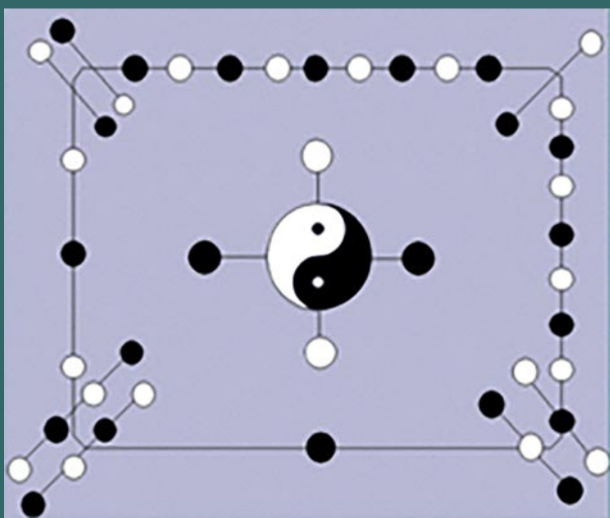
ISBN 978-1-59973-661-7

VOLUME 2, 2020

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2020

Vol.2, 2020 ISBN 978-1-59973-661-7

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO

(www.mathcombin.com)

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June, 2020

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **Mathematical Combinatorics (International Book Series)** is the book format of International Journal of Mathematical Combinatorics (ISSN 1937-1055), sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Directory of Open Access (DoAJ), EBSCO (USA), International Scientific Indexing (ISI, impact factor 2.012), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by an email directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA
Email: maolinfan@163.com

Price: US$48.00

Famous Words:

The ideals which have lighted my way, and time after time have given me new courage to face life cheerfully have been kindness, beauty and truth.

By Albert Einstein, an American scientist

Math.Combin.Book Ser. Vol.2(2020), 1-17

Mannheim Partner D-Curves in Minkowski 3-Space

Tanju Kahraman[1], Mehmet Önder[3], Mustafa Kazaz[1], H. Hüseyin Uğurlu[2]

1. Celal Bayar University, Department of Mathematics, Faculty of Arts and Sciences, Manisa, Turkey

2. Gazi University, Gazi Faculty of Education, Department of Secondary Education Science and Mathematics Teaching, Mathematics Teaching Program, Ankara, Turkey

3. Delibekirli Village, 31440, Kırıkhan, Hatay, Turkey

E-mail: tanju.kahraman@cbu.edu.tr, mehmetonder197999@gmail.com
mustafa.kazaz@cbu.edu.tr, hugurlu@gazi.edu.tr

Abstract: In this paper, we give the definition, different types and characterizations of Mannheim partner D-curves in Minkowski 3-space E_1^3. We find the relations between the geodesic curvatures, the normal curvatures and the geodesic torsions of these associated curves. Furthermore, we show that the definition and the characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases in Minkowski 3-space E_1^3.

Key Words: Minkowski 3-space, Mannheim partner D-curves, Darboux frame.

AMS(2010): 53A35, 53B30, 53C50.

§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the related curves for which there exist corresponding relations between the curves are very interesting and an important problem. The most fascinating examples of such curves are associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. The well known of the associated curves is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. The relation is that the principal normal of a curve is the principal normal of another curve i.e, the Bertrand curve is a curve which shares the normal line with another curve. Over years many mathematicians have studied on Bertrand curves in different spaces and consider the properties of these curves [1-6].

Furthermore, Bertrand curves are not only the example of associated curves. Recently, a new definition of the associated curves was given by Liu and Wang [9,17]. They called these new curves as Mannheim partner curves: Let x and x_1 be two curves in the three dimensional Euclidean E^3. If there exists a corresponding relationship between the space curves x and x_1 such that, at the corresponding points of the curves, the principal normal lines of x coincides with the binormal lines of x_1, then x is called a Mannheim curve, and x_1 is called a Mannheim partner curve of x. The pair $\{x, x_1\}$is said to be a Mannheim pair. They showed that the curve

[1]Received March 13, 2020, Accepted June 2, 2020.

$x_1(s_1)$ is the Mannheim partner curve of the curve $x(s)$ if and only if the curvature κ_1 and the torsion τ_1 of $x_1(s_1)$ satisfy the following equation

$$\dot{\tau} = \frac{d\tau}{ds_1} = \frac{\kappa_1}{\lambda}(1 + \lambda^2\tau_1^2)$$

for some non-zero constant λ. They also study the Mannheim curves in Minkowski 3-space[9,16]. Some different characterizations of Mannheim partner curves are given by Orbay and others [12]. The differential geometry of the curves fully lying on a surface in Minkowski 3-spaceE_1^3 is given by Ugurlu, Kocayigit and Topal [8,14,15]. They have given the Darboux frame of the curves according to the Lorentzian characters of surfaces and the curves. Finally, in the Euclidean 3-space, Mannheim partner D -curves is defined by Kazaz, M. and others [7]

In this paper we consider the notion of the Mannheim partner curve for the curves lying on the surfaces. We call these new associated curves as Mannheim partner D-curves and by using the Darboux frame of the curves we give the definition, different types and the characterizations of these curves in Minkowski 3-spaceE_1^3.

§2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space IR^3 provided with the standard flat metric given by

$$\langle \ , \ \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike)[11]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, Lorentz vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

$$e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \text{ and } e_1 \times e_2 = -e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = -e_2.$$

Denote by $\left\{\vec{T},\ \vec{N},\ \vec{B}\right\}$ the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski space E_1^3. For an arbitrary spacelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $\left\langle \vec{T},\vec{T}\right\rangle = 1,\ \left\langle \vec{N},\vec{N}\right\rangle = \varepsilon = \pm 1,\ \left\langle \vec{B},\vec{B}\right\rangle = -\varepsilon,\ \left\langle \vec{T},\vec{N}\right\rangle = \left\langle \vec{T},\vec{B}\right\rangle = \left\langle \vec{N},\vec{B}\right\rangle = 0$ and k_1 and k_2 are curvature and torsion of the spacelike curve $\alpha(s)$ respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike first principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$. Furthermore, for a timelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

where $\left\langle \vec{T},\vec{T}\right\rangle = -1,\ \left\langle \vec{N},\vec{N}\right\rangle = \left\langle \vec{B},\vec{B}\right\rangle = 1,\ \left\langle \vec{T},\vec{N}\right\rangle = \left\langle \vec{T},\vec{B}\right\rangle = \left\langle \vec{N},\vec{B}\right\rangle = 0$ and k_1 and k_2 are curvature and torsion of the timelike curve $\alpha(s)$ respectively [14,15].

Definition 2.1([11]) (i) (Hyperbolic angle) *Let $\vec{x}$ and $\vec{y}$ be future pointing (or past pointing) timelike vectors in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= -|\vec{x}|\ |\vec{y}| \cosh\theta$. This number is called the hyperbolic angle between the vectors $\vec{x}$ and $\vec{y}$.*

(ii) (Central angle) *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \cosh\theta$. This number is called the central angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iii) (Spacelike angle) *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \cos\theta$. This number is called the spacelike angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iv) (Lorentzian timelike angle) *Let $\vec{x}$ be a spacelike vector and $\vec{y}$ be a timelike vector in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \sinh\theta$. This number is called the Lorentzian timelike angle between the vectors $\vec{x}$ and $\vec{y}$.*

Definition 2.2([11]) *A surface in the Minkowski 3-space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and it is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector, respectively.*

Lemma 2.1([11]) *In the Minkowski 3-space IR_1^3, the following properties are satisfied:*

(i) *Two timelike vectors are never orthogonal;*

(ii) *Two null vectors are orthogonal if and only if they are linearly dependent;*

(iii) *A timelike vector is never orthogonal to a null (lightlike) vector .*

§3. Darboux Frame of a Curve Lying on a Surface in Minkowski 3-space E_1^3

Let S be an oriented surface in three-dimensional Minkowski space E_1^3 and let consider a non-null curve $x(s)$ lying on S fully. Since the curve $x(s)$ is also in space, there exists Frenet frame $\left\{\vec{T},\vec{N},\vec{B}\right\}$ at each points of the curve where $\vec{T}$ is unit tangent vector, $\vec{N}$ is principal normal vector and $\vec{B}$ is binormal vector, respectively.

Since the curve $x(s)$ lies on the surface S there exists another frame of the curve $x(s)$which is called Darboux frame and denoted by $\left\{\vec{T},\vec{g},\vec{n}\right\}$. In this frame $\vec{T}$ is the unit tangent of the curve, $\vec{n}$ is the unit normal of the surface S and $\vec{g}$ is a unit vector given by $\vec{g}=\vec{n}\times\vec{T}$. Since the unit tangent $\vec{T}$ is common in both Frenet frame and Darboux frame, the vectors $\vec{N}$, $\vec{B}$, $\vec{g}$ and $\vec{n}$ lie on the same plane. Then, if the surface S is an oriented timelike surface, the relations between these frames can be given as follows:

If the curve $x(s)$ is timelike, If the curve $x(s)$ is spacelike

$$\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\varphi&\sin\varphi\\0&-\sin\varphi&\cos\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix},\quad \begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cosh\varphi&\sinh\varphi\\0&\sinh\varphi&\cosh\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

If the surface S is an oriented spacelike surface, then the curve $x(s)$lying on S is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cosh\varphi&\sinh\varphi\\0&\sinh\varphi&\cosh\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

In all cases, φ is the angle between the vectors $\vec{g}$ and $\vec{N}$.

According to the Lorentzian causal characters of the surface S and the curve $x(s)$ lying on S, the derivative formulae of the Darboux frame can be changed as follows:

(i) If the surface S is a timelike surface, then the curve $x(s)$ lying on S can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ is given by

$$\begin{bmatrix}\dot{\vec{T}}\\ \dot{\vec{g}}\\ \dot{\vec{n}}\end{bmatrix}=\begin{bmatrix}0&k_g&-\varepsilon k_n\\k_g&0&\varepsilon\tau_g\\k_n&\tau_g&0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix},\left\langle\vec{T},\vec{T}\right\rangle=\varepsilon=\pm1,\quad \langle\vec{g},\vec{g}\rangle=-\varepsilon,\quad \langle\vec{n},\vec{n}\rangle=1. \qquad (1)$$

(ii) If the surface S is a spacelike surface, then the curve $x(s)$ lying on S is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ is given by

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}, \left\langle \vec{T}, \vec{T} \right\rangle = 1, \quad \langle \vec{g}, \vec{g} \rangle = 1, \quad \langle \vec{n}, \vec{n} \rangle = -1. \tag{2}$$

In these formulae k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows (See [9,14,15]):

- if both S and $x(s)$ are timelike or spacelike,

$$k_g = \kappa \cos \varphi, k_n = \kappa \sin \varphi, \tau_g = \tau + \frac{d\varphi}{ds}; \tag{3}$$

- if S is timelike and $x(s)$ is spacelike

$$k_g = \kappa \cosh \varphi, k_n = \kappa \sinh \varphi, \tau_g = \tau + \frac{d\varphi}{ds}. \tag{4}$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $x(s)$ can be calculated as follows:

$$k_g = -\left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2} \times n \right\rangle, \tau_g = -\varepsilon \left\langle \frac{dx}{ds}, n \times \frac{dn}{ds} \right\rangle, \tag{5}$$

$$k_g = -\left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2} \times n \right\rangle, \tau_g = \left\langle \frac{dx}{ds}, n \times \frac{dn}{ds} \right\rangle. \tag{6}$$

where $\varepsilon = \left\langle \vec{T}, \vec{T} \right\rangle = \pm 1$.

In the differential geometry of surfaces, for a curve $x(s)$lying on a surface S the followings are well-known

- $x(s)$is a geodesic curve $\Leftrightarrow k_g = 0$,
- $x(s)$is an asymptotic line $\Leftrightarrow k_n = 0$,
- $x(s)$is a principal line $\Leftrightarrow \tau_g = 0$ [10].

Along every point of the surface passes a geodesic in every direction. A geodesic is uniquely determined by an initial point and tangent at that point. All straight lines on a surface are geodesics. Along all curved geodesics the principal normal coincides with the surface normal. Along asymptotic lines osculating planes and tangent planes coincide, along geodesics they are normal. Through a point of a nondevelopable surface pass two asymptotic lines which can be real or imaginary [13].

§4. Mannheim Partner D-Curves in Minkowski 3-Space E_1^3

In this section, by considering the Darboux frame, we define Mannheim partner D-curves and give the characterizations of these curves in Minkowski 3-space E_1^3.

Definition 4.1 *Let S and S_1 be oriented surfaces in three-dimensional Minkowski space E_1^3 and let consider the curves $x(s)$ and $x_1(s_1)$ parametrized by the arc-lenght lying fully on S and S_1, respectively. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. If there exists a corresponding relationship between the curves x and x_1 such that, at the corresponding points of the curves, the Darboux frame element g of x coincides with the Darboux frame element n_1 of x_1, then x is called a Mannheim D-curve, and x_1 is a Mannheim partner D-curve of x. Then, the pair $\{x, x_1\}$is said to be a Mannheim D-pair. If there exist such curves lying on the oriented surfaces S and S_1, respectively, we call the pair $\{S, S_1\}$ as Mannheim pair surfaces.*

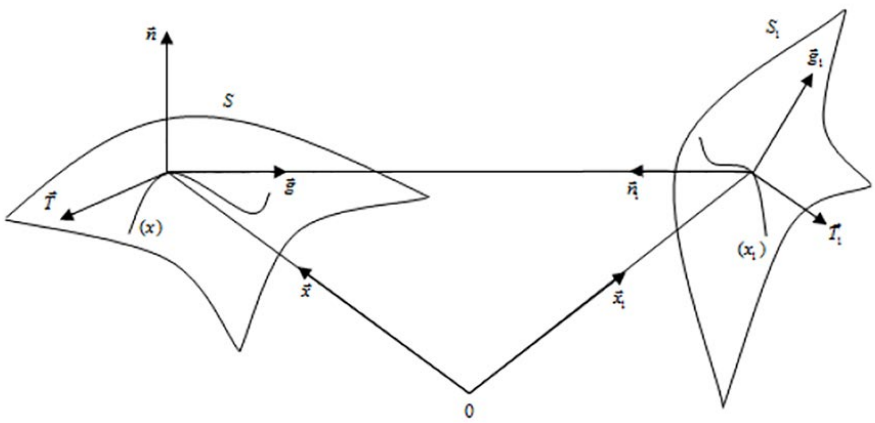


Figure 1 Mannheim partner D-curves

By considering the Lorentzian casual characters of the surfaces and the curves, from Definition 4.1, it is easily seen that there are five different types of the Mannheim D-curves in Minkowski 3-space. Let the pair $\{x, x_1\}$ be a Mannheim D-pair. Then according to the character of the surface S we have the followings:

Case 1. The oriented surface S is spacelike.

If both the surface S and the curve $x(s)$lying on S are spacelike then, there are two cases; first one is that the surface S_1 is a timelike surface and the curve $x_1(s_1)$ fully lying on S_1 is spacelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1. The second one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 2.

Case 2. The oriented surface S is timelike.

If the curve $x(s)$lying on S is a timelike curve then there are two cases; one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 3. The other case is that the curve $x_1(s_1)$ fully lying on S_1 is a spacelike curve. Then the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 4. If the curve $x(s)$lying on S is a spacelike curve then both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are spacelike. Then we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 5.

Theorem 4.1 *Let S be an oriented surface and $x(s)$ be a Mannheim D-curve in E_1^3 with arc length parameter s fully lying on S. If S_1 is another oriented surface and $x_1(s_1)$ is a curve with arc length parameter s_1 fully lying on S_1, then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if the normal curvature k_n of $x(s)$ and the geodesic curvature k_{g_1}, the normal curvature k_{n_1} and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equations.*

(i) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1 or 3, then

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\cosh\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

(ii) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 2 or 4, then

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\sinh\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

(iii) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 5, we have

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 + \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(-k_n\frac{1+\lambda k_{n_1}}{\cos\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

for some nonzero constants λ, where θ is the angle between the tangent vectors T and T_1 at the corresponding points of the curves x and x_1.

Proof (*i*) Suppose that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$by $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. Then by the definition we can assume that

$$x(s) = x_1(s_1) + \lambda(s_1)n_1(s_1) \tag{7}$$

for some function $\lambda(s_1)$. By taking derivative of (7) with respect to s_1 and applying the Darboux formulas (1) we have

$$T\frac{ds}{ds_1} = (1+\lambda k_{n_1})T_1 + \dot{\lambda}n_1 + \lambda\tau_{g_1}g_1. \tag{8}$$

Since the direction of n_1 coincides with the direction of g, we get

$$\dot{\lambda}(s_1) = 0.$$

This means that λ is a nonzero constant. Thus, the equality (8) can be written as follows

$$T\cdot\frac{ds}{ds_1}=(1+\lambda k_{n_1})T_1+\lambda\tau_{g_1}g_1. \tag{9}$$

On the other hand we have

$$T=\cosh\theta\,T_1+\sinh\theta\,g_1, \tag{10}$$

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1. By differentiating this last equation with respect to s_1, we get

$$(k_g g+k_n n)\frac{ds}{ds_1}=(\dot\theta+k_{g_1})\sinh\theta T_1+(\dot\theta+k_{g_1})\cosh\theta g_1+(-k_{n_1}\cosh\theta+\tau_{g_1}\sinh\theta)n_1. \tag{11}$$

From this equation and the fact that

$$n=\sinh\theta\,T_1+\cosh\theta\,g_1, \tag{12}$$

we get

$$\begin{aligned}(k_g g+k_n\sinh\theta T_1+k_n\cosh\theta g_1)\frac{ds}{ds_1} &= (\dot\theta+k_{g_1})\sinh\theta T_1+(\dot\theta+k_{g_1})\cosh\theta g_1\\ &\quad+(-k_{n_1}\cosh\theta+\tau_{g_1}\sinh\theta)n_1.\end{aligned} \tag{13}$$

Since the direction of n_1 is coincident with g we have

$$\dot\theta=k_n\frac{ds}{ds_1}-k_{g_1}. \tag{14}$$

From (9) and (10) and notice that T_1 is orthogonal to g_1 we obtain

$$\frac{ds}{ds_1}=\frac{1+\lambda k_{n_1}}{\cosh\theta}=\frac{\lambda\tau_{g_1}}{\sinh\theta}. \tag{15}$$

Equality (15) gives us

$$\tanh\theta=\frac{\lambda\tau_{g_1}}{1+\lambda k_{n_1}}. \tag{16}$$

By taking the derivative of this equation and applying (15) we get

$$\dot\tau_{g_1}=\frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\cosh\theta}-k_{g_1}\right)+\frac{\lambda^2\tau_{g_1}\dot k_{n_1}}{1+\lambda k_{n_1}}\right], \tag{17}$$

that is desired.

Conversely, assume that the equation (17) holds for some nonzero constants λ. Then by using (14) and (15), (16) gives us

$$k_n\left(\frac{ds}{ds_1}\right)^3=\lambda\dot\tau_{g_1}(1+\lambda k_{n_1})-\lambda^2\tau_{g_1}\dot k_{n_1}+\left((1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2\right)k_{g_1}. \tag{18}$$

Define a curve
$$x(s)=x_1(s_1)+\lambda n_1(s_1), \tag{19}$$
where λ is non-zero constant. We will prove that x is a Mannheim D-curve and x_1 is the Mannheim partner D-curve of x. By taking the derivative of (19) with respect to s_1 twice, we get
$$\frac{ds}{ds_1}T=(1+\lambda k_{n_1})T_1+\lambda\tau_{g_1}g_1 \tag{20}$$
and
$$\begin{aligned}(k_g g+k_n n)\left(\frac{ds}{ds_1}\right)^2+T\frac{d^2s}{ds_1^2} &= (\lambda\dot{k}_{n_1}+\lambda\tau_{g_1}k_{g_1})T_1+\left((1+\lambda k_{n_1})k_{g_1}+\lambda\dot{\tau}_{g_1}\right)g_1\\ &\quad+\left(-(1+\lambda k_{n_1})k_{n_1}+\lambda\tau_{g_1}^2\right)n_1,\end{aligned} \tag{21}$$
respectively. Taking the cross product of (20) with (21) we have
$$\begin{aligned}&[k_g n+k_n g]\left(\frac{ds}{ds_1}\right)^3\\ &=\left(\lambda\tau_{g_1}k_{n_1}(1+\lambda k_{n_1})-\lambda^2\tau_{g_1}^3\right)T_1-\left[-(1+\lambda k_{n_1})^2k_{n_1}+\lambda\tau_{g_1}^2(1+\lambda k_{n_1})\right]g_1\\ &\quad+\left[-k_{g_1}(1+\lambda k_{n_1})^2-\lambda\dot{\tau}_{g_1}(1+\lambda k_{n_1})+\lambda^2\tau_{g_1}\dot{k}_{n_1}+\lambda^2\tau_{g_1}^2k_{g_1}\right]n_1.\end{aligned} \tag{22}$$

By substituting (18) in (22) we get
$$\begin{aligned}[k_g n+k_n g]\left(\frac{ds}{ds_1}\right)^3 &= \left(\lambda\tau_{g_1}k_{n_1}(1+\lambda k_{n_1})-\lambda^2\tau_{g_1}^3\right)T_1\\ &\quad-\left(-k_{n_1}(1+\lambda k_{n_1})^2+\lambda\tau_{g_1}^2(1+\lambda k_{n_1})\right)g_1-k_n\left(\frac{ds}{ds_1}\right)^3n_1.\end{aligned} \tag{23}$$

Taking the cross product of (20) with (23) we have
$$\begin{aligned}[k_g g+k_n n]\left(\frac{ds}{ds_1}\right)^4 &= k_n\left(\frac{ds}{ds_1}\right)^3\lambda\tau_{g_1}T_1+k_n\left(\frac{ds}{ds_1}\right)^3(1+\lambda k_{n_1})g_1\\ &\quad+\left((1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2\right)\left(\lambda\tau_{g_1}^2-k_{n_1}(1+\lambda k_{n_1})\right)n_1.\end{aligned} \tag{24}$$

From (23) and (24) we have
$$\begin{aligned}&\left(k_g^2-k_n^2\right)\left(\frac{ds}{ds_1}\right)^4\vec{n}\\ &=\left[k_g\frac{ds}{ds_1}\left(\lambda\tau_{g_1}k_{n_1}(1+\lambda k_{n_1})-\lambda^2\tau_{g_1}^3\right)-\lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{T}_1\\ &\quad-\left[k_g\frac{ds}{ds_1}\left(-k_{n_1}(1+\lambda k_{n_1})^2+\lambda\tau_{g_1}^2(1+\lambda k_{n_1})\right)+(1+\lambda k_{n_1})k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{g}_1\\ &\quad-\left[k_nk_g\left(\frac{ds}{ds_1}\right)^4+k_n\left((1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2\right)\left(\lambda\tau_{g_1}^2-k_{n_1}(1+\lambda k_{n_1})\right)\right]\vec{n}_1.\end{aligned} \tag{25}$$

Furthermore, from (20) and (23) we get

$$\begin{cases} \left(\lambda^2\tau_{g_1}^2 - (1+\lambda k_{n_1})^2\right) = \left(\frac{ds}{ds_1}\right)^2 , \\ k_g\left(\frac{ds}{ds_1}\right)^2 - \lambda\tau_{g_1}^2 + k_{n_1}(1+\lambda k_{n_1}) = 0 \end{cases} \tag{26}$$

respectively. Substituting (26) in (25) we obtain

$$\begin{aligned} &\left(k_g^2 - k_n^2\right)\left(\frac{ds}{ds_1}\right)^4 \vec{n} \\ &= \left[k_g\frac{ds}{ds_1}\left(\lambda\tau_{g_1}k_{n_1}(1+\lambda k_{n_1}) - \lambda^2\tau_{g_1}^3\right) - \lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{T}_1 \\ &\quad - \left[k_g\frac{ds}{ds_1}\left(-k_{n_1}(1+\lambda k_{n_1})^2 + \lambda\tau_{g_1}^2(1+\lambda k_{n_1})\right) + (1+\lambda k_{n_1})k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{g}_1. \end{aligned} \tag{27}$$

Equality (20) and (27) shows that the vectors $\vec{T}$ and $\vec{n}$ lie on the plane $sp\left\{\vec{T}_1, \vec{g}_1\right\}$. So, at the corresponding points of the curves, the Darboux frame element $\vec{g}$ of x coincides with the Darboux frame element $\vec{n}_1$ of x_1, i.e, the curves x and x_1 are Mannheim D-pair curves of the type 1. □

Let now give the characterizations of Mannheim partner D-curves in some special cases. Let the pair $\{x, x_1\}$ be a Mannheim D-pair of the type 1 or 3 in Minkowski 3-space E_1^3. Assume that $x(s)$ be an asymptotic Mannheim D-curve. Then, from (16) we have the following special cases:

(i) Consider that $x_1(s_1)$ is a geodesic curve. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the following equation holds,

$$\dot{\tau}_{g_1} = -\frac{\lambda\tau_{g_1}\dot{k}_{n_1}}{1-\lambda k_{n_1}}.$$

(ii) Assume that $x_1(s_1)$ is an asymptotic line. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the geodesic curvature $k_{g_1}$and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equation,

$$\lambda\dot{\tau}_{g_1} = (1+\lambda^2\tau_{g_1}^2)k_{g_1}.$$

In this case, the Frenet frame of the curve $x_1(s_1)$ coincides with its Darboux frame. Thus, we have $k_{g_1} = \kappa_1$ and $\tau_{g_1} = \tau_1$. So, in Minkowski 3-space the Mannheim partner D-curves become the Mannheim partner curves, i.e., if both $x(s)$ and $x_1(s_1)$ are asymptotic lines then, the definition and the characterizations of the Mannheim partner D-curves involve those of the Mannheim partner curves in Minkowski 3-space.

(iii) If $x_1(s_1)$ is a principal line then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if the geodesic curvature $k_{g_1} = 0$, that is $x_1(s_1)$ is also a geodesic curve or $k_{n_1} = -1/\lambda = const.$

The proofs of the statement (ii) and (iii) of Theorem 4. 1 and the particular cases given above can be given by the same way of the proof of statement (i).

Theorem 4.2 *Let the pair $\{x, x_1\}$ be a Mannheim D-pair in Minkowski 3-space E_1^3. Then the relation between geodesic curvature k_g, geodesic torsion τ_g of $x(s)$ and the normal curvature k_{n_1}, the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given as follows:*

(i) *if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1, 3, 4 or 5 then*

$$k_g - k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}),$$

(ii) *if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 2, then*

$$k_g + k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}).$$

Proof (i) Let $x(s)$ be a Mannheim D-curve and $x_1(s_1)$ be a Mannheim partner D-curve of $x(s)$ in Minkowski 3-space E_1^3 and the pair $\{x, x_1\}$ be of the type 1. Then by definition we can write

$$x_1(s_1) = x(s) - \lambda(s) g(s) \tag{28}$$

for some constants λ. By differentiating (28) with respect to s_1 we have

$$T_1 = (1 + \lambda k_g) \frac{ds}{ds_1} T - \lambda \tau_g \frac{ds}{ds_1} n. \tag{29}$$

By definition we have

$$T_1 = \cosh\theta T - \sinh\theta n. \tag{30}$$

From (29) and (30) we obtain

$$\cosh\theta = (1 + \lambda k_g) \frac{ds}{ds_1}, \quad \sinh\theta = \lambda \tau_g \frac{ds}{ds_1}. \tag{31}$$

Using (13) and (31) it is easily seen that

$$k_g - k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}) \tag{32}$$

This completes the proof. □

Let the pair $\{x, x_1\}$be a Mannheim D-pair of the type 1 in Minkowski 3-space E_1^3. Then, we obtain the following special cases by Theorem 4.2.

(i) If x_1 is an asymptotic line, then

$$k_g = \lambda \tau_g \tau_{g_1}$$

(ii) If x_1 is a principal line, then

$$k_g - k_{n_1} = -\lambda k_g k_{n_1}$$

(iii) If x is a geodesic curve, then

$$k_{n_1} = -\lambda \tau_g \tau_{g_1}$$

(iv) If x is a principal line then

$$k_g - k_{n_1} = -\lambda k_g k_{n_1}$$

The proof of the cases that the pair $\{x, x_1\}$be a Mannheim D-pair of the type 2, 3, 4 or 5 can be given by a similar procedure used in the proof of the case that the pair $\{x, x_1\}$ is of the type 1.

Theorem 4.3 *Let $\{x, x_1\}$ be Mannheim D-pair of the type 1. Then the following relations hold:*

(i) $k_{g_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$;
(ii) $\tau_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta$;
(iii) $k_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta$;
(iv) $\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta) \frac{ds}{ds_1}$.

