



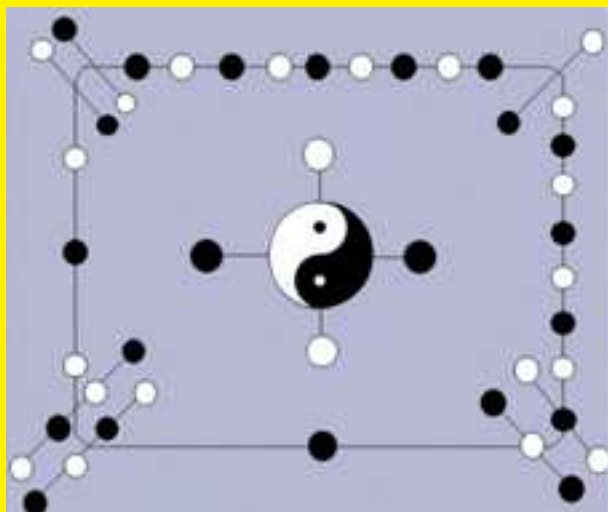
ISBN 978-1-59973-558-0

VOLUME 1, 2018

# MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND  
ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

March, 2018

Vol.1, 2018

ISBN 978-1-59973-558-0

# MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO

([www.mathcombin.com](http://www.mathcombin.com))

The Madis of Chinese Academy of Sciences and  
Academy of Mathematical Combinatorics & Applications, USA

March, 2018

**Aims and Scope:** The **Mathematical Combinatorics (International Book Series)** is a fully refereed international book series with ISBN number on each issue, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 110-160 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces, ..., etc.. Smarandache geometries;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds; Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics; Mathematical theory on gravitational fields; Mathematical theory on parallel universes; Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing

10 Estes St. Ipswich, MA 01938-2106, USA

Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371

<http://www.ebsco.com/home/prints subs/priceproj.asp>

and

*Gale Directory of Publications and Broadcast Media*, Gale, a part of Cengage Learning

27500 Drake Rd. Farmington Hills, MI 48331-3535, USA

Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075

<http://www.gale.com>

**Indexing and Reviews:** Mathematical Reviews (USA), Zentralblatt Math (Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Directory of Open Access (DoAJ), International Statistical Institute (ISI), International Scientific Indexing (ISI, impact factor 1.743), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

**Subscription** A subscription can be ordered by an email directly to

**Prof.Linfan Mao, PhD**

The Editor-in-Chief of *International Journal of Mathematical Combinatorics*

Chinese Academy of Mathematics and System Science

Beijing, 100190, P.R.China

Email: [maolinfan@163.com](mailto:maolinfan@163.com)

**Price:** US\$48.00

# Editorial Board (4th)

## Editor-in-Chief

### **Linfan MAO**

Chinese Academy of Mathematics and System  
Science, P.R.China

and

Academy of Mathematical Combinatorics &  
Applications, USA

Email: maolinfan@163.com

### **Shaofei Du**

Capital Normal University, P.R.China

Email: dushf@mail.cnu.edu.cn

### **Xiaodong Hu**

Chinese Academy of Mathematics and System  
Science, P.R.China

Email: xdhu@amss.ac.cn

## Deputy Editor-in-Chief

### **Guohua Song**

Beijing University of Civil Engineering and  
Architecture, P.R.China

Email: songguohua@bucea.edu.cn

### **Yuanqiu Huang**

Hunan Normal University, P.R.China

Email: hyqq@public.cs.hn.cn

### **H.Iseri**

Mansfield University, USA

Email: hiseri@mnsfld.edu

## Editors

### **Arindam Bhattacharyya**

Jadavpur University, India

Email: bhattachar1968@yahoo.co.in

### **Said Broumi**

Hassan II University Mohammedia

Hay El Baraka Ben M'sik Casablanca

B.P.7951 Morocco

### **Xueliang Li**

Nankai University, P.R.China

Email: lxl@nankai.edu.cn

### **Guodong Liu**

Huizhou University

Email: lgd@hzu.edu.cn

### **W.B.Vasantha Kandasamy**

Indian Institute of Technology, India

Email: vasantha@iitm.ac.in

### **Junliang Cai**

Beijing Normal University, P.R.China

Email: caijunliang@bnu.edu.cn

### **Ion Patrascu**

Fratii Buzesti National College

Craiova Romania

### **Yanxun Chang**

Beijing Jiaotong University, P.R.China

Email: yxchang@center.njtu.edu.cn

### **Han Ren**

East China Normal University, P.R.China

Email: hren@math.ecnu.edu.cn

### **Jingan Cui**

Beijing University of Civil Engineering and  
Architecture, P.R.China

Email: cuijingan@bucea.edu.cn

### **Ovidiu-Ilie Sandru**

Politehnica University of Bucharest  
Romania

**Mingyao Xu**

Peking University, P.R.China

Email: xumy@math.pku.edu.cn

**Guiying Yan**

Chinese Academy of Mathematics and System

Science, P.R.China

Email: yanguiying@yahoo.com

**Y. Zhang**

Department of Computer Science

Georgia State University, Atlanta, USA

**Famous Words:**

*The problems that exist in the world today cannot be solved by the level of thinking that created them.*

By Albert Einstein, an American physicist.

## Some Fixed Point Results for Contractive Type Conditions in Cone $b$ -Metric Spaces and Applications

G.S.Saluja

(Department of Mathematics, Govt. Kaktiya P. G. College, Jagdalpur, Jagdalpur - 494001 (C.G.), India)

saluja1963@gmail.com

**Abstract:** In this paper, we establish some fixed point results for contractive type conditions in the framework of complete cone  $b$ -metric spaces and give some applications of our results. The results presented in this paper generalize, extend and unify several well-known comparable results in the existing literature.

**Key Words:** Fixed point, contractive type condition, cone  $b$ -metric space, cone.

**AMS(2010):** 47H10, 54H25.

### §1. Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding fixed point of contractive mappings becomes the center of strong research activity. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle [2] in 1922.

In [3], Bakhtin introduced  $b$ -metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in  $b$ -metric spaces that generalized the famous contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in  $b$ -metric spaces (see [4, 5, 11] and references therein). In recent investigation, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [14, 15, 17, 20]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces.

In 2007, Huang and Zhang [14] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 16, 20, 23] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. In 2008, Rezapour and Hambarani [20] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

Recently, Hussain and Shah [15] introduced the concept of cone  $b$ -metric space as a general-

---

<sup>1</sup>Received July 23, 2017, Accepted February 8, 2018.

ization of  $b$ -metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about  $KKM$  mappings in the setting of a cone  $b$ -metric space. In this paper, we give some examples in cone  $b$ -metric spaces, then obtain some fixed theorems for contractive type conditions in the setting of cone  $b$ -metric spaces.

**Definition 1.1**([14]) *Let  $E$  be a real Banach space. A subset  $P$  of  $E$  is called a cone whenever the following conditions hold:)*

- ( $c_1$ )  $P$  is closed, nonempty and  $P \neq \{0\}$ ;
- ( $c_2$ )  $a, b \in \mathbb{R}$ ,  $a, b \geq 0$  and  $x, y \in P$  imply  $ax + by \in P$ ;
- ( $c_3$ )  $P \cap (-P) = \{0\}$ .

Given a cone  $P \subset E$ , we define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  to indicate that  $x \leq y$  but  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in P^0$ , where  $P^0$  stands for the interior of  $P$ . If  $P^0 \neq \emptyset$  then  $P$  is called a solid cone (see [22]).

There exist two kinds of cones- normal (with the normal constant  $K$ ) and non-normal ones ([12]).

Let  $E$  be a real Banach space,  $P \subset E$  a cone and  $\leq$  partial ordering defined by  $P$ . Then  $P$  is called normal if there is a number  $K > 0$  such that for all  $x, y \in P$ ,

$$0 \leq x \leq y \text{ imply } \|x\| \leq K\|y\|, \quad (1.1)$$

or equivalently, if  $(\forall n) x_n \leq y_n \leq z_n$  and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x \text{ imply } \lim_{n \rightarrow \infty} y_n = x. \quad (1.2)$$

The least positive number  $K$  satisfying (1.1) is called the normal constant of  $P$ .

**Example 1.2**([22]) Let  $E = C_{\mathbb{R}}^1[0, 1]$  with  $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$  on  $P = \{x \in E : x(t) \geq 0\}$ . This cone is not normal. Consider, for example,  $x_n(t) = \frac{t^n}{n}$  and  $y_n(t) = \frac{1}{n}$ . Then  $0 \leq x_n \leq y_n$ , and  $\lim_{n \rightarrow \infty} y_n = 0$ , but  $\|x_n\| = \max_{t \in [0, 1]} |\frac{t^n}{n}| + \max_{t \in [0, 1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$ ; hence  $x_n$  does not converge to zero. It follows by (1.2) that  $P$  is a non-normal cone.

**Definition 1.3**([14, 24]) *Let  $X$  be a nonempty set. Suppose that the mapping  $d: X \times X \rightarrow E$  satisfies:*

- ( $d_1$ )  $0 \leq d(x, y)$  for all  $x, y \in X$  with  $x \neq y$  and  $d(x, y) = 0 \Leftrightarrow x = y$ ;
- ( $d_2$ )  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- ( $d_3$ )  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space [14].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where  $E = \mathbb{R}$  and  $P = [0, +\infty)$ .

**Example 1.4**([14]) Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d: X \times X \rightarrow E$

defined by  $d(x, y) = (|x - y|, \alpha|x - y|)$ , where  $\alpha \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space with normal cone  $P$  where  $K = 1$ .

**Example 1.5**([19]) Let  $E = \ell^2$ ,  $P = \{\{x_n\}_{n \geq 1} \in E : x_n \geq 0, \text{ for all } n\}$ ,  $(X, \rho)$  a metric space, and  $d: X \times X \rightarrow E$  defined by  $d(x, y) = \{\rho(x, y)/2^n\}_{n \geq 1}$ . Then  $(X, d)$  is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

**Definition 1.6**([15]) Let  $X$  be a nonempty set and  $s \geq 1$  be a given real number. A mapping  $d: X \times X \rightarrow E$  is said to be cone  $b$ -metric if and only if, for all  $x, y, z \in X$ , the following conditions are satisfied:

- (b<sub>1</sub>)  $0 \leq d(x, y)$  with  $x \neq y$  and  $d(x, y) = 0 \Leftrightarrow x = y$ ;
- (b<sub>2</sub>)  $d(x, y) = d(y, x)$ ;
- (b<sub>3</sub>)  $d(x, y) \leq s[d(x, z) + d(z, y)]$ .

The pair  $(X, d)$  is called a cone  $b$ -metric space.

**Remark 1.7** The class of cone  $b$ -metric spaces is larger than the class of cone metric space since any cone metric space must be a cone  $b$ -metric space. Therefore, it is obvious that cone  $b$ -metric spaces generalize  $b$ -metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone  $b$ -metric space instead of a cone metric space is meaningful since there exist cone  $b$ -metric space which are not cone metric space.

**Example 1.8**([13]) Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset E$ ,  $X = \mathbb{R}$  and  $d: X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$ , where  $\alpha \geq 0$  and  $p > 1$  are two constants. Then  $(X, d)$  is a cone  $b$ -metric space with the coefficient  $s = 2^p > 1$ , but not a cone metric space.

**Example 1.9**([13]) Let  $X = \ell^p$  with  $0 < p < 1$ , where  $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ . Let  $d: X \times X \rightarrow \mathbb{R}_+$  defined by  $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$ , where  $x = \{x_n\}$ ,  $y = \{y_n\} \in \ell^p$ . Then  $(X, d)$  is a cone  $b$ -metric space with the coefficient  $s = 2^{1/p} > 1$ , but not a cone metric space.

**Example 1.10**([13]) Let  $X = \{1, 2, 3, 4\}$ ,  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$ . Define  $d: X \times X \rightarrow E$  by

$$d(x, y) = \begin{cases} (|x - y|^{-1}, |x - y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then  $(X, d)$  is a cone  $b$ -metric space with the coefficient  $s = \frac{6}{5} > 1$ . But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1, 2) > d(1, 4) + d(4, 2), \quad d(3, 4) > d(3, 1) + d(1, 4).$$

**Definition 1.11**([15]) Let  $(X, d)$  be a cone  $b$ -metric space,  $x \in X$  and  $\{x_n\}$  be a sequence in  $X$ . Then



- $\{x_n\}$  is a Cauchy sequence whenever, if for every  $c \in E$  with  $0 \ll c$ , then there is a natural number  $N$  such that for all  $n, m \geq N$ ,  $d(x_n, x_m) \ll c$ ;
- $\{x_n\}$  converges to  $x$  whenever, for every  $c \in E$  with  $0 \ll c$ , then there is a natural number  $N$  such that for all  $n \geq N$ ,  $d(x_n, x) \ll c$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .
- $(X, d)$  is a complete cone  $b$ -metric space if every Cauchy sequence is convergent.

In the following  $(X, d)$  will stand for a cone  $b$ -metric space with respect to a cone  $P$  with  $P^0 \neq \emptyset$  in a real Banach space  $E$  and  $\leq$  is partial ordering in  $E$  with respect to  $P$ .

**Definition 1.12**([10]) Let  $(X, d)$  be a metric space. A self mapping  $T: X \rightarrow X$  is called quasi contraction if it satisfies the following condition:

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \right\}$$

for all  $x, y \in X$  and  $h \in (0, 1)$  is a constant.

**Definition 1.13**([10]) Let  $(X, d)$  be a metric space. A self mapping  $T: X \rightarrow X$  is called Ciric quasi-contraction if it satisfies the following condition:

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

for all  $x, y \in X$  and  $h \in (0, 1)$  is a constant.

The following lemmas are often used (in particular when dealing with cone metric spaces in which the cone need not be normal).

**Lemma 1.14**([17]) Let  $P$  be a cone and  $\{a_n\}$  be a sequence in  $E$ . If  $c \in \text{int } P$  and  $0 \leq a_n \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists  $N$  such that for all  $n > N$ , we have  $a_n \ll c$ .

**Lemma 1.15**([17]) Let  $x, y, z \in E$ , if  $x \leq y$  and  $y \ll z$ , then  $x \ll z$ .

**Lemma 1.16**([15]) Let  $P$  be a cone and  $0 \leq u \ll c$  for each  $c \in \text{int } P$ , then  $u = 0$ .

**Lemma 1.17**([8]) Let  $P$  be a cone, if  $u \in P$  and  $u \leq ku$  for some  $0 \leq k < 1$ , then  $u = 0$ .

**Lemma 1.18**([17]) Let  $P$  be a cone and  $a \leq b + c$  for each  $c \in \text{int } P$ , then  $a \leq b$ .

## §2. Main Results

In this section we shall prove some fixed point theorems of contractive type conditions in the framework of cone  $b$ -metric spaces.

**Theorem 2.1** Let  $(X, d)$  be a complete cone  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose

that the mapping  $T: X \rightarrow X$  satisfies the contractive type condition:

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &\quad + \mu[d(x, Ty) + d(y, Tx)] \end{aligned} \quad (2.1)$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma, \mu \geq 0$  are constants such that  $s\alpha + \beta + s\gamma + (s^2 + s)\mu < 1$ . Then  $T$  has a unique fixed point in  $X$ .

*Proof* Choose  $x_0 \in X$ . We construct the iterative sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$ ,  $n \geq 1$ , that is,  $x_{n+1} = Tx_n = T^{n+1}x_0$ . From (2.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, Tx_n) + \gamma d(x_{n-1}, Tx_{n-1}) \\ &\quad + \mu[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] \\ &= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) \\ &\quad + \mu[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ &= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) \\ &\quad + \mu d(x_{n-1}, x_{n+1}) \\ &\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) \\ &\quad + s\mu[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= (\alpha + \gamma + s\mu)d(x_n, x_{n-1}) \\ &\quad + (\beta + s\mu)d(x_n, x_{n+1}). \end{aligned} \quad (2.2)$$

This implies that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \left( \frac{\alpha + \gamma + s\mu}{1 - \beta - s\mu} \right) d(x_n, x_{n-1}) \\ &= \lambda d(x_n, x_{n-1}) \end{aligned} \quad (2.3)$$

where

$$\lambda = \left( \frac{\alpha + \gamma + s\mu}{1 - \beta - s\mu} \right).$$

As  $s\alpha + \beta + s\gamma + (s^2 + s)\mu < 1$ , it is clear that  $\lambda < 1/s$ .

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1}). \quad (2.4)$$

Using (2.4) in (2.3), we get

$$d(x_{n+1}, x_n) \leq \lambda^2 d(x_{n-1}, x_{n-2}). \quad (2.5)$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0). \quad (2.6)$$

Let  $m \geq 1, p \geq 1$ , we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^2d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^3d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1}d(x_{m+p-1}, x_{m+p}) \\
&\leq s\lambda^m d(x_1, x_0) + s^2\lambda^{m+1}d(x_1, x_0) + s^3\lambda^{m+2}d(x_1, x_0) \\
&\quad + \cdots + s^p\lambda^{m+p-1}d(x_1, x_0) \\
&= s\lambda^m[1 + s\lambda + s^2\lambda^2 + s^3\lambda^3 + \cdots + (s\lambda)^{p-1}]d(x_1, x_0) \\
&\leq \left[ \frac{s\lambda^m}{1 - s\lambda} \right] d(x_1, x_0).
\end{aligned}$$

Let  $0 \ll \varepsilon$  be given. Notice that  $\left[ \frac{s\lambda^m}{1 - s\lambda} \right] d(x_1, x_0) \rightarrow 0$  as  $m \rightarrow \infty$  for any  $p$  since  $0 < s\lambda < 1$ . Making full use of Lemma 1.14, we find  $m_0 \in \mathbb{N}$  such that

$$\left[ \frac{s\lambda^m}{1 - s\lambda} \right] d(x_1, x_0) \ll \varepsilon$$

for each  $m > m_0$ . Thus

$$d(x_m, x_{m+p}) \leq \left[ \frac{s\lambda^m}{1 - s\lambda} \right] d(x_1, x_0) \ll \varepsilon$$

for all  $m \geq 1, p \geq 1$ . So, by Lemma 1.15,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is a complete cone  $b$ -metric space, there exists  $u \in X$  such that  $x_n \rightarrow u$  as  $n \rightarrow \infty$ . Take  $n_0 \in \mathbb{N}$  such that  $d(x_n, u) \ll \frac{\varepsilon(1-s(\beta+\mu))}{s(\alpha+\mu+1)}$  for all  $n > n_0$ . Hence,

$$\begin{aligned}
d(Tu, u) &\leq s[d(Tu, Tx_n) + d(Tx_n, u)] \\
&= sd(Tu, Tx_n) + sd(Tx_n, u) \\
&\leq s\left\{ \alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, Tx_n) \right. \\
&\quad \left. + \mu[d(u, Tx_n) + d(x_n, Tu)] \right\} + sd(x_{n+1}, u) \\
&= s\left\{ \alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, x_{n+1}) \right. \\
&\quad \left. + \mu[d(u, x_{n+1}) + d(x_n, Tu)] \right\} + sd(x_{n+1}, u) \\
&= s(\alpha + \mu + 1)d(x_n, u) + s(\beta + \mu)d(Tu, u). \tag{2.7}
\end{aligned}$$

This implies that

$$d(Tu, u) \leq \left( \frac{s(\alpha + \mu + 1)}{1 - s(\beta + \mu)} \right) \ll \varepsilon,$$

for each  $n > n_0$ . Then, by Lemma 1.16, we deduce that  $d(Tu, u) = 0$ , that is,  $Tu = u$ . Thus  $u$  is a fixed point of  $T$ .

Now, we show that the fixed point is unique. If there is another fixed point  $u^*$  of  $T$  such

that  $Tu^* = u^*$ , then from (2.1), we have

$$\begin{aligned}
 d(u, u^*) &= d(Tu, Tu^*) \\
 &\leq \alpha d(u, u^*) + \beta d(u, Tu) + \gamma d(u^*, Tu^*) \\
 &\quad + \mu [d(u, Tu^*) + d(u^*, Tu)] \\
 &\leq \alpha d(u, u^*) + \beta d(u, u) + \gamma d(u^*, u^*) \\
 &\quad + \mu [d(u, u^*) + d(u^*, u)] \\
 &= (\alpha + 2\mu) d(u, u^*) \\
 &\leq (s\alpha + \beta + s\gamma + (s^2 + s)\mu) d(u, u^*).
 \end{aligned}$$

By Lemma 1.17, we have  $u = u^*$ . This completes the proof.  $\square$

**Remark 2.2** Theorem 2.1 extends Theorem 2.1 of Huang and Xu in [13] to the case of weaker contractive condition considered in this paper.

From Theorem 2.1, we obtain the following result as corollaries.

**Corollary 2.3** *Let  $(X, d)$  be a complete cone b-metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the contractive condition:*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

*for all  $x, y \in X$ , where  $\alpha \in [0, \frac{1}{s})$  is a constant. Then  $T$  has a unique fixed point in  $X$ .*

*Proof* The proof of Corollary 2.3 is immediately follows from Theorem 2.1 by taking  $\beta = \gamma = \mu = 0$ . This completes the proof.  $\square$

**Corollary 2.4** *Let  $(X, d)$  be a complete cone b-metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the contractive condition:*

$$d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)]$$

*for all  $x, y \in X$ , where  $\beta \in [0, \frac{1}{1+s})$  is a constant. Then  $T$  has a unique fixed point in  $X$ .*

*Proof* The proof of Corollary 2.4 is immediately follows from Theorem 2.1 by taking  $\alpha = \mu = 0$  and  $\beta = \gamma$ . This completes the proof.  $\square$

**Corollary 2.5** *Let  $(X, d)$  be a complete cone b-metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the contractive condition:*

$$d(Tx, Ty) \leq \mu [d(x, Ty) + d(y, Tx)]$$

*for all  $x, y \in X$ , where  $\mu \in [0, \frac{1}{s+s^2})$  is a constant. Then  $T$  has a unique fixed point in  $X$ .*

*Proof* The proof of Corollary 2.5 is immediately follows from Theorem 2.1 by taking

$\alpha = \beta = \gamma = 0$ . This completes the proof.  $\square$

**Remark 2.6** Corollaries 2.3, 2.4 and 2.5 extend Theorem 1, 3 and 4 of Huang and Zhang [14] to the case of cone  $b$ -metric space without normal constant considered in this paper.

**Remark 2.7** Corollary 2.3 also extends the well known Banach contraction principle [2] to that in the setting of cone  $b$ -metric spaces.

**Remark 2.8** Corollary 2.4 also extends the Kannan contraction [18] to that in the setting of cone  $b$ -metric spaces.

**Remark 2.9** Corollary 2.5 also extends the Chatterjea contraction [7] to that in the setting of cone  $b$ -metric spaces.

**Remark 2.10** Theorem 2.1 also extends several results from the existing literature to the case of weaker contractive condition considered in this paper in the setting of cone  $b$ -metric spaces.

**Theorem 2.11** *Let  $(X, d)$  be a complete cone  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the contractive type condition:*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx) \quad (2.8)$$

for all  $x, y \in X$  and  $\alpha, \beta, \gamma \geq 0$  are constants such that  $s\alpha + s(1+s)\gamma < 1$ . Then  $T$  has a unique fixed point in  $X$ .

*Proof* Choose  $x_0 \in X$ . We construct the iterative sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$ ,  $n \geq 1$ , that is,  $x_{n+1} = Tx_n = T^{n+1}x_0$ . From (2.8), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, Tx_{n-1}) + \gamma d(x_{n-1}, Tx_n) \\ &= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_n) + \gamma d(x_{n-1}, x_{n+1}) \\ &= \alpha d(x_n, x_{n-1}) + \gamma d(x_{n-1}, x_{n+1}) \\ &\leq \alpha d(x_n, x_{n-1}) + s\gamma [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= (\alpha + s\gamma) d(x_n, x_{n-1}) + s\gamma d(x_n, x_{n+1}). \end{aligned} \quad (2.9)$$

This implies that

$$d(x_{n+1}, x_n) \leq \left( \frac{\alpha + s\gamma}{1 - s\gamma} \right) d(x_n, x_{n-1}) = \rho d(x_n, x_{n-1}), \quad (2.11)$$

where

$$\rho = \left( \frac{\alpha + s\gamma}{1 - s\gamma} \right).$$

As  $s\alpha + s(s+1)\gamma < 1$ , it is clear that  $\rho < 1/s$ .

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq \rho d(x_{n-2}, x_{n-1}). \quad (2.11)$$

Using (2.11) in (2.10), we get

$$d(x_{n+1}, x_n) \leq \rho^2 d(x_{n-1}, x_{n-2}). \quad (2.12)$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq \rho^n d(x_1, x_0). \quad (2.13)$$

Let  $m, n \geq 1$  and  $m > n$ , we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1}d(x_{n+m-1}, x_m) \\ &\leq s\rho^n d(x_1, x_0) + s^2\rho^{n+1}d(x_1, x_0) + s^3\rho^{n+2}d(x_1, x_0) \\ &\quad + \cdots + s^m\rho^{n+m-1}d(x_1, x_0) \\ &= s\rho^n[1 + s\rho + s^2\rho^2 + s^3\rho^3 + \cdots + (s\rho)^{m-1}]d(x_1, x_0) \\ &\leq \left[ \frac{s\rho^n}{1 - s\rho} \right] d(x_1, x_0). \end{aligned}$$

Let  $0 \ll \varepsilon_1$  be given. Notice that  $\left[ \frac{s\rho^n}{1 - s\rho} \right] d(x_1, x_0) \rightarrow 0$  as  $n \rightarrow \infty$  since  $0 < s\rho < 1$ . Making full use of Lemma 1.14, we find  $n_0 \in \mathbb{N}$  such that

$$\left[ \frac{s\rho^n}{1 - s\rho} \right] d(x_1, x_0) \ll \varepsilon_1$$

for each  $n > n_0$ . Thus

$$d(x_n, x_m) \leq \left[ \frac{s\rho^n}{1 - s\rho} \right] d(x_1, x_0) \ll \varepsilon_1$$

for all  $n, m \geq 1$ . So, by Lemma 1.15,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is a complete cone  $b$ -metric space, there exists  $v \in X$  such that  $x_n \rightarrow v$  as  $n \rightarrow \infty$ . Take  $n_1 \in \mathbb{N}$  such that  $d(x_n, v) \ll \frac{\varepsilon_1(1-s\gamma)}{s(\alpha+1)}$  for all  $n > n_1$ . Hence,

$$\begin{aligned} d(Tv, v) &\leq s[d(Tv, Tx_n) + d(Tx_n, v)] \\ &= sd(Tv, Tx_n) + sd(Tx_n, v) \\ &\leq s[\alpha d(v, x_n) + \beta d(v, Tx_n) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\ &= s[\alpha d(v, x_n) + \beta d(v, x_{n+1}) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\ &= s(\alpha + 1)d(v, x_n) + s\gamma d(Tv, v). \end{aligned} \quad (2.14)$$

This implies that

$$d(Tv, v) \leq \left( \frac{s(\alpha + 1)}{1 - s\gamma} \right) d(x_n, v) \ll \varepsilon_1,$$

for each  $n > n_1$ . Then, by Lemma 1.16, we deduce that  $d(Tv, v) = 0$ , that is,  $Tv = v$ . Thus  $v$  is a fixed point of  $T$ .

Now, we show that the fixed point is unique. If there is another fixed point  $v^*$  of  $T$  such that  $Tv^* = v^*$ , then from (2.8), we have

$$\begin{aligned} d(v, v^*) &= d(Tv, Tv^*) \\ &\leq \alpha d(v, v^*) + \beta d(v, Tv^*) + \gamma d(v^*, Tv) \\ &= \alpha d(v, v^*) + \beta d(v, v^*) + \gamma d(v^*, v) \\ &= (\alpha + \beta + \gamma) d(v, v^*) \\ &\leq (s\alpha + s(1 + s)\gamma) d(v, v^*). \end{aligned}$$

By Lemma 1.17, we have  $v = v^*$ . This completes the proof.  $\square$

**Theorem 2.12** *Let  $(X, d)$  be a complete cone b-metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the following contractive condition: there exists*

$$u(x, y) \in \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2s}, \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}$$

such that

$$d(Tx, Ty) \leq k u(x, y), \quad (2.15)$$

for all  $x, y \in X$ , where  $k \in [0, 1)$  is a constant with  $ks < 1$ . Then  $T$  has a unique fixed point in  $X$ .

*Proof* Choose  $x_0 \in X$ . We construct the iterative sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$ ,  $n \geq 1$ , that is,  $x_{n+1} = Tx_n = T^{n+1}x_0$ . From (2.15), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq k u(x_n, x_{n-1}) \leq \cdots \leq k^n u(x_1, x_0). \end{aligned} \quad (2.16)$$

Let  $m, n \geq 1$  and  $m > n$ , we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1}d(x_{n+m-1}, x_m) \end{aligned}$$

$$\begin{aligned}
&\leq sk^n u(x_1, x_0) + s^2 k^{n+1} u(x_1, x_0) + s^3 k^{n+2} u(x_1, x_0) \\
&\quad + \cdots + s^m k^{n+m-1} u(x_1, x_0) \\
&= sk^n [1 + sk + s^2 k^2 + s^3 k^3 + \cdots + (sk)^{m-1}] u(x_1, x_0) \\
&\leq \left[ \frac{sk^n}{1 - sk} \right] u(x_1, x_0).
\end{aligned}$$

Let  $0 \ll r$  be given. Notice that

$$\left[ \frac{sk^n}{1 - sk} \right] u(x_1, x_0) \rightarrow 0$$

as  $n \rightarrow \infty$  since  $0 < k < 1$ . Making full use of Lemma 1.14, we find  $n_0 \in \mathbb{N}$  such that

$$\left[ \frac{sk^n}{1 - sk} \right] u(x_1, x_0) \ll r$$

for each  $n > n_0$ . Thus

$$d(x_n, x_m) \leq \left[ \frac{sk^n}{1 - sk} \right] u(x_1, x_0) \ll r$$

for all  $n, m \geq 1$ . So, by Lemma 1.15,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is a complete cone  $b$ -metric space, there exists  $p \in X$  such that  $x_n \rightarrow p$  as  $n \rightarrow \infty$ . Take  $n_1 \in \mathbb{N}$  such that  $d(x_n, p) \ll \frac{r}{s(k+1)}$  for all  $n > n_1$ . Hence,

$$\begin{aligned}
d(Tp, p) &\leq s[d(Tp, Tx_n) + d(Tx_n, p)] \\
&= sd(Tp, Tx_n) + sd(Tx_n, p) \\
&\leq sk u(p, x_n) + s d(x_{n+1}, p) \\
&\leq sk d(p, x_n) + s d(x_n, p) \\
&= s(k+1) d(x_n, p).
\end{aligned}$$

This implies that

$$d(Tp, p) \ll r,$$

for each  $n > n_1$ . Then, by Lemma 1.16, we deduce that  $d(Tp, p) = 0$ , that is,  $Tp = p$ . Thus  $p$  is a fixed point of  $T$ .

Now, we show that the fixed point is unique. If there is another fixed point  $q$  of  $T$  such that  $Tq = q$ , then by the given condition (2.15), we have

$$d(p, q) = d(Tp, Tq) \leq k u(p, q) = k d(p, q).$$

By Lemma 1.17, we have  $p = q$ . This completes the proof.  $\square$

**Theorem 2.13** *Let  $(X, d)$  be a complete cone  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies the following contractive condition:*

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty)\} \quad (2.17)$$



for all  $x, y \in X$ , where  $h \in [0, 1)$  is a constant with  $sh < 1$ . Then  $T$  has a unique fixed point in  $X$ .

*Proof* Choose  $x_0 \in X$ . We construct the iterative sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$ ,  $n \geq 1$ , that is,  $x_{n+1} = Tx_n = T^{n+1}x_0$ . From (2.17), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq h \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &= h \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\leq h d(x_n, x_{n-1}). \end{aligned} \quad (2.18)$$

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq h d(x_{n-2}, x_{n-1}). \quad (2.19)$$

Using (2.19) in (2.18), we get

$$d(x_{n+1}, x_n) \leq h^2 d(x_{n-1}, x_{n-2}). \quad (2.20)$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq h^n d(x_1, x_0). \quad (2.21)$$

Let  $m, n \geq 1$  and  $m > n$ , we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1}d(x_{n+m-1}, x_m) \\ &\leq sh^n d(x_1, x_0) + s^2h^{n+1}d(x_1, x_0) + s^3h^{n+2}d(x_1, x_0) \\ &\quad + \cdots + s^mh^{n+m-1}d(x_1, x_0) \\ &= sh^n[1 + sh + s^2h^2 + s^3h^3 + \cdots + (sh)^{m-1}]d(x_1, x_0) \\ &\leq \left[ \frac{sh^n}{1 - sh} \right] d(x_1, x_0). \end{aligned}$$

Let  $0 \ll c$  be given. Notice that

$$\left[ \frac{sh^n}{1 - sh} \right] d(x_1, x_0) \rightarrow 0$$

as  $n \rightarrow \infty$  since  $0 < h < 1$ . Making full use of Lemma 1.14, we find  $N_0 \in \mathbb{N}$  such that

$$\left[ \frac{sh^n}{1 - sh} \right] d(x_1, x_0) \ll c$$

for each  $n > N_0$ . Thus

$$d(x_n, x_m) \leq \left[ \frac{sh^n}{1-sh} \right] d(x_1, x_0) \ll c$$

for all  $n, m \geq 1$ . So, by Lemma 1.15,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is a complete cone  $b$ -metric space, there exists  $q \in X$  such that  $x_n \rightarrow q$  as  $n \rightarrow \infty$ . Take  $N_1 \in \mathbb{N}$  such that  $d(x_n, q) \ll \frac{c}{s(h+1)}$  for all  $n > N_1$ . Hence,

$$\begin{aligned} d(Tq, q) &\leq s[d(Tq, Tx_n) + d(Tx_n, q)] \\ &= sd(Tq, Tx_n) + sd(Tx_n, q) \\ &\leq sh \max\{d(q, x_n), d(x_n, Tx_n), d(q, Tq)\} + s d(x_{n+1}, q) \\ &= sh \max\{d(q, x_n), d(x_n, x_{n+1}), d(q, Tq)\} + s d(x_{n+1}, q) \\ &\leq sh d(q, x_n) + s d(x_n, q) \\ &= s(h+1) d(x_n, q). \end{aligned}$$

This implies that

$$d(Tq, q) \ll c,$$

for each  $n > N_1$ . Then, by Lemma 1.16, we deduce that  $d(Tq, q) = 0$ , that is,  $Tq = q$ . Thus  $q$  is a fixed point of  $T$ .

Now, we show that the fixed point is unique. If there is another fixed point  $q'$  of  $T$  such that  $Tq' = q'$ , then by the given condition (2.17), we have

$$\begin{aligned} d(q, q') &= d(Tq, Tq') \\ &\leq h \max\{d(q, q'), d(q, Tq), d(q', Tq')\} \\ &= h \max\{d(q, q'), d(q, q), d(q', q')\} \\ &= h \max\{d(q, q'), 0, 0\} \\ &\leq h d(q, q') \end{aligned}$$

By Lemma 1.17, we have  $q = q'$ . This completes the proof.  $\square$

**Example 2.14**([13]) Let  $X = [0, 1]$ ,  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset E$  and  $d: X \times X \rightarrow E$  defined by  $d(x, y) = (|x-y|^p, |x-y|^p)$  for all  $x, y \in X$  where  $p > 1$  is a constant. Then  $(X, d)$  is a complete cone  $b$ -metric space. Let us define  $T: X \rightarrow X$  as  $T(x) = \frac{1}{2}x - \frac{1}{4}x^2$  for all  $x \in X$ . Therefore,

$$\begin{aligned} d(Tx, Ty) &= (|Tx - Ty|^p, |Tx - Ty|^p) \\ &= \left( \left| \frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y) \right|^p, \left| \frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y) \right|^p \right) \\ &= \left( |x-y|^p \cdot \left| \frac{1}{2} - \frac{1}{4}(x+y) \right|^p, |x-y|^p \cdot \left| \frac{1}{2} - \frac{1}{4}(x+y) \right|^p \right) \\ &\leq \frac{1}{2^p} (|x-y|^p, |x-y|^p) = \frac{1}{2^p} d(x, y). \end{aligned}$$

Hence  $0 \in X$  is the unique fixed point of  $T$ .

Other consequence of our result for the mapping involving contraction of integral type is the following.

Denote  $\Lambda$  the set of functions  $\varphi: [0, \infty) \rightarrow [0, \infty)$  satisfying the following hypothesis:

( $h_1$ )  $\varphi$  is a Lebesgue-integrable mapping on each compact subset of  $[0, \infty)$ ;

( $h_2$ ) for any  $\varepsilon > 0$  we have  $\int_0^\infty \varphi(t)dt > 0$ .

**Theorem 2.15** *Let  $(X, d)$  be a complete cone  $b$ -metric space (CCbMS) with the coefficient  $s \geq 1$ . Suppose that the mapping  $T: X \rightarrow X$  satisfies:*

$$\int_0^{d(Tx, Ty)} \psi(t)dt \leq \beta \int_0^{d(x, y)} \psi(t)dt$$

for all  $x, y \in X$ , where  $\beta \in [0, 1)$  is a constant with  $s\beta < 1$  and  $\psi \in \Lambda$ . Then  $T$  has a unique fixed point in  $X$ .

**Remark 2.16** Theorem 2.15 extends Theorem 2.1 of Branciari [6] from complete metric space to that setting of complete cone  $b$ -metric space considered in this paper.

### §3. Applications

In this section we shall apply Theorem 2.1 to the first order differential equation.

**Example 3.1**  $X = C([1, 3], \mathbb{R})$ ,  $E = \mathbb{R}^2$ ,  $\alpha > 0$  and

$$d(x, y) = \left( \sup_{t \in [1, 3]} |x(t) - y(t)|^2, \alpha \sup_{t \in [1, 3]} |x(t) - y(t)|^2 \right)$$

for every  $x, y \in X$ , and  $P = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0\}$ . Then  $(X, d)$  is a cone  $b$ -metric space. Define  $T: X \rightarrow X$  by

$$T(x(t)) = 4 + \int_1^t (x(u) + u^2) e^{u-5} du.$$

For  $x, y \in X$ ,

$$\begin{aligned} d(Tx, Ty) &= \left( \sup_{t \in [1, 3]} |T(x(t)) - T(y(t))|^2, \alpha \sup_{t \in [1, 3]} |T(x(t)) - T(y(t))|^2 \right) \\ &\leq \left( \int_1^3 |(x(u) - y(u))|^2 e^{-2} du, \alpha \int_1^3 |(x(u) - y(u))|^2 e^{-2} du \right) \\ &= 2e^{-2} d(x, y) \\ &\leq 2e^{-1} d(x, y). \end{aligned}$$

Thus for  $\alpha = \frac{2}{e} < 1$ ,  $\beta = \gamma = \mu = 0$ , all conditions of Theorem 2.1 are satisfied and so  $T$

has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t (x(u) + u^2) e^{u-5} du,$$

or the differential equation:

$$x'(t) = (x(t) + t^2) e^{t-5}, \quad t \in [1, 3], \quad x(1) = 4.$$

Hence, the use of Theorem 2.1 is a delightful way of showing the existence and uniqueness of solutions for the following class of integral equations:

$$q + \int_p^t K(x(u), u) du = x(t) \in C([p, q], \mathbb{R}^n).$$

Now, we shall apply Corollary 2.3 to the first order periodic boundary problem

$$\begin{cases} \frac{dx}{dt} = F(t, x(t)), \\ x(0) = \mu, \end{cases} \quad (3.1)$$

where  $F: [-h, h] \times [\mu - \theta, \mu + \theta]$  is a continuous function.

**Example 3.2**([13]) Consider the boundary problem (3.1) with the continuous function  $F$  and suppose  $F(x, y)$  satisfies the local Lipschitz condition, i.e., if  $|x| \leq h$ ,  $y_1, y_2 \in [\mu - \theta, \mu + \theta]$ , it induces

$$|F(x, y_1) - F(x, y_2)| \leq L |y_1 - y_2|.$$

Set  $M = \max_{[-h, h] \times [\mu - \theta, \mu + \theta]} |F(x, y)|$  such that  $h^2 < \min\{\theta/M^2, 1/L^2\}$ , then there exists a unique solution of (3.1).

*Proof* Let  $X = E = C([-h, h])$  and  $P = \{u \in E : u \geq 0\}$ . Put  $d: X \times X \rightarrow E$  as  $d(x, y) = f(t) \max_{-h \leq t \leq h} |x(t) - y(t)|^2$  with  $f: [-h, h] \rightarrow \mathbb{R}$  such that  $f(t) = e^t$ . It is clear that  $(X, d)$  is a complete cone  $b$ -metric space.

Note that (3.1) is equivalent to the integral equation

$$x(t) = \mu + \int_0^t F(u, x(u)) du.$$

Define a mapping  $T: C([-h, h]) \rightarrow \mathbb{R}$  by  $x(t) = \mu + \int_0^t F(u, x(u)) du$ . If

$$x(t), y(t) \in B(\mu, f\theta) = \{\varphi(t) \in C([-h, h]) : d(\mu, \varphi) \leq f\theta\},$$

then from

$$\begin{aligned}
d(Tx, Ty) &= f(t) \max_{-h \leq t \leq h} \left| \int_0^t F(u, x(u)) du - \int_0^t F(u, y(u)) du \right|^2 \\
&= f(t) \max_{-h \leq t \leq h} \left| \int_0^t [F(u, x(u)) - F(u, y(u))] du \right|^2 \\
&\leq h^2 f(t) \max_{-h \leq t \leq h} |F(u, x(u)) - F(u, y(u))|^2 \\
&\leq h^2 L^2 f(t) \max_{-h \leq t \leq h} |x(u) - y(u)|^2 = h^2 L^2 d(x, y),
\end{aligned}$$

and

$$\begin{aligned}
d(Tx, \mu) &= f(t) \max_{-h \leq t \leq h} \left| \int_0^t F(u, x(u)) du \right|^2 \\
&\leq h^2 f \max_{-h \leq t \leq h} |F(u, x(u))|^2 \leq h^2 M^2 f \leq f\theta,
\end{aligned}$$

we speculate  $T: B(\mu, f\theta) \rightarrow B(\mu, f\theta)$  is a contraction mapping.

Lastly, we prove that  $(B(\mu, f\theta), d)$  is complete. In fact, suppose  $\{x_n\}$  is a Cauchy sequence in  $B(\mu, f\theta)$ . Then  $\{x_n\}$  is a Cauchy sequence in  $X$ . Since  $(X, d)$  is complete, there is  $q \in X$  such that  $x_n \rightarrow q$  ( $n \rightarrow \infty$ ). So, for each  $c \in \text{int } P$ , there exists  $N$ , whenever  $n > N$ , we obtain  $d(x_n, q) \ll c$ . Thus, it follows from

$$d(\mu, q) \leq d(x_n, \mu) + d(\mu, q) \leq f\theta + c$$

and Lemma 1.18 that  $d(\mu, q) \leq f\theta$ , which means  $q \in B(\mu, f\theta)$ , that is,  $(B(\mu, f\theta), d)$  is complete. Thus, from the above statement, all the conditions of Corollary 2.3 are satisfied. Hence  $T$  has a unique fixed point  $x(t) \in B(\mu, f\theta)$  or we say that, there exists a unique solution of (3.1).  $\square$

## References

- [1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, **341** (2008), 416-420.
- [2] S. Banach, Sur les operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.*, **3** (1922), 133-181.
- [3] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal. Gos. Ped. Inst. Unianowsk*, **30** (1989), 26-37.
- [4] M. Boriceanu, M. Bota and A. Petrusel, A multivalued fractals in  $b$ -metric spaces, *Cent. Eur. J. Math.*, **8(2)** (2010), 367-377.
- [5] M. Bota, A. Moinar and V. Csaba, On Ekeland's variational principle in  $b$ -metric spaces, *Fixed Point Theory*, **12** (2011), 21-28.
- [6] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, **29** (2002), 531-536.
- [7] S. K. Chatterjaj, Fixed point theorems compacts, *Rend. Acad. Bulgare Sci.*, **25** (1972), 727-730.

- [8] S-H Cho and J-S Bae, Common fixed point theorems for mappings satisfying property (E.A) on cone metric space, *Math. Comput. Model.*, **53** (2011), 945-951.
- [9] L. B. Ćirić, Generalized contractions and fixed point theorems, *Publications de l'Institut Mathématique*, Nouvelle Serie, **12(26)** (1971), 19-26.
- [10] L. B. Ćirić, A generalization of Banach principle, *Proc. Amer. Math. Soc.*, **45** (1974), 727-730.
- [11] S. Czerwik, Nonlinear set-valued contraction mappings in  $b$ -metric spaces, *Atti. Semin. Mat. Fis. Univ. Modena*, **46** (1998), 263-276.
- [12] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, (1985).
- [13] H. Huang and S. Xu, Fixed point theorems of contractive mappings in cone  $b$ -metric spaces and applications, *Fixed Point Theory Appl.*, **112(2013)**.
- [14] L. -G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332(2)** (2007), 1468-1476.
- [15] N. Hussain and MH. Shah, KKM mappings in cone  $b$ -metric spaces, *Comput. Math. Appl.*, **62** (2011), 1677-1684.
- [16] D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.*, **341** (2008), 876-882.
- [17] S. Janković, Z. Kadelburg and S. Radenović, On cone metric spaces: A survey, *Nonlinear Anal.*, **4(7)** (2011), 2591-2601.
- [18] R. Kannan, Some results on fixed point theorems, *Bull. Calcutta Math. Soc.*, **60** (1969), 71-78.
- [19] Sh. Rezapour, A review on topological properties of cone metric spaces, in *Proceedings of the International Conference on Analysis, Topology and Appl.*, (ATA 08), Vrinjacka Banja, Serbia, May-June (2008).
- [20] Sh. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.*, **345(2)** (2008), 719-724.
- [21] B. E. Rhoades, Two fixed point theorems for mapping satisfying a general contractive type condition of integral type, *Int. J. Math. Math. Sci.*, **63** (2003), 4007-4013.
- [22] J. Vandergraft, Newton method for convex operators in partially ordered spaces, *SIAM J. Numer. Anal.*, **4** (1967), 406-432.
- [23] P. Vetro, Common fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo*, (2) **56(3)** (2007), 464-468.
- [24] P. P. Zabrejko,  $K$ -metric and  $K$ -normed linear spaces: survey, *Collectanea Mathematica*, **48(4-6)** (1997), 825-859.

## A Note on Slant and Hemislant Submanifolds of an $(\epsilon)$ -Para Sasakian Manifold

Barnali Laha and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata, India)

E-mail: barnali.laha87@gmail.com, bhattachar1968@yahoo.co.in

**Abstract:** In the present note we have derived some basic results pertaining to the geometry of slant and hemislant submanifolds of an  $(\epsilon)$ -para Sasakian manifold. In particular, we have obtained few results on a totally umbilical slant submanifold of an  $(\epsilon)$ -para Sasakian manifold. In the next section, we have obtained the integrability conditions of the involved distributions of hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold. Finally we have verified the theorems by providing an example of three dimensional hemislant submanifold of  $(\epsilon)$ -para Sasakian manifold.

**Key Words:**  $(\epsilon)$ -Para Sasakian manifold, totally umbilical submanifold, slant submanifold, hemislant submanifold.

**AMS(2010):** 53B25, 53C15, 53C40, 53C42.

### §1. Introduction

Connected almost contact metric manifold was classified by S.Tanno [13], as those automorphism group has maximum dimension. He has given following classifications:

- (i) Homogeneous normal contact Riemannian manifolds with constant  $\phi$  holomorphic sectional curvature if the sectional curvature of the plane section containing  $\xi$ , say  $K(X, \xi) > 0$ ;
- (ii) Global Riemannian product of a line (or a circle) and a Kaehlerian manifold with constant holomorphic sectional curvature, if  $K(X, \xi) = 0$ ;
- (iii) A warped product space  $RX_f C_n$ , if  $K(X, \xi) < 0$ .

Manifold of class (i) has Sasakian structure. The manifold of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. The manifold of class (iii) are characterized by some tensorial equations, attaining a Kenmotsu structure.

An almost paracontact structure  $(\phi, \xi, \eta)$  satisfying  $\phi^2 = I - \eta \otimes \xi$  and  $\eta(\xi) = 1$  on a differentiable manifold was introduced by Sato [11] in 1976. After him Takahashi [14] in 1969, gave the notion of almost contact manifold equipped with an associated pseudo-Riemannian metric. Later on, motivated by these circumstances, M.M.Tripathi et.al.([15]) has drawn a relation between a semi-Riemannian metric ( not necessarily Lorentzian ) and an almost paracontact structure, and he named this indefinite almost paracontact metric structure an  $(\epsilon)$ -

---

<sup>1</sup>Received August 2, 2017, Accepted February 10, 2018.

almost paracontact structure, where the structure vector field  $\xi$  will be spacelike or timelike according as  $\epsilon = 1$  or  $\epsilon = -1$ . Authors have discussed  $(\epsilon)$ - almost paracontact manifolds and in particular  $(\epsilon)$ - Sasakian manifolds in ([15]).

On the other hand, the study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([4]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [8]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [10]. These submanifolds are a generalized version of CR-submanifolds. After him, J.L.Cabrerizo et.al. ([2]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [3]. Recently, B.Sahin extended the study of pseudo-slant submanifolds in Kaehler setting for their warped product. Totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [12].

This paper contains the analysis about slant and pseudo-slant submanifolds of an  $(\epsilon)$ -para Sasakian manifold. Section (1) is introductory. Section (2) gives us a view of  $(\epsilon)$ -para Sasakian manifold. In section (3) we have obtained some results on a totally umbilical proper slant submanifold  $M$  of an  $(\epsilon)$ -para Sasakian manifold. Finally, in section (4) we have derived some conditions for the integrability of the distributions on the hemislant submanifolds of an  $(\epsilon)$ -para Sasakian manifold.

## §2. Preliminaries

Let  $\tilde{M}$  be an  $n$ -dimensional almost paracontact manifold [11] endowed with an almost paracontact structure  $(\phi, \xi, \eta)$  consisting of a tensor field  $\phi$  of type  $(1, 1)$ , a structure vector field  $\xi$  and 1-form  $\eta$  satisfying:

$$\phi^2 = I - \eta \otimes \xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

$$\phi(\xi) = 0 \quad (2.3)$$

and

$$\eta \circ \phi = 0 \quad (2.4)$$

for any vector field  $X, Y \in \tilde{M}$ . A semi-Riemannian metric [9] on a manifold  $\tilde{M}$ , is a non-degenerate symmetric tensor field  $g$  of type  $(0, 2)$ . If this metric is of index 1 then it is called Lorentzian metric ([1]). Let  $g$  be semi-Riemannian metric with index 1 in an  $n$ -dimensional almost paracontact manifold  $\tilde{M}$  such that,

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon \eta(X) \eta(Y), \quad (2.5)$$

where  $\epsilon = +1$  or  $-1$ . Then  $\tilde{M}$  is called an almost paracontact metric manifold associated with an  $(\epsilon)$ -almost paracontact metric structure  $(\phi, \xi, \eta, g, \epsilon)$  ([?]). In case, if index  $(g) = 1$  then an



$(\epsilon)$ -almost paracontact metric manifold is defined as a Lorentzian paracontact manifold and if the metric is positive definite, then an  $(\epsilon)$ -almost paracontact metric manifold is the usual almost paracontact metric manifold [?].

