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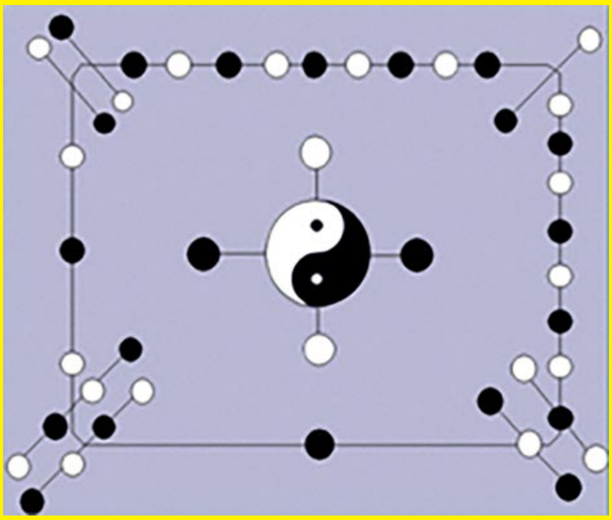

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Famous Words:

We know nothing of what will happen in future, but by the analogy of past experience.

By Abraham Lincoln, an American president

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On k-Type Slant Helices Due to Bishop Frame in Euclidean 4-Space $\mathbb{E}^4$

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Abstract: In this work, we study $k-$type ($k \in \{0, 1, 2, 3\}$) slant helices with non-zero Bishop curvature functions due to Bishop frame in $\mathbb{E}^4$. General helix is a $0-$type slant helix within the notation of this study. We characterize all of slant helices in terms of Bishop curvatures in $\mathbb{E}^4$.

Key Words: Bishop frame, regular curves, general helix, slant helix, Euclidean 4-space.

AMS(2010): 53A04.

§1. Introduction

In the local differential geometry of space curves, it is well-known that a general helix is a curve whose tangent makes a constant angle with a non-zero constant vector field (the axis of the helix). Moreover, the necessary and sufficient condition for a curve is a general helix if and only if the ratio of the curvature and the torsion of that curve is constant. A slant helix is defined as a curve whose principal normal vector makes a constant angle with a fixed direction by Izumiya and Takeuchi in $\mathbb{E}^3$ [7]. Some characterizations of a slant helix are investigated in [9]. Ali and Turgut have generalized the slant helix to Euclidean n-space $\mathbb{E}^n$ and have given some properties for a non-degenerate slant helix [2]. Öztürk et.al. have considered the focal representation and some properties of focal curves with their curvatures of k-slant helices in $\mathbb{E}^{m+1}$ [11]. Further, some characterizations of slant helices in different spaces such as Minkowski and Galilean are studied [12, 13, 14, 16].

Most of the study of curves are done by using Frenet-Serret frame in classical differential geometry in Euclidean space. In [4], Bishop defined an alternative over Frenet frame for a curve and called it Bishop frame. The advantage of Bishop frame is well-defined when the curve has a vanishing second derivative in 3-dimensional Euclidean space $\mathbb{E}^3$ unlike Frenet frame. Also, Bishop frame is used in many applications such as engineering, computer aided design, DNA analysis etc. After defining this useful alternative frame, many studies have been done by

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mathematicians using it [3, 8, 15]. Özçelik et. al. have been introduced the parallel transport frame of the curve in 4-dimensional Euclidean space $\mathbb{E}^4$ [10].

The present study aims to determine the characterize all of slant helices in terms of Bishop curvatures in $\mathbb{E}^4$ with the help of the literature.

§2. Preliminaries

Here, the basic definitions and theorems for the theory of curves in Euclidean 4-space $\mathbb{E}^4$ are given for the next section (A more complete elementary treatment can be found in [5], [6]).

The standard flat metric in Euclidean 4-space $\mathbb{E}^4$ is given by

$$\langle \quad , \quad \rangle = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of Euclidean 4-space $\mathbb{E}^4$. The norm of an arbitrary vector $a \in \mathbb{E}^4$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. The curve α is called an unit speed curve if a velocity vector v of α satisfies $\|v\| = 1$. For vectors $v, w \in \mathbb{E}^4$, it is said to be orthogonal if and only if $\langle v, w\rangle = 0$. Let $\alpha = \alpha(s)$ be a regular curve in Euclidean 4-space $\mathbb{E}^4$. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve.

Denote by $\{T, N, B, E\}$ the moving Frenet-Serret frame along the curve α in the space $\mathbb{E}^4$. For an arbitrary curve α in Euclidean 4-space $\mathbb{E}^4$, the following Frenet-Serret formulae is given with respect to the first curvature κ, the second curvature τ and the third curvature σ in [6]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix},$$

where T, N, B and E are called the tangent, the principal normal, the first and the second binormal vectors of the curve α, respectively.

Theorem 2.1([14]) *Let $\alpha = \alpha(t)$ be an arbitrary curve in Euclidean 4-space $\mathbb{E}^4$ with above Frenet-Serret equations. Frenet-Serret apparatus of α can be written as follows:*

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{2.1}$$

$$N = \frac{\|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha''\rangle \alpha'}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha''\rangle \alpha' \right\|}, \tag{2.2}$$

$$B = \mu N \wedge T \wedge B_2, \tag{2.3}$$

$$E = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{2.4}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|}{\|\alpha'\|^4} \tag{2.5}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha' \right\|} \tag{2.6}$$

and

$$\sigma = \frac{\left\langle \alpha^{(IV)}, E \right\rangle}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}, \tag{2.7}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of the matrix $[T, N, B, E]$.

Bishop frame is also referred to as parallel transport that is an orthonormal frame formed by transporting in parallel each component of the frame. The parallel transport is formed with tangent vector and any convenient arbitrary basis for the remainder of the frame (for details, see [4], [10]). Then, the relations between Frenet-Serret frame and parallel transport frame for the curve $\alpha : I \subset R \to \mathbb{E}^4$ are given as follows:

$$\begin{aligned}
T(s) &= T(s), \\
N(s) &= \cos\theta(s)\cos\psi(s)M_1 + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2 \\
&\quad + (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3, \\
B(s) &= \cos\theta(s)\sin\psi(s)M_1 + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2 \\
&\quad + (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3, \\
E(s) &= -\sin\theta(s)M_1 + \sin\phi(s)\cos\theta(s)M_2 + \cos\phi(s)\cos\theta(s)M_3.
\end{aligned}$$

The parallel transport frame equations are expressed as [10]

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{2.8}$$

where k_1, k_2, k_3 are curvature functions according to parallel transport frame of the curve α, and their expression as follows:

$$\begin{aligned}
k_1 &= \kappa\cos\theta(s)\cos\psi(s), \\
k_2 &= \kappa(-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s)), \\
k_3 &= \kappa(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s)),
\end{aligned}$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}, \quad \psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}, \quad \phi' = -\frac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta},$$

and Frenet curvature functions are given as follows:

$$\kappa(s) = \sqrt{k_1^2+k_2^2+k_3^2}, \quad \tau(s) = -\psi' + \phi'\sin\theta, \quad \sigma(s) = \frac{\theta'}{\sin\psi},$$

and

$$\phi'\cos\theta + \theta'\cot\psi = 0,$$

in terms of the invariants of parallel transport frame.

§3. On k-Type Slant Helices Due to Bishop Frame in Euclidean 4-Space $\mathbb{E}^4$

Definition 3.1 *Let $\alpha = \alpha(s)$ be a curve parametrized by arc-length with $\{T, M_1, M_2, M_3\}$ a Bishop frame in $\mathbb{E}^4$. If there exists a non-zero constant vector field U in $\mathbb{E}^4$ such that $\langle M_k, U\rangle \neq 0$ is a constant for all $s \in I$, where $M_0 = T$, then α is said to be $k-$type ($k \in \{0,1,2,3\}$) slant helix, and U is called axis of α.*

Theorem 3.2 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. There is no $0-$type slant helix (general helix) due to Bishop frame in $\mathbb{E}^4$.*

Proof Let $\alpha = \alpha(s)$ be $0-$type slant helix in $\mathbb{E}^4$ and the axis of the curve be U. Then, we have that

$$\langle T, U\rangle = c_1(s) = constant \tag{3.1}$$

along the curve α. Differentiating (3.1) with respect to s and using Bishop frame, we know that

$$k_1\langle M_1, U\rangle + k_2\langle M_2, U\rangle + k_3\langle M_3, U\rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{T\}$ and therefore, it can be written as

$$U = c_1(s)T. \tag{3.2}$$

Differentiation of (3.2) gives

$$c_1k_1M_1 + c_1(s)k_2M_2 + c_1(s)k_3M_3 = 0.$$

Since the vectors $\{M_1, M_2, M_3\}$ are linearly independent, we have $c_1 = 0$ which yields

$$U = 0. \tag{3.3}$$

Since the result (3.3) contradicts with the definition of U, we claim that there is no $0-$type slant helix (general helix) due to Bishop frame in $\mathbb{E}^4$. $\square$

Theorem 3.3 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then α is a $1-$type slant helix if and only if the function*

$$-c_0 \frac{k_1}{k_3} - c_1 \frac{k_2}{k_3} \tag{3.4}$$

is a constant, and $c_0 = const.$ and $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $1-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero direction. Then we have

$$\langle M_1, U \rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.5}$$

along the curve α. Using (3.5) and Bishop frame formulae, we have

$$-k_1 \langle T, U \rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and it can be written as

$$U = c_0(s)M_1 + a(s)M_2 + b(s)M_3. \tag{3.6}$$

Differentiation of (3.6) gives

$$(-c_0k_1 - ak_2 - bk_3)T + a'M_2 + b'M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -c_0k_1 - ak_2 - bk_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.7}$$

From (3.7), we obtain

$$a = c_1, \quad b = -c_0 \frac{k_1}{k_3} - c_1 \frac{k_2}{k_3},$$

where c_1 is constant.

Conversely, if (3.4) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U \rangle$ =constant. We consider the axis as

$$U = M_1 + M_2 - \left(\frac{k_1}{k_3} + \frac{k_2}{k_3} \right) M_3. \tag{3.8}$$

Differentiating U with the help of (3.4) gives $U' = 0$. This means that U is a constant vector. As a result, α is a $1-$type slant helix in $\mathbb{E}^4$. □

Using Theorem 3.3, we have the following result.

Corollary 3.4 *Let $\alpha = \alpha(s)$ be a $1-$type slant helix with non zero Bishop curvatures k_1, k_2,* *and k_3 due to Bishop frame in $\mathbb{E}^4$. Then the axes of α are obtained by*

$$U = c_0 M_1 + c_1 M_2 + (-c_0 \frac{k_1}{k_3} - c_1 \frac{k_2}{k_3}) M_3,$$

where c_0, c_1 are constants.

Theorem 3.5 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$.Then α is a $2-$type slant helix if and only if the function*

$$-c_0 \frac{k_2}{k_3} - c_1 \frac{k_1}{k_3} \tag{3.9}$$

is a constant, and $c_0 = const.$ and $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $2-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero constant direction. Then we have

$$\langle M_2, U \rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.10}$$

along the curve α. Differentiating (3.10) with respect to s and using Bishop frame, we have

$$-k_2 \langle T, U \rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and can be decomposed as

$$U = a(s) M_1 + c_0(s) M_2 + b(s) M_3. \tag{3.11}$$

Differentiation of (3.11) gives

$$(-ak_1 - c_0 k_2 - bk_3)T + a' M_2 + b' M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -ak_1 - c_0 k_2 - bk_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.12}$$

From (3.12), we obtain

$$a = c_1, \quad b = -c_0 \frac{k_2}{k_3} - c_1 \frac{k_1}{k_3},$$

where c_1 is constant.

Conversely, if (3.9) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U \rangle$ =constant. We consider the axis as

$$U = M_1 + M_2 + \left(\frac{k_2}{k_3} + \frac{k_1}{k_3} \right) M_3. \tag{3.13}$$

Differentiating U with the help of (3.9) gives $U' = 0$. This means that U is a constaant vector. As a result, α is a $2-$type slant helix in $\mathbb{E}^4$. □

Using Theorem 3.5, we have the following result.

Corollary 3.6 *Let $\alpha = \alpha(s)$ be a $2-$type slant helix with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then the axes of α are obtained by*

$$U = c_1 M_1 + c_0 M_2 + (-c_0 \frac{k_2}{k_3} - c_1 \frac{k_3}{k_3}) M_3,$$

where c_0, c_1 are constants.

Theorem 3.7 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then α is a $3-$type slant helix if and only if the function*

$$-c_1 \frac{k_1}{k_2} - c_0 \frac{k_3}{k_2} \tag{3.14}$$

is a constant, and $c_0 = const.$ and $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $3-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero direction. Then we have

$$\langle M_3, U \rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.15}$$

along the curve α. Using (3.15) and Bishop frame formulae, we have

$$-k_3 \langle T, U \rangle = 0,$$

which implies that the vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and can be written as

$$U = a(s) M_1 + b(s) M_2 + c_0(s) M_3. \tag{3.16}$$

Differentiation of (3.16) gives

$$(-c_0 k_1 - a k_2 - b k_3) T + a' M_2 + b' M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -a k_1 - b k_2 - c_0 k_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.17}$$

From (3.17), we obtain

$$a = c_1, \quad b = -c_1 \frac{k_1}{k_2} - c_0 \frac{k_3}{k_2},$$

where c_1 is constant.

Conversely, if (3.4) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U \rangle$ =constant.

We consider the axis as

$$U = M_1 + \left(-\frac{k_1}{k_2} - \frac{k_3}{k_2}\right) M_2 + M_3. \tag{3.18}$$

Differentiating U with the help of (3.14) gives $U' = 0$. This means that U is a constant vector. As a result, α is a $3-$type slant helix in $\mathbb{E}^4$. $\square$

From the above theorem, we have the following result.

Corollary 3.8 *Let $\alpha = \alpha(s)$ be a $3-$type slant helix with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then the axes of α are obtained by*

$$U = c_1 M_1 + \left(-c_1\frac{k_1}{k_2} - c_0\frac{k_3}{k_2}\right) M_2 + c_0 M_3,$$

where c_0, c_1 are constants.

§4. Conclusion

The properties of $k-$type ($k \in \{0, 1, 2, 3\}$) slant helices with non-zero Bishop curvature functions with Bishop frame in $\mathbb{E}^4$ are obtained. General helix ($0-$type slant helix) that does not exist according to Bishop frame in $\mathbb{E}^4$ is given. All of slant helices are characterized in terms of Bishop curvatures in $\mathbb{E}^4$ in this paper.

References

[1] Ali A.T., (2012), Position vectors of slant helices in Euclidean 3-space, *J.Egyptian Math Soc.*, 20(1), pp. 1-6.

[2] Ali A. T. and Turgut M., (2010), Some characterizations of slant helices in the Euclidean space $\mathbb{E}^n$, *Hacet J.Math.Stat.*, 39(3), pp. 327-336.

[3] Bukcu B. and Karacan M. K., (2009), *The Slant Helices According to Bishop Frame*, World Academy of Science, Engineering and Technology, 59, pp. 1039-1042.

[4] Bishop, L. R., (1975), There's more than one way to frame a curve, *Amer. Math. Monthly*, 82(3), pp. 246–251.

[5] Do Carmo M. P., (1976), *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ.

[6] Hacisalihoğlu H.H., (2000), *Differential Geometry I*, Ankara University Faculty of Science Press, Ankara.

[7] Izumiya S. and Takeuchi N., (2004), New special curves and developable surfaces, *Turk J. Math.*, 28(2) , pp. 531–537.

[8] Kişi İ. and Öztürk G., (2013), AW(k)-type curves according to the Bishop frame, *arXiv* math.DG1305.3381v1.

[9] Kula L., Ekmekçi N., YaylıY. and İlarslan K., (2010), Characterizations of slant helices in Euclidean 3-space, *Turkish J Math.*, 34, pp. 261-273.

[10] Özçelik F., Bozkurt Z., Gök İ., Ekmekçi F. N. and YaylıY., (2014), Parallel transport frame

in 4-dimensional Euclidean space $\mathbb{E}^4$, *Casp J.Math. Scie.*, 3(1), pp. 91-102.

[11] Öztürk G., Bulca B., Bayram B. and Arslan K., (2015), Focal representation of k-slant helices in $\mathbb{E}^{m+1}$, *Acta Univ Sapientiae, Mathematica*, 7(2), pp. 200-209.

[12] Qian J. and Kim Y. H., (2016), Null helix and k-type null slant helıces in $\mathbb{E}_1^4$, *Revista De La Union Matematica Argentina*, 57(1), pp. 71-83.

[13] Tawfık A., Lopez R. and Turgut M., (2012), k-type partially null and pseudo null slant helices in Minkowski 4-space, *Math. Commun.*, 17, pp. 93-103.

[14] Turgut M., Yılmaz S., (2009), Some characterizations of type-3 slant helices in Minkowski space-time, *Involve J. Math.*, 2, pp. 115-120.

[15] Ünlütürk Y. and Çimdiker M., (2014), Some characterizations of curves of AW(k)-type according to the Bishop frame, *NTMSCI*, 2(3), pp. 206-213.

[16] Yoon, D. W., (2013), On the inclined curves in Galilean 4-space, *Appl. Math. Sci.*, 7(44), pp. 2193-2199.

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Left Centralizers on
Lie Ideals in Prime and Semiprime Gamma Rings

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Abstract: Let U be a Lie ideal of a 2-torsion free prime Γ-ring M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping satifying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then we prove that $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$. Also this result is extended to semiprime Γ-rings.

Key Words: Prime Γ-ring, semiprime Γ-ring, left centralizer, Lie ideal.

AMS(2010): 16N60, 16W25, 16W10.

§1. Introduction

The concept of a gamma ring was presented as a generalization of the classical rings. This gamma ring was first introduced by Nobusawa [1] in 1964, which is currently known as Γ_N-ring. After two years it was more broadly generalized by Bernes [2] in the sense of Nobusawa [1], which is now known as the Γ-ring. It is shown that a Γ-ring need not be a ring, but a Γ-ring is more general than rings [1], and also that every Γ_N-ring is a Γ-ring [2]. From its beginning, the various important theories of the classical rings were extended and generalized to the theories of Γ-rings [3,4]. Such theories have been attracted much international attentions as an emerging areas of research to the modern algebraists to enrich the areas of algebras. Recently, many researchers determine a number of basic properties of Γ-rings with creative and productive remarkable results [5, 6, 7, 8, 9, 10, 11].

Borut Zalar [12] studied on centralizers of semiprime rings and shown that Jordan centralizers and centralizers of this rings coincide. Using the concept of centralizers, Vukman [13,14] established a number of results on prime and semiprime rings. Such results and ideas have been extended to prime and semiprime Γ-rings in different aspects such as centralizers and θ-centralizers [7, 9, 11, 15, 16], Jordan centralizers and Jordan θ-centralizers [17,18] and centralizers with involutions [19].

The Lie ideals and Jordan derivations of prime rings was studied in [20, 21, 22, 23], and

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these works have been extended to Lie ideals of prime Γ-rings of Jordan derivations [8, 23, 24] and Jordan k-derivations [25]. In fact, a number of significant results of classical ring theories were developed in prime and semiprime Γ-rings with Lie ideals and Jordan structures [6,26]. However, the research on centralizers of prime and semiprime gamma rings with Lie ideals is still an unexplored area and it would be of interest to further works. Thus the aim of this article is to extend the results of [7] to the Lie ideals in prime and semiprime Γ-rings.

§2. Preliminaries

Let M and Γ be additive abelian groups. If there exists a mapping $(a, \alpha, b) \to a\alpha b$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

- (i) $a\alpha b \in M$;
- (ii) $(a+b)\alpha c$=$a\alpha c$+$b\alpha c$, $a(\alpha+\beta)c$=$a\alpha c$+$a\beta c$, $a\alpha(b+c)$=$a\alpha b$+$a\alpha c$;
- (iii) $(a\alpha b\beta c$=$a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring.

Let M be a Γ-ring. Then M is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies a=0 or b=0, and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies a=0. An additive subgroup U of M is said to be a Lie ideal of M if $[u, x]_\alpha \in U$ for all $u \in U$, $x \in M$ and $\alpha \in \Gamma$. Furthermore, M is said to be commutative Γ-ring if $a\alpha b$=$b\alpha a$ for all $a, b \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M)$ =$\{a \in M : a\alpha b = b\alpha a$ for all $\alpha \in \Gamma, b \in M\}$ is called the centre of the Γ-ring M.

If M is a Γ-ring, then $[a, b]_\alpha$=$a\alpha b-b\alpha a$ is known as the commutator of a and b with respect to α, where $a, b \in M$ and $\alpha \in \Gamma$. It has the basic commutator identities

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a[\alpha, \beta]_c b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b[\alpha, \beta]_a c + b\alpha[a, c]_\beta,$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. One consider the following assumption [7],

$$(A) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \alpha b\beta c = a\beta b\alpha c,$$

for all $a, b, c \in M$, and $\alpha, \beta \in \Gamma$, which will extensively use through the paper. According to the assumption (A), the above two identities reduce to

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta,$$
$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta.$$

For existence of such a Γ-ring M, we present the following example.

Example 2.1([5], Example 1.1) Let R be an associative ring with the unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : \text{n is an integer} \right\}$. Then M is a Γ-ring. A simple verification shows that M satisfies the assumption (A).

An additive mapping $T : M \to M$ is called a left (right) centralizer if $T(a\alpha b) = T(a)\alpha b$ (resp. $T(a\alpha b) = a\alpha T(b)$) for all $a, b \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer, and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $T : M \to M$ is Jordan left(right) centralizer if $T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$ for all $x \in M$ and $\alpha \in \Gamma$. Every left centralizer is a Jordan left centralizer but the converse is not in general true. An additive mappings $T : M \to M$ is called a Jordan centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

§3. Left Centralizers of Prime Gamma Rings

Lemma 3.1 *Let M be a Γ-ring and U a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then*

(a) $T(u\alpha v + v\alpha u) = T(u)\alpha v + T(v)\alpha u$;
(b) $T(u\alpha v\beta u + u\beta v\alpha u) = T(u)\alpha v\beta u + T(u)\beta v\alpha u$;
(c) $T(u\alpha v\beta u) = T(u)\alpha v\beta u$;
(d) $T(u\alpha v\beta w + w\beta v\alpha u) = T(u)\alpha v\beta w + T(w)\beta v\alpha u$,

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Proof By the definition of Lie ideal U, $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Thus we have, $u\beta v + v\beta u = (u + v)\beta(u + v) - u\beta u - vv \in U$ for all $u, v \in U$ and $\beta \in \Gamma$. Therefore

$$\begin{aligned} T(u\alpha v + v\alpha u) \quad &= T((u + v)\alpha(u + v)) - T(u\alpha u) - T(v\alpha v) \\ &= T(u + v)\alpha(u + v) - T(u)\alpha u - T(v)\alpha v \\ &= T(u)\alpha u + T(u)\alpha v + T(v)\alpha u + T(v)\alpha v - T(u)\alpha - T(v)\alpha v \\ &= T(u)\alpha v + T(v)\alpha u. \end{aligned}$$

Hence

$$T(u\alpha v + v\alpha u) = T(u)\alpha v + T(v)\alpha u. \tag{3.1}$$

Since $u\beta v + v\beta u \in U$ for all $u, v \in U$ and $\beta \in \Gamma$, we replace v by $u\beta v + v\beta u$ in relation (3.1), we obtain

$$\begin{aligned} &T(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) = T(u)\alpha(u\beta v + v\beta u) + T(u\beta v + v\beta u)\alpha u \\ &\Rightarrow T(u\alpha u\beta v + u\alpha v\beta u + u\beta v\alpha u + v\beta u\alpha u) = T(u)\alpha(u\beta v + v\beta u) + T(u)\beta v\alpha u + T(v)\beta u\alpha u \\ &\Rightarrow T(u\alpha v\beta u + u\beta v\alpha u) + T(u\alpha u)\beta v + T(v)\beta u\alpha u \\ &\quad = T(u)\alpha u\beta v + T(u)\alpha v\beta u + T(u)\beta v\alpha u + T(v)\beta u\alpha u \\ &\Rightarrow T(u\alpha v\beta u + u\beta v\alpha u) = T(u)\alpha v\beta u + T(u)\beta v\alpha u. \end{aligned} \tag{3.2}$$

Hence (b) proved.

By using the assumption (A) in the above relation (3.2), we obtain

$$2T(u\alpha v\beta u) \quad = \quad 2T(u)\alpha v\beta u.$$

Thus for the 2-torsion freeness of M, we have

$$T(u\alpha v\beta u) = T(u)\alpha v\beta u. \tag{3.3}$$

Putting $u = u + w$ in the relation (3.3), we obtain the result (d). □

Define $B_\alpha(u, v) = T(u\alpha v) - T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$. Then we have the following remarks and lemmas.

Remark 3.1 It is clear that $B_\alpha(u, v)$ is an additive mapping such that $B_\alpha(u, v) + B_\alpha(v, u) = 0$.

Remark 3.2 It is also clear that T is a left centralizer if and only if $B_\alpha(u, v) = 0$.

Lemma 3.2 *Let M be a 2-torsion free Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then $B_\alpha(u, v)\beta w\gamma[u, v]_\delta = 0$ and $[u, v]_\delta\beta w\gamma B_\alpha(u, v) = 0$.*

Proof First, we compute

$$x = T(2(u\alpha v)\beta w\gamma 2(v\delta u) + 2(v\alpha u)\beta w\gamma 2(u\delta v)) \tag{3.4}$$

in two different ways. Then using Lemma 3.1 (c) in (3.4), we have

$$x = 4T(u)\alpha v\beta w\gamma v\delta u + 4T(v)\alpha u\beta w\gamma u\delta v, \tag{3.5}$$

and using Lemma 3.1 (d) in (3.4), we have

$$x = 4T(u\alpha v)\beta w\gamma v\delta u + 4T(v\alpha u)\beta w\gamma u\delta v. \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$\begin{aligned}
&4\{(T(u\alpha v) - T(u)\alpha v)\beta w\gamma v\delta u + (T(v\alpha u) - T(v)\alpha u)\beta w\gamma u\delta v\} = 0\\
&\Rightarrow 4\{B_\alpha(u, v)\beta w\gamma v\delta u + B_\alpha(v, u)\beta w\gamma u\delta v\} = 0\\
&\Rightarrow 4\{B_\alpha(u, v)\beta w\gamma v\delta u - B_\alpha(u, v)\beta w\gamma u\delta v\} = 0.
\end{aligned}$$

Hence, we have

$$4B_\alpha(u, v)\beta w\gamma[u, v]_\delta \quad = \quad 0.$$

Therefore, by the semiprimeness of M, we obtain

$$B_\alpha(u, v)\beta w\gamma[u, v]_\delta \quad = \quad 0$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Similarly, we can easily prove that

$$[u, v]_\delta \beta w \gamma B_\alpha(u, v) \quad = \quad 0$$

for all $u, v, w \in U$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. $\square$

Lemma 3.3([8],Lemma 2) *Let $U \nsubseteq Z(M)$ be a Lie ideal of a 2-torsion free prime Γ-ring M and $a, b \in M$ such that $a\alpha U\beta b = 0$. Then $a = 0$ or $b = 0$.*

Lemma 3.3 *Let U be a commutative Lie ideal of a 2-torsion free prime Γ-ring M. Then $U \subseteq Z(M)$.*

Proof For $u \in U$, $m \in M$ and $\alpha \in \Gamma$, we have $[u, m]_\alpha \in U$. Since U is commutative, $[u, [u, m]_\alpha]_\beta = 0$ for all $\beta \in \Gamma$. Now for $x, y \in M$ and $\gamma \in \Gamma$, replace $x\gamma y$ for m, we obtain $[u, [u, x\gamma y]_\alpha]_\beta = 0$. By using (A), we have $[u, x\gamma[u, y]_\alpha + [u, x]_\alpha\gamma y]_\beta = 0$, $\Rightarrow [u, x\gamma[u, y]_\alpha]_\beta + [u, [u, x]_\alpha\gamma y]_\beta = 0$, $\Rightarrow x\gamma[u, [u, y]_\alpha]_\beta + [u, x]_\beta\gamma[u, y]_\alpha + [u, x]_\alpha\gamma[u, y]_\beta + [u, [u, x]_\alpha]_\beta\gamma y = 0$, $\Rightarrow [u, x]_\beta\gamma[u, y]_\alpha + [u, x]_\alpha\gamma[u, y]_\beta = 0$. After some calculation and using the assumption (A), we have $2[u, x]_\alpha\gamma[u, y]_\beta = 0$. Since M is 2-torsion free, thus $[u, x]_\alpha\gamma[u, y]_\beta = 0$.
Putting y by $y\delta m$ for all $m \in M$, we have $[u, x]_\alpha\gamma[u, y\delta m]_\beta = 0$. This implies $[u, x]_\alpha\gamma y\delta[u, m]_\beta + [u, x]_\alpha\gamma[u, y]_\beta\delta m = 0$, $\Rightarrow [u, x]_\alpha\gamma y\delta[u, m]_\beta = 0$. Hence by primeness of M, $[u, x]_\alpha = 0$ or $[u, m]_\beta = 0$. If $[u, x]_\alpha = 0$, then $U \subseteq Z(M)$ and if $[u, m]_\beta = 0$, then also $U \subseteq Z(M)$. $\square$

Theorem 3.1 *Let U be a Lie ideal of a 2-torsion free prime Γ-ring M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping such that $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof If U is a commutative Lie ideal of M, then by Lemma 3.4, $U \subseteq Z(M)$. Therefore, by Lemma 3.1(d), we have

$$T(u\alpha v\beta w + w\beta v\alpha u) = T(u)\alpha v\beta w + T(w)\beta v\alpha u. \tag{3.7}$$

Since U is commutative, we have $u\alpha v = v\alpha u$. Therefore

$$T((u\alpha v)\beta w + w\beta(u\alpha v)) = T(u\alpha v)\beta w + T(w)\beta u\alpha v. \tag{3.8}$$

Comparing (3.7) and (3.8), and using $u\alpha v = v\alpha u$, we obtain

$$(T(u\alpha v) - T(u)\alpha v)\beta w \quad = 0,$$

which yields that $G_\alpha(u, v)\beta w = 0$. Since $w \in U$, $[w, m]_\gamma \in U$ for all $m \in M$ and $\gamma \in \Gamma$. Replacing w by $[w, m]_\gamma$, we obtain $G_\alpha(u, v)\beta[w, m]_\gamma = 0$. This implies $G_\alpha(u, v)\beta w\gamma m - G_\alpha(u, v)\beta m\gamma w = 0$. Hence the relation becomes $G_\alpha(u, v)\beta m\gamma w = 0$ for all $u, v, w \in U$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since $U \neq 0$, in view of the Lemma 3.3, $G_\alpha(u, v) = 0$.

If U is not commutative, then $U \nsubseteq Z(M)$. In this case, we have from Lemma 3.2,

$B_\alpha(u,v)\beta w\gamma[u,v]_\delta = 0$. Putting $u = u+x$ for all $u \in U$, we have $(B_\alpha(u,v)+B_\alpha(x,v))\beta w\gamma([u,v]_\delta + [x,v]_\delta) = 0$. This implies $B_\alpha(u,v)\beta w\gamma[x,v]_\delta + B_\alpha(x,v)\beta w\gamma[u,v]_\delta = 0$.

Now,

$$\begin{aligned}&B_\alpha(u,v)\beta w\gamma[x,v]_\delta\mu z\gamma B_\alpha(u,v)\beta w\gamma[x,v]_\delta\\&= -B_\alpha(u,v)w\gamma[x,v]_\delta\mu z\gamma B_\alpha(x,v)\beta w\gamma[u,v]_\delta = 0.\end{aligned}$$

Therefore, by Lemma 3.3, we have $B_\alpha(u,v)\beta w\gamma[x,v]_\delta = 0$ for all $x \in U$. Similarly, using $v = v+y$, we obtain $B_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0$ for all $y \in U$. And by Lemma 3.3, we obtain $B_\alpha(u,v) = 0$ or $[x,y]_\delta = 0$. If $[x,y]_\delta = 0$, then U is commutative which shows a contradiction that $U \not\subseteq Z(M)$. Therefore $B_\alpha(u,v) = 0$. □

Corollary 3.1 *Let M be a 2-torsion free prime Γ-ring and $T : M \to M$ be a Jordan left centralizer. Then T is a left centralizer.*

§4. Left Centralizers of Semiprime Gamma Rings

Lemma 4.1 *Let U be a commutative Lie ideal of a 2-torsion free semiprime Γ-ring M. Then $U \subseteq Z(M)$.*

Proof For $u \in U$ and $x \in M$, we have $[u,[u,x]_\alpha]_\alpha = 0$. Repacing $x = x\gamma y$, we have $[u,[u,x\gamma y]_\alpha]_\alpha = 0$. This implies $[u, x\gamma[u,y]_\alpha + [u,x]_\alpha\gamma y]_\alpha = 0$, $\Rightarrow$ $[u,x]_\alpha\gamma[u,y]_\alpha + x\gamma[u,[u,y]_\alpha]_\alpha + [u,[u,x]_\alpha]_\alpha\gamma y + [u,x]_\alpha\gamma[u,y]_\alpha = 0$, $\Rightarrow$ $[u,x]_\alpha\gamma[u,y]_\alpha + [u,x]_\alpha\gamma[u,y]_\alpha = 0$, i.e., $2[u,x]_\alpha\gamma[u,y]_\alpha = 0$. Hence by 2-torsion freeness, we have $[u,x]_\alpha\gamma[u,y]_\alpha = 0$. Replacing y by $y\delta x$, we have $[u,x]_\alpha\gamma[u,y\delta x]_\alpha = 0$. This implies, $[u,x]_\alpha\gamma[u,y]_\alpha\delta x + [u,x]_\alpha\gamma y\delta[u,x]_\alpha = 0$, that is, $[u,x]_\alpha\gamma y\delta[u,x]_\alpha = 0$ for all $y \in M$. Since M is semiprime, $[u,x]_\alpha = 0$, which shows that $U \in Z(M)$. □

Lemma 4.2 *Let U be a Lie ideal of a 2-torsion free Γ-ring M satisfying the assumption (A), then $T(U) = \{x \in M : [x,M]_\Gamma \subseteq U\}$ is both a subring and a Lie ideal of M such that $U \subseteq T(M)$.*

Proof Since U is a Lie ideal of M, so we have $[U,M]_\Gamma \subseteq U$. Thus $U \subseteq T(U)$. Also, we have $[T(U),M]_\Gamma \subseteq U \subseteq T(U)$. Hence $T(U)$ is a Lie ideal of M.

Suppose that $x,y \in T(U)$, then $[x,m]_\alpha \in U$ and $[y,m]_\alpha \in U$, for all $m \in M$ and $\alpha \in \Gamma$.

Now, $[x\alpha y, m]_\beta = x\alpha[y,m]_\beta + [x,m]_\beta\alpha y \in U$. Hence $[x\alpha y, m]_\beta \in U$, for all $x,y \in T(U)$, $m \in M$ and $\alpha,\beta \in \Gamma$. Therefore, $x\alpha y \in T(U)$. □

Lemma 4.3 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ-ring M. Then there exists a nonzero ideal $K = M\Gamma[U,U]_\Gamma\Gamma M$ of M generated by $[U,U]_\Gamma$ such that $[K,M]_\Gamma \subseteq U$.*

Proof First, we have to prove that if $[U,U]_\Gamma = 0$, then $U \subseteq Z(M)$. Let $[U,U]_\Gamma = 0$. Then for all $a \in U$, $\alpha \in \Gamma$, we have $[u,[u,x]_\alpha]_\alpha = 0$ for all $x \in M$. Then using the proof of Lemma 4.1, we obtain $U \subseteq Z(M)$, which is a contradiction. Thus, let $[U,U]_\Gamma \neq 0$. Then

$K = M\Gamma[U,U]_\Gamma\Gamma M$ is a nonzero ideal of M generated by $[U,U]_\Gamma$. Let $x, y \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$, we have $[x, y\beta m]_\alpha, y, [x,m]_\alpha \in U \subseteq T(U)$. Hence by Lemma 4.2, $[x,y]_\alpha\beta m = [x, y\beta m]_\alpha - y\beta[x,m]_\alpha \in T(U)$.

