrac{n^2 + 9n}{2} < \dbinom{n+4}{2}$. Therefore, $\chi_c[M(K_{1,n,n})] \le n + 3$. Hence $\chi_c[M(K_{1,n,n})] = n + 3$. □

Example 3.3

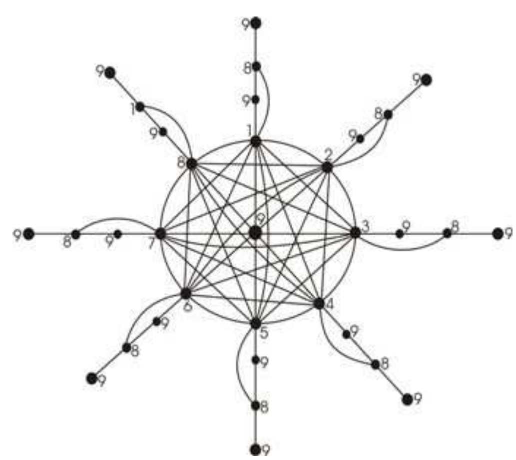

Fig.5

Middle graph of $M(K_{1,8,8})$

$\chi_c[M(K_{1,8,8})] = 11$

§4. Achromatic Coloring on Total Graph of Double Star Graph

Algorithm 4.1

Input: The number "n" of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $T(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{v_i\}$;
$C(v_i) = n + 2$;
$V_3 = \{s_i\}$;
$C(s_i) = n + 3$;
$V_4 = \{u_i\}$;
$C(u_i) = n + 1$;
}
$V_5 = \{v\}$;
$C(v) = n + 1$;
$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$;
end

Theorem 4.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[T(K_{1,n,n})] = n + 3.$$

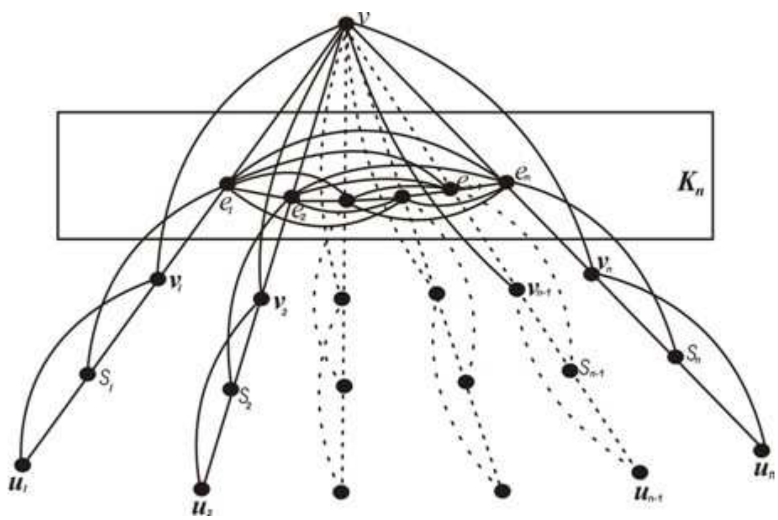

Fig.6

Total graph of double star graph $T(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v, v_1, v_2, .., v_n\} \cup \{u_1, u_2, .., u_n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, ..., e_n\} \cup \{s_1, s_2, s_3, ..., s_n\}$. By the definition of total graph, we have $V[T(K_{1,n,n})] = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\} \cup \{e_i/1 \leq i \leq n\} \cup \{s_i/1 \leq i \leq n\}$, in which the vertices v, e_1, e_2,...,e_n induce a clique of order $n+1$ (say K_{n+1}). Now consider the vertex set $V[T(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, ...c_n, c_{n+1}, c_{n+2}, c_{n+3}\}$, assign a proper coloring to $T(K_{1,n,n})$ by Algorithm 4.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, ...n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..n - 1$, then the edges joining the vertex pair (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will stand for the color pair (c_i, c_j).

Thus any pair in the color class is adjacent by at least one edge. Thus we have $\chi_c[T(K_{1,n,n})] \geq n + 3$. The number of edges of $T[K_{1,n,n}] = \dfrac{n^2 + 13n}{2} < \dbinom{n+4}{2}$. Therefore, $\chi_c[(K_{1,n,n})] \leq n + 3$. Hence $\chi_c[T(K_{1,n,n})] = n + 3$. □

Example 4.3

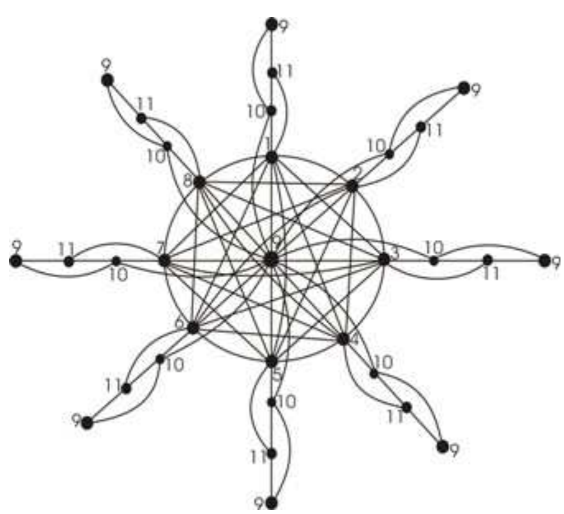

Fig.7

Total graph of $T(K_{1,8,8})$
$\chi_c[T(K_{1,8,8})] = 11$

§5. Achromatic Coloring on Line Graph of Double Star Graphs

Algorithm 5.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $L(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = n + 1$;
}
$V = V_1 \cup V_2$;
end

Theorem 5.1 *For any double star graph $K_{1,n,n}$,the achromatic number,*

$$\chi_c[L(K_{1,n,n})] = n + 1.$$

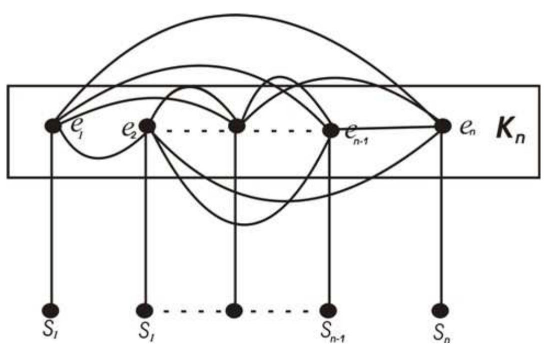

Fig.8

Line graph of double star graph $L(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v\} \cup \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, ...e_n\} \cup \{s_1, s_2, s_3, ..., s_n\}$ By the definition of Line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $L(K_{1,n,n})$. The vertices e_1, e_2, ..., e_n induce a clique of order n in $L(K_{1,n,n})$.i.e., $V[L(K_{1,n,n})] = \{e_i/1 \le i \le n\} \cup \{s_i/1 \le i \le n\}$. Now consider the vertex set $V[L(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, ...c_n, c_{n+1}\}$, assigned a proper coloring to $L(K_{1,n,n})$ by Algorithm 5.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), and (e_i, s_i) will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, ...n$. The edges joining the vertices (e_i, e_j), and (e_i, s_i), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, s_i) will accommodate the color pair(c_i, c_j). Similarly if $i = n$,$j = 1, 2, ..n - 1$, then the edges joining the vertex pair (e_i, e_j), (e_i, s_i) will stand for the color pair (c_i, c_j).

Thus any pair in the color class is adjacent by at least one edge we have $\chi_c[L[K_{1,n,n})] \geq n+1$. The number of edges of edges of $L[K_{1,n,n}] = \frac{n^2+n}{2} < \binom{n+2}{2}$. Therefore, $\chi_c[L(K_{1,n,n})] \leq n+1$. Hence $\chi_c[L(K_{1,n,n})] = n+1$. □

Example 5.3

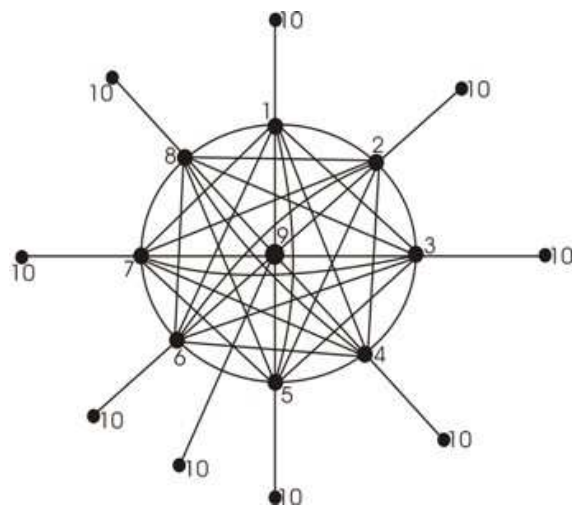

Fig.9

Line graph of $L(K_{1,9,9})$
$\chi_c[L(K_{1,9,9})] = 10$

§6. Main Theorems

Theorem 6.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[L(K_{1,n,n})] = \chi[M(K_{1,n,n})] = \chi[T[K_{1,n,n})] = n+1.$$

Theorem 6.2 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[M(K_{1,n,n})] = \chi_c[T(K_{1,n,n})] = n+3.$$

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Some Results on Super Mean Graphs

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Abstract: Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced *Smarandachely edge m-labeling* f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. Particularly, in the case of $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a *super mean labeling.* A graph that admits a Smarandachely super mean m-labeling is called *Smarandachely super m-mean graph.* In this paper, we prove that the H-graph, corona of a H-graph, $G \odot S_2$ where G is a H-graph, the cycle C_{2n} for $n \geq 3$, corona of the cycle C_n for $n \geq 3$, mC_n-snake for $m \geq 1, n \geq 3$ and $n \neq 4$, the dragon $P_n(C_m)$ for $m \geq 3$ and $m \neq 4$ and $C_m \times P_n$ for $m = 3, 5$ are super mean graphs, i.e., Smarandachely super 2-mean graphs.

Keywords: Labeling, Smarandachely super mean labeling, Smarandachely super m-mean graph, super mean labeling, super mean graphs

AMS(2000): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the Cartesian

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product $G_1 \times G_2$ has p_1p_2 vertices which are $\{(u,v)/u \in G_1, v \in G_2\}$. The edges are obtained as follows: (u_1,v_1) and (u_2,v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$.

The corona of a graph G on p vertices $v_1, v_2, \ldots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \ldots, u_p$ and the new edges u_iv_i for $1 \le i \le p$, denoted by $G \odot K_1$. For a graph G, the 2-corona of G is the graph obtained from G by identifying the center vertex of the star S_2 at each vertex of G, denoted by $G \odot S_2$. The baloon of a graph G, $P_n(G)$ is the graph obtained from G by identifying an end vertex of P_n at a vertex of G. $P_n(C_m)$ is called a dragon. The join of two graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G with each vertex of H by means of an edge and it is denoted by $G + H$.

A path of n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. $K_{1,m}$ is called a star, denoted by S_m. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively, denoted by $B(m)$. A triangular snake T_n is obtained from a path $v_1v_2 \ldots v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $1 \le i \le n-1$, that is, every edge of a path is replaced by a triangle C_3.

We define the H-graph of a path P_n to be the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even and a cyclic snake mC_n the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k+1$ and identifying the vertex $v_{(k+1)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k$.

A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced *Smarandachely edge m-labeling* f_S^* for an edge $e = uv$, an integer $m \ge 2$ is defined by

$$f_S^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. Particularly, in the case of $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a *super mean labeling.* A graph that admits a Smarandachely super mean m-labeling is called *Smarandachely super m-mean graph*, particularly, *super mean graph* if $m = 2$. A super mean labeling of the graph P_6^2 is shown in Fig.1.1.

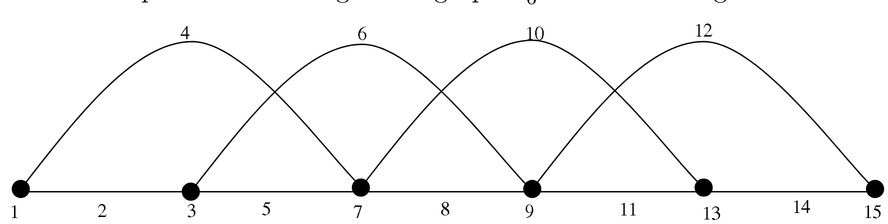


Fig.1.1

The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [7]. They have studied in [4,5,7,8] the mean labeling of some standard graphs.