The condition (2.5) is equivalent to

$$g(X, \phi Y) = g(\phi X, Y) \quad (2.6)$$

equipped with

$$g(X, \xi) = \epsilon \eta(X). \quad (2.7)$$

From (2.7), it can be easily observed that

$$g(\xi, \xi) = \epsilon, \quad (2.8)$$

i.e. structure vector field  $\xi$  is never lightlike. We define

$$\Phi(X, Y) = g(X, \phi Y) \quad (2.9)$$

and we can obtain

$$\Phi(X, \xi) = 0. \quad (2.10)$$

From (2.9), we can also calculate

$$(\nabla_X \Phi)(Y, Z) = g(\nabla_X \phi)(Y, Z) = (\nabla_X \Phi)(Z, Y). \quad (2.11)$$

An  $(\epsilon)$ -almost paracontact metric manifold  $\tilde{M}$  satisfying

$$2\Phi(X, Y) = (\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) \quad (2.12)$$

$\forall X, Y \in TM$ , then  $\tilde{M}$  is called an  $(\epsilon)$ -paracontact metric manifold ([15]).

An  $(\epsilon)$ -almost paracontact metric structure  $(\phi, \xi, \eta, g, \epsilon)$  is called an  $(\epsilon)$ - $S$ -paracontact metric structure if

$$\tilde{\nabla}_X \xi = \epsilon \phi X \quad (2.13)$$

for  $\forall X \in T\tilde{M}$ . A manifold endowed with an  $(\epsilon)$ - $S$ -paracontact metric structure is called an  $(\epsilon)$ - $S$ -paracontact metric manifold. Equation (2.13) can be written as

$$\Phi(X, Y) = g(\phi X, Y) = \epsilon g(\tilde{\nabla}_X \xi, Y) = (\nabla_X \eta)(Y) \quad (2.14)$$

for  $\forall X, Y \in TM$ .

An  $(\epsilon)$ -almost paracontact metric structure is called an  $(\epsilon)$ -para Sasakian structure if the following relation holds

$$(\tilde{\nabla}_X \phi)(Y) = -g(\phi X, \phi Y)\xi - \epsilon \eta(Y)\phi^2 X, \quad (2.15)$$

where  $\tilde{\nabla}$  is the Levi-Civita connection with respect to  $g$  on  $\tilde{M}$ . A manifold equipped with an

$(\epsilon)$ -para Sasakian structure is called an  $(\epsilon)$ -para Sasakian manifold.

From the definition of contact CR-submanifolds of an  $(\epsilon)$ -paracontact Sasakian manifold we have

**Definition 2.1**([7]) *An  $n$ -dimensional Riemannian submanifold  $M$  of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  is called a contact CR-submanifold if*

(i)  $\xi$  is tangent to  $M$ ;

(ii) *there exists on  $M$  a differentiable distribution  $D : x \longrightarrow D_x \subset T_x(M)$ , such that  $D_x$  is invariant under  $\phi$ ; i.e.,  $\phi D_x \subset D_x$ , for each  $x \in M$  and the orthogonal complementary distribution  $D^\perp : x \longrightarrow D_x^\perp \subset T_x^\perp(M)$  of the distribution  $D$  on  $M$  is totally real; i.e.,  $\phi D_x^\perp \subset T_x^\perp(M)$ , where  $T_x(M)$  and  $T_x^\perp(M)$  are the tangent space and the normal space of  $M$  at  $x$ .  $D$  (resp.  $D^\perp$ ) is the horizontal (resp. vertical) distribution. The contact CR-submanifold of an  $(\epsilon)$ -para Sasakian manifold is called  $\xi$ -horizontal (resp.  $\xi$ -vertical) if  $\xi_x \in D_x$  (resp.  $\xi_x \in D_x^\perp$ ) for each  $x \in M$ .*

Let  $TM$  and  $T^\perp M$  be the Lie algebras of vector fields tangential to  $M$  and normal to  $M$  respectively.  $h$  and  $A$  denote the second fundamental form and the shape operator of the immersion of  $M$  into  $\tilde{M}$  respectively. If  $\nabla$  is the induced connection on  $M$ , the Gauss and Weingarten formulae of  $M$  into  $\tilde{M}$  are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.16)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp N \quad (2.17)$$

for any  $X, Y$  in  $TM$  and  $N$  in  $T^\perp M$ .  $\nabla^\perp$  is the connection on normal bundle and  $A_N$  is the Weingarten endomorphism associated with  $N$  by

$$g(A_V X, Y) = g(h(X, Y), V). \quad (2.18)$$

For any  $x \in M$  and  $X \in T_x M$  we decompose it as

$$\phi X = TX + NX, \quad (2.19)$$

where  $TX \in T_x M$  and  $NX \in T_x^\perp M$ .

Similarly for  $V \in T_x^\perp M$  we know

$$\phi V = tV + nV, \quad (2.20)$$

where  $tV$  (resp.  $nV$ ) is vertical (resp. normal) component of  $\phi V$ .

Now, for any  $X, Y \in TM$ , comparing the tangential and normal parts of  $(\tilde{\nabla}_X \phi)Y$  by  $\mathbf{P}_X Y$  and  $\mathbf{Q}_X Y$  respectively. After having some brief calculation, we obtain

$$\mathbf{P}_X Y = (\tilde{\nabla}_X T)Y - A_{NY} X - th(X, Y), \quad (2.21)$$

$$\mathbf{Q}_X Y = (\tilde{\nabla}_X N)Y + h(X, TY) - nh(X, Y) \quad (2.22)$$

for any  $X, Y \in TM$ .

Again for any  $V \in T^\perp M$ , denoting tangential and normal parts of  $(\tilde{\nabla}_X \phi)V$  by  $\mathbf{P}_X V$  and  $\mathbf{Q}_X V$  respectively, we have

$$\mathbf{P}_X V = (\tilde{\nabla}_X t)V - A_{nV}X + TA_V X, \quad (2.23)$$

$$\mathbf{Q}_X V = (\tilde{\nabla}_X n)V + h(tV, X) + NA_V X, \quad (2.24)$$

where the covariant derivatives of  $T, N, t$  and  $n$  are given by

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (2.25)$$

$$(\tilde{\nabla}_X N)Y = \nabla_X^\perp NY - N\nabla_X Y, \quad (2.26)$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla_X^\perp V, \quad (2.27)$$

$$(\tilde{\nabla}_X n)V = \nabla_X^\perp nV - n\nabla_X^\perp V \quad \forall \quad X, Y \in TM, V \in T^\perp M. \quad (2.28)$$

A submanifold  $M$  of an almost contact metric manifold  $\tilde{M}$  is called totally umbilical if

$$h(X, Y) = g(X, Y)H \quad (2.29)$$

for any  $X, Y \in \Gamma(TM)$ , where  $H$  is the mean curvature. A submanifold  $M$  is said to be totally geodesic if  $h(X, Y) = 0$  for each  $X, Y \in \Gamma(TM)$  and is minimal if  $H = 0$  on  $M$ .

### §3. Slant Submanifolds

The slant submanifold of a para contact Lorentzian manifold were first defined by [5]. Hereafter, for a submanifold  $M$  of an almost contact manifold, authors in [6] assumed that the structure vector field  $\xi$  is tangential to the submanifold  $M$ , whence the tangent bundle  $TM$  can be decomposed as

$$(a) TM = D \oplus \langle \xi \rangle,$$

where the orthogonal complementary distribution  $D$  of  $\langle \xi \rangle$  is known as the slant distribution on  $M$  and  $\langle \xi \rangle$  is the 1-dimensional distribution on  $M$  spanned by the structure vector field  $\xi$ , and they also assumed that  $g(X, X) \geq 0 \quad \forall X \in TM \setminus \xi$ . Let  $M$  be an immersed submanifold of  $\tilde{M}$ . For any  $x \in M$  and  $X \in T_x M$ , if the vectors  $X$  and  $\xi$  are linearly independent, then the angle  $\theta(X) \in [0, \pi/2]$  between  $\phi X$  and  $T_x M$  is well defined, if  $\theta(X)$  does not depend on the choice of  $x \in M$  and  $X \in T_x M$ , then  $M$  is slant in  $\tilde{M}$ . The constant angle  $\theta(X)$  is then called the slant angle of  $M$  in  $\tilde{M}$  by [5] and which in short we denote by  $Sl\alpha(M)$ . If  $\mu$  is  $\phi$ -invariant of the normal bundle  $T^\perp M$ , then

$$(b) T^\perp M = FTM \oplus \langle \mu \rangle.$$

Defining the endomorphism  $P : TM \longrightarrow TM$ , whose square  $P^2$  will be denoted by  $Q$ . Then the tensor fields on  $M$  of type  $(1, 1)$  determined by these endomorphism will be denoted by the

same letters, respectively  $P$  and  $Q$ .

It is proved the following theorem in [6]:

*For a proper slant submanifold  $M$  of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  with slant angle  $\theta$ , then*

$$QX = \lambda(X - \eta(X)\xi). \quad (3.1)$$

From this theorem we can state our next theorem,

**Theorem 3.1** *Let  $M$  be a submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  such that  $\xi \in TM$ . Then,  $M$  is slant iff there exists a constant  $\lambda \in [0, 1]$  such that*

$$T^2 = \lambda(I - \eta \otimes \xi). \quad (3.2)$$

*Furthermore, in such case, if  $\theta$  is the slant angle of  $M$ , then  $\lambda = \cos^2\theta$ . Hence for a slant manifold we have*

$$g(TX, TX) = \cos^2\theta(g(X, Y) - \epsilon\eta(X)\eta(Y)), \quad (3.3)$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \epsilon\eta(X)\eta(Y)) \quad (3.4)$$

for  $\forall X, Y \in TM$ .

*Proof* Follows from [5]. □

Assuming  $M$  to be totally umbilical proper slant submanifold of an  $(\epsilon)$ -para Sasakian manifold, we can obtain the following theorem.

**Theorem 3.2** *Let  $M$  be a totally umbilical proper slant submanifold of an  $(\epsilon)$ -paracontact Sasakian manifold  $\tilde{M}$ , then for any  $X \in TM$  following conditions are equivalent:*

- (i)  $H \in \mu$ ;
- (ii)  $g(\nabla_{TX}\xi, X) = \epsilon[||X||^2 - \eta^2(X)]$ .

*Proof* For any  $X \in TM$  we know  $h(X, TX) = g(X, TX)H = 0$ . Then from (2.16) and (2.17) and the structure equation of  $(\epsilon)$ -para Sasakian manifold for any vector field  $X \in TM$ , we calculate

$$\begin{aligned} 0 &= \phi(\nabla_X X + h(X, X)) - \nabla_X TX + A_{NX}X - \nabla_X^\perp NX \\ &\quad - g(\phi X, \phi X)\xi - \epsilon\eta(X)\phi^2 X. \end{aligned} \quad (3.5)$$

After using (2.19), and on comparing the tangential component we obtain

$$\begin{aligned} 0 &= ZT\nabla_X X - \nabla_X TX + th(X, X) + A_{NX}X - g(X, X)\xi \\ &\quad + 2\epsilon\eta^2(X)\xi - \epsilon\eta(X)X. \end{aligned} \quad (3.6)$$

As  $M$  is totally umbilical submanifold then the term  $A_{NX}X$  becomes  $Xg(H, NX)$ , so using

this fact above equation takes the form

$$\begin{aligned} 0 &= T\nabla_X X - \nabla_X TX + th(X, X) + Xg(NX, H) + g(X, X)tH \\ &\quad + g(X, X)\xi - 2\epsilon\eta^2(X)\xi + \epsilon\eta(X)X. \end{aligned} \quad (3.7)$$

If  $H \in \mu$  then from (3.6) we get

$$T\nabla_X X - \nabla_X TX = -\|X\|^2\xi + 2\epsilon\eta(X)[2\eta(X)\xi - X]. \quad (3.8)$$

Taking the inner product in (3.7) by  $\xi$  we obtain

$$g(\nabla_X TX, \xi) = \eta^2(X) - \epsilon\|X\|^2. \quad (3.9)$$

Replacing  $X$  by  $TX$ , we derive

$$g(\nabla_{TX} T^2 X, \xi) = \eta^2(TX) - \epsilon\|TX\|^2. \quad (3.10)$$

Then from equation (3.3) and (3.4) we calculate

$$\cos^2\theta g(X, \nabla_{TX}\xi) - \cos^2\theta \eta(X)g(\xi, \nabla_{TX}\xi) = -\cos^2\theta[\epsilon\|X\|^2 - \eta^2(X)]. \quad (3.11)$$

Therefore we can conclude that

$$g(X, \nabla_{TX}\xi) - \eta(X)g(\xi, \nabla_{TX}\xi) = \epsilon\|X\|^2 - \eta^2(X). \quad (3.12)$$

Now we know that  $g(\xi, \xi) = \epsilon$ . Taking the covariant derivative of this equation with respect to  $TX$  for any  $X \in TM$ , we obtain

$$g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0, \quad (3.13)$$

which implies  $g(\nabla_{TX}\xi, \xi) = 0$ . Hence (3.8) becomes

$$g(X, \nabla_{TX}\xi) = \epsilon\|X\|^2 - \eta^2(X). \quad (3.14)$$

This proves part (ii) of the theorem. If (3.9) holds then equation (3.6) implies  $H \in \mu$ . This proves theorem (3.2).

Now if  $\epsilon\|X\|^2 - \eta^2(X) = 0$ , then from (3.9), we conclude

$$g(X, \nabla_{TX}\xi) = 0. \quad (3.15)$$

Replacing  $X$  by  $TX$  we have by using (3.3), we get

$$g(TX, \nabla_{T^2 X}\xi) = g(\nabla_{\cos^2\theta(X-\eta(X)\xi)}\xi, TX) = 0. \quad (3.16)$$

Then the above equation becomes

$$\cos^2\theta g(\nabla_X \xi, TX) + \cos^2\theta \eta(X)g(\nabla_X \xi, TX) = 0. \quad (3.17)$$

From the structure equation (2.4) we have  $\nabla_X \xi = 0$ . Thus we can write

$$\cos^2\theta g(\nabla_X \xi, TX) = 0. \quad (3.18)$$

Thus from equation (3.10) we get either  $M$  is an anti-invariant submanifold or  $\nabla_X \xi = 0$  i.e.  $\xi$  is a Killing vector field on  $M$  or  $M$  is trivial. If  $\xi$  is not Killing then we can take at least two linearly independent vectors  $X$  and  $TX$  to span  $D_\theta$  i.e. the  $\dim M \geq 3$ .

From above discussion we can conclude the following theorem.

**Theorem 3.3** *Let  $M$  be a totally umbilical slant submanifold of an  $\epsilon$ -para Sasakian manifold  $\tilde{M}$  such that  $\epsilon||X||^2 = \eta^2(X)$  on  $M$  then one of the following statements is true:*

- (i)  $H \in \Gamma(\mu)$ ;
- (ii)  $M$  is an anti-invariant submanifold;
- (iii) If  $M$  is a proper slant submanifold then  $\dim M \geq 3$ ;
- (iv)  $M$  is trivial;
- (v)  $\xi$  is a Killing vector field on  $M$ .

Next we prove

**Theorem 3.4** *A totally umbilical proper slant submanifold  $M$  of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  is totally geodesic if  $\nabla_X^\perp H \in \Gamma(\mu)$  for any  $X \in TM$ .*

*Proof* As  $\tilde{M}$  is an  $(\epsilon)$ -paracontact Sasakian manifold we have

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y. \quad (3.19)$$

From the fact that  $\phi Y = TY + NY$  and  $\tilde{M}$  is an  $(\epsilon)$ -paracontact Sasakian manifold we infer

$$\begin{aligned} \tilde{\nabla}_X TY + \tilde{\nabla}_X NY &= T\nabla_X Y + N\nabla_X Y + \phi h(X, Y) - g(\phi X, \phi Y)\xi \\ &\quad - \epsilon \eta(Y)\phi^2 X. \end{aligned} \quad (3.20)$$

Using (2.25), (2.26) and (2.29) we obtain

$$\begin{aligned} \nabla_X TY + h(X, TY) - A_{NY}X &+ \nabla_X^\perp NY = T\nabla_X Y + N\nabla_X Y \\ &+ g(X, Y)\phi H - g(\phi X, \phi Y)\xi - \epsilon \eta(Y)\phi^2 X. \end{aligned} \quad (3.21)$$

Taking inner product with  $\phi H$  and using the fact that  $H \in \Gamma(\mu)$ , from (2.5) and (2.29) we

get

$$\begin{aligned} g(X, TY)g(H, \phi H) + g(\nabla_X^\perp NY, \phi H) &= g(X, Y)\|H\|^2 \\ &\quad - g(\phi X, \phi Y)g(\phi H, \xi) - \epsilon\eta(Y)g(\phi^2 X, \phi H). \end{aligned} \quad (3.22)$$

Now we consider

$$\tilde{\nabla}_X \phi H = \phi \tilde{\nabla}_X H + (\tilde{\nabla}_X \phi)H. \quad (3.23)$$

Using the covariant derivative of  $\nabla$

$$\begin{aligned} -A_{\phi H}X + \nabla_X^\perp \phi H &= -TA_HX - NA_HX + t\nabla_X^\perp H \\ &\quad + n\nabla_X^\perp H - g(\phi X, \phi H)\xi - \epsilon\eta(H)\phi^2 X. \end{aligned} \quad (3.24)$$

Taking inner product with  $NY$ , for any  $Y \in \Gamma(TM)$  and as the submanifold considered is always tangent to  $\xi$  we obtain

$$g(\nabla_X^\perp \phi H, NY) = -g(NA_HX, NY) + g(n\nabla_X^\perp H, NY) - \epsilon\eta(H)g(\phi^2 X, NY). \quad (3.25)$$

Since  $n\nabla_X^\perp H \in \Gamma(\mu)$ , then by (3.5) the above equation takes the form

$$\begin{aligned} g(\nabla_X^\perp \phi H, NY) &= -\sin^2\theta [g(A_HX, Y) - \epsilon\eta(A_HX)\eta(Y)] \\ &\quad - \epsilon\eta(H)g(\phi^2 X, NY). \end{aligned} \quad (3.26)$$

Using (2.17), (2.18) and (2.29) and having some brief calculations we obtain

$$\begin{aligned} g(\tilde{\nabla}_X \phi H, NY) &= -\sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 \\ &\quad - \epsilon\eta(H)g(\phi^2 X, NY). \end{aligned} \quad (3.27)$$

The above equation can be written as

$$\begin{aligned} g(\tilde{\nabla}_X NY, \phi H) &= \sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 \\ &\quad + \epsilon\eta(H)g(\phi^2 X, NY). \end{aligned} \quad (3.28)$$

Again using the fact that  $H \in \Gamma(\mu)$  and by Weingarten formula we have

$$\begin{aligned} g(\nabla_X^\perp NY, \phi H) &= \sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 \\ &\quad + \epsilon\eta(H)g(\phi^2 X, NY). \end{aligned} \quad (3.29)$$

From (3.14) and (3.21) we get

$$\begin{aligned} \sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 + \epsilon\eta(H)g(\phi^2 X, NY) &= g(X, Y)\|H\|^2 \\ &\quad - \epsilon\eta(Y)g(\phi^2 X, \phi H). \end{aligned} \quad (3.30)$$

The equation (3.22) has a solution if  $H = 0$ . Hence  $M$  is totally geodesic in  $\tilde{M}$ .  $\square$

#### §4. Hemislant Submanifolds

A.Carriazo [3] introduced hemi-slant submanifolds as a special case of bislant submanifolds and he called them pseudo-slant submanifolds. This section deals with a special case of hemislant submanifolds which are totally umbilical.

**Definition 4.1** ([5,16]) *A submanifold  $M$  of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  is said to be a hemislant submanifold if there exist two orthogonal complementary distributions  $D_1$  and  $D_2$  satisfying the following properties:*

- (i)  $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$ ;
- (ii)  $D_1$  is a slant distribution with slant angle  $\theta \neq \pi/2$ ;
- (iii)  $D_2$  is totally real i.e.,  $\phi D_2 \subseteq T^\perp M$ .

A hemislant submanifold is called proper hemislant submanifold if  $\theta \neq 0, \frac{\pi}{2}$ . Further if  $\mu$  is  $\phi$ -invariant subspace of the normal bundle  $T^\perp M$ , then for pseudo-slant submanifold, the normal bundle  $T^\perp M$  can be decomposed as

$$T^\perp M = ND_1 \oplus ND_2 \oplus \langle \mu \rangle.$$

In this section we will derive some of the integrability conditions of the involved distributions of a hemislant submanifold, which play a crucial role from a geometrical point of view.

**Theorem 4.1** *Let  $M$  be a hemislant submanifold of an  $(\epsilon)$ -paracontact Sasakian manifold  $\tilde{M}$  then  $g([X, Y], \xi) = 0$  for any  $X, Y \in D_1 \oplus D_2$ .*

*Proof* We know

$$g(X, \phi Y) = g(Y, \phi X), \quad \nabla_X \xi = \epsilon \phi X. \quad (4.1)$$

Taking inner product with  $Y$  we obtain

$$g(\nabla_X \xi, Y) = \epsilon g(\phi X, Y). \quad (4.2)$$

We can write

$$g(\nabla_X Y, \xi) = -\epsilon g(\phi X, Y). \quad (4.3)$$

Interchanging  $X, Y$  we get

$$g(\nabla_Y X, \xi) = -\epsilon g(\phi Y, X). \quad (4.4)$$

Subtracting equations (4.3) and (4.4) and using (4.1) we have

$$g([X, Y], \xi) = 0. \quad (4.5)$$

This completes the proof.  $\square$

From Theorem (4.1) we can deduce the following corollaries.



**Corollary 4.1** *The distribution  $D_1 \oplus D_2$  on a hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  is integrable.*

**Corollary 4.2** *The distribution  $D_1$  and  $D_2$  on a hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$  is integrable.*

**Proposition 4.1** *Let  $M$  be a hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$ , then for any  $Z, W \in D_2$ , the anti-invariant distribution  $D_2 \oplus \langle \xi \rangle$  is integrable iff*

$$A_{\phi Z}W - A_{\phi W}Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z - \epsilon\eta(W)Z + \epsilon\eta(Z)W = 0.$$

*Proof* For any  $Z, W \in D_2 \oplus \xi$  we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi W) + \phi \tilde{\nabla}_Z W = (\tilde{\nabla}_Z \phi W) + \phi \nabla_Z W + \phi h(Z, W). \quad (4.6)$$

Using (2.16) and (2.17) we have

$$-A_{\phi W}Z + \nabla_Z^\perp \phi W = (\tilde{\nabla}_Z \phi W) + \phi \tilde{\nabla}_Z W = (\tilde{\nabla}_Z \phi W) + \phi \nabla_Z W + \phi h(Z, W). \quad (4.6)$$

Interchanging  $Z$  and  $W$ , we obtain

$$-A_{\phi Z}W + \nabla_W^\perp \phi Z = (\tilde{\nabla}_W \phi Z) + \phi \tilde{\nabla}_W Z = (\tilde{\nabla}_W \phi Z) + \phi \nabla_W Z + \phi h(W, Z). \quad (4.8)$$

Then from (4.7) and (4.8) we calculate

$$A_{\phi Z}W - A_{\phi W}Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z = (\tilde{\nabla}_Z \phi W) - (\tilde{\nabla}_W \phi Z) + \phi[Z, W]. \quad (4.9)$$

From (2.15) we obtain

$$A_{\phi Z}W - A_{\phi W}Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z = \phi[Z, W] - \epsilon\eta(W)\phi^2 Z + \epsilon\eta(Z)\phi^2 W. \quad (4.10)$$

Taking inner product with  $\phi X$ , for any  $X \in D_1$  we obtain

$$\begin{aligned} & g(A_{\phi Z}W - A_{\phi W}Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z, \phi X) \\ &= g(\phi[Z, W], \phi X) - \epsilon\eta(W)g(\phi^2 Z, \phi X) + \epsilon\eta(Z)g(\phi^2 W, \phi X). \end{aligned} \quad (4.11)$$

Thus from (2.5) the above equation takes the form

$$\begin{aligned} g(\phi[Z, W], \phi X) &= g(A_{\phi Z}W - A_{\phi W}Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z - \epsilon\eta(W)Z \\ &\quad + \epsilon\eta(Z)W, \phi X). \end{aligned} \quad (4.12)$$

The distribution  $D_2 \oplus \langle \xi \rangle$  is integrable iff the right hand side of the above equation is zero.

**Proposition 4.2** *Let  $M$  be a hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$ , then the anti-invariant distribution  $D_1 \oplus \langle \xi \rangle$  is integrable iff*

$$h(Y, TX) + \nabla_Y^\perp NX - h(X, TY) - \nabla_X^\perp NY \in \mu$$

for all  $X, Y \in D_1 \oplus \langle \xi \rangle$ .

*Proof* For any  $X, Y \in D_1 \oplus \langle \xi \rangle$ , we have

$$\begin{aligned} \phi[X, Y] &= \phi[\tilde{\nabla}_Y X - \tilde{\nabla}_X Y] = \tilde{\nabla}_Y TX + \tilde{\nabla}_Y NX - \tilde{\nabla}_X TY \\ &\quad - \tilde{\nabla}_X NY - \epsilon\eta(Y)\phi^2 X + \epsilon\eta(X)\phi^2 Y. \end{aligned} \quad (4.13)$$

$$\begin{aligned} \phi[X, Y] &= \nabla_Y TX + h(Y, TX) - A_{NX}Y + \nabla_Y^\perp NX - \nabla_X TY - h(X, TY) \\ &\quad + A_{NY}X - \nabla_X^\perp NY - \epsilon\eta(Y)\phi^2 X + \epsilon\eta(X)\phi^2 Y. \end{aligned} \quad (4.14)$$

Taking the product with  $\phi Z$ , for any  $Z \in D_2$ , we obtain on solving

$$\begin{aligned} g(\phi[X, Y], \phi Z) &= g(h(Y, TX), \phi Z) + g(\nabla_Y^\perp NX, \phi Z) - g(h(X, TY), \phi Z) \\ &\quad - g(\nabla_X^\perp NY, \phi Z) - \epsilon\eta(Y)g(X - \eta(X)\xi, \phi Z) \\ &\quad + \epsilon\eta(Y)g(Y - \eta(Y)\xi, \phi Z). \end{aligned} \quad (4.15)$$

$$g([X, Y], Z) = g(h(Y, TX) + \nabla_Y^\perp NX - h(X, TY) - \nabla_X^\perp NY, \phi Z). \quad (4.16)$$

Thus our assertion follows from equation (4.16).  $\square$

**Theorem 4.2** *Let  $M$  be a hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$ , then at least one of the following statements is true:*

- (i)  $\dim D_2 = 1$ ;
- (ii)  $H \in \mu$ ;
- (iii)  $M$  is proper slant.

*Proof* For any  $Z, W \in TM$ , we have

$$(\tilde{\nabla}_Z \phi W) + (\tilde{\nabla}_W \phi Z) = -2g(\phi Z, \phi W)\xi - \epsilon\eta(Z)\phi^2 W + \epsilon\eta(W)\phi^2 Z. \quad (4.17)$$

If we assume the vector fields  $Z, W \in D_2$ , then the above equation reduces to

$$(\tilde{\nabla}_Z \phi)W + (\tilde{\nabla}_W \phi)Z + 2g(\phi Z, \phi W)\xi = 0. \quad (4.18)$$

In particular if we take the above equation for one vector  $Z \in D_2$ , i.e

$$(\tilde{\nabla}_Z \phi)Z + g(\phi Z, \phi Z)\xi = 0. \quad (4.19)$$

Again using (2.6) we have

$$(\tilde{\nabla}_Z \phi)Z + \|Z\|^2 \xi = 0. \quad (4.20)$$

Therefore the tangential and normal components of the above equation are  $\mathbf{P}_Z Z = \|Z\|^2 \xi$  and  $\mathbf{Q}_Z Z = 0$  respectively. From (2.21) and tangential component of (4.20) we get

$$(\tilde{\nabla}_Z T)Z = -T\nabla_Z Z = A_{NZ}Z + th(Z, Z) - \|Z\|^2 \xi. \quad (4.21)$$

Taking the product with  $W \in D_2$ , we get from (2.18)

$$g(T\nabla_Z Z, W) = g(h(Z, W), NZ) + g(th(Z, Z), W). \quad (4.22)$$

Using the fact that  $M$  is totally umbilical submanifold and for any  $W \in D_2$ , then the above equation takes the form

$$g(Z, W)g(H, NZ) + \|Z\|^2 g(tH, W) = -g(T\nabla_Z Z, W) = 0. \quad (4.23)$$

Thus the equation (4.10) has a solution if either  $\dim D_2 = 1$  or  $H \in \mu$  or  $D_2 = 0$ , i.e.  $M$  is proper slant.  $\square$

From the above conclusions we can obtain the following theorem

**Theorem 4.3** *Let  $M$  be a totally umbilical hemislant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$ . Then at least one of the following statements is true:*

- (i)  $M$  is an anti-invariant submanifold;
- (ii)  $g(\nabla_{TX} \xi, X) = \epsilon[\|X\|^2 - \eta^2(X)]$ ;
- (iii)  $M$  is totally geodesic submanifold;
- (iv)  $\dim D_2 = 1$ ;
- (v)  $M$  is a proper slant submanifold.

*Proof* If  $H \neq 0$  then from equation (3.19), we can conclude that the slant distribution  $D_1 = 0$  i.e.  $M$  is anti-invariant submanifold which is case (i). If  $D_1 \neq 0$  and  $H \in \mu$ , then from theorem (3.2) we get (ii) for any  $X \in TM$ . Again if  $H \in \mu$  then by theorem (3.4),  $M$  is totally geodesic. Lastly if  $H \notin \mu$ , then the equation (4.23) has a solution if either  $\dim D_2 = 1$  or  $D_2 = 0$ . Hence the theorem follows.  $\square$

Next we have the following theorem.

**Theorem 4.4** *Let  $M$  be a submanifold of an almost contact metric manifold  $\tilde{M}$ , such that  $\xi \in TM$ . Then  $M$  is a pseudo-slant submanifold iff there exists a constant  $\lambda \in (0, 1]$  such that*

- (i)  $D = \{X \in TM | T^2 X = -\lambda X\}$  is a distribution on  $M$ ;
- (ii) For case  $X \in TM$ , orthogonal to  $D$ ,  $TX = 0$ .

Furthermore in this case  $\lambda = \cos^2 \theta$ , where  $\theta$  denotes the slant angle of  $D$ .

**Proof:** Follows from [11].  $\square$

Again we prove

**Theorem 4.5** *Let  $M$  be a hemi-slant submanifold of an  $(\epsilon)$ -para Sasakian manifold  $\tilde{M}$ . Then  $\nabla Q = 0$  iff  $M$  is an anti-invariant submanifold.*

*Proof* Considering the distribution  $D_2 \oplus \langle \xi \rangle$ , from (4.4) we can write

$$T^2X = \lambda(X - \eta(X)\xi). \quad (4.24)$$

Denoting the slant angle of  $M$  by  $\theta$ . Then, replacing  $X$  by  $\nabla_X Y$ , we obtain from (4.24)

$$Q\nabla_X Y = \cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi]. \quad (4.25)$$

for any  $X, Y \in D_2 \oplus \langle \xi \rangle$ . After taking the covariant derivative of equation (4.24) we have

$$\nabla_X QY = \cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi] - g(Y, \nabla_X \xi)\xi - \eta(Y)\nabla_X \xi. \quad (4.26)$$

Adding equations (4.25) and (4.26) we obtain

$$\begin{aligned} (\tilde{\nabla}_X Q)Y &= \cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi] + g(Y, \epsilon TX)\xi + \eta(Y)\epsilon TX \\ &\quad - \cos^2\theta\nabla_X Y + \cos^2\theta\eta(\nabla_X Y)\xi \end{aligned} \quad (4.27)$$

for any  $X, Y \in D_2 \oplus \langle \xi \rangle$ .

Here we observe that  $g(Y, TX)\xi + TX\eta(Y) \neq 0$ . Therefore  $(\tilde{\nabla}_X Q) = 0$  iff  $\theta = \frac{\pi}{2}$  holds in  $D_2 \oplus \langle \xi \rangle$ . Again  $D_1$  is anti-invariant by definition. Thus, the theorem follows.  $\square$

## §5. An Example

Let us give an example of a three dimensional submanifold of  $(\epsilon)$ -paracontact Sasakian manifold which is pseudo slant so as to verify the above results. Let  $\mathbf{R}^3$  be a 3-dimensional Euclidean space with a rectangular coordinates  $(x, y, z)$ , we put

$$\eta = dy \quad \xi = \frac{\partial}{\partial y}. \quad (5.1)$$

We define the  $(1, 1)$  tensor  $\phi$  as:

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial z} \quad \phi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x} \quad \phi\left(\frac{\partial}{\partial y}\right) = 0 \quad (5.2)$$

and we define the Riemannian metric  $g$  as

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we can easily see that  $(\phi, \xi, \eta, g)$  is an  $(\epsilon)$ -paracontact Sasakian manifold on  $\mathbf{R}^3$ .

The vector fields  $e_1 = \frac{\partial}{\partial x}$ ,  $e_2 = \frac{\partial}{\partial y}$ ,  $e_3 = \frac{\partial}{\partial z}$ , forms a frame of  $TM$ . We have

$$\phi e_1 = e_3, \quad \phi e_2 = 0, \quad \phi e_3 = e_1.$$

Let  $D_1 = \langle e_2 \rangle$ ,  $D_2 = \langle e_1 \rangle$  and  $\xi = \langle e_3 \rangle$ . We know

$$\cos \angle(\phi X, Y) = \frac{g(\phi X, Y)}{|\phi X||Y|}.$$

Suppose  $X \in D_1$  and  $Y \in TM$ . Then we can write  $X = Ke_2$  where  $K$  is a scalar and  $Y = re_1 + se_2 + te_3$  where  $r, s, t$  are scalars. Notice that

$$g(\phi X, Y) = g(\phi e_2, re_1 + se_2 + te_3) = rg(0, e_1) + sg(0, e_2) + tg(0, e_3) = 0.$$

Hence  $\cos \angle(\phi X, Y) = 0$  implies  $\theta = \frac{\pi}{2}$ . Hence the distribution  $D_1$  is anti-invariant.

Again let us assume  $U \in D_1$  and  $V \in TM$ . Then we can write  $U = ae_1$ , where  $a$  is a scalar and  $V = ke_1 + le_2 + me_3$  where  $k, l, m$  are scalars. Using the formula above we get that

$$g(\phi U, V) = g(\phi(ae_1), ke_1 + le_2 + me_3) = am.$$

Hence  $\cos \angle(\phi U, V) = \text{constant}$ . So we have obtained that the distribution  $D_2$  is slant.

In this case, the distribution  $D_1$  is anti-invariant while  $D_2$  is slant. Hence the submanifold under consideration is hemislant.

## References

- [1] J.K.Beem and P.E..Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, New York, 1981.
- [2] J.L.Cabrero, A.Carriazo, L.M.Fernandez, M.Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow. Math. J.* 42(2000).
- [3] A.Carriazo, *New Developments in Slant Submanifolds Theory*, Narosa Publishing House, New Delhi, India, 2002.
- [4] B.Y.Chen, Slant immersions, *Bulletin of the Australian Mathematical Society*, Vol.41, no.1, pp. 135-147, 1990.
- [5] M.A.Khan, Khushwant singh and V.A.Khan, Slant submanifold of LP-contact manifolds, *Differential Geometry-Dynamical System, Balkan Society of Geometers*, Vol.12, pp. 102-108, 2010.
- [6] M.A.Khan, K.Singh and V.A.Khan, Slant Submanifolds of LP-contact manifolds, *Differential Geometry Dynamical Systems*, Vol.12, pp. 102-108, 2010.
- [7] M. Kobayashi, CR-submanifolds of a Sasakian manifold, *Tensor N.S.* Vol. 35, 297-307, 1981.
- [8] A.Lotta, Slant Submanifolds in contact geometry, *Bull.Math.Soc. Roumania*, 39, 183-198, 1996.

- [9] B.O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, 1983.
- [10] N.Papaghiuc, Semi-slant submanifolds of Kaehlerian manifold, *An.Stiint.Univ.Al.I.CuzaIasi*, Ser.Noua, mat. 40(1), 55-61, 1994.
- [11] I.Sato, On a structure similar to the almost contact structure, *Tensor N.S.* 30(3), 219-224, 1976.
- [12] B.Sahin, Every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic, *Results in Mathematics*, Vol.54, No.1-2, pp. 167-172, 2009.
- [13] S.Tanno, The automorphism groups of almost contact Riemannian manifolds, *Tohoku Math. J.*, 21, 21-38, 1969.
- [14] T.Takahashi, Sasakian manifold with Pseudo-Riemannian metric, *Tohoku Math.J.*, 21(2), 644-653, 1969.
- [15] M.M.Tripathi, E.Kilic, S.Y.Perktas and S.Keles, Indefinite almost paracontact metric manifolds, *arXiv: 0908.2729 vi[math.DG]*, 2009.
- [16] Siraj Uddin, Meraaj Ali Khan, Khuswant Singh, *Totally Umbilical Proper Slant and Hemislant Submanifolds of an LP-Cosymplectic Manifold*, Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2011, Article ID 516238, 9 pages, 2011.

## A Study on Set-Valuations of Signed Graphs

P.K.Ashraf<sup>1</sup>, K.A.Germina<sup>2</sup> and N.K.Sudev<sup>3</sup>

1. Department of Mathematics, Government Arts and Science College, Koduvally,  
Kozhikkode - 673572, Kerala, India

2. Department of Mathematics, Central University of Kerala, Kasaragod-671316, Kerala, India

3. Centre for Studies in Discrete Mathematics, Vidya Academy of Science & Technology,  
Thalakkottukara, Thrissur - 680501, Kerala, India

E-mail: ashrafkalanthod@gmail.com, srgerminaka@gmail.com, sudevnk@gmail.com

**Abstract:** Let  $X$  be a non-empty ground set and  $\mathcal{P}(X)$  be its power set. A set-labeling (or a set-valuation) of a graph  $G$  is an injective set-valued function  $f : V(G) \rightarrow \mathcal{P}(X)$  such that the induced function  $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$  is defined by  $f^\oplus(uv) = f(u) \oplus f(v)$ , where  $f(u) \oplus f(v)$  is the symmetric difference of the sets  $f(u)$  and  $f(v)$ . A graph which admits a set-labeling is known to be a set-labeled graph. A set-labeling  $f$  of a graph  $G$  is said to be a set-indexer of  $G$  if the associated function  $f^\oplus$  is also injective. In this paper, we define the notion of set-valuations of signed graphs and discuss certain properties of signed graphs which admits certain types of set-valuations.

**Key Words:** Signed graphs, balanced signed graphs, clustering of signed graphs, set-labeled signed graphs.

**AMS(2010):** 05C78, 05C22.

### §1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4, 8, 13] and for the topics in signed graphs we refer to [14, 15]. Unless mentioned otherwise, all graphs considered here are simple, finite, undirected and have no isolated vertices.

#### 1.1 An Overview of Set-Valued Graphs

Let  $X$  be a non-empty set and  $\mathcal{P}(X)$  be its power set. A *set-labeling* (or a *set-valuation*) of a graph  $G$  is an injective function  $f : V(G) \rightarrow \mathcal{P}(X)$  such that the induced function  $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$  is defined by  $f^\oplus(uv) = f(u) \oplus f(v) \forall uv \in E(G)$ , where  $\oplus$  is the symmetric difference of two sets. A graph  $G$  which admits a set-labeling is called an *set-labeled graph* (or a set-valued graph)(see [1]).

A *set-indexer* of a graph  $G$  is an injective function  $f : V(G) \rightarrow \mathcal{P}(X)$  such that the induced function  $f^\oplus : E(G) \rightarrow \mathcal{P}(X)$  is also injective. A graph  $G$  which admits a set-indexer is called a *set-indexed graph* (see [1]).

---

<sup>1</sup>Received July 27, 2017, Accepted February 12, 2018.

Several types of set-valuations of graphs have been introduced in later studies and their properties and structural characteristics of such set-valued graphs have been done extensively.

## 1.2 Preliminaries on Signed Graphs

An edge of a graph  $G$  having only one end vertex is known as a *half edge* of  $G$  and an edge of  $G$  without end vertices is called *loose edge* of  $G$ .

A *signed graph* (see [14, 15]), denoted by  $\Sigma(G, \sigma)$ , is a graph  $G(V, E)$  together with a function  $\sigma : E(G) \rightarrow \{+, -\}$  that assigns a sign, either  $+$  or  $-$ , to each ordinary edge in  $G$ . The function  $\sigma$  is called the *signature* or *sign function* of  $\Sigma$ , which is defined on all edges except half edges and is required to be positive on free loops.

An edge  $e$  of a signed graph  $\Sigma$  is said to be a *positive edge* if  $\sigma(e) = +$  and an edge  $\sigma(e)$  of a signed graph  $\Sigma$  is said to be a *negative edge* if  $\sigma(e) = -$ . The set  $E^+$  denotes the set of all positive edges in  $\Sigma$  and the set  $E^-$  denotes the set of negative edges in  $\Sigma$ .

A simple cycle (or path) of a signed graph  $\Sigma$  is said to be *balanced* (see [3, 9]) if the product of signs of its edges is  $+$ . A signed graph  $\Sigma$  is said to be a *balanced signed graph* if it contains no half edges and all of its simple cycles are balanced. It is to be noted that the number of all negative signed graph is balanced if and only if it is bipartite.

Balance or imbalance is the basic and the most important property of a signed graph. The following theorem, popularly known as *Harary's Balance Theorem*, establishes a criteria for balance in a signed graph.

**Theorem 1.1**([9]) *The following statements about a signed graph are equivalent.*

- (i) *A signed graph  $\Sigma$  is balanced;*
- (ii)  *$\Sigma$  has no half edges and there is a partition  $(V_1, V_2)$  of  $V(\Sigma)$  such that  $E^- = E(V_1, V_2)$ ;*
- (iii)  *$\Sigma$  has no half edges and any two paths with the same end points have the same sign.*

Some balancing properties of certain types of signed graphs have been studied in [6, 7].

A signed graph  $\Sigma$  is said to be *clusterable* or *partitionable* (see [14, 15]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. If  $V(\Sigma)$  can be partitioned in to  $k$  subsets with the above mentioned conditions, then the signed graph  $\Sigma$  is said to be *k-clusterable*. In this paper, we study the 2-clusterability of signed graphs only.

Note that 2-clusterability always implies balance in a signed graph  $\Sigma$ . But, the converse need not be true. If all edges in  $\Sigma$  are positive edges, then  $\Sigma$  is balanced but not 2-clusterable.

In this paper, we introduce the notion of set-valuations of signed graphs and study the properties and characteristics of such signed graphs.

## §2. Set-Labeled Signed Graphs

Motivated from the studies on set-valuations of signed digraphs in [2], and the studies on integer additive set-labeled signed graphs in [11], we define the notion of a set-labeling of a signed graph as follows.



**Definition 2.1** Let  $X$  be a non-empty set and let  $\Sigma$  be a signed graph, with corresponding underlying graph  $G$  and the signature  $\sigma$ . An injective function  $f : V(\Sigma) \rightarrow \mathcal{P}(X)$  is said to be a set-labeling (or set-valuation) of  $\Sigma$  if  $f$  is a set-labeling of the underlying graph  $G$  and the signature of  $\Sigma$  is defined by  $\sigma(uv) = (-1)^{|f(u) \oplus f(v)|}$ . A signed graph  $\Sigma$  together with a set-labeling  $f$  is known as a set-labeled signed graph (or set valued signed graph) and is denoted by  $\Sigma_f$ .

**Definition 2.2** A set-labeling  $f$  of a signed graph  $\Sigma$  is said to be a set-indexer of  $\Sigma$  if  $f$  is a set-indexer of the underlying graph  $G$ .

If the context is clear, we can represent a set-valued signed graph or a set-indexed signed graph simply by  $\Sigma$  itself. In this section, we discuss the 2-clusterability and balance of set-valued signed graphs.

The following theorem establishes the existence of set-valuations for all signed graphs.

**Theorem 2.3** Every signed graph admits a set-labeling (and a set-indexer).

*Proof* Let  $\Sigma$  be a signed graph whose vertex set is given by  $V(\Sigma) = \{v_1, v_2, \dots, v_n\}$ . Let  $X = \{1, 2, 3, \dots, n\}$ . Define a set-valued function  $f : V(\Sigma) \rightarrow \mathcal{P}(X)$  such that  $f(v_i) = \{i\}$ , where  $1 \leq i \leq n$ . Clearly,  $f$  is an injective function. Then,  $f^\oplus(v_i v_j) = \{i, j\}, \forall uv \in E(G)$ . Note that  $f^\oplus$  is also an injective function and hence  $f$  is a set-indexer of  $\Sigma$ .  $\square$

We say that two sets are of *same parity* if they are simultaneously even or simultaneously odd. If two sets are not of same parity, then they are said to be the sets of *opposite parity*. The signature of an edge of a set-valued signed graph can be determined in terms of the set-labels of its end vertices, as described in the following theorem.

**Theorem 2.4** An edge  $e$  of a set-labeled signed graph is a positive edge if and only if the set-labels of its end vertices are of the same parity.

*Proof* Let  $f$  be a set-labeling of a given signed graph  $\Sigma$ . Assume that, an edge  $e = v_i v_j$  be a positive edge in  $\Sigma$ . Then,  $|f(v_i) \oplus f(v_j)| = |f(v_i) - f(v_j)| + |f(v_j) - f(v_i)|$  is an even number. That is,  $|f(v_i) - f(v_j)|$  and  $|f(v_j) - f(v_i)|$  are simultaneously even or simultaneously odd. Hence, we need to consider the following cases.

**Case 1.** Assume that both  $|f(v_i) - f(v_j)|$  and  $|f(v_j) - f(v_i)|$  are even. That is, both  $|f(v_i) - f(v_i) \cap f(v_j)|$  and  $|f(v_j) - f(v_i) \cap f(v_j)|$  are even. Then, we have

**Subcase 1.1** Let  $|f(v_i)|$  be an even integer. Then, since  $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$ , we have  $|f(v_i) \cap f(v_j)|$  must also be even. Hence, as  $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$  is even, we have  $|f(v_j)|$  is even.

**Subcase 1.2** Let  $|f(v_i)|$  be an odd integer. Then, since  $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$  is even, we have  $|f(v_i) \cap f(v_j)|$  must be odd. Hence, as  $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$  is even, we have  $|f(v_j)|$  is odd.

**Case 2.** Assume that both  $|f(v_i) - f(v_j)|$  and  $|f(v_j) - f(v_i)|$  are odd. That is, both  $|f(v_i) - f(v_i) \cap f(v_j)|$  and  $|f(v_j) - f(v_i) \cap f(v_j)|$  are odd. Then, we have

**Subcase 2.1** Let  $|f(v_i)|$  be an even integer. Then, since  $|f(v_i) - f(v_i) \cap f(v_j)|$  is odd, we have  $|f(v_i) \cap f(v_j)|$  must be odd. Hence, as  $|f(v_j) - f(v_i) \cap f(v_j)|$  is odd, we have  $|f(v_j)|$  is even.

**Subcase 2.2** Let  $|f(v_i)|$  be an odd integer. Then, since  $|f(v_i) - f(v_i) \cap f(v_j)|$  is odd, we have  $|f(v_i) \cap f(v_j)|$  must be even. Then, as  $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$  is odd, we have  $|f(v_j)|$  is odd.  $\square$

As a contrapositive of Theorem 2.4, we can prove the following theorem also.

**Theorem 2.5** *An edge  $e$  of a set-labeled signed graph is a negative edge if and only if the set-labels of its end vertices are of the opposite parity.*

The following result is an immediate consequence of Theorems 2.4 and 2.5.

**Corollary 2.6** *A set-valued signed graph  $\Sigma$  is balanced if and only if every cycle in  $\Sigma$  has even number of edges whose end vertices have opposite parity set-labels.*

*Proof* Note that the number of negative edges in any cycle of a balanced signed graph is even. Hence, the proof is immediate from Theorem 2.5.  $\square$

The following theorem discusses a necessary and sufficient condition for a set-valued signed graph to be 2-clusterable.

**Theorem 2.7** *A set-valued signed graph is 2-clusterable if and only if at least two adjacent vertices in  $\Sigma$  have opposite parity set-labels.*

*Proof* First, assume that at least two adjacent vertices in the set-valued signed graph  $\Sigma$  have opposite parity set-labels. If  $e = v_i v_j$  be an edge of  $\Sigma$  such that  $f(v_i)$  and  $f(v_j)$  are of opposite parity, then  $\sigma(v_i v_j) = -$ . Then, we can find  $(U_1, U_2)$  be a partition of  $V(\Sigma)$  such that  $U_1$  contains one end vertex of every negative edge and  $U_2$  contains the other end vertex of every negative edge. Therefore,  $\Sigma$  is 2-clusterable.

Conversely, assume that  $\Sigma$  is 2-clusterable. Then, there exist two non-empty subsets  $U_1$  and  $U_2$  of  $V(\Sigma)$  such that  $U_1 \cup U_2 = V(\Sigma)$ . Since  $\Sigma$  is a connected signed graph, at least one vertex in  $U_1$  is adjacent some vertices in  $U_2$  and vice versa. Let  $e = v_i v_j$  be such an edge in  $\Sigma$ . Since  $\Sigma$  is 2-clusterable,  $e$  is a negative edge and hence  $f(v_i)$  and  $f(v_j)$  are of opposite parity. This completes the proof.  $\square$

**Theorem 2.8** *Let  $f$  be a set-indexer defined on a signed graph  $\Sigma$  whose underlined graph  $G$  is an Eulerian graph. If  $\Sigma$  is balanced, then*

$$\sum_{e \in E(\Sigma)} |f^\oplus(e)| \equiv 0 \pmod{2}.$$

*Proof* Let the underlying graph  $G$  of  $\Sigma$  is Eulerian. Then,  $G = \bigcup_{i=1}^k C_i$ , where each  $C_i$  is a cycle such that  $C_i$  and  $C_j$  are edge-disjoint for  $i \neq j$ . Let  $E_i$  be the edge set of the cycle  $C_i$ .

Since  $f$  is a set-indexer of  $\Sigma$ , we have  $f(E_i) \cap f(E_j) = \emptyset$ , for  $i \neq j$ . Hence, we have

$$\sum_{e \in E(\S)} |f(e)| = \sum_{i=1}^k \sum_{e_i \in E_i} f(e_i). \quad (1)$$

Consider the cycle  $C_i$ . Let  $A_i$  be the set all positive edges and  $B_i$  be the set of all negative edges in the cycle  $C_i$ . Then, the set-labels of edges in  $A_i$  are of even parity and those of edges in  $B_i$  are of odd parity. That is,  $|f^\oplus(e)| \equiv 0 \pmod{2}$  for all  $e \in A_i$  and hence we have

$$\sum_{e \in A_i} |f(e)| \equiv 0 \pmod{2}. \quad (2)$$

Since  $\Sigma$  is balanced, the number of negative edges in  $C_i$  is even. Therefore, the number of elements in  $B_i$  must be even. That is, the number of edges having odd parity set-labels in  $C_i$  is even. Therefore, being a sum of even number of odd integers, we have

$$\sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod{2}. \quad (3)$$

From Equations (2) and (3), we have

$$\sum_{e \in E_i} |f(e)| = \sum_{e_i \in A_i} |f(e_i)| + \sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod{2}. \quad (4)$$

Therefore, by Equations (1) and (4), we can conclude that

$$\sum_{e \in E(\S)} |f^\oplus(e)| \equiv 0 \pmod{2}. \quad \square$$

From the above results, we infer the most important result on a set-valued signed graph as follows.

**Theorem 2.9** *If a signed graph  $\Sigma$  admits a vertex set-labeling, then  $\Sigma$  is balanced.*

*Proof* Let  $\Sigma$  be a signed graph which admits a set-labeling. If all vertices of  $\Sigma$  have the same parity set-labels, then by Theorem 2.4, all edges of  $\Sigma$  are positive edges and hence  $\Sigma$  is balanced.