Proof (i) Since the pair $\{x, x_1\}$ is of the type 1, we have $\langle T, T_1\rangle = \cosh\theta$. By differentiating this equality with respect to s_1 we have

$$\left\langle (k_g g + k_n n)\frac{ds}{ds_1}, T_1 \right\rangle + \langle T, k_{g_1} g_1 - k_{n_1} n_1\rangle = \sinh\theta \frac{d\theta}{ds_1}.$$

Using the fact that the direction of n_1 coincides with the direction of g and

$$\begin{cases} T_1 = \cosh\theta T - \sinh\theta n, \\ g_1 = -\sinh\theta T + \cosh\theta n, \end{cases} \tag{33}$$

we easily get that

$$k_{g_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}.$$

(ii) By definition we get $\langle n, n_1\rangle = 0$. Differentiating this equality with respect to s_1 we have

$$\left\langle (k_n T + \tau_g g)\frac{ds}{ds_1}, n_1 \right\rangle + \langle n, k_{n_1} T_1 + \tau_{g_1} g_1\rangle = 0.$$

By (29) we obtain

$$\tau_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta.$$

(iii) By differentiating the equation $\langle T, n_1\rangle = 0$ with respect to s_1 we get

$$\left\langle (k_g g + k_n n)\frac{ds}{ds_1}, n_1 \right\rangle + \langle T, k_{n_1} T_1 + \tau_{g_1} g_1\rangle = 0.$$

From (29) it follows that

$$k_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta.$$

(iv) By differentiating the equation $\langle g, g_1 \rangle = 0$ with respect to s_1 we obtain

$$\left\langle (-k_g T + \tau_g n) \frac{ds}{ds_1}, g_1 \right\rangle + \langle g, k_{g_1} T_1 + \tau_{g_1} n_1 \rangle = 0.$$

By considering (29) we get

$$\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta) \frac{ds}{ds_1}. \tag{34}$$

This completes the proof. □

The statements of Theorem 4.3 for the pairs $\{x, x_1\}$ of the type 2, 3, 4, and 5 are given in Tables 1 and 2, and the proofs can be easily done by the same way of the case the pairs $\{x, x_1\}$ is of the type 1.

For the pair $\{x, x_1\}$ of the type 2	For the pair $\{x, x_1\}$ of the type 3
$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = k_{n_1} \cosh\theta - \tau_{g_1} \sinh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \sinh\theta - \tau_{g_1} \cosh\theta$ $(iv)\ \tau_{g_1} = (-k_g \cosh\theta + \tau_g \sinh\theta) \frac{ds}{ds_1}$	$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = k_{n_1} \sinh\theta - \tau_{g_1} \cosh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \cosh\theta - \tau_{g_1} \sinh\theta$ $(iv)\ \tau_{g_1} = (k_g \sinh\theta - \tau_g \cosh\theta) \frac{ds}{ds_1}$

Table 1

For the pair $\{x, x_1\}$ of the type 4	For the pair $\{x, x_1\}$ of the type 5
$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta$ $(iv)\ \tau_{g_1} = (k_g \cosh\theta - \tau_g \sinh\theta) \frac{ds}{ds_1}$	$(i)\ k_{g_1} = -k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = -k_{n_1} \cos\theta + \tau_{g_1} \sin\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \sin\theta + \tau_{g_1} \cos\theta$ $(iv)\ \tau_{g_1} = (k_g \cos\theta + \tau_g \sin\theta) \frac{ds}{ds_1}$

Table 2

Let now x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and the pair $\{x, x_1\}$ be of the type 1. From (5) and by using the fact that n_1 is coincident with g we have

$$\begin{aligned} k_{g_1} &= -\langle \dot{x}_1, \ddot{x}_1 \times n_1 \rangle = -\langle \dot{x}_1, \ddot{x}_1 \times g \rangle \\ &= k_n \left[-(1+\lambda k_g)^2 + \lambda^2 \tau_g^2 \right] \left(\frac{ds}{ds_1} \right)^3 + \left[\lambda \dot{\tau}_g (1+\lambda k_g) - \lambda^2 \tau_g \dot{k}_g \right] \left(\frac{ds}{ds_1} \right)^2. \end{aligned}$$

Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g, the normal curvature $k_n$and the geodesic torsion τ_g of $x(s)$ are given as follows:

If $k_g = 0$ then from (33) the geodesic curvature k_{g_1} of $x_1(s_1)$ is

$$k_{g_1} = \left(\frac{ds}{ds_1}\right)^3 (-1 + \lambda^2\tau_g^2)k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda\dot{\tau}_g. \tag{34}$$

If $k_n = 0$ then the relation between k_g, τ_g and k_{g_1} is

$$k_{g_1} = \lambda\left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g(1 + \lambda k_g) - \lambda\tau_g\dot{k}_g\right). \tag{35}$$

If $\tau_g = 0$ then, for the geodesic curvature k_{g_1}, we have

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1 + \lambda k_g)^2 k_n. \tag{36}$$

From (34),(35) and (36) we give the following corollary.

Corollary 4.1 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and the pair the pair $\{x, x_1\}$ be of the type 1. Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature, k_g, the normal curvature k_n and the geodesic torsion τ_g of $x(s)$ are given as follows*

(*i*) *If x is a geodesic curve, then the geodesic curvature k_{g_1} of $x_1(s_1)$is*

$$k_{g_1} = \left(\frac{ds}{ds_1}\right)^3 (-1 + \lambda^2\tau_g^2)k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda\dot{\tau}_g$$

(*ii*) *If x is an asymptotic line, then the equation of k_{g_1} is*

$$k_{g_1} = \lambda\left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g(1 + \lambda k_g) - \lambda\tau_g\dot{k}_g\right)$$

(*iii*) *If x is a principal line, then the geodesic curvature k_{g_1} of $x_1(s_1)$ is*

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1 + \lambda k_g)^2 k_n.$$

If the pair $\{x, x_1\}$is of the type 2, 3, 4 or 5 then the geodesic curvature of the curve $x_1(s_1)$ is given in Tables 3 and 4 following.

If the pair $\{x, x_1\}$is of the type 2	If the pair $\{x, x_1\}$ is of the type 3
$k_{g_1} = k_n\left[(1+\lambda k_g)^2 + \lambda^2\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda\dot{\tau}_g(1+\lambda k_g) + \lambda^2\tau_g\dot{k}_g\right]\left(\frac{ds}{ds_1}\right)^2$	$k_{g_1} = -k_n\left[(1-\lambda k_g)^2 + \lambda^2\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda\dot{\tau}_g(1-\lambda k_g) - \lambda^2\tau_g\dot{k}_g\right]\left(\frac{ds}{ds_1}\right)^2$

Table 3

If the pair $\{x, x_1\}$ is of the type 4	If the pair $\{x, x_1\}$is of the type 5
$k_{g_1} = k_n \left[(1-\lambda k_g)^2 - \lambda^2\tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[\lambda\dot{\tau}_g(1-\lambda k_g) + \lambda^2\tau_g\dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$	$k_{g_1} = -k_n \left[(1-\lambda k_g)^2 + \lambda^2\tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda\dot{\tau}_g(1-\lambda k_g) - \lambda^2\tau_g\dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$

Table 4

and the statements in Corollary 4.1 are obtained by the same way.

Similarly, if the pair $\{x, x_1\}$ is of the type 1, from(6) and by using the fact that g is coincident with n_1, the relation between the geodesic torsion $\tau_{g_1}$of $x_1(s_1)$ and the geodesic torsion $\tau_g$of $x(s)$ is given by

$$\tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2. \tag{37}$$

Furthermore, by using (15), from (37) we have

$$\tau_g\tau_{g_1} = \frac{\sinh^2\theta}{\lambda^2}. \tag{38}$$

Then, from (37) and (38) we can give the following corollary.

Corollary 4.2 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and $\{x, x_1\}$ be of the type 1. Then the relation between the geodesic torsion τ_{g_1} of $x_1(s_1)$ and the geodesic torsion τ_g of $x(s)$ is given by one of the followings:*

(i) $\tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$;

(ii) $\tau_g\tau_{g_1} = \frac{\sinh^2\theta}{\lambda^2}$

and so, the Mannheim partner D-curve x_1 is a principal line when the Mannheim D-curve x is a principal line.

Similarly, from (15) and (37) we get

(iii) $\frac{\tau_g}{\tau_{g_1}} = \frac{\cosh^2\theta}{(1+\lambda k_{n_1})^2}$.

Then, if $x_1(s_1)$ is an asymptotic curve, i.e.,$k_{n_1} = 0$, we have

$$\tau_g = \cosh^2\theta\tau_{g_1}. \tag{39}$$

From (39) we have the following corollary.

Corollary 4.3 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and $\{x, x_1\}$ be of the type 1. If $x_1(s_1)$ is an asymptotic curve then the relation between the geodesic torsion τ_g of $x(s)$ and the geodesic torsion $\tau_{g_1}$of $x_1(s_1)$ is given as follows:*

(iv) $\tau_g = \cosh^2\theta\ \tau_{g_1}$,

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1.

When the pair $\{x, x_1\}$ is of the type 2, 3, 4 or 5, then the relations which give the geodesic torsion τ_{g_1} of $x_1(s_1)$ are given in Tables 5 and 6 following.

For the pair $\{x, x_1\}$ of the type 2	For the pair $\{x, x_1\}$ of the type 3
$(i)\ \tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g\tau_{g_1} = \frac{\cosh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = \frac{\sinh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = \sinh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.	$(i)\ \tau_{g_1} = -\tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g\tau_{g_1} = -\frac{\sinh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = -\frac{\cosh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = -\cosh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.

Table 5

For the pair $\{x, x_1\}$ of the type 4	For the pair $\{x, x_1\}$ of the type 5
$(i)\ \tau_{g_1} = -\tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g\tau_{g_1} = -\frac{\cosh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = -\frac{\sinh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = -\sinh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.	$(i)\ \tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g\tau_{g_1} = \frac{\sin^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = \frac{\cos^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = \cos^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.

Table 6

§5. Conclusions

In this paper, in Minkowski 3-space E_1^3, the definition and characterizations of Mannheim partner D-curves are given which is a new study of associated curves lying on surfaces. It is shown that in Minkowski 3-space E_1^3, the definition and the characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases. Furthermore, the relations between the geodesic curvature, the normal curvature and the geodesic torsion of these curves are given.

References

[1] Blaschke, W., *Differential Geometrie and Geometrischke Grundlagen ven Einsteins Relativitast theorie Dover*, New York, (1945)

[2] Burke J. F., Bertrand Curves Associated with a Pair of Curves, *Mathematics Magazine*, Vol. 34(1) (1960) 60-62.

[3] Görgülü, E., Özdamar, E., A generalizations of the Bertrand curves as general inclined curves in E^n, *Communications de Fac. Sci.la Fac. Sci. Uni. Ankara*, Series A1, 35 (1986) 53-60.

[4] Hacisalihoglu, H.H., Diferansiyel Geometri, Inönü Üniversitesi Fen Edebiyat Fakültesi Yay?nlar? No:2 (1983).

[5] Izumiya, S., Takeuchi, N., Special Curves and Ruled surfaces, *Beitrage zur Algebra und*

Geometrie Contributions to Algebra and Geometry, Vol. 44(1) (2003) 203-212.

[6] Izumiya, S., Takeuchi, N., Generic properties of helices and Bertrand curves, *Journal of Geometry*, 74 (2002) 97–109.

[7] Kazaz, M., Ugurlu, H. H., Önder, M., Kahraman, T., Mannheim partner D-curves in the Euclidean 3-space E3, *New Trends Math. Sci.*, 3(2) (2015) 24-35.

[8] Kocayigit, H., *Minkowski 3-Uzayında Timelike Asal Normalli Spacelike Egrilerin Frenet ve Darboux vektörleri*, CBU Fen Bilimleri Enstitüsü Yüksek Lisans Tezi (2004).

[9] Liu, H., Wang, F., Mannheim partner curves in 3-space, *Journal of Geometry*, Vol. 88(1-2), (2008) 120-126.

[10] O'Neill, B., *Elementary Differential Geometry*, Academic Press Inc. New York, 1966.

[11] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, (1983).

[12] Orbay, K., Kasap, E., On Mannheim Partner Curves in E^3, *International Journal of Physical Sciences*, Vol. 4(5), (2009) 261-264.

[13] Struik, D. J., *Lectures on Classical Differential Geometry*, 2^{nd} ed. Addison Wesley, Dover, (1988).

[14] Ugurlu, H. H., Kocayigit, H., The Frenet and Darboux instantaneous rotation vectors of curves on time-like surface, *Mathematical & Computational Applications*, Vol.1(2) (1996) 133-141.

[15] Ugurlu, H. H., Topal, A., Relation between Darboux instantaneous rotation vectors of curves on a time-like surfaces, *Mathematical & Computational Applications*, Vol.1(2) (1996) 149-157.

[16] Wang, F., Liu, H., Mannheim partner curves in 3-Euclidean space, *Mathematics in Practice and Theory*, 37(1) (2007) 141-143.

[17] Whittemore, J. K., Bertrand curves and helices, *Duke Math. J.*, Vol. 6(1) (1940), 235-245.

Math.Combin.Book Ser. Vol.2(2020), 18-32

On Solutions of Second-Order Fuzzy Initial Value Problem by Fuzzy Laplace Transform

H. Gültekin Çitil

(Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey)

E-mail: hulya.citil@giresun.edu.tr

Abstract: In this paper, we investigate the solutions of second-order fuzzy initial value problem with positive coefficient using the fuzzy Laplace transform under the approach of generalized differentiability. Several theorems are given on the studied problem. The problem is shown on an example.

Key Words: Fuzzy initial value problems, second-order fuzzy differential equation, generalized differentiability, fuzzy Laplace transform.

AMS(2010): 35G10, 35J05, 62A86.

§1. Introduction

Fuzzy differential equations are important topic. Especially, fuzzy initial value problems and its applications. For example, real-word problems, mathematical models in science and technology, population models, civil engineering. So, many researchers have studied fuzzy differential equations.

There are several approach solving the fuzzy differential equations. The first is Hukuhara differentiability [7,14]. The second approach is generalized differentiability [9,15]. The third generate the fuzzy solution from the crips solution. These are extension principle [7,8], the concept of differential inclusion [13] and the fuzzy problem to be a set of crips problem [11]. But, many fuzzy initial and boundary value problems can not be solved as analytically. Therefore, the another approach is to find approximate solutions. The numeric methods are introduced and studied [1-4,12]. The another approach is the fuzzy Laplace transform. The solutions of fuzzy differential equation is studied by fuzzy Laplace transform [5,18,19,21]. One of the most important applications fuzzy Laplace transform is to solve fuzzy initial value problems.

In this paper, the solutions of second-order fuzzy initial value problem with positive coefficient are investigated by fuzzy Laplace transform. Generalized differentiability, fuzzy Laplace transform, Hukuhara difference and fuzzy arithmetic are used. The aim of this study is to investigate solutions using the properties fuzzy Laplace transform by generalized differentiability for second-order fuzzy initial value problem.

[1]Received January 13, 2020, Accepted June 5, 2020.

§2. Preliminaries

Definition 2.1([17]) *A fuzzy number is a mapping* $u:\mathbb{R}\to[0,1]$ *satisfying the following properties:*

(1) u *is normal:* $\exists x_0\in\mathbb{R}$ *for which* $u(x_0)=1$*;*
(u) u *is convex fuzzy set:* $u(\lambda x+(1-\lambda)y)\geqslant\min\{u(x),u(y)\}$ *for all* $x,y\in\mathbb{R},\lambda\in[0,1]$*;*
(3) u *is upper semi-continuous on* $\mathbb{R}$*,*
(4) $cl\{x\in\mathbb{R}\mid u(x)>0\}$ *is compact, where cl denotes the closure of a subset.*

Let $\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2.2([15]) *Let* $u\in\mathbb{R}_F$. *The* α*-level set of* u*, denoted ,* $[u]^\alpha$*,* $0<\alpha\leq1$*, is* $[u]^\alpha=\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$*. If* $\alpha=0$*,* $[u]^0=cl\{suppu\}=cl\{x\in\mathbb{R}\mid u(x)>0\}$*. The notation,* $[u]^\alpha=[\underline{u}_\alpha,\overline{u}_\alpha]$ *denotes explicitly the* α*-level set of* u*, where* $\underline{u}_\alpha$ *and* $\overline{u}_\alpha$ *denote the left-hand endpoint and the right-hand endpoint of* $[u]^\alpha$*, respectively.*

The following remark shows when $[\underline{u}_\alpha,\overline{u}_\alpha]$ is a valid α-level set.

Remark 2.1([10,15]) The sufficient and necessary conditions for $[\underline{u}_\alpha,\overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(1) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

(2) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

(3) $\underline{u}_\alpha\leq\overline{u}_\alpha$, $0\leq\alpha\leq1$.

Definition 2.3([17]) *If* A *is a symmetric triangular fuzzy number with support* $[\underline{a},\overline{a}]$*, the* $\alpha-$*level set of* A *is*

$$[A]^\alpha=[\underline{A}_\alpha,\overline{A}_\alpha]=\left[\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right],\quad(\underline{A}_1=\overline{A}_1,\ \underline{A}_1-\underline{A}_\alpha=\overline{A}_\alpha-\overline{A}_1).$$

Definition 2.4([12,15,20]) *Let* $u,v\in\mathbb{R}_F$*. If there exists* $w\in\mathbb{R}_F$ *such that* $u=v+w$*, then* w *is called the Hukuhara difference of fuzzy numbers* u *and* v*, and it is denoted by* $w=u\ominus v$*.*

Definition 2.5([6,12,15]) *Let* $f:[a,b]\to\mathbb{R}_F$ *and* $t_0\in[a,b]$*. We say that* f *is Hukuhara differentiable at* t_0*, if there exists an element* $f'(t_0)\in\mathbb{R}_F$ *such that for all* $h>0$ *sufficiently small,* $\exists f(t_0+h)\ominus f(t_0)$*,* $f(t_0)\ominus f(t_0-h)$ *and the limits*

$$\lim_{h\to0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0).$$

Definition 2.6([15]) *Let* $f:[a,b]\to\mathbb{R}_F$ *and* $t_0\in[a,b]$*. We say that* f *is (1)-differentiable at* t_0*, if there exists an element* $f'(t_0)\in\mathbb{R}_F$ *such that for all* $h>0$ *sufficiently small near to 0,* *exist* $f(t_0+h)\ominus f(t_0)$, $f(t_0)\ominus f(t_0-h)$ *and the limits*

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0),$$

and f is (2)-differentiable if for all $h>0$ *sufficiently small near to 0, exist* $f(t_0)\ominus f(t_0+h)$, $f(t_0-h)\ominus f(t_0)$ *and the limits*

$$\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0+h)}{-h}=\lim_{h\to 0}\frac{f(t_0-h)\ominus f(t_0)}{-h}=f'(t_0).$$

Theorem 2.7([16]) *Let* $f:[a,b]\to\mathbb{R}_F$ *be fuzzy function, where* $[f(t)]^\alpha=\left[\underline{f}_\alpha(t),\overline{f}_\alpha(t)\right]$, *for each* $\alpha\in[0,1]$.

(*i*) *If f is (1)-differentiable then* $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are differentiable functions and* $\left[f'(t)\right]^\alpha=\left[\underline{f}'_\alpha(t),\overline{f}'_\alpha(t)\right]$,

(*ii*) *If f is (2)-differentiable then* $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are differentiable functions and* $\left[f'(t)\right]^\alpha=\left[\overline{f}'_\alpha(t),\underline{f}'_\alpha(t)\right]$.

Theorem 2.8([16]) *Let* $f':[a,b]\to\mathbb{R}_F$ *be fuzzy function, where* $[f(t)]^\alpha=\left[\underline{f}_\alpha(t),\overline{f}_\alpha(t)\right]$, *for each* $\alpha\in[0,1]$, *f is (1)-differentiable or (2)-differentiable.*

(*i*) *If f and* f' *are (1)-differentiable then* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable functions and* $\left[f''(t)\right]^\alpha=\left[\underline{f}''_\alpha(t),\overline{f}''_\alpha(t)\right]$,

(*ii*) *If f is (1)-differentiable and* f' *is (2)-differentiable then* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable functions and* $\left[f''(t)\right]^\alpha=\left[\overline{f}''_\alpha(t),\underline{f}''_\alpha(t)\right]$,

(*iii*) *If f is (2)-differentiable and* f' *is (1)-differentiable then* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable functions and* $\left[f''(t)\right]^\alpha=\left[\overline{f}''_\alpha(t),\underline{f}''_\alpha(t)\right]$,

(*iv*) *If f and* f' *are (2)-differentiable then* $\underline{f}'_\alpha$ *and* $\overline{f}'_\alpha$ *are differentiable functions and* $\left[f''(t)\right]^\alpha=\left[\underline{f}''_\alpha(t),\overline{f}''_\alpha(t)\right]$.

Definition 2.9([18,21]) *The fuzzy Laplace transform of fuzzy-valued function f is defined as follows:*

$$F(s)=L(f(t))=\int_0^\infty e^{-st}f(t)\,dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}f(t)\,dt.$$

$$F(s)=L(f(t))=\left[\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t)\,dt,\ \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t)\,dt\right],$$

$$F(s,\alpha)=L(f(t,\alpha))=\left[L\left(\underline{f}(t,\alpha)\right),L\left(\overline{f}(t,\alpha)\right)\right]$$

where,

$$L\left(\underline{f}(t,\alpha)\right)=\int_0^\infty e^{-st}\underline{f}(t,\alpha)\,dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t,\alpha)\,dt,$$

$$L\left(\overline{f}(t,\alpha)\right)=\int_0^\infty e^{-st}\overline{f}(t,\alpha)\,dt=\lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t,\alpha)\,dt.$$

Theorem 2.10([5,18,21]) *Suppose that f is continuous fuzzy-valued function on $[0,\infty)$ and exponential order α and that f' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then*

$$L\left(f'(t)\right)=sL(f(t))\ominus f(0),$$

if f is (1)-differentiable,

$$L\left(f'(t)\right)=(-f(0))\ominus(-sL(f(t))),$$

if f is (2)-differentiable.

Theorem 2.11([18,21]) *Suppose that f and f' are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order α and that f'' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then*

$$L\left(f''(t)\right)=s^2L(f(t))\ominus sf(0)\ominus f'(0)$$

if f and f' are (1)-differentiable,

$$L\left(f''(t)\right)=-f'(0)\ominus\left(-s^2\right)L(f(t))-sf(0)$$

if f is (1)-differentiable and f' is (2)-differentiable,

$$L\left(f''(t)\right)=-sf(0)\ominus\left(-s^2\right)L(f(t))\ominus f'(0)$$

if f is (2)-differentiable and f' is (1)-differentiable,

$$L\left(f''(t)\right)=s^2L(f(t))\ominus sf(0)-f'(0)$$

if f and f' are (2)-differentiable.

Theorem 2.12([5,18]) *Let $f(x)$, $g(x)$ be continuous fuzzy-valued functions suppose that c_1 and c_2 are constant, then*

$$L\left(c_1f(x)+c_2g(x)\right)=\left(c_1L(f(x))\right)+\left(c_2L(g(x))\right).$$

Theorem 2.13([5]) *Let $f(x)$ be continuous fuzzy-valued function on $[0,\infty)$ and $\lambda \geq 0$, then*

$$L(\lambda f(x)) = \lambda (L(f(x))).$$

§3. Main Results

In this section, we consider solutions of the fuzzy initial value problem

$$y''(t) = \lambda y(t), \tag{1}$$

$$y(0) = [A]^\alpha, \quad y'(0) = [B]^\alpha, \tag{2}$$

by Laplace transform, where $\lambda > 0$, A and B are symmetric triangular fuzzy numbers with supports $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$,

$$[A]^\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right],$$

$$[B]^\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right] = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right],$$

where, (i,j) solution $(i,j = 1,2)$ means that y is (i)-differentiable, y' is (j)-differentiable.

(1,1) solution: Since y and y' are (1)-differentiable, taking the fuzzy Laplace transform of the equation (1),

$$s^2 L(y(t,\alpha)) \ominus sy(0,\alpha) \ominus y'(0,\alpha) = \lambda L(y(t,\alpha))$$

is obtained. From this, we have the equations

$$s^2 L\left(\underline{y}(t,\alpha)\right) - s\underline{y}(0,\alpha) - \underline{y}'(0,\alpha) = \lambda L\left(\underline{y}(t,\alpha)\right),$$

$$s^2 L(\overline{y}(t,\alpha)) - s\overline{y}(0,\alpha) - \overline{y}'(0,\alpha) = \lambda L(\overline{y}(t,\alpha)).$$

Using the initial values (2), we obtain

$$L\left(\underline{y}(t,\alpha)\right) = \frac{s}{s^2 - \lambda}\underline{A}_\alpha + \frac{1}{s^2 - \lambda}\underline{B}_\alpha,$$

$$L(\overline{y}(t,\alpha)) = \frac{s}{s^2 - \lambda}\overline{A}_\alpha + \frac{1}{s^2 - \lambda}\overline{B}_\alpha.$$

From here, taking the inverse Laplace transform of these equations, it gives

$$\underline{y}(t,\alpha) = L^{-1}\left(\frac{s}{s^2 - \lambda}\right)\underline{A}_\alpha + L^{-1}\left(\frac{1}{s^2 - \lambda}\right)\underline{B}_\alpha,$$

$$\overline{y}(t,\alpha) = L^{-1}\left(\frac{s}{s^2 - \lambda}\right)\overline{A}_\alpha + L^{-1}\left(\frac{1}{s^2 - \lambda}\right)\overline{B}_\alpha.$$

Thus, $(1,1)$ solution is

$$\underline{y}(t,\alpha)=\frac{1}{2}\underline{A}_\alpha\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{2\sqrt{\lambda}}\underline{B}_\alpha\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right), \tag{3}$$

$$\overline{y}(t,\alpha)=\frac{1}{2}\overline{A}_\alpha\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{2\sqrt{\lambda}}\overline{B}_\alpha\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right), \tag{4}$$

$$[y(t)]^\alpha=\left[\underline{y}(t,\alpha),\overline{y}(t,\alpha)\right]. \tag{5}$$

(1,2) solution: Since y is (1)-differentiable and y' is (2)-differentiable, taking the fuzzy Laplace transform of the equation (1),

$$-y'(0,\alpha)\ominus\left(-s^2\right)L(y(t,\alpha))-sy(0,\alpha)=\lambda L(y(t,\alpha))$$

is obtained. Then, we have the equations

$$-\overline{y}'(0,\alpha)+s^2L(\overline{y}(t,\alpha))-s\overline{y}(0,\alpha)=\lambda L\left(\underline{y}(t,\alpha)\right), \tag{6}$$

$$-\underline{y}'(0,\alpha)+s^2L\left(\underline{y}(t,\alpha)\right)-s\underline{y}(0,\alpha)=\lambda L(\overline{y}(t,\alpha)). \tag{7}$$

If $L(\overline{y}(t,\alpha))$ in the equation (7) is replaced by the equation (6) and using the initial conditions, we have

$$L\left(\underline{y}(t,\alpha)\right)=\frac{s^2}{s^4-\lambda^2}\underline{B}_\alpha+\frac{\lambda}{s^4-\lambda^2}\overline{B}_\alpha+\frac{s^3}{s^4-\lambda^2}\underline{A}_\alpha+\frac{\lambda s}{s^4-\lambda^2}\overline{A}_\alpha.$$

From this, the lower solution is obtained as

$$\begin{aligned}\underline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha+\overline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha+\overline{A}_\alpha\right)\\ &+\frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_\alpha+\overline{B}_\alpha}{\sqrt{\lambda}}\right)+\underline{A}_\alpha+\overline{A}_\alpha\right)\\ &+\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha-\overline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha-\overline{A}_\alpha\right).\end{aligned} \tag{8}$$

Similarly, the upper solution is obtained as

$$\begin{aligned}\overline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha+\overline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha+\overline{A}_\alpha\right)\\ &+\frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_\alpha+\overline{B}_\alpha}{\sqrt{\lambda}}\right)+\underline{A}_\alpha+\overline{A}_\alpha\right)\\ &+\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{B}_\alpha-\underline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{A}_\alpha-\underline{A}_\alpha\right).\end{aligned} \tag{9}$$

That is, $(1,2)$ solution is

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right], \tag{10}$$

where $\underline{y}(t,\alpha)$ is the equation (8) and $\overline{y}(t,\alpha)$ is the equation (9).

(2,1) solution: Since y is (2)-differentiable and y' is (1)-differentiable, from the equation

$$-sy(0,\alpha) \ominus \left(-s^2\right) L(y(t,\alpha)) \ominus y'(0,\alpha) = \lambda L(y(t,\alpha))$$

and $y'(0,\alpha) = \left[\overline{y}'(0,\alpha), \underline{y}'(0,\alpha)\right]$, we have the equations

$$-s\overline{y}(0,\alpha) + s^2 L(\overline{y}(t,\alpha)) - \overline{y}'(0,\alpha) = \lambda L\left(\underline{y}(t,\alpha)\right), \tag{11}$$

$$-s\underline{y}(0,\alpha) + s^2 L\left(\underline{y}(t,\alpha)\right) - \underline{y}'(0,\alpha) = \lambda L(\overline{y}(t,\alpha)). \tag{12}$$

If $L(\overline{y}(t,\alpha))$ in the equation (12) is replaced by the equation (11) and using the initial conditions, it gives the equation

$$L\left(\underline{y}(t,\alpha)\right) = \frac{\lambda s}{s^4-\lambda^2}\overline{A}_{\alpha} + \frac{s^3}{s^4-\lambda^2}\underline{A}_{\alpha} + \frac{s^2}{s^4-\lambda^2}\overline{B}_{\alpha} + \frac{\lambda}{s^4-\lambda^2}\underline{B}_{\alpha}.$$

From this, we have the lower solution

$$\begin{aligned} \underline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_{\alpha}+\overline{B}_{\alpha}}{\sqrt{\lambda}} + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\ &\quad + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_{\alpha}+\overline{B}_{\alpha}}{\sqrt{\lambda}}\right) + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\ &\quad + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{B}_{\alpha} - \underline{B}_{\alpha}\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_{\alpha} - \overline{A}_{\alpha}\right). \end{aligned} \tag{13}$$

Similarly, the upper solution is obtained as

$$\begin{aligned} \overline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_{\alpha}+\overline{B}_{\alpha}}{\sqrt{\lambda}} + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\ &\quad + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_{\alpha}+\overline{B}_{\alpha}}{\sqrt{\lambda}}\right) + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\ &\quad + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_{\alpha} - \overline{B}_{\alpha}\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{A}_{\alpha} - \underline{A}_{\alpha}\right). \end{aligned} \tag{14}$$

That is, $(2,1)$ solution is

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right], \tag{15}$$

where $\underline{y}(t,\alpha)$ is the equation (13) and $\overline{y}(t,\alpha)$ is the equation (14).