The concept of super mean labeling was first introduced by R. Ponraj and D. Ramya [2]. They have studied in [2,3] the super mean labeling of some standard graphs like $P_n, C_{2n+1}, n \geq 1$, $K_n(n \leq 3)$, $K_{1,n}(n \leq 3)$, T_n, $C_m \cup P_n(m \geq 3, n \geq 1)$, $B_{m,n}(m = n, n+1)$ etc. They have proved that the union of two super mean graph is super mean graph and C_4 is not a super mean graph. Also they determined all super mean graph of order ≤ 5.

In this paper, we establish the super meanness of the graph C_{2n} for $n \geq 3$, the H-graph, Corona of a H- graph, 2-corona of a H-graph, corona of cycle C_n for $n \geq 3$, mC_n-snake for $m \geq 1$, $n \geq 3$ and $n \neq 4$, the dragon $P_n(C_m)$ for $m \geq 3$ and $m \neq 4$ and $C_m \times P_n$ for $m = 3, 5$.

§2. Results

Theorem 2.1 *The H-graph G is a super mean graph.*

Proof Let $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of the graph G. We define a labeling $f : V(G) \to \{1, 2, \ldots, p+q\}$ as follows:

$f(v_i) = 2i - 1, \quad 1 \leq i \leq n$

$f(u_i) = 2n + 2i - 1, \quad 1 \leq i \leq n$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$f^*(v_i v_{i+1}) = 2i, \quad 1 \leq i \leq n-1$

$f^*(u_i u_{i+1}) = 2n + 2i, \quad 1 \leq i \leq n-1$

$f^*(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}) = 2n$ if n is odd

$f^*(v_{\frac{n}{2}+1} u_{\frac{n}{2}}) = 2n$ if n is even

Then clearly it can be verified that the H-graph G is a super mean graph. For example the super mean labelings of H-graphs G_1 and G_2 are shown in Fig.2.1. □

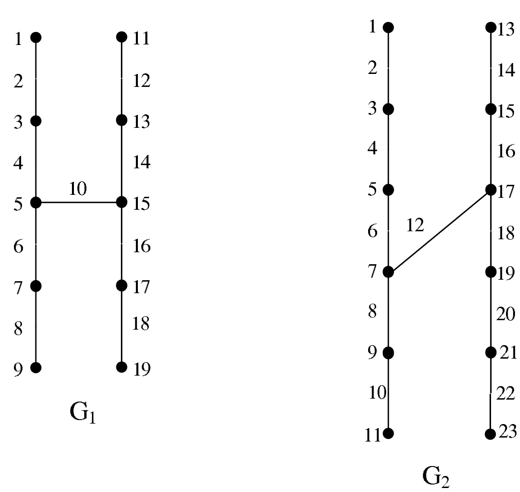


Fig.2.1

Theorem 2.2 *If a H-graph G is a super mean graph, then $G \odot K_1$ is a super mean graph.*

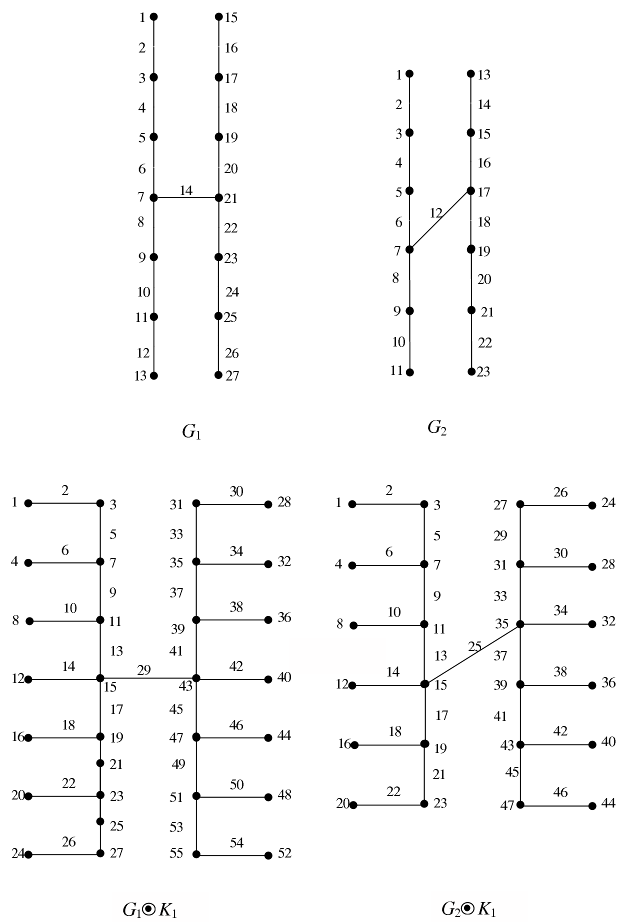


Fig.2.2

Proof Let f be a super mean labeling of G with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$. Let $v'_1, v'_2, \ldots, v'_n$ and $u'_1, u'_2, \ldots, u'_n$ be the corresponding new vertices in $G \odot K_1$.

We define a labeling $g : V(G \odot K_1) \rightarrow \{1, 2, \ldots, p+q\}$ as follows:

$$\begin{aligned} g(v_i) &= f(v_i) + 2i, && 1 \leq i \leq n \\ g(u_i) &= f(u_i) + 2n + 2i, && 1 \leq i \leq n \\ g(v'_1) &= f(v_1) \\ g(v'_i) &= f(v_i) + 2i - 3, && 2 \leq i \leq n \\ g(u'_i) &= f(u_i) + 2n + 2i - 3, && 1 \leq i \leq n \end{aligned}$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$\begin{array}{lll}
g^*(v_iv_{i+1}) & = f^*(v_iv_{i+1}) + 2i + 1, & 1 \leq i \leq n-1 \\
g^*(u_iu_{i+1}) & = f^*(u_iu_{i+1}) + 2n + 2i + 1, & 1 \leq i \leq n-1 \\
g^*(v_iv_i') & = f(v_i) + 2i - 1, & 1 \leq i \leq n \\
g^*(u_iu_i') & = f(u_i) + 2n + 2i - 1, & 1 \leq i \leq n \\
g^*(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}) & = 2f^*((v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}) + 1 & \text{if } n \text{ is odd} \\
g^*(v_{\frac{n}{2}+1}u_{\frac{n}{2}}) & = 2f^*((v_{\frac{n}{2}+1}u_{\frac{n}{2}}) + 1 & \text{if } n \text{ is even}
\end{array}$$

It can be easily verified that g is a super mean labeling and hence $G \odot K_1$ is a super mean graph. For example the super mean labeling of H-graphs G_1, G_2, $G_1 \odot K_1$ and $G_2 \odot K_1$ are shown in Fig.2.2. □

Theorem 2.3 *If a H-graph G is a super mean graph, then $G \odot S_2$ is a super mean graph.*

Proof Let f be a super mean labeling of G with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$. Let $v_1', v_2', \ldots, v_n'$, $v_1'', v_2'', \ldots, v_n''$, $u_1', u_2', \ldots, u_n'$ and $u_1'', u_2'', \ldots, u_n''$ be the corresponding new vertices in $G \odot S_2$.

We define $g : V(G \odot S_2) \to \{1, 2, \ldots, p + q\}$ as follows:

$$\begin{array}{lll}
g(v_i) & = f(v_i) + 4i - 2, & 1 \leq i \leq n \\
g(v_i') & = f(v_i) + 4i - 4, & 1 \leq i \leq n \\
g(v_i'') & = f(v_i) + 4i, & 1 \leq i \leq n \\
g(u_i) & = f(u_i) + 4n + 4i - 2, & 1 \leq i \leq n \\
g(u_i') & = f(u_i) + 4n + 4i - 4, & 1 \leq i \leq n \\
g(u_i'') & = f(u_i) + 4n + 4i, & 1 \leq i \leq n
\end{array}$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$\begin{array}{lll}
g^*(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}) & = 3f^*(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}) & \text{if } n \text{ is odd} \\
g^*(v_{\frac{n}{2}+1}u_{\frac{n}{2}}) & = 3f^*(v_{\frac{n}{2}+1}u_{\frac{n}{2}}) & \text{if } n \text{ is even}
\end{array}$$

$$\begin{array}{lll}
g^*(v_iv_{i+1}) & = f^*(v_iv_{i+1}) + 4i, & 1 \leq i \leq n-1 \\
g^*(v_iv_i') & = f(v_i) + 4i - 3, & 1 \leq i \leq n \\
g^*(v_iv_i'') & = f(v_i) + 4i - 1, & 1 \leq i \leq n \\
g^*(u_iu_{i+1}) & = f^*(u_iu_{i+1}) + 4n + 4i & 1 \leq i \leq n-1 \\
g^*(u_iu_i') & = f(u_i) + 4n + 4i - 3, & 1 \leq i \leq n \\
g^*(u_iu_i'') & = f(u_i) + 4n + 4i - 1, & 1 \leq i \leq n
\end{array}$$

It can be easily verified that g is a super mean labeling and hence $G \odot S_2$ is a super mean graph. For example the super mean labelings of $G_1 \odot S_2$ and $G_2 \odot S_2$ are shown in Fig.2.3. □

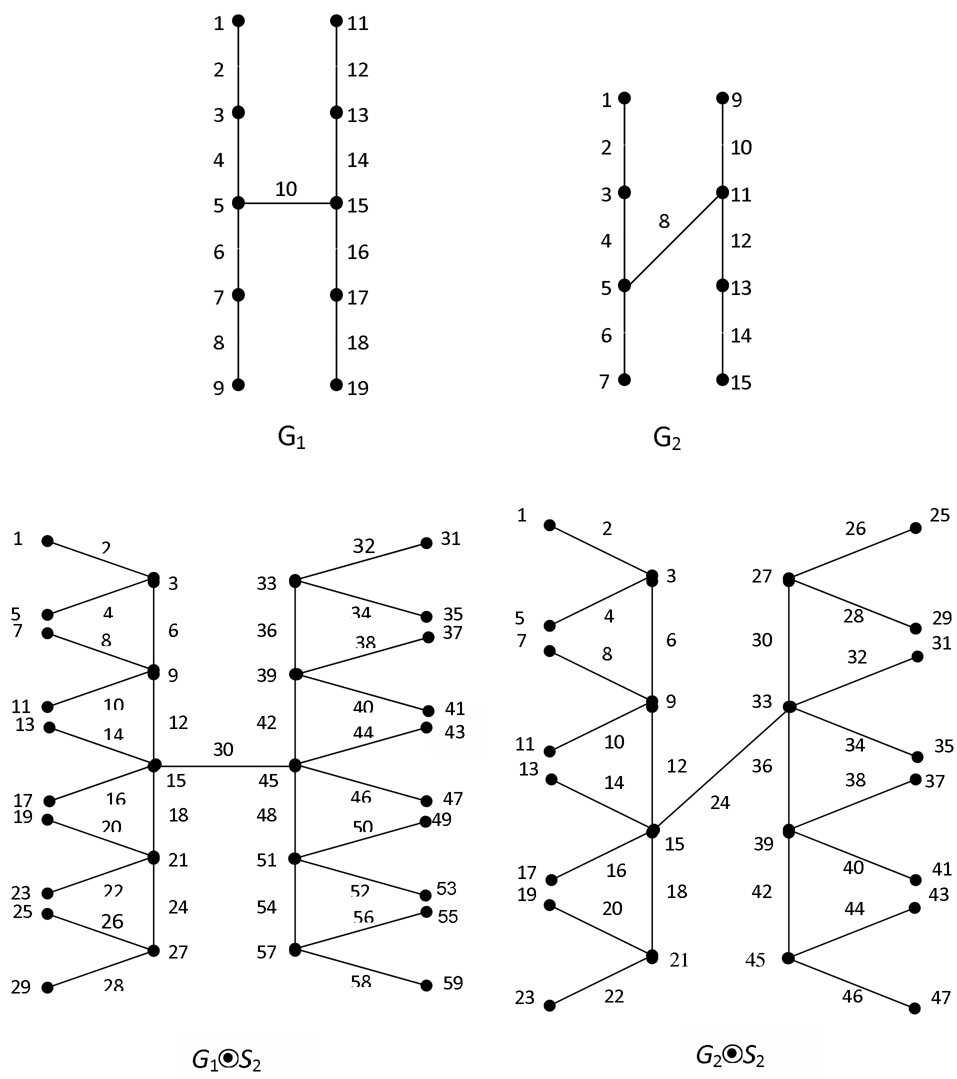


Fig.2.3

Theorem 2.4 *Cycle C_{2n} is a super mean graph for $n \geq 3$.*

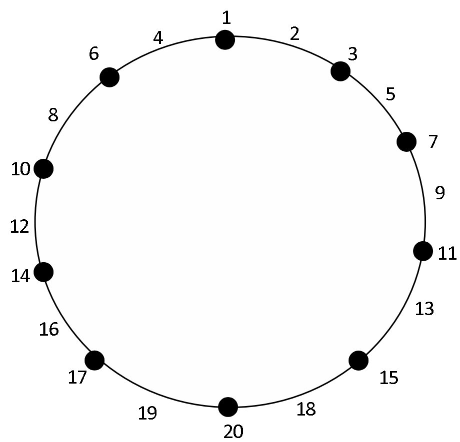
Proof Let C_{2n} be a cycle with vertices $u_1, u_2, \ldots, u_{2n}$ and edges $e_1, e_2, \ldots, e_{2n}$. Define $f : V(C_{2n}) \to \{1, 2, \ldots, p+q\}$ as follows:

$$\begin{aligned} f(u_1) &= 1 \\ f(u_i) &= 4i - 5, && 2 \leq i \leq n \\ f(u_{n+j}) &= 4n - 3j + 3, && 1 \leq j \leq 2 \\ f(u_{n+j+2}) &= 4n - 4j - 2, && 1 \leq j \leq n-2 \end{aligned}$$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(e_1) &= 2 \\ f^*(e_i) &= 4i-3, \qquad 2 \le i \le n-1 \\ f^*(e_n) &= 4n-2, \\ f^*(e_{n+1}) &= 4n-1, \\ f^*(e_{n+j+1}) &= 4n-4j, \qquad 1 \le j \le n-1 \end{aligned}$$

It can be easily verified that f is a super mean labeling and hence C_{2n} is a super mean graph. For example the super mean labeling of C_{10} is shown in Fig.2.4. □



C_{10}

Fig.2.4

Remark 2.5 In [2], it was proved that $C_{2n+1}, n \geq 1$ is a super mean graph and C_4 is not a super mean graph and hence the cycle C_n is a super mean graph for $n \geq 3$ and $n \neq 4$.