Also, we can show that $m\beta[x,y]_\alpha \in T(U)$ and therefore, we obtain $[[U,U]_\Gamma, M]_\Gamma \subseteq U$. That is, $[[[[x,y]_\alpha, m]_\alpha, s]_\alpha, t]_\alpha \in U$ for all $m, s, t \in M$ and $\alpha \in \Gamma$. Hence $[[x,y]_\alpha\alpha m\alpha s - m\alpha[x,y]_\alpha\alpha s + [s,m]_\alpha\alpha[x,y]_\alpha - [s\alpha[x,y]_\alpha, m]_\alpha, t]_\alpha \in T(U)$. Since $[x,y]_\alpha\alpha m\alpha s$, $s\alpha[x,y]_\alpha$, $[s,m]_\alpha\alpha[x,y]_\alpha \in T(U)$. Thus, $[m\alpha[x,y]_\alpha\alpha s, t]_\alpha \in U$ for all $m, s, t \in M$ and $\alpha \in \Gamma$. Hence $[K,M]_\Gamma \subseteq U$. □

Lemma 4.4 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprme Γ-ring M and $a \in U$. If $a\alpha U\beta a = \{0\}$ for all $\alpha, \beta \in \Gamma$, then $a\alpha a = 0$ and there exists a nonzero ideal $K = M\Gamma[U,U]_\Gamma\Gamma M$ of M generated by $[U,U]_\Gamma$ such that $[K,M]_\Gamma \subseteq U$ and $K\Gamma a = a\Gamma K = \{0\}$.*

Proof If $a\alpha U\beta a = \{0\}$ for all $\alpha, \beta \in \Gamma$, then $a\alpha[a, a\delta m]_\alpha\beta a = 0$ for all $m \in M$ and $\delta \in \Gamma$. Therefore,

$$\begin{aligned} 0 &= a\alpha(a\alpha a\delta m - a\delta m\alpha a)\beta a \\ &= a\alpha a\alpha a\delta m\beta a - a\alpha a\delta m\alpha a\beta a \\ &= a\alpha a\delta a\alpha m\beta a - a\alpha a\delta m\beta a\alpha a \end{aligned}$$

Since $a\alpha a\delta a = 0$, we have $(a\alpha a)\delta m\beta(a\alpha a) = 0$ and hence $a\alpha a = 0$ for semiprimeness of M. Now, we obtain $a\alpha[k\gamma a, m]_\mu\beta u\alpha a = 0$ for all $k \in K$, $m \in M$, $u \in U$ and $\alpha, \beta, \mu \in \Gamma$. Therefore

$$\begin{aligned} 0 &= a\alpha(k\gamma a\mu m - m\mu k\gamma a)\beta u\alpha a \\ &= a\alpha k\gamma a\mu m\beta u\alpha a - a\alpha m\mu k\gamma a\beta u\alpha a \\ &= a\alpha k\gamma a\mu m\beta u\alpha a \end{aligned}$$

Thus, we have $a\alpha k\gamma a\mu m\beta[k,a]_\gamma\alpha a = 0$. This implies that $a\alpha k\gamma a\mu m\beta(k\gamma a - a\gamma k)\alpha a = 0$ and hence $a\alpha k\gamma a\mu m\beta k\gamma a\alpha a - a\alpha k\gamma a\mu m\beta a\gamma k\alpha a = 0$. Hence by using (A) and $a\alpha a = 0$, we have $(a\alpha k\gamma a)\mu m\beta(a\alpha k\gamma a) = 0$. Hence $a\alpha k\gamma a = 0$ for semiprimeness of M. Thus we find that $(a\alpha k)\Gamma M\Gamma(a\alpha k) = 0$. Hence $a\alpha k = 0$ for all $k \in K$, that is, $a\alpha K = \{0\}$. Similarly, we have $K\alpha a = \{0\}$. □

Lemma 4.5 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprme Γ-ring M and $a, b \in U$ and $\alpha, \beta \in \Gamma$.*

(*i*) *If $a\alpha U\beta a = \{0\}$, then $a = 0$;*

(*ii*) *If $a\alpha U = \{0\}$ ($U\alpha a = \{0\}$), then $a = 0$;*

(*iii*) *If $u\alpha u \in U$ for all $u \in U$ and $a\alpha U\beta b = \{0\}$, then $a\alpha b = 0$ and $b\alpha a = 0$ for all $\alpha \in \Gamma$.*

Proof (*i*) By Lemma 4.4, we have $K\alpha a = M\Gamma[U,U]_\Gamma\Gamma M\alpha a = \{0\}$ and $a\alpha a = 0$ for all

$\alpha \in \Gamma$. Thus for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned} 0 &= [[a,x]_\alpha, a]_\gamma \beta y \alpha a \\ &= [a\alpha x - x\alpha a, a]_\gamma \beta y \alpha a \\ &= a\alpha [x,a]_\gamma \beta y \alpha a - [x,a]_\gamma \alpha a \beta y \alpha a \\ &= a\alpha x\gamma a\beta y\alpha a - a\alpha a\gamma x\beta y\alpha a - x\gamma a\alpha a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a \\ &= a\alpha x\gamma a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a \\ &= 2a\alpha x\gamma a\beta y\alpha a. \end{aligned}$$

By the 2-torsion freeness of M, we have, $a\alpha x\gamma a\beta y\alpha a = 0$. Hence, we obtain $a\alpha x\gamma a\beta y\alpha a\delta x\gamma a = 0$. By using the assumption (A), we have $(a\alpha x\gamma a)\beta y\delta(a\alpha x\gamma a) = 0$ for all $y \in M$. This implies $(a\alpha x\gamma a)\beta M\delta(a\alpha x\gamma a) = 0$. Hence, by semiprimeness of M, we have $a\alpha x\gamma a = 0$ for all $x \in M$ and $\alpha, \gamma \in \Gamma$. Again, by semiprimeness of M, we obtain $a = 0$.

(ii) If $a\alpha U = \{0\}$, then $a\alpha U\beta a = \{0\}$ for all $\beta \in \Gamma$. Thus, by (i), we have $a = 0$. Semilarly, if $U\alpha a = \{0\}$, then $a = 0$.

(iii) If $a\alpha U\beta b = \{0\}$, then we have $(b\gamma a)\alpha U\beta(b\gamma a) = \{0\}$ and hence, by (i), $b\gamma a = 0$, for all $\gamma \in \Gamma$. Also, $(a\gamma b)\alpha U\beta(a\gamma b) = \{0\}$ if $a\alpha U\beta b = \{0\}$ and hence $a\gamma b = 0$. □

Theorem 4.1 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ-ring such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ be an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof Since $U \not\subseteq Z(M)$, we have from Lemma 3.2,

$$B_\alpha(u,v)\beta w\gamma[u,v]_\delta \quad = 0$$

By linearing u, we obtain

$$B_\alpha(u,v)\beta w\gamma[x,v]_\delta + B_\alpha(x,v)\beta w\gamma[u,v]_\delta \quad = 0$$

for all $x \in U$. Now,

$$\begin{aligned} &B_\alpha(u,v)\beta w\gamma[x,v]_\delta \mu z\nu B_\alpha(u,v)\beta w\gamma[x,v]_\delta \\ &\qquad = -B_\alpha(u,v)\beta w\gamma[x,v]_\delta \mu z\nu B_\alpha(x,v)\beta w\gamma[u,v]_\delta \\ &\qquad = 0 \end{aligned}$$

for all $z \in U$. Hence, by Lemma 4.5(i),

$$B_\alpha(u,v)\beta w\gamma[x,v]_\delta \quad = 0.$$

Similarly, linearizing v, we obtain

$$B_\alpha(u,v)\beta w\gamma[x,y]_\delta \quad = 0.$$

for all $y \in U$. Hence the similar proof of the Theorem 2.1 in [7], we obtain the required result. $\square$

Corollary 4.1 *Let M be a 2-torsion free semiprime Γ-ring and $T : M \to M$ be a Jordan left centralizer. Then T is a left centralizer.*

Example 4.1 Let R be a commutative ring with a unity element 1 having the characteristic 2. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in Z, \text{n is not divisible by 2} \right\}$. Then M is a Γ-ring. Let $N = \{(x,x) : x \in R\} \subseteq M$.
Now for all $(x,x) \in N$, $(a,b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have

$$\begin{aligned} &(x,x)\begin{pmatrix} n \\ n \end{pmatrix}(a,b) - (a,b)\begin{pmatrix} n \\ n \end{pmatrix}(x,x) \\ &= (xna - bnx, xnb - anx) \\ &= (xna - 2bnx + bnx, bnx - 2anx + xna) \\ &= (xna + bnx, bnx + xna) \in N. \end{aligned}$$

Therefore, N is a Lie ideal of M.

References

[1] N. Nobusawa, On the Generalization of the Ring Theory, *Osaka J. Math.*, 1(1964), 81-89.

[2] W.E. Bernes, On the Γ-rings of Nobusawa, *Pacific J.Math.*, 18(1966), 411-422.

[3] L. Luh, On the theory of simple Gamma rings, *Michigan Math.J.*, 16(1969), 65-75.

[4] S. Kyuno, On prime Gamma ring, *Pacific J.Math.*, 75(1978), 185-190.

[5] Y. Ceven, Jordan left derivations on completely prime gamma rings, *C.U.Fen-Edebiyat Fakultesi,Fen Bilimleri Dergisi*(2002)Cilt 23 Sayi 2.

[6] A.C.Paul and S Uddin, Lie and Jordan structure in simple gamma rings, *J. Physical Sciences*, Vol.14(2010), 77-86.

[7] M.F. Hoque and A.C. Paul, On Centralizers of Semiprime Gamma Rings, *International Mathematical Forum*, Vol.6(2011), No.13, 627-638.

[8] A.K. Halder and A.C. Paul, Jordan left derivations on Lie ideals of prime gamma rings, *Punjab Uni. J. Math.*, Vol. 44(2012), 23-29.

[9] Z. Ullah and M. A. Chaudhary, *On K-centralizers of semiprime gamma rings, International J. of Algebra*, 6(2012), 1001-1010.

[10] K. K. Dey, A.C.Paul and I. S. Rakhimove, Generalized derivations in semiprime gamma rings, *International J. of Math. Mathematical Sci.*, Vol.2012, 270132.

[11] M.F. Hoque and A.C. Paul, Prime Gamma Rings with Centralizing and Commuting Generalized derivations, *International Journal of Algebra*, Vol. 7(13) (2013), 645-651.

[12] B. Zalar, On centralizers of semiprime rings, *Comment.Math. Univ. Carolinae*, 32(1991), 609-614.

[13] J. Vukman, Centralizers in prime and semiprime rings, *Comment. Math. Univ. Carolinae*, 38, (1997), 231-240.

[14] J. Vukman, Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, 42, 2(2001), 237-245.

[15] M.F. Hoque and A.C. Paul, Centralizers on prime and semiprime gamma rings, *Italian J. Of Pure and Applied Mathematics*, Vol. 35 (2015), 575-586.

[16] M.F. Hoque and A.C. Paul, The θ-Centralizers of Semiprime Gamma Rings, *Research Journal of Applied Sciences, Engineering and Technology*, 6(22), (2013) , 4129-4137.

[17] M.F. Hoque and A.C. Paul, The Jordan θ-centralizers of semiprime gamma rings with involution, *International J. Math. Combin.*, Vol.4 (2013), 15-30.

[18] S. M. Slih and B M. Hammad, Jordan higher centralizer on Γ-rings, *International Mathematical Forum*, 8(11) (2013), 517-526.

[19] M.F. Hoque, F. S. Alshammari and A.C. Paul, Left centralizers of semiprime gamma rings with involution, *Applied Mathematical Sciences*, Vol. 8, (2014) , 4713-4722.

[20] R. Awtar, Lie ideals and Jordan derivations of prime rings, *Proc. Amer. Math. Soc.*, 90(1984) 1, 9-14.

[21] M. Ashraf adn N. U. Rehman, On Lie ideals and Jordan left derivations of prime rings, *Arch. Math. (Brno)*, 36(2000), 201-206.

[22] W. Cortes and C. Haetinger, On Lie ideals and left Jordan σ-centralizers of 2-torsion free rings, *Math. J. Okayama Univ.*, Vol. 51(2009), 111-119.

[23] M. Hongan, N.U. Rehman and R.M. Al-Omary, Lie ideals and Jordan triple derivations in rings, *Rend. Sem. Mat. Uni. Padova*, Vol.120 (2011), 147-156.

[24] N.U. Rehman and M. Hongan, Generalized Jordan derivations on Lie ideals associate with Hochschild 2-cocycles of rings, *Rend. Circ. Mat. Palermo*, Vol.60(2011), 437-444.

[25] A.C.Paul and A. Nazneen, Jordan k-derivations on Lie ideals of prime Γ-rings, *Universal J. Comp. Math.*, Vol.2(1)(2014), 17-21.

[26] M.M. Rahman and A.C.Paul, Jordan derivations on Lie ideals of prime Γ-rings, *Mathematical Theory and Modelling*, Vol.3(3) (2013), 128-135.

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Ruled Surfaces According to Parallel Trasport Frame in $\mathbb{E}^4$

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Abstract: In this paper, we studied the ruled surface generated by a straight line in parallel transport frame moving along a curve in four dimensional Euclidean space and we obtained Gaussian and mean curvatures. Some results and theorems related to be developable and Chen surfaces were given. As a result we gave a special example of ruled surfaces in $\mathbb{E}^4$.

Key Words: Ruled surface, Gaussian curvature, developable surface.

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§1. Introduction

Differential geometry of ruled surfaces has been studied in classical geometry using various approaches (see [11] and [15]). They have also been studied in kinematics by many investigators based primarily on line geometry (see [4], [20] and [21]). Developable surfaces are special ruled surfaces [14]. On the motion of the Frenet vectors and ruled surfaces in the Minkowskı 3- space have been investigated and developable ruled surfaces are given for the special cases in [22]: Later ruled surfaces in the Minkowskı 3-space according to Bishop frame has been studied in [23].

The study of ruled hypersurfaces in higher dimensions have also been studied by many authors (see, e.g. [1]). Although ruled hypersurfaces have singularities, in general there have been very few studies of ruled hypersurfaces with singularities [13]. The 2-ruled hypersurfaces in $\mathbb{E}^4$ is a oneparameter family of planes in $\mathbb{E}^4$, which is a generation of ruled surfaces in $\mathbb{E}^3$ (see [19]). In 1936 Plass studied ruled surfaces imbedded in a Euclidean space of four dimensions. Curvature properties of the surface are investigated with respect to the variation of normal vectors and a curvature conic along a generator of the surface [17]. A theory of ruled surface in $\mathbb{E}^4$ was developed by T. Otsuiki and K. Shiohamain [16]. Afterwards Superconformal ruled surfaces in $\mathbb{E}^4$ have been investigated by Bayram et all (see [2]). The authors agree to the terms of this Copyright Notice, which will apply to this submission if and when it is published by this journal (comments to the editor can be added below). The Frenet frame is constructed for the curve of 3 -time continuously differentiable non-degenerate curves. But, curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In

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this situation, we need an alternative frame in $\mathbb{E}^3$. Therefore In [3], Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in 3 dimensional Euclidean space.

The advantages of the Bishop frame and the comparable Bishop frame with the Frenet frame in Euclidean 3 space was given by Bishop [3] and Hanson [10]. In Euclidean 4-space $\mathbb{E}^4$, we have the same problem for a curve like being in Euclidean 3 space. That is, one of the ith $(1 < i < 4)$ derivative of the curve may be zero. In this situation, we need an alternative frame. So, Using the similar idea, Gökçelik et all. Considered such curves and construct an alternative frame [9]. They gave parallel transport frame of a curve and introduced the relations between the frame and Frenet frame of the curve in 4 dimensional Euclidean space.

In this paper, using the method in a paper of Yüksel [23], we obtained some characterization for ruled surfaces according to parallel transport frame in 4-dimensional Euclidean space.

§2. Basic Concepts

A normal vector field $V(s)$ which is perpendicular to its tangent vector field $T(s)$ said to be relatively parallel vector field if its derivative is tangential along the curve $\alpha(s)$. For a given curve if $T(s)$ is given unique, we choose any convenient arbitrary basis which consist of relatively parallel vector field $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, they are perpendicular to $T(s)$ at each point [9].

Let $\alpha : I \to R \subset \mathbb{E}^4$ be arbitrary curve in the Euclidean 4-space $\mathbb{E}^4$. Recall that the curve $\alpha(s)$ is parameterized by arclength function s if $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where $\langle\ ,\ \rangle$ is the inner product of $\mathbb{E}^4$ given by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. In particular, the norm of a vector $X \in \mathbb{E}^4$ is given by $\|X\| = \sqrt{\langle X, X\rangle}$. Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed curve α. Then T, N, B_1 and B_2 are the tangent, the principal normal, first and second binormal vectors of the curve, respectively. If it is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where, κ, τ and σ denote principal curvature functions according to Serret- Frenet frame of the curve α, respectively. We use the tangent vector $T(s)$ and three relatively parallel vector fields $M_1(s), M_2(s)$ and $M_3(s)$ to construct an alternative frame. We call this frame a parallel transport frame along the curve α. The reason for the name parallel transport frame is because the normal component of the derivatives of the normal vector field is zero. We shall call the set

$\{T, M_1, M_2, M_3\}$ as parallel transport frame and

$$k_1 = \langle T', M_1 \rangle\,,\ k_2 = \langle T', M_2 \rangle\,,\ k_3 = \langle T', M_3 \rangle$$

as parallel transport curvatures. Parallel transport frame of a curve and the relations between the frame and Frenet frame of the curve in 4 dimensional Euclidean space using the Euler angles are given as follow.

Theorem 2.1([9]) *Let $\{T, N, B_1, B_2\}$ be a Frenet frame along a unit speed curve $\alpha : I \to R \subset \mathbb{E}^4$ and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame of the curve α. The relation may be expressed as*

$$\begin{aligned}
T &= T(s),\\
N &= \cos\theta(s)\cos\psi(s)M_1 + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2\\
&\quad +(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3,\\
B_1 &= \cos\theta(s)\sin\psi(s)M_1 + (cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2\\
&\quad +(-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3,\\
B_2 &= -\sin\theta(s)M_1 + \sin\phi(s)\cos\theta(s)M_2 + \cos\phi(s)\cos\theta(s)M_3.
\end{aligned}$$

The alternative parallel frame equations are

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{1}$$

where k_1, k_2, k_3 are principal curvature functions according to parallel transport frame of the curve α and their expression as follows

$$\begin{aligned}
k_1 &= \kappa\cos\theta(s)\cos\psi(s),\\
k_2 &= \kappa(-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s)),\\
k_3 &= \kappa(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s)),
\end{aligned}$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}},\ \psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta^2}}{\sqrt{\kappa^2+\tau^2}},\ \phi' = -\frac{\sqrt{\sigma^2-\theta^2}}{\cos\theta}$$

and the following equalities hold

$$\begin{cases} \kappa(s) = \sqrt{k_1^2+k_2^2+k_3^2}, \\ \tau(s) = -\psi' + \phi'\sin\theta, \\ \sigma(s) = \frac{\theta'}{\sin\psi}, \\ \theta ion\cos\theta(s) + \theta'\cot\psi = 0. \end{cases} \tag{2}$$

§3. Ruled Surfaces in $\mathbb{E}^4$ According to Parallel Transport Frame

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v):(u,v)\in D\subset\mathbb{E}^2$. The tangent space to M at an arbitrary point $p=(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E=\langle X_u, X_u\rangle\,,\ F=\langle X_u, X_v\rangle\,,\ G=\langle X_v, X_v\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. We assume that $g=EG-F^2\neq 0$, i.e., the surface patch $X(u,v)$ is regular.

A ruled surface M in a Euclidean space of four dimension $\mathbb{E}^4$ may be considered as generated by a vector moving along a curve. If the curve C is represented by

$$\alpha(u)=(f_1(u), f_2(u), f_3(u), f_4(u))$$

and the moving vector by

$$\beta(u)=(g_1(u), g_2(u), g_3(u), g_4(u)),$$

where the functions of the parameter u sufficiently regular to permit differentiation as may be required, of any point p on the surface, with the coordinates X_i, will be given by

$$M:\ X_i(u,v)=\alpha(u)+v\beta(u), \tag{3}$$

where if $\beta(u)$ is a unit vector (i.e. $\langle\beta,\beta\rangle=1$), v is the distance of p from the curve C in the positive direction of $\beta(u)$. Curve C is called directrix of the surface and vector $\beta(u)$ is the rulling of generators [18]. If all the vectors $\beta(u)$ are moved to the same point, they form a cone which cuts a unit hypersphere on the origin in a curve. This cone is called a director-cone of the surface. From now on we assume that $\alpha(u)$ is a unit speed curve and $\langle\alpha'(u),\beta(u)\rangle=0$.

Proposition 3.1 ([2]) *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization* (3)*. Then the Gaussian curvature of M at point p is*

$$K=-\frac{1}{g}\left\{\langle X_{uv}, X_{uv}\rangle-\frac{1}{E}\langle X_{uv}, X_u\rangle^2\right\}. \tag{4}$$

Proposition 3.2([2]) *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization* (3)*. Then the mean curvature of M at point p is*

$$4\|H\|=\frac{1}{g^2}\left\{\begin{array}{c}\langle X_{uu}, X_{uu}\rangle-\frac{1}{E}\langle X_{uu}, X_u\rangle^2\\ +\frac{1}{G}\langle X_{uv}, X_u\rangle\left[2\langle X_{uv}, X_v\rangle+\langle X_{uv}, X_u\rangle\right]\\ -\frac{2}{EG}\langle X_{uv}, X_u\rangle\langle X_{uv}, X_v\rangle\langle X_u, X_v\rangle\end{array}\right\}. \tag{5}$$

For a vanishing mean curvature of M, we have the following result of ([17], page 17).

Corollary 3.3([17]) *The only minimal surfaces in $\mathbb{E}^4$ are those of $\mathbb{E}^3$, namely the right helicoid.*

In [5], B.Y. Chen defined the allied vector field $a(v)$ of a normal vector field v. In particular, the allied mean curvature vector field is orthogonal to H. Further, B.Y. Chen defined the A-surface to be the surfaces for which $a(H)$ vanishes identically. Such surfaces are also called Chen surfaces [7]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which $\dim N_1 \leq 1$, in particular all hypersurfaces. These Chen surfaces are said to be trivial A-surfaces [8]. In [18], B. Rouxel considered ruled Chen surfaces in $\mathbb{E}^n$. For more details, see also [6] and [12].

Theorem 3.4([19]) *A ruled surface in $\mathbb{E}^n$ $(n > 3)$ is a Chen surface if and only if it is one of the following surfaces:*

(i) *a developable ruled surface;*

(ii) *a ruled surface generated by the n-th vector of the Frenet frame of a curve in $\mathbb{E}^n$ with constant $(n-1)$-st curvature;*

(iii) *a "helicoid" with a constant distribution parameter.*

3.1 One Parameter Spatial Motion in $\mathbb{E}^4$

Let $\alpha : I \to \mathbb{E}^4$ be a unit speed curve and $\{T, M_1, M_2, M_3\}$ g be its parallel transport frame where $\{T, M_1, M_2, M_3\}$ are the tangent, principal normal, first binormal, second binormal vectors of the curve α, respectively. The two coordinate systems $\{O; T, M_1, M_2, M_3\}$ and $\{O'; e_1, e_2, e_3, e_4\}$ are orthogonal coordinate systems in $\mathbb{E}^4$ which represent the moving space H and the fixed space H' respectively.

Let A be a unit vector

$$A \in Sp\{T, M_1, M_2, M_3\} \quad \text{and} \quad A = a_1 T + a_2 M_1 + a_3 M_2 + a_4 M_3$$

such that

$$\langle A, A \rangle = 1$$

We can obtain the Gaussian curvature of the ruled surface generated by a straight line A of the moving space H. The tangent space to M at an arbitrary point $P = X(u, v)$ of M is spanned by

$$X_u = \alpha'(u) + v\beta'(u), \quad X_v = \beta(u).$$

Further, the coefficient of the first fundamental form becomes

$$\begin{aligned}
E &= \langle X_u, X_u \rangle = (1 - v(a_2 k_1 + a_3 k_2 + a_4 k_3))^2 + v^2 a_1^2 \left(k_1^2 + k_2^2 + k_3^2\right), \\
F &= \langle X_u, X_v \rangle = a_1, \\
G &= \langle X_v, X_v \rangle = 1
\end{aligned}$$

and

$$g = EG - F^2 = (1 - v(a_2 k_1 + a_3 k_2 + a_4 k_3))^2 + a_1^2 \left(v^2 \left(k_1^2 + k_2^2 + k_3^2\right) - 1\right).$$

Hence, taking into account (4), the Gauss curvature is an obtain as follow.

$$K = \frac{W_1}{W_2}, \tag{6}$$

where

$$\begin{aligned} W_1 &= -\left[(a_2k_1 + a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right]\left[(1 - v\,(a_2k_1 + a_3k_2 + a_4k_3))^2 + v^2a_1^2\kappa^2\right] \\ &\quad + \left[-\,(a_2k_1 + a_3k_2 + a_4k_3) + v\left((a_2k_1 + a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right)\right]^2 \\ W_2 &= (1 - v\,(a_2k_1 + a_3k_2 + a_4k_3))^2 \\ &\quad + a_1^2\left(v^2\kappa^2 - 1\right)\left((1 - v\,(a_2k_1 + a_3k_2 + a_4k_3))^2 + v^2a_1^2\kappa^2\right). \end{aligned}$$

3.2 Special Cases

Let M be a ruled surface given by the parametrization (3) and A be the director vector of the base curve α.

3.2.1 The Case $A = T$

In this case, $a_1 = 1$, $a_2 = a_3 = a_4 = 0$. Thus, from (6) we obtain Gaussian curvature as follows

$$K_T = -\frac{1}{v^2(1 + v^2\kappa^2)}.$$

Hence the following corollary is hold.

Corollary 3.5 *According to the parallel transport frame in* $\mathbb{E}^4$ *there is no developable ruled surface.*

3.2.2 The Case $A = M_1$

In this case, $a_2 = 1$, $a_1 = a_3 = a_4 = 0$. Thus, from (6)

$$K_{M_1} = 0.$$

Hence the following theorem is hold.

Theorem 3.6 *During the one-parameter spatial motion* H/H' *the ruled surface in the fixed space* H' *generated by the* M_1 *line of the curve* $\alpha(s)$ *in the moving space* H *is developable.*

We have the following result of Theorem 3.4.

Corollary 3.7 *During the one-parameter spatial motion* H/H' *the ruled surface in the fixed space* H' *generated by the* M_1 *line of the curve* $\alpha(s)$ *in the moving space* H *is Chen surface.*

3.2.3 The Case $A = M_2$

In this case, $a_3 = 1$, $a_1 = a_2 = a_4 = 0$. Thus, from (6)

$$K_{M_2} = 0.$$

Hence the following theorem is hold.

Theorem 3.8 *During the one-parameter spatial motion H/H' the ruled surface in the fixed space H' generated by the M_2 line of the curve $\alpha(s)$ in the moving space H is developable.*

We have the following result of Theorem 3.4.

Corollary 3.9 *During the one-parameter spatial motion H/H' the ruled surface in the fixed space H' generated by the M_2 line of the curve $\alpha(s)$ in the moving space H is Chen surface.*

3.2.4 The Case $A = M_3$

In this case, $a_4 = 1$, $a_1 = a_2 = a_3 = 0$. Thus, from (6)

$$K_{M_3} = 0.$$

Hence the following theorem is hold.

Theorem 3.10 *During the one -parameter spatial motion H/H' the ruled surface in the fixed space H' generated by the M_3 line of the curve $\alpha(s)$ in the moving space H is developable.*

We have the following result of Theorem 3.4.

Corollary 3.11 *During the one-parameter spatial motion H/H' the ruled surface in the fixed space H' generated by the M_3 line of the curve $\alpha(s)$ in the moving space H is Chen surface.*

3.2.5 The Case $A \in Sp\{T(u), M_1(u)\}$

In this case, $a_3 = a_4 = 0$. So, the director vector A is given by

$$A = a_1 T + a_2 M_1, \quad a_1^2 + a_2^2 = 1.$$

In this case, Gauss Curvature of the ruled surface is given by

$$K_A = \frac{W_1^{12}}{W_2^{12}},$$

where,

$$\begin{aligned} W_1^{12} &= -\left(a_2^2 k_1^2 + a_1^2 \kappa^2\right)\left[\left(1 - v a_2 k_1\right)^2 + v^2 a_1^2 \kappa^2\right] + \left[-a_2 k_1 + v\left(a_2^2 k_1^2 + a_1^2 \kappa^2\right)\right]^2 \\ W_2^{12} &= \left(\left(1 - v a_2 k_1\right)^2 + a_1^2\left(v^2 \kappa^2 - 1\right)\right)\left(\left(1 - v a_2 k_1\right)^2 + v^2 a_1^2 \kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A = 0$. Thus,

$$-\left(a_2^2 k_1^2 + a_1^2 \kappa^2\right)\left[(1 - v a_2 k_1)^2 + v^2 a_1^2 \kappa^2\right] + \left[-a_2 k_1 + v\left(a_2^2 k_1^2 + a_1^2 \kappa^2\right)\right]^2 = 0$$

or

$$a_1^2 \kappa^2 = 0,$$

which implies that $a_1 = 0$ because of $\kappa \neq 0$. However, if $a_1 = 0$, this is the case of $A = M_1$.

3.2.6 The Case $A \in Sp\{T(u), M_2(u)\}$

In this case, $a_2 = a_4 = 0$. So, the director vector A is given by

$$A = a_1 T + a_3 M_2, \quad a_1^2 + a_3^2 = 1.$$

In this case, Gauss Curvature of the ruled surface is given by

$$K_A = \frac{W_1^{13}}{W_2^{13}},$$

where,

$$\begin{aligned} W_1^{13} &= -\left(a_3^2 k_2^2 + a_1^2 \kappa^2\right)\left[(1 - v a_3 k_2)^2 + v^2 a_1^2 \kappa^2\right] + \left[-a_3 k_2 + v\left(a_3^2 k_2^2 + a_1^2 \kappa^2\right)\right]^2 \\ W_2^{13} &= \left((1 - v a_3 k_2)^2 + a_1^2\left(v^2 \kappa^2 - 1\right)\right)\left((1 - v a_3 k_2)^2 + v^2 a_1^2 \kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A = 0$. Thus

$$-\left(a_3^2 k_2^2 + a_1^2 \kappa^2\right)\left[(1 - v a_3 k_2)^2 + v^2 a_1^2 \kappa^2\right] + \left[-a_3 k_2 + v\left(a_3^2 k_2^2 + a_1^2 \kappa^2\right)\right]^2 = 0$$

or

$$a_1^2 \kappa^2 = 0,$$

which implies that $a_1 = 0$ because of $\kappa \neq 0$. However, if $a_1 = 0$, this is the case of $A = M_2$.

3.2.7 The Case $A \in Sp\{T(u), M_3(u)\}$

In this case, $a_2 = a_3 = 0$. So, the director vector A is given by

$$A = a_1 T + a_4 M_3, \quad a_1^2 + a_4^2 = 1.$$

In this case, Gauss Curvature of the ruled surface is given by

$$K_A = \frac{W_1^{14}}{W_2^{14}},$$

where,

$$\begin{aligned} W_1^{14} &= -\left(a_4^2k3^2+a_1^2\kappa^2\right)\left[(1-va_4k_3)^2+v^2a_1^2\kappa^2\right]+\left[-a_4k_3+v\left(a_4^2k_3^2+a_1^2\kappa^2\right)\right]^2 \\ W_2^{14} &= \left((1-va_4k_3)^2+a_1^2\left(v^2\kappa^2-1\right)\right)\left((1-va_4k_3)^2+v^2a_1^2\kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A=0$. Thus

$$-\left(a_4^2k3^2+a_1^2\kappa^2\right)\left[(1-va_4k_3)^2+v^2a_1^2\kappa^2\right]+\left[-a_4k_3+v\left(a_4^2k_3^2+a_1^2\kappa^2\right)\right]^2=0$$

or

$$a_1^2\kappa^2=0,$$

which implies that $a_1=0$ because of $\kappa\neq 0$. However, if $a_1=0$, this is the case of $A=M_3$.

3.2.8 The Cases $A\in Sp\{M_1(u),M_2(u)\}\bigcap Sp\{M_1(u),M_3(u)\}\bigcap Sp\{M_2(u),M_3(u)\}$

In this cases, the Gauss curvatures are zero. Hence the ruled surfaces generated by $A\in Sp\{M_1(u),M_2(u)\}$, $A\in Sp\{M_1(u),M_3(u)\}$ and $A\in Sp\{M_2(u),M_3(u)\}$ Bishop vectors respectively are developable.

3.2.9 The Case $A\in Sp\{T(u),M_1(u),M_2(u)\}$

In this cases, a_4 is zero. So, the director vector A is given by

$$A=a_1T+a_2M_1+a_3M_2,\quad a_1^2+a_2^2+a_3^2=1.$$

In this case,

$$K_A=\frac{W_1^{123}}{W_2^{123}},$$

where,

$$\begin{aligned} W_1^{123} &= -\left[(a_2k_1+a_3k_2)^2+a_1^2\kappa^2\right]\left[(1-v\left(a_2k_1+a_3k_2\right))^2+v^2a_1^2\kappa^2\right] \\ &\quad+\left[-\left(a_2k_1+a_3k_2\right)+v\left((a_2k_1+a_3k_2)^2+a_1^2\kappa^2\right)\right]^2 \\ W_2^{123} &= \left((1-v\left(a_2k_1+a_3k_2\right))^2+a_1^2\left(v^2\kappa^2-1\right)\right)\left((1-v\left(a_2k_1+a_3k_2\right))^2+v^2a_1^2\kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A=0$. Thus

$$\begin{aligned} -\left[(a_2k_1+a_3k_2)^2+a_1^2\kappa^2\right]\left[(1-v\left(a_2k_1+a_3k_2\right))^2+v^2a_1^2\kappa^2\right] \\ +\left[-\left(a_2k_1+a_3k_2\right)+v\left((a_2k_1+a_3k_2)^2+a_1^2\kappa^2\right)\right]^2=0 \end{aligned}$$

or

$$a_1^2\kappa^2=0,$$

which implies that $a_1=0$ because of $\kappa\neq 0$. However, if $a_1=0$, this is the case of $A\in Sp\{M_1(u),M_2(u)\}$.

3.2.10 The Case $A \in Sp\{T(u), M_1(u), M_3(u)\}$

In this cases, a_3 is zero. So, the director vector A is given by

$$A = a_1 T + a_2 M_1 + a_4 M_3, \quad a_1^2 + a_2^2 + a_4^2 = 1.$$

In this case,

$$K_A = \frac{W_1^{124}}{W_2^{124}},$$

where,

$$\begin{aligned} W_1^{124} &= -\left[(a_2k_1 + a_4k_3)^2 + a_1^2\kappa^2\right]\left[(1 - v(a_2k_1 + a_4k_3))^2 + v^2a_1^2\kappa^2\right] \\ &\quad + \left[-(a_2k_1 + a_4k_3) + v\left((a_2k_1 + a_4k_3)^2 + a_1^2\kappa^2\right)\right]^2 \\ W_2^{124} &= \left((1 - v(a_2k_1 + a_4k_3))^2 + a_1^2\left(v^2\kappa^2 - 1\right)\right)\left((1 - v(a_2k_1 + a_4k_3))^2 + v^2a_1^2\kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A = 0$. Thus

$$\begin{aligned} &-\left[(a_2k_1 + a_4k_3)^2 + a_1^2\kappa^2\right]\left[(1 - v(a_2k_1 + a_4k_3))^2 + v^2a_1^2\kappa^2\right] \\ &\qquad + \left[-(a_2k_1 + a_4k_3) + v\left((a_2k_1 + a_4k_3)^2 + a_1^2\kappa^2\right)\right]^2 = 0 \end{aligned}$$

or

$$a_1^2\kappa^2 = 0,$$

which implies that $a_1 = 0$ because of $\kappa \neq 0$. However, if $a_1 = 0$, this is the case of $A \in Sp\{M_1(u), M_3(u)\}$.