Next, let that  $\Sigma$  contains vertices with opposite parity set-labels. Let  $A_i$  be the set of all vertices with odd parity set-labels and  $B_i$  be the set of all even parity set-labels. First, assume  $v_i$  be a vertex in  $A_i$  whose adjacent vertices are in  $B_i$ . Then,  $v_i$  is one end vertex of some negative edges in  $\Sigma$ . If  $v_i$  is not in a cycle of  $\Sigma$ , then none of these negative edges will be a part in any cycle of  $\Sigma$ .

If  $v_i$  is an internal vertex of a cycle  $C$ , then it is adjacent to two vertices, say  $v_j$  and  $v_k$ , which are in  $B_i$ . Hence, the edges  $v_i v_j$  and  $v_i v_k$  are negative edges. If two vertices  $v_i$  and  $v_j$  are adjacent in the cycle  $C$ , then  $v_i$  is adjacent to one more vertex, say  $v_k$  and the vertex  $v_j$  is also adjacent to one more vertex  $v_l$  and in the cycle  $C$ , the edges  $v_i v_k$  and  $v_j v_l$  are negative

edges and the edge  $v_i v_j$  is a positive edge. If the vertices  $v_i$  and  $v_j$  are not adjacent, then also each of them induce two negative, which may not be distinct always. However, in each case the number of negative edges will be even. This condition can be verified in all cases when any number element of  $A_i$  are the vertices of any cycle  $C$  in  $\Sigma$ . Hence, the number of negative edges in any cycle of a set-labeled signed graph is even. Hence,  $\Sigma$  is balanced.

Hence, in this case, the number negative edges in  $C$  will always be even.  $\square$

It is interesting to check whether the converse of the above theorem is valid. In context of set-labeling of signed graphs, a necessary and sufficient condition for a signed graph  $\Sigma$  is to be balanced is given in the following theorem.

**Theorem 2.10** *A signed graph  $\Sigma$  is balanced if and only if it admits a set-labeling.*

*Proof* The proof is an immediate consequence of Theorems 2.4 and 2.9.  $\square$

In view of Theorems 2.4 and 2.10, we have

**Theorem 2.11** *Any set-labeled signed graph is balanced.*

### §3. Conclusion

In this paper, we have discussed the characteristics and properties of the signed graphs which admit set-labeling with a focus on 2-clusterability and balance of these signed graphs. There are several open problems in this area. Some of the open problems that seem to be promising for further investigations are following.

**Problem 3.1** *Discuss the  $k$ -clusterability of different types of set-labeled signed graphs for  $k > 2$ .*

**Problem 3.2** *Discuss the balance, 2-clusterability and general  $k$ -clusterability of other types of set-labeling of signed graphs such as topological set-labeling, topogenic set-labeling, graceful set-labeling, sequential set-labeling etc.*

**Problem 3.3** *Discuss the balance and 2-clusterability and general  $k$ -clusterability of different set-labeling of signed graphs, with different set operations other than the symmetric difference of sets.*

Further studies on other characteristics of signed graphs corresponding to different set-labeled graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

### Acknowledgement

The authors would like to dedicate this work to (Late) Prof. (Dr.) Belamannu Devadas Acharya, who had been the motivator and the role model for them and who have introduced the concept of set-valuations of graphs.

## References

- [1] B. D. Acharya, *Set-Valuations and Their Applications*, MRI Lecture notes in Applied Mathematics, No.2, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1983.
- [2] B. D. Acharya, Set-valuations of signed digraphs, *J. Combin. Inform. System Sci.*, **37**(2-4)(2012), 145-167.
- [3] J Akiyama, D. Avis, V. Chavátal and H. Era, Balancing signed graphs, *Discrete Appl. Math.*, **3**(4)(1981), 227-233., DOI: 10.1016/0166-218X(81)90001-9.
- [4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Application*, North-Holland, New York, 1982.
- [5] J. A. Gallian, A dynamic survey of graph labelling, *Electron. J. Combin.*, (2015), (#DS-6).
- [6] K. A. Germina and S. Hameed, On signed paths, signed cycles and their energies, *Appl. Math. Sci.*, **4**(70)(2010), 3455-3466.
- [7] S. Hameed and K. A. Germina, On composition of signed graphs, *Discuss. Math. Graph Theory*, **32**(3)(2012), 507-516., DOI: 10.7151/dmgt.1615.
- [8] F. Harary, *Graph Theory*, Addison-Wesley Pub. Co. Inc., 1969.
- [9] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, **2**(2)(1953), 143-146.
- [10] F. Harary and J A. Kabell, A simple algorithm to detect balance in signed graphs, *Math. Soc. Sci.*, **1**(1)(1980), 131-136., DOI: 10.1016/0165-4896(80)90010-4.
- [11] N. K. Sudev and K. A. Germina, A study on integer additive set-valuations of signed graphs, *Carpathian Math. Publ.*, **7**(2)(2015), 236-246., DOI:10.15330/cmp.7.2.236-246.
- [12] N. K. Sudev, P. K. Ashraf and K. A. Germina, Some new results on integer additive set-valued signed graphs, *TWMS J. Appl. Eng. Math.*, to appear.
- [13] D. B. West, *Introduction to Graph Theory*, Pearson Education Inc., 2001.
- [14] T. Zaslavsky, Signed graphs, *Discrete Appl. Math.*, **4**(1)(1982), 47-74., DOI: 10.1016/0166-218X(82)90033-6.
- [15] T. Zaslavsky, Signed graphs and geometry, *J. Combin. Inform. System Sci.*, **37**(2-4)(2012), 95-143.

## Extended Quasi Conformal Curvature Tensor on $N(k)$ -Contact Metric Manifold

Venkatesha, H.Aruna Kumara

(Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India)

R.T.Naveen Kumar

(Department of Mathematics, Siddaganga Institute of Technology, Tumakuru - 572 103, Karnataka, India)

vensmath@gmail.com, arunkumarh4665@gmail.com, rtnaveenkumar@gmail.com

**Abstract:** In this paper certain results on  $N(k)$ -contact metric manifold endowed with a extended quasi conformal curvature tensor are formulated. First, we consider  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifold. Next we describe extended quasi-conformally semi-symmetric and extended quasi conformal pseudo-symmetric  $N(k)$ -contact metric manifold. Finally, we study the conditions  $\tilde{C}_e(\xi, X) \cdot R = 0$  and  $\tilde{C}_e(\xi, X) \cdot S = 0$  on  $N(k)$ -contact metric manifold.

**Key Words:**  $N(k)$ -contact metric manifold, quasi conformal curvature tensor, extended quasi conformal curvature tensor,  $\eta$ -Einstein.

**AMS(2010):** 53C10, 53C15, 53C25.

### §1. Introduction

In 1968, Yano and Sawaki [20] introduced the notion of quasi conformal curvature tensor  $\tilde{C}$  on a Riemannian manifold  $M$  and is given by

$$\begin{aligned}\tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &- \frac{r}{2n+1} \left[ \frac{a}{2n} + 2b \right] \{g(Y, Z)X - g(X, Z)Y\}\end{aligned}\quad (1.1)$$

for all  $X, Y \in TM$ , where  $a$  and  $b$  are constants and  $r$  is a scalar curvature. If  $a = 1$  and  $b = -\frac{1}{2n-1}$ , then quasi conformal curvature tensor reduces to conformal curvature tensor.

The extended form of quasi conformal curvature tensor [8] is given by

$$\tilde{C}_e(X, Y)Z = \tilde{C}(X, Y)Z - \eta(X)\tilde{C}(\xi, Y)Z - \eta(Y)\tilde{C}(X, \xi)Z - \eta(Z)\tilde{C}(X, Y)\xi. \quad (1.2)$$

On the other hand Tanno [19] introduced a class of contact metric manifolds for which the characteristic vector field  $\xi$  belongs to the  $k$ -nullity distribution for some real number  $k$ . Such manifolds are known as  $N(k)$ -contact metric manifolds. The authors Blair, Kim and Tripathi [2]

---

<sup>1</sup>Received August 20, 2017, Accepted February 15, 2018.

gave the classification of  $N(k)$ -contact metric manifold satisfying the condition  $Z(\xi, X) \cdot Z = 0$ . Also quasi conformal curvature tensor on a sasakian manifold has been studied by De et al., [11]. Recently in [10], the authors study certain properties of  $N(k)$ -contact metric manifold endowed with a concircular curvature tensor.

Motivated by these studies the present paper is organized as follows: After giving preliminaries and basic formulas in Section 2, we study  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifolds in Section 3 and we found that the manifold is  $\eta$ -Einstein and also it admits a  $\eta$ -parallel Ricci tensor. In fact Section 4 is devoted to the study of extended quasi-conformally semi-symmetric  $N(k)$ -contact metric manifold and proved that the manifold is either locally isometric to  $E^{n+1} \times S^n(4)$  or it is extended quasi-conformally flat. Then, in Section 5, we consider extended quasi conformal pseudo-symmetric  $N(k)$ -contact metric manifold and we found that the manifold reduces to  $\eta$ -Einstein. Finally in the last section, we discuss  $N(k)$ -contact metric manifolds satisfying conditions  $\tilde{C}_e(\xi, X) \cdot R = 0$  and  $\tilde{C}_e(\xi, X) \cdot S = 0$

## §2. Preliminaries

A  $(2n+1)$ -dimensional smooth manifold  $M$  is said to be a contact manifold if it carries a global differentiable 1-form  $\eta$  which satisfies the condition  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M$ . Also a contact manifold admits an almost contact structure  $(\phi, \xi, \eta)$ , where  $\phi$  is  $(1, 1)$ -tensor field,  $\xi$  is a characteristic vector field and  $\eta$  is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \cdot \phi = 0. \quad (2.1)$$

An almost contact metric structure is said to be normal if the induced complex structure  $J$  on the product manifold  $M \times R$  is defined by,

$$J \left( X, f \frac{d}{dt} \right) = \left( \phi X - f\xi, \eta(X) \frac{d}{dt} \right),$$

is integrable, where  $X$  is tangent to  $M$ ,  $t$  is the coordinate of  $R$  and  $f$  is a smooth function on  $M \times R$ . Let  $g$  be the Riemannian metric with almost contact structure  $(\phi, \xi, \eta)$  i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

From (2.1), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad (2.2)$$

for all  $X, Y \in TM$ . An almost contact metric structure is called contact metric structure if  $g(X, \phi Y) = d\eta(X, Y)$ . Moreover, if  $\nabla$  denotes the Riemannian connection of  $g$ , then the following relation holds;

$$\nabla_X \xi = -\phi X - \phi hX. \quad (2.3)$$

A normal contact metric manifold is a Sasakian manifold. An almost metric manifold is

Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

where  $\nabla$  is the Levi-Civita connection of the Riemannian metric  $g$ .

As a generalization of both  $R(X, Y)\xi = 0$  and Sasakian case, the authors Blair, Koufogiorgos and Papantoniou [4] introduced the idea of  $(k, \mu)$ -nullity distribution on a contact metric manifold and gave several reasons for studying it. The  $(k, \mu)$ -nullity distribution  $N(k, \mu)$  of a contact metric manifold  $M$  is defined by

$$N(k, \mu) : p \rightarrow N_p(k, \mu) = \{Z \in T_p M : R(X, Y)Z = (kI + \mu h)(g(Y, Z)X - g(X, Z)Y)\},$$

where  $(k, \mu) \in \mathbb{R}^2$ . A contact metric manifold with  $\xi \in N(k, \mu)$  is called a  $(k, \mu)$ -contact metric manifold. If  $\mu = 0$ , the  $(k, \mu)$ -nullity distribution reduces to  $k$ -nullity distribution [19]. The  $k$ -nullity distribution  $N(k)$  of a Riemannian manifold is defined by

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \quad (2.4)$$

$k$  being a constant. If the characteristic vector field  $\xi \in N(k)$ , then we call a contact metric manifold as  $N(k)$ -contact metric manifold [2]. If  $k = 1$ , then the manifold is Sasakian and if  $k = 0$ , then the manifold is locally isometric to the product  $E^{n+1} \times S^n(4)$  for  $n > 1$  and flat for  $n = 1$  [3]. In an  $N(k)$ -contact metric manifold, the following relations holds:

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad (2.5)$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \quad (2.6)$$

$$\begin{aligned} S(X, Y) &= 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) \\ &\quad + [2nk - 2(n-1)]\eta(X)\eta(Y), \end{aligned} \quad (2.7)$$

$$S(X, \xi) = 2nk\eta(X), \quad Q\xi = 2nk\xi. \quad (2.8)$$

Also in an  $N(k)$ -contact metric manifold, extended quasi conformal curvature tensor satisfies the following:

$$\begin{aligned} \tilde{C}_e(X, Y)\xi &= \left[ a \left( \frac{r}{2n(2n+1)} - k \right) + 2b \left( \frac{r}{2n+1} - nk \right) \right] (\eta(Y)X - \eta(X)Y) \\ &\quad - b[\eta(Y)QX - \eta(X)QY], \end{aligned} \quad (2.9)$$

$$\begin{aligned} \tilde{C}_e(\xi, X)Y &= \left[ a \left( k - \frac{r}{2n(2n+1)} \right) + 2b \left( nk - \frac{r}{2n+1} \right) \right] (\eta(Y)X \\ &\quad - \eta(X)\eta(Y)\xi) + b[\eta(Y)QX - 2nk\eta(X)\eta(Y)\xi] = -\tilde{C}_e(X, \xi)Y, \end{aligned} \quad (2.10)$$

$$\tilde{C}_e(\xi, \xi)X = 0, \quad (2.11)$$

$$\eta(\tilde{C}_e(X, Y)\xi) = \eta(\tilde{C}_e(\xi, X)Y) = \eta(\tilde{C}_e(X, \xi)Y) = \eta(\tilde{C}_e(X, Y)Z) = 0. \quad (2.12)$$

By virtue of (1.2), let  $\{e_i\}$  be an orthonormal basis of the tangent space at each point of



the manifold and using (1.1) and (2.8), we get

$$\sum_{i=1}^{2n} g(\tilde{C}_e(e_i, Y)Z, e_i) = Lg(Y, Z) + MS(Y, Z) + N\eta(Y)\eta(W), \quad (2.13)$$

where,

$$\begin{aligned} L &= b(r - 2nk) - \left[ \frac{r(2n-1)}{2n+1} \left( \frac{a}{2n} + 2b \right) \right] - \left[ ka + 2nkb - \frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) \right], \\ M &= a + b(2n-3) \end{aligned}$$

and

$$N = 4nkb - (4n-3) \left[ ak + 2nkb - \frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) \right] - 2b(r - 2nk).$$

**Definition 2.1** A  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold  $M$  is said to be  $\eta$ -Einstein if its Ricci tensor  $S$  is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y),$$

for any vector fields  $X$  and  $Y$ , where  $\alpha$  and  $\beta$  are constants. If  $\beta=0$ , then the manifold  $M$  is an Einstein manifold.

**Definition 2.2** In a  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold, if the Ricci tensor  $S$  satisfies  $(\nabla_W S)(\phi X, \phi Y) = 0$ , then the Ricci tensor is said to be  $\eta$ -parallel.

In [1], Baikoussis and Koufogiorgos proved the following lemma.

**Lemma 2.1** Let  $M$  be an  $\eta$ -Einstein manifold of dimension  $(2n+1)$  ( $n \geq 1$ ). If  $\xi$  belongs to the  $k$ -nullity distribution, then  $k = 1$  and the structure is Sasakian.

### §3. $\xi$ -Extended Quasi Conformally Flat $N(k)$ -Contact Metric Manifolds

**Definition 3.1** A  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold is said to be  $\xi$ -extended quasi conformally flat if

$$\tilde{C}_e(X, Y)\xi = 0 \quad \text{for all } X, Y \in TM. \quad (3.1)$$

Let us consider  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifold. Then from (3.1) and (2.9), it can be easily seen that

$$\begin{aligned} 0 &= \left[ a \left( \frac{r}{2n(2n+1)} - k \right) + 2b \left( \frac{r}{2n+1} - nk \right) \right] (\eta(Y)X - \eta(X)Y) \\ &\quad - b[\eta(Y)QX - \eta(X)QY]. \end{aligned} \quad (3.2)$$

Taking inner product of (3.2) with respect to  $W$ , we get

$$\begin{aligned} 0 &= \left[ a \left( \frac{r}{2n(2n+1)} - k \right) + 2b \left( \frac{r}{2n+1} - nk \right) \right] (\eta(Y)g(X, W) - \eta(X)g(Y, W)) \\ &\quad - b[\eta(Y)S(X, W) - \eta(X)S(Y, W)]. \end{aligned}$$

On plugging  $Y = \xi$  in above equation, gives

$$S(X, W) = Ag(X, W) + B\eta(X)\eta(W), \quad (3.3)$$

where

$$A = \left[ \frac{r}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) - 2nk - \frac{ka}{b} \right] \quad \text{and} \quad B = \left[ 4nk + \frac{ka}{b} - \frac{r}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) \right].$$

Hence we can state the following:

**Theorem 3.1** *A  $(2n+1)$ -dimensional  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifold is an  $\eta$ -Einstein manifold.*

Hence in view of Lemma 2.1 and above result, we can state the following result:

**Corollary 3.1** *Let  $M$  be a  $(2n+1)$ -dimensional  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifold, then  $k = 1$  and the structure is Sasakian.*

Replacing  $X$  and  $W$  by  $\phi X$  and  $\phi W$  in (3.3) and using (2.1), we obtain

$$S(\phi X, \phi W) = M'g(\phi X, \phi W). \quad (3.4)$$

Now taking the covariant derivative of (3.4) with respect to  $U$ , yields

$$(\nabla_U S)(\phi X, \phi W) = \frac{dr(U)}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) g(\phi X, \phi W).$$

If we consider  $N(k)$ -contact metric manifold with constant scalar curvature, then above equation becomes

$$(\nabla_U S)(\phi X, \phi W) = 0.$$

Hence this leads us to the following result:

**Corollary 3.2** *A  $(2n+1)$ -dimensional  $\xi$ -extended quasi conformally flat  $N(k)$ -contact metric manifold with constant scalar curvature admits a  $\eta$ -parallel Ricci tensor.*

#### §4. Extended Quasi-Conformally Semi-Symmetric $N(k)$ -Contact Metric Manifold

Let us consider an extended quasi-conformally semi-symmetric  $N(k)$ -contact metric manifold

i.e.,

$$R(\xi, X) \cdot \tilde{C}_e = 0.$$

Then the above condition turns into,

$$\begin{aligned} 0 &= R(\xi, X)\tilde{C}_e(U, V)W - \tilde{C}_e(R(\xi, X)U, V)W \\ &\quad - \tilde{C}_e(U, R(\xi, X)V)W - \tilde{C}_e(U, V)R(\xi, X)W. \end{aligned} \quad (4.1)$$

In view of (2.6), equation (4.1) can be written as

$$\begin{aligned} 0 &= k \left[ g(X, \tilde{C}_e(U, V)W)\xi - \eta(\tilde{C}_e(U, V)W)X - g(X, U)\tilde{C}_e(\xi, V)W \right. \\ &\quad + \eta(U)\tilde{C}_e(X, V)W - g(X, V)\tilde{C}_e(U, \xi)W + \eta(V)\tilde{C}_e(U, X)W \\ &\quad \left. - g(X, W)\tilde{C}_e(U, V)\xi + \eta(W)\tilde{C}_e(U, V)X \right]. \end{aligned} \quad (4.2)$$

Which implies that either  $k = 0$  or

$$\begin{aligned} &\left[ g(X, \tilde{C}_e(U, V)W)\xi - \eta(\tilde{C}_e(U, V)W)X - g(X, U)\tilde{C}_e(\xi, V)W + \eta(U)\tilde{C}_e(X, V)W \right. \\ &\quad \left. - g(X, V)\tilde{C}_e(U, \xi)W + \eta(V)\tilde{C}_e(U, X)W - g(X, W)\tilde{C}_e(U, V)\xi + \eta(W)\tilde{C}_e(U, V)X \right] = 0. \end{aligned}$$

Now taking inner product of above equation with  $\xi$  and then using (2.12), we get

$$g(X, \tilde{C}_e(U, V)W) = 0.$$

Which implies that  $\tilde{C}_e(U, V)W = 0$ . Hence we can state the following:

**Theorem 4.1** *An extended quasi-conformally semi-symmetric  $N(k)$ -contact metric manifold is either locally isometric to  $E^{n+1} \times S^n(4)$  for  $n > 1$  and flat for  $n = 1$  or the manifold is extended quasi-conformally flat.*

## §5. Extended Quasi Conformal Pseudo-Symmetric $N(k)$ -Contact Metric Manifold

**Definition 5.1** *A  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold  $M$  is said to be extended quasi conformal pseudo-symmetric if*

$$(R(X, Y) \cdot \tilde{C}_e)(U, V)W = L_{\tilde{C}_e}[(X \wedge Y) \cdot \tilde{C}_e](U, V)W, \quad (5.1)$$

*holds for any vector fields  $X, Y, U, V, W \in TM$ , where  $L_{\tilde{C}_e}$  is function of  $M$ . The endomorphism  $X \wedge Y$  is defined by*

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \quad (5.2)$$

Now we prove the following result:

**Theorem 5.1** *Let  $M$  be a  $(2n+1)$ -dimensional extended quasi conformal pseudo-symmetric*

$N(k)$ -contact metric manifold. Then either  $L_{\tilde{C}_e} = k$  or the manifold is  $\eta$ -Einstein.

*Proof* Let us consider a  $(2n+1)$ -dimensional extended quasi conformal pseudo symmetric  $N(k)$ -contact metric manifold. Taking  $Y = \xi$  in (5.1), we get

$$(R(X, \xi) \cdot \tilde{C}_e)(U, V)W = L_{\tilde{C}_e}[(X \wedge \xi) \cdot \tilde{C}_e](U, V)W. \quad (5.3)$$

By virtue of (5.2) and (2.12), right hand side of above equation becomes

$$\begin{aligned} L_{\tilde{C}_e}[-g(X, \tilde{C}_e(U, V)W)\xi - \eta(U)\tilde{C}_e(X, V)W + g(X, U)\tilde{C}_e(\xi, V)W - \eta(V)\tilde{C}_e(U, X)W \\ + g(X, V)\tilde{C}_e(U, \xi)W - \eta(W)\tilde{C}_e(U, V)X + g(X, W)\tilde{C}_e(U, V)\xi]. \end{aligned} \quad (5.4)$$

In view of (2.5), left hand side of (5.3) gives

$$\begin{aligned} k[-g(X, \tilde{C}_e(U, V)W)\xi - \eta(U)\tilde{C}_e(X, V)W + g(X, U)\tilde{C}_e(\xi, V)W - \eta(V)\tilde{C}_e(U, X)W \\ + g(X, V)\tilde{C}_e(U, \xi)W - \eta(W)\tilde{C}_e(U, V)X + g(X, W)\tilde{C}_e(U, V)\xi]. \end{aligned} \quad (5.5)$$

By considering (5.5) and (5.4) in (5.3) with  $V = \xi$ , we get

$$\begin{aligned} (L_{\tilde{C}_e} - k)[-g(X, \tilde{C}_e(U, \xi)W)\xi - \eta(U)\tilde{C}_e(X, \xi)W + g(X, U)\tilde{C}_e(\xi, \xi)W \\ - \eta(\xi)\tilde{C}_e(U, X)W + g(X, \xi)\tilde{C}_e(U, \xi)W - \eta(W)\tilde{C}_e(U, \xi)X + g(X, W)\tilde{C}_e(U, \xi)\xi] = 0. \end{aligned} \quad (5.6)$$

By using (2.9)-(2.11) in (5.6), we have either  $(L_{\tilde{C}_e} - k) = 0$  or

$$\begin{aligned} \tilde{C}_e(U, X)W &= \left[ a \left( k - \frac{r}{2n(2n+1)} \right) + 2b \left( nk - \frac{r}{2n+1} \right) \right] \{ \eta(W)g(X, U)\xi \\ &- 2\eta(U)\eta(X)\eta(W)\xi + \eta(U)\eta(W)X + g(X, W)\eta(U)\xi - g(X, W)U \} \\ &+ b[\eta(W)S(X, U)\xi - 4nk\eta(U)\eta(X)\eta(W)\xi + \eta(U)\eta(W)QX \\ &+ 2nk g(X, W)\eta(U)\xi - g(X, W)QU]. \end{aligned} \quad (5.7)$$

On contracting (5.7) with respect to  $U$  and then using (2.13), we have

$$S(X, W) = A'g(X, W) + B'\eta(X)\eta(W),$$

where,

$$\begin{aligned} A' &= \frac{1}{a+b(2n-3)} \left[ (2-2n)ka + (6-2n)2nkb - \frac{r(3-4n)}{2n+1} \left( \frac{a}{2n} + 2b \right) - 2rb \right], \\ B' &= \frac{1}{a+b(2n-3)} \left[ (4n-3) \left( ka + 2nkb - \frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) \right) + 2br - 12nkb \right]. \end{aligned}$$

Thus  $M$  is a  $\eta$ -Einstein manifold.  $\square$

### §6. $N(k)$ -Contact Metric Manifold Satisfying $\tilde{C}_e(\xi, X) \cdot R = 0$ and $\tilde{C}_e(\xi, X) \cdot S = 0$

First we consider an  $N(k)$ -contact metric manifold satisfying  $\tilde{C}_e(\xi, X) \cdot R = 0$ . Now it follows from above condition that

$$0 = \tilde{C}_e(\xi, X)R(U, V)Y - R(\tilde{C}_e U, V)Y - R(U, \tilde{C}_e(\xi, X)V)Y - R(U, V)\tilde{C}_e(\xi, X)Y. \quad (6.1)$$

By virtue of (2.10) in (6.1), gives

$$\begin{aligned} 0 &= \left[ a \left( k - \frac{r}{2n(2n+1)} \right) + 2b \left( nk - \frac{r}{2n+1} \right) \right] \{ \eta(R(U, V)Y)[X - \eta(X)\xi] \\ &- \eta(U)[R(X, V)Y - \eta(X)R(\xi, V)Y] - \eta(V)[R(U, X)Y - \eta(X)R(U, \xi)Y] \\ &- \eta(Y)[R(U, V)X - \eta(X)R(U, V)\xi] \} + b \{ \eta(R(U, V)Y)[QX - 2nk\eta(X)\xi] \\ &- \eta(U)[R(QX, V)Y - 2nk\eta(X)R(\xi, V)Y] - \eta(V)[R(U, QX)Y \\ &- 2nk\eta(X)R(U, \xi)Y] - \eta(Y)[R(U, V)QX - 2nk\eta(X)R(U, V)\xi] \}. \end{aligned} \quad (6.2)$$

Considering  $U = \xi$  in (6.2), gives

$$\begin{aligned} 0 &= \left[ a \left( k - \frac{r}{2n(2n+1)} \right) + 2b \left( nk - \frac{r}{2n+1} \right) \right] \{ \eta(R(\xi, V)Y)[X - \eta(X)\xi] \\ &- [R(X, V)Y - \eta(X)R(\xi, V)Y] - \eta(V)[R(\xi, X)Y - \eta(X)R(\xi, \xi)Y] \\ &- \eta(Y)[R(\xi, V)X - \eta(X)R(\xi, V)\xi] \} + b \{ \eta(R(\xi, V)Y)[QX - 2nk\eta(X)\xi] \\ &- [R(QX, V)Y - 2nk\eta(X)R(\xi, V)Y] - \eta(V)[R(\xi, QX)Y \\ &- 2nk\eta(X)R(\xi, \xi)Y] - \eta(Y)[R(\xi, V)QX - 2nk\eta(X)R(\xi, V)\xi] \}. \end{aligned} \quad (6.3)$$

Taking inner product of (6.3) with respect to  $\xi$  and then by virtue of (2.5) and (2.6), we obtain

$$0 = k\eta(Y) [S(V, X) - A''g(V, X) - B''\eta(V)\eta(X)], \quad (6.4)$$

where  $A'' = \left[ \frac{r}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) - 2nk - \frac{ka}{b} \right]$  and  $B'' = \left[ 4nk + \frac{ka}{b} - \frac{r}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) \right]$ .

Since for an  $N(k)$ -contact metric manifolds  $\eta(Y) \neq 0$ , then (6.4) yields either  $k = 0$  or

$$S(V, X) = A''g(V, X) + B''\eta(V)\eta(X).$$

This leads us to the following:

**Theorem 6.1** *Let  $M$  be a  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold satisfying the condition  $\tilde{C}_e(\xi, X) \cdot R = 0$ . Then  $M$  reduces to  $\eta$ -Einstein manifold or it is locally isometric to  $E^{n+1} \times S^n(4)$  for  $n > 1$  and flat for  $n = 1$ .*

Next we prove the following result:

**Theorem 6.2** *Let  $M$  be an  $(2n+1)$ -dimensional  $N(k)$ -contact metric manifold satisfying  $\tilde{C}_e(\xi, X) \cdot S = 0$ . Then the Ricci tensor  $S$  is given by the equation (6.7).—*

*Proof* Let us consider an  $N(k)$ -contact metric manifold satisfying the condition  $\tilde{C}_e(\xi, X) \cdot S = 0$ . Then it can be easily seen that

$$S(\tilde{C}_e(\xi, X)Y, W) + S(Y, \tilde{C}_e(\xi, X)W) = 0. \quad (6.5)$$

By virtue of (2.10), it follows from above equation that

$$\begin{aligned} 0 = & \left[ a \left( k - \frac{r}{2n(2n+1)} \right) + 2b \left( nk - \frac{r}{2n+1} \right) \right] \{ \eta(Y)[S(X, W) - \eta(X)S(\xi, W)] \\ & + \eta(W)[S(Y, X) - \eta(X)S(Y, \xi)] \} + b \{ \eta(Y)[S(QX, W) - 2nk\eta(X)S(\xi, W)] \\ & + \eta(W)[S(Y, QX) - 2nk\eta(X)S(Y, \xi)] \}. \end{aligned} \quad (6.6)$$

On plugging  $Y = \xi$  in (6.6) and making necessary calculation, we have

$$S(QX, W) = MS(X, Y) + N\eta(X)\eta(W), \quad (6.7)$$

where,

$$\begin{aligned} M &= \left[ \frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) - \frac{ak}{b} - 2nk \right], \\ N &= \left[ \frac{2nk^2a}{b} + 8n^2k^2 - \frac{2nkr}{b(2n+1)} \left( \frac{a}{2n} + 2b \right) \right]. \end{aligned}$$

Hence the proof.  $\square$

## References

- [1] C. Baikoussis and T. Koufogiorgos, On a type of contact manifold, *Journal of Geometry*, 1-2 (1993), 19.
- [2] D.E. Blair, Two remarks on Contact manifold, *Tohoku Math. J.*, 29 (1977), 319-324.
- [3] D.E. Blair, J.S. Kim and M.M. Tripathi, On the concircular curvature tensor of a contact metric manifold, *J. Korean Math. Soc.*, 42 (2005), 5, 883-892.
- [4] D.E. Blair, T. Koufogiorgos and Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.*, 91, 189 (1995).
- [5] W.M. Boothby and H.C. Wang, On contact manifolds, *Ann. of Math.*, 68 (1958), 721-734.
- [6] Cihan Ozgur and Sibel Sular, On  $N(k)$ -contact metric manifolds satisfying certain conditions, *SUT Journal of Mathematics*, 44, 1 (2008), 89-99.
- [7] Cihan Ozgur and U.C. De, On the quasi-conformal curvature tensor of a kenmotsu manifold, *Mathematica Pannonica*, 17 (2) (2006), 221-228.
- [8] U.C. De and Avijit Sarkar, On the quasi-conformal curvature tensor of a  $(\kappa, \mu)$  contact metric manifold, *Math. Reports*, 14(64), 2(2012), 115-129
- [9] U.C. De and Y. Matsuyama, Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor, *SUT J. Math.*, 42 (2006), 295-303.
- [10] U.C. De, Ahmet Yildiz and Sujith Ghosh, On a class of  $N(k)$ -contact metric manifolds,

- Lecture Notes in Mathematics*, Math. Reports, 16 (66), 2 (2014), 207-217.
- [11] U.C. De, J.B. Jun and A.K. Gazi, Sasakian manifolds with quasi-conformal curvature tensor, *Bull. Korean Math. Soc.*, 45 (2008), 313-319.
  - [12] J-S. Kim, J. Choi, C. Ozgur and M.M. Tripathi, On the contact conformal curvature tensor of a contact metric manifold, *Indian J. Pure Appl. Math.*, 37 (2006), 4, 199-207.
  - [13] M. Kon, Invariant submanifolds in Sasakian manifolds, *Mathematische Annalen*, 219, 3 (1976), 277-290.
  - [14] B.J. Papantoniou, Contact Riemannian manifolds satisfying  $R(\xi, X) \cdot R = 0$  and  $k \in (k, \mu)$ -nullity distribution, *Yokohama Math. J.*, 40 (1993), 149-161.
  - [15] J.S. Pak and Y.J. Shin, A note on contact conformal curvature tensor, *Korean Math. Soc.*, 13 (1998), 2, 337-343.
  - [16] D. Perrone, Contact Riemannian manifolds satisfying  $R(\xi, X) \cdot R = 0$ , *Yokohama Math. J.*, 39 (1992), 141-149.
  - [17] D.G. Prakasha, C.S. Bagewadi and Venkatesha, Conformally and quasi-conformally conservative curvature tensors on a trans-Sasakian manifold with respect to semi-symmetric metric connections, *Differential Geometry - Dynamical Systems*, Vol. 10, (2008), 263-274.
  - [18] S. Sasaki, *Lecture Notes on Almost Contact Manifold*, Tohoku University I, 1965.
  - [19] S. Tanno, Ricci curvature of contact riemannian manifolds, *Tohoku Math. J.*, 40 (1988), 441-448.
  - [20] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Differential Geom.*, 2 (1968), 161-184.

## The Examination of Eigenvalues and Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to Boundary Conditions

Hülya GÜLTEKİN ÇİTİL

(Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, Turkey)

Nihat ALTINIŞIK

(Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Samsun, Turkey)

hulya.citil@giresun.edu.tr, anihat@omu.edu.tr

**Abstract:** In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the three different situations according to the boundary conditions. This examination is studied under the approach of Hukuhara differentiability. The examples are given for this situations.

**Key Words:** Fuzzy boundary value problems, second-order fuzzy differential equations, Hukuhara differentiability, eigenvalue, eigenfunction.

**AMS(2010):** 03E72, 34A07, 34B24.

### §1. Introduction

Approaches to fuzzy boundary value problems can be of two types. The first approach assumes that the derivative in the boundary value problem can be considered as a derivative of fuzzy function. This derivative can be Hukuhara derivative or a derivative in generalized sense [1,2,11,12]. The second approach is based on generating the fuzzy solution from crisp solution. In particular case, this approach can be of three ways. The first one uses the extension principle [3]. The second way uses the concept of differential inclusion [10]. In the third way the fuzzy problem is considered to be a set of crisp problem [6,7].

In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the approach of Hukuhara differentiability. It is seen that applied method for this examination is different according to given boundary conditions. Therefore, the eigenvalues and the eigenfunctions of the problem are examined under the three different situation and the examples are given for this situations.

### §2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

---

<sup>1</sup>Received June 28, 2017, Accepted February 16, 2018.



**Definition 2.1**([11]) A fuzzy number is a function  $u : \mathbb{R} \rightarrow [0, 1]$  satisfying the following properties:

- (i)  $u$  is normal;
- (ii)  $u$  is convex fuzzy set;
- (iii)  $u$  is upper semi-continuous on  $\mathbb{R}$ ;
- (iv)  $cl \{x \in \mathbb{R} \mid u(x) > 0\}$  is compact where  $cl$  denotes the closure of a subset.

Let  $\mathbb{R}_F$  denote the space of fuzzy numbers.

**Definition 2.2**([12]) Let  $u \in \mathbb{R}_F$ . The  $\alpha$ -level set of  $u$ , denoted  $[u]^\alpha$ ,  $0 < \alpha \leq 1$ , is

$$[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}.$$

If  $\alpha = 0$ , the support of  $u$  is defined

$$[u]^0 = cl \{x \in \mathbb{R} \mid u(x) > 0\}.$$

The notation, denotes explicitly the  $\alpha$ -level set of  $u$ . The notation,  $[u]^\alpha = [\underline{u}_\alpha, \bar{u}_\alpha]$  denotes explicitly the  $\alpha$ -level set of  $u$ . We refer to  $\underline{u}$  and  $\bar{u}$  as the lower and upper branches of  $u$ , respectively.

The following remark shows when  $[\underline{u}_\alpha, \bar{u}_\alpha]$  is a valid  $\alpha$ -level set.

**Remark 2.3**([11]) The sufficient and necessary conditions for  $[\underline{u}_\alpha, \bar{u}_\alpha]$  to define the parametric form of a fuzzy number as follows:

- (i)  $\underline{u}_\alpha$  is bounded monotonic increasing (nondecreasing) left-continuous function on  $(0, 1]$  and right-continuous for  $\alpha = 0$ ,
- (ii)  $\bar{u}_\alpha$  is bounded monotonic decreasing (nonincreasing) left-continuous function on  $(0, 1]$  and right-continuous for  $\alpha = 0$ ,
- (iii)  $\underline{u}_\alpha \leq \bar{u}_\alpha$ ,  $0 \leq \alpha \leq 1$ .

**Definition 2.4**([13]) If  $A$  is a symmetric triangular numbers with supports  $[\underline{a}, \bar{a}]$ , the  $\alpha$ -level sets of  $[A]^\alpha$  is  $[A]^\alpha = \left[ \underline{a} + \left( \frac{\bar{a} - \underline{a}}{2} \right) \alpha, \bar{a} - \left( \frac{\bar{a} - \underline{a}}{2} \right) \alpha \right]$ .

**Definition 2.5**([12]) For  $u, v \in \mathbb{R}_F$  and  $\lambda \in \mathbb{R}$ , the sum  $u + v$  and the product  $\lambda u$  are defined by  $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ ,  $[\lambda u]^\alpha = \lambda [u]^\alpha$ ,  $\forall \alpha \in [0, 1]$ , where means the usual addition of two intervals (subsets) of  $\mathbb{R}$  and  $\lambda [u]^\alpha$  means the usual product between a scalar and a subset of  $\mathbb{R}$ .

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}, \quad ([10])$$

by

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max \{ |\underline{u}_\alpha - \underline{v}_\alpha|, |\bar{u}_\alpha - \bar{v}_\alpha| \}.$$

**Definition 2.6**([13]) Let  $u, v \in \mathbb{R}_F$ . If there exists  $w \in \mathbb{R}_F$  such that  $u = v + w$ , then  $w$  is called the Hukuhara difference of fuzzy numbers  $u$  and  $v$ , and it is denoted by  $w = u \ominus v$ .

**Definition 2.7**([11]) Let  $f : [a, b] \rightarrow \mathbb{R}_F$  and  $t_0 \in [a, b]$ . We say that  $f$  is Hukuhara differential at  $t_0$ , if there exists an element  $f'(t_0) \in \mathbb{R}_F$  such that for all  $h > 0$  sufficiently small,  $\exists f(t_0 + h) \ominus f(t_0)$ ,  $f(t_0) \ominus f(t_0 - h)$  and the limits

$$\lim_{h \rightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Here the limits are taken in the metric space  $(\mathbb{R}_F, D)$ .

**Theorem 2.8**([5]) Let  $f : I \rightarrow \mathbb{R}_F$  be a function and denote  $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$ , for each  $\alpha \in [0, 1]$ . If  $f$  is Hukuhara differentiable, then  $\underline{f}_\alpha$  and  $\bar{f}_\alpha$  are differentiable functions and  $[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \bar{f}'_\alpha(t)]$ .

**Definition 2.9**([9]) Let  $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$  be a function.  $f^+$  and  $f^-$  are not the negative function defined as

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases}, \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

The function  $f^+$  and  $f^-$  are called the positive piece and negative piece of  $f$ , respectively.

### §3. The Eigenvalues and the Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to the Boundary Conditions

Let

$$\begin{aligned} Ly &= p(x)y'' + q(x)y, \quad p'(x) = 0, \\ A, B, C, D &\geq 0, \quad A^2 + B^2 \neq 0 \text{ and } C^2 + D^2 \neq 0. \end{aligned}$$

(I) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \quad x \in [a, b] \quad (3.1)$$

$$Ay(a) + By'(a) = 0, \quad (3.2)$$

$$Cy(b) + Dy'(b) = 0. \quad (3.3)$$

Let be functions  $\underline{\phi}_\alpha, \underline{\psi}_\alpha, \bar{\phi}_\alpha, \bar{\psi}_\alpha$  the solution of the fuzzy boundary value problem (3.1)-(3.3). The eigenvalues of the fuzzy boundary value problem (3.1)-(3.3) if and only if are consist of the zeros of functions  $\underline{W}_\alpha(\lambda)$  and  $\bar{W}_\alpha(\lambda)$ , where ([8])

$$\underline{W}_\alpha(\lambda) = W(\underline{\phi}_\alpha, \underline{\psi}_\alpha)(x, \lambda) = \underline{\phi}_\alpha(x, \lambda) \underline{\psi}'_\alpha(x, \lambda) - \underline{\psi}_\alpha(x, \lambda) \underline{\phi}'_\alpha(x, \lambda),$$

$$\overline{W}_\alpha(\lambda) = W(\overline{\phi}_\alpha, \overline{\psi}_\alpha)(x, \lambda) = \overline{\phi}_\alpha(x, \lambda) \overline{\psi}'_\alpha(x, \lambda) - \overline{\psi}_\alpha(x, \lambda) \overline{\phi}'_\alpha(x, \lambda).$$

**Example 3.1** Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0. \quad (3.4)$$

Let be  $\lambda = k^2$ ,  $k > 0$ ,

$$\phi(x, \lambda) = \sin(kx)$$

be the solution of the classical differential equation  $y'' + \lambda y = 0$  satisfying the condition  $y(0) = 0$  and

$$\psi(x, \lambda) = \left( \cos(k) + \frac{\sin(k)}{k} \right) \cos(kx) + \left( \sin(k) - \frac{\cos(k)}{k} \right) \sin(kx)$$

be the solution satisfying the condition  $y(1) = 1$ ,  $y'(1) = -1$ . Then,

$$[\phi(x, \lambda)]^\alpha = [\underline{\phi}_\alpha(x, \lambda), \overline{\phi}_\alpha(x, \lambda)] = [\alpha, 2 - \alpha] \sin(kx) \quad (3.5)$$

is the solution of the fuzzy differential equation  $y'' + \lambda y = 0$  satisfying the condition  $y(0) = 0$  and

$$[\psi(x, \lambda)]^\alpha = [\underline{\psi}_\alpha(x, \lambda), \overline{\psi}_\alpha(x, \lambda)] = [\alpha, 2 - \alpha] \psi(x, \lambda) \quad (3.6)$$

is the solution satisfying the condition  $y(1) = 1$ ,  $y'(1) = -1$ . Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.4) are zeros the functions  $\underline{W}_\alpha(\lambda)$  and  $\overline{W}_\alpha(\lambda)$ ,  $\underline{W}_\alpha(\lambda)$  is obtained as

$$\begin{aligned} \underline{W}_\alpha(\lambda) &= \alpha^2 \{ (-k \cos(k) - \sin(k)) \sin^2(kx) + (k \sin(k) - \cos(k)) \cos(kx) \sin(kx) + \\ &\quad + (-k \cos(k) - \sin(k)) \cos^2(kx) - (k \sin(k) - \cos(k)) \cos(kx) \sin(kx) \} \\ \underline{W}_\alpha(\lambda) &= -\alpha^2 (k \cos(k) + \sin(k)) \end{aligned}$$

and similarly  $\overline{W}_\alpha(\lambda)$  is obtained as

$$\overline{W}_\alpha(\lambda) = -(2 - \alpha)^2 (k \cos(k) + \sin(k)).$$

From here, yields

$$\underline{W}_\alpha(\lambda) = 0 \Rightarrow k \cos(k) + \sin(k) = 0,$$

$$\overline{W}_\alpha(\lambda) = 0 \Rightarrow k \cos(k) + \sin(k) = 0.$$

Computing the values  $k$  satisfying the equation  $k \cos(k) + \sin(k) = 0$ , we have

$$k_1 = 2.028757838, \quad k_2 = 4.913180439, \quad k_3 = 7.978665712, \quad k_4 = 11.08553841, \dots$$

We show that this values are  $k_n$ ,  $n = 1, 2, \dots$  Substituting this values in (3.5), (3.6), we obtain

$$[\phi_n(x)]^\alpha = [\underline{\phi}_{n_\alpha}(x), \overline{\phi}_{n_\alpha}(x)] = [\alpha, 2 - \alpha] \sin(k_n x),$$

$$[\psi_n(x)]^\alpha = [\underline{\psi}_{n_\alpha}(x), \overline{\psi}_{n_\alpha}(x)] = [\alpha, 2 - \alpha] \left( \sin(k_n) - \frac{\cos(k_n)}{k_n} \right) \sin(k_n x).$$

As

$$[\alpha, 2 - \alpha] (\sin(k_n x))^+$$

and

$$[\alpha, 2 - \alpha] \left( \left( \sin(k_n) - \frac{\cos(k_n)}{k_n} \right) \sin(k_n x) \right)^+, [\phi_n(x)]^\alpha$$

and  $[\psi_n(x)]^\alpha$  are a valid  $\alpha$ -level set. Let be  $k_n x \in [(n-1)\pi, n\pi]$ ,  $n = 1, 2, \dots$

(i) If  $n$  is single,  $\sin(k_n x) \geq 0$ . Then  $[\phi_n(x)]^\alpha$  is a valid  $\alpha$ -level set.

(ii) If  $n$  is double,  $\sin(k_n x) \leq 0$ . Also, since  $x \in [0, 1]$ ,  $k_n \in [(n-1)\pi, n\pi]$ , and according to Fig.1,  $\sin(k_n) - \frac{\cos(k_n)}{k_n} < 0$  for  $n$  is double. Then  $[\psi_n(x)]^\alpha$  is a valid  $\alpha$ -level set.

Consequently,  $k_n x \in [(n-1)\pi, n\pi]$ ,  $n = 1, 2, \dots$

(i) If  $n$  is single, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

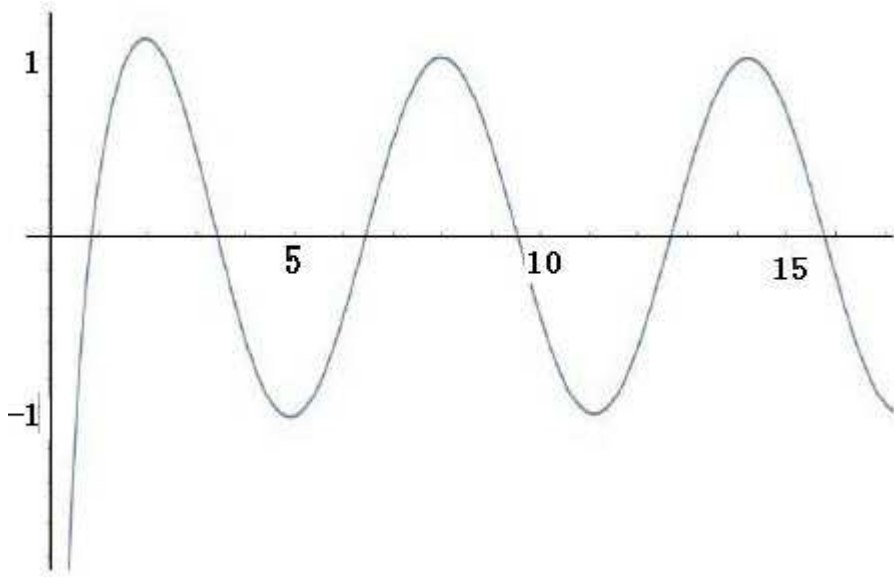
$$[y_{1n}(x)]^\alpha = [\alpha, 2 - \alpha] \sin(k_n x),$$

(ii) If  $n$  is double, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2 - \alpha] \left( \sin(k_n) - \frac{\cos(k_n)}{k_n} \right) \sin(k_n x),$$

(iii) If  $\alpha = 1$ , the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$



**Fig.1** The graphic of the function  $f(k) = \sin(k) - \frac{\cos(k)}{k}$

(II) Consider the eigenvalues of the fuzzy boundary value problem,

$$Ly + \lambda y = 0, \quad x \in (a, b) \quad (3.7)$$

$$Ay(a) = By'(a), \quad (3.8)$$

$$Cy(b) = Dy'(b). \quad (3.9)$$

$$[\phi(x, \lambda)]^\alpha = [\underline{\phi}_\alpha(x, \lambda), \overline{\phi}_\alpha(x, \lambda)]$$

is the solution of the fuzzy differential equation (3.7) satisfying the conditions  $y(a) = B$ ,  $y'(a) = A$  and

$$[\psi(x, \lambda)]^\alpha = [\underline{\psi}_\alpha(x, \lambda), \overline{\psi}_\alpha(x, \lambda)]$$

is the solution satisfying the conditions

$$y(b) = D, \quad y'(b) = C.$$

Hence, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is valid for the problem (3.7)-(3.9).

**Example 3.2** Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = y(1). \quad (3.10)$$

Let  $\lambda = k^2$ ,  $k > 0$ . Then,

$$[\phi(x, \lambda)]^\alpha = [\underline{\phi}_\alpha(x, \lambda), \overline{\phi}_\alpha(x, \lambda)] = [\alpha, 2 - \alpha] \sin(kx) \quad (3.11)$$

is the solution of the fuzzy differential equation  $y'' + \lambda y = 0$  satisfying the condition  $y(0) = 0$  and

$$[\psi(x, \lambda)]^\alpha = [\underline{\psi}_\alpha(x, \lambda), \overline{\psi}_\alpha(x, \lambda)] = [\alpha, 2 - \alpha] \psi(x, \lambda) \quad (3.12)$$

is the solution satisfying the condition  $y'(1) = y(1)$ , where

$$\psi(x, \lambda) = \left( \cos(k) - \frac{\sin(k)}{k} \right) \cos(kx) + \left( \sin(k) + \frac{\cos(k)}{k} \right) \sin(kx).$$

Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.10) are zeros the functions  $\underline{W}_\alpha(\lambda)$  and  $\overline{W}_\alpha(\lambda)$ , we obtained

$$\underline{W}_\alpha(\lambda) = -\alpha^2(k \cos(k) - \sin(k)), \quad \overline{W}_\alpha(\lambda) = -(2 - \alpha)^2(k \cos(k) - \sin(k)).$$

Computing the values  $k$  satisfying the equation  $k \cos(k) - \sin(k) = 0$ , we have

$$k_1 = 4.493409458, \quad k_1 = 7.725251837, \quad k_3 = 10.90412166, \quad k_4 = 14.06619391, \dots$$

We show that this values are  $k_n$ ,  $n = 1, 2, \dots$ . Substituting this values in (3.11), (3.12),

we obtain

$$[\phi_n(x)]^\alpha = [\underline{\phi}_{n\alpha}(x), \overline{\phi}_{n\alpha}(x)] = [\alpha, 2 - \alpha] \sin(k_n x),$$

$$[\psi_n(x)]^\alpha = [\underline{\psi}_{n\alpha}(x), \overline{\psi}_{n\alpha}(x)] = [\alpha, 2 - \alpha] \left( \sin(k_n) + \frac{\cos(k_n)}{k_n} \right) \sin(k_n x).$$

Let  $k_n x \in [(n-1)\pi, n\pi]$ ,  $n = 1, 2, \dots$

(i) If  $n$  is single,  $\sin(k_n x) \geq 0$ . Then  $[\phi_n(x)]^\alpha$  is a valid  $\alpha$ -level set.

