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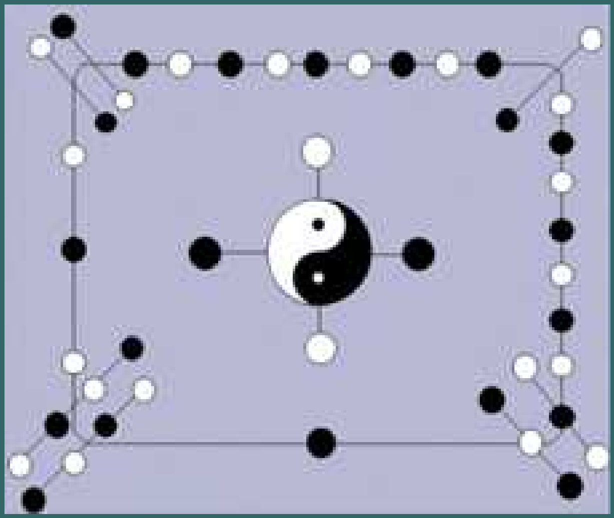

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Ricci Soliton and Conformal Ricci Soliton in Lorentzian β-Kenmotsu Manifold

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Abstract: In this paper we have studied quasi conformal curvature tensor, Ricci tensor, projective curvature tensor, pseudo projective curvature tensor in Lorentzian β-Kenmotsu manifold admitting Ricci soliton and conformal Ricci soliton.

Key Words: Trans-Sasakian manifold, β-Kenmotsu manifold, Lorentzian β-Kenmotsu manifold, Ricci soliton, conformal Ricci flow.

AMS(2010): 53C25, 35K65, 53C44, 53D10, 53D15.

§1. Introduction

Hamilton started the study of Ricci flow [12] in 1982 and proved its existence. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [11]classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman also did an excellent work on Ricci flow [15], [16].

The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling.Ramesh Sharma [18], M. M. Tripathi [19], Bejan, Crasmareanu [4]studied Ricci soliton in contact metric manifolds also. The Ricci soliton equation is given by

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady and expanding according as

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λ is negative, zero and positive respectively.

In 2005, A.E. Fischer [10] introduced the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation scalar curvature R is considered as constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. The conformal Ricci flow equation on M where M is considered as a smooth closed connected oriented n-manifold($n > 3$), is defined by the equation [10]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $r = -1$, where p is a scalar non-dynamical field(time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.4}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called a trans-Sasakian structure [14] if the product manifold belongs to the class W_4 where W_4 is a class of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [6]. A trans-Sasakian structure of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic [5], $\beta-$Kenmotsu [13], and $\alpha-$Sasakian [13], respectively.

§2. **Preliminaries**

A differentiable manifold of dimension n is called Lorentzian Kenmotsu manifold [2] if it admits a $(1,1)$ tensor field ϕ, a covarient vector field ξ, a 1-form η and Lorentzian metric g which satisfy on M respectively such that

$$\phi^2 X = X + \eta(X)\xi, g(X,\xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = -1, \eta(\phi X) = 0, \phi\xi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all $X, Y \in \chi(M)$.

If Lorentzian Kenmotsu manifold M satisfies

$$\nabla_X \xi = \beta[X - \eta(X)\xi], (\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Then the manifold M is called Lorentzian $\beta-$Kenmotsu manifold.

Furthermore, on an Lorentzian β-Kenmotsu manifold M the following relations hold [1], [17]:

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{2.6}$$

$$R(\xi, X)Y = \beta^2[\eta(Y)X - g(X,Y)\xi], \tag{2.7}$$

$$R(X,Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X], \tag{2.8}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), \tag{2.9}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.10}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{2.11}$$

where β is some constant, R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$ for all $X, Y \in \chi(M)$.

Now from definition of Lie derivative we have

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y) + g(\beta[X - \eta(X)\xi], Y) + g(X, \beta[Y - \eta(Y)\xi]) \\ &= 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y). \end{aligned} \tag{2.12}$$

Applying Ricci soliton equation (1.2) in (2.12) we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[-2\lambda g(X,Y)] - \frac{1}{2}[2\beta g(X,Y) - 2\beta\eta(X)\eta(Y)] \\ &= -\lambda g(X,Y) - \beta g(X,Y) + \beta\eta(X)\eta(Y) \\ &= \acute{A} g(X,Y) + \beta\eta(X)\eta(Y), \end{aligned} \tag{2.13}$$

where $\acute{A} = (-\lambda - \beta)$, which shows that the manifold is η-Einstein.

Also

$$QX = \acute{A}X + \beta\eta(X)\xi, \tag{2.14}$$

$$S(X,\xi) = (\acute{A} + \beta)\eta(X) = A\eta(X). \tag{2.15}$$

If we put $X = Y = e_i$ in (2.13) where $\{e_i\}$ is the orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$R(g) = \acute{A}n + \beta.$$

Proposition 2.1 *A Lorentzian β-Kenmotsu manifold admitting Ricci soliton is η-Einstein.*

Again applying conformal Ricci soliton (1.4) in (2.12) we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[2\lambda-(p+\frac{2}{n})]g(X,Y)-\frac{1}{2}[2\beta g(X,Y)-2\beta\eta(X)\eta(Y)] \\ &= \acute{B}g(X,Y)+\beta\eta(X)\eta(Y), \end{aligned} \tag{2.16}$$

where

$$\acute{B}=\frac{1}{2}[2\lambda-(p+\frac{2}{n})]-\beta, \tag{2.17}$$

which also shows that the manifold is η-Einstein.

Also

$$QX=\acute{B}X+\beta\eta(X)\xi, \tag{2.18}$$

$$S(X,\xi)=(\acute{B}+\beta)\eta(X)=B\eta(X). \tag{2.19}$$

If we put $X=Y=e_i$ in (2.16) where $\{e_i\}$ is the orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$r=\acute{B}n+\beta.$$

For conformal Ricci soliton $r(g)=-1$. So

$$-1=\acute{B}n+\beta$$

which gives $B=\frac{1}{n}(-\beta-1)$.

Comparing the values of B from (2.17) with the above equation we get

$$\lambda=\frac{1}{n}(\beta(n-1)-1)+\frac{1}{2}(p+\frac{2}{n})$$

Proposition 2.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton is η-Einstein and the value of the scalar*

$$\lambda=\frac{1}{n}(\beta(n-1)-1)+\frac{1}{2}(p+\frac{2}{n}).$$

§3. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).\tilde{C}=0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). Quasi conformal curvature tensor $\tilde{C}$ on M is defined by

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n}+2b][g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{3.1}$$

where r is scalar curvature.

Putting $Z = \xi$ in (3.1) we have

$$\begin{aligned}\tilde{C}(X,Y)\xi &= aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n} + 2b][g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.2}$$

Using (2.1), (2.8), (2.14), (2.15) in (3.2) we get

$$\tilde{C}(X,Y)\xi = [-a\beta^2 + Ab + \acute{A}b - [\frac{r}{2n+1}][\frac{a}{2n} + 2b]](\eta(Y)X - \eta(X)Y).$$

Let

$$D = -a\beta^2 + Ab + \acute{A}b - [\frac{r}{2n+1}][\frac{a}{2n} + 2b],$$

so we have

$$\tilde{C}(X,Y)\xi = D(\eta(Y)X - \eta(X)Y). \tag{3.3}$$

Taking inner product with Z in (3.3) we get

$$-\eta(\tilde{C}(X,Y)Z) = D[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.4}$$

Now we consider that the Lorentzian β-Kenmotsu manifold M which admits Ricci soliton is quasi conformally semi symmetric i.e. $R(\xi, X).\tilde{C} = 0$ holds in M, which implies

$$R(\xi,X)(\tilde{C}(Y,Z)W) - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W - \tilde{C}(Y,Z)R(\xi,X)W = 0, \tag{3.5}$$

for all vector fields X, Y, Z, W on M.

Using (2.7) in (3.5) and putting $W = \xi$ we get

$$\begin{aligned}&\eta(\tilde{C}(Y,Z)\xi)X - g(X,\tilde{C}(Y,Z)\xi)\xi - \eta(Y)\tilde{C}(X,Z)\xi + g(X,Y)\tilde{C}(\xi,Z)\xi \\ &-\eta(Z)\tilde{C}(Y,X)\xi + g(X,Z)\tilde{C}(Y,\xi)\xi - \eta(\xi)\tilde{C}(Y,Z)X + g(X,\xi)\tilde{C}(Y,Z)\xi = 0. \end{aligned} \tag{3.6}$$

Taking inner product with ξ in (3.6) and using (2.2), (3.3) we obtain

$$g(X,\tilde{C}(Y,Z)\xi) + \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.7}$$

Putting $Z = \xi$ in (3.7) and using (3.3) we get

$$-Dg(X,Y) - D\eta(X)\eta(Y) + \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.8}$$

Now from (3.1) we can write

$$\begin{aligned}\tilde{C}(Y,\xi)X &= aR(Y,\xi)X + b[S(\xi,X)Y - S(Y,X)\xi + g(\xi,X)QY - g(Y,X)Q\xi] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n} + 2b][g(\xi,X)Y - g(Y,X)\xi]. \end{aligned} \tag{3.9}$$

Taking inner product with ξ and using (2.2), (2.7), (2.9), (2.10) in (3.9) we get

$$\begin{aligned}\eta(\tilde{C}(Y,\xi)X) &= a\eta(\beta^2(g(X,Y)\xi-\eta(X)Y))+b[A\eta(X)\eta(Y)+S(X,Y)+\eta(X)(\acute{A}\eta(Y)\\ &\quad -\beta\eta(Y))-g(X,Y)(-\acute{A}+\beta)]-[\frac{r}{2n+1}][\frac{a}{2n}+2b][\eta(X)\eta(Y)+g(X,Y)].\end{aligned}$$

After a long simplification we have

$$\begin{aligned}\eta(\tilde{C}(Y,\xi)X) &= g(X,Y)[\acute{A}b-b\beta-a\beta^2-[\frac{r}{2n+1}][\frac{a}{2n}+2b]]\\ &\quad +\eta(X)\eta(Y)[2\acute{A}b-a\beta^2-[\frac{r}{2n+1}][\frac{a}{2n}+2b]]+bS(X,Y). \qquad (3.10)\end{aligned}$$

Putting (3.10) in (3.5) we get

$$\rho g(X,Y)+\sigma\eta(X)\eta(Y)=S(X,Y), \qquad (3.11)$$

where

$$\rho=\frac{1}{b}[D+b\beta+a\beta^2-\acute{A}b+[\frac{r}{2n+1}][\frac{a}{2n}+2b]]$$

and

$$\sigma=\frac{1}{b}[D+a\beta^2-2\acute{A}b+[\frac{r}{2n+1}][\frac{a}{2n}+2b]].$$

So from (3.11) we conclude that the manifold becomes η-Einstein manifold. Thus we can write the following theorem:

Theorem 3.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is quasi conformally semi symmetric i.e. $R(\xi,X).\tilde{C}=0$, then the manifold is η-Einstein manifold where $\tilde{C}$ is quasi conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then after a brief calculation we can also establish that the manifold becomes η-Einstein, only the values of constants ρ,σ will be changed which would not hamper our main result.

Hence we can state the following theorem:

Theorem 3.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is quasi conformally semi symmetric i.e. $R(\xi,X).\tilde{C}=0$, then the manifold is η-Einstein manifold where $\tilde{C}$ is quasi conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§4. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).S=0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). Now we consider that the tensor derivative of S by $R(\xi,X)$ is zero i.e. $R(\xi,X).S=0$. Then the

Lorentzian β-Kenmotsu manifold admitting Ricci soliton is Ricci semi symmetric which implies

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0. \tag{4.1}$$

Using (2.13) in (4.1) we get

$$\acute{A}g(R(\xi,X)Y,Z)+\beta\eta(R(\xi,X)Y)\eta(Z)+\acute{A}g(Y,R(\xi,X)Z)+\beta\eta(Y)\eta(R(\xi,X)Z)=0. \tag{4.2}$$

Using (2.7) in (4.2) we get

$$\acute{A}g(\beta^2[\eta(Y)X-g(X,Y)\xi],Z)+\acute{A}g(Y,\beta^2[\eta(Z)X-g(X,Z)\xi])+\beta\eta(\beta^2[\eta(Y)X-$$
$$g(X,Y)\xi])\eta(Z)+\beta\eta(Y)\eta(\beta^2[\eta(Z)X-g(X,Z)\xi])=0. \tag{4.3}$$

Using (2.2) in (4.3) we have

$$\acute{A}\beta^2\eta(Y)g(X,Z)-\acute{A}\beta^2\eta(Z)g(X,Y)+\acute{A}\beta^2\eta(Z)g(X,Y)-\acute{A}\beta^2\eta(Y)g(X,Z)$$
$$+\beta^3\eta(Y)\eta(X)\eta(Z)+\beta^3g(X,Y)\eta(Z)+\beta^3\eta(Y)\eta(X)\eta(Z)+\beta^3g(X,Z)\eta(Y)=0. \tag{4.4}$$

Putting $Z=\xi$ in (4.4) and using (2.2) we obtain

$$g(X,Y)=-\eta(X)\eta(Y).$$

Hence we can state the following theorem:

Theorem 4.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is Ricci semi symmetric i.e. $R(\xi,X).S=0$, then $g(X,Y)=-\eta(X)\eta(Y)$ where S is Ricci tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then by similar calculation we can obtain the same result. Hence we can state the following theorem:

Theorem 4.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is Ricci semi symmetric i.e. $R(\xi,X).S=0$, then $g(X,Y)=-\eta(X)\eta(Y)$ where S is Ricci tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§5. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).P=0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). The projective curvature tensor P on M is defined by

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Z)Y]. \tag{5.1}$$

Here we consider that the manifold is projectively semi symmetric i.e. $R(\xi,X).P=0$ holds.

So

$$R(\xi, X)(P(Y, Z)W) - P(R(\xi, X)Y, Z)W - P(Y, R(\xi, X)Z)W - P(Y, Z)R(\xi, X)W = 0, \quad (5.2)$$

for all vector fields X, Y, Z, W on M.

Using (2.7) and putting $Z = \xi$ in (5.2) we have

$$\eta(P(Y, \xi)W)X - g(X, P(Y, \xi)W)\xi - \eta(Y)P(X, \xi)W + g(X, Y)P(\xi, \xi)W$$
$$-\eta(\xi)P(Y, X)W + g(X, \xi)P(Y, \xi)W - \eta(W)P(Y, \xi)X + g(X, W)P(Y, \xi)\xi = 0. \quad (5.3)$$

Now from (5.1) we can write

$$P(X, \xi)Z = R(X, \xi)Z - \frac{1}{n-1}[S(\xi, Z)X - S(X, Z)\xi]. \quad (5.4)$$

Using (2.7), (2.15) in (5.4) we get

$$P(X, \xi)Z = \beta^2 g(X, Z)\xi + \frac{1}{n-1}S(X, Z)\xi + (\frac{A}{n-1} - \beta^2)\eta(Z)X. \quad (5.5)$$

Putting (5.5) and $W = \xi$ in (5.3) and after a long calculation we get

$$\frac{1}{n-1}S(X, Y)\xi + (\frac{A}{n-1} + \beta^2)\eta(X)Y - \frac{A}{n-1}g(X, Y)\xi$$
$$-(\frac{A}{n-1} + \beta^2)\eta(Y)X = 0. \quad (5.6)$$

Taking inner product with ξ in (5.6) we obtain

$$S(X, Y) = -Ag(X, Y),$$

which clearly shows that the manifold in an Einstein manifold.

Thus we can conclude the following theorem:

Theorem 5.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is projectively semi symmetric i.e. $R(\xi, X).P = 0$ holds, then the manifold is an Einstein manifold where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then using the same calculation we can obtain similar result, only the value of constant A will be changed which would not hamper our main result. Hence we can state the following theorem:

Theorem 5.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is projectively semi symmetric i.e. $R(\xi, X).P = 0$ holds, then the manifold is an Einstein manifold where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§6. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi, X).\tilde{P} = 0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g, V, λ). The pseudo projective curvature tensor $\tilde{P}$ on M is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y]. \quad (6.1)$$

Here we consider that the manifold is pseudo projectively semi symmetric i.e. $R(\xi, X).\tilde{P} = 0$ holds.

So

$$R(\xi,X)(\tilde{P}(Y,Z)W) - \tilde{P}(R(\xi,X)Y,Z)W - \tilde{P}(Y,R(\xi,X)Z)W - \tilde{P}(Y,Z)R(\xi,X)W = 0, \quad (6.2)$$

for all vector fields X, Y, Z, W on M.

Using (2.7) and putting $W = \xi$ in (6.2) we have

$$\eta(\tilde{P}(Y,Z)\xi)X - g(X,\tilde{P}(Y,Z)\xi)\xi - \eta(Y)\tilde{P}(X,Z)\xi + g(X,Y)\tilde{P}(\xi,Z)\xi - \eta(Z)\tilde{P}(Y,X)\xi + g(X,Z)\tilde{P}(Y,\xi)\xi - \eta(\xi)\tilde{P}(Y,Z)X + \eta(X)\tilde{P}(Y,Z)\xi = 0. \quad (6.3)$$

Now from (6.1) we can write

$$\tilde{P}(X,Y)\xi = aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] + \frac{r}{n}[\frac{a}{n-1} + b][g(Y,\xi)X - g(X,\xi)Y]. \quad (6.4)$$

Using $(2.1), (2.8), (2.15)$ in (6.4) and after a long calculation we get

$$\tilde{P}(X,Y)\xi = \varphi(\eta(X)Y - \theta(Y)X), \quad (6.5)$$

where $\varphi = (a\beta^2 - Ab - \frac{r}{n}[\frac{a}{n-1} + b])$.

Using (6.5) and putting $Z = \xi$ in (6.3) we obtain

$$\tilde{P}(Y,\xi)X + \varphi\eta(X)Y - \varphi g(X,Y)\xi = 0. \quad (6.6)$$

Taking inner product with ξ in (6.6) we get

$$\eta(\tilde{P}(Y,\xi)X) + \varphi\eta(X)\eta(Y) - \varphi g(X,Y) = 0. \quad (6.7)$$

Again from (6.1) we can write

$$\tilde{P}(X,\xi)Z = a(X,\xi)Z + b[S(\xi,Z)X - S(X,Z)\xi] + \frac{r}{n}[\frac{a}{n-1} + b][g(\xi,Z)X - g(X,Z)\xi]. \quad (6.8)$$

Using $(2.1), (2.7), (2.15)$ in (6.8) we get

$$\tilde{P}(X,\xi)Z = a\beta^2[g(X,Z)\xi - \eta(Z)X] + b[A\eta(Z)X - S(X,Z)\xi] + \frac{r}{n}[\frac{a}{n-1} + b][g(\xi,Z)X - g(X,Z)\xi]. \quad (6.9)$$

Taking inner product with ξ and replacing X by Y, Z by X in (6.9) we have

$$\eta(\tilde{P}(Y,\xi)X) = a\beta^2[-g(X,Y) - \eta(X)\eta(Y)] + b[A\eta(X)\eta(Y) + S(X,Y)] + \frac{r}{n}[\frac{a}{n-1} + b][\eta(X)\eta(Y) - g(X,Y)]. \quad (6.10)$$

Using (6.10) in (6.7) and after a brief simplification we obtain

$$S(X,Y) = Tg(X,Y) + U\eta(X)\eta(Y), \quad (6.11)$$

where $T = -\frac{1}{b}[-a\beta^2 - \frac{r}{n}[\frac{a}{n-1} + b] - \varphi]$ and $U = -\frac{1}{b}[\varphi + \frac{r}{n}[\frac{a}{n-1} + b] + Ab - a\beta^2]$.

From (6.11) we can conclude that the manifold is η-Einstein. Thus we have the following theorem:

Theorem 6.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is pseudo projectively semi symmetric i.e. $R(\xi, X).\tilde{P} = 0$ holds, then the manifold is η Einstein manifold where $\tilde{P}$ is pseudo projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then by following the same calculation we would obtain the same result, only the constant value of T and U will be changed. Hence we can state the following theorem:

Theorem 6.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is pseudo projectively semi symmetric i.e. $R(\xi, X).\tilde{P} = 0$ holds, then the manifold is η Einstein manifold where $\tilde{P}$ is pseudo projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§7. An Example of a 3-Dimensional Lorentzian β-Kenmotsu Manifold

In this section we construct an example of a 3-dimensional Lorentzian β-kenmotsu manifold.To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = e^{-z}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = 1, g(e_2, e_2) = 1, g(e_3, e_3) = -1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form which satisfies the relation

$$\eta(e_3) = -1.$$

Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then we have

$$\phi^2(Z) = Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after calculating we have

$$[e_1, e_3] = e^{-z}e_1, [e_1, e_2] = 0, [e_2, e_3] = e^{-z}e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$
$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (7.1)$$

By Koszul's formula we get

$$\nabla_{e_1} e_1 = e^{-z}e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0,$$

$$\nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = \text{‘}e^{-z}e_3, \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_1} e_3 = e^{-z}e_1, \nabla_{e_2} e_3 = e^{-z}e_2, \nabla_{e_3} e_3 = 0.$$

From the above we have found that $\beta = e^{-z}$ and it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a Lorentzian β-kenmotsu manifold. The results established in this note can be verified on this manifold.

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Some Properties of Conformal β -Change

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Abstract: We have considered the conformal β-change of the Finsler metric given by

$$L(x,y) \to \bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)),$$

where $\sigma(x)$ is a function of x, $\beta(x,y) = b_i(x)y^i$ is a 1-form on the underlying manifold M^n,and $f(L(x,y), \beta(x,y))$ is a homogeneous function of degree one in L and β.We have studied quasi-C-reducibility, C-reducibility and semi-C-reducibility of the Finsler space with this metric. We have also calculated V-curvature tensor and T-tensor of the space with this changed metric in terms of v-curvature tensor and T-tensor respectively of the space with the original metric.

Key Words: Conformal change, β-change, Finsler space, quasi-C-reducibility, C-reducibility, semi-C-reducibility, V-curvature tensor, T-tensor.

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§1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentialble manifold M^n equipped with the fundamental function L(x,y).B.N.Prasad and Bindu Kumari and C. Shibata [1,2] have studied the general case of β-change,that is, $L^*(x,y) = f(L,\beta)$,where f is positively homogeneous function of degree one in L and β, and β given by $\beta(x,y) = b_i(x)y^i$ is a one- form on M^n. The β-change of special Finsler spaces has been studied by H.S.Shukla, O.P.Pandey and Khageshwar Mandal [7].

The conformal theory of Finsler space was initiated by M.S. Knebelman [12] in 1929 and has been investigated in detail by many authors (Hashiguchi [8] ,Izumi[4,5] and Kitayama [9]). The conformal change is defined as $L^*(x,y) = e^{\sigma(x)}L(x,y)$, where $\sigma(x)$ is a function of position only and known as conformal factor. In 2008, Abed [15,16] introduced the change $\bar{L}(x,y) = e^{\sigma(x)}L(x,y) + \beta(x,y)$, which he called a β-conformal change, and in 2009 and 2010,Nabil L.Youssef, S.H.Abed and S.G. Elgendi [13,14] introduced the transformation $\bar{L}(x,y) = f(e^{\sigma}L, \beta)$, which is β-change of conformally changed Finsler metric L. They have not only established the relationships between some important tensors of (M^n, L) and the corresponding tensors of $(M^n, \bar{L})$, but have also studied several properties of this change.

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We have changed the order of combination of the above two changes in our paper [6], where we have applied β-change first and conformal change afterwards, i.e.,

$$\bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)), \tag{1.1}$$

where $\sigma(x)$ is a function of x, $\beta(x,y) = b_i(x)y^i$ is a 1-form. We have called this change as conformal β-change of Finsler metric. In this paper we have investigated the condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space.We have also found the necessary and sufficient conditions for this change to be a projective change.

In the present paper,we investigate some properties of conformal β-change. The Finsler space equipped with the metric $\bar{L}$ given by (1.1) will be denoted by $\bar{F^n}$.Throughout the paper the quantities corresponding to $\bar{F^n}$ will be denoted by putting bar on the top of them.We shall denote the partial derivatives with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively. The Fundamental quantities of F^n are given by

$$g_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{L^2}{2} = h_{ij} + l_i l_j, \quad l_i = \dot{\partial}_i L.$$

Homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{1.2}$$

where subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively. Differentiating above equations with respect to L and β respectively, we get

$$Lf_{12} + \beta f_{22} = 0 \text{ and } Lf_{11} + \beta f_{21} = 0. \tag{1.3}$$

Hence we have

$$f_{11}/\beta^2 = (-f_{12})/L\beta = f_{22}/L^2, \tag{1.4}$$

which gives

$$f_{11} = \beta^2\omega, f_{12} = -L\beta\omega, f_{22} = L^2\omega, \tag{1.5}$$

where Weierstrass function ω is positively homogeneous of degree -3 in L and β. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0, \tag{1.6}$$

where ω_1 and ω_2 are positively homogeneous of degree -4 in L and β. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not linear function of L and β so that $\omega \neq 0$.

The concept of concurrent vector field has been given by Matsumoto and K. Eguchi [11] and S. Tachibana [17], which is defind as follows:

The vector field b_i is said to be a concurrent vector field if

$$b_{i|j} = -g_{ij} \qquad b_i|_j = 0, \tag{1.7}$$

where small and long solidus denote the h- and v-covariant derivatives respectively. It has been proved by Matsumoto that b_i and its contravariant components b^i are functions of coordinates alone. Therefore from the second equation of (1.7),we have $C_{ijk}b^i = 0$.

The aim of this paper is to study some special Finsler spaces arising from conformal β-change of Finsler metric,viz., quasi-C-reducible, C-reducible and semi-C-reducible Finsler spaces. Further, we shall obtain v-curvature tensor and T-tensor of this space and connect them with v-curvature tensor and T-tensor respectively of the original space.

§2. Metric Tensor and Angular Metric Tensor of $\bar{F}^n$

Differentiating equation (1.1) with respect to y^i we have

$$\bar{l}_i = e^{\sigma}(f_1 l_i + f_2 b_i). \tag{2.1}$$

Differentiating (2.1) with respect to y^j, we get

$$\bar{h}_{ij} = e^{2\sigma}\left(\frac{ff_1}{L}h_{ij} + fL^2\omega m_i m_j\right), \tag{2.2}$$

where $m_i = b_i - \dfrac{\beta}{L}L_i$.

From (2.1) and (2.2) we get the following relation between metric tensors of F^n and $\bar{F}^n$:

$$\bar{g}_{ij} = e^{2\sigma}\left[\frac{ff_1}{L}g_{ij} - \frac{p\beta}{L}l_i l_j + (fL^2\omega + f_2^2)b_i b_j + p(b_i l_j + b_j l_i)\right], \tag{2.3}$$

where $p = f_1 f_2 - f\beta L\omega$.

The contravariant components $\bar{g}^{ij}$ of the metric tensor of $\bar{F}^n$, obtainable from $\bar{g}^{ij}\bar{g}_{jk} = \delta^i_k$, are as follows:

$$\bar{g}^{ij} = e^{-2\sigma}\left[\frac{L}{ff_1}g^{ij} + \frac{pL^3}{f^3 f_1 t}\left(\frac{f\beta}{L^2} - \Delta f_2\right)l^i l^j - \frac{L^4\omega}{ff_1 t}b^i b^j - \frac{pL^2}{f^2 f_1 t}(l^i b^j + l^j b^i)\right], \tag{2.4}$$

where $l^i = g^{ij}l_j$, $b^2 = b_i b^i$, $b^i = g^{ij}b_j$, g^{ij} is the reciprocal tensor of g_{ij} of F^n, and

$$t = f_1 + L^3\omega\Delta, \Delta = b^2 - \frac{\beta^2}{L^2}. \tag{2.5}$$

$$\begin{aligned}
&(a)\ \dot{\partial}_i f = e^{\sigma}\left(\frac{f}{L}l_i + f_2 m_i\right), \quad (b)\ \dot{\partial}_i f_1 = -e^{\sigma}\beta L\omega m_i,\\
&(c)\dot{\partial}_i f_2 = e^{\sigma}L^2\omega m_i, \quad (d)\ \dot{\partial}_i p = -\beta q L m_i,\\
&(e)\ \dot{\partial}_i \omega = -\frac{3\omega}{L}l_i + \omega_2 m_i, \quad (f)\dot{\partial}_i b^2 = -2C_{..i},\\
&(g)\ \dot{\partial}_i \Delta = -2C_{..i} - \frac{2\beta}{L^2}m_i,
\end{aligned} \tag{2.6}$$

$$(a)\ \dot{\partial}_i q = -\frac{3q}{L}l_i, \quad (b)\ \dot{\partial}_i t = -2L^3\omega C_{..i} + [L^3\Delta\omega_2 - 3\beta L\omega]m_i,$$
$$(c)\ \dot{\partial}_i q = -\frac{3q}{L}l_i + (4f_2\omega_2 + 3\omega^2L^2 + f\omega_{22})m_i. \qquad (2.7)$$

§3. Cartan's C-Tensor and C-Vectors of $\bar{F}^n$

Cartan's covariant C-tensor C_{ijk} of F^n is defined by

$$\bar{C}_{ijk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2 = \dot{\partial}_k g_{ij}$$

and Cartan's C-vectors are defined as follows:

$$C_i = C_{ijk}g^{jk}, C^i = C^i_{jk}g^{jk}. \qquad (3.1)$$

We shall write $C^2 = C^iC_i$. Under the conformal β-chang (1.1) we get the following relation between Cartan's C-tensors of F^n and $\bar{F}^n$:

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2}m_im_jm_k\right]. \qquad (3.2)$$

We have

$$(a)\ m_il^i = 0,$$
$$(b)\ m_ib^i = b^2 - \frac{\beta^2}{L^2} = \Delta = b_im^i,$$
$$(c)\ g_{ij}m^i = h_{ij}m^i = m_j. \qquad (3.3)$$

From (2.1), (2.3), (2.4) and (3.2), we get

$$\begin{aligned}\bar{C}^h_{ij} \;=\; & C^h_{ij} + \frac{p}{2ff_1}(h_{ij}m^h + h^h_jm_i + h^h_im_j) + \frac{qL^3}{2ff_1}m_jm_km^h \\ & -\frac{L}{ft}C_{\cdot jk}n^h - \frac{pL\Delta}{2f^2f_1t}h_{jk}n^h - \frac{2pL + qL^4\Delta}{2f^2f_1t}m_jm_kn^h, \end{aligned} \qquad (3.4)$$

where $n^h = fL^2\omega b^h + pl^h$ and $h^i_j = g^{il}h_{lj}, C_{.ij} = C_{rij}b^r, C_{..i} = C_{rji}b^rb^j$ and so on.