Theorem 2.6 *Corona of a cycle C_n is a super mean graph for $n \geq 3$.*

Proof Let C_n be a cycle with vertices $u_1, u_2, \ldots, u_n$ and edges $e_1, e_2, \ldots, e_n$. Let $v_1, v_2, \ldots, v_n$ be the corresponding new vertices in $C_n \odot K_1$ and E_i be the edges joining $u_iv_i, i = 1$ to n.

Define $f : V(C_n \odot K_1) \to \{1, 2, \ldots, p+q\}$ as follows:

Case i When n is odd, $n = 2m+1, m = 1, 2, 3, \ldots$

$$f(u_1) = 3$$

$$f(u_i) = \begin{cases} 5 + 8(i-2) & 2 \le i \le m+1 \\ 12 + 8(2m+1-i) & m+2 \le i \le 2m+1 \end{cases}$$

$$f(v_1) = 1$$

$$f(v_i) = \begin{cases} 7 + 8(i-2) & 2 \le i \le m+1 \\ 10 + 8(2m+1-i) & m+2 \le i \le 2m+1 \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$$f^*(e_1) = 4$$

$$f^*(e_i)=\begin{cases}9+8(i-2) & 2\le i\le m+1\\ 8+8(2m+1-i) & m+2\le i\le 2m+1\end{cases}$$

$$f^*(E_1)=2$$

$$f^*(E_i)=\begin{cases}6+8(i-2) & 2\le i\le m+1\\ 11+8(2m+1-i) & m+2\le i\le 2m+1\end{cases}$$

Case ii When n is even, $n=2m, m=2,3,\ldots$

$$\begin{aligned}
f(u_1) &= 3\\
f(u_i) &= 5+8(i-2), && 2\le i\le m\\
f(u_{m+1}) &= 8m-2,\\
f(u_i) &= 12+8(2m-i), && m+2\le i\le 2m\\
f(v_1) &= 1\\
f(v_i) &= 7+8(i-2), && 2\le i\le m\\
f(v_{m+1}) &= 8m,\\
f(v_{m+2}) &= 8m-7,\\
f(v_i) &= 10+8(2m-i), && m+3\le i\le 2m
\end{aligned}$$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$$\begin{aligned}
f^*(e_1) &= 4\\
f^*(e_i) &= 9+8(i-2), && 2\le i\le m-1\\
f^*(e_m) &= 8m-6,\\
f^*(e_{m+1}) &= 8m-3,\\
f^*(e_i) &= 8+8(2m-i), && m+2\le i\le 2m\\
f^*(E_1) &= 2\\
f^*(E_i) &= 6+8(i-2), && 2\le i\le m\\
f^*(E_{m+1}) &= 8m-1\\
f^*(E_i) &= 11+8(2m-i), && m+2\le i\le 2m
\end{aligned}$$

It can be easily verified that f is a super mean labeling and hence $C_n\odot K_1$ is a super mean graph. For example the super mean labelings of $C_7\odot K_1$ and $C_8\odot K_1$ are shown in Fig.2.5.□

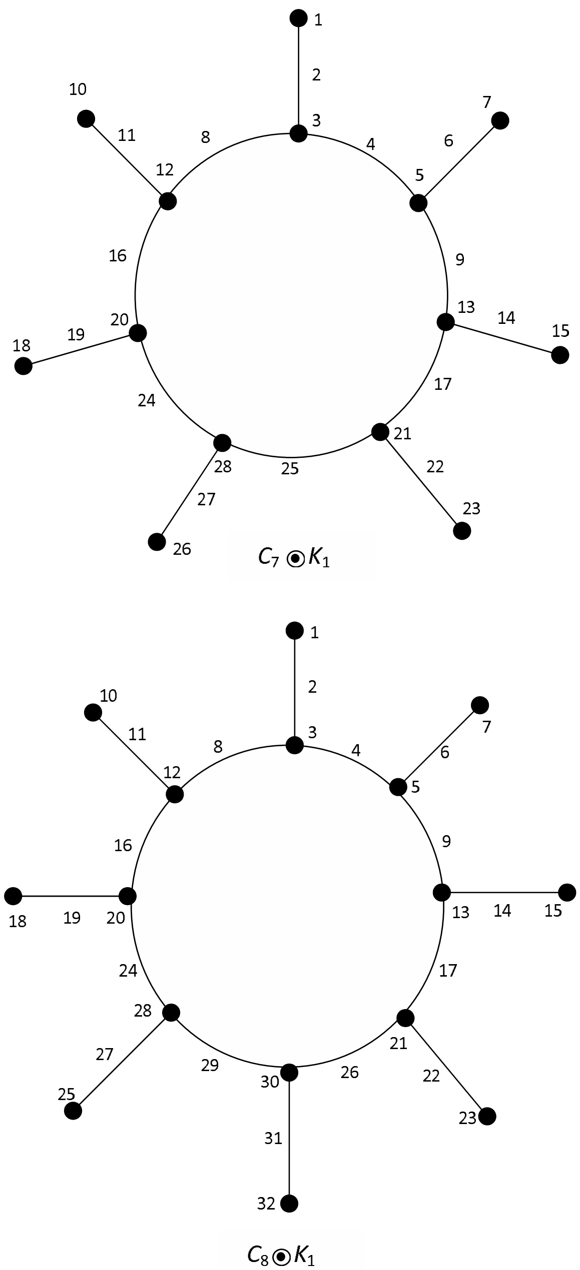


Fig.2.5

Remark 2.7 C_4 is not a super mean graph, but $C_4 \odot K_1$ is a super mean graph.

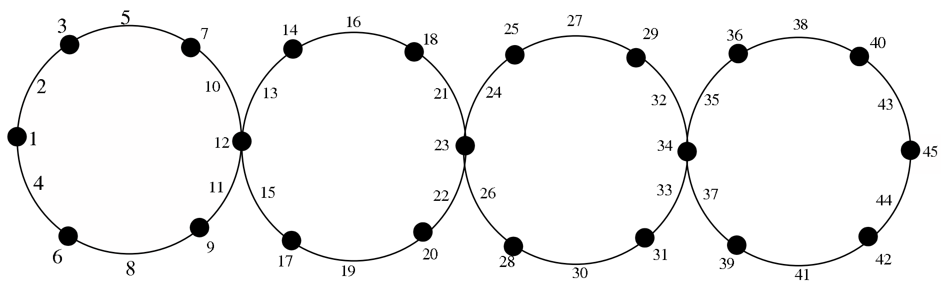
Theorem 2.8 *The graph mC_n- snake, $m \geq 1, n \geq 3$ and $n \neq 4$ has a super mean labeling.*

Proof We prove this result by induction on m.

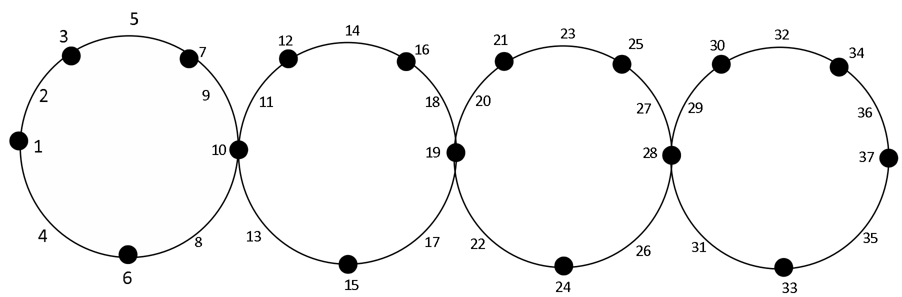
Let $v_{1_j}, v_{2_j}, \ldots, v_{n_j}$ be the vertices and $e_{1_j}, e_{2_j}, \ldots, e_{n_j}$ be the edges of mC_n for $1 \leq j \leq m$. Let f be a super mean labeling of the cycle C_n.

When $m = 1$, by Remark 1.5, C_n is a super mean graph, $n \geq 3, n \neq 4$. Hence the result is true when $m = 1$.

Let $m = 2$. The cyclic snake $2C_n$ is the graph obtained from 2 copies of C_n by identifying the vertex $v_{(k+2)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n when $n = 2k+1$ and identifying the vertex $v_{(k+1)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n when $n = 2k$.



$4C_6$-snake



$4C_5$-snake

Fig.2.6

Define a super mean labeling g of $2C_n$ as follows:

For $1 \leq i \leq n$,

$$
\begin{aligned}
g(v_{i_1}) &= f(v_{i_1}) \\
g(v_{i_2}) &= f(v_{i_1}) + 2n - 1 \\
g^*(e_{i_1}) &= f^*(e_{i_1}) \\
g^*(e_{i_2}) &= f^*(e_{i_1}) + 2n - 1.
\end{aligned}
$$

Thus, $2C_n$-snake is a super mean graph.

Assume that mC_n-snake is a super mean graph for any $m \geq 1$. We will prove that $(m+1)C_n$-snake is a super mean graph. Super mean labeling g of $(m+1)C_n$ is defined as follows:

$$
\begin{aligned}
g(v_{i_j}) &= f(v_{i_1}) + (j-1)(2n-1), && 1 \leq i \leq n, 2 \leq j \leq m \\
g(v_{i_{m+1}}) &= f(v_{i_1}) + m(2n-1), && 1 \leq i \leq n
\end{aligned}
$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$g^*(e_{i_j}) = f^*(e_{i_1}) + (j-1)(2n-1), \quad 1 \le i \le n, 2 \le j \le m$$
$$g^*(e_{i_{m+1}}) = f^*(e_{i_1}) + m(2n-1), \quad 1 \le i \le n$$

Then it is easy to check the resultant labeling g is a super mean labeling of $(m+1)C_n$-snake. For example the super mean labelings of $4C_6$-snake and $4C_5$- snake are shown in Fig.2.6. □

Theorem 2.9 *If G is a super mean graph then $P_n(G)$ is also a super mean graph.*

Proof Let f be a super mean labeling of G. Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_1, e_2, \ldots, e_q$ be the edges of G and let $u_1, u_2, \ldots, u_n$ and $E_1, E_2, \ldots, E_{n-1}$ be the vertices and edge of P_n respectively.