3.2.11 The Case $A \in Sp\{T(u), M_2(u), M_3(u)\}$

In this cases, a_2 is zero. So, the director vector A is given by

$$A = a_1 T + a_3 M_2 + a_4 M_3, \quad a_1^2 + a_3^2 + a_4^2 = 1.$$

In this case,

$$K_A = \frac{W_1^{134}}{W_2^{134}},$$

where,

$$\begin{aligned} W_1^{134} &= -\left[(a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right]\left[(1 - v(a_3k_2 + a_4k_3))^2 + v^2a_1^2\kappa^2\right] \\ &\quad + \left[-(a_3k_2 + a_4k_3) + v\left((a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right)\right]^2 \\ W_2^{134} &= \left((1 - v(a_3k_2 + a_4k_3))^2 + a_1^2\left(v^2\kappa^2 - 1\right)\right)\left((1 - v(a_3k_2 + a_4k_3))^2 + v^2a_1^2\kappa^2\right). \end{aligned}$$

The ruled surfaces is developable if and only if $K_A = 0$. Thus

$$-\left[(a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right]\left[(1 - v(a_3k_2 + a_4k_3))^2 + v^2a_1^2\kappa^2\right] + \left[-(a_3k_2 + a_4k_3) + v\left((a_3k_2 + a_4k_3)^2 + a_1^2\kappa^2\right)\right]^2 = 0$$

or

$$a_1^2\kappa^2 = 0,$$

which implies that $a_1 = 0$ because of $\kappa \neq 0$. However, if $a_1 = 0$, this is the case of $A \in Sp\{M_2(u), M_3(u)\}$.

3.2.12 The Case $A \in Sp\{M_1(u), M_2(u), M_3(u)\}$

In this cases, a_1 is zero. So, the director vector A is given by

$$A = a_2M_1 + a_3M_2 + a_4M_3, \quad a_2^2 + a_3^2 + a_4^2 = 1.$$

In this case,

$$K_A = \frac{W_1^{234}}{W_2^{234}},$$

where,

$$\begin{aligned} W_1^{234} &= -(a_2k_1 + a_3k_2 + a_4k_3)^2(1 - v(a_2k_1 + a_3k_2 + a_4k_3))^2 \\ &\quad + \left[-(a_2k_1 + a_3k_2 + a_4k_3) + v(a_2k_1 + a_3k_2 + a_4k_3)^2\right]^2 \\ W_2^{234} &= (1 - v(a_2k_1 + a_3k_2 + a_4k_3))^2(1 - v(a_2k_1 + a_3k_2 + a_4k_3))^2. \end{aligned}$$

The ruled surfaces is developable if and only if $K_A = 0$. Hence the following theorem is hold.

Theorem 3.12 *During the one-parameter spatial motion H/H' the ruled surface in the fixed space H' generated by $A \in Sp\{M_1(u), M_2(u), M_3(u)\}$ line of the curve $\alpha(s)$ in the moving space H is developable.*

We have the following result of Theorem 3.4.

Corollary 3.13 *During the one-parameter spatial motion H/H' the ruled surface in the fixed space H' generated by $A \in Sp\{M_1(u), M_2(u), M_3(u)\}$ line of the curve $\alpha(s)$ in the moving space H is Chen surface.*

Example 3.14 Let α be a smooth closed regular curve in $\mathbb{E}^4$ given by the arclength parameter with curvatures k_1, k_2, k_3 and the parallel transport frame $\{T, N_1, N_2, N_3\}$. The parallel transport equation of α are given as follows:

$$\begin{cases} T' = k_1 M_1 + k_2 M_2 + k_3 M_3 \\ M_1' = -k_1 T \\ M_2' = -k_2 T \\ M_3' = -k_3 T \end{cases}$$

Let α be a curve as above and consider the ruled surface

$$M_i = X(u,v) = \alpha(u) + vN_i, \quad i = 1,2,3. \tag{7}$$

Then, by using of (1) it is easy to calculate that mean curvatures of these surfaces. See Table 1 for details.

Surface	Mean Curvature (H)
M_1	$\dfrac{k_2^2 + k_3^2}{4(1 - vk_1)^2}$
M_2	$\dfrac{k_1^2 + k_3^2}{4(1 - vk_2)^2}$
M_3	$\dfrac{k_1^2 + k_2^2}{4(1 - vk_3)^2}$

Table 1

Hence following results are obtained.

Corollary 3.15 *Let α be a smooth closed regular curve in $\mathbb{E}^4$, and let M_1, M_2, M_3 be ruled surfaces given by the parametrization* (7). *In that case there is no minimal surfaces among the surfaces of M_1, M_2, M_3.*

References

[1] N. H.Abdel All, H. N. Abd-Ellah, The tangential variation on hyperruled surfaces, *Applied Mathematics and Computation*, 149(2004), 475- 492.

[2] B.Bayram, B.Bulca, K.Arslan,G. Öztürk, Superconformal ruled surfaces in $\mathbb{E}^4$, *Mathematical Communications*, 14(2009), 235-244.

[3] R. L.Bishop, There is more than one way to frame a curve, *The American Mathematical Monthly*, 82(1975), 246-251.

[4] O. Bottema, B.Roth, *Theoretical Kinematics*, North-Holland Press, New York, 1979.

[5] B.Y.Chen, *Geometry of Submanifols*, Dekker, New York, 1973.

[6] U.Dursun, On product k-Chen type submanifolds, *Glasgow Math. J.*, 39(1997), 243-249.

[7] F.Geysens, L.Verheyen, L.Verstraelen, Sur les surfaces A on les surfaces de Chen, *C. R. Acad. Sc. Paris*, 211(1981).

[8] F.Geysens, L.Verheyen, L.Verstraelen, Characterization and examples of Chen submanifolds, *Journal of Geometry*, 20(1983), 47-62.

[9] F.Gökçelik, Z.Bozkurt, İ.Gök, F.N. Ekmekci, Y.Yaylı, Parallel transport frame in 4-dimensional Euclidean space $\mathbb{E}^4$, *Caspian Journal of Mathematical Sciences*, Iran, 3(1)(2014), 91-102.

[10] A. J.Hanson, H.Ma, Parallel transport approach to curve framing, *Indıana University Department of Computer Science Department*, 425(1995).

[11] J. Hoschek, *Liniengeometrie*, Bibliographisches Institut AG, ZuriÄch, 1971.

[12] E.İyigün, K. Arslan, G.Öztüurk, A characterization of Chen surfaces in $\mathbb{E}^4$, *Bull. Malays. Math. Sci. Soc.*, 31(2008), 209-215.

[13] S.Izumiya, N.Takeuchi, Singularities of ruled surfaces in $\mathbb{R}^3$, *Math. Proc. Camb.Phil. Soc.*, 130(2001), 1-11.

[14] S.Izumiya, N.Takeuchi, New special curves and developable surfaces, *Turk. J.Math.*, 28(2004), 153-163.

[15] E.Kruppa, *Analytische und konstruktive Differentialgeometrie*, Springer, Wien, 1957.

[16] T.Otsuiki, K.Shiohama, A theory of ruled surfaces in $\mathbb{E}^4$, *Kodai Math. Sem. Rep.*, 19(1967), 370-380.

[17] M.H.Plass, *Ruled Surfaces in Euclidean Four Space*, Ph.D. thesis, Massachusetts Institute of Technology, 1939.

[18] B.Rouxel, Ruled A-submanifolds in Euclidean Space $\mathbb{E}^4$, *Soochow J. Math.*, 6(1980),117-121.

[19] K.Saji, Singularities of non-degenerate 2-ruled hypersurfaces in 4- space, *Hiroshima Math. J.*, 32(2002), 301-323.

[20] G.R.Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics, *Mechanisms and Machine Theory*, 11(1976), 141-156.

[21] A.T.Yang, Y.Kirson, B.Roth, On a kinematic curvature theory for ruled surfaces, in: *Proceedings of the Fourth World Congress on the Theory of Machines and Mech-anisms*, (F. Freudenstein and R. Alizade, Eds.), Mechanical Engineering Publications,1975, 737-742.

[22] Y.Yaylı, On the motion of the Frenet vectors and spacelike ruled surfaces in the minkowski 3-space, *Mathematical & Computational Applications*, 5(2000), 49-55.

[23] N.Yuksel, The ruled surfaces according to Bishop frame in Minkowski 3-space, *Abstract and Applied Analysis*, (2013).

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Cayley Fuzzy Digraph Structure Induced by Groups

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Abstract: In this paper we introduce a class of Cayley fuzzy digraph structure induced by groups. Further many graph properties are expressed in terms of algebraic properties.

Key Words: Fuzzy graph, Cayley digraph structure, vertex transitive graph.

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§1. Introduction

Digraph Structure Let V be a non-empty set and $S_1, S_2, \cdots, S_k$ are relations on V which are mutually disjoint, then $G' = (V, S_1, S_2, \cdots, S_n)$ is a *digraph structure*. In addition, if $S_1, S_2, \cdots, S_k$ are symmetric and irreflexive, then $G' = (V, S_1, S_2, \cdots, S_k)$ is a *graph structure*, see [2] for details.

Let G be a group and $S_1, S_2, \cdots, S_n$ be mutually disjoint subsets of G. Then the *Cayley digraph structure* of G with respect to $S_1, S_2, \cdots, S_n$ is defined as the graph structure $X = (G; E_1, E_2, \cdots, E_n)$, where $E_i = \{(x, y) : x^{-1}y \in S_i\}$ [1]. In case, a digraph structure with only one connection set is the usual Cayley digraph. So a Cayley digraph structure is a generalization of the Cayley digraph.

Fuzzy Digraph Structure Let $G' = (V, S_1, S_2, \cdots, S_k)$ be a graph (digraph) structure and $\mu, \rho_1, \rho_2, \cdots, \rho_k$ be fuzzy subsets of $V, S_1, S_2, \cdots, S_k$ respectively such that $\rho_i(x, y) \leq \mu(x) \wedge \mu(y)$, for all $x, y \in V$ and $i = 1, 2, \cdots, k$. Then $G = (\mu, \rho_1, \rho_2, \cdots, \rho_k)$ is a *fuzzy graph (digraph) structure* of G' [8], such that $\rho_i(x, y) \leq \mu(x) \wedge \mu(y)$, for all $x, y \in V$ and $i = 1, 2, \cdots, n$. Then $G = (\mu, \rho_1, \rho_2, \cdots, \rho_n)$ is a *fuzzy digraph structure* of G'.

Let V be a non-empty set, μ be fuzzy subset of V and $R_1, R_2, \ldots, R_n$ be mutually disjoint fuzzy relations on μ. Then $G = (\mu, R_1, R_2, \cdots, R_n)$ is a fuzzy digraph structure on V. In case $\mu = \chi_V$, where χ_V is the characteristic function on V, then the fuzzy digraph structure $(\mu, R_1, R_2, \cdots, R_n)$ is simply denoted by $G = (V; R_1, R_2, \cdots, R_n)$.

A fuzzy digraph structure $G = (V; R_1, R_2, \cdots, R_n)$ is called (i) trivial if $R_i \equiv 0$ for all i, (ii) reflexive if for all $x \in V, R_i(x, x) = 1$ for some i, (iii) symmetric if $R_i = R_i^{-1}$ for all i, (iv) transitive if for every i and j, $R_i \wedge R_j \leq R_k$ for some k, (v) a Hasse diagram if for every positive integer $m \geq 2$ and for every $x_1, x_2, \cdots, x_m$ of V with $R_i(x_j, x_{j+1}) > 0$ for all $j = 0, 1, 2, \cdots, m-1$, implies $R_i(x_0, x_m) = 0$ for all i, and (vi) complete if for any

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$x, y \in V, R_i(x, y) > 0$, for some $i = 1, 2, \cdots, n$. A walk of length k in a digraph structure is an alternating sequence $W = x_0, e_0, x_1, \cdots, e_k, x_k$, where $e_j = (x_j, x_{j+1})$ and $R_i(e_j) > 0$ for some i. A walk W is called a path if all the vertices are distinct. We use notation $x_0, x_1, x_2, \cdots, x_k$ for the walk W. A walk is called a circuit if its first and last vertices are the same, but no other vertex is repeated. A weak path is a sequence $x_1, x_2, \cdots, x_m$ of distinct vertices of V such that for $j = 1, 2, \cdots, m-1$, $R_i \vee R_i^{-1}(x_j, x_{j+1}) > 0$ for some $i = 1, 2, \cdots, n$. Distance between two vertices x and y in G is the length of the shortest path from x to y and is denoted by $d(x, y)$. Diameter of the fuzzy digraph structure G, denoted by $d(G)$, is defined by $d(G) = max_{x,y \in G} d(x, y)$. A fuzzy digraph structure $G = (V; R_1, R_2, \cdots, R_n)$ is called (i) connected (strongly connected) if y is connected to x for all $x, y \in V$, and (ii) weakly connected if any two vertices can be joined by a weak path, that is, the fuzzy digraph structure $G' = (V; R_1 \vee R_1^{-1}, R_2 \vee R_2^{-1}, \cdots, R_n \vee R_n^{-1})$ is connected. A weakly connected fuzzy digraph structure $G = (V; R_1, R_2, \cdots, R_n)$ with out any circuits is called a tree.

The present work is a generalisation of the work in [6] in which Madhavan Namboothiri N.M. et al. introduced a class of Cayley fuzzy graphs induced by groups.

§2. Cayley Fuzzy Digraph Structure

Definition 2.1 *Let V be a group and $\nu_1, \nu_2, \cdots, \nu_n$ be mutually disjoint fuzzy subsets of V. Then, Cayley Fuzzy Digraph Structure of V with respect to $\nu_1, \nu_2, \cdots, \nu_n$ is defined as $(V; R_1, R_2, \cdots, R_n)$ where $R_i(x, y) = \nu_i(x^{-1}y)$ and is denoted by $CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$. The subsets $\nu_1, \nu_2, \cdots, \nu_n$ are called connection fuzzy subsets of $CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$. In case, a Cayley fuzzy digraph structure with only one connection set is usual Cayley fuzzy graph.*

Theorem 2.2 *$G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is vertex-transitive.*

Proof Let a and b be any two arbitrary elements in G. Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x$ for all $x \in V$. Clearly, ψ is a bijection onto itself. Furthermore, we have, for each $x, y \in V$,

$$\begin{aligned} R_i(\psi(x), \psi(y)) &= R_i(ba^{-1}x, ba^{-1}y) \\ &= \nu_i((ba^{-1}x)^{-1}(ba^{-1}y)) \\ &= \nu_i(x^{-1}y) = R_i(x, y). \end{aligned}$$

Hence, the proof is complete. □

Theorem 2.3 *Cayley fuzzy digraph structures are regular.*

Proof Let $G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ be a cayley fuzzy digraph structure. Let $u, v \in V$. Since Cayley fuzzy digraph structures are vertex transitive, there exist an automorphism say, f on G such that, $f(u) = v$ and $R_i(f(x), f(y)) = R_i(x, y)$ for any $x, y \in V$ and $i = 1, 2, \cdots, n$.

Then the in-degree of u,

$$\begin{aligned} ind(u) &= \sum_{x\in V}\sum_{i=1}^{n} R_i(x,u) = \sum_{x\in V}\sum_{i=1}^{n} R_i(f(x),f(u)) \\ &= \sum_{x\in V}\sum_{i=1}^{n} R_i(f(x),v) = \sum_{f(x)\in V}\sum_{i=1}^{n} R_i(f(x),v) \\ &= \sum_{y\in V}\sum_{i=1}^{n} R_i(y,v) = ind(v). \end{aligned}$$

Similarly, we can prove that $outd(u) = outd(v)$. Therefore, G is in-regular and out-regular. Now to prove that G is regular we just need to show that $ind(1) = outd(1)$.

$$\begin{aligned} ind(1) &= \sum_{x\in V}\sum_{i=1}^{n} R_i(x,1) = \sum_{x\in V}\sum_{i=1}^{n} \nu_i(x^{-1}) \\ &= \sum_{x\in V}\sum_{i=1}^{n} \nu_i(x) = \sum_{x\in V}\sum_{i=1}^{n} R_i(1,x) = outd(1). \end{aligned}$$

Therefore, G is regular. $\square$

Theorem 2.4 *$G = CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is a trivial graph if and only if $\nu_i \equiv 0$ for all i.*

Proof By definition, G is trivial if and only if $R_i \equiv 0$ for all i. This implies that $\nu_i \equiv 0$ for all i. $\square$

Theorem 2.5 *$G = CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is reflexive if and only if $\nu_i(1) = 1$ for some i.*

Proof Assume that $G = CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is reflexive. Then for every $x \in V$, $R_i(x,x) = 1$ for some i. This implies that $\nu_i(x^{-1}x) = \nu_i(1) = 1$ for some i.

Conversely, let $\nu_i(1) = 1$ for some i, say $i = k$. This implies that for each $x \in V$, $R_k(x,x) = \nu_k(x^{-1}x) = \nu_k(1) = 1$. That is G is reflexive. $\square$

Theorem 2.6 *$G = CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is symmetric if and only if $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V$, $i = 1,2,\cdots,n$.*

Proof Suppose that G is symmetric. Then for any $x \in V$,

$$\nu_i(x) = \nu(x^{-1}x^2) = R_i(x,x^2) = {R_i}^{-1}(x,x^2) = R_i(x^2,x) = \nu_i(x^{-1}x^{-1}x) = \nu_i(x^{-1}).$$

Therefore, $\nu_i(x) = \nu_i(x^{-1})$.

Conversely, suppose that $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V$. Then for any $x, y \in V$, $R_i(x,y) = \nu_i(x^{-1}y) = \nu_i((x^{-1}y)^{-1}) = \nu_i(y^{-1}x) = R_i(y,x)$. This implies that, R is symmetric. Hence the proof is complete. $\square$

Theorem 2.7 *$G = CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is transitive if and only if for every i,j and for* *any* $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ *for some* k.

Proof First assume that G is transitive. That is, for every i, j, $R_i \circ R_j \leq R_k$ for some k. For $x, y \in V$,

$$\begin{aligned}\nu_i(x) \wedge \nu_j(y) &\leq \vee\{\nu_i(z) \wedge \nu_j(z^{-1}(xy)) : z \in V\}\\ &= \vee\{R_i(1, z) \wedge R_j(z, xy) : z \in V\}\\ &= R_i \circ R_j(1, xy)\\ &\leq R_k(1, xy) = \nu_k(xy).\end{aligned}$$

That is, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

Now let for any $x, y \in V$ and i, j, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k. Then,

$$\begin{aligned}(R_i \circ R_j)(x, y) &= \vee\{R_i(x, z) \wedge R_j(z, y) : z \in V\}\\ &= \vee\{\nu_i(x^{-1}z) \wedge \nu_j(z^{-1}y) : z \in V\}\\ &\leq \vee\{\nu_k((x^{-1}z)(z^{-1}y)) : z \in V\}\\ &= \nu_k(x^{-1}y) = R_k(x, y).\end{aligned}$$

Thus, $R_i \circ R_j \leq R_k$ for some k. This completes the proof. □

Theorem 2.8 $G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ *is complete if and only if* $\cup {\nu_i}_\circ^+ = V$.

Proof First assume that G is complete. That is $\cup {R_i}_\circ^+ = V \times V$. Clearly, $\cup {\nu_i}_\circ^+ \subseteq V$. Now let $x \in V$. Then $(1, x) \in {R_i}_\circ^+$ for some i. That is, $R_i(1, x) \geq 0$, which implies, $\nu_i(x) \geq 0$. Thus, $x \in \cup {\nu_i}_\circ^+$. Therefore, $V \subseteq \cup {\nu_i}_\circ^+$. That is, $\cup {\nu_i}_\circ^+ = V$.

Conversely, assume $\cup {\nu_i}_\circ^+ = V$. Let $(x, y) \in V \times V$. Then $x, y \in V \Rightarrow x^{-1}y \in V \Rightarrow x^{-1}y \in \cup {\nu_i}_\circ^+ \Rightarrow x^{-1}y \in {\nu_i}_\circ^+$ for some i. Then, $\nu_i(x^{-1}y) \geq 0$. That is, $R_i(x, y) \geq 0$ which implies $(x, y) \in {R_i}_\circ^+$. Hence, $V \times V \subseteq \cup {R_i}_\circ^+$. Therefore,

$$\bigcup {R_i}_\circ^+ = V \times V.$$

This completes the proof. □

Let A_k be the set of all elements $x \in V$ of the form $x = x_1 x_2 \cdots x_k$, where $x_j \in {\nu_i}_0^+$ for some $i = 1, 2, \cdots, n$. Then $[\vartheta]$ is defined as $[\vartheta] = \bigcup\limits_{k=1}^{n} A_k$. Let B_k be the set of all elements $y \in V$ of the form $y = y_1 y_2 \cdots y_k$, where $y_j \in (\nu_i \wedge \nu_i^{-1})_0^+$ for some $i = 1, 2, \cdots, n$. Then $[[\vartheta]]$ is defined as $[[\vartheta]] = \bigcup\limits_{k=1}^{n} B_k$.

Theorem 2.9 $G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ *is connected if and only if* $V = [\vartheta]$.

Proof First assume that $G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is connected. Clearly, $[\vartheta] \subseteq V$. Now let $x \in V$. Then there exists a path from 1 *to* x say, $(1, y_1, y_2, \cdots, y_k = x)$. Then, for some i, $R_{i_1}(1,y_1)>0$, that is, $y_1\in\nu^+_{i_10}$. Also, $y^{-1}_{j-1}y_j\in\nu^+_{i_j0}$,for $j=2,3,\cdots,k$. This implies that $x\in A_k$, since, $x=(1.y_1)(y_1^{-1}y_2)(y_2^{-1}y_3)\cdots(y^{-1}_{k-1}y_k)$. Therefore, $x\in\bigcup\limits_{k=1}^{n}A_k=[\vartheta]$. Hence, $V=[\vartheta]$.

Conversely, assume that $V=[\vartheta]$. Let $x,y\in V$. Then $z=x^{-1}y\in V$, implies, $z\in[\vartheta]=\bigcup\limits_{k=1}^{n}A_k$. Then $z=z_1z_2\cdots z_k$. Then $1,z_1,z_1z_2,\cdots,z_1z_2\cdots z_k=z$ is a path from 1 to z. Then $x,xz_1,xz_1z_2,\cdots,xz_1z_2\cdots z_k=xz=y$ is a path from x to y, implies G is connected. This completes the proof. □

Theorem 2.10 *$G=CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is weakly connected if and only if $V=[[\vartheta]]$.*

Proof Assume G be weakly connected. Clearly, $[[\vartheta]]\subseteq V$. Let $x\in V$. Then there exist a weak path say, $1,x_1,x_2,\cdots,x_k=x$ from 1 to x. Then, $1x_1\in(\nu_{i_1}\vee\nu^{-1}_{i_1})^+_0$, $x_1^{-1}x_2\in(\nu_{i_2}\vee\nu^{-1}_{i_2})^+_0,\cdots,$ $x^{-1}_{k-1}x_k\in(\nu_{i_k}\vee\nu_{i_k})^+_0$, which clearly implies that

$$x\in\bigcup_k B_k=[[\vartheta]].$$

Hence, $V\in[[\vartheta]]$.

Conversely, assume that $V=[[\vartheta]]$. Let $x,y\in V$, implies $z=x^{-1}y\in V$. Therefore, $z\in[[\vartheta]]$. Then there exist elements $z_j\in(\nu_{i_j}\vee\nu^{-1}_{i_j})^+_0$, $j=1,2,\cdots,k$, such that $z=z_1z_2\cdots z_k$, for some $k\in\{1,2,\cdots,n\}$. Then $1,z_1,z_1z_2,\cdots,z_1z_2\cdots z_k=z$ is a weak path from 1 to z and hence $x,xz_1,xz_1z_2,\cdots,xz_1z_2\cdots z_k=xz=y$ is a weak path from x to y. Therefore, G is weakly connected. This completes the proof. □

Theorem 2.11 *$G=CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is partially ordered if and only if*

(*i*) *$\nu_i(1)=1$ for some i;*

(*ii*) *for every i,j and for any $x,y\in V,\nu_i(x)\wedge\nu_j(y)\le\nu_k(xy)$ for some k;*

(*iii*) *$\{x:\nu(x)=\nu(x^{-1})\}=\{1\}$ for all $i=1,2,\cdots,n$.*

Theorem 2.12 *$G=CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is quasi-ordered if and only if*

(*i*) *$\nu_i(1)=1$ for some i;*

(*ii*) *for every i,j and for any $x,y\in V,\nu_i(x)\wedge\nu_j(y)\le\nu_k(xy)$ for some k.*

Theorem 2.13 *$G=CayF_D(V;\nu_1,\nu_2,\cdots,\nu_n)$ is a Hasse diagram if and only if G is connected and $\nu_k(x_1x_2\cdots x_m)=0,\ k=1,2,\cdots,n$, for any collection $x_1,x_2,\cdots,x_m$ of vertices in V with $m\ge2$ and $\nu_{i_j}(x_j)>0$ for $j=1,2,\cdots,m$.*

Proof Suppose G is a Hasse diagram. Since $\nu_{i_j}(x_j)>0$ for $j=1,2,\cdots,m$, $(1,x_1,x_1x_2,\cdots,x_1x_2\cdots x_m)$ is a path from 1 to $x_1x_2\cdots x_m$. Now since G is a Hasse diagram, $R_k(1,x_1x_2\cdots x_m)=0$ for all k. Therefore $\nu_k(x_1x_2\cdots x_m)=0$ for all $k=1,2,\cdots,n$.

Conversely suppose, G is connected and $\nu_k(x_1x_2\cdots x_m)=0,\ k=1,2,\cdots,n$, for any collection $x_1,x_2,\cdots,x_m$ of vertices in V with $m\ge2$ and $\nu_{i_j}(x_j)>0$ for $j=1,2,\cdots,m$. Let

$(x_0, x_1, \cdots, x_m)$ be a path in G from x_1 to x_m, $m \geq 2$. Then $R_{i_1}(x_0, x_1) > 0$, $R_{i_2}(x_1, x_2) > 0$, $\cdots, R_{i_m}(x_{m-1}, x_m) > 0$ which implies, $\nu_{i_1}(x_0^{-1}x_1) > 0$, $\nu_{i_2}(x_1^{-1}x_2) > 0$, $\cdots, \nu_{i_m}(x_{m-1}^{-1}x_m) > 0$. Thus, by assumption, $\nu_k(x_0^{-1}x_1x_1^{-1}x_2 \cdots x_{m-1}^{-1}\, x_m) = \nu_k(x_0^{-1}x_m) = 0$. Therefore, $R_k(x_0, x_m) = 0$ for all $k = 1, 2, \cdots, n$. Hence, G is a Hasse diagram. This completes the proof. $\square$

Theorem 2.14 *For $k = 1, 2, \cdots, n$, let A_k be the set of all products of the form $\nu_{i_1}\nu_{i_2}\cdots\nu_{i_k} = \{x_1x_2\cdots x_k : x_j \in \nu_{i_j}{}_0^+, j = 1, 2, \cdots, k\}$. If $G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ has finite diameter, then the diameter of G is the least positive integer m such that*

$$G = \bigcup_{A \in A_m} A.$$

Theorem 2.15 *$G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is a tree if and only if $V = [[\vartheta]]$ and $1 \notin [\vartheta]$.*

Definition 2.16([6]) *Let $(S, *)$ be a semigroup. Let A be a fuzzy subset of S. Then A is said to be fuzzy sub-semigroup of S if for all $a, b \in S$, $A(ab) \geq A(a) \wedge A(b)$.*

Definition 2.17 *Let $(S, *)$ be a semigroup and let $\nu_1, \nu_2, \cdots, \nu_n$ be mutually disjoint fuzzy subsets of S. The fuzzy sub-semigroup generated by $\nu_1, \nu_2, \cdots, \nu_n$ is the smallest fuzzy sub-semigroup of S which contains $\nu_1, \nu_2, \cdots, \nu_n$. Let us denote it by $\langle \nu_{(123\cdots n)} \rangle$.*

Theorem 2.18 *Let $(S, *)$ be a semigroup and let $\nu_1, \nu_2, \cdots, \nu_n$ be mutually disjoint fuzzy subsets of S. Then the fuzzy subset $\langle \nu_{(123\cdots n)} \rangle$ is precisely given by $\langle \nu_{(123\cdots n)} \rangle(x) = \vee\big\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \nu_{jk}(x_k) : x = x_1x_2\cdots x_k$ with a finite positive integer $k, x_i \in S$ and $\nu_{j_i}(x_i) > 0$ for some $j_i = 1, 2, \cdots, n\big\}$ for any $x \in S$.*

Proof Let ν' be the fuzzy subset of V defined by $\nu'(x) = \vee\big\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3\cdots x_m,\ xj_i \in \nu_{j_{i_\circ}}^+, m \in \{1, 2, 3, \cdots, n\}\big\}$ for any $x \in V$. If $y \in V$, by definition of ν', it is clear that $\nu'(y) \geq \nu_{j_k}(y)$ where $j_k \in \{1, 2, \cdots, n\}$ and $\nu_{j_k}(y) \geq 0$. Thus, we have $\nu_{j_k} \leq \nu'$ for all j_i. This implies that ν' contains $\nu_1, \nu_2, \cdots, \nu_n$. Let $x, y \in V$. If $\nu_{j_i}(x) = 0$ or $\nu_{j_i}(y) = 0$, then $\nu_{j_i}(x) \wedge \nu_{j_i}(y) = 0$. Then, $\nu'(xy) \geq \nu_{j_i}(x) \wedge \nu_{j_i}(y)$. Again, if $\nu_{j_i}(x) \neq 0$ and $\nu_{j_i}(y) \neq 0$, then by definition of ν', we have $\nu'(xy) \geq \nu_{j_i}(x) \wedge \nu_{j_i}(y)$. Hence ν' is a fuzzy sub semigroup of V containing $\nu_i, i \in \{1, 2, \cdots, n\}$. Now let A be any fuzzy sub semigroup of V containing $\nu_i, i \in \{1, 2, \cdots, n\}$. Then, for any $x \in V$ with $x = x_1x_2x_3\ldots x_m$, $x_i \in \nu_{j_{i_\circ}}^+$, for $i = 1, 2, \cdots, n, m \in \{1, 2, 3, \cdots, n\}$ we have $A(x) \geq A(x_1) \wedge A(x_2) \wedge \cdots \wedge A(x_m) \geq \nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \wedge \nu_{j_m}(x_m)$, which implies that $A(x) \geq \vee\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3\cdots x_m,\ x_{j_i} \in \nu_{j_{i_\circ}}^+, m \in \{1, 2, 3, \cdots, n\}$ for $j_i \in \{1, 2, \ldots, n\}$ for any $x \in V$. Therefore, $A(x) \geq \nu'(x)$ for all $x \in V$. Thus, $\nu' = \langle \nu_{(123\cdots n)} \rangle$. That is, $\langle \nu_{(123\cdots n)} \rangle(x) = \vee\big\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3\cdots x_m,\ x_{j_i} \in \nu_{j_{i_\circ}}^+, m \in \{1, 2, 3, \cdots, n\}\big\}$ for any $x \in V$. $\square$

References

[1] Anil Kumar V. and Parameswaran Ashok Nair, A class of Cayley digraph structures induced by groups, *Journal of Mathematics Research*, Vol. 4, No.2, 28-33, 2012.

[2] E. Sampathkumar, Generalised graph structures, *Bull. Kerala Math. Assoc.*, Vol. 3, No.2 (2006), 65-123.

[3] H.B. Richard, *A Survey of Binary Systems*, Springer- Verlag New York, 1971.

[4] J. N. Mordeson and Premchand S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, Springer, 2000.

[5] K. R. Parthasarathy, *Basic Graph Theory*, Tata-McGraw-Hill Pub., New Delhi, 1994.

[6] Madhavan Namboothiri N. M., Anil Kumar V. and Ramachandran P. T., Cayley Fuzzy graphs, *Far East Journal of Mathematical Sciences*, Vol. 73, No.1, 2013, 15-39.

[7] Cayley, Desiderata and suggestions: No. 2, the theory of groups: graphical representation, *Amer. J. Math.*, 1 (1878).

[8] T. Dinesh and T. V. Ramakrishnan, On generalised Fuzzy graph structures, *Applied Mathematical Sciences*, Vol. 5, 2011, No.173-180.

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Arctangent Finsler Spaces With Reversible Geodesics

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Abstract: A Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In this work, we study a class of special Finsler metrics F called arctangent Finsler metric, which is a special (α,β)-metric, $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$ ($\epsilon \neq 0$ are constant), where α is a Riemannian metric and β is a 1-form. The conditions for an arctangent Finsler space (M, F) to be with reversible geodesic are obtained. Further, we study some geometrical properties of F and prove that the arctangent metric F induces a generalized weighted quasi-distance d_F on M.

Key Words: Reversible Geodesics, weighted quasi metric.

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§1. Introduction

An interesting topic in Finsler geometry is to study the reversible geodesics of a Finsler metric. Recall that, a Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In the last decade many interesting and applicable results have been obtained on the theory of Finsler spaces with reversible geodesics. In [7], Crampin gives necessary and sufficient conditions for a Finsler metric (M, F) to be with reversible and strictly reversible geodesics, respectively.

Reversible geodesic of (α,β)-metric and two dimensional Finsler spaces with (α,β)-metric were studied by Masca, Sabau and Shimada ([8],[9]). In [6], Sabau and Shimada have given some important results on reversible geodesics. In [10], Shanker and Baby have exhaust reversible geodesics for generalized (α,β)-metric.

§2. Preliminaries

Let $F^n = (M, F)$ be a connected n-dimensional Finsler manifold and let $TM = \cup_{x\in M} T_xM$ denotes the tangent bundle of M with local coordinates $u = (X, Y) = (x^i, y^i) \in TM$, where $i = 1, \cdots, n$, $y = y^i \frac{\partial}{\partial x^i}$.

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If $\gamma : [0,1] \to M$ is a piecewise C^∞ curve on M, then its Finslerian length is defined as

$$L_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt \tag{2.1}$$

and the Finslerian distance function $d_F : M \times M \to [0, \infty)$ is defined by $d_F(p,q) = inf_\gamma L$, where infimum is taken over all piecewise C^∞ curves γ om M joining the points $p, q \in M$. In general, this is not symmetric.

A curve $\gamma : [0,1] \to M$ is called a geodesic of (M, F) if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their endpoints fixed. We denote the reverse Finsler metric of F as $\tilde{F} : TM \to (0, \infty)$, given by $\tilde{F}(x,y) = F(x,-y)$. One can easily see that $\tilde{F}$ is also a Finsler metric.

Lemma 2.1 *A Finsler metric is with a reversible geodesic if and only if for any geodesic $\gamma(t)$ of F, the reverse curve $\tilde{\gamma}(t) = \gamma(1-t)$ is also a geodesic of F.*

Lemma 2.2 *Let (M, F) be a connected, complete Finsler manifold with associated distance function $dF : M \times M \to [0, \infty)$. Then, d_F is a symmetric distance function on $M \times M$ if and only if F is a reversible Finsler metric, i.e., $F(x,y) = F(x,-y)$.*

Lemma 2.3 *A smooth curve $\gamma : [0,1] \to M$ is a constant Finslerian speed geodesic of (M, F) if and only if it satisfies $\ddot{\gamma} + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$, $i = 1, ..., n$, where the functions $G^i : TM \to \boldsymbol{R}$ given by*

$$G^i(x,y) = \Gamma^i_{jk}(x,y)y^i y^j \tag{2.2}$$

with

$$\Gamma^i_{jk}(x,y) = \frac{g^{is}}{2}(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}).$$

Remark 2.4 It is well known [3] that the vector field

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

is a vector field on TM, whose integral lines are the canonical lifts $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ of the geodesics of γ. This vector field Γ is called the canonical geodesics spray of the Finsler space (M, F) and G^i are called the coefficients of the geodesics spray Γ.