Proposition 3.1 *Let $\bar{F}^n = (M^n, \bar{L})$ be an n-dimensional Finsler space obtained from the conformal β-change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element $\bar{l}_i$, angular metric tensor $\bar{h}_{ij}$, fundamental metric tensor $\bar{g}_{ij}$ and (h)hv-torsion tensor $\bar{C}_{ijk}$ of $\bar{F}^n$ are given by (2.1), (2.2), (2.3) and (3.2), respectively.*

From (2.4),(3.1),(3.2) and (3.4) we get the following relations between the C-vectors of of F^n and $\bar{F}^n$ and their magnitudes

$$\bar{C}_i = C_i - L^3\omega C_{i..} + \mu m_i, \qquad (3.5)$$

where

$$\mu = \frac{p(n+1)}{2ff_1} - \frac{3pL^3\omega\Delta}{2ff_1} + \frac{qL^3\Delta(1-L^3\omega\Delta)}{2ff_1};$$

$$\bar{C}^i = \frac{e^{-2\sigma}L}{ff_1}C^i + M^i, \tag{3.6}$$

where

$$M^i = \frac{\mu e^{-2\sigma}L}{ff_1}m^i - \frac{L^4\omega}{ff_1}C^i_{..} - \left(C_i - e^{2\sigma}L^3\omega C_{i..} + \mu\Delta\right)\left(\frac{L^3\omega}{ff_1}b^i + \frac{L}{ft}y^i\right)$$

and

$$\bar{C}^2 = \frac{e^{-2\sigma}}{p}C^2 + \lambda, \tag{3.7}$$

where

$$\begin{aligned}\lambda \;=\; & \left(\frac{e^{-2\sigma}L}{ff_1} - L^3\omega\Delta\right)\mu^2\Delta + \frac{2\mu e^{-2\sigma}L}{ff_1}C_{.} \\ & - (1+2\mu\Delta)\,L^3\omega + \left(1 - 3\mu + e^{2\sigma}L^2\omega ff_1 C_{.}\right)L^3\omega C_{...} \\ & + L^3\omega C_{..r}\left(\left(e^{4\sigma}L\omega f^2 f_1^2 C_{i..} - \mu\Delta\right)L^3\omega b^r - e^{2\sigma}L^2\omega ff_1 C^r_{..} - 2C^r\right).\end{aligned}$$

§4. Special Cases of $\bar{F}^n$

In this section, following Matsumoto [10], we shall investigate special cases of $\bar{F}^n$ which is conformally β-changed Finsler space obtained from F^n.

Definition 4.1 *A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be quasi-C-reducible if the Cartan tensor C_{ijk} satisfies*

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j, \tag{4.1}$$

where Q_{ij} is a symmetric indicatory tensor.

The equation (3.2) can be put as

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j\right)m_k\right\}\right],$$

where $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices i, j and k.

Putting the value of m_k from equation (3.5) in the above equation, we get

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j)(\bar{C}_k - C_k + L^3\omega C_{k..}\right)\right\}\right].$$

Rearranging this equation, we get

$$\begin{aligned}\bar{C}_{ijk} &= e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j\right)\bar{C}_k\right\}\right.\\ &\left. + \frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j\right)\left(L^3\omega C_{k..} - C_k\right)\right\}\right].\end{aligned}$$

Further rearrangment of this equations gives

$$\bar{C}_{ijk} = \pi_{(ijk)}(\bar{H}_{ij}\bar{C}_k) + U_{ijk}, \tag{4.2}$$

where $\bar{H}_{ij} = \frac{e^{2\sigma}}{6\mu}\{(\frac{3p}{L}h_{ij} + qL^2m_im_j)$, and

$$U_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j\right)\left(L^3\omega C_{k..} - C_k\right)\right\}\right] \tag{4.3}$$

Since $\bar{H}_{ij}$ is a symmetric and indicatory tensor,therefore from equation (4.2) we have the following theorem.

Theorem 4.1 *Conformally β-changed Finsler space $\bar{F}^n$ is quasi-C-reducible iff the tensor U_{ijk} of equation (4.3) vanishes identically.*

We obtain a generalized form of Matsumoto's result [10] as a corollary of the above theorem.

Corollary 4.1 *If F^n is Reimannian space, then the conformally β-changed Finsler space $\bar{F}^n$ is always a quasi-C-reducible Finsler space.*

Definition 4.2 *A Finsler space (M^n, L) of dimension $n \geq 3$ is called C-reducible if the Cartan tensor C_{ijk} is written in the form*

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i). \tag{4.4}$$

Define the tensor $G_{ijk} = C_{ijk} - \frac{1}{(n+1)}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i)$. It is clear that G_{ijk} is symmetric and indicatory. Moreover, G_{ijk} vanishes iff F^n is C-reducible.

Proposition 4.1 *Under the conformal β-change(1.1), the tensor $\bar{G}_{ijk}$ associated with the space $\bar{F}^n$ has the form*

$$\bar{G}_{ijk} = e^{2\sigma}\frac{ff_1}{L}G_{ijk} + V_{ijk} \tag{4.5}$$

where

$$\begin{aligned}V_{ijk} &= \frac{1}{(n+1)}\pi_{(ijk)}\{(e^{2\sigma}(n+1)(\alpha_1h_{ij} + \alpha_2m_im_j)m_k + e^{2\sigma}\omega L^2m_im_jC_k\\ &+ e^{2\sigma}L^2\omega(ff_1h_{ij} + L^3\omega m_im_j)C_{k..}\},\end{aligned} \tag{4.6}$$

$$\alpha_1 = \frac{e^{2\sigma}p}{2L} - \frac{\mu ff_1e^{2\sigma}}{L(n+1)}, \quad \alpha_2 = \frac{e^{2\sigma}qL^2}{6} - \frac{\mu e^{2\sigma}\omega L^2}{(n+1)}.$$

From (4.5) we have the following theorem.

Theorem 4.2 *Conformally β-changed Finsler space $\bar{F}^n$ is C-reducible iff F^n is C-reducible and the tensor V_{ijk} given by (4.6) vanishes identically.*

Definition 4.3 *A Finsler space (M^n, L) of dimension $n \geq 3$ is called semi-C-reducible if the Cartan tensor C_{ijk} is expressible in the form:*

$$C_{ijk} = \frac{r}{n+1}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) + \frac{s}{C^2}C_iC_jC_k, \tag{4.7}$$

where r and s are scalar functions such that $r + s = 1$.

Using equations (2.2), (3.5) and (3.7) in equation (3.2), we have

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{p}{2\mu ff_1}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{ki}\bar{C}_j + \bar{h}_{jk}\bar{C}_i) + \frac{\Delta L(f_1q - 3p\omega)}{2ff_1\mu t\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k\right].$$

If we put

$$r' = \frac{p(n+1)}{2\mu ff_1}, s' = \frac{\Delta L(f_1q - 3p\omega)}{2ff_1\mu t},$$

we find that $r' + s' = 1$ and

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{r'}{n+1}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{ki}\bar{C}_j + \bar{h}_{jk}\bar{C}_i) + \frac{s'}{\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k\right]. \tag{4.8}$$

From equation (4.8) we infer that $\bar{F}^n$ is semi-C- reducible iff $C_{ijk} = 0$, i.e. iff F^n is a Reimannian space. Thus we have the following theorem.

Theorem 4.3 *Conformally β-changed Finsler space $\bar{F}^n$ is semi-C-reducible iff F^n is a Riemannian space.*

§5. v-Curvature Tensor of $\bar{F}^n$

The v-curvature tensor [10] of Finsler space with fundamental function L is given by

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj}$$

Therefore the v-curvature tensor of conformally β-changed Finsler space $\bar{F}^n$ will be given by

$$\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj}. \tag{5.1}$$

From equations (3.2)and(3.4), we have

$$\begin{aligned}\bar{C}_{ijr}\bar{C}^r_{hk} \quad = \quad & e^{2\sigma}\left[\frac{ff_1}{L}C_{ijr}C^r_{hk} + \frac{p}{2L}(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j\right. \\ & +C_{hjk}m_i) + \frac{pf_1}{2Lt}(C_{.ij}h_{hk} + C_{hk}h_{ij}) - \frac{ff_1L^2\omega}{t}C_{.ij}C_{.hk}\end{aligned}$$

$$+\frac{p^2\Delta}{4fLt}h_{hk}h_{ij}+\frac{L^2(qf_1-2p\omega)}{2t}(C_{.ij}m_km_h+C_{.hk}m_im_j)$$
$$+\frac{p(p+L^3q\Delta)}{4Lft}(h_{ij}m_hm_k+h_{hk}m_im_j)+\frac{p^2}{4Lff_1}(h_{ij}m_hm_k$$
$$+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h+h_{ik}m_hm_j)$$
$$\left.+\frac{L^2(2pqt+(qf_1-2p\omega)(2p+L^3q\Delta))}{4ff_1t}m_im_jm_hm_k\right]. \qquad (5.2)$$

We get the following relation between v-curvature tensors of (M^n, L) and $(M^n, \bar{L})$:

$$\bar{S}_{hijk}=e^{2\sigma}\left[\frac{ff_1}{L}S_{hijk}+d_{hj}d_{ik}-d_{hk}d_{ij}+E_{hk}E_{ij}-E_{hj}E_{ik}\right], \qquad (5.3)$$

where

$$d_{ij}=PC_{.ij}-Qh_{ij}+Rm_im_j, \qquad (5.4)$$

$$E_{ij}=Sh_{ij}+Tm_im_j, \qquad (5.5)$$

$$P=L\left(\frac{s}{t}\right)^{1/2},\quad Q=\frac{pg}{2L^2\sqrt{st}},R=\frac{L\,(2\omega p-f_1q)}{2\sqrt{st}},S=\frac{p}{2L^2\sqrt{f\omega}},\quad T=\frac{L\,(qf_1-\omega p)}{2f_1\sqrt{f\omega}}.$$

Proposition 5.1 *The relation between v-curvature tensors of F^n and $\bar{F}^n$ is given by (5.3).*

When b_i in β is a concurrent vector field,then $C_{.ij}=0$. Therefore the value of v-curvature tensor of $\bar{F}^n$ as given by (5.3) is reduced to the extent that $d_{ij}=Rm_im_j-Qh_{ij}$.

§6. The T-Tensor T_{hijk}

The T-tensor of F^n is defined in [3] by

$$T_{hijk}=LC_{hij}\mid_k+C_{hij}l_k+C_{hik}l_j+C_{hjk}l_i+C_{ijk}l_h, \qquad (6.1)$$

where

$$C_{hij}\mid_k=\dot{\partial}_kC_{hij}-C_{rij}C^r_{hk}-C_{hrj}C^r_{ik}-C_{hir}C^r_{jk}. \qquad (6.2)$$

In this section we compute the T-tensor of $\bar{F}^n$, which is given by

$$\bar{T}_{hijk}=\bar{L}\bar{C}_{hij}\bar{|}_k+\bar{C}_{hij}\bar{l}_k+\bar{C}_{hik}\bar{l}_j+\bar{C}_{hjk}\bar{l}_i+\bar{C}_{ijk}\bar{l}_h, \qquad (6.3)$$

where

$$\bar{C}_{hij}\bar{|}_k=\dot{\partial}_k\bar{C}_{hij}-\bar{C}_{rij}\bar{C}^r_{hk}-\bar{C}_{hrj}\bar{C}^r_{ik}-\bar{C}_{hir}\bar{C}^r_{jk}. \qquad (6.4)$$

The derivatives of m_i and h_{ij} with respect to y^k are given by

$$\dot{\partial}_km_i=-\frac{\beta}{L^2}h_{ik}-\frac{1}{L}(l_im_k),\quad \dot{\partial}_kh_{ij}=2C_{ijk}-\frac{1}{L}(l_ih_{jk}+l_jh_{ki}) \qquad (6.5)$$

From equations (3.2)and (6.5), we have

$$\begin{aligned}\dot{\partial}_k\bar{C}_{hij} &= e^{2\sigma}\left[\frac{ff_1}{L}\partial_k C_{hij} + \frac{p}{L}(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j + C_{hjk}m_i)\right.\\ &\quad -\frac{p\beta}{2L^3}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) + \frac{p}{2L^2}(h_{jk}l_h m_i + h_{hk}l_j m_i\\ &\quad +h_{hk}l_i m_j + h_{ik}l_h m_j + h_{jk}l_i m_h + h_{jk}l_h m_i + h_{ij}l_h m_k + h_{hj}l_i m_k\\ &\quad +h_{ik}l_j m_k + h_{ij}l_k m_h + h_{jh}l_k m_i + h_{hi}l_k m_j) - \frac{\beta q}{2}(h_{ij}m_h m_k\\ &\quad +h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k + h_{jk}m_i m_h + h_{ik}m_h m_j)\\ &\quad -\frac{qL}{2}(l_i m_j m_h m_k + l_j m_i m_h m_k + l_h m_i m_j m_k + h_k m_i m_j m_h)\\ &\quad \left. + \frac{L^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})m_h m_i m_j m_k\right]. \qquad (6.6)\end{aligned}$$

Using equations (6.5) and (5.2) in equation(6.4), we get

$$\begin{aligned}\bar{C}_{hij}\bar{|}_k &= e^{2\sigma}\frac{ff_1}{L}C_{hij}\ |_k - \frac{e^{2\sigma}p}{2L}(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j + C_{hjk}m_i)\\ &\quad -pe^{2\sigma}\left(\frac{2f\beta t}{4fL^3t} + \frac{L^2p\Delta}{4fL^3t}\right)(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) - e^{2\sigma}\left(\frac{\beta q}{2}\right.\\ &\quad \left.+\frac{p^2f_1 + pqf_1L^3\Delta + 3p^2}{4Lff_1t}\right)(h_{ij}m_h m_k + h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k\\ &\quad +h_{jk}m_i m_h + h_{ik}m_h m_j) - \frac{e^{2\sigma}p}{2L^2}[l_h(h_{jk}m_i + h_{ij}m_k\\ &\quad +h_{ik}m_j) + l_j(h_{hk}m_i + h_{ik}m_{kh} + h_{ih}m_k) + l_i(h_{hk}m_j + h_{jk}m_h\\ &\quad +h_{hj}m_k) + l_k(h_{ij}m_h + h_{jh}m_i + h_{hi}m_j)] - \frac{e^{2\sigma}qL}{2}(l_i m_j m_h m_k\\ &\quad +l_j m_i m_h m_k + l_h m_i m_j m_k + h_k m_i m_j m_h) - \frac{pf_1e^{2\sigma}}{2Lt}(C_{.ij}h_{hk}\\ &\quad +C_{.hj}h_{ik} + C_{.hk}h_{ij} + C_{.ik}h_h + C_{.hi}h_{jk} + C_{.jk}h_{hi}) + \frac{e^{2\sigma}ff_1L^2\omega}{t}(C_{.ij}C_{.hk}\\ &\quad +C_{.hj}C_{.ik} + C_{.hi}C_{.jk}) - \frac{e^{2\sigma}L^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_k m_h\\ &\quad +C_{.hk}m_i m_j + C_{.hj}m_i m_k + C_{.ik}m_j m_h\\ &\quad +C_{.hi}m_j m_k + C_{.jk}m_h m_i) + e^{2\sigma}\left[\frac{L^2(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})}{2}\right.\\ &\quad \left. - \frac{3L^2(2pqt + (qf_1 - 2p\omega)(2p + L^3q\Delta)}{4ff_1t}\right]m_i m_j m_h m_k. \qquad (6.7)\end{aligned}$$

Using equations (2.1), (3.2) and (6.6) in equation (6.3), we get the following relation between T-tensors of Finsler spaces F^n and $\bar{F}^n$:

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[\frac{f^2 f_1}{L^2}T_{hijk} + \frac{f(f_1 f_2 + f\beta L\omega)}{2L}\,(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j \\ &+C_{hjk}m_i) + \frac{f^2 f_1 L^2\omega}{t}(C_{.ij}C_{.hk} + C_{.hj}C_{.ik} + C_{.hi}C_{.jk}) \\ &-\frac{pf_1}{2Lt}(C_{.ij}h_{hk} + C_{.hj}h_{ik} + C_{.hk}h_{ij} + C_{.ik}h_h + C_{.hi}h_{jk} + C_{.jk}h_{hi}) \\ &-\frac{fL^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_k m_h + C_{.hk}m_i m_j + C_{.hj}m_i m_k \\ &+C_{.ik}m_j m_h + C_{.hi}m_j m_k + C_{.jk}m_h m_i) - \frac{p(2f\beta t + L^2 p\Delta)}{4L^3 t}(h_{ij}h_{hk} \\ &+h_{hj}h_{ik} + h_{ih}h_{jk}) - \left(\frac{p^2 f_1 + pqf_1 L^3\Delta + 3p^2}{4Lf_1 t} + \frac{\beta qf}{2} - \frac{pf_2}{L}\right) \\ &(h_{ij}m_h m_k + h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k + h_{jk}m_i m_h \\ &+h_{ik}m_h m_j) + \Big[\frac{L^2(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})}{2} + 2L^2 f_2 q \\ &-\frac{3L^2(2pqt + (qf_1 - 2p\omega)(2p + L^3 q\Delta)}{4f_1 t}\Big]m_i m_j m_h m_k\Big]. \end{aligned} \tag{6.8}$$

Proposition 6.1 *The relation between T-tensors of F^n and $\bar{F}^n$ is given by (6.7).*

If bi is a concurrent vector field in F^n, then $C_{.ij} = 0$. Therefore from(6.8), we have

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[\frac{f^2 f_1}{L^2}T_{hijk} - \frac{p(2f\beta t + L^2 p\Delta)}{4L^3 t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) \\ &-\left(\frac{p^2 f_1 + pqf_1 L^3\Delta + 3p^2 t}{4Lf_1 t} + \frac{\beta qf}{2} - \frac{pf_2}{L}\right)(h_{ij}m_h m_k + h_{hk}m_i m_j \\ &+h_{hj}m_i m_k + h_{hi}m_j m_k + h_{jk}m_i m_h + h_{ik}m_h m_j) \\ &+\Big[2L^2 f_2 q + \frac{L^2(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})}{2} + \frac{3L^2(qf_1 - 2p\omega)(2p + L^3 q\Delta)}{4Lff_1 t} \\ &-\frac{3L^2 2pqt}{4Lff_1 t}\Big]m_i m_j m_h m_k\Big]. \end{aligned} \tag{6.9}$$

If bi is a concurrent vector field in F^n, with vanishing T-tensor then T-tensor of F^n is given by

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[-\frac{p(2f\beta t + L^2 p\Delta)}{4L^3 t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) \\ &-\left(\frac{p^2 f_1 + pqf_1 L^3\Delta + 3p^2 t}{4Lf_1 t} + \frac{\beta qf}{2} - \frac{pf_2}{L}\right)(h_{ij}m_h m_k \\ &+h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k + h_{jk}m_i m_h + h_{ik}m_h m_j) \\ &+\Big[\frac{L^2(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})}{2} - \frac{3L^2 2pqt}{4Lff_1 t} \\ &+\frac{3L^2(qf_1 - 2p\omega)(2p + L^3 q\Delta)}{4Lff_1 t} + 2L^2 f_2 q\Big]m_i m_j m_h m_k\Big]. \end{aligned} \tag{6.10}$$

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Equitable Coloring on Triple Star Graph Families

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Abstract: An equitable k-coloring of a graph G is a proper k-coloring of G such that the sizes of any two color class differ by at most one. In this paper we investigate the equitable chromatic number for the Central graph, Middle graph, Total graph and Line graph of Triple star graph $K_{1,n,n,n}$ denoted by $C(K_{1,n,n,n})$, $M(K_{1,n,n,n})$, $T(K_{1,n,n,n})$ and $L(K_{1,n,n,n})$ respectively.

Key Words: Equitable coloring, Smarandachely equitable k-coloring, triple star graph, central graph, middle graph, total graph and line graph.

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§1. Introduction

A graph consist of a vertex set $V(G)$ and an edge set $E(G)$. All Graphs in this paper are finite, loopless and without multiple edges. We refer the reader [8] for terminology in graph theory. Graph coloring is an important research problem [7, 10]. A proper k-coloring of a graph is a labelling $f : V(G) \to \{1, 2, \cdots, k\}$ such that the adjacent vertices have different labels. The labels are colors and the vertices with same color form a color class. The chromatic number of a graph G, written as $\chi(G)$ is the least k such that G has a proper k-coloring.

Equitable colorings naturally arise in some scheduling, partitioning and load balancing problems [11,12]. In 1973, Meyer [4] introduced first the notion of equitable colorability. In 1998, Lih [5] surveyed the progress on the equitable coloring of graphs.

We say that a graph $G = (V, E)$ is equitably k-colorable if and only if its vertex set can be partitioned into independent sets $\{V_1, V_2, \cdots, V_k\} \subset V$ such that $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j). The smallest integer k for which G is equitable k-colorable is known as the equitable chromatic number [1,3] of G and denoted by $\chi_=(G)$. On the other hand, if V can be partitioned into independent sets $\{V_1, V_2, \cdots, V_k\} \subset V$ with $||V_i| - |V_j|| \geq 1$ holds for every pair (i, j), such a k-coloring is called a *Smarandachely equitable k-coloring*.

In this paper, we find the equitable chromatic number $\chi_=(G)$ for central, line, middle and total graphs of triple star graph.

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§2. Preliminaries

For a given graph $G = (V, E)$ we do a operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called *central graph* of G [1] and is denoted by $C(G)$.

The *line graph* [6] of a graph G, denoted by $L(G)$ is a graph whose vertices are the edges of G and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *middle graph* [2] of G denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$ in which two vertices x, y are adjacent in $M(G)$ if the following condition hold:

(1) $x, y \in E(G)$ and x, y are adjacent in G;

(2) $x \in V(G),\ y \in E(G)$ and they are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *total graph* [1,2] of G is denoted by $T(G)$ and is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ is adjacent in $T(G)$, if one of the following holds:

(1) x, y are in $V(G)$ and x is adjacent to y in G;

(2) x, y are in $E(G)$ and x, y are adjacent in G;

(3) x is in $V(G)$, y is in $E(G)$ and x, y are adjacent in G.

Triple star $K_{1,n,n,n}$ [9] is a tree obtained from the double star [2] $K_{1,n,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $3n + 1$ vertices and $3n$ edges.

§3. Equitable Coloring on Central Graph of Triple Star Graph

Algorithm 1.

Input: The number ‘n’ of $k_{1,n,n,n}$;
Output: Assigning equitable colouring for the vetices in $C(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$
$C(e_i) = i$;
$V_2 = \{a_i\}$;
$C(a_i) = i$;
}
$V_3 = \{v\}$;
$C(v) = n + 1$;

for $i = 2$ to n

{

$V_4 = \{v_i\}$;

$C(v_i) = i - 1$;

$V_5 = \{w_i\}$;

$C(w_i) = i - 1$;

$V_6 = \{u_i\}$;

$C(u_i) = i - 1$;

}

$C(v_1) = n$;

$C(w_1) = n$;

$C(u_1) = n$;

for $i = 1$ to 5

{

$V_7 = \{s_i\}$;

$C(s_i) = n + 1$;

}

for $i = 6$ to n

{

$V_8 = \{s_i\}$;

$C(s_i) = i$;

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8$;

end

Theorem 3.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number*

$$\chi_{=}[C(K_{1,n,n,n})] = n + 1.$$

Proof Let $\{v_i : 1 \leq i \leq n\}$, $\{w_i : 1 \leq i \leq n\}$ and $\{u_i : 1 \leq i \leq n\}$ be the vertices in $K_{1,n,n,n}$. The vertex v is adjacent to the vertices $v_i (1 \leq i \leq n)$. The vertices $v_i (1 \leq i \leq n)$ is adjacent to the vertices $w_i (1 \leq i \leq n)$ and the vertices $w_i (1 \leq i \leq n)$ is adjacent to the vertices $u_i (1 \leq i \leq n)$.

By the definition of central graph on $K_{1,n,n,n}$, let the edges vv_i, v_iw_i and w_iu_i $(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i, a_i, s_i (1 \leq i \leq n)$ respectively.

Clearly,

$$\begin{aligned} V[C(K_{1,n,n,n})] &= \{v\}\bigcup\{v_i : 1\le i\le n\}\bigcup\{w_i : 1\le i\le n\} \\ &\bigcup\{u_i : 1\le i\le n\}\bigcup\{e_i : 1\le i\le n\} \\ &\bigcup\{a_i : 1\le i\le n\}\cup\{s_i : 1\le i\le n\} \end{aligned}$$

The vertices v and $u_i(1 \le i \le n)$ induces a clique of order $n+1$ (say k_{n+1}) in $[C[K_{1,n,n,n}]]$. Therefore

$$\chi_=[C(K_{1,n,n,n})] \ge n+1$$

Now consider the vertex set $V[C(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, c_3, \cdots c_{n+1}\}$. Assign an equitable coloring to $C(K_{1,n,n,n})$ by Algorithm 1. Therefore

$$\chi_=[C(K_{1,n,n,n})] \le n+1.$$

An easy check shows that $||v_i| - |v_j|| \le 1$. Hence

$$\chi_=[C(K_{1,n,n,n})] = n+1.$$ □

§4. Equitable Coloring on Line graph of Triple Star Graph

Algorithm 2.

Input: The number 'n' of $K_{1,n,n,n}$;
Output: Assigning equitable coloring for the vertices in $L(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = i$;
}
for $i = 2$ to n
{
$V_3 = \{a_i\}$;
$C(a_i) = i - 1$;
}

$C(a_1) = n$;

$V = V_1 \cup V_2 \cup V_3$;

end

Theorem 4.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[L(K_{1,n,n,n})] = n.$$

Proof Let $\{v_i : 1 \leq i \leq n\}$, $\{w_i : 1 \leq i \leq n\}$ and $\{u_i : 1 \leq i \leq n\}$ be the vertices in $K_{1,n,n,n}$. The vertex v is adjacent to the vertices $v_i(1 \leq i \leq n)$ with edges $e_i(1 \leq i \leq n)$. The vertices $v_i(1 \leq i \leq n)$ is adjacent to the vertices $w_i(1 \leq i \leq n)$ with edges $a_i(1 \leq i \leq n)$. The vertices $w_i(1 \leq i \leq n)$ is adjacent to the vertices $u_i(1 \leq i \leq n)$ with edges $s_i(1 \leq i \leq n)$.

By the definition of line graph on $K_{1,n,n,n}$ the edges $e_i, a_i, s_i(1 \leq i \leq n)$ of $K_{1,n,n,n}$ are the vertices of $L(K_{1,n,n,n})$. Clearly

$$V[L(K_{1,n,n,n})] = \{e_i : 1 \leq i \leq n\} \bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\}$$

The vertices $e_i(1 \leq i \leq n)$ induces a clique of order n (say K_n) in $L(K_{1,n,n,n})$. Therefore

$$\chi_=[L(K_{1,n,n,n})] \geq n.$$

Now consider the vertex set $V[L(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, \cdots c_n\}$.

Assign an equitable coloring to $L(K_{1,n,n,n})$ by Algorithm 2. Therefore

$$\chi_=[L(K_{1,n,n,n})] \leq n.$$

An easy check shows that $||v_i| - |v_j|| \leq 1$. Hence

$$\chi_=[L(K_{1,n,n,n})] = n. \qquad \square$$

§5. Equitable Coloring on Middle and Total Graphs of Triple Star Graph

Algorithm 3.

Input: The number 'n' of $K_{1,n,n,n}$;

Output: Assigning equitable coloring for the vertices in $M(K_{1,n,n,n})$ and $T(K_{1,n,n,n})$.

begin

for $i = 1$ to n

{

$V_1 = \{e_i\}$;

$C(e_i) = i$;

$V_2 = \{s_i\}$;

$C(s_i) = i$;

}

$V_3 = \{v\}$;

$C(v) = n + 1$;

for $i = 2$ to n

{

$V_4 = \{v_i\}$;

$C(v_i) = i - 1$;

}

$C(v_1) = n$;

for $i = 3$ to n

{

$V_5 = \{a_i\}$;

$C(a_i) = i - 2$;

}

$C(a_1) = n + 1$;

$C(a_2) = n + 1$;

for $i = 4$ to n

{

$V_6 = \{w_i\}$

$C(w_i) = i - 3$;

}

$C(w_1) = n - 1$;

$C(w_2) = n$;

$C(w_3) = n + 1;$

for $i = 1$ to n

{

$V_7 = \{u_i\};$

$C(u_i) = i + 1;$

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7$

end

Theorem 5.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[M(K_{1,n,n,n})] = n + 1, \; n \geq 4.$$

Proof Let $V(K_{1,n,n,n}) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\}$.

By the definition of middle graph on $K_{1,n,n,n}$ each edge vv_i, v_iw_i and $w_iu_i (1 \leq i \leq n)$ in $K_{1,n,n,n}$ are subdivided by the vertices e_i, w_i, $s_i (1 \leq i \leq n)$ respectively. Clearly

$$\begin{aligned} V[M(K_{1,n,n,n})] &= \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \\ &\bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \\ &\bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\} \end{aligned}$$

The vertices v and $e_i (1 \leq i \leq n)$ induces a clique of order $n+1$ (say k_{n+1}) in $[M(K_{1,n,n,n})]$. Therefore

$$\chi_=[M(K_{1,n,n,n})] \geq n + 1.$$

Now consider the vertex set $V[M(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, \cdots c_{n+1}\}$. Assign an equitable coloring to $M(K_{1,n,n,n})$ by Algorithm 3. Therefore

$$\chi_= M[(K_{1,n,n,n})] \leq n + 1, \quad ||v_i| - |v_j|| \leq 1.$$

Hence

$$\chi_=[M(K_{1,n,n,n})] = n + 1 \; \forall n \geq 4. \qquad \square$$

Theorem 5.2 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[T(K_{1,n,n,n})] = n + 1, \quad n \geq 4.$$

Proof Let $V(K_{1,n,n,n}) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\}$ and $E(K_{1,n,n,n}) = \{e_i : 1 \leq i \leq n\} \bigcup \{a_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

By the definition of Total graph, the edge vv_i, v_iw_i and $w_iu_i(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices e_i, a_i and $s_i(1 \leq i \leq n)$ respectively. Clearly

$$\begin{aligned} V[T(K_{1,n,n,n})] &= \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \\ &\quad \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \\ &\quad \bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\}. \end{aligned}$$

The vertices v and $e_i(1 \leq i \leq n)$ induces a clique of order $n+1$ (say k_{n+1}) in $T(K_{1,n,n,n})$. Therefore

$$\chi_{=}[T(K_{1,n,n,n})] \geq n+1, \quad n \geq 4.$$

Now consider the vertex set $V(T(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, \cdots, c_{n+1}\}$. Assign an equitable coloring to $T(K_{1,n,n,n})$ by Algorithm 3. Therefore

$$\chi_{=}[T(K_{1,n,n,n})] \leq n+1, \quad n \geq 4, \quad ||v_i| - |v_j|| \leq 1.$$

Hence

$$\chi_{=}[T(K_{1,n,n,n})] = n+1, \forall n \geq 4. \qquad \square$$

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On the Tangent Vector Fields of Striction Curves Along the Involute and Bertrandian Frenet Ruled Surfaces

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Abstract: In this paper we consider nine special ruled surfaces associated to an involute of a curve α and its Bertrand mate α^{**} with $k_1 \neq 0$. They are called as involute Frenet ruled and Bertrandian Frenet ruled surfaces, because of their generators which are the Frenet vector fields of curve α. First we give the striction curves of all Frenet ruled surfaces. Then the striction curves of involute and Bertrandian Frenet ruled surfaces are given in terms of the Frenet apparatus of the curve α. Some results are given on the striction curves of involute and Bertrand Frenet ruled surfaces based on the tangent vector fields in E^3.

Key Words: Frenet ruled surface, involute Frenet ruled surface, Bertrandian Frenet ruled surface, evolute-involute curve, Bertrand curve pair, striction curves.