(2,2) solution: Since y and $y^{'}$ are (2)-differentiable,

$$s^2 L\left(y\left(t,\alpha\right)\right)\ominus sy\left(0,\alpha\right)-y^{'}\left(0,\alpha\right)=\lambda L\left(y\left(t,\alpha\right)\right).$$

From this, we have the equations

$$s^2 L\left(\underline{y}\left(t,\alpha\right)\right)-s\underline{y}\left(0,\alpha\right)-\underline{y}^{'}\left(0,\alpha\right)=\lambda\underline{y}\left(t,\alpha\right),$$

$$s^2 L\left(\overline{y}\left(t,\alpha\right)\right)-s\overline{y}\left(0,\alpha\right)-\overline{y}^{'}\left(0,\alpha\right)=\lambda\overline{y}\left(t,\alpha\right).$$

Then, (2,2) solution is

$$\underline{y}\left(t,\alpha\right)=\frac{1}{2}\underline{A}_\alpha\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{2\sqrt{\lambda}}\overline{B}_\alpha\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right), \tag{16}$$

$$\overline{y}\left(t,\alpha\right)=\frac{1}{2}\overline{A}_\alpha\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{2\sqrt{\lambda}}\underline{B}_\alpha\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right), \tag{17}$$

$$\left[y\left(t\right)\right]^\alpha=\left[\underline{y}\left(t,\alpha\right),\overline{y}\left(t,\alpha\right)\right]. \tag{18}$$

Theorem 3.1 *The* $(1,1)$ *solution of the initial value problem* $(1)-(2)$ *is a valid* $\alpha-$*level set.*

Proof Since

$$\frac{\partial\underline{y}\left(t,\alpha\right)}{\partial\alpha}=\frac{1}{4}\left(\overline{a}-\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{4\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\geq 0,$$

$$\frac{\partial\overline{y}\left(t,\alpha\right)}{\partial\alpha}=-\frac{1}{4}\left(\overline{a}-\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)-\frac{1}{4\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\leq 0,$$

$$\begin{aligned}\overline{y}\left(t,\alpha\right)-\underline{y}\left(t,\alpha\right) &= \left(1-\alpha\right)\left(\frac{1}{2}\left(\overline{a}-\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e-^{\sqrt{\lambda}t}\right)\right.\\ &\quad \left.\frac{1}{2\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\right)\geq 0,\end{aligned}$$

Thus,(1,1) solution of the initial value problem (1)-(2) is a valid $\alpha-$level set. □

Theorem 3.2 *The* $(1,2)$ *solution of the initial value problem* $(1)-(2)$ *is valid* $\alpha-$*level set, when*

$$t\geq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right)$$

for $t\in\left(0,\frac{\pi}{2\sqrt{\lambda}}\right)$.

Proof If

$$\frac{\partial\underline{y}\left(t,\alpha\right)}{\partial\alpha}\geq 0,\quad\frac{\partial\overline{y}\left(t,\alpha\right)}{\partial\alpha}\leq 0,\quad\underline{y}\left(t,\alpha\right)\leq\overline{y}\left(t,\alpha\right),$$

the (1,2) solution of the initial value problem (1)-(2) is valid $\alpha-$level set. Thus, it must be

$$\sin\left(\sqrt{\lambda}t\right)\left(\overline{b}-\underline{b}\right)+\sqrt{\lambda}\cos\left(\sqrt{\lambda}t\right)\left(\overline{a}-\underline{a}\right)\geq 0.$$

For $\sqrt{\lambda}t\in\left(0,\frac{\pi}{2}\right)\Rightarrow t\in\left(0,\frac{\pi}{2\sqrt{\lambda}}\right)$, we have

$$\sqrt{\lambda}t\geq\tan\left(-\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right)\Rightarrow t\geq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right).$$

This completes the proof. □

Theorem 3.3 *The* $(2,1)$ *solution of the initial value problem* $(1)-(2)$ *is valid* $\alpha-$*level set, when*

$$t\leq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right)$$

for $t\in\left(0,\frac{\pi}{2\sqrt{\lambda}}\right)$.

Proof The proof is similar to Theorem 3.1 and 3.2. □

Theorem 3.4 *The* $(2,2)$ *solution of the initial value problem* $(1)-(2)$ *is a valid* $\alpha-$*level set for* $t>0$ *satisfying the inequality*

$$\frac{e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}}{e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}}\geq\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}$$

Proof The proof is similar to Theorem 3.1 and 3.2. □

Theorem 3.5 *All of the solutions are symmetric triangular fuzzy numbers for any* $t>0$.

Proof For (1,1) solution, since

$$\underline{y}\left(t,1\right)=\frac{1}{4}\left(\overline{a}+\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)=\overline{y}\left(t,1\right)$$

and

$$\begin{aligned}\underline{y}\left(t,1\right)-\underline{y}\left(t,\alpha\right) &= \left(1-\alpha\right)\left(\frac{1}{4}\left(\overline{a}-\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)\right.\\ &\quad\left.\frac{1}{4\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\right)\\ &= \overline{y}\left(t,\alpha\right)-\overline{y}\left(t,1\right),\end{aligned}$$

the (1,1) solution of the initial value problem (1)-(2) is a symmetric triangular fuzzy number for any $t>0$.

For (1,2) solution, since

$$\begin{aligned} \underline{y}(t,1) &= e^{\sqrt{\lambda}t}\left(\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)+\frac{1}{4}\left(\overline{a}+\underline{a}\right)\right) \\ &\quad +e^{-\sqrt{\lambda}t}\left(-\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)+\frac{1}{4}\left(\overline{a}+\underline{a}\right)\right) \\ &= \overline{y}(t,1) \end{aligned}$$

and

$$\begin{aligned} \underline{y}(t,1)-\underline{y}(t,\alpha) &= (1-\alpha)\left(\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{a}-\underline{a}\right)\right) \\ &= \overline{y}(t,\alpha)-\overline{y}(t,1), \end{aligned}$$

the (1,2) solution of the initial value problem (1)-(2) is a symmetric triangular fuzzy number for any $t > 0$.

For the cases of (1,2) and (2,2) solutions, the proof is similar. □

Example 3.6 Consider the solutions of the fuzzy initial value problem

$$y''(t) = y(t), \quad y(0) = [1]^{\alpha}, \quad y^{'}(0) = [0]^{\alpha} \tag{19}$$

by fuzzy Laplace transform, where $[1]^{\alpha} = [\alpha, 2-\alpha]$, $[0]^{\alpha} = [-1+\alpha, 1-\alpha]$.

Its (1,1) solution is

$$\underline{y}(t,\alpha) = \frac{1}{2}\left(\alpha\left(e^{t}+e^{-t}\right)+(-1+\alpha)\left(e^{t}-e^{-t}\right)\right),$$

$$\overline{y}(t,\alpha) = \frac{1}{2}\left((2-\alpha)\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(e^{t}-e^{-t}\right)\right),$$

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

Its (1,2) solution is

$$\underline{y}(t,\alpha) = \frac{1}{2}\left(e^{t}+e^{-t}\right)+(\alpha-1)\left(\sin(t)+\cos(t)\right),$$

$$\overline{y}(t,\alpha) = \frac{1}{2}\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(\sin(t)+\cos(t)\right),$$

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

Its (2,1) solution is

$$\underline{y}(t,\alpha) = \frac{1}{2}\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(\sin(t)-\cos(t)\right),$$

$$\overline{y}(t,\alpha) = \frac{1}{2}\left(e^{t}+e^{-t}\right)+(\alpha-1)\left(\sin(t)-\cos(t)\right),$$

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

and its (2,2) solution is

$$\underline{y}(t,\alpha) = \frac{1}{2}\left(\alpha\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(e^{t}-e^{-t}\right)\right),$$

$$\overline{y}(t,\alpha) = \frac{1}{2}\left((2-\alpha)\left(e^{t}+e^{-t}\right)+(-1+\alpha)\left(e^{t}-e^{-t}\right)\right).$$

$$[y(t)]^{\alpha} = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

According to Theorem 3.1 and 3.2, (1,1) solution is a valid $\alpha-$level set, and (1,2) solution is a valid $\alpha-$level set since the function $f(t) > 0$ for $t \in \left(0, \frac{\pi}{2}\right)$ in Figure 1. By Theorem 3.3, (2,1) solution is a valid $\alpha-$level set for $t \in [0, 0.785398]$ since the function $g(t) \leq 0$ in Figure 2. That is, (1,2) solution is not solution of the problem. Also by Theorem 3.4, (2,2) solution is a valid $\alpha-$level set since the function $h(t) > 0$ in Figure 3. All of the solutions are symmetric triangular fuzzy numbers for any $t > 0$. We can see that the graphics of solutions in Figure 4-Figure 7. Also, we can see that in Figure 4, (1,1) solution is fuzzier as time goes by and (2,1) is not a valid fuzzy function for $t \geq 0.785398$ in Figure 6.

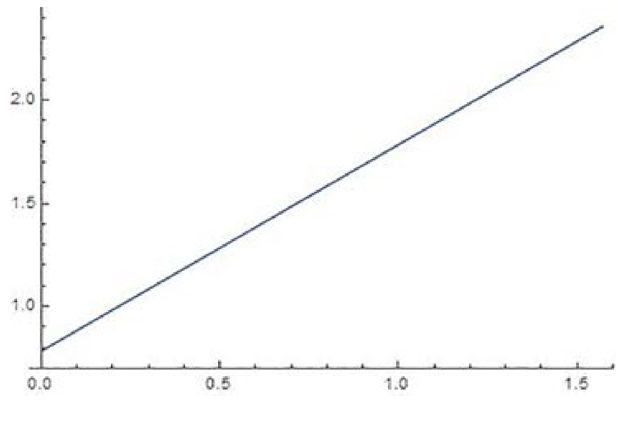


Figure 1 Graphic of the function $f(t) = t - \tan^{-1}(-1)$

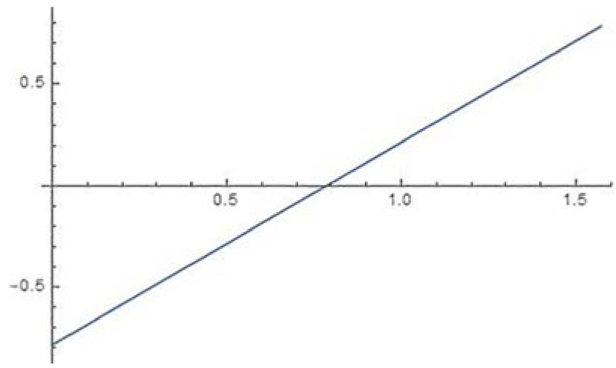


Figure 2 Graphic of the function $g(t) = t - \tan^{-1}(1)$

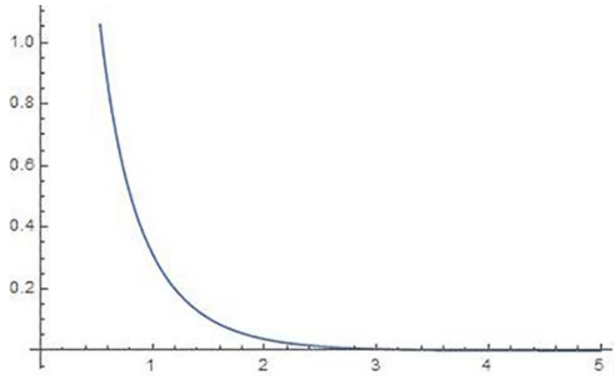


Figure 3 Graphic of the function $h(t) = \frac{e^t+e^{-t}}{e^t-e^{-t}} - 1$

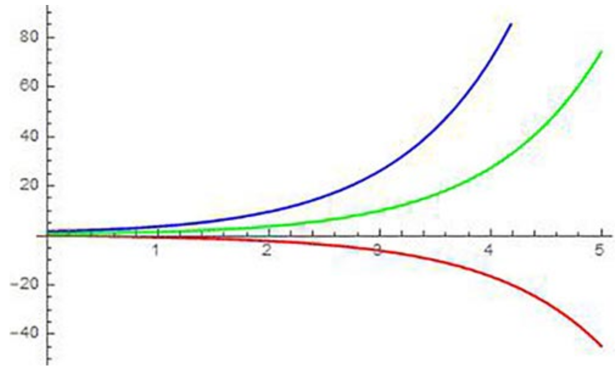


Figure 4 Graphic of (1,1) solution for $\alpha = 0.2$

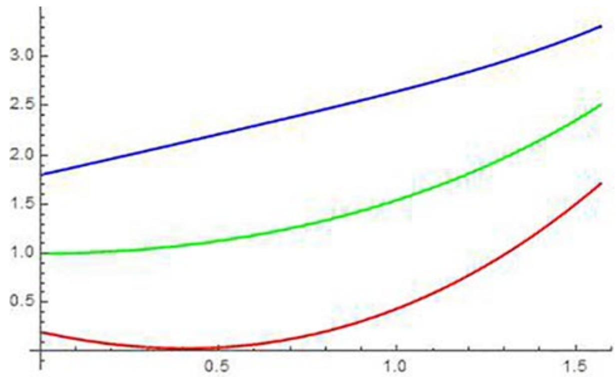


Figure 5 Graphic of (1,2) solution for $\alpha = 0.2$

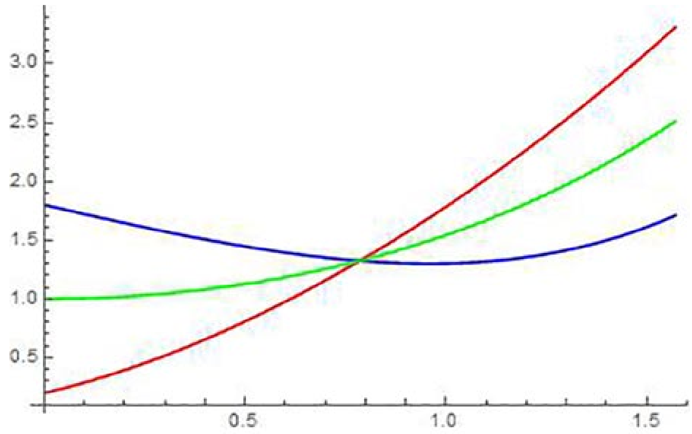


Figure 6 Graphic of (2,1) solution for $\alpha = 0.2$

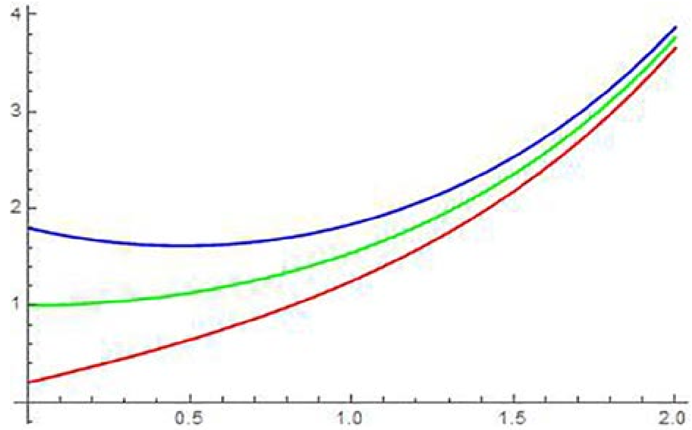


Figure 7 Graphic of (2,2) solution for $\alpha = 0.2$

Blue $\rightarrow \overline{y}_\alpha(t)$
Red $\rightarrow \underline{y}_\alpha(t)$
Green $\rightarrow \overline{y}_1(t) = \underline{y}_1(t)$

§4. Conclusions

In this paper, the solutions of second-order fuzzy initial value problem with positive coefficient are investigated by fuzzy Laplace transform. Generalized differentiability, Hukuhara difference and fuzzy arithmetic are used. Solutions are found by fuzzy Laplace transform using the generalized differentiability. It is shown that whether the solutions valid fuzzy functions or not. Studied problem is shown on an example. Graphics of found solutions are drawn. It is found that (1,1), (1,2) and (2,2) solutions are valid fuzzy level sets and (2,1) solution is a valid fuzzy level set for $t \in [0, 0.785398]$. But (1,1) solution has a drawback : it is fuzzier as time goes by.

References

[1] Abbasbandy S, Allahviranloo T., Numerical solutions of fuzzy differential equations by Taylor method, *Compututational Methods in Applied Mathematics*, 2002; 2(2):113–124.

[2] Abbasbandy S, Allahviranlo T., Lopez-Pouso O. et al, Numeric methods for fuzzy differential inclusions, *Computers and Mathematics with Applications*, 2004; 48(10-11):1633–1641.

[3] Allahviranloo T, Ahmady N., Ahmady E., Numerical solution of fuzzy differential equations by predictor-corrector method, *Information Sciences*, 2007; 177(7):1633–1647.

[4] Allahviranloo T, Ahmady E., Ahmady N., Nth-order fuzzy linear differential equations, *Information Sciences*, 2008; 178(5):1309–1324.

[5] Allahviranloo T., Barkhordari Ahmadi M., Fuzzy Laplace transforms, *Soft Computing*, 2010, 14(3):235-243.

[6] Bede B., Note on "Numerical solutions of fuzzy differential equations by predictor-corrector method", *Information Sciences*, 2008; 178(7):1917–1922.

[7] Buckley JJ, Feuring T., Fuzzy differential equations, *Fuzzy Sets and Systems*, 2000; 110(1):43–54.

[8] Buckley JJ, Feuring T., Fuzzy initial value problem for Nth-order linear differential equations, *Fuzzy Sets and Systems*, 2001; 121(2):247–255.

[9] Ceylan T., Altınışık N., Eigenvalue problem with fuzzy coefficients of boundary conditions, *Scholars Journal of Physics, Mathematics and Statistics*, 2018; 5(2):187–193.

[10] Dubois D., Prade H., Operations on fuzzy numbers, *International Journal of Systems Sciences*, 1978; 9(6):613–626.

[11] Gasilov N., Amrahov SE, Fatullayev AG, A geometric approach to solve fuzzy linear systems of differential equations, *Applied Mathematics and Information Sciences*, 2011; 5(3):484-499.

[12] Guo X., Shang D., Lu X., Fuzzy approximate solutions of second-order fuzzy linear boundary value problems, *Boundary Value Problems*, 2013; doi: 10.1186/1687-2770-2013-212, 1-17.

[13] Hüllermeier E., An approach to modelling and simulation of uncertain dynamical systems, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 1997; 5(2):117-137.

[14] Kaleva O., Fuzzy differential equations, *Fuzzy Sets and Systems*, 1987; 24(3):301-317.

[15] Khastan A., Nieto JJ, A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis*, 2010, 72(9-10):3583-3593.

[16] Khastan A., Bahrami F., Ivaz K., New results on multiple solutions for nth-order fuzzy differential equations under generalized differentiability, *Boundary Value Problems*, 2009, doi:10.1155/2009/395714, 1-13.

[17] Liu HK, Comparison results of two-point fuzzy boundary value problems, *International Journal of Computational and Mathematical Sciences*, 2011, 5(1):1-7.

[18] Patel KR, Desai NB, Solution of variable coefficient fuzzy differential equations by fuzzy Laplace transform, *Internatinal Journal on Recent and Innovation Trends in Computing and Communication*, 2017, 5(6):927-942.

[19] Patel KR, Desai NB, Solution of fuzzy initial value problems by fuzzy Laplace transform, *Kalpa Publications in Computing*, 2017, 2:25-37.

[20] Puri ML, Ralescu DA, Differentials for fuzzy functions, *Journal of Mathematical Analysis and Applications*, 1983, 91(2):552-558.

[21] Salahshour S., Allahviranloo T., Applications of fuzzy Laplace transforms, *Soft Computing*, 2013, 17(1):145-158.

Math.Combin.Book Ser. Vol.2(2020), 33-42

On the Uniqueness and Value Distribution of Entire Functions With Their Derivatives

Ashok Rathod

Department of Mathematics

KLE Society's G I Bagewadi Arts Science and Commerce College, Nippani-591237, India

Naveenkumar S.H.

Department of Mathematics, GITAM University, Bangalore-562163, India

E-mail: ashokmrmaths@gmail.com, naveenkumarsh.220@gmail.com

Abstract: In this article, we study the uniqueness problem of entire functions sharing a value with their derivatives. The result of this paper extends the result due to Zhang.

Key Words: Value distribution, entire functions, weighted sharing, Nevanlinna theory, multiplicity.

AMS(2010): 30D35.

§1. Introduction

The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory and also the uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. Ming-Liang Fang [11] and Q Zhang [9] and several other authors proved some interesting results on uniqueness and value sharing of entire functions and also meromorphic function that shares one small function with its derivative (see [3-5, 7-8, 10,12-31).

Let f be a transcendental meromorphic function in the plane and $m(r, f)$, $N(r, f)$ and $T(r, f)$ be the usual notations used in the Nevanlinna theory. Let $S(r, f)$ denote any quantity satisfying $S(r, f)=o[T(r, f)]$ as $r \longrightarrow \infty$ except possibly for a set of r of finite linear measure. Throughout this paper we denote by $a, a_0, a_1, \cdots, a_n$ meromorphic functions (or constants) for smaller growth than f, that is $T(r, f) = S(r, f)$.

Let f and g be two non-constant meromorphic functions. Let a be a finite complex number. We denote by $E(a, f)$ the set of zeros of $f - a$ (counting multiplicity), by $\overline{E}(a, f)$ the set of zeros of $f - a$ (ignoring multiplicity). We say f and g share a CM (IM), if $E(a, f) = E(a, g)$ ($\overline{E}(a, f) = \overline{E}(a, g)$). Similarly, we define that f and g share a small function $a(z)$ CM (IM), if $E(a(z), f) = E(a(z), g)$ ($\overline{E}(a(z), f) = \overline{E}(a(z), g)$). Moreover, $GCD(n_1, n_2, \cdots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \cdots, n_k$.

In 2005, Zhang [9] obtained the following result.

[1]Received December 27, 2019, Accepted June 6, 2020.

Theorem A. *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+k)\Theta(\infty,f) + 2\delta_{2+k}(0,f) > k+4, \tag{1.1}$$

or if $l = 1$ and

$$(4+k)\Theta(\infty,f) + 3\delta_{2+k}(0,f) > k+6, \tag{1.2}$$

or if $l = 0$ and

$$(6+2k)\Theta(\infty,f) + 5\delta_{2+k}(0,f) > 2k+10, \tag{1.3}$$

then $f \equiv f^{(k)}$.

Let

$$\mathcal{P}(w) = a_{n+m}w^{n+m} + \cdots + a_n w^n + \cdots + a_0 = a_{n+m}\prod_{i=1}^{s}(w - w_{p_i})^{p_i},$$

where $a_j(j = 0,1,2,\cdots,n+m-1)$, $a_{n+m} \neq 0$ and $w_{p_i}(i = 1,2,\cdots,s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, \cdots, p_s$, $s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Let $p > \max_{p \neq p_i, i=1,2,\cdots,r}\{p_i\}$, $r = s-1$, where s and r are two positive integers.

Let

$$P(w_{1)} = a_{n+m}\prod_{i=1}^{s-1}(w_1 + w_p - w_{p_i})^{p_i} = b_q w_1^q + b_{q-1}w_1^{q-1} + \cdots + b_0,$$

where $a_{n+m} = b_q$, $w_1 = w - w_p$, $q = n+m-p$. Therefore, $\mathcal{P}(w) = w_1^p P(w_1)$. We assume $P(w_1) = b_q \prod_{i=1}^{r}(w_1 - \alpha_i)^{p_i}$, where $\alpha_i = w_{p_i} - w_p$, $(i = 1,2,\cdots,r)$, be distinct zeros of $P(w_1)$.

Definition 1.1([2]) *For two positive integers n, p we define $\mu_p = \min\{n,p\}$ and $\mu_p^* = p+1-\mu_p$. Then it is clear that*

$$N_p\left(r, \frac{1}{f^n}\right) \leq \mu_p N_{\mu_p^*}\left(r, \frac{1}{f}\right). \tag{1.4}$$

In the present paper, we extend Theorem A by investigating the uniqueness of meromorphic functions of the form $f_1^p P(f_1) - a$ and $(f_1^p P(f_1))^{(k)} - a$ and obtain the following result.

Theorem 1.1 *Let $k(\geq 1)$, $l(\geq 0)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers, f and $f_1 = f - w_p$ be two non-constant entire functions. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \cdots + a_n z^n + \cdots + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Suppose $\mathcal{P}(f)$ and $(\mathcal{P}(f))^{(k)}$ share $(1,l)$.*

If $l \geq 2$ and

$$\mu_2\delta_{\mu_2^*}(w_p,f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f) > m+n-2p+\mu_2+\mu_{k+2} \tag{1.5}$$

or $l = 1$ *and*

$$\frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > \frac{3(m+n)-5p}{2} + \mu_2 + \mu_{k+2} + \frac{1}{2} \tag{1.6}$$

or $l = 0$ *and*

$$2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+1}\delta_{\mu_{k+1}^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > 4(m+n) - 5p + 2 + \mu_2 + \mu_{k+1} + \mu_{k+2} \tag{1.7}$$

then $\mathcal{P}(f) \equiv (\mathcal{P}(f))^{(k)}$.

§2. Preliminary Lemmas

Let F and G be two non-constant meromorphic functions. We denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 ([9]) *Let f be a non constant meromorphic function, k, p, be two positive integers, then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Clearly,

$$\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right).$$

Lemma 2.2 ([6] *Let H be defined as in* (2.1). *If F and G share 1 IM and $H \not\equiv 0$, then*

$$N_{11}\left(r, \frac{1}{F-1}\right) \le N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.3 ([1]) *Let F and G share $(1, l)$ and $\overline{N}(r, F) = \overline{N}(r, G)$ and $H \not\equiv 0$, then*

$$\begin{aligned} N(r, H) \quad \le \quad & \overline{N}(r, F) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) \\ & + \overline{N}_0\left(r, \frac{1}{G'}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned}$$

§3. Proof of Theorem 1.1

Proof of Theorem 1.1 Let $F = \mathcal{P}(f) = f_1^p P(f_1)$ and $G = (\mathcal{P}(f))^{(k)} = (f_1^p P(f_1))^{(k)}$. Since $\mathcal{P}(f)$ and $[\mathcal{P}(f)]^{(k)}$ share $(1, l)$, F, G share $(1, l)$ except the zeros and poles of $a(z)$. Also, let's note that

$$\overline{N}(r, F) = \overline{N}(r, f) + S(r, f) \quad \text{and} \quad \overline{N}(r, G) = \overline{N}(r, f) + S(r, f).$$

Let H be defined as in (2.1). We consider the following cases.

Case 1. Suppose $H \not\equiv 0$.

By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right) \\ &+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) \\ &-\overline{N}_0\left(r,\frac{1}{F'}\right)-\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G), \quad (3.1) \end{aligned}$$

where $\overline{N}_0\left(r,\frac{1}{F'}\right)$ denotes the reduced counting function of the zeros of F' which are not the zeros of $F(F-1)$.

Since F and G share 1 IM, it is easy to verify that

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &= N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right) \\ &+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)=\overline{N}\left(r,\frac{1}{G-1}\right). \quad (3.2) \end{aligned}$$

Using Lemmas 2.2 and 2.3, (3.1) and (3.2), we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right) \\ &+N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right) \\ &+3\overline{N}_L\left(r,\frac{1}{F-1}\right) \\ &+3\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G). \quad (3.3) \end{aligned}$$

Subcase 1.1 $l \geq 2.$.

Obviously,

$$\begin{aligned} N_{11}\left(r,\frac{1}{F-1}\right) &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right)+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right) \\ &\leq N\left(r,\frac{1}{G-1}\right)+S(r,F) \\ &\leq T(r,G)+S(r,F)+S(r,G). \quad (3.4) \end{aligned}$$

Using (3.3) and (3.4), we get

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,F)+S(r,F). \quad (3.5)$$

Using Lemma 2.1, (1.4) and (3.5), we get

$$\begin{aligned}(n+m)T(r,f) &\leq N_2\left(r,\frac{1}{f_1^pP(f_1)}\right)+N_2\left(r,\frac{1}{(f_1^pP(f_1))^{(k)}}\right)+3\overline{N}(r,f)+S(r,f)\\ &\leq 3\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)+(n+m-p)T(r,f)\\ &\quad +N_{k+2}\left(r,\frac{1}{f_1^pP(f_1)}\right)+k\overline{N}(r,f)+S(r,f)\\ &\leq (k+3)\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)+2(n+m-p)T(r,f)\\ &\quad +\mu_{k+2}N_{\mu_{k+2}^*}\left(r,\frac{1}{f-w_p}\right)+S(r,f).\end{aligned}$$

So, $\mu_2\delta_{\mu_2^*}(w_p,f)+\mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f)\leq m+n-2p+\mu_2+\mu_{k+2}$, which contradicts with (1.5).

Subcase 1.2 $l=1$.

It is easy to verify that

$$\begin{aligned}N_{11}\left(r,\frac{1}{F-1}\right) &+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)+S(r,F)\\ &\leq T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{3.6}$$

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r,\frac{F}{F'}\right)\\ &\leq \frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,F)\\ &\leq \frac{1}{2}\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right)+S(r,F).\end{aligned}\tag{3.7}$$

Using (3.3), (3.6) and (3.7), we get

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{7}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F).\tag{3.8}$$

Using Lemma (2.1), (1.4) and (3.8), we get

$$\begin{aligned}(n+m)T(r,f)\leq &\left(k+\frac{7}{2}\right)\overline{N}(r,f)+\mu_2N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right)+\mu_{k+2}N_{\mu_{k+2}^*}\left(r,\frac{1}{f-w_p}\right)\\ &+\frac{1}{2}\overline{N}\left(r,\frac{1}{f-w_p}\right)+\frac{5}{2}(n+m-p)T(r,f)+S(r,f).\end{aligned}$$

So,

$$\frac{1}{2}\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f)\le\frac{3(m+n)-5p}{2}+\mu_2+\mu_{k+2}+\frac{1}{2}$$

which contradicts with (1.6).