We define g on $P_n(G)$ as follows:

$$g(v_i) = f(v_i), \quad 1 \le i \le p.$$
$$g(u_j) = p + q + 2j - 2, \quad 1 \le j \le n.$$

For the vertex labeling g, the induced edge labeling g^* is defined as follows:

$$g^*(e_i) = f(e_i) \quad 1 \le i \le p.$$
$$g^*(E_j) = p + q + 2j - 1, \quad 1 \le j \le n-1.$$

Then g is a super mean labeling of $P_n(G)$. □

Corollary 1.10 *Dragon $P_n(C_m)$ is a super mean graph for $m \ge 3$ and $m \ne 4$.*

Proof Since C_m is a super mean graph for $m \ge 3$ and $m \ne 4$, by using the above theorem, $P_n(C_m)$ for $m \ge 3$ and $m \ne 4$ is also a super mean graph. For example, the super mean labeling of $P_5(C_6)$ is shown in Fig.2.7. □

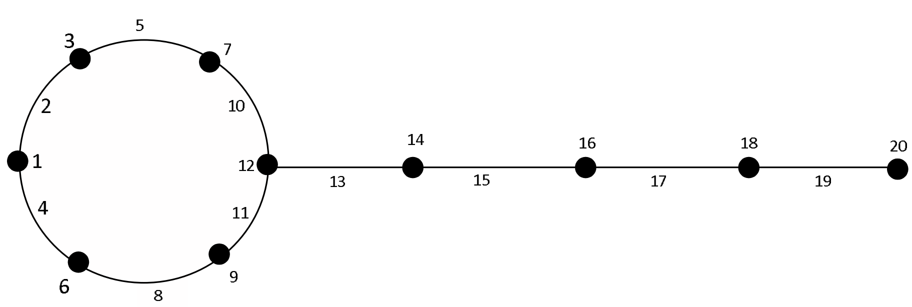


Fig.2.7

Remark 2.11 The converse of the above theorem need not be true. For example consider the graph C_4. $P_n(C_4)$ for $n \ge 3$ is a super mean graph but C_4 is not a super mean graph. The super mean labeling of the graph $P_4(C_4)$ is shown in Fig.2.8

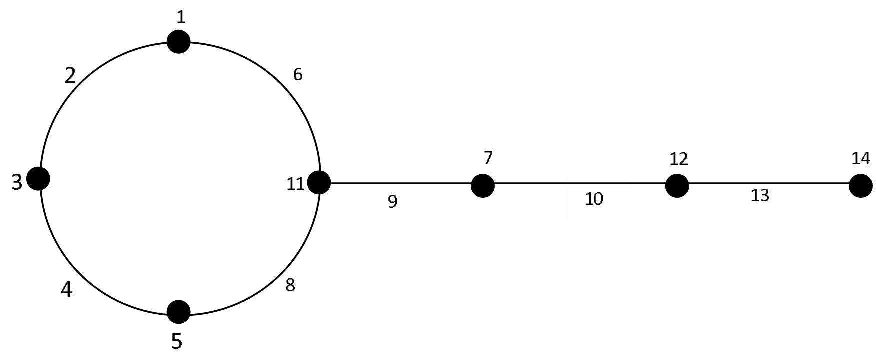


Fig.2.8

Theorem 2.12 *$C_m \times P_n$ for $n \geq 1, m = 3, 5$ are super mean graphs.*

Proof Let $V(C_m \times P_n) = \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(C_m \times P_n) = \{e_{i_j} : e_{i_j} = v_{i_j} v_{(i+1)_j}, 1 \leq j \leq n, 1 \leq i \leq m\} \cup \{E_{i_j} : E_{i_j} = v_{i_j} v_{i_{j+1}}, 1 \leq j \leq n-1, 1 \leq i \leq m\}$ where $i+1$ is taken modulo m.

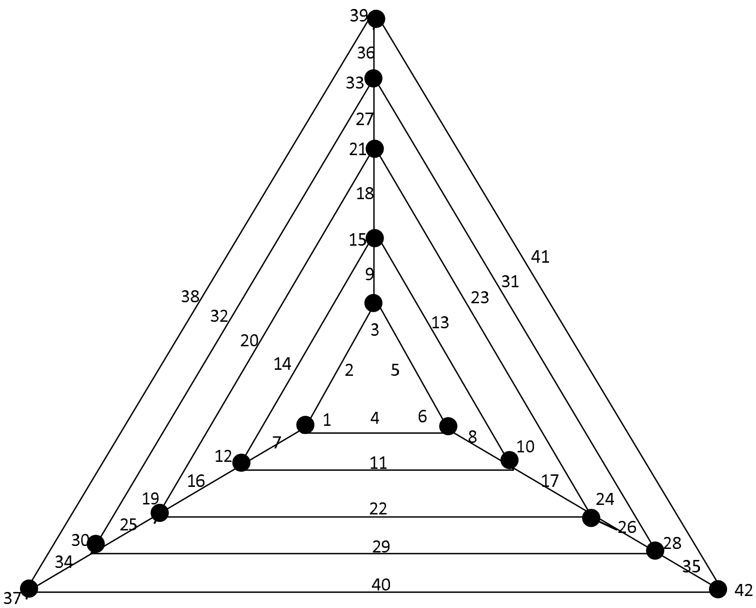
Case i $m = 3$

First we label the vertices of C_3^1 and C_3^2 as follows:

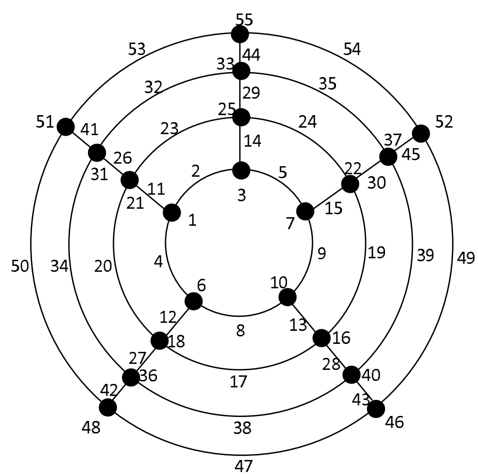
$$
\begin{aligned}
f(v_{1_1}) &= 1 \\
f(v_{i_1}) &= 3i - 3, && 2 \leq i \leq 3 \\
f(v_{i_2}) &= 12 + 3(i-1), && 1 \leq i \leq 2 \\
f(v_{3_2}) &= 10
\end{aligned}
$$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$$
\begin{aligned}
f^*(e_{i_1}) &= 2 + 3(i-1), && 1 \leq i \leq 2 \\
f^*(e_{3_1}) &= 4 \\
f^*(e_{1_2}) &= 14 \\
f^*(e_{i_2}) &= 13 - 2(i-2), && 2 \leq i \leq 3 \\
f^*(E_{i_1}) &= 7 + 2(i-1), && 1 \leq i \leq 2 \\
f^*(E_{3_1}) &= 8
\end{aligned}
$$



$C_3 \times P_5$



$C_5 \times P_4$

Fig.2.9

If the vertices and edges of C_3^{2j-1} and C_3^{2j} are labeled then the vertices and edges of C_3^{2j+1} and C_3^{2j+2} are labeled as follows:

$$f(v_{i_{2j+1}}) = f(v_{i_{2j-1}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-1}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even}$$

$$f(v_{i_{2j+2}}) = f(v_{i_{2j}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even.}$$

$$f^*(e_{i_{2j+1}}) = f^*(e_{i_{2j-1}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-1}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even}$$

$$f^*(e_{j_{2j+2}}) = f^*(e_{i_{2j}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even}$$

$$f^*(E_{i_{2j+1}}) = f^*(E_{i_{2j-1}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even}$$

$$f^*(E_{i_{2j+2}}) = f^*(E_{i_{2j}}) + 18, \quad 1 \le i \le 3, 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and } 1 \le j \le \tfrac{n-4}{2} \text{ if } n \text{ is even}$$

Case ii $m = 5$.

First we Label the vertices of C_5^1 and C_5^2 as follows:

$$f(v_{1_1}) = 1$$

$$f(v_{i_1}) = \begin{cases} 4i-5, & 2 \le i \le 3 \\ 10-4(i-4), & 4 \le i \le 5 \end{cases}$$

$$f(v_{1_2}) = 21$$

$$f(v_{i_2}) = \begin{cases} 25-3(i-2), & 2 \le i \le 3 \\ 16+2(i-4) & 4 \le i \le 5 \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is defined as follows:

$$f^*(e_{i_1}) = 2+3(i-1), \quad 1 \le i \le 2$$

$$f^*(e_{3_1}) = 9$$

$$f^*(e_{i_1}) = 8-4(i-4), \quad 4 \le i \le 5$$

$$f^*(e_{i_2}) = \begin{cases} 23+(i-1), & 1 \le i \le 2 \\ 19-2(i-3), & 3 \le i \le 4 \end{cases}$$

$$f^*(e_{5_2}) = 20,$$

$$f^*(E_{1_1}) = 11$$

$$f^*(E_{i_1}) = \begin{cases} 14+(i-2), & 2 \le i \le 3 \\ 13-(i-4), & 4 \le i \le 5 \end{cases}$$

If the vertices and edges of C_5^{2j-1} and C_5^{2j} are labeled then the vertices and edges of C_5^{2j+1} and C_5^{2j+2} are labeled as follows:

$$
\begin{aligned}
f(v_{i_{2j+1}}) &= l(v_{i_{2j-1}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even and} \\
& && 1 \le j \le \tfrac{n-1}{2} \text{ if } n \text{ is odd} \\
f(v_{i_{2j+2}}) &= l(v_{i_{2j}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even and} \\
& && 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd.} \\
f^*(E_{i_{2j+1}}) &= f^*(E_{i_{2j-1}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and} \\
& && 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even} \\
f^*(E_{i_{2j+2}}) &= f^*(E_{i_{2j}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd and} \\
& && 1 \le j \le \tfrac{n-4}{2} \text{ if } n \text{ is even} \\
f^*(e_{i_{2j+1}}) &= f^*(e_{i_{2j-1}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even and} \\
& && 1 \le j \le \tfrac{n-1}{2} \text{ if } n \text{ is odd} \\
f^*(e_{i_{2j+2}}) &= f^*(e_{i_{2j}}) + 30, 1 \le i \le 5, && 1 \le j \le \tfrac{n-2}{2} \text{ if } n \text{ is even and} \\
& && 1 \le j \le \tfrac{n-3}{2} \text{ if } n \text{ is odd.}
\end{aligned}
$$

Then it is easy to check that the labeling f is a super mean labeling of $C_3 \times P_n$ and $C_5 \times P_n$. For example the super mean labeling of $C_3 \times P_5$ and $C_5 \times P_4$ are shown in Fig.2.9. □

§3. Open Problems

We present the following open problem for further research.

Open Problem. *For what values of m (except 3,5) the graph $C_m \times P_n$ is super mean graph.*

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Chromatic Polynomial of Smarandache ν_E-Product of Graphs

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Abstract: Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. For a chosen edge set $E \subset E_2$, the *Smarandache ν_E-product* $G_1 \times_{\nu_E} G_2$ of $G_1,\ G_2$ is defined by

$$V(G_1 \times_{\nu_E} G_2) = V_1 \times V_2,$$

$$E(G_1 \times_{\nu_E} G_2) = \{(a,b)(a',b')|a = a', (b,b') \in E_2, \text{or } b = b', (a,a') \in E_1\}$$
$$\cup\{(a,b)(a',b')|(a,a') \in E_1 \text{ and } (b,b') \in E\}.$$

Particularly, if $E = \emptyset$ or E_2, then $G_1 \times_{\nu_E} G_2$ is the *Cartesian product* $G_1 \times G_2$ or *strong product* $G_1 * G_2$ of G_1 and G_2 in graph theory. Finding the chromatic polynomial of Smarandache ν_E-product of two graphs is an unsolved problem in general, even for the Cartesian product and strong product of two graphs. In this paper we determine the chromatic polynomial in the case of the Cartesian and strong product of a tree and a complete graph.

Keywords: Coloring graph, Smarandache ν_E-product graph, strong product graph, Cartesian product graph, chromatic polynomial.

AMS(2000): 05C15

§1. Introduction

Sabidussi and Vizing defined Graph products first time in [4] [5]. A lot of works has been done on various topics related to graph products, however there are still many open problems [3]. Generally, we can construct Smarandache ν_E-product of graphs G_1 and G_2 for $E \subset E(G_2)$ as follows.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. For a chosen edge set $E \subset E_2$, the *Smarandache ν_E-product* $G_1 \times_{\nu_E} G_2$ of $G_1,\ G_2$ is defined by

$$V(G_1 \times_{\nu_E} G_2) = V_1 \times V_2,$$

$$E(G_1 \times_{\nu_E} G_2) = \{(a,b)(a',b')|a = a', (b,b') \in E_2, \text{or } b = b', (a,a') \in E_1\}$$
$$\cup\{(a,b)(a',b')|(a,a') \in E_1 \text{ and } (b,b') \in E\}.$$

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Particularly, if $E = \emptyset$ or E_2, then $G_1 \times_{\nu_E} G_2$ is nothing but the *Cartesian product* $G_1 \times G_2$ or *strong product* $G_1 * G_2$ of G_1 and G_2 in graph theory.