Definition 2.5 *If F and $\tilde{F}$ are two different fundamental Finsler functions on the same manifold M, then they are said to be projectively equivalent if their geodesics coincide as set points.*

Lemma 2.6 *A Finsler structure (M, F) is with a reversible geodesic if and only if F and its reverse function $\tilde{F}$ are projectively equivalent.*

The main purpose of the current paper is to determine the reversible geodesics for special

Finsler metrics F called arctangent Finsler metric, which is a special (α, β)-metric,

$$F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta \ (\epsilon \neq 0 \text{are constant}),$$

where α is a Riemannian metric and β is a 1-form. The paper is organized as follows:

Starting with preliminary definitions on reversible geodesics in section two, in section three, we obtain the conditions for an arctangent Finsler space to be with reversible geodesics (see Theorem 3.1). in section four, we prove that if F is projectively flat then it is with reversible geodesics (see Theorem 4.2). In section five, we study metric structures associated to F and prove that the arctangent metric F induces a generalized weighted quasi-distance function d_F on the manifold M (see Theorem 5.2).

§3. Reversible Geodesics of Arctangent Finsler Metric

Consider a Finsler space (M, F) with a special (α, β)-metric

$$F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta. \ (\epsilon \neq 0 \text{ are constant}),$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form.

The necessary and sufficient condition for F to have a reversible geodesic is [6]

$$\tilde{\Gamma}\frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i} = 0, \tag{3.1}$$

where $\tilde{\Gamma}$ is the reverse of Γ, the geodesic spray of F, moreover $\tilde{\Gamma}$ is geodesic spray for F. The necessary and sufficient condition for F to have strictly reversible geodesics is $\tilde{\Gamma}F = 0$. Now $F = \alpha + \beta\arctan(\frac{\beta}{\alpha}) + \epsilon\beta$. Then $\tilde{F} = \alpha + \beta\arctan(\frac{\beta}{\alpha}) - \epsilon\beta$, so $F = \tilde{F} + 2\epsilon\beta$.

We have

$$\begin{aligned}
\tilde{\Gamma}F_{y^i} - F_{x^i} =& \arctan(\frac{\beta}{\alpha})(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big)(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}) + \epsilon(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}) \\
=& \Big(\arctan(\frac{\beta}{\alpha}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big) + \epsilon\Big)(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}),
\end{aligned}$$

Notice that for the spray $\tilde{\Gamma}$, we have

$$\tilde{\Gamma}\beta_{y^i} - \beta_{x^i} = (\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j,$$

So we get

$$\tilde{\Gamma}F_{y^i} - F_{x^i} = \Big(\arctan(\frac{\beta}{\alpha}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big) + \epsilon\Big)(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j. \tag{3.2}$$

Now,

$$\left\{\arctan(\frac{\beta}{\alpha})+\frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big)+\epsilon\right\}$$

can not be zero. Therefore, from equation (3.1) and (3.2) we conclude that F is with reversible geodesics if and only if

$$\left(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\right)y^j=0,$$

i.e., $\tilde{F}$ is with reversible geodesic if and only if β is closed 1-form. Hence, we have the following theorem

Theorem 3.1 *Let (M,F) be an arctangent Finsler space with F defined by the Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and 1-form $\beta=b_iy^i$. Then F is with a reversible geodesic if and only if β is a closed 1-form on M.*

§4. Projective Flatness of Arctangent Finsler metric

A Finsler space (M,F) is called (locally) projectively flat if all its geodesics are straight lines [4]. An equivalent condition is that the spray coefficients G^i of F can be expressed as $G^i=P(x,y)y^i$, where $P(x,y)=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k$. An equivalent characterization of projective flatness is the Hamels relation [5].

$$\frac{\partial^2 F}{\partial x^m\partial y^k}y^m-\frac{\partial F}{\partial x^k}=0.$$

Proposition 4.1 *Let (M,F) be an arctangent space with F defined by the Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and 1-form $\beta=b_iy^i$. Then F is projectively flat if and only if $\tilde{F}$ is projectively flat.*

Outline of the Proof Recall (see Theorem 3.1 of [6]) that if $F=F_0+\epsilon\beta$ is a Finsler metric, where F_0 is an absolute homogeneous (α,β)-metric, then any two of the following properties imply the third one:

(a) F is projectively flat;

(b) F_0 is projectively flat;

(c) β is closed.

In our case

$$F=\alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\beta=\tilde{F}+2\epsilon\beta,$$

where

$$\tilde{F}=\alpha+\beta\arctan(\frac{\beta}{\alpha})-\epsilon\beta$$

which is absolute homogeneous.

Proof of Proposition 4.1 Let (M,F) be projectively flat, then by Hamels relation for projective flatness, we have

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.$$

Thus we have

$$\begin{aligned}
&\frac{\partial^2 (\tilde{F} + 2\epsilon\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial (\tilde{F} + 2\epsilon\beta)}{\partial x^k} = 0, \\
&\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} + 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big) = 0, \\
&\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = -2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big).
\end{aligned}$$

Since β is closed, Then we have

$$\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = 0$$

Therefore $\tilde{F}$ is projectively flat.

Conversely, suppose that $\tilde{F}$ is projectively flat. Since $\tilde{F}$ is projectively flat, therefore $\tilde{F}$ will satisfy Hamels equation

$$\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = 0,$$

So we have

$$\begin{aligned}
&\frac{\partial^2 (F - 2\epsilon\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial (F - 2\epsilon\beta)}{\partial x^k} = 0, \\
&\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} - 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big) = 0, \\
&\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big).
\end{aligned}$$

Since β is closed, Then we get:

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0$$

Therefore F is projectively flat. □

Theorem 4.2 *Let (M, F) be be an arctangent space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ and 1-form $\beta = b_i y^i$. If F is projectively flat, then it is with a reversible geodesic.*

Proof By Hamels relation, we can see that F is projectively flat if and only if $\tilde{F}$ is projectively flat. This implies that F and $\tilde{F}$ both are projectively equivalent to the standard Euclidean metric and therefore F must be projective to $\tilde{F}$. Thus F must be with a reversible geodesic. □

§5. **Weighted Quasi Metric Associated with Arctangent Finsler Metric**

It is well known that the Riemannian spaces can be represented as metric spaces. Indeed, for a Riemannian space (M, α), one can define the induced metric space (M, d_α) with the metric

$$d_\alpha : M \times M \to [0, \infty) \quad d_\alpha(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b \alpha(\gamma(t), \dot{\gamma}(t))dt, \tag{5.1}$$

where $\Gamma_{xy} = \{\gamma : [a, b] \to M | \gamma \text{ is piecewise}, \ \gamma(a) = x, \ \gamma(b) = y\}$ is the set of curves joining x and y, $\dot{\gamma}(t)$ is the tangent vector to γ at $\gamma(t)$. Then d_α is a metric on M satisfying the following conditions:

(1) Positiveness: $d_\alpha(x, y) > 0$ if $x \neq y, \ d_\alpha(x, x) = 0, \ x, y \in X$;

(2) Symmetry: $d_\alpha(x, y) = d_\alpha(y, x), \ \ \forall x, y \in M$;

(3) Triangle inequality: $d_\alpha(x, y) \leq d_\alpha(x, z) + d_\alpha(z, y), \ \ \forall x, y, z \in M$.

Similar to the Riemannian space, one can induce the metric d_F to a Finsler space (M, F), given by

$$d_F : M \times M \to [0, \infty) \quad d_F(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b F(\gamma(t), \dot{\gamma}(t))dt, \tag{5.2}$$

but unlike the Riemannian case, here d_F lacks the symmetric condition. In fact, d_F is a special case of quasi metric defined below.

Definition 5.1([1]) *A quasi metric d on a set X is a function $d : X \times X \to [0, \infty)$ that satisfies the following axioms:*

(1) *Positiveness:* $d(x, y) > 0$ *if* $x \neq y$, $d(x, x) = 0$, $x, y \in X$*;*

(2) *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in X$*;*

(3) *Separation axiom:* $d(x, y) = d(y, x) = 0 \ \Rightarrow \ x = y, \forall x, y \in X$.

One special class of quasi metric spaces are the so called weighted quasi metric spaces (M, d, w), where d is a quasi-metric on M and for each d, there exists a function $w : M \to [0, \infty)$, called the weight of d, that satisfies

(4) *Weightability:* $d(x, y) + w(x) = d(y, x) + w(y), \ \forall x, y \in M$.

In this case, the weight function w is R-valued, and is called generalized weight.

Theorem 5.2 *Let M be an n-dimensional simply connected smooth manifold. Arctangent Finsler metric $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$ ($\epsilon \neq 0$ are constant) induces a generalized weighted quasi-distance d_F on M.*

Proof Consider an arctangent Finsler space (M, F) with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ and 1-form $\beta = b_i y^i$. ($\epsilon \neq 0$ are constant) Let $\gamma_{xy} \in \Gamma_{xy}$ be a Finslerian geodesic, then from (5.2), we have

$$\begin{aligned} d_F(x,y) &= \int_a^b F(\gamma(t),\dot\gamma(t))dt \\ &= \int_a^b (\alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\beta)dt \\ &= \int_a^b (\alpha+\beta\arctan(\frac{\beta}{\alpha}))dt+\epsilon\int_{\gamma_{xy}}\beta \\ &= \int_{\gamma_{xy}} \alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\int_{\gamma_{xy}}\beta. \end{aligned} \tag{5.3}$$

Let us consider a fixed point $a\in M$ and define the function $w_a: M\to R, w_a(x)=d_F(a,x)-d_F(x,a)$. From (5.3) it follows that

$$\begin{aligned} w_a(x) &= d_F(a,x)-d_F(x,a) \\ &= \int_{\gamma_{ax}} \alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\int_{\gamma_{ax}}\beta \\ &\quad -\int_{\gamma_{xa}} \alpha+\beta\arctan(\frac{\beta}{\alpha})-\epsilon\int_{\gamma_{xa}}\beta \\ &= \epsilon\int_{\gamma_{ax}}\beta-\epsilon\int_{\gamma_{xa}}\beta \\ &= -2\int_{\gamma_{xa}}\beta, \end{aligned} \tag{5.4}$$

where we have used the Stokes theorem for the 1-form β on the closed domain D with boundary $\partial D=\gamma_{ax}\cup\gamma_{xa}$. One can see that w_a is the anti derivative of β. This is well defined if and only if the path integral in right hand side of (5.4) is path independent, that is, β must be exact. Then d_F is a weighted quasi-metric with generalized weight w_a. Indeed, we have

$$\begin{aligned} d_F(x,y)+w_a(x) &= \int_{\gamma_{xy}}(\alpha+\beta\arctan(\frac{\beta}{\alpha}))+\epsilon\int_{\gamma_{xy}}\beta+\epsilon\int_{\gamma_{ax}}\beta-\epsilon\int_{\gamma_{xa}}\beta \\ &= \int_{\gamma_{xy}}(\alpha+\beta\arctan(\frac{\beta}{\alpha}))-\epsilon\int_{\gamma_{xa}}\beta-\epsilon\int_{\gamma_{ya}}\beta, \end{aligned} \tag{5.5}$$

where we have again used the Stokes theorem for the 1-form β on the closed domain with boundary $\gamma_{ax}\cup\gamma_{xy}\cup\gamma_{ya}$.

Similarly,

$$d_F(y,x)+w_a(y)=\int_{\gamma_{yx}}(\alpha+\beta\arctan(\frac{\beta}{\alpha}))-\int_{\gamma_{ya}}\beta-\int_{\gamma_{xa}}\beta. \tag{5.6}$$

From equations (5.5) and (5.6) we conclude that d_F is weighted quasimetric with generalized weight w_a. This completes the proof. □

Next, recall the following result.

Lemma 5.3([1],[2]) *Let (M,d) be any quasi-metric space. Then d is weightable if and only if there exists $w : M \to [0,\infty)$ such that*

$$d(x,y) = \rho(x,y) + \frac{1}{2}[w(x) - w(y)], \quad \forall x, y \in M, \tag{5.7}$$

where ρ is the symmetrized distance function of d. Moreover, we have

$$\frac{1}{2}[w(x) - w(y)] \leq \rho(x,y), \quad \forall x, y \in M. \tag{5.8}$$

The proof is trivial from the definition of weighted quasi-metric.

Remark 5.4 If (M,F) is an arctangent Finsler space with $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$, ($\epsilon \neq 0$ are constant). then the induced quasi-metric d_F and the symmetrized metric ρ induce the same topology on M. This follows immediately from ([3],[4]).

Remark 5.5 From Lemma 5.3, it can be seen that the assumption of w to be smooth is not essential.

Next, we discuss an interesting geometric property concerning the geodesic triangles.

Proposition 5.6 *Let (M,F) be an arctangent space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ and 1-form $\beta = b_i y^i$. ($\epsilon \neq 0$ are constant). Then the perimeter length of any geodesic triangle on M does not depend on the orientation, that is,*

$$d_F(x,y) + d_F(y,z) + d_F(z,x) = d_F(x,z) + d_F(z,y) + d_F(y,x), \ \forall x, y, z \in M. \tag{5.9}$$

Proof From Theorem 3.1, it follows that the quasi-metric is weightable and therefore (5.7) holds good. By using this formula an elementary computation proves (5.9). □

References

[1] H. P. A. Kunzi, V. Vajner,Weighted quasi-metrics, *Annals New York Acad. Sci.*, 728(1994), 64-77.

[2] P. Vitolo,A representation theorem for quasi-metric space, *Topology Appl.*, 65(1995), 101-104.

[3] Z. Shen, *Differential Geometry of Sprays and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, (2001).

[4] Z. Shen,On projectively flat (α,β)-metrics, *Canad. Math. Bull.*, 52(2009), 132-144.

[5] G. Hamel,*Ü*ber die Geomtrieen in denen die Geraden die Kürzeintensind, *Math. Ann.*, 57(1903).

[6] S. V. Sabau, H. Shimada,Finsler manifolds with reversible geodesics, *Rev. Roumaine Math. Pures Appl.*, 57(2012), 91-103.

[7] M. Crampin,Randers spaces with reversible geodesics, *Publ. Math. Debrecen*, 67/3(2005), 401-409.

[8] I. Masca, S. V. Sabau, H.Shimada, Reversible geodesics for (α, β)- metrics, *Intl. J. Math.*, 21(2010), 1071-1094.

[9] I. Masca, S. V. Sabau, H. Shimada,Two dimensional (α, β)-metrics with reversible geodesic-s, *Publ. Math. Debrecen*, 82(2013), 485-501.

[10] G. Shanker, S. A. Baby,Reversible geodesics of Finsler spaces with a special (α, β)-metric, *Bull. Cal. Math. Soc.*, 109(2017), 183-188.

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Laplacian and Signless Laplacian Degree Sum Distance Energy of Some Graphs

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Abstract: In this paper we define the Laplacian degree sum distance matrix and signless Laplacian degree sum distance matrix from the well known degree sum distance matrix and define Laplacian degree sum distance energy and signless Laplacian degree sum distance energy. We evaluate the Laplacian degree sum distance energy and signless Laplacian degree sum distance energy of some graphs. We also obtain some bounds for Laplacian degree sum distance energy.

Key Words: Laplacian degree sum distance energy, average degree, signless Laplacian degree sum distance energy.

AMS(2010): 05C50.

§1. Introduction

All the graphs considered here are finite simple, connected and undirected. Let G be such a graph of order n.The degree of a vertex v_i, $d(v_i)$ is the number of edges incident on it and we denote d_{ij} as the distance between vertex v_i and v_j.

Several results on Laplacian energy of graph G are reported in the literature [2, 3, 4, 5, 6]. The signless Laplacian energy is also studied in the literature rigorously [7, 8, 9]. We have discussed degree sum distance matrix in [1], as DSD(G).

We now define the Laplacian and the signless Laplacian degree sum distance matrix of a connected graph G as

$$L_{DSD}(G) = [ldsd_{ij}]$$

where,

$$\begin{aligned} ldsd_{ij} &= -d_{ij}(d(v_i) + d(v_j)) \text{ if } i \neq j \\ &= 2d_i \text{ if } i = j \end{aligned}$$

and $Q_{DSD}(G) = [qdsd_{ij}]$ where,

$$\begin{aligned} qdsd_{ij} &= d_{ij}(d(v_i) + d(v_j)) \text{ if } i \neq j \\ &= 2d_i \text{ if } i = j \end{aligned}$$

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The following are obvious for $L_{DSD}(G)$ and $Q_{DSD}(G)$.

(1) Both $L_{DSD}(G)$ and $Q_{DSD}(G)$ are real symmetric, hence their eigenvalues are real.

(2) If β_i and γ_i, i=$1, 2, 3 \cdots, n$ are eigenvalues of $L_{DSD}(G)$ and $Q_{DSD}(G)$ respectively then they can be arranged in non-increasing order as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ respectively.

Definition 1.1 *Let G be graph of order n and size m.If $2avd(G)$ denotes double average degree of a graph given by, $2\frac{\sum_1^n d_i(G)}{n}$, then analogous to usual Laplacian and signless Laplacian energy we define the Laplacian and signless Laplacian degree sum distance energy as,*

$$LE_{DSD}(G) = \sum_{i=1}^{n} |\beta_i - 2avd(G)| \ \ and \ \ QE_{DSD}(G) = \sum_{i=1}^{n} |\gamma_i - 2avd(G)|.$$

The double average degree is taken to be consistent with the degree sum entries defined in the matrix defined

Example 1.2 For the graph G given in Figure 1,

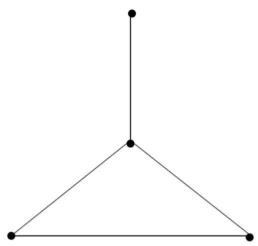

Figure 1

The double average degree of G is $2 \times \frac{4}{2} = 4$.

$L_{DSD}(G) = \begin{bmatrix} 2 & -4 & -6 & -6 \\ -4 & 6 & -5 & -5 \\ -6 & -5 & 4 & -4 \\ -6 & -5 & -4 & 4 \end{bmatrix}$	$Q_{DSD}(G) = \begin{bmatrix} 2 & 4 & 6 & 6 \\ 4 & 6 & 5 & 5 \\ 6 & 5 & 4 & 4 \\ 6 & 5 & 4 & 4 \end{bmatrix}$
Laplacian degree sum distance eigenvalues are $\beta_1 = 11.1686$, $\beta_2 = 8.065$, $\beta_3 = 8$,$\beta_4 = -11.2342$	signless Laplacian degree sum distance eigenvalues are $\gamma_1 = 19.0266$, $\gamma_2 = 1.0761$, $\gamma_3 = 0$,$\gamma_4 = -4.1027$
$LE_{DSD}(G) = \|11.1686 - 4\| + \|8.065 - 4\|$ $+\|8 - 4\| + \|11.2342 + 4\| = 30.4678.$	$QE_{DSD}(G) = \|19.0266 - 4\| + \|4 - 1.0761\|.$ $+\|0 + 4\| + \|4 + 4.1027\| = 30.0532.$

Table 1

§2. Bounds on Laplacian Degree Sum Distance Energy

We introduce the auxiliary Laplacian degree sum distance eigenvalue μ_i, defined as,

$$\mu_i = \beta_i - 2\frac{1}{n}\sum_{i=1}^{n} d_i.$$

Lemma 2.1 *Let G be a graph of order n, then we have*

$$\sum_{i=1}^{n} \mu_i = 0 \quad and \quad \sum_{i=1}^{n} \mu_i^2 = 2R,$$

where,

$$R = \frac{1}{2}\left(\frac{4}{n}\sum_{i=1}^{n} d_i^2 + 2\sum_{i=1, i<j}^{n}((d_i + d_j)d_{ij})^2\right).$$

The Laplacian degree sum distance energy of G can be another as

$$LE_{DSD}(G) = \sum_{i=1}^{n} |\mu_i| = \sum_{i=1}^{n}\left|\beta_i - 2\frac{1}{n}\sum_{i=1}^{n} d_i\right|.$$

Proposition 2.2 *Let G be a graph of order $n \geq 2$, then, $2\sqrt{R} \leq LE_{DSD}(G) \leq \sqrt{2nR}$, where R is defined above.*

Proof Consider the equation

$$\begin{aligned} N &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i| + |\mu_j|)^2 = 2n\sum_{i=1}^{n}|\mu_i|^2 - 2\left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right) \\ &= 2n2R - 2(LE_{DSD}(G))^2 = 4nR - 2(LE_{DSD}(G))^2 \end{aligned}$$

from Lemma[2.1].

Note that, $N \geq 0$, i.e, $4nR - 2(LE_{DSD}(G))^2 \geq 0$ which implies, $LE_{DSD}(G) \leq \sqrt{2nR}$. Again from Lemma 2.1 we have $\left(\sum_{i=1}^{n}\mu_i\right)^2 = 0$, the fact that R$\geq$0 and

$$\begin{aligned} \sum_{i=1}^{n}\mu_i^2 &= \left(\sum_{i=1}^{n}\mu_i\right)^2 - 2\sum_{1\leq i<j\leq n}\mu_i\mu_j \\ &\leq 2\sum_{1\leq i<j\leq n}|\mu_i\mu_j| \leq \sum_{1\leq i<j\leq n}|\mu_i|.|\mu_j|, \\ 2R &\leq 2\sum_{1\leq i<j\leq n}|\mu_i|.|\mu_j| \end{aligned}$$

so that

$$\begin{aligned} LE^2_{DSD}(G) &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \\ &= \sum_{i=1}^{n} |\mu_i|^2 + 2\sum_{1\le i<j\le n} |\mu_i|.|\mu_j| \\ &= 2R + 2R = 4R. \end{aligned}$$

Then, $LE_{DSD}(G) \geq 2\sqrt{R}$. □

Lemma 2.3([10]) *Let $a_1, a_2, \cdots, a_n$ be non-negative numbers, then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}\right].$$

Proposition 2.4 *Let G be a graph with n vertices and m edges, then*

$$\sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}} \leq LE_{DSD}(G) \leq \sqrt{2(n-1)R + n\Delta^{\frac{2}{n}}},$$

where,

$$\Delta = \left| det\left(L_{DSD}(G) - \left[\frac{2}{n}\sum_{i=1}^{n} d_i\right] I_n\right)\right|.$$

Proof We assume that $\Delta \neq 0$, by setting $a_i = \mu_i^2$ where $i = 1, 2, \cdots, n$, and

$$P = n\left[\frac{1}{n}\sum_{i=1}^{n} \mu_i^2 - \left(\prod_{i=1}^{n} \mu_i^2\right)^{\frac{1}{n}}\right] \geq 0$$

From Lemma 2.3, we have

$$P \leq n\sum_{i=1}^{n} \mu_i^2 - \left(\prod_{i=1}^{n} |\mu_i|^2\right) \leq (n-1)P,$$

which can be further expressed as $P \leq 2nR - (LE(G))^2 \leq (n-1)P$.

Hence,

$$P = n\left[\frac{1}{n}\sum_{i=1}^{n} \mu_i^2 - \left(\prod_{i=1}^{n} \mu_i^2\right)^{\frac{1}{n}}\right] = n\left[\frac{1}{n}2P - \Delta^{\frac{2}{n}}\right] = 2P - n\Delta^{\frac{2}{n}}$$

By substituting in the above inequality, we obtain desired result. □

Proposition 2.5 *If G is any graph of order n and Δ as defined above, then*

$$\sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}} \leq LE_{DSD}(G) \leq \sqrt{2Rn}.$$

Proof For lower bound consider,

$$[LE_{DSD}(G)]^2 = \sum_{i=1}^{n}(|\mu_i|)^2 = \sum_{i=1}^{n}(\mu_i)^2 + 2\sum_{i<j}|\mu_i||\mu_j|$$

.

Since $AM \geq GM$, we have

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq (\prod_{i\neq j}|\mu_i||\mu_j|)^{\frac{1}{n(n-1)}} \\
&= \prod_{i=1}^{n}(|\mu_i|^{2n-2})^{\frac{1}{n(n-1)}} \\
&= (\prod_{i=1}^{n}|\mu_i|^{\frac{2}{n}}) = \Delta^{\frac{2}{n}}.
\end{aligned}$$

Therefore,

$$\prod_{i\neq j}|\mu_i||\mu_j| \geq n(n-1)\Delta^{\frac{2}{n}}.$$

Hence,

$$[LE_{DSD}(G)]^2 \geq 2R + n(n-1)\Delta^{\frac{2}{n}},$$

i.e.,

$$LE_{DSD}(G) \geq \sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}}. \tag{1}$$

For upper bound we define

$$\begin{aligned}
X &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i| + |\mu_j|)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(|\mu_i|^2 + |\mu_j|^2\right) + 2\left(\sum_{i=1}^{n}|\mu_i||\mu_j|\right) \\
&= n\sum_{i=1}^{n}(\mu_i)^2 + n\sum_{i=1}^{n}(\mu_j)^2 - 2\left(\sum_{i=1}^{n}|\mu_i||\mu_j|\right) \\
&= 2nR + 2nR - 2[LE_{DSD}(G)]^2 = 4nR - 2[LE_{DSD}(G)]^2
\end{aligned}$$

Since $X \geq 0$, we get that

$$LE_{DSD}(G) \leq \sqrt{2Rn}. \tag{2}$$

Combining (1) and (2) we obtain the desired result. □

§3. LE_{DSD} of Some Graphs

Lemma 3.1 *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of any matrix P of order n, then the eigenvalues of the matrix $kI_n \pm P$ are $k \pm \lambda_1, k \pm \lambda_2, \cdots, k \pm \lambda_n$.*

Using Lemma[3.1] one can directly obtain the Laplacian and signless Laplacian degree sum distance eigenvalues from the degree sum distance eigenvalues for a regular graph G.The degree sum distance energy is already discussed by the present authors in [1].

In general the Laplacian degree sum distance energy and signless Laplacian degree sum distance energy are equal for a regular graph G.This is consistent with the equality of Laplacian and signless Laplacian energy for regular graph G. Hence we discuss graphs which are not regular.

Lemma 3.2 *Let a and b be two arbitrary constants, I is the identity matrix and J is $n \times n$ matrix whose all entries are $1's$. If $A = (a-b)I + bJ$ then the characteristic polynomial of A, is, $|\lambda I - A| = [\lambda - a + b]^{n-1}[\lambda - a - (n-1)b]$.*

Theorem 3.3 *The Laplacian degree sum distance energy of the complete bipartite graph $K_{m,n}$ is*

$$\begin{aligned} LE_{DSD}(K_{m,n}) &= \left|\frac{2n(m+3n)(m-1)}{m+n}\right| + \left|\frac{2m(n+3m)(n-1)}{m+n}\right| \\ &+ \left|\beta_1 + \frac{4mn}{m+n}\right| + \left|\frac{4mn}{m+n} - \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are the roots of the equation

$$\beta^2 + 2(4mn - 3n - 3m)\beta + mn(14mn - 24m - 24n - m^2 - n^2 + 36) = 0.$$

Proof In $K_{m,n}$, m vertices have degree n and n vertices have degree m.The diameter being 2 the structure of the degree sum distance matrix is

$$L_{DSD}(K_{m,n}) = \begin{pmatrix} 2nI_m - 4mA(K_m) & -(m+n)J_{m\times n} \\ -(m+n)J_{n\times m} & 2mI_n - 4nA(K_n) \end{pmatrix}$$

where J is matrix of all 1's and A the adjacency matrix.The Laplacian degree sum distance polynomial is then given by

$$|\beta I - L_{DSD}(K_{m,n})| = \begin{vmatrix} (\beta - 2n)I_m + 4mA(K_m) & (m+n)J_{m\times n} \\ (m+n)J_{n\times m} & (\beta - 2m)I_n + 4nA(K_n) \end{vmatrix}$$

Using Lemma 3.2 with

$$a = \beta - 2m - \frac{(m+n)^2(n-1)}{\beta - 4m(n-1)} \quad \text{and} \quad b = -4n - \frac{(m+n)^2(n-1)}{\beta - 4m(n-1)},$$

we get that

$$\begin{aligned}|\beta I - L_{DSD}(K_{m,n})| &= [\beta - 6n]^{m-1}[\beta - 6m]^{n-1}[\beta^2 + 2(4mn - 3n - 3m)\beta \\ &\quad + mn(14mn - 24m - 24n - m^2 - n^2 + 36)].\end{aligned}$$

Using double average degree as, $doubleavd(K_{m,n}) = \dfrac{4mn}{m+n}$, we get desired result. □

Corollary 3.4 *If $m = 1$, we get star graph $K_{1,n}$ whose Laplacian degree sum distance energy is,*

$$LE_{DSD}(K_{1,n}) = \frac{(2n+6)(n-1)}{n+1} + |\frac{4n}{n+1} - \beta_1| + |\frac{4n}{n+1} - \beta_2|,$$

where β_1 and β_2 are roots of the equation,

$$[\beta^2 + 2(n-3)\beta - n(n^2 - 6n + 5)] = 0.$$

Let K_n+e and K_n-e denote the graph obtained by adding or deleting an edge e respectively to complete graph K_n, both are of diameter 2.

Theorem 3.5 *The LE_{DSD} of $K_n + e$ is*

$$\begin{aligned}LE_{DSD}(K_n+e) &= \left|\frac{2(n^2-n-2)}{n+1} - (2(n-1) - (2n-2))(2n-2)\right| + \left|\frac{2(n^2-n-2)}{n+1} + \beta_1\right| \\ &\quad + \left|\frac{2(n^2-n-2)}{n+1} - \beta_2\right| + \left|\frac{2(n^2-n-2)}{n+1} - \beta_3\right|,\end{aligned}$$

where β_1,β_2 and β_3 are roots of the equation,

$$\begin{aligned}&\beta^3 - ((2n+2) - 2(n-1)(n-3))\beta^2 - (4(n^2-1)(n-3) + 4n^2(n-1) + (2n-1)^2(n-1) \\ &+ (n-1)^2)\beta + 8n^3(n-1) + 2(2n-1)^2(n-1) + 4n(n^2-1)(2n-1) - 2(n-1)^3(n-3) = 0.\end{aligned}$$

Proof In $K_n + e$ one vertex has degree n, one vertex has degree 1 remaining having degree $n-1$, then we have

$$L_{DSD}(K_n+e) = \begin{bmatrix} 2n & -(n+1) & -(2n-1) & \dots & -(2n-1) & \dots & -(2n-1) \\ -(n+1) & 2 & -2n & \dots & -2n & \dots & -2n \\ -(2n-1) & -2n & 2(n-1) & \ddots & -(2n-2) & \dots & -(2n-2) \\ -(2n-1) & -2n & -(2n-2) & \dots & 2(n-1) & \dots & -(2n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(2n-1) & -2n & -(2n-2) & \dots & -(2n-2) & \dots & 2(n-1) \end{bmatrix}$$

The Laplacian degree sum distance polynomial is

$$|\beta I - L_{DSD}(K_n+e)| = \begin{bmatrix} \beta-2n & (n+1) & (2n-1) & \ldots & (2n-1) & \ldots & (2n-1) \\ (n+1) & \beta-2 & 2n & \ldots & 2n & \ldots & 2n \\ (2n-1) & 2(n-1) & \beta-2(n-1) & \ddots & (2n-2) & \ldots & (2n-2) \\ (2n-1) & 2n & (2n-2) & \ldots & \beta-2(n-1) & \ldots & (2n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (2n-1) & 2(n-1) & (2n-2) & \ldots & (2n-2) & \ldots & \beta-2(n-1) \end{bmatrix}$$

Hence,

$$\begin{aligned} |\beta I - L_{DSD}(K_n+e)| &= [\beta-2(n-1)-(2n-2)]^{n-2}\left[\beta^3-((2n+2)-2(n-1)(n-3))\beta^2\right. \\ &\quad -(4(n^2-1)(n-3)+4n^2(n-1)+(2n-1)^2(n-1)+(n-1)^2)\beta \\ &\quad +8n^3(n-1)+2(2n-1)^2(n-1)+4n(n^2-1)(2n-1) \\ &\quad \left.-2(n-1)^3(n-3)\right]. \end{aligned}$$

Using double average degree as

$$doubleavd(K_n+e) = \frac{2(n^2-n-2)}{n+1},$$

we get the desired result. $\square$

Theorem 3.6 *The LE_{DSD} of $K_n - e$ is*

$$\begin{aligned} LE_{DSD}(K_n-e) &= \left|\frac{2(n-2)(n+1)}{n} - 4(n-1)\right|(n-3) + \left|\frac{2(n-2)(n+1)}{n} - 6(n-2)\right| \\ &\quad + \left|\frac{2(n-2)(n+1)}{n} - \beta_1\right| + \left|\frac{2(n-2)(n+1)}{n} + \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are roots of the equation

$$[\beta^2 + (2n^2-8n+4)\beta - 2(2n^3-6n^2+5n-2)] = 0.$$

Proof In $K_n - e$ two vertices are of degree $n-2$ and remaining are of degree $n-1$. Proceeding in a way similar to Theorem 3.5, whose Laplacian degree sum distance polynomial of $K_n - e$ is

$$|\beta I - L_{DSD}(K_n-e)| = [\beta-2(2n-2)]^{n-3}[\beta-6(n-2)][\beta^2+(2n^2-8n+4)\beta-2(2n^3-6n^2+5n-2)].$$

Using double average degree as

$$doubleavd(K_n-e) = \frac{2(n-2)(n+1)}{n},$$

we get the desired result. $\square$

Let K_n be a complete graph of order n then the vertex coalescence of K_n with K_n will be denoted by $K_nO_vK_n$ and the edge coalescence by $K_nO_eK_n$. $K_nO_vK_n$ has $2n-1$ vertices and $2\times(nC_2)$ edges whereas $K_nO_eK_n$ has $2n-2$ vertices and $2\times(nC_2-1)$ edges.

Theorem 3.7 *The LE_{DSD} of $K_nO_vK_n$ is*

$$\begin{aligned} LE_{DSD}(K_nO_vK_n) &= \left|\frac{4n(n-1)}{2n-1}-2(n-1)(n+1)\right|+\left|\frac{4n(n-1)}{2n-1}-4(n-1)\right|(2n-4) \\ &\quad+\left|\frac{4n(n-1)}{2n-1}-\beta_1\right|+\left|\beta_2-\frac{4n(n-1)}{2n-1}\right|, \end{aligned}$$

where β_1 and β_2 are the roots of the equation,

$$\beta^2+(6n^2-20n+14)\beta-2(n-1)(21n^2-50n+29)=0.$$

Proof The graph $K_nO_vK_n$ is of diameter 2 has two sets of vertices one at a distance 2 from each other and other at 1.There is one vertex of degree $(2n-2)$ and remaining $(2n-2)$ of degree $(n-1)$. With suitable labeling the L_{DSD} of $K_nO_vK_n$ takes the form

$L_{DSD}(K_nO_vK_n)=$

$$\begin{bmatrix} 2(2n-2) & -(3n-3) & -(3n-3) & \dots & -(3n-3) & -(3n-3) & -(3n-3) & \dots & -(3n-3) \\ -(3n-3) & 2(n-1) & -2(n-1) & \dots & -2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-3) & -2(n-1) & 2(n-1) & \dots & -2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(3n-3) & -2(n-1) & -2(n-1) & \dots & 2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & 2(n-1) & -2(n-1) & \dots & -2(n-1) \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & 2(n-1) & \dots & -2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & -2(n-1) & \dots & 2(n-1) \end{bmatrix}$$

So that the Laplacian degree sum distance polynomial is

$|\beta I-L_{DSD}(K_nO_vK_n)|=$

$$\begin{bmatrix} \beta-2(2n-2) & (3n-3) & (3n-3) & \dots & (3n-3) & (3n-3) & (3n-3) & \dots & (3n-3) \\ (3n-3) & \beta-2(n-1) & 2(n-1) & \dots & 2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ (3n-3) & 2(n-1) & \beta-2(n-1) & \dots & 2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (3n-3) & 2(n-1) & 2(n-1) & \dots & \beta-2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & \beta-2(n-1) & 2(n-1) & \dots & 2(n-1) \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & 2(n-1) & \beta-2(n-1) & \dots & 2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & 2(n-1) & 2(n-1) & \dots & \beta-2(n-1) \end{bmatrix}$$

$$\begin{aligned} |\beta I-L_{DSD}(K_nO_vK_n)| &= [\beta-2(n-1)(n+1)][\beta-4(n-1)]^{2n-4} \\ &\quad\times[\beta^2+(6n^2-20n+14)\beta-2(n-1)(21n^2-50n+29)]. \end{aligned}$$

Using double average degree as

$$doubleavd(K_nO_vK_n) = \frac{4n(n-1)}{2n-1},$$

we get the desired result. □

On similar lines we get the Laplacian degree sum distance energy of $K_nO_eK_n$.