Subcase 1.3 $l=0$.

It is easy to verify that

$$\begin{aligned}&N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\quad\le N\left(r,\frac{1}{G-1}\right)+S(r,F)\le T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{3.9}$$

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right)&\le N\left(r,\frac{1}{F-1}\right)-\overline{N}\left(r,\frac{1}{F-1}\right)\\&\le N\left(r,\frac{F}{F'}\right)\le N\left(r,\frac{F'}{F}\right)+S(r,F)\\&\le\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+S(r,F).\end{aligned}\tag{3.10}$$

Using (3.3), (3.9) and (3.10), we get

$$\begin{aligned}T(r,F)\ \le\ &N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\&+6\overline{N}(r,F)+2\overline{N}\left(r,\frac{1}{F}\right)+N_1\left(r,\frac{1}{G}\right)+S(r,F).\end{aligned}\tag{3.11}$$

Using Lemma 2.1 and (3.11), we get

$$\begin{aligned}(n+m)T(r,f)\ \le\ &N_2\left(r,\frac{1}{f_1^pP(f_1)}\right)+N_2\left(r,\frac{1}{(f^pP(f_1))^{(k)}}\right)+6\overline{N}(r,f)\\&+2\overline{N}\left(r,\frac{1}{f_1^pP(f_1)}\right)+N_1\left(r,\frac{1}{(f_1^pP(f_1))^{(k)}}\right)+S(r,f).\end{aligned}$$

So,

$$\begin{aligned}&2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\mu_{k+1}\delta_{\mu_{k+1}^*}(w_p,f)+\mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f)\\&\qquad\le 4(m+n)-5p+2+\mu_2+\mu_{k+1}+\mu_{k+2}.\end{aligned}$$

which contradicts with (1.7).

Case 2. Suppose $H\equiv 0$.

Using (2.1), we get

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}.$$

Hence,

$$\frac{1}{F-1} \equiv C\frac{1}{G-1} + D, \tag{3.12}$$

where C, D are constants and $C \neq 0$.

We discuss the following three cases:

Subcase 2.1 $D \neq 0, -1$.

Rewrite (3.12) as,

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF},$$

we have,

$$\overline{N}(r,G) = \overline{N}\left(r, \frac{1}{F - \frac{(D+1)}{D}}\right).$$

By using second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}
(n+m)T(r,f) &= T(r,F) + S(r,f) \\
&\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{(D+1)}{D}}\right) + S(r,f) \\
&\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,G) + S(r,f) \\
&\leq 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f_1^p P(f_1)}\right) + S(r,f) \\
&\leq 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-w_p}\right) + (n+m-p)T(r,f) + S(r,f).
\end{aligned}$$

So, $\Theta(w_p, f) \leq 1-p$,

which contradicts with (1.5), (1.6) and (1.7).

Subcase 2.2 $D = 0$.

Then from (3.12), we get

$$G = CF - (C-1). \tag{3.13}$$

If $C \neq 1$, then

$$\overline{N}\left(r,\frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \frac{(C-1)}{C}}\right).$$

Proceeding as in Subcase 2.1, we get

$$\begin{aligned}(n+m)T(r,f) &\le (k+1)\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-w_p}\right)\\ &+2(n+m-p)T(r,f)+N_{k+1}\left(r,\frac{1}{f-w_p}\right)+S(r,f).\end{aligned}$$

So,

$$\Theta(w_p,f)+\mu_{k+1}\delta_{\mu^*_{k+1}}(w_p,f)\le 1+\mu_{k+1}+n+m-2p,$$

which contradicts with (1.5), (1.6) and (1.7).

Therefore, $C=1$. By using (3.13), we get $F\equiv G$ and so, $f_1^pP(f_1)=(f_1^pP(f_1))^{(k)}$.

Subcase 2.3 $D=-1$.

Then from (3.12) we get

$$\begin{aligned}\frac{1}{F-1} &= \frac{C}{G-1}-1\\ &\Rightarrow \frac{F}{F-1}=\frac{C}{G-1}.\end{aligned}$$

Hence we have $\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}(r,G)=S(r,f)$ and hence $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$.

If $C\neq -1$, then

$$\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F-\frac{C}{C+1}}\right).$$

Proceeding as in Subcase 2.1, we get

$$\begin{aligned}(n+m)T(r,f)\le (k+1)\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-w_p}\right)\\ +2(n+m-p)T(r,f)+N_{k+1}\left(r,\frac{1}{f-w_p}\right)+S(r,f).\end{aligned}$$

So,

$$\Theta(w_p,f)+\mu_{k+1}\delta_{\mu^*_{k+1}}(w_p,f)\le 1+\mu_{k+1}+n+m-2p$$

which contradicts with (1.5), (1.6) and (1.7).

Therefore, $C=-1$. By using (3.13), we get $FG\equiv 1$. Hence, $\mathcal{P}(f)(\mathcal{P}(f))^{(k)}=1$. Thus in this case,

$$\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)=S(r,f).$$

We have,

$$\frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)} = \frac{1}{(\mathcal{P}(f))^2}. \tag{3.14}$$

From first fundamental theorem and (3.14), we get

$$\begin{aligned}
2T(r,\mathcal{P}(f)) &\le T\left(r, \frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)}\right)\\
&\le N\left(r, \frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)}\right) + S(r,f)\\
&\le k\left(\overline{N}(r,\mathcal{P}(f)) + \overline{N}\left(r, \frac{1}{\mathcal{P}(f)}\right)\right) + S(r,f)\\
&\le S(r,f),
\end{aligned}$$

which is impossible. This completes the Proof. □

References

[1] A. Banerjee and S. Majumder, On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative, *Comment. Math. Univ. Carolin.*, 51 (2010), No. 4, 565–576.

[2] A. Banerjee and B. Chakraborty, Further investigations on a question of Zhang and Lü, *Ann. Univ. Paedagog. Crac. Stud. Math.*, 14 (2015), 105–119.

[3] R. Brück, On entire functions which share one value CM with their first derivative, *Results Math.*, 30 (1996), No. 1-2, 21–24.

[4] W. K. Hayman, *Meromorphic Functions,* Oxford, Clarendon Press, 1964.

[5] I. Lahiri and A. Sarkar, Uniqueness of a meromorphic function and its derivative, *J. Inequal. Pure Appl. Math.*, 5 (2004), No. 1, Article 20, 9 pp.

[6] J.D. Li and G.-X. Huang, On meromorphic functions that share one small function with their derivatives, *Palest. J. Math.*, 4 (2015), No. 1, 91–96.

[7] L. Liu and Y. Gu, Uniqueness of meromorphic functions that share one small function with their derivatives, *Kodai Math. J.*, 27 (2004), No. 3, 272–279.

[8] L.Z. Yang, Solution of a differential equation and its applications, *Kodai Math. J.*, 22 (1999), No. 3, 458–464.

[9] Q. Zhang, Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure Appl. Math.*, 6 (2005), No. 4, Article 116, 13 pp.

[10] T. C. Alzahary and H. X. Yi, Weighted value sharing and a question of I. Lahiri, *Complex Var. Theory Appl.*, 49 (2004), No. 15, 1063–1078.

[11] Ming-Liang Fang, Uniqueness and value-sharing of entire functions, *Computers and Mathematics with Applications*, 44(2002), 823-831.

[12] R. S. Dyavanal, Ashok Rathod,Uniqueness theorems for meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences*, 12:1(2016), 1-10.

[13] R. S. Dyavanal, Ashok Rathod,Multiple values and uniqueness of meromorphic functions

on annuli, *International Journal Pure and Applied Mathematics*, 107:4(2016), 983-995.

[14] R. S. Dyavanal, Ashok Rathod,On the value distribution of meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences*, 12:2(2016), 203-217.

[15] R. S. Dyavanal, Ashok Rathod,Some generalisation of Nevanlinna's five-value theorem for algebroid functions on annuli, *Asian Journal of Mathematics and Computer Research*, 20:2(2017), 85-95.

[16] R. S. Dyavanal, Ashok Rathod,Nevanlinna's five-value theorem for derivatives of meromorphic functions sharing values on annuli, *Asian Journal of Mathematics and Computer Research*, 20:1(2017), 13-21.

[17] R. S. Dyavanal, Ashok Rathod,Unicity theorem for algebroid functions related to multiple values and derivatives on annuli, *International Journal of Fuzzy Mathematical Archive*, 13:1(2017), 25-39.

[18] R. S. Dyavanal, Ashok Rathod,General Milloux inequality for algebroid functions on annuli, *International Journal of Mathematics and Applications*, 5:3 (2017), 319-326.

[19] Ashok Rathod, The multiple values of algebroid functions and uniqueness, *Asian Journal of Mathematics and Computer Research*, 14:2(2016), 150-157.

[20] Ashok Rathod,The uniqueness of meromorphic functions concerning five or more values and small functions on annuli, *Asian Journal of Current Research*, 1:3(2016), 101-107.

[21] Ashok Rathod,Uniqueness of algebroid functions dealing with multiple values on annuli, *Journal of Basic and Applied Research International*, 19:3(2016), 157-167.

[22] Ashok Rathod,On the deficiencies of algebroid functions and their differential polynomials, *Journal of Basic and Applied Research International*, 1:1(2016), 1-11.

[23] Ashok Rathod,The multiple values of algebroid functions and uniqueness on annuli, *Konuralp Journal of Mathematics*, 5:2(2017), 216-227.

[24] Ashok Rathod,Several uniqueness theorems for algebroid functions, *J. Anal.*, 25:2(2017), 203-213.

[25] Ashok Rathod,Nevanlinnas five-value theorem for algebroid functions, *Ufa Mathematical Journal*, 10:2(2018), 127-132.

[26] Ashok Rathod,Nevanlinnas five-value theorem for derivatives of algebroid functions on annuli, *Tamkang Journal of Mathematics*, 49:2(2018), 129-142.

[27] S. S. Bhoosnurmath, R. S. Dyavanal, Mahesh Barki, Ashok Rathod,Value distribution for nth difference operator of meromorphic functions with maximal deficiency sum, *J Anal*, 27(2019), 797-811.

[28] Ashok Rathod,Characteristic function and deficiency of algebroid functions on annuli, *Ufa Mathematical Journal*, 11:1(2019), 121-132.

[29] Ashok Rathod,Value distribution of a algebroid function and its linear combination of derivatives on annuli, *Electronic Journal of Mathematical Analysis and Applications*, 8:1(2020), 129-142.

[30] Ashok Rathod,Uniqueness theorems for meromorphic functions on annuli, *Ufa Mathematical Journal*, 12(2020), 115-121.

[31] Ashok Rathod,Exceptional values of algebroid functions on annuli, *The Journal of Analysis* (Accepted).

Math.Combin.Book Ser. Vol.2(2020), 43-67

Number of Spanning Trees of Some of Pyramid Graphs Generated by a Wheel Graph

Salama Nagy Daoud[1,2] and Wedad Saleh[1]

1. Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah 41411, Saudi Arabia

2. Department of Mathematics and Computer Science
Faculty of Science, Menoufia University, Shebin El Kom 32511, Egypt

E-mail: salamadaoud@gmail.com, wed_10_777@hotmail.com

Abstract: In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we define some classes of pyramid graphs and we derive simple formulas of the complexity, number of spanning trees, of these graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Key Words: Number of spanning tree, Chebyshev Polynomial, pyramid graph.

AMS(2010): 05C05, 05C50.

§1. Introduction

The study of the number of spanning trees in a graph has a long history and has been very active because computing this number is important: (1) in analyzing energy of masers in investigating the possible particle transitions; (2) in estimating the reliability of a network; (3) in designing electrical circuits; (4) in enumerating certain chemical isomers; (5) in counting the number of Eulerian circuits in a graph. See [1]-[7], [20],[22] and [24]. For a graph G , a spanning tree in G is a tree which has the same vertex set as G . The number of spanning trees of, also known as, the complexity of the graph, denoted by $\tau(G)$, this quantity is a well-studied quantity for long time. A classical result of Kirchhoff [19] can be used to determine the number of spanning trees for a graph G with p vertices. If $V = u_1, u_2, c \ldots, u_p$, then the Kirchhoff matrix L defined as $p \times p$ characteristic matrix $L = D - A$ where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G ,$L = [X_{ij}]$ defined as follows:

(i) $X_{ij} = -1$ when U_i and U_j are adjacent and $i \neq j$;
(ii) X_{ij} equals the degree of vertex U_i if $i = j$, and
(iii) $X_{ij} = 0$ otherwise.

All of co-factors of L are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p$ denote the eignvalues of L matrix of a p point graph. It is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [18] have shown that $\tau(G) = \frac{1}{p}\prod_{j=1}^{p-1}\sigma_j$.

[1]Received February 20, 2020, Accepted June 7, 2020.

For more results in this field, see [10]-[17].

Now, we introduce the following lemma.

Lemma 1.1([8]) *$\tau(G) = \frac{1}{p^2} det(pI - \bar{D} + \bar{A})$ where $\bar{A}, \bar{D}$ are the adjacency and degree matrices of $\bar{G}$, the complement of G, respectively, and I is the $p \times p$ unit matrix.*

The advantage of this formula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

§2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, see Yuanping, et. al. [23].

Let $M_p(Z)$ be $p \times p$ matrix such that:

$$M_p(Z) = \begin{pmatrix} 2Z & -1 & 0 & \cdots & 0 \\ -1 & 2Z & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2Z & -1 \\ 0 & \cdots & 0 & -1 & 2Z \end{pmatrix},$$

Further, we recall that the Chebyshev polynomials of the first kind are defined by

$$T_p(Z) = \cos(p \arccos Z). \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{p-1}(Z) = \frac{1}{p}\frac{d}{dz}T_p(Z) = \frac{\sin(p \arccos Z)}{\cos(\arccos Z)}. \tag{2}$$

It is easily verified that

$$U_p(Z) - 2ZU_{p-1}(Z) + U_{p-2}(z) = 0. \tag{3}$$

It can then be shown from this recursion that by expanding one gets

$$U_p(Z) = det(M_p(z)), p \geqslant 1. \tag{4}$$

Furthermore, by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_p(Z) = \frac{1}{2\sqrt{Z^2-1}}\left[(Z+\sqrt{Z^2-1})^{p+1} - (Z-\sqrt{Z^2-1})^{p+1}\right], p \geqslant 1 \tag{5}$$

where the identity is true for all complex Z (except at $Z = \pm 1$, where the function can be taken as the limit). The definition of $U_p(Z)$ easily yields its zeros and it can therefore be verified that

$$U_{p-1}(Z) = 2^{p-1} \prod_{i=1}^{p-1} \left(Z - \cos \frac{i\pi}{p} \right). \tag{6}$$

One further note that

$$U_{p-1}(-Z) = (-1)^{p-1} U_{p-1}(Z). \tag{7}$$

These two results yield another formula for

$$U_{p-1}^2(Z) = 4^{p-1} \prod_{i=1}^{p-1} \left(Z^2 - \cos^2 \frac{i\pi}{p} \right). \tag{8}$$

Finally, a simple manipulation of the above formula yields the following formula (9), which is extremely useful to us latter:

$$U_{p-1}^2(\sqrt{\frac{Z+2}{4}}) = \prod_{i=1}^{p-1} \left(Z - 2\cos \frac{2i\pi}{p} \right). \tag{9}$$

Furthermore, one can show that

$$U_{p-1}^2(Z) = \frac{1}{2(1-z^2)}[1 - T_{2p}] = \frac{1}{2(1-z^2)} \left[1 - T_p \left(2Z^2 - 1\right)\right], \tag{10}$$

$$T_p(Z) = \frac{1}{2} \left[(z + \sqrt{Z^2-1})^p + (z - \sqrt{z^2-})^p \right]. \tag{11}$$

Now we introduce the following important two lemmas.

Lemma 2.1([8]) *Let $A_p(Z)$ be $p \times p$ circulant matrix such that:*

$$A_p(Z) = \begin{pmatrix} Z & 0 & 1 & \cdots & 1 & 0 \\ 0 & Z & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & Z & 1 \\ 0 & 1 & \cdots & 1 & 0 & Z \end{pmatrix},$$

Then for $p \geqslant 3$, $Z \geqslant 4$ we have:

$$\det(A_p(Z)) = \frac{2(Z+p-3)}{Z-3} \left[T_p(\frac{Z-1}{2}) - 1 \right].$$

Lemma 2.2([21]) *If $X \in F^{p\times p}$, $Y \in F^{p\times q}$, $Z \in F^{q\times p}$ and$W \in F^{q\times q}$. If X and W are* *nonsingular matrices, then*

$$\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(X - YW^{-1}Z)\det W = \det X \det(W - ZX^{-1}Y).$$

This Lemma give a sort of symmetry for some matrices which facilitates our calculations of the complexities of some special graphs.

§3. Main Results

Definition 3.1([9]) *The pyramid graph $P_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ and m sets of vertices, say, $V_1^1, V_2^1, \cdots, V_p^1, \cdots, V_1^q, V_2^q, \cdots, V_p^q$ such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q See Figure 1.*

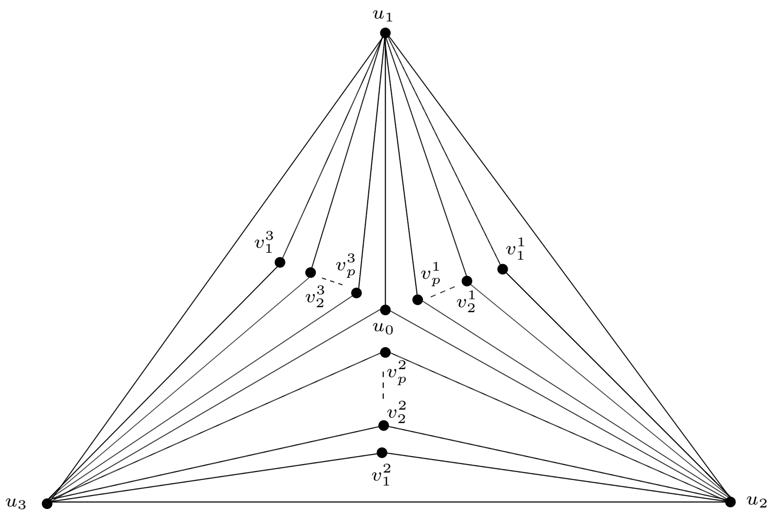


Figure 1 The pyramid graph $P_p^{(3)}$

Theorem 3.2([9]) *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(P_p^{(q)}) = 2^{pq-q}\left[(p+3+\sqrt{2p+5})^q + \left[(p+3-\sqrt{2p+5})^q - 2(p+2)^q\right]\right].$$

Definition 3.3 *The pyramid graph $A_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1} ,where $j = 1, 2, \cdots, q-1$

and v_i^q is adjacent to u_1 and u_q See Figure 2.

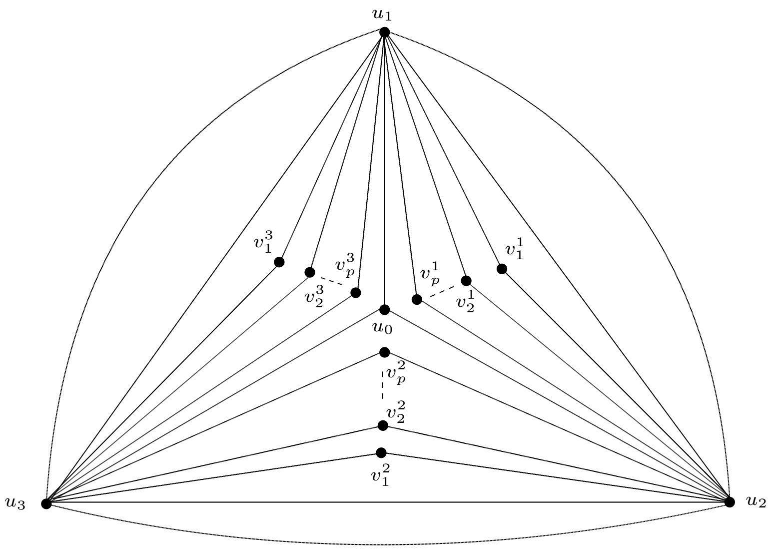


Figure 2 The pyramid graph $A_p^{(3)}$

Theorem 3.4 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(A_p^{(q)}) = 2^{pq-q}\left[(p+5+\sqrt{2p+9})^q + \left[(p+5-\sqrt{2p+9})^q - 2(p+4)^q\right]\right].$$

Proof Applying Lemma 1.1, We have

$$\begin{aligned}\tau(A_p^{(q)}) &= \frac{1}{(pq+q+1)^2} det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}\end{aligned}$$

where,

$$A = \begin{pmatrix} q+1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 2(p+3) & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 \\ \cdots & -1 & 2(p+3)-1 & \cdots & \cdots & \cdots & \cdots & 1 & \\ \cdots & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ 0 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 & 2(p+3) \end{pmatrix}$$

$$B=\begin{pmatrix}
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

$$C=\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}$$

$$D=\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\
\cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}, \quad G=\begin{pmatrix}
\cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\
\cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0
\end{pmatrix}$$

$$\begin{pmatrix} E & F \\ H & I \end{pmatrix}=\begin{pmatrix}
3 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & 1 \\
1 & \cdots & 1 & 3
\end{pmatrix}$$

Let $j=(1\cdots 1)$ be the $1\times p$ matrix with all one, and J_p the $p\times p$ matrix with all one. Set $k=2p+4$ and $h=pq+q+1$. Then we have

$\tau(A_p^{(q)})$

$$= \frac{1}{h^2}\det\begin{pmatrix}
(q+1) & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
0 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h^2}\det\begin{pmatrix}
h & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
h & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

Hence, we know that

$$
\tau(A_p^{(q)})
$$

$$
= \frac{1}{h}\det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{h}\det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
0 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
0 & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
0 & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}
$$

$$= \frac{1}{h}\det\begin{pmatrix} k & -1 & 1 & \cdots & 1 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\ -1 & k & -1 & \ddots & \ddots & 1 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & -j & 0 \\ -1 & 1 & \cdots & 1 & -1 & k & 0 & \cdots & \cdots & 0 & -j & -j \\ 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\ j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & 2I_{pq} & & & \\ j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\ 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & & \end{pmatrix}$$

Using Lemma 2.2, yields

$$\tau(A_p^{(q)}) = \frac{1}{h}\begin{pmatrix} X & Y \\ Z & 2I_{pq} \end{pmatrix} = \frac{1}{h}det(X - Y\frac{1}{2I_{pq}}Z)2^{pq}$$

$$= \frac{1}{h}2^{pq}2^{-q}\det\begin{pmatrix} 2k & p-2 & 2(p+1) & \cdots & 2(p+1) & p-2 \\ p-2 & 2k & p-2 & 2(p+1) & \cdots & 2(p+1) \\ 2(p+1) & p-2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2(p+1) \\ 2(p+1) & \ddots & \ddots & \ddots & \ddots & p-2 \\ p-2 & 2(n+1) & \cdots & 2(p+1) & p-2 & 2k \end{pmatrix}$$

Straightforward induction using properties of determinants, we have

$$\tau(A_p^{(q)}) = \frac{1}{b}2^{pq-q}\frac{2k+p(2q-4)+(2q-10)}{2k+p(q-4)+(4q-10)}$$

$$\times\det\begin{pmatrix} (2k-p+2) & 0 & 2(p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+1) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= \frac{1}{h} 2^{pq-q} \frac{2h}{pq+4q+2} \det \begin{pmatrix} (2k-p+2) & 0 & 2(p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+1) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= 2^{pq-q+1} \frac{(p+4)^q}{pq+4q+2} \det \begin{pmatrix} \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p+2)}{(p+4)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\begin{aligned} \tau(A_p^{(q)}) &= 2^{pq-q+1} \times \frac{(p+4)^q}{pq+4q+2} \times \frac{2(\frac{2k-p+2}{p+4}+q-3)}{\frac{2k-p+2}{p+4}-3} \times \left[T_p(\frac{\frac{2k-p+2}{p+4}-1}{2}) - 1 \right] \\ &= 2^{pq-q+1} \times \frac{(p+4)^q}{pq+4q+2} \times (pq+4q+2) \times \left[T_q(\frac{2k-2p-2}{2(p+4)}) - 1 \right] \\ &= 2^{pq-p+1} \times (p+4)^q \times \left[T_q(\frac{p+5}{p+4}) - 1 \right]. \end{aligned}$$

Using Equation (11), yields the result. □

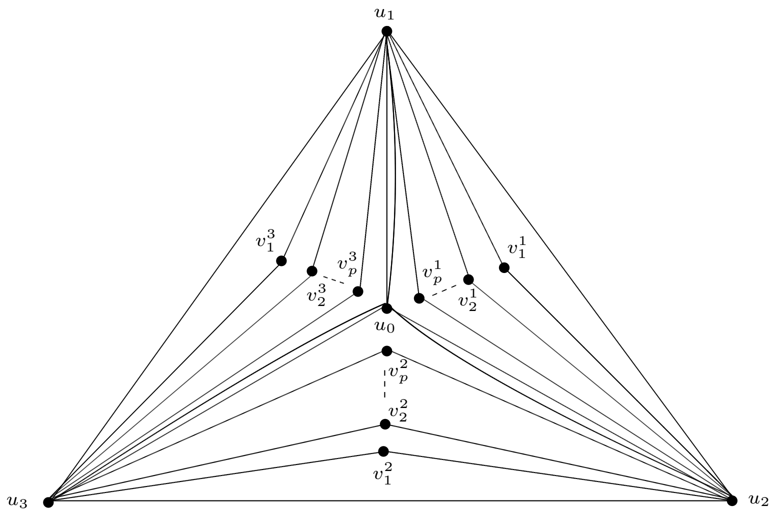


Figure 3 The pyramid graph $B_p^{(3)}$

Definition 3.5 *The pyramid graph $B_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q. See Figure 3.

Theorem 3.6 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(B_p^{(q)}) = 2^{pq-q}\left[(p+4+2\sqrt{p+3})^q + \left[(p+4-2\sqrt{p+3})^q - 2(p+2)^q\right]\right].$$

Proof Applying Lemma 2.1, we get

$$\begin{aligned}\tau(B_p^{(q)}) &= \frac{1}{(pq+q+1)^2}\det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}\end{aligned}$$

where,

$$A = \begin{pmatrix} 2q+1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2p+5 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\ \cdots & 0 & 2p+5 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 2p+5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$C = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$G = \begin{pmatrix} \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ H & I \end{pmatrix} = \begin{pmatrix} 3 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times p$ matrix with all one, and J_p the $p \times p$ matrix with all one. Set $k = 2p + 5$ and $h = pq + q + 1$. Then we have

$$\tau(B_p^{(q)}) = \frac{1}{h^2} \det \begin{pmatrix}
2q+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \ddots & \cdots & j & 0 \\
\vdots & 0 & k & 0 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & 0 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
-1 & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h^2} \det \begin{pmatrix}
h & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & k & 0 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
h & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \ldots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

Therefore,

$$\tau(B_p^{(q)})$$

$$= \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & k & 0 & \ddots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0 \\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (k+1) & 1 & 2 & \cdots & 2 & 1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & (k+1) & 1 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & 2 & \cdots & 2 & 1 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
0 & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h}\det\begin{pmatrix}
(k+1) & 1 & 2 & \cdots & 2 & 1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & (k+1) & 1 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 2 & \cdots & 2 & 1 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & 2I_{pq} & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}$$

Using Lemma 2.2 yields

$$\tau(B_p^{(q)}) = \frac{1}{h}\begin{pmatrix} X & Y \\ Z & 2I_{pq}\end{pmatrix} = \frac{1}{b} det\left(X - Y\frac{1}{2I_{pq}}Z\right)2^{pq}$$

$$= \frac{1}{h}2^{pq}2^{-q}\det\begin{pmatrix}
(2k+2p+2) & (3p+2) & 4(p+1) & \cdots & 4(p+1) & (3p+2) \\
(3p+2) & (2k+2p+2) & (3p+2) & \ddots & \ddots & 4(p+1) \\
4(p+1) & (3p+2) & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(p+1) \\
4(p+1) & \ddots & \ddots & \ddots & \ddots & (3p+2) \\
(3p+2) & 4(p+1) & \cdots & 4(p+1) & (3p+2) & (2k+2p+2)
\end{pmatrix}$$

Straightforward induction using properties of determinants, we have

$$\begin{aligned}
\tau(B_p^{(q)}) \quad = \quad & \frac{1}{h}2^{pq-q}\frac{2k+p(4q-4)+(4q-6)}{2k+p(q-4)+(2q-6)} \\
& \times\det\begin{pmatrix}
(2k-p) & 0 & (p+2) & \cdots & (p+2) & 0 \\
0 & (2k-p) & 0 & (p+2) & \cdots & (p+2) \\
(p+2) & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (p+2) \\
(p+2) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (p+2) & \cdots & (p+2) & 0 & (2k-p)
\end{pmatrix}
\end{aligned}$$

$$= \frac{1}{h} 2^{pq-q} \frac{4h}{pq+2q+4} \det \begin{pmatrix} (2k-p) & 0 & (p+2) & \cdots & (p+2) & 0 \\ 0 & (2k-p) & 0 & (p+2) & \cdots & (p+2) \\ (p+2) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+2) \\ (p+2) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+2) & \cdots & (p+2) & 0 & (2k-p) \end{pmatrix}$$

$$= 2^{pq-q+2} \frac{(p+2)^q}{pq+2q+4} \det \begin{pmatrix} \frac{(2k-p)}{(p+2)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p)}{(p+2)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p)}{(p+2)} \end{pmatrix}$$

Using Lemma 2.1 yields

$$\begin{aligned} \tau(B_p^{(q)}) &= 2^{pq-q+2} \times \frac{(p+2)^q}{pq+2q+4} \times \frac{2(\frac{2k-p}{p+2}+q-3)}{\frac{2k-p}{p+2}-3} \times \left[T_p(\frac{\frac{2k-p}{p+2}-1}{2}) - 1 \right] \\ &= 2^{pq-q+1} \times \frac{(p+2)^q}{pq+2q+4} \times (pq+2q+4) \times \left[T_q(\frac{2k-2p-2}{2(p+2)}) - 1 \right] \\ &= 2^{pq-p+1} \times (p+2)^q \times \left[T_q(\frac{p+4}{p+2}) - 1 \right]. \end{aligned}$$

Using Equation (11), yields the result. □

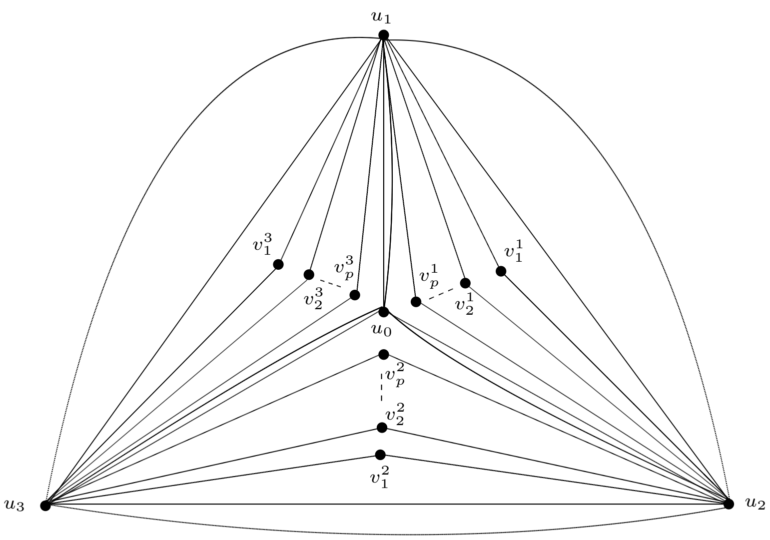


Figure 4 The pyramid graph $C_p^{(3)}$

Definition 3.7 *The pyramid graph $C_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say,*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1} ,where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q. See Figure 4.