The chromatic polynomial of graph G, $\chi(G,k)$ is the number of different coloring ways, with at most k color. The chromatic number of G is the smallest integer k such that $\chi(G,k)$ be positive [2, 6]. Thus we can determine $\chi(G)$ by calculating $\chi(G,k)$. Let $G-e$ be a graph obtained by deleting an edge e from G. An edge e of G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G \cdot e$. The following theorems are well known.

Theorem 1.1([2, 6], Chromatic recurrence) *If G is a simple graph and $e \in E(G)$, then*

$$\chi(G,k) = \chi(G-e,k) - \chi(G \cdot e,k).$$

Theorem 1.2([2]) *If G has n components $G_1, G_2, \cdots, G_n$, then*

$$\chi(G,k) = \chi(G_1,k)\chi(G_2,k)\cdots\chi(G_n,k).$$

Theorem 1.3([2]) *Let G be a graph with subgraphs G_1, G_2 such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = K_n$. Then*

$$\chi(G,k) = \frac{\chi(G_1,k)\chi(G_2,k)}{\chi(k_n,k)}.$$

Example 1.1 If K_n is a complete graph with n vertices then

$$\chi(K_n,k) = (k)_n = k(k-1)(k-2)\cdots(k-n+1).$$

Example 1.2 If C_n is a cycle graph with n vertices then

$$\chi(C_n,k) = (k-1)^n + (-1)(k-1).$$

§2. Cartesian Product

In this section, we consider the chromatic polynomial of Cartesian product, i.e., Smarandache ν_E-product graph of two graphs G_1, G_2 with $E = \emptyset$.

Theorem 2.1 *Let K_2 be a complete graph with two vertices and P_n be a path with $n \geq 3$ vertices, then $\chi(P_n \times K_2,k) = (k^2-3k+3)\chi(P_{n-1} \times K_2,k)$.*

Proof If $G_1 = P_{n-1} \times K_2, G_2 = C_4$, we have $G_1 \cup G_2 = P_n \times K_2, G_1 \cap G_2 = K_2$. then by Theorem 1.3, we have

$$\begin{aligned}\chi(P_n \times K_2, k) &= \frac{\chi(P_{n-1} \times K_2, k)\chi(C_4, k)}{\chi(K_2, k)} = \frac{\chi(P_{n-1} \times K_2, k)((k-1)^4 + (-1)(k-1))}{K(k-1)} \\ &= (k^2 - 3k + 3)\chi(P_{n-1} \times K_2, k).\end{aligned}$$

□

By continue above recursive relation, we have following result.

Corollary 2.1 *For $n \geq 3$, $\chi(P_n \times K_2, k) = k(k-1)(k^2 - 3k + 3)^{n-1}$.*

Theorem 2.2 *For each path P_n, $n \geq 3$ and complete graph K_m, $m \geq 2$,*

$$\chi(P_n \times K_m, k) = (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m, i)}{\chi(K_i, k)})^{n-1},$$

where $C(m, i)$ is choice of m vertices for i.

Proof If we consider $G_1 = P_{n-1} \times K_m$, $G_2 = P_2 \times K_m$, then G_1, G_2 are subgraphs of $P_n \times K_m$ such that $G_1 \cup G_2 = P_n \times K_m, G_1 \cap G_2 = K_m$. Therefore, by Theorem 1.3 it follows that,

$$\chi(P_n \times K_m, k) = \frac{\chi(P_{n-1} \times K_m, k)\ \chi(P_2 \times K_m, k)}{\chi(K_m, k)}$$

and with a recursive use of this relation, we have

$$\chi(P_n \times K_m, k) = \frac{\chi(P_2 \times K_m, k)^{n-1}}{\chi(K_m, k)^{n-2}}.$$

Then is sufficient to compute the chromatic polynomial of $P_2 \times K_m$. By using Theorem 1.1 and deleting and contracting the edges of P_2 in this product, at the end, we obtain 2^m graphs. Each of these graphs consist of two copy of K_m which have a $K_i, (0 \leq i \leq m)$ in their intersection, and there is no other edges than these. The chromatic polynomial of these graphs, by using Theorem 1.2 is

$$\phi_i(k) = \frac{(\chi(K_m, k))^2}{\chi(K_i, k)}.$$

Here we define $\chi(K_0, k) = 1$. On the other hand we have a choice of m vertices for i, so the number of these graphs is equal to $C(m, i)$. Since for each i these graphs have $2m - i$ vertices each, thus in summation a coefficient $(-1)^i$ appears,

$$\chi(P_2 \times K_m, k) = \sum_{i=0}^{m} (-1)^i C(m, i) \phi_i(k),$$

then

$$\begin{aligned}\chi(P_n \times K_m, k) &= \frac{\chi(P_2 \times K_m, k)^{n-1}}{\chi(K_m, k)^{n-2}} = \frac{(\sum_{i=0}^{m}(-1)^i C(m,i)\frac{(\chi(K_m,\ k))^2}{\chi(K_i,\ k)})^{n-1}}{\chi(K_m, k)^{n-2}} \\ &= (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.\end{aligned}$$

□

Note that in the steps of this proof we have not used the structure of P_n. But we use only the existence of a vertex of degree one in each step in finding recursive relation. So we can use this argument alternatively for each tree with n vertices. In fact P_n is a special case of tree. Therefore we can obtain a more general result following.

Corollary 2.2 *Let T_n be a tree with n vertices and K_m be a complete graph of m vertices, then*

$$\chi(T_n \times K_m, k) = (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.$$

Corollary 2.3 $\chi(C_n \times K_m,\ k) = (k-1)^m (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i,k)})^{n-1}.$

Proof Let $G_1 = P_n \times K_m,\ G_2 = P_2 \times K_m - E(K_m)$, then $G_1,\ G_2$ are subgraphs of $C_n \times K_m$ such that $G_1 \cup G_2 = C_n \times K_m, G_1 \cap G_2 = K_m$. Therefore, by Theorem 1.3 it follows that

$$\chi(C_n \times K_m, k) = \frac{\chi(P_n \times K_m, k)\ \chi(G_2, k)}{\chi(K_m, k)}$$

But $\chi(G_2, k) = (k-1)^m\ \chi(K_m, k)$ then

$$\begin{aligned}\chi(C_n \times K_m, k) &= \frac{\chi(P_n \times K_m, k)\ (k-1)^m \chi(K_m, k)}{\chi(K_m, k)} \\ &= (k-1)^m (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.\end{aligned}$$

□

Thus for the Cartesian product of a complete graph and a tree and a cycle, the chromatic polynomial is found. However this for two complete graphs is open. If the chromatic polynomial of Cartesian product of two complete graphs of order n is found, we can determine the number of Latin squares of order n. Moreover $\chi(P_n \times P_m)$ is not yet known [1].

§3. Strong Products

In this section, we consider the chromatic polynomial of strong product, i.e., Smarandache ν_E-product graph of two graphs $G_1,\ G_2$ with $E = E(G_2)$. We get some theorems for chromatic polynomial of strong products same as Cartesian product.

Theorem 3.1 *Let K_2 be a complete graph with two vertices and P_n be a path with $n \geq 3$ vertices, then*

$$\chi(P_n * k_2, k) = (k^2 - 3k + 3)\chi(P_{n-1} * k_2, k).$$

So we have the following theorem by above recursive relation.

Corollary 3.1 *For $n \geq 3$, $\chi(P_n * K_2, k) = k(k-1)(k^2 - 5k + 6)^{n-1}$.*

Theorem 3.2 *For each path P_n, $n \geq 3$ and complete graph K_m, $m \geq 2$*

$$\chi(P_n * K_m, k) = \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}.$$

Proof If we consider $G_1 = P_{n-1} * K_m$, $G_2 = P_2 * K_m$, by Theorem 1.3 it follows that

$$\chi(P_n * K_m, k) = \frac{\chi(P_{n-1} * K_m, k)\ \chi(P_2 * K_m, k)}{\chi(K_m, k)}$$

and with a recursive use of this relation we have

$$\chi(P_n * K_m, k) = \frac{(\chi(P_2 * K_m, k)^{n-1}}{\chi(K_m, k)},$$

but $P_2 * K_m = K_{2m}$ and thus

$$\begin{aligned}
\chi(P_n * K_m, k) &= \frac{[(k)_{2m}]^{n-1}}{[(k)_m]^{n-2}} = \frac{[k(k-1)\cdots(k-2m+1)]^{n-1}}{[k(k-1)\cdots(k-m+1)]^{n-2}} \\
&= k(k-1)...(k-m+1)[(k-m)(k-m-1)...(k-2m+1)]^{n-1} \\
&= \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}.
\end{aligned}$$

□

Therefore we obtain a more general result as follows.

Corollary 3.2 *Let T_n be a tree with n vertices and K_m be a complete graph with m vertices, then*

$$\chi(T_n * K_m, k) = \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}.$$

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Open Distance-Pattern Uniform Graphs

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Abstract: Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f^o_M(u) = \{d(u, v) \,:\, v \in M,\ u \neq v\}$, called its open M-distance-pattern. A graph G is called a *Smarandachely uniform k-graph* if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f^o_{M_i}(u) = f^o_{M_j}(u)$ and $f^o_{M_i}(u) = f^o_{M_j}(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a *k-family of open distance-pattern uniform (odpu-) set of G* and the minimum cardinality of odpu-sets in G, if they exist, is called the *Smarandachely odpu-number* of G, denoted by $od^S_k(G)$. Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph*. In this case, its odpu-number $od^S_k(G)$ of G is abbreviated to $od(G)$. In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph.

Key Words: Smarandachely uniform k-graph, open distance-pattern, open distance-pattern, uniform graphs, open distance-pattern uniform (odpu-) set, Smarandachely odpu-number, odpu-number.

AMS(2000): 05C12

§1. Introduction

All graphs considered in this paper are finite, simple, undirected and connected. For graph theoretic terminology we refer to Harary [6].

The concept of open distance-pattern and open distance-pattern uniform graphs were suggested by B.D. Acharya. Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, the open M-distance-pattern of a vertex u in G is defined to be the set $f^o_M(u) = \{d(u, v) : v \in M,\ u \neq v\}$, where $d(x, y)$ denotes the distance between the vertices x and y in G. A graph G is called a *Smarandachely uniform k-graph* if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f^o_{M_i}(u) = f^o_{M_j}(u)$ and $f^o_{M_i}(u) = f^o_{M_j}(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a *k-family of open distance-pattern uniform (odpu-) set of G* and the minimum cardinality of odpu-sets in G, if they exist, is called the *Smarandachely odpu-number* of G, denoted by $od^S_k(G)$. Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph*. In this case, its odpu-number $od^S_k(G)$ of G is abbreviated to $od(G)$. We need the following theorem.

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Theorem 1.1([5]) *Let G be a graph of order $n, n \geq 4$. Then the following conditions are equivalent.*

(i) The graph G is self-centred with radius $r \geq 2$ and for every $u \in V(G)$, there exists exactly one vertex v such that $d(u, v) = r$.

(ii) The graph G is r-decreasing.

(iii) There exists a decomposition of $V(G)$ into pairs $\{u, v\}$ such that $d(u, v) = r(G) > max(d(u, x), d(x, v))$ for every $x \in V(G) - \{u, v\}$.

In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph G.

§2. Odpu-Sets in Graphs

It is clear that an odpu-set in any nontrivial graph must have at least two vertices. The following theorem gives a basic property of odpu-sets.

Theorem 2.1 *In any graph G, if there exists an odpu-set M, then $M \subseteq Z(G)$ where $Z(G)$ is the center of the graph G. Also $M \subseteq Z(G)$ is an odpu-set if and only if $f_M^o(v) = \{1, 2, \ldots, r(G)\}$, for all $v \in V(G)$.*

proof Let G have an odpu-set $M \subseteq V(G)$ and let $v \in M$. Suppose $v \notin Z(G)$. Then $e(v) > r(G)$. Hence there exists a vertex $u \in V(G)$ such that $d(u, v) > r(G)$. Since $v \in M$, $f_M^o(u)$ contains an element, which is greater than $r(G)$. Now let $w \in V(G)$ be such that $e(w) = r(G)$. Then $d(w, v) \leq r(G)$ for all $v \in M$. Hence $f_M^o(w)$ does not contain an element greater than $r(G)$, so that $f_M^o(u) \neq f_M^o(w)$. Thus M is not an odpu-set, which is a contradiction. Hence $M \subseteq Z(G)$.