Theorem 3.8 *The LE_{DSD} of $K_nO_eK_n$ is,*

$$\begin{aligned} LE_{DSD}(K_nO_eK_n) &= \left|2\left(\frac{n^2-n-1}{n-1}\right) - 4(n-1)\right|(2n-4) \\ &+ \left|2\left(\frac{n^2-n-1}{n-1}\right) - 4(2n-3)\right| \\ &+ \left|2\left(\frac{n^2-n-1}{n-1}\right) - \beta_1\right| + \left|2\left(\frac{n^2-n-1}{n-1}\right) - \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are roots of the equation, $\beta^2 + 2(n-1)(3n-8)\beta - 4(3n-4)^2(n-2) = 0$.

Proof The graph $K_nO_eK_n$ has two sets of vertices one at a distance 2 from each other and another at distance 1, being of diameter 2. There are two vertices of degree $(2n-3)$ and remaining $(2n-4)$ of degree $(n-1)$. With suitable labeling the L_{DSD} of $K_nO_eK_n$ takes the form

$L_{DSD}(K_nO_eK_n) =$

$$\begin{bmatrix} 2(2n-3) & -2(2n-3) & -(3n-4) & \ldots & -(3n-4) & -(3n-4) & -(3n-4) & \ldots & -(3n-4) \\ -2(2n-3) & 2(2n-3) & -(3n-4) & \ldots & -(3n-4) & -(3n-4) & -(3n-4) & \ldots & -(3n-4) \\ -(3n-4) & -(3n-4) & 2(n-1) & \ldots & -2(n-1) & -4(n-1) & -4(n-1) & \ldots & -4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ -(3n-4) & -(3n-4) & -2(n-1) & \ldots & 2(n-1) & -4(n-1) & -4(n-1) & \ldots & -4(n-1) \\ -(3n-4) & -(3n-4) & -4(n-1) & \ldots & -4(n-1) & 2(n-1) & -4(n-1) & \ldots & -4(n-1) \\ -(3n-4) & -(3n-4) & -4(n-1) & \ldots & -4(n-1) & -2(n-1) & 2(n-1) & \ldots & -2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ -(3n-4) & -(3n-4) & -4(n-1) & \ldots & -4(n-1) & -2(n-1) & -2(n-1) & \ldots & 2(n-1) \end{bmatrix}$$

So that the Laplacian degree sum distance polynomial is

$$|\beta - L_{DSD}(K_nO_eK_n)| = [\beta-4(n-1)]^{2n-6}[\beta-4(2n-3)][\beta^2+2(n-1)(3n-8)\beta-4(3n-4)^2(n-2)]$$

Using double average degree as

$$doubleavd(K_nO_eK_n) = 2\left(\frac{n^2-n-1}{n-1}\right),$$

we get the desired result. □

§4. QE_{DSD} of Some Graphs

We now state without proof results on QE_{DSD} which follow on similar lines like LE_{DSD} proved in previous section.

Theorem 4.1 *The signless Laplacian degree sum distance energy of the complete bipartite graph $K_{m,n}$ is*

$$\begin{aligned} QE_{DSD}(K_{m,n}) &= \left|\frac{2n(m-n)(m-1)}{m+n}\right| + \left|\frac{2m(m-n)(n-1)}{m+n}\right| \\ &+ \left|\frac{4mn}{m+n} - \gamma_1\right| + \left|\frac{4mn}{m+n} - \gamma_2\right|, \end{aligned}$$

where γ_1 and γ_2 are the roots of the equation

$$\gamma^2 + 2(n+m-4mn)\gamma + mn(14mn - m^2 - n^2 - 8m - 8n + 4) = 0.$$

Corollary 4.2 *If $m = 1$ we get star graph $K_{1,n}$ whose signless Laplacian degree sum distance energy is*

$$QE_{DSD}(K_{1,n}) = \left|\frac{4n}{n+1} - 2\right|(n-1) + \left|\gamma_1 - \frac{4n}{n+1}\right| + \left|\gamma_2 - \frac{4n}{n+1}\right|,$$

where γ_1 and γ_1 are roots of the equation, $\gamma^2 - 2(3n-1)\gamma - n(n^2 - 6n + 5) = 0$.

Theorem 4.3 *The QE_{DSD} of $K_n - e$ is*

$$\begin{aligned} QE_{DSD}(K_n - e) &= \left|\frac{2(n-2)(n+1)}{n}\right|(n-3) + \left|\frac{2(n-2)(n+1)}{n} + 2(n-1)\right| \\ &+ \left|\frac{2(n-2)(n+1)}{n} - \gamma_1\right| + \left|\frac{2(n-2)(n+1)}{n} - \gamma_2\right|, \end{aligned}$$

where γ_1 and γ_2 are roots of the equation,$\gamma^2 + (2n^2 - 8)\gamma + (4n^3 - 20n^2 + 30n - 12) = 0$.

Theorem 4.4 *The QE_{DSD} of $K_n + e$ is*

$$\begin{aligned} QE_{DSD}(K_n + e) &= \left|\frac{2(n^2-n+2)}{n+1} + 2(n-1) - (2n-2)\right|(n-2) \\ &+ \left|\frac{2(n^2-n+2)}{n+1} + 2(n-1) + (2n-2)(n-2)\right| \\ &+ \left|\frac{2(n^2-n+2)}{n+1} - \gamma_1\right| + \left|\frac{2(n^2-n+2)}{n+1} - \gamma_2\right| + \left|\frac{2(n^2-n+2)}{n+1} - \gamma_3\right|, \end{aligned}$$

where γ_1, γ_2 *and* γ_3 *are roots of the equation*

$$\gamma^3 - 2(2n + (n-1)(n-2))\gamma^2 + (4(n^2-1)(n-1) - 4n^2(n-1)$$
$$-(2n-1)^2(n-1) - (n+1)^2 + 4n)\gamma + 2((2n-1)^2(n-1) + 4n^3(n-1)$$
$$+(n-1)^4 - 2n(n^2-1)(2n-1)) = 0.$$

Theorem 4.5 *The* QE_{DSD} *of* $K_nO_vK_n$ *is*

$$\begin{aligned} QE_{DSD}(K_nO_vK_n) &= \left|\frac{4n(n-1)}{2n-1}\right|(2n-4) + \left|2(n-1)^2 - \frac{4n(n-1)}{2n-1}\right| \\ &\quad + \left|\frac{4n(n-1)}{2n-1} - \gamma_1\right| + \left|\frac{4n(n-1)}{2n-1} - \gamma_2\right|, \end{aligned}$$

where γ_1 *and* γ_2 *are roots of the equation* $\gamma^2 - 2(n-1)(3n-1)\gamma + 6(n-1)^3 = 0$.

Theorem 4.6 *The* QE_{DSD} *of* $K_nO_eK_n$ *is*

$$\begin{aligned} QE_{DSD}(K_nO_eK_n) &= \left|2\left(\frac{n^2-n-1}{n-1}\right)\right|(2n-5) + \left|2\left(\frac{n^2-n-1}{n-1}\right) - (2n-4)(n-1)\right| \\ &\quad + \left|\gamma_1 - 2\left(\frac{n^2-n-1}{n-1}\right)\right| + \left|\gamma_2 - 2\left(\frac{n^2-n-1}{n-1}\right)\right|, \end{aligned}$$

where γ_1 *and* γ_2 *are the roots of equation* $\gamma^2 - 2n(3n-5)\gamma + 4(n-2)(3n^2-6n+2) = 0$.

§5. Conclusion

We defined the L_{DSD} and Q_{DSD} of a graph, obtained expressions for energy of some graphs.

References

[1] Sudhir.R.Jog and Jeetendra.R.Gurjar, Degree sum distance energy of some graphs, Communicated.

[2] Ivan Gutman, Bo Zhou, Laplacian energy of a group, *Linear Algebra and its Application*, 414 (2006), 29-37.

[3] B. Mohar, The Laplacian spectrum of graphs, in *Graph Theory,Combinatorics and Applications*, Y. Alavi, G. Chartrand, O. E.Ollerman, and A. J. Schwenk, Eds., 871-898, John Wiley and Sons, New York, NY, USA, 1991.

[4] R. Li, Some lower bounds for Laplacian energy of graphs, *International Journal of Contemporary Mathematical Sciences*, Vol.4, No. 5(2009), 219-223.

[5] Bo Zhou, Ivan Gutman and Tatzana Aleksic, A note on Laplacian energy of graphs, *MATCH Commun.Math.Comput. Chem.*, Vol.60 (2008), 441-446.

[6] Bo Zhou, More on energy and Laplacian energy, *MATCH Commun.Math.Comput. Chem.*, Vol.64 (2010), 75-84.

[7] D.M. Cvetkovic, P. Rowlinson, and S. K. Simi, Signless Laplacians of finite graphs, *Linear Algebra and Its Applications*, Vol. 423, No.1(2007), 155-171.

[8] D.M. Cvetkovic, P. Rowlinson, and S. K. Simi, Eigenvalue bounds for the signless Laplacian, *Publications de lInstitut Mathmatique*, Vol. 81, No. 95(2007), 11-27.

[9] D.M. Cvetkovic, Signless Laplacians and line graphs, *Bulletin,Classe des Sciences Mathmatiques et Naturelles, Sciences Mathmatiques Naturelles/Sciences Mathematiques*, Vol. 131, No. 30 (2005), 85-92.

[10] H. Kober, On the arithmetic and geometric means and on Holders inequality, *Proc. Amer. Math. Soc.*, 9 (1958), 452-459.

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Common Fixed Point of Five Self Maps for a Class of A-Contraction on 2-Metric Space

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Abstract: In this paper we prove some common fixed points for self mappings in a complete 2-metric space using A-contraction and weakly compatible mappings which is a generalization of many results and improve of earlier results in this literature.

Key Words: Common fixed point, self maps, A-contraction, weak compatible mapping, 2-metric space.

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§1. Introduction

Fixed point theory is an important part of mathematics. Moreover, it's well known that the contraction mapping principle, which is introduced by S. Banach in 1922. During the last few decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.Gahler [4] first introduce the concept of 2-metric space. Then many authors like Iseki [6], Rhodes [11], Simoniya [13], etc. investigating the existence of fixed point and common fixed point for various contractive mappings. Naidu and Prasad [10] prove that every convergent sequences in a 2-metric space need not be a Cauchy sequence. Recently, M. Akram [2] defined A-contraction on a metric space and proved some common fixed points theorems. G. Akinbo [1] generalize the result using the concept of weakly compatible mapping. V.Gupta et al. [5] , M.Saha et al. [12] also proved fixed point theorems on A-contraction in the 2-metric space. In this paper we prove some common fixed point theorem for five self mappings by using A-contraction and weak compatibility.

§2. Definitions and Preliminaries

Definition 2.1([4]) *Gahler introduce 2-metric space as:*

Let X be a non-empty set and let $d : X \times X \times X \to [0, \infty)$ be such that

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(1) *To each pair of point* $x, y \in X$ *with* $x \neq y$ *there exists a point* $z \in X$ *such that* $d(x, y, z) \neq 0$*;*

(2) $d(x, y, z) = 0$ *when at least two of the three points are equal;*

(3) *For any* $x, y, z \in X$, $d(x, y, z) = d(x, z, y) = d(y, z, x)$*;*

(4) *For any* x, y, z, w *in* X ,$d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$.

Then d *is called a 2-metric and* (X, d) *is called a 2-metric space.*

Let X denote a complete 2-metric space unless or otherwise stated instead of (X, d).

Definition 2.2 *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence when* $d(x_n, x_m, a) \to 0$ *as* $n, m \to 0$.

Definition 2.3 *A sequence* $\{x_n\}$ *in* X *is said to be converge to an element* x *in* X *when* $d(x_n, x, a) \to 0$ *as* $n \to \infty$.

Definition 2.4 *A 2-metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to a point of* X.

Definition 2.5 *Two self maps* A *and* B *of a 2-metric space* (X, d) *are said to be compatible if* $\lim\limits_{n\to\infty} d(ABx_n, BAx_n, a) = 0$ *for all* $a \in X$ *, where* $\{x_n\}$ *is a sequence in*X *such that if* $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = x$ *for some* x *in* X.

Definition 2.6 *Let* A *and* B *be mappings from a metric space* (X, d) *into it-self.* A *and* B *are said to be weakly compatible if they commute at their coincidence point i.e,* $Ax = Bx$ *for some* x *in* X *implies* $ABx = BAx$.

Definition 2.7 *On the other hand Akram* [2] *defined* A*-contraction as follows:*
Let a non-empty set A *consisting of all functions* $\alpha : R_+^3 \to R_+$ *satisfying*

(1) A *is continuous on the set* R_+^3 *of all triplet of non negative real's (with respect to the Euclidean metric on* R^3*);*

(2) $a \leq kb$ *for some* $k \in [0, 1)$ *whenever* $a \leq \alpha(a, b, b)$*or* $a \leq \alpha(b, a, b)$ *or* $a \leq \alpha(b, b, a)$ *for all* $a, b \in R_+$.

Definition 2.8 *A self map* T *on a 2-metric space* X *is said to be A-contractions if for each* $u \in X$,

$$d(Tx, Ty, u) \leq \alpha\{d(x, y, u), d(x, Tx, u), d(y, Ty, u)\}$$

holds for all $x, y \in X$ *and* $\alpha \in A$.

§3. Main Results

Theorem 3.1 *Let* F, G, S, T *and* h *be five continuous self mappings of a complete 2-metric space* (X, d)*, such that* $T(X) \subset G(X)$ *and* $S(X) \subset F(X)$*. Assume* h *is an injective mapping.*

If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy

$$d(hSx, hTy, u) \leq \alpha\{d(hGx, hFy, u), d(hGx, hSx, u), d(hFy, hTy, u)\} \tag{3.1}$$

where $\alpha \in A$ and for all $x, y, u \in X$. Suppose further that (T, F) and (S, G) are weakly compatible subspace of X, then (S, G, h) and (T, F, h) have a coincidence point in X. Also, F, G, S, T and h have a uniquely common fixed point in X.

Proof Here, F, G, S, T and h be self maps of 2-metric space . Let x_0 be any point in X and as $S(X) \subset FX), T(X) \subset G(X)$ then there exists x_1, x_2 in X such that $Sx_0 = Fx_1$, $Tx_1 = Gx_2$, $Sx_2 = Fx_3 \cdots$. Inductively, we can construct sequences $\{x_n\}$ and$\{y_n\}$ in X such that

$y_n = hSx_n = hFx_{n+1}$ when n is even,

$y_n = hTx_n = hGx_{n+2}$, when n is odd.

Now, we will prove that $\{y_n\}$ is a Cauchy sequence. Assuming $n \in \mathbb{N}$ is even, then

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(hSx_n, hTx_{n+1}, u) \\ &\leq \alpha\{d(hGx_n, hFx_{n+1}, u), d(hGx_n, hSx_n, u), d(hFx_{n+1}, hTx_{n+1}, u)\} \text{ (by (3.1))} \\ &= \alpha\{d(y_{n-1}, y_n, u), d(y_{n-1}, y_n, u), d(y_n, y_{n+1}, u)\} \leq kd(y_{n-1}, y_n, u), \end{aligned} \tag{3.2}$$

where $k \in [0, 1)$ as $\alpha \in A$.

When, $n \in \mathbb{N}$ is odd, then

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(hTx_n, hSx_{n+1,} u) = d(hSx_{n+1}, hTx_n, u)) \\ &\leq \alpha\{d(hGx_{n+1}, hFx_n, u), d(hGx_{n+1}, hSx_{n+1}, u), d(hFx_n, hTx_n, u)\} \text{ (by (3.1))} \\ &= \alpha\{d(y_n, y_{n-1}, u), d(y_n, y_{n+1}, u), d(y_{n-1}, y_n, u)\} \leq kd(y_{n-1}, y_n, u), \end{aligned} \tag{3.3}$$

where $k \in [0, 1)$ (as $\in A$).

Thus, whether n is even or odd, we have

$$d(y_n, y_{n+1}, u) \leq kd(y_{n-1}, y_n, u)$$

for some $k \in [0, 1)$.

Inductively,

$$\begin{aligned} d(y_n, y_{n+1}, u) &\leq kd(y_{n-1}, y_n, u) \leq k^2 d(y_{n-2}, y_{n-1}, u) \\ &\leq \cdots \leq k^n d(y_0, y_1, u)) \end{aligned} \tag{3.4}$$

for some $k \in [0, 1)$.

Now,

$$\begin{aligned} d(y_n, y_{n+2}, u) &\leq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, u) + d(y_{n+1}, y_{n+2}, u) \\ &= d(y_n, y_{n+2}, y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u). \end{aligned} \tag{3.5}$$

When n is even,

$$\begin{aligned} d(y_n, y_{n+2}, y_{n+1}) &= d(y_{n+1}, y_{n+2}, y_n) = d(hTx_{n+1}, hSx_{n+2,}y_n) \\ &= d(hSx_{n+2,}hTx_{n+1,}y_n) \\ &\leq \alpha\{d(hGx_{n+2}, hFx_{n+1}, y_n), d(hGx_{n+2}, hSx_{n+2}, y_n), d(hFx_{n+1}, hTx_{n+1}, y_n)\} \\ &\quad (by\ (3.1)) \\ &= \alpha\{d(y_{n+1}, y_n, y_n), d(y_{n+1}, y_{n+2}, y_n), d(y_n, y_{n+1}, y_n)\} \\ &= \alpha\{0, d(y_{n+1}, y_{n+2}, y_n), 0\} \leq k \cdot 0 = 0. \end{aligned}$$

Similarly, when n is odd we can find $d(y_n, y_{n+2}, y_{n+1}) = 0$.

So, from (3.5) we get

$$d(y_n, y_{n+2}, u) \leq \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u).$$

Similarly proceeding as above we will get

$$\begin{aligned} d(y_n, y_{n+p}, u) &\leq \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u) \\ &= (k^n + k^{n+1} + \cdots + k^{n+p-1}) d(y_{n+r}, y_{n+r+1}, u) \\ &= k^n(1 - k^p)/(1 - k) d(y_{n+r}, y_{n+r+1}, u) \\ &\leq k^n/(1 - k) d(y_{n+r}, y_{n+r+1}, u). \end{aligned}$$

Taking $\lim\limits_{n\to\infty}$ on the above inequality we get, $\lim_{n\to\infty} d(y_n, y_{n+p}, u) \leq 0$ as $k \in [0, 1)$, Which shows that $\{y_n\}$ is a Cauchy sequence in X.

Since, $T(X)$ is complete then $\{y_n\}$ converges to a point z in $T(X)$, and since, $T(X) \subset G(X)$, then there exists a point q in X such that $Gq = z$. and as h is injective so

$$hGq = hz. \tag{3.6}$$

Now, let $hSq \neq hz$ and

$$\lim_{n\to\infty} y_n = z \text{ (as } y_n \text{ converges to } z). \tag{3.7}$$

Now,

$$d(hSq, hz, u) \leq d(hSq, hz, hTy_m) + d(hSq, hTy_m, u) + d(hTy_m, hz, u). \tag{3.8}$$

Now,

$$\begin{aligned} d(hSq, hTy_m, u) &\leq \alpha\{d(hGq, hFy_m, u), d(hGq, hSq, u), d(hFy_m, hTy_m, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hz, hz, u), d(hz, hSq, u), d(hz, hz, u)\} \\ &= \alpha\{0, d(hz, hSq, u), 0\} \leq k \cdot 0 = 0. \end{aligned}$$

Using the above value and taking $\lim_{n\to\infty}$ we get from (3.8)

$$d(hSq, hz, u) \leq d(hSq, hz, hz) + 0 + d(hz, hz, u) = 0 + 0 + 0 = 0.$$

So,

$$hSq = hz. \tag{3.9}$$

Now, from (3.6) and (3.8) we get that

$$hGq = hz = hSq. \tag{3.10}$$

Since $S(X) \subset F(X)$, we know that there exists a point $v \in X$ such that $Fv = z$ or $hFv = hz$, i.e.,

$$hFv = hSq = hz = hGq. \tag{3.11}$$

Now,

$$\begin{aligned} d(hz, hTv, u) &= d(hSq, hTv, u) \\ &\leq \alpha\{d(hGq, hFv, u), d(hGq, hSq, u), d(hFv, hTv, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hz, hz, u), d(hz, hz, u), d(hz, hTv, u)\} \\ &= \alpha\{0, 0, d(hz, hTv, u)\} \leq k \cdot 0 = 0. \end{aligned}$$

So, $hTv = hz$ as u is a arbitrary point in X. Thus,

$$hFv = hTv = hz = hGq = hSq. \tag{3.12}$$

As, (F, T) and (S, G) are weakly compatible pair, then F, T are commute at v and S, G are commute at q. So that,

$$hFz = F(hz) = F(hTv) = T(hFv) = T(hz) = hTz$$

and

$$hSz = S(hz) = S(hGq) = G(hSq) = G(hz) = hGz(\text{ by } (3.12)). \tag{3.13}$$

Now,

$$\begin{aligned} d(hSz, hz, u) &= d(hSz, hTv, u) \leq \alpha\{d(hGz, hFv, u), d(hGz, hSz, u), d(hFv, hTv, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hsz, hz, u), d(hSz, hSz, u), d(hz, hz, u)\} \ (by\ (3.12)\ \&\ (3.13)) \\ &= \alpha\{d(hsz, hz, u), 0, 0\} \leq k \cdot 0 = 0. \end{aligned}$$

So, $hSz = hz$, i.e., $Shz = hz$..

Thus,

$$hz \text{ is a fixed point of } S. \tag{3.14}$$

From (3.13) we get, $hSz = hz = hGz = Ghz$, i.e.,

$$hz \text{ is a fixed point of } G. \tag{3.15}$$

Now,

$$\begin{aligned} d(hz, hTz, u) &= d(hSz, hTz, u) \\ &\leq \alpha\{d(hGz, hFz, u), d(hGz, hSz, u), d(hFz, hTz, u)\} \ (by(3.1)) \\ &= \alpha\{d(hz, hTz, u), d(hz, hz, u), d(hTz, hTv, u)\} \\ &= \alpha\{d(hz, hTz, u), 0, 0\} \leq k \cdots 0 = 0. \end{aligned}$$

So, $hTz = hz$ and $Thz = hz = hFz = Fhz$ (by (3.13)). Thus,

$$hz \text{ is a fixed point of } T \text{ and } F. \tag{3.16}$$

As h is an injective function so, $hz = z$ i.e.,

$$z \text{ is a fixed point of } h. \tag{3.17}$$

From (3.14), (3.15), (3.16) and (3.17), we get that z is a fixed point of S, G, T, F and h.

To prove the uniqueness, let r be another fixed point of S, G, T, F and h such that $r \neq z$. Then,

$$\begin{aligned} d(z, r, u) &= d(hSz, hTr, u) \leq \alpha\{d(hGz, hFr, u), d(hGz, hSz, u), d(hFr, hTr, u)\} \\ &= \alpha\{(z, r, u), (z, z, u), (r, r, u)\} = \alpha\{(z, r, u), 0, 0\} \leq k \cdot 0 = 0. \end{aligned}$$

So $z = r$, i.e.,z is a unique common fixed point of S, G, T, F and h. □

Theorem 3.2([1],[5]) *Let F, G, S and T be continuous self mappings of a complete 2-metric space (X, d), such that $T(X) \subset G(X)$ and $S(X) \subset F(X)$. If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy*

$$d(Sx, Ty, u) \leq \alpha\{d(Gx, Fy, u), d(Gx, Sx, u), d(Fy, Ty, u)\}, \tag{3.18}$$

where$\alpha \in A$ and for all $x, y, u \in X$. Suppose further that (T, F) and (S, G) are weakly compatible subspace of X, then (S, G) and (T, F) have a coincidence point in X. Also, F, G, S and T have a common unique fixed point in X.

Proof If we put $h = I$, the identity mapping in our main results, then the theorem immediately follows. □

Theorem 3.3 *Let* F, G, S, T *and* h *be five continuous self mappings of a complete 2-metric space* (X, d) *and let* $\{S_n\}_{n=1}^{\infty}$ *and* $\{T_n\}_{n=1}^{\infty}$ *be sequence on* S *and* T *such that* $T_n(X) \subset G(X)$ *and* $S_n(X) \subset F(X)$. *Assume* h *is an injective mapping. If* $S(X)$ *or* $T(X)$ *is a complete subspace of* X *and satisfy*

$$d(hS_ix, hT_jy, u) \leq \alpha\{d(hGx, hFy, u), d(hGx, hS_ix, u), d(hFy, hT_jy, u)\}, \tag{3.19}$$

where $\alpha \in A$ *and for all* $x, y, u \in X$. *Suppose further that* (T_n, F) *and* (S_n, G) *are weakly compatible subspace of* X, *then* (S_n, G, h) *and* (T_n, F, h) *have a coincidence point in* X. *Also,* F, G, S_n, T_n *and* h *have a common unique fixed point in* X.

Proof For any arbitrary $x_0 \in X$ and $n = 0, 1, 2, 3 \cdots$ following a similar argument as in Theorem 3.1 we can define a sequence $\{y'_n\}$ in X such that

$$y'_n = hS_nx_n = hFx_{n+1}$$

when n is even and

$$y'_n = hT_nx_n = hGx_{n+2},$$

when n is odd.

Now, for each $i = 1, 3, 5, \cdots$ and $j = 2, 4, 6, \cdots$, we get from (3.19)

$$d(y'_i, y'_{i+1}, u) \leq kd(y'_{i-1}, y'_i, u)$$

and

$$d(y'_j, y'_{j+1}, u) \leq kd(y'_{j-1}, y'_j, u),$$

i.e.,

$$d(y'_n, y'_{n+1}, u) \leq kd(y'_{n-1}, y'_n, u), \ n = 1, 2, \cdots$$

By induction (as in the proof of Theorem 3.1) we have

$$d(y'_n, y'_{n+1}, u) \leq k^n d(y'_0, y'_1, u)$$

for some $k \in [0, 1)$. Consequently sequence $\{y'_n\}$ is Cauchy in X. The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1. □

Conclusion Our main result is a generalization and improve result of the existing results in this literature. We generalize the results of G. Akinbo [1], M.Saha and D. Dey [2], and many others.

References

[1] G.Akinbo et al., A note on A-Contractions and common fixed points, *Acta Universitatis Apulensis*, 23(2010), 91-98.

[2] M. Akram et al., A general class of contractions: A-contractions, *Novi Sad J. Math.*,

38(1)(2008), 25-33.

[3] S. De Sarkar and Kalishankar Tiwary, Some common fixed point theorems for contractive type compatible mappings in 2-metric spaces, *Journal of Mathematics*, II, 2(2009), 181-194.

[4] S.Gahler, 2-metrische raume and ther topogosche struktur, *Math.Nache.*, 26(1963), 115-148.

[5] V.Gupta et.al., Some common fixed point theorems for a class of A-contractions on 2-metric space, *Int. J. pure and appl. Math.*, 78(6)(2012),909-916.

[6] K.Iseki, Fixed point theorems in 2-metric spaces, *Math.Sem.Notes. Kobe. Univ.*, 3(1975), 133-136.

[7] G.Jungck, Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.*, 9(1986), 771-779.

[8] G.Jungck, Common fixed points for commuting and compatible mappings on compacts, *Proc. Amer. Math. Soc.*, 103(1988), 977-985.

[9] G.Jungck and B.E.Rhodes, Fixed point theorems for compatible mappings, *Internat. J. Matha and Math. Sci.*, (3) 16(1993), 417-428.

[10] S.V.R.Naidu and J.Rajendra Prasad, Fixed point in 2-metric spaces, *Ind. J. Pure and Appl. Math.*, (8)17(1986), 974-993.

[11] B.E.Rhoades, Contraction type mapping on 2-metric spaces, *Math.Nacher*, 91(1979), 151-155.

[12] M.Saha, D.Dey, Fixed point theorems for a class of A-contractions on 2-metric space, *Novi sad J. Math.*, 40(1),(2010), 3-8.

[13] N.S.Simoniya , An approach to fixed point theorem in a 2-metric apace, *The Math. Education*, 27(1993), 8-10.

[14] Kalishankar Tiwary, T.Basu, S.Sen, Some common fixed point theorems in complete metric spaces, 21(4)(1995), 451-459.

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On Some Properties of Mixed Super Quasi Einstein Manifolds

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Abstract: In this paper we have studied Ricci-pseudo symmetric mixed super quasi Einstein manifolds and Ricci-semi symmetric mixed super quasi Einstein manifolds. Finally, we get relation mixed super quasi Einstein manifold.

Key Words: Mixed super quasi Einstein manifold, Ricci-pseudo symmetric, Ricci-semi symmetric manifold.

AMS(2010): 53C25, 53D10, 53C44.

§1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2)satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

and is not identically zero,where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \qquad \forall X \in TM, \tag{1.2}$$

U being a unit vector field. In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.

Again, in [15], U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \tag{1.3}$$

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where a, b, c are non-zero scalars and A, B are two 1-forms such that

$$g(X,U) = A(X) \qquad and \qquad g(X,V) = B(X), \tag{1.4}$$

where, U, V being unit vectors which are orthogonal, i.e,

$$g(U,V) = 0. \tag{1.5}$$

This type of manifold are denoted by $G(QE)_n$.

Chaki introduced super quasi Einstein manifold [10], denoted by S$(QE)_n$, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y) \tag{1.6}$$

where a, b, c, d are non-zero scalars of which, A, B are two non zero 1-forms defined as (1.4) and U, V being mutually orthogonal unit vector fields, D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0 \qquad \forall\ X \in TM. \tag{1.7}$$

In such case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, D is called the associated tensor of the manifold. Such an n-dimensional manifold shall be denoted by the symbol $S(QE)_n$.

In the recent papers [2],[4], A.Bhattacharyya and T.De introduced the notion of mixed generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a mixed generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + B(X)A(Y)] \tag{1.8}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \ \text{ and } \ g(X,V) = B(X) \tag{1.9}$$

$$g(U,V) = 0 \tag{1.10}$$

where, A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

We introduced mixed super quasi Einstein manifolds [5],[11],[18]. A non-flat Riemannian manifold (M^n, g),(n≥3) is called mixed super quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &\quad + d[A(X)B(Y) + B(X)A(Y)] + eD(X,Y) \end{aligned} \tag{1.11}$$

where a, b, c, d, e are non-zero scalars, A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \quad and \quad g(X,V) = B(X) \qquad \forall X \in TM, \tag{1.12}$$

and U, V being mutually orthogonal unit vector fields, D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0. \qquad \forall \ X. \tag{1.13}$$

In such case a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, D is called the associated tensor of the manifold.Such an n-dimensional manifold shall be denoted by the symbol $MS(QE)_n$.

§2. Preliminaries

We know in a n-dimensional $(n > 2)$ Riemannian manifold the covariant quasi conformal curvature tensor is defined as ([3],[7],[17])

$$\begin{aligned}\tilde{C}(X,Y,Z,W) &= \acute{a}\acute{R}(X,Y,Z,W) + \acute{b}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad + g(Y,Z)g(QX,W) - g(X,W)g(QY,W)] - \frac{r}{n}[\frac{\acute{a}}{n-1} \\ &\quad + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned} \tag{2.1}$$

where

$$g(C(X,Y)Z,W) = \tilde{C}(X,Y,Z,W). \tag{2.2}$$

The projective curvature tensor is denoted by $\tilde{P}(X,Y,Z,W)$ and in a $V_n(n > 2)$ it is defined as

$$\tilde{P}(X,Y,Z,W) = \acute{R}(X,Y,Z,W) - \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,W)g(X,W)]. \tag{2.3}$$

§3. Ricci-Pseudo Symmetric Mixed Super Quasi Einstein Manifold

An n-dimensional semi-Riemannian manifold (M^n, g) is called Ricci -pseudo symmetric [12] if the tensor $R.S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{3.1}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W) \tag{3.2}$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{3.3}$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n [19].

The condition of Ricci-pseudo symmetricity is equivalent to the relation

$$(R(X,Y).S)(Z,W) = L_sQ(g,S)(Z,W;X,Y), \tag{3.4}$$

which holds on the set

$$U_s = \{x \in M : S \neq \frac{r}{n} g \ at \ x\},$$

where L_s is some function on U_s [19]. If $R.S = 0$ then the manifold is called Ricci-semi symmetric. Every Ricci-semi symmetric manifold is Ricci-pseudo symmetric but the converse is not true [1],[6],[9],[12],[13],[19].

Theorem 3.1 *In a Ricci-pseudo symmetric mixed super quasi Einstein manifold the following relation holds*

$$R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)].$$

Proof Let the manifold be Ricci-pseudo symmetric. Then from (3.1) and (3.4) we get

$$L_sQ(g,S)(Z,W;X,Y) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W). \tag{3.5}$$

Using (3.2) in (3.5), we get

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) &= L_s[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad +g(Y,W)S(X,Z) - g(X,W)S(Y,Z)]. \end{aligned}$$

Since the manifold is a mixed super quasi Einstein manifold, Using the well - known properties of curvature tensor R we obtain

$$\begin{aligned} & b[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)] + c[B(R(X,Y)Z)B(W) + B(R(X,Y)W)B(Z)] \\ & +d[A(R(X,Y)Z)B(W) + B(R(X,Y)Z)A(W) + A(R(X,Y)W)B(Z) + B(R(X,Y)W)A(Z)] \\ & +e[D(R(X,Y)Z,W) + D(R(X,Y)W,Z)] \\ & = L_s\{b[g(Y,Z)A(X)A(W) - g(X,Z)a(Y)A(W) + g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)] \\ & +c[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) + g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)] \\ & +d[g(Y,Z)A(X)B(W) + g(Y,Z)B(X)A(W) - g(X,Z)A(Y)B(W) - g(X,Z)B(Y)A(W) \\ & +g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X) - g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)] \\ & +e[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)]\}. \end{aligned} \tag{3.7}$$

Putting $Z = W = U$ in (3.7) we obtain

$$2d\{R(X,Y,U,V)\} = L_s\{2d[A(Y)B(X) - A(X)B(Y)]\} = 0.$$

Since $d \neq 0$, we get

$$R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)]. \tag{3.8}$$

Hence the theorem follows. $\square$

§4. Ricci-Semi Symmetric Mixed Super Quasi-Einstein Manifold

An n-dimensional manifold (M^n, g) is called semi-symmetric [13] if $R(X;Y)\cdot S = 0, \forall\ X, Y$, where $R(X;Y)$ denotes the curvature operator.