Theorem 3.8 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(C_p^{(q)}) = 2^{pq-q}\left[(p+6+2\sqrt{p+5})^q + \left[(p+6-2\sqrt{p+5})^q - 2(p+4)^q\right]\right].$$

Proof Applying Lemma 1.1, We have

$$\begin{aligned}\tau(C_p^{(q)}) &= \frac{1}{(pq+q+1)^2} det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}\end{aligned}$$

where,

$$A = \begin{pmatrix} 2q+1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2p+7 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 \\ \cdots & -1 & 2p+7 & -1 & \ddots & \ddots & \ddots & \ddots & 1 \\ \cdots & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 & 2p+7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$C = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$G = \begin{pmatrix} \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} E & F \\ H & I \end{pmatrix} = \begin{pmatrix} 3 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times p$ matrix with all one, and J_p the $p \times p$ matrix with all one. Set $k = 2p + 7$ and $h = pq + q + 1$. Then we have

$$\tau(C_p^{(q)}) = \frac{1}{h^2} \det \begin{pmatrix}
2q+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \ddots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h^2} \det \begin{pmatrix}
h & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \cdots & \cdots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
h & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \ldots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

Clearly,

$$\tau(C_p^{(q)}) = \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & \ddots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (k+1) & 0 & 2 & \cdots & 2 & 0 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 0 & (k+1) & 0 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 2 & \cdots & 2 & 0 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
0 & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}$$

$$
= \frac{1}{h}\det\begin{pmatrix}
(k+1) & 0 & 2 & \cdots & 2 & 0 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & (k+1) & 0 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & 2 & 0 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & 2I_{pq} & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}
$$

Using Lemma 2.2, yields

$$
\begin{aligned}
\tau(C_p^{(q)}) &= \frac{1}{h}\begin{pmatrix} X & Y \\ Z & 2I_{pq} \end{pmatrix} \\
&= \frac{1}{h}det(X - Y\frac{1}{2I_{pq}}Z)2^{pq} = \frac{1}{h}2^{pq}2^{-q} \\
&\quad \times \det\begin{pmatrix}
(2k+2p+2) & 3p & 4(p+1) & \cdots & 4(p+1) & 3p \\
3p & (2k+2p+2) & 3p & \ddots & \ddots & 4(p+1) \\
4(p+1) & 3p & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(p+1) \\
4(p+1) & \ddots & \ddots & \ddots & \ddots & 3p \\
3p & 4(p+1) & \cdots & 4(p+1) & 3n & (2k+2p+2)
\end{pmatrix}.
\end{aligned}
$$

Straightforward induction using properties of determinants, we have

$$
\begin{aligned}
\tau(C_p^{(q)}) &= \frac{1}{h}2^{pq-q}\frac{2k+p(4q-4)+(4q-10)}{2k+p(q-4)+(2q-10)} \\
&\quad \times \det\begin{pmatrix}
(2k-p+2) & 0 & (p+4) & \cdots & (p+4) & 0 \\
0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\
(p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\
(p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (p+4) & \cdots & (p+4) & 0 & (2k-p+2)
\end{pmatrix}
\end{aligned}
$$

$$= \frac{1}{h}2^{pq-q}\frac{4h}{pq+4q+4}$$

$$\times \det\begin{pmatrix} (2k-p+2) & 0 & (p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+4) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= 2^{pq-q+2}\frac{(p+4)^q}{pq+4q+4}\det\begin{pmatrix} \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p+2)}{(p+4)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\begin{aligned} \tau(C_p^{(q)}) &= 2^{pq-q+2}\times\frac{(p+4)^q}{pq+4q+4}\times\frac{2(\frac{2k-p+2}{p+4}+q-3)}{\frac{2k-p+2}{p+4}-3}\times\left[T_p(\frac{\frac{2k-p+2}{p+4}-1}{2})-1\right] \\ &= 2^{pq-q+1}\times\frac{(p+4)^q}{pq+4q+4}\times(pq+4q+4)\times\left[T_q(\frac{2k-2p-2}{2(p+4)})-1\right] \\ &= 2^{pq-p+1}\times(p+4)^q\times\left[T_q(\frac{p+6}{p+4})-1\right]. \end{aligned}$$

By Equation (11), yields the result. □

§4. Numerical Results

The following tables presents some number of spanning trees of studied pyramid graphs.

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
3	0	16	49	50	128
3	1	242	578	676	1444
3	2	3136	6400	8192	15488
3	3	36992	67712	92416	160000
3	4	409600	692224	991232	1605632
3	5	4333568	6889472	10240000	15745024

Table 1

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
4	0	45	225	192	720
4	1	1792	6336	6400	18816
4	2	57600	163072	184320	458752
4	3	1622016	3932160	4816896	10616832
4	4	41746432	90243072	117440512	235929600
4	5	1006632960	19992294400	2717908992	5075107840

Table 2

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
5	0	121	961	722	3872
5	1	12482	64082	58564	232324
5	2	984064	3810304	3964928	12781568
5	3	65619968	208406528	237899776	658640896
5	4	3901751296	10696523776	13088325632	32245809152
5	5	213408284672	522192945152	674448277504	1514986799104

Table 3

§5. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and Lemmas and their proofs.

References

[1] D. L. Applegate, R. E. V. Bixby, Chvtal, and W. J. Cook, *The Traveling Salesman Problem: A Computational Study*, Princeton University Press, (2006).

[2] T. Atajan and H. Inaba, Network reliability analysis by counting the number of spanning trees, ISCIT 2004, *IEEE International symposium on Communication and Information Technology*, 1 (2004), 601-604.

[3] F. T. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory*, 10 (1986), 339-352.

[4] F. T. Boesch and A. Salyanarayana in C.L. Suffel, A survey of some network reliability

analysis and synthesis results, *Networks*, 54 (2009), 99-107.

[5] T. J. N. Brown, R. B. Mallion, P. Pollak, and A. Roth, Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes, *Discrete Appl. Math.*, 67 (1996), 51-66.

[6] G. Chen, B. Wu, and Z. Zhang, Properties and applications of Laplacian spectra for Koch networks, *J. Phys. A: Math. Theor.*, 45 (2012), 025102.

[7] D. Cvetkovi, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Applications*(Third Edition), Johann Ambrosius Barth, Heidelberg, (1995).

[8] S.N. Daoud, Chebyshev polynomials and spanning tree formulas, *International J. Math. Combin.*, Vol.4 (2013), 68-79.

[9] S.N.Daoud, On a class of some pyramid graphs and Chebyshev polynomials, *Journal of Mathematical Problems in Engineering*, Hindawi Publishing Corporation Vol. 2013, Article ID 820549, 11 pages.

[10] S.N. Daoud, Number of Spanning Trees of Different Products of Complete and Complete Bipartite Graphs, *Journal of Mathematical Problems in Engineering*, Hindawi pub. Cor.Vol. 2014 (2014) Article ID 965105, 20 pages.

[11] S.N. Daoud, The deletion-contraction method for counting the number of spanning trees of graphs, *Eur. Phys. J. Plus*, 130 (2015), 1- 14.

[12] S.N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *Journal of the Egyptian Mathematical Society*, Volume 25, Issue 4, 424-433(2017).

[13] S.N. Daoud, Number of spanning trees in different products of complete and complete tripartite graphs, *Ars Cmbinatoria*, Vol. 139. 85-103 (2018).

[14] S.N. Daoud, Complexity of join and corona graphs and Chebyshev polynomials. *Journal of Taibah University for Sciences*,Vol. 12, NO. 5, (2018),557 -572.

[15] S.N. Daoud, Number of Spanning trees of cartesian and composition products of graphs and Chebyshev polynomials, *IEEE Access*, Vol.7(2019) 71142 -71157.

[16] S.N. Daoud, Number of spanning trees of some families of graphs generated by a triangle, *Journal of Taibah University for Sciences*, Vol.13, NO. 1, (2019)731-739.

[17] Jia-Bao Liu and S. N. Daoud, Number of spanning trees in the sequence of some graphs, *Complecity, Complexity*, Hindawi Publ. Corp. Volume 2019, Article ID 4271783, 22. pages (2019).

[18] A. K. Kelmans and V. M. Chelnokov, A certain polynomials of a graph and graphs with an extermal number of trees, *J. Comb. Theory* (B) 16, (1974), 197-214.

[19] G. G. Kirchhoff, Uber die Auflosung der Gleichungen, auf welche man be ider Untersuchung der Linearen Verteilung galvanischer Storme gefuhrt wird, *Ann. Phys. Chem.*, 72, (1847) 497-508.

[20] E. C. Kirby, D. J. Klein, R. B. Mallion, P. Pollak, and H. Sachs, A theorem for counting spanning trees in general chemical graphs and its particular application to toroidal fullerenes, *Croat. Chem. Acta*, 77 (2004), 263-278.

[21] M. Marcus, *A Survey of Matrix Theory and Matrix Inequalities*, Unvi. Allyn and Bacon. Inc. Boston (1964).

[22] F.Y. Wu, Number of spanning trees on a lattice, *J. Phys.* A 10, (1977), 113-115.

[23] Z. Yuanping, Y.Xuerong and J. Mordecai, Chebyshev polynomials and spanning trees formulas for circulant and related graphs, *Discrete Mathematics*, 298 (2005), 334-364.

[24] F. Zhang and X. Yong, Asymptotic enumeration theorems for the number of spanning trees and Eulerian trail in circulant digraphs & graphs, *Sci. China*, Ser. A43, 2(1999), 264-271.

Math.Combin.Book Ser. Vol.2(2020), 68-73

Complementary Distance Energy of Complement of Line Graphs of Regular Graphs

Harishchandra S. Ramane and Daneshwari Patil

(Department of Mathematics, Karnatak University, Dharwad - 580003, India)

E-mail: hsramane@yahoo.com, daneshwarip@gmail.com

Abstract: The complementary distance (CD) matrix of a graph G is defined as $CD(G) = [c_{ij}]$, where $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and $c_{ij} = 0$, otherwise, where D is the diameter of G and d_{ij} is the distance between the vertices v_i and v_j in G. The CD-energy of G is defined as the sum of the absolute values of the eigenvalues of CD-matrix. Two graphs are said to be CD-equienergetic if they have same CD-energy. In this paper we obtain the CD-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus construct pairs of CD-equienergetic graphs of same order and having different CD-eigenvalues.

Key Words: Complementary distance eigenvalues, energy, equienergetic graphs.

AMS(2010): 05C50.

§1. Introduction

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of a graph G is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [4].

The *distance* between the vertices v_i and v_j, denoted by d_{ij}, is the length of the shortest path joining v_i and v_j. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G [3]. A graph G is said to be *r-regular graph* if all of its vertices have same degree equal to r.

The *complementary distance* between the vertices v_i and v_j, denoted by c_{ij} is defined as

$$c_{ij} = 1 + D - d_{ij},$$

where D is the diameter of G and d_{ij} is the distance between v_i and v_j in G.

The *complementary distance matrix* or *CD-matrix* [7] of a graph G is an $n \times n$ matrix

[1]Received December 12, 2019, Accepted June 8, 2020.

$CD(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 + D - d_{ij}, & \text{if} \quad i \neq j \\ \\ 0, & \text{if} \quad i = j. \end{cases}$$

The complementary distance matrix is an important source of structural descriptors in the quantitative structure property relationship (QSPR) model in chemistry [7,9].

The eigenvalues of $CD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are said to be the *complementary distance eigenvalues* or *CD-eigenvalues* of G and their collection is called *CD-spectra* of G. Two non-isomorphic graphs are said to be *CD-cospectral* if they have same CD-spectra.

The *complementary distance energy* or CD*-energy* of a graph G denoted by $CDE(G)$ is defined as [10]

$$CDE(G) = \sum_{i=1}^{n} |\mu_i| \ . \tag{1}$$

The Eq. (1) is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [5]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{2}$$

Two connected graphs G_1 and G_2 are said to be *complementary distance equienergetic* or CD*-equienergetic* if $CDE(G_1) = CDE(G_2)$. The CD-equienergetic graphs are reported in [10]. In this paper we obtain the CD-energy of the complement of line graphs of certain regular graphs and thus construct CD-equienergetic graphs having different CD-spectra.

The *line graph* of G, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [6].

For $k = 1, 2, \cdots$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [6]. The line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (n_0 r_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently the order and degree of $L^k(G)$ are [1,2]

$$n_k = \frac{r_{k-1} n_{k-1}}{2} \tag{3}$$

and

$$r_k = 2r_{k-1} - 2, \tag{4}$$

where n_i and r_i stands for order and degree of $L^i(G)$, $i = 0, 1, \cdots$.

Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{5}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \tag{6}$$

We need following results.

Theorem 1.1([4]) *If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

Theorem 1.2([12]) *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\begin{array}{lll} \lambda_i + r - 2, & i = 1, 2, \cdots, n, & \textit{and} \\ -2, & (n(r-2)/2 \ \textit{times}). & \end{array}$$

Theorem 1.3([11]) *Let G be an r-regular graph of order n. If $r, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of G, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \cdots, n$.*

Theorem 1.4([10]) *Let G be an r-regular graph on n vertices and $diam(G) = 2$. If $r, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of G, then CD-eigenvalues of G are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \cdots, n$.*

Lemma 1.5([8]) *Let G be an r-regular graph on n vertices. If $r \leq \frac{n-1}{2}$ then*

$$diam\left(\overline{L^k(G)}\right) = 2, \quad k \geq 1.$$

§2. CD-Energy

Theorem 2.1 *Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then*

$$CDE\left(\overline{L(G)}\right) = 2r(n-2).$$

Proof Let the adjacency eigenvalues of G be $r, \lambda_2, \cdots, \lambda_n$. From Theorem 1.3 , the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{lll} 2r - 2, & \text{and} & \\ \lambda_i + r - 2, & i = 2, 3, \cdots, n, & \text{and} \\ -2, & n(r-2)/2 \ \text{times}. & \end{array}\right\} \tag{7}$$

From Theorem 1.4 and the Eq. (3), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{lll} (nr/2)-2r+1, & \text{and} & \\ -\lambda_i-r+1, & i=2,3,\cdots,n, & \text{and} \\ 1, & n(r-2)/2 \ \text{times.} & \end{array}\right\} \quad (8)$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since $r \leq \frac{n-1}{2}$, by Lemma 1.5, $diam\left(\overline{L(G)}\right) = 2$. Therefore by Theorem 1.4 and Eq. (8), the CD-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{lll} nr-2r, & \text{and} & \\ -\lambda_i-r, & i=2,3,\cdots,n, & \text{and} \\ 0, & n(r-2)/2 \ \text{times.} & \end{array}\right\} \quad (9)$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \cdots, n$. Hence from Eq. (9),

$$\begin{aligned} CDE\left(\overline{L(G)}\right) &= nr-2r+\sum_{i=2}^{n}(\lambda_i+r)+|0|\times\frac{n(r-2)}{2} \\ &= 2r(n-2) \end{aligned}$$

because of

$$\sum_{i=2}^{n}\lambda_i = -r.$$

This completes the proof. □

Corollary 2.2 *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$CDE\left(\overline{L^k(G)}\right) = 2r_{k-1}(n_{k-1}-2).$$

Proof If $r_0 \leq \frac{n_0-1}{2}$, then by Eqs. (3) and (4), we have

$$r_1 = 2r_0 - 2 \leq n_0 - 3 \leq \frac{1}{2}\left(\frac{n_0r_0}{2} - 1\right) = \frac{n_1-1}{2}.$$

Hence

$$r_{k-1} \leq \frac{n_{k-1}-1}{2}.$$

Therefore, by Theorem 2.1,

$$CDE\left(\overline{L^k(G)}\right) = CDE\left(\overline{L(L^{k-1}(G))}\right) = 2r_{k-1}(n_{k-1}-2). \qquad \square$$

Corollary 2.3 *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$CDE\left(\overline{L^k(G)}\right) = \left[\frac{2n_0}{2^{k-1}}\prod_{i=0}^{k-1}(2^i r_0 - 2^{i+1}+2)\right] - 4(2^{k-1}r_0 - 2^k + 2).$$

§3. CD-Equienergetic Graphs

If G_1 and G_2 are the regular graphs of same order and of same degree. Then $L(G_1)$ and $L(G_2)$ are of the same order and of same degree. Further their complements are also of same order and of same degree.

Lemma 3.1 *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Proof The result follows from Eqs. (7), (8) and (9). $\square$

Lemma 3.1 can be extended for k-iterated line graph as given below.

Lemma 3.2 *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Theorem 3.3 *Let G_1 and G_2 be regular, non CD-cospectral graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non CD-cospectral, CD-equienergetic graphs of equal order and of equal number of edges.*

Proof The result follows from Lemma 3.2 and Corollary 2.3. $\square$

Acknowledgement

H. S. Ramane is thankful to the University Grants Commission (UGC), New Delhi for support through grant under UGC-SAP DRS-III, 2016-2021: F.510/3/DRS-III/2016 (SAP-I). D. Patil is thankful to the Karnataka Science and Technology Promotion Society, Bengaluru for fellowship No. DST/KSTePS/Ph.D Fellowship/OTH-01:2018-19.

References

[1] F. Buckley, Iterated line graphs, *Congr. Numer.*, 33 (1981), 390–394.

[2] F. Buckley, The size of iterated line graphs, *Graph Theory Notes N. Y.*, 25 (1993), 33–36.

[3] F. Buckley, F. Harary, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.

[4] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.

[5] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103 (1978), 1–22.

[6] F. Harary, *Graph Theory*, Addison–Wesley, Reading, 1969.

[7] O. Ivanciuc, T. Ivanciuc, A. T. Balaban, The complementary distance matrix, a new molecular graph metric, *ACH-Models Chem.*, 137 (2000), 57–82.

[8] G. Indulal, D-spectrum and D-energy of complements of iterated line graphs of regular graphs, *J. Alg. Stru. Appl.*, 4 (2017), 51–56.

[9] D. Jenežić, A. Miličević, S. Nikolić, N. Trinajstić, *Graph Theoretical Matrices in Chemistry*, Uni. Kragujevac, Kragujevac, 2007.

[10] H. S. Ramane, K. C. Nandeesh, Complementary distance spectra and complementary distance energy of line graphs of regular graphs, *J. Indones. Math. Soc.*, 22 (2016), 27–35.

[11] H. Sachs, Über selbstkomplementare Graphen, *Publ. Math. Debrecen*, 9 (1962), 270–288.

[12] H. Sachs, Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II, *Wiss. Z. TH Ilmenau*, 13 (1967), 405–412.

Math.Combin.Book Ser. Vol.2(2020), 74-79

Reversible DNA Codes Over a Family of the Finite Rings

Abdullah Dertli

(Ondokuz Mayıs University, Faculty of Arts and Sciences, Mathematics Department, Samsun, Turkey)

Yasemin Cengellenmis

(Trakya University, Faculty of Sciences, Mathematics Department, Edirne, Turkey)

E-mail: abdullah.dertli@gmail.com, ycengellenmis@gmail.com

Abstract: In this paper, the results introduced in [4] are extended. We define a non trivial automorphism θ_a over $R_a = F_4[u_1, \cdots, u_a]/\left\langle u_i^2 = u_i, u_i u_j = u_j u_i \right\rangle$, where $i, j = 1, 2, \cdots, a$ and a generalized Gray map over R_a which preserves DNA reversibility. The reversibility problem for DNA codes over a family of the finite rings R_a is solved, by using the skew cyclic codes over R_a. 2^a-mers are matched with the elements of the finite ring R_a.

Key Words: Reversibility, DNA codes, finite ring, Gray map.

AMS(2010): 68P30.

§1. Introduction

The reversibility problem is very important in DNA computing. Let $(\alpha_1, \alpha_2) \in R_1^2$ be a codeword corresponding to $ATGC$. The reverse of (α_1, α_2) is (α_2, α_1). The vector (α_2, α_1) corresponding to $GCAT$. It is not reverse of $ATGC$. The reverse of $ATGC$ is $CGTA$.

Some authors use the different approachers in order to solve this problem [1-11].

In [4], by defining a nontrivial automorphism, the skew cyclic codes over the finite ring R_2 were introduced DNA 4-bases were matched with the elements 256 of the finite R_2. The reversible DNA codes were obtained.

In this paper, motivated by the previous work [4], we study the reversibility problem for DNA 2^a-bases, by using the skew cyclic codes over the finite ring R_a.

§2. Preliminaries

A family of the finite rings $R_a = F_4[u_1, \cdots, u_a]/\left\langle u_i^2 = u_i, u_i u_j = u_j u_i \right\rangle$, where $i, j = 1, 2, \cdots, a$ contains the commutative the finite rings with characteristic 2 and cardinality 4^{2^a}. The finite rings of the family are written as recursively

$$R_j = R_{j-1} + u_j R_{j-1}$$

[1]Received November 13, 2019, Accepted June 9, 2020.

where $j = 1, 2, \cdots, a$ and $R_1 = F_4 + u_1 F_4, u_1^2 = u_1$, where $R_0 = F_4 = \{0, 1, w, w^2 = w + 1\}$.

We defined the Gray map as follows,

$$\begin{array}{rcl} \phi_i & : & R_i \longrightarrow R_{i-1}^2 \\ x_{i-1} + u_i y_{i-1} & \longmapsto & (x_{i-1} + y_{i-1}, x_{i-1}) \end{array}$$

where $i = 1, 2, \cdots, a$ and

$$\begin{array}{rcl} \phi_1 & : & R_1 \longrightarrow R_0^2 \\ x_0 + u_1 y_0 & \longmapsto & (x_0 + y_0, x_0) \end{array}$$

where $R_0 = F_4$.

In [4], by using the matching the elements of R_0 and $S_{D_4} = \{A, T, C, G\}$ which is given as $\xi_0(0) = A, \xi_0(1) = T, \xi_0(w) = C, \xi_0(w^2) = G$ by using the Gray map from $R_1 = F_4 + u_1 F_4$ to F_4^2, it is defined a ξ_1 correspondence between the elements of the finite ring $R_1 = F_4 + u_1 F_4$ and DNA double pairs as follows

elements α	DNA double pairs $\xi_1(\alpha)$
0	AA
1	TT
w	CC
$1+w$	GG
u_1	TA
$1+u_1$	AT
u_1+w	GC
$1+u_1+w$	CG
$u_1 w$	CA
$1+u_1 w$	GT
$w+u_1 w$	AC
$1+w+u_1 w$	TG
$u_1+u_1 w$	GA
$1+u_1+u_1 w$	CT
$w+u_1+u_1 w$	TC
$1+w+u_1+u_1 w$	AG

§3. Skew Cyclic Codes over R_a

Definition 3.1 *Let B be a finite ring and θ be a non trivial automorphism on B. A subset C*

of B^n is called a skew cyclic code of length n if C satisfies the following conditions:

(1) *C is a submodule of B^n;*

(2) *if $c = (c_0, c_1, \cdots, c_{n-1}) \in C$, then $\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})) \in C$, where σ_θ is the skew cyclic shift operator.*

By defining a non trivial automorphism on R_a as follows, we can define the skew cyclic codes over R_a.

$$\begin{array}{rcl} \theta_i & : & R_i \longrightarrow R_i \\ x_{i-1} + u_i y_{i-1} & \longmapsto & \theta_{i-1}(x_{i-1} + y_{i-1}) + u_i \theta_{i-1}(y_{i-1}) \end{array}$$

and

$$\begin{array}{rcl} \theta_1 & : & R_1 \longrightarrow R_1 \\ x_0 + u_1 y_0 & \longmapsto & (x_0 + y_0) + u_1 y_0 \end{array}$$

where $i = 2, 3, \cdots, a$. The order of θ_i is 2, where $i = 1, 2, \cdots, a$.

The rings

$$R_i[x, \theta_i] = \{b_0^i + b_1^i x + ... + b_{n-1}^i x^{n-1} : b_j^i \in R_i, n \in N, i = 1, \cdots, a, j = 0, \cdots, n-1\}$$

are called skew polynomial rings with the usual polynomial addition and the multiplication as follows

$$(\varrho x^s)(\eta x^t) = \varrho \theta_i^s(\eta) x^{s+t}$$

where $i = 1, \cdots, a$. They are non commutative rings.

In polynomial representation, a skew cyclic code of length n over R_i is defined as a left ideal of the quotient ring $R_{\theta_i,n} = R_i[x, \theta_i] / \langle x^n - 1 \rangle$, if the order of θ_i divides n, that is n is even. If the order of θ_i does not divides n, a skew cyclic code of length n over R_i is defined as a left $R_i[x, \theta_i]$-submodule of $R_{\theta_i,n}$, since the set $R_{\theta_i,n} = R_i[x, \theta_i] / \langle x^n - 1 \rangle = \{f_i(x) + \langle x^n - 1 \rangle : f_i(x) \in R_i[x, \theta_i]\}$ is a left $R_i[x, \theta_i]$-module with the multiplication from left defined by

$$r_i(x)(f_i(x) + \langle x^n - 1 \rangle) = r_i(x) f_i(x) + \langle x^n - 1 \rangle$$

where for any $r_i(x) \in R_i[x, \theta_i]$, for $i = 1, \cdots, a$.

In both case, the following is hold.

Theorem 3.2 *Let C_i be a skew cyclic code over R_i and let $f_i(x)$ be a polynomial in C_i of minimal degree, $i = 1, \cdots, a$. If the leading coefficient of $f_i(x)$ is a unit in R_i, then $C_i = \langle f_i(x) \rangle$, where $f_i(x)$ is a right divisor of $x^n - 1$.*

Definition 3.3 *For $\mathbf{x} = (x_0^i, x_1^i, \cdots, x_{n-1}^i) \in R_i^n$, the vector $(x_{n-1}^i, x_{n-2}^i, \cdots, x_1^i, x_0^i)$ is called the reverse of $\mathbf{x}$ and is denoted by $\mathbf{x}^r$. A linear code C_i of length n over R_i is called reversible if $\mathbf{x}^r \in C_i$ for every $\mathbf{x} \in C_i$, where $i = 1, \cdots, a$.*

We can express the matching the elements R_1 and $S_{D_{16}} = \{AA, TT, \cdots, GG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1 = x_0 + u_1 y_0 \in R_1$ and $\theta_1(\alpha_1)$ are mapped to DNA 2-mers which are reverse of each other. Let ξ_1 be a correspondence the elements of the finite ring R_1 and DNA 2-mers. For example

$$\xi_1(u_1) = TA, \text{ while } \xi_1(\theta_1(u_1)) = AT$$

This can be extended to a map γ_i from R_{i-1}^2 to 2^i-mers as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in R_{i-1}$, for $i = 1, \cdots, a$.

By using a map $\Psi_i = \gamma_i \circ \phi_i$, we can explain a relationship between skew cyclic codes and DNA codes. $\Psi_i(r_i)$ and $\Psi_i(\theta_i(r_i))$ are DNA reverse of each other, where $r_i = a_{i-1} + u_i b_{i-1}$, $a_{i-1}, b_{i-1} \in R_{i-1}$, where $i = 1, \cdots, a$.