Now, let $M \subseteq Z(G)$ be an odpu-set. Then $max \quad f_M^o(v) = r(G)$. Let $u \in M$ be such that $d(u, v) = r(G)$. Let the shortest $u - v$ path be $(u = v_1, v_2, \cdots, v_{r(G)} = v)$. Then v_1 is adjacent to u. Therefore, $1 \in f_M^o(v_1)$. Since M is an odpu-set, $1 \in f_M^o(x)$ for all $x \in V(G)$. Now, $d(v_2, u) = 2$, whence $2 \in f_M^o(v_2)$. Since M is an odpu-set, $2 \in f_M^o(x)$ for all $x \in V(G)$. Proceeding like this, we get $\{1, 2, 3, \cdots, r(G)\} \subseteq f_M^o(x)$ and since $M \subseteq Z(G)$, $f_M^o(x) = \{1, 2, 3, \cdots, r(G)\}$ for all $x \in V$. The converse is obvious. □

Corollary 2.2 *A connected graph G is an odpu-graph if and only if the center $Z(G)$ of G is an odpu-set.*

Proof Let G be an odpu-graph with an odpu-set M. Then $f_M^o(v) = \{1, 2, \ldots, r(G)\}$ for all $v \in V(G)$. Since $f_{Z(G)}^o(v) \supseteq f_M^o(v)$ and $d(u, v) \leq r(G)$ for every $v \in V$ and $u \in Z(G)$, it follows that $Z(G)$ is an odpu set of G. The converse is obvious. □

Corollary 2.3 *Every self-centered graph is an odpu-graph.*

Proof Let G be a self-centered graph. Take $M = V(G)$. Since G is self-centered, $e(v) = r(G)$ for all $v \in V(G)$. Therefore, $f^o_M(v) = \{1, 2, \cdots, r(G)\}$ for all $v \in V(G)$, so that M is an odpu-set for G. □

Remark 2.4 The converse of Corollary 2.3 is not true. For example the graph $K_2 + \overline{K_2}$, is not self-centered but it is an odpu-graph. Moreover, there exist self-centered graphs having a proper subset of $Z(G) = V(G)$ as an odpu-set.

Theorem 2.5 *If G is an odpu-graph with $n \geq 3$, then $\delta(G) \geq 2$ and G is 2-connected.*

Proof Let G be an odpu-graph with $n \geq 3$ and let M be an odpu-set of G. If G has a pendant vertex v, it follows from Theorem 2.1 that $v \notin M$. Also, v is adjacent to exactly one vertex $w \in V(G)$. Since M is an odpu-set, $max \ \ f^o_M(w) = r(G)$. Therefore, there exists $u \in M$ such that $d(u, w) = r(G)$. Now $d(u, v) = r(G) + 1$ and $f^o_M(v)$ contains $r(G) + 1$. Hence $f^o_M(v) \neq f^o_M(w)$, a contradiction. Thus $\delta(G) \geq 2$.

Now suppose G is not 2-connected. Let B_1 and B_2 be blocks in G such that $V(B_1) \cap V(B_2) = \{u\}$. Since, the center of a graph lies in a block, we may assume that the center $Z(G) \subseteq B_1$. Let $v \in B_2$ be such that $uv \in E(G)$. Then there exists a vertex $w \in M$ such that $d(u, w) = r(G)$ and $d(v, w) = r(G) + 1$, so that $r(G) + 1 \in f^o_M(u)$, which is a contradiction. Hence G is 2-connected. □

Corollary 2.6 *A tree T has an odpu-set M if and only if T is isomorphic to P_2.*

Corollary 2.7 *If G is a unicyclic odpu-graph, then G is isomorphic to a cycle.*

Corollary 2.8 *A block graph G is an odpu-graph if and only if G is complete.*

Corollary 2.9 *In any graph G, if there exists an odpu-set M, then every subset M' of $Z(G)$ such that $M \subseteq M'$ is also an odpu-set.*

Thus Corollary 2.9 shows that in a limited sense the property of subsets of $V(G)$ being odpu-sets is *super-hereditary* within $Z(G)$. The next remark gives an algorithm to recognize odpu-graphs.

Remark 2.10 Let G be a finite simple connected graph. The the following algorithm recognizes odpu-graphs.

Step-1: Determine the center of the graph G.

Step-2: Generate the $c \times n$ distance matrix $D(G)$ of G where $c = |Z(G)|$.

Step-3: Check whether each column C_i has the elements $1, 2, \ldots, r$.

Step-4: If then, G is an odpu-graph.

Or else G is not an odpu-graph.

The above algorithm is efficient since we have polynomial time algorithm to determine $Z(G)$ and to compute the matrix $D(G)$.

Theorem 2.11 *Every odpu-graph G satisfies, $r(G) \leq d(G) \leq r(G)+1$. Further for any positive integer r, there exists an odpu-graph with $r(G) = r$ and $d(G) = r+1$.*

Proof Let G be an odpu-graph. Since $r(G) \leq d(G)$ for any graph G, it is enough to prove that $d(G) \leq r(G) + 1$. If G is a self-centered graph, then $r(G) = d(G)$. Assume G is not self-centered and let u and v be two antipodal vertices of G. Since G is an odpu-graph, $Z(G)$ is an odpu-set and hence there exist vertices $u', v' \in Z(G)$ such that $d(u, u') = 1$ and $d(v, v') = 1$. Now, G is not self-centered, and $d(u, v) = d$, implies $u, v \notin Z(G)$. If $d > r + 1$; since $d(u, u') = d(v, v') = 1$, the only possibility is $d(u', v') = r$, which implies $d(u, v') = r + 1$. But $v' \in Z(G)$ and hence $r + 1 \in f^o_M(u)$, which is not possible. Hence $d(u, v) = d \leqslant r + 1$ and the result follows.

Now, let r be any positive integer. For $r = 1$ take $G = K_2 + \bar{K_n}, n \geqslant 2$. For $r \geqslant 2$, let G be the graph obtained from C_{2r} by adding a vertex v_e corresponding to each edge e in C_{2r} and joining v_e to the end vertices of e. Then, it is easy to check that an odpu-set of the resulting graph is $V(C_{2r})$. □

However, it should be noted that $d = r+1$ is not a sufficient condition for the graph to be an odpu-graph. For the graph G consisting of the cycle C_r with exactly one pendent edge at one of its vertices, $d = r + 1$ but G is not an odpu-graph.

Remark 2.12 Theorem 2.11 states that there are only two classes of odpu-graphs, those which are self-centered or those for which $d(G) = r(G) + 1$. Hence, the problem of characterizing odpu-graphs reduces to the problem of characterizing odpu-graphs with $d(G) = r(G) + 1$.

The following theorem gives a complete characterization of odpu-graphs with radius one.

Theorem 2.13 *A graph with radius 1 and diameter 2 is an odpu-graph if and only if there exists a subset $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M\rangle$ is complete, $\langle V - M\rangle$ is not complete and any vertex in $V - M$ is adjacent to all the vertices of M.*

Proof Assume that G is an odpu-graph with radius $r = 1$ and diameter $d = 2$. Then, $f^o_M(v) = \{1\}$ for all $v \in V(G)$. If $\langle M\rangle$ is not complete, then there exist two vertices $u, v \in M$ such that $d(u, v) \geq 2$. Hence, both $f^o_M(u)$ and $f^o_M(v)$ contains a number greater than 1, which is not possible. Therefore, $\langle M\rangle$ is complete. Next, if $x \in V - M$ then, since $f^o_M(x) = \{1\}$, x is adjacent to all the vertices of $\langle M\rangle$. Now, if $\langle V - M\rangle$ is complete, then since $\langle M\rangle$ is complete the above argument implies that G is complete, whence diameter of G would be one, a contradiction. Thus, $\langle V - M\rangle$ is not complete.

Conversely assume $\langle M\rangle$ is complete with $|M| \geq 2$, $\langle V - M\rangle$ is not complete and every vertex of $\langle V - M\rangle$ is adjacent to all the vertices in $\langle M\rangle$. Then, clearly, the diameter of G is two and radius of G is one. Also, since $|M| \geq 2$, there exist at least two universal vertices in M (i.e. Each is adjacent to every other vertices in M). Therefore $f^o_M(v) = \{1\}$ for every $v \in V(G)$. Hence G must be an odpu-graph with M as an odpu-set. □

Theorem 2.14 *Let G be a graph of order $n \geq 3$. Then the following are equivalent.*

(*i*) *Every k-element subset of $V(G)$ forms an odpu-set, where $2 \leq k \leq n$.*

(*ii*) *Every* 2*-element subset of* $V(G)$ *forms an odpu-set.*

(*iii*) G *is complete.*

Proof Trivially (*i*) implies (*ii*)

If every 2-element subset M of $V(G)$ forms an odpu-set, then $f^o_M(v) = \{1\}$ for all $v \in V(G)$ and hence G is complete.

Obviously (*iii*) implies (*i*). □

Theorem 2.15 *Any graph* G*(may or may not be connected) with* $\delta(G) \geq 1$ *and having no vertex of full-degree can be embedded into an odpu-graph* H *with* G *as an induced subgraph of* H *of order* $|V(G)| + 2$ *such that* $V(G)$ *is an odpu-set of the graph* H.

Proof Let G be a graph with $\delta(G) \geq 1$ and having no vertex of full-degree. Let $u, v \in V(G)$ be any two adjacent vertices and let $a, b \notin V(G)$. Let H be the graph obtained by joining a to b and also, joining a to all vertices of G except u and joining the vertex b to all vertices of G except v. Let $M = V(G) \subset V(H)$. Since a is adjacent to all the vertices except u and $d(a, u) = 2$, implies $f^o_M(a) = \{1, 2\}$. Similarly $f^o_M(b) = \{1, 2\}$. Since u is adjacent to v, $1 \in f^o_M(u)$. Since u does not have full degree, there exists a vertex x, which is not adjacent to u. But (u, b, x) is a path in H and hence $d(u, x) = 2$ in H for all such $x \in V(G)$. Therefore $f^o_M(u) = \{1, 2\}$. Similarly $f^o_M(v) = \{1, 2\}$. Now let $w \in V(G)$, $w \neq u, v$. Now since no vertex w is an isolated vertex and w does not have full-degree, there exist vertices x and y in $V(G)$ such that $wx \in E(H)$ and $wy \notin E(H)$. But then, there exists a path (w, a, y) or (w, b, y) with length 2 in H. Also every vertex which is not adjacent to w is at a distance 2 in H. Therefore $f^o_M(w) = \{1, 2\}$. Hence $f^o_M(x) = \{1, 2\}$ for all $x \in V(H)$. Hence H is an odpu-graph and $V(G)$ is an odpu-set of H. □

Remark 2.16 Bollobás [1] proved that almost all graphs have diameter 2 and almost no graph has a node of full degree. Hence almost no graph has radius one. Since $r(G) \leq d(G)$, almost all graphs have $r(G) = d(G) = 2$, that is, almost all graphs are self-centered with diameter 2. Since self-centered graphs are odpu-graphs, the following corollary is immediate.

Corollary 2.17 *Almost all graphs are odpu-graphs.*

§3. Odpu-Number of a Graph

As we have observed in section 2, if G has an odpu-set M then $M \subseteq Z(G)$ and if $M \subseteq M' \subseteq Z(G)$, then M' is also an odpu-set. This motivates the definition of odpu-number of an odpu-graph.

Definition 3.1 *The Odpu-number of a graph* G, *denoted by* $od(G)$, *is the minimum cardinality of an odpu-set in* G.

In this section we characterize odpu-graphs which have odpu-number 2 and also prove that there is no graph with odpu-number 3 and for any positive integer $k \neq 1, 3$, there exists a graph with odpu-number k. We also present several embedding theorems. Clearly,

$$2 \leqslant od(G) \leqslant |Z(G)| \text{ for any odpu} - \text{graph } G. \quad (3.1)$$

Since the upper bound for $|Z(G)|$ is $|V(G)|$, the above inequality becomes,

$$2 \leqslant od(G) \leqslant |V(G)|. \quad (3.2)$$

The next theorem gives a characterization of graphs attaining the lower bound in the above inequality.

Theorem 3.2 *For any graph G, $od(G) = 2$ if and only if there exist at least two vertices $x, y \in V(G)$ such that $d(x) = d(y) = |V(G)| - 1$.*

Proof Suppose that the graph G has an odpu-set M with $|M| = 2$. Let $M = \{x, y\}$. We claim that $d(x) = d(y) = n - 1$, where $n = |V(G)|$. If not, there are two possibilities.

Case 1. $d(x) = n - 1$ and $d(y) < n - 1$.

Since $d(x) = n - 1$, x is adjacent to y. Therefore, $f^o_M(x) = \{1\}$. Also, since $d(y) < n - 1$, it follows that $2 \in f^o_M(w)$ for any vertex w not adjacent to v, which is a contradiction.