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§1. Introduction

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$ [2]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\left\langle \alpha', v \right\rangle = 0$ where $\alpha_s = \alpha'$. The fundamental forms of the $B-scroll$ with null directrix and Cartan frame in the Minkowskian $3-space$ are examined in [5]. The properties of some ruled surfaces are also examined in $\mathbb{E}^3$ [6] , [7] ,[9] and [11]. A striction point on a ruled surface $\varphi(s,v) = \alpha(s) + v.e(s)$ is the foot of the common normal between two consecutive generators (or ruling). To illustrate the current situation, we bring here the famous example of L. K. Graves [3], so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [1]. Deriving a curve based on an other curve is one of the main subjects in geometry. Involute-evolute curves and Bertrand curves are of these kinds. An involute of a given curve is well-known concept in Euclidean $3-space$. We can say that evolute

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and involute are methods of deriving a new curve based on a given curve. The involute of a curve is called sometimes *evolvent* and evolvents play a part in the construction of gears. The evolute is the locus of the centers of osculating circles of the given planar curve [12]. Let α and α^* be the curves in Euclidean $3-space$. The tangent lines to a curve α generate a surface called the tangent surface of α. If a curve α^* is an involute of α, then by definition α is an evolute of α^*. Hence if we are given a curve α, then its evolutes are the curves whose tangent lines intersect α orthogonally. By using a similar method we produce a new ruled surface based on an other ruled surface. The differential geometric elements of the *involute* $\tilde{D}$ *scroll* are examined in [10]. It is well-known that if a curve is differentiable in an open interval at each point then a set of three mutually orthogonal unit vectors can be constructed. We say the set of these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curve. The set whose elements are frame vectors and curvatures of a curve α is called Frenet-Serret apparatus of the curve. Let Frenet vector fields of α be $V_1(s), V_2(s), V_3(s)$ and let first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. Then the quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are called the Frenet-Serret apparatus of the curves. If a rigid object moves along a regular curve described parametrically by $\alpha(s)$. then we know that this object has its own intrinsic coordinate system. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where curvature functions are defined by $k_1(s) = \|V_1(s)\|$, $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$.

Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. If the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* then the curve α^* is called the involute of the curve α, for $\forall s \in I$ provided that $\langle V_1, V_1^* \rangle = 0$. We can write

$$\alpha^*(s) = \alpha(s) + (c - s)V_1(s) \tag{1.1}$$

the distance between corresponding points of the involute curve in $\mathbb{E}^3$ is ([4],[8])

$$d\big(\alpha(s), \alpha^*(s)\big) = |c - s|, c = constant, \forall s \in I.$$

Theorem 1.1([4],[8]) *The Frenet vectors of the involute α^*, based on its evolute curve α are*

$$\begin{cases} V_1^* = V_2, \\ V_2^* = \dfrac{-k_1 V_1 + k_2 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \\ V_3^* = \dfrac{k_2 V_1 + k_1 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}}. \end{cases} \tag{1.2}$$

The first and the second curvatures of involute α^* are

$$k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{\lambda k_1}, \quad k_2^* = \frac{k_2' k_1 - k_1' k_2}{\lambda k_1 \left(k_1^2 + k_2^2\right)} = \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{\lambda k_1 \left(k_1^2 + k_2^2\right)}, \tag{1.3}$$

where $(\sigma - s)k_1 > 0,\ k_1 \neq 0.$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be two C^2- class differentiable unit speed curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^{**}(s), V_2^{**}(s), V_3^{**}(s)$ be the Frenet frames of the curves α and α^{**}, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^{**} of the curve α^{**}, then the pair (α, α^{**}) is called a Bertrand curve pair [4], [8]. Also α^{**} is called a Bertrand mate. If the curve α^{**} is a Bertrand mate of α then we may write

$$\alpha^{**}(s) = \alpha(s) + \lambda V_2(s) \tag{1.4}$$

If the curve α^{**} is Bertrand mate $\alpha(s)$ then we have

$$\langle V_1^{**}(s), V_1(s)\rangle = \cos\theta = constant.$$

Theorem 1.2([4],[8]) *The distance between corresponding points of the Bertrand curve pair in* $\mathbb{E}^3$ *is constant.*

Theorem 1.3([4]) *If the second curvature $k_2(s) \neq 0$ along a curve $\alpha(s)$ then $\alpha(s)$ is called a Bertrand curve provided that nonzero real numbers λ and β $\lambda k_1 + \beta k_2 = 1$ hold along the curve $\alpha(s)$ where $s \in I$. It follows that a circular helix is a Bertrand curve.*

Theorem 1.4([4]) *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be two C^2- class differentiable unit speed curves and let the quantities $\{V_1, V_2, V_3, k_1, k_2\}$ and $\{V_1^{**}, V_2^{**}, V_3^{**}, k_1^{**}, k_2^{**}\}$ be Frenet-Serret apparatus of the curves α and its Bertrand mate α^{**} respectively, then*

$$\begin{cases} V_1^{**} = \dfrac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \\ \\ V_2^{**} = V_2, \\ \\ V_3^{**} = \dfrac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}};\ \lambda k_2 > 0 \end{cases} \tag{1.5}$$

The first and the second curvatures of the offset curve α^{**} are given by

$$\begin{cases} k_1^{**} = \dfrac{\beta k_1 - \lambda k_2}{\left(\lambda^2 + \beta^2\right) k_2} = \dfrac{k_1 - \lambda\left(k_1^2 + k_2^2\right)}{\left(\lambda^2 + \beta^2\right) k_2^2}, \\ k_2^{**} = \dfrac{1}{\left(\lambda^2 + \beta^2\right) k_2}. \end{cases} \tag{1.6}$$

Due to this theorem, we can write

$$\beta k_1 - \lambda k_2 = m \Longrightarrow \frac{k_2^{**}}{k_1^{**}} = \frac{1}{\beta k_1 - \lambda k_2} = \frac{1}{m},$$

$$\left(\frac{k_2^{**}}{k_1^{**}}\right)' = \frac{-m'}{m^2 k_2 \sqrt{\lambda^2 + \beta^2}} \Longrightarrow \frac{ds}{ds^{**}} = \frac{1}{k_2 \sqrt{\lambda^2 + \beta^2}}.$$

A differentiable one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns to each $u \in I$ a point $\alpha(u) \in \mathbb{R}^3$ and a vector $X(u) \in \mathbb{R}^3, X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiable on u. For each $u \in I$, the line L which passes through $\alpha(u)$ and is parallel to $X(u)$ is called the line of the family at u. Given a one-parameter family of lines $\{\alpha(u), X(u)\}$ the parameterized surface

$$\varphi(u,v) = \alpha(u) + v.X(u) \ \ where\ u \in I \ \ and\ v \in \mathbb{R} \tag{1.7}$$

is called the *ruled surface* generated by the family $\{\alpha(u), X(u)\}$. The lines L are called the rulings and the curve $\alpha(u)$ is called an anchor of the surface φ, [2].

Theorem 1.5([2]) *The striction point on a ruled surface $\varphi(u,v) = \alpha(u) + v.X(u)$ is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given by*

$$c(u) = \alpha(u) - \frac{\langle \alpha_u', X_u' \rangle}{\langle X_u', X_u' \rangle}.X(u) \tag{1.8}$$

where $X_u' = D_T X(u)$.

§2. On the Tangent Vector Fields of Striction Curves Along the Involute and Bertrandian Frenet Ruled Surfaces

Definition 2.1 *In the Euclidean $3-$ space, let $\alpha(s)$ be the arc length curve. The equations*

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \end{cases} \tag{2.1}$$

are the parametrization of the ruled surface which is called $V_1 - scroll$ (tangent ruled surface), $V_2 - scroll$ (normal ruled surface) and $V_3 - scroll$ (binormal ruled surface) respectively in [6].

Theorem 2.1([6]) *The striction curves of Frenet ruled surfaces are given by the following* *matrix*

$$\begin{bmatrix} c_1-\alpha \\ c_2-\alpha \\ c_3-\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{k_2^2+k_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Theorem 2.2 *The tangent vector fields T_1, T_2 and T_3 belonging to striction curves of Frenet ruled surface is given by*

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where

$$a = \frac{k_2^2}{\eta\|c_2'(s)\|}, \quad b = \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}, \quad c = \frac{k_1k_2}{\eta\|c_2'(s)\|} \text{ and } \eta = k_1^2 + k_2^2.$$

Proof It is easy to give this matrix because we have already got the following equalities

$$T_1(s) = T_3(s) = \alpha'(s) = V_1.$$

Since $c_2(s) = \alpha(s) + \dfrac{k_1}{k_1^2+k_2^2}V_2$, where $k_1^2 + k_2^2 = \eta \neq 0$, hence we have

$$\begin{aligned} c_2'(s) &= \frac{k_2^2}{\eta}V_1 + \left(\frac{k_1}{\eta}\right)' V_2 + \frac{k_1k_2}{\eta}V_3, \\ T_2(s) &= \frac{c_2'(s)}{\|c_2'(s)\|} = \frac{\eta k_2^2 V_1 + (k_1'\eta - k_1\eta')V_2 + \eta k_2 k_1 V_3}{\left(\eta^3 k_2^4 + (k_1'\eta k_1 \eta')^2\right)^{\frac{1}{2}}}. \end{aligned}$$

□

2.1 Involute Frenet Ruled Surfaces

In this subsection, first we give the tangent, normal and binormal Frenet ruled surfaces of the involute-evolute curves. Further we write their parametric equations in terms of the Frenet apparatus of the involute-evolute curves. Hence they are called *involute Frenet ruled surfaces* as in the following way.

Definition 2.2([6]) *In the Euclidean* 3*-space, let* $\alpha(s)$ *be the arc length curve. The equations*

$$\begin{aligned}
\varphi_1^*(s,v_1) &= \alpha^*(s) + v_1 V_1^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_1 V_2(s),\\
\varphi_2^*(s,v_2) &= \alpha^*(s) + v_2 V_2^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_2\left(\frac{-k_1V_1 + k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right),\\
\varphi_3^*(s,v_3) &= \alpha^*(s) + v_3 V_3^*(s) = \alpha(s) + (\sigma - s)V_1(s) + v_3\left(\frac{k_2V_1 + k_1V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right)
\end{aligned}$$

are the parametrization of the ruled surfaces which are called involute tangent ruled surface, involute normal ruled surface and involute binormal ruled surface, respectively.

We can deduce from Theorem 2.1 striction curves of the involute Frenet ruled surfaces are given by the following matrix

$$\begin{bmatrix} c_1^* - \alpha^* \\ c_2^* - \alpha^* \\ c_3^* - \alpha^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^*}{k_1^{*2}+k_2^{*2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

It is easy to give the following matrix for the striction curves of four Frenet ruled surfaces along the *involute* curve α^*.

$$\begin{aligned}
c_1^*(s) &= c_3^*(s) = \alpha^*(s),\\
c_2^*(s) &= \alpha^*(s) + \frac{k_1^*}{k_1^{*2}+k_2^{*2}} V_2^*(s).
\end{aligned}$$

Also we can write explicit equations of the striction curves on involute Frenet ruled surfaces in terms of Frenet apparatus of an evolute curve α.

Theorem 2.3 *The equations of the striction curves on involute Frenet ruled surfaces in terms of Frenet apparatus of an evolute curve* α *are given by*

$$\begin{bmatrix} c_1^* - \alpha \\ c_2^* - \alpha \\ c_3^* - \alpha \end{bmatrix} = \begin{bmatrix} \sigma - s & 0 & 0 \\ (\sigma - s)\left(1 - \frac{k_1^2}{(k_1^2+k_2^2)(1+m)}\right) & 0 & \dfrac{(\sigma - s)k_1k_2}{(k_1^2+k_2^2)(1+m)} \\ \sigma - s & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Theorem 2.4 *The tangent vector fields* ${T_1}^*, {T_2}^*, {T_3}^*$ *of striction curves belonging to an involute Frenet ruled surface in terms of Frenet apparatus by themselves are given by*

$$[T^*] = \begin{bmatrix} {T_1}^* \\ {T_2}^* \\ {T_3}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} {V_1}^* \\ {V_2}^* \\ {V_3}^* \end{bmatrix}.$$

$$a^* = \frac{k_2^{*2}}{\eta^* \left\| c_2^{*\prime}(s) \right\|}, \;\; b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\| c_2^{*\prime}(s) \right\|}, \;\; c^* = \frac{k_1^* k_2^*}{\eta^* \left\| c_2^{*\prime}(s) \right\|}, \;\; \eta^* = k_1^{*2} + k_2^{*2}, \;\; \mu^* = \left(\frac{k_2^*}{k_1^*}\right)'.$$

2.2 Bertrandian Frenet ruled surfaces

In this subsection, first we give the tangent, normal and binormal Frenet ruled surfaces of the Bertrand mate α^{**}. Further we write their parametric equations in terms of the Frenet apparatus of the Bertrand curve α. Hence they are called *Bertrandian Frenet ruled surfaces* as in the following way.

Definition 2.3([6]) *In the Euclidean $3-$space, let $\alpha(s)$ be the arc length curve. The equations*

$$\begin{aligned} \varphi_1^{**}(s, w_1) &= \alpha^{**}(s) + w_1 V_1^{**}(s) = \alpha + \lambda V_2 + w_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \\ \varphi_2^{**}(s, w_2) &= \alpha^{**}(s) + w_2 V_2^{**}(s) = \alpha + (\lambda + w_2) V_2, \\ \varphi_3^{**}(s, w_3) &= \alpha^{**}(s) + w_3 V_3^{**}(s) = \alpha + \lambda V_2 + w_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}} \right), \end{aligned} \tag{2.2}$$

are the parametrization of the ruled surfaces which are called Bertrandian tangent ruled surface, Bertrandian normal ruled surface and Bertrandian binormal ruled surface, respectively.

We can also deduce from Theorem 2.1 the striction curves of Bertrand Frenet ruled surfaces are given by the following matrix

$$\begin{bmatrix} c_1^{**} - \alpha^{**} \\ c_2^{**} - \alpha^{**} \\ c_3^{**} - \alpha^{**} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^{**}}{k_1^{**2} + k_2^{**2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}.$$

It is easy to give the following matrix for the striction curves belonging to Bertrand Frenet ruled surfaces

$$\begin{aligned} c_1^{**}(s) &= c_3^{**}(s) = \alpha^{**}(s) \\ c_2^{**}(s) &= \alpha^{**}(s) + \frac{k_1^{**}}{k_1^{**2} + k_2^{**2}} V_2^{**}(s) \end{aligned}$$

Theorem 2.5 *The equations of the striction curves on Bertrandian Frenet ruled surfaces in terms of Frenet apparatus of curve α*

$$\begin{bmatrix} c_1^{**} - \alpha \\ c_2^{**} - \alpha \\ c_3^{**} - \alpha \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & \left(\lambda + \frac{m(\lambda^2 + \beta^2) k_2}{(m^2 + 1)} \right) & 0 \\ 0 & \lambda & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Proof Since the equations of the striction curves on Bertrandian Frenet ruled surfaces in terms of Frenet apparatus of curve α are

$$c_1^{**}(s) \quad = \quad c_3^{**}(s) = \alpha^{**}(s) = \alpha(s) + \lambda V_2(s)$$

the first and the second curvatures of the curve α^{**} are given by $k_1^{**} = \dfrac{\beta k_1 - \lambda k_2}{(\lambda^2+\beta^2)\,k_2}$ and $k_2^{**} = \dfrac{1}{(\lambda^2+\beta^2)\,k_2}$. Also $k_2 k_2^{**} = \dfrac{1}{(\lambda^2+\beta^2)}$ and

$$c_2^{**}(s) = \alpha^{**}(s) + \frac{k_1^{**}}{k_1^{**2}+k_2^{**2}} V_2^{**}(s) = \alpha + \left(\lambda + \frac{\left(\lambda^2+\beta^2\right)k_2}{\left((\beta k_1 - \lambda k_2)^2 + 1\right)}\right) V_2. \qquad \square$$

Theorem 2.6 *The tangent vector fields* T_1^{**}, T_2^{**} *and* T_3^{**} *of striction curves belonging to Bertrandian Frenet ruled surface are given by*

$$\begin{bmatrix} T_1^{**} \\ T_2^{**} \\ T_3^{**} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}$$

where

$$a^{**} = \frac{k_2^{**2}}{\eta^{**}\left\|c_2^{**\prime}(s)\right\|},\ \ b^{**} = \frac{\left(\frac{k_1^{**}}{\eta^{**}}\right)'}{\left\|c_2^{**\prime}(s)\right\|},\ \ c^{**} = \frac{k_1^{**}k_2^{**}}{\eta^{**}\left\|c_2^{**\prime}(s)\right\|}\ \ \text{and}\ \ \eta^{**} = k_1^{**2} + k_2^{**2}.$$

Theorem 2.7 *The product of tangent vector fields* ${T_1}^*, {T_2}^*, {T_3}^*$ *and tangent vector fields* ${T_1}^{**}, {T_2}^{**}, {T_3}^{**}$ *of striction curves on an involute and Bertrandian Frenet ruled surface respectively, are given by*

$$[T^*]\,[T^{**}]^{\mathbf{T}} = A\begin{bmatrix} 0 & Ab^{**} & 0 \\ B & a^{**}B + b^{**}a^*A + c^{**}C & B \\ 0 & b^{**}A & 0 \end{bmatrix}$$

where the coefficients are

$$A = \sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)},\ \ B = b^*(-\beta k_1 + \lambda k_2) + c^*,\ \ C = b^* + c^*(-\lambda k_2 + \beta k_1).$$

Proof Let $[T^*] = [A^*]\,[V^*]$ and $[T^{**}] = [A^{**}]\,[V^{**}]$ be given. By using the properties of a matrix following result can be obtained:

$$
\begin{aligned}
[T^*]\,[T^{**}]^{\mathbf{T}} &= [A^*]\,[V^*]\,([A^{**}]\,[V^{**}])^{\mathbf{T}} \\
&= [A^*]\left([V^*]\,[V^{**}]^{\mathbf{T}}\right)[A^{**}]^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} {V_1}^* \\ {V_2}^* \\ {V_3}^* \end{bmatrix}
\left(\begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} {V_1}^{**} \\ {V_2}^{**} \\ {V_3}^{**} \end{bmatrix}\right)^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} {V_1}^* \\ {V_2}^* \\ {V_3}^* \end{bmatrix}
\begin{bmatrix} {V_1}^{**} \\ {V_2}^{**} \\ {V_3}^{**} \end{bmatrix}^{\mathbf{T}}
\begin{bmatrix} 1 & 0 & 0 \\ a^{**} & b^{**} & c^{**} \\ 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}} \\
&= A \begin{bmatrix} 0 & b^{**}A & 0 \\ B & a^{**}B + b^{**}a^*A + c^{**}C & B \\ 0 & b^{**}A & 0 \end{bmatrix}. \qquad \square
\end{aligned}
$$

As a result of Theorem 2.1 we can write that in the Euclidean $3-space$, the position of the unit tangent vector field T_1^*, T_2^*, T_3^* and $T_1^{**}, T_2^{**}, T_3^{**}$ of striction curves belonging to ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*$ and $\varphi_1^{**}, \varphi_2^{**}, \varphi_3^{**}$ respectively, along the curve α^* and α^{**}, can be expressed by the following equations

$$
[T^*]\,[T^{**}]^T = \begin{bmatrix} \langle T_1^*, T_1^{**}\rangle & \langle T_1^*, T_2^{**}\rangle & \langle T_1^*, T_3^{**}\rangle \\ \langle T_2^*, T_1^{**}\rangle & \langle T_2^*, T_2^{**}\rangle & \langle T_2^*, T_3^{**}\rangle \\ \langle T_3^*, T_1^{**}\rangle & \langle T_3^*, T_2^{**}\rangle & \langle T_3^*, T_3^{**}\rangle \end{bmatrix},
$$

here $[T^{**}]^T$ is the transpose matrix of $[T^{**}]$.

Hence we may write that, there are four tangent vector fields on striction curves which are perpendicular to each other, for the involute and Bertrandian Frenet ruled surfaces given above. Since $\langle T_1^*, T_1^{**}\rangle = \langle T_1^*, T_3^{**}\rangle = \langle T_3^*, T_1^{**}\rangle = \langle T_3^*, T_3^{**}\rangle = 0$, it is trivial.

Theorem 2.8 (i) *The tangent vector fields of striction curves on an involute tangent and Bertrandian normal ruled surfaces are perpendicular under the condition*

$$
\left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2 + \beta^2)k_2}{(\beta k_1 - \lambda k_2)^2 + 1}\right]' = 0, \;\; \lambda^2 = -\beta^2 \;\; or \;\; {k_1}^2 = -{k_2}^2.
$$

(ii) *The tangent vector fields of striction curves on an involute binormal and Bertrandian normal ruled surfaces are perpendicular under the condition*

$$
\left[\frac{(\beta k_1 - \lambda k_2)(\lambda^2 + \beta^2)k_2}{(\beta k_1 - \lambda k_2)^2 + 1}\right]' = 0, \;\; \lambda^2 = -\beta^2 \;\; or \;\; {k_1}^2 = -{k_2}^2.
$$

Proof (*i*) Since $\langle T_1^*, T_2^{**}\rangle = b^{**}A$ and $\langle T_1^*, T_2^{**}\rangle = 0$

$$\begin{aligned} b^{**}A &= 0 \\ \left[\frac{(\beta k_1-\lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1-\lambda k_2)^2+1}\right]'\sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)} &= 0 \end{aligned}$$

$$\left[\frac{(\beta k_1-\lambda k_2)(\lambda^2+\beta^2)k_2}{(\beta k_1-\lambda k_2)^2+1}\right]' = 0 \ \ or \ \ \sqrt{(\lambda^2+\beta^2)({k_1}^2+{k_2}^2)} = 0,$$

this completes the proof.

(*ii*) Since $\langle T_1^*, T_2^{**}\rangle = \langle T_3^*, T_2^{**}\rangle = b^{**}A$, the proof is trivial. □

Theorem 2.9 (*i*) *The tangent vector fields of striction curves on an involute normal and Bertrandian tangent ruled surfaces are perpendicular under the condition*

$$-\beta k_1+\lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'}.$$

(*ii*) *The tangent vector fields of striction curves on an involute normal and Bertrandian binormal ruled surfaces are perpendicular under the condition*

$$-\beta k_1+\lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'}.$$

Proof (*i*) Since $\langle T_2^*, T_1^{**}\rangle = B = b^*(-\beta k_1+\lambda k_2)+c^*$ and $\langle T_2^*, T_1^{**}\rangle = 0$

$$\begin{aligned} B = b^*(-\beta k_1+\lambda k_2)+c^* &= 0 \\ \beta k_1-\lambda k_2+\frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'} &= 0 \end{aligned}$$

$$-\beta k_1+\lambda k_2 = \frac{k_2^2(\frac{k_1}{k_2})'(k_1^2+k_2^2)^{\frac{3}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]\left(\frac{\lambda k_1(k_1^2+k_2^2)^{\frac{5}{2}}}{\left[(k_1^2+k_2^2)^3+k_2^4(\frac{k_1}{k_2})'^2\right]}\right)'},$$

this completes the proof.

(*ii*) Since $\langle T_2^*, T_1^{**}\rangle = \langle T_2^*, T_3^{**}\rangle = B = b^*(-\beta k_1+\lambda k_2)+c^*$, the proof is trivial. □

Corollary 2.1 *The inner product between tangent vector fields of striction curves on an involute* *normal and Bertrandian normal ruled surfaces of the* (α^*, α^{**}) *is*

$$\langle T_2^*, T_2^{**}\rangle = a^{**}B + b^{**}a^*A + c^{**}C.$$

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On the Leap Zagreb Indices of Generalized xyz-Point-Line Transformation Graphs $T^{xyz}(G)$ when $z=1$

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Abstract: For a graph G, the first, second and third leap Zagreb indices are the sum of squares of 2-distance degree of vertices of G; the sum of product of 2-distance degree of end vertices of edges in G and the sum of product of 1-distance degree and 2-distance degrees of vertices of G, respectively. In this paper, we obtain the expressions for these three leap Zagreb indices of generalized xyz point line transformation graphs $T^{xyz}(G)$ when $z=1$.

Key Words: Distance, degree, diameter, Zagreb index, leap Zagreb index, reformulated Zagreb index.

AMS(2010): 05C90, 05C35, 05C12, 05C07.

§1. Introduction

Let $G=(V,E)$ be a simple graph of order n and size m. The *k-distance degree of a vertex* $v \in V(G)$, denoted by $d_k(v/G) = |N_k(v/G)|$ where $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ [17] in which $d(u,v)$ is the distance between the vertices u and v in G that is the length of the shortest path joining u and v in G. The *degree of a vertex* v in a graph G is the number of edges incident to it in G and is denoted by $d_G(v)$. Here $N_1(v/G)$ is nothing but $N_G(v)$ and $d_1(v/G)$ is same as $d_G(v)$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree of an edge* $e=uv$ in G, denoted by $d_1(e/G)$ (or $d_G(e)$), is defined by $d_1(e/G) = d_1(u/G) + d_1(v/G) - 2$.

The *complement* of a graph G is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G. $\overline{G}$ has n vertices and $\frac{n(n-1)}{2} - m$ edges. The *line graph* $L(G)$ of a graph G with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex incident in common. The *complement of line graph* $\overline{L(G)}$ *or jump graph* $J(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $J(G)$ are adjacent whenever the corresponding edges in G have no vertex incident in common. The *subdivision graph* $S(G)$ of a graph G whose vertex set is $V(G)\bigcup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is

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an edge of G incident with it. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G whose vertex set is $V(G)\bigcup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G non incident with it.

We follow [11] and [13] for unexplained graph theoretic terminologies and notations.

The first and second Zagreb indices [9] of a graph G are defined as follows:

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v),$$

respectively. These are widely studied degree based topological indices due to their applications in chemistry. For details see the papers [5, 7, 8, 10, 18]. The first Zagreb index [15] can also be expressed as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]$$

Ashrafi et al. [1] defined the first and second Zagreb coindices as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M_2}(G) = \sum_{uv\notin E(G)} [d_G(u)d_G(v)],$$

respectively.

In 2004, Milićević et al. [14] reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The first and second reformulated Zagreb indices are defined, respectively, as

$$EM_1(G) = \sum_{e\in E(G)} d_G(e)^2 \quad \text{and} \quad EM_2(G) = \sum_{e\sim f} [d_G(e)d_G(f)]$$

In [12], Hosamani and Trinajstić defined the first and second reformulated Zagreb coindices respectively as

$$\begin{aligned} \overline{EM_1}(G) &= \sum_{e\sim f} [d_G(e) + d_G(f)], \\ \overline{EM_2}(G) &= \sum_{e\nsim f} [d_G(e) + d_G(f)]. \end{aligned}$$

In 2017, Naji et al. [16] introduced the leap Zagreb indices. For a graph G, the first, second, and third leap Zagreb indices [16] are denoted and defined respectively as:

$$\begin{aligned} LM_1(G) &= \sum_{v\in V(G)} d_2(v/G)^2, \\ LM_2(G) &= \sum_{uv\in E(G)} d_2(u/G)d_2(v/G), \\ LM_3(G) &= \sum_{v\in V(G)} d_1(v/G)d_2(v/G). \end{aligned}$$

Throughout this paper, in our results we write the notations $d_1(v)$ and $d_1(e)$ respectively for degree of a vertex v and degree of an edge e of a graph.

§2. Generalized xyz-Point-Line Transformation Graph $T^{xyz}(G)$

The procedure of obtaining a new graph from a given graph by using incidence (or nonincidence) relation between vertex and an edge and an adjacency (or nonadjacency) relation between two vertices or two edges of a graph is known as *graph transformation* and the graph obtained by doing so is called a transformation graph. For a graph $G = (V, E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The following graph operations depending on $x, y, z \in \{0, 1, +, -\}$ induce functions $T^{xyz} : \mathcal{G} \to \mathcal{G}$. These operations are introduced by Deng et al. in [6]. They called these resulting graphs as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$ and studied the Laplacian characteristic polynomials and some other Laplacian parameters of xyz-transformations of an r-regular graph G. In [2], Wu Bayoindureng et al. introduced the total transformation graphs and studied the basic properties of total transformation graphs. Motivated by this, Basavanagoud [3] studied the basic properties of the xyz-transformation graphs by calling them xyz-point-line transformation graphs by changing the notion of xyz-transformations of a graph G as $T^{xyz}(G)$ to avoid confusion between parent graph G and its xyz-transformations.

Definition 2.1([6]) *Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.*

Since there are 64 distinct 3 - permutations of $\{0, 1, +, -\}$. Thus obtained 64 kinds of generalized xyz-point-line transformation graphs. There are 16 different graphs for each case when $z = 0$, $z = 1$, $z = +$, $z = -$.

In this paper, we consider the xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = 1$.

Example 2.1 Let $G = K_2 \cdot K_3$ be a graph. Then G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$ which are depicted in the following Figure 1.

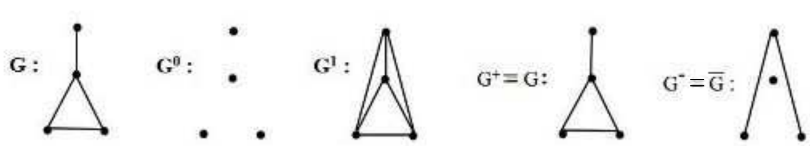

Figure 1

The self-explanatory examples of the path P_4 and its *xyz-point-line transformation graphs* $T^{xy1}(P_4)$ are depicted in Figure 2.