(ii) If  $n$  is double,  $\sin(k_n x) \leq 0$ . Also, since  $x \in [0, 1]$ ,  $k_n \in [(n-1)\pi, n\pi]$  and according to Fig.2,  $\sin(k_n) + \frac{\cos(k_n)}{k_n} < 0$  for  $n$  is double. Then  $[\psi_n(x)]^\alpha$  is a valid  $\alpha$ -level set.

Consequently,  $k_n x \in [(n-1)\pi, n\pi]$ ,  $n = 1, 2, \dots$

(i) If  $n$  is single, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

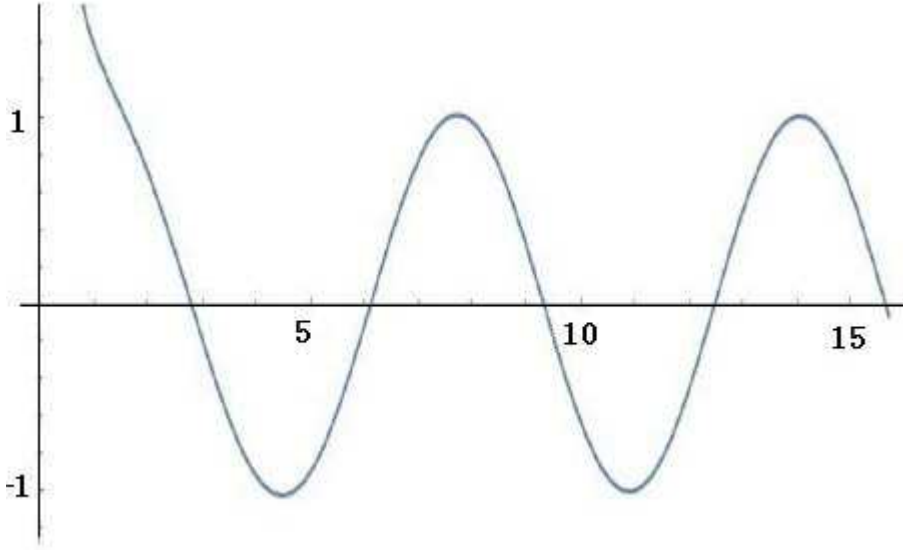
$$[y_{1n}(x)]^\alpha = [\alpha, 2 - \alpha] \sin(k_n x),$$

(ii) If  $n$  is double, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2 - \alpha] \left( \sin(k_n) + \frac{\cos(k_n)}{k_n} \right) \sin(k_n x),$$

(iii) If  $\alpha = 1$ , the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$



**Fig.2** The graphic of the function  $f(k) = \sin(k) + \frac{\cos(k)}{k}$

(III) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \quad x \in (a, b), \quad (3.13)$$

$$-Ay(a) + By'(a) = 0, \quad (3.14)$$

$$-Cy(b) + Dy'(b) = 0. \quad (3.15)$$

Let be  $[y]^\alpha = [\underline{y}_\alpha, \overline{y}_\alpha]$  the general solution of the fuzzy differential equation (3.13). From the boundary condition (3.14)

$$-A [\underline{y}_\alpha(a, \lambda), \overline{y}_\alpha(a, \lambda)] + B [\underline{y}_\alpha'(a, \lambda), \overline{y}_\alpha'(a, \lambda)]' = 0.$$

Using the Hukuhara differentiability, the fuzzy arithmetic and  $-[y]^\alpha = [-\overline{y}_\alpha, -\underline{y}_\alpha]$ , we obtained

$$[-A\overline{y}_\alpha(a, \lambda) + B\underline{y}_\alpha'(a, \lambda), -A\underline{y}_\alpha(a, \lambda) + B\overline{y}_\alpha'(a, \lambda)] = 0.$$

From here, the equations

$$-A\overline{y}_\alpha(a, \lambda) + B\underline{y}_\alpha'(a, \lambda) = 0 \quad (3.16)$$

$$-A\underline{y}_\alpha(a, \lambda) + B\overline{y}_\alpha'(a, \lambda) = 0 \quad (3.17)$$

are obtained. So we can not decompose the lower solution and upper solution. Therefore we can not find the function  $\phi_\alpha(x, \lambda)$  satisfying the condition (3.16) and the function  $\overline{\phi}_\alpha(x, \lambda)$  satisfying the condition (3.17) of the fuzzy differential equation (3.13). Consequently, there is not the function

$$[\phi(x, \lambda)]^\alpha = [\underline{\phi}_\alpha(x, \lambda), \overline{\phi}_\alpha(x, \lambda)]$$

satisfying the condition (3.14) of the fuzzy differential equation (3.13). Similarly, there is not the function

$$[\psi(x, \lambda)]^\alpha = [\underline{\psi}_\alpha(x, \lambda), \overline{\psi}_\alpha(x, \lambda)]$$

satisfying the condition (3.15) of the fuzzy differential equation (3.13).

Therefore, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is not valid for the problem (3.13)-(3.15).

**Example 3.3** Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) - y(1) = 0. \quad (3.18)$$

Let be  $[y]^\alpha = [\underline{y}_\alpha, \overline{y}_\alpha]$  and  $\lambda = k^2$ ,  $k > 0$ . Then, the lower and upper solutions of the fuzzy differential equation in (3.18) are

$$\underline{y}_\alpha(x, \lambda) = c_1(\alpha) \cos(kx) + c_2(\alpha) \sin(kx), \quad (3.19)$$

$$\overline{y}_\alpha(x, \lambda) = c_3(\alpha) \cos(kx) + c_4(\alpha) \sin(kx). \quad (3.20)$$

From the boundary conditon  $y(0) = 0$

$$\underline{y}_\alpha(0, \lambda) = c_1(\alpha) = 0, \quad \overline{y}_\alpha(0, \lambda) = c_3(\alpha) = 0.$$

From the boundary conditon  $y'(1) - y(1) = 0$ ,

$$\underline{y}'_{\alpha}(1, \lambda) - \overline{y}_{\alpha}(1, \lambda) = 0, \quad \overline{y}'_{\alpha}(1, \lambda) - \underline{y}_{\alpha}(1, \lambda) = 0$$

and from here, we obtain the system of equations

$$\underline{y}'_{\alpha}(1, \lambda) - \overline{y}_{\alpha}(1, \lambda) = kc_2(\alpha) \cos(k) - c_4(\alpha) \sin(k) = 0,$$

$$\overline{y}'_{\alpha}(1, \lambda) - \underline{y}_{\alpha}(1, \lambda) = kc_4(\alpha) \cos(k) - c_2(\alpha) \sin(k) = 0.$$

If

$$\begin{vmatrix} k \cos(k) & -\sin(k) \\ -\sin(k) & k \cos(k) \end{vmatrix} = k^2 \cos^2(k) - \sin^2(k) = 0,$$

there is the non-zero solution of the system of equation. Computing the values  $k$  satisfying this equation, we have

$$k_1 = 2.028757838, \quad k_2 = 4.493409458, \quad k_3 = 4.913180439, \quad k_4 = 7.725251837, \dots$$

We show that this values are  $k_n, n = 1, 2, \dots$ . Substituing this values in (3.19), (3.20), we obtain

$$\underline{y}_{n\alpha}(x) = c_2(\alpha) \sin(k_n x), \quad \overline{y}_{n\alpha}(x) = c_4(\alpha) \sin(k_n x),$$

$$[y_n(x)]^{\alpha} = [\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)].$$

As

$$\frac{\partial \underline{y}_{n\alpha}(x)}{\partial \alpha} \geq 0, \quad \frac{\partial \overline{y}_{n\alpha}(x)}{\partial \alpha} \leq 0 \text{ and } \underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x),$$

$[y_n(x)]^{\alpha}$  is valid  $\alpha$ -level set. Then  $k_n x \in [m\pi, (m+1)\pi], m = 0, 1, \dots$

(i) If  $m$  is double, since  $\sin(k_n x) \geq 0$ , it must be  $\frac{\partial c_2(\alpha)}{\partial \alpha} \geq 0, \frac{\partial c_4(\alpha)}{\partial \alpha} \leq 0$  and  $c_2(\alpha) \leq c_4(\alpha)$ ,

(ii) If  $m$  is single, since  $\sin(k_n x) \leq 0$ , it must be  $\frac{\partial c_2(\alpha)}{\partial \alpha} \leq 0, \frac{\partial c_4(\alpha)}{\partial \alpha} \geq 0$  and  $c_2(\alpha) \geq c_4(\alpha)$ .

Consequently,  $k_n x \in [m\pi, (m+1)\pi], m = 0, 1, \dots$

(i) If  $m$  is double, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_{1n}(x)]^{\alpha} = [c_2(\alpha) \sin(k_n x), c_4(\alpha) \sin(k_n x)],$$

for  $c_2(\alpha)$  and  $c_4(\alpha)$  satisfying the inequalities  $\frac{\partial c_2(\alpha)}{\partial \alpha} > 0, \frac{\partial c_4(\alpha)}{\partial \alpha} < 0$  and  $c_2(\alpha) < c_4(\alpha)$ .

(ii) If  $m$  is single, the eigenvalues are  $\lambda_n = k_n^2$ , with associated eigenfunctions

$$[y_{2n}(x)]^{\alpha} = [c_2(\alpha) \sin(k_n x), c_4(\alpha) \sin(k_n x)],$$

for  $c_2(\alpha)$  and  $c_4(\alpha)$  satisfying the inequalities  $\frac{\partial c_2(\alpha)}{\partial \alpha} < 0, \frac{\partial c_4(\alpha)}{\partial \alpha} > 0$  and  $c_2(\alpha) > c_4(\alpha)$ .

(iii) If  $\frac{\partial c_2(\alpha)}{\partial \alpha} = 0, \frac{\partial c_4(\alpha)}{\partial \alpha} = 0$  and  $c_2(\alpha) = c_2 = c_4 = c_4(\alpha)$ , the eigenvalues are  $\lambda_n = k_n^2$ ,



with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

## References

- [1] B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151 (2005) 581-599.
- [2] B. Bede, A note on two-point boundary value problems associated with non-linear fuzzy differential equations, *Fuzzy Sets and Systems*, 157 (2006) 986-989.
- [3] J. J. Buckley, T. Feuring, Fuzzy differential equations, *Fuzzy Sets and Systems*, 110 (2000) 43-54.
- [4] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, *Chaos, Solitons & Fractals*, 38 (2008) 112-119.
- [5] O. S. Fard, A. Esfahani, A. V. Kamyad, On Solution Of A Class Of Fuzzy BVPs, *Iranian of Fuzzy Systems*, 9 (1) (2012), 49-60.
- [6] N. A. Gasilov, Ş. E. Amrahov, A. G. Fatullayev, A geometric approach to solve fuzzy linear systems of differential equations, *Appl. Math. Inf. Sci.*, 5 (2011) 484-495.
- [7] N. A. Gasilov, Ş. E. Amrahov, A. G. Fatullayev, Linear differential equations with fuzzy boundary values, in: *Proceedings of the 5th International Conference on Application of Information and Communication Technologies*, IEEE, 2011, 696–700.
- [8] H. Gültekin Çitil, N. Altınışık, On the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem, *Journal of Mathematical and Computational Science*, 7(4) (2017), 786-805.
- [9] J.-K. Hunter, B. Nachtergaele, *Applied Analysis*, World Scientific, River Edge, NJ, Singapore, 2001.
- [10] E. Hüllermeier, An approach to modelling and simulation of uncertain systems, *Int. J. Uncertain. Fuzz., Knowledge-Based System*, 5 (1997) 117-137.
- [11] A. Khastan, F. Bahrami, K. Ivaz, New results on multiple solutions for nth-order fuzzy differential equations under generalized differentiability, *Boundary Value Problems*, Article ID 395714, (2009), 13 pp.
- [12] A. Khastan, J. J. Nieto, A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis*, 72 (2010) 3583-3593.
- [13] H.-K. Liu, Comparations results of two-point fuzzy boundary value problems, *International Journal of Computational and Mathematical Sciences*, 5(1) (2011) 1-7.

## Nonholonomic Frames for Finsler Space with Deformed Special $(\alpha, \beta)$ Metric

Brijesh Kumar Tripathi

(Department of Mathematics, L.E. College Morbi, Morbi-2, (Gujarat), 363642, India)

E-mail: brijeshkumartripathi4@gmail.com

**Abstract:** The purpose of present paper to determine the Finsler spaces due to deformation of special Finsler  $(\alpha, \beta)$  metrics. Consequently, we obtain the nonholonomic frame with the help of Riemannian metric  $\alpha^2 = a_{ij}(x)y^i y^j$ , one form metric  $\beta = b_i(x)y^i$  and Douglas metric  $L(\alpha, \beta) = \left(\alpha + \frac{\beta^2}{\alpha}\right)$  such as forms

$$\begin{aligned} \text{I. } L(\alpha, \beta) &= \alpha\beta, \\ \text{II. } L(\alpha, \beta) &= \left(\alpha + \frac{\beta^2}{\alpha}\right)\beta = \alpha\beta + \frac{\beta^3}{\alpha}. \end{aligned}$$

The first metric of the above deformation is obtained by the product of Riemannian metric and one form and second one is the product of Douglas metric and 1-form metric.

**Key Words:** Finsler space,  $(\alpha, \beta)$ -metrics, Riemannian metric, one form metric, Douglas metric, GL-metric, nonholonomic Finsler frame.

**AMS(2010):** 53C60, 53B40.

### §1. Introduction

P.R. Holland [1], [2] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field in the year 1982. In the year 1987, R.S. Ingarden [3] was the first person, to point out that the Lorentz force law can be written in above case as geodesic equation on a Finsler space called Randers space. Further in 1995, R.G. Beil [5], [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered the fundamental tensor field might be the deformation of two different special Finsler spaces from the  $(\alpha, \beta)$ - metrics. First we consider a nonholonomic frame for a Finsler space with  $(\alpha, \beta)$ - metrics such as first product of Riemannian metric and 1-form metric and second is the product of Douglas metric and 1-form metric. Further we obtain

---

<sup>1</sup>Received August 2, 2017, Accepted February 20, 2018.

a corresponding frame for each of these two Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [10], Tripathi [14, 16] and Narasimhamurthy [15].

## §2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with  $(\alpha, \beta)$ -metrics [11].

**Definition 2.1** A Finsler space  $F^n = \{M, F(x, y)\}$  is called with  $(\alpha, \beta)$ -metric if there exists a 2-homogeneous function  $L$  of two variables such that the Finsler metric  $F : TM \rightarrow R$  is given by

$$F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}, \quad (2.1)$$

where  $\alpha^2(x, y) = a_{ij}(x)y^i y^j$ ,  $\alpha$  is a Riemannian metric on the manifold  $M$ , and  $\beta(x, y) = b_i(x)y^i$  is a 1-form on  $M$ .

The first Finsler spaces with  $(\alpha, \beta)$ -metrics were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case by considering,  $a_{ij}(x)$  the components of a Riemannian metric on the base manifold  $M$ ,  $a(x, y) > 0$  and  $b(x, y) \geq 0$  Two functions on  $TM$  and  $B(x, y) = B_i(x, y)(dx^i)$  a vertical 1-form on  $TM$ . Then

$$g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x, y)B_j(x, y).$$

Now a days the above generalized Lagrange metric is known as the Beil metric. The metric tensor  $g_{ij}$  is also known as a Beil deformation of the Riemannian metric  $a_{ij}$ . It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for  $a(x, y) = \exp(2\sigma(x, y))$  and  $b = 0$ . The case  $a(x, y) = 1$  with various choices of  $b$  and  $B_i$  was introduced and studied by R.G. Beil for constructing a new unified field theory [6]. Further Bucataru [12] considered the class of Lagrange spaces with  $(\alpha, \beta)$ -metric and obtained some new and interesting results in the year 2002.

A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland. If we do not ask for the function  $L$  to be homogeneous of order two with respect to the  $(\alpha, \beta)$  variables, then we have a Lagrange space with  $(\alpha, \beta)$ -metric. Next we defined some different Finsler space with  $(\alpha, \beta)$ -metrics.

Further consider  $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$  the fundamental tensor of the Randers space  $(M, F)$ . Taking into account the homogeneity of  $\alpha$  and  $F$  we have the following formulae:

$$\begin{aligned} p^i &= \frac{1}{a} y^i = a^{ij} \frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij} p^j = \frac{\partial \alpha}{\partial y^i}; \\ l^i &= \frac{1}{L} y^i = g^{ij} \frac{\partial L}{\partial y^j}; \quad l_i = g_{ij} l^j = \frac{\partial L}{\partial y^i} = P_i + b_i, \\ l^i &= \frac{1}{L} p^i; \quad l^i l_i = p^i p_i = 1; \quad l^i p_i = \frac{\alpha}{L}; \quad p^i l_i = \frac{L}{\alpha}; \end{aligned} \quad (2.2)$$

$$b_i P^i = \frac{\beta}{\alpha}; b_i l^i = \frac{\beta}{L}$$

with respect to these notations, the metric tensors  $a_{ij}$  and  $g_{ij}$  are related by [13],

$$g_{ij}(x, y) = \frac{L}{\alpha} a_{ij} + b_i P_j + P_i b_j - \frac{\beta}{\alpha} p_i p_j = \frac{L}{\alpha} (a_{ij} - p_i p_j) + l_i l_j. \quad (2.3)$$

**Theorem 2.1**([10]) *For a Finsler space  $(M, F)$  consider the metric with the entries:*

$$Y_j^i = \sqrt{\frac{\alpha}{L}} (\delta_j^i - l^i l_j + \sqrt{\frac{\alpha}{L}} p^i p_j) \quad (2.4)$$

defined on  $TM$ . Then  $Y_j = Y_j^i (\frac{\partial}{\partial y^i})$ ,  $j \in 1, 2, 3, \dots, n$  is a nonholonomic frame.

**Theorem 2.2**([7]) *With respect to frame the holonomic components of the Finsler metric tensor  $a_{\alpha\beta}$  is the Randers metric  $g_{ij}$ , i.e.,*

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}. \quad (2.5)$$

Throughout this section we shall rise and lower indices only with the Riemannian metric  $a_{ij}(x)$  that is  $y_i = a_{ij} y^j$ ,  $\beta^i = a^{ij} b_j$ , and so on. For a Finsler space with  $(\alpha, \beta)$ -metric  $F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}$  we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2} \frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta}; \rho_{-2} = \frac{1}{2\alpha^2} \left( \frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha} \frac{\partial L}{\partial \alpha} \right), \quad (2.6)$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with  $(\alpha, \beta)$ -metric we have

$$\rho_{-1} \beta + \rho_{-2} \alpha^2 = 0 \quad (2.7)$$

with respect to the notations we have that the metric tensor  $g_{ij}$  of a Finsler space with  $(\alpha, \beta)$ -metric is given by [13]

$$g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1} \{b_i(x) y_j + b_j(x) y_i\} + \rho_{-2} y_i y_j. \quad (2.8)$$

From (2.8) we can see that  $g_{ij}$  is the result of two Finsler deformations

$$\begin{aligned} I. \quad a_{ij} &\rightarrow h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}} (\rho_{-1} b_i + \rho_{-2} y_i) (\rho_{-1} b_j + \rho_{-2} y_j), \\ II. \quad h_{ij} &\rightarrow g_{ij} = h_{ij} + \frac{1}{\rho_{-2}} (\rho_0 \rho_{-1} - \rho_{-1}^2) b_i b_j. \end{aligned} \quad (2.9)$$

The nonholonomic Finsler frame that corresponding to the  $I^{st}$  deformation (2.9) is according to the theorem (7.9.1) in [10], given by

$$X_j^i = \sqrt{\rho} \delta_j^i - \frac{1}{\beta^2} \{ \sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}} \} (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b_j + \rho_{-2} y_j), \quad (2.10)$$

where  $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2b^2 + \beta\rho_{-1}\rho_{-2}$ .

This metric tensor  $a_{ij}$  and  $h_{ij}$  are related by

$$h_{ij} = X_i^k X_j^l a_{kl}. \quad (2.11)$$

Again the frame that corresponds to the  $II_{nd}$  deformation (2.9) given by

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \left\{ 1 \pm \sqrt{1 + \left( \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2} \right)} \right\} b^i b_j, \quad (2.12)$$

where  $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$ .

The metric tensor  $h_{ij}$  and  $g_{ij}$  are related by the formula

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \quad (2.13)$$

**Theorem 2.3**([10]) *Let  $F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}$  be the metric function of a Finsler space with  $(\alpha, \beta)$  metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

*is a nonholonomic Finsler frame with  $X_k^i$  and  $Y_j^k$  are given by (2.10) and (2.12) respectively.*

### §3. Nonholonomic Frames for Finsler Space with Deformed $(\alpha, \beta)$ Metric

In this section we consider two cases of nonholonomic Finsler frames with special  $(\alpha, \beta)$ -metrics, such an  $I^{st}$  Finsler frame product of Riemannian metric, one form metric and  $II^{nd}$  Finsler frame product of Douglas metric and 1-form metric.

#### 3.1 Nonholonomic Frame for $L = \alpha\beta$

In the first case, for a Finsler space with the fundamental function  $L = \alpha\beta$  the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\beta}{2\alpha}, & \rho_0 &= 0, \\ \rho_{-1} &= \frac{1}{2\alpha}, & \rho_{-2} &= -\frac{\beta}{2\alpha^3}, \\ B^2 &= \frac{1}{4\alpha^4}(\alpha^2 b^2 - \beta^2). \end{aligned} \quad (3.1)$$

Using (3.1) in (2.10) we have

$$X_j^i = \sqrt{\frac{\beta}{2\alpha}} \delta_j^i - \frac{1}{4\alpha^2 \beta^2} \left[ \sqrt{\frac{\beta}{2\alpha}} + \sqrt{\frac{\beta - 4\alpha^4 \beta}{2\alpha}} \right] (b^i - \frac{\beta}{\alpha^2} y^i) (b_j - \frac{\beta}{\alpha^2} y_j). \quad (3.2)$$

Again using (3.1) in (2.12) we have

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \{1 \pm \sqrt{1 + \frac{2\beta C^2}{\alpha}}\} b^i b_j, \quad (3.3)$$

where  $C^2 = \frac{\beta}{2\alpha} b^2 - \frac{1}{2\alpha^3\beta} (\alpha^2 b^2 - \beta^2)^2$ .

**Theorem 3.1** *Let  $L = \alpha\beta$  be the metric function of a Finsler space with  $(\alpha, \beta)$  metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

*is nonholonomic Finsler Frame with  $X_k^i$  and  $Y_j^k$  are given by (3.2) and (3.3) respectively.*

### 3.2 Nonholonomic Frame for $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$

In the second case, for a Finsler space with the fundamental function  $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$  the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}, & \rho_0 &= \frac{3\beta}{\alpha}, \\ \rho_{-1} &= \frac{\alpha^2 - 3\beta^2}{2\alpha^3}, & \rho_{-2} &= \frac{3\beta^3 - \alpha^2\beta}{2\alpha^5}, \\ B^2 &= \frac{(\alpha^2 - 3\beta)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^8}. \end{aligned} \quad (3.4)$$

Using (3.4) in (2.10) we have

$$\begin{aligned} X_j^i &= \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}} \delta_j^i - \frac{(\alpha^2 - 3\beta^2)^2}{4\alpha^6\beta^2} \left[ \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}} \right. \\ &\quad \left. + \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3} + \frac{2\alpha^5\beta}{(3\beta^2 - \alpha^2)}} \right] (b^i - \frac{\beta}{\alpha^2} y^i) (b_j - \frac{\beta}{\alpha^2} y_j). \end{aligned} \quad (3.5)$$

Again using (3.4) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \{1 \pm \sqrt{1 + \frac{\alpha\beta C^2}{\alpha^2 - 3\beta^2}}\} b^i b_j, \quad (3.6)$$

where  $C^2 = \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3} b^2 + \frac{(3\beta^2 - \alpha^2)}{2\alpha^3\beta} (\alpha^2 b^2 - \beta^2)^2$ .

**Theorem 3.2** *Let  $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$  be the metric function of a Finsler space with  $(\alpha, \beta)$  metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

*is nonholonomic Finsler Frame with  $X_k^i$  and  $Y_j^k$  are given by (3.5) and (3.6) respectively.*

#### §4. Conclusions

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru ([7], [8]) has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with  $(\alpha, \beta)$ -metric, it appears a natural question: does how many Finsler space with  $(\alpha, \beta)$ -metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with  $(\alpha, \beta)$ -metrics.

In this work, we consider the Douglas metric, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many  $(\alpha, \beta)$ -metrics in future work we can determine the frames for them also.

#### References

- [1] Holland P.R., Electromagnetism, particles and anholonomy, *Physics Letters*, **91**(6) (1982), 275-278.
- [2] Holland P.R., Anholonomic deformations in the ether: a significance for the electrodynamic potentials, In: Hiley B.J., Peat F.D. (eds.), *Quantum Implications*, Routledge and Kegan Paul, London and New York, 295-311 (1987).
- [3] Ingarden R.S., On physical interpretations of Finsler and Kawaguchi spaces, *Tensor N.S.*, **46**,(1987), 354-360.
- [4] Randers G., On asymmetric metric in the four space of general relativity, *Phys. Rev.*, **59**,(1941), 195-199.
- [5] Beil R.G., Comparison of unified field theories, *Tensor N.S.*, **56**,(1995), 175-183.
- [6] Beil R.G., Finsler and Kaluza-Klein gauge theories, *Intern. J. Theor. Phys.*, **32**, 6(1993), 1021-1031.
- [7] Miron R. and Anastasiei M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., FTPH, No.59, (1994).
- [8] Antonelli P.L., Bucataru I., Finsler connections in anholonomic geometry of a Kropina space, —it Nonlinear Studies, **8** (1), (2001), 171-184.
- [9] Hrimiuc D., Shimada H., On the L-duality between Lagrange and Hamilton manifolds, *Nonlinear World*, **3**, (1996), 613-641.
- [10] Ioan Bucataru, Radu Miron, *Finsler-Lagrange Geometry: Applications to Dynamical Systems* CEEX ET 3174/2005-2007, and CEEX M III 12595/2007 (2007).
- [11] Matsumoto M., Theory of Finsler spaces with  $(\alpha, \beta)$ -metric, *Rep. Math. Phys.*, **31**, (1992), 43-83.
- [12] Bucataru I., Nonholonomic frames on Finsler geometry, *Balkan Journal of Geometry and its Applications*, **7** (1), (2002), 13-27.
- [13] Matsumoto M., *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaishisha Press, Otsu, Japan, 1986.

- [14] Tripathi Brijesh Kumar, Pandey K. B. and Tiwari R.B., Nonholonomic frame for Finsler space with  $(\alpha, \beta)$ -metric, *International J. Math. Combin.*, Vol.1, (2016), 109-115.
- [15] Narasimhamurthy S.K., Mallikarjun Y. Kumar, Kavyashree A. R., Nonholonomic frames for Finsler Space with special  $(\alpha, \beta)$ -metric, *International Journal of Scientific and Research Publications*, 4(1), (2014), 1-7.
- [16] Tripathi Brijesh Kumar and Chaubey V. K., Nonholonomic frames for Finsler space with generalized Kropina metric, *International Journal of Pure and Applied Mathematics*, Vol. 108(4), (2016) 921-928, doi:10.12732/ijpam.v108i4.17.



## Homology of a Type of Octahedron

Shouvik Datta Choudhury, Santu Dey and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata-700032, India)

E-mail: shouvikdc8645@gmail.com, santu.mathju@gmail.com, bhattachar1968@yahoo.co.in

**Abstract:** In this paper we find out the homology of a type of octahedron with six vertices, twelve edges and eight faces and have shown that it is analogous with the homology of a chain complex  $0 \longrightarrow \mathbf{R}^6 \longrightarrow \mathbf{R}^{12} \longrightarrow \mathbf{R}^8 \longrightarrow 0$  and also find out the singular homology and the Euler characteristic of this type of octahedron which is equal to  $\sum_{n=0}^{\infty} \dim_{\mathbf{R}}(H_n(S))$ , where  $S$  is a octahedron.

**Key Words:** Homology module, singular homology, homotopy.

**AMS(2010):** 13D05, 18G55, 18G35, 55U15, 18G20.

### §1. Introduction

Homology classes were first defined rigorously by Henri Poincaré in his seminal paper “Analysis situs” in 1895 referring to the work of Riemann, Betti and von Dyck. The homology group was further developed by Emmy Noether [1] and, independently, by Leopold Vietoris and Walther Mayer [2] in the period 1925-28.

In mathematics (especially algebraic topology and abstract algebra), homology is a certain general procedure to associate a sequence of abelian groups or modules with a given mathematical object such as a topological space or a group. So, in algebraic topology, singular homology refers to the study of a certain set of algebraic invariants of a topological space  $X$ , the so-called homology groups  $H_n(X)$ . Intuitively spoken, singular homology counts, for each dimension  $n$ , the  $n$ -dimensional holes of a space.

The abstract algebra invariants such as ring, field were used to make concept of homology more rigorous and these developments give rise to mathematical branches such as homological algebra and K-Theory.

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. Finally, it is detailed enough so that actual calculations may be performed in important cases.

Let  $f$  and  $g$  be matrices whose product is zero. If  $g.v = 0$  for some column vector  $v$ , say, of length  $n$ , we can not always write  $v = f.u$  for some row vector  $u$ . This failure is measured by the defect

$$d = n - \text{rank}(f) - \text{rank}(g).$$

---

<sup>1</sup>Received August 4, 2017, Accepted February 23, 2018.

Let  $R$  be a ring and let  $U$ ,  $V$  and  $W$  be the modules over  $R$ . In modern language,  $f$  and  $g$  represent linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

with  $gf = 0$ , and  $d$  is the dimension of the homology module

$$H = \ker(g)/f(U)$$

Given an  $R$ -module homomorphism  $f : A \longrightarrow B$ , one is immediately led to study the kernel  $\ker f$  and image  $\text{im} f$ . Given another map  $g : B \longrightarrow C$ , we can form the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C, \quad (1)$$

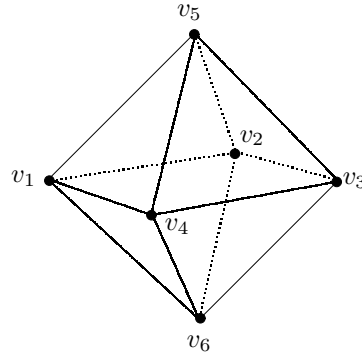
where  $A$ ,  $B$  and  $C$  are the modules over  $R$ . We say that such a sequence is exact (at  $B$ ) if  $\ker(g) = \text{im}(f)$ . This implies in particular that the composite  $gf : A \longrightarrow C$  is zero and finally brings our attention to sequence (1) such that  $gf = 0$ .

The word polyhedron has slightly different meanings in geometry and algebraic geometry. In elementary geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges. In [4], S. Dey et al. studied homology of a type of heptahedron. Here we consider polyhedron octahedron with eight faces, six vertices and twelve edges.

In this paper, first we find out that homology of a type of octahedron is analogous to the homology of a chain complex,  $0 \longrightarrow \mathbf{R}^7 \longrightarrow \mathbf{R}^{12} \longrightarrow \mathbf{R}^7 \longrightarrow 0$  and we also find out the matrices of this complex. Next we show computationally,  $H_2(S) \cong H_0(S) \cong \mathbf{R}$  and  $H_1(S) = 0$  and the Euler characteristic of this type of octahedron which is equal to  $\sum_{n=0}^{\infty} \dim_R(H_n(S))$ .

## §2. Homology of a Octahedron

We can obtain a chain complex from a geometric object. We refer to the Weibel's book [3] for some details of the construction. We illustrate it with a octahedron  $S$  in Fig.1 following.



**Fig.1**

Level the vertex set of S as  $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and then the twelve edges  $e_{12}, e_{23}, e_{34}, e_{41}, e_{15}, e_{25}, e_{35}, e_{45}, e_{16}, e_{26}, e_{36}, e_{46}$ , where  $e_{ij}=e_{ji}$  for  $i, j = 1, 2, 3, 4, 5, 6$  can be ordered as

$$\begin{aligned} E = & \{ \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_5\}, \{v_2, v_5\}, \\ & \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_6\}, \{v_2, v_6\}, \{v_3, v_6\}, \{v_4, v_6\} \} \end{aligned}$$

and seven faces  $f_1, f_2, f_3, f_4, f_5, f_6, f_7$  can be ordered as

$$\begin{aligned} F = & \{ \{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \\ & \{v_4, v_1, v_5\}, \{v_1, v_2, v_6\}, \{v_2, v_3, v_6\}, \{v_4, v_1, v_6\} \}. \end{aligned}$$

Let R be a ring and let  $C_i(S)$  be the free R-module on the set V, E, F, respectively. Define maps  $\partial_0, \partial_1, \partial_2: F \rightarrow E$  by removing the first, second, and third vertices, respectively except first face for each map. For the first face we define each map in such a way so that we can construct the homology. So,  $\partial_0, \partial_1, \partial_2$  are given by

$\partial_0 : f_1 \longrightarrow e_{25}$	$\partial_1 : f_1 \longrightarrow e_{15}$	$\partial_2 : f_1 \longrightarrow e_{12}$
$f_2 \longrightarrow e_{35}$	$f_2 \longrightarrow e_{25}$	$f_2 \longrightarrow e_{23}$
$f_3 \longrightarrow e_{45}$	$f_3 \longrightarrow e_{35}$	$f_3 \longrightarrow e_{34}$
$f_4 \longrightarrow e_{15}$	$f_4 \longrightarrow e_{45}$	$f_4 \longrightarrow e_{41}$
$f_5 \longrightarrow e_{26}$	$f_5 \longrightarrow e_{16}$	$f_5 \longrightarrow e_{12}$
$f_6 \longrightarrow e_{36}$	$f_6 \longrightarrow e_{26}$	$f_6 \longrightarrow e_{23}$
$f_7 \longrightarrow e_{46}$	$f_7 \longrightarrow e_{36}$	$f_7 \longrightarrow e_{34}$
$f_7 \longrightarrow e_{16}$	$f_7 \longrightarrow e_{46}$	$f_7 \longrightarrow e_{41}$

The set maps  $\partial_i$  yield  $k + 1$  module maps  $C_k \longrightarrow C_{k-1}$ , which we also call  $\partial_i$ , their alternating sum  $d_i = \Sigma(-1)^i \partial_i$  is the map  $C_k \longrightarrow C_{k-1}$ , where  $(0 \leq i \leq k \leq n)$  in the chain complex C. We can then define the map

$$d_2 = \partial_0 - \partial_1 + \partial_2 : C_2 \longrightarrow C_1,$$

which is given by

$$\begin{aligned} f_1 & \longrightarrow e_{25} - e_{15} + e_{12} \\ f_2 & \longrightarrow e_{35} - e_{25} + e_{23} \\ f_3 & \longrightarrow e_{45} - e_{35} + e_{34} \\ f_4 & \longrightarrow e_{15} - e_{45} + e_{41} \\ f_5 & \longrightarrow e_{26} - e_{16} + e_{12} \\ f_6 & \longrightarrow e_{36} - e_{26} + e_{23} \end{aligned}$$

$$\begin{aligned} f_7 &\longrightarrow e_{46} - e_{36} + e_{34} \\ f_8 &\longrightarrow e_{16} - e_{46} + e_{41} \end{aligned}$$

We can define maps  $\partial_0, \partial_1 : E \longrightarrow V$  by removing the first, second vertices, respectively. Therefore we have

$\begin{aligned} \partial_0 : e_{12} &\longrightarrow v_2 \\ e_{23} &\longrightarrow v_3 \\ e_{34} &\longrightarrow v_4 \\ e_{41} &\longrightarrow v_1 \\ e_{15} &\longrightarrow v_5 \\ e_{25} &\longrightarrow v_5 \\ e_{35} &\longrightarrow v_5 \\ e_{45} &\longrightarrow v_5 \\ e_{16} &\longrightarrow v_6 \\ e_{26} &\longrightarrow v_6 \\ e_{36} &\longrightarrow v_6 \\ e_{46} &\longrightarrow v_6 \end{aligned}$	$\begin{aligned} \partial_1 : e_{12} &\longrightarrow v_1 \\ e_{23} &\longrightarrow v_2 \\ e_{34} &\longrightarrow v_3 \\ e_{41} &\longrightarrow v_4 \\ e_{15} &\longrightarrow v_1 \\ e_{25} &\longrightarrow v_2 \\ e_{35} &\longrightarrow v_3 \\ e_{45} &\longrightarrow v_4 \\ e_{16} &\longrightarrow v_1 \\ e_{26} &\longrightarrow v_2 \\ e_{36} &\longrightarrow v_3 \\ e_{46} &\longrightarrow v_4 \end{aligned}$
---	---

We can define map  $d_1 = \partial_0 - \partial_1$  from  $C_1$  to  $C_0$ , and it is given by

$$\begin{aligned} e_{12} &\longrightarrow v_2 - v_1 \\ e_{23} &\longrightarrow v_3 - v_2 \\ e_{34} &\longrightarrow v_4 - v_3 \\ e_{41} &\longrightarrow v_1 - v_4 \\ e_{15} &\longrightarrow v_5 - v_1 \\ e_{25} &\longrightarrow v_5 - v_2 \\ e_{35} &\longrightarrow v_5 - v_3 \\ e_{45} &\longrightarrow v_5 - v_4 \\ e_{16} &\longrightarrow v_6 - v_1 \\ e_{26} &\longrightarrow v_6 - v_2 \\ e_{36} &\longrightarrow v_6 - v_3 \\ e_{46} &\longrightarrow v_6 - v_4 \end{aligned}$$

By viewing  $C_0 = \mathbf{R}^6$ ,  $C_1 = \mathbf{R}^{12}$ , and  $C_2 = \mathbf{R}^8$ , the maps  $d_1$  and  $d_2$  are given by the

following matrices

$$d_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$d_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Because  $d_1 d_2$  is easily computed to be zero matrix, the sequence

$$\cdots C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

is a complex. We compute the homology  $V_n(S)$  of this complex with the help of Matlab. By Finding the column space of  $d_1$ , we find  $\text{im}(d_1)$ . This space has a basis consisting of the vectors

$$\{(-1, 1, 0, 0, 0, 0), (0, -1, 1, 0, 0, 0), (0, 0, -1, 1, 0, 0), (-1, 0, 0, 0, 1, 0), (-1, 0, 0, 0, 0, 1)\}.$$

We note that by adding  $(0, 0, 0, 0, 0, 1)$  that we get a basis for  $\mathbf{R}^6$ . Therefore

$$C_0/\text{im}(d_1) \cong \mathbf{R}.$$

Thus

$$V_0(S) = \mathbf{R}.$$

Now,  $\ker(d_1)$  has a basis

$$\left\{ \begin{array}{ll} (1, 1, 1, 1, 1, 1, 1), & (1, 0, 1, 1, 0, 1, 1), \\ (1, 0, 0, 1, 0, 0, 1), & (1, 0, 0, 0, 0, 0, 0), \\ (0, -1, -1, -1, 0, 0, 0), & (0, 1, 0, 0, 0, 0, 0), \\ (0, 0, 1, 0, 0, 0, 0), & (0, 0, 0, 1, 0, 0, 0), \\ (0, 0, 0, 0, -1, -1, -1), & (0, 0, 0, 0, 1, 0, 0), \\ (0, 0, 0, 0, 0, 1, 0), & (0, 0, 0, 0, 0, 0, 1) \end{array} \right\}$$

Again,  $\text{im}(d_2)$  has a basis

$$\left\{ \begin{array}{ll} \{(1, 0, 0, 0, 0, 0, 0), & (0, 1, 0, 0, 0, 0, 0), \\ (0, 0, 1, 0, 0, 0, 0), & (0, 0, 0, 1, 0, 0, 0), \\ (0, 0, 0, 0, 1, 0, 0), & (0, 0, 0, 0, 0, 1, 0), \\ (0, 0, 0, 0, 0, 0, 1), & (0, 0, 0, 0, -1, -1, -1) \\ (-1, 0, 0, 1, -1, 0, 0), & (1, -1, 0, 0, 0, -1, 0), \\ (0, 1, -1, 0, 0, 0, -1), & (0, 0, 1, -1, 1, 1, 1)\} \end{array} \right\}$$

If  $u_i$  are the basis vectors of  $\ker(d_1)$ , then the following vectors of  $\text{im}(d_2)$  can be constructed in the following way:

$$\begin{aligned} (-1, 0, 0, 1, -1, 0, 0) & \text{ is } u_9 - u_4 + u_{12} + u_{10} + u_{11}, \\ (1, -1, 0, 0, 0, -1, 0) & \text{ is } u_4 - u_6 - u_{11}, \\ (0, 1, -1, 0, 0, 0, -1) & \text{ is } u_6 - u_7 - u_{12} \quad \text{and} \\ (0, 0, 1, -1, 1, 1, 1) & \text{ is } u_7 - u_8 + u_{10} + u_{12} + u_{11}. \end{aligned}$$

The rest of the elements of  $\text{im}(d_2)$  can be found in  $\ker(d_1)$ .

Thus we see that  $\text{im}(d_2) = \ker(d_1)$ . Therefore,  $V_1(S) = 0$ . Finally,  $\ker(d_2)$  has a basis of one element  $\{(-1, -1, -1, -1, 1, 1, 1)\}$ . So,  $V_2(S) = \ker d_2 = \mathbf{R}$ . To summarize, the singular homology  $V_n(S)$  of the Octahedron is

$$\begin{aligned} V_0(S) &= V_2(S) = \mathbf{R}, \\ V_1(S) &= 0, \\ V_n(S) &= 0 \text{ if } n \geq 3. \end{aligned}$$

The *Euler characteristic* is a fundamental invariant for the classification of surfaces, so it is particularly useful that it can be calculated with homological algebra. The Euler characteristic of such surface  $H$  is  $v - e + f$ , where  $v$  is the number of vertices,  $e$  is the number of edges and  $f$  is the number of faces. Now, the Euler characteristic of octahedron is 2, which is equal to  $\sum_{n=0}^{\infty} \dim_R(V_n(S))$ . This is the same as the Euler Characteristic of a sphere as a octahedron is

homeomorphic to a sphere, so it is homotopic to a sphere.

### §3. Conclusion

One can find the homology of other polyhedron like prism, decahedron etc.

### Acknowledgement

The second author is supported by DST/INSPIRE Fellowship/2013/1041, Govt. of India.

### References

- [1] Henri Paul Cartan and Samuel Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.
- [2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
- [3] Charles A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.
- [4] S. Dey, B. Pal and A. Bhattacharyya, *Homology of a type of Heptahedron*, Journal of the Calcutta Mathematical Society, Vol.13(1), 35-40(2017).

## Even Modular Edge Irregularity Strength of Graphs

K.Muthugurupackiam

(Department of Mathematics, Rajah Serfoji Government College, Thanjavur, India)

S.Ramya

(Department of Mathematics, Bharathidasan University Constituent College, Orathanadu, India)

gurupackiam@yahoo.com, manoramya.8036@gmail.com

**Abstract:** A new graph characteristic, even modular edge irregularity strength of graphs is introduced. Estimation on this parameter is obtained and the precise values of this parameter are obtained for some families of graphs.

**Key Words:** Irregular labeling, modular irregular labeling, even modular edge irregular labeling, vertex  $k$ -labeling, irregularity strength, modular irregularity strength, even modular edge irregularity strength, Smarandachely  $p$ -modular edge irregularity strength.

**AMS(2010):** 05C78.

### §1. Introduction

Let  $G = (V, E)$  be a simple graph, having at most one isolated vertex and no component of order 2. A map that carries vertex set (edge set or both) as domain to the positive integers  $\{1, 2, \dots, k\}$  is called *vertex  $k$ -labeling* (*edge  $k$ -labeling* or *total  $k$ -labeling*). Well-known parameter irregularity strength of a graph introduced by Chartrand et al. [6]. A simple graph  $G$  is called irregular if there exists an edge  $k$ -labeling  $\lambda : E(G) \rightarrow \{1, 2, \dots, k\}$  such that the weight of a vertex  $v$  under the labeling defined by  $w_\lambda(v) = \sum \lambda(uv)$ , are pairwise distinct. The minimum value of  $k$ , for which  $G$  is irregular, called irregularity strength of  $G$  denoted by  $s(G)$ .

The parameter irregularity strength of a graph is attracted by numerous authors. Aigner and Triesch [1] proved that  $s(G) \leq n - 1$  if  $G$  is a connected graph of order  $n$ , and  $s(G) \leq n + 1$  otherwise. Nierhoff [15] refined their method and showed that  $s(G) \leq n - 1$  for all graphs with finite irregularity strength, except for  $K_3$ . This bound is tight e.g. for stars. In particular Faudree and Lehel [8] showed that if  $G$  is  $d$ -regular ( $d \geq 2$ ), then  $\lceil \frac{n+d-1}{d} \rceil \leq s(G) \leq \lceil \frac{n}{2} \rceil + 9$ , and they conjectured that  $s(G) \leq \lceil \frac{n}{d} \rceil + c$  for some constant  $c$ . Przybylo in [16] proved that  $s(G) \leq 16\frac{n}{\delta} + 6$ . Kalkowski, Karonski and Pfender [12] showed that  $s(G) \leq 6\frac{n}{\delta} + 6$ , where  $\delta$  is the minimum degree of graph  $G$ . Currently Majerski and Przybylo [13] proved that  $s(G) \leq (4 + o(1))\frac{n}{\delta} + 4$  for graphs with minimum degree  $\delta \geq \sqrt{n} \ln n$ . Other interesting results on the irregularity strength can be found in [3, 4, 5, 7, 9]. For recent survey of graph labeling refer the paper [10].

---

<sup>1</sup>Received April 18, 2017, Accepted February 25, 2018.



Ali Ahmad et al.[2] introduced edge irregularity strength of a graph as follows: Consider a simple graph  $G$  together with a vertex  $k$ -labeling  $\chi : V(G) \rightarrow \{1, 2, \dots, k\}$ . The weight of an edge  $xy$  in  $G$ , denoted by  $wt(xy) = \chi(x) + \chi(y)$ . A vertex  $k$ -labeling is defined to be an edge irregular  $k$ -labeling of the graph  $G$  if for every two different edges  $e$  and  $f$  there is  $wt(e) \neq wt(f)$ . The minimum  $k$  for which the graph  $G$  has an edge irregular  $k$ -labeling is called the edge irregularity strength of  $G$ , denoted by  $es(G)$ . The lower bound of  $es(G)$  was given by the following inequality

$$es(G) \geq \max\left\{\left\lceil \frac{E(G) + 1}{2} \right\rceil, \Delta\right\}$$

where  $\Delta$  is the maximum degree of graph  $G$ . Ibrahim Tarawneh et al. [11], determined the exact value of edge irregularity strength of corona graphs of path  $P_n$  with  $P_2$ ,  $P_n$  with  $K_1$  and  $P_n$  with  $S_m$ .

Martin Bača et al. [14] introduced modular irregularity strength of a graph. An edge labeling  $\psi : E(G) \rightarrow \{1, 2, \dots, k\}$  is called *modular irregular  $k$ -labeling* if there exists a bijective weight function  $\sigma : V(G) \rightarrow Z_n$  defined by  $\sigma(x) = \sum \psi(xy)$  called *modular weight* of the vertex  $x$ , where  $Z_n$  is the group of integers modulo  $n$  and the sum is over all vertices  $y$  adjacent to  $x$ . They defined the *modular irregularity strength* of a graph  $G$ , denoted by  $ms(G)$ , as the minimum  $k$  for which  $G$  has a modular irregular  $k$ -labeling.

Motivated by the edge irregularity strength of graphs we introduce a new parameter, an even modular edge irregularity strength of graph, a modular version of edge irregularity strength.

Let  $G = (V, E)$  be a  $(n, m)$ -graph together with a vertex  $k$ -labeling  $\rho : V(G) \rightarrow \{1, 2, \dots, k\}$ . Define a set of edge weight  $W = \{wt(uv) : wt(uv) = \rho(u) + \rho(v), \forall uv \in E\}$ . Vertex labeling  $\rho$  is called *even modular edge irregular labeling* if there exists a bijective map  $\sigma : W \rightarrow M$  defined for each edge weight  $wt(uv)$  there corresponds an element  $x \in M$  such that  $wt(uv) \equiv x \pmod{2m}$ , where  $M = \{0, 2, 4, \dots, 2(m-1)\}$ . We define the *even modular edge irregularity strength* of a graph  $G$ , denoted by  $emes(G)$ , as the minimum  $k$  for which  $G$  has an even modular edge irregular labeling. If there doesn't exist an even modular edge irregular labeling for  $G$ , we define  $emes(G) = \infty$ . Generally, if  $M = \{0, p, 2p, \dots, (m-1)p\}$  for a prime number  $p$ , such a modular edge irregular labeling is called a *Smarandache  $p$ -modular edge irregular labeling* and the minimum  $k$  for which  $G$  has a Smarandachely  $p$ -modular edge irregular labeling is denoted by  $emes^p(G)$ . Clearly,  $emes^2(G) = emes(G)$ .

The main aim of this paper is to show a lower bound of the even modular edge irregularity strength and determine the precise values of this parameter for some families of graphs.

## §2. Main Results

Following theorem gives the lower bound of even modular edge irregularity strength of a graph.

**Theorem 2.1** *Let  $G$  be a  $(n, m)$ -graph. Then  $emes(G) \geq m$ .*

*Proof* Let  $G$  be a  $(n, m)$ -graph together with an even modular edge irregular labeling

$\rho : V(G) \rightarrow \{1, 2, \dots, k\}$ . Consider the even edge weights of  $G$ , there should be an edge  $e$  such that  $wt(e) \equiv 0 \pmod{2m}$ . Since the weight of  $e$  must be at least  $2m$ ,  $emes(G) \geq m$ .  $\square$

**Lemma 2.1** *Let  $(d_1, d_2, \dots, d_n)$  be the degree sequence of a graph  $G$  and let  $(l_1, l_2, \dots, l_n)$  be the corresponding vertex labels of an even modular edge irregular labeling of  $G$ . Then the sum of all the edge weights denoted as  $S$  is equal to the sum of the product of degree with its corresponding labels, that is,*

$$S = \sum_{e \in E} wt(e) = \sum_{i=1}^n d_i l_i.$$

**Lemma 2.2** *In any even modular edge irregular labeling of  $C_n$ , labels of all vertices are of same parity.*

*Proof* By definition, weight of an edge is sum of the labels of its end vertices. To obtain an even edge weight, both the labels must be either odd or even, and hence all the vertex labels of  $C_n$  are of same parity.  $\square$

**Theorem 2.2** *Let  $C_n$  be a cycle of order  $n \geq 3$ . Then*

$$emes(C_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod{4}, \\ n, & \text{if } n \equiv 1 \pmod{4}, \\ n+2, & \text{if } n \equiv 3 \pmod{4}, \\ \infty, & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

*Proof* Let  $V(C_n) = \{v_i : i = 1, 2, \dots, n\}$  be the vertex set and let  $E(C_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \dots, n\}$  be the edge set of the cycle  $C_n$ . Define the vertex labeling  $\rho : V \rightarrow \{1, 2, \dots, n+2\}$  as follows:

$$\rho(v_i) = 2i - 1, \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$

If  $n \equiv 0, 1 \pmod{4}$ , then, for  $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ ,

$$\rho(v_{n+1-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

If  $n \equiv 3 \pmod{4}$ , then for  $2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ ,

$$\rho(v_{n+2-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

We can easily check that the above labeling  $\rho$ , is an even modular edge irregular labeling of  $C_n$ . Thus,

$$emes(C_n) \leq \begin{cases} n+1, & \text{if } n \equiv 0 \pmod{4} \\ n, & \text{if } n \equiv 1 \pmod{4} \\ n+2, & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

Now let us find the lower bound of  $emes(C_n)$  as follows:

**Case 1.** Suppose  $n \equiv 0 \pmod{4}$ . Consider the set of even edge weights  $W(C_n) = \{2, 4, 6, \dots, 2n\}$ . To obtain the weight 2 for an edge, we must assign label 1 to both of its end vertices, and hence all the vertices of  $C_n$  must receive odd labels by Lemma 2.2. Since the heaviest weight is  $2n$ ,  $emes(C_n) \geq n+1$ .