Case 2. $d(x) < n - 1$ and $d(y) < n - 1$.

If $xy \in E(G)$, then $f^o_M(x) = f^o_M(y) = \{1\}$ and for any vertex w not adjacent to u, $f^o_M(w) \neq \{1\}$.

If $xy \notin E(G)$, then $1 \notin f^o_M(x)$ and for any vertex w which is adjacent to x, $1 \in f^o_M(w)$, which is a contradiction. Hence $d(x) = d(y) = n - 1$.

Conversely, let G be a graph with $u, v \in V(G)$ such that $d(u) = d(v) = n-1$. Let $M = \{u, v\}$. Then $f^o_M(x) = \{1\}$ for all $x \in V(G)$ and hence M is an odpu-set with $|M| = 2$. □

Corollary 3.3 *For any odpu-graph G if $|M| = 2$, then $\langle M \rangle$ is isomorphic to K_2.*

Corollary 3.4 *$od(K_n) = 2$ for all $n \geqslant 2$.*

Corollary 3.5 *If a (p, q)-graph has an odpu-set M with odpu-number 2, then $2p - 3 \leq q \leq \frac{p(p-1)}{2}$.*

Proof By Theorem 3.2, there exist at least two vertices having degree $p - 1$ and hence $q \geq 2p - 3$. The other inequality is trivial. □

Theorem 3.6 *There is no graph with odpu-number three.*

Proof Suppose there exists a graph G with $od(G) = 3$ and let $M = \{x, y, z\}$ be an odpu-set in G. Since G is connected, $1 \in f^o_M(x) \cap f^o_M(y) \cap f^o_M(z)$.

We claim that x, y, z form a triangle in G. Since $1 \in f^o_M(x)$, and $1 \in f^0_M(z)$, we may assume that $xy, yz \in E(G)$. Now if $xz \notin E(G)$, then $d(x, z) = 2$ and hence $2 \in f^o_M(x) \cap f^0_M(Z)$ and $f^o_M(y) = \{1\}$, which is not possible. Thus $xz \in E(G)$ and x, y, z forms a triangle in G.

Now $f^o_M(w) = \{1\}$ for any $w \in V(G) - M$ and hence w is adjacent to all the vertices of M. Thus G is complete and $od(G) = 2$, which is again a contradiction. Hence there is no graph G with $od(G) = 3$. □

Next we prove that the existence of graph with odpu-numbers $k \neq 1, 3$. We need the following definition.

Definition 3.7 *The shadow graph $S(G)$ of a graph G is obtained from G by adding for each vertex v of G a new vertex v', called the shadow vertex of v, and joining v' to all the neighbors of v in G.*

Theorem 3.8 *For every positive integer $k \neq 1, 3$, there exists a graph G with odpu-number k.*

Proof Clearly $od(P_2) = 2$ and $od(C_4) = 4$. Now we will prove that the shadow graph of any complete graph K_n, $n \geq 3$ is an odpu-graph with odpu-number $n + 2$.

Let the vertices of the complete graph K_n be $v_1, v_2, \ldots, v_n$ and the corresponding shadow vertices be $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$. Since the shadow graph $S(K_n)$ of K_n is self-centered with radius 2 and $n \geq 3$, by Corollary 2.3, it is an odpu-graph. Let M be the smallest odpu-set of $S(K_n)$. We establish that $|M| = n + 2$ in the following three steps.

First, we show $\{v_1^{'}, v_2^{'}, \cdots, v_n^{'}\} \subseteq M$. If there is a shadow vertex $v_i^{'} \notin M$, then $2 \notin f^o_M(v_i)$ since v_i is adjacent to all the vertices of $S(K_n)$ other than $v_i^{'}$, implying thereby that M is not an odpu-set, contrary to our assumption. Thus, the claim holds.

Now, we show that $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$ is not an odpu-set of $S(K_n)$. Note that $v_1^{'}, v_2^{'}, \ldots, v_n^{'}$ are pairwise non-adjacent and if $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$, then $1 \notin f^o_M(v_i^{'})$ for all $v_i^{'} \in M$. But $1 \in f^o_M(v_i)$, $1 \leq i \leq n$, and hence M is not an odpu-set.

From the above two steps, we conclude that $|M| > n$. Now, $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\} \cup \{v_i\}$ where v_i is any vertex of K_n is not an odpu-set. Further, since all the shadow vertices are pairwise nonadjacent and v_i is not adjacent to $v_i^{'}$, $1 \notin f^o_M(v_i^{'})$. Hence $|M| > n + 1$. Let v_i, $v_j \in V(K_n)$ be any two vertices of K_n and let $M = \{v_i, v_j, v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$. We prove that M is an odpu-set and thereby establish that $od(G) = n + 2$. Now, $d(v_i, v_j) = 1$ and $d(v_i, v_i^{'}) = d(v_j, v_j^{'}) = 2$, so that $f^o_M(v_i) = f^o_M(v_j) = \{1, 2\}$. Also, for any vertex $v_k \in V(K_n)$, $d(v_k, v_i) = 1$ and $d(v_k, v_k^{'}) = 2$, so that $f^o_M(v_k) = \{1, 2\}$. Again, $d(v_i^{'}, v_j) = d(v_j^{'}, v_i) = 1$ and for any shadow vertex $v_k^{'} \in V(S(K_n))$, $d(v_k^{'}, v_i) = 1$ and since all the shadow vertices are pairwise non-adjacent, $f^o_M(v_k^{'}) = \{1, 2\}$. Thus, M is an odpu-set and $od(G) = n + 2$. □

Remark 3.9 We have proved that 3 cannot be the odpu number of any graph. Hence, by the above theorem, for an odpu-graph the numbers 1 and 3 are the only two numbers forbidden as odpu-numbers of any graph.

Theorem 3.10 $od(C_{2k+1}) = 2k$.

Proof Let $C_{2k+1} = (v_1, v_2, \ldots, v_{2k+1}, v_1)$. Clearly $M = \{v_1, v_2, \ldots, v_{2k}\}$ is an odpu-set of C_{2k+1}. Now, let M be any odpu-set of C_{2k+1}. Then, there exists a vertex $v_i \in V(C_{2k+1})$ such that $v_i \notin M$. Without loss of generality, assume that $v_i = v_{2k+1}$. Then, since $1 \in f^o_M(v_{2k+1})$, either $v_{2k} \in M$ or $v_1 \in M$ or both $v_1, v_{2k} \in M$. Without loss of generality, let $v_1 \in M$. Since

$d(v_1, v_{2k+1}) = 1$ and $v_{2k+1} \notin M$, and v_2 is the only element other than v_{2k+1} at a distance 1 from v_1, we see that $v_2 \in M$. Now, $d(v_2, v_{2k+1}) = 2$ and $v_{2k+1} \notin M$, and v_4 is the only element other than v_{2k+1} at a distance 2; this implies $v_4 \in M$. Proceeding in this manner, we get $v_2, v_4 \ldots, v_{2k} \in M$. Now since $d(v_{2k}, v_{2k+1}) = 1$ and $v_{2k+1} \notin M$, and v_{2k-1} is the only element other than v_{2k+1} at a distance 1 from v_{2k}, we get $v_{2k-1} \in M$. Next, since $d(v_{2k-1}, v_{2k+1}) = 2$ and $v_{2k+1} \notin M$, and v_{2k-3} is the only element other than v_{2k+1} at a distance 2 from v_{2k-1}, we get $v_{2k-3} \in M$. Proceeding like this, we get $M = \{v_1, v_2, \ldots, v_{2k}\}$. Hence $od(C_{2k+1}) = 2k$. □

Definition 3.11([2]) *A graph is an r-decreasing graph if $r(G - v) = r(G) - 1$ for all $v \in V(G)$.*

We now proceed to characterize odpu-graphs G with $od(G) = |V(G)|$. We need the following lemma.

Lemma 3.12 *Let G be a self-centered graph with $r(G) \geq 2$. Then for each $u \in V(G)$, there exist at least two vertices in every i^{th} neighborhood $N_i(u) = \{v \in V(G) : d(u, v) = i\}$ of u, $i = 1, 2, \ldots, r - 1$.*

Proof Let G be a self-centered graph and let u be any arbitrary vertex of G. If possible, let for some i, $1 \leq i \leq r - 1$, $N_i(u)$ contains exactly one vertex, say w. Then, since $e(w) = r$, there exists $x \in V(G)$ such that $d(x, w) = r$.

If $x \in N_j(u)$ for some $j > i$, then $d(u, x) > r$, which is a contradiction. Again if $x \in N_j(u)$ for some $j < i$, then $d(x, w) = r < i \leq r - 1$, which is again a contradiction. Hence $N_i(u)$ contains at least two vertices. □

Theorem 3.13 *Let G be a graph of order n, $n \geq 4$. Then the following conditions are equivalent.*

(i) $od(G) = n$.

(ii) *the graph G is self-centered with radius $r \geq 2$ and for every $u \in V(G)$, there exists exactly one vertex v such that $d(u, v) = r$.*

(iii) *the graph G is r-decreasing.*

(iv) *there exists a decomposition of $V(G)$ into pairs $\{u, v\}$ such that $d(u, v) = r(G) >$ max $(d(u, x), d(x, v))$ for every $x \in V(G) - \{u, v\}$.*

Proof Let G be a graph of order n, $n \geq 4$. The equivalence of $(ii), (iii)$ and (iv) follows from Theorem 1.1. We now prove that (i) and (ii) are equivalent.

$(i) \Rightarrow (ii)$

Let G be a graph with $od(G) = n = |V(G)|$. Hence, $e(u) = r$ for all $u \in V(G)$ so that G is self-centered. Now, we show that for every $u \in V(G)$, there exists exactly one vertex $v \in V(G)$ such that $d(u, v) = r$.

First, we show that for some vertex $u_0 \in V(G)$, there exists exactly one vertex $v_0 \in V(G)$ such that $d(u_o, v_0) = r$. Suppose for every vertex $x \in V(G)$, there exist at least two vertices x_1 and x_2 in $V(G)$ such that $d(x, x_1) = r$ and $d(x, x_2) = r$. Let $M = V(G) - \{x_1\}$. Then, since $d(x, x_2) = r$, $f_M^o(x) = \{1, 2, \ldots, r\}$. Further, since $d(x, x_1) = r$, $f_M^o(x_1) = \{1, 2, \ldots, r\}$. Also, since $d(x, x_2) = r$, and by Lemma 3.12, $f_M^o(x_2) = \{1, 2, \cdots, r\}$. Let y be any vertex other than

x, x_1 and x_2. Let $1 \leqslant k \leqslant r$, and if $d(y, x) = k$, then by Lemma 3.12 and by assumption, there exists another vertex $z \in M$ such that $d(y, z) = k$. Therefore, $f^o_M(y) = \{1, 2, \ldots, r\}$. Thus $M = V(G) - \{x_1\}$ is an odpu-set for G, which is a contradiction to the hypothesis. Thus, there exists a vertex $u_0 \in V(G)$ such that there is exactly one vertex $v_0 \in V(G)$ with $d(u_0, v_0) = r$. Next, we claim that u_0 is the unique vertex for v_0 such that $d(u_0, v_0) = r$. Suppose there is a vertex $w_0 \neq u_0$ with $d(w_0, v_0) = r$. Let $M = V(G) - \{u_0\}$. Then, $d(u_0, v_0) = r$ implies $f^o_M(u_0) = \{1, 2, \ldots, r\}$ and $d(v_0, w_0) = r$ imply $f^o_M(v_0) = \{1, 2, \ldots, r\}$. Also, since $d(v_0, w_0) = r$, by Lemma 3.12, it follows that $f^o_M(w_0) = \{1, 2, \ldots, r\}$. Now let $x \in V(G) - \{u_0, v_0, w_0\}$. Since $d(x, u_0) < r$, we get $f^o_M(x) = \{1, 2, \ldots, r\}$. Hence, $M = V(G) - \{u_0\}$ is an odpu-set for G, which is a contradiction. Therefore, for the vertex v_0, u_0 is the unique vertex such that $d(u_0, v_0) = r$.

Next, we claim that there is some vertex $u_1 \in V(G) - \{u_0, v_0\}$ such that there is exactly one vertex $v_1 \in V(G)$ at a distance r from u_1. If for every vertex $u_1 \in V(G) - \{u_0, v_0\}$, there are at least two vertices v_1 and w_1 in $V(G)$ at a distance r from u_1, then proceeding as above, we can prove that $M = V(G) - \{v_1\}$ is an odpu-set of G, a contradiction. Therefore, v_1 is the only vertex at a distance r from u_1. Continuing the above procedure we conclude that for every vertex $u \in V(G)$ there exists exactly one vertex $v \in V(G)$ at a distance r from u and for the vertex v, u is the only vertex at a distance r. Thus (i) implies (ii).