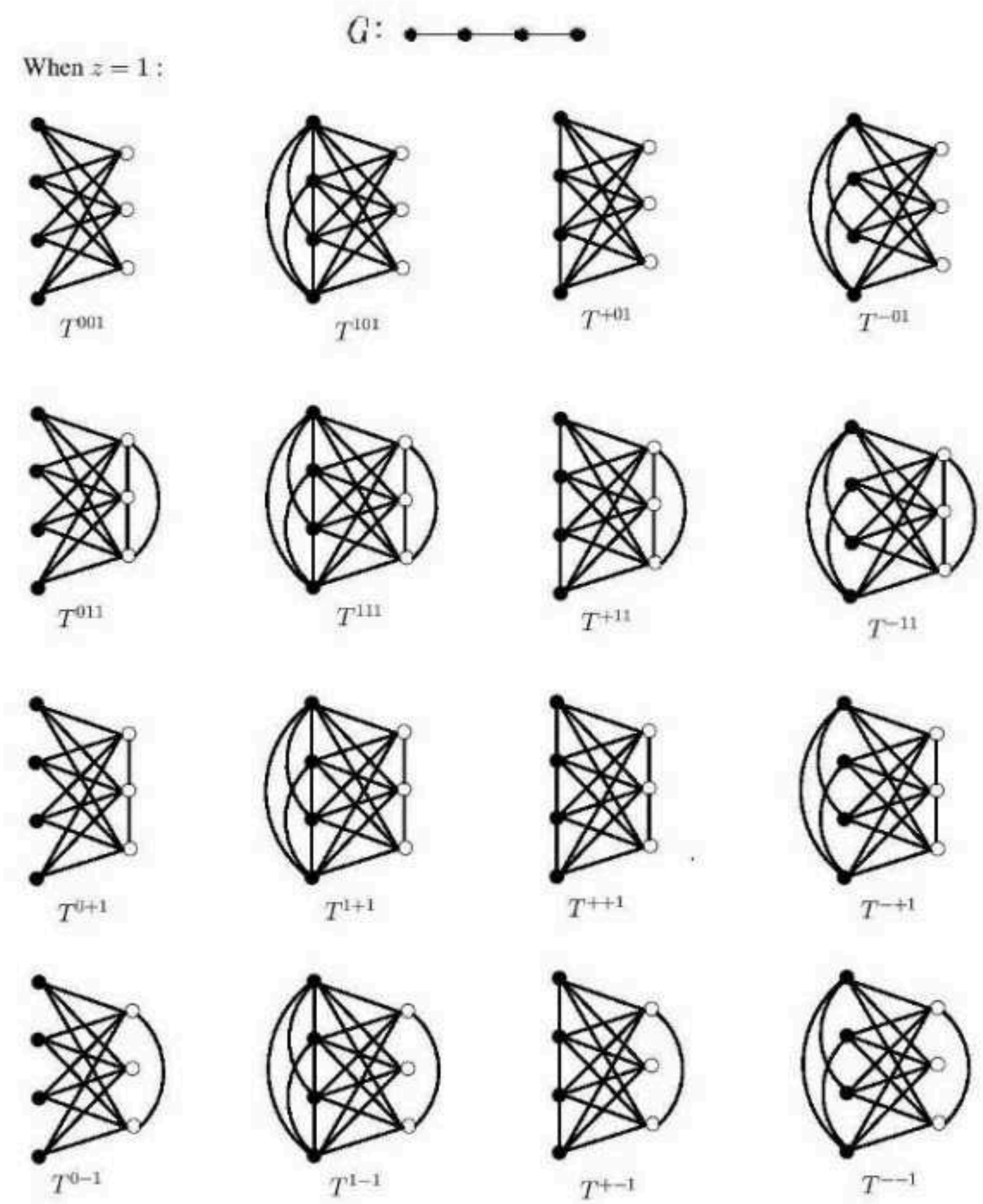
G:
When z = 1 :
T001
T101
T+01
T−01
T011
T111
T+11
T−11
T0+1
T1+1
T++1
T−+1
T0−1
T1−1
T+−1
T−−1

Figure 2

§3. Leap Zagreb Indices of $T^{xy1}(G)$

Theorem 3.1([3]) *Let G be a graph of order n and size m. Then*

(1) $|V(T^{xyz}(G))| = n + m$;

(2) $|E(T^{xyz}(G))| = |E(G^x)| + |E(L(G)^y)| + |E(W)|$, *where*

$$|E(G^x)| = \begin{cases} 0 & if\ x = 0. \\ \binom{n}{2} & if\ x = 1. \\ m & if\ x = +. \\ \binom{n}{2} - m & if\ x = -. \end{cases}$$

$$|E(L(G)^y)| = \begin{cases} 0 & if\ y = 0. \\ \binom{m}{2} & if\ y = 1. \\ -m + \frac{1}{2}M_1 & if\ y = +. \\ \binom{m+1}{2} - \frac{1}{2}M_1 & if\ y = -. \end{cases}$$

$$|E(W)| = \begin{cases} 0 & if\ z = 0. \\ mn & if\ z = 1. \\ m & if\ z = +. \\ m(n-2) & if\ z = -. \end{cases}$$

The following Propositions are useful for calculating $d_2(T^{xy1}(G))$ in Observation 3.4.

Proposition 3.2([4]) *Let G be a graph of order n and size m. Let v be a vertex of G. Then*

$$d_{T^{xy1}(G)}(v) = \begin{cases} m & if\ x = 0, y \in \{0, 1, +, -\} \\ n + m - 1 & if\ x = 1, y \in \{0, 1, +, -\} \\ m + d_G(v) & if\ x = +, y \in \{0, 1, +, -\} \\ n + m - 1 - d_G(v) & if\ x = -, y \in \{0, 1, +, -\} \end{cases}$$

Proposition 3.3([4]) *Let G be a graph of order n and size m. Let e be an edge of G. Then*

$$d_{T^{xy1}(G)}(e) = \begin{cases} n & if\ y = 0, x \in \{0, 1, +, -\} \\ n + m - 1 & if\ y = 1, x \in \{0, 1, +, -\} \\ n + d_G(e) & if\ y = +, x \in \{0, 1, +, -\} \\ n + m - 1 - d_G(e) & if\ y = -, x \in \{0, 1, +, -\} \end{cases}$$

Observation 3.4 Let G be a connected (n, m) graph. Then

(1) $d_2(v/T^{001})(G) = \begin{cases} (n-1) & if\ v \in V(G) \\ (m-1) & if\ v = e \in E(G) \end{cases}$

(2) $d_2(v/T^{101})(G)=\begin{cases} 0 & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(3) $d_2(v/T^{+01})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(4) $d_2(v/T^{-01})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(5) $d_2(v/T^{011})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(6) $d_2(v/T^{111})(G)=\begin{cases} 0 & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(7) $d_2(v/T^{+11})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(8) $d_2(v/T^{-11})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(9) $d_2(v/T^{0+1})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(10) $d_2(v/T^{1+1})(G)=\begin{cases} 0 & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(11) $d_2(v/T^{++1})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(12) $d_2(v/T^{-+1})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(13) $d_2(v/T^{0-1})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(14) $d_2(v/T^{1-1})(G)=\begin{cases} 0 & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(15) $d_2(v/T^{+-1})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(16) $d_2(v/T^{+-1})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

The above Observation 3.4 is useful for computing leap Zagreb indices of transformation graphs $T^{xy1}(G)$ in the forthcoming theorems.

Theorem 3.5 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{001}(G)) = n(n-1)^2 + m(m-1)^2$;

(2) $LM_2(T^{001}(G)) = mn(m-1)(n-1)$;

(3) $LM_3(T^{001}(G)) = mn(m+n-2)$.

Proof The graph $T^{001}(G)$ has $n+m$ vertices and mn edges, refer Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned} LM_1(T^{001}(G)) &= \sum_{v\in V(T^{001}(G))} d_2(v/T^{001}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{001}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{001}(G))^2 \\ &= n(n-1)^2 + m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{001}(G)) &= \sum_{uv\in E(T^{001}(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= \sum_{uv\in E(S(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= (n-1)(m-1)2m + (n-1)(m-1)(mn-2m) = mn(n-1)(m-1). \end{aligned}$$

$$\begin{aligned} LM_3(T^{001}(G)) &= \sum_{v\in V(T^{001}(G))} \left[d_1(v/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{001}(G))\right]\left[d_2(e/T^{001}(G))\right] \\ &= mn(n-1) + mn(m-1) = mn(m+n-2). \end{aligned}$$

□

Theorem 3.6 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{101}(G)) = m(m-1)^2$;

(2) $LM_2(T^{101}(G)) = 0$;

(3) $LM_3(T^{101}(G)) = mn(m-1)$.

Proof Notice that the graph $T^{101}(G)$ has $n+m$ vertices and $mn + \frac{n(n-1)}{2}$ edges by Theorem 3.1. According to the definitions of first, second and third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4, calculation shows the following.

$$\begin{aligned}
LM_1(T^{101}(G)) &= \sum_{v\in V(T^{101}(G))} d_2(v/T^{101}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{101}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{101}(G))^2 \\
&= m(m-1)^2.
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{101}(G)) &= \sum_{uv\in E(T^{101}(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&\quad + \sum_{uv\notin E(G)} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] = 0.
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{101}(G)) &= \sum_{v\in V(T^{101}(G))} \left[d_1(v/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{101}(G))\right]\left[d_2(e/T^{101}(G))\right] = mn(m-1). \qquad \square
\end{aligned}$$

Theorem 3.7 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+01}(G)) = n(n-1)^2 + m(m-1)^2 + M_1(G) - 4m(n-1)$;

(2) $LM_2(T^{+01}(G)) = M_2(G) - (n-1)M_1(G) + m[(n-1)^2 + (m-1)(n^2-n-2m)]$;

(3) $LM_3(T^{+01}(G)) = m[n(n+m) - 2(m+1)] - M_1(G)$.

Proof By Theorem 3.1, we know that the graph $T^{+01}(G)$ has $n+m$ vertices and $m(n+1)$ edges. By using the definitions of first, second and third leap Zagreb indices and applying

Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned} LM_1(T^{+01}(G)) &= \sum_{v\in V(T^{+01}(G))} d_2(v/T^{+01}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{+01}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{+01}(G))^2 \\ &= \sum_{v\in V(G)} (n-1-d_1(v/G))^2 + \sum_{e\in E(G)} (m-1)^2 \\ &= \sum_{v\in V(G)} \left[(n-1)^2 + d_1(v/G)^2 - 2(n-1)d_1(v/G)\right] + \sum_{e\in E(G)} (m-1)^2 \\ &= n(n-1)^2 + m(m-1)^2 + M_1(G) - 4m(n-1). \end{aligned}$$

$$\begin{aligned} LM_2(T^{+01}(G)) &= \sum_{uv\in E(T^{+01}(G))} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} [d_2(u/T^{+01}(G))][d_2(v/T^{+01}(G))] \\ &= \sum_{uv\in E(G)} \left[(n-1)^2 - (n-1)(d_1(u/G)+d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right] \\ &\quad + \sum_{uv\in E(S(G))} (m-1)(n-1-d_1(u/(G)) \\ &\quad + \sum_{uv\in E(\overline{S}(G))} (m-1)(n-1-d_1(u/(G)) \\ &= M_2(G) - (n-1)M_1(G) + m[(n-1)^2 + (m-1)(n^2-n-2m)]. \end{aligned}$$

$$\begin{aligned} LM_3(T^{+01}(G)) &= \sum_{v\in V(T^{+01}(G))} \left[d_1(v/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+01}(G))\right]\left[d_2(e/T^{+01}(G))\right] \\ &= \sum_{v\in V(G)} [(m+d_1(v/G))(n-1-d_1(v/G))] + \sum_{e\in E(G)} n(m-1) \\ &= m[n(n+m) - 2(m+1)] - M_1(G). \qquad \square \end{aligned}$$

Theorem 3.8 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{-01}(G)) = M_1(G) + m(m-1)^2$;

(2) $LM_2(T^{-01}(G)) = \overline{M_2}(G) + 2m^2(m-1)$;

(3) $LM_3(T^{-01}(G)) = m[n(m+1) + 2(m-1)] - M_1(G)$.

Proof We know the graph $T^{-01}(G)$ has $n+m$ vertices and $(n-1)(\frac{n}{2}+m)$ edges, refer Theorem 3.1. By definitions of the first, second and third leap Zagreb indices and applying Propositions 3.2, 3.3 and Observation 3.4 we have the following.

$$\begin{aligned} LM_1(T^{-01}(G)) &= \sum_{v\in V(T^{-01}(G))} d_2(v/T^{-01}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{-01}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-01}(G))^2 \\ &= M_1(G) + m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{-01}(G)) &= \sum_{uv\in E(T^{-01}(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{uv\notin E(G)} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{uv\notin E(G)} [d_1(u/G)][d_1(v/G)] + \sum_{uv\in E(S(G))} (m-1)d_1(u/G) \\ &\quad + \sum_{uv\in E(\overline{S}(G))} (m-1)d_1(u/G) \\ &= \overline{M_2}(G) + 2m^2(m-1). \end{aligned}$$

$$\begin{aligned} LM_3(T^{-01}(G)) &= \sum_{v\in V(T^{-01}(G))} \left[d_1(v/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{-01}(G))\right]\left[d_2(e/T^{-01}(G))\right] \\ &= m[n(m+1) + 2(m-1)] - M_1(G). \end{aligned}$$

□

Theorem 3.9 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{011}(G)) = n(n-1)^2$;

(2) $LM_2(T^{011}(G)) = 0$;

(3) $LM_3(T^{011}(G)) = mn(n-1)$.

Proof We are easily know that the graph $T^{011}(G)$ has $n+m$ vertices and $m(\frac{m-1}{2}+n)$

edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions $3.2, 3.3$ and Observation 3.4 we know the following.

$$\begin{aligned} LM_1(T^{011}(G)) &= \sum_{v\in V(T^{011}(G))} d_2(v/T^{011}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{011}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{011}(G))^2 \\ &= n(n-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{011}(G)) &= \sum_{uv\in E(T^{011}(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &= \sum_{uv\in E(L(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] = 0. \end{aligned}$$

$$\begin{aligned} LM_3(T^{011}(G)) &= \sum_{v\in V(T^{011}(G))} \left[d_1(v/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{011}(G))\right]\left[d_2(e/T^{011}(G))\right] = mn(n-1). \end{aligned}$$ □

Theorem 3.10 *Let G be (n,m) graph. Then*

$$LM_1(T^{111}(G)) = LM_2(T^{111}(G)) = LM_3(T^{111}(G)) = 0.$$

Proof Notice that the graph $T^{111}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2}+\frac{m(m-1)}{2}+mn$ edges by Theorem 3.1. By definitions of the first, second and third leap Zagreb indices along with Propositions $3.2, 3.3$ and Observation 3.4, we get similarly the desired result as the proof of above theorems. □

Theorem 3.11 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+11}(G)) = (n-1)(n^2-n-4m)+M_1(G)$;
(2) $LM_2(T^{+11}(G)) = m(n-1)^2-(n-1)M_1(G)+M_2(G)$;
(3) $LM_3(T^{+11}(G)) = m[(n-1)(n+2)-2m]-M_1(G)$.

Proof Clearly, the graph $T^{+11}(G)$ has $n+m$ vertices and $\frac{m(m+1)}{2}+mn$ edges by Theorem

3.1. By definitions of the first, second and the third leap Zagreb indices, we get the following by applying Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned}
LM_1(T^{+11}(G)) &= \sum_{v\in V(T^{+11}(G))} d_2(v/T^{+11}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{+11}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{+11}(G))^2 \\
&= \sum_{v\in V(G)} [(n-1)^2 + d_1(v/G)^2 - 2(n-1)d_1(v/G)] \\
&= (n-1)(n^2-n-4m) + M_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{+11}(G)) &= \sum_{uv\in E(T^{+11}(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&= m(n-1)^2 - (n-1)M_1(G) + M_2(G).
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{+11}(G)) &= \sum_{v\in V(T^{+11}(G))} \left[d_1(v/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+11}(G))\right]\left[d_2(e/T^{+11}(G))\right] \\
&= m[(n-1)(n+2)-2m] - M_1(G).
\end{aligned}$$

□

Theorem 3.12 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{-11}(G)) = M_1(G)$;

(2) $LM_2(T^{-11}(G)) = \overline{M_2}(G)$;

(3) $LM_3(T^{-11}(G)) = 2m(n+m-1) - M_1(G)$.

Proof Obviously, the graph $T^{-11}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + \frac{m(m-3)}{2} + mn$ edges, refer Theorem 3.1. Similarly, by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we know the following.

$$\begin{aligned}
LM_1(T^{-11}(G)) &= \sum_{v\in V(T^{-11}(G))} d_2(v/T^{-11}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{-11}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-11}(G))^2 \\
&= M_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{-11}(G)) &= \sum_{uv\in E(T^{-11}(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&= \sum_{uv\notin E(G)} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} [d_2(u/T^{-11}(G))][d_2(v/T^{-11}(G))] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} [d_2(u/T^{-11}(G))][d_2(v/T^{-11}(G))] \\
&= \overline{M_2}(G).
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{-11}(G)) &= \sum_{v\in V(T^{-11}(G))} \left[d_1(v/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{-11}(G))\right]\left[d_2(e/T^{-11}(G))\right] \\
&= 2m(n+m-1) - M_1(G).
\end{aligned}$$

□

Theorem 3.13 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{0+1}(G)) = n(n-1)^2 + m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G)$;

(2) $LM_2(T^{0+1}(G)) = [\frac{(m-1)^2}{2} - n(n-1)]M_1(G) - (m-1)EM_1(G) + EM_2(G)$
$+ m(m-1)[n(n-1) - (m-1)] + 2mn(n-1)$;

(3) $LM_3(T^{0+1}(G)) = (m+n-1)M_1(G) - EM_1(G) + m[n(n+m) - 2(m-1)]$.

Proof Notice that the graph $T^{0+1}(G)$ has $n+m$ vertices and $m(n-1) + \frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices we get the following by applying Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned} LM_1(T^{0+1}(G)) &= \sum_{v\in V(T^{0+1}(G))} d_2(v/T^{0+1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{0+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{0+1}(G))^2 \\ &= \sum_{v\in V(G)} (n-1)^2 + \sum_{e\in E(G)} [(m-1)^2 + d_1(e/G)^2 - 2(m-1)d_1(e/G)] \\ &= n(n-1)^2 + m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{0+1}(G)) &= \sum_{uv\in E(T^{0+1}(G))} \left[d_2(u/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &= \sum_{uv\in E(L(G))} \left[d_2(u/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &= \sum_{uv\in E(L(G))} \left[(m-1)^2 - (m-1)(d_1(u/G) + d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right] \\ &\quad + \sum_{uv\in E(S(G))} [(n-1)(m-1-d_1(v/G))] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} [(n-1)(m-1-d_1(v/G))] \\ &= [\frac{(m-1)^2}{2} - n(n-1)]M_1(G) - (m-1)EM_1(G) + EM_2(G) \\ &\quad + m(m-1)[n(n-1) - (m-1)] + 2mn(n-1). \end{aligned}$$

$$\begin{aligned} LM_3(T^{0+1}(G)) &= \sum_{v\in V(T^{0+1}(G))} \left[d_1(v/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{0+1}(G))\right] \left[d_2(v/T^{0+1}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{0+1}(G))\right] \left[d_2(e/T^{0+1}(G))\right] \\ &= \sum_{v\in V(G)} [m(n-1)] + \sum_{e\in E(G)} [(n + d_1(e/G))(m-1-d_1(e/G))] \\ &= (m+n-1)M_1(G) - EM_1(G) + m[n(n+m) - 2(m-1)]. \end{aligned}$$

□

Theorem 3.14 *Let G be (n, m) graph. Then*

(1) $LM_1(T^{1+1}(G)) = m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G)$;
(2) $LM_2(T^{1+1}(G)) = \frac{(m-1)^2}{2}M_1(G) - (m-1)EM_1(G) + EM_2(G) - m(m-1)^2$;

(3) $LM_3(T^{1+1}(G)) = (m-n-1)M_1(G) - EM_1(G) + m[n(m+1) - 2(m-1)]$.

Proof Clearly, the graph $T^{1+1}(G)$ has $n+m$ vertices and $(n-1)(\frac{n}{2}+m)+\frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices we therefore get the following by Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned}
LM_1(T^{1+1}(G)) &= \sum_{v\in V(T^{1+1}(G))} d_2(v/T^{1+1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{1+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{1+1}(G))^2 \\
&= \sum_{e\in E(G)} \left[(m-1)^2 + d_1(e/G)^2 - 2(m-1)d_1(e/G)\right] \\
&= m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{1+1}(G)) &= \sum_{uv\in E(T^{1+1}(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&\quad + \sum_{uv\notin E(G)} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&= \sum_{uv\in E(L(G))} \left[(m-1)^2 - (m-1)(d_1(u/G)+d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right] \\
&= \frac{(m-1)^2}{2}M_1(G) - (m-1)EM_1(G) + EM_2(G) - m(m-1)^2.
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{1+1}(G)) &= \sum_{v\in V(T^{1+1}(G))} \left[d_1(v/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{1+1}(G))\right]\left[d_2(e/T^{1+1}(G))\right] \\
&= (m-n-1)M_1(G) - EM_1(G) + m[n(m+1) - 2(m-1)]. \quad \square
\end{aligned}$$

Theorem 3.15 *Let* G *be* (n, m) *graph. Then*

(1) $LM_1(T^{++1}(G)) = (n-1)[n(n-1) - 4m] + m(m-1)(m+3)$

$$-(2m-3)M_1(G)+EM_1(G);$$

(2) $LM_2(T^{++1}(G)) = \Big[\frac{(m-1)^2}{2}-(n-1)(n+1)\Big)M_1(G)+M_2(G)-(m-1)EM_1(G)$
$+EM_2(G)+m[n(2n-3)-m(3m-4)+mn(n-1)]$
$+\sum\limits_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G)$
$+\sum\limits_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G);$

(3) $LM_3(T^{++1}(G)) = mn(m+n-2)+(m-n-2)M_1(G)-EM_1(G)$.

Proof Clearly, the graph $T^{++1}(G)$ has $n+m$ vertices and $mn+\frac{M_1(G)}{2}$ edges by Theorem 3.1. Now by definitions of the first, second and the third leap Zagreb indices, applying Propositions 3.2, 3.3 and Observation 3.4 we have the following.

$$\begin{aligned}
LM_1(T^{++1}(G)) &= \sum_{v\in V(T^{++1}(G))} d_2(v/T^{++1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{++1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{++1}(G))^2 \\
&= \sum_{v\in V(G)} [n-1-d_1(v/G)]^2 + \sum_{e\in E(G)} [m-1-d_1(e/G)]^2 \\
&= (n-1)[n(n-1)-4m]+m(m-1)(m+3)-(2m-3)M_1(G) \\
&\quad +EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{++1}(G)) &= \sum_{uv\in E(T^{++1}(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&= \left(\frac{(m-1)^2}{2}-(n-1)(n+1)\right)M_1(G)+M_2(G)-(m-1)EM_1(G) \\
&\quad +EM_2(G)+m[n(2n-3)-m(3m-4)+mn(n-1)] \\
&\quad + \sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G) \\
&\quad + \sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G).
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{++1}(G)) &= \sum_{v\in V(T^{++1}(G))} \left[d_1(v/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{++1}(G))\right]\left[d_2(e/T^{++1}(G))\right] \\
&= mn(m+n-2) + (m-n-2)M_1(G) - EM_1(G). \qquad \square
\end{aligned}$$

Theorem 3.16 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{-+1}(G)) = m(m-1)(m+3) - (2m-3)M_1(G) + EM_1(G)$;

(2) $LM_2(T^{-+1}(G)) = \frac{(m-1)^2}{2}M_1(G) + \overline{M_2}(G) - (m-1)EM_1(G) + EM_2(G) + m(m-1)(m+1)$ $- \left[\sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G)\right]$;

(3) $LM_3(T^{-+1}(G)) = (m-n-2)M_1(G) - EM_1(G) + mn(m+3)$.

Proof Notice that the graph $T^{-+1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(n-2) + \frac{M_1(G)}{2}$ edges, refer Theorem 3.1. We are easily get the following by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned}
LM_1(T^{-+1}(G)) &= \sum_{v\in V(T^{-+1}(G))} d_2(v/T^{-+1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{-+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-+1}(G))^2 \\
&= m(m-1)(m+3) - (2m-3)M_1(G) + EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{-+1}(G)) &= \sum_{uv\in E(T^{-+1}(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \sum_{uv\notin E(G)} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \frac{(m-1)^2}{2}M_1(G) + \overline{M_2}(G) - (m-1)EM_1(G) \\
&\quad + EM_2(G) + m(m-1)(m+1) \\
&\quad - \left[\sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G)\right].
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{-+1}(G)) &= \sum_{v \in V(T^{-+1}(G))} \left[d_1(v/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \sum_{v \in V(G)} \left[d_1(v/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{e \in E(G)} \left[d_1(e/T^{-+1}(G))\right]\left[d_2(e/T^{-+1}(G))\right] \\
&= (m-n-2)M_1(G) - EM_1(G) + mn(m+3). \qquad \square
\end{aligned}$$

Theorem 3.17 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{0-1}(G)) = n(n-1)^2 + EM_1(G)$;

(2) $LM_2(T^{0-1}(G)) = \overline{EM_2}(G) + n(n-1)M_1(G) - 2mn(n-1)$;

(3) $LM_3(T^{0-1}(G)) = (n+m-1)M_1(G) - EM_1(G) + m(n^2 - 3n - 2m + 2)$.

Proof Notice that the graph $T^{0-1}(G)$ has $n+m$ vertices and $m(\frac{m+1}{2} + n) - \frac{M_1(G)}{2}$ edges, refer Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned}
LM_1(T^{0-1}(G)) &= \sum_{v \in V(T^{0-1}(G))} d_2(v/T^{0-1}(G))^2 \\
&= \sum_{v \in V(G)} d_2(v/T^{0-1}(G))^2 + \sum_{e \in E(G)} d_2(e/T^{0-1}(G))^2 \\
&= n(n-1)^2 + EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{0-1}(G)) &= \sum_{uv \in E(T^{0-1}(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&= \sum_{uv \notin E(L(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&\quad + \sum_{uv \in E(S(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&\quad + \sum_{uv \in E(\overline{S}(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&= \sum_{uv \notin E(L(G))} [d_1(u/G) \cdot d_1(v/G)] + \sum_{uv \in E(S(G))} (n-1)d_1(v/G) \\
&\quad + \sum_{uv \in E(\overline{S}(G))} (n-1)d_1(v/G) \\
&= \overline{EM_2}(G) + n(n-1)M_1(G) - 2mn(n-1).
\end{aligned}$$

$$\begin{aligned} LM_3(T^{0-1}(G)) &= \sum_{v \in V(T^{0-1}(G))} \left[d_1(v/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\ &= \sum_{v \in V(G)} \left[d_1(v/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\ &\quad + \sum_{e \in E(G)} \left[d_1(e/T^{0-1}(G))\right]\left[d_2(e/T^{0-1}(G))\right] \\ &= (n+m-1)M_1(G) - EM_1(G) + m(n^2 - 3n - 2m + 2). \qquad \square \end{aligned}$$

Theorem 3.18 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{1-1}(G)) = EM_1(G)$;

(2) $LM_2(T^{1-1}(G)) = \overline{EM_2}(G)$;

(3) $LM_3(T^{1-1}(G)) = (n+m-1)M_1(G) - EM_1(G) - 2m(n+m-1)$.

Proof Clearly, the graph $T^{1-1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(\frac{m+1}{2} + n) - \frac{M_1(G)}{2}$ edges by Theorem 3.1. Whence, by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned} LM_1(T^{1-1}(G)) &= \sum_{v \in V(T^{1-1}(G))} d_2(v/T^{1-1}(G))^2 \\ &= \sum_{v \in V(G)} d_2(v/T^{1-1}(G))^2 + \sum_{e \in E(G)} d_2(e/T^{1-1}(G))^2 \\ &= EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{1-1}(G)) &= \sum_{uv \in E(T^{1-1}(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &= \sum_{uv \in E(G)} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{uv \notin E(G)} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{uv \notin E(L(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{uv \in E(S(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{uv \in E(\overline{S}(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &= \overline{EM_2}(G). \end{aligned}$$

$$\begin{aligned} LM_3(T^{1-1}(G)) &= \sum_{v\in V(T^{1-1}(G))}\left[d_1(v/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &= \sum_{v\in V(G)}\left[d_1(v/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{e\in E(G)}\left[d_1(e/T^{1-1}(G))\right]\left[d_2(e/T^{1-1}(G))\right] \\ &= (n+m-1)M_1(G)-EM_1(G)-2m(n+m-1). \end{aligned}$$ □

Theorem 3.19 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+-1}(G)) = M_1(G)+EM_1(G)+(n-1)[n(n-1)-4m]$;

(2) $LM_2(T^{+-1}(G)) = (n-1)^2M_1(G)+M_2(G)+\overline{EM_2}(G)-m(n-1)(n+1)$
$-\left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G)+\sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right]$;

(3) $LM_3(T^{+-1}(G)) = (n+m-2)M_1(G)-EM_1(G)+m[(n-1)(n+2)-2(2m+n-1)]$.

Proof Clearly, the graph $T^{+-1}(G)$ has $n+m$ vertices and $m(\frac{m+3}{2}+n)-\frac{M_1(G)}{2}$edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we therefore get the following.

$$\begin{aligned} LM_1(T^{+-1}(G)) &= \sum_{v\in V(T^{+-1}(G))} d_2(v/T^{+-1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{+-1}(G))^2+\sum_{e\in E(G)} d_2(e/T^{+-1}(G))^2 \\ &= M_1(G)+EM_1(G)+(n-1)[n(n-1)-4m]. \end{aligned}$$

$$\begin{aligned} LM_2(T^{+-1}(G)) &= \sum_{uv\in E(T^{+-1}(G))}\left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= \sum_{uv\in E(G)}\left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))}\left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))}\left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))}\left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= (n-1)^2M_1(G)+M_2(G)+\overline{EM_2}(G)-m(n-1)(n+1) \\ &\quad -\left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G)+\sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right]. \end{aligned}$$

$$\begin{aligned} LM_3(T^{+-1}(G)) &= \sum_{v\in V(T^{+-1}(G))} \left[d_1(v/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+-1}(G))\right]\left[d_2(e/T^{+-1}(G))\right] \\ &= (n+m-2)M_1(G) - EM_1(G) + m[(n-1)(n+2) - 2(2m+n-1)]. \quad \square \end{aligned}$$

Theorem 3.20 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{--1}(G)) = M_1(G) + EM_1(G)$;

(2) $LM_2(T^{--1}(G)) = \overline{M_1}(G) + \overline{EM_2}(G) + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G)$;

(3) $LM_3(T^{--1}(G)) = (n+m-2)M_1(G) - EM_1(G)$.

Proof Notice that the graph $T^{--1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(\frac{m-1}{2} + n) - \frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices, Propositions 3.2, 3.3 and Observation 3.4, we are easily get the following.

$$\begin{aligned} LM_1(T^{--1}(G)) &= \sum_{v\in V(T^{--1}(G))} d_2(v/T^{--1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{--1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{--1}(G))^2 \\ &= M_1(G) + EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{--1}(G)) &= \sum_{uv\in E(T^{--1}(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &= \sum_{uv\notin E(G)} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &= \overline{M_1}(G) + \overline{EM_2}(G) + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) \\ &\quad + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G). \end{aligned}$$

$$
\begin{aligned}
LM_3(T^{--1}(G)) &= \sum_{v\in V(T^{--1}(G))} \left[d_1(v/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{--1}(G))\right]\left[d_2(e/T^{--1}(G))\right] \\
&= (n+m-2)M_1(G) - EM_1(G). \qquad \square
\end{aligned}
$$

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A Generalization on
Product Degree Distance of Strong Product of Graphs

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Abstract: In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of strong product of a connected graph and the complete multipartite graph with partite sets of sizes $m_0, m_1, \cdots, m_{r-1}$ are obtained.

Key Words: Reciprocal product degree distance, product degree distance, strong product.

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§1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u,v) : u \in V(G), v \in V(H)\}$ and $(u,x)(v,y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [12]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Let G be a connected graph. Then *Wiener index* of G is defined as

$$W(G) = \frac{1}{2} \sum_{u,\, v\, \in\, V(G)} d_G(u,v)$$

with the summation going over all pairs of distinct vertices of G. This definition can be further

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generalized in the following way:

$$W_\lambda(G) = \frac{1}{2} \sum_{u,\, v \in V(G)} d_G^\lambda(u,v),$$

where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and λ is a real number [13, 14]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is Harary index of G. In the chemical literature also $W_{\frac{1}{2}}$ [29] as well as the general case W_λ were examined [10, 15].

Dobrynin and Kochetova [6] and Gutman [11] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as

$$DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v),$$

where $d_G(u)$ is the degree of the vertex u in G. Similarly, the *product degree distance* or *Gutman index* of a connected graph G is defined as

$$DD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u) d_G(v) d_G(u,v).$$

The *additively weighted Harary index*(H_A) or *reciprocal degree distance*(RDD) is defined in [3] as

$$H_A(G) = RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u,v)}.$$

Similarly, Su et al. [28] introduce the *reciprocal product degree distance* of graphs, which can be seen as a product-degree-weight version of Harary index

$$RDD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u,v)}.$$

In [16], Hamzeh et al. recently introduced generalized degree distance of graphs. Hua and Zhang [18] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. Pattabiraman et al. [22, 23] have obtained the reciprocal degree distance of join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [3, 20, 27].