Theorem 4.1 *In a Ricci-semi symmetric mixed super quasi Einstein manifold satisfy the condition*

$$(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0.$$

it Proof We know

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \quad and\ R(X,Y).S = 0$$

which implies

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W = 0.$$

i.e.,

$$\begin{aligned}
&ag(R(X,Y)Z,W) + bA(R(X,Y)Z)A(W) + cB(R(X,Y)Z)B(W)\\
&+d\{A(R(X,Y)Z)B(W) + B(R(X,Y)Z)A(W)\} + eD(R(X,Y)Z,W)\\
&+ag(R(X,Y)W,Z) + bA(R(X,Y)W)A(Z) + cB(R(X,Y)W)B(Z)\\
&+d\{A(R(X,Y)W)B(Z) + B(R(X,Y)W)A(Z)\} + eD(R(X,Y)W,Z) = 0. \qquad (4.1)
\end{aligned}$$

Putting $Z = W = U$ in (4.1) we get

$$(a+b)A(R(X,Y)U) + dB(R(X,Y)U) = 0.$$

$$(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0. \qquad (4.2)$$

Hence the theorem follows. □

§5. Mixed Super Quasi Manifolds Satisfying the Condition $\tilde{C}\cdot S = 0$

Theorem 5.1 *In a mixed super quasi Einstein manifold with the condition $\tilde{C}\cdot S = 0$ satisfy*

$$\begin{aligned}
&(a+b)\acute{a}\acute{R}(X,Y,U,U) + d\acute{a}\acute{R}(X,Y,U,V) + d(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)]\\
&+de_1(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)] = 0.
\end{aligned}$$

Proof From condition $\tilde{C}.S = 0$ we get

$$S(\tilde{C}(X,Y)Z,W) + S(Z,\tilde{C}(X,Y)W = 0$$

for all vector fields X, Y, Z, W on (M^n, g), i.e.,

$$\begin{aligned}
&ag(\tilde{C}(X,Y)Z,W) + bA(\tilde{C}(X,Y)Z)A(W) + cB(\tilde{C}(X,Y)Z)B(W)\\
&+d\{A(\tilde{C}(X,Y)Z)B(W) + B(\tilde{C}(X,Y)Z)A(W)\} + eD(\tilde{C}(X,Y)Z,W)\\
&+ag(\tilde{C}(X,Y)W,Z) + bA(\tilde{C}(X,Y)W)A(Z) + cB(\tilde{C}(X,Y)W)B(Z)\\
&+d\{A(\tilde{C}(X,Y)W)B(Z) + B(\tilde{C}(X,Y)W)A(Z)\} + eD(\tilde{C}(X,Y)W,Z) = 0. \qquad (5.1)
\end{aligned}$$

Putting $Z = W = U$ in (2.7) we obtain

$$(a+b)\acute{C}(X,Y,U,U) + d\acute{C}(X,Y,U,V) = 0. \qquad (5.2)$$

Using (1.11) in (2.1) and putting $Z = W = U$ we get

$$\acute{C}(X,Y,U,U) = \acute{a}\acute{R}(X,Y,U,U). \qquad (5.3)$$

Using (1.11) in (2.1) and putting $Z = U$ and $W = V$ we obtain

$$\begin{aligned}
\acute{C}(X,Y,U,V) \quad = \quad & \acute{a}\acute{R}(X,Y,U,V) + (a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)]\\
& +e_1[A(X)D(Y,V) - A(Y)D(X,V)], \qquad (5.4)
\end{aligned}$$

where

$$\begin{aligned}
a_1 \quad &= \quad -\{\frac{r}{n}[\frac{\acute{a}}{n-1} + 2\acute{b}] - 2a\acute{b}\},\\
b_1 \quad &= \quad b\acute{b}, c_1 = c\acute{b}, d_1 = d\acute{b} \text{ and } e_1 = e\acute{b}.
\end{aligned}$$

Using (5.3) and (5.4) in (5.2) we get

$$\begin{aligned}
(a+b)\acute{a}\acute{R}(X,Y,U,U) \quad + \quad & d\acute{a}\acute{R}(X,Y,U,V) + d(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)]\\
+ \quad & de_1(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)] = 0. \qquad (5.5)
\end{aligned}$$

Corollary 5.1 *In a mixed super quasi Einstein manifold with the condition $\tilde{P}.S = 0$ satisfying the condition $(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0$.*

References

[1] Blair, B. E(1976), Contact manifolds in Riemannian geometry, *Lecture notes on Mathematics* 509, Springer Verlag.

[2] Bhattacharyya A. and De T., On mixed generalized quasi-Einstein manifold, *Diff.Geometry and Dynamical System*, Vol. 2007, 40-46.

[3] Bhattacharyya A. and Debnath D., On some types of quasi Einstein manifolds and generalized quasi Einstein manifolds, —it Ganita, Vol. 57, No. 2, 2006, 185-191.

[4] Bhattacharyya A., De T. and Debnath D., On mixed generalized quasi-Einstein manifold

and some properties, *An. St. Univ. "Al.I.Cuza" Iasi S.I.a Mathematica*(N.S), 53(2007), No.1, 137-148.

[5] Bhattacharyya A., Tarafdar M. and Debnath D., On mixed super quasi-Einstein manifold, *Diff.Geometry and Dynamical System*, Vol.10, 2008, 44-57.

[6] Bhattacharyya A. and Debnath D., Some types of generalized quasi Einstein, pseudo Ricci-symmetric and weakly symmetric manifold, *An. St. Univ. ""Al.I.Cuza" Din Iasi (S.N) Mathematica*, Tomul LV, 2009, f.1145-151.

[7] Chaki M.C and Ghosh M.L., On quasi conformally flat and quasiconformally conservative Riemannian manifolds, *An. St. Univ. "AL.I.CUZA" IASI Tomul XXXVIII S.I.a Mathematica*, f2 (1997), 375-381.

[8] Chaki M.C and Maity R.K., On quasi Einstein manifold, *Publ. Math.Debrecen*, 57(2000), 297-306.

[9] Chen B.Y., *Geometry of Submanifolds,* Marcel Dekker, 1973.

[10] Chaki M.C., On super quasi Einstein manifolds, *Publ.Math.Debrecen*, 64/3-4(2004), 481-488.

[11] Debnath D. and Bhattacharyya A. Some global properties of mixed super quasi-Einstein manifold, —it Diff.Geometry and Dynamical System, Vol.11, 2009, 105-111.

[12] Deszcz R. On pseudo symmetric spaces, *Bull. Soc. Belg. Math.*, Ser. A., 44: 1-34, 1992.

[13] Debnath D and Bhattacharyya A.,Some Ricci-semi symmertic spaces, *Journal of Mathematics*, Vol.II, No.2(2009), 111-118.

[14] De U.C. and Ghosh G.C., On quasi Einstein manifolds, *Periodica Mathematica Hungarica*, Vol. 48(1-2). 2004, 223-231.

[15] De U.C and Ghosh G.C., On generalized quasi Einstein manifolds, *KYUNGPOOK Math. J.*, 44(2004), 607-615.

[16] Debnath D and Bhattacharyya A., On some types of quasi Eiestein, generalized quasi Einstein and super quasi Einstein manifolds, *J.Rajasthan Acad. Phy. Sci.*, Vol. 10, No.1, 2011,33-40.

[17] Debnath D., Quasi conformally flat generalized quasi Einstein space, *Journal of Mathematics*, Vol.4, No.1(2012), 41-46.

[18] Debnath D., Some properties of mixed generalized and mixed super quasi Einstein manifolds, *Journal of Mathematics*, Vol.II, No.2(2009),pp-147-158.

[19] Özgür C., On some classes of super quasi Einstein manifolds, *Chaos, Solitons and Fractals*, 40: 1156-1161, 2009.

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K-Number of Special Family of Graphs

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Abstract: An $L(3, 2, 1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geqslant 3$ if x and y are adjacent, $|f(x)-f(y)| \geqslant 2$ if x and y are at distance 2, and $|f(x)-f(y)| \geqslant 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3, 2, 1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3, 2, 1)$-labeling with k as the maximum label. In this paper, we determine the $L(3, 2, 1)$-labeling number for fan, double fan, wheel, friendship graph in terms of the maximum degree of the graphs.

Key Words: $L(3, 2, 1)$-labeling, channel assignment, wheel, k-number.

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§1. Introduction

In this era of communication technology, the growth of FM radio and televisions are increasing day by day. We have to assign a channel (non-negative integer) to each television station in a set of given stations such that there is no interference between stations and the span of the assigned channels is minimized. The level of interference between any two television stations correlate with their locations. If stations are closer then the interference is more and their broadcast will be disturbed. Thus, the channel assignment problem is a mathematics problem in which we have to assign a channel to each station in a set of given stations such that there is no interference in the broadcast, which is optimal.

Hale introduced a graph model of the channel assignment problem in 1980 [1]. Robert modified this with stations which are "close" and "very close" which correspond to stations at distance two and stations at distance one in graph theoretic terms [2]. The mathematical abstraction of this problem was introduced by Griggs and Yeh as $L(2, 1)$ problem [3]. An $L(2, 1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geqslant 2$ if x and y are adjacent and $|f(x) - f(y)| \geqslant 1$ if x and y are at distance 2, for all x and y in $V(G)$. But, practically, interference among channels may go beyond two levels. Liu and Shao modified the above problem, by considering stations

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at distance 1, 2 and 3 and it was called as $L(3,2,1)$ problem [4].

An $L(3,2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geqslant 3$ if x and y are adjacent, $|f(x)-f(y)| \geqslant 2$ if x and y are at distance 2, and $|f(x)-f(y)| \geqslant 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label.

Here, we consider only a simple, finite, connected, undirected graph without loops or multiple edges. For standard terminology and notation, we follow Bondy and Murty [5] or Murugan [6].

§2. Known Results

Jean Clipperton et al., [7] determined the $L(3,2,1)$-labeling number for paths, cycles, caterpillars, n-array trees, complete graphs and complete bipartite graphs. Ma-Lian Chia et al.,[8] determined the $L(3,2,1)$-labeling number for Cartesian product of paths and cycles, and the power of paths. Also, they presented upper bounds for the $L(3,2,1)$-labeling numbers of general graphs and trees. Shao [9] determined bounds for the $L(3,2,1)$-labeling numbers for Kneser graphs, extremely irregular graphs, Halin graphs. Liu and Shao [4] proved that for a planar graph G, $k(G) \leqslant 15(\Delta^2-\Delta+1)$, where Δ is the maximum degree of G.

§3. Results

Theorem 3.1 *The $L(3,2,1)$-labeling number of a fan F_n, $n \geqslant 5$, is $k(F_n)=2n+1$.*

Proof The proof is divided into two cases following.

Case 1. n is even.

Consider a fan $F_n = P_n + K_1$, where $v_1, v_2, \cdots, v_{\frac{n}{2}}$, $w_1, w_2, \cdots, w_{\frac{n}{2}}$ are the vertices of P_n and u denotes the vertex of K_1.

Define $f : V(F_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(v_i) &= 4i-1 \quad \text{if} \quad i = 1, 2, \cdots, \tfrac{n}{2}, \\ f(w_i) &= 4i+1 \quad \text{if} \quad i = 1, 2, \cdots, \tfrac{n}{2} \ \text{ and} \\ f(u) &= 0. \end{aligned}$$

First, we consider adjacent vertices.

$$
\begin{aligned}
|f(v_i)-f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = 4 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n}{2}-1,\\
|f(v_i)-f(u)| &= 4i-1 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n}{2},\\
|f(w_i)-f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = 4 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n}{2}-1,\\
|f(w_i)-f(u)| &= 4i+1 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n}{2},\\
|f(v_{\frac{n}{2}})-f(w_1)| &= |4\left(\tfrac{n}{2}\right)-1-(4+1)| = |2n-6| \geqslant 3 \text{ (since } n \geqslant 5).
\end{aligned}
$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(F_n) \leqslant 2n+1$. Since the maximum degree of F_n is n, $k(F_n) \geqslant 2n+1$ [8]. Hence $k(F_n) = 2n+1$.

Case 2. n is odd.

Consider a fan $F_n = P_n + K_1$, where $v_1, v_2, \cdots, v_{\frac{n+1}{2}}$, $w_1, w_2, \cdots, w_{\frac{n-1}{2}}$ are the vertices of P_n and u denotes the vertex of K_1.

Define $f : V(F_n) \to \mathbb{N} \cup \{0\}$ such that

$$
\begin{aligned}
f(v_i) &= 4i-1 \quad \text{if} \quad i=1,2,\cdots,\tfrac{n+1}{2},\\
f(w_i) &= 4i+1 \quad \text{if} \quad i=1,2,\cdots,\tfrac{n-1}{2} \quad \text{and}\\
f(u) &= 0.
\end{aligned}
$$

First we consider adjacent vertices.

$$
\begin{aligned}
|f(v_i)-f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = 4 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n+1}{2}-1,\\
|f(v_i)-f(u)| &= 4i-1 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n+1}{2},\\
|f(w_i)-f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = 4 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n-1}{2}-1,\\
|f(w_i)-f(u)| &= 4i+1 \geqslant 3, \quad i=1,2,\cdots,\tfrac{n-1}{2},\\
|f(v_{\frac{n+1}{2}})-f(w_1)| &= |4\left(\tfrac{n+1}{2}\right)-1-(4+1)| = |2n-4| \geqslant 3 \text{ (since } n \geqslant 5).
\end{aligned}
$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(F_n) \leqslant 2n+1$. Since the maximum degree of F_n is n, $k(F_n) \geqslant 2n+1$ [8]. Hence $k(F_n) = 2n+1$. □

Theorem 3.2 *The $L(3,2,1)$-labeling number of a double fan DF_n, $n \geqslant 5$, is $k(DF_n) = 2n+3$.*

Proof The proof is divided into two cases following.

Case 1. n is even.

Consider a double fan $DF_n = P_n + 2K_1$, where $v_1, v_2, \cdots, v_{\frac{n}{2}}$, $w_1, w_2, \cdots, w_{\frac{n}{2}}$ are the vertices of P_n, u and w denote the vertices of $2K_1$.

Define $f : V(DF_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{array}{rclll} f(v_i) & = & 4(i-1) & \text{if} & i = 1, 2, \cdots, \frac{n}{2}, \\ f(w_i) & = & 2 + 4(i-1) & \text{if} & i = 1, 2, \cdots, \frac{n}{2}, \\ f(u) & = & 2n+1 & \text{and} & \\ f(w) & = & 2n+3. & & \end{array}$$

First we consider adjacent vertices. For $i = 1, 2, \cdots, \frac{n}{2} - 1$,

$$|f(v_i) - f(v_{i+1})| = |4(i-1) - \{(4(i+1-1) - 1)\}| = |-4| \geqslant 3.$$

For $i = 1, 2, \ldots, \frac{n}{2}$,

$$|f(u) - f(v_i)| \geqslant |f(u) - f(v_{\frac{n}{2}})| = \left|2n + 1 - \left\{4\left(\frac{n}{2} - 1\right)\right\}\right| = 5 \geqslant 3.$$

For $i = 1, 2, \ldots, \frac{n}{2}$,

$$|f(w) - f(v_i)| \geqslant |f(w) - f(v_{\frac{n}{2}})| = |2n + 3 - 2n + 4| = 7 \geqslant 3.$$

For $i = 1, 2, \ldots, \frac{n}{2} - 1$,

$$|f(w_i) - f(w_{i+1})| = |2 + 4(i-1) - \{2 + 4(i+1-1)\}| = |-4| \geqslant 3.$$

For $i = 1, 2, \ldots, \frac{n}{2}$,

$$|f(u) - f(w_i)| \geqslant |f(u) - f(w_{\frac{n}{2}})| = \left|2n + 1 - \left\{2 + 4\left(\frac{n}{2} - 1\right)\right\}\right| = 3 \geqslant 3.$$

For $i = 1, 2, \ldots, \frac{n}{2}$,

$$|f(w) - f(w_i)| \geqslant |f(w) - f(w_{\frac{n}{2}})| = \left|2n + 3 - \left\{2 + 4\left(\frac{n}{2} - 1\right)\right\}\right| = 5 \geqslant 3.$$

$$|f(v_{\frac{n}{2}}) - f(w_1)| = \left|4\left(\frac{n}{2} - 1\right) - 2\right| = |2n - 6| \geqslant 3 \quad \text{since } n \geqslant 5.$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3, 2, 1)$-labeling and so $k(DF_n) \leqslant 2n + 3$.

Consider the subgraph $K_{1,n}$ such that u is the central vertex and $v_1, v_2, \cdots, v_{\frac{n}{2}}, w_1, w_2, \cdots, w_{\frac{n}{2}}$ are the end vertices of $K_{1,n}$. Since the k-number of $K_{1,n}$ is $2n + 1$ and in any optimal labeling, u receives 0 or $2n + 1$ [8] and so, in DF_n, w cannot assume any integer in $[0, 2n + 1]$ as label. Thus, w has to be labeled at least with $2n + 3$. That is, $k(DF_n) \geqslant 2n + 3$. Hence, $k(DF_n) = 2n + 3$.

Case 2. n is odd.

Consider a double fan $DF_n = P_n + 2K_1$, where $v_1, v_2, \cdots, v_{\frac{n+1}{2}}$, $w_1, w_2, \cdots, w_{\frac{n-1}{2}}$ are the vertices of P_n, u and w denote the vertices of $2K_1$.

Define $f : V(DF_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(v_i) &= 4(i-1) && \text{if} \quad i = 1, 2, \cdots, \tfrac{n+1}{2}, \\ f(w_i) &= 2 + 4(i-1) && \text{if} \quad i = 1, 2, \cdots, \tfrac{n-1}{2}, \\ f(u) &= 2n+1 && \text{and} \\ f(w) &= 2n+3. \end{aligned}$$

First we consider adjacent vertices. For $i = 1, 2, \cdots, \frac{n+1}{2} - 1$,

$$|f(v_i) - f(v_{i+1})| = |4(i-1) - \{4(i+1-1)\}| = |-4| \geqslant 3.$$

For $i = 1, 2, \cdots, \frac{n+1}{2}$,

$$|f(u) - f(v_i)| \geqslant |f(u) - f(v_{\frac{n+1}{2}})| = \left| 2n+1 - \left\{ 4\left(\frac{n+1}{2} - 1\right) \right\} \right| = 3.$$

For $i = 1, 2, \cdots, \frac{n+1}{2}$,

$$|f(w) - f(v_i)| \geqslant |f(w) - f(v_{\frac{n+1}{2}})| = \left| 2n+3 - \left\{ 4\left(\frac{n+1}{2} - 1\right) \right\} \right| = 5 \geqslant 3.$$

For $i = 1, 2, \cdots, \frac{n-1}{2} - 1$,

$$|f(w_i) - f(w_{i+1})| = |2 + 4(i-1) - \{2 + 4(i+1-1)\}| = |-4| \geqslant 3.$$

For $i = 1, 2, \cdots, \frac{n-1}{2}$,

$$|f(u) - f(w_i)| \geqslant |f(u) - f(w_{\frac{n-1}{2}})| = \left| 2n+1 - \left\{ 2 + 4\left(\frac{n-1}{2} - 1\right) \right\} \right| = 5 \geqslant 3.$$

For $i = 1, 2, \cdots, \frac{n-1}{2}$,

$$|f(w) - f(w_i)| \geqslant |f(w) - f(w_{\frac{n-1}{2}})| = \left| 2n+3 - \left\{ 2 + 4\left(\frac{n-1}{2} - 1\right) \right\} \right| = 7 \geqslant 3.$$

$$|f(v_{\frac{n+1}{2}}) - f(w_1)| = \left| 4\left(\frac{n+1}{2} - 1\right) - 2 \right| = |2n-4| \geqslant 3 \quad \text{since } n \geqslant 5.$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(DF_n) \leqslant 2n+3$.

Consider a subgraph $K_{1,n}$ such that u is the central vertex and $v_1, v_2, \cdots, v_{\frac{n+1}{2}}, w_1, w_2, \cdots, w_{\frac{n-1}{2}}$ are the end vertices of $K_{1,n}$. Since the k-number of $K_{1,n}$ is $2n+1$ and in any optimal labeling, u receives 0 or $2n+1$ [8] and so, in DF_n, w cannot assume any integer in $[0, 2n+1]$ as label. Thus, w has to be labeled at least with $2n+3$. That is, $k(DF_n) \geqslant 2n+3$. Hence, $k(DF_n) = 2n+3$. $\square$

Theorem 3.3 *The $L(3,2,1)$-labeling number of a wheel W_n, $n \geqslant 6$, is $k(W_n) = 2n+1$.*

Proof The proof is divided into two cases following.

Case 1. n is even.

Consider a wheel $W_n = C_n + K_1$, where $v_1, v_2, \cdots, v_{\frac{n}{2}}$, $w_1, w_2, \cdots, w_{\frac{n}{2}}$ are the vertices of C_n and u denotes the vertex of K_1.

Define $f : V(W_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(v_i) &= 4i-1 \quad \text{if} \quad i = 1,2,\cdots,\tfrac{n}{2}, \\ f(w_i) &= 4i+1 \quad \text{if} \quad i = 1,2,\cdots,\tfrac{n}{2} \ \text{ and} \\ f(u) &= 0. \end{aligned}$$

First we consider adjacent vertices.

$$\begin{aligned} |f(v_i) - f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = |-4| \geqslant 3, & i &= 1,2,\cdots,\tfrac{n}{2}, \\ |f(w_i) - f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = |-4| \geqslant 3, & i &= 1,2,\cdots,\tfrac{n}{2}, \\ |f(v_{\frac{n}{2}}) - f(w_1)| &= |2n-1-5| = 2n-6 \geqslant 3 & &(\text{since } n \geqslant 6), \\ |f(w_{\frac{n}{2}}) - f(v_1)| &= |2n+1-3| = 2n-2 \geqslant 3 & &(\text{since } n \geqslant 6), \\ |f(u) - f(v_i)| &\geqslant |f(u) - f(v_1)| = 3, & i &= 1,2,\cdots,\tfrac{n}{2}, \\ |f(u) - f(w_i)| &\geqslant |f(u) - f(w_1)| = 5, & i &= 1,2,\cdots,\tfrac{n}{2}. \end{aligned}$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(W_n) \leqslant 2n+1$. Since the maximum degree of W_n is n, $k(W_n) \geqslant 2n+1$. Hence $k(W_n) = 2n+1$.

Case 2. n is odd.

Consider a wheel $W_n = C_n + K_1$, where $v_1, w, v_2, \cdots, v_{\frac{n-1}{2}}$, $w_1, w_2, \cdots, w_{\frac{n-1}{2}}$ are the vertices of C_n and u denotes the vertex of K_1.

Define $f : V(W_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(v_i) &= 4i-1 \quad \text{if} \quad i = 1,2,\cdots,\tfrac{n-1}{2}, \\ f(w_i) &= 4i+1 \quad \text{if} \quad i = 1,2,\cdots,\tfrac{n-1}{2}, \\ f(w) &= 2n+1, \\ f(u) &= 0. \end{aligned}$$

First we consider adjacent vertices.

$$\begin{aligned}
|f(v_i) - f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = 4 \geqslant 3, \\
&\qquad\qquad\qquad\qquad\qquad i = 1, 2, \cdots, \tfrac{n-1}{2} - 1, \\
|f(w_i) - f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = |-4| \geqslant 3, \\
&\qquad\qquad\qquad\qquad\qquad i = 1, 2, \cdots, \tfrac{n-1}{2} - 1, \\
|f(v_{\frac{n-1}{2}}) - f(w_1)| &= |2n-2-1-5| = 2n-8 \geqslant 3, \quad (\text{since } n \geqslant 6), \\
|f(w_{\frac{n-1}{2}}) - f(v_1)| &= |2n-2+1-3| = 2n-4 \geqslant 3, \quad (\text{since } n \geqslant 6) \\
|f(u) - f(v_i)| &\geqslant |f(u) - f(v_1)| = 3, \qquad i = 1, 2, \cdots, \tfrac{n-1}{2}, \\
|f(u) - f(w_i)| &\geqslant |f(u) - f(w_1)| = 5, \qquad i = 1, 2, \cdots, \tfrac{n-1}{2}.
\end{aligned}$$

Also,

$$\begin{aligned}
|f(u) - f(w)| &= 2n+1, \\
|f(w) - f(v_1)| &= 2n+1-3 = 2n-2, \\
|f(w) - f(v_2)| &= 2n+1-7 = 2n-6.
\end{aligned}$$

Also, since the labels of v_is and w_is are increasing by 4, and w lies between v_1 and v_2 with label $2n+1$, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(W_n) \leqslant 2n+1$. Since the maximum degree of W_n is n, $k(W_n) \geqslant 2n+1$ [8]. Hence $k(W_n) = 2n+1$. $\square$

Theorem 3.4 *The $L(3,2,1)$-labeling number of a friendship graph FS_n is $k(FS_n) = 4n+1$.*

Proof The proof is divided into two cases following.

Case 1. n is even.

Consider a friendship graph $FS_n = nK_2 + K_1$, where $v_1, v_2, \cdots, v_n$, $w_1, w_2, \cdots, w_n$ are the vertices of FS_n such that v_i and v_{i+1}, $i = 1, 3, \cdots, n-1$ are adjacent and w_i and w_{i+1}, $i = 1, 3, \cdots, n-1$ are adjacent. Also u denote the vertex of K_1.

Define $f : V(FS_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned}
f(v_i) &= 4i-1 \quad \text{if} \quad i = 1, 2, \cdots, n, \\
f(w_i) &= 4i+1 \quad \text{if} \quad i = 1, 2, \cdots, n, \\
f(u) &= 0.
\end{aligned}$$

First we consider adjacent vertices.

$$\begin{aligned}
|f(v_i) - f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = |-4| \geqslant 3, \quad i = 1,2,\cdots,n-1,\\
|f(w_i) - f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = |-4| \geqslant 3, \quad i = 1,2,\cdots,n-1,\\
|f(u) - f(v_i)| &\geqslant |f(u) - f(v_1)| = 3, \qquad i = 1,2,\cdots,n,\\
|f(u) - f(w_i)| &\geqslant |f(u) - f(w_1)| = 5, \qquad i = 1,2,\cdots,n.
\end{aligned}$$

Also, since the labels of v_is and w_is are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(FS_n) \leqslant 4n+1$. Since the maximum degree of FS_n is $2n$,

$$k(FS_n) \geqslant 2(2n)+1 = 4n+1.$$

Hence $k(FS_n) = 4n+1$.

Case 2. n is odd.

Consider a friendship graph $FS_n = nK_2 + K_1$, where $v_1, v_2, \cdots, v_{n-1}$, $w_1, w_2, \cdots, w_{n+1}$ are the vertices of FS_n such that v_i and v_{i+1}, $i = 1,3,\cdots,n-1$ are adjacent and w_i and w_{i+1}, $i = 1,3,\cdots,n$ are adjacent. Also u denote the vertex of K_1.

Define $f : V(FS_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned}
f(v_i) &= 4i-1 \quad \text{if} \quad i = 1,2,\cdots,n-1,\\
f(w_i) &= 4i+1 \quad \text{if} \quad i = 1,2,\cdots,n-2,\\
f(w_{n-1}) &= 4n-1,\\
f(w_n) &= 4n-3,\\
f(w_{n+1}) &= 4n+1,\\
f(u) &= 0.
\end{aligned}$$

First we consider adjacent vertices.

$$\begin{aligned}
|f(v_i) - f(v_{i+1})| &= |4i-1-(4(i+1)-1)| = |-4| \geqslant 3,\\
&\qquad\qquad i = 1,3,5,\ldots,n-2,\\
|f(w_i) - f(w_{i+1})| &= |4i+1-(4(i+1)+1)| = |-4| \geqslant 3,\\
&\qquad\qquad i = 1,3,5,\ldots,n-4,\\
|f(w_{n-2}) - f(w_{n-1})| &= |4n-7-(4n-1)| = |-6| \geqslant 3,\\
|f(w_n) - f(w_{n+1})| &= |4n-3-(4n+1)| = |-4| \geqslant 3,\\
|f(u) - f(v_i)| &\geqslant |f(u) - f(v_1)| = 3, \quad i = 1,2,\cdots,n-1,\\
|f(u) - f(w_i)| &\geqslant |f(u) - f(w_1)| = 5, \quad i = 1,2,\cdots,n+1.
\end{aligned}$$

Also, since the labels of v_is ($i \leqslant n-1$) and w_is ($i \leqslant n-2$) are increasing by 4, distance 2 and distance 3 conditions are satisfied. Therefore, f is a $L(3,2,1)$-labeling and so $k(FS_n) \leqslant 4n+1$. Since the maximum degree of FS_n is $2n$, $k(FS_n) \geqslant 2(2n)+1 = 4n+1$. Hence

$$k(FS_n) = 4n+1. \qquad \square$$

§4. Conclusion

Understanding the importance of channel assignment problem, researchers have contributed more on $L(2,1)$-labeling. In a pragmatic approach, conditions on distance two and distance one is not sufficient for channel assignment problem and so $L(3,2,1)$-labeling has its own importance and this work will encourage researchers towards this.

References

[1] W.K.Hale, Frequency assignment: theory and application, *Proc.IEEE*, 68(1980), 1497-1514.

[2] F.S.Roberts, T-colorings of graphs: recent results and open problems, *Discrete Math.*, 93(1991), 229-245.

[3] J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance 2, *SIAM J. Discrete Math.*, 5 (1992), 586-595.

[4] Jia-Zhuang Liu and Zhen-dong Shao, The $L(3,2,1)$-labeling problem on graphs, *Mathematica Applicata*, 17(4) (2004), 596-602.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, 2008.

[6] M. Murugan, *Topics in Graph Theory and Algorithms*, Muthali Publishing House, Chennai, Revised Edition 2018.

[7] Jean Clippeton, Jessica Gehrtz, Zsuzsanna Szaniszlo and Desmond Torkornoo, $L(3,2,1)$-labeling of Simple Graphs, *VERUM*, Valparaiso University, 2006.

[8] Ma-Lian Chia, David Kuo, Hong-ya Liao, Cian-Hui Yang and Roger K.Yeh, $L(3,2,1)$-labeling of Graphs, *Taiwanese Journal of Mathematics*, Vol.15, No.6 (2011), 2439-2457.

[9] Zhen-dong Shao, The $L(3,2,1)$-labeling problem on graphs, *J. Qufu Normal University*, 30(3) (2004), 24-28.

[10] M. Murugan, Radio Number for special family of graphs with diameter 2, 3 and 4, *International J.Math. Combin.*, Vol.3(2015), 99-104.

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Computing the Number of Distinct Fuzzy Subgroups for the Nilpotent p-Group of $D_{2^n} \times C_4$

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Abstract: In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^n with a cyclic group of order four, where $n > 3$.

Key Words: Finite p-Groups, nilpotent Group, fuzzy subgroups, dihedral Group, inclusion-exclusion principle, maximal subgroups.

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§1. Introduction

In the fuzzy group theory , the classification of the fuzzy subgroups, most especially the finite p-groups cannot be underestimated. This aspect of pure Mathematics has undergone a dynamic developments over the years. For instance, many researchers have treated cases of finite abelian groups (see [2], [3]). The starting point for this concept all started as presented in [5] and [6] . Since then, the study has been extended to some other important classes of finite abelian and nonabelian groups such as the dihedral , quaternion, semidihedral, and hamiltonian groups.

Although, the natural equivalence relation was introduced in [7], where a method to determine the number and nature of fuzzy subgroups of a finite group G was developed with respect to the natural equivalence. In [1] and [3], a different approach was applied for the classification. In this work , an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are then finally computed with ease.

§2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if

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(*i*) $\lambda(ab) \geq min\{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;

(*ii*) $\lambda(a^{-1}) \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ (see Marius [6]), which implies

Theorem 2.1 *The set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.*

We define the level subset

$$\lambda G_\beta = \{a \in G/\lambda(a) \geq \beta\} \ \text{ for each } \beta \in [0, 1]$$

The fuzzy subgroups of a finite p-group G are thus, characterized, based on these level subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. Theorem 2.1 gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of G.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite p-group G (see [6]). One of them is defined by: $\lambda \sim \gamma$ if and only if $(\lambda(a) > \lambda(b) \Longleftrightarrow v(a) > v(b), \ \forall\, a, b \in G)$. Also, two fuzzy subgroups λ, γ of G and said to be distinct if $\lambda \times v$.

As a result of this development, let G be a finite p-group and suppose that $\lambda : G \longrightarrow [0, 1]$ is a fuzzy subgroup of G. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G \tag{2 − 1}$$

Also, we have that

$$\lambda(a) = \beta_t \Longleftrightarrow t = \max\{r/a \in \lambda G_{\beta_r}\} \Longleftrightarrow a \in \lambda G_{\beta_t} \backslash \lambda G_{\beta_{t-1}},$$

for any $a \in G$ and $t = 1, \cdots, k$, where by convention, set $\lambda G_{\beta_0} = \phi$.

§3. Methodology

In the sequel, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the *inclusion-exclusion principle*, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \tag{3 − 1}$$

In [5], the formula (3-1) was used to obtain the explicit formulas for some positive integers n.

Theorem 3.1([6]) *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is*

(i) $h(\mathbb{Z}_{p^n}) = 2^n$;

(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§4. The number of Fuzzy Subgroups for $\mathbb{Z}_4 \times \mathbb{Z}_4$

Lemma 4.1 *Let G be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$.*

Proof By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that

$$\begin{aligned} \frac{1}{2}h(G) &= 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_4). \end{aligned}$$

Applying Theorem 3.1, $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24. \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48.$ □

Corollary 4.2 *Following Lemma* 4.1*, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) =$ 1536, 4096, 10496 and 26112, respectively.*

Proposition 4.3 *Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \geq 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$.*

Proof G has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$. Hence,

$$\begin{aligned} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}((n+1) - j) \\ &= 2^{n+1}(2(n+1) + \frac{1}{2}(n-2)(n+3)) = 2^n(n^2 + 5n - 2) \end{aligned}$$

for $n \geq 2$. We therefore know that

$$\begin{aligned} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) &= 2^{n-1}((n-1)^2 + 5(n-1) - 2) \\ &= 2^{n-1}(n^2 + 3n - 6) \end{aligned}$$

for $n > 2$. This completes the proof. □

Theorem 4.4([4]) *Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by $h(G) = 2^{2n}(2n+1) - 2^{n+1}$ for $n > 3$.*

§5. The Number of Fuzzy Subgroups for $D_{2^n} \times C_4$

Proposition 5.1 *Suppose that $G = D_{2^n} \times C_4$. Then, the number of distinct fuzzy subgroups of G is given by*

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

for $n \geq 3$.

Proof Calculation shows that

$$\begin{aligned}
\frac{1}{2}h(D_{2^n} \times C_4) &= h(D_{2^n} \times C_2) + 2h(D_{2^{n-1}} \times C_4) - 4h(D_{2^{n-1}} \times C_2) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\
&\quad -2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 8h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + h(\mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^{n-2}}) \\
&= (n-3)2^{2n+2} + 2^{2(n-3)}(1460) + 3(2^n(2n-1) + 2^{n+1}(2n-3) + 2^{n+2}(2n-5) \\
&\quad + \cdots + 7(2^{2(n-2)})) \\
&= (n-3).2^{2n+2} + 2^{2(n-3)}(1460) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j) \\
&= 2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j).
\end{aligned}$$

This completes the proof. □

References

[1] Mashinchi M., Mukaidono M., A classification of fuzzy subgroups, *Ninth Fuzzy System Symposium* (1992), 649-652.

[2] Mashinchi M., Mukaidono M., On fuzzy subgroups classification, *Research Reports of Meiji Univ.*, 9(1993), 31-36.

[3] Murali V., Makamba B. B., On an equivalence of fuzzy subgroups III, *Int. J. Math. Sci.*, 36(2003), 2303- 2313.

[4] Adebisi S.A., Enioluwafe M., An explicit formula for the number of distinct fuzzy subgroups of the cartesian product of the Dihedral group of order 2^n with a cyclic group of order 2, *Universal J.of Mathematics and Mathematical Sciences*, Vol.13, No.1(2020), 1-7.

[5] Tarnauceanu M., The classification of fuzzy subgroups of finite cyclic groups and Delannoy number, *European J.Combin.* 30(2009), 283-289.

[6] Tarnauceanu M., Classifying fuzzy subgroups of finite nonabelian groups, *J. Fuzzy Syst.*, 9(2012), 33 - 43.

[7] Tarnauceanu M., Classifying fuzzy subgroups for a class of finite p-groups, "ALL CUZa" Univ. Iasi,(2011).

[8] Tarnauceanu M., Bentea L., On the number of fuzzy subgroups of finite abelian groups, *Fuzzy Sets and Systems*, 159(2008), 1084-1096.

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On Helix in Minkowski 3-Space and its Retractions

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Abstract: Our aim in the present article is to introduce and study new types of retractions of helix $\mathbb{H}$ in Minkowski 3-space. The isometric and topological folding of $\mathbb{H}$ are achieved. The Frenet equations of the helix before and after retractions and foldings are achieved.