For $r_i = a_{i-1} + u_i b_{i-1} \in R_i$, we have

$$\begin{aligned} \Psi_i(r_i) &= \gamma_i(\phi_i(a_{i-1} + u_i b_{i-1})) = \gamma_i(a_{i-1} + b_{i-1}, a_{i-1}) \\ &= (\xi_{i-1}(a_{i-1} + b_{i-1}), \xi_{i-1}(a_{i-1})) \end{aligned}$$

On the other hand,

$$\begin{aligned} \Psi_i(\theta_i(r_i)) &= \Psi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + u_i \theta_{i-1}(b_{i-1})) \\ &= \gamma_i(\phi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + u_i \theta_{i-1}(b_{i-1}))) \\ &= \gamma_i(\theta_{i-1}(a_{i-1}), \theta_{i-1}(a_{i-1} + b_{i-1})) \\ &= (\xi_{i-1}(\theta_{i-1}(a_{i-1})), \xi_{i-1}(\theta_{i-1}(a_{i-1} + b_{i-1}))) \end{aligned}$$

where $i = 1, \cdots, a$. This map can be extended as follows.

For any $r_i = (r_0^i, \cdots, r_{n-1}^i) \in R_i^n$, where $i = 1, 2, \cdots, a$.

$$\left(\Psi_i\left(r_0^i\right), \Psi_i\left(r_1^i\right), ..., \Psi_i\left(r_{n-1}^i\right)\right)^r = \left(\Psi_i\left(\theta_i\left(r_{n-1}^i\right)\right), ..., \Psi_i\left(\theta_i\left(r_1^i\right)\right), \Psi_i\left(\theta_i\left(r_0^i\right)\right)\right)$$

Example 3.4 If $r_3 = u_3((w + u_1) + u_2(1 + u_1 w)) \in R_3$, then we have

$$\begin{aligned} \Psi_3(r_3) &= \gamma_3(\phi_3(r_3)) = \gamma_3(w + u_1 + u_2(1 + u_1 w), 0) \\ &= (\xi_2(w + u_1 + u_2(1 + u_1 w)), \xi_2(0)) = (AGGC, AAAA) \end{aligned}$$

On the other hand,

$$\begin{aligned}\Psi_3\left(\theta_3(r_3)\right) &= \Psi_3\left(\theta_2(w+u_1+u_2\left(1+u_1w\right))+u_3\theta_2(w+u_1+u_2\left(1+u_1w\right))\right)\\ &= \gamma_3\left(\theta_2(0),\theta_2(w+u_1+u_2\left(1+u_1w\right))\right)\\ &= \left(\xi_2\left(\theta_2(0)\right),\xi_2\left(\theta_2(w+u_1+u_2\left(1+u_1w\right))\right)\right)\\ &= (AAAA, CGGA)\end{aligned}$$

Definition 3.5 *Let C_i be a code of length n over R_i, for $i=1,\cdots,a$. If $\Psi_i(\mathbf{c})^r \in \Psi_i(C_i)$ for all $\mathbf{c}\in C_i$, then C_i or equivalently $\Psi_i(C_i)$ is called a reversible DNA code.*

Definition 3.6 *Let $g_i(x)=b_0^i+b_1^ix+b_2^ix^2+\cdots+b_s^ix^s$ be a polynomial of degree s over R_i, for $i=1,\cdots,a$. $g_i(x)$ is called a palindromic polynomial if $b_j^i=b_{s-j}^i$ for all $j\in\{0,1,\cdots,s\}$. $g_i(x)$ is called a θ_i-palindromic polynomial if $b_j^i=\theta_i(b_{s-j}^i)$ for all $j\in\{0,1,\cdots,s\}$, for $i=1,\cdots,a$.*

As the order of θ_i is 2, a skew cyclic code of odd length n over R_a with respect to θ_i is an ordinary cyclic code. So we will take the length n to be even.

Theorem 3.7 *Let $C_i=\langle f_i\left(x\right)\rangle$ be a skew cyclic code of length n over R_i, for $i=1,\cdots,a$, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial then $\Psi_i(C_i)$ is a reversible DNA code.*

Proof Let $f_i(x)$ be a θ_i-palindromic polynomial and $f_i(x)=a_0^i+a_1^ix+\cdots+a_{2s-1}^ix^{2s-1}$. So $a_d^i=\theta_i(a_{2s-1-d}^i)$, for all $d=0,1,\cdots,s-1$. Let $h_i(x)=h_0^i+h_1^ix+\cdots+h_{2k-1}^ix^{2k-1}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l=1,\cdots,n-1$. For any $t<n/2$, the coefficient of x^t in $h_i(x)f_i(x)$ is

$$b_t^i=\sum_{j=0}^{t}h_j^i\theta_i^j(a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i=\sum_{j=0}^{t}h_{2k-1-j}^i\theta_i^{2k-1-j}(a_{2s-1-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x)=\sum_{p=0}^{2k-1}h_p^ix^pf_i(x)$ corresponds a vector $b=(b_0^i,b_1^i,\cdots,b_{n-1}^i)\in C_i$ for $i=1,\cdots,a$.

The vector $\Psi_i\left(b\right)^{\ r}=\left(\left(\Psi_i\left(b_0^i\right),\Psi_i\left(b_1^i\right),\cdots,\Psi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\Psi_i\left(z\right)$, where the vector z corresponds the polynomial $\sum_{p=0}^{2k-1}\theta_i(h_p^i)x^{2k-1-p}f_i(x)$,for $i=1,\cdots,a$.

Since $z=(z_1^i,\cdots,z_n^i)\in C_i$, then $\Psi_i(C_i)$ is a reversible DNA code, for $i=1,\cdots,a$. □

Theorem 3.8 *Let $C_i=\langle f_i\left(x\right)\rangle$ be a skew cyclic code of length n over R_i, for $i=1,\cdots,a$, where $f_i(x)$ is a right divisor of x^n-1 and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial then $\Psi_i(C_i)$ is a reversible DNA code.*

Proof Let $f_i(x)$ be a palindromic polynomial with even degree. $f_i(x)=a_0^i+a_1^ix+\cdots+a_{2s}^ix^{2s}$ and $a_d^i=a_{2s-d}^i$, for all $d=0,1,\cdots,s$. Let $h_i(x)=h_0^i+h_1^ix+\cdots+h_{2k}^ix^{2k}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l=1,\cdots,n-1$. For any $t<n/2$, the coefficient of x^t

in $h_i(x)f_i(x)$ is

$$b_t^i = \sum_{j=0}^{t} h_j^i \theta_i^j (a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i = \sum_{j=0}^{t} h_{2k-j}^i \theta_i^{2k-j}(a_{2s-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k} h_p^i x^p f_i(x)$ corresponds a vector $b = (b_0^i, b_1^i, \cdots, b_{n-1}^i) \in C_i$ for $i = 1, \cdots, a$.

The vector $\Psi_i(b)^r = \left(\left(\Psi_i\left(b_0^i\right), \Psi_i\left(b_1^i\right), \cdots, \Psi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\Psi_i(z)$, where the vector z corresponds the polynomial $\sum_{p=0}^{2k} \theta_i(h_p^i) x^{2k-p} f_i(x)$,for $i = 1, \cdots, a$.

Since $z = (z_1^i, \cdots, z_n^i) \in C_i$, then $\Psi_i(C_i)$ is a reversible DNA code, for $i = 1, \cdots, a$. □

§4. Conclusion

We have shown that skew cyclic codes over the ring R_a can be used to construct the reversible DNA codes.

References

[1] Adleman L. , Molecular computation of the solutions to combinatorial problems, *Science*, 266,1021-1024, 1994.

[2] Bennenni N., Guenda K., and Mesnager S., DNA cyclic codes over rings, *Advances in Mathematics of Communications*, 11.1, 2017.

[3] Cengellenmis Y., Dertli A., On the cyclic DNA codes over the finite ring $F_2 + uF_2 + vF_2 + wF_2 + uvF_2 + uwF_2 + vwF_2 + uvwF_2$, *Acta Universitatis Apulensis*, 2019.

[4] Cengellenmis Y., Dertli A., On the skew cyclic codes and the reversibility problem for DNA 4-bases, —it Mathematics in Computer Science, DOI: 10.1007/s11786-019-00424-5, 2019.

[5] Dertli A. , Cengellenmis Y., On cyclic DNA codes over the rings $Z_4 + wZ_4$ and $Z_4 + wZ_4 + vZ_4 + wvZ_4$, *arXiv*:1605.02968, 2016.

[6] Guenda K., Aaron Gulliver, T., Construction of cyclic codes over $F_2 + uF_2$ for DNA computing, *AAECC*, 24(6), 445-459, 2013.

[7] Gursoy Fatma., Oztas S. E., Siap I., Reversible DNA codes over $F_{16} + uF_{16} + vF_{16} + uvF_{16}$, *arXiv*:1703.10189v1, 2017.

[8] Liang J., Wang L., On cyclic DNA codes over $F_2 + uF_2$, *J.Appl Math Comput.*, DOI :10.1007/s12190-015-0892-8.

[9] Limbachiya D. , Rao B. and Gupta Manish K., The art of DNA strings: sixteen years of DNA coding theory, *arXiv*:1607.00266, 2016.

[10] Pattanayak S., Singh A. K., Construction of cyclic DNA codes over the Ring $Z_4[u]/ < u^2 - 1 >$ Based on the deletion distance, *arXiv*:1603.04055, 2016.

[11] Zhu S., Chen X., Cyclic DNA codes over $F_2 + uF_2 + vF_2 + uvF_2$ and their applications, *J. Appl.Math Comput.*, 55, 479-493, DOI: 10.1007/s12190-016-1046-3.

Math.Combin.Book Ser. Vol.2(2020), 80-87

A Note on Torian Algebras

Ilojide Emmanuel

(Department of Mathematics, Federal University of Agriculture, Abeokuta 110101, Nigeria)

E-mail: emmailojide@yahoo.com, ilojidee@funaab.edu.ng

Abstract: In [6], obic algebras were introduced. In this paper, a class of obic algebras is studied. It is shown that with a suitably defined binary relation, this class of obic algebras are partially ordered sets. The partial ordering is used to investigate some of their properties.

Key Words: Torian algebra, obic algebra, harmonic torian algebra, Smarandachely torian algebra.

AMS(2010): 20N02, 20N05, 06F35.

§1. Introduction

Algebras of type $(2,0)$ are well known types of algebraic structures. They comprise non-empty sets, some constant element together with a binary operation. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and So, in [2] and [4] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties.

In [6], obic algebras were introduced. Homomorphisms and krib maps as well as monics of obic algebras were studied. In this paper, a class of obic algebras is studied. It is shown that with a suitably defined binary relation, this class of obic algebras are partially ordered sets. The partial ordering is used to investigate some of their properties.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms*

(1) $x*0=x$;
(2) $[x*(y*z)]*x = x*[y*(z*x)]$;
(3) $x*x=0$

hold for all $x,y,z\in X$.

Example 2.1([6]) Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

[1]Received January 24, 2020, Accepted June 10, 2020.

Lemma 2.1([6]) *Let X be an obic algebra. Then,*

$$x * y = [x * (y * x)] * x$$

hold for all $x, y \in X$.

Definition 2.2([6]) *An obic algebra X is said to have the weak property (WP) if $x * y = 0$ and $y * x = 0$ imply that $x = y$.*

Definition 2.3([6]) *Let $(X; *, 0)$ and $(Y; \circ, 0')$ be obic algebras. A function $f : X \to Y$ is called an obic homomorphism if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in X$.*

Lemma 2.2([6]) *Let $f : X \to Y$ be an obic homomorphism. The equivalence relation $\sim$ defined by $(x \sim y) \Rightarrow f(x) = f(y)$ is a congruence.*

Theorem 2.1([6]) *Let $f : X \to Y$ be an obic homomorphism. Then $(\overline{X}; \diamond, [0])$ is an obic algebra.*

§3. Main Results

Definition 3.1 *An obic algebra X is called torian if $[(x * y) * (x * z)] * (z * y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$ such that $[(x * y) * (x * z)] * (z * y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Example 3.1 Let $X = \{0, 1\}$. Define a binary operation $*$ on X by the multiplication table below

$*$	0	1
0	0	1
1	1	0

Then, $(X, *, 0)$ is torian.

Lemma 3.1 *Let X be a torian algebra. Then, the following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * (x * y)] * (y * x) = 0$;
(2) $[(x * y) * x] * (0 * y) = 0$;
(3) $[(0 * y) * (0 * z)] * (z * y) = 0$;
(4) $[x * (x * z)] * z = 0$;
(5) $[0 * (x * y)] * (y * x) = [(x * y) * x] * (0 * y)$;
(6) $[0 * (x * y)] * (y * x) = [(0 * y) * (0 * z)] * (z * y)$;
(7) $[0 * (x * y)] * (y * x) = [x * (x * z)] * z$;
(8) $[(x * y) * x] * (0 * y) = [(0 * y) * (0 * z)] * (z * y)$;
(9) $[(x * y) * x] * (0 * y) = [x * (x * z)] * z$;
(10) $[(0 * y) * (0 * z)] * (z * y) = [x * (x * z)] * z$.

Proof The proof follows immediately from definition. □

The following corollary follows from Lemmas 2.1 and 3.1.

Corollary 3.1 *Let X be a torian algebra. The following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = 0$;
(2) $[[[x * (y * x)] * x] * x] * (0 * y) = 0$;
(3) $[(0 * y) * (0 * z)] * [[z * (y * z)] * z] = 0$;
(4) $[x * [[x * (z * x)] * x]] * z = 0$;
(5) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [[[x * (y * x)] * x] * x] * (0 * y)$;
(6) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [(0 * y) * (0 * z)] * [[z * (y * z)] * z]$;
(7) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [x * [[x * (z * x)] * x]] * z$;
(8) $[[[x * (y * x)] * x] * x] * (0 * y) = [(0 * y) * (0 * z)] * [[z * (y * z)] * z]$;
(9) $[[[x * (y * x)] * x] * x] * (0 * y) = [x * [[x * (z * x)] * x]] * z$;
(10) $[(0 * y) * (0 * z)] * [[z * (y * z)] * z] = [x * [[x * (z * x)] * x]] * z$.

A torian algebra which has the weak property is called a weak property torian algebra (WPTA).

Lemma 3.2 *Let X be a WPTA. Define a relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for $x * y \in X$. Then $(X; \sim)$ is a partially ordered set.*

Proof The reflexivity and anti-symmetry follow from definition. Now let $x, y, z \in X$ such that $x \sim y$ and $y \sim z$. Then $x * z = [(x * z) * 0] * 0 = [(x * z) * (x * y)] * (y * z) = 0$. So, the transitivity holds. □

Proposition 3.1 *A torian algebra X is a WPTA if and only if there exists a partial ordering $\sim$ on X such that for all $x, y, z \in X$ hold with:*

(1) $(x * y) * (x * z) \sim (z * y)$;
(2) $[x * (x * y)] \sim y$;
(3) $x * y = 0 \Leftrightarrow x \sim y$.

Proof Suppose X is a WPTA. By Lemma 3.2, X is equipped with a partial ordering. Clearly, $(x * y) * (x * z) \sim (z * y)$ holds. Also, by Lemma 3.1(4), $[x * (x * y)] \sim y$ holds. Clearly, $x * y = 0 \Leftrightarrow x \sim y$ holds.

Conversely, suppose X is a torian algebra with partial ordering $\sim$ satisfying $(x*y)*(x*z) \sim (z * y)$, $[x * (x * y)] \sim y$ and $x * y = 0 \Leftrightarrow x \sim y$ for all $x, y, z \in X$. Let $x, y \in X$ such that $x * y = 0$ and $y * x = 0$. Then $x \sim y$ and $y \sim x$. By anti-symmetry, $x = y$ as required. □

The following corollary follows from Proposition 3.1 and Lemma 2.1.

Corollary 3.2 *A torian algebra X is a WPTA if and only if there exists a partial ordering $\sim$ on X such that for all $x, y, z \in X$ hold with:*

(1) $[[x * (y * x)] * x] * [x * (z * x)] * x \sim [z * (y * z)] * z$;

(2) $[x * [x * (y * x)] * x] \sim y$;

(3) $[x * (y * x)] * x = 0 \Leftrightarrow x \sim y$.

Proposition 3.2 *Let X be a torian algebra with partial ordering $\sim$. Then for all $x, y, z \in X$ hold with:*

(1) $z \sim y \Rightarrow (x * y) \sim (x * z)$;

(2) $x \sim z \Rightarrow (x * y) \sim (z * y)$;

(3) $x \sim y \Rightarrow [0 * (x * z)] \sim (z * y)$.

Proof The proof follows from definition immediately. □

The following corollary follows from Proposition 3.2 and Lemma 2.1.

Corollary 3.3 *Let X be a torian algebra with partial ordering $\sim$. Then, for all $x, y, z \in X$ hold with:*

(1) $z \sim y \Rightarrow [[x * (y * x)] * x] \sim [[x * (z * x)] * x]$;

(2) $x \sim z \Rightarrow [[x * (y * x)] * x] \sim [z * (y * z)] * z$;

(3) $x \sim y \Rightarrow [0 * [x * (z * x)] * x] \sim [[z * (y * z)] * z]$.

Proposition 3.3 *Let X be a torian algebra with partial ordering $\sim$. Then, the following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * (x * y)] \sim (y * x)$;

(2) $[(x * y) * x] \sim (0 * y)$;

(3) $[(0 * y) * (0 * z)] \sim (z * y)$;

(4) $[x * (x * z)] \sim z$.

Proof The proof follows from definition immediately. □

The following corollary follows from Proposition 3.3 and Lemma 2.1.

Corollary 3.4 *Let X be a torian algebra with partial ordering $\sim$. Then the following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * [x * (y * x)] * x] \sim [[y * (x * y)] * y]$;

(2) $[[[x * (y * x)] * x] \sim (0 * y)$;

(3) $[(0 * y) * (0 * z)] \sim [[z * (y * z)] * z]$

(4) $[x * [x * (z * x)] * x] \sim z$.

Proposition 3.4 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with*

$$x * [x * (x * y)] = x * y.$$

Proof Since X is torian, we have $(x * y) * [x * (x * (x * y))] \sim [x * (x * y)] * y$. Also, $[x * (x * y)] \sim y$ (by Proposition 3.3(4)). So, we now have

$$[x * (x * y)] * y = 0.$$

This gives us $(x*y)*[x*[x*(x*y)]]\sim 0$. Therefore, $(x*y)*[x*[x*(x*y)]]=0$.

Also, $[x*[x*(x*y)]]*(x*y)=0$ (by Proposition 3.3(4)). Hence, $x*[x*(x*y)]=x*y$ as required. □

The following corollary follows from Proposition 3.4 and Lemma 2.1.

Corollary 3.5 *Let X be a WPTA. Then, for all $x,y\in X$ hold with:*

$$x*[x*[x*(y*x)]*x]=[[x*(y*x)]*x].$$

Lemma 3.3 *Let X be a WPTA. Then, for all $x,y,z\in X$ hold with*

$$(x*y)*z=(x*z)*y.$$

Proof The proof follows from Proposition 3.3(4) and Proposition 3.1(1). □

Proposition 3.5 *Let X be a WPTA. Then, for all $x,y\in X$ hold with*

$$(0*x)*(0*y)=0*(x*y).$$

Proof Notice that

$$\begin{aligned}
(0*x)*(0*y) &= [[(x*y)*(x*y)]*x]*(0*y)\\
&= [[(x*y)*x]*(x*y)]*(0*y) \text{ (by Lemma3.3)}\\
&= [[(x*x)*y]*(x*y)]*(0*y)\\
&= [(0*y)*(x*y)]*(0*y)\\
&= [(0*y)*(0*y)]*(x*y)\\
&= 0*(x*y)
\end{aligned}$$

as required. □

The following corollary follows from Proposition 3.5 and Lemma 2.1.

Corollary 3.6 *Let X be a WPTA. Then, for all $x,y\in X$ hold with*

$$(0*x)*(0*y)=0*[[x*(y*x)]*x].$$

Proposition 3.6 *Let X be a WPTA. Then, for all $x\in X$ hold with*

$$0*[x*[0*(0*x)]]=0.$$

Proof Notice that

$$\begin{aligned} 0*[x*[0*(0*x)]] &= (0*x)*[0*[0*(0*x)]] \quad \text{(by Proposition 3.5)} \\ &= (0*x)*(0*x) \quad \text{(by Proposition 3.4)} \\ &= 0 \end{aligned}$$

as required. □

Proposition 3.7 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with:*

(1) $[0*(y*x)]*(x*y) = 0$;
(2) $[(x*y)*(0*y)]*x = 0$;
(3) $[(0*y)*(z*y)]*(0*z) = 0$;
(4) $[0*(y*x)]*(x*y) = [(x*y)*(0*y)]*x$;
(5) $[0*(y*x)]*(x*y) = [(0*y)*(z*y)]*(0*z)$;
(6) $[(x*y)*(0*y)]*x = [(0*y)*(z*y)]*(0*z)$.

Proof The proof follows from Lemmas 3.1 and 3.3. □

The following corollary follow from Proposition 3.7 and Lemma 2.1.

Corollary 3.7 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with:*

(1) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x] = 0$;
(2) $[[[x*(y*x)]*x]*(0*y)]*x = 0$;
(3) $[(0*y)*[z*(y*z)]*z]*(0*z) = 0$;
(4) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x] = [[[x*(y*x)]*x]*(0*y)]*x$;
(5) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x] = [(0*y)*[z*(y*x)]*z]*(0*z)$;
(6) $[[[x*(y*x)]*x]*(0*y)]*x = [(0*y)*[z*(y*z)]*z]*(0*z)$.

Definition 3.2 *A torian algebra X is said to be harmonic if $0*x = x$ for all $x \in X$.*

Corollary 3.8 *Let X be a harmonic WPTA. Then, for all $x, , y, z \in X$ hold with:*

(1) $(y*x)*(x*y) = 0$;
(2) $[(x*y)*y]*x = 0$;
(3) $[y*(z*y)]*z = 0$;
(4) $(y*x)*(x*y) = [(x*y)*y]*x$;
(5) $(y*x)*(x*y) = [y*(z*y)]*z$;
(6) $[(x*y)*y]*x = [y*(z*y)]*z$.

Proof The proof follows from Proposition 3.7. □

Definition 3.3 *Let X be a WPTA. An element $x \in X$ is said to fix 0 if $0*x = 0$. If every element in X fixes 0, then X is said to fix 0.*

Proposition 3.8 *Let X be WPTA which fixes* 0*. Then, for all $x, y \in X$ hold with*

$$(x * y) * x = 0.$$

Proof The proof is straightforward by definition. □

Proposition 3.9 *Let X be a WPTA which fixes* 0*. Then if $x, y \in X$ such that $x * (x * y) = 0$, then $x = x * y$.*

Proof The proof is straightforward by definition. □

Theorem 3.1 *A WPTA X fixes* 0 *if and only if $(x * y) * x = 0$ for all $x, y \in X$.*

Proof Notice that $0 = (x*y)*x = (x*x)*y = 0*y$, and the converse follows by Proposition 3.8. □

Proposition 3.10 *Let X be a WPTA. Let $x, y, z \in X$ such that $x * y = x * z$, then,*

$$0 * y = 0 * z.$$

Proof Notice that $(x * y) * x = 0 * y$. Similarly, $(x * z) * x = 0 * z$. Then, the conclusion follows. □

Corollary 3.9 *Let X be a harmonic WPTA. Let $x, y, z \in X$ such that $x * y = x * z$. Then $y = z$.*

Proof The proof follows from Proposition 3.10. □

Proposition 3.11 *A torian algebra X fixes* 0 *if and only if $x * (0 * y) = x$ for all $x, y \in X$.*

Proof Suppose $x * (0 * y) = x$. Since X is torian, we have

$$\begin{aligned} 0 &= [(0 * x) * (0 * 0)] * (0 * x) \\ &= (0 * x) * (0 * x) = 0 * x \quad \text{(by the hypothesis).} \end{aligned}$$

The converse is obvious. □

Proposition 3.12 *Let X be a WPTA. Then $x * [0 * (0 * x)]$ fixes* 0 *for any $x \in X$.*

Proof Notice that

$$\begin{aligned} [x * [0 * (0 * x)]] &= (0 * x) * [0 * [0 * (0 * x)]] \quad \text{(by Proposition 3.5)} \\ &= (0 * x) * (0 * x) \quad \text{(by Proposition 3.4)} \\ &= 0 \end{aligned}$$

as required. □

Proposition 3.13 *Let a be a fixed element of a WPTA, X. If $x * a = 0 \Rightarrow x = a$ for any $x \in X$, then $0 * (0 * a) = a$.*

Proof Notice that $[0 * (0 * a)] * a = 0$ by Lemma 3.1(4). So, $0 * (0 * a) = a$ as required. □

Proposition 3.14 *Let a be a fixed element of a WPTA, X. If $0 * (0 * a) = a$, then $0 * x = a$ for some $x \in X$.*

Proof Put $0 * a = x$ in $0 * (0 * a) = a$. Then the conclusion follows. □

Proposition 3.15 *Let a be a fixed element of a WPTA, X. If $0 * x = a$ for some $x \in X$, then $x * a = 0 \Rightarrow x = a$.*

Proof Let $y \in X$ such that $y * a = 0$. Then we have $y * (0 * x) = 0$ and

$$\begin{aligned} a * y &= (0 * x) * y \\ &= [0 * [0 * (0 * x)]] * y \quad \text{(by Proposition 3.4)} \\ &= (0 * y) * [0 * (0 * x)] \quad \text{(by Lemma 3.3)} \\ &= 0 * [y * (0 * x)] \quad \text{(by Proposition 3.5)} \\ &= 0 * (y * a) = 0 * 0 = 0 \end{aligned}$$

Since $a * y = 0$ and $y * a = 0$, then $y = a$ as required. This completes the proof. □

By Propositions 3.13, 3.14 and 3.15, we have the following theorem.

Theorem 3.2 *Let a be a fixed element of a WPTA X. Then the following conclusions are equivalent:*

(1) $x * a = 0 \Rightarrow x = a$ *for any* $x \in X$*;*
(2) $0 * (0 * a) = a$*;*
(3) $0 * x = a$ *for some* $x \in X$*.*

References

[1] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, 66(2007), 113–116.
[2] S.S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, 68(2008), 351–357.
[3] S.S. Ahn and K. S. So, On generalized upper sets in BE-algebras, *Bull. Korean Math. Soc.*, 46(2009), 281–287.
[4] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185pp.
[5] J. Dene and A. D. Keedwell, *Latin Squares and Their Applications*, the English University press Ltd, 1974, 549pp.
[6] E. Ilojide, On obic algebras, *International J. Math. Combin.*, 4(2019), 80–88.

Math.Combin.Book Ser. Vol.2(2020), 88-100

On Quotient of Randić and Sum-Connectivity Energy of Graphs

Puttaswamy and C. A. Bhavya

(Department of Mathematics, P.E.S. college of Engineering, Mandya-571401, India)

E-mail: prof.puttaswamy@gmail.com, cabhavya212@gmail.com

Abstract: In this paper we define the quotient of Randić and sum-connectivity energy of a graph. Then we obtain upper and lower bounds for $E_{qrs}(G)$, quotient of Randić and sum-connectivity energy of a graph. Further we compute the quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Key Words: Quotient of Randić, eigenvalues of the sum-connectivity matrix, sum-connectivity energy.

AMS(2010): 05C50.

§1. Introduction

In 2010, Bo Zhou and Nenad Trinajstic [3] have introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \cdots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i. Then the sum-connectivity matrix of G is defined as $R = (R_{ij})$, where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G arranged in a non-increasing order.

In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [2], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

[1]Received March 13, 2020, Accepted June 11, 2020.

Motivated by these works, we introduce the Quotient of Randić and sum-connectivity energy of a simple graph G as follows. Let a and b be two nonnegative real number with $a \neq 0$. The quotient of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{qrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph G are the eigenvalues of A_{qrs}. Since A_{qrs} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$. Then the Quotient of Randić and sum-connectivity energy of G is defined as

$$E_{qrs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

Since A_{qsr} is a real symmetric matrix, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A_{qsr}) = 0 \tag{1}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A_{qsr}^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = 2\sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)} \tag{2}$$

In this paper we obtain the upper and lower bounds for $E_{qsr}(G)$ and compute the $E_{qrs}(G)$ of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

§2. Upper and Lower Bounds for $E_{qrs}(G)$

In this section we obtain Upper and lower bounds for $E_{qsr}(G)$.