**Case 2.** Suppose  $n \equiv 1 \pmod{4}$ . By Theorem 2.1,  $emes(C_n) \geq n$ .

**Case 3.** Suppose  $n \equiv 3 \pmod{4}$ . Assume that the cycle  $C_n$  has the set of even edge weights  $W(C_n) = \{2, 4, \dots, 2n\}$ , then  $\frac{S}{2}$  is even, where  $S$  is the sum of the weights. Since the least weight is 2, by Lemma 2.2 all the vertex labels of  $C_n$  must be odd and hence  $\sum_{i=1}^n l_i$  is odd, which is a contradiction to  $\sum_{i=1}^n l_i = \frac{S}{2}$  by Lemma 2.1.

Assume that  $C_n$  has the set of even edge weights  $W(C_n) = \{4, 6, \dots, 2n+2\}$ . Now the sum of the labels,

$$\sum_{i=1}^n l_i = \frac{S}{2} = \frac{(n+1)(n+2)}{2} - 1$$

is odd and hence each label must odd. Heaviest weight  $2n+2$  can be obtained by assigning the label at least  $n+2$ . Thus,  $emes(C_n) \geq n+2$ .

**Case 4.** Suppose  $n \equiv 2 \pmod{4}$ . If the cycle  $C_n$  has an even modular edge irregular labeling, then the sum of the edge weights  $S \equiv 2 \pmod{4}$ , and hence  $\frac{S}{2}$  is odd. When  $n \equiv 2 \pmod{4}$ , sum of the labels  $\sum_{i=1}^n l_i$  is even, which is a contradiction to  $\sum_{i=1}^n l_i = \frac{S}{2}$ . Thus,  $emes(C_n) = \infty$ , if  $n \equiv 2 \pmod{4}$ .  $\square$

**Theorem 2.3** Let  $P_n$  be a path of order  $n \geq 2$ . Then

$$emes(P_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

*Proof* Let  $V(P_n) = \{v_i : i = 1, 2, \dots, n\}$  be the vertex set and let  $E(P_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \dots, n-1\}$  be the edge set of the path  $P_n$ .

Define the vertex  $n$ -labeling  $\theta : V \rightarrow \{1, 2, \dots, n\}$  as follows:

For  $1 \leq i \leq n$ ,

$$\theta(v_i) = \begin{cases} i, & \text{if } i \text{ is odd} \\ i-1, & \text{if } i \text{ is even.} \end{cases}$$

Clearly,  $\theta$  is an even modular edge irregular labeling of  $P_n$ . Thus, the upper bound of

$emes(P_n)$  can be obtained as follows:

$$emes(P_n) \leq \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Let us find the lower bound of  $emes(P_n)$ .

**Case 1.** Assume that  $n$  is odd. Consider the optimal even edge weight  $W(P_n) = \{2, 4, \dots, 2(n-1)\}$ . Since the least weight is 2, all the vertices of  $P_n$  must receive odd labels. To obtain the heaviest weight  $2(n-1)$ , we must assign vertex label at least  $n$ . Thus,  $emes(P_n) \geq n$ .

**Case 2.** Assume that  $n$  is even. In this case, the lower bound can be obtain directly from Theorem 2.1.  $\square$

**Theorem 2.4** Let  $K_{1,n}$  be a star graph of order  $n+1, n \geq 1$ . Then  $emes(K_{1,n}) = 2n-1$ .

*Proof* Let  $V(K_{1,n}) = \{x, v_i : i = 1, 2, \dots, n\}$  be the vertex set and let  $E(K_{1,n}) = \{e_i = xv_i : i = 1, 2, \dots, n\}$  be the edge set of the path  $K_{1,n}$ .

Define the vertex labeling  $\lambda_1 : V \rightarrow \{1, 2, \dots, 2n-1\}$  as follows:

$$\lambda_1(x) = 1,$$

$$\lambda_1(v_i) = 2i-1, 1 \leq i \leq n.$$

From the above even modular edge irregular labeling  $\lambda_1$ , upper bound of  $emes(K_{1,n})$  is obtained as follows,  $emes(K_{1,n}) \leq 2n-1$ .

Consider the optimal even edge weights  $W(K_{1,n}) = \{2, 4, \dots, 2n\}$ . Since the least weight is 2, the vertex  $x$  must be label with 1. To obtain the heaviest weight  $2n$ , we must assign label at least  $2n-1$  to other end vertex. Thus,  $emes(K_{1,n}) \geq 2n-1$ . Hence the theorem.  $\square$

**Theorem 2.5** Let  $K_{2,n}$  be the complete bipartite graph of order  $n+2, n \geq 2$ . Then  $emes(K_{2,n}) = 2n+1$ .

*Proof* Let  $V(K_{2,n}) = \{x, y, v_i : i = 1, 2, \dots, n\}$  be the vertex set and let  $E(K_{2,n}) = \{xv_i, yv_i : i = 1, 2, \dots, n\}$  be the edge set of the complete bipartite graph  $K_{2,n}$ .

Define the vertex labeling  $\lambda_2 : V \rightarrow \{1, 2, \dots, 2n+1\}$  as follows:

$$\lambda_2(x) = 1, \quad \lambda_2(y) = 2n+1$$

$$\lambda_2(v_i) = 2i-1, 1 \leq i \leq n.$$

From the above even modular edge irregular labeling  $\lambda_2$ , upper bound of  $emes(K_{2,n})$  is obtained as follows:  $emes(K_{2,n}) \leq 2n+1$ .

Consider the even edge weights of  $K_{2,n}$  as  $2, 4, \dots, 4n$ . Since the least edge weight is 2, all the vertices must receive odd labels. Therefore, we must assign label at least  $2n+1$ , to obtain the heaviest weight  $4n$ . Hence  $emes(K_{2,n}) \geq 2n+1$ .  $\square$

A rectangular graph  $R_n, n \geq 2$ , is a graph obtained from the path  $P_{n+1}$  by replacing each

edge of the path by a rectangle  $C_4$ . Let

$$V(R_n) = \{v_i : i = 1, 2, \dots, 2n\} \cup \{u_j : j = 1, 2, \dots, n+1\}$$

be the vertex set and let

$$\begin{aligned} E(R_n) = & \{v_{2i-1}v_{2i} : i = 1, 2, \dots, n\} \cup \{u_i u_{i+1} : i = 1, 2, \dots, n\} \\ & \cup \{v_{2i-1}u_i : i = 1, 2, \dots, n\} \cup \{v_{2i-2}u_i : i = 2, 3, \dots, n+1\} \end{aligned}$$

be the edge set of the the rectangular graph  $R_n$ . The following theorem gives the precise value of even modular edge irregularity strength of rectangular graph.

**Theorem 2.6** *Let  $R_n$  be a rectangular graph of order  $3n+1$ ,  $n \geq 2$ . Then  $emes(R_n) = 4n+1$ .*

*Proof* Define the vertex labeling  $\alpha : V \rightarrow \{1, 2, \dots, 4n+1\}$  as follows:

$$\begin{aligned} \alpha(v_i) &= 2i-1, 1 \leq i \leq 2n, \\ \alpha(u_i) &= 4i-3, 1 \leq i \leq n+1. \end{aligned}$$

Upper bound  $emes(R_n) \leq 4n+1$  can be obtained from the above labeling  $\alpha$ .

Consider the even edge weights of  $R_n$  as  $2, 4, \dots, 8n$ . Since the least weight is 2, all the vertex labels must be odd. Therefore, we must assign label at least  $4n+1$ , to obtain the heaviest weight  $8n$ . Hence,  $emes(R_n) \geq 4n+1$ .  $\square$

**Theorem 2.7** *Let  $tP_4$ ,  $t \geq 1$ , denote the disjoint union of  $t$  copies of path  $P_4$ . Then  $emes(tP_4) = 3t$ .*

*Proof* Let  $V(tP_4) = \{u_{ij} : 1 \leq i \leq t, 1 \leq j \leq 4\}$  be the vertex set and let  $E(tP_4) = \{u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4} : 1 \leq i \leq t\}$  be the edge set of  $tP_4$ . Define the vertex labeling  $\beta : V \rightarrow \{1, 2, \dots, 3t\}$  as follows:

$$\begin{aligned} \beta(u_{i1}) &= \beta(u_{i2}) = 3i-2, 1 \leq i \leq t, \\ \beta(u_{i3}) &= \beta(u_{i4}) = 3i, 1 \leq i \leq t. \end{aligned}$$

Clearly,  $\beta$  is an even modular edge irregular labeling of  $tP_4$  and hence  $emes(tP_4) \leq 3t$ . The lower bound  $emes(tP_4) \geq 3t$  can be obtained directly from Theorem 2.1. Hence, we get that  $emes(tP_4) = 3t$ .  $\square$

**Theorem 2.8** *Let  $tC_3$ ,  $t \geq 2$ , denote the disjoint union of  $t$  copies of cycle  $C_3$ . Then  $emes(tC_3) = 3t+2$ .*

*Proof* Let  $V(tC_3) = \{v_{ij} : 1 \leq i \leq t, 1 \leq j \leq 3\}$  be the vertex set and let  $E(tC_3) = \{v_{i1}v_{i2}, v_{i2}v_{i3}, v_{i1}v_{i3} : 1 \leq i \leq t\}$  be the edge set of  $tC_3$ . Define the vertex labeling  $\theta : V \rightarrow \{1, 2, \dots, 3t+2\}$  as follows:

$$\theta(v_{i1}) = \begin{cases} 1, & i = 1 \\ 3t, & 2 \leq i \leq t, \end{cases} \quad \theta(v_{i2}) = 3i+2, 1 \leq i \leq t, \quad \text{and} \quad \theta(v_{i3}) = \begin{cases} 3, & i = 1 \\ 3i+1, & 2 \leq i \leq t, \end{cases}$$

Clearly,  $\theta$  is an even modular edge irregular labeling of  $tP_4$  and hence  $emes(tC_3) \leq 3t + 2$ .

Consider the optimal edge weights of  $tC_3$  as  $4, 6, 8, \dots, 6t + 2$ . Since any two adjacent vertices of  $tC_3$  can not receive the same labels, we must assign label at least  $3t + 2$  to get the heaviest label  $6t + 2$ . Hence,  $emes(tC_3) \geq 3t + 2$ .  $\square$

Ladder graph  $L_n = K_2 \times P_n, n \geq 3$  is formed by taking two isomorphic copies of  $P_n$  and joining the corresponding vertices by an edge. Let  $V = \{u_i, v_i : 1 \leq i \leq n\}$  be the vertex set and let

$$E = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$$

be the edge set of  $L_n$ . The following theorem gives the precise value of even modular edge irregularity strength of ladder graph.

**Theorem 2.9** *Let  $L_n = K_2 \times P_n, n \geq 3$  be the ladder graph. Then*

$$emes(L_n) = \begin{cases} 3n - 2, & \text{if } n \text{ is odd,} \\ 3n - 1, & \text{if } n \text{ is even.} \end{cases}$$

*Proof* Defined the vertex labeling  $\phi : V \rightarrow \{1, 2, \dots, 3n - 1\}$  as follows:

$$\phi(u_i) = \begin{cases} 3i - 2, & \text{if } i \text{ is odd} \\ 3i - 3, & \text{if } i \text{ is even} \end{cases} \quad 1 \leq i \leq n,$$

$$\phi(v_i) = \begin{cases} 3i - 2, & \text{if } i \text{ is odd} \\ 3i - 1, & \text{if } i \text{ is even.} \end{cases} \quad 1 \leq i \leq n.$$

Clearly,  $\phi$  is an even modular edge irregular labeling of  $L_n$  and hence

$$emes(L_n) \leq \begin{cases} 3n - 2, & \text{if } n \text{ is odd,} \\ 3n - 1, & \text{if } n \text{ is even.} \end{cases}$$

Lower bound  $emes(L_n) \geq 3n - 2$ , can be obtained directly from Theorem 2.1, when  $n$  is odd.

Suppose  $n$  is even. Consider optimal edge weights of  $L_n$  as  $2, 4, \dots, 6n - 4$ . Since  $L_n$  has a span cycle, all the vertices of  $L_n$  must receive the labels of same parity. Furthermore, to obtain the edge weight 2, the corresponding end vertices must be label 1, and hence all the labels must be odd. Thus  $emes(L_n) \geq 3n - 1$ . Hence the theorem.  $\square$

### §3. Conclusion

In this paper we introduced a new graph parameter, the even modular edge irregularity strength,  $emes(G)$ , as a modular version of edge irregularity strength. We determined the exact value of even modular edge irregularity strength of some families of graphs and a lower bound of  $emes$  is obtained. However, the determination of upper bound is still open.

## References

- [1] M. Aigner and E. Triesch, Irregular assignments of trees and forests, *SIAM J. Discrete Math.*, **3** (1990), 439–449.
- [2] Ali Ahmad, Omar Bin Saeed Al-Mushayt, Martin Bača, On edge irregularity strength of graphs, *Appl. Math. Comput.*, **243** (2014), 607–610.
- [3] D. Amar and O. Togni, Irregularity strength of trees, *Discrete Math.*, **190** (1998), 15–38.
- [4] M. Anholcer and C. Palmer, Irregular labellings of circulant graphs, *Discrete Math.*, **312** (2012), 3461–3466.
- [5] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory*, **45** (2004), 241–254.
- [6] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.*, **64** (1988), 187–192.
- [7] R.J. Faudree, M.S. Jacobson, J. Lehel and R.H. Schlep, Irregular networks, regular graphs and integer matrices with distinct row and column sums, *Discrete Math.*, **76** (1988), 223–240.
- [8] R.J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs, *Combinatorica*, **52** (1987), 247–256.
- [9] A. Frieze, R.J. Gould, M. Karonski and F. Pfender, On graph irregularity strength, *J. Graph Theory*, **41** (2002), 120–137.
- [10] J. Gallian, A dynamic survey of graph labeling, *The Electronic J. Combin.*, **19** (2012), #DS6.
- [11] Ibrahim Tarawneh, Roslan Hasni, Ali Ahmad, On the edge irregularity strength of corona product of graphs with paths, *Appl. Math. E-Notes*, **16** (2016), 80–87.
- [12] M. Kalkowski, M. Karonski and F. Pfender, A new upper bound for the irregularity strength of graphs, *SIAM J. Discrete Math.*, **25**(3) (2011), 1319–1321.
- [13] P. Majerski and J. Przybylo, On the irregularity strength of dense graphs, *SIAM J. Discrete Math.*, **28**(1) (2014), 197–205.
- [14] Martin Bača, K. Muthugurupackiam, KM. Kathiresan, S. Ramya, Modular irregularity strength of graphs, *Communicated*.
- [15] T. Nierhoff, A tight bound on the irregularity strength of graphs, *SIAM J. Discrete Math.*, **13** (2000), 313–323.
- [16] J. Przybylo, Irregularity strength of regular graphs, *Electron. J. Combin.*, **15** (2008), R82.

## Finslerian Hypersurfaces and Quartic Change of Finsler Metric

Gauree Shanker

(Department of Mathematics and Statistics, Central University of Punjab, Bathinda-151001, India)

Ramdayal Singh Kushwaha

(Department of Mathematics and Statistics, Banasthali University, Banasthali, Rajasthan-304022, India)

E-mail: gsp.math.1978@gmail.com, sruthymuthu123@gmail.com

**Abstract:** In the present paper we have studied the Finslerian hypersurfaces and quartic change of a Finsler metric. The relationship with Finslerian hypersurface and the other which is finslerian hypersurface given by quartic change have been obtained. We have also proved that quartic change makes three type of hyper surfaces invariant under certain condition. These three type of hyper surfaces are hyperplanes of first, second, and third kind.

**Key Words:** Finsler metric, Finslerian hyperspaces, quartic change, hyperplanes of first, second, and third kind.

**AMS(2010):** 53B40, 53C60.

### §1. Introduction

Let  $(M^n, L)$  be an  $n$ - dimensional Finsler space on a differential Manifold  $M^n$ , equipped with fundamental function  $L(x, y)$ . In 1984 C. Shibata [11] introduced the transformation of Finsler metric:

$$L^*(x, y) = f(L, \beta), \quad (1.1)$$

where  $\beta = b_i y^i$ ,  $b_i(x)$  are the components of a covariant vector in  $(M^n, L)$  and  $f$  is positively homogeneous function of degree one in  $L$  and  $\beta$ . The change of metric is called a  $\beta$ - change. A particular  $\beta$ - change of a Finsler metric function is a quartic change of metric function is defined as

$$L^* = (L^4 + \beta^4)^{1/4}. \quad (1.2)$$

If  $L(x, y)$  reduces to the metric function of Riemann space then  $L^*(x, y)$  reduces to the metric function of space generated by quartic metric. Due to this reason this transformation (1.2) has been called the quartic change of Finsler metric.

On the other hand, in 1985, M. Matsumoto investigated the theory of Finslerian hypersurface [4]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. These names were given by Rapesak [8]. Kikuchi [3] gave other name following haimovichi [1]. In the year 2005, Prasad and Tripathi [7] studied the Finslerian Hy-

---

<sup>1</sup>Received July 9, 2017, Accepted February 26, 2018.



persurfaces and Kropina change of a Finsler metric and obtained different results in his paper. Again, in the year 2005, Prasad, Chaubey and Patel [6] studied the Finslerian Hypersurfaces and Matsumoto change of a Finsler metric and obtained different results.

In the present paper, using the field of linear frame ([2, 3, 5]) we shall consider Finslerian hypersurfaces given by a quartic change of a Finsler metric. Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finslerian hypersurface given by quartic change. We also show that a quartic change makes three types of hypersurfaces invariant under certain condition.

## §2. Preliminaries

Let  $M^n$  be an  $n$ -dimensional smooth manifold and  $F^n = (M^n, L)$  be an  $n$ -dimensional Finsler space equipped with the fundamental function  $L(x, y)$  on  $M^n$ . The metric tensor  $g_{ij}(x, y)$  and Cartan's C-tensor  $C_{ijk}(x, y)$  are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

respectively and we can introduce the Cartan's Connection

$$CT = (F_{jk}^i, N_j^i, C_{jk}^i) \text{ in } F^n.$$

A hypersurface  $M^{n-1}$  of the underlying smooth manifold  $M^n$  may be parametrically represented by the equation  $x^i = x^i(u^\alpha)$ , where  $u^\alpha$  are Gaussian coordinates on  $M^{n-1}$  and Greek indices vary from 1 to  $n-1$ . Here we shall assume that the matrix consisting of the projection factors  $B_\alpha^i = \partial x^i / \partial u^\alpha$  is of rank  $n-1$ . The following notations are also employed:  $B_{\alpha\beta}^i = \partial^2 x^i / \partial u^\alpha \partial u^\beta$ ,  $B_{o\beta}^i = v^\alpha B_{\alpha\beta}^i$ . If the supporting element  $y^i$  at a point  $(u^\alpha)$  of  $M^{n-1}$  is assumed to be tangential to  $M^{n-1}$ , we may then write  $y^i = B_\alpha^i(u) v^\alpha$  i.e.  $v^\alpha$  is thought of as the supporting element of  $M^{n-1}$  at the point  $(u^\alpha)$ . Since the function  $\bar{L}(u, v) = L(x(u), y(u, v))$  gives to a Finsler metric of  $M^{n-1}$ , we get a  $n-1$ -dimensional Finsler space  $F^{n-1} = (M^{n-1}, \bar{L}(u, v))$ .

At each point  $(u^\alpha)$  of  $F^{n-1}$ , the unit normal vector  $N^i(u, v)$  is defined by

$$g_{ij} B_\alpha^i N^j = 0, \quad g_{ij} N^i N^j = 1. \quad (2.1)$$

If  $(B_i^\alpha, N_i)$  is the inverse matrix of  $(B_\alpha^i, N^i)$ , we have

$$B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i N_i = 0, \quad N^i N_i = 1 \text{ and } B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i.$$

Making use of the inverse matrix  $(g^{\alpha\beta})$  of  $(g_{\alpha\beta})$ , we get

$$B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad N_i = g_{ij} N^j. \quad (2.2)$$

For the induced Cartan's connection  $ICT = (F_{\beta\gamma}^\alpha, N_\beta^\alpha, C_{\beta\gamma}^\alpha)$  on  $F^{n-1}$ , the second funda-

mental  $h$ -tensor  $H_{\alpha\beta}$  and the normal curvature vector  $H_\alpha$  are respectively given by [8]

$$H_{\alpha\beta} = N_i(B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k) + M_\alpha H_\beta, \quad H_\beta = N_i(B_{0\beta}^i + N_j^i B_\beta^j), \quad (2.3)$$

where

$$M_\alpha = C_{ijk} B_\alpha^i N^j N^k. \quad (2.4)$$

Contracting  $H_{\alpha\beta}$  by  $v^\alpha$ , we immediately get  $H_{0\beta} = H_{\alpha\beta} v^\alpha = H_\beta$ . Furthermore the second fundamental  $v$ -tensor  $M_{\alpha\beta}$  is given by [10]

$$M_{\alpha\beta} = C_{ijk} B_\alpha^i B_\beta^j N^k. \quad (2.5)$$

### §3. Quartic Changed Finsler Space

Let  $F^n = (M^n, L)$  be a given Finsler space and let  $b_i(x)dx^i$  be a one form on  $M^n$ . We shall define on  $M^n$  a function  $L^*(x, y) (> 0)$  by the equation (1.2) where we put  $\beta(x, y) = b_i(x)y^i$ . To find the metric tensor  $g_{ij}^*$ , the angular metric tensor  $h_{ij}^*$ , the Cartan's  $C$ -tensor  $C_{ijk}^*$  of  $F^{*n} = (M^n, L^*)$  we use the following results.

$$\partial\beta/\partial y^i = b_i, \quad \partial L/\partial y^i = l_i, \quad \partial l_j/\partial y^i = L^{-1}h_{ij}, \quad (3.1)$$

where  $h_{ij}$  are components of angular metric tensor of  $F^n$  given by

$$h_{ij} = g_{ij} - l_i l_j = L \partial^2 L / \partial y^i \partial y^j.$$

The successive differentiation of (1.2) with respect to  $y^i$  and  $y^j$  give

$$l_i^* = \frac{L^3 l_i + \beta^3 b_i}{(L^4 + \beta^4)^{3/4}} \quad (3.2)$$

$$\begin{aligned} h_{ij}^* &= \frac{1}{(L^4 + \beta^4)^{3/2}} [L^2(L^4 + \beta^4)h_{ij} + 3\beta^4 L^2 l_i l_j \\ &\quad + 3L^4 \beta^2 b_i b_j - 3L^3 \beta^3 (l_i b_j + l_j b_i)] \end{aligned} \quad (3.3)$$

From (3.2) and (3.3) we get the following relation between metric tensors of  $F^n$  and  $F^{*n}$

$$\begin{aligned} g_{ij}^* &= \frac{1}{(L^4 + \beta^4)^{3/2}} [L^2(L^4 + \beta^4)g_{ij} + 2\beta^4 L^2 l_i l_j \\ &\quad + \beta^2(3L^4 + \beta^4)\beta^2 b_i b_j - 2L^3 \beta^3 (l_i b_j + l_j b_i)]. \end{aligned} \quad (3.4)$$

Differentiating (3.4) with respect to  $y^k$  and using (3.1) we get the following relation between

the Cartan's  $C$ -tensor of  $F^n$  and  $F^{*n}$

$$\begin{aligned} C_{ijk}^* &= \left( \frac{L^2}{\sqrt{L^4 + \beta^4}} \right) C_{ijk} - \left( \frac{L^2 \beta^3}{(L^4 + \beta^4)^{3/2}} \right) (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) \\ &\quad - 3 \left( \frac{(\beta^4 - L^4)}{(L^4 + \beta^4)^{5/2}} \right) m_i m_j m_k, \end{aligned} \quad (3.5)$$

where  $m_i = b_i - (\beta/L)l_i$ . It is to be noted that

$$m_i l^i = 0, \quad m_i b^i = b^2 - \beta^2/L^2, \quad h_{ij} l^j = 0, \quad h_{ij} m^j = h_{ij} b^i = m_i, \quad (3.6)$$

where  $m^i = g^{ij} m_j = b^i - (\beta/L)l^i$ .

#### §4. Hypersurface Given by a Quartic Change

Consider a Finslerian hypersurface  $F^{n-1} = (M^{n-1}, \bar{L}(u, v))$  of the  $F^n$  and another Finslerian hypersurface  $F^{*n-1} = (M^{n-1}, \bar{L}^*(u, v))$  of the  $F^{*n}$  given by the quartic change. Let  $N^i$  be the unit normal vector at each point of  $F^{n-1}$  and  $(B_\alpha^i, N_i)$  be the inverse matrix of  $(B_\alpha^i, N^i)$ . The functions  $B_\alpha^i$  may be considered as components of  $n-1$  linearly independent tangent vectors of  $F^{n-1}$  and they are invariant under quartic change. Thus we shall show that a unit normal vector  $N^{*i}(u, v)$  of  $F^{*n-1}$  is uniquely determined by

$$g_{ij}^* B_\alpha^i N^{*j} = 0, \quad g_{ij}^* N^{*i} N^{*j} = 1. \quad (4.1)$$

Contracting (3.4) by  $N^i N^j$  and paying attention to (2.1) and  $l_i N^i = 0$ , we have

$$g_{ij}^* N^i N^j = \frac{L^2(L^4 + \beta^4) + \beta^2(3L^4 + \beta^4)(b_i N^i)^2}{(L^4 + \beta^4)^{3/2}}.$$

Therefore we obtain

$$g_{ij}^* \left( \pm \frac{(L^4 + \beta^4)^{3/4} N^i}{\sqrt{L^2(L^4 + \beta^4) + \tau(b_i N^i)^2}} \right) \left( \pm \frac{(L^4 + \beta^4)^{3/4} N^j}{\sqrt{L^2(L^4 + \beta^4) + \tau(b_i N^i)^2}} \right) = 1.$$

where

$$\tau = \beta^2(3L^4 + \beta^4).$$

Hence we can put

$$N^{*i} = \frac{(L^4 + \beta^4)^{3/4} N^i}{\sqrt{L^2(L^4 + \beta^4) + \tau(b_i N^i)^2}}, \quad (4.2)$$

where we have chosen the positive sign in order to fix an orientation.

Using (2.1), (3.4), (4.1) and (4.2) we obtain from the first condition of (4.1),

$$\left[ -\frac{2L^3}{(L^4 + \beta^4)^{3/2}} l_i B_\alpha^i + \frac{\beta^2(3L^4 + \beta^4)}{(L^4 + \beta^4)^{3/2}} l_i B_\alpha^i \right] \frac{(L^4 + \beta^4)^{3/4} b_j N^j}{\sqrt{L^2(L^4 + \beta^4) + \tau(b_i N^i)^2}} = 0$$

If

$$-2L^3 \beta l_i B_\alpha^i + (3L^4 + \beta^4) b_i B_\alpha^i = 0$$

then contracting it by  $v^\alpha$  and using  $y^i = B_\alpha^i v^\alpha$  we get  $L = 0$  which is a contradiction with assumption that  $L > 0$ . Hence  $b_i N^i = 0$ . Therefore (4.2) is rewritten as

$$N^{*i} = \frac{(L^4 + \beta^4)^{1/4} N^i}{L}. \quad (4.3)$$

Summary the above, we obtain

**Proposition 4.1** *For a field of linear frame  $(B_1^i, B_2^i, \dots, B_{n-1}^i, N^i)$  of  $F^n$  there exists a field of linear frame  $(B_1^i, B_2^i, \dots, B_{n-1}^i, N^{*i} = \frac{(L^4 + \beta^4)^{1/4} N^i}{L})$  such that (4.1) is satisfied along  $F^{n-1}$  and then  $b_i$  is tangential to both the hypersurface  $F^{n-1}$  and  $F^{*n-1}$ .*

The quantities  $B_i^{*\alpha}$  are uniquely defined along  $F^{n-1}$  by

$$B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_\beta^j,$$

where  $(g^{*\alpha\beta})$  is the inverse matrix of  $(g_{\alpha\beta}^*)$ . Let  $(B_i^{*\alpha}, N_i^*)$  be the inverse of  $(B_\alpha^i, N^i)$ , then we have  $B_\alpha^i B_i^{*\beta} = \delta_\alpha^\beta$ ,  $B_\alpha^i N_i^* = 0$ ,  $N^{*i} N_i^* = 1$  and furthermore  $B_\alpha^i B_j^{*\alpha} + N^{*i} N_j^* = \delta_j^i$ . We also get  $N_i^* = g_{ij}^* N^{*j}$  which in view of (3.4), (2.2) and (4.3) gives

$$N_i^* = \frac{L}{(L^4 + \beta^4)^{1/4}} N_i. \quad (4.4)$$

We denote the Cartan's connection of  $F^n$  and  $F^{*n}$  by  $(F_{jk}^i, N_j^i, C_{jk}^i)$  and  $(F_{jk}^{*i}, N_j^{*i}, C_{jk}^{*i})$  respectively and put  $D_{jk}^i = F_{jk}^{*i} - F_{jk}^i$  which will be called difference tensor. We choose the vector field  $b_i$  in  $F^n$  such that

$$D_{jk}^i = A_{jk} b^i - B_{jk} l^i, \quad (4.5)$$

where  $A_{jk}$  and  $B_{jk}$  are components of a symmetric covariant tensor of second order. Since  $N_i b^i = 0$  and  $N_i l^i = 0$ , from (4.5) we get

$$N_i D_{jk}^i = 0, \quad N_i F_{jk}^{*i} = N_i F_{jk}^i, \quad \text{and} \quad N_i D_{0k}^i = 0. \quad (4.6)$$

Therefore from (3.3) and (4.3) we get

$$H_\alpha^* = \frac{L}{(L^4 + \beta^4)^{1/4}} H_\alpha. \quad (4.7)$$

If each path of a hypersurface  $F^{n-1}$  with respect to the induced connection is also a path

of enveloping space  $F^n$ , then  $F^{n-1}$  is called a hyperplane of the first kind [?]. A hyperplane of the first kind is characterized by  $H_\alpha = 0$ . Hence from (4.7), we have

**Theorem 4.1** *If  $b_i(x)$  be a vector field in  $F^n$  satisfying (4.5), then a hypersurface  $F^{n-1}$  is a hyperplane of the first kind if and only if the hypersurface  $F^{*n-1}$  is a hyperplane of the first kind.*

Next contracting (3.5) by  $B_i^\alpha N^{*j} N^{*k}$  and paying attention to (4.4),  $m_i N^i = 0$ ,  $h_{jk} N^j N^k = 1$  and  $h_{ij} B_\alpha^i N^j = 0$  we get

$$M_\alpha^* = M_\alpha - \frac{\beta^3}{(L^4 + \beta^4)^{3/2}} m_i B_\alpha^i. \quad (4.8)$$

From (2.3), (4.4), (4.5), (4.6), (4.7) and (4.8), we have

$$H_{\alpha\beta}^* = \frac{L}{(L^4 + \beta^4)^{1/4}} (H_{\alpha\beta} - \frac{\beta^3}{(L^4 + \beta^4)^{3/2}} m_i B_\alpha^i H_\beta). \quad (4.9)$$

If each  $h$ -path of a hypersurface  $F^{n-1}$  with respect to the induced connection is also  $h$ -path of the enveloping space  $F^n$ , then  $F^{n-1}$  is called a hyperplane of the second kind [?]. A hyperplane of the second kind is characterized by  $H_{\alpha\beta} = 0$ . Since  $H_{\alpha\beta} = 0$  implies that  $H_\alpha = 0$ , from (4.7) and (4.9) we have the following

**Theorem 4.2** *If  $b_i(x)$  be a vector field in  $F^n$  satisfying (4.5), then a hypersurface  $F^{n-1}$  is a hyperplane of the second kind if and only if the hypersurface  $F^{*n-1}$  is a hyperplane of the second kind.*

Finally contracting (3.5) by  $B_\alpha^i B_\beta^j N^{*k}$  and paying attention to (4.3) we have

$$M_{\alpha\beta}^* = \frac{L}{(L^4 + \beta^4)^{1/4}} M_{\alpha\beta}. \quad (4.10)$$

If the unit normal vector of  $F^{n-1}$  is parallel along each curve of  $F^{n-1}$ , the  $F^{n-1}$  is called a hyperplane of the third kind [8]. A hyperplane of the third kind is characterized by  $H_{\alpha\beta} = 0$ ,  $M_{\alpha\beta} = 0$ . From (4.7), (4.9) and (4.10) we have

**Theorem 4.3** *If  $b_i(x)$  be a vector field in  $F^n$  satisfying (4.5), then a hypersurface  $F^{n-1}$  is a hyperplane of the third kind if and only if the hypersurface  $F^{*n-1}$  is a hyperplane of the third kind.*

## References

- [1] M. Haimovici, Varietes totalement extremales et varieties totalement geodesiques dans les espaces de Finsler, *Ann. Sc. Univ. Jassy*, **25**, **1**, (1939), 359-644.
- [2] M. Kitayama, Finslerian hypersurfaces and metric transformation, *Tensor, N. S.*, **60**, (1998), 171-178.

- [3] S. Kikuchi, On the theory of subspace, *Tensor, N. S.*, **2**, (1952), 67-79.
- [4] M. Matsumoto, The induced and intrinsic connections of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ.*, **25**, (1985), 107-144.
- [5] A Moor, Finslererraume von identischer Torsion, *Acta Sci. Math.*, **34**, (1973), 279-288.
- [6] B. N. Prasad and G. C. Chaubey and G. S. Patel, Finslerian hypersurfaces and Matsumoto change of Finsler metric, *Varahmihir Journal of Mathematical sciences*, **5**, 2 (2005), 337-342.
- [7] B. N. Prasad and B. K. Tripathi, Finslerian hypersurfaces and Kropina change of Finsler metric, *Journal of The Tensor Society of India*, **23**, (2005), 49-58.
- [8] A. Rapesak, Eine neue charakterisierung Finslershen Taume Konstanter Krümmung and projectiveben Raume, *Acta Math. Acad. Sci. Hungar.*, **8**, (1957), 1-18.
- [9] C. Shibata, On Finsler spaces with Kropina metric, *Rep. Math. Phys.*, **13**, (1978), 117-128.
- [10] C. Shibata, U. P. Singh and A. K. Singh, On induced and intrinsic theories of hypersurface of Kropina space, *Hokkaido Univ. of Education (II A)*, **34**, (1983), 1-11.
- [11] C. Shibata, On invariant tensors of  $\beta$ -change of Finsler metrics, *J. Math. kyoto Univ.*, **24**, (1984), 163-188.
- [12] U. P. Singh, B. N. Prasad and Bindu Kumari, On Kropina change of Finsler metric, *Tensor, N. S.*, **64**, (2003), 181-188.

## Adjacency Matrices of Some Directional Paths and Stars

Rajesh Kumar T.J and Mathew Varkey T.K

(Department of Mathematics, T.K.M College of Engineering, Kollam, Kerala, India)

vptjrk@gmail.com, mathewvarkeytk@gmail.com

**Abstract:** Graph labeling where vertices and edges are assigned values subject to certain conditions have been motivated by various applied mathematical fields. Most of the problems in graph labeling are discussed on undirected graphs. Bloom G.S. and Hsu D.F. defined the labeling on directed graphs. In this paper, we discuss the adjacency matrices of graceful digraphs such as unidirectional paths, alternating paths, many orientations of directed star and a class of directed bistar. We also discuss the adjacency matrices of unidirectional paths and alternating paths if they are odd digraceful.

**Key Words:** Digraceful labeling, digraceful graph, Smarandachely  $H$ -digraceful graph, adjacency matrix, unidirectional paths, alternating paths.

**AMS(2010):** 05C78.

### §1. Introduction

Most of the applications in Engineering and Science have undirected graphs are in the problems. The graph labeling problems are widely focus on undirected graphs. There exists situations in which directed graphs are playing a key role in some problems. There we using graph labeling in directed graphs. In 1980, Bloom G.S. and Hsu D.F. ([3], [4], [5]) extend the concept of graceful labeling of undirected graphs to digraphs. They investigate the graceful labeling problems of digraphs. Graceful digraphs are related in a variety ways to other areas of Mathematics.

The underlying graph  $UG(D)$  of a digraph  $D$  is obtained from  $D$  by removing the direction of each arc in  $D$ . Here we consider  $UG(D)$  is connected and has no self loops or multiple edges. In other words the digraph  $D$  is simply connected. The vertex set and edge set of a simple connected digraph are denoted by  $V(D)$  and  $E(D)$  where  $E(D) = \overrightarrow{uv}, uv \in V(D)$ . For an arc  $\overrightarrow{uv}$  the first vertex  $u$  is its tail and second vertex  $v$  is its head.

For all terminology and notations in graph theory, we follow Harary [1] and for all terminology regarding graceful labeling, we follow [2]. A connected graph with  $p$  vertices and  $q$  edges is called graceful if it is possible to label the vertices of  $x$  with pairwise distinct integers  $f(x)$  in  $\{0, 1, 2, 3, \dots, q\}$  so that each edge,  $xy$ , is labeled  $|f(x) - f(y)|$ , the resulting edge labels are pairwise distinct (and thus from the entire set  $\{1, 2, 3, \dots, q\}$ ). A connected graph with  $p$  vertices and  $q$  edges is called odd graceful if it is possible to label the vertices of  $x$  with pairwise

---

<sup>1</sup>Received August 1, 2017, Accepted February 28, 2018.

distinct integers  $f(x)$  in  $\{0, 1, 2, 3, \dots, 2q-1\}$  so that each edge,  $xy$ , is labeled  $|f(x) - f(y)|$ , the resulting edge labels are pairwise distinct (and thus from the entire set  $\{1, 3, 5, \dots, 2q-1\}$ ).

**Definition 1.1**([4]) *A digraph  $D$  with  $p$  vertices and  $q$  edges is said to be digraceful if there exists an injection  $f : V(D) \rightarrow \{0, 1, 2, \dots, q\}$  such that the induced function  $f' : E(D) \rightarrow \{1, 2, 3, \dots, q\}$  which is denoted by  $f'(u, v) = [f(v) - f(u)](\text{mod } q + 1)$  for every directed edge  $(u, v)$  is a bijection, where  $[v](\text{mod } n)$  denotes the least positive integer of  $v$  modulo  $n$ . If the edge values are all distinct then the labeling is called a digraceful labeling of a digraph. A digraph is called a graceful digraph if it has a digraceful labeling.*

Generally, a *Smarandachely  $H$ -digraceful graph* is such a digraceful graph  $G$  that each subgraph  $H' \prec G$  is a graceful digraph if  $H' \cong H$ . Clearly, a Smarandachely  $G$ -digraceful graph is nothing else but a digraceful graph. Let  $f$  be a labeling of a digraph  $D$  from  $V(D)$  to  $\{0, 1, 2, \dots, q\}$  such that for each arc  $\vec{uv} \in D$ ,  $f(\vec{uv}) = f(u) - f(v)$  if  $f(u) > f(v)$ , otherwise  $f(\vec{uv}) = q + 1 + f(u) - f(v)$ . We call  $f$  a digraceful labeling of  $D$  if the arc label set  $\{f(\vec{uv}) : \vec{uv} \in E(D)\} = \{1, 2, 3, \dots, q\}$ . Therefore  $D$  is called a graceful digraph.

**Definition 1.2** *A digraph  $D$  with  $p$  vertices and  $q$  edges is said to be odd digraceful if there exists an injection  $f : V(D) \rightarrow \{0, 1, 2, \dots, 2q-1\}$  such that the induced function  $f' : E(D) \rightarrow \{1, 2, 3, \dots, 2q-1\}$  which is denoted by  $f'(u, v) = [f(v) - f(u)](\text{mod } 2q)$  for every directed edge  $(u, v)$  is a bijection, where  $[v](\text{mod } n)$  denotes the least positive integer of  $v$  modulo  $n$ . If the edge values are all distinct then the labeling is called an odd digraceful labeling of a digraph. A digraph is called an odd graceful digraph if it has an odd digraceful labeling.*

Let  $f$  be a labeling of a digraph  $D$  from  $V(D)$  to  $\{0, 1, 2, \dots, 2q-1\}$  such that for each arc  $\vec{uv} \in D$ ,  $f(\vec{uv}) = f(u) - f(v)$  if  $f(u) > f(v)$ , otherwise  $f(\vec{uv}) = q + 1 + f(u) - f(v)$ . We call  $f$  an odd digraceful labeling of  $D$  if the arc label set  $\{f(\vec{uv}) : \vec{uv} \in E(D)\} = \{1, 2, 3, \dots, q\}$ .

Therefore  $D$  is called an odd graceful digraph.

In next two sections, we develop a generalized adjacency matrix of unidirectional paths and alternating paths based on graceful labeling and odd graceful labeling.

## §2. Adjacency Matrices of Unidirectional Paths

**Definition 2.1** *Let  $D$  be a graph with  $V(D) = \{v_1, v_2, v_3, \dots, v_p\}$ . Then the matrix  $A_D = [x_{ij}]$  defined by*

$$x_{ij} = \begin{cases} +1, & \text{if node } v_i \text{ is connected to } v_j \text{ and directed from } v_i \text{ to } v_j \\ -1, & \text{if node } v_j \text{ is connected to } v_i \text{ and directed from } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases} \quad (0.1)$$

*is called the adjacency matrix of  $D$ .*

**Definition 2.2** *Let  $D$  be a digraph with  $q$  edges and a labeling*



$f : V(D) \rightarrow \{0, 1, 2, \dots, q\}$ . Then the  $(q+1) \times (q+1)$  matrix  $M_D = [x_{ij}]$  defined by

$$x_{ij} = \begin{cases} +1, & \text{if } xy \in E(D) \text{ for } f(x) = i \text{ and } f(y) = j \text{ directed from } x \text{ to } y \\ -1, & \text{if } yx \in E(D) \text{ for } f(y) = i \text{ and } f(x) = j \text{ directed from } x \text{ to } y \\ 0, & \text{otherwise} \end{cases} \quad (0.2)$$

is called the generalized adjacency matrix of  $D$  induced by the labeling  $f$ .

The generalized adjacency matrix of the digraph is skew symmetric. The generalized adjacency matrix allows all zeros for rows and columns corresponding to missing labels, while in the adjacency matrix such rows corresponding to vertices of degree zero. If two graphs have the same adjacency matrix, then they are isomorphic, but if two graphs have the same generalized adjacency matrix they may not be isomorphic.

**Definition 2.3** Let  $A$  be a  $p \times p$  matrix. Then the  $k^{th}$  diagonal line of  $A$  is the collection of entries  $D_k = \{a_{ij} / j - i = k\}$ , counting multiplicity.

It is clear that  $0 \leq |k| \leq p - 1$ . When  $M = M_D$  is a generalized adjacency matrix,  $M$  is symmetric and  $D_k = D_{-k}$ . The entry  $\pm 1$  corresponding to an edge between vertices  $v_i$  and  $v_j$  lies on the  $(j - i)^{th}$  diagonal line. The main diagonal line is  $D_k$  for  $k = 0$ . The odd diagonal lines are  $D_k$  for  $k = \pm 1, \pm 3, \pm 5, \dots, \pm 2q - 1$  and the even diagonal line are  $D_k$  for  $k = \pm 2, \pm 4, \pm 6, \dots, \pm 2q - 2$ .

**Definition 2.4** ([6]) A finite group  $(G, \cdot)$  of order  $n$  is said to be sequenceable if its elements can be arranged in a sequence  $a_0 = e, a_1, a_2, \dots, a_{n-1}$  in such a way that the partial products  $b_0 = a_0, b_1 = a_0 a_1, b_2 = a_0 a_1 a_2, \dots, b_{n-1} = a_0 a_1 a_2 \dots a_{n-1}$  are all distinct.

**Definition 2.7** If both the indegree and outdegree of all the internal vertices of a directed path are one, then it is called unidirectional path and is denoted by  $\vec{P}_n$ .

**Theorem 2.1** ([4])  $\vec{P}_n$  on  $n$  vertices is graceful if and only if  $z_n$  is sequenceable.

**Theorem 2.2** Let  $D$  be a labeled unidirectional path with  $q$  edges and let  $[M_D]$  be the generalized adjacency matrix for  $D$ . Then  $D$  is digraceful if and only if  $[M_D]$  has exactly one entry  $\pm 1$  in each diagonal lines, except the main diagonal line of zeros.

*Proof* It is to be noted that the matrix  $[M_D]$  is a staircase shaped matrix and skew symmetric. The main diagonal line is  $D_k$  for  $k = 0$ .

Suppose that  $[M_D]$  has exactly one entry  $\pm 1$  in each diagonal lines except the main diagonal. Suppose to the contrary that the labeling of  $D$  that induces  $[M_D]$  is not digraceful. Then there are distinct edges  $v_r v_s$  and  $v_t v_u$  with edge labels  $s - r = t - u = k > 0$ . This implies that the sum of all the elements in  $a_{ij}$  if  $j - i = \pm k$  is either 2 or 0. In the upper triangular matrix the sum of  $a_{ij}$  is 0 and in lower triangular matrix the sum of  $a_{ij}$  is 2 contradicting the assumption that  $[M_D]$  has exactly one entry  $\pm 1$  in each diagonal  $D_k$ . Therefore labeling of  $D$  is graceful.

Let  $f$  be an digraceful labeling on  $D$  and consider  $[M_D]$ . Then, for all  $k = \pm 1, \pm 3, \pm 5, \dots, \pm q$ , there is exactly one nonzero entry  $a_{ij} = \pm 1$  for  $j - i = k$ , contributing to  $D_k$  ( $k \neq 0$ ) since each edge has a unique label. Then  $[M_D]$  has exactly one entry one in each diagonal line except the main diagonal line. This completes the proof.  $\square$

**Theorem 2.3** *Unidirectional path  $\overrightarrow{P_n}$  on  $n$  vertices is odd digraceful if  $n$  is even.*

**Theorem 2.4** *Let  $D$  be a labeled unidirectional path with  $q$  edges and let  $[M_D]$  be the generalized adjacency matrix for  $D$ . Then  $D$  is odd digraceful if and only if  $[M_D]$  has exactly one entry  $\pm 1$  in each odd diagonal lines and all the entries are 0 in the even diagonal lines including the main diagonal line of zeros.*

*Proof* It is to be noted that the matrix  $[M_D]$  is a staircase shaped matrix and skew symmetric. The main diagonal line is  $D_k$  for  $k = 0$ . The odd diagonal lines are  $D_k$  for  $k = \pm 1, \pm 3, \pm 5, \dots, \pm 2q - 1$  and the even diagonal line are  $D_k$  for  $k = \pm 2, \pm 4, \pm 6, \dots, \pm 2q - 2$ .

Suppose that  $[M_D]$  has exactly one entry 1 in each odd diagonal lines and all the diagonal entries are zero in the even diagonal lines including the main diagonal. Suppose to the contrary that the labeling of  $G$  that induces  $[M_D]$  is not odd digraceful. Then there are distinct edges  $v_r v_s$  and  $v_t v_u$  with edge labels  $s - r = t - u = k > 0$ . This implies that there exists atleast one even diagonal which has a non zero entry and at least one odd diagonal which has all entries are zero contradicting the assumption that  $[M_D]$  has exactly one entry 1 in each odd diagonal  $D_k$  for  $k = \pm 1, \pm 3, \pm 5, \dots, \pm 2q - 1$ . Therefore labeling of is odd graceful.

Let  $f$  be an odd graceful labeling on  $G$  and consider  $[M_D]$ . Then for all  $k = \pm 1, \pm 3, \pm 5, \dots, \pm 2q - 1$ , there is exactly one nonzero entry  $a_{ij} = \pm 1$  such that  $j - i = k$ , contributing to  $D_k$  since each edge has a unique label. Then  $[M_D]$  has exactly one entry one in each odd diagonal line. Also in odd graceful labeling there is no even number edge labeling, so the diagonal line  $D_k$  are 0 for  $k = \pm 2, \pm 4, \pm 6, \dots, \pm 2q - 2$ . This completes the proof.  $\square$

### §3. Adjacency Matrices of Alternating Paths

Now we consider the adjacency matrix in digraceful labeling of alternating path  $\overrightarrow{AP_p}$ . An alternating path  $\overrightarrow{AP_p}$  with  $p$  vertices is an oriented path in which any two consecutive arcs have opposite directions. Let  $v_1, v_2, v_3, \dots, v_p$  be the vertices of  $\overrightarrow{AP_p}$  and the arcs of  $\overrightarrow{AP_p}$  are  $\overrightarrow{v_1 v_2}, \overrightarrow{v_3 v_2}, \overrightarrow{v_3 v_4}, \dots, \overrightarrow{v_{p-1} v_p}$  when  $p$  is even or  $\overrightarrow{v_1 v_2}, \overrightarrow{v_3 v_2}, \overrightarrow{v_3 v_4}, \dots, \overrightarrow{v_p v_{p-1}}$  when  $p$  is odd. An alternating path  $\overrightarrow{AP_p}$  is digraceful based on the following labeling  $f : V(\overrightarrow{AP_p}) \rightarrow \{0, 1, 2, 3, \dots, p - 1\}$  defined by

$$\begin{aligned} f(v_1) &= 0; \\ f(v_{2i}) &= p - i \quad \text{for } i = 1, 2, 3, \dots, \left\lfloor \frac{p}{2} \right\rfloor; \\ f(v_{2i+1}) &= i \quad \text{for } i = 1, 2, 3, \dots, \left\lfloor \frac{p-1}{2} \right\rfloor. \end{aligned}$$

We have the following theorem.

**Theorem 3.1** *Let  $D = \overrightarrow{AP_p}$  be a labeled alternating path with  $q$  edges and let  $[M_D]$  be the*

generalized adjacency matrix for  $D$ . Then  $[M_D]$  has exactly one entry  $+1$  in each diagonals of the upper triangular matrix and exactly one entry  $-1$  in each diagonals of the lower triangular matrix if and only if  $D$  is digraceful.

*Proof* It is to be noted that the matrix is skew symmetric. First assume that  $[M_D]$  has exactly one entry  $+1$  in each diagonals of upper triangular matrix and exactly one entry  $-1$  in each diagonals of lower triangular matrix. Suppose to the contrary that the  $D$  is not digraceful. Then there are distinct edge labels  $h - g = f - q = k > 0$ . This implies that the sum of all the elements in  $a_{ij}$  is 0 if  $j - i = \pm k$ , contradicting the assumption. So  $D$  is digraceful.

Let  $f$  be a digraceful labeling on  $D$  and consider  $[M_D]$ . consider the arc  $\overrightarrow{v_i v_j}$  on  $D$ . Since  $D$  is digraceful the labeling on the vertex  $v_j$  is greater than labeling on  $v_i$ . Then for all diagonals  $D_k$  for  $k = \pm 1, \pm 2, \pm 3, \dots, \pm q$  there is exactly one nonzero entry  $a_{ij} = 1$  for  $j > i$  and  $a_{ij} = -1$  for  $j < i$ .  $\square$

Now we consider the adjacency matrix in odd digraceful labeling of alternating path  $\overrightarrow{AP_p}$ . An alternating path  $\overrightarrow{AP_p}$  is odd digraceful based on the following labeling  $f : V(\overrightarrow{AP_p}) \rightarrow \{0, 1, 2, 3, \dots, 2p - 3\}$  defined by

$$\begin{aligned} f(v_1) &= 0; \\ f(v_{2i}) &= 2p - 2i - 1, \text{ for } i = 1, 2, 3, \dots, \lfloor \frac{p}{2} \rfloor; \\ f(v_{2i+1}) &= 2i, \text{ for } i = 1, 2, 3, \dots, \lfloor \frac{p-1}{2} \rfloor. \end{aligned}$$

**Corollary 3.1** Let  $D = \overrightarrow{AP_p}$  be a labeled alternating path with  $q$  edges and let  $[M_D]$  be the generalized adjacency matrix for  $D$ . Then  $[M_D]$  has exactly one entry  $+1$  in each odd diagonals of the upper triangular matrix and exactly one entry  $-1$  in each odd diagonals of the lower triangular matrix if and only if  $D$  is odd digraceful. Also all the entries are zero for even diagonal lines.