Key Words: Retraction, folding, Frenet equations, helix $\mathbb{H}$ in Minkowski 3-space.

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§1. Introduction

It is not uncommon that, the theory of retraction has always been one of the interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry [1-6].

Minkowski space is originally derived from the relativity in physics. In fact, a time like curve corresponds to the path of an observer moving at less than the speed of the light, a light-like curves correspond to moving at the speed of the light and a space like curves moving faster than light. The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by $g = dx_1^2 + dx_2^2 - dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system in E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters, it can be space like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and light-like if $g(v, v) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can be locally space-like, time like or light-like, if all of its velocity vectors $\alpha'(s)$ are respectively, space-like, time like or light-like respectively. A curve in Lorentzian space L^n is a smooth map $\alpha : I \to L^n$ where I is the open interval in the real line $\mathbb{R}$. The interval I has a coordinate system consisting of the identity map u of I. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. The velocity of α at $t \in I$ is $\alpha' = \left.\frac{d\alpha(u)}{du}\right|_t$. Next, v, w in E_1^3 are said to be orthogonal if $g(v, w) = 0$. Vectors A curve α is said to be regular if $\alpha'(t)$ does not vanish for all in t in I, $\alpha \in L^n$ is space like if its velocity vectors α' are space like for all $t \in I$, similarly for time like and null. If α is a null curve, we can re-parameterize it such that $\langle \alpha'(t), \alpha'(t) \rangle = 0$ and $\alpha'(t) \neq 0$ [3-8].

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§2. **Preliminary Notes**

Let $\alpha(t)$ be a curve in the space-time in parameterized by arc length function s Lopez [9]. Then for the unit speed curve $\alpha(t)$ with non-null frame vectors, we distinguish three cases depending on the causal character of $T(t)$ and its Frenet equations are as follows

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \mu_1\upsilon\kappa & 0 & \mu_2\upsilon\tau \\ 0 & \mu_3\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We write the following subcases.

Case 1. If $\alpha(s)$ is time-like curve in E_1^3, then T is time-like vector and T' is space-like vector. Then $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = \mu_2 = 1, \mu_3 = -1$ and T, B and N are mutually orthogonal vectors satisfying the equations $g(N,N) = g(B,B) = 1,\ g(T,T) = -1$.

Case 2. If $\alpha(s)$ is space-like curve in E_1^3, then T is space-like vector, since $T'(s)$ is orthogonal to the space-like vector $T(s)$, $T'(s)$ is space-like, time-like or light-like. Thus we distinguish three cases according to $T'(s)$.

Subcase 2.1. If the vector $T'(s)$ is space-like, N is space-like vector and B is time-like vector. Then, $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = -1,\ \mu_2 = \mu_3 = 1$, where T, N and B are mutually orthogonal vectors satisfying equations $g(T,T) = g(N,N) = 1$ and $g(B,B) = -1$.

Subcase 2.2 If the vector $T'(s)$ is time-like, N is time-like vector and B is space-like vector. Then, $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = \mu_2 = \mu_3 = 1$, where T, N and B are mutually orthogonal vectors satisfying equations $g(T,T) = g(B,B) = 1$ and $g(N,N) = -1$.

Subcase 2.3 If the vector $T'(t)$ is light-like for all t, $N(t) = T'(t)$ is light-like vector and $B(s)$ is unique light-like vector with $g(N,B) = -1$ and it is orthogonal to T. The Frenet equations have

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \upsilon\tau & 0 \\ 1 & 0 & -\upsilon\tau \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Case 3. If $\alpha(s)$ is light-like curve in E_1^3 and $B(s)$ is unique light-like vector such that $g(T,B) = -1$ and it is orthogonal to N, the pseudo-torsion is $\tau = -\langle N', B\rangle$. Then the Frenet equations

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \upsilon\tau & 0 & 1 \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let $\alpha = \alpha(t)$ be an arbitrary space-like curve in Minkowski space E_1^3. We have $\acute{\alpha} = \upsilon T$, $T = \frac{\acute{\alpha}}{\upsilon}$. From Frenet equations we have $\acute{T} = \upsilon\kappa N$. Then $\kappa = \frac{\|\acute{T}\|}{\upsilon}$ and $N = \frac{\acute{T}}{\upsilon\kappa} = \frac{\acute{T}}{\|\acute{T}\|}$. The vector product of $N \times T$, gives us $B = N \times T$. Using Frenet equations $\acute{N} = \upsilon\kappa T - \upsilon\tau B$. Thus

$\upsilon = \|\acute{\alpha}(t)\|$ to have the second curvature, and by inner product we obtain $\tau = g\left(-\frac{\acute{N}-\upsilon\kappa T}{\upsilon}, B\right)$.

A subset A of a topological space X is called retract of X if there exists a continuous map $r : X \to A$ called a retraction such that $r(a) = a$ for any $a \in A$ [2, 3, 6].

Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to N$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ, if f does not preserve the length it is called topological folding [4, 5, 6].

Definition 2.1([9-13]) *We name that a helix is a curve where the tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio τ/κ is constant along the curve, where τ and κ denote the curvature and the torsion, respectively.*

Definition 2.2 *Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors in E_1^3, the vector product in Minkowski space-time E_1^3 is defined by the determinant*

$$u \wedge v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where, e_1, e_2 and e_3 are mutually orthogonal vectors (coordinate direction vectors), υ denotes speed of the curve.

§3. Main Results

Lemma 3.1 *Let $H = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Since $\left\langle \dot{H}, \dot{H} \right\rangle = a^2 - b^2$,*

$$\upsilon = |\dot{H}(t)| = \sqrt{|a^2 - b^2|}, \quad T = \left(\frac{-a\sin t}{\sqrt{|a^2-b^2|}}, \frac{a\cos t}{\sqrt{|a^2-b^2|}}, \frac{b}{\sqrt{|a^2-b^2|}} \right),$$

$$|a^2 - b^2| = \begin{cases} a^2 - b^2 & if\ a^2 > b^2, \\ b^2 - a^2 & if\ a^2 < b^2 \end{cases},$$

and $\langle T, T\rangle = 1$ if $a^2 > b^2$, then the helix H is space-like, and $\langle T, T\rangle = -1$ if $a^2 < b^2$, then the helix H is time-like and a null curve if $a^2 = b^2$.

Proof Consider three cases following.

Case 1. Let $H = \{(a\cos t, a\sin t, bt)\}$ be helix in E_1^3. If $a^2 > b^2$, then $H(t)$ is a space like curve. Since $T = \frac{\dot{H}}{\upsilon}$ then

$$T = \left(\frac{-a\sin t}{\sqrt{|a^2-b^2|}}, \frac{a\cos t}{\sqrt{|a^2-b^2|}}, \frac{b}{\sqrt{|a^2-b^2|}} \right),$$

and so $\langle T, T\rangle = 1$, $B = T \times N$,

$$B = \left\{ \frac{1}{\sqrt{a^2 - b^2}} (b \sin t, -b \cos t, -a) \right\}, \quad N = \{(-\cos t, -\sin t, 0)\}$$

with curvature $\kappa = \frac{a}{a^2 - b^2}$ and torsion $\tau = \frac{-b}{a^2 - b^2}$. Also,

$$\left\langle \dot{T}, \dot{T} \right\rangle = \frac{a^2}{a^2 - b^2} > 0, \quad \left| \dot{T} \right| = \frac{a}{\sqrt{a^2 - b^2}}$$

and $\dot{T}$ is a space-like vector and the Frenet equations in matrix notions are

$$\begin{aligned}
\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a}{\sqrt{a^2 - b^2}} & 0 \\ \frac{-a}{\sqrt{a^2 - b^2}} & 0 & \frac{-b}{\sqrt{a^2 - b^2}} \\ 0 & \frac{-b}{\sqrt{a^2 - b^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a \sin t}{\sqrt{a^2 - b^2}}, \frac{a \cos t}{\sqrt{a^2 - b^2}}, \frac{b}{\sqrt{a^2 - b^2}} \right) \\ (-\cos t, -\sin t, 0) \\ \frac{1}{\sqrt{a^2 - b^2}} (b \sin t, -b \cos t, -a) \end{pmatrix}.
\end{aligned}$$

Case 2. If $a^2 < b^2$, then the helix $H(t)$ is a time-like curve with $\langle T, T\rangle = -1$. Since

$$T = \left(\frac{-a \sin t}{\sqrt{|a^2 - b^2|}}, \frac{a \cos t}{\sqrt{|a^2 - b^2|}}, \frac{b}{\sqrt{|a^2 - b^2|}} \right),$$

and so $N = \{(-\cos t, -\sin t, 0)\}$, $B = T \times N$, then $B = \frac{1}{\sqrt{a^2 - b^2}} (b \sin t, -b \cos t, -a)$ with curvature $\kappa = \frac{a}{b^2 - a^2}$ and torsion $\tau = \frac{b}{b^2 - a^2}$. Also, $\left\langle \dot{T}, \dot{T} \right\rangle = \frac{a^2}{b^2 - a^2} > 0$, and $\dot{T}$ is a space-like vector and the Frenet equations in matrix notions are

$$\begin{aligned}
\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a}{\sqrt{b^2 - a^2}} & 0 \\ \frac{a}{\sqrt{b^2 - a^2}} & 0 & \frac{b}{\sqrt{b^2 - a^2}} \\ 0 & \frac{-b}{\sqrt{b^2 - a^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a \sin t}{\sqrt{b^2 - a^2}}, \frac{a \cos t}{\sqrt{b^2 - a^2}}, \frac{b}{\sqrt{b^2 - a^2}} \right) \\ (-\cos t, -\sin t, 0) \\ \frac{1}{\sqrt{b^2 - a^2}} (b \sin t, -b \cos t, -a) \end{pmatrix}.
\end{aligned}$$

Case 3. The curve is light-like if $a^2 = b^2$ and we have $\upsilon = \sqrt{a^2 - b^2} = 0$, then the tangent vector T undefined, and so to calculate the Frenet equation of $H(t)$ re-parameterization by the pseudo arc length s. Let $a = b = \frac{1}{c^2}$. Thus, the equation of the light-like helix is $H(s) = \left\{ \frac{1}{c^2} (\cos(cs), \sin(cs), cs) \right\}$, where the curvature $\kappa = 1$, the tangent vector $T(s) = \left\{ \frac{1}{c} (-\sin(cs), \cos(cs), 1) \right\}$, the normal vector $N(s) = T'(s) = \{(-\cos(cs), -\sin(cs), 0)\}$. The bi-normal vector $B(s)$ is define as follows, since $B(s)$ is a unique light-like vector then (1)

$\langle B,B\rangle = 0$. Also, (2) $g(T,B) = -1$, and since B is orthogonal to N then (3) $\langle B,N\rangle = 0$. From (1), (2) and (3), then $B(s) = \frac{c}{2}(\sin(cs), -\cos(cs), 1)$, $\tau = -\langle N', B\rangle = \frac{c^2}{2}$. Then, the Frenet equations of $H(s)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \tau & 0 & 1 \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{-c^2}{2} & 0 & 1 \\ 0 & \frac{-c^2}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{c}(-\sin(cs), \cos(cs), 1) \\ (-\cos(cs), -\sin(cs), 0) \\ \frac{c}{2}(\sin(cs), -\cos(cs), 1) \end{pmatrix}.$$

This completes the proof. □

Definition 3.2 *Assume that* $H(t) = \{(x_1(t), x_2(t), x_3(t), \cdots, x_n(t))\}$ $t \in \mathrm{domain}H(t)$ *is any non-null curve in Minkowski n-space* E_1^n. *Then the retraction map of* $H(t)$ *called projection retraction with* $n-1$ *dimension defined as* $r_i(H) : \{(x_1(t), x_2(t), x_3(t), \cdots, x_n(t))\} \to (x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) - \{(x_i)\}$ *with* $\{(x_i)\} = \{(l_1x_1(t), l_2x_2(t), \cdots, l_nx_n(t))\}$, *where* $l_n = 1$ *when* $n = i$ *and* $l_n = 0$ *when* $n \neq i$, $i, n \in \mathbb{N}$, $t \in \mathrm{Domain}H$.

Theorem 3.3 *Let* $P(t)$ *be a non-pseudo null space-like curve in* E_1^3 *with curvature* $\kappa = 1$ *and constant torsion* τ, *the projection retraction* $r_i(P) = (x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) - \{(x_i)\}$, $i \in \{1,2,3\}$, $s \in \mathrm{Domain}r_i(P)$, *then the Frenet apparatus of* $r_i(P)$ *can be formed by Frenet apparatus of* $P(t)$.

Proof Let $P(t) = \{(a\cos t, a\sin t, bt)\}$ be a non-pseudo null space-like curve P in E_1^3 with constant curvature. For the curve $r_1(P(t)) = (0, a\sin t, bt)$, $\upsilon_r = \sqrt{a^2\cos^2(t) - b^2}$. So

$$r_1(P(t)) = (a\cos t, a\sin t, bt).$$

Differentiating this equation with respect to t, the tangent vector

$$T_r(t) = \frac{\upsilon}{\upsilon_r}T(t) - \left(\frac{-a\sin t}{\upsilon_r}, 0, 0\right),$$

where, $T = \frac{\dot{r}}{\upsilon}, T_r = \frac{\dot{r}_1}{\upsilon_r}$, $\dot{T}_r = \dot{T}(t) - (-a\cos t, 0, 0)$. Then, $\kappa = \frac{\|\dot{T}\|}{\upsilon}$, $N = \frac{\dot{T}}{\upsilon\kappa} = \frac{\dot{T}}{\|\dot{T}\|}$, the normal vector of $r_1(P)$,

$$N_r(t) = \left(\frac{\upsilon}{\upsilon_r}\right)^2 \frac{\kappa}{\kappa_r}N(t) - \frac{1}{\kappa_r\upsilon_r^2}(-a\cos t, 0, 0) - \dot{\upsilon}_rT(t).$$

And the bi-normal vector of $r_1(P)$ is

$$B_r(s) = \frac{\tau}{\kappa}T_r(s) - \frac{1}{\kappa}N_r'(s) = \frac{\tau}{\kappa}P'(s) - \frac{1}{\kappa}P'''(s).$$

If $r_1(P)$ is a null curve and $B_r(t) = T_r \wedge N_r$, if $r_1(P)$ is a non-null curve. Similarly, we have the same proof for $r_2(P(t))$ and $r_3(P(t))$. $\square$

Theorem 3.4 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then the Frenet equations of the y-z retraction projection $r_1(H) = \{(0, a\sin t, bt)\}$ and the x-z retraction projection $r_2(H) = \{(a\cos t, 0, bt)\}$ can be formed by the Frenet equations of $H(t)$.*

Proof Let $H = \{(a\cos t, a\sin t, bt)\}$ and $r_1(H) = \{(0, a\sin t, bt)\}$ be y-z retraction projection of $H(t)$ with constant curvature $\kappa \neq 0$ and $\tau = 0$ then the ratio $\tau/\kappa = 0$ for any projection plan of $r_i(H)$. Hence, $r_1(H)$ is not a helix, $\dot{r}_1(H) = (0, a\cos t, b)$ and $\langle \dot{r}_1, \dot{r}_1\rangle = a^2\cos^2(t) - b^2$ for all $t \in$ domain$r_1(H)$, and we have three cases following.

Case 1. If $a^2\cos^2(t) - b^2 > 0$, the retraction $r_1(H)$ is a space-like curve with time like vector N_r,

$$\upsilon_r = \sqrt{a^2\cos^2 t - b^2}, \quad \kappa_r = \frac{ab\sin t}{(a^2\cos^2(t) - b^2)^{\frac{3}{2}}}.$$

From Lemma 3.1 and Theorem 3.3 we get

$$\begin{aligned} T_r &= \left(0, \frac{a\cos t}{\sqrt{a^2\cos^2(t) - b^2}}, \frac{b}{\sqrt{a^2\cos^2(t) - b^2}}\right), \\ N_r &= \left(0, \frac{b}{\sqrt{a^2\cos^2(t) - b^2}}, \frac{a\cos t}{\sqrt{a^2\cos^2(t) - b^2}}\right) \end{aligned}$$

and the torsion is $\tau_r = 0$, $\kappa_r > 0$, $B_r = (1, 0, 0)$. Then, the Frenet equations of the retraction $r_1(H)$ in matrix notation are

$$\begin{aligned} \begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa_r & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{ab\sin t}{a^2\cos^2(t)-b^2} & 0 \\ \frac{ab\sin t}{a^2\cos^2(t)-b^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(0, \frac{a\cos t}{\sqrt{a^2\cos^2(t)-b^2}}, \frac{b}{\sqrt{a^2\cos^2(t)-b^2}}\right) \\ (0, \frac{b}{\sqrt{a^2\cos^2(t)-b^2}}, \frac{a\cos t}{\sqrt{a^2\cos^2(t)-b^2}}) \\ (1,0,0) \end{pmatrix}. \end{aligned}$$

If $a^2\cos^2(t) - b^2 < 0$, the retraction $r_1(H)$ is space-like curve with time like vector N_r, $\upsilon_r = \sqrt{b^2 - a^2\cos^2 t}$, $\kappa_r = \frac{ab\sin t}{(b^2 - a^2\cos^2(t))^{\frac{3}{2}}}$. From Lemma 3.1 and Theorem 3.3 we get

$$\begin{aligned} T_r &= \left(0, \frac{a\cos t}{\sqrt{b^2 - a^2\cos^2(t)}}, \frac{b}{\sqrt{b^2 - a^2\cos^2(t)}}\right), \\ N_r &= \left(0, \frac{-b}{\sqrt{b^2 - a^2\cos^2(t)}}, \frac{-a\cos t}{\sqrt{b^2 - a^2\cos^2(t)}}\right) \end{aligned}$$

and the torsion is $\tau_r = 0,\ \kappa_r > 0,\ B_r = (1,0,0)$. Then, the Frenet equations of the retraction $r_1(H)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa_r & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{ab\sin t}{b^2-a^2\cos^2(t)} & 0 \\ \frac{ab\sin t}{b^2-a^2\cos^2(t)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(0, \frac{a\cos t}{\sqrt{b^2-a^2\cos^2(t)}}, \frac{b}{\sqrt{b^2-a^2\cos^2(t)}}\right) \\ (0, \frac{-b}{\sqrt{b^2-a^2\cos^2(t)}}, \frac{-a\cos t}{\sqrt{b^2-a^2\cos^2(t)}}) \\ (1,0,0) \end{pmatrix}.$$

If $a^2\cos^2(t) - b^2 = 0$, then the retraction $r_1(H) = \{(0, a\sin t, bt)\}$ is a light like curve and the Frenet equations of the retraction $r_1(H)$ cannot appoints.

Case 2. The helix $H = \{(a\cos t, a\sin t, bt)\}$ and $r_2(H) = \{(a\cos t, 0, bt)\}$ is x-z retraction projection of the curve $H(t)$, and $\langle \dot{r}, \dot{r}\rangle = a^2\sin^2(t) - b^2$ for all $t \in$ domain$r_2(H)$. WE have three cases should be discussed.

If $a^2\sin^2(t) - b^2 > 0$, then the retraction $r_2(H)$ is space-like, $\upsilon = |\dot{r}| = \sqrt{a^2\sin^2(t) - b^2}$,

$$T = \frac{\dot{r}_2}{\upsilon} = \left(\frac{-a\sin t}{\sqrt{a^2\sin^2(t) - b^2}}, 0, \frac{b}{\sqrt{a^2\sin^2(t) - b^2}}\right)$$

and

$$\left\langle \dot{T}, \dot{T} \right\rangle = \frac{-a^2b^2\cos^2(t)}{\left(a^2\sin^2(t) - b^2\right)^2} < 0, \quad \left|\dot{T}\right| = \frac{ab\cos t}{a^2\sin^2(t) - b^2},$$

then $\dot{T}$ is time-like, $\langle T, T\rangle = 1$. So, T is space-like with curvature

$$\kappa = \frac{ab\cos t}{\left(a^2\sin^2(t) - b^2\right)^{\frac{3}{2}}}, \quad N = \left(\frac{b}{\sqrt{a^2\sin^2(t) - b^2}}, 0, \frac{-a\sin t}{\sqrt{a^2\sin^2(t) - b^2}}\right)$$

and $B = (0,-1,0)$, $\langle B, B\rangle = 1 > 0$, which is positively oriented, and the torsion $\tau = 0$. Then, the Frenet equations of $r_2(H)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{ab\cos t}{a^2\sin^2(t)-b^2} & 0 \\ \frac{ab\cos t}{a^2\sin^2(t)-b^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{a\sin t}{\sqrt{a^2\sin^2(t)-b^2}}, 0, \frac{b}{\sqrt{a^2\sin^2(t)-b^2}}\right) \\ (\frac{b}{\sqrt{a^2\sin^2(t)-b^2}}, 0, \frac{-a\sin t}{\sqrt{a^2\sin^2(t)-b^2}}) \\ (0,-1,0) \end{pmatrix}.$$

If $a^2\sin^2(t)-b^2<0$, then the retraction $r_2(H)$ is a time-like curve. Since $T=\frac{\dot r_2}{\upsilon}$, so

$$T=\left(\frac{-a\sin t}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{b}{\sqrt{b^2-a^2\sin^2(t)}}\right),\langle T,T\rangle=-1,\upsilon=|\dot r_2|=\sqrt{b^2-a^2\sin^2(t)}$$

and

$$\left\langle \dot T,\dot T\right\rangle=\frac{a^2b^2\cos^2(t)}{\left(b^2-a^2\sin^2(t)\right)^2}>0,\quad \left|\dot T\right|=\frac{ab\cos t}{b^2-a^2\sin^2(t)}.$$

Hence, $\dot T$ is space-like, $\langle T,T\rangle=1$, T is time-like with curvature

$$\kappa=\frac{ab\cos t}{\left(b^2-a^2\sin^2(t)\right)^{\frac{3}{2}}},\quad N=\left(\frac{-b}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{a\sin t}{\sqrt{b^2-a^2\sin^2(t)}}\right)$$

and $B=(0,-1,0)$, $\langle B,B\rangle=1>0$, which is positively oriented, so the torsion $\tau=0$. Then, the Frenet equations of $r_2(H)$ in matrix notation are

$$\begin{pmatrix}\dot T\\ \dot N\\ \dot B\end{pmatrix}=\begin{pmatrix}0&\upsilon\kappa_r&0\\ \upsilon\kappa&0&\upsilon\tau\\ 0&-\upsilon\tau&0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}$$

$$=\begin{pmatrix}0&\frac{ab\cos t}{b^2-a^2\sin^2(t)}&0\\ \frac{ab\cos t}{b^2-a^2\sin^2(t)}&0&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{b}{\sqrt{b^2-a^2\sin^2(t)}}\right)\\ (\frac{-b}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{a\sin t}{\sqrt{b^2-a^2\sin^2(t)}})\\ (0,-1,0)\end{pmatrix}.$$

If $a^2\sin^2(t)-b^2=0$, then the retraction $r_2(H)$ is a light-like curve and the tangent vector T is undefined, and then the Frenet equations cannot be appointed, see Figure 1 for details.

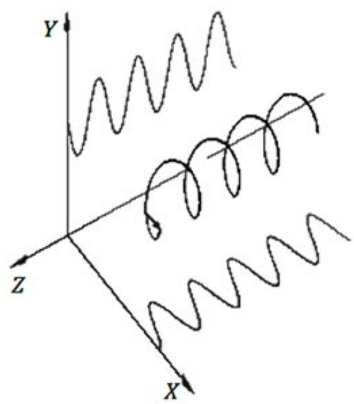

Figure 1

This completes the proof. □

Theorem 3.5 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then the Frenet equations are variants under the x-y retraction projection $r_3(H) = \{(a\cos t, a\sin t, 0)\}$,*

Proof Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ and let $r_3(H) = \{(a\cos t, a\sin t, 0)\}$ be x-y retraction projection of the curve $H(t)$. Clearly, $\langle \dot r_3, \dot r_3\rangle = a^2 > 0$ for all $t \in$ domain$r_3(H)$, and $|\dot r_3| = a$,

$$T = \frac{\dot r_3}{\upsilon} = (-\sin t, \cos t, 0),$$

where $\upsilon = a$, $\langle T, T\rangle = 1$. Then, $r_3(H)$ is a space-like curve. And $\left\langle \dot T, \dot T\right\rangle = 1$, i.e., $|\dot T| = 1$. Thus $\dot T$ is space-like with curvature $\kappa = \frac{\|\dot T\|}{\upsilon} = \frac{1}{a} > 0$ and $\tau = 0$, $N = (-\cos t, -\sin t, 0)$, $B = (0, 0, -1)$. Then, the Frenet equations in matrix notation are

$$\begin{aligned}\begin{pmatrix}\dot T\\ \dot N\\ \dot B\end{pmatrix} &= \begin{pmatrix}0 & \upsilon\kappa_r & 0\\ -\upsilon\kappa & 0 & \upsilon\tau\\ 0 & \upsilon\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &= \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}(-\sin t, \cos t, 0)\\ (-\cos t, -\sin t, 0)\\ (0, 0, -1)\end{pmatrix}.\end{aligned}$$

This completes the proof. □

Corollary 3.6 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then, the Frenet equations can not be appointed under the retraction projection $r_{1,2}(H) = \{((l_1)a\cos t, (l_2)a\sin t, (l_3)bt)\} = \{(0, 0, bt)\}$, $i_1 = l_2 = 0$, $l_3 = 1$, and $r_{1,2}(H)$ is a straight line $z = bt$.*

Proof Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3 and let $r_{1,2}(H) = \{(0, 0, bt)\}$ be a retraction of the curve $H(t)$ and $\langle \dot r_{1,2}, \dot r_{1,2}\rangle = -b^2 < 0$ for all $t \in$ domain$H$, then $r_{1,2}(H)$ is a time-like curve and $|\dot r_{1,2}| = b$, $T = \frac{\dot r_{1,2}}{\upsilon} = (0, 0, 1)$, where $\upsilon = a$ and $\left\langle \dot T, \dot T\right\rangle = 0, |\dot T| = 0$. Thus, $\dot T$ is light-like and T is time-like, the normal and bi-normal vectors N, B are undefined. And the Frenet equations cannot be appointed. The ratio τ/κ is undefined. Since $\kappa = \tau = 0$ then $r_{1,2}(H)$ is a straight line. □

Definition 3.7 *Let $P(t), P(t) \subset E_1^3$ be any curve in Minkowski 3-space $E_1^3, t \in \mathbb{R}$, we have an arbitrary point, $t_0 \in$ Domain$P \ni I = (\delta - t_0, \delta + t_0)$ with $\delta > 0$, $I \subset \mathbb{R}$ and $\delta \in \mathbb{R}$, then there exists related retractions for every interval I define as $r_I(P(t)) \subset P(t) \subset E_1^3$.*

Theorem 3.8 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(t))$ are different from the Frenet equations of $H(t)$.*

Proof Notice that $r_i(H) = \{(a\cos f(t), a\sin f(t), bf(t))\}$. Since $\langle \dot r, \dot r\rangle = \dot f^2\left(a^2 - b^2\right)$, $\dot f^2 > 0$ for all t, the retraction $r_i(H)$ of the helix $H(t)$ is a space-like curve when $a^2 - b^2 > 0$, a time-like curve if $a^2 - b^2 < 0$ and a null-like curve if $a^2 = b^2$. And since the ratio $\tau/\kappa = c$ is a constant for all t, then the retraction $r_i(H)$ is a helix. We have three cases should be discussed.

Case 1. The curve $r_i(H)$ is a space-like retraction helix if $a^2 > b^2$, and we have

$$\langle \dot{r}, \dot{r} \rangle = \dot{f}^2(a^2 - b^2),\ \upsilon = \dot{f}\sqrt{a^2 - b^2},\ T_r = \left(\frac{-a\sin f}{\sqrt{a^2 - b^2}}, \frac{a\cos f}{\sqrt{a^2 - b^2}}, \frac{b}{\sqrt{a^2 - b^2}} \right)$$

and $\langle T_r, T_r \rangle = 1$. Hence, $r_i(H)$ is space-like, $N_r = \{(-\cos f, -\sin f, 0)\}$, $B = T \times N$ and

$$B_r = \left\{ \frac{1}{\sqrt{a^2 - b^2}}(b\sin f, -b\cos f, -a) \right\}$$

with curvature $\kappa_r = \frac{a}{a^2 - b^2}$ and torsion $\tau_r = \frac{-b}{a^2 - b^2}$. Also,

$$\left\langle \dot{T}_r, \dot{T}_r \right\rangle = \frac{\dot{f}^2 a^2}{a^2 - b^2} > 0,\quad |\dot{T}| = \frac{\dot{f}a}{\sqrt{a^2 - b^2}}$$

and $\dot{T}_r$ is a space-like vector, and the Frenet equations of $r_i(H)$ in matrix notation are

$$\begin{aligned}
\begin{pmatrix} \dot{T}_r \\ \dot{N}_r \\ \dot{B}_r \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T_r \\ N_r \\ B_r \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a\dot{f}}{\sqrt{a^2-b^2}} & 0 \\ \frac{-a\dot{f}}{\sqrt{a^2-b^2}} & 0 & \frac{-b\dot{f}}{\sqrt{a^2-b^2}} \\ 0 & \frac{-b\dot{f}}{\sqrt{a^2-b^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin f}{\sqrt{a^2-b^2}}, \frac{a\cos f}{\sqrt{a^2-b^2}}, \frac{b}{\sqrt{a^2-b^2}} \right) \\ (-\cos f, -\sin f, 0) \\ \frac{1}{\sqrt{a^2-b^2}}(b\sin f, -b\cos f, -a) \end{pmatrix}.
\end{aligned}$$

Case 2. The curve $r_i(H)$ is a time-like retraction helix if $a^2 < b^2$, and then we have

$$|\dot{r}(H)| = \dot{f}^2(|a^2 - b^2|),\ \upsilon = \dot{f}\sqrt{b^2 - a^2},\ T_r = \left(\frac{-a\sin f}{\sqrt{b^2 - a^2}}, \frac{a\cos f}{\sqrt{b^2 - a^2}}, \frac{b}{\sqrt{b^2 - a^2}} \right)$$

and $\langle T_r, T_r \rangle = -1$.

So, $N_r = \{(-\cos f, -\sin f, 0)\}$, $B = T \times N$, $B_r = \left\{ \frac{1}{\sqrt{b^2-a^2}}(b\sin f, -b\cos f, -a) \right\}$, the basis $\{T, N, B\}$ is positive oriented because $\langle B, B \rangle = 1$, the curvature is $\kappa_r = \frac{a}{b^2-a^2}$ and the torsion is $\tau_r = \frac{b}{b^2-a^2}$. Also, $\left\langle \dot{T}_r, \dot{T}_r \right\rangle = \frac{\dot{f}^2 a^2}{b^2-a^2} > 0$, and $\dot{T}_r$ is a space-like vector. So, the Frenet equations of $r_i(H)$ in matrix notation are

$$\begin{aligned}
\begin{pmatrix} \dot{T}_r \\ \dot{N}_r \\ \dot{B}_r \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T_r \\ N_r \\ B_r \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a\dot{f}}{\sqrt{b^2-a^2}} & 0 \\ \frac{a\dot{f}}{\sqrt{b^2-a^2}} & 0 & \frac{b\dot{f}}{\sqrt{b^2-a^2}} \\ 0 & \frac{-b\dot{f}}{\sqrt{b^2-a^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin f}{\sqrt{b^2-a^2}}, \frac{a\cos f}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}} \right) \\ (-\cos f, -\sin f, 0) \\ \frac{1}{\sqrt{b^2-a^2}}(b\sin f, -b\cos f, -a) \end{pmatrix}.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.9 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a null helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(t))$ are different from the Frenet equations of $H(t)$.*

Proof Let $r_i(H)$ be a light-like retraction helix of $H(t)$, re-parameterization by the pseudo arc length s. So, $a = b = \frac{1}{c^2}$, $f(t) = cf(s)$ and the equation of a light-like helix is $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$, where $r_i(H) = \left\{\frac{1}{c^2}(\cos(cf(s)), \sin(cf(s)), cf(s))\right\}$ (I) with curvature $\kappa_r = 1$, the tangent vector

$$T_r(s) = \left\{\frac{1}{c}(f'(-\sin(cf(s))), \cos(cf(s)), 1)\right\}, \quad \langle T_r, T_r\rangle = 0.$$

Then $r_i(H)$ is a light-like helix and the norma vector is

$$N_r(s) = T_r'(s) = \left\{\left(-\frac{f''}{c}\sin(cf) - f'^2\cos(cf), \frac{f''}{c}\cos(cf) - f'^2, \frac{f''}{c}\right)\right\},$$

the bi-normal vector $B_r(s)$ is defined as follows:

Notice that $B(s)$ is a unique light-like vector then (1) $\langle B, B\rangle = 0$; Also, (2) $g(T, B) = 1$ and since B is orthogonal to N, then (3) $\langle B, N\rangle = 0$. From (1),(2) and (3), the bi-normal vector is

$$B_r(s) = \{(J, K, L)\}$$

$$\begin{aligned}
J &= \frac{f''}{f'^3}(\tan^2(cf)\sin(cf) - \sec(cf)) + \left(\frac{c}{2f'} - \frac{f''}{2cf'^5}\right)\sin(cf),\\
K &= \frac{f''}{2cf'^5}\cos(cf) - \frac{f''}{f'^3}\tan^2(cf)\cos(cf) - \frac{c}{2f'}\cos(cf),\\
L &= \frac{f''}{2f'^3}(\tan(cf) - \tan^2(cf)) + \frac{f''}{2cf'^5} + \frac{c}{2f'}
\end{aligned}$$

with torsion

$$\tau_r = -\langle N_r', B_r\rangle = \frac{f'''}{f'} + \frac{f''}{2f'} - \frac{3f''^2}{f'^2} + (\tan(cf) - \tan^2(cf))cf'' - \frac{c^2}{2}f'^2$$

This completes the proof. □

Corollary 3.10 *Let $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ be a null helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(s))$ can be formed by the Frenet equations of $H(s)$ if $f''(s) = 0$.*

Proof Let $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ be a null helix in E_1^3, where

$$r_i(H) = \left\{\frac{1}{c^2}(\cos(cf(s)), \sin(cf(s)), cf(s))\right\}$$

with curvature $\kappa_r = 1$. If $f''(s) = 0$ then $f'(s) = c_1$ and $f(s) = c_1 s + c_2$ be a linear function, where $c_1 = 1$ and $c_2 < 0$ are constants. By substituting there conditions in the equations of

$T_r(s), N_r(s)$ and $B_r(s)$ in Theorem 3.9 and applying the value of $T(s), N(s)$ and $B(s)$ following

$$\begin{cases} T(s) = \left\{\frac{1}{c}(-\sin(cs), \cos(cs), 1)\right\} \\ N(s) = (-\cos(cs), -\sin(cs), 0) \\ B(s) = \frac{c}{2}\{(\sin(cs), -\cos(cs), 1)\} \end{cases}$$

Then ,we have $r_i(H) = \left\{\frac{1}{c^2}(\cos(cs + c_1)), \sin(cs + c_1, cs + c_1)\right\}$. The tangent vector of $r_i(H)$ is $T_r(s) = \lambda_1 T + \lambda_2 N + \lambda$, where $\lambda = (0, 0, 1 - \lambda_1)$ and λ_1, λ_2 are constants. Then the normal vector of $r_i(H)$ is $N_r(s) = \nu_1 N + \nu_2 B + \nu$, where $\nu = (0, 0, -\nu_2)$ and ν_1, ν_2 are constants, the bi-normal vector of $r_i(H)$ is $B_r(s) = \eta_1 B - \eta_2 N + \eta$, where $\eta = (0, 0, 1 - \eta_1)$ and η_1, η_2 are constants and the curve $r_i(H)$ has torsion $\tau_r = \frac{-c^2}{2} f'^2 = f'^2 \tau$, curvature $\kappa_r = \kappa = 1$. Then, the Frenet equations of the retraction $r_i(H) = H(f(s))$ can be formed by the Frenet equations of $H(s)$ if $f''(s) = 0$ and so

$$\begin{cases} T_r(s) = \lambda_1 T(s) + \lambda_2 N(s) + \lambda \\ N_r(s) = \nu_N(s) + \nu_2 B(s) + \nu \\ B_r(s) = \eta_1 B(s) - \eta_2 N(s) + \eta \end{cases}$$

This completes the proof. □

Now, we introduce types of conditional foldings of the helix $H = \{(a\cos t, a\sin t, bt)\}$ in E_1^3. Clearly, $H' = \{(a\sin t, a\cos t, b)\}$. Define

$$\Psi : \{(a\cos t, a\sin t, bt)\} \to \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}, \quad m > 1.$$

Theorem 3.11 *The Frenet equations of the non-null helix $H = \{(a\cos t, a\sin t, bt)\}$ in E_1^3 are invariant under the folding $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ for integers $m > 1$.*

Proof Let $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ be a folding of the helix $H = \{(a\cos t, a\sin t, bt)\}$ and $\left\langle \dot{\Psi}, \dot{\Psi} \right\rangle = \frac{1}{m^2}\left(a^2 - b^2\right)$. Since $\frac{1}{m^2} > 0$, this folding is a space-like curve if $a^2 > b^2$, a time-like curve if $a^2 < b^2$ and a null curve if $a^2 = b^2$. Since $H(t)$ is a helix then the ratio $\tau/\kappa = c$ is a constant. So the folding $\Psi(t)$ has the ratio $\tau_f/\kappa_f = \tau/\kappa = -\frac{b}{a}$, is also a constant and then $\Psi(t)$ is a helix. We have three cases should be discussed.