Theorem 2.1 *Let G be a simple graph of order n with no isolated vertices and let a, b be two nonnegative real number with $a \neq 0$. Then*

$$E_{qrs}(G) \leq \sqrt{2n\sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \tag{3}$$

Proof Let $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, be the eigenvalues of A_{qrs}. Then using (2) and the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

with $a_i = 1$, $b_i =| \lambda_i |$. We obtain

$$E_{qrs}(G) = \sum_{i=1}^{n} |\lambda_i| = \sqrt{(\sum_{i=1}^{n} |\lambda_i|)^2} \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2} = \sqrt{2n \sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \qquad \square$$

Theorem 2.2 *Let G be a simple graph of order n with no isolated vertices and let a, b be two nonnegative real number with $a \neq 0$. Then*

$$E_{qrs}(G) \geq 2\sqrt{\sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \tag{4}$$

Proof From (1), we have

$$\sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\leq i<j\leq n} \lambda_i \lambda_j = 0$$

and therefore

$$-\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1\leq i<j\leq n} \lambda_i \lambda_j. \tag{5}$$

Thus

$$\begin{aligned} (E_{qrs}(G))^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\leq i<j\leq n} |\lambda_i \lambda_j| \\ &\geq \sum_{i=1}^{n} \lambda_i^2 + 2 \left| \sum_{1\leq i<j\leq n} \lambda_i \lambda_j \right| = 2 \sum_{i=1}^{n} \lambda_i^2 \end{aligned}$$

on using (5). This together with (2) implies that

$$(E_{qrs}(G))^2 \geq 4 \sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)},$$

which gives (4). $\square$

§3. Quotient of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 3.1([4]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2([4]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a* *bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3([5]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute Quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Theorem 3.4 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of the complete bipartite graph $K_{m,n}$ is*

$$2\sqrt{\frac{b(mn)^2}{a(m+n)}}.$$

Proof Let the vertex set of the complete bipartite graph be $V(K_{m,n}) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. Then the Quotient of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{qrs} = \begin{pmatrix} 0 & \cdots & 0 & \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \sqrt{\frac{b(nm)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} \\ \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda I_m & -\sqrt{\frac{b(mn)}{a(m+n)}} J^T \\ -\sqrt{\frac{b(mn)}{a(m+n)}} J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\sqrt{\frac{b(mn)}{a(m+n)}} J^T \\ -\sqrt{\frac{b(mn)}{a(m+n)}} J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m|\left|\lambda I_n - \left(-\sqrt{\frac{b(mn)}{a(m+n)}}J\right)\frac{I_m}{\lambda}\left(-\sqrt{\frac{b(mn)}{a(m+n)}}J^T\right)\right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{\left(\frac{(a(m+n)}{b(mn)}\right)^n}\left|\frac{a(m+n)}{b(mn)}\lambda^2 I_n - JJ^T\right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{\left(\frac{a(m+n)}{b(mn)}\right)^n}P_{JJ^T}\left(\frac{a(m+n)}{b(mn)}\lambda^2\right) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{\left(\frac{a(m+n)}{b(mn)}\right)^n}\left(\left(\frac{a(m+n)}{b(mn)}\right)\lambda^2 - mn\right)\left(\frac{a(m+n)}{b(mn)}\lambda^2\right)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}\left(\lambda^2 - \frac{b(mn)^2}{a(m+n)}\right) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & \sqrt{\frac{b(mn)^2}{a(m+n)}} & -\sqrt{\frac{b(mn)^2}{a(m+n)}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{qrs}(K_{m,n}) = 2\sqrt{\frac{b(mn)^2}{a(m+n)}},$$

as desired. □

Theorem 3.5 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of the S_n is*

$$2\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then the quotient of Randić and sum-connectivity matrix of the star graph S_n is given by

$$A_{qrs} = \begin{pmatrix} 0 & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & -\sqrt{\frac{b(n-1)}{an}} & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ -\sqrt{\frac{b(n-1)}{an}} & \lambda & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where, $\mu = \lambda\sqrt{\frac{an}{b(n-1)}}$. Then

$$|\lambda I - A_{qrs}| = \phi_n(\mu)\left(\sqrt{\frac{b(n-1)}{a(m+n)}}\right)^n,$$

where

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)).$$

Therefore

$$|\lambda I - A_{qrs}| = \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \left[\left((\frac{an}{b(n-1)})\lambda^2 - (n-1)\right)\left(\lambda\sqrt{\frac{b(n-1)}{an}}\right)^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 - \frac{b(n-1)^2}{an}\right) = 0.$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 0 & \sqrt{\frac{b(n-1)^2}{an}} & -\sqrt{\frac{b(n-1)^2}{an}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of S_n is

$$E_{qrs}(S_n) = 2\sqrt{\frac{b(n-1)^2}{an}}.$$

□

Theorem 3.6 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of K_n is*

$$2(n-1)\sqrt{\frac{(n-1)b}{a2}}.$$

Proof Let the vertex set of a complete graph be given by

$$V(K_n) = \{v_1, v_2, \cdots, v_n\}.$$

Then, the quotient of Randić and sum-connectivity energy of matrix of the complete graph K_n is given by

$$A_{qrs} = \begin{pmatrix} 0 & \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \cdots & \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} \\ \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & 0 & \cdots & \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \cdots & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \lambda & \cdots & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & -\sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & \lambda \end{vmatrix}$$

$$= \left(\sqrt{\frac{(n-1)^2b}{a2(n-1)}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{\frac{a2(n-1)}{(n-1)^2b}}$. Then

$$|\lambda I - A_{qrs}| = \phi_n(\mu)\left(\sqrt{\frac{(n-1)^2b}{a2(n-1)}}\right)^n,$$

where

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix} = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}.$$

A calculation shows that

$$\begin{aligned} \phi_n(\mu) &= -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right] \\ &= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)). \end{aligned}$$

Iterating this, we obtain $\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1))$. Thus, the characteristic equation is given by

$$\left(\sqrt{\frac{(n-1)^2 b}{a2(n-1)}}\right)^n (\mu+1)^{n-1}(\mu - (n-1)) = 0.$$

Hence the quotient of Randić and sum-connectivity energy of K_n is

$$E_{qrs}(K_n) = 2(n-1)\sqrt{\frac{(n-1)b}{a2}}. \qquad \square$$

Theorem 3.7 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is*

$$4\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \cdots, v_{2m+2}\}$. Then the quotient of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{qrs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where, $m+1 = n$.

Its characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ 0 & \lambda & \cdots & 0 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} & \lambda & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where

$$\Lambda = \sqrt{\frac{na}{(n-1)b}}\lambda.$$

Let

$$\begin{aligned} \phi_{2n}(\Lambda) &= \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} \\ &= (-1)^{2n+2n}\Lambda\begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)} \end{aligned}$$

$$+(-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

where,

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda)\\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda).\end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where,

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

A calculation shows that

$$\begin{aligned}\phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda)\\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda)\\ &= \left(\Lambda^n - (n-1)\Lambda^{n-2}\right)\Theta_n(\Lambda).\end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^n - (n-1)\Lambda^{n-2}.$$

Thus,

$$\phi_{2n}(\Lambda) = \left(\Lambda^n - (n-1)\Lambda^{n-2}\right)^2.$$

Hence characteristic equation becomes

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\phi_{2n}(\Lambda) = 0,$$

which is the same as

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\left(\Lambda^n - (n-1)\Lambda^{n-2}\right)^2 = 0$$

and reduces to

$$\lambda^{2n-4}\left(\frac{na}{b(n-1)}\lambda^2 - (n-1)\right)^2 = 0.$$

Therefore,

$$Spec\left((S_m \wedge P_2)\right) = \begin{pmatrix} 0 & \sqrt{\frac{b(n-1)^2}{na+(n-1)b}} & -\sqrt{\frac{b(n-1)^2}{na}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{qrs}((S_m \wedge P_2)) = 4\sqrt{\frac{b(n-1)^2}{na}}. \qquad \square$$

References

[1] C. Adiga, Abdelmejid Bayad, Ivan Gutman and Shrikanth Avant Srinivas , The minimum covering energy of a graphs, *Kragujevac j. Sci.*, 34 (2012), 39-56.

[2] Ş. Burcu Bozkurt, A. Dilek Güngör, I. Gutman and A. Sinan Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 239-250.

[3] Bo Zhou and Nenad Trinajstic , On sum-connectivity matrix and sum-connectivity energy of (molecular) graphs *Acta Chim. Slov.*, 57 (2010), 518-523.

[4] F. Harary, *Graph Theory*, 10^{th} Reprint, Narosa Publishing House, New Delhi, 2001.

[5] Ş. M. A. Soeud, Huda Jaber, Some families of strongly*-graphs(Communicated).

Math.Combin.Book Ser. Vol.2(2020), 101-108

On Ideals of Torian Algebras

Ilojide Emmanuel

(Department of Mathematics, Federal University of Agriculture, Abeokuta 110101, Nigeria)

E-mail: emmailojide@yahoo.com, ilojidee@funaab.edu.ng

Abstract: In this paper, the notion of ideals in torian algebras is introduced. Their properties are investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras are studied.

Key Words: Ideals, torian algebras, dual, congruences, Smarandachely torian algebra.

AMS(2010): 20N02, 20N05, 06F35.

§1. Introduction

An algebra of type (2,0) is a well known type of algebraic structures. It comprises a non-empty set, some constant element together with a binary operation and interesting behaviors. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and So, in [2] and [3] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties.

In [6], obic algebras were introduced. Homomorphisms and krib maps as well as monics of obic algebras were studied in this paper. Some properties of a class of obic algebras were studied in [7]. In this paper, the notion of ideals in torian algebras is introduced. Their properties are investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras are studied.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x,y,z\in X$:*

(1) $x*0=x$;
(2) $[x*(y*z)]*x=x*[y*(z*x)]$;
(3) $x*x=0$.

Lemma 2.1([6]) *Let X be an obic algebra. Then, for all $x,y\in X$ hold with*

$$x*y=[x*(y*x)]*x.$$

[1]Received February 22, 2020, Accepted June 12, 2020.

Definition 2.2([6]) *Let $(X;*,0)$ and $(Y;\circ,0')$ be obic algebras. A function $f:X\to Y$ is called an obic homomorphism if $f(a*b)=f(a)\circ f(b)$ for all $a,b\in X$.*

Definition 2.3([6]) *Let $f:X\to Y$ be an obic homomorphism. The set $\{x\in X: f(x)=0'\}$ is called the kernel of f. It is denoted by $Ker(f)$.*

Let $f:X\to Y$ be an obic homomorphism. If f is injective, then it is called a monomorphism. If f is surjective, then it is called an epimorphism. If f is both injective and surjective, then it is called an isomorphism.

Definition 2.4([6]) *An obic algebra X is said to have the weak property (WP) if $x*y=0$ and $y*x=0$ imply that $x=y$.*

Theorem 2.1 *Let $\phi: X\to X$ be an obic homomorphism; where X has the weak property. Then ϕ is injective if and only if $ker(\phi)=\{0\}$.*

Definition 2.5([6]) *An equivalence relation $\sim^*$ on an obic algebra X is called a congruence if $(x\sim^* y)$ and $(u\sim^* v)\Rightarrow (x*u)\sim^*(y*v)$.*

Lemma 2.2([6]) *Let $f:X\to Y$ be an obic homomorphism. The equivalence relation $\sim^*$ defined by $(x\sim^* y)\Rightarrow f(x)=f(y)$ is a congruence.*

Definition 2.6([7]) *An obic algebra X is called torian if $[(x*y)*(x*z)]*(z*y)=0$ for all $x,y,z\in X$. Otherwise, if there are $x,y,z\in X$ such that $[(x*y)*(x*z)]*(z*y)\neq 0$, such an obic algebra X is called Smarandachely torian.*

Example 2.1 Let $X=\{0,1\}$. Define a binary operation $*$ on X by the multiplication table below

$*$	0	1
0	0	1
1	1	0

Then, $(X,*,0)$ is torian algebra.

Example 2.2 Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is torian.

From now on, X will denote a torian algebra equipped with the weak property.

Lemma 2.3([7]) *Let X be a torian algebra. Then, for all $x,y,z\in X$ hold with*

$$(x*y)*z=(x*z)*y.$$

Proposition 2.1([7]) *Let X be a torian algebra. Then, for all $x, y \in X$ hold with*

$$(0 * x) * (0 * y) = 0 * (x * y).$$

Definition 2.7([7]) *Let X be a torian algebra. An element $x \in X$ is said to fix 0 if $0 * x = 0$. If every element in X fixes 0, then X is said to fix 0.*

The set of all elements of X which fix 0 is denoted by 0^*.

Lemma 2.4([7]) *Let X be a torian algebra. Define the relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for all $x, y \in X$. Then $(X; \sim)$ is a partially ordered set.*

The following Lemma follows from definition.

Lemma 2.4([7]) *Let X be a torian algebra with the partial ordering $\sim$. Then,*

$$[(x * y) * (z * y)] \sim (x * z)$$

for all $x, y, z \in X$.

§3. Main Results

Definition 3.1 *Let X be a torian algebra. A non-empty set S of X is called a left ideal of X if the following hold*

(1) $0 \in S$;

(2) *if $x, y \in X$ such that $x, [[y * (x * y)] * y] \in S$, then $y \in S$.*

Definition 3.2 *Let X be a torian algebra. A non-empty set S of X is called a right ideal of X if the following hold*

(1) $0 \in S$;

(2) *if $x, y \in X$ such that $x, [[x * (y * x)] * x] \in S$, then $y \in S$.*

Remark 3.1 If S is both a left ideal and a right ideal of X, then S is called an ideal of X.

Example 3.1 Every torian algebra X has at least two left ideals, namely, $\{0\}$ and X.

Example 3.2 The subset $S = \{1, -1\}$ is a left ideal of the torian algebra in example 2.2.

Proposition 3.1 *Let X be a torian algebra. The collection of all elements in X which fix 0 is a left ideal of X.*

Proof Now, $0 \in 0^*$. Let $x, y \in X$ such that $x, [[y * (x * y)] * y] \in 0^*$. Then we show that

$y \in 0^*$. Since $[y * (x * y)] * y \in 0^*$, we have $0 * [[y * (x * y)] * y] = 0$. But

$$\begin{aligned} 0 * y &= (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) \quad \text{(by Proposition 2.1)} \\ &= 0 * [[y * (x * y)] * y] = 0 \end{aligned}$$

as required. □

Remark 3.2 If a left ideal S of X is such that $[[x * (y * x)] * x] \in S$ for all $x, y \in X$, then S is said to be a complete left ideal of X or that S is complete in X.

Proposition 3.2 *Let S be a left ideal of a torian algebra X. If $0 * x \in S$ for all $x \in S$, then S is a complete left ideal.*

Proof Let $x, y \in S$. Then notice that

$$[[x * (y * x)] * x] * x = (x * y) * x = 0 * y \in S.$$

And since S is a left ideal, we have that $[[x * (y * x)] * x] \in S$. This completes the proof. □

The following theorem follows from Propositions 3.1 and 3.2.

Theorem 3.1 *Let X be a torian algebra. Then 0^* is a complete left ideal of X if and only if*

$$0 * x \in 0^*$$

for all $x \in 0^*$.

Corollary 3.1 *Let X be a torian algebra which fixes 0. Then every left ideal of X is complete.*

The following proposition follows from definition.

Proposition 3.3 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras. Let $f : X \to Y$ be a homomorphism. Then $Ker(f)$ is a complete left ideal of X.*

Theorem 3.2 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras such that $[[x*[(x*y)*z]]*y]*z = 0$ for all $x, y, z \in X$. Let $f : X \to Y$ be an epimorphism. If S is a left ideal of X, then $f(S)$ is left ideal of Y.*

Proof Let $x', y' \in Y$ such that $x', [[y' \odot (x' \odot y')] \odot y'] \in f(S)$. Now, there exist $x, y, z \in S$ such that $f(x) = x', f(y) = [[y' \odot (x' \odot y')] \odot y'], f(z) = y'$. Clearly, $z * [(z * x) * y] \in X$. Let $z * [(z * x) * y] = w$; so that $[[z * [(z * x) * y] * x] * y = 0 \in S; \Rightarrow w \in S$. Hence, $f(w) \in f(S)$. We show that $y' = f(w)$. Notice that $f[(z * x) * y] = [f(z * x) \odot f(y)] = [f(z) \odot f(x)] \odot f(y) = (y' \odot x') \odot (y' \odot x') = 0'$.

Notice that $f(w) = f[z * [(z * x) * y]] = f(z) \odot f[(z * x) * y] = y' \odot 0' = y'$ as required. □

Corollary 3.2 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras such that $[[x*[(x*y)*z]]*y]*z = 0$ for all $x, y, z \in X$. Let $f : X \to Y$ be an epimorphism. If S is a complete left ideal of X, then*

$f(S)$ is a complete left ideal of Y.

Theorem 3.3 *A left ideal S of a torian algebra X is complete in X if and only if the following hold*

(1) $0 \in S$;

(2) $[[x * (z * x)] * x], [[y * (z * y)] * y], z \in S \Rightarrow x * y \in S$.

Proof Suppose S is a left ideal of X satisfying (1) and (2). Now, $[[0 * (0 * 0)] * 0], [[x * (0 * x)] * x], 0 \in S$. So, $0 * x \in S$; and by Proposition 3.2, S is complete.

Conversely, suppose S is complete in X. Clearly, $0 \in S$. Let $[[x*(z*x)]*x], [[y*(z*y)]*y], z \in S$. Then $x, y \in S$. So, $x * y \in S$ as required. □

Definition 3.3 *Let S be a left ideal of a torian algebra X. The set $S^* = \{x \in S : [[0*(x*0)]*0] \in S\}$ is called the dual of S.*

The following proposition follows from definition.

Proposition 3.4 *Let S be a left ideal of a torian algebra X. Then the dual of S is a complete left ideal of X.*

Proposition 3.5 *Let X be a torian algebra. Let S be a left ideal of X. If T is a complete left ideal of X such that $T \subseteq S$, then $T \subseteq S^*$.*

Proof Let $x \in T$. Then $0 * x \in T$. Since $T \subseteq S$, then $x, 0 * x = [[0 * (x * 0)] * 0] \in S$. Therefore, $x \in S^*$ as required. The proof is complete. □

By Propositions 3.4 and 3.5, we have the following Theorem.

Theorem 3.4 *Let S be a left ideal of a torian algebra X. Then the dual of S is complete in X. Moreover, S^* is the largest complete left ideal of X that contains S.*

Definition 3.4 *Let X be a torian algebra. The set $x_\lambda = \{x \in X : (x * y) * z = 0;\ y, z \in X\}$ is called a left nucleus of X.*

Proposition 3.6 *Let S be a left ideal of a torian algebra X. Then S is the union of left nuclei of X.*

Proof Let $x \in X$. Notice that $(x * 0) * x = 0$. So, x belongs to a left nucleus of X. Now, let x be in the union of left nuclei of X. There exist $y, z \in S$ such that $(x * y) * z = 0 \in S$. It follows that $x \in S$ as required. The proof is complete. □

The following proposition is straightforward.

Proposition 3.7 *Let S be a non-empty subset of a torian algebra X with $0 \in S$ such that S is the union of left nuclei of X. Then S is a left ideal of X.*

By Propositions 3.6 and 3.7, we have the following Theorem.

Theorem 3.5 *Let S be a non-empty subset of a torian algebra X with $0 \in S$. Then S is a left ideal of X if and only if S is the union of left nuclei of X.*

Definition 3.5 *Let X be a torian algebra. An element $a \in X$ is said to be palindromic if there exists an element $x \in X$ such that $a * x = a$. The element x is then said to be palindromic to a.*

The collection of all elements in X that are palindromic to a is denoted by a^*.

Proposition 3.8 *Let a be a fixed element of a torian algebra X. Then $0 * x = 0$ for all $x \in a^*$.*

Proof Clearly, $0 \in a^*$. So, a^* is not empty. Now, let $x \in a^*$. Notice that $0*x = (a*a)*x = (a * x) * a = 0$ as required. □

Theorem 3.6 *Let a be a fixed element of a torian algebra X with the partial ordering $\sim$. Then a^* is a complete left ideal of X.*

Proof Let $x, y \in X$ such that $x, [[y * (x * y)] * y] \in a^*$. Notice that

$$\begin{aligned} (a * y) * a &= (a * a) * y \\ &= 0 * y = (0 * y) * 0 \\ &= (0 * y) * (0 * x) = 0 * (y * x) \\ &= 0 * [[y * (x * y)] * y] = 0. \end{aligned}$$

So, $(a * y) \sim a$.

Notice also that

$$\begin{aligned} a &= a * [[y * (x * y)] * y] = a * (y * x) \\ &= (a * x) * (y * x) \sim (a * y) \quad \text{(by Lemma 2.4)}. \end{aligned}$$

So, $y \in a^*$. The completeness of a^* follows from Proposition 3.2. □

Proposition 3.9 *Let X be a torian algebra. Let S and T be left ideals of X. If $[[x*(y*x)]*x = x$ for all $x \in S, y \in T$, then $S \cap T = \{0\}$.*

Proof Let $x \in S \cap T$. Then $x \in S, x \in T$. Now, notice that

$$x = [x * (x * x)] * x = x * x = 0$$

as required. □

Theorem 3.7 *Let X be a torian algebra equipped with a congruence $\sim^*$. Then $\overline{0} = \{x \in X : x \sim^* 0\}$ is a complete left ideal of X.*

Proof Clearly, $0 \in \overline{0}$. Now, let $x, y \in X$ such that $x, [[y*(x*y)]*y] \in \overline{0}$. Then, $x \sim^* 0$ and $[[y * (x * y)] * y] = y * x \sim^* 0$. Also, $y \sim^* y$. We therefore have that $y * x \sim^* y$. So $y \sim^* y * x$. Hence, $y \sim^* 0$; giving us that $y \in \overline{0}$ as required.

Now, let $x, y \in \overline{0}$. We show that $[[x * (y * x)] * x] \in \overline{0}$. Notice that $x \sim^* 0$ and $y \sim^* 0$. So, $x * y \sim^* 0$; which gives $[[x * (y * x)] * x] = x * y \sim^* 0$ as required. The proof is complete. □

Theorem 3.8 *Let S be a left ideal of a torian algebra X. Let $\sim^1$ be a relation on X defined by X by $x \sim^1 y \Leftrightarrow [[x * (y * x)] * x] \in S$ and $[[y * (x * y)] * y] \in S$ for all $x, y \in X$. Then $\sim^1$ is a congruence on X.*

Proof We first show that $\sim^1$ is an equivalence relation. Clearly, $[[x*(x*x)]*x] = 0 \in S$. so, $\sim^1$ is reflexive. Let $x, y \in X$ such that $x \sim^1 y$. Then $[[x*(y*x)]*x] \in S$ and $[[y*(x*y)]*y] \in S$; which implies that $y \sim^1 x$. So, $\sim^1$ is symmetric. Let $x, y, z \in X$ such that $x \sim^1 y$ and $y \sim^1 z$. Then, $[[x * (y * x)] * x], [[y * (x * y)] * y], [[y * (z * y)] * y], [[z * (y * z)] * z] \in S$. Now, since X is torian, we have

$$[[[x * (z * x)] * x] * [[x * (y * x)] * x]] * [[y * (z * y)] * y]$$
$$= [(x * z) * (x * y)] * (y * z) = 0 \in S.$$

So, $[[x * (z * x)] * x] \in S$ by virtue of S being a left ideal.

Also,

$$[[[z * (x * z)] * z] * [[z * (y * z)] * z]] * [[y * (x * y)] * y]$$
$$= [(z * x) * (z * y)] * (y * x) = 0 \in S.$$

Hence, $x \sim^1 z$. So, $\sim^1$ is transitive.

Now let $x, y, u, v \in X$ such that $x \sim^1 y$ and $u \sim^1 v$. Then,

$$[[x * (y * x)] * x], [[y * (x * y)] * y], [[u * (v * u)] * u], [[v * (u * v)] * v] \in S.$$

Notice that by Lemma 2.5, we have

$$[[[x * (u * x)] * x] * [[y * (u * y)] * y]] * [[x * (y * x)] * x]$$
$$= [(x * u) * (y * u)] * (x * y) = 0 \in S$$

and

$$[[[y * (u * y)] * y] * [[x * (u * x)] * x]] * [[y * (x * y)] * y]$$
$$= [(y * u) * (x * u)] * (y * x) = 0 \in S.$$

So, $[(x * u) * (y * u)] \in S$ and $[(y * u) * (x * u)] \in S$. Hence, $(x * u) \sim^1 (y * u)$.

Similar argument gives $(y * u) \sim^1 (y * v)$. Since, $(x * u) \sim^1 (y * u)$ and $(y * u) \sim^1 (y * v)$, then $(x * u) \sim^1 (y * v)$ as required. The proof is complete. □

Corollary 3.3 *Let S be a left ideal of a torian algebra X. Let $\sim^2$ be a relation on X defined by $x \sim^2 y$ if and only if S is complete in X and $[[y * (x * y)] * y] \in S$ for all $x, y \in X$. Then $\sim^2$ is a congruence on X.*

References

[1] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, 66(2007),113–116.

[2] S.S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, 68(2008), 351–357.

[3] S.S. Ahn and K. S. So, On generalized upper sets in BE-algebras, *Bull. Korean Math. Soc.*, 46(2009), 281–287.

[4] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185pp.

[5] J. Dene and A. D. Keedwell, *Latin Squares and Their Applications*, the English University press Ltd, 1974, 549pp.

[6] E. Ilojide, On obic algebras, *International J. Math. Combin.*, 4(2019), 80–88.

[7] E. Ilojide, A note on Torian algebras, *International J. Math. Combin.*, 2(2020), 80–87.

Math.Combin.Book Ser. Vol.2(2020), 109-117

Triangular Difference Mean Graphs

P. Jeyanthi[1], M. Selvi[2] and D. Ramya[3]

1.Research Centre, Department of Mathematics, Govindammal Aditanar College for Women
Tiruchendur-628 215, Tamil Nadu, India

2.Department of Mathematics, Dr. Sivanthi Aditanar College of Engineering
Tiruchendur-628 215, Tamil Nadu, India

3.Department of Mathematics, Government Arts College (Autonomous)
Salem-7, Tamil Nadu, India

E-mail: jeyajeyanthi@rediffmail.com, selvm80@yahoo.in, aymar_ padma@yahoo.co.in

Abstract: In this paper, we define a new labeling namely triangular difference mean labeling and investigate triangular difference mean behaviours of some standard graphs. A triangular difference mean labeling of a graph $G = (p, q)$ is an injection $f : V \longrightarrow Z^+$, where Z^+ is a set of positive integers such that for each edge $e = uv$, the edge labels are defined as

$$f^*(e) = \left\lceil \frac{|f(u) - f(v)|}{2} \right\rceil$$

such that the values of the edges are the first q triangular numbers. A graph that admits a triangular difference mean labeling is called a triangular difference mean graph.

Key Words: Mean labeling, triangular difference mean labeling, Smarandachely k-triangular labeling, triangular difference mean graph.

AMS(2010): 05C78.

§1. Introduction

By a graph, we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Terms and notations not defined here are used in the sense of Harary [2] and for number theory we follow Burton[1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of graph labeling and an excellent survey on graph labeling can be found in [3]. The notion of triangular mean labeling was due to Seenivasan et al. [7]. Let $G = (V, E)$ be a graph with p vertices and q edges. Consider an injection $f : V(G) \longrightarrow \{0, 1, 2, \cdots, T_q\}$,where T_q is the q^{th} triangular number. Define $f^* : E(G) \longrightarrow \{1, 3, \cdots, T_q\}$ such that $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ for all edges $e = uv$. If $f^*(E(G))$ is a sequence of consecutive triangular numbers $T_1, T_2, \cdots, T_q$, then the function f is said to be triangular mean.Generally, If there are only k consecutive triangular numbers $T_i, T_{i+1}, \cdots, T_{i+k-1}$ with $k \leq q$ in $f^*(E(G))$, such a f is called a Smarandachely k-triangular labeling. A graph that admits a triangular mean labeling or Smarandachely k-triangular labeling is called a triangular mean graph or a Smarandachely k-triangular mean

[1]Received November 21, 2019, Accepted June 13, 2020.

graph.

Murugan et al.[4] introduced skolem difference mean labeling and some standard results on skolem difference mean labeling were proved in [5] and [6]. A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $\{1, 2, 3, \cdots, p + q\}$ in such a way that for each edge $e = uv$, let $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the resulting labels of the edges are distinct and are $1, 2, 3, \cdots, q$. A graph that admits a skolem difference mean labeling is called a skolem difference mean graph.

Motivated by the concepts in [7] and [4], we define a new labeling namely triangular difference mean labeling. A triangular difference mean labeling of a graph $G = (p, q)$ is an injection $f : V \longrightarrow Z^+$, where Z^+ is a set of positive integers such that for each edge $e = uv$, the edge labels are defined as $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ such that the values of the edges are the first q triangular numbers. A graph that admits a triangular difference mean labeling is called a triangular difference mean graph. We use the following definitions in the subsequent sequel.

Definition 1.1 *A vertex of degree one is called a pendant vertex and a pendant edge is an edge incident with a pendant vertex. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then join the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.*

Definition 1.2 *The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.*

Definition 1.3 *The graph $C_n@P_m$ is obtained by identifying one pendant vertex of the path P_m to a vertex of the cycle C_n.*

Definition 1.4 *A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If the n^{th} triangular number is denoted by T_n, then $T_n = \frac{1}{2}n(n + 1)$.*

§2. Triangular Difference Mean Graphs

In this section, we establish that path $P_n(n \geq 1)$, $K_{1,n}(n \geq 1)$, $P_n \odot K_1(n \geq 2)$, $B_{m,n}(m \geq 1,\ n \geq 1)$, $T(n, m)$, $S(\underbrace{n, n, \cdots, n}_{\text{m times}})$, $C_n(n > 3)$ and $C_n@P_m(n \geq 4, m \geq 2)$ admit triangular difference mean labeling . Further, we prove that C_3 is not a triangular difference mean graph.

Theorem 2.1 *Any path $P_n(n \geq 1)$ is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Then $E(P_n) = \{e_i = v_iv_{i+1} : 1 \leq i \leq n - 1\}$. Define $f : V(P_n) \longrightarrow Z^+$ as follows:

$f(v_1) = 1$ and $f(v_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 1$ for $2 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(e_i) = T_i$ for $1 \leq i \leq n-1$. Hence P_n is a triangular difference mean graph. □

The triangular difference mean labeling of P_5 is given in Figure 1.

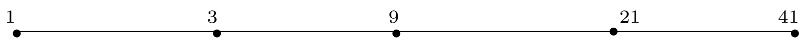


Figure 1

Theorem 2.2 *The star graph $K_{1,n} (n \geq 1)$ admits triangular difference mean labeling.*

Proof Let v be the apex vertex and $v_1, v_2, \cdots, v_n$ be the pendant vertices of the star $K_{1,n}$. Then $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Define $f : V(K_{1,n}) \longrightarrow Z^+$ as follows:

$f(v) = 1, f(v_i) = 2T_i + 1$ for $1 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(vv_i) = T_i$ for $1 \leq i \leq n$.