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v),$$

where λ is a real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) =$

$RDD(G)$. Similarly, *generalized product degree distance*, denoted by $H^*_\lambda(G)$, is defined as

$$H^*_\lambda(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u)d_G(v)d^\lambda_G(u,v).$$

If $\lambda = 1$, then $H^*_\lambda(G) = DD_*(G)$ and if $\lambda = -1$, then $H^*_\lambda(G) = RDD_*(G)$. Therefore the study of the above topological indices are important and we try to obtain the results related to these indices. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et al. [16, 17]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The generalized degree distance and generalized product degree distance of some classes of graphs are obtained in [24, 25, 26]. In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of strong product $G \boxtimes K_{m_0,m_1,\cdots,m_{r-1}}$, where $K_{m_0,m_1,\cdots,m_{r-1}}$ is the complete multipartite graph with partite sets of sizes m_0, m_1, $\cdots$, m_{r-1} are obtained.

The *first Zagreb index* is defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2$$

and the *second Zagreb index* is defined as

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

In fact, one can rewrite the first Zagreb index as

$$M_1(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v)).$$

The Zagreb indices were found to be successful in chemical and physico-chemical applications, especially in QSPR/QSAR studies, see [8, 9].

For $S \subseteq V(G)$, $\langle S\rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum_{s\in S, t\in T} d_G(s,t)$.

§2. Generalized Product Degree Distance of Strong Product of Graphs

In this section, we obtain the Generalized product degree distance of $G \boxtimes K_{m_0,m_1,\cdots,m_{r-1}}$. Let G be a simple connected graph with $V(G) = \{v_0, v_1, \cdots, v_{n-1}\}$ and let $K_{m_0,m_1,\cdots,m_{r-1}}$, $r \geq 2$, be the complete multiparite graph with partite sets V_0, V_1, $\cdots$, V_{r-1} and let $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0,m_1,\cdots,m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$.

For our convenience, the vertex set of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$ is written as

$$V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) = \bigcup_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}.$$

Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\cdots,n-1 \\ j=0,1,\cdots,r-1}}$. Let $X_i = \bigcup_{j=0}^{r-1} B_{ij}$ and $Y_j = \bigcup_{i=0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, respectively. If we denote $V(B_{ij}) = \{x_{i1}, x_{i2}, \cdots, x_{im_j}\}$ and $V(B_{kp}) = \{x_{k1}, x_{k2}, \cdots, x_{k\, m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the *corresponding vertices* of B_{ij} and B_{kp}. Further, if $v_i v_k \in E(G)$, then the induced subgraph $\langle B_{ij} \bigcup B_{kp} \rangle$ of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according as $j \neq p$ or $j = p$, respectively.

The following remark is follows from the structure of the graph $K_{m_0, m_1, \cdots, m_{r-1}}$.

Remark 2.1 *Let n_0 and q be the number of vertices and edges of $K_{m_0, m_1, \cdots, m_{r-1}}$. Then the sums*

$$\begin{aligned} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p &= 2q, \\ \sum_{j=0}^{r-1} m_j^2 &= n_0^2 - 2q, \\ \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j^2 m_p = n_0 q - 3t &= \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p^2, \\ \sum_{j=0}^{r-1} m_j^3 &= n_0^3 - 3n_0 q + 3t \end{aligned}$$

and

$$\sum_{j=0}^{r-1} m_j^4 = n_0^4 - 4n_0^2 q + 2q^2 + 4n_0 t - 4\tau,$$

where t and τ are the number of triangles and $K_4^{'s}$ in $K_{m_0, m_1, \cdots, m_{r-1}}$.

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Lemma 2.2 *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$, where $r \geq 2$. Then*

(i) If $v_i v_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, *then* $d_{G'}(x_{it}, x_{k\ell}) = 1$.

(ii) If $v_i v_k \notin E(G)$, *then for any two vertices* $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.

(iii) For any two distinct vertices in B_{ij}, *their distance is 2.*

The proof of the following lemma follows easily from Lemma 2.2, which is used in the proof of the main theorems of this section.

Lemma 2.3 *Let* G *be a connected graph and let* $B_{ij},\ B_{kp} \in \mathscr{B}$ *of the graph* $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, *where* $r \geq 2$.

(i) If $v_i v_k \in E(G)$, *then*

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_j m_p, & if\ \ j \neq p, \\ \frac{m_j(m_j+1)}{2}, & if\ \ j = p, \end{cases}$$

(ii) If $v_i v_k \notin E(G)$, *then*

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_j m_p}{d_G(v_i, v_k)}, & if\ j \neq p, \\ \frac{m_j^2}{d_G(v_i, v_k)}, & if\ j = p. \end{cases}$$

(iii) $$d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} m_j m_p, & if\ j \neq p, \\ \frac{m_j(m_j-1)}{2}, & if\ j = p. \end{cases}$$

Lemma 2.4 *Let* G *be a connected graph and let* B_{ij} *in* $G' = G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$. *Then the degree of a vertex* $(v_i, u_j) \in B_{ij}$ *in* G' *is*

$$d_{G'}((v_i, u_j)) = d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j),$$

where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.

Now we obtain the generalized product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.5 *Let* G *be a connected graph with* n *vertices and* m *edges. Then*

$$\begin{aligned}
&H^*_\lambda(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\
&= (4q^2 + n_0^2 + 4n_0 q)H^*_\lambda(G) + 4q^2 W_\lambda(G) + (4q^2 + 2n_0 q)H_\lambda(G) + \frac{n}{2}(4q^2 - n_0 q - 3t) \\
&+ \frac{M_1(G)}{2}\Big[4n_0^2 q - 2q^2 + 4n_0 t + 9t + 7n_0 q - n_0 - 3n_0^2 - 2n_0^3 + 8\tau\Big] \\
&+ m\Big[3n_0 q + 2n_0 t - 2q^2 - 3t - 4q + 4\tau\Big] \\
&+ 2^\lambda\Big[M_1(G)(2q^2 - 2n_0 t - 6t - 2q - 4\tau) + m(2q^2 - 2n_0 t - n_0 q - 3t - 4\tau)\Big] \\
&+ (2^\lambda - 1)M_2(G)\big[2q^2 - 2n_0 t - 3n_0^3 + 10n_0 q + n_0^2 - 18t - 6q - n_0 - 4\tau\Big].
\end{aligned}$$

Proof Let $G'=G\boxtimes K_{m_0,\,m_1,\,\ldots,\,m_{r-1}}$. Clearly,

$$\begin{aligned}
H^*_\lambda(G') &= \frac{1}{2}\sum_{B_{ij},\,B_{kp}\in\mathscr{B}} d_{G'}(B_{ij})d_{G'}(B_{kp})d^\lambda_{G'}(B_{ij},B_{kp})\\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d^\lambda_{G'}(B_{ij},B_{ip})\\
&\quad+\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^\lambda_{G'}(B_{ij},B_{kj})\\
&\quad+\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d^\lambda_{G'}(B_{ij},B_{kp})\\
&\quad+\sum_{i=0}^{n-1}\sum_{j=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ij})d^\lambda_{G'}(B_{ij},B_{ij})\Bigg). \qquad (2.1)
\end{aligned}$$

We shall obtain the sums of (2.1) are separately.

First we calculate $A_1=\sum\limits_{i=0}^{n-1}\sum\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d^\lambda_{G'}(B_{ij},B_{ip})$. For that first we find T'_1. By Lemma 2.4, we have

$$\begin{aligned}
T'_1 &= d_{G'}(B_{ij})d_{G'}(B_{ip})\\
&= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)\Big(d_G(v_i)(n_0-m_p+1)+(n_0-m_p)\Big)\\
&= \Big((n_0+1)^2-(n_0+1)m_j-(n_0+1)m_p+m_jm_p\Big)d^2_G(v_i)\\
&\quad+\Big(2n_0(n_0+1)-(2n_0+1)m_j-(2n_0+1)m_p+2m_jm_p\Big)d_G(v_i)\\
&\quad+\Big(n_0^2-n_0m_p-n_0m_j+m_jm_p\Big).
\end{aligned}$$

From Lemma 2.3, we have $d^\lambda_{G'}(B_{ij},B_{ip})=m_jm_p$. Thus

$$\begin{aligned}
&T'_1d^\lambda_{G'}(B_{ij},B_{ip})=T'_1m_jm_p\\
&=\Big((n_0+1)^2m_jm_p-(n_0+1)m_j^2m_p-(n_0+1)m_jm_p^2+m_j^2m_p^2\Big)d^2_G(v_i)\\
&\quad+\Big(2n_0(n_0+1)m_jm_p-(2n_0+1)m_j^2m_p-(2n_0+1)m_jm_p^2+2m_j^2m_p^2\Big)d_G(v_i)\\
&\quad+\Big(n_0^2m_jm_p-n_0m_j^2m_p-n_0m_jm_p^2+m_j^2m_p^2\Big).
\end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} T_1 &= \sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} T_1' d^{\lambda}_{G'}(B_{ij}, B_{ip}) \\ &= \Big(2q^2 + 2qn_0 + 2n_0t + 2q + 4\tau + 6t\Big) d^2_G(v_i) \\ &\quad + \Big(2qn_0 + 4n_0t - 4q^2 + 6t + 8\tau\Big) d_G(v_i) \\ &\quad + \Big(2n_0t + 2q^2 + 4\tau\Big). \end{aligned}$$

From the definition of the first Zagreb index, we have

$$\begin{aligned} A_1 &= \sum_{i=0}^{n-1} T_1 \\ &= \Big(2q^2 + 2qn_0 + 2n_0t + 2q + 4\tau + 6t\Big) M_1(G) \\ &\quad + 2m\Big(2qn_0 + 4n_0t - 4q^2 + 6t + 8\tau\Big) \\ &\quad + n\Big(2n_0t + 2q^2 + 4\tau\Big). \end{aligned}$$

Next we obtain $A_2 = \sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1} \sum\limits_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kj}) d^{\lambda}_{G'}(B_{ij}, B_{kj})$. For that first we find T_2'.

By Lemma 2.4, we have

$$\begin{aligned} T_2' &= d_{G'}(B_{ij}) d_{G'}(B_{kj}) \\ &= \Big(d_G(v_i)(n_0 - m_j + 1) + (n_0 - m_j)\Big)\Big(d_G(v_k)(n_0 - m_j + 1) + (n_0 - m_j)\Big) \\ &= (n_0 - m_j + 1)^2 d_G(v_i) d_G(v_k) + (n_0 - m_j)(n_0 - m_j + 1)(d_G(v_i) + d_G(v_k)) \\ &\quad + (n_0 - m_j)^2. \end{aligned}$$

Thus

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) + \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(1-2^{\lambda}+2^{\lambda}m_j\Big)m_j + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k),\\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(\Big(1-2^{\lambda}+2^{\lambda}m_j\Big)m_j + m_j^2 - m_j^2\Big) + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k)\\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2'(2^{\lambda}-1)\Big(m_j^2-m_j\Big) + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k)\\ &= S_1+S_2, \end{aligned} \tag{2.2}$$

where S_1 and S_2 are the sums of the terms of the above expression, in order.

Now we calculate S_1. For that first we find the following.

$$\begin{aligned} &(2^{\lambda}-1)T_2'\Big(m_j^2-m_j\Big) = (2^{\lambda}-1)\Big[\Big(m_j^4-(2n_0+3)m_j^3+(n_0^2+4n_0+3)m_j^2\\ &-(n_0+1)^2m_j\Big)d_G(v_i)d_G(v_k)\\ &+\Big(m_j^4-(2n_0+2)m_j^3+(n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j\Big)(d_G(v_i)+d_G(v_k))\\ &+\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+2n_0)m_j^2-n_o^2m_j\Big)\Big]. \end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} T_2'' &= \sum_{j=0}^{r-1}(2^{\lambda}-1)T_2'\Big(m_j^2-m_j\Big)\\ &= (2^{\lambda}-1)\Big[\Big(2q^2-2n_0t-4\tau-3n_0^3+10n_0q-18t+n_0^2-6q-n_0\Big)d_G(v_i)d_G(v_k)\\ &\quad+\Big(2q^2-4\tau-2n_0t-6t-2q\Big)(d_G(v_i)+d_G(v_k))\\ &\quad+\Big(2q^2-4\tau-2n_0t-n_0q-3t\Big)\Big]. \end{aligned}$$

Hence

$$\begin{aligned} S_1 &= \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} T_2''\\ &= (2^{\lambda}-1)\Big[\Big(2q^2-2n_0t-4\tau-3n_0^3+10n_0q-18t+n_0^2-6q-n_0\Big)2M_2(G)\\ &\quad+\Big(2q^2-4\tau-2n_0t-6t-2q\Big)2M_1(G)\\ &\quad+2m\Big(2q^2-4\tau-2n_0t-n_0q-3t\Big)\Big]. \end{aligned}$$

Next we calculate S_2. For that we need the following.

$$\begin{aligned} T_2'm_j^2 &= \Big(m_j^4-(2n_0+2)m_j^3+(n_0+1)^2m_j^2\Big)d_G(v_i)d_G(v_k) \\ &+\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+n_0)m_j^2\Big)(d_G(v_i)+d_G(v_k)) \\ &+\Big(m_j^4-2n_0m_j^3+n_0^2m_j^2\Big). \end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} T_2 &= \sum_{j=0}^{r-1} T_2'm_j^2 \\ &= \Big(2q^2-4\tau-2n_0t-6t+2n_0q-2q+n_0^2\Big)d_G(v_i)d_G(v_k) \\ &+\Big(2q^2-4\tau-2n_0t-3t+n_0q\Big)(d_G(v_i)+d_G(v_k)) \\ &+\Big(2q^2-4\tau-2n_0t\Big). \end{aligned}$$

From the definitions of H_λ^*,H_λ and W_λ, we obtain

$$\begin{aligned} S_2 &= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} T_2 d_G^\lambda(v_i,v_k) \\ &= 2\Big(2q^2-4\tau-2n_0t-6t+2n_0q-2q+n_0^2\Big)H_\lambda^*(G) \\ &+2\Big(2q^2-4\tau-2n_0t-3t+n_0q\Big)H_\lambda(G) \\ &+2\Big(2q^2-4\tau-2n_0t\Big)W_\lambda(G). \end{aligned}$$

Now we calculate $A_3=\sum\limits_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum\limits_{\substack{j,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^\lambda(B_{ij},B_{kp})$. For that first we compute T_3'. By Lemma 2.4, we have

$$\begin{aligned} T_3' &= d_{G'}(B_{ij})d_{G'}(B_{kp}) \\ &= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)\Big(d_G(v_k)(n_0-m_p+1)+(n_0-m_p)\Big) \\ &= d_G(v_i)d_G(v_k)(n_0-m_j+1)(n_0-m_p+1)+d_G(v_i)(n_0-m_j+1)(n_0-m_p) \\ &+d_G(v_k)(n_0-m_p+1)(n_0-m_j)+(n_0-m_j)(n_0-m_p). \end{aligned}$$

Since the distance between B_{ij} and B_{kp} is $m_jm_pd_G^{\lambda}(v_i,v_k)$. Thus

$$\begin{aligned}T_3'm_jm_p &= d_G(v_i)d_G(v_k)\Big((n_0^2+2n_0+1)m_jm_p-(n_0+1)m_j^2m_p-(n_0+1)m_jm_p^2+m_j^2m_p^2\Big)\\&+d_G(v_i)\Big((n_0^2+n_0)m_jm_p-(n_0+1)m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big)\\&+d_G(v_k)\Big((n_0^2+n_0)m_jm_p-n_0m_jm_p^2-(n_0+1)m_j^2m_p+m_j^2m_p^2\Big)\\&+\Big(n_0^2m_jm_p-n_0m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big).\end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned}T_3 &= \sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}T_3'm_jm_p = d_G(v_i)d_G(v_k)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big)\\&+(d_G(v_i)+d_G(v_k))\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big)\\&+\Big(2n_0t+2q^2+4\tau\Big).\end{aligned}$$

Hence

$$\begin{aligned}A_3 &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}T_3d_G^{\lambda}(v_i,v_k) = 2H_{\lambda}^*(G)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big)\\&+2H_{\lambda}(G)\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big)\\&+2W_{\lambda}(G)\Big(2n_0t+2q^2+4\tau\Big).\end{aligned}$$

Finally, we obtain $A_4=\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{r-1}d_{G'}(B_{ij})d_{G'}(B_{ij})d_{G'}^{\lambda}(B_{ij},B_{ij})$. For that first we calculate T_4'. By Lemma 2.4, we have

$$\begin{aligned}T_4' &= d_{G'}(B_{ij})d_{G'}(B_{ij})\\&= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)^2\\&= d_G^2(v_i)(n_0-m_j+1)^2+2d_G(v_i)(n_0-m_j)(n_0-m_j+1)+(n_0-m_j)^2.\end{aligned}$$

From Lemma 2.3, the distance between $(B_{ij}$ and $(B_{ij}$ is $m_j(m_j-1)$. Thus

$$\begin{aligned}T_4'm_j(m_j-1) &= d_G^2(v_i)\Big(m_j^4-(2n_0+3)m_j^3+((n_0+1)^2+2)m_j^2-(n_0+1)^2m_j\Big)\\&+2d_G(v_i)\Big(m_j^4-(2n_0+2)m_j^3+(n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j\Big)\\&+\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+2n_0)m_j^2-n_0^2m_j\Big).\end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned} T_4 &= \sum_{j=0}^{r-1} T_4' m_j(m_j-1) \\ &= d_G^2(v_i)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big) \\ &\quad +2d_G(v_i)\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) \\ &\quad +\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big). \end{aligned}$$

Hence

$$\begin{aligned} A_4 &= \sum_{i=0}^{n-1} T_4 d_{G'}^{\lambda}(B_{ij}, B_{ij}) \\ &= M_1(G)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big) \\ &\quad +4m\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) \\ &\quad +n\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big). \end{aligned}$$

Adding A_1,S_1,S_2,A_3 and A_4 we get the required result. □

If we set $\lambda = 1$ in Theorem 2.5, we obtain the product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.6 *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} &DD_*(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\ &= (4q^2 + n_0^2 + 4n_0q)DD_*(G) + 4q^2W(G) \\ &\quad +(4q^2 + 2n_0q)DD(G) + \frac{n}{2}(4q^2 - n_0q - 3t) \\ &\quad +\frac{M_1(G)}{2}\Big[4n_0^2q + 6q^2 - 4n_0t - 15t + 7n_0q - n_0 - 3n_0^2 - 2n_0^3 - 8\tau\Big] \\ &\quad +m\Big[n_0q - 2n_0t + 2q^2 - 9t - 4q - 4\tau\Big] \\ &\quad +M_2(G)\Big[2q^2 - 2n_0t - 3n_0^3 + 10n_0q + n_0^2 - 18t - 6q - n_0 - 4\tau\Big] \end{aligned}$$

for $r \geq 2$.

Setting $\lambda = -1$ in Theorem 2.5, we obtain the reciprocal product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.7 *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} &RDD_*(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\ &= (4q^2 + n_0^2 + 4n_0q)RDD_*(G) + 4q^2H(G) \\ &\quad +(4q^2 + 2n_0q)RDD(G) + \frac{n}{2}(4q^2 - n_0q - 3t) \end{aligned}$$

$$+\frac{M_1(G)}{2}\Big[4n_0^2q+2n_0t+3t+7n_0q-n_0-3n_0^2-2n_0^3-2q+4\tau\Big]$$
$$+m\Big[\frac{5n_0q}{2}+n_0t-q^2-\frac{9t}{2}-4q+2\tau\Big]$$
$$-\frac{M_2(G)}{2}\Big[2q^2-2n_0t-3n_0^3+10n_0q+n_0^2-18t-6q-n_0-4\tau\Big]$$

for $r \geq 2$.

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Semifull Line (Block) Signed Graphs

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Abstract: In this paper we introduced the new notions semifull signed graph and semifull line (block) signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, semifull signed graph, semifull line (block) signed graph, negation of a signed graph, semifull Smarandachely graph.

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§1. Introduction

For all terminology and notation in graph theory we refer the reader to consult any one of the standard text-books by Chartrand and Zhang [2], Harary [3] and West [12].

If $B = \{u_1, u_2, \cdots, u_r, r \geq 2\}$ is a block of a graph Γ, then we say that vertex u_1 and block B are incident with each other, as are u_2 and B and so on. If two blocks B_1 and B_2 of G are incident with a common cut vertex, then they are adjacent blocks. If $B = \{e_1, e_2, \cdots, e_s, s \geq 1\}$ is a block of a graph Γ, then we say that an edge e_1 and block B are incident with each other, as are e_2 and B and so on. This concept was introduced by Kulli [7]. The vertices, edges and blocks of a graph are called its members.

The line graph $L(\Gamma)$ of a graph Γ is the graph whose vertex set is the set of edges of Γ in which two vertices are adjacent if the corresponding edges are adjacent (see [3]).

The semifull graph $\mathcal{SF}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of vertices, edges and blocks of Γ in which two vertices are adjacent if the corresponding members of Γ are adjacent or one corresponds to a vertex and the other to an edge incident with it or one corresponds to a block B of Γ and the other to a vertex v of Γ and v is in B. In fact, this notion was introduced by Kulli [8]. Generally, for a subset $B' \subset B$, a semifull Smarandachely graph $\mathcal{SSF}(\Gamma)$ of a graph Γ on B' is the graph with $V(\mathcal{SSF}(\Gamma)) = V(\Gamma) \bigcup E(\Gamma) \bigcup B'$, and two vertices are adjacent in $\mathcal{SSF}(\Gamma)$ if the corresponding members of Γ are adjacent or one corresponds to a vertex and the other to an edge incident with it or one corresponds to a block

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B' of Γ and the other to a vertex v of Γ with $v \in B'$. Clearly, $\mathcal{SSF}(\Gamma) = \mathcal{SF}(\Gamma)$ if $B' = B$.

In [9], the author introduced the new notions called "*semifull line graphs and semifull block graphs*" as follows: The semifull line graph $\mathcal{SFL}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of the set of vertices, edges and blocks of Γ in which two vertices are adjacent $\mathcal{SFL}(\Gamma)$ if the corresponding vertices and edges of Γ are adjacent or one corresponds to a vertex of Γ and other to an edge incident with it or one corresponds to a block B of Γ and other to a vertex v of Γ and v is in B.

The semifull block graph $\mathcal{SFB}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of the set of vertices, edges and blocks of Γ in which two vertices are adjacent in $\mathcal{SFB}(\Gamma)$ if the corresponding vertices and blocks of Γ are adjacent or one corresponds to a vertex of Γ and other to an edge incident with it or one corresponds to a block B of Γ and other to a vertex v of Γ and v is in B.

A *signed graph* is an ordered pair $\Sigma = (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is a graph called *underlying graph of* Σ and $\sigma : E \to \{+, -\}$ is a function. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced*, if every cycle in Σ has an even number of negative edges (See [4]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of Σ is positive.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [6] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [4] provided a mathematical model for balance through graphs.

A *marking* of Σ is a function $\zeta : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(1)(Harary [4]) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2.*

(2)(Sampathkumar [10]) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$.*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.1.

A switching function for Σ is a function $\zeta : V \rightarrow \{+,-\}$. The switched signature is $\sigma^{\zeta}(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^{\zeta} := (\Sigma|\sigma^{\zeta})$. We say that Σ switched by ζ. Note that $\Sigma^{\zeta} = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^{ζ} where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^{\zeta} \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\}$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [11]) or *cycle isomorphic* (see [13]) if there exists an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [13]):

Theorem 1.2(T. Zaslavsky, [13]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

§2. Semifull Line Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion called semifull line graphs to realm of signed graphs: the *semifull line signed graph* $\mathcal{SFL}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SFL}(\Sigma) = (\mathcal{SFL}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SFL}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull line signed graph, if $\Sigma \cong \mathcal{SFL}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull line signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull line signed graphs.

Theorem 2.1 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull line signed graph $\mathcal{SFL}(\Sigma)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $\mathcal{SFL}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SFL}(\Sigma)$. Then by definition of $\mathcal{SFL}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SFL}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull line signed graph, $\mathcal{SFL}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SFL}^0(\Sigma) = \Sigma,\ \mathcal{SFL}^k(\Sigma) = \mathcal{SFL}(\mathcal{SFL}^{k-1}(\Sigma))$$

Corollary 2.2 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SFL}^k(\Sigma)$ is balanced.*

Proposition 2.3 *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull line signed graphs are switching equivalent.*

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.1, $\mathcal{SFL}(\Sigma_1)$ and $\mathcal{SFL}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

The *semifull signed graph* $\mathcal{SF}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SF}(\Sigma) = (\mathcal{SF}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SF}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull signed graph, if $\Sigma \cong \mathcal{SF}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull signed graphs.

Theorem 2.4 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull signed graph $\mathcal{SF}(\Sigma)$ is balanced.*

Proof Let σ' denote the signing of $\mathcal{SF}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SF}(\Sigma)$. Then by definition of $\mathcal{SF}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SF}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull line signed graph, $\mathcal{SF}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SF}^0(\Sigma) = \Sigma,\ \mathcal{SF}^k(\Sigma) = \mathcal{SF}(\mathcal{SF}^{k-1}(\Sigma))$$

Corollary 2.5 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SF}^k(\Sigma)$ is balanced.*

Proposition 2.6 *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull signed graphs are switching equivalent.*

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.4, $\mathcal{SF}(\Sigma_1)$ and $\mathcal{SF}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

In [9], the author characterizes graphs such that semifull line graphs and semifull graphs are isomorphic.

Theorem 2.7 *Let Γ be a nontrivial connected graph. The graphs $\mathcal{SFL}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic if and only if Γ is a block.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$.

Theorem 2.8 *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$ if and only if Γ is a block.*

Proof Suppose that $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$. Then clearly, $\mathcal{SFL}(\Gamma) \cong \mathcal{SF}(\Gamma)$. Hence by Theorem 2.7, Γ is a block.

Conversely, suppose that Σ is a signed graph whose underlying graph is a block. Then by Theorem 2.7, $\mathcal{SFL}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic. Since for any signed graph Σ, both $\mathcal{SFL}(\Sigma)$ and $\mathcal{SF}(\Sigma)$ are balanced, the result follows by Theorem 1.2. □

The following result characterize signed graphs which are semifull line signed graphs.

Theorem 2.9 *A signed graph $\Sigma = (\Gamma, \sigma)$ is a semifull line signed graph if and only if Σ is balanced signed graph and its underlying graph Γ is a semifull line graph.*

Proof Suppose that Σ is balanced and Γ is a semifull line graph. Then there exists a graph Γ' such that $\mathcal{SFL}(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1.1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $\mathcal{SFL}(\Sigma') \cong \Sigma$. Hence Σ is a semifull line signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a semifull line signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that

$$\mathcal{SFL}(\Sigma') \cong \Sigma.$$

Hence, Γ is the semiful line graph of Γ' and by Theorem 2.1, Σ is balanced. □

In view of the above result, we can easily characterize signed graphs which are semifull signed graphs.

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [5] as follows:

$\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $\mathcal{SFL}(\Sigma)$ ($\mathcal{SF}(\Sigma)$) is balanced. We now examine, the conditions under which negation $\eta(\Sigma)$ of $\mathcal{SFL}(\Sigma)$ ($\mathcal{SF}(\Sigma)$) is balanced.

Theorem 2.10 *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $\mathcal{SFL}(\Gamma)$ ($\mathcal{SF}(\Gamma)$) is bipartite then $\eta(\mathcal{SFL}(\Sigma))$ ($\eta(\mathcal{SF}(\Sigma))$) is balanced.*

Proof Since $\mathcal{SFL}(\Sigma)$ ($\mathcal{SF}(\Sigma)$) is balanced, if each cycle C in $\mathcal{SFL}(\Sigma)$ ($\mathcal{SF}(\Sigma)$) contains even number of negative edges. Also, since $\mathcal{SFL}(\Gamma)$ ($\mathcal{SF}(\Gamma)$) is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{SFL}(\Sigma)$ ($\mathcal{SF}(\Sigma)$) is also even. Hence $\eta(\mathcal{SFL}(\Sigma))$ ($\eta(\mathcal{SF}(\Sigma))$) is balanced. □

§3. Semifull Block Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion called semifull block graphs to realm of signed graphs: the *semifull block signed graph* $\mathcal{SFB}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SFB}(\Sigma) = (\mathcal{SFB}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SFB}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull block signed graph, if $\Sigma \cong \mathcal{SFL}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull block signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull block signed graphs.

Theorem 3.1 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull block signed graph $\mathcal{SFB}(\Sigma)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $\mathcal{SFB}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SFB}(\Sigma)$. Then by definition of $\mathcal{SFB}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SFB}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull block signed graph, $\mathcal{SFB}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SFB}^0(\Sigma) = \Sigma,\ \mathcal{SFB}^k(\Sigma) = \mathcal{SFB}(\mathcal{SFB}^{k-1}(\Sigma))$$

Corollary 3.2 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SFB}^k(\Sigma)$ is balanced.*

Proposition 3.3 —it For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull block signed graphs are switching equivalent.

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 3.1, $\mathcal{SFB}(\Sigma_1)$ and $\mathcal{SFB}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

In [9], the author characterizes graphs such that semifull block graphs and semifull graphs are isomorphic.

Theorem 3.4 *Let Γ be a nontrivial connected graph. The graphs $\mathcal{SFB}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic if and only if Γ is P_2.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$.

Theorem 3.5 *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$ if and only if Γ is P_2.*

Proof Suppose that $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$. Then clearly, $\mathcal{SFB}(\Gamma) \cong \mathcal{SF}(\Gamma)$. Hence by Theorem 16, Γ is P_2.

Conversely, suppose that Σ is a signed graph whose underlying graph is P_2. Then by Theorem 16, $\mathcal{SFB}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic. Since for any signed graph Σ, both $\mathcal{SFB}(\Sigma)$ and $\mathcal{SF}(\Sigma)$ are balanced, the result follows by Theorem 2. □

In view of the Theorem 2.9, we can easily characterize signed graphs which are semifull block signed graphs.

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Accurate Independent Domination in Graphs

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Abstract: A dominating set D of a graph $G = (V, E)$ is an *independent dominating set*, if the induced subgraph $\langle D\rangle$ has no edges. An independent dominating set D of G is an *accurate independent dominating set* if $V - D$ has no independent dominating set of cardinality $|D|$. The *accurate independent domination number* $i_a(G)$ of G is the minimum cardinality of an accurate independent dominating set of G. In this paper, we initiate a study of this new parameter and obtain some results concerning this parameter.

Key Words: Domination, independent domination number, accurate independent domination number, Smarandache H-dominating set.

AMS(2010): 05C69.

§1. Introduction

All graphs considered here are finite, nontrivial, undirected with no loops and multiple edges. For graph theoretic terminology we refer to Harary [1].

Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. Let $\Delta(G)(\delta(G))$ denote the *maximum (minimum)* degree and $\lceil x\rceil(\lfloor x\rfloor)$ the *least (greatest)* integer greater(less) than or equal to x. The *neighborhood* of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent with u. The *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_o(G)$. For any set S of vertices of G, the *induced subgraph* $\langle S\rangle$ is maximal subgraph of G with vertex set S.

The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. A *wounded spider* is the graph formed by subdividing at most $n-1$ of the edges of a star $K_{1,n}$ for $n \geq 0$. Let $\Omega(G)$ be the set of all pendant vertices of G, that is the set of vertices of degree 1. A vertex v is called a support vertex if v is neighbor of a pendant vertex and $d_G(v) > 1$. Denote by $X(G)$ the set of all support vertices in G, $M(G)$ be the set

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of vertices which are adjacent to support vertex and $J(G)$ be the set of vertices which are not adjacent to a support vertex. The diameter $diam(G)$ of a connected graph G is the maximum distance between two vertices of G, that is $diam(G) = max_{u,v\in V(G)}\ d_G(u,v)$. A set $B \subseteq V$ is a *2-packing* if for each pair of vertices $u, v \in B, N_G[u] \cap N_G[v] = \phi$

A *proper coloring* of a graph $G = (V(G), E(G))$ is a function from the vertices of the graph to a set of *colors* such that any two adjacent vertices have different colors. The chromatic number $\chi(G)$ is the minimum number of colors needed in a proper coloring of a graph. A *dominator coloring* of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G. This concept was introduced by R. Gera at.al [3].