Now, suppose (ii) holds. Then M is the unique odpu-set of G and hence $od(G) = n$. □

Corollary 3.14 *If G is an odpu-graph with $od(G) = |V(G)| = n$, then G is self-centered and n is even.*

Corollary 3.15 *If G is an odpu-graph with $od(G) = |V(G)| = n$ then $r(G) \geq 3$ and u_1, u_2 are different vertices of G, then, $N(u_1) \neq N(u_2)$.*

Proof If $N(u_1) = N(u_2)$, then $d(u_1, v_1) = d(u_2, v_1)$, which contradicts Theorem 3.13. □

Corollary 3.16 *The odpu-number $od(G) = |V(G)|$ for the n-dimensional cube and for even cycle C_{2n}.*

Corollary 3.17 *Let G be a graph with $r(G) = 2$. Then $od(G) = |V(G)|$ if and only if G is isomorphic to $K_{2,2,\ldots,2}$.*

Proof If $G = K_{2,2,\ldots,2}$, then $r(G) = 2$ and G is self-centered and by Theorem 3.13, $od(G) = |V(G)| = 2n$.

Conversely, let G be a graph with $r(G) = 2$. Then G is self-centered and it follows from Theorem 3.13 that for each vertex, there exists exactly one vertex at a distance 2. Hence $G \cong K_{2,2,\ldots,2}$. □

Problem 3.1 *Characterize odpu-graphs for which $od(G) = |Z(G)|$.*

Theorem 3.18 *If a graph G has odpu-number 4, then $r(G) = 2$.*

Proof Let G be an odpu-graph with odpu-number 4. Let $M = \{u, v, x, y\}$ be an odpu-set of G. If $r(G) = 1$, then $f^o_M(x) = \{1\}$ for all $x \in V(G)$. Therefore, $\langle M \rangle$ is complete. Hence, any two elements of M forms an odpu-set of G which implies $od(G) = 2$, which is a contradiction.

Hence $r(G) \geq 2$.

Since $r(G) \geq 2$, none of the vertices in M is adjacent to all the other vertices in M and $\langle M \rangle$ has no isolated vertex. Hence $\langle M \rangle = P_4$ or C_4 or $2K_2$.

If $\langle M \rangle = P_4$ or C_4 then the radius of $\langle M \rangle$ is 2. Hence, there exists a vertex v in M such that $f_M^o(v) = \{1, 2\}$ so that $r(G) = 2$.

Suppose $\langle M \rangle = 2P_2$ and let $E(\langle M \rangle) = \{uv, xy\}$. Since $|M| = 4$, $r(G) \leq 3$. If $r(G) = 3$, then $3 \in f_M^o(x)$ and $3 \in f_M^o(u)$. Hence, there exists a vertex $w \notin M$ such that $xw, uw \in E(G)$. Hence, $d(x, w) = d(u, w) = 1$. Also, $d(y, w) = d(v, w) = 2$. Therefore, $3 \notin f_M^o(w)$, which is a contradiction. Thus, $r(G) = 2$. □

A set S of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex of G is either in S or is adjacent to a vertex in S; further, if $\langle S \rangle$ is isolate-free then S is called a *total dominating set* of G (see Haynes *et al*[7]). The next result establishes the relation between odpu-sets and total dominating sets in an odpu-graph.

Theorem 3.19 *For any odpu-graph G, every odpu-set in G is a total dominating set of G.*

Proof Let M be an odpu-set of the graph G. Since $1 \in f_M^o(u)$, for all $u \in V(G)$, for any vertex $u \in V(G)$ there exists a vertex $v \in M$ such that $uv \in E(G)$. Hence, M is a total dominating set of G. □

Recall that the total domination number $\gamma_t(G)$ of a graph G is the least cardinality of a total dominating set in G.

Corollary 3.20 *For any odpu-graph G, $\gamma_t(G) \leqslant od(G)$.*

Problem 3.2 *Characterize odpu-graphs G such that $\gamma_t(G) = od(G)$.*

Let H be a graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and let $G_1, G_2, \ldots, G_n$ be a set of vertex disjoint graphs. Then the graph obtained from H by replacing each vertex x_i of H by the graph G_i and joining all the vertices of G_i to all the vertices of G_j if and only if $x_i x_j \in E(H)$, is denoted as $H[G_1, G_2, \ldots, G_n]$.

Theorem 3.21 *Let H be a connected odpu-graph of order $n \geq 2$ and radius $r \geq 2$. Let $K = H[G_1, G_2, \ldots, G_n]$. Then $od(H) = od(K)$.*

Proof Let $V(H) = \{x_1, x_2, \ldots, x_n\}$. Let G_i be the graph replaced at the vertex x_i in H. It follows from the definition of K that if $(x_{i1}, x_{i2}, \ldots, x_{ir})$ is a shortest path in H, then $(x_{i1,j1}, x_{i2,j2}, \ldots, x_{ir,jr})$ is a shortest path in K where $x_{ik,jk}$ is an arbitrary vertex in G_{ik}. Hence $M \subseteq V(H)$is odpu-set in H if and only if the set $M_1 \subseteq V(K)$, where M_1 has exactly one vertex from G_i if and only if $x_i \in M$, is an odpu-set for K. Hence $od(H) = od(K)$. □

Corollary 3.22 *A graph G with radius $r(G) \geq 2$ is an odpu-graph if and only if its shadow graph is an odpu-graph.*

Theorem 3.23 *Given a positive integer $n \neq 1, 3$, any graph G can be embedded as an induced subgraph into an odpu-graph K with odpu-number n.*

Proof If $n = 2$, then $K = C_3[G, K_1, K_1]$ is an odpu-graph with $od(K) = od(C_3) = 2$ and G is an induced subgraph of K. Suppose $n \geq 4$. Then by Theorem 3.8, there exists an odpu-graph H with $od(H) = n$. Now by Theorem 3.21, $K = H[G, K_1, K_1, \cdots, K_1]$ is an odpu-graph with $od(K) = od(H) = n$ and G is an induced subgraph of K. □

Remark 3.24 If G and K are as in Theorem 3.23, we have

(1) $\omega(H) = \omega(G) + 2$,

(2) $\chi(H) = \chi(G) + 2$,

(3) $\beta_1(H) = \beta_1(G) + 1$ and

(4) $\beta_0(H) = \beta_0(G)$

where $\omega(G)$ is the clique number, $\chi(G)$ is the chromatic number, $\beta_1(G)$ is the matching number and $\beta_0(G)$ is the independence number of G. Since finding these parameters are NP-complete for graphs, finding these four parameters for an odpu-graph is also NP-complete.

§4. Bipartite Odpu-Graphs

In this section we characterize complete multipartite odpu-graphs and bipartite odpu-graphs with odpu-number 2 and 4. Further we prove that there are no bipartite graph with odpu-number 5.

Theorem 4.1 *The complete n-partite graph $K_{a_1,a_2,\cdots,a_n}$ is an odpu-graph if and only if either $a_i = a_j = 1$ for some i and j or $a_1, a_2, a_3, \cdots a_n \geq 2$. Hence $od(K_{a_1,a_2,\cdots,a_n}) = 2$ or $2n$.*

Proof Suppose $G = K_{a_1,a_2,\cdots,a_n}$ is an odpu-graph. If $a_1 = 1$ for exactly one i, then $|Z(K_{a_1,a_2,\cdots,a_n})| = 1$. Hence G is not an odpu-graph, which is a contradiction.

Conversely assume, either $a_i = a_j = 1$ for some i and j or $a_1, a_2, a_3, \cdots a_n \geq 2$. If $a_i = a_j = 1$ for some i and j, then there exist two vertices of full degree and hence G is an odpu-graph with odpu-number 2. If $a_1, a_2, a_3, \cdots a_n \geq 2$, then for any set M which contains exactly two vertices from each partite set, we have $f^o_M(v) = \{1, 2\}$ for all $v \in V(G)$ ane hence M is an odpu-set with $|M| = 2n$. Further if M is any subset of $V(G)$ with $|M| < 2n$, there exists a partite set V_i such that $|M \cap V_i| \leq 1$ and $f^0_M(v) = \{1\}$ for some $v \in V_i$ and M is not an odpu-set. Hence $od(G) = 2n$. □

Theorem 4.2 *Let G be a bipartite odpu-graph. Then $od(G) = 2$ if and only if G is isomorphic to P_2.*

Proof Let G be a bipartite odpu-graph with bipartition (X, Y). Let $od(G) = 2$. Then, by Theorem 3.2, there exist at least two vertices of degree $n - 1$. Hence $|X| = |Y| = 1$ and G is isomorphic to P_2. The converse is obvious. □

Theorem 4.3 *A bipartite odpu-graph G with bipartition (X, Y) has odpu-number 4 if and only if the set X has at least two vertices of degree $|Y|$ and the set Y has at least two vertices of degree $|X|$.*

Proof Suppose $od(G) = 4$. Let M be an odpu-set of G with $|M| = 4$. Then, by Theorem 3.18, $r(G) = 2$ and hence $f^o_M(x) = \{1, 2\}$ for all $x \in V(G)$.

First, we show that $|M \cap X| = |M \cap Y| = 2$. If $|M \cap X| = 4$, then $1 \notin f^o_M(v)$ for all $v \in M$. If $|M \cap X| = 3$ and $|M \cap Y| = 1$ then $2 \notin f^o_M(v)$ for the vertex $v \in M \cap Y$. Hence it follows that $|M \cap X| = |M \cap Y| = 2$. Let $M \cap X = \{u, v\}$ and $M \cap Y = \{x, y\}$. Since $f^0_M(w) = \{1, 2\}$ for all $w \in V$, it follows that every vertex in X is adjacent to both x and y and every vertex in Y is adjacent to both u and v. Hence, $deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = |X|$.

Conversely, suppose $u, v \in X$, $x, y \in Y$, $deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = |X|$. Let $M = \{u, v, x, y, \}$. Clearly $f^0_M(w) = \{1, 2\}$ for all $w \in V$. Hence M is an odpu-set. Also, since there exists no full degree vertex in G, by Theorem 3.2 the odpu-number cannot be equal to 2. Also, since 3 is not the odpu-number of any graph. Hence the odpu-number of G is 4. □

Theorem 4.4 *The number 5 cannot be the odpu-number of a bipartite graph.*

Proof Suppose there exists a bipartite graph G with bipartition (X, Y) and $od(G) = 5$. Let $M = \{u, v, x, y, z\}$ be a odpu-set for G.

First, we shall show that $|X \cap M| \geq 2$ and $|Y \cap M| \geq 2$. Suppose, on the contrary, one of these inequalities fails to hold, say $|X \cap M| \leq 1$. If X has no element in M, then $1 \notin f^o_M(a)$ for all $\in M$, which is a contradiction. Therefore, $|X \cap M| = 1$. Without loss of generality, let $\{u\} = X \cap M$. Then, since $1 \in f^o_M(v) \cap f^o_M(x) \cap f^o_M(y) \cap f^o_M(z)$, all the vertices v, x, y, z should be adjacent to u. Hence $2 \notin f^o_M(u)$, a contradiction. Thus, we see that each of X and Y must have at least two vertices in M. Without loss of generality, we may assume $u, v \in X$ and $x, y, z \in Y$.

Case 1. $r(G) = 2$.

Then $f^o_M(w) = \{1, 2\}$ for all $w \in Y$. Then proceeding as in Theorem 4.3, we get $deg(u) = deg(v) = |Y|$ and $deg(x) = deg(y) = deg(z) = |X|$. Therefore, by Theorem 4.3, $\{u, v, x, y\}$ forms an odpu-set of G, a contradiction to our assumption that M is a minimum odpu-set of G. Therefore, $r = 2$ is not possible.

Case 2. $r(G) \geq 3$.

Since M is an odpu-set of G, $f^o_M(a) = \{1, 2, \ldots, r\}$ for all $a \in V(G)$. Then, since $2 \in f^o_M(u)$, there exists a vertex $b \in Y$ such that $ub, bv \in E(G)$. But since $b \in Y$ and $ub, bv \in E(G)$, $3 \notin f^o_M(b)$, which is a contradiction. Hence the result follows. □

Conjecture 4.5 *For a bipartite odpu-graph the odpu-number is always even.*

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Man's greatness lies in his power of thought.

By Blaise Pascal, a French scientist.

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