#### §4. Adjacency Matrices of Many Orientations of Directed Star

In this section we consider the adjacency matrices of many orientations of directed star. Let  $K_{1,m}$  be an orientations of a star  $K_{1,m}$  on  $(m + 1)$  vertices. In ([7]) Bing Yao, Ming Yao and Hui Cheng proved the following conditions on gracefulness of directed star.

- (1) A directed star  $\overrightarrow{K_{1,m}}$  is digraceful if  $m$  is odd;
- (2) A directed star  $\overrightarrow{K_{1,m}}$  is digraceful if  $m$  is even and one of out-degree and in-degree of the center  $w$  of  $\overrightarrow{K_{1,m}}$  must be even.

We have the following theorem based on adjacency matrices on graceful directed stars.

**Theorem 4.1** The directed star  $\overrightarrow{K_{1,m}}$  is digraceful if and only if the adjacency matrices are any one of the following two forms:

- (1) There exists exactly one entry  $1$  or  $-1$  in each diagonal lines except the main diagonal;
- (2) There exists both  $\pm 1$  in the diagonals  $D_{\pm 1}$  and the remaining diagonal lines have exactly one entry  $\pm 1$ . The main and last diagonal lines have all entries are zeros.

*Proof* The adjacency matrix is skew symmetric. The directed star have no self loops, the entries in the main diagonal line are all zeros. First suppose that the directed star  $\overrightarrow{K_{1,m}}$  is digraceful. The center vertex in the directed star is connected to all the other vertices. The labeling on the center vertex has either zero or any one of the numbers from 1 to  $m-1$ . If the labeling on the center vertex is zero, then it is connected to all the  $m$  labeled vertices. Since each edge has a unique label, in each edge of the adjacency matrix there must exist an entry  $\pm 1$ . Suppose the labeling on the center vertex is other than zero. The labeling on the center vertex is any one of the numbers from 1 to  $m-1$ . If  $r$  is the labeling on the center vertex, then there exists at least two vertices connected to the center vertex have labelings  $r-1$  and  $r+1$ . The diagonal lines in the adjacency matrix corresponding to these vertices are  $D_{\pm 1}$  and they have both  $\pm 1$ . Also there exists a labeling from center vertex to all other remaining vertices. So the remaining diagonal lines have exactly one entry  $\pm 1$ . Since there is no connection between the labeling zero and  $m$ , the last diagonal is zero. Conversely suppose that the adjacency matrix satisfies any one of the given conditions. The directed star  $\overrightarrow{K_{1,m}}$  is digraceful followed by Theorem 2.1.  $\square$

## §5. Adjacency Matrices of a Class of Directed Bistar

A ditree  $H$  with diameter three is called a directed bistar. In ([7]), we have the following description about a class of directed bistars  $T(s, t)$  Dibistar(I): The vertex set and arc set of a directed bistar  $T(s, t)$  are defined as  $V(T(s, t)) = u_i, u, v, v_j : i \in [1, s-1], j \in [1, t]$  and  $A(T(s, t)) = \overrightarrow{uu_i}, \overrightarrow{uv}, \overrightarrow{vv_j} : i \in [1, s-1], j \in [1, t]$ , respectively, where  $u$  is the root of  $T(s, t)$ . Clearly, the in-degrees  $d_T^-(s, t)(u) = 0$  and  $d_T^-(s, t)(v) = 1$  and the out-degrees  $d_T^+(s, t)(u) = s$  and  $d_T^+(s, t)(v) = t$ . In ([7]) it is to be proved that every directed bistar  $T(2l+1, 2k-2l)$  defined by Dibistar(I) is digraceful for integers  $k > l \geq 0$ .

**Theorem 5.1** *The directed bistar  $T(2l+1, 2k-2l)$  defined by dibistar(I) is digraceful if and only if the adjacency matrices have at least  $\pm 1$  in each diagonal lines except  $4l$  diagonal lines in which*

- (1)  $2l$  diagonal entries contains both 1 and -1;
- (2)  $2l$  diagonal entries are all zero except the main diagonal.

*Proof* If  $l = 0$ , the directed bistar  $T(1, 2k)$  is a directed star  $K_{1, 2k+1}$ . The result follows from theorem(6). Let  $v_1$  and  $v_2$  be the centers of the directed bistar  $T(2l+1, 2k-2l)$ . Suppose the center  $v_1$  is connected to  $2l$  vertices and the center  $v_2$  is connected to  $2k-2l+1$  vertices. Since the bistar is digraceful, each edge has a unique labeling. Let the labeling on  $v_2$  is zero which is not connected to  $2l$  vertices. The labeling contributed a non zero entry  $\pm 1$  in each diagonal lines of the adjacency matrix except the  $2l$  diagonal lines and the main diagonal. Also the labeling on  $v_1$  is any integer from 1 to  $2k+1$  and it contributed a non zero entry  $\pm 1$  in  $2l$  diagonal lines of the adjacency matrix. So the  $2l$  diagonal entries contains both 1 and -1. The converse follows from Theorem 2.1.  $\square$

## References

- [1] F.Harary, *Graph Theory*, Addison Wesley, Reading, M.A,1969.
- [2] Joseph A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 17(2014).
- [3] G.S.Bloom and D.F.Hsu, On graceful digraphs and a problem in network addressing, *Congr.Numer.*, 35(1982),91-103.
- [4] G.S.Bloom and D.F.Hsu, On graceful directed graphs, *SIAM J.Algebraic Discrete Methods*, 6(1985),No.3,519-536.
- [5] G.S.Bloom and D.F.Hsu, *On Graceful directed graphs that are computational models of some algebraic systems in graph theory with applications to algorithms and computer science*, Wiley-Intersci. Publ, Wiley, New York(1984) 89-102.
- [6] D.F.Hsu, Keedwell A.D., Generalized complete mappings, neofields, sequenceable groups and block designs, *Pac.J.Math.*, 117(1985),291-312.
- [7] Bing Yao, Ming Yao and Hui Cheng, On gracefulness of Directed Trees with Short Diameters, *Bull.Malays.Math.Sci.Soc.*, (2)35(1)(2012),133-146.

## Minimum Equitable Dominating Randić Energy of a Graph

Rajendra P.

(Bharathi College, PG and Research Centre, Bharathinagara, 571 422, India)

R.Rangarajan

(DOS in Mathematics, University of Mysore, Mysuru, 570 006, India)

prajumaths@gmail.com, rajra63@gmail.com

**Abstract:** Let  $G$  be a graph with vertex set  $V(G)$ , edge set  $E(G)$  and  $d_i$  is the degree of its  $i$ -th vertex  $v_i$ , then the Randić matrix  $R(G)$  of  $G$  is the square matrix of order  $n$ , whose  $(i, j)$ -entry is equal to  $\frac{1}{\sqrt{d_i d_j}}$  if the  $i$ -th vertex  $v_i$  and  $j$ -th vertex  $v_j$  of  $G$  are adjacent, and zero otherwise. The Randić energy [3]  $RE(G)$  of the graph  $G$  is defined as the sum of the absolute values of the eigenvalues of the Randić matrix  $R(G)$ . A subset  $ED$  of  $V(G)$  is called an equitable dominating set [11], if for every  $v_i \in V(G) - ED$  there exists a vertex  $v_j \in ED$  such that  $v_i v_j \in E(G)$  and  $|d_i(v_i) - d_j(v_j)| \leq 1$ . In the contrast, such a dominating set  $ED$  is Smarandachely if  $|d_i(v_i) - d_j(v_j)| \geq 1$ . Recently, Adiga, et.al. introduced, the minimum covering energy  $Ec(G)$  of a graph [1] and S. Burcu Bozkurt, et.al. introduced, Randić Matrix and Randić Energy of a graph [3]. Motivated by these papers, Minimum equitable dominating Randić energy of a graph  $RE_{ED}(G)$  of some graphs are worked out and bounds on  $RE_{ED}(G)$  are obtained.

**Key Words:** Randić matrix and its energy, minimum equitable dominating set and minimum equitable dominating Randić energy of a graph.

**AMS(2010):** 05C50.

### §1. Introduction

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The adjacency matrix  $A(G)$  of the graph  $G$  is a square matrix, whose  $(i, j)$ -entry is equal to 1 if the vertices  $v_i$  and  $v_j$  are adjacent, otherwise zero [12]. Since  $A(G)$  is symmetric, its eigenvalues are all real. Denote them by  $\lambda_1, \lambda_2, \dots, \lambda_n$ , and as a whole, they are called the spectrum of  $G$  and denoted by  $\text{Spec}(G)$ . The energy of graph [12]  $G$  is

$$\varepsilon(G) = \sum_{i=1}^n |\lambda_i|.$$

The literature on energy of a graph and its bounds can refer [4,8,9,10,12]. The Randić matrix  $R(G) = (r_{ij})$  of  $G$  is the square matrix of order  $n$ , where

---

<sup>1</sup>Received July 28, 2017, Accepted March 1, 2018.

$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 0, & \text{otherwise.} \end{cases}$$

The Randic energy [3]  $RE(G)$  of the graph  $G$  is defined as the sum of the absolute values of the eigenvalues of the Randić matrix  $R(G)$ . Let  $\rho_1, \rho_2, \dots, \rho_n$  be the eigenvalues of the Randić matrix  $R(G)$ . Since  $R(G)$  is symmetric matrix, these eigenvalues are real numbers and their sum is zero. Randić energy [3] can be defined as

$$RE(G) = \sum_{i=1}^n |\rho_i|$$

For details of Randić energy and its bounds, can refer [2, 3, 5, 6, 7].

Let  $G$  be a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $ED$  is minimum equitable dominating set of  $G$ . Minimum equitable dominating Randić matrix of  $G$  is  $n \times n$  matrix  $R_{ED}(G) = (r_{ij})$ , where

$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 1, & \text{if } i = j \text{ and } v_i \in ED; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of  $R_{ED}(G)$  is denoted by  $\det(\rho I - R_{ED}(G)) = |\rho I - R_{ED}(G)|$ . Since  $R_{ED}(G)$  is symmetric, its eigenvalues are real numbers. If the distinct eigenvalues of  $R_{ED}(G)$  are  $\rho_1 > \rho_2 > \dots > \rho_r$  with their multiplicities are  $m_1, m_2, \dots, m_r$  then spectrum of  $R_{ED}(G)$  is denoted by

$$SpecR_{ED}(G) = \begin{pmatrix} \rho_1 & \rho_2 & \dots & \rho_r \\ m_1 & m_2 & \dots & m_r \end{pmatrix}.$$

The minimum equitable dominating Randić energy of  $G$  is defined as

$$RE_{ED}(G) = \sum_{i=1}^n |\rho_i|.$$

**Example 1.1** Let  $W_5$  be a wheel graph, with vertex set  $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ , and let its minimum equitable dominating set be  $ED = \{v_1\}$ . Then minimum equitable dominating Randić matrix  $R_{ED}(W_5)$  is

$$R_{ED}(W_5) = \begin{bmatrix} 1 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$SpecR_{ED}(W_5) = \begin{pmatrix} -0.6666 & 0 & 0.2324 & 1.4342 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of  $W_5$  is  $RE_{ED}(W_5) = 2.3332$ .

## §2. Bounds for the Minimum Equitable Dominating Randić Energy of a Graph

**Lemma 2.1** *If  $\rho_1, \rho_2, \dots, \rho_n$  are the eigenvalues of  $R_{ED}(G)$ . Then*

$$\sum_{i=1}^n \rho_i = |ED|$$

and

$$\sum_{i=1}^n \rho_i^2 = |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j},$$

where  $ED$  is minimum equitable dominating set.

*Proof* (i) The sum of eigenvalues of  $R_{ED}(G)$  is

$$\sum_{i=1}^n \rho_i = \sum_{i=1}^n r_{ii} = |ED|.$$

(ii) Consider, the sum of squares of  $\rho_1, \rho_2, \dots, \rho_n$  is,

$$\begin{aligned} \sum_{i=1}^n \rho_i^2 &= \sum_{i=1}^n \sum_{j=1}^n r_{ij} r_{ji} = \sum_{i=1}^n (r_{ii})^2 + \sum_{i \neq j} r_{ij} r_{ji} \\ &= \sum_{i=1}^n (r_{ii})^2 + 2 \sum_{i < j} (r_{ij})^2 \\ \sum_{i=1}^n \rho_i^2 &= |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}. \quad \square \end{aligned}$$

Upper and lower bounds for  $RE_{ED}(G)$  is similar proof to McClelland's inequalities [10], are given below

**Theorem 2.2**(Upper Bound) *Let  $G$  be a graph with  $ED$  is minimum equitable dominating set. Then*

$$RE_{ED}(G) \leq \sqrt{n \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]}.$$

*Proof* Let  $\rho_1, \rho_2, \dots, \rho_n$  be the eigenvalues of  $R_{ED}(G)$ . By Cauchy-Schwartz inequality, we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right), \quad (1)$$

where  $a$  and  $b$  are any real numbers.



If  $a_i = 1$ ,  $b_i = |\rho_i|$  in (1), we get

$$\left( \sum_{i=1}^n |\rho_i| \right)^2 \leq \left( \sum_{i=1}^n 1^2 \right) \left( \sum_{i=1}^n |\rho_i|^2 \right), \quad [RE_{ED}(G)]^2 \leq n \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]$$

by Lemma 2.1,

$$RE_{ED}(G) \leq \sqrt{n \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]}.$$

**Theorem 2.3**(Lower Bound) *Let  $G$  be a graph with  $|ED|$  is minimum equitable dominating set and  $d_i$  is degree of  $v_i$ . Then*

$$RE_{ED}(G) \geq \sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}},$$

where  $D = \prod_{i=1}^n |\rho_i|$ .

*Proof* Consider

$$[RE_{ED}(G)]^2 = \left[ \sum_{i=1}^n |\rho_i| \right]^2 = \sum_{i=1}^n |\rho_i|^2 + \sum_{i \neq j} |\rho_i| |\rho_j|. \quad (2)$$

By using arithmetic and geometric mean inequality, we have

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} |\rho_i| |\rho_j| &\geq \left( \prod_{i \neq j} |\rho_i| |\rho_j| \right)^{\frac{1}{n(n-1)}} \\ \sum_{i \neq j} |\rho_i| |\rho_j| &\geq n(n-1) \left( \prod_{i=1}^n |\rho_i|^{2(n-1)} \right)^{\frac{1}{n(n-1)}} \\ \sum_{i \neq j} |\rho_i| |\rho_j| &\geq n(n-1) \left( \prod_{i=1}^n |\rho_i| \right)^{2/n}. \end{aligned} \quad (3)$$

Now using (3) in (2), we get

$$\begin{aligned} [RE_{ED}(G)]^2 &\geq \sum_{i=1}^n |\rho_i|^2 + n(n-1) \left( \prod_{i=1}^n |\rho_i| \right)^{2/n}, \\ [RE_{ED}(G)]^2 &\geq |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}, \quad \text{where } D = \prod_{i=1}^n |\rho_i|, \\ RE_{ED}(G) &\geq \sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}}. \end{aligned}$$

### §3. Bounds for Largest Eigenvalue of $R_{ED}(G)$ and its Energy

**Proposition 3.1** *Let  $G$  be a graph and  $\rho_1(G) = \max_{1 \leq i \leq n} \{|\rho_i|\}$  be the largest eigenvalue of  $R_{ED}(G)$ . Then*

$$\sqrt{\frac{1}{n} \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}.$$

*Proof* Consider,

$$\begin{aligned} \rho_1^2(G) &= \max_{1 \leq i \leq n} \{|\rho_i|^2\} \leq \sum_{i=1}^n |\rho_i|^2 = |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \\ \rho_1(G) &\leq \sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}} \end{aligned}$$

Next,

$$\begin{aligned} n \rho_1^2(G) &\geq \sum_{i=1}^n \rho_i^2 = |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \\ \rho_1^2(G) &\geq \frac{1}{n} \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right] \\ \rho_1(G) &\geq \sqrt{\frac{1}{n} \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]} \end{aligned}$$

Therefore,

$$\sqrt{\frac{1}{n} \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}.$$

**Proposition 3.2** *If  $G$  is a graph and  $n \leq |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}$ , then*

$$RE_{ED}(G) \leq \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} + \sqrt{(n-1) \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - \left( \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} \right)^2 \right]}.$$

*Proof* We know that,

$$\sum_{i=1}^n \rho_i^2 = |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}, \quad \sum_{i=2}^n \rho_i^2 = |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - \rho_1^2 \quad (4)$$

By Cauchy-Schwarz inequality, we have

$$\left( \sum_{i=2}^n a_i b_i \right)^2 \leq \left( \sum_{i=2}^n a_i^2 \right) \left( \sum_{i=2}^n b_i^2 \right).$$

If  $a_i = 1$  and  $b_i = |\rho_i|$ , we have

$$\begin{aligned} \left( \sum_{i=2}^n |\rho_i| \right)^2 &\leq (n-1) \sum_{i=2}^n |\rho_i|^2 \\ [RE_{ED}(G) - |\rho_1|]^2 &\leq (n-1) \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - \rho_1^2 \right] \\ RE_{ED}(G) &\leq \rho_1 + \sqrt{(n-1) \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - \rho_1^2 \right]} \end{aligned} \quad (5)$$

Consider the function,

$$F(x) = x + \sqrt{(n-1) \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - x^2 \right]}.$$

Then,

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - x^2}}.$$

Here  $F(x)$  is decreasing in

$$\left( \frac{1}{n} \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right], |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \right).$$

We know that  $F'(x) \leq 0$ ,

$$1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - x^2}} \leq 0.$$

We have

$$x \geq \sqrt{\frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n}}.$$

Since,

$$n \leq |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} \quad \text{and} \quad \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} \leq \rho_1,$$

we have

$$\sqrt{\frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n}} \leq \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} \leq \rho_1 \leq |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}.$$

Then, equation (5) become

$$RE_{ED}(G) \leq \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} + \sqrt{(n-1) \left[ |ED| + 2 \sum_{i < j} \frac{1}{d_i d_j} - \left( \frac{|ED| + 2 \sum_{i < j} \frac{1}{d_i d_j}}{n} \right)^2 \right]}.$$

□

#### §4. Minimum Equitable Dominating Randić Energy of Some Graphs

**Theorem 4.1** *If  $K_n$  is complete graph with  $n$  vertices, then minimum equitable dominating Randić energy of  $K_n$  is*

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}.$$

*Proof* Let  $K_n$  be the complete graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and minimum equitable dominating set is  $ED = \{v_1\}$ , we have characteristic polynomial of  $R_{ED}(K_n)$  is

$$|\rho I - R_{ED}(K_n)| = \begin{vmatrix} \rho - 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \rho & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \rho & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \rho & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \rho \end{vmatrix}_{n \times n}.$$

$R'_k = R_k - R_2$ ,  $k = 3, 4, \dots, n-1, n$ . Then, we get  $\left(\rho + \frac{1}{n-1}\right)$  common from  $R_3$  to  $R_n$  and we have

$$|\rho I - R_{ED}(K_n)| = \left(\rho + \frac{1}{n-1}\right)^{n-2} \begin{vmatrix} \rho - 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n \times n}.$$

$C'_2 = C_2 + C_3 + \cdots + C_n$ , we get the characteristic polynomial

$$\begin{aligned} |\rho I - R_{ED}(K_n)| &= \left( \rho + \frac{1}{n-1} \right)^{n-2} \left[ \rho^2 - \left( \frac{2n-3}{n-1} \right) \rho + \left( \frac{n-3}{n-1} \right) \right], \\ \text{Spec} R_{ED}(K_n) &= \begin{pmatrix} \frac{-1}{n-1} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The minimum equitable dominating Randić energy of  $K_n$  is

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}. \quad \square$$

**Theorem 4.2** *If  $S_n, (n \geq 4)$  is star graph with  $n$  vertices, then minimum equitable dominating Randić energy of  $S_n$  is*

$$RE_{ED}(S_n) = n.$$

*Proof* Let  $S_n, (n \geq 4)$  be the star graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and minimum equitable dominating set is  $ED = \{v_1, v_2, \dots, v_n\}$ , we have characteristic polynomial of  $R_{ED}(S_n)$  is

$$|\rho I - R_{ED}(S_n)| = \begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & \rho-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & \rho-1 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & \rho-1 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & 0 & \rho-1 \end{vmatrix}_{n \times n}.$$

$R'_k = R_k - R_2, k = 3, 4, \dots, n$ . Then taking  $(\rho-1)$  common from  $R_3$  to  $R_n$ , we get

$$|\rho I - R_{ED}(S_n)| = (\rho-1)^{n-2} \begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n \times n}.$$

$C'_2 = C_2 + C_3 + \cdots + C_n$ . Then, the characteristic polynomial

$$|\rho I - R_{ED}(S_n)| = \rho(\rho-1)^{n-2}(\rho-2),$$

$$\text{Spec}R_{ED}(S_n) = \begin{pmatrix} 0 & 1 & 2 \\ & & \\ 1 & n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of  $S_n$  is  $RE_{ED}(S_n) = n$ .  $\square$

**Theorem 4.3** *If  $K_{m,n}$ , where  $m < n$  and  $|m - n| \geq 2$  is complete bipartite graph with  $m + n$  vertices, then minimum equitable dominating Randić energy of  $K_{m,n}$  is  $RE_{ED}(K_{m,n}) = m + n$ .*

*Proof* Let  $K_{m,n}$ , where  $m < n$  and  $|m - n| \geq 2$  be the complete bipartite graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$  and minimum equitable dominating set is  $ED = V(G)$ , we have characteristic polynomial of  $R_{ED}(K_{m,n})$  is

$$|\rho I - R_{ED}(K_{m,n})| = \begin{vmatrix} \rho - 1 & 0 & \dots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & \rho - 1 & \dots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho - 1 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & 0 & \dots & 0 & \rho - 1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho - 1 & 0 & \dots & 0 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & \rho - 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \dots & \rho - 1 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \dots & 0 & \rho - 1 \end{vmatrix}.$$

$R'_k = R_k - R_m$ ,  $k = 1, 2, 3, \dots, m - 1$  and  $R'_d = R_d - R_{m+1}$ ,  $d = m + 2, m + 3, \dots, m + n$ . Then, taking  $(\rho - 1)$  common from  $R_1$  to  $R_{m-1}$  and  $R_{m+2}$  to  $R_{m+n}$ , we get

$$(\rho - 1)^{m+n-2} \begin{vmatrix} 1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \rho - 1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho - 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

$$C'_{m+1} = C_{m+1} + C_{m+2} + \cdots + C_{m+n},$$

$$(\rho - 1)^{m+n-2} \begin{vmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \rho - 1 & \frac{-n}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho - 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

The characteristic polynomial  $|\rho I - R_{ED}(K_{m,n})| = \rho(\rho - 1)^{m+n-2}(\rho - 2)$ ,

$$\text{Spec} R_{ED}(K_{m,n}) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of  $K_{m,n}$  is  $RE_{ED}(K_{m,n}) = m + n$ .  $\square$

**Theorem 4.4** *If  $K_{n \times 2}$ , ( $n \geq 3$ ) is cocktail party graph with  $2n$  vertices, then minimum equitable dominating Randić energy of  $K_{n \times 2}$  is  $\frac{4n-6}{n-1}$ .*

*Proof* Let  $K_{n \times 2}$ , ( $n \geq 3$ ) be the cocktail party graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$  and minimum equitable dominating set is  $ED = \{v_1, u_1\}$ , we have characteristic polynomial of  $R_{ED}(K_{n \times 2})$  is

$$|\rho I - R_{ED}(K_{n \times 2})| = \begin{vmatrix} \lambda - 1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda - 1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda \end{vmatrix}.$$

$R'_k = R_k - R_3$ ,  $k = 4, 5, \dots, n$  and  $R'_{n+k} = R_{n+k} - R_{k+1}$ ,  $k = 2, 3, \dots, n$ , we get

$$|\rho I - R_{ED}(K_{n \times 2})| = \begin{vmatrix} \lambda - 1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda - 1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & \frac{-1}{2n-2} - \lambda & \lambda + \frac{1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \frac{-1}{2n-2} - \lambda & 0 & \cdots & \lambda + \frac{1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & -\lambda & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

$C'_3 = C_3 + C_4 + \cdots + C_n + C_{n+1} + \cdots + C_{2n}$  and  $C'_k = C_k + C_{n+(k-1)}$ ,  $k = 4, 5, \dots, n+1$ . We get

$$|\rho I - R_{ED}(K_{n \times 2})| = \begin{vmatrix} \lambda - 1 & 0 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda - 1 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda - \frac{(2n-4)}{(2n-2)} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & 0 & \lambda + \frac{1}{n-1} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda + \frac{1}{n-1} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

The characteristic polynomial

$$\begin{aligned} |\rho I - R_{ED}(K_{n \times 2})| &= \rho^{n-1}(\rho - 1) \left( \rho + \frac{1}{n-1} \right)^{n-2} \left[ \rho^2 - \left( \frac{2n-3}{n-1} \right) \rho + \left( \frac{n-3}{n-1} \right) \right], \\ \text{Spec} R_{ED}(K_{n \times 2}) &= \begin{pmatrix} \frac{-1}{n-1} & 0 & 1 & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & n-1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The minimum equitable dominating Randić energy of  $K_{n \times 2}$  is

$$RE_{ED}(K_{n \times 2}) = \frac{4n-6}{n-1},$$

where  $n \geq 3$ . □



## References

- [1] C. Adiga, A. Bayad, I. Gutman and Shrikanth A. S, The minimum covering energy of a graph, *Krag. J. Sci.*, 34 (2012), 39-56.
- [2] Ş. Burcu Bozkurt, A. Dilek Güngör and Ivan Gutman, Randić, Spectral radius and Randić energy, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 321-334.
- [3] Ş. Burcu Bozkurt, A. Dilek Güngör, Ivan Gutman and A. Sinan Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 239-250.
- [4] D. M. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, New York, USA, 1980.
- [5] K. C. Das and S. Sorgun: On Randić energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 72 (2014), 227-238.
- [6] A. Dilek Maden, New bounds on the incidence Eenergy, Randić energy and Randić Estrada index, *MATCH Commun. Math. Comput. Chem.*, 74, (2015), 367-387.
- [7] B. Furtula and I. Gutman, Comparing energy and Randić energy, *Macedonian Journal of Chemistry and Chemical Engineering*, 32(1), (2013), 117-123.
- [8] J.H. Koolen and V. Moulton, Maximal energy graphs. *Advances in Applied Mathematics*, 26, (2001), 47-52.
- [9] I. Gutman, The energy of a graph, *Ber. Math-Statist. Sect. Forschungsz. Graz*, 103 (1978), 1-22.
- [10] B. J. McClelland, Properties of the latent roots of a matrix: The estimation of  $\pi$ -electron energies, *J. Chem. Phys.*, 54 (1971), 640-643.
- [11] V. Swaminathan and K. M. Dharmalingam, Degree equitable domination on graphs, *Krag. J. of Math.*, 35 (1) (2011), 191-197.
- [12] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer New York, 2012.

## Linear Cyclic Snakes as Super Vertex Mean Graphs

A.Lourdusamy and Sherry George

Department of Mathematics, Manonmaniam Sundaranar University  
St. Xavier's College (Autonomous), Palayamkottai, Tirunelveli - 627002, India

lourdusamy15@gmail.com

**Abstract:** A super vertex mean labeling  $f$  of a  $(p, q)$  - graph  $G(V, E)$  is defined as an injection from  $E$  to the set  $\{1, 2, 3, \dots, p+q\}$  that induces for each vertex  $v$  the label defined by the rule  $f^v(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right)$ , where  $E_v$  denotes the set of edges in  $G$  that are incident at the vertex  $v$ , such that the set of all edge label and the induced vertex labels is  $\{1, 2, 3, \dots, p+q\}$ . All the cycles,  $C_n$ ,  $n \geq 3$  and  $n \neq 4$  are super vertex mean graphs. Our attempt in this paper is to show that all the linear cyclic snakes, including  $kC_4$ , are also super vertex mean graphs, even though  $C_4$  is not an SVM graph. We also define the term Super Vertex Mean number of graphs.

**Key Words:** Super vertex mean label, Smarandachely super  $H$ -vertex mean labeling, linear cyclic snakes, SVM number.

**AMS(2010):** 05C78.

### §1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols  $V(G)$  and  $E(G)$  will denote the vertex set and edge set of a graph  $G$  respectively.  $p$  and  $q$  denote the number of vertices and edges of  $G$  respectively. A graph of order  $p$  and size  $q$  is often called a  $(p, q)$  - graph.

A labeling of a graph  $G$  is an assignment of labels either to the vertices or edges. There are varieties of vertex as well as edge labeling that are already in the literature. Mean labeling was introduced by Somasundaram and Ponraj [14]. Super mean labeling was introduced by D.Ramya et al.[10]. Some results on mean labeling and super mean labeling are given in [5, 6, 9, 10, 11, 13, 14] and [15]. Lourdusamy and Seenivasan [5] introduced vertex mean labeling as an edge analogue of mean labeling.

Continuing on the same line and inspired by the above mentioned concepts, Lourdusamy et al. [7] brought in a new extension of mean labeling, called Super vertex mean labeling of graphs.

### §2. Super Vertex Mean Labeling

**Definition 2.1** A super vertex mean labeling  $f$  of a  $(p, q)$  - graph  $G(V, E)$  is defined as an

---

<sup>1</sup>Received February 4, 2017, Accepted March 3, 2018.

injection from  $E$  to the set  $\{1, 2, 3, \dots, p + q\}$  that induces for each vertex  $v$  the label defined by the rule  $f^v(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right)$ , where  $E_v$  denotes the set of edges in  $G$  that are incident at the vertex  $v$ , such that the set of all edge labels and the induced vertex labels is  $\{1, 2, 3, \dots, p + q\}$ .

Furthermore, a super vertex mean labeling  $f$  on  $G$  is Smarandachely super  $H$ -vertex mean labeling if the induced vertex labels on vertex in  $H'$  is  $\{1, 2, 3, \dots, p(H') + q(H')\}$  for each subgraph  $H' \cong H$  in  $G$ , where  $p(H')$  and  $q(H')$  are respectively the order and the size of  $H'$ .

A graph that accepts super vertex mean labeling is called a super vertex mean (hereafter, SVM) graph. The following results have already been proved in [7] and [8]. We use them for our further study on the super vertex mean behavior of linear cyclic snakes. Before entering into the results, we define the term cyclic snakes.

**Definition 2.2** A  $kC_n$ -snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle  $C_n$  and the block-cut point graph is a path  $P$ , where  $P$  is the path of minimum length that contains all the cut vertices of a  $kC_n$ -snake. Barrientos [13] has proved that any  $kC_n$ -snake is represented by a string  $s_1, s_2, s_3, \dots, s_{k-2}$  of integers of length  $k - 2$ , where the  $i^{\text{th}}$  integer,  $s_i$  on the string is the distance between  $i^{\text{th}}$  and  $i + 1^{\text{th}}$  cut vertices along the path,  $P$ , from one extreme and is taken from  $S_n = \{1, 2, 3, \dots, \lfloor \frac{n}{2} \rfloor\}$ . A  $kC_n$ -snake is said to be linear if each integer  $s_i$  of its string is equal to  $\lfloor \frac{n}{2} \rfloor$ .

**Remark 2.3** The strings obtained from both the extremes are assumed to be the same. A linear cyclic snake,  $kC_n$  is obtained from  $k$  copies of  $C_n$  by identifying the vertex  $v_{i,r+1}$  in the  $i^{\text{th}}$  copy of  $C_n$  at a vertex  $v_{i+1,1}$  in the  $(i + 1)^{\text{th}}$  copy of  $C_n$ , where  $1 \leq i \leq k - 1$  and  $n = 2r$  or  $n = 2r + 1$ , depending upon whether  $n$  is even or odd respectively.

**Note 2.4** that  $v_{i,r+1} = v_{i+1,1}$  for  $1 \leq i \leq k - 1$ , and we consider this vertex as  $v_{i+1,1}$  throughout this paper.

### §3. Known Results

We have known results listed in the following.

- (1) All the cycles except  $C_4$  are SVM graphs([8]);
- (2) All odd cycles,  $C_n$  can be SVM labeled as many as  $\lfloor \frac{n}{2} \rfloor$  ways and every even cycle,  $C_n$ , except  $C_4$  can have  $\lfloor \frac{n}{2} \rfloor - 1$  types of SVM labelings([8]);
- (3) A linear triangular snake,  $kC_3$  with  $k$  blocks is an SVM graph([7]).

### §4. Linear Cyclic Snakes of Higher Orders

We proceed to prove that other linear cyclic snakes too are super vertex mean graphs.

**Theorem 4.1** Linear quadrilateral snakes,  $kC_4$  with  $k \geq 2$  blocks are SVM graphs.

*Proof* Let  $kC_4$  be a linear quadrilateral snake with  $p$  vertices and  $q$  edges. Then  $p = 3k + 1$  and  $q = 4k$ . Define  $f : E(kC_4) \rightarrow \{1, 2, 3, \dots, 7k + 1\}$  to be  $f(u_i u_{i+1}) = 7i$  if  $1 \leq i \leq k - 1$  and  $7k + 1$  if  $i = k$ . When  $1 \leq i \leq k - 1$  and  $k \geq 2$ ,

$$f(e_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j = 1 \\ 7i - 5, & \text{if } 2 \leq i \leq k - 1, \text{ and } j = 1 \\ 7i - 1, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 2 \\ 7i, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 3 \\ 7i - 4, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 4. \end{cases}$$

When  $i = k$ ,  $k \geq 2$  and  $k$  is even,

$$f(e_{i,j}) = \begin{cases} 7k - 6, & \text{if } j = 1, \\ 7k - 3, & \text{if } j = 2, \\ 7k - 1, & \text{if } j = 3, \\ 7k + 1, & \text{if } j = 4. \end{cases}$$

When  $i = k$ ,  $k \geq 3$  and  $k$  is odd,

$$f(e_{i,j}) = \begin{cases} 7k - 5, & \text{if } j = 1, \\ 7k - 2, & \text{if } j = 2, \\ 7k + 1, & \text{if } j = 3, \\ 7k - 4, & \text{if } j = 4. \end{cases}$$

It can be easily verified that  $f$  is injective. The induced vertex labels are as follows:

When  $1 \leq i \leq k - 1$  and  $k \geq 2$ ,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 7i - 6, & \text{if } 2 \leq i \leq k - 1, \text{ and } j = 1, \\ 7i - 3, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 2, \\ 7i - 2, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 4. \end{cases}$$

When  $i = k$ ,  $k \geq 2$ , and  $k$  is even,

$$f^v(v_{i,j}) = \begin{cases} 7k - 5, & \text{if } j = 1, \\ 7k - 4, & \text{if } j = 2, \\ 7k - 2, & \text{if } j = 3, \\ 7k, & \text{if } j = 4. \end{cases}$$

When  $i = k$ ,  $k \geq 3$ , and  $k$  is odd,

$$f^v(v_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k, & \text{if } j = 3, \\ 7k-1, & \text{if } j = 4. \end{cases}$$

It can be easily verified that the set of edge labels and induced vertex labels is  $\{1, 2, 3, \dots, 7k+1\}$  in two following cases:

**Case 1.** When  $k$  is even,

$$\begin{aligned} f(E) = & \{1, 6, 7, 3, 9, 13, 14, 10, 16, 20, 21, 27, \dots, 7k-12, \\ & 7k-8, 7k-7, 7k-11, 7k-6, 7k-3, 7k-1, 7k+1\} \end{aligned}$$

and,

$$f^v(V) = \{2, 4, 8, 5, 11, 15, 12, 18, 22, 19, \dots, 7k-10, 7k-5, 7k-9, 7k-4, 7k-2, 7k\}.$$

Therefore,

$$\begin{aligned} f(E) \cup f^v(V) = & \{1, 2, 3, 4, \dots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ & 7k-7, 7k-6, 7k-5, 7k-4, 7k-3, 7k-2, 7k-1, 7k, 7k+1\}. \end{aligned}$$

**Case 2.** When  $k$  is odd,

$$\begin{aligned} f(E) = & \{1, 6, 7, 3, 9, 13, \dots, 7k-12, 7k-8, \\ & 7k-7, 7k-11, 7k-5, 7k-2, 7k+1, 7k-4\} \end{aligned}$$

and,

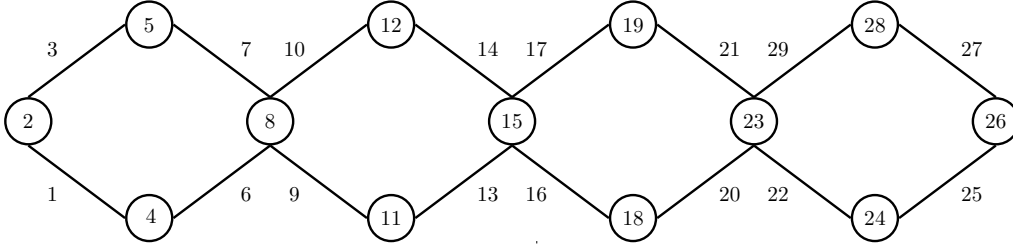
$$f^v(V) = \{2, 4, 8, 5, \dots, 7k-10, 7k-6, 7k-9, 7k-3, 7k, 7k-1\}.$$

Therefore,

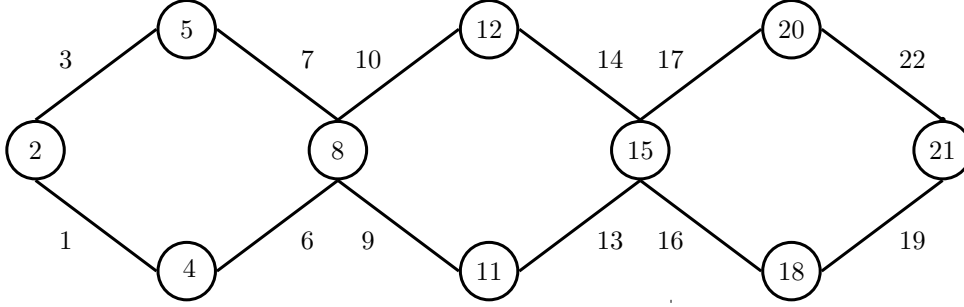
$$\begin{aligned} f(E) \cup f^v(V) = & \{1, 2, 3, 4, \dots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ & 7k-7, 7k-6, 7k-5, 7k-4, 7k-3, 7k-2, 7k-1, 7k, 7k+1\}. \end{aligned}$$

Thus it has been proved that the labeling  $f : E(kC_4) \rightarrow \{1, 2, 3, \dots, 7k+1\}$  is a super vertex mean labeling.  $\square$

**Example 4.2** The super vertex-mean labelings of two linear quadrilateral snakes with 4 and 3 blocks are shown in figures 1 and 2 respectively.



**Figure 1** A Super vertex-mean labeling of a linear quadrilateral snake with 4 blocks



**Figure 2** A super vertex-mean labeling of a linear quadrilateral snake with 3 blocks

**Theorem 4.3** All linear pentagonal snakes,  $kC_5$  with  $k, k \geq 2$  blocks are SVM graphs.

*Proof* Let  $kC_5$  be a linear pentagonal snake with  $k, k \geq 2$  blocks of  $C_5$ . Here  $p = 4k + 1$ ,  $q = 5k$  and  $p + q = 9k + 1$ . Define  $f : E(kC_5) \rightarrow \{1, 2, 3, \dots, 9k + 1\}$  as follows:

When  $i = 1$ ,

$$f(e_{i,j}) = \begin{cases} 5, & \text{if } j = 1, \\ 2j + 4, & \text{if } 2 \leq j \leq 3, \\ 1, & \text{if } j = 4, \\ 3, & \text{if } j = 5. \end{cases}$$

And When  $2 \leq i \leq k$ ,

$$f(e_{i,j}) = \begin{cases} 9i - 9, & \text{if } j = 1, \\ 9i + 2j - 5, & \text{if } 2 \leq j \leq 3, \\ 9i + 3j - 18, & \text{if } 4 \leq j \leq 5. \end{cases}$$

It can be easily verified that  $f$  is injective. Then, the induced vertex labels are as follows:

When  $i = 1$ ,

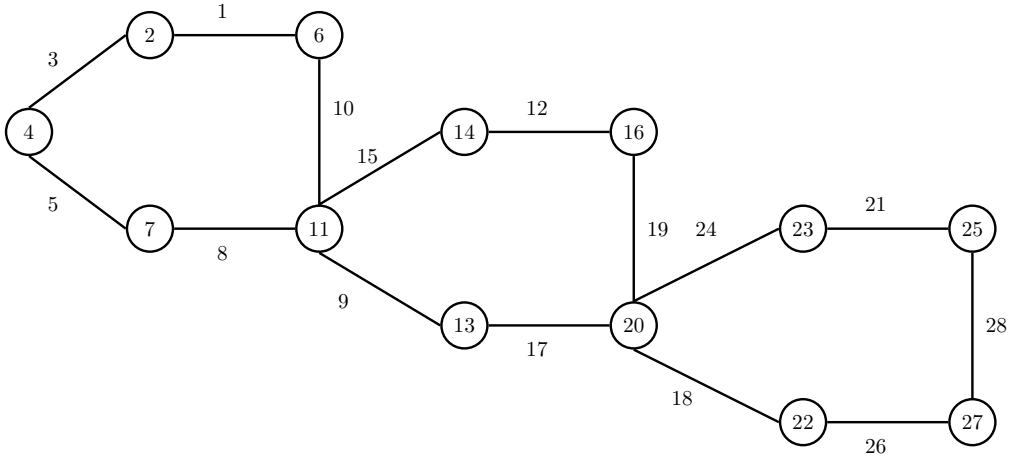
$$f^v(v_{i,j}) = \begin{cases} 4, & \text{if } j = 1, \\ 7, & \text{if } j = 2, \\ 6, & \text{if } j = 4, \\ 2, & \text{if } j = 5. \end{cases}$$

When  $2 \leq i \leq k$ ,

$$f^v(v_{i,j}) = \begin{cases} 9i + 2j - 9, & \text{if } 1 \leq j \leq 2 \\ 9i - 2j + 6, & \text{if } 4 \leq j \leq 5 \\ 9k, & \text{if } i = k, \text{ and } j = 3. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is  $\{1, 2, 3, \dots, 9k + 1\}$ . Therefore, all linear pentagonal snakes  $kC_5$  are super vertex mean graphs.  $\square$

**Example 4.4** A graph given in Figure 3 is an SVM labeling of a linear pentagonal snake with 3 blocks.



**Figure 3** An SVM labeling of a linear pentagonal snake,  $3C_5$ .

**Theorem 4.5** All linear hexagonal snakes,  $kC_6, k \geq 2$  are super vertex mean graphs.

*Proof* Let  $kC_6$  be a hexagonal snake with  $k, k \geq 2$  blocks of  $C_6$ .  $p = 5k + 1$  and  $q = 6k$  and  $p + q = 11k + 1$ . Define  $f : E(G_n) \rightarrow \{1, 2, 3, \dots, 11k + 1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4, \\ 7, & \text{if } i = 1, \text{ and } j = 5, \\ 1, & \text{if } i = 1, \text{ and } j = 6, \\ 11i - 4j, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\ 11i + 3j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 4, \\ 11i - 8j + 37, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 6. \end{cases}$$

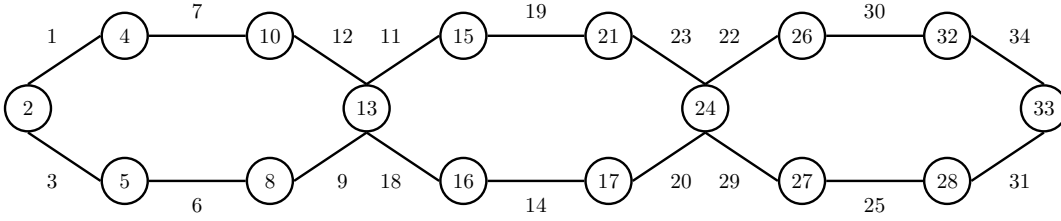
It can be easily verified that  $f$  is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 11i + 3j - 12, & \text{if } 1 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\ 8, & \text{if } i = 1, \text{ and } j = 3, \\ 11i - 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ 11k, & \text{if } i = k, \text{ and } j = 4, \\ 11i - 1, & \text{if } 1 \leq i \leq k, \text{ and } j = 5, \\ 11i - 7, & \text{if } 1 \leq i \leq k, \text{ and } j = 6. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is  $\{1, 2, 3, \dots, 11k + 1\}$ .

Therefore, linear hexagonal snakes, all  $kC_6$  with  $k$  blocks of  $C_6$  are super vertex mean graphs.  $\square$

**Example 4.6** A graph given in Figure 4 is an SVM labeling of a linear hexagonal snake with 3 blocks.



**Figure 4** An SVM labeling of a linear hexagonal snake,  $3C_6$ .

**Theorem 4.7** All linear heptagonal snakes,  $kC_7, k \geq 2$  are super vertex mean graphs.

*Proof* Let  $kC_7$  be a linear heptagonal snake with  $k, k \geq 2$  blocks of  $C_7$ .  $p = 6k + 1$  and  $q = 7k$  and  $p + q = 13k + 1$ . Define  $f : E(G_n) \rightarrow \{1, 2, 3, \dots, 13k + 1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} 4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 30 - 4j, & \text{if } i = 1, \text{ and } 4 \leq j \leq 6, \\ 1, & \text{if } i = 1, \text{ and } j = 7, \\ 13i + 2j - 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 4, \\ 13i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\ 13i + 2j - 21, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7. \end{cases}$$



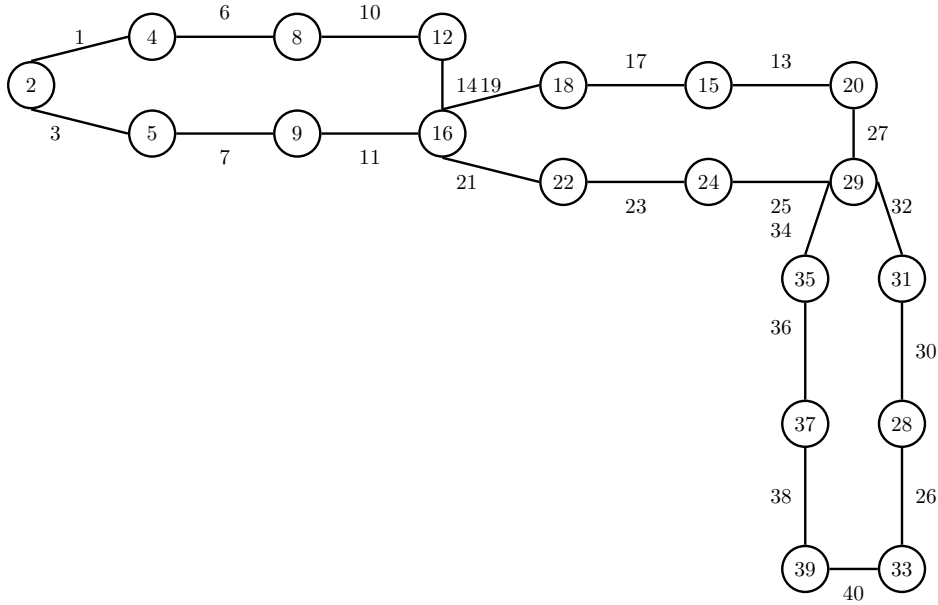
It can be easily verified that  $f$  is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq 3, \\ 32 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 7, \\ 13i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 13i + 2j - 8, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\ 13i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\ 13i + 3j - 29, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\ 13k, & \text{if } i = k, \text{ and } j = 4. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is  $\{1, 2, 3, \dots, 13k + 1\}$ .

Therefore, all linear heptagonal snakes,  $kC_7$  with  $k$  blocks of  $C_7$  are super vertex mean graphs.  $\square$

**Example 4.8** A graph given in Figure 5 is an SVM labeling of a linear heptagonal snake,  $3C_7$ .



**Figure 4** A Super Vertex Mean Labeling of  $3C_7$  linear snake.

**Theorem 4.9** Let  $kC_n$  be a linear cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 8$  and  $n \equiv 0(mod 2)$ . Then  $kC_n$  is a super vertex mean graph.

*Proof* Let  $kC_n$  be a linear cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 8$  and  $n \equiv 0(mod 2)$ . Let  $n = 2r, r \geq 4$ . Now,  $p = (n - 1)k + 1$  and  $q = nk$  and  $p + q = (2n - 1)k + 1$ . Define

$f : E(kC_n) \rightarrow \{1, 2, 3, \dots, (2n-1)k+1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 4n - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq n - 2, \\ 7, & \text{if } i = 1, \text{ and } j = n - 1, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n - 1)i - 2n + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n - 1)i - 2n + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (2n - 1)i - 2n + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (2n - 1)i - 2n + 4j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n - 1)i + 2n - 4j + 5, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq n - 1, \\ (2n - 1)(i - 1), & \text{if } i = k, \text{ and } j = n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 4n - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n - 1, \\ 4, & \text{if } i = 1, \text{ and } j = n, \\ (2n - 1)i - 2n + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n - 1)i - 2n + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (2n - 1)i - 2n + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (2n - 1)i - 2n + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n - 1)i + 2n - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq n - 1, \\ (2n - 1)i - 2n + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = n, \\ (2n - 1)k, & \text{if } i = k, \text{ and } j = r + 1. \end{cases}$$

We prove the theorem by mathematical induction on  $r$ , where  $n = 2r, r \geq 4$ . The above edge labeling function  $f(e)$  and the induced vertex labeling function  $f^v(v)$  are expressed in terms of  $r$  as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 8r - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 2, \\ 7, & \text{if } i = 1, \text{ and } j = 2r - 1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r \end{cases}$$

and

$$f(e_{i,j}) = \begin{cases} (4r-1)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-1)i - 4r + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-1)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-1)i - 4r + 4j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r-1)i + 4r - 4j + 5, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq 2r-1, \\ (4r-1)(i-1), & \text{if } i = k, \text{ and } j = 2r. \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r+2 \leq j \leq 2r-1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r-1)i - 4r + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-1)i - 4r + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-1)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-1)i - 4r + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r-1)i + 4r - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq 2r-1, \\ (4r-1)i - 4r + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (4r-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We prove that the theorem is true when  $r = 4$ ,  $n = 8$ .

When  $r = 4$  the linear cyclic snake is a linear octagonal snake with  $k, k \geq 2$  cycles of  $C_8$ .