A set D of vertices in a graph $G = (V, E)$ is a *dominating set* of G, if every vertex in $V - D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. For a comprehensive survey of domination in graphs, see [4, 5, 7].

Generally, if $\langle D\rangle \simeq H$, such a dominating set D is called a *Smarandache H-dominating set*. A dominating set D of a graph $G = (V, E)$ is an *independent dominating set*, if the induced subgraph $\langle D\rangle$ has no edges, i.e., a Smarandache H-dominating set with $E(H) = \emptyset$. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set.

A dominating set D of $G = (V, E)$ is an *accurate dominating set* if $V - D$ has no dominating set of cardinality $|D|$. The *accurate domination number* $\gamma_a(G)$ of G is the minimum cardinality of an accurate dominating set. This concept was introduced by Kulli and Kattimani [6, 9].

An independent dominating set D of G is an *accurate independent dominating set* if $V - D$ has no independent dominating set of cardinality $|D|$. The *accurate independent domination number* $i_a(G)$ of G is the minimum cardinality of an accurate independent dominating set of G. This concept was introduced by Kulli [8].

For example, we consider the graph G in Figure 1. The accurate independent dominating sets are $\{1, 2, 6, 7\}$ and $\{1, 3, 6, 7\}$. Therefore $i_a(G) = 4$.

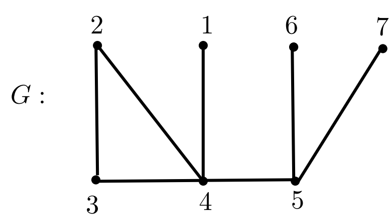


Figure 1

§2. Results

Observation 2.1

1. Every accurate independent dominating set is independent and dominating. Hence it is a minimal dominating set.

2. Every minimal accurate independent dominating set is a maximal independent dominating set.

Proposition 2.1 *For any nontrivial connected graph G, $\gamma(G) \leq i_a(G)$.*

Proof Clearly, every accurate independent dominating set of G is a dominating set of G. Thus result holds. □

Proposition 2.2 *If G contains an isolated vertex, then every accurate dominating set is an accurate independent dominating set.*

Now we obtain the exact values of $i_a(G)$ for some standard class of graphs.

Proposition 2.3 *For graphs P_p, W_p and $K_{m,n}$, there are*

(1) $i_a(P_p) = \lceil p/3 \rceil \; if \; p \geq 3$;
(2) $i_a(W_p) = 1 \; if \; p \geq 5$;
(3) $i_a(K_{m,n}) = m \; for \; 1 \leq m < n$.

Theorem 2.1 *For any graph G, $i_a(G) \leq p - \gamma(G)$.*

Proof Let D be a minimal dominating set of G. Then there exist at least one accurate independent dominating set in $(V - D)$ and by proposition 2.1,

$$i_a(G) \leq |V| - |D| \leq p - \gamma(G).$$

Notice that the path P_4 achieves this bound. □

Theorem 2.2 *For any graph G,*

$$\lceil p/ \triangle +1 \rceil \leq i_a(G) \leq \lfloor p \triangle / \triangle +1 \rfloor$$

and these bounds are sharp.

Proof It is known that $p/ \triangle +1 \leq \gamma(G)$ and by proposition 2.1, we see that the lower bound holds. By Theorem 2.1,

$$\begin{aligned} i_a(G) &\leq p - \gamma(G), \\ &\leq p - p/ \triangle +1 \\ &\leq p \triangle / \triangle +1. \end{aligned}$$

Notice that the path $P_p, p \geq 3$ achieves the lower bound. This completes the proof. □

Proposition 2.4 *If $G = K_{m_1,m_2,m_3,\cdots,m_r}, r \geq 3$, then*

$$i_a(G) = m_1 \quad if \quad m_1 < m_2 < m_3 \cdots < m_r.$$

Theorem 2.3 *For any graph G without isolated vertices $\gamma_a(G) \leq i_a(G)$ if $G \neq K_{m_1,m_2,m_3,\cdots,m_r}, r \geq 3$. Furthermore, the equality holds if $G = P_p(p \neq 4, p \geq 3), W_p(p \geq 5)$ or $K_{m,n}$ for $1 \leq m < n$.*

Proof Since we have $\gamma(G) \leq \gamma_a(G)$ and by Proposition 2.1,$\gamma_a(G) \leq i_a(G)$.

Let $\gamma_a(G) \leq i_a(G)$. If $G = K_{m_1,m_2,m_3,\cdots,m_r}$, $r \geq 3$ then by Proposition 2.4, $i_a(G) = m_1$ if $m_1 < m_2 < m_3 \cdots < m_r$ and also accurate domination number is $\lfloor p/2 \rfloor + 1$ i.e., $\gamma_a(G) = \lfloor p/2 \rfloor + 1 > m_1 = i_a(G)$, a contradiction. □

Corollary 2.1 *For any graph G, $i_a(G) = \gamma(G)$ if $diam(G) = 2$.*

Proposition 2.5 *For any graph G without isolated vertices $i(G) \leq i_a(G)$. Furthermore, the equality holds if $G = P_p$ $(p \geq 3)$, W_p $(p \geq 5)$ or $K_{m,n}$ for $1 \leq m < n$.*

Proof Every accurate independent dominating set is a independent dominating set. Thus result holds. □

Definition 2.1 *The double star $S_{n,m}$ is the graph obtained by joining the centers of two stars $K_{1,n}$ and $K_{1,m}$ with an edge.*

Proposition 2.6 *For any graph G, $i_a(G) \leq \beta_o(G)$. Furthermore, the equality holds if $G = S_{n,m}$.*

Proof Since every minimal accurate independent dominating set is an maximal independent dominating set. Thus result holds. □

Theorem 2.4 *For any graph G, $i_a(G) \leq p - \alpha_0(G)$.*

Proof Let S be a vertex cover of G. Then $V - S$ is an accurate independent dominating set. Then $i_a(G) \leq |V - S| \leq p - \alpha_0(G)$. □

Corollary 2.2 *Fr any graph G, $i_a(G) \leq p - \beta_0(G) + 2$.*

Theorem 2.5 *If G is any nontrivial connected graph containing exactly one vertex of degree $\triangle$ $(G) = p - 1$, then $\gamma(G) = i_a(G) = 1$.*

Proof Let G be any nontrivial connected graph containing exactly one vertex v of degree $deg(v) = p - 1$. Let D be a minimal dominating set of G containing vertex of degree $deg(v) = p = 1$. Then D is a minimum dominating set of G i.e.,

$$|D| = \gamma(G) = 1. \tag{1}$$

Also $V - D$ has no dominating set of same cardinality $|D|$. Therefore,

$$|D| = i_a(G). \tag{2}$$

Hence, by (1) and (2) $\gamma(G) = i_a(G) = 1$. □

Theorem 2.6 *If G is a connected graph with p vertices then $i_a(G) = p/2$ if and only if $G = H \circ K_1$,where H is any nontrivial connected graph.*

Proof Let D be any minimal accurate independent dominating set with $|D| = p/2$. If $G \neq H \circ K_1$ then there exist at least one vertex $v_i \in V(G)$ which is neither a pendant vertex nor a support vertex. Then there exist a minimal accurate independent dominating set D' containing v_i such that

$$|D'| \leq |D| - \{v_i\} \leq p/2 - \{v_i\} \leq p/2 - 1,$$

which is a contradiction to minimality of D.

Conversely, let l be the set of all pendant vertices in $G = H \circ K_1$ such that $|l| = p/2$. If $G = H \circ K_1$, then there exist a minimal accurate independent dominating set $D \subseteq V(G)$ containing all pendant vertices of G. Hence $|D| = |l| = p/2$. □

Now we characterize the trees for which $i_a(T) = p - \Delta(T)$.

Theorem 2.7 *For any tree T, $i_a(T) = p - \Delta(T)$ if and only if T is a wounded spider and $T \neq K_1,\ K_{1,1}$.*

Proof Suppose T is wounded spider. Then it is easy to verify that
$i_a(T) = p - \Delta(T)$.

Conversely, suppose T is a tree with $i_a(T) = p - \Delta(T)$. Let v be a vertex of maximum degree $\Delta(T)$ and u be a vertex in $N(v)$ which has degree 1. If $T - N[v] = \phi$ then T is the star $K_{1,n}, n \geq 2$. Thus T is a double wounded spider. Assume now there is at least one vertex in $T - N[v]$. Let S be a maximal independent set of $\langle T - N[v]\rangle$. Then either $S \cup \{v\}$ or $S \cup \{u\}$ is an accurate independent dominating set of T. Thus $p = i_a(T) + \Delta(T) \leq |S| + 1 + \Delta(T) \leq p$. This implies that $V - N(v)$ is an accurate independent dominating set. Furthermore, $N(v)$ is also an accurate independent dominating set.

The connectivity of T implies that each vertex in $V - N[v]$ must be adjacent to at least one vertex in $N(v)$. Moreover if any vertex in $V - N[v]$ is adjacent to two or more vertices in $N(v)$, then a cycle is formed. Hence each vertex in $V - N[v]$ is adjacent to exactly one vertex in $N(v)$. To show that $\Delta(T) + 1$ vertices are necessary to dominate T, there must be at least one vertex in $N(v)$ which are not adjacent to any vertex in $V - N[v]$ and each vertex in $N(v)$ has either 0 or 1 neighbors in $V - N[v]$. Thus T is a wounded spider. □

Proposition 2.7 *If G is a path P_p, $p \geq 3$ then $\gamma(P_p) = i_a(P_p)$.*

We characterize the class of trees with equal domination and accurate independent domination number in the next section.

§3. Characterization of (γ, i_a)-Trees

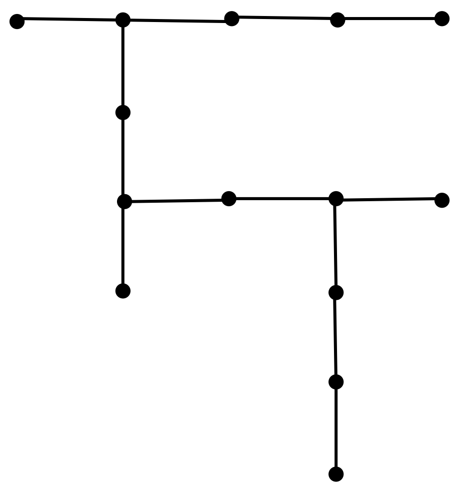
For any graph theoretical parameter λ *and* μ, we define G to be (λ, μ)-graph if $\lambda(G) =$

$\mu(G)$. Here we provide a constructive characterization of (γ, i_a)-trees.

To characterize (γ, i_a)-trees we introduce family τ_1 of trees $T = T_k$ that can be obtained as follows. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by the following operation.

Operation O Attach a path P_3(x,y,z) and an edge mx, where m is a support vertex of a tree T.

$$\tau = \{T/\text{obtained from } P_5 \text{ by finite sequence of operations of } O\}$$



Tree T belonging to family τ_1

Observation 3.1 If $T \in \tau$, then

1. $i_a(T) = \lceil p + 1/3 \rceil$;
2. $X(T)$ is a minimal dominating set as well as a minimal accurate independent dominating set of T;
3. $\langle V - D \rangle$ is totally disconnected.

Corollary 3.1 *If tree T with $p \geq 5$ belongs to the family τ then $\gamma(T) = |X(T)|$ and $i_a(T) = |X(T)|$.*

Lemma 3.1 *If a tree T belongs to the family τ then T is a $(\gamma, i_a) - tree$.*

Proof If $T = P_p,\ p \geq 3$ then from proposition 2.7 T is a $(\gamma, i_a) - tree$. Now if $T = P_p,\ p \geq 3$ then we proceed by induction on the number of operations $n(T)$ required to construct the tree T. If $n(T) = 0$ then $T \in P_5$ by proposition 2.7 T is a (γ, i_a)-tree.

Assume now that T is a tree belonging to the family τ with $n(T) = k$, for some positive integer k and each tree $T' \in \tau$ with $n(T') < k$ and with $V(T') \geq 5$ is a (γ, i_a)-tree in which $X(T')$ is a minimal accurate independent dominating set of T'. Then T can be obtained from a tree T' belonging to τ by operation O where $m \in V(T') - (M(T') - \Omega(T'))$ and we add path (x, y, z) and the edge mx. Then z is a pendant vertex in T and y is a support vertex and $x \in M(T)$. Thus $S(T) = X(T') \cup \{y\}$ is a minimal accurate independent dominating set of T. Therefore $i_a(T) \geq |X(T)| = |X(T')| + 1$. Hence we conclude that $i_a(T) = i_a(T') + 1$. By the induction hypothesis and by observation 3.1(2) $i_a(T') = \gamma(T') = |X(T')|$. In this way $i_a(T) = |X(T)|$ and in particular $i_a(T) = \gamma(T)$. □

Lemma 3.2 *If T is a $(\gamma, i_a) - tree$, then T belongs to the family τ.*

Proof If T is a path $P_p,\ p \geq 3$ then by proposition 2.7 T is a $(\gamma, i_a) - tree$. It is easy to verify that the statement is true for all trees T with diameter less than or equal to 4. Hence we may assume that $diam(T) \geq 4$. Let T be rooted at a support vertex m of a longest path P. Let P be a $m - z$ path and let y be the neighbor of z. Further, let x be a vertex belongs to $M(T)$. Let T be a (γ, i_a)-tree. Now we proceed by induction on number of vertices $|V(T)|$ of a (γ, i_a)-tree. Let T be a (γ, i_a)-tree and assume that the result holds good for all trees on $V(T) - 1$ vertices. By observation 3.1(2) since T is (γ, i_a)-tree it contains minimal accurate independent dominating set D that contains all support vertices of a tree. In particular $\{m, y\} \subset D$ and the vertices x and z are independent in $\langle V - D \rangle$.

Let $T' = T - (x, y, z)$. Then $D - \{y\}$is dominating set of T' and so $\gamma(T') \leq \gamma(T) - 1$. Any dominating set can be extended to a minimal accurate independent dominating set of T by adding to it the vertices (x, y, z) and so $i_a(T) \leq i_a(T') + 1$. Hence, $i_a(T') \leq \gamma(T') \leq \gamma(T) + 1 \leq i_a(T) - 1 \leq i_a(T')$. Consequently, we must have equality throughout this inequality chain. In particular $i_a(T') = \gamma(T')$ and $i_a(T) = i_a(T') + 1$. By inductive hypothesis any minimal accurate independent dominating set of a tree T' can be extended to minimal accurate independent dominating set of a tree T by operation O. Thus $T \in \tau$. □

As an immediate consequence of lemmas 3.1 and 3.2, we have the following characterization of trees with equal domination and accurate independent domination number.

Theorem 3.1 *Let T be a tree. Then $i_a(T) = \gamma(T)$ if and only if $T \in \tau$.*

§4. Accurate Independent Domination of Some Graph Families

In this section accurate independent domination of *fan graph*,*double fan graph*, *helm graph* and *gear graph* are considered. We also obtain the corresponding relation between other dominating parameters and dominator coloring of the above graph families.

Definition 4.1 *A fan graph, denoted by F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex.*

Observation 4.1 Let F_n be a fan. Then,

1. F_n is a planar undirected graph with $2n + 1$ vertices and $3n$ edges;
2. F_n has exactly one vertex with $\Delta(F_n) = p - 1$;
3. $Diam(F_n) = 2$.

Theorem 4.1([2]) *For a fan graph $F_n, n \geq 2,\ \chi_d(F_n) = 3$.*

Proposition 4.1 *For a fan graph $F_n, n \geq 2,\ i_a(F_n) = 1$.*

Proof By Observation 4.1(2) and Theorem 2.5 result holds. □

Proposition 4.2 *For a fan graph $F_n, n \geq 2,\ i_a(F_n) < \chi_d(F_n)$.*

Proof By Proposition 4.1 and Theorem 4.1, we know that $\chi_d(F_n) = 3$. This implies that $i_a(F_n) < \chi_d(F_n)$. □

Definition 4.2 *A double fan graph, denoted by $F_{2,n}$ isomorphic to $P_n + 2K_1$.*

Observation 4.2

1. $F_{2,n}$ is a planar undirected graph with $(n+2)$ vertices and $(3n-1)$ edges;
2. $Diam(G) = 2$.

Theorem 4.2([2]) *For a double fan graph $F_{2,n}, n \geq 2,\ \chi_d(F_{2,n}) = 3$.*

Theorem 4.3 *For a double fan graph $F_{2,n}$, $n \geq 2$, $i_a(F_{2,2}) = 2$, $i_a(F_{2,3}) = 1$, $i_a(F_{2,5}) = 3$ and $i_a(F_{2,n}) = 2$ if $n \geq 7$.*

Proof Our proof is divided into cases following.

Case 1. If $n = 2$ *and* $n \geq 7$, then $F_{2,n},\ n \geq 2$ has only one accurate independent dominating set D of $|D| = 2$. Hence, $i_a(F_{2,n}) = 2$.

Case 2. If $n = 3$, $thenF_{2,3}$ has exactly one vertex of $\Delta(G) = p - 1$. Then by Theorem 2.5, $i_a(F_{2,n}) = 1$.

Case 3. If n=5 and D be a independent dominating set of G with $|D| = 2$, then $(V - D)$ also has an independent dominating set of cardinality 2. Hence D is not accurate.

Let D_1 be a independent dominating set with $|D_1| = 3$, then $V - D_1$ has no independent dominating set of cardinality 3. Then D_1 is accurate. Hence, $i_a(F_{2,n}) = 3$.

Case 4. If n=4 and 6, there does not exist accurate independent dominating set. □

Proposition 4.3 *For a double fan graph $F_{2,n}$, $n \geq 7$,*

$$\gamma(F_{2,n}) = i(F_{2,n}) = \gamma_a(F_{2,n}) = i_a(F_{2,n}) = 2$$

.

Proof Let $F_{2,n},\ n \geq 7$ be a Double fan graph. Then $2k_1$ forms a minimal dominating set of $F_{2,n}$ such that $\gamma(F_{2,n}) = 2$. Since this dominating set is independent and in $(V - D)$ there is no independent dominating set of cardinality 2 it is both independent and accurate independent dominating set. Also it is accurate dominating set. Hence,

$$\gamma(F_{2,n}) = i(F_{2,n}) = \gamma_a(F_{2,n}) = i_a(F_{2,n}) = 2.$$ □

Proposition 4.4 *For Double fan graph* $F_{2,n}$, $n \geq 7$

$$i_a(F_{2,n}) \leq \chi_d(F_{2,n}).$$

Proof The proof follows by Theorems 4.2 and 4.3. □

Definition 4.3([1]) *For* $n \geq 4$,*the wheel* W_n *is defined to be the graph* $W_n = C_{n-1} + K_1$. *Also it is defined as* $W_{1,n} = C_n + K_1$.

Definition 4.4 *A helm* H_n *is the graph obtained from* $W_{1,n}$ *by attaching a pendant edge at each vertex of the n-cycle.*

Observation 4.3 A helm H_n is a planar undirected graph with (2n+1) vertices and 3n edges.

Theorem 4.4([2]) *For Helm graph* H_n , $n \geq 3, \chi_d(H_n) = n + 1$.

Proposition 4.5 *For a helm graph* H_n, $n \geq 3$, $i_a(H_n) = n$.

Proof Let H_n , $n \geq 3$ be a helm graph. Then there exist a minimal independent dominating set D with $|D| = n$ and $(V - D)$ has no independent dominating set of cardinality n. Hence D is accurate. Therefore $i_a(H_n) = n$. □

Proposition 4.6 *For a helm graph* H_n , $n \geq 3$

$$\gamma(H_n) = i(H_n) = \gamma_a(H_n) = i_a(H_n) = n.$$

Proposition 4.7 *For a helm graph* H_n , $n \geq 3$

$$i_a(H_n) = \chi_d(H_n) - 1.$$

Proof Applying Proposition 4.5 , $i_a(H_n) = n = n + 1 - 1 = \chi_d(H_n) - 1$ by Theorem 4.4, $\chi_d(H_n) = n + 1$. Hence the proof. □

Definition 4.5 *A gear graph* G_n *also known as a bipartite wheel graph, is a wheel graph* $W_{1,n}$ *with a vertex added between each pair of adjacent vertices of the outer cycle.*

Observation 4.4 *A gear graph* G_n *is a planar undirected graph with* $2n + 1$ *vertices and* $3n$ *edges.*

Theorem 4.5([2]) *For a gear graph* G_n , $n \geq 3$,

$$\chi_d(G_n) = \lceil 2n/3 \rceil + 2.$$

Theorem 4.6 *For a gear graph* G_n , $n \geq 3$, $i_a(G_n) = n$.

Proof It is clear from the definition of gear graph G_n is obtained from wheel graph $W_{1,n}$ with a vertex added between each pair of adjacent vertices of the outer cycle of wheel graph $W_{1,n}$. These n vertices forms an independent dominating set in G_n such that $(V - D)$ has no independent dominating set of cardinality n. Therefore, the set D with cardinality n is accurate independent dominating set of G_n. Therefore $i_a(G_n) = n$. □

Corollary 4.1 *For any gear graph* G_n, $n \geq 3, \gamma(G_n) = i(G_n) = n - 1$.

Proposition 4.8 *For a gear graph* G_n , $n \geq 3$,

$$i_a(G_n) = \gamma_a(G_n).$$

Proposition 4.9 *For a graph* G_n , $n \geq 3$

$$i_a(G_n) = \gamma(G_n) + 1 = i(G_n) + 1.$$

Proof Applying Theorem 4.6 and Corollary 4.1, we know that $i_a(G_n) = n = n - 1 + 1 = \gamma(G_n) + 1 = i(G_n) + 1$. □

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On r-Dynamic Coloring of the Triple Star Graph Families

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Abstract: An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq \min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper we investigate the r-dynamic chromatic number of the central graph, middle graph, total graph and line graph of the triple star graph $K_{1,n,n,n}$ denoted by $C(K_{1,n,n,n})$, $M(K_{1,n,n,n})$, $T(K_{1,n,n,n})$ and $L(K_{1,n,n,n})$ respectively.

Key Words: Smarandachely r-dynamic coloring, r-dynamic coloring, triple star graph, central graph, middle graph, total graph and line graph.

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§1. Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5, 17]. Thus for a graph $G, \delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G respectively. When the context is clear we write, δ, Δ and χ for brevity. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v in G and $d(v) = |N(v)|$. The r-dynamic chromatic number was first introduced by Montgomery [14].

An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. Generally, for a subgraph $G' \prec G$ and a coloring c on G if $|c(N(v))| \geq min\{r, d(v)\}$ for $v \in V(G \setminus G')$ but $|c(N(v))| \leq min\{r, d(v)\}$ for $u \in V(G')$, such a r coloring is called a *Smarandachely r-dynamic coloring* on G. Clearly, if $G' = \emptyset$, a Smarandachely r-dynamic coloring is nothing else but the r-dynamic coloring.

The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number denoted by $\chi_d(G)$ [1-4, 8]. By simple observation, we can show that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ can

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be arbitrarily large, for example $\chi(Petersen) = 2$, $\chi_d(Petersen) = 3$, but $\chi_3(Petersen) = 10$. Thus, finding an exact values of $\chi_r(G)$ is not trivially easy.

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, for a graph G with $\Delta(G) \geq 3$, Lai et al. [8] proved that $\chi_d(G) \leq \Delta(G) + 1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [7]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2log_2\alpha(G) + 3$. Taherkhani gave in [15] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e., $\chi_2(G) - \chi(G) \leq \lceil (\Delta e)/\delta log\left(2e\left(\Delta^2+1\right)\right)\rceil$.

Li et al. proved in [10] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Li and Zhou [9] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

N.Mohanapriya et al. [11, 12] studied the dynamic chromatic number for various graph families. Also, it was proven in [13] that the $r-$ dynamic chromatic number of line graph of a helm graph H_n is

$$\chi_r(L(H_n)) = \begin{cases} n-1, & \delta \leq r \leq n-2, \\ n+1, & r = n-1, \\ n+2, & r = n \;\; and \;\; n \equiv 1 \;\; mod \;\; 3, \\ n+3, & r = n \;\; and \;\; n \not\equiv 1 \;\; mod \;\; 3, \\ n+4, & r = n+1 = \Delta, n \geq 6 \;\; and \;\; 2n-2 \equiv 0 \;\; mod \;\; 5, \\ n+5, & r = n+1 = \Delta, n \geq 6 \;\; and \;\; 2n-2 \not\equiv 0 \;\; mod \;\; 5. \end{cases}$$

In this paper, we study $\chi_r(G)$, the r- dynamic chromatic number of the middle, central, total and line graphs of the triple star graphs are discussed.

§2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [6] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The central graph [16] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [6, 16] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The line graph [13] of G denoted by $L(G)$ is the graph with vertices are the edges of G

with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Theorem 2.1 *For any triple star graph $K_{1,n,n,n}$, the r-dynamic chromatic number*

$$\chi_r(C(K_{1,n,n,n})) = \begin{cases} 2n+1, & r = 1 \\ 3n+1, & 2 \leq r \leq \Delta - 1 \\ 4n+1, & r \geq \Delta \end{cases}$$

Proof First we apply the definition of central graph on $K_{1,n,n,n}$. Let the edge vv_i, v_iw_i and w_iu_i be subdivided by the vertices $e_i(1 \leq i \leq n)$, $e'_i(1 \leq i \leq n)$ and $e''_i(1 \leq i \leq n)$ in $K_{1,n,n,n}$.

Clearly $V(C(K_{1,n,n,n})) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\} \bigcup \{e''_i : 1 \leq i \leq n\}$. The vertices $v_i(1 \leq i \leq n)$ induce a clique of order n (say K_n) and the vertices $v, u_i(1 \leq i \leq n)$ induce a clique of order $n+1$ (say K_{n+1}) in $C(K_{1,n,n,n})$ respectively. Thus, we have $\chi_r(C(K_{1,n,n,n})) \geq n+1$.

Case 1. $r = 1$.

Consider the color class $C_1 = \{c_1, c_2, c_3, \cdots, c_{(2n+1)}\}$ and assign the r-dynamic coloring to $C(K_{1,n,n,n})$ by Algorithm 2.1.1. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(C(K_{1,n,n,n})) = 2n+1$.

Case 2. $2 \leq r \leq \Delta - 1$.

Consider the color class $C_2 = \{c_1, c_2, c_3, \cdots, c_{(3n+1)}\}$ and assign the r-dynamic coloring to $C(K_{1,n,n,n})$ by Algorithm 2.1.2. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(C(K_{1,n,n,n})) = 3n+1$.

Case 3. $r \geq \Delta$.

Consider the color class $C_3 = \{c_1, c_2, c_3, \cdots, c_{(4n+1)}\}$ and assign the r-dynamic coloring to $C(K_{1,n,n,n})$ by Algorithm 2.1.3. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence $\chi_r(C(K_{1,n,n,n})) = 4n+1$. □

Algorithm 2.1.1

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $C(K_{1,n,n,n})$.

begin

for $i = 1$ to n

{

$V_1 = \{e_i\}$;

$C(e_i) = i$;

}

$V_2 = \{v\}$;

$C(v) = n+1$;

for $i = 1$ to n
{
$V_3 = \{v_i\}$;
$C(v_i) = n + i + 1$;
}
for $i = 1$ to n
{
$V_4 = \{e'_i\}$;
$C(e'_i) = n + 1$;
}
for $i = 1$ to n
{
$V_5 = \{w_i\}$;
$C(w_i) = i$;
}
for $i = 1$ to n
{
$V_6 = \{e''_i\}$;
$C(e''_i) = n + 1$;
}
for $i = 1$ to n
{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Algorithm 2.1.2

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $C(K_{1,n,n,n})$.

begin

for $i = 1$ to n
{
$V_1 = \{u_i\}$;
$C(u_i) = i$;
}
for $i = 1$ to n
{
$V_2 = \{e''_i\}$;
$C(e''_i) = n + 1$;
}

for $i = 1$ to n
{
$V_3 = \{w_i\}$;
$C(w_i) = n + i + 1$;
}
for $i = 1$ to n
{
$V_4 = \{e'_i\}$;
$C(e'_i) = i$;
}
for $i = 1$ to n
{
$V_5 = \{v_i\}$;
$C(v_i) = 2n + i + 1$;
}
for $i = 1$ to $n - 1$
{
$V_6 = \{e_i\}$;
$C(e_i) = 2n + i + 2$;
}
$C(e_n) = 2n + 2$;
$V_7 = \{v\}$;
$C(v) = n + 1$;
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Algorithm 2.1.3

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $C(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{u_i\}$;
$C(u_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;
for $i = 1$ to n
{
$V_3 = \{w_i\}$;
$C(w_i) = n + i + 1$;
}

for $i = 1$ to n
{
$V_4 = \{v_i\}$;
$C(v_i) = 2n + i + 1$;
}
for $i = 1$ to n
{
$V_5 = \{e_i\}$;
$C(e_i) = 3n + i + 1$;
}
for $i = 1$ to n
{
$V_6 = \{e'_i\}$;
$C(e'_i) = i$;
}
for $i = 1$ to n
{
$V_7 = \{e''_i\}$;
$C(e''_i) = 3n + 2$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Theorem 2.2 *For any triple star graph $K_{1,n,n,n}$, the r-dynamic chromatic number*

$$\chi_r(M(K_{1,n,n,n})) = \begin{cases} n+1, & 1 \leq r \leq n \\ n+2, & r = n+1 \\ n+3, & r \geq \Delta \end{cases}$$

Proof By definition of middle graph, each edge vv_i, v_iw_i and w_iu_i be subdivided by the vertices $e_i (1 \leq i \leq n)$, $e'_i(1 \leq i \leq n)$ and $e''_i(1 \leq i \leq n)$ in $K_{1,n,n,n}$ and the vertices v, e_i induce a clique of order $n+1$(say K_{n+1}) in $M(K_{1,n,n,n})$. i.e.,$V(M(K_{1,n,n,n})) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\} \bigcup \{e''_i : 1 \leq i \leq n\}$. Thus we have $\chi_r(M(K_{1,n,n,n})) \geq n+1$.

Case 1. $1 \leq r \leq n$.

Consider the color class $C_1 = \{c_1, c_2, c_3, \cdots, c_{(n+1)}\}$ and assign the r-dynamic coloring to $M(K_{1,n,n,n})$ by Algorithm 2.2.1. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(M(K_{1,n,n,n})) = n+1$, for $1 \leq r \leq n$.

Case 2. $r = n+1$.

Consider the color class $C_2 = \{c_1, c_2, c_3, \cdots, c_{(n+1)}, c_{(n+2)}\}$ and assign the r-dynamic coloring to $M(K_{1,n,n,n})$ by Algorithm 2.2.2. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(M(K_{1,n,n,n})) = n + 2$, for $r = n + 1$.

Case 3. $r = \Delta$.