Then the induced edge labels are the triangular numbers $T_1, T_2, \cdots, T_n$. Hence $K_{1,n}$is a triangular difference mean graph. □

The triangular difference mean labeling of $K_{1,8}$ is shown in Figure 2.

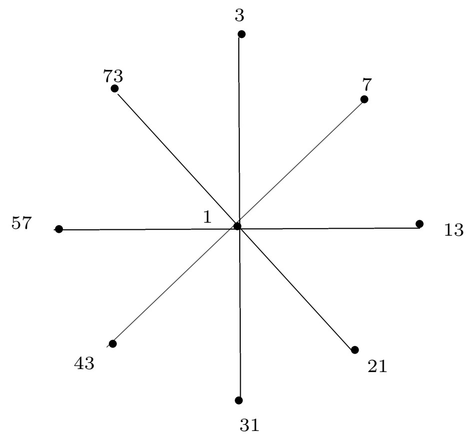


Figure 2

Theorem 2.3 *The comb graph $P_n \odot K_1 (n \geq 2)$ admits triangular difference mean labeling.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n and $u_1, u_2, \cdots, u_n$ be the pendant vertices adjacent to $v_1, v_2, \cdots, v_n$ respectively. Then $E(P_n \odot K_1) = \{e_i = v_i v_{i+1}, e'_j = u_j v_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Define $f : V(P_n \odot K_1) \longrightarrow Z^+$ as follows:

$f(v_1) = 1, f(v_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 1$ for $2 \leq i \leq n, f(u_1) = 2T_n$;
$f(u_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 2T_{n+i-1} + 1$ for $2 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(e_i) = T_i$ for $1 \le i \le n-1$, $f^*(e'_j) = T_{n+j-1}$ for $1 \le j \le n$.

Then the edge labels are the triangular numbers: $T_1, T_2, \cdots, T_{2n-1}$. Hence $P_n \odot K_1$ is a triangular difference mean graph. □

The triangular difference mean labeling of $P_5 \odot K_1$ is shown in Figure 3.

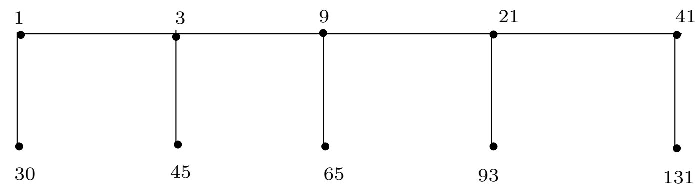


Figure 3

Theorem 2.4 *The bistar $B_{m,n}(m \ge 1,\ n \ge 1)$ is a triangular difference mean graph.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m,\ 1 \le j \le n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m,\ 1 \le j \le n\}$. Define $f : V(B_{m,n}) \longrightarrow Z^+$ as follows:

$f(u) = 1, f(v) = 3, f(u_i) = 2T_{i+1} + 1$ for $1 \le i \le m$;
$f(v_j) = 2T_{m+j+1} + 3$ for $1 \le j \le n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(uv) = T_1,\ f^*(uu_i) = T_{i+1}$ for $1 \le i \le m$;
$f^*(vv_j) = T_{m+j+1}$ for $1 \le j \le n$.

The induced edge labels are the first $m+n+1$ triangular numbers and hence $B_{m,n}$ is a triangular difference mean graph. □

The triangular difference mean labeling of $B_{4,5}$ is shown in Figure 4.

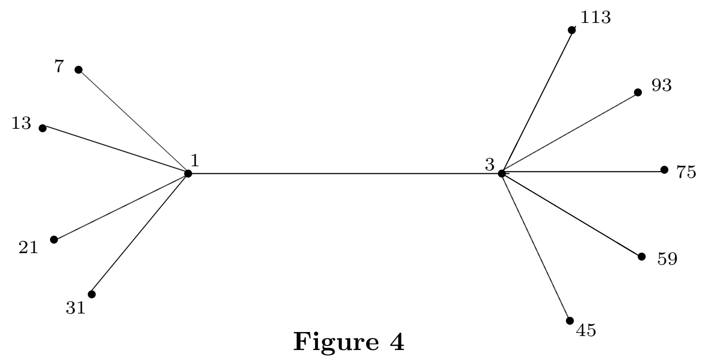


Figure 4

Theorem 2.5 *A graph obtained by joining the roots of different stars to a new vertex, is a triangular difference mean graph.*

Proof Let $K_{1,n_1}, K_{1,n_2}, \cdots, K_{1,n_k}$ be k stars. Let G be a graph obtained by joining the central vertices of the stars to a new vertex u.

Assign 1 to u; $2T_1+1, 2T_2+1, \cdots, 2T_k+1$ to the central vertices of the stars; $2T_{k+1}+2T_1+1, 2T_{k+2}+2T_1+1, \cdots, 2T_{k+n_1}+2T_1+1$ to the pendant vertices of the first star; $2T_{k+n_1+1}+2T_2+1, 2T_{k+n_1+2}+2T_2+1, \cdots, 2T_{k+n_1+n_2}+2T_2+1$ to the pendant vertices of the second star and so on, finally assign the numbers $2T_{k+n_1+n_2+\cdots+n_{k-1}+1}+2T_k+1, 2T_{k+n_1+n_2+\cdots+n_{k-1}+2}+2T_k+1, \cdots, 2T_{k+n_1+n_2+\cdots,+n_{k-1}+n_k}+2T_k+1$ to the pendant vertices of the last star. Then, the edge labels are the triangular numbers $T_1, T_2, \cdots, T_{k+n_1+n_2+\cdots+n_{k-1}+n_k}$ and also the vertex labels are all different. □

The triangular difference mean labeling of the tree given in Theorem 2.5 with $k=3, n_1=4, n_2=5$ and $n_3=4$ is shown in Figure 5.

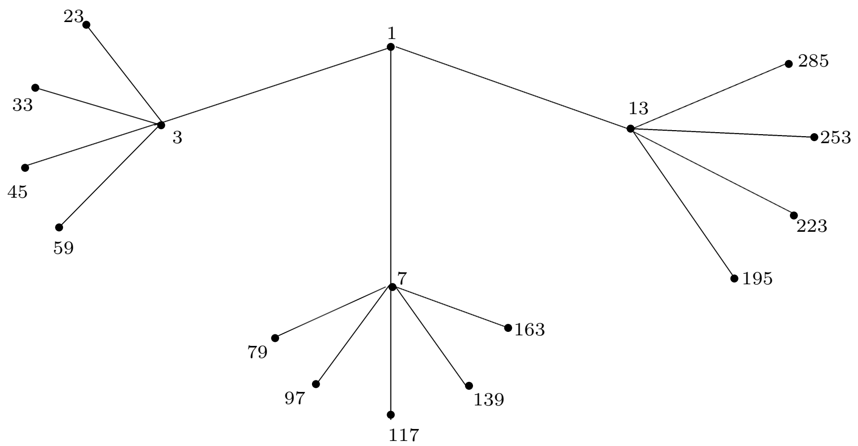


Figure 5

Theorem 2.6 *A tree $T(n,m)$, obtained by identifying a central vertex of a star with a pendant vertex of a path , is a triangular difference mean graph.*

Proof Let $v_0, v_1, v_2, \cdots, v_n$ be the vertices of the path $P_n$having path length $n(n \geq 1)$ and $u, u_1, u_2, \cdots, u_m$ be the vertices of the star $K_{1,m}$. Let $T(n,m)$ be a tree obtained by identifying v_0 with u.

Define $f : V(T(n,m)) \longrightarrow Z^+$ as follows:

$f(v_0) = 1$, $f(u_i) = 2T_i + 1$ for $1 \leq i \leq m$,

$f(v_j) = 2(T_{m+1} + T_{m+2} + \ldots + T_{m+j}) + 1$ for $1 \leq j \leq n$.

For a vertex label f, the induced edge label f^* is as follows:

$f^*(v_0u_i) = T_i$ for $1 \leq i \leq m$;

$f^*(v_{j-1}v_j) = T_{m+j}$ for $1 \leq j \leq n$.

Then the induced edge labels are the first $m+n$ triangular numbers. Hence the tree $T(n,m)$ admits a triangular difference mean labeling. □

The triangular difference mean labeling of a tree $T(3,7)$ is shown in Figure 6.

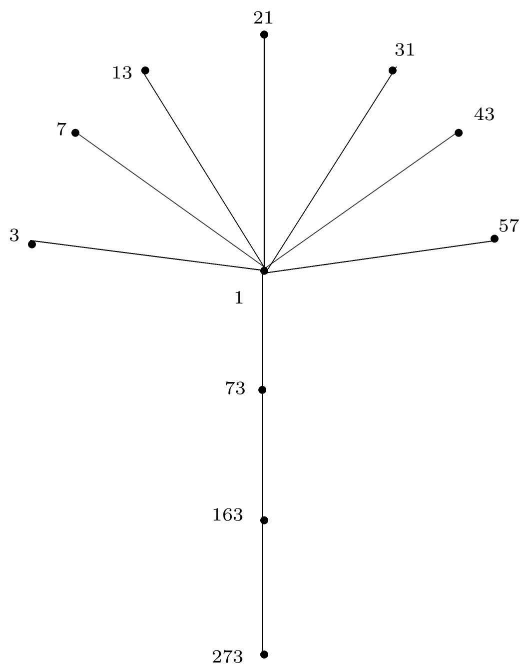


Figure 6

Theorem 2.7 *The caterpillar* $S(\underbrace{n, n, \cdots, n}_{mtimes})$ *is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_m$ be the vertices of the path P_m and $v_j^i(1 \leq i \leq n,\ 1 \leq j \leq m)$ be the pendant vertices incident with $v_j(1 \leq j \leq m)$.
Then $V(S(\underbrace{n, n, \cdots, n}_{\text{m times}})) = \{v_j, v_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(S(\underbrace{n, n, \cdots, n}_{\text{m times}})) = \{v_t v_{t+1},\ v_j v_j^i : 1 \leq t \leq m-1,\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$.

Define $f : V(S(\underbrace{n, n, \cdots, n}_{\text{m times}})) \longrightarrow Z^+$ as follows:

$f(v_1) = 1,\ f(v_j) = 2(T_1 + T_2 + \cdots + T_{j-1}) + 1$ for $2 \leq j \leq m$;
$f(v_j^i) = f(v_j) + 2T_{m+(j-1)n+i-1}$ for $1 \leq j \leq m$ and $1 \leq i \leq n$.

For each vertex label f, the induced edge label f^* is as follows:

$f^*(v_j v_{j+1}) = T_j$ for $1 \leq j \leq m-1$;
$f^*(v_j v_j^i) = T_{m+(j-1)n+i-1}$ for $1 \leq j \leq m$ and $1 \leq i \leq n$.

Then the edge labels are the triangular numbers $T_1, T_2, \cdots, T_{m-1}, T_m, \cdots, T_{m+n-1}$ and also the vertex labels are different. Hence $S((\underbrace{n, n, \cdots, n}_{\text{m times}}))$ is a triangular difference mean graph. □

Theorem 2.8 *Every cycle* $C_n(n > 3)$ *is a triangular difference mean graph.*

Proof We prove this theorem in two cases.

Case 1. $n = 4m + 1$.

Let $S = \left\lceil \frac{1}{2}\sum\limits_{i=1}^{n} T_i \right\rceil$. Select some of the $T_i^{'s}$ namely $T_{l_1}, T_{l_2}, \cdots, T_{l_k}$ from $T_1, T_2, \cdots, T_n$ such that $\sum\limits_{i=1}^{k} T_{l_i} = S$, where $k < n$ and assume $T_{l_1} > T_{l_2} > \cdots > T_{l_k}$. Then the remaining $T_i^{'s}$ namely, $T_{l_{k+1}}, T_{l_{k+2}}, \cdots, T_{l_n}$ are such that $T_{l_{k+1}} > T_{l_{k+2}} > \cdots, > T_{l_n}$ and $\sum\limits_{i=k+1}^{n} T_{l_i} = S - 1$. Let $v_1, v_2, \cdots, v_{k-1}, v_k, v_{k+1}, \cdots, v_n$ be the vertices of C_n. Label the first $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ as follows:

$f(v_1) = 1,\ f(v_2) = 2T_{l_1},\ f(v_3) = 2T_{l_1} + 2T_{l_2} - 1;$

$f(v_4) = 2T_{l_1} + 2T_{l_2} + 2T_{l_3} - 1, \cdots, f(v_{k+1}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 1$ and then,

$f(v_{k+2}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 1;$

$f(v_{k+3}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - 1, \cdots;$

$f(v_n) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - \cdots - 2T_{l_{n-1}} - 1.$

Hence, the edge labels are the triangular numbers $\{T_{l_1}, T_{l_2}, \cdots, T_{l_{k-1}}, T_{l_k}, T_{l_{k+1}}, \cdots, T_{l_n}\} = \{T_1, T_2, \cdots, T_n\}$ and also the vertex labels are all different.

Case 2. $n \neq 4m + 1, m \geqslant 1$.

Let $S = \left\lceil \frac{1}{2}\sum\limits_{i=1}^{n} T_i \right\rceil$. Select some of the T'^{s}_i namely $T_{l_1}, T_{l_2}, \cdots, T_{l_k}$ from $T_1, T_2, \cdots, T_n$ such that $\sum\limits_{i=1}^{k} T_{l_i} = S$, where $k < n$ and assume $T_{l_1} > T_{l_2} > \cdots > T_{l_k}$. Then the remaining T'^{s}_i namely, $T_{l_{k+1}}, T_{l_{k+2}}, \cdots, T_{l_n}$ are such that $T_{l_{k+1}} > T_{l_{k+2}} > \cdots > T_{l_n}$ and $\sum\limits_{i=k+1}^{n} T_{l_i} = S$. Let $v_1, v_2, \cdots, v_{k-1}, v_k, v_{k+1}, \cdots, v_n$ be the vertices of C_n. We label the vertices $v_1, v_2, \cdots, v_n$ as follows:

$f(v_1) = 1,\ f(v_2) = 2T_{l_1} + 1,\ f(v_3) = 2T_{l_1} + 2T_{l_2} + 1;$

$f(v_4) = 2T_{l_1} + 2T_{l_2} + 2T_{l_3} + 1, \cdots;$

$f(v_{k+1}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} + 1;$

$f(v_{k+2}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} + 1;$

$f(v_{k+3}) = 2T_{l_1} + 2T_{l_2} + \cdots, T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} + 1, \cdots;$

$f(v_n) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - \cdots - 2T_{l_{n-1}} + 1.$

Thus, the edge labels are the triangular numbers $\{T_{l_1}, T_{l_2}, \cdots, T_{l_{k-1}}, T_{l_k}, T_{l_{k+1}}, \cdots, T_{l_n}\}$ and also the vertex labels are all different. □

The triangular difference mean labeling of C_6 is shown in Figure 8.

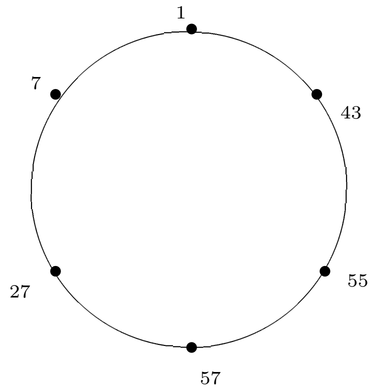


Figure 8

Theorem 2.9 *The graph $C_n@P_m (n \geq 4, m \geq 2)$ is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and $u_1, u_2, \cdots, u_m$ be the vertices of the path P_m. The graph $C_n@P_m$ is obtained by identifying the vertexu_1 with the vertex v_1. We label the vertices of C_n as in Theorem 2.9 and assign the number $2T_{n+1}+2T_{n+2}+\cdots+2T_{n+j-1}+1$ to vertex u_j of the path P_m for $2 \leq j \leq m$. Then the induced edge labels are the first $m+n-1$ triangular numbers. Hence, $C_n@P_m$ is a triangular difference mean graph. □

The triangular difference mean labeling of $C_4@P_3$ is shown in Figure 9.

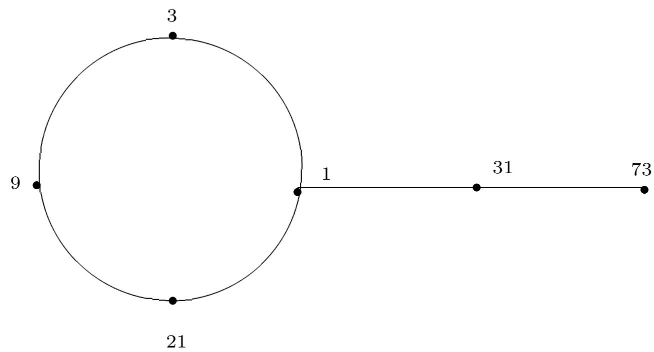


Figure 9

Theorem 2.10 *The cycle C_3 is not a triangular difference mean graph.*

Proof Suppose C_3 is a triangular difference mean graph with triangular difference mean labeling f. Let the vertices of C_3 be u, v, w. Let $f(u) = x$. Then to get 1 as an edge label we must have $f(v) \in \{x+1, x+2, x-1, x-2\}$. To get $T_2, f(w) \in \{x+5, x+6, x-5, x-6\}$ or $f(w) \in \{x-6, x-7, x+4, x+5\}$. Then we get either $\{1, 3, 2\}$ or $\{1, 3, 4\}$ as the set of induced edge labels. Therefore, $T_3 = 6$ can not be an edge label of C_3. Hence C_3 is not a triangular difference mean graph. □

References

[1] David M. Burton, *Elementary Number Theory*, Second Edition, Wm. C. Brown Company Publishers, 1980.

[2] F. Harary, *Graph Theory*, Addison Wesley, Massachusetts, 1972.

[3] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, (2019), # DS6.

[4] K. Murugan, A. Subramanian, Skolem difference mean labeling of H- graphs, *International Journal of Mathematics and Soft Computing*, Vol.1, No.1(2011), 115 - 129.

[5] D.Ramya, M. Selvi and R. Kalaiyarasi, On skolem difference mean labeling of graphs, *International Journal of Mathematical Archive*, Vol.4, 12(2013), 73 - 79.

[6] D. Ramya and M. Selvi, On skolem difference mean labeling of some trees, *International Journal of Mathematics and Soft Computing*, Vol.4, No.2. (2014), 11 - 18.

[7] M. Seenivasan, A. Lourdusamy and M. Ravi Ramasubramanian, Triangular mean labeling of graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, Vol.10, 6(2007), 815 - 822.

Math.Combin.Book Ser. Vol.2(2020), 118-120

Uni-Distance Domination of Square of Paths

Kishori P. Narayankar, Denzil Jason Saldanha

(Department of Mathematics, Mangalore University, Mangalagangothri, Mangalore-574199, India)

John Sherra

(St Aloysius College (Autonomous), Mangalore-575003, India)

E-mail: kishori_pn@yahoo.co.in, denzil53@gmail.com, johnsherra@gmail.com

Abstract: A dominating set D of G which is also a resolving set of G is called a *metro dominating set.* A metro dominating set D of a graph $G(V,E)$ is a *uni-distance dominating set* (in short an *UDD-set*) if $\mid N(v) \cap D \mid= 1$ for each vertex $v \in V - D$ and the minimum of cardinalities of an *UDD-set of G* is the *uni-distance domination number of G* denoted by $\gamma_{\mu\beta}(G)$. In this paper we determine unique distance domination number of P_n^2 graphs.

Key Words: Domination, metric dimension, metro domination, uni-distance domination, Smarandachely distance k dominating set.

AMS(2010): 05C69.

§1. Introduction

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u,v)$. For a vertex v of a graph, $N(v)$ denote the set of all vertices adjacent to v and is called open neighborhood of v. Similarly, the closed neighborhood of v is defined as $N[v] = N(v) \cup \{v\}$.

Let $G(V,E)$ be a graph. For each ordered subset $S = \{v_1, v_2, v_3, \cdots, v_k\}$ of V, each vertex $v \in V$ can be associated with a vector of distances denoted by $\Gamma(v/S) = (d(v_1,v), d(v_2,v), \cdots, d(v_k,v))$. The set S is said to be a *resolving set* of G if $\Gamma(v/S) \neq \Gamma(u/S)$ for every $u, v \in V - S$. A resolving set of minimum cardinality is a *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The k-tuple, $\Gamma(v/S)$ associated to the vertex $v \in V$ with respect to a metric basis S, is referred as a *code generated by S* for that vertex v. If $\Gamma(v/S) = (c_1, c_2, \cdots, c_k)$, then $c_1, c_2, c_3, \cdots, c_k$ are called components of the code of v generated by S and in particular c_i, $1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A dominating set D of a graph $G(V,E)$ is the subset of V having the property that for each vertex $v \in V - D$, there exists a vertex $u \in D$ such that uv is in E. A dominating set D of G which is also a resolving set of G is called a *metro dominating set.*

A metro dominating set D of a graph $G(V,E)$ is a *uni-distance dominating set* (in short an *UDD-set*) if $\mid N(v) \cap D \mid= 1$ for each vertex $v \in V - D$. Generally, if $\mid N(v) \cap D \mid= k$

[1]Received November 14, 2019, Accepted June 14, 2020.

for each vertex $v \in V - D$, $k \geq 1$, such a metro dominating set D is called a *Smarandachely distance k dominating set* (Smarandachely k DD-sets of G) and the minimum of cardinalities of the Smarandachely DD-sets of G is the number of Smarandachely k UDD-sets of G, denoted by $\gamma^k_{S\mu\beta}(G)$. Particularly, if $k = 1$, i.e., the *uni-distance domination number of G* denoted by $\gamma_{\mu\beta}(G)$. For an integer $n \geq 3$, we determine the uni-distance domination number $\gamma_{\mu\beta}\left(P_n^2\right)$ of P_n^2 in this paper.

§2. Main Results

Consider P_n, $n \geq 3$. Join v_i to v_{i+2} for $1 \leq i \leq n-2$. The resulting graph is denoted by P_n^2.

Lemma 2.1 *For any positive integer n, $\gamma_{\mu\beta}(P_n^2) \geq \left\lceil \frac{n}{5} \right\rceil$.*

Proof A vertex v_i dominates five vertices $v_i, v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}$. Therefore, if D is minimal dominating set then $| D | \geq \frac{n}{5}$. Hence we have $\gamma(P_n^2) \geq \left\lceil \frac{n}{5} \right\rceil$.

End vertex v_1 of P_n^2 can dominate only 3 vertices v_1, v_2 and v_3. As we have to minimize $| D |$, we include v_3 in D, which dominates v_1, v_2, v_3, v_4 and v_5. □

Lemma 2.2 *If $n = 5k$, $k \in \mathbb{N}$ then $\gamma_{\mu\beta}\left(P_n^2\right) = k = \left\lceil \frac{n}{5} \right\rceil$.*

Proof When $k = 1$, v_3 dominates all vertices of P_5^2. Hence $\gamma(P_5^2) = 1$.

Let $n = 5k$. Then $D = \{v_3, v_8, v_{13}, \cdots, v_{5k-2}\}$ and $| D |$=k. When n=5(k+1), take $D' = D \cup \{v_{5k+3}\}$. Observe that $| D' | = k + 1$ and D' dominates all vertices. From Lemma 2.1, we have

$$\gamma(P^2_{5(k+1)}) \geq \left\lceil \frac{5(k+1)}{5} \right\rceil = k + 1$$

and $| D' | = k + 1$. Therefore we conclude that $\gamma(P_5(k+1))^2 = k + 1$. Thus by induction

$$\gamma\left(P_n^2\right) = k = \left\lceil \frac{n}{5} \right\rceil.$$

In P_n^2, consider any v_j and v_{j+5} in D. Vertex v_j dominates $v_{j-2}, v_{j-1}, v_{j+1}, v_{j+2}$. Vertex v_{j+5} dominates $v_{j+3}, v_{j+4}, v_{j+6}$ and v_{j+7}. These vertices are uniquely dominated by v_j and v_{j+5}. The vertices v_1 and v_2 are uniquely dominated by v_3. The vertex v_{5k} and v_{5k-1} are uniquely dominated by v_{5k-2}.

In P_n^2, we observe that

$$d(v_i, v_j) = d(v_i, v_{j-1}) = \frac{j-i}{2}$$

where i and j are both even and $j \geqslant i$. When i is odd and j is even

$$d(v_i, v_{j+1}) = d(v_i, v_j) = \frac{j-i+1}{2}.$$

We take $D = \{v_3, v_8, v_{13}, \cdots\}$. Note that $d(v_3, v_{j+1}) = d(v_3, v_j)$, $j \geqslant 3$ and j even. Also $d(v_3, v_2) = d(v_3, v_1)$. Now when $j \geqslant 8$ and j is even,

$$d(v_8, v_j) = \frac{j-8}{2} \quad \text{and} \quad d(v_8, v_{j+1}) = \frac{(j+2)-8}{2}.$$

Hence $d(v_8, v_j) \neq d(v_8, v_{j+1})$. Therefore $\{v_3, v_8\}$ resolve all vertices v_j, $j \geqslant 8$, Now $d(v_3, v_1) = 1$ but $d(v_8, v_1) = 4$ and $d(v_3, v_2) = 1$ but $d(v_8, v_2) = 3$. Hence$\{v_3, v_8\}$ resolve v_1 and v_2.

If $3 \leqslant j \leqslant 8$ then $\{v_3, v_8\}$ generate the same code (1,2) for v_4 and v_5. Also $\{v_3, v_8\}$ generate the same code (2,1) for v_6 and v_7. We have $d(v_{13}, v_4) = 5$ and $d(v_{13}, v_5) = 4$. Also $d(v_{13}, v_6) = 4$ and $d(v_{13}, v_7) = 3$. Hence $\{v_3, v_8, v_{13}\}$ resolves all vertices of P_n^2. Therefore to resolve all vertices of P_n^2 we take $n \geq 11$. We observe that $\{v_3, v_8, \cdots, v_{5k-2}\}$ uniquely dominates all vertices in $V - D$. Hence we have the conclusion. $\square$

If $n = 5k+1$, $n = 5k+2$, $n = 5k+3$, $D = \{v_1, v_6, v_{11}, \cdots, v_{5k-4}, v_{5k+1}\}$ is a UDD set. Therefore $\gamma_{\mu\beta}\left(P_n^2\right) = k+1$. If $n = 5k+4$, $D = \{v_2, v_7, v_{12}, \cdots, v_{5k-3}, v_{5k+2}\}$ is a UDD set and we have $\gamma_{\mu\beta}\left(P_n^2\right) = k+1 = \left\lceil \frac{n}{5} \right\rceil$. Thus we obtain $\gamma_{\mu\beta}\left(P_n^2\right) = \left\lceil \frac{n}{5} \right\rceil$ for $\forall n \geq 11$. If $n < 11$, then we observe that $\gamma_{\mu\beta}(P_n^2) = n$. Hence, we have

Theorem 2.3 *For an integer $n \geq 3$,*

$$\gamma_{\mu\beta}(P_n^2) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil, & \text{for } n \geq 11 \\ n, & \text{for } n < 11. \end{cases}$$

References

[1] E. J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs, *NetWorks*, 7: 247 - 261, (1977).

[2] Gary Chartrand, Linda Eroh, Mark A. Johnson and Ortrud R.Oellermann, Resolvability in graphs and the metric dimension of a graph,*Discrete Appl. Math.,* 105(1-3)(2000),99-113.

[3] Harary F,Melter R.A., On the metric dimension of a graph *Ars Combinatoria* ,2(1976),191-195 .

[4] S.Kuller, B.Raghavachari and A.Rosenfied, Land marks in graphs, *Disc. Appl. Math.,*70 (1996), 217-229.

[5] C. Poisson and P. Zhang, The metric dimension of unicyclic graphs, *J. Comb. Math Comb.Compu.,* 40(2002), 17-32.

[6] P. J. Slater, Domination and location in acyclic graphs, *Networks*, 17(1987), 55-64.

[7] P. J. Slater, Locating dominating sets, in Y. Alavi and A. Schwenk ed.,*Graph Theory, Combinatorics, and Applications,* Proc. Seventh Quad International Conference on the theory and applications of Graphs. John Wiley and Sons, Inc. (1995), 1073-1079.

[8] B. Sooryanarayana and John Sherra, Unique metro domination in graphs,*Adv Appl Discrete Math.,* Vol 14(2), (2014), 125-149.

[9] H.B.Walikar, Kishori P. Narayankar and Shailaja S. Shirakol, The number of minimum dominating sets in $P_n \times P_2$, *International J.Math. Combin.* Vol.3 (2010), 17-21.

[10] B. Sooryanarayana and John Sherra, Unique metro domination number of circulant graphs, *International J.Math. Combin.*, Vol.1(2019), 53-61.

Famous Words

There are no stable antimatter unless free antimatter such as those of positrons, free antiprotons, free antineutrons, free antihydrogens and antimatter stars, i.e., antiprotron star, antineutron star or their combination in universe. It is completely different from the normal matters world. There are no possibility for the birth of living antibeings, no antipeoples, and it is only a symmetrical mirror of elementary particles but with a different mechanism on composing antimatters. (Extracted from the paper: A new understanding on the asymmetry of matter-antimatter, *Progress in Physics*, Vol.15, 3(2019), 156-162)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in the **Mathematical Combinatorics (International Book Series)**. An effort is made to publish a paper duly recommended by a referee within a period of $3-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

June 2020

Contents

A Book Series on Mathematical Combinatorics

ISBN 978-1-59973-661-7