Now,  $p = 7k + 1$  and  $q = 8k$  and  $p + q = 15k + 1$ . Define  $f : E(kC_n) \rightarrow \{1, 2, 3, \dots, 15k + 1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 13, & \text{if } i = 1, \text{ and } j = 4, \\ 36 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 6, \\ 7, & \text{if } i = 1, \text{ and } j = 7, \\ 1, & \text{if } i = 1, \text{ and } j = 8, \\ 15i - 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 15i - 12, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ 15i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ 15i - 2, & \text{if } 2 \leq i \leq k, \text{ and } j = 4, \\ 15i - 4j + 21, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 7, \\ 15i - 15, & \text{if } i = k, \text{ and } j = 8. \end{cases}$$

It can be easily verified that  $f$  is injective. The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 38 - 4j, & \text{if } i = 1, \text{ and } 6 \leq j \leq 7, \\ 4, & \text{if } i = 1, \text{ and } j = 8, \\ 15i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 15i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ 15i - 9, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ 15i - 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 4, \\ 15i - 4j + 23, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\ 15i - 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 8, \\ 15k, & \text{if } i = k, \text{ and } j = 5. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is  $\{1, 2, 3, \dots, 15k + 1\}$  as follows:

$$\left\{ \begin{array}{ll} f(E) & = \{3, 6, 9, 13, 16, 12, 7, 1\} \cup \\ & \{22, 18, 24, 28, 31, 27, 23, 15\} \cup \dots, \\ & \{15k - 8, 15k - 12, 15k - 6, 15k - 2, 15k + 1, \\ & 15k - 3, 15k - 7, 15k - 15\}. \\ f^v(V) & = \{2, 5, 8, 11, 14, 10, 4\} \cup \\ & \{17, 20, 21, 26, 29, 25, 19\} \cup \dots, \\ & \{15k - 13, 15k - 10, 15k - 9, 15k - 4, 15k - 1, \\ & 15k - 5, 15k - 11, 15k\}. \\ f(E) \cup f^v(V) & = \{1, 3, 6, 7, 9, 12, 13, 16\} \cup \\ & \{2, 4, 5, 8, 10, 11, 14\} \cup \\ & \{15, 18, 22, 23, 24, 27, 28, 31\} \cup \\ & \{17, 19, 20, 21, 25, 26, 29\} \cup \dots, \\ & \{15k - 15, 15k - 12, 15k - 8, 15k - 7, 15k - 6, \\ & 15k - 3, 15k - 2, 15k + 1\} \cup \\ & \{15k - 13, 15k - 11, 15k - 10, 15k - 9, 15k - 5, \\ & 15k - 4, 15k - 1, 15k\}. \\ & = \{1, 2, 3, 4, \dots, 29, 30, 31, \dots, 15k - 2, 15k - 1, 15k, 15k + 1\}. \end{array} \right.$$

Thus the theorem is true when  $r = 4$ .

Now we assume that the theorem is true for  $r - 1, r \geq 5$  (i.e., for  $n - 2, n \geq 10$ ). In this case,  $p = (n - 3)k + 1 = (2r - 3)k + 1$  and  $q = (n - 2)k = (2r - 2)k$  and  $p + q = (2n - 5)k + 1 = (4r - 5)k + 1$ .

The induction hypothesis is that the edge labeling,

$$f : E(kC_{2r-2}) \rightarrow \{1, 2, 3, \dots, (4r-5)k+1\}$$

defined as follows, is a super vertex mean labeling, where  $r \geq 5, n \geq 10, n \equiv 0(mod 2)$  and  $k \geq 2$ .

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r - 1, \\ 8r - 4j - 4, & \text{if } i = 1, \text{ and } r \leq j \leq 2r - 4, \\ 7, & \text{if } i = 1, \text{ and } j = 2r - 3, \\ 1, & \text{if } i = 1, \text{ and } j = 2r - 2, \\ (4r - 5)i - 4r + 12, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 5)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 5)i - 4r + 14, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 5)i - 4r + 4j + 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r - 1, \\ (4r - 5)i + 4r - 4j + 1, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq 2r - 3, \\ (4r - 5)(i - 1), & \text{if } i = k, \text{ and } j = 2r - 2. \end{cases}$$

and the induced vertex labeling is,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r - 1, \\ 8r - 4j - 2, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 3, \\ 4, & \text{if } i = 1, \text{ and } j = 2r - 2, \\ (4r - 5)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 5)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 5)i - 4r + 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 5)i - 4r + 4j, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r - 1, \\ (4r - 5)i + 4r - 4j + 3, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq 2r - 3, \\ (4r - 5)i - 4r + 9, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r - 2, \\ (4r - 5)k, & \text{if } i = k, \text{ and } j = r. \end{cases}$$

Now we prove that the result is true for any  $r$ . If we replace  $r$  with  $r + 1$  in the above mapping we get,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 8r - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq 2r - 2 \end{cases}$$

and

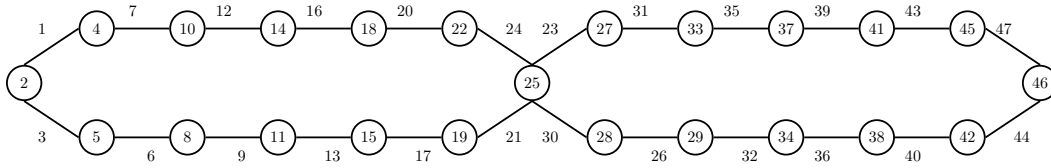
$$f(e_{i,j}) = \begin{cases} 7, & \text{if } i = 1, \text{ and } j = 2r - 1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r - 1)i - 4r + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 1)i - 4r + 4j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r - 1)i + 4r - 4j + 5, & \text{if } 2 \leq i \leq k, \text{ and } r + 1 \leq j \leq 2r - 1, \\ (4r - 1)(i - 1), & \text{if } i = k, \text{ and } j = 2r. \end{cases}$$

and,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq 2r - 1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r - 1)i - 4r + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r - 1)i - 4r + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r - 1)i + 4r - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq 2r - 1, \\ (4r - 1)i - 4r + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (4r - 1)k, & \text{if } i = k, \text{ and } j = r + 1. \end{cases}$$

This is equivalent to the original labeling in terms of  $n$ , which is given in the beginning of the proof, and it is clear that  $f(E) \cup f^v(V) = \{1, 2, 3, \dots, (4r - 1)k - 1, (4r - 1)k, (4r - 1)k + 1\}$ . Thus the theorem is proved by mathematical induction.  $\square$

**Example 4.10** A graph given in Figure 6 is an SVM labeling of a linear cyclic snake  $2C_{12}$ .



**Figure 5** A super vertex mean labeling of a linear cyclic snake  $2C_{12}$ .

**Theorem 4.11** Let  $kC_n$  be a linear cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 9$  and  $n \equiv 1(\text{mod } 4)$ . Then  $kC_n$  is a super vertex mean graph.

*Proof* Let  $kC_n$  be a cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 9$  and  $n \equiv 1(\text{mod } 4)$ . Let

$n = 2r + 1, r \geq 4$ , and  $r = 2s, s \geq 2$  so that  $n = 4s + 1$ . Now  $p = (n - 1)k + 1$  and  $q = nk$  and  $p + q = (2n - 1)k + 1$ . Define  $f : E(kC_n) \rightarrow \{1, 2, 3, \dots, (2n - 1)k + 1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} n, & \text{if } i = 1, \text{ and } j = 1, \\ 2j + n - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq r + 1, \\ 2j - n - 2, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n, \\ (2n - 1)i - n - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n - 1)i - n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq r - 3, \\ (2n - 1)i - n + 2j - 1, & \text{if } 2 \leq i \leq k, \text{ and } r - 2 \leq j \leq r - 1, \\ (2n - 1)i - n + 2j, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r + 1, \\ (2n - 1)i - 8, & \text{if } 2 \leq i \leq k, n \neq 9 \text{ and } j = r + 2, \\ (2n - 1)i - 7, & \text{if } 2 \leq i \leq k, n = 9 \text{ and } j = r + 2, \\ (2n - 1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r + 3, \\ (2n - 1)i - 3n + 2j - 3, & \text{if } 2 \leq i \leq k, \text{ and } r + 4 \leq j \leq n + 1 - s, \\ (2n - 1)i - 3n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } n + 2 - s \leq j \leq n - 1, \\ (2n - 1)i - n, & \text{if } 2 \leq i \leq k, \text{ and } j = n. \end{cases}$$

and, the induced vertex labels are as follows:

When  $i = 1$ , and  $n \geq 9$ ,

$$f^v(v_{i,j}) = \begin{cases} n - 1, & \text{if } j = 1, \\ 2j + n - 2, & \text{if } 2 \leq j \leq r, \\ n + 1, & \text{if } j = r + 2, \\ 2j - 16, & \text{if } r + 3 \leq j \leq n. \end{cases}$$

and when  $2 \leq i \leq k$ , and  $n = 9$ ,

$$f^v(v_{i,j}) = \begin{cases} 17i - 13, & \text{if } j = 1, \\ 17i + 3j - 14, & \text{if } 2 \leq j \leq 4, \\ 17i, & \text{if } j = 5 \text{ and } i = k, \\ 17i - 3, & \text{if } j = 6, \\ 22, & \text{if } j = 7, \\ 19, & \text{if } j = 8, \\ 23, & \text{if } j = 9. \end{cases}$$

and when  $2 \leq i \leq k$ , and  $n \geq 13$ ,

$$f^v(v_{i,j}) = \begin{cases} (2n - 1)i - 3r - 1, & \text{if } j = 1, \\ (2n - 1)i - n + 2j - 3, & \text{if } 2 \leq j \leq r - 3 \end{cases}$$

and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i - n + 2j - 2, & \text{if } r-2 \leq j \leq r-1, \\ (2n-1)i - 2, & \text{if } j = r, \\ (2n-1)i, & \text{if } i = k, \text{ and } j = r+1, \\ (2n-1)i - 3, & \text{if } j = r+2, \\ (2n-1)i - n - 3, & \text{if } j = r+3, \\ (2n-1)i - 3n + 2j - 4, & \text{if } r+4 \leq j \leq n+1-s, \\ (2n-1)i - 3n + 2j - 3, & \text{if } n+2-s \leq j \leq n-1, \\ (2n-1)i - n - 2, & \text{if } j = n. \end{cases}$$

We can easily prove the theorem by the technique of mathematical induction on  $s$ . The remaining of the proof is left as an exercise.  $\square$

**Theorem 4.13** *Let  $kC_n$  be a linear cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 11$  and  $n \equiv 3(\text{mod } 4)$ . Then  $kC_n$  is a super vertex mean graph.*

*Proof* Let  $kC_n$  be a linear cyclic snake with  $k, k \geq 2$  blocks of  $C_n, n \geq 11$  and  $n \equiv 3(\text{mod } 4)$ . Let  $n = 2r + 1, r \geq 5$ , and  $r = 2s + 1, s \geq 2$  so that  $n = 4s + 3$ . Now,  $p = (n-1)k + 1$  and  $q = nk$  and  $p + q = (2n-1)k + 1$ . Define  $f : E(kC_n) \rightarrow \{1, 2, 3, \dots, (2n-1)k + 1\}$  as follows:

$$f(e_{i,j}) = \begin{cases} 4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq r, \\ 4n - 4j + 2, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i - 2n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 3, \\ (2n-1)i - 2n + 4j + 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (2n-1)i - 3r + 3j - 2, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r+1, \\ (2n-1)i + 2n - 4j + 2, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq 2r-2, \\ (2n-1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-1, \\ (2n-1)i - 4n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 2r \leq j \leq n. \end{cases}$$

and, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq r, \\ 4n - 4j + 4, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n, \\ (2n-1)i - 2n + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i - 2n + 2j + 6, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\ (2n-1)i - 2n + 4j - 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r+2 \end{cases}$$



and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i + 2n - 4j + 4, & \text{if } 2 \leq i \leq k, \text{ and } r+3 \leq j \leq 2r-2, \\ (2n-1)i - 2n + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-1, \\ (2n-1)i - 2n + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (2n-1)i - 2n + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r+1, \\ (2n-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We can easily prove that the above labeling is an SVM labeling of  $kC_n$ , where  $k \geq 2$  blocks of  $C_n$ ,  $n \geq 11$  and  $n \equiv 3 \pmod{4}$ , by using the technique of mathematical induction on  $s$ , where  $n = 4s + 3$ . Thus the theorem can easily be proved.  $\square$

## §5. Super Vertex Mean Number

The concept of super vertex mean number arises from the earlier concepts such as, mean number, super mean number etc. M.Somasundaram and R.Ponraj have introduced the term mean number of a graph [16] and they have found the mean number of many standard graphs. Later on, A.Nagarajan et.al. introduced the concept of super mean number of a graph [9] and proved the existence of it for any graph by finding out the limit values of it. Encouraged by their works we introduce this new concept which we like to name as super vertex mean number or SVM number.

**Definition 5.1** Let  $f$  be a an injective function of a  $(p, q)$  - graph  $G(V, E)$  defined from  $E$  to the set  $\{1, 2, 3, \dots, n\}$  that induces for each vertex  $v$  the label defined by the rule  $f^v(v) = \text{Round}\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$ , where  $E_v$  denotes the set of edges in  $G$  that are incident at the vertex  $v$ . Let  $f(E) \cup f^v(V) \subseteq \{1, 2, 3, \dots, n\}$ . If  $n$  is the smallest positive integer satisfying these conditions together with the condition that all the vertex labels as well as the edge labels are distinct, then  $n$  is called the super vertex mean number (or SVM number) of the graph  $G(V, E)$ , and is denoted by  $SV_m(G)$ .

**Observation 5.2** It is observed that  $SV_m(G) = p + q$ , for all SVM graphs  $G$  whose order is  $p$  and size is  $q$ . And for other graphs  $G$ ,  $SV_m(G) \geq p + q + 1$ . For graphs containing an isolated vertex or a leaf, the super vertex mean number is not defined.

Therefore, for any  $(p, q)$  - graph  $G$ ,  $p + q \leq SV_m(G) \leq \infty$ . For example, the SVM number of  $C_4$ ,  $SV_m(C_4) = 9$ .

## §6. Conclusion

In this paper, we have proved that all the linear cyclic snakes are super vertex mean graphs. In the case of super mean labeling, the vertex analogue of SVM, it was easy to obtain a general formula for linear cyclic snakes as well as other cyclic snakes represented the string  $s_1, s_2, s_3, \dots, s_{k-2}$ , where each  $s_i$  need not be equal. This is because when we calculate the induced edge label for an edge, by finding the average of the labels of the two vertices which are

the end points of that particular edge, we need to consider only those two vertices. Therefore the average remains the same as in the case of cycles.

But for super vertex mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula for cyclic snakes represented the string  $s_1, s_2, s_3, \dots, s_{k-2}$ , where each  $s_i$  need not be equal. Another possibility emerges is that we try to explore the SVM labeling of  $KC$  – snakes, which is defined as a connecting graph in which each of the  $k$  many blocks is isomorphic to a cycle  $C_n$  for some  $n$  and the block - cut point graph is a path. As in the case of  $kC_n$  – snakes, a  $kC$  –snake too can be represented by a string of integers,  $s_1, s_2, \dots, s - k - 2$ . It remains still an open problem to label a  $kC$  –snake which has either equal  $s_i$  or different  $s_i$ .

## References

- [1] B.D.Acharya and K.A.Germina, Vertex-graceful Graphs, *Journal of Discrete Mathematical Science and Chryptography*, Vol.13(2010), No.5, pp.453-463.
- [2] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 16, 2013.
- [3] S.W.Golomb, How to number a graph, —it Graph Theory and Computing, pp. 23-27, 1972.
- [4] R.L.Graham and N.J.A.Solane, On additive bases and harmonious graphs, *SIAM, J.Alg. Discrete Methods*, 1:382-404, 1980.
- [5] A.Lourdusamy and M.Seenivasan, Vertex-mean graphs, *International Journal of Mathematical Combinatorics*, 3:114-120, 2011.
- [6] A.Lourdusamy and M.Seenivasan, Mean labelings of cyclic snakes, *AKCE International Journal of Graphs and Combinatorics*, 8(2): 105-113, 2011.
- [7] A.Lourdusamy, M.Seenivasan, Sherry George and R.Revathy, Super vertex-mean graphs, *Sciencia Acta Xaveriana*, Vol.5(No.2), 39-46, 2014.
- [8] A.Lourdusamy, Sherry George and M.Seenivasan, *An Extension of Mean Labeling*, Communicated to publication.
- [9] A.Nagrajan and R.Vasuki, Super mean number of a graph, *Kragujevac Journal of Mathematics*, Vol.36 (No.1, 2012) 93-107.
- [10] R.Ponraj, *Studies in Labelings of Graphs*, Ph.D.thesis, Manonmaniam Sundaranar University, India, 2004.
- [11] R.Ponraj and D.Ramya, On super mean graphs of order 5, *Bulletin of Pure and Applied Sciences*, 25(1), 143-148,2006.
- [12] D.Ramya, R.Ponraj, and P.Jeyanthi, Super mean labeling of graphs, *Ars Combin.*, 112(2013), 65-72.
- [13] A. Rosa, On certain valuations of the vertices of a graph, In *Theory of Graphs*, pp. 349-355, 1967.
- [14] M.Seenivasan, *Studies in Graph Theory; Some New Labeling Concepts*, Ph.D.thesis, Manonmaniam Sundaranar University, India, 2013.

- [15] S.Somasundaram and R.Ponraj, Super mean labeling of graphs, *National Academy, Science Letters*, 26(2003) 210-213.
- [16] S.Somasundaram and R.Ponraj, Mean number of a graph, *Pure and Applied Mathematika Sciences*, LXV, (12)(2007), 93-102.
- [17] R.Vasuki and A.Nagrajan, Some results on super mean labeling of graphs, *International Journal of Mathematical Combinatorics*, Vol.3 (2009) 82-96.

## Equal Degree Graphs of Simple Graphs

T.Chalapathi

(Department of Mathematics, Sree Vidyanikethan Eng.College, Tirupati-517502, Andhra Pradesh, India)

R.V M S S Kiran Kumar

(Department of Mathematics, S.V.University, Tirupati-517502, Andhra Pradesh, India)

chalapathi.tekuri@gmail.com, kksaisiva@gmail.com

**Abstract:** This paper introduces equal degree graphs of simple existed graphs. These graphs exhibited some properties which are co-related with the older one. We characterize graphs for which their equal degree graphs are connected, completed, disconnected but not totally disconnected. We also obtain several properties of equal degree graphs and specify which graphs are isomorphic to equal degree graphs and complement of equal degree graphs. Furthermore, the relation between equal degree graphs and degree Prime graphs is determined.

**Key Words:** Parameters of the graph, simple graphs, equal degree graphs, degree prime graphs, degree graph isomorphism, Smarandachely  $k$ -degree graph, Smarandachely degree graph.

**AMS(2010):** 05C25.

### §1. Introduction

The evaluations of new graphs are involving sets of objects and binary relations among them. So the construction and preparation of graphs varies author to author, and thus it is difficult to pin point its formulation to a single source. Thus the graphs discovered many times, and each discovery being independent of the other. For this reason, there are various types of graphs each with its own definition.

Many authors, starting from 2003, the parameters vertex degree and degree sequence of graphs were used again in Graph theory, and several types of graphs have been introduced. In this sequel we have introduced equal degree graphs of various simple graphs and characterized their properties.

For any finite group  $G$ , the definition and notation of degree graph of a simple group was introduced by Lewis and white [5]. This graph is defined as follows: Let  $G$  be a finite group and let  $cd(G)$  be the set of irreducible character degrees of  $G$ . Then the degree graph  $\Delta(G)$  is the graph whose set of vertices is set of primes that divide degrees in  $cd(G)$ , with an edge between  $p$  and  $q$  if  $pq$  divides  $a$  for some  $a \in cd(G)$ .

---

<sup>1</sup>Received September 25, 2017, Accepted March 6, 2018.

In [6], the authors introduced the degree pattern of a finite group  $G$  and denoted by  $D(G)$ , where  $D(G) = (deg(p_1), deg(p_2), \dots, deg(p_k))$ , here  $p_1 < p_2 < \dots < p_k$  are distinct primes in prime decomposition of  $n$ . Further, the authors S.F.Kapur, Albert and Curtiss [7] introduced the notation  $D(G)$  for degree sets of connected graphs, trees, planner and outer planner graphs. According to these authors, the notation  $D(G)$  is a degree set of degrees of vertices of  $G$ .

The authors Manoussakis, Patil and Sankar [8] was proved that for any finite non empty set  $S$  of non negative integers, there exist a disconnected graph  $G$  such that  $D(G) = S$ , and also the minimum order of such a graph is determined.

There are several ways to produce new graphs from the existing graphs in Graph theory. Recently the authors M.Sattanathan and R.Kala [1] introduced a special way to produce the degree prime graph  $DP(G)$  for any finite simple undirected graph  $G$ . The  $PD(G)$  of a graph  $G$  having the same vertex set of  $G$  and two vertices are adjacent in  $PD(G)$  if and only if their unequal degrees are relatively prime in  $G$ . By the motivation of these degree prime graphs, we construct and study the equal degree graphs of simple graphs with usual notation  $D(G)$ . We suspect that these graphs will be used to solve many computational problems in computer engineering and applied sciences.

Throughout this paper,  $G$  and  $D(G)$  represent finite simple undirected graphs having without loops and without multiple edges of same order. We have introduced degree graph  $D(G)$  of  $G$ , which is defined as a graph with same vertex set as  $G$  and two vertices of  $G$  are adjacent in  $D(G)$  if and only if their degrees are equal in  $G$ . In this paper we studied interrelations between  $G$  and  $D(G)$ , and hence we obtain several properties and their consequences of  $D(G)$  with illustrations and examples. Further we characterize  $G$  for which  $D(G)$  either is connected, disconnected, totally disconnected or complete.

## §2. Basic Definitions and Notations

In this section we consider basic definitions and their graph theoretical notations. Throughout the text, we consider  $G$  is an abstract graph structure which is a finite undirected graph without loops and multiple edges. We represent  $G$  as  $G = G(V, E)$  with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . We are only going to deal with finite graphs, so we define  $|V| = n$  to be the order of  $G$  and  $|E| = m$  to be the size of  $G$  where  $n$  and  $m$  are called graph parameters. Further if there is an edge  $e$  in  $G$  between the vertices  $u$  and  $v$ , we briefly write  $e = uv$  or  $e = (u, v)$  and say edge  $e$  joins the vertices  $u$  and  $v$ . A vertex is said to be isolated if it is not adjacent to any other vertex.

The complement of a graph  $G(V, E)$  is the graph  $G^c(V, E^c)$  having the same vertex set as  $G$ , and its edge set  $E^c$  is the complement of  $E$ , that is,  $uv$  is an edge of  $G^c$  if and only if  $uv$  is not an edge of  $G$ . A complete graph of order  $n$  is denoted by  $K_n$ . A graph of order  $n$  with no edges in an empty graph and is denoted by  $N_n = K_n^c$  which isomorphic to totally disconnected. A path of length  $n$  is denoted by  $P_n$ . A cycle of length  $n$  ( $n \geq 3$ ) is a cycle of length  $n$  and is denoted by  $C_n$ .

We now turn to graphs whose vertex sets can be partitioned in special ways. A graph  $G$  is a partite graph if  $V(G)$  can be partitioned into subsets, called partite sets. A graph  $G$  is

a  $k$ -partite graph if  $V(G)$  can be partitioned into  $k$  subsets  $V_1, V_2, \dots, V_k$  (partite sets) such that  $uv$  is an edge of  $G$  if  $u$  and  $v$  belongs to different partite sets. In addition, if every two vertices in different partite sets are joined by an edge, then  $G$  is a complete  $k$ -partite graph. If  $|V_i| = n_i$  for  $1 \leq i \leq k$ , then we denote this complete  $k$ -partite graph by  $K_{n_1, n_2, \dots, n_k}$ . Thus

- (1)  $K_{1,1,\dots,1} \cong K_n$ ;
- (2)  $K_{n,n}$  is a complete bipartite;
- (3)  $K_{1,n}$  is a Star;
- (4)  $K_{n_1, n_2}$  is a complete bipartite.

In a graph  $G$ , the degree of a vertex  $u$  is the number of edges of  $G$  which are incident to  $u$  and denoted by  $d(u)$ ,  $\deg(u)$  or  $\deg_G(u)$ . A graph is regular if all its vertices are of the same degree and  $r$ -regular if all of its vertices are of degree  $r$ . A 3-regular graph is a cubic graph  $Q_8$ . If  $d_i, 1 \leq i \leq n$ , be the degree of the vertices  $v_i$  of a graph  $G$  then the sequence  $d(G) = (d_1, d_2, \dots, d_n)$  is the degree sequence of  $G$ . Usually, we order the vertices so that the degree sequence is monotonically increasing, that is,  $\delta(G) = d_1 \leq d_2 \leq \dots \leq d_n = \Delta(G)$ . Also two graphs with the same degree sequence are said to be degree equivalent.

We need the following results for preparing equal degree graphs.

**Theorem 2.1**([2]) *The sum of the degrees of graphs is even, being twice the number of edges.*

**Theorem 2.2**([2]) *In any graph there is an even number of vertices of odd degree.*

**Theorem 2.3**([2]) *If  $d_1, d_2, \dots, d_n$  is the degree sequence of some graph, then, necessarily  $\sum_{i=1}^n d_i$  is even, and  $0 \leq d_i \leq n-1$  for  $1 \leq i \leq n$ . But converse is not true.*

**Theorem 2.5**([2]) *Let  $G$  be a graph of order  $n \geq 2$ , then  $d(G)$  contains at least two numbers are same.*

**Theorem 2.6**([2]) *Let  $G$  be a  $r$ -regular graph of order  $n$ . Then  $|E(G)| = \frac{rn}{2}$ .*

**Theorem 2.7**([3]) *Let  $G$  be a  $r$ -regular graph of order  $n$ . Then  $G^c$  is  $(n-r-1)$ -regular graph of order  $n$ .*

**Theorem 2.8**([3]) *A graph is 1-regular if and only if it is of even order and is the disjoint union of some  $K_2$ 's.*

If  $n = 1$ , then  $G$  is called trivial graph, otherwise  $G$  is called non-trivial graph. In this paper we consider non-trivial graphs only. For further details and notations we refer [4].

### §3. Equal Degree Graphs

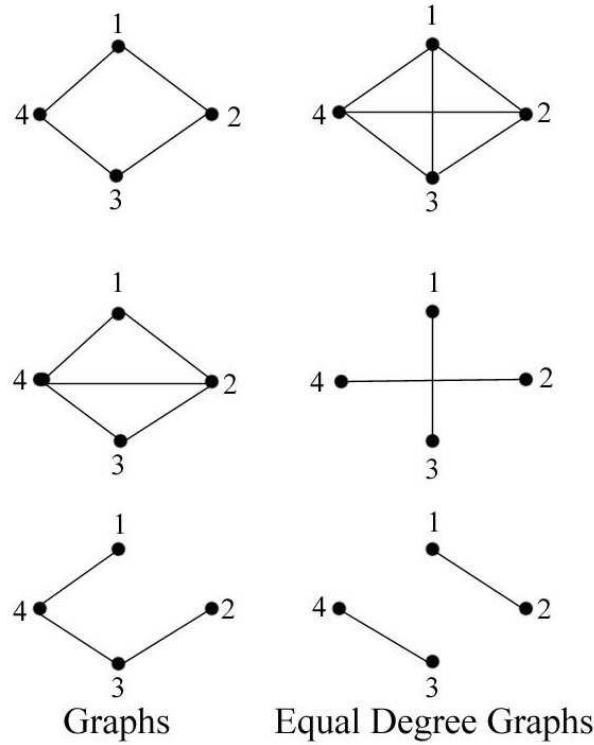
We have already mentioned that the best known parameters of a graph are order and size. But another parameter of a graph is degree of a vertex which is a meaningful to have a term for the number of edges meeting at a vertex. By using this parameter we define equal degree graph as follows.

**Definition 3.1** Let  $G$  be a simple graph with vertex set  $V = \{1, 2, \dots, n\}$ ,  $n > 1$ . Then the equal degree graph of  $G$  is  $D(G)$  having the same vertex set as  $G$  and two vertices  $u, v \in V$  are adjacent in  $D(G)$  if and only if  $\deg_G(u) = \deg_G(v)$ .

Generally, we know *Smarandachely  $k$ -degree graphs*  $D_k G$  of a graph  $G$  in which vertices  $u, v \in V$  are adjacent if and only if  $|\deg_G(u) - \deg_G(v)| = k$  for integers  $k \leq \Delta(G) - \delta(G)$  and *Smarandachely degree graph*  $SG$  in which  $u, v$  are adjacent if  $|\deg_G(u) - \deg_G(v)| \geq 1$ . Clearly,  $D_0 G = DG$ . The definition of equal degree graphs should be noted that the following.

1.  $D(G)$  is a non-trivial graph;
2.  $D(G)$  has at least one edge;
3.  $D(G)$  is also a simple undirected graph having without multiple edges.

The following Fig.1 shows that simple graphs and their equal degree graphs of order 4.



**Fig.1** Graphs and their equal degree graphs

We now characterize graphs  $G$  for which  $D(G)$  is either regular or not. The following propositions are immediate.

**Proposition 3.2** For any graph  $G$ , the graph  $D(G)$  is never a 0-regular graph.

*Proof* Suppose  $D(G)$  is a 0-regular graph. Then its degree sequence is  $d(D(G)) = (0, 0, \dots, 0)$  for any graph  $G$ . This shows that  $D(G)$  is isomorphic to empty graph, and thus  $D(G)$  is a trivial graph, which is a contradiction to the definition of equal degree graphs. Thus the result follows.  $\square$

**Remark 3.3** (i)  $D(G)$  is 1-regular if and only if  $D(G)$  is a graph of order 2. (ii)  $D(G)$  is 1-regular if and only if  $G$  is a graph of order 4 and  $d(G) = (0, 0, 1, 1)$  or  $(2, 2, 3, 3)$ .

**Proposition 3.4** For any graph  $G$  of order  $n > 4$ ,  $D(G)$  is never 1-regular. In particular,  $D(G)$  is never  $2, 3, \dots, (n-2)$ -regular.

*Proof* Suppose  $G$  is a graph with size  $n > 4$ . We show that  $D(G)$  is not 1-regular. Assume that  $D(G)$  is 1-regular. Then the degree sequence of  $G$  is  $d(D(G)) = (1, 1, \dots, 1)(n \text{ times})$ .

$$(1 + 1 + \dots + 1)(n \text{ times}) = \begin{cases} \text{even,} & \text{if } n \text{ is even} \\ \text{odd,} & \text{if } n \text{ is odd} \end{cases}$$

**Case 1.** If  $n$  is odd, then  $1 + 1 + \dots + 1(n \text{ times})$  is odd, which contradicts to the Theorem 2.2. So in this case the result is not true.

**Case 2.** If  $n$  is even, then  $1 + 1 + \dots + 1(n \text{ times})$  is odd, which is impossible because the graph  $D(G)$  does not contain an even number of vertices of same odd degree.

From the above cases our assumption is not true, and hence  $D(G)$  is never 1-regular for any graph  $G$  of order  $n > 4$ .  $\square$

**Remark 3.5** Similarly we can show that  $D(G)$  is never 2-regular, 3-regular,  $\dots$ ,  $(n-2)$ -regular but it should be  $(n-1)$ -regular.

**Theorem 3.6**(Fundamental Theorem) For any graph  $G$  of order  $n$ , the degree graph  $D(G)$  is either complete or disconnected but not totally disconnected.

*Proof* Suppose  $D(G)$  is totally disconnected. Then obviously  $D(G)$  is isomorphic to  $N_n$ . But by the definition of degree graph,  $D(G)$  is never isomorphic to  $N_n$ . Hence  $D(G)$  is not totally disconnected.

Now we prove that  $D(G)$  is either complete or disconnected. Suppose  $D(G)$  is disconnected. Then there is nothing to prove. If possible assume that  $D(G)$  is connected, then the Proposition 3.4 and Remark 3.5 shows that  $D(G)$  is  $(n-1)$ -regular, and hence  $D(G)$  is complete.  $\square$

#### §4. Complete Equal Degree Graphs

In this section we are going to prove that the equal degree graphs are complete. Further we characterize graphs  $G$  for which  $D(G)$  is complete and also we show that  $G \cong D(G)$ .

**Proposition 4.1** The degree graph of regular graph is complete.

*Proof* Let  $V$  be a vertex set of  $r$ -regular graph  $G$ . Then the degree sequence of  $G$  is  $d(G) = (r, r, \dots, r)$ , that is,  $d(i) = r$  for each  $1 \leq i \leq n$ . We show that  $D(G)$  is complete. For this let  $i, j \in V$ ,  $i \neq j$ , then  $\deg(i) = r$  and  $\deg(j) = r$ . Therefore,  $\deg(i) = \deg(j)$ , for all  $i \neq j$  in  $V(G)$ . Thus  $i$  and  $j$  are adjacent in  $D(G)$ . This shows that every two distinct pair of



vertices is joined by an edge in  $D(G)$ . Hence  $D(G)$  is complete.  $\square$

This proposition has a number of useful consequences.

**Corollary 4.2** *Let  $G$  be a connected graph of order  $n > 4$ . Then  $D(G)$  is either complete or disconnected.*

*Proof* The proof is divided into cases following.

**Case 1.** Suppose  $G$  is a connected regular graph of order  $n > 4$ . Then, by the Proposition 4.1,  $D(G)$  is complete.

**Case 2.** Suppose  $G$  is a connected but not regular. Then the degree sequence of  $G$  contains at least two distinct positive integers, say  $s$  and  $t$ . That is, if  $u, v \in G$ , then  $\deg(u) \neq \deg(v)$  implies  $u$  is not adjacent to  $v$  in  $D(G)$ . Hence  $D(G)$  is disconnected.  $\square$

**Corollary 4.3** *For each  $n > 1$ , we have*

- (i)  $D(K_n) = K_n$ ;
- (ii)  $D(C_n) = K_n$ ;
- (iii)  $D(N_n) = K_n$ ;
- (iv)  $D(K_{n,n}) = K_{2n}$ ;
- (v)  $D(Q_8) = K_8$ .

*Proof* (i) The complete graph  $K_n$  is  $(n-1)$ -regular, and thus  $D(K_n) = K_n$ .

(ii) For each  $n \geq 3$ , the cycle  $C_n$  is 2-regular. Hence  $D(C_n) = K_n$ .

(iii) For each, the empty graph  $N_n$  is 0-regular graph. Hence  $D(N_n) = K_n$ .

(iv) Since the completed bipartite graph  $K_{n,n}$  is  $n$ -regular and the order of  $K_{n,n}$  is  $2n$ .

Thus  $D(K_{n,n}) = K_{2n}$ .

(v)  $Q_8$  is 3-regular of order 8, and thus  $D(Q_8) = K_8$ .  $\square$

**Corollary 4.4** *Let  $G^c$  be the complement of  $r$ -regular graph  $G$  of order  $n$ , then  $D(G) = D(G^c) \cong K_n$ .*

*Proof* We deduces this consequence from the Theorem 2.6 and Proposition 4.1 as follows.

We know that  $G$  is  $r$ -regular graph of order  $n$  if and only if  $G^c$  is  $(n-r-1)$ -regular graph of same order  $n$ . Thus the Proposition 4.1 shows that  $D(G) \cong K_n$  if and only if  $D(G^c) \cong K_n$ . Hence  $D(G) = D(G^c) \cong K_n$ .  $\square$

**Corollary 4.5** *Let  $G$  be any graph of order  $n$ . For fixed  $m \in \mathbb{Z}$ , if there exists an integer  $n$  such that  $\deg(v) = mn$  for each vertex  $v$  of  $G$ . Then,  $D(G) = K_n$ .*

*Proof* Obviously follows from Proposition 4.1, since  $G$  is  $mn$ -regular graph of order  $n$ .

We now characterize the graphs  $G$  which attain bounds for  $|E(D(G))|$ . We know that

$$0 \leq |E(G)| \leq \frac{n(n-1)}{2}$$

for any simple graph  $G$  of order  $n \geq 1$ . But the following result specifies that the bounds for  $|E(D(G))|$ .  $\square$

**Theorem 4.6** *If  $G$  is any graph of order  $n > 1$ , then  $0 < |E(G)| \leq \frac{n(n-1)}{2}$ .*

*Proof* From the definition of equal degree graphs,  $|V(G)| = |V(D(G))| = n$ , and  $n > 1$ . For any non-trivial graph  $G$ , we have  $\deg_G u \leq (n-1)$  for each  $u \in V(G)$ . This is also true in  $D(G)$ , that is,  $\deg_{D(G)} u \leq (n-1)$  for each  $u \in V(D(G))$ . From Theorem 2.1 we have

$$2|E(D(G))| = \sum_{d \in V(D(G))} \deg(d) \Rightarrow 2|E(D(G))| \leq n(n-1) \Rightarrow |E(D(G))| \leq \frac{n(n-1)}{2}.$$

It is one extreme of the required inequality. At the other extreme, a degree graph  $D(G)$  may possess at least one edge at all. That is,  $|E(D(G))| \neq 0$ . Hence

$$0 < |E(G)| \leq \frac{n(n-1)}{2}. \quad \square$$

**Remark 4.7** The above inequality says that the following two specifications for  $D(G)$ :

- (1)  $D(G)$  or  $D(G^c)$  has at least one edge or at most  $\binom{n}{2}$  edges;
- (2)  $D(G)$  is never totally disconnected. In particular,  $D(G) \not\cong N_n$  for each  $G$  of order  $n > 1$ .

**Corollary 4.8** *Let  $G$  be a  $r$ -regular graph of order  $n > 1$ . Then*

$$|E(G)| = \frac{rn}{2} \quad \text{and} \quad |E(D(G))| = \binom{n}{2}.$$

**Proposition 4.9** *Let  $G$  be a graph of order  $n$ . Then  $|E(G)| = |E(D(G))|$  if and only if  $G$  and  $D(G)$  are  $(r-1)$ -regular graphs.*

*Proof* Let  $G$  be a  $r$ -regular graph of order  $n > 1$ . By the Corollary 4.7,

$$|E(G)| = \frac{rn}{2} \quad \text{and} \quad |E(D(G))| = \binom{n}{2}.$$

Therefore,

$$|E(G)| = |E(D(G))| \Leftrightarrow \frac{rn}{2} = \binom{n}{2} \Leftrightarrow \frac{rn}{2} = \frac{n(n-1)}{2} \Leftrightarrow r = n-1 \Leftrightarrow G$$

is  $(n-1)$ -regular, and hence  $D(G)$  is also  $(n-1)$ -regular.  $\square$

**Proposition 4.10** *If  $G^c$  is a complement of  $G$ , then  $D(G^c) = D(G)$ .*

*Proof* Let  $G^c$  be the complement of a graph  $G$  of order  $n > 1$ . Then the following cases arise on the regularity of  $G$ .

**Case 1.** Suppose  $G$  is a regular graph. Then the result obviously follows from Proposition 4.1.

**Case 2.** Suppose  $G$  is not a regular graph. We show that  $D(G^c) = D(G)$ . If possible assume that  $D(G^c) \neq D(G)$ , then the following three subcases arise.

**Subcase 2.1** If  $V(D(G^c)) \neq V(D(G))$  then obviously  $V(G^c) \neq V(G)$ , which is a contradiction to the fact that  $V(G^c) = V(G)$ .

**Subcase 2.2** If  $E(D(G^c)) \neq E(D(G))$ , then  $V(D(G^c)) + V(D(G)) = \frac{n(n-1)}{2}$ . This shows that either  $D(G)$  or  $D(G^c)$  has at most  $\frac{n(n-1)}{4}$  edges, which is a contradiction to the fact that  $D(G)$  or  $D(G^c)$  has at most  $\frac{n(n-1)}{2}$  edges.

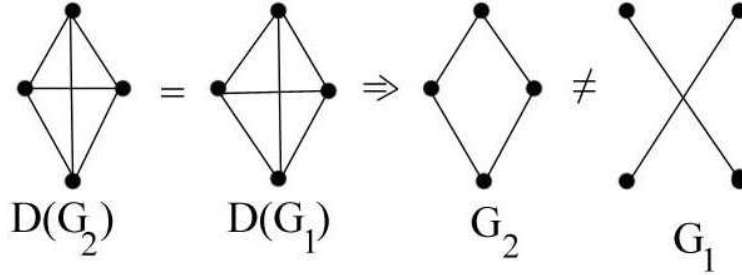
**Subcase 2.3** If  $V(D(G^c)) \neq V(D(G))$ ,  $E(D(G^c)) \neq E(D(G))$ , then trivially it is not true from cases 2.1 and 2.2.

From the above three subcases 2.1, 2.2 and 2.3, we conclude that  $D(G^c) \neq D(G)$  is not true. Hence  $D(G^c) = D(G)$ .  $\square$

**Remark 4.11** The converse of the above result is not true. For example,  $D(N_n) = D(N_n^c)$  but  $N_n \neq N_n^c$ .

**Theorem 4.12** Let  $G_1$  and  $G_2$  be same regular graphs of order  $n$ . Then  $D(G_1) = D(G_2)$ . But converse is not true.

*Proof* Suppose  $G_1$  and  $G_2$  be regular graphs of same order  $n > 1$ . By the Proposition 4.1,  $D(G_1) \cong K_n$  and  $D(G_2) \cong K_n$ , and thus  $D(G_1) = D(G_2)$ . But converse of this result is not true. This illustrates the Figure 2. Consider the graphs  $G_1$  and  $G_2$  on four vertices and their degree graphs.



**Fig.2** Graphs  $G_1$ ,  $G_2$ ,  $D(G_1)$  and  $D(G_2)$ .

**Theorem 4.13** Let  $G_1$  and  $G_2$  be graphs of same order  $n > 1$  such that  $d(G_1) = d(G_2)$ . Then  $D(G_1) = D(G_2)$ . But converse is not true.

*Proof* Suppose  $G_1$  and  $G_2$  be non-regular degree equivalent graphs of same order  $n > 1$ . Then their degree sequences are equal. That is,  $d(G_1) = d(G_2) = (d_1, d_2, \dots, d_n)$  where  $d_1 \leq d_2 \leq \dots \leq d_n$ ,  $0 \leq d_i \leq n-1$  for each  $1 \leq i \leq n$ . By the definition of degree graphs,  $G_1$  and  $G_2$  ( $G_1 \neq G_2$ ) are both realize same degree graph, that is,  $D(G_1) = D(G_2)$ .

Converse of the Theorem 4.13 is not true, in general. For the degree sequences  $d(G_1) = (2, 2, 2, 2)$  and  $d(G_2) = (3, 2, 3, 2)$ , we have  $D(G_1) = D(G_2)$  implies that  $d(G_1) \neq d(G_2)$ .  $\square$

**Theorem 4.14** *If  $G_1$  and  $G_2$  are two graphs such that  $G_1 \cong G_2$ , then  $D(G_1) \cong D(G_2)$ . But the converse is not true.*

*Proof* Suppose  $G_1 \cong G_2$ . Then there exists an isomorphism  $\varphi$  from  $G_1$  onto  $G_2$ . We show that  $D(G_1) \cong D(G_2)$ . For this let  $(u, v) \in E(D(G_1))$ , then by the definition of degree graphs,  $\deg_{G_1} u = \deg_{G_1} v \Rightarrow \deg_{G_2} \varphi(u) = \deg_{G_2} \varphi(v) \Rightarrow (\varphi(u), \varphi(v)) \in E(D(G_2)) \Rightarrow D(G_1) \cong D(G_2)$ .

But converse of this result is not true. For example,  $D(N_4) \cong D(C_4)$  but  $N_4 \not\cong C_4$ .  $\square$

## §5. Disconnected Equal Degree Graphs

In this section we characterize the graphs  $G$  for which  $D(G)$  is disconnected.

**Theorem 5.1** *Let  $G$  be a graph of order  $n + k$ . Then  $D(G) \cong K_n \cup N_k$  where  $n$  is the number of vertices of same degree and  $k$  is the number vertices of unequal degree.*

*Proof* Let  $|V(G)| = n + k$ . Then  $V$  can be partitioned into two subsets  $S_1$  and  $S_2$  such that  $S_1 = \{u_1, u_2, \dots, u_n\}$  and  $S_2 = \{v_1, v_2, \dots, v_k\}$  where  $\deg_G u_i = \deg_G u_j$  for all  $1 \leq i \neq j \leq n$  and  $\deg_G v_i \neq \deg_G v_j$  for all  $1 \leq i \neq j \leq k$ . By the definition of Degree graphs,  $\langle S_1 \rangle \cong K_n$  and  $\langle S_2 \rangle \cong N_k$ . Hence  $D(G) \cong K_n \cup N_k$ .  $\square$

This theorem gives the following consequences.

**Corollary 5.2**  $D(P_n) = \langle S_1 \rangle \cup \langle S_2 \rangle$  where  $\langle S_1 \rangle \cong K_2$  and  $\langle S_2 \rangle \cong K_{n-2}$ .

*Proof* Let  $v_1$  and  $v_{n+1}$  be the internal and external vertices of the path  $P_n : v_1 e_1 v_2 e_2 v_3 \dots v_n e_n v_{n+1}$ . Then the vertex set  $V = V(P_n)$  can be partitioned two disjoint sets  $S_1$  and  $S_2$  such that  $S_1 = \{v_1, v_{n+1}\}$  and  $S_2 = \{v_1, v_2, \dots, v_n\}$ .

**Case 1.** In this case  $\deg_G v_1 = \deg_G v_{n+1} \neq \deg_G v_j$  for  $j = 2, 3, \dots, n$ . This shows that there exists only one edge between  $v_1$  and  $v_{n+1}$  in  $S_1$ , and which are not adjacent to the vertices in  $S_2$ . Thus the degree graph of  $S_1$  is an induced sub graph  $\langle S_1 \rangle$  which is isomorphic to  $K_2$ .

**Case 2.** Suppose  $v_i, v_j \in S_2$  for every  $i \neq j$ . Then  $\deg_G v_i = \deg_G v_j = 2 \neq 1 = \deg_G v_1 = \deg_G v_{n+1}$  for each  $i \neq j$  such that  $2 \leq i, j \leq n$ . Thus the degree graph of  $S_2$  is also an induced subgraph  $\langle S_2 \rangle$  which is isomorphic to  $K_{n-2}$ .

**Case 3.** Suppose  $u \in S_1$  and  $v \in S_2$ . Then there is no edge between  $u$  and  $v$  in the equal degree graph whose vertex set is  $V = S_1 \cup S_2$  since  $\deg_G u = 1 \neq 2 = \deg_G v$ .

From the cases 1, 2 and 3 we conclude that  $D(P_n)$  is disconnected with two disjoint components  $\langle S_1 \rangle$  and  $\langle S_2 \rangle$ .  $\square$

The proofs of the following consequences are obvious.

**Corollary 5.3** *The following results on  $D(G)$  are true:*

- (1)  $D(K_{1,n}) = K_1 \cup K_{n-1}$ ;
- (2)  $D(K_{n_1, n_2, \dots, n_k}) = K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}$ ;
- (3)  $D(W_n) = K_1 \cup K_{n-1}$ .

**Corollary 5.4** *Let  $G$  be a graph of order  $p + q$ . Then  $D(G) = K_p \cup K_q$  where  $p$  is the number of vertices of same degree and  $q$  is the number of vertices of another same degree.*

**Corollary 5.5** *Let  $T$  be a tree of order  $n_1 + n_2 + n_3$ . Then  $D(T) = K_{n_1} \cup K_{n_2} \cup K_{n_3}$  where  $n_1$  is the number of pendent vertices,  $n_2$  is the number of non-pendent vertices of same degree and  $n_3$  is another non-pendent vertex of another same degree.*

**Theorem 5.6** *Let  $D$  be a connected Euler graph. Then  $D(G)$  is either complete or disjoint union of complete components.*

*Proof* Consider the two cases on the regularity and non-regularity of a connected Euler graph  $G$ .

**Case 1.** Let  $G$  be a regular graph. The Proposition 4.1 shows that  $D(G)$  is complete.

**Case 2.** Let  $G$  be a non-regular graph of order  $n_1 + n_2 + \dots + n_k$ . Then  $n_1$  is the number of vertices of degree 2,  $n_2$  is the number of vertices of degree 4, and so on  $n_k$  is the number of vertices of degree  $2k$ . The Theorem 5.1 shows that  $D(G)$  is isomorphic to the disjoint union of complete components  $K_{n_1}, K_{n_2}, \dots, K_{n_k}$ . Hence  $D(G) = K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}$ .  $\square$

**Example 5.7** For each  $n \geq 3$ , the cycle  $C_n$  is a regular Euler graph, and thus  $D(C_n) = K_n$ .

**Example 5.8([3])** The graphs  $D_{12}$  and  $M_{16}$  are the Davids and Mohammeds graphs of order 12 and 16 respectively, which are non-regular Euler graphs, and their Degree graphs  $D(D_6) = K_6 \cup K'_6$  and  $D(M_{11}) = K_4 \cup K'_7$ , which are disconnected graphs.

**Theorem 5.9** *Let  $G$  be a simple graph of order  $n > 4$ . Then  $D(D(G)) \cong D(G) \Leftrightarrow G$  is  $r$ -regular graph.*

**Proof** For any simple graph  $G$  of order  $n > 4$ , we have  $G$  is  $r$ -regular  $\Leftrightarrow D(G) \cong K_n$ , by the Proposition 4.1  $\Leftrightarrow D(D(G)) \cong D(K_n) \Leftrightarrow D(D(G)) \cong K_n$  (since  $D(K_n) = K_n$ )  $\Leftrightarrow D(D(G)) \cong D(G)$ .  $\square$

## §6. Relation Between $D(G)$ and $DP(G)$

In [1], the authors M. Sattanathan and R. Kala introduced Degree prime graphs and studied their characterizations. According to these authors  $DP(G)$  is a graph whose vertex set is same as  $V(G)$  and  $u, v \in V(G)$  are adjacent in  $DP(G)$  if and only if

$$\deg_G u \neq \deg_G v, \quad \gcd(\deg_G u, \deg_G v) = 1.$$

In this paper, we are going to study the relation between  $D(G)$  and  $DP(G)$ . Fundamentally we observe that the following:

For any graph  $G$ , we have

- (1)  $D(G) \neq DP(G)$ ;
- (2)  $D(DP(G)) \neq DP(D(G))$ .

But the following result specifies that the relation between  $D(G)$  and  $DP(G)$  for some graphs  $G$ .

**Theorem 6.1** *If  $G$  is either totally disconnected, regular or complete, then  $D(G) \cong (DP(G))^c$ .*

*Proof.* For any totally disconnected, regular or complete graph  $G$ , we know that  $D(G) \cong K_n$  and  $DP(G) \cong N_n$ . But  $K_n = N_n^c$ . This shows that the required result is obviously true.  
Box

Here, we present an open problem following:

**Problem 6.2** *Let  $G$  be a graph. Then*

- (1) *Find the cardinality of the set  $S = \{G : D(G) \text{ is complete} \}$ ;*
- (2) *Find the cardinality of the set  $S = \{G : D(D(G)) = D(G)\}$ ;*
- (3) *For the finite family of graphs  $\{G_i\}$ , show that*

$$\bigcup_{i=1}^n D(G_i) = D\left(\bigcup_{i=1}^n G_i\right) \quad \text{and} \quad \bigcap_{i=1}^n D(G_i) = D\left(\bigcap_{i=1}^n G_i\right);$$

- (4) *Find the graph  $G$  such that  $D(D(\cdots(D(G))\cdots))(n \text{ times}) = G$ .*

## References

- [1] M.Sattanathan and R.Kala, Degree prime graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 2009, Vol.12, 167-173.
- [2] F.Harary, *Graph Theory*, Addition-Wesley, Reading, Mass., 1972.
- [3] G Suresh Singh, *Graph Theory*, PHI New Delhi, 2010.
- [4] J.A. Bondy and U.S.R. Mutry, *Graph Theory*, Springer India, 2013.
- [5] Mark L. Lewis, Donald L. White, Diameters of degree graphs of nonsolvable groups, *Journal of Algebra*, 283(2005), pp.8092.
- [6] Yan Yanxiong, Chen Guiyun, XU Haijing, A new Characterization of certain symmetric and alternating groups. *Advances in mathematics*. Vol.42, No.5, 2013.pp.601-610.
- [7] Kapoor S., Polimeni Albert and Wall Curtiss, Degree sets for graphs, *Fundamenta Mathematicae*, Vol.95, No.3(1977), pp. 189-194.
- [8] Y. Manoussakis, H.P. Patil and V. Sankar, *Further results on degree sets for graphs*, *Akce j. Graphs. Combin.*, Vol.1, No. 2(2004), pp. 77-82.