Case 1. The folded helix is space-like if $a^2 > b^2$. So, $|\Psi(t)| = v = \sqrt{\left(\frac{a}{m}\right)^2 - \left(\frac{b}{m}\right)^2}$ and

$$\begin{aligned} T_\Psi &= \left(\frac{-a\sin t}{\sqrt{a^2 - b^2}}, \frac{a\cos t}{\sqrt{a^2 - b^2}}, \frac{b}{\sqrt{a^2 - b^2}}\right), \quad \langle T, T\rangle = 1, \\ B_\Psi &= T_\Psi \times N_\Psi = \left\{\frac{1}{\sqrt{a^2 - b^2}}(b\sin t, -b\cos t, -a)\right\}, \\ N_\Psi &= \{(-\cos t, -\sin t, 0)\} \end{aligned}$$

with curvature $\kappa_\Psi = \frac{ma}{a^2 - b^2}$ and torsion $\tau_\Psi = \frac{-mb}{a^2 - b^2}$. Also, $\left\langle \dot{T}_\Psi, \dot{T}_\Psi \right\rangle = \frac{a^2}{a^2 - b^2} > 0$, $|\dot{T}_\Psi| =$

$\frac{a}{\sqrt{a^2-b^2}}$. Thus, $\dot{T}_\Psi$ is a space-like vector, and the Frenet equations in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{a}{\sqrt{a^2-b^2}} & 0 \\ \frac{-a}{\sqrt{a^2-b^2}} & 0 & \frac{-b}{\sqrt{a^2-b^2}} \\ 0 & \frac{-b}{\sqrt{a^2-b^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin t}{\sqrt{a^2-b^2}}, \frac{a\cos t}{\sqrt{a^2-b^2}}, \frac{b}{\sqrt{a^2-b^2}}\right) \\ (-\cos t, -\sin t, 0) \\ \frac{1}{\sqrt{a^2-b^2}}(b\sin t, -b\cos t, -a) \end{pmatrix}.$$

Case 2. The folded helix is time-like if $a^2 < b^2$. Then, we have $|\dot{H}(t)| = \sqrt{|a^2-b^2|}$, where, $\upsilon = \sqrt{\left(\frac{b}{m}\right)^2 - \left(\frac{a}{m}\right)^2}$ and

$$T_\Psi = \left(\frac{-a\sin t}{\sqrt{b^2-a^2}}, \frac{a\cos t}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}}\right),$$
$$N_\Psi = \{(-\cos t, -\sin t, 0)\}.$$

So, $\langle T, T\rangle = -1$ and

$$B_\Psi = T_\Psi \times N_\Psi = \left\{\frac{1}{\sqrt{b^2-a^2}}(b\sin t, -b\cos t, -a)\right\},$$

the basis $\{T, N, B\}$ is positive oriented because $\langle B, B\rangle = 1$, $\dot{T} = \left(\frac{-a\cos t}{\sqrt{b^2-a^2}}, \frac{-a\sin t}{\sqrt{b^2-a^2}}, 0\right)$. The curvature $\kappa_\Psi = \frac{ma}{b^2-a^2}$ and the torsion $\tau_\Psi = \frac{-mb}{b^2-a^2}$. Also, $\left\langle \dot{T}_\Psi, \dot{T}_\Psi \right\rangle = \frac{a^2}{b^2-a^2} > 0$. So, $\dot{T}_\Psi$ is a space-like vector, and the Frenet equations in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{a}{\sqrt{b^2-a^2}} & 0 \\ \frac{a}{\sqrt{b^2-a^2}} & 0 & \frac{b}{\sqrt{b^2-a^2}} \\ 0 & \frac{-b}{\sqrt{b^2-a^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin t}{\sqrt{b^2-a^2}}, \frac{a\cos t}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}}\right) \\ (-\cos t, -\sin t, 0) \\ \frac{1}{\sqrt{b^2-a^2}}(b\sin t, -b\cos t, -a) \end{pmatrix}.$$

This completes the proof. □

Theorem 3.12 *Let $H = \{(a\cos t, a\sin t, bt)\}$ be a null helix in E_1^3, $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ a folding of $H(t)$ for integers $m > 1$. Then, the Frenet equation of $\Psi(t)$ can be formed by the Frenet equation of $H(t)$.*

Proof Let $\Psi(t) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ be a folding of the null helix $H = \{(a\cos t, a\sin t, bt)\}$, $\upsilon = 0$. Then, T is undefined, re-parameterization $H(t)$ by the pseudo arc length s. Let $a = b = \frac{1}{c^2}, t = cs$. Thus the equation of the light-like helix is $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$,

$\Psi(s) = \frac{1}{m}\left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ with curvature $\kappa = 1$. The tangent vector is $T_\Psi(s) = \dot{\Psi}(s) = \frac{1}{m}\left\{\frac{1}{c}(-\sin(cs), \cos(cs), 1)\right\}$, the normal vector is $N_\Psi(s) = T'_\Psi(s) = \frac{1}{m}\{(-\cos(cs), \sin(cs), 0)\}$ with the the bi-normal vector defined as follows.

Since $B_\Psi(s)$ is a unique light-like vector, we know that (1) $\langle B, b\rangle = 0$. Also, (2) $g(T_\Psi, B_\Psi) = -1$. Notice that B_Ψ is orthogonal to N_Ψ, there must be (3) $\langle B_\Psi, N_\Psi\rangle = 0$. From (1), (2) and (3),

$$B_\Psi(s) = \frac{mc}{2}(\sin(cs), -\cos(cs), 1)$$

and $\tau_\Psi = -\langle N'_\Psi, B_\Psi\rangle = \frac{c^2}{2}$. So, the Frenet equations of folded curve $\Psi(H)$ in matrix notation are

$$\begin{aligned}\begin{pmatrix} T'_\Psi \\ N'_\Psi \\ B'_\Psi \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ \tau_\Psi & 0 & 1 \\ 0 & \tau_\Psi & 0 \end{pmatrix}\begin{pmatrix} T_\Psi \\ N_\Psi \\ B_\Psi \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{-c^2}{2} & 0 & 1 \\ 0 & \frac{-c^2}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{mc}(-\sin(cs), \cos(cs), 1) \\ \frac{1}{m}(-\cos(cs), -\sin(cs), 0) \\ \frac{mc}{2}(\sin(cs), -\cos(cs), 1) \end{pmatrix},\end{aligned}$$

where, $\kappa_\Psi = \kappa = 1$, $\tau_\Psi = \tau = \frac{-c^2}{2}$ and

$$\begin{cases} T_\Psi = \frac{1}{m}T \\ N_\Psi = \frac{1}{m}N \\ B_\Psi = mB \end{cases}$$

This completes the proof. □

References

[1] A.E.El-Ahmady, The variation of the density on chaotic spheres in chaotic space-like Minkowski space time, *Chaos, Solitons and Fractals*, Vol.31 (2007), 1272-1278.

[2] A.E.El-Ahmady, The deformation retracts and topological folding of Kottler space, *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie Nouvelle Série*, Vol. 36 (84), No. 3/4 (1992), 195-202.

[3] A.E.El-Ahmady, Fuzzy deformation retract of fuzzy spheres in fuzzy Minkowski space $\check{M}^{n+1}$, *The Journal of Fuzzy Mathematics*, Vol. 22,No.3(2014), 555-563.

[4] A.E.El-Ahmady and E.Al-Hesiny, On the geometry of curves in Minkowski 3-space and its folding, *Applied Mathematics*, Vol.4, No. 5(2013), 746-752.

[5] A.E.El-Ahmady and A.El-Araby, On fuzzy spheres in fuzzy Minkowski space, *Nuovo. Cimento.*, Vol.125B (2010).

[6] A.E.El-Ahmady and A.T.M.Zidan, On hyperbola in Minkowski 3-space and its deformations, *International Journal of Applied Mathematics and Statistics*, Vol.57, No.5(2018), 115-127.

[7] J.Walrave, *Curves and Surfaces in Minkowski Space*, Ph.D. Thesis, ProQuest LLC, AnnArbor, Mich,USA, Katholieke Universiteit, Leuven, Belgium (1995).

[8] R.Lopez, *Differential Geometry of Curves and Surfaces in LorentzCMinkowski space*, Instituto de Matematica Estatistica, University of Sao Paulo, Brazil, (2008).

[9] M.do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.

[10] Ahmad T. Ali and Rafael Lopez, On slant helices in Minkowski space in E_1^3, *J. Korean Math. Soc.*, 48(2011), No.1, 159-167.

[11] S.Yilmaz and M.Turgut, Smarandache curves in Minkowski space-time, *International J.Math. Combin.*, Vol.3 (2008), 51-55.

[12] Ferrandez A., Gimenez A., Luca P., Null helices in Lorentzian space forms, *Int. J. Mod. Phys.*, A 16(2001), 4845-4863.

[13] K.İlarslan, O. Boyacıoglǔ, Position vectors of a time-like and a null helix in Minkowski 3-space, *Chaos Solitons Fractals*, 38 (2008) 1383-1389.

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Some Results on 4-Total Difference Cordial Graphs

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Abstract: Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ be a map where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $|f(u) - f(v)|$. f is called k-total difference cordial labeling of G if $|t_{df}(i) - t_{df}(j)| \leq 1$, $i, j \in \{0, 1, 2, \cdots, k-1\}$ where $t_{df}(x)$ denotes the total number of vertices and the edges labeled with x. A graph with admits a k-total difference cordial labeling is called k-total difference cordial graphs.

Key Words: Difference cordial labeling, Smarandachely difference cordial labeling, star, path, cycle, bistar, crown, comb.

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§1. Introduction

We consider here finite, simple and undirected graphs only. Ponraj etl., has been introduced the concept of k-total difference cordial graph in [4]. In [4,5], 3-total difference cordial labeling path , complete graph,comb ,armed crown, crown , wheel, star etc have been investigate and also we prove that every graph is a subgraph of a connected k-total difference cordial graphs in .In this paper we investigate 4-total difference of cordial labeling of some graphs like star, path, cycle, bistar, crown, comb, etc.

§2. K-Total Difference Cordial Labeling

Definition 2.1 *Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $|f(u) - f(v)|$. f is called k-total difference cordial labeling of G if $|t_{df}(i) - t_{df}(j)| \leq 1$, $i, j \in \{0, 1, 2, \cdots, k-1\}$ where $t_{df}(x)$ denotes the total number of vertices and the edges labelled with x. A graph with a k-total difference cordial labeling is called k-total difference cordial graph. Otherwise, if there is a pair $\{i, j\} \subset \{0, 1, 2, \cdots, k-1\}$ such that $|t_{df}(i) - t_{df}(j)| > 1$,such a labeling is called a Smarandachely k-total difference cordial labeling of G.*

Remark 2.2([6]) 2-total difference cordial graph is 2-total product cordial graph.

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§3. Preliminaries

Definition 3.1 *The corona of G_1 with G_2,$G_1 \odot G_2$ is the graph obtained by taking one copy of G_2 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 3.2 *Armed crown AC_n is the graph obtained from the cycle $C_n : u_1u_2 \cdots u_nu_1$ with $V(AC_n) = V(C_n) \bigcup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(AC_n) = E(C_n) \bigcup \{u_iv_i, v_iw_i : 1 \leq i \leq n\}$.*

Definition 3.3 *An edge $x = uv$ of G is said to be subdivided if it is replaced by the edges uw and wv where w is a vertex not in $V(G)$.If every edge of G is subdivided,the resulting graph is called the subdivision graph $S(G)$.*

Definition 3.4 *Jelly fish graphs $J(m, n)$ obtained from a cycle $C_4 : uxvyu$ by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.*

Definition 3.5 *Triangular snake T_n is obtained from the path $P_n : u_1u_2 \cdots u_n$ with $V(T_n) = V(P_n) \bigcup \{v_i : 1 \leq i \leq n-1\}$ and $E(T_n) = E(P_n) = \bigcup \{u_iv_i, u_{i+1}v_i : (1 \leq i \leq n-1)\}$.*

Definition 3.6 *Double Triangular snake $D(T_n)$ is obtained from the Path $P_n : u_1u_2 \cdots u_n$ with $V(D(T_n)) = V(P_n) \bigcup \{v_i, w_i : 1 \leq i \leq n-1\}$ and $E(D(T_n)) = E(P_n) \bigcup \{u_iv_i, u_iw_i : 1 \leq i \leq n-1\} \bigcup \{v_iu_{i+1}, w_iu_{i+1} : 1 \leq i \leq n-1\}$.*

§4. Main Results

Theorem 4.1 *Any star $K_{1,n}$ is 4-total difference cordial.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_1, n) = \{uu_i : 1 \leq i \leq n\}$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$,$r \in N$. Assign the label 1 to the central vertex. Next assign the label 0 to the vertices $u_1, u_2, \ldots, u_{2r}$ and assign the label 3 to the remaining vertices.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$,$r \in N$. Assign the label 1 to the central vertex u .We now move to the pendent vertices.Assign the label 0 to the vertices $u_1, u_2, \cdots, u_{2r}$ and assign the label 3 to the next remaining vertices $u_{2r+1}, u_{2r+2}, \cdots, u_{4r}$ and u_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$,$r \in N$.In this case assign the label 0 to the vertices $u_1, u_2, \cdots, u_{2r}$ and u_{2r+1}.Next assign the label 3 to the vertices $u_{2r+2}, u_{2r+3}, \cdots, u_{4r+2}$.Finally assign 1 to the central vertex u.

Case 4. $n \equiv 3 \pmod 4$.

As in case (3) assign the label to u, $u_1, u_2, \cdots, u_{n-1}$. Next assign the label 3 to the vertex

u_n.

Table 1 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

Values of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
$n \equiv 0 \pmod 4$	$2r$	$2r+1$	$2r$	$2r$
$n \equiv 1 \pmod 4$	$2r$	$2r+1$	$2r+1$	$2r+1$
$n \equiv 2 \pmod 4$	$2r+1$	$2r+2$	$2r+1$	$2r+1$
$n \equiv 3 \pmod 4$	$2r+1$	$2r+2$	$2r+2$	$2r+2$

Table 1

A 4-total difference cordial labeling of $K_{1,n}(n = 1, 2, 3)$ is given in Table 2.

Values of n	u	u_1	u_2	u_3
1	1	3		
2	1	0	3	
3	1	0	3	3

Table 2

This completes the proof. □

Theorem 4.2 *The path P_n is 4-total difference cordial for all values of n.*

Proof Let P_n be the path $u_1, u_2, \cdots, u_n$.

Case 1. $n \equiv 0 \pmod 4$ $n > 3$.

Let $n = 4r$, $r \in \mathbb{N}$, Assign the labels $3, 1, 1$ and 3 respectively to the vertices u_1,u_2,u_3,u_4.Next assign the labels $3, 1, 1$ and 3 to the next 4 vertices u_5,u_6,u_7,u_8 respectively. Proceeding like this until we reach the vertex u_n. That is in this process the last 4 vertices u_{n-3},u_{n-2}, u_{n-1} and u_n receive the labels $3, 1, 1$ and 3.

Case 2. $n \equiv 1 \pmod 4$ $n > 3$.

Let $n = 4r + 1$,$r \in \mathbb{N}$. As in Case 1, assign the label to the vertices $u_1, u_2, \cdots, u_{n-1}$. Next assign the label 3 to the vertex u_n.

Case 3. $n \equiv 2 \pmod 4$, $n > 3$.

Let $n = 4r + 2$,$r \in \mathbb{N}$. Assign the label to the vertices $u_1, u_2, \cdots, u_{n-1}$ as in Case 2. Next assign the label 1 to the vertices u_n.

Case 4. $n \equiv 3 \pmod 4$, $n > 3$.

Let $n = 4r + 3$,$r \in \mathbb{N}$. Assign the label to the vertices $u_1, u_2, \cdots, u_{n-1}$ as in Case 3. Next assign the label 1 to the vertex u_n. This vertex labels is a 4-total difference cordial labels follows from Table 3 for $n > 3$.

Values of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
$n \equiv 0 \pmod 4$	$2r-1$	$2r$	$2r$	$2r$
$n \equiv 1 \pmod 4$	$2r$	$2r$	$2r$	$2r+1$
$n \equiv 2 \pmod 4$	$2r$	$2r+1$	$2r+1$	$2r+1$
$n \equiv 3 \pmod 4$	$2r+1$	$2r+2$	$2r+1$	$2r+1$

Table 3

A 4-total difference cordial labeling of $P_n(n = 1, 2, 3)$ is given in Table 4.

Values of n	u_1	u_2	u_3
1	0		
2	0	2	
3	0	2	3

Table 4

This completes the proof. □

Theorem 4.3 *The cycle C_n is 4-total difference cordial if $n \equiv 0, 1, 3 \pmod 4$*

Proof Let C_n be the cycle $u_1u_2 \cdots u_nu_1$. Assign the label to the vertices $u_1, u_2, \cdots, u_n$ as in Theorem 4.2. Table 5 given below shows that this labeling of C_n is a 4-total difference cordial.

Values of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
$n \equiv 0 \pmod 4$	$2r$	$2r$	$2r$	$2r$
$n \equiv 1 \pmod 4$	$2r$	$2r$	$2r$	$2r+1$
$n \equiv 3 \pmod 4$	$2r+1$	$2r+2$	$2r+1$	$2r+1$

Table 5

This completes the proof. □

Theorem 4.4 *The bistar $B_{n,n}$ is 4-total different cordial for all n.*

Proof Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv : (1 \leq i \leq n)\}$. Clearly $B_{n,n}$ has $2n+2$ vertices and $2n+1$ edges. Assign the label 1 to the central vertices u and v. Assign the label 3 to the vertices $u_1, u_2, \cdots, u_n$ and v_1. We now assign the label 1 to the vertices $v_2, v_3, \cdots, v_n$. Clearly $t_{df}(0) = n$, $t_{df}(1) = t_{df}(2) = t_{df}(3) = n+1$. Therefore f is a 4-total difference cordial labeling. □

Theorem 4.5 *The crown $C_n \odot K_1$ is 4-total difference cordial labeling for all values of n.*

Proof Let C_n be the cycle $u_1u_2 \cdots u_nu_1$ .Let $V(C_n \odot K_1)V(C_n) \bigcup \{v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = E(C_n) \bigcup \{u_iv_i : 1 \leq i \leq n\}$. Assign the label 1 to the cycle vertices $u_1, u_2, \cdots, u_n$. Next we move to the pendent vertices v_i. Assign the label 3 to all pendent vertices $v_1, v_2, \cdots, v_n$.

Clearly $t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = n$. Hence f is a 4-total difference cordial labeling. □

Corollary 4.1 *All combs are* 4*-total difference cordial labeling.*

Proof Clearly the vertex labeling in theoeam 4.5 is also a 4-total difference cordial labeling of $P_n \odot K_1$. □

Theorem 4.6 *The armed crown* AC_n *is* 4*-total difference cordial for all* n*.*

Proof Clearly AC_n has 3 vertices and $3n$ edges. Let the vertex set and edge set as in Definition 3.2. Assign the label 1 to the all the cycle vertices $u_1, u_2, \cdots, u_n$. Next we assign the label 3 to the vertices $v_1, v_2, \cdots, v_n$.

Case 1. n is even.

In this case assign the label 3 to the pendent vertices $w_1 w_2 \cdots w_{\frac{n}{2}}$ and 1 to the remaining pendent vertices $w_{\frac{n}{2}+1}, w_{\frac{n}{2}+2}, \cdots, w_n$.

Case 2. n is odd.

Assign the label 3 to the vertices $w_1, w_2, \cdots, w_{\frac{n+1}{2}}$ and 1 to the vertices $w_{\frac{n+3}{2}}, w_{\frac{n+5}{2}}, \cdots, w_n$. The table 6 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

Values of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
n is odd	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 6

This completes the proof. □

Theorem 4.7 *The double triangular snake* DT_n *is* 4*-total difference cordial for all* n*.*

Proof Let the vertex set and edge set as in Definition 3.6.

Case 1. $n \equiv 0 \pmod 3$.

Assign the labels $3, 2, 3$ to the path vertices u_1, u_2, u_3. Next assign the labels $3, 2, 3$ to the next 3 vertices u_4, u_5, u_6 respectively. Proceeding like this until we reach the vertices u_n. That is in the process the last three vertices u_{n-2}, u_{n-1}, u_n receive the label $3, 2, 3$. Next assign the label 0 to the vertices $v_1, v_2, \cdots, v_n$ and assign the label 2 to the vertices $w_1, w_2, \cdots, w_n$.

Case 2. $n \equiv 1 \pmod 3$.

In this case assign the labels to the vertices $u_i, (1 \le i \le n-1)$,$v_i, w_i, (1 \le i \le n-1)$ as in Case 1. Next assign the labels $3, 0, 2$ respectively to the vetices u_n, v_{n-1}, w_n.

Case 3. $n \equiv 2 \pmod 3$.

As in Case 2 assign the labels to the vertices $u_1, u_2, \cdots, u_{n-1}$,$v_1, v_2, \cdots v_{n-2}$ and $w_1, w_2, \cdots w_{n-2}$.

Finally assign the label $2, 0$ and 2 to the vertices u_n,v_{n-1} and w_{n-1}. Table 7 given below establish that this labeling scheme is a 4-total difference cordial labeling of DT_n.

Nature of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
$n \equiv 0 \pmod 3$	$2n-2$	$2n-2$	$2n-1$	$2n-2$
$n \equiv 1 \pmod 3$	$2n-2$	$2n-2$	$2n-2$	$2n-1$
$n \equiv 2 \pmod 3$	$2n-2$	$2n-2$	$2n-1$	$2n-2$

Table 7

This completes the proof. □

Example 4.1 A 4-total difference cordial labeling of $D(T_6)$ is shown in Figure 1.

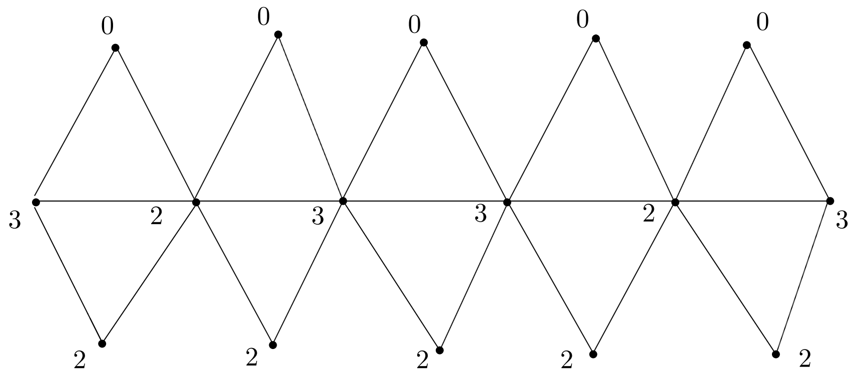

Figure 1

Theorem 4.8 *The jelly fish $J(n,n)$ is 4-total difference cordial for all n.*

Proof Let C_4 be a cycle $uxvyu$. Let $V(J(n,n))$=$V(C_4) \bigcup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(J(n,n))$ $= E(C_4) \cup \{xy, xu_i, yv_i : 1 \leq i \leq n\}$. Assign the label 1 to the all cycle vertices u,x,y,v. Next we move to the pendent vertices. Assign the label 3 to the $u_1, u_2, \cdots, u_n$ and v_1,v_2. Assign the label 1 to the $v_3, v_4, \cdots, v_n$. Since $t_{df}(0) = n+3$,$t_{df}(1) = t_{df}(2) = t_{df}(3) = n+2$,f is a 4-total difference cordial labeling. □

Theorem 4.9 *The subdivision of the bistar $B_{n,n}$, $S(B_{n,n})$ is 4-total different cordial for all n.*

Proof Let $V(S(B_{n,n})) = \{u, w, v, u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{uu_i, u_ix_i, uw,$ $wv, vv_i, v_iy_i : 1 \leq i \leq n\}$. Assign the label 1 to the vertices u,w and v.Next assign the label 3 to the vertices $u_1, u_2, \cdots, u_i$,
$x_1, x_2, \cdots, x_i$ and v_1. We now assign the label 2 to the vertices $y_1, y_2, \cdots, y_n$ and v_2. Finally assign the label 1 to the vertices $v_3, v_4, \cdots, v_n$. Since $t_{df}(0) = t_{df}(1) = t_{df}(3) = 2n+1$,$t_{df}(2) =$ $2n+2$. The labeling f is a 4-total difference cordial labeling. □

Theorem 4.10 *$P_n \odot 2K_1$ is 4-total difference cordial for all n.*

Proof Let P_n be the path $u_1, u_2, \cdots, u_n$.Let v_i,w_i be the pendent vertices adjacent to u_i $(1 \leq i \leq n)$. Assign the label 1 to the path vertices $u_1, u_2, \cdots, u_n$.

Case 1. n is even.

Assign the label 3 to all the vertices v_1,v_2, $\cdots$, v_n and w_1,w_2, $\cdots$, $w_{\frac{n}{2}}$. We now assign the label 1 to the vertices $w_{\frac{n}{2}+1}, w_{\frac{n}{2}+2}, \cdots, w_n$.

Case 2. n is odd.

As in Case 1 assign the label to the vertices u_i,v_i,w_i $(1 \leq i \leq n)$. Next assign the label 3 to the vertices u_n and assign the label 1 to the vertex w_n.

Table 8 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

Values of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
n is even	$\frac{3n}{2} - 1$	$\frac{3n}{2}$	$\frac{3n}{2}$	$\frac{3n}{2}$
n is odd	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$

Table 8

This completes the proof. $\Box$

Theorem 4.11 *$S(P_n \odot K_1)$ is 4-total difference cordial for all n.*

Proof Let P_n be the path $u_1, u_2, \cdots u_n$.Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i : 1 \leq i \leq n\}$. Let x_i be the vertex which subdivide the edge u_iu_{i+1},$\{1 \leq i \leq n-1\}$ and y_i be the vertex which subdivide u_iv_i : $\{1 \leq i \leq n\}$. Assign the label 3 to the all path vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$ and v_2. Next we assign the label 1 to the vertices $y_1, y_2, \ldots y_n$ and v_1. Finally we assign the label 2 to the remaining vertices $v_3, v_4, \cdots v_n$. Clearly $t_{df}(0) = t_{df}(1) = t_{df}(2) = 2n - 1$,$t_{df}(3) = 2n$. Therefore, f is a 4-total difference cordial labeling of $S(P_n \odot K_1)$. $\Box$

Theorem 4.12 *$S(C_n \odot K_1)$ is 4-total difference cordial for all values of n.*

Proof Let C_n : $u_1u_2 \cdots u_nu_1$ be the cycle. Let $V(C_n \odot K_1)$=$V(C_n) \bigcup \{v_i : (1 \leq i \leq n)\}$ and $E(C_n \odot K_1) = E(C_n) \bigcup \{u_iv_i : 1 \leq i \leq n\}$. Let x_i, y_i be the vertices which subdivide the edges $u_iu_{i+1}(1 \leq i \leq n-1)$, $u_iv_i(1 \leq i \leq n)$ respectively. First we assign the label 3 to the cycle vertices $u_1, u_2, \cdots u_n$ and $x_1, x_2, \cdots x_n$. Next we assign the label 1 to the $y_1, y_2, \cdots y_n$. Finally assign the label 2 to the all pendent vertices $v_1, v_2, \cdots v_n$. Clearly $t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = 2n$. Therefore f is a 4-total difference cordial labeling of $S(C_n \odot K_1)$. $\Box$

Theorem 4.13 *$S(AC_n)$ is 4-total difference cordial for all n.*

Proof Let the vertex set and edge set of AC_n as in definition 3.2.let x_i : $(1 \leq i \leq n-1)$,y_i : $(1 \leq i \leq n-1)$ and z_i : $(1 \leq i \leq n-1)$ be the vertex which subdivide the edges u_iu_{i+1} : $(1 \leq i \leq n-1)$,u_iv_i : $(1 \leq i \leq n-1)$ and v_iw_i : $(1 \leq i \leq n-1)$ respectively. Assign the label 3 to the vertices $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$ and $w_1, w_2, \cdots, w_n$. Next assign the label 1 to the vertices $y_1, y_2, \cdots, y_n$.Then assign the label 2 to the vertices $v_1, v_2, \cdots, v_n$ and $z_1, z_2, \cdots z_n$. obviously $t_{df}(0) = t_{df}(1) = t_{df}(2) = t_{df}(3) = 3n$. Therefore f is a 4-total difference cordial labeling of $S(AC_n)$. $\square$

Example 4.2 A 4-total difference cordial labeling of $S(AC_n)$ is shown in Figure 2.

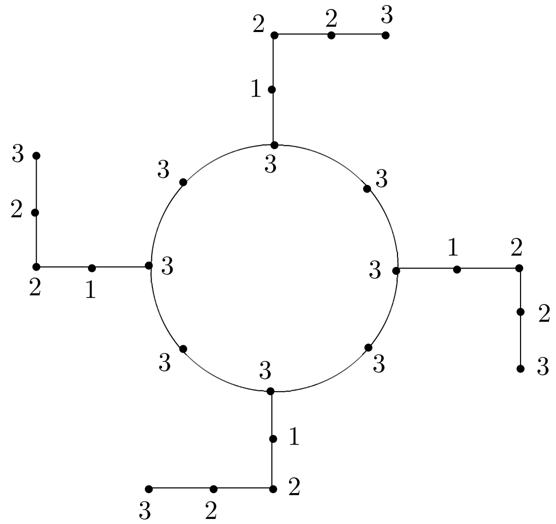

Figure 2

Theorem 4.14 *$S(T_n)$ is 4-total difference cordial.*

Proof Let the vertex set and edge set of T_n as in Definition 3.7. Let x_i, y_i and z_i be the vertices which subdivide the edges u_iu_{i+1},u_i, v_i and $u_{i+1}v_i$, $(1 \leq i \leq n)$.

Case 1. $n \equiv 0 \pmod 4)$.

Assign the label 3 to the vertices $u_1, u_2, \cdots u_n$ and $x_1, x_2, \cdots x_{n-1}$. Assign the label 1 to the vertices $y_1, y_2 \cdots y_{n-1}$. Next assign the label $2, 3, 1$ and 3 to the vertices z_1, z_2, z_3, z_4 then assign the label $2, 3, 1$ and 3 to the next 4 vertices z_5, z_6, z_7, z_8 respectively. Proceeding like this until we reach the vertices z_{n-1}. That is in the process the last four vertices are $z_{n-4}, z_{n-3}, z_{n-2}, z_{n-1}$ receive the label $2, 3, 1, 3$. Next assign the label $0, 2, 3, 2$ to the vertices $0, 2, 3, 2$ to the vertices v_1, v_2, v_3, v_4 then assign the label $0, 2, 3, 2$ to the next 4 vertices v_5, v_6, v_7, v_8 respectively. Proceeding like this until we reach the vertices v_{n-1}.That is in the process the last 4 vertices $v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$ receive the label $0, 2, 3, 2$.

Case 2. $n \equiv 1 \pmod 4$.

As in Case 1 assign the labels to the vertices $u_i, (1 \leq i \leq n-1)$,$v_i, x_i, y_i, z_i, (1 \leq i \leq n-2)$. Next assign the labels $3, 0, 3, 1$ and 2respect to the vertices $u_n, v_{n-1}, x_{n-1}, y_{n-1}$ and z_{n-1}.

Case 3. $n \equiv 2 \pmod 4$.

In this case, assign the labels to the vertices $u_i, (1 \leq i \leq n-1)$,$v_i, x_i, y_i, z_i, (1 \leq i \leq n-2)$ as in Case 2. Finally assign the labels $3, 2, 3, 1$ and 3 to the vertices $u_n, v_{n-1}, x_{n-1}, y_{n-1}$ and

z_{n-1} respectively.

Case 4. $n \equiv 3 \pmod 4$.

As in Case 3, assign the label to $u_i, (1 \le i \le n-1)$,$v_i, x_i, y_i, z_i, (1 \le i \le n-2)$. Next assign the labels $3, 3, 3, 1$ and 1 to the vertices $u_n, v_{n-1}, x_{n-1}, y_{n-1}$ and z_{n-1} respectively. Table 9 given below establish that this vertex labeling pattern is a 4-total difference cordial labeling.

Nature of n	$t_{df}(0)$	$t_{df}(1)$	$t_{df}(2)$	$t_{df}(3)$
$n \equiv 0 \pmod 4$	$\frac{11n-8}{4}$	$\frac{11n-8}{4}$	$\frac{11n-12}{4}$	$\frac{11n-12}{4}$
$n \equiv 1 \pmod 4$	$\frac{11n-9}{4}$	$\frac{11n-9}{4}$	$\frac{11n-5}{4}$	$\frac{11n-9}{4}$
$n \equiv 2 \pmod 4$	$\frac{11n-10}{4}$	$\frac{11n-10}{4}$	$\frac{11n-10}{4}$	$\frac{11n-10}{4}$
$n \equiv 3 \pmod 4$	$\frac{11n-9}{4}$	$\frac{11n-9}{4}$	$\frac{11n-9}{4}$	$\frac{11n-13}{4}$

Table 9

This completes the proof. □

References

[1] I.Cahit, Cordial graphs: A weaker version of graceful and harmonious graphs, *Ars Combinatoria*, 23(1987), 201-207.

[2] J.A.Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19 (2017) #Ds6.

[3] F.Harary, *Graph theory*, Addision wesley, New Delhi (1969).

[4] R.Ponraj, S.Yesu Doss Philip and R.Kala, k-total difference cordial graphs, *Journal of Algorithms and Combutation*, 51(2019), 121-128.

[5] R.Ponraj, S.Yesu Doss Philip and R.Kala, 3-total difference cordial graphs, *Global Engineering Science and research*, 6(2019), 46-51.

[6] M.Sundaram, R.Ponraj and S.Somasundaram, Total product cordial labeling of graphs, *Bulletin of Pure and Applied Sciences* (Section E - Maths and Statistics), 25E(2005), 199-203.

Famous Words

We usually understand the universe by matter, not including antimatter. However, the universe consists of matter and antimatter, and the matter contributes only 4.6% to the whole matter/energy distribution of the universe as physicist verified. Thus, we always understand the universe by the known matter, i.e., 4.6% not the whole 100% consisting of the universe. However, we can not conclude the universe is dominated by the matter, and can not claim that we have hold on the truth face of the universe because all known of humans are a partial or local true on matters. (Extracted from the paper: Mathematical Elements on Natural Reality, *Bull. Cal. Math. Soc.*, 111, (6) 597-618 (2019))

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

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Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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