Consider the color class $C_3 = \{c_1, c_2, c_3, \cdots, c_n, c_{(n+1)}, c_{(n+2)}, c_{(n+3)}\}$ and assign the r-dynamic coloring to $M(K_{1,n,n,n})$ by Algorithm 2.2.3. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(M(K_{1,n,n,n})) = n + 3$, for $r \geq \Delta$. □

Algorithm 2.2.1

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $M(K_{1,n,n,n})$.

begin

for $i = 1$ to n

{

$V_1 = \{e_i\}$;

$C(e_i) = i$;

}

$V_2 = \{v\}$;

$C(v) = n + 1$;

for $i = 1$ to n

{

$V_3 = \{v_i\}$;

$C(v_i) = n + 1$;

}

for $i = 1$ to $n - 1$

{

$V_4 = \{e'_i\}$;

$C(e'_i) = i + 1$;

}

$C(e'_n) = 1$;

for $i = 1$ to $n - 2$

{

$V_5 = \{w_i\}$;

$C(w_i) = i + 2$;

}

$C(w_{n-1}) = 1$;

$C(w_n) = 2$;

for $i = 1$ to n

{

$V_6 = \{e''_i\}$;

$C(e''_i) = n + 1$;

}
for $i = 1$ to n
{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Algorithm 2.2.2

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $M(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;
for $i = 1$ to n
{
$V_3 = \{v_i\}$;
$C(v_i) = n + 2$;
}
for $i = 1$ to n
{
$V_4 = \{e'_i\}$;
$C(e'_i) = n + 1$;
}
for $i = 1$ to $n - 1$
{
$V_5 = \{w_i\}$;
$C(w_i) = i + 1$;
}
$C(w_n) = 1$;
for $i = 1$ to $n - 2$
{
$V_6 = \{e''_i\}$;
$C(e''_i) = i + 2$;
}
$C(e''_{n-1}) = 1$;

$C(e''_n) = 2;$
for $i = 1$ to n
{
$V_7 = \{u_i\};$
$C(u_i) = n + 1;$
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7;$
end

Algorithm 2.2.3

Input: The number ”n” of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $M(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\};$
$C(e_i) = i;$
}
$V_2 = \{v\};$
$C(v) = n + 1;$
for $i = 1$ to n
{
$V_3 = \{v_i\};$
$C(v_i) = n + 2;$
}
for $i = 1$ to n
{
$V_4 = \{e'_i\};$
$C(e'_i) = n + 3;$
}
for $i = 1$ to n
{
$V_5 = \{w_i\};$
$C(w_i) = n + 1;$
}
for $i = 1$ to $n - 1$
{
$V_6 = \{e''_i\};$
$C(e''_i) = i + 1;$
}
$C(e''_n) = 1;$
for $i = 1$ to n

{
$V_7 = \{u_i\}$;
$C(u_i) = n + 2$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Theorem 2.3 *For any triple star graph $K_{1,n,n,n}$, the r-dynamic chromatic number,*

$$\chi_r(T(K_{1,n,n,n})) = \begin{cases} n+1, & 1 \leq r \leq n \\ r+1, & n+1 \leq r \leq \Delta - 2 \\ 2n, & r = \Delta - 1 \\ 2n+1, & r \geq \Delta \end{cases}$$

Proof By definition of total graph, each edge vv_i, v_iw_i and w_iu_i be subdivided by the vertices $e_i(1 \leq i \leq n)$, $e'_i(1 \leq i \leq n)$ and $e''_i(1 \leq i \leq n)$ in $K_{1,n,n,n}$ and the vertices v, e_i induce a clique of order $n+1$(say K_{n+1}) in $T(K_{1,n,n,n})$. i.e.,$V(T(K_{1,n,n,n})) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\} \bigcup \{e''_i : 1 \leq i \leq n\}$. Thus, we have $\chi_r(T(K_{1,n,n,n})) \geq n+1$.

Case 1. $1 \leq r \leq n$.

Consider the color class $C_1 = \{c_1, c_2, c_3, \cdots, c_{(n+1)}\}$ and assign the r-dynamic coloring to $T(K_{1,n,n,n})$ by Algorithm 2.3.1. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(T(K_{1,n,n,n})) = n+1$, for $1 \leq r \leq n$.

Case 2. $n+1 \leq r \leq \Delta - 2$.

Consider the color class $C_2 = \{c_1, c_2, c_3, \cdots, c_{(2n-1)}\}$ and assign the r-dynamic coloring to $T(K_{1,n,n,n})$ by Algorithm 2.3.2. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(T(K_{1,n,n,n})) = r+1$, for $n+1 \leq r \leq \Delta - 2$.

Case 3. $r = \Delta - 1$.

Consider the color class $C_3 = \{c_1, c_2, c_3, \cdots, c_{2n}\}$ if $r = \Delta - 1$ and assign the r-dynamic coloring to $T(K_{1,n,n,n})$ by Algorithm 2.3.3. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(T(K_{1,n,n,n})) = 2n$ for $r = \Delta - 1$.

Case 4. $r = \Delta$.

Consider the color class $C_4 = \{c_1, c_2, c_3, \cdots, c_{2n+1}\}$ if $r = \Delta$ and assign the r-dynamic coloring to $T(K_{1,n,n,n})$ by Algorithm 2.3.4. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(T(K_{1,n,n,n})) = 2n+1$ for $r \geq \Delta$. □

Algorithm 2.3.1

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $T(K_{1,n,n,n})$.

begin

for $i = 1$ to n

{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;

for $i = 1$ to $n - 3$

{
$V_3 = \{v_i\}$;
$C(v_i) = i + 3$;
}
$C(v_{n-2}) = 1$;
$C(v_{n-1}) = 2$;
$C(v_n) = 3$;

for $i = 1$ to $n - 2$

{
$V_4 = \{e'_i\}$;
$C(e'_i) = i + 2$;
}
$C(e'_{n-1}) = 1$;
$C(e'_n) = 2$;

for $i = 1$ to $n - 1$

{
$V_5 = \{w_i\}$;
$C(w_i) = i + 1$;
}
$C(w_n) = 1$;

for $i = 1$ to n

{
$V_6 = \{e''_i\}$;
$C(e''_i) = n + 1$;
}

for $i = 1$ to n

{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}

$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;

end

Algorithm 2.3.2

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $T(K_{1,n,n,n})$.

begin

for $i = 1$ to n

{

$V_1 = \{e_i\}$;

$C(e_i) = i$;

}

$V_2 = \{v\}$;

$C(v) = n + 1$;

for $i = 1$ to $n - 2$

{

$V_3 = \{v_i\}$;

$C(v_i) = r + 1$;

}

$C(v_{n-1}) = n + 2$;

$C(v_n) = n + 3$;

for $i = 1$ to $n - 3$

{

$V_4 = \{e'_i\}$;

$C(e'_i) = n + i + 2$;

}

$C(e'_{n-2}) = n + 2$;

$C(e'_{n-1}) = n + 3$;

$C(e'_n) = n + 4$;

for $i = 1$ to $n - 1$

{

$V_5 = \{w_i\}$;

$C(w_i) = i + 1$;

}

$C(w_n) = 1$;

for $i = 1$ to n

{

$V_6 = \{e''_i\}$;

$C(e''_i) = n + 1$;

}

for $i = 1$ to n

{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Algorithm 2.3.3

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $T(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;
for $i = 1$ to $n - 1$
{
$V_3 = \{v_i\}$;
$C(v_i) = n + i + 1$;
}
$C(v_n) = n + 2$;
for $i = 1$ to $n - 2$
{
$V_4 = \{e'_i\}$;
$C(e'_i) = n + i + 2$;
}
$C(e'_{n-1}) = n + 2$;
$C(e'_n) = n + 3$;
for $i = 1$ to $n - 1$
{
$V_5 = \{w_i\}$;
$C(w_i) = i + 1$;
}
$C(w_n) = 1$;
for $i = 1$ to n
{
$V_6 = \{e''_i\}$;
$C(e''_i) = n + 1$;
}

for $i = 1$ to n
{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Algorithm 2.3.4

Input: The number "n" of $K_{1,n,n,n}$.
Output: Assigning r-dynamic coloring for the vertices in $T(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;
for $i = 1$ to n
{
$V_3 = \{v_i\}$;
$C(v_i) = n + i + 1$;
}
for $i = 1$ to $n - 1$
{
$V_4 = \{e'_i\}$;
$C(e'_i) = n + i + 2$;
}
$C(e'_n) = n + 2$;
for $i = 1$ to $n - 1$
{
$V_5 = \{w_i\}$;
$C(w_i) = i + 1$;
}
$C(w_n) = 1$;
for $i = 1$ to n
{
$V_6 = \{e''_i\}$;
$C(e''_i) = n + 1$;
}
for $i = 1$ to n

{
$V_7 = \{u_i\}$;
$C(u_i) = i$;
}
$V = V_1 \bigcup V_2 \bigcup V_3 \bigcup V_4 \bigcup V_5 \bigcup V_6 \bigcup V_7$;
end

Theorem 2.4 *For any triple star graph $K_{1,n,n,n}$, the r-dynamic chromatic number,*

$$\chi_r(L(K_{1,n,n,n})) = \begin{cases} n, & 1 \leq r \leq n-1 \\ n+1, & r \geq \Delta \end{cases}$$

Proof First we apply the definition of line graph on $K_{1,n,n,n}$. By the definition of line graph, each edge of $K_{1,n,n,n}$ taken to be as vertex in $L(K_{1,n,n,n})$.The vertices $e_1, e_2, \cdots, e_n$ induce a clique of order n in $L(K_{1,n,n,n})$. i.e., $V(L(K_{1,n,n,n})) = E(K_{1,n,n,n}) = \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\} \bigcup \{e''_i : 1 \leq i \leq n\}$. Thus, we have $\chi_r(L(K_{1,n,n,n})) \geq n$.

Case 1. $1 \leq r \leq \Delta - 1$.

Now consider the vertex set $V(L(K_{1,n,n,n}))$ and color class $C_1 = \{c_1, c_2, \cdots, c_n\}$, assign r dynamic coloring to $L(K_{1,n,n,n})$ by Algorithm 2.4.1. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(L(K_{1,n,n,n})) = n$, for $1 \leq r \leq \Delta - 1$.

Case 2. $r \geq \Delta$.

Now consider the vertex set $V(L(K_{1,n,n}))$ and color class $C_2 = \{c_1, c_2, \cdots, c_n, c_{n+1}\}$, assign r dynamic coloring to $L(K_{1,n,n,n})$ by Algorithm 2.4.2. Thus, an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(L(K_{1,n,n,n})) = n+1$ for $r \geq \Delta$. □

Algorithm 2.4.1

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $L(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
for $i = 1$ to $n-1$
{
$V_2 = \{e'_i\}$;
$C(e'_i) = i+1$;
}
$C(e'_n) = 1$;

for $i = 1$ to $n-2$
{
$V_3 = \{e''_i\}$;
$C(e''_i) = i+2$;
}
$C(e''_{n-1}) = 1$;
$C(e''_n) = 2$;
$V = V_1 \bigcup V_2 \bigcup V_3$;
end

Algorithm 2.4.2

Input: The number "n" of $K_{1,n,n,n}$.

Output: Assigning r-dynamic coloring for the vertices in $L(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
for $i = 1$ to n
{
$V_2 = \{e'_i\}$;
$C(e'_i) = n+1$;
}
for $i = 1$ to $n-1$
{
$V_3 = \{e''_i\}$;
$C(e''_i) = i+1$;
}
$C(e''_n) = 1$;
$V = V_1 \bigcup V_2 \bigcup V_3$;
end

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(1,N)-Arithmetic Labelling of Ladder and Subdivision of Ladder

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Abstract: A (p, q)-graph G is said to be $(1, N)$-arithmetic labelling if there is a function ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$. In this paper we prove that ladder and subdivision of ladder are $(1, N)$-arithmetic labelling for every positive integer $N > 1$.

Key Words: Ladder, subdivision of ladder, one modulo N graceful, Smarandache k modulo N graceful.

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§1. Introduction

V.Ramachandran and C. Sekar [8, 9] introduced one modulo N graceful where N is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. Generally, a graph G with q edges is called to be *Smarandache k modulo N graceful* if one replacing N by kN in the definition of one modulo N graceful graph. Clearly, a graph G is Smarandache k modulo N graceful if and only if it is one modulo kN graceful by definition.

B. D. Acharya and S. M. Hegde [2] introduced (k, d)- arithmetic graphs. A (p, q)- graph G is said to be (k, d)- arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k+d, k+2d, \ldots, k+(q-1)d$. Joseph A. Gallian [4] surveyed numerous graph labelling methods.

V.Ramachandran and C. Sekar [10] introduced $(1, N)$-Arithmetic labelling. We proved that stars, paths, complete bipartite graph $K_{m,n}$, highly irregular graph $H_i(m, m)$ and cycle C_{4k} are $(1, N)$-Arithmetic labelling,C_{4k+2} is not $(1, N)$-Arithmetic labelling. We also proved that no graph G containing an odd cycle is $(1, N)$-arithmetic labelling for every positive integer

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N. A (p, q)-graph G is said to be $(1, N)$-Arithmetic labelling if there is a function $\phi : V(G) \to \{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$.

In this situation the induced mapping ϕ^* to the edges is given by $\phi^*(uv)=\phi(u)+\phi(v)$. If the values of $\phi(u)+\phi(v)$ are $1, N+1, 2N+1, \ldots, N(q-1)+1$ all distinct, then we call the labelling of vertices as $(1, N)$- Arithmetic labelling. In case if the induced mapping ϕ^* is defined as $\phi^*(uv)=|\phi(u)-\phi(v)|$ and if the resulting edge labels are are distinct and equal to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$. We call it as one modulo N graceful. In this paper we prove that Ladder and Subdivision of Ladder are $(1, N)$-Arithmetic labelling for every positive integer $N > 1$.

§2. Main Results

Definition 2.1 *A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u)-\phi(v)|$.*

Definition 2.2 *A (p, q)-graph G is said to be $(1, N)$-Arithmetic labelling if there is a function ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$.*

Definition 2.3 *A (p, q)- graph G is said to be (k, d)- arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k+d, k+2d, \cdots, k+(q-1)d$.*

Definition 2.4([7]) *Let G be a graph with p vertices and q edges. A graph H is said to be a subdivision of G if H is obtained from G by subdividing every edge of G exactly once. H is denoted by $S(G)$.*

Definition 2.5 *The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is a path with $\times$ denotes the cartesian product. L_n has $2n$ vertices and $3n-2$ edges.*

Theorem 2.6 *For every positive integer n, ladder L_n is $(1, N)$-Arithmetic labelling, for every positive integer $N > 1$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of L_n, respectively, and let $u_i v_{i+1}, i = 1, 2, \cdots, n-1$. $v_i u_{i+1}, i = 1, 2, \cdots, n-1$ and $u_i v_i, i = 1, 2, \cdots, n$ be the edges of L_n. The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is a path with $\times$ denotes the cartesian product. Then the ladder L_n has $2n$vertices and $3n-2$ edges as shown in figures following. Define $\phi(u_i) = N(i-1)$ for $i = 1, 2, 3, \cdots, n$, $\phi(v_i) = 2N(i-1)+1$ for $i = 1, 2, 3, \cdots, n$.

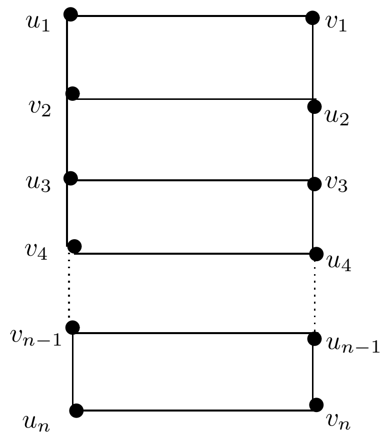

Figure 1 Ladder L_n where n is odd

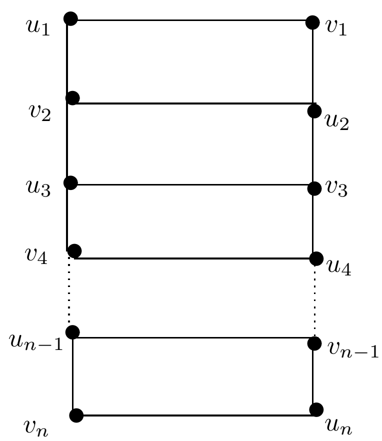

Figure 2 Ladder L_n where n is even

From the definition of ϕ it is clear that

$$\{\phi(u_i), i = 1, 2, \cdots, n\} \bigcup \{\phi(v_i), i = 1, 2, \cdots, n\}$$
$$= \{0, N, 2N, \ldots, N(n-1)\} \cup \{1, 2N+1, 4N+1, \cdots, 2N(n-1)+1\}$$

It is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

for $i = 1, 2, \cdots, n$, $\phi^*(v_i u_i) = \mid \phi(v_i) + \phi(u_i) \mid = 3N(i-1)+1$; for $i = 1, 2, \cdots, n-1$, $\phi^*(v_{i+1}u_i) = \mid \phi(v_{i+1}) + \phi(u_i) \mid = N(3i-1)+1$, $\phi^*(v_i u_{i+1}) = \mid \phi(v_i) + \phi(u_{i+1}) \mid = N(3i-2)+1$.

This shows that the edges have the distinct labels $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $q = 3n-2$. Hence L_n is $(1, N)$-Arithmetic labelling for every positive integer $N > 1$. □

Example 2.7. A $(1, 5)$-Arithmetic labelling of L_6 is shown in Figure 3.

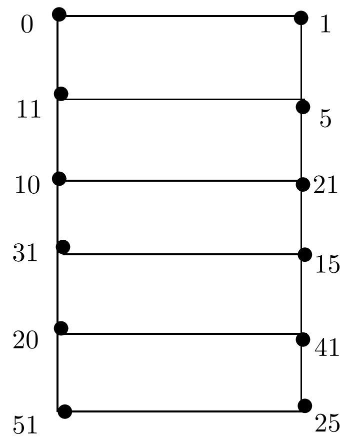

Figure 3

Example 2.8 A $(1,2)$-Arithmetic labelling of L_7 is shown in Figure 4.

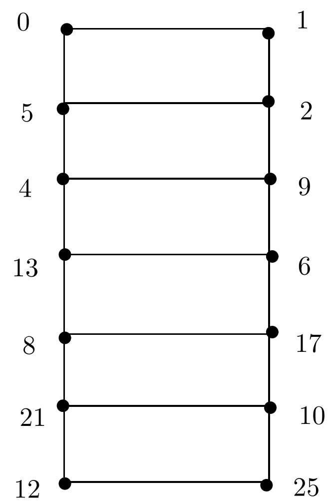

Figure 4

Theorem 2.9 *A subdivision of ladder L_n is $(1,N)$-Arithmetic labelling for every positive integer $N > 1$.*

Proof Let $G = L_n$. The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is a path with $\times$ denotes the cartesian product. L_n has $2n$ vertices and $3n-2$ edges. A graph H is said to be a subdivision of G if H is obtained from G by subdividing every edge of G exactly once. H is denoted by $S(G)$. Then the subdivision of ladder L_n has $5n-2$ vertices and $6n-4$ edges as shown in Figure 5. Let $H = S(L_n)$.

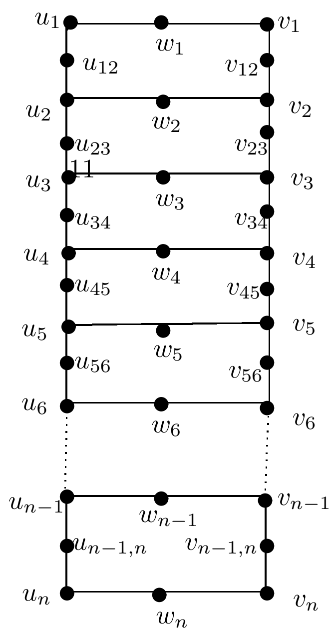

Figure 5 Subdivision of ladder L_n

Define the following functions:

$\eta : N \rightarrow N$ by

$$\eta(i) = \begin{cases} N(2i-1) & \text{if } i \text{ is even} \\ 2N(i-1) & \text{if } i \text{ is odd} \end{cases}$$

and $\gamma : N \rightarrow N$ by

$$\gamma(i) = \begin{cases} 2N(i-1) & \text{if } i \text{ is even} \\ N(2i-1) & \text{if } i \text{ is odd} \end{cases}$$

Define $\phi : V \rightarrow \{0, 1, 2, \cdots, q\}$ by

$$\phi(u_i) = \eta(i), i = 1, 2, \cdots, n$$
$$\phi(v_i) = \gamma(i), i = 1, 2, \cdots, n.$$

Define

$$\phi(u_{i,i+1}) = \begin{cases} 1 + (i-1)4N & \text{if } i \text{ is odd} \\ (4i-1)N + 1 & \text{if } i \text{ is even.} \end{cases}$$

For $i = 1, 2, \cdots, n-2$, define

$$\phi(v_{i,i+1}) = \phi(u_{i+1,i+2}) - 4N, \quad \phi(v_{n-1,n}) = \phi(u_{n-2,n-1}) + 4N,$$

$$\phi(w_i) = \begin{cases} 1 + (4i-3)N & \text{if } i = 1, 2, \cdots, n-1 \\ 4Nn - 4N + 1 & \text{if } i = n\ . \end{cases}$$

It is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

$$\phi^*(w_n u_n) = \mid \phi(w_n) + \phi(u_n) \mid = 6Nn - 6N + 1,$$

$$\phi^*(w_n v_n) = \mid \phi(w_n) + \phi(v_n) \mid = 6Nn - 5N + 1,$$

$$\phi^*(v_{n-1,n} v_{n-1}) = \mid \phi(v_{n-1,n}) + \phi(v_{n-1}) \mid = \begin{cases} 6Nn - 12N + 1 & \text{if } n \text{ is odd} \\ 6Nn - 8N + 1 & \text{if } n \text{ is even.} \end{cases},$$

$$\phi^*(v_{n-1,n} v_n) = \mid \phi(v_{n-1,n}) + \phi(v_n) \mid = \begin{cases} 6Nn - 9N + 1 & \text{if } n \text{ is odd} \\ 6Nn - 7N + 1 & \text{if } n \text{ is even.} \end{cases}$$

For $i = 1, 2, \cdots, n-1$,

$$\phi^*(w_i u_i) = \mid \phi(w_i) + \phi(u_i) \mid = \begin{cases} N(6i-4) + 1 & \text{if } i \text{ is even} \\ N(6i-5) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$\phi^*(w_i v_i) = \mid \phi(w_i) + \phi(v_i) \mid = \begin{cases} N(6i-5) + 1 & \text{if } i \text{ is even} \\ N(6i-4) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

For $i = 1, 2, \cdots, n-1$,

$$\phi^*(u_{i,i+1} u_i) = \mid \phi(u_{i,i+1}) + \phi(u_i) \mid = \begin{cases} N(6i-2) + 1 & \text{if } i \text{ is even} \\ N(6i-6) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$\phi^*(u_{i,i+1} u_{i+1}) = \mid \phi(u_{i,i+1}) + \phi(u_{i+1}) \mid = \begin{cases} N(6i-3) + 1 & \text{if } i \text{ is odd} \\ N(6i-1) + 1 & \text{if } i \text{ is even.} \end{cases}$$

For $i = 1, 2, \cdots, n-2$,

$$\phi^*(v_{i,i+1} v_i) = \mid \phi(v_{i,i+1}) + \phi(v_i) \mid = \begin{cases} N(6i-6) + 1 & \text{if } i \text{ is even} \\ N(6i-2) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

$$\phi^*(v_{i,i+1} v_{i+1}) = \mid \phi(v_{i,i+1}) + \phi(v_{i+1}) \mid = \begin{cases} N(6i-3) + 1 & \text{if } i \text{ is even} \\ N(6i-1) + 1 & \text{if } i \text{ is odd.} \end{cases}$$

This shows that the edges have distinct labels $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$ with $q = 6n-4$. Hence $S(L_n)$ is $(1, N)$-Arithmetic labelling for every positive integer $N > 1$. □

Example 2.10 A $(1,3)$-Arithmetic labelling of $S(L_5)$ is shown in Figure 6.

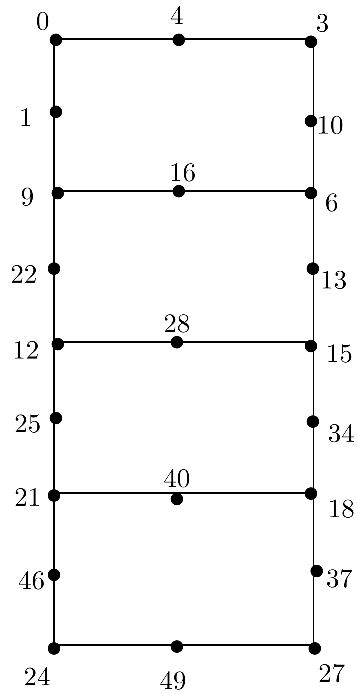

Figure 6

Example 2.11 A $(1,10)$-Arithmetic labelling of $S(L_6)$ is shown in Figure 7.

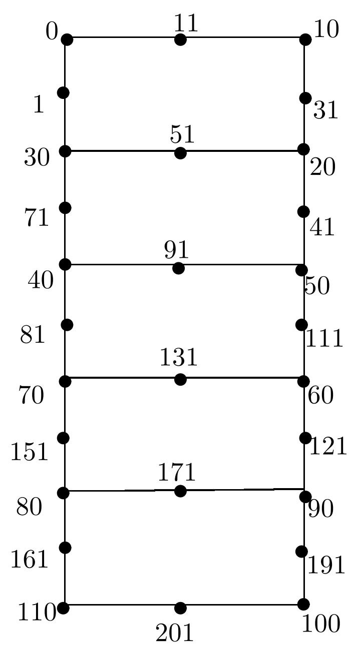

Figure 7

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3-Difference Cordial Labeling of Corona Related Graphs

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Abstract: Let G be a (p,q) graph. Let $f: V(G) \to \{1,2,\cdots,k\}$ be a map where k is an integer $2 \leq k \leq p$. For each edge uv, assign the label $|f(u)-f(v)|$. f is called k-difference cordial labeling of G if $|v_f(i)-v_f(j)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labelled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a k-difference cordial labeling is called a k-difference cordial graph. In this paper we investigate 3-difference cordial labeling behavior of $DT_n \odot K_1$ $DT_n \odot 2K_1$,$DT_n \odot K_2$ and some more graphs.

Key Words: Difference cordial labeling, Smarandachely k-difference cordial labeling, path, complete graph, triangular snake, corona.

AMS(2010): 05C78.

§1. Introduction

All Graphs in this paper are finite ,undirect and simple. Let G_1, G_2 respectively be (p_1,q_1), (p_2,q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. Ponraj et al. [3], has been introduced the concept of k-difference cordial labeling of graphs and studied the 3-difference cordial labeling behavior of of some graphs. In [4,5,6,7] they investigate the 3-difference cordial labeling behavior of path, cycle, complete graph, complete bipartite graph, star, bistar, comb, double comb, quadrilateral snake, $C_4^{(t)}$, $S(K_{1,n})$, $S(B_{n,n})$ and carona of some graphs with double alternate triangular snake double alternate quadrilateral snake . In this paper we examine the 3-difference cordial labeling behavior of $DT_n \odot K_1$ $DT_n \odot 2K_1$,$DT_n \odot K_2$ etc. Terms are not defined here follows from Harary [2].

§2. k-Difference Cordial Labeling

Definition 2.1 *Let G be a (p,q) graph and let $f: V(G) \to \{1,2,\cdots,k\}$ be a map. For*

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each edge uv, assign the label $|f(u) - f(v)|$. f is called a k-difference cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labelled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a k-difference cordial labeling is called a k-difference cordial graph.

On the other hand, if $|v_f(i) - v_f(j)| \geq 1$ or $|e_f(0) - e_f(1)| \geq 1$, such a labeling is called a Smarandachely k-difference cordial labeling of G.

A double triangular snake DT_n consists of two triangular snakes that have a common path. That is a double triangular snake is obtained from a path $u_1u_2\cdots u_n$ by joining u_i and u_{i+1} to two new vertices v_i $(1 \leq i \leq n-1)$ and w_i $(1 \leq i \leq n-1)$.

First we investigate the 3-difference cordial labeling behavior of $DT_n \odot K_1$.

Theorem 2.1 *$DT_n \odot K_1$ is 3-difference cordial.*

Proof Let $V(DT_n \odot K_1) = V(DT_n) \bigcup \{x_i : 1 \leq i \leq n\} \bigcup \{v_i', w_i' : 1 \leq i \leq n-1\}$ and $E(DT_n \odot K_1) = E(DT_n) \bigcup \{u_ix_i : 1 \leq i \leq n\} \bigcup \{v_iv_i', w_iw_i' : 1 \leq i \leq n-1\}$.

Case 1. n is even.

First we consider the path vertices u_i. Assign the label 1 to all the path vertices u_i $(1 \leq i \leq n)$. Then assign the label 2 to the path vertices $v_1, v_3, v_5, \cdots$ and assign the label 1 to the path vertices $v_2, v_4, v_6, \cdots$. Now we consider the vertices w_i. Assign the label 2 to all the vertices w_i $(1 \leq i \leq n-1)$. Next we move to the vertices v_i' and w_i'. Assign the label 2 to the vertices v_{2i+1}' for all the values of $i = 0, 1, 2, 3, \cdots$ and assign the label 1 to the vertices v_{2i} for $i = 1, 2, 3, \cdots$. Next we assign the label 1 to the vertex w_1' and assign the label 3 to the vertices $w_2', w_3', w_4', \cdots$ Finally assign the label 3 to all the vertices of x_i $(1 \leq i \leq n)$. The vertex condition and the edge conditions are $v_f(1) = v_f(2) = \frac{6n-3}{3}$ and $v_f(3) = \frac{6n-2}{3}$ and $e_f(0) = 4n-4$ and $e_f(1) = 4n-3$.

Case 2. n is odd.

Assign the label to the path vertices u_i $(1 \leq i \leq n)$, v_i $(1 \leq i \leq n-1)$, w_i $(1 \leq i \leq n-1)$, v_i' $(1 \leq i \leq n-1)$, x_i $(1 \leq i \leq n)$ as in case 1. Then assign the label 3 to all the vertices w_i' $(1 \leq i \leq n-1)$. Since $e_f(0) = 4n-3$, $e_f(1) = 4n-4$ and $v_f(1) = v_f(3) = 2n-1$ and $v_f(2) = 2n-2$, $DT_n \odot K_1$ is 3-difference cordial. □

Next investigation about $DT_n \odot 2K_1$.

Theorem 2.2 *$DT_n \odot 2K_1$ is 3-difference cordial.*

Proof Let $V(DT_n \odot 2K_1) = V(DT_n) \bigcup \{x_i, y_i : 1 \leq i \leq n\} \bigcup \{v_i', v_i'', w_i', w_i'' : 1 \leq i \leq n-1\}$ and $E(DT_n \odot 2K_1) = E(DT_n) \bigcup \{u_ix_i, u_iy_i : 1 \leq i \leq n\} \bigcup \{v_iv_i', v_iv_i'', w_iw_i', w_iw_i'' : 1 \leq i \leq n-1\}$.

Case 1. n is even.

Consider the path vertices u_i. Assign the label 1 to the path vertex u_1. Now we assign the labels 1,1,2,2 to the vertices u_2, u_3, u_4, u_5 respectively. Then we assign the labels 1,1,2,2 to the next four vertices u_6, u_7, u_8, u_9 respectively. Proceeding like this we assign the label to the next four vertices and so on. If all the vertices are labeled then we stop the process. Otherwise there are some non labeled vertices are exist. If the number of non labeled vertices are less than or equal to 3 then assign the labels 1,1,2 to the non labeled vertices. If it is two then assign the label 1,1 to the non labeled vertices. If only one non labeled vertex is exist then assign the label 1 only. Next we consider the label v_i. Assign the label 2 to the vertex v_1. Then we assign the label 2 to the vertices $v_2, v_4, v_6, \cdots$ and assign the label 3 to the vertices $v_3, v_5, v_7, \cdots$. Next we move to the vertices x_i and y_i. Assign the label 2 to the vertices x_1 and x_2 and we assign the label 3 to the vertices y_1 and y_2. Now we assign the label 1 to the vertices x_{4i+1} and x_{4i} for all the values of $i = 1, 2, 3, \cdots$. Then we assign the label 1 to the vertices x_{4i+3} for $i = 0, 1, 2, 3, \cdots$. Next we assign the label 2 to the vertices x_{4i+2} for all the values of $i = 1, 2, 3, \cdots$. Now we assign the label 3 to the vertices y_{4i+3} for i=0,1,2,3,... For all the values of $i = 1, 2, 3, \cdots$. assign the label 3 to the vertices y_{4i+1} and y_{4i+2}. Then we assign the label 2 to the vertices y_{4i} for $i = 1, 2, 3, \cdots$. Now we consider the vertices v'_i and v''_i. For all the values of i=1,2,3... assign the label 1 to the vertices v'_{4i+1}, v'_{4i+2}. Assign the label 1 to the vertices v'_{4i} for $i = 1, 2, 3, \cdots$. Then we assign the label 2 to the vertices v'_{4i+3} for all the values of $i = 0, 1, 2, 3, \cdots$. Consider the vertices v''_i. Assign the label 3 to the path vertex v''_{4i+1}, v''_{4i+2} and v''_{4i+3} for all the values of $i = 0, 1, 2, 3, \cdots$. Next we assign the label 2 to the vertices v''_{4i} for $i = 1, 2, 3, \cdots$. Now we assign the label 3 to the vertices $w_i (1 \le i \le n-1)$. Next we move to the vertices w'_i and w''_i. Assign the label 1 to all the vertices of $w'_i(1 \le i \le n-1)$ and we assign the label 2 to all the vertices of $w''_i(1 \le i \le n-1)$. Since $v_f(1) = v_f(2) = v_f(3) = 3n-2$ and $e_f(0) = \frac{11n-10}{2}$ and $e_f(1) = \frac{11n-8}{2}$, this labeling is 3-difference cordial labeling.

Case 2. n is odd.

First we consider the path vertices u_i. Assign the label 1,1,2,2 to the first four path vertices u_1, u_2, u_3, u_4 respectively. Then we assign the labels 1,1,2,2 to the next four vertices u_5, u_6, u_7, u_8 respectively. Continuing like this assign the label to the next four vertices and so on. If all the vertices are labeled then we stop the process. Otherwise there are some on labeled vertices are exist. If the number of non labeled vertices are less than or equal to 3 then assign the labels 1,1,2 to the non labeled vertices. If it is 2 assign the labels 1,1 to the non labeled vertices. If only one non labeled vertex exist then assign the label 1 to that vertex. Consider the vertices v_i. Assign the label 2 to the vertices $v_1, v_3, v_5, \cdots$ and we assign the label 3 to the vertices $v_2, v_4, v_6, \cdots$. Next we move to the vertices w_i. Assign the label to the vertices w_i $(1 \le i \le n-)$ as in case 1. Now we consider the vertices x_i and y_i. Assign the label 2 to the vertices x_{4i+1} for all the values of $i = 0, 1, 2, 3, \cdots$. For all the values of i=0,1,2,3,... assign the label 1 to the vertices x_{4i+2} and x_{4i+3}. Then we assign the label 1 to the vertices x_{4i} for all the values of $i = 1, 2, 3, \cdots$. Next we assign the label 3 to the vertices y_{4i+1} and y_{4i+2} for all the values of $i = 0, 1, 2, 3, \cdots$ and we assign the label 3 to the vertices y_{4i} for $i = 1, 2, 3, \cdots$. Then we assign the label 2 to the vertices y_{4i+3} for all values $i = 0, 1, 2, 3, \cdots$. Next we move to the vertices v'_i and v''_i. For all the values of $i = 0, 1, 2, 3, \cdots$ assign the label 1 to the vertices v'_{4i+1} and v'_{4i+3}. Now we assign the label 1 to the vertices v'_{4i} for $i = 1, 2, 3, \cdots$. Next we assign the label 2 to the vertices v'_{4i+2} for $i = 01, 2, 3, \cdots$. Consider the vertices v'_i. Assign the label

3 to the vertices v''_{4i+1} and v''_{4i+2} for all the values of $i = 0, 1, 2, 3, \cdots$ and we assign the label 1 to the vertices v_{4i} for $i = 1, 2, 3, \cdots$. For the values of $i = 0, 1, 2, 3, \cdots$ assign the label 2 to the vertices v_{4i+3}. Finally we consider the vertices w'_i and w''_i. Assign the label to the vertices w'_i $(1 \le i \le n-1)$ and w''_i $(1 \le i \le n-1)$ as in case 1. The vertex and edge condition are $v_f(1) = v_f(2) = v_f(3) = 3n-2$ and $e_f(0) = e_f(1) = \frac{11n-9}{2}$. □

We now investigate the graph $DT_n \odot K_2$.

Theorem 2.3 *$DT_n \odot K_2$ is 3-difference cordial.*

Proof Let $V(DT_n \odot K_2) = V(DT_n) \bigcup \{x_i, y_i : 1 \le i \le n\} \bigcup \{v'_i, v''_i, w'_i, w''_i : 1 \le i \le n-1\}$ and $E(DT_n \odot K_2) = E(DT_n) \bigcup \{u_i x_i, u_i y_i, x_i y_i : 1 \le i \le n\} \bigcup \{v_i v'_i, v_i v''_i, v'_i v''_i, w_i w'_i, w_i w''_i, w'_i w''_i : 1 \le i \le n-1\}$.

Case 1. n is even.

Consider the path vertices u_i. Assign the label 1 to the path vertices $u_1, u_2, u_3, \cdots$. Then we assign the labels 2 to the vertices $v_1, v_2, v_3, \cdots$. Next we assign the labels 3 to the vertices w_1, w_2, w_3, w_4. Now we consider the vertices v'_i and v''_i. Assign the label 2 to the vertex v'_1. Then we assign the label 1 to the vertices $v'_2, v'_3, v'_4, v'_6, \cdots$. Now we assign the label 3 to the vertices $v''_1, v''_2, v''_3, v''_4, \cdots$. Next we move to the vertices w'_i and w''_i. Assign the label 1 to the vertex w'_1. Then we assign the label 1 to the vertices $w'_2, w'_4, w'_6, \cdots$ and assign the label 2 to the vertices $w'_3, w'_5, w'_7, \cdots$. Assign the label 2 to the vertices $w''_1, w''_2, w''_3, w''_4, \cdots$. Finally we move to the vertices x_i and y_i. Assign the label 1 to the vertices $x_1, x_3, x_5, \cdots$ and we assign the label 2 to the vertices $x_2, x_4, x_6, \cdots$ then we assign the label 3 to the vertices $y_1, y_2, y_3, \cdots$. Clearly in this case the vertex and edge condition is given in $v_f(1) = v_f(2) = v_f(3) = 3n-2$ and $e_f(0) = 7n-5$ and $e_f(1) = 7n-6$.

Case 2. n is odd.

Assign the label to the vertices u_i $(1 \le i \le n)$, v_i $(1 \le i \le n-1)$ and w_i $(1 \le i \le n-1)$ as in case 1. Consider the vertices v'_i and v''_i. Assign the label 1 to the vertices $v'_1, v'_2, v'_3, v'_4, \cdots$. Then assign the label to the vertices v''_i $(1 \le i \le n-1)$ as in case 1. Now we move to the vertices w'_i and w''_i. Assign the label 1 to the vertices $w'_1, w'_3, w'_5, \cdots$ and we assign the label 3 to the vertices $w'_2, w'_4, w'_6, \cdots$. Next we assign the label to the vertices w''_i $(1 \le i \le n-1)$ as in case 1. Now we consider the vertices x_i and y_i. Assign the label 2 to the vertices $x_1, x_3, x_5, \cdots$ and we assign the label 1 to the vertices $x_2, x_4, x_6, \cdots$. Then we assign the label to the vertices y_i $(1 \le i \le n)$ as in case 1. Since $v_f(1) = v_f(2) = v_f(3) = 3n-2$ and $e_f(0) = 7n-6$ and $e_f(1) = 7n-5$, this labeling is 3-difference cordial labeling. □

A double quadrilateral snake DQ_n consists of two quadrilateral snakes that have a common path. Let $V(DQ_n) = \{u_i : 1 \le i \le n\} \bigcup \{v_i, w_i, xi, y_i : 1 \le i \le n-1\}$ and $E(DQ_n) = \{u_i u_{i+1}, v_i w_i, x_i y_i, w_i u_{i+1}, y_i u_{i+1} : 1 \le i \le n-1\}$.

Now we investigate the graphs $DQ_n \odot K_1$,$DQ_n \odot 2K_1$ and $DQ_n \odot K_2$.

Theorem 2.4 *$DQ_n \odot K_1$ is 3-difference cordial.*

Proof Let $V(DQ_n \odot K_1) = V(DQ_n) \bigcup \{u'_i : 1 \leq i \leq n\} \bigcup \{v'_i, w'_i, x'_i, y'_i : 1 \leq i \leq n-1\}$ and $E(DQ_n \odot K_1) = E(DQ_n) \bigcup \{u_i u'_i : 1 \leq i \leq n\} \bigcup \{u_i v'_i, w_i w'_i, x_i x'_i, y_i, y'_i : 1 \leq i \leq n-1\}$. Assign the label 1 to the path vertex u_1. Next we assign the labels 1,1,2 to the vertices u_2, u_3, u_4 respectively. Then we assign the labels 1,1,2 to the next three path vertices u_5, u_6, u_7 respectively. Proceeding like this we assign the label to the next three vertices and so on. If all the vertices are labeled then we stop the process. Otherwise there are some non labeled vertices are exist. If the number of non labeled vertices are less than or equal to 2 then assign the labels 1,1 to the non labeled vertices. If only one non labeled vertex exist then assign the label 1 only. Now we consider the vertices v_i and w_i. Assign the label 2 to the vertices v_{3i+1} and v_{3i+2} for all the values of i=0,1,2,3... For all the vales of $i = 1, 2, 3, \cdots$ assign the label 1 to the vertices v_{3i}. Then we assign the label 3 to the vertices w_i $(1 \leq i \leq n)$. Next we move to the vertices x_i and y_i. Assign the labels 2,3 to the vertices x_1 and y_1 respectively. Then we assign the label 2 to the vertices $x_2, x_5, x_8, \cdots$. Now we assign the label 1 to the vertices $x_3, x_6, x_9, \cdots$ and the vertices $x_4, x_7, x_{10}, \cdots$. Assign the label 3 to the vertices $y_1, y_2, y_3, \cdots$. We consider the vertices u'_i. Assign the labels 2,3 to the vertices u'_1 and u'_2 respectively. Now we assign the label 1 to the vertices $u'_3, u'_6, u'_9, \cdots$ and we assign the label 3 to the vertices $u'_4, u'_7, u'_{10}, \cdots$. Then we assign the label 2 to the vertices $u'_5, u'_8, u'_{11}, \cdots$. Next we move to the vertices v'_i and w'_i. Assign the the label 3 to the vertex w'_1. Now assign the label 1 to all the vertices of v'_i $(1 \leq i \leq n-1)$ and we assign the label 2 to the vertices $w'_2, w'_3, w'_4, \cdots$. We consider the vertices x'_i and y'_i. Assign the label 2,1 to the vertices x'_1 and y'_1 respectively. Also we assign the label 2 to the vertices $x'_2, x'_5, x'_8, \cdots$ and the vertices $x'_3, x'_6, x'_9, \cdots$. Then we assign the label 1 to the vertices $x'_4, x'_7, x'_{10}, \cdots$. Next we assign the label 3 to the vertices $y'_2, y'_3, y'_4, ...$ The vertex condition is $e_f(0) = 6n-5$ and $e_f(1) = 6n-6$. Also the edge condition is given in Table 1 following. □

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{10n-9}{3}$	$\frac{10n-6}{3}$	$\frac{10n-9}{3}$
$n \equiv 1 \pmod 3$	$\frac{10n-10}{3}$	$\frac{10n-7}{3}$	$\frac{10n-7}{3}$
$n \equiv 2 \pmod 3$	$\frac{10n-8}{3}$	$\frac{10n-8}{3}$	$\frac{10n-8}{3}$

Table 1

Theorem 2.5 *$DQ_n \odot 2K_1$ is 3-difference cordial.*

Proof Let $V(DQ_n \odot 2K_1) = V(DQ_n) \bigcup \{u'_i, u''_i : 1 \leq i \leq n\} \bigcup \{v'_i, v''_i, w'_i, w''_i, x'_i, x''_i, y'_i, y''_i : 1 \leq i \leq n-1\}$ and $E(DQ_n \odot 2K_1) = E(DQ_n) \bigcup \{u_i u'_i, u_i u''_i : 1 \leq i \leq n\} \bigcup \{v_i v'_i, v_i v''_i, w_i w'_i, w_i w''_i, x_i x'_i, x_i x''_i, y_i y''_i, y_i y''_i : 1 \leq i \leq n-1\}$. First we consider the path vertices u_i. Assign the label 1 to the path vertices $u_1, u_3, u_5, \cdots$ and we assign the label 2 to the path vertices $u_2, u_4, u_6, \cdots$. Clearly the last vertex u_n received the label 2 or 1 according as $n \equiv 0 \text{mod } 2$ or $n \equiv 1 \pmod 2$. Next we move to the vertices v_i and w_i. Assign the label 1 to all the vertices of v_i $(1 \leq i \leq n)$ and we assign the label 3 to the vertices $w_1, w_2, w_3, ...$ Then we assign the label to the vertices x_i $(1 \leq i \leq n-1)$ is same as assign the label to the vertices v_i $(1 \leq i \leq n-1)$ and we assign the label to the vertices y_i $(1 \leq i \leq n-1)$ is same as assign the label to the vertices w_i $(1 \leq i \leq n-1)$. Next we move to the vertices u'_i and v''_i. Assign the label 2 to the vertices

$v'_1, v'_2, v'_3, \cdots$ then we assign the label 3 to the vertex v''_1. Assign the label 3 to the vertices v''_{2i} for all the values of $i = 1, 2, 3, \cdots$ and we assign the label 2 to the vertices v''_{2i+1} for $i = 1, 2, 3, \cdots$. Next we consider the vertices w'_i and w''_i. Assign the label 1 to the vertices $w'_1, w'_2, w'_3, \cdots$ and we assign the label 3 to the vertices $w''_1, w''_2, w''_3, \cdots$. Next we move to the vertices x'_i and x''_i. Assign the label 1 to all the vertices of x'_i $(1 \leq i \leq n-1)$ and we assign the label 2 to all the vertices of x''_i $(1 \leq i \leq n-1)$. Now we assign the label 2 to the vertices $y'_1, y'_2, y'_3, \cdots$ and we assign the label 3 to the vertices $y''_1, y''_2, y''_3, \cdots$. Finally we move to the vertices u'_i and u''_i. Assign the label 2 to the vertices $u'_1, u'_3, u'_5, \cdots$ and we assign the label 1 to the vertices $u'_2, u'_4, u'_6, \cdots$. Next we assign the label 2 to the vertices $u''_1, u''_3, u''_5, \cdots$ and we assign the label 3 to the vertices $u''_2, u''_4, u''_6, \cdots$. The vertex condition is $v_f(1) = v_f(2) = v_f(3) = \frac{15n-12}{3}$. Also the edge condition is given in Table 2. □

Values of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{17n-16}{2}$	$\frac{17n-14}{2}$
$n \equiv 1 \pmod 2$	$\frac{17n-15}{2}$	$\frac{17n-15}{2}$

Table 2

Theorem 2.6 *$DQ_n \odot K_2$ is 3-difference cordial.*

Proof Let $V(DQ_n \odot K_2) = V(DQ_n) \bigcup \{u'_i, u''_i : 1 \leq i \leq n\} \bigcup \{v'_i, v''_i, w'_i, w''_i, x'_i, x''_i, y'_i, y''_i : 1 \leq i \leq n-1\}$ and $E(DQ_n \odot K_2) = E(DQ_n) \bigcup \{u_i u'_i, u_i u''_i, u'_i u''_i : 1 \leq i \leq n\} \bigcup \{v_i v'_i, v_i v''_i, v'_i v''_i, w_i w'_i, w_i w''_i, w'_i w''_i, x_i x'_i, x_i x''_i, x'_i x''_i, y_i y''_i, y_i y''_i, y'_i y''_i : 1 \leq i \leq n-1\}$. First we consider the path vertices u_i. Assign the label 1 to the vertex u_1. Then we assign the label 1 to the vertices $u_2, u_4, u_6, \cdots$ and we assign the label 2 to the path vertices $u_1, u_3, u_5, \cdots$. Note that in this case the last vertex u_n received the label 1or 2 according as $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 2$. Next we move to the vertices v_i and w_i. Assign the label 2 to the vertex v_1. Then we assign the label 3 to all the vertices of w_i $(1 \leq i \leq n-1)$. Assign the label 1 to the vertices $v_2, v_3, v_4, ...$ We consider the vertices x_i and y_i. Assign the label to the vertices x_i $(1 \leq i \leq n-1)$ is same as assign the label to the vertices v_i $(1 \leq i \leq n-1)$ and assign the label to the vertices y_i $(1 \leq i \leq n-1)$ is same as assign the label to the vertices w_i $(1 \leq i \leq n-1)$. Next we move to the vertices v'_i and v''_i. Assign the label 2 to the vertices $v'_1, v'_2, v'_3, \cdots$ and assign the label 3 to the vertices $v''_1, v''_2, v''_3, \cdots$. Consider the vertices x'_i and x''_i. Assign the label 1 to all the vertices of x'_i $(1 \leq i \leq n-1)$. Assign the label 2 to the vertex x''_1. Then we assign the label 3 to the vertices $x''_2, x''_3, x''_4, \cdots$. Now we assign the label 2 to all the vertices of w'_i $(1 \leq i \leq n-1)$ and assign the label 3 to all the vertices of w''_i $(1 \leq i \leq n-1)$. Now we move to the vertices y'_i and y''_i. Assign the label 1 to the vertices $y'_1, y'_2, y'_3, \cdots$ and we assign the label 2 to the vertices $y''_1, y''_2, y''_3, \cdots$. Next we move to the vertices u'_i and u''_i. Assign the label 1,3 to the vertices u'_1 and u''_1 respectively. Assign the label 1 to the vertices u'_{2i} for all the values of $i = 1, 2, 3, \cdots$ and assign the label 2 to the vertices u_{2i+1} for $i = 1, 2, 3, \cdots$ then we assign the label 2 to the vertices $u''_2, u''_3, u''_4, \cdots$. The vertex and edge conditions are

$$v_f(1) = v_f(2) = v_f(3) = \frac{15n-12}{3}$$

and

$$e_f(0) = 11n - 9, \quad e_f(1) = 11n - 10. \qquad \square$$

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Graph Operations on Zero-Divisor Graph of Posets

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Abstract: We know that some large graphs can be constructed from some smaller graphs by using graphs operations. Many properties of such large graphs are closely related to those of the corresponding smaller ones. In this paper we investigate some operations of zero-divisor graph of posets.

Key Words: Poset, zero-divisor graph, graph operation.

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§1. Introduction

In [2], Beck, for the first time, studied zero-divisor graphs of the commutative rings. Later, D. F. Anderson and Livingston investigated nonzero zero-divisor graphs of the rings (see [1]). Some researchers also studied the zero-divisor graph of the commutative rings. Subsequently, others extended the study to the commutative semigroups with zero. These can be seen in [3, 5, 7, 8].

Assume $(P, \leq)$ is a poset (i.e., P is a partially ordered set) with the least element 0. For every $x, y \in P$, defined of $L(x, y) = \{z \in P | z \leq x \text{ and } z \leq y\}$. x is a zero-divisor element of P if $l(x, y) = 0$, for some $0 \neq y \in P$. $\Gamma(P)$ is the zero-divisor graph of poset P, where the its vertex set consists of nonzero zero-divisors elements of P and x is adjacent to y if only if $L(x, y) = \{0\}$.

In this paper, P denotes a poset with the least element 0 and $Z(P)$ is nonzero zero-divisor elements of P. The zero-divisor graph is undirected graph with vertices $Z(S)$ such that for every distinct $x, y \in Z(S)$, x and y are adjacent if only if $L(x, y) = \{0\}$. Throughout this paper, G always denotes a zero-divisor graph which is a simple graph (i.e., undirected graph without loops and multiple) and the set vertices of G show $V(G)$ and the set edges of G denotes $E(G)$. The degree of vertex x is the number of edges of G intersecting x. $N(x)$, which is the set of vertices adjacent to vertex x, is called the neighborhood of vertex x. If n is a (finite or infinite) natural number, then an n-partite graph is a graph, which is a set of vertices that can be partitioned into subsets, each of which edges connects vertices of two different sets. A complete n- partite graph is a n- partite graph such that every vertex is adjacent to the vertices which are in a different part. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is called an induced subgraph of G if for every $x, y \in V(H)$, $\{x, y\} \in E(G)$. A subgraph H of G is called a clique if H is a complete graph. The clique number $\omega(G)$ of G is the least upper

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bound of the cliques sizes of G.

Many large graphs can be constructed by expanding small graphs, thus it is important to know which properties of small graphs can be transfered to the expanded ones, for example Wang in [6] proved that the lexicographof vertex transitive graphs is also vertex transitive as well as the lexicographic product of edge transitive graphs. Specapan in [9] found the fewest number of vertices for Cartesian product of two graphs whose removal from the graph results in a disconnected or trivial graph. Motivated by these, we consider five kinds of graph products as the expander graphs which is described below and we can verify if regard the product of them can be regarded as a Cayley graph of the semigroup which is made by their product underlying semigroup and if the answer is positive does it inherit Col-Aut-vertex property of from the precedents. Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of G. An edge joins the vertex u to the vertex v is denoted by (u, v).

In [10], the authors described the following definition:

Definition 1.1 *A simple graph G is called a compact graph if G does not contain isolated vertices and for each pair x and y of non-adjacent vertices of G, there exists a vertex z with $N(x) \cup N(y) \subseteq N(z)$.*

Definition 1.2 *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. $\Gamma = (V, E)$, the product of them is a graph with vertex set $V = V_1 \times V_2$, and two vertices (u_1, u_2) is adjacent to (v_1, v_2) in Γ if one of the relevant conditions happen depending on the product.*

(1) Cartesian product. *u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;*

(2) Tensor product. *u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;*

(3) Strong product. *u_1 is adjacent to v_1 in Γ_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2 or u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2;*

(4) Lexicographic. *u_1 is adjacent to v_1 in Γ_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2;*

(5) Co-normal product. *u_1 is adjacent to v_1 in Γ_1 or u_2 is adjacent to v_2 in Γ_2;*

(6) Modular product. *u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2 or u_1 is not adjacent to v_1 in Γ_1 and u_2 is not also adjacent to v_2 in Γ_2.*

§2. Preliminary Notes

In this section, we recall some lemmas and definitions. from Dancheny Lu and Tongsue We in [10], the authors described following definition.

Definition 2.1 *A simple graph G is called a compact graph if G does not contain isolated vertices and for each pair x and y of non-adjacent vertices of G, there exists a vertex z with $N(x) \cup N(y) \subseteq N(z)$.*

It has been showed the following theorem in [10].

Theorem 2.2 *A simple graph G is the zero-divisor graph of a poset if and only if G is a* *compact graph.*

§3. Cartesian Product

Through this section, we assume that P and Q are two posets with the least element 0. Assume G and H are in zero-divisors graphs of P and Q, respectively. $N(x)$ and $N(a)$ are neighborhoods in G and H , respectively, where $x \in V(P)$ and $a \in V(Q)$.

Theorem 3.1 *Let Γ be the Cartesian product of two zero-divisor graph of G and H. Then $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a))$, for any $(x,r) \in V(G \times H)$.*

Proof Let $(s,r) \in N(x,a)$. Therefore, (s,r) is adjacent to (x,a). Thus, s is adjacent to x in G and $r = a$ or $s = x$ and r is adjacent to a in H. Hence, $s \in N(x)$ and $r = a$ or $s = x$ and $r \in N(a)$. It can be concluded that to $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a))$. □

Theorem 3.2 *Let G and H be two compact graphs. Then Γ the cartesian product of them is not a compact graph.*

Proof Let (x,a) and (y,b) be two arbitrary vertices not being adjacent of the graph Γ, where $(x,a) \neq (y,b)$. Therefore, x and y are not adjacent in G or $a \neq b$ in H and $x \neq y$ in G or a, b are not adjacent in H. Assume that there exists $(z,c) \in V(\Gamma)$ such that $N(x,a) \cup N(y,b) \subseteq N(z,c)$. That is,

$$\begin{aligned}(N(x) \times \{a\}) \cup (\{x\} &\times N(a)) \cup (N(y) \times \{b\}) \cup (\{y\} \times N(b))\\ &\subseteq N(z) \times \{c\}) \cup (\{z\} \times N(c)).\end{aligned}$$

Assume that $(m,a), (n,a) \in (N(x) \times \{a\})$ such that $(m,a) \in N(z) \times \{c\}$ and $(n,a) \in \{z\} \times N(c)$. Then, $m \in N(z), a = c, a \in N(c)$. Hence, $ac = 0$ and $c^2 = 0$. That is a contradiction. Therefore, $N(x) \times \{a\}$ has intersection only one of $N(z) \times \{c\}$ and $\{z\} \times N(c)$. Similary, we get this subject for $(\{x\} \times N(a)), (N(y) \times \{b\})$ and $(\{y\} \times N(b))$.

Now, suppose $N(x) \times \{a\} \subseteq N(z) \times \{c\}($ i.e., $a = c, N(x) \subseteq N(z))$. If $\{x\} \times N(a) \subseteq N(z) \times \{c\}$, we have $N(a) = \{c\}$. Hence, $ac = 0$. On the other hand $a = c$, then $c^2 = 0$. That is a contradiction. Therefore, $\{x\} \times N(a) \subseteq \{z\} \times N(c)$, that is $x = z$ and $N(a) \subseteq N(c)$. Then, $N(x) = N(z)$ and $N(a) = N(c)$.

Suppose $N(y) \times \{b,\} \subseteq N(z) \times \{c\}$. Thus, $N(y) \subseteq N(z) = N(x), b = c$. Hence, $a = b = c, N(a) = N(b) = N(c)$.

If $\{y\} \times N(b) \subseteq N(z) \times \{c\}, y \in N(z)$ and $N(b) = c$. Then, $bc = c^2 = 0$. That is a contract. Therefore, $\{y\} \times N(b) \subseteq \{c\} \times N(z)$. We get $y = z$ and $N(b) \subseteq N(c)$. It leads to $a = b = c$ and $x = y = z$. That is a contradiction. □

Corollary 3.2 *Let G and H be two compact graphs of two poset. Then, the cartesian product of them is not a graph of a poset.*

Proof Referring to the theorem above and [10], it is clear. □

§4. Tensor Product

Through this section, we assume that G and H are two zero-divisor graphs of poset P and Q with the least element 0, respectively.

Theorem 4.1 *Γ is the tensor product of the graphs G and H. Then $N(x,a) = (N(x) \times N(a))$ for any $(x,a) \in V(G \times H)$.*

Proof Assume $(s,r) \in N(x,a)$. Then, (s,r) is adjacent to (x,a). By Definition 1.2, s and x are adjacent and r and a are adjacent too. Therefore, $s \in N(x)$ and $r \in N(a)$. It leads to $N(x,a) = N(x) \times N(a)$. □

§5. Strong Product

Through this section, we assume that H and K are two zero-divisor graphs of poset P and Q with the least element 0, respectively.

By Definition 1.2, Theorems 3.1 and 4.1, we conclude the following theorems.

Theorem 5.1 *Γ the strong product of two zero-divisors graphs G and H of posets. Then, if runs for any $(x,a) \in V(\Gamma)$, $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a)) \cup (N(x) \times N(a))$*

Proof By Definition 1.2, for any $(r,s) \in N(x,a)$, where $(x,a) \in V(\Gamma)$, r is adjacent to x in$G$ and $s = a$ or $r = x$ and s is adjacent to a in H or r is adjacent to x in G and s is adjacent to aiH. Therefore, $N(x,a) = (N(x) \times \{a\}) \cup (\{x\} \times N(a)) \cup (N(x) \times N(a))$. □

§6. Co-normal Product

Theorem 6.1 *Γ is the co-normal product of two graphs G and H of two the posets of P and Q, respectively. Then for any $(x,a) \in V(\Gamma)$, $N(x,a) = (N(x) \times V(H)) \cup (V(H) \times N(a))$.*

Proof By Definition 1.2, if $((s,r)$ is adjacent to (x,a), s and x are adjacent in G or r,a are adjacent in H. Thus, $N(x,a) = (N(x) \times V(H)) \cup (V(H) \times N(a))$. □

Theorem 6.2 *If Γ is the co-normal product of two compact graphs G and H, then Γ is a compact graph.*

Proof Let (x,a) and (y,b) not be in Γ and $(x,a) \neq (y,b)$. By referring the virtue of Definition 1.2, we get x and$y$ are not adjacent in G and a and b are not adjacent in H. Then there exist $z \in G$ and $s \in H$ such that $N(x) \cup N(y) \subseteq N(z)$ and $N(a) \cup N(b) \subseteq N(c)$. Hence,

$$\begin{aligned} N(x,a) \cup N(y,b) &= (N(x) \times N(a)) \cup (N(y) \times N(b) \\ &\subseteq (N(z) \times N(c)) \cup (N(z) \times N(c)) = N(z) \times N(c) \end{aligned}$$ □

Now, we get the following corollary.

Corollary 6.3 *The co-product of two zero-divisor graphs of posets is a zero-divisor graph of a poset.*

Proof By the above theorem and [10], it is clear. □

§7. Lexicographic Product

Theorem 7.1 *Γ the lexicographic product of two zero-divisor graphs G and H of the two posets P and Q, respectively. Then, $N(x,a) = (N(x) \times V(H)) \cup (\{x\} \times N(a))$, for any $(x,a) \in V(\Gamma)$.*

Proof By Definition 1.2, assume $(s,r) \in N(x,a)$. Therefore, s and x are adjacent in G or $s = x$ in G and r and a are adjacent in H. Therefore, $N(x,a) = (N(x) \times V(H)) \cup (\{x\} \times N(a))$, for any $(x,a) \in V(\gamma)$. □

§8. Modular Product

Theorem 8.1 *Γ the Modular product of two zero-divisor graphs G and H of the two posets P and Q respectively. Then, $N(x,a) = (N(x) \times N(a)) \cup (N^c(x) \times N^c(a))$, for any $(x,a) \in V(\Gamma)$.*

Proof By Definition 1.2, assume $(s,r) \in N(x,a)$. Therefore, s and x are adjacent in G while r and a are adjacent in H or s and x are not adjacent in G whereas r and a are not adjacent in H. Thus, $N(x,a) = (N(x) \times N(a)) \cup (N^c(x) \times N^c(a))$. □

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The tragedy of the world is that those who are imaginative have but slight experience, and those who are experienced have feeble imaginations.

By Alfred North Whitehead, A British philosopher and mathematician

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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