## 4-Remainder Cordial Labeling of Some Graphs

R.Ponraj<sup>1</sup>, K.Annathurai<sup>2</sup> and R.Kala<sup>3</sup>

1. Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India

2. Department of Mathematics, Thiruvalluvar College, Papanasam -627425, India

3. Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, India

ponrajmaths@gmail.com, kannathuraitvcmaths@gmail.com, karthipyi91@yahoo.co.in

**Abstract:** Let  $G$  be a  $(p, q)$  graph. Let  $f$  be a function from  $V(G)$  to the set  $\{1, 2, \dots, k\}$  where  $k$  is an integer  $2 < k \leq |V(G)|$ . For each edge  $uv$  assign the label  $r$  where  $r$  is the remainder when  $f(u)$  is divided by  $f(v)$  (or)  $f(v)$  is divided by  $f(u)$  according as  $f(u) \geq f(v)$  or  $f(v) \geq f(u)$ .  $f$  is called a  $k$ -remainder cordial labeling of  $G$  if  $|v_f(i) - v_f(j)| \leq 1$ ,  $i, j \in \{1, \dots, k\}$ , where  $v_f(x)$  denote the number of vertices labeled with  $x$  and  $|e_f(0) - e_f(1)| \leq 1$  where  $e_f(0)$  and  $e_f(1)$  respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with admits a  $k$ -remainder cordial labeling is called a  $k$ -remainder cordial graph. In this paper we investigate the 4- remainder cordial behavior of grid, subdivision of crown, Subdivision of bistar, book, Jelly fish, subdivision of Jelly fish, Mongolian tent graphs.

**Key Words:**  $k$ -Remainder cordial labeling, Smarandache  $k$ -remainder cordial labeling, grid, subdivision of crown, subdivision of bistar, book, Jelly fish, subdivision of Jelly fish, Mongolian tent.

**AMS(2010):** 05C78.

### §1. Introduction

We considered only finite and simple graphs. The subdivision graph  $S(G)$  of a graph  $G$  is obtained by replacing each edge  $uv$  by a path  $uvw$ . The product graph  $G_1 \times G_2$  is defined as follows:

Consider any two points  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  in  $V = V_1 \times V_2$ . Then  $u$  and  $v$  are adjacent in  $G_1 \times G_2$  whenever  $[u_1 = v_1 \text{ and } u_2 \text{ adj } v_2]$  or  $[u_2 = v_2 \text{ and } u_1 \text{ adj } v_1]$ . The graph  $P_m \times P_n$  is called the planar grid. Let  $G_1, G_2$  respectively be  $(p_1, q_1), (p_2, q_2)$  graphs. The corona of  $G_1$  with  $G_2$ ,  $G_1 \odot G_2$  is the graph obtained by taking one copy of  $G_1$  and  $p_1$  copies of  $G_2$  and joining the  $i^{th}$  vertex of  $G_1$  with an edge to every vertex in the  $i^{th}$  copy of  $G_2$ . A mongolian tent  $M_{m,n}$  is a graph obtained from  $P_m \times P_n$  by adding one extra vertex above the grid and joining every other of the top row of  $P_m \times P_n$  to the new vertex. Cahit [1], introduced the concept of cordial labeling of graphs. Ponraj et al. [4, 6], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of path, cycle,

---

<sup>1</sup>Received February 4, 2017, Accepted March 8, 2018.

star, bistar, complete graph,  $S(K_{1,n})$ ,  $S(B_{n,n})$ ,  $S(W_n)$ ,  $P_n^2$ ,  $P_n^2 \cup K_{1,n}$ ,  $P_n^2 \cup B_{n,n}$ ,  $P_n \cup B_{n,n}$ ,  $P_n \cup K_{1,n}$ ,  $K_{1,n} \cup S(K_{1,n})$ ,  $K_{1,n} \cup S(B_{n,n})$ ,  $S(K_{1,n}) \cup S(B_{n,n})$ , etc., and also the concept of  $k$ -remainder cordial labeling introduced in [5]. In this paper we investigate the 4-remainder cordial labeling behavior of Grid, Subdivision of crown, Subdivision of bistar, Book, Jelly fish, Subdivision of Jelly fish, Mongolian tent, etc.,. Terms are not defined here follows from Harary [3] and Gallian [2].

## §2. $k$ -Remainder Cordial Labeling

**Definition 2.1** Let  $G$  be a  $(p, q)$  graph. Let  $f$  be a function from  $V(G)$  to the set  $\{1, 2, \dots, k\}$  where  $k$  is an integer  $2 < k \leq |V(G)|$ . For each edge  $uv$  assign the label  $r$  where  $r$  is the remainder when  $f(u)$  is divided by  $f(v)$  (or)  $f(v)$  is divided by  $f(u)$  according as  $f(u) \geq f(v)$  or  $f(v) \geq f(u)$ . The labeling  $f$  is called a  $k$ -remainder cordial labeling of  $G$  if  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$ , otherwise, Smarandachely if  $|v_f(i) - v_f(j)| \geq 1$  or  $|e_f(0) - e_f(1)| \geq 1$  for integers  $i, j \in \{1, \dots, k\}$ , where  $v_f(x)$  and  $e_f(0)$ ,  $e_f(1)$  respectively denote the number of vertices labeled with  $x$ , the number of edges labeled with even integers and the number of edges labelled with odd integers. Such a graph with a  $k$ -remainder cordial labeling is called a  $k$ -remainder cordial graph.

First we investigate the 4-remainder cordial labeling behavior of the planar grid.

**Theorem 2.2** The planar grid  $P_m \times P_n$  is 4-remainder cordial.

*Proof* Clearly this grid has  $m$ -rows and  $n$ -columns. We assign the labels to the vertices by row wise.

**Case 1.**  $m \equiv 0 \pmod{4}$

Let  $m = 4t$ . Then assign the label 1 to the vertices of  $1^{st}, 2^{nd}, \dots, t^{th}$  rows. Next we move to the  $(t+1)^{th}$  row. Assign the label 4 to the vertices of  $(t+1)^{th}, (t+2)^{th}, \dots, (2t)^{th}$  rows. Next assign the label to the vertices  $(2t+1)^{th}$  row. Assign the labels 2 and 3 alternatively to the vertices of  $(2t+1)^{th}$  row. Next move to  $(2t+2)^{th}$  row. Assign the labels 3 and 2 alternatively to the vertices of  $(2t+2)^{th}$  row. In general  $i^{th}$  row is called as in the  $(i-2)^{th}$  row, where  $2t+1 \leq i \leq 3t$ . This procedure continued until we reach the  $(4t)^{th}$  row.

**Case 2.**  $m \equiv 1 \pmod{4}$

As in Case 1, assign the labels to the vertices of the first, second,  $\dots, (m-1)^{th}$  row. We give the label to the  $m^{th}$  row as in given below.

**Subcase 2.1**  $n \equiv 0 \pmod{4}$

Rotate the row and column and result follows from Case 1.

**Subcase 2.2**  $n \equiv 1 \pmod{4}$

Assign the labels 4, 3, 4, 3,  $\dots$ , 4, 3 to the vertices of the first, second,  $\dots, (\frac{n-1}{2})^{th}$  columns. Next assign the label 2 to the vertices of  $(\frac{n+1}{2})^{th}$  column. Then next assign the labels 2, 1, 2, 1,  $\dots$ ,



2, 1 to the vertices of  $\frac{n+3}{2}, \frac{n+5}{2}, \dots, (\frac{2n}{2} - 2)^{th}$  columns. Assign the remaining vertices.

**Subcase 2.3**  $n \equiv 2 \pmod{4}$

Assign the labels 4, 3, 4, 3,  $\dots$ , 4, 3 to the vertices of  $1^{st}, 2^{nd}, \dots, (\frac{n-2}{2})^{th}$  columns. Next assign the label 2 to the vertices of  $(\frac{n}{2})^{th}$  column. Then next assign the labels 2, 1, 2, 1,  $\dots$ , 2, 1 to the vertices of  $\frac{n}{2}+1, \frac{n}{2}+2, \dots, (\frac{2n}{2} - 1)^{th}$  columns. Finally assign the label 1 to the remaining vertices of  $n^{th}$  column.

**Subcase 2.4**  $n \equiv 3 \pmod{4}$

Assign the labels 4, 3, 4, 3,  $\dots$ , 4, 3 alternatively to the vertices of  $1^{st}, 2^{nd}, \dots, (\frac{n+1}{2})^{th}$  columns. Then next assign the labels 1, 2, 1, 2,  $\dots$  to the vertices of  $\frac{n+3}{2}, \frac{n+5}{2}, \dots, (\frac{2n}{2} - 1)^{th}$  columns. Finally assign the label 1 to the remaining vertices of  $n^{th}$  column. Hence  $f$  is a 4-remainder cordial labeling of  $P_m \times P_n$ .

All other cases follow by symmetry.  $\square$

Next is the graph  $K_2 + mK_1$ .

**Theorem 2.3** *If  $m \equiv 0, 1, 3 \pmod{4}$  then  $K_2 + mK_1$  is 4-remainder cordial.*

*Proof* It is easy to verify that  $K_2 + mK_1$  has  $m + 2$  vertices and  $2m$  edges. Let  $V(K_2 + mK_1) = \{u, u_i, v : 1 \leq i \leq m\}$  and  $E(K_2 + mK_1) = \{uv, uu_i, vu_i : 1 \leq i \leq m\}$ .

**Case 1.**  $m \equiv 0 \pmod{4}$

Let  $m = 4t$ . Then assign the label 3, 3 respectively to the vertices  $u, v$ . Next assign the label 1 to the vertices  $u_1, u_2, \dots, u_{t+1}$ . Then next assign the label 2 to the vertices  $u_{t+2}, u_{t+3}, \dots, u_{2t+1}$ . Then followed by assign the label 3 to the vertices  $u_{2t+2}, u_{2t+3}, \dots, u_{3t}$ . Finally assign the label 4 to the remaining non-labelled vertices  $u_{3t+1}, u_{3t+2}, \dots, u_{4t}$ .

**Case 2.**  $m \equiv 1 \pmod{4}$

As in Case 1, assign the labels to the vertices  $u, v, u_i, (1 \leq i \leq m - 1)$ . Next assign the label 2 to the vertex  $u_m$ .

**Case 3.**  $m \equiv 3 \pmod{4}$

Assign the labels to the vertices  $u, v, u_i, (1 \leq i \leq m - 2)$  as in case(ii). Finally assign the labels 3, 4 respectively to the vertices  $u_{m-1}, u_m$ . The table given below establish that this labeling  $f$  is a 4-remainder cordial labeling.

Nature of $m$	$e_f(0)$	$e_f(1)$
$m \equiv 0 \pmod{4}$	$m + 1$	$m$
$m \equiv 1 \pmod{4}$	$m$	$m + 1$
$m \equiv 3 \pmod{4}$	$m$	$m + 1$

**Table 1**

This completes the proof.  $\square$

The next graph is the book graph  $B_n$ .

**Theorem 2.4** *The book  $B_n$  is 4-remainder cordial for all  $n$ .*

*Proof* Let  $V(B_n) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$  and  $E(B_n) = \{uv, uu_i, vv_i, u_i v_i : 1 \leq i \leq n\}$ .

**Case 1.**  $n$  is even

Assign the labels 3, 4 to the vertices  $u$  and  $v$  respectively. Assign the label 1 to the vertices  $u_1, u_2, \dots, u_{\frac{n}{2}}$  and assign 4 to the vertices  $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \dots, u_n$ . Next we consider the vertices  $v_i$ . Assign the label 2 to the vertices  $v_1, v_2, \dots, v_{\frac{n}{2}}$ . Next assign the label 3 to the remaining vertices  $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \dots, v_n$ , respectively.

**Case 2.**  $n$  is odd

Assign the labels 3, 4 to the vertices  $u$  and  $v$  respectively. Fix the labels 4, 2, 1 to the vertices  $u_1, u_2, \dots, u_{\frac{n}{2}+1}$  and also fix the labels 3, 1, 2 respectively to the vertices  $v_1, v_2, \dots, v_{\frac{n}{2}+1}$ . Assign the labels to the vertices  $u_4, u_5, \dots, u_n$  as in the sequence 2, 1, 2, 1, ..., 2, 1. In similar fashion, assign the labels to the vertices  $v_4, v_5, \dots, v_n$  as in the sequence 3, 4, 3, 4, ..., 3, 4. The table 2 shows that this vertex labeling  $f$  is a 4-remainder cordial labeling.

Nature of $n$	$e_f(0)$	$e_f(1)$
$n$ is even	$m + 1$	$m$
$n$ is odd	$m$	$m + 1$

**Table 2**

This completes the proof. □

Now we consider the subdivision of  $B_{n,n}$ .

**Theorem 2.5** *The subdivision of  $B_{n,n}$  is 4-remainder cordial.*

*Proof* Let  $V(S(B_{n,n})) = \{u, v, u_i, v_i, w_i, x, x_i : 1 \leq i \leq n\}$  and  $E(S(B_{n,n})) = \{uu_i, vv_i, u_i w_i, v_i x_i, ux, xv : 1 \leq i \leq n\}$ . It is clearly to verify that  $S(B_{n,n})$  has  $4n + 3$  vertices and  $4n + 2$  edges.

Assign the labels 1, 4, 3 to the vertices  $u, x$  and  $v$  respectively. Assign the labels 1, 3 alternatively to the vertices  $u_1, u_2, \dots, u_n$ . Next assign the labels 2, 4 alternatively to the vertices  $w_1, w_2, \dots, w_n$ . We now consider the vertices  $v_i$  and  $x_i$ . Assign the labels 2, 4 alternatively to the vertices  $v_1, v_2, \dots, v_n$ . Then finally assign the labels 3, 1 alternatively to the vertices  $x_1, x_2, \dots, x_n$ . Obviously this vertex labeling is a 4-remainder cordial labeling. □

Next, we consider the subdivision of crown graph.

**Theorem 2.5** *The subdivision of  $C_n \odot K_1$  is 4-remainder cordial.*

*Proof* Let  $u_1 u_2 \dots u_n$  be a cycle  $C_n$ . Let  $V(C_n \odot K_1) = V(C_n \cup \{v_i : 1 \leq i \leq n\})$  and  $E(C_n \odot K_1) = E(C_n \cup \{u_i v_i : 1 \leq i \leq n\})$ . The subdivide edges  $u_i u_{i+1}$  and  $u_i v_i$  by  $x_i$  and  $y_i$  respectively. Assign the label 2 to the vertices  $u_i, (1 \leq i \leq n)$  and 3 to the vertices

$x_i, (1 \leq i \leq n)$ . Next assign the label 1 to the vertices  $y_i, (1 \leq i \leq n)$ . Finally assign the label 4 to the vertices  $v_i, (1 \leq i \leq n)$ . Clearly, this labeling  $f$  is a 4-remainder cordial labeling.  $\square$

Now we consider the Jelly fish  $J(m, n)$ .

**Theorem 2.6** *The Jelly fish  $J(m, n)$  is 4-remainder cordial.*

*Proof* Let  $V(J(m, n)) = \{u, v, x, y, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$  and  $E(J(m, n)) = \{uu_i, vv_j, ux, uy, vx, vy : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ . Clearly  $J(m, n)$  has  $m + n + 4$  vertices and  $m + n + 5$  edges.

**Case 1.**  $m = n$  and  $m$  is even.

Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{\frac{n}{2}}$  and assign the label 4 to the vertices  $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \dots, u_n$ . Next assign the label 1 to the vertices  $v_1, v_2, \dots, v_{\frac{n}{2}}$  and assign 3 to the vertices  $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \dots, v_n$ . Finally assign the labels 3, 4, 2, 1 respectively to the vertices  $u, x, y, v$ .

**Case 2.**  $m = n$  and  $m$  is odd.

In this case assign the labels to the vertices  $u_i, v_i (1 \leq i \leq m - 1)$  and  $u, v, x, y$  as in Case 1. Next assign the labels 2, 1 respectively to the vertices  $u_n$  and  $v_n$ .

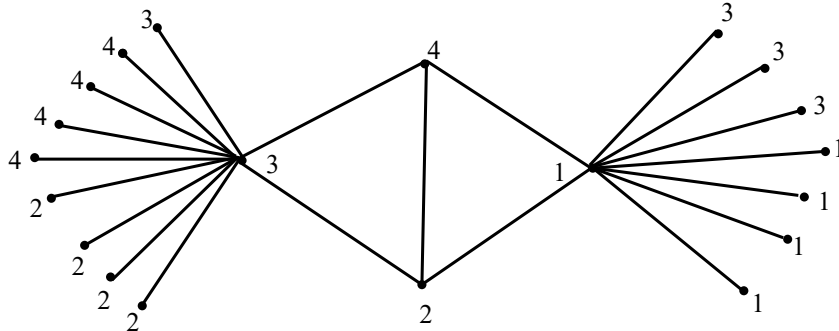
**Case 3.**  $m \neq n$  and assume  $m > n$ .

Assign the labels 3, 4, 1, 2 to the vertices  $u, x, y, v$  respectively. As in Case 1 and 2, assign the labels to the vertices  $u_i, v_i (1 \leq i \leq n)$ .

**Subcase 3.1**  $m - n$  is even. Assign the labels to the vertices  $u_{n+1}, u_{n+2}, \dots, u_m$  as in the sequence 3, 4, 2, 1; 3, 4, 2, 1;  $\dots$ . It is easy to verify that  $u_n$  is received the label 1 if  $m - n \equiv 0 \pmod{4}$ .

**Subcase 3.2**  $m - n$  is odd. Assign the labels to the vertices  $u_i (n \leq i \leq m)$  as in the sequence 4, 3, 2, 1; 4, 3, 2, 1;  $\dots$ . Clearly,  $u_n$  is received the label 1 if  $m - n \equiv 0 \pmod{4}$ .  $\square$

For illustration, a 4-remainder cordial labeling of Jelly fish  $J(m, n)$  is shown in Figure 1.



**Figure 1**

**Theorem 2.8** *The subdivision of the Jelly fish  $J(m, n)$  is 4-remainder cordial.*

*Proof* Let  $V(S(J(m, n))) = \{u, u_i, x_i, v, v_j, y_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{w_i : 1 \leq i \leq 7\}$  and  $E(S(J(m, n))) = \{uu_i, u_i x_i, vv_j, v_j y_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{uw_1, uw_2, w_1 w_5, w_5 w_6, w_6 w_7, w_5 w_3, w_3 v, v w_4, w_4 w_7, w_2 w_7\}$ .

**Case 1.**  $m = n$ .

Assign the label 2 to the vertices  $u_1, u_2, \dots, u_m$  and 3 to the vertices  $x_1, x_2, \dots, x_m$ . Next assign the label 1 to the vertices  $v_1, v_2, \dots, v_m$  and assign the label 4 to the vertices  $y_1, y_2, \dots, y_m$ . Finally assign the labels 3, 2, 3, 2, 3, 1, 4, 4 and 1 respectively to the vertices  $u, w_1, w_5, w_6, w_7, w_2, w_3, v$  and  $w_4$ .

**Case 2.**  $m > n$ .

Assign the labels to the vertices  $u, u_i, v, v_i, x_i, y_i, w_1, w_2, w_3, w_4, w_5, w_6, w_7, (1 \leq i \leq n)$  as in case(i). Next assign the labels 1, 4 to the next two vertices  $x_{n+1}, x_{n+2}$  respectively. Then next assign the labels 1, 4 respectively to the vertices  $x_{n+3}, x_{n+4}$ . Proceeding like this until we reach the vertex  $x_n$ . That is the vertices  $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \dots$  are labelled in the pattern 1, 4; 1, 4; 1, 4; 1, 4;  $\dots$ . Similarly the vertices  $u_{n+1}, u_{n+2}, \dots$  are labelled as 2, 3; 2, 3; 2, 3;  $\dots$ , 2, 3. The Table 3, establish that this vertex labeling  $f$  is a 4-remainder cordial labeling of  $S(J(m, n))$ .

Nature of $m$ and $n$	$e_f(0)$	$e_f(1)$
$m = n$	$2n + 5$	$2n + 5$
$m > n$	$m + n + 5$	$m + n + 5$

This completes the proof.  $\square$

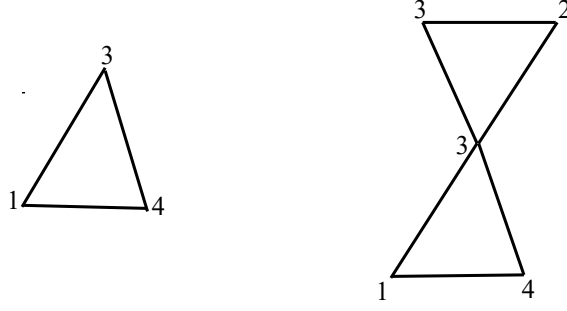
**Theorem 2.9** The graph  $C_3^{(t)}$  is 4-remainder cordial.

*Proof* Let  $V(C_3^{(t)}) = \{u, u_i, v_i : 1 \leq i \leq n\}$  and  $E(C_3^{(t)}) = \{uu_i, vv_i, u_i v_i : 1 \leq i \leq n\}$ . Assume  $t \geq 3$ . Fix the label 3 to the central vertex  $u$  and fix the labels 1, 2, 2, 4, 3, 4 respectively to the vertices  $u_1, u_2, u_3, v_1, v_2$  and  $v_3$ . Next assign the labels 1, 2 to the vertices  $u_4, u_5$ . Then assign the labels 1, 2 respectively to the next two vertices  $u_6, u_7$  and so on. That is the vertices  $u_4, u_5, u_6, u_7$  are labelled as in the pattern 1, 2, 1, 2  $\dots$ , 1, 2. Note that the vertex  $u_n$  received the label 1 or 2 according as  $n$  is even or odd. In a similar way assign the labels to the vertices  $v_4, v_5, v_6, v_7$  as in the sequence 4, 3, 4, 3, 4, 3,  $\dots$ . Clearly 4 is the label of  $u_n$  according as  $n$  is even or odd. The Table 4 establish that this vertex labeling is a 4-remainder cordial labeling of  $C_3^{(t)}$ ,  $t \geq 3$ .

Nature of $n$	$e_f(0)$	$e_f(1)$
$n$ is even	$\frac{3n}{2}$	$\frac{3n}{2}$
$n$ is odd	$n + 1$	$n + 2$

Table 4

For  $t = 1, 2$  the remainder cordial labeling of graphs  $C_3^{(1)}$  and  $C_3^{(2)}$  are given below in Figure 2.

**Figure 2**

This completes the proof.  $\square$

**Theorem 2.10** *The Mongolian tent  $M_{m,n}$  is 4-remainder cordial.*

*Proof* Assign the label 3 to the new vertex.

**Case 1.**  $m \equiv 0 \pmod{4}$  and  $n \equiv 0, 2 \pmod{4}$ .

Consider the first row of  $M_n$ . Assign the labels  $2, 3, 2, 3, \dots, 2, 3$  to the vertices in the first row. Next assign the labels  $3, 2, 3, 2, \dots, 3, 2$  to the vertices in the second row. This procedure is continue until reach the  $\frac{n}{2}^{th}$  row. Next assign the labels  $1, 4, 1, 4, \dots, 1, 4$  to the vertices in the  $\frac{n}{2} + 1^{th}$  row. Then next assign the labels  $4, 1, 4, 1, \dots, 4, 1$  to the vertices in the  $\frac{n}{2} + 2^{th}$  row. This proceedings like this assign the labels continue until reach the last row.

**Case 2.**  $m \equiv 2 \pmod{4}$  and  $n \equiv 0, 2 \pmod{4}$ .

In this case assign the labels to the vertices as in Case 1.

**Case 3.**  $m \equiv 1 \pmod{4}$  and  $n \equiv 0, 2 \pmod{4}$ .

Here assign the labels by column wise to the vertices of  $M_n$ . Assign the labels  $2, 3, 2, 3, \dots, 2, 3$  to the vertices in the first column. Next assign the labels  $3, 2, 3, 2, \dots, 3, 2$  to the vertices in the second column. This method is continue until reach the  $\frac{n}{2}^{th}$  column. Next assign the labels  $1, 4, 1, 4, \dots, 1, 4$  to the vertices in the  $\frac{n}{2} + 1^{th}$  column. Then next assign the labels  $4, 1, 4, 1, \dots, 4, 1$  to the vertices in the  $\frac{n}{2} + 2^{th}$  column. This procedure is continue until reach the last column.

**Case 4.**  $m \equiv 3 \pmod{4}$  and  $n \equiv 0, 2 \pmod{4}$ .

As in Case 3 assign the labels to the vertices in this case. The remainder cordial labeling of graphs  $M_{7,4}$  is given below in Figure 3..

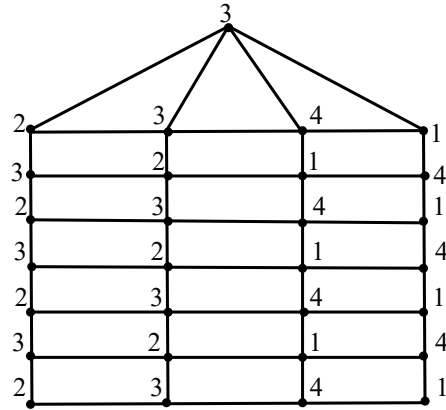


Figure 3

This completes the proof. □

## References

- [1] Cahit I., Cordial Graphs : A weaker version of Graceful and Harmonious graphs, *Ars Combin.*, **23** (1987), 201–207.
- [2] Gallian J.A., A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **19**, (2017).
- [3] Harary F., *Graph Theory*, Addison Wesley, New Delhi, 1969.
- [4] Ponraj R., Annathurai K. and Kala R., Remainder cordial labeling of graphs, *Journal of Algorithms and Computation*, **Vol.49**, (2017), 17–30.
- [5] Ponraj R., Annathurai K. and Kala R.,  $k$ -Remaider cordial graphs, *Journal of Algorithms and Computation*, **Vol.49(2)**, (2017), 41-52.
- [6] Ponraj R., Annathurai K. and Kala R., Remainder cordiality of some graphs, *Accepted for publication in Palestin Journal of Mathematics* .

## Motion Planning in Certain Lexicographic Product Graphs

A. D Akwu and O. Oyewumi

(Department of Mathematics, Federal University of Agriculture, Makurdi, Nigeria)

abolaopeyemi@yahoo.co.uk, opeyemioluwaoyewumi@gmail.com

**Abstract:** Let  $G$  be an undirected graph with  $n$  vertices in which a robot is placed at a vertex say  $v$ , and a hole at vertex  $u$  and in all other  $(n - 2)$  vertices are obstacles. We refer to this assignment of robot and obstacles as a configuration  $C_u^v$  of  $G$ . Suppose we have a one player game in which the robot or obstacle can be slide to an adjacent vertex if it is empty i.e. if it has a hole. The goal is to take the robot to a particular destination vertex using minimum number of moves. In this article, we give the minimum number of moves required for the motion planning problem in Lexicographic products of some graphs. In addition, we proved the necessary and sufficient condition for the connectivity of the lexicographic product of two graphs.

**Key Words:** Motion in a graph, lexicographic product graphs, k-factor of graphs.

**AMS(2010):** 05C85, 05C75, 68R10, 91A43.

### §1. Introduction

Given a graph  $G$ , with a robot placed at one of it's vertices and movable obstacles at some other vertices. Assuming that we are allowed to slide the robot and obstacles to an adjacent vertex if it is empty. Let  $u, v \in V(G)$ , and suppose that the robot is at  $v$  and the hole at  $u$  and obstacles at other vertices we refer to this as a configuration  $C_u^v$ . Now we use  $u \xleftarrow{r} v$  and  $u \xleftarrow{o} v$  to denote respectively, the robot move and the obstacle move from vertex  $v$  to an adjacent vertex  $u$  where  $u, v \in E(G)$ . A simple move is referred to as moving an obstacle or the robot to an adjacent empty vertex while a graph  $G$  is  $k$ -reachable if there exists a  $k$ -configuration such that the robot can reach any vertex of the graph in a finite number of simple moves. The objective is to find a minimum sequence of moves that takes the robot from (source) vertex  $p$  to a (destination) vertex  $t$ .

For two vertices  $u, v \in V(G)$ , let  $d_G(u, v)$  denotes the distance between  $u$  and  $v$  in  $G$ . Most of the distances used in this article are in  $G$  so we use  $d(u, v)$  instead of  $d_G(u, v)$  to represent the distance between the vertices  $u$  and  $v$  in  $G$ . We denote the complete, complement of a complete, cycle, path graph and complete graph minus one factor on  $n$  vertices by  $K_n$ ,  $\overline{K}_n$ ,  $C_n$ ,  $P_n$  and  $K_n - I$  respectively.

The motion planning problem in graph was proposed by Papadimitriou et al [9] where it was shown that with arbitrary number of holes, the decision version of such problem is NP-

---

<sup>1</sup>Received August 26, 2017, Accepted March 10, 2018.

complete and that the problem is complex even when it is restricted to planar graphs. They also gave time algorithm for trees. The result in [9] was improve in [2]. Robot motion planning on graphs (RMPG) is a graph with a robot placed at one of its vertices and movable obstacle at some of the other vertices while generalization of RMPG problem is the Multiple robot motion planning in graph (MRMPG) whereby we have  $k$  different robots with respective destinations. Ellips and Azadeh [5] studied MRMPG on trees and introduced the concept of minimal solvable trees. Auletta et al [1] also studied the feasibility of MRMPG problem on trees and gave an algorithm that, on input of two arrangements of  $k$  robots on a tree of order  $n$ , decides in time  $O(n)$  whether the two arrangements are reachable from one another. Parberry [8] worked on grid of order  $n^2$  with multiple robots while Deb and Kapoor [4,3] generalized and apply the technique used in [8] to calculate the minimum number of moves for the motion planning problem for the cartesian product of two given graphs.

The MRMPG problem of grid graph of order  $n^2$  with  $n^2 - 1$  robots is known as  $(n^2 - 1)$ -puzzle. The objective of  $(n^2 - 1)$ -puzzle is to verify whether two given configuration of the grid graph of order  $n^2$  are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. The  $(n^2 - 1)$ -puzzle have been studied extensively in [7, 8, 10, 11].

Our work was motivated by Deb and Kapoor [3] whereby they gave minimum sequence of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. They also proved that the path traced by the robot coincides with a shortest path in case of Cartesian product graphs of graphs. In this paper, we consider the case of lexicographic product graphs. Here we give the minimum number of moves required for the motion planning problem in the Lexicographic product of two graphs say  $G$  and  $H$ , where  $G$  and  $H$  are specified in each of our cases.

### 1.1 Lexicographic Product of Graphs

**Definition 1.1** *The lexicographic product  $G \circ H$  of two graphs  $G$  and  $H$  is a graph with vertex set  $V(G) \times V(H)$  in which  $(u_i, v_j)$  and  $(u_p, v_q)$  are adjacent if one of the following condition holds:*

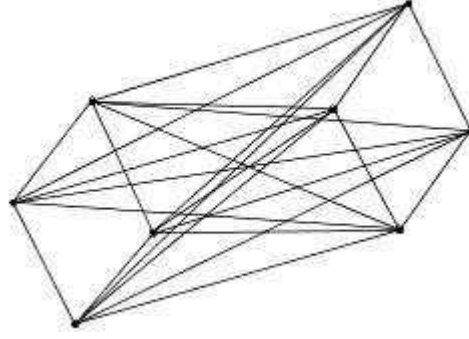
- (i)  $\{u_i, u_p\} \in E(G)$ ;
- (ii)  $u_i = u_p$  and  $\{v_j, v_q\} \in E(H)$ .

The graphs  $G$  and  $H$  are known as the factors of  $G \circ H$ . Now onwards  $G$  and  $H$  are simple graphs with  $V(G) = \{1, 2, 3, \dots, m\}$  unless otherwise stated.

Suppose we are dealing with  $p$ -copies of a graph  $G$  and we are denoting these  $p$ -copies of  $G$  by  $G^i$ , where  $i = \{1, 2, 3, \dots, p\}$ . Then for each vertex  $u \in V(G)$  we denote the corresponding vertex in the  $i^{th}$  copy  $G^i$  by  $u^i$ . The girth of a graph  $G$ , denoted by  $g(G)$  is the length of the shortest cycle contained in graph  $G$ .

**Example 1.2** Let  $G = P_2$  and  $H = C_4$ . The graph  $G \circ H$  is shown in Figure 1 below.





**Figure 1**  $P_2 \circ C_4$

**Remark 1.3** The Lexicographic product  $G \circ H$ , of graphs  $G$  and  $H$ , is the graph obtained by replacing each vertex of  $G$  by a copy of  $H$  and every edge of  $G$  by the complete bipartite graph  $K_{[H],[H]}$ .

## 1.2 Connectivity of Lexicographic Product of Two Graphs

Here we aim at proving a corollary which Deb and Kapoor [3] mentioned as concerning the condition for which the lexicographic product  $G \circ H$  is connected.

**Proposition 1.4** (See [6]) *Suppose that  $u^i$  and  $v^j$  are two vertices in  $G \circ H$ . Then*

$$d_{(G \circ H)}(u^i, v^j) = \begin{cases} d_H(u, v), & \text{if } i = j \text{ and } d_G(i) = 0 \\ \min\{d_H(u, v), 2\}, & \text{if } i = j \text{ and } d_G(i) \neq 0 \\ d_G(i, j), & \text{if } i \neq j \end{cases}$$

**Theorem 1.5** *Let  $G$  and  $H$  be two non-trivial graphs. Then  $G \circ H$  is connected if and only if  $G$  is connected.*

*Proof* Assume that  $G \circ H$  is connected. We only need to show that graph  $G$  is connected. Given that  $u, v \in V(H)$  and  $i, j \in V(G)$ . Let  $u^i$  be an arbitrary vertex in  $G \circ H$ . Since  $G \circ H$  is connected it means that the vertex  $u^i$  has an edge with at least a vertex in  $G \circ H$  (especially in one of the factors in the product graph), let this vertex be  $v^j$ . Next, by definition of the lexicographic product graph, for  $u^i$  and  $v^j$  to have an edge in  $G \circ H$  then  $\{i, j\} \in E(G)$ . Which implies that there is a path between  $i$  and  $j$  in  $G$ . But  $u^i$  is an arbitrary vertex in  $G \circ H$  we conclude that there is a path between each pair of vertices in graph  $G$ . Therefore  $G$  is connected. Conversely, suppose that graph  $G$  is connected. It suffices us to show that  $G \circ H$  is connected. Since  $G$  is connected it implies that for all  $i, j \in V(G)$  where  $i$  and  $j$  are distinct  $d(i, j) \neq 0$ . We shall prove this by contradiction. Assume that the graph  $G \circ H$  is disconnected. If  $G \circ H$  is disconnected it means that there exist an arbitrary pair of vertices

$(u^i, v^j)$  in  $G \circ H$  such that  $d_{G \circ H}(u^i, v^j) = 0$  for all  $i, j \in V(G)$  and  $u, v \in V(H)$ . Since  $i$  and  $j$  are distinct then by Proposition 1.4 we have that  $d_{G \circ H}(u^i, v^j) = d_G(i, j)$ . But  $d_G(i, j) \neq 0$  therefore  $d_{G \circ H}(u^i, v^j) \neq 0$  a contradiction. Since the pair  $(u^i, v^j)$  is arbitrary we conclude that the product graph  $G \circ H$  is connected. This completes the proof.  $\square$

## §2. Robot Moves in Lexicographic Product of a Graph and Complement of Complete Graph

**Definition 2.1** An edge  $u^p, v^q$  in  $G \circ H$  is said to be a  $G$ -edge if  $u = v$  and  $\{p, q\} \in E(G)$ .

**Definition 2.2** Given two graphs  $G$  and  $H$ . For any  $u^p, v^q \in V(G \circ H)$ , we call the distance between  $u$  and  $v$  in  $H$  to be the  $H$ -distance between  $u^p$  and  $v^q$  in  $G \circ H$ . We use  $d_G(u^p, v^q)$  and  $d_H(u^p, v^q)$  to denote the  $G$ -distance and  $H$ -distance between  $u^p, v^q$  in  $G \circ H$ , respectively.

Now when there is no confusion about the graph in question  $G$ , we use  $d(u, v)$  instead of  $d_G(u, v)$  to represent the distance between  $u$  and  $v$  in  $G$ .

In view of the above definition, we now have this proposition.

**Proposition 2.3** Given two graphs  $G$  and  $H$ . Let  $\{i, j\}, \{j, k\} \in E(G)$  and  $u, v \in V(H)$ . Then  $d_{G \circ H - u^i}(v^i, u^j) = d_{G \circ H}(v^i, u^j)$ .

*Proof* To prove this, notice that  $d_{G \circ H - u^i}(v^i, u^j) = 1$  which is same as  $d_{G \circ H}(v^i, u^j)$ .

Each vertex set of copy  $H^i$  is adjacent to all other vertices in copy  $H^j$  for all  $\{i, j\} \in E(G)$  in Lexicographic product graphs.  $\square$

The following two results was given in [4].

**Lemma 2.4**([4]) Given two graphs  $G$  and  $H$ . Let  $\{i, j\}, \{j, k\} \in E(G)$  such that  $\{i, k\} \notin E(G)$  and  $u, v, w \in V(H)$ . Consider the configuration  $C_{u^i}^{v^j}$  of  $G \circ H$ . Then we require at least three moves to move the robot from  $v^j$  to  $w^k$ .

**Lemma 2.5**([4]) Given two graphs  $G$  and  $H$ . Let  $u, v, w \in V(H)$  such that  $\{u, v\}, \{v, w\} \in E(H)$  and  $\{u, w\} \notin E(H)$ . For some  $i \in V(G)$ , consider the configuration  $C_{u^i}^{v^i}$  of  $G \circ H$ . Then minimum of three moves are required to move the robot from  $v^i$  to  $w^i$ .

**Proposition 2.6** Let  $G$  be a graph and  $H$  a complement of a complete graph on  $n$  vertices. Let  $\{i, j\}, \{j, k\} \in E(G)$  and  $u, v \in V(H)$ . Then starting from the configuration  $C_{v^i}^{u^i}$  we need at least two moves to move the robot to  $u^j$ .

*Proof* To move the robot from  $u^i$  to  $u^j$  before it, the hole is required to move from  $v^i$  to  $u^j$ . This takes just a move since  $d_{G \circ H - u^i}(v^i, u^j) = 1$ . Then the move  $u^j \xleftarrow{r} u^i$  takes the robot to  $u^j$ . Hence the result follows.  $\square$

**Corollary 2.7** Let  $G$  be any graph and  $H$  a complement of a complete graph on  $n$  vertices. Let  $\{i, j\}, \{j, k\} \in E(G)$  and  $u, v \in V(H)$ , where  $u$  and  $v$  are distinct. Then starting from the configuration  $C_{v^i}^{v^j}$  we need at least three moves to move the robot to  $v^k$ .

*Proof* Observe that  $\{v^i, v^j\}, \{v^j, v^k\} \in E(G \circ H)$ . For the robot to move to  $v^k$  before it, the hole must move from  $v^i$  to  $v^k$ . This takes  $d_{G \circ H}(v^i, v^k) = 2$ . Then the move  $v^k \xleftarrow{r} v^j$  moves the robot from  $v^j$  to  $v^k$ . Hence the result follows.  $\square$

**Definition 2.8** A robot move in  $G \circ H$  is called a *G-move* if the edge along which the move take place is a *G-edge*.

**Definition 2.9** Let  $T$  be a sequence of moves that take the robot from  $u^p$  to  $v^q$  in  $G \circ H$ . An *H-move* (respectively *G-move*) in  $T$  of the robot is said to be a secondary *H-move* (respectively *G-move*) if it is preceded by an *H-move* (respectively *G-move*). An *H-move* (respectively *G-move*) in  $T$  of the robot is said to be a primary *H-move* (respectively *G-move*) if it is preceded by a *G-move* (respectively *H-move*). Also the edge corresponding to a primary *G-move* (respectively *H-move*) in  $T$  is said to be a primary *G-edge* (respectively *H-edge*). In view of the above definitions we have the following remark.

**Remark 2.10** Given graph  $G$  and  $H$  a complement of a complete graph on  $n$  vertices.

(i) In view of Proposition 2.6, to perform the first move of the robot we require at least 2 moves;

(ii) In view of Corollary 2.7, to perform each secondary move of the robot we require at least 3 moves.

**Theorem 2.11** Let  $G$  be a graph and  $H$  a complement of a complete graph on  $n$  vertices. Let  $i, j, k \in V(G)$  and  $u, v, w \in V(H)$ . Then each robot and obstacle moves in a minimum sequence of moves that takes  $C_{u^i}^{v^i}$  to  $C_{u^j}^{w^j}$  in  $G \circ H$  is a *G-move*. Also such a minimum sequence involves exactly a number of *G-moves* of the robot and  $3a$  moves in total, where  $a = d(i, j) \geq 1$ .

*Proof* Since  $\{u, v\}, \{v, w\} \notin E(H)$  it means  $d_H(u^i, v^i) = d_H(v^i, w^i) = 0$ . The first part of this lemma follows. Now, let  $T$  be a sequence of moves that takes  $C_{u^i}^{v^i}$  to  $C_{u^j}^{w^j}$  in  $G \circ H$ . First, let  $z$  be the number of robot moves in  $T$ . By Proposition 2.6, we need at least two moves to accomplish the first move of the robot. Observe that each remaining  $z - 1$  robot moves in  $T$  is a secondary *G-move*. So by Remark 2.10, we need minimum of  $3(z - 1)$  moves to accomplish the  $z - 1$  secondary *G-moves* (since the first robot move places the hole at a preceding copy of the robot). Now, if  $w^j \xleftarrow{r} w^a$  is the  $z^{th}$  robot move in  $T$ , it will leave the graph  $G \circ H$  with the configuration  $C_{w^a}^{w^j}$ . Since  $d_{G \circ H}(w^a, u^j) = 1$ , so we need minimum of one more move to take the hole from  $w^a$  to  $u^j$ . Hence  $T$  involves minimum  $3z$  moves. Notice that, the expression  $3z$  takes the minimum value when  $z$  is minimum.

Next, let  $d(i, j) = a$  and  $[i = i_0, i_1, i_2, \dots, i_a = j]$  be a path of length  $a$  connecting  $i$  and  $j$  in  $G$ . Then  $[v^i = v^{i_0}, v^{i_1}, v^{i_2}, \dots, v^{i_a} = w^j]$  is a path of length  $a$  in  $G \circ H$  joining  $v^i$  to  $w^j$ . So the sequence of moves  $u^i \xleftarrow{o^*} v^{i_1} \xleftarrow{r} v^{i_0} \xleftarrow{o^*} v^{i_2} \xleftarrow{r} v^{i_1} \xleftarrow{o^*} v^{i_3} \xleftarrow{r} v^{i_2} \xleftarrow{o^*} v^{i_4} \dots v^{i_{a-2}} \xleftarrow{o^*} v^{i_{a-1}} \xleftarrow{o^*} u^j$  takes the robot from  $v^i$  to  $w^j$  along this path. Also it involves exactly  $a$  number of *G-moves* of the robot. Furthermore, from the given sequence above obstacle moves involves exactly  $2 + 2(a - 1)$  moves. Therefore a minimum sequence of moves  $T$  that takes the configuration  $C_{u^i}^{v^i}$  to  $C_{u^j}^{w^j}$  involves exactly  $3a$  moves.  $\square$

**Theorem 2.12** *Let  $G$  be a graph and  $H$  a complement of a complete graph on  $n$  vertices. Let  $\{i, j\} \in E(G)$  and  $u, v, w \in V(H)$ . Then each robot move in a minimum sequence of moves that takes  $C_{u^i}^{v^i}$  to  $C_{w^j}^{w^i}$  in  $G \circ H$  involves exactly 5 moves.*

*Proof* Combining Proposition 2.6 and Corollary 2.7 gives the result.  $\square$

The above Lemma gives the minimum number of moves required to take the robot from a given factor to itself and to another factor  $G \circ H$ . The proof of this lemma is immediate from Theorems 2.11 and 2.12.

**Lemma 2.13** *Consider the graph  $G \circ H$ . Let  $u, v \in V(H)$  with the initial configuration  $C_{u^i}^{v^i}$ , where  $G$  is any graph and  $H$  a complement of a complete graph on  $n$  vertices. Then*

- (i) *to move the robot from  $H^i$  to  $H^i$  we require at least 5  $G$ -moves;*
- (ii) *to move the robot from  $H^i$  to  $H^j$  we require at least  $1 + 2(a - 1) + a$   $G$ -moves. Where  $a = d(i, j) \geq 1$ .*

**Corollary 2.14** *Let  $G$  be a path and  $H$  a complement of a complete graph on  $n$  vertices. Let  $i, j, k \in V(G)$  and  $u, v, w \in V(H)$ . Then each robot and obstacle moves in a minimum sequence of moves that takes  $C_{u^i}^{v^i}$  to  $C_{w^j}^{w^i}$  in  $G \circ H$  is a  $G$ -move. Also such a minimum sequence involves exactly  $3a$  moves in total, where  $a = d(i, j) \geq 1$ .*

*Proof* The proof of this corollary is immediate from Lemma 2.13.  $\square$

**Corollary 2.15** *Let  $G$  be a path and  $H$  a complement of a complete graph on  $n$  vertices. Let  $i, j \in V(G)$  and  $u, v, w \in V(H)$ . Then each robot move in a minimum sequence of moves that takes  $C_{u^i}^{v^i}$  to  $C_{w^j}^{w^i}$  in  $G \circ H$  involves exactly 5 moves.*

*proof* The proof of this corollary is immediate from Lemma 2.13.  $\square$

Note that for such a product (Corollaries 2.14 and 2.15) there is no shortest path. Next, we consider the case when graph  $G$  is a complete graph. We would do this by stating the lemma below without proof.

**Lemma 2.16** *Let  $G$  be complete and  $H$  a complement of a complete graph on  $n$  vertices. Let  $i, j \in V(G)$  and  $u, v, w \in V(H)$ . Then starting from configuration  $C_{u^i}^{v^i}$ ,*

- (i) *we require at least 5 moves to move the robot to  $w^i$ ;*
- (ii) *we require a minimum of 2 moves to move the robot to  $w^k$ .*

### §3. Robot Moves in $K_n - I \circ H$

In this section we investigate a case where graph  $G$  is a complete graph minus a 1-factor and  $H$  a complete graph or it's complement.

**Lemma 3.1** *Let  $G$  be a complete graph minus a 1-factor and  $H$  a complete graph (or it's complement). Let  $i, j \in V(G)$  but  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . Then each robot move in a*

minimum sequence of moves that takes  $C_{v^i}^{u^i}$  to  $C_{v^j}^{u^j}$  is a  $G$ -move. Also such a minimum sequence involves exactly 6 moves.

*Proof* let  $T$  be a sequence of moves that takes  $C_{v^i}^{u^i}$  to  $C_{v^j}^{u^j}$ . By Proposition 2.6, two moves is required to move the robot to an adjacent vertex. Next by Corollary 2.7, four additional moves is required to take the robot and hole to their required destination, while the remaining move is the last move of the hole. So, the sequence  $T$  of moves  $v^i \xleftarrow{o^*} u^k \xleftarrow{r} u^i \xleftarrow{o^*} v^k \xleftarrow{o^*} u^j \xleftarrow{r} u^k \xleftarrow{o^*} v^j$  takes the robot and the hole to the required destination, and each move in this sequence is a  $G$ -move. Also  $T$  involves exactly six moves.  $\square$

**Lemma 3.2** *Let  $G$  be a complete graph minus a 1-factor and  $H$  a complete graph (or it's complement). Let  $i, j \in V(G)$  but  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . Then each robot move in a minimum sequence of moves that takes  $C_{v^j}^{u^i}$  to  $C_{v^i}^{u^j}$  is a  $G$ -move. Also such a minimum sequence involves exactly 6 moves.*

*Proof* The proof can be drawn in the same line as that of Lemma 3.1.  $\square$

**Lemma 3.3** *Let  $G$  be a complete graph minus a 1-factor and  $H$  a complement of a complete graph with  $n$  vertices. Let  $i, j \in V(G)$  but  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . Then each robot move in a minimum sequence of moves that takes  $C_{v^j}^{u^i}$  to  $C_{v^i}^{u^j}$  is a  $G$ -move. Also such a minimum sequence involves exactly 6 moves.*

*Proof* Let  $T$  be a sequence of moves that takes  $C_{v^j}^{u^i}$  to  $C_{v^i}^{u^j}$ . By Proposition 2.6, we need at least 2 moves to accomplish the first  $G$ -move of the robot. Notice that the last move of the robot is also a  $G$ -move. Now if  $v^i \xleftarrow{r} v^m$  is the last move of the robot, it will leave the graph  $G \circ H$  with the configuration  $C_{v^m}^{u^i}$  and this would require 3 moves. Since  $d_{G \circ H}(v^m, v^j) = 1$  so we need minimum of one more move to take the hole from  $v^m$  to  $v^j$ . Therefore  $T$  involves exactly 6 moves.  $\square$

**Lemma 3.4** *Let  $G$  be a complete graph minus a 1-factor and  $H$  a complete graph. Let  $i, j \in V(G)$  but  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . Then starting from the configuration  $C_{u^j}^{u^i}$  of  $G \circ H$ , we require at least 3 moves to move the robot to  $v^i$ .*

*Proof* For the robot to move to  $v^i$  before it, the hole must be moved from  $u^j$  to  $v^i$ . This takes 2 moves since  $d_{G \circ H}(u^j, v^i) = 2$ . Then the move  $v^i \xleftarrow{r} u^i$  takes the robot from  $u^i$  to  $v^i$ . Hence the result follows  $\square$

In view of Lemmas 3.1, 3.2 and 3.3 we have the following theorem.

**Theorem 3.5** *Consider the graph  $G \circ H$ . Let  $G$  be a complete graph minus a 1-factor and  $H$  a graph. Let  $i, j \in V(G)$  where  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . According as the hole is either at the  $i^{\text{th}}$  or  $j^{\text{th}}$  copy of  $G \circ H$ . Then to move the robot from*

- (i)  $H^i$  to  $H^j$  we require 5 moves if  $H$  is either  $K_n$  or  $\overline{K_n}$ ;
- (ii)  $H^i$  to  $H^i$  we require 5 moves if  $H$  is  $\overline{K_n}$ .

Finally, we have the corollary below which is as a result of Lemma 3.4.

**Corollary 3.6** *Consider the graph  $G \circ H$ . Let  $G$  be a complete graph minus a 1-factor and  $H$  a complete graph. Let  $i, j \in V(G)$  but  $\{i, j\} \notin E(G)$  and  $u, v \in V(H)$ . Then to move the robot from  $H^i$  to  $H^j$  we require 3 moves according as the hole is either at the  $j^{\text{th}}$  or  $i^{\text{th}}$  copy of  $G \circ H$  respectively.*

## References

- [1] V. Auletta, A. Monti, M. Parente, and P. Persiano, A linear time algorithm for the feasibility of pebble motions on trees, *Proceedings of Scandinavian Workshop on Algorithm Theory, (SWAT) Lecture Notes in Computer Science*, 1097, pages 259-270, 1996.
- [2] V. Auletta and P. Persiano, Optimal pebble motion on a tree, *Information and Computation*, 165:42-68, February 2001.
- [3] B. Deb, K. Kapoor, Motion planning in Cartesian product graphs, *Disc. Math. Graph Theory*, 34 (2014) 207-221.
- [4] B. Deb, K. Kapoor, *On Motion Planning in Graphs*, Ph.D thesis, Indian Institute of Technology, Guwahati, 2012.
- [5] M. Ellips and H. Azadeh, Solvability of multi robot motion planning problems on trees, *Intelligent Robots and Systems*, 2009. IROS 2009. IEEE/RSJ International Conference on, pages 5936-5941, Oct. 2009.
- [6] R. Hammack, W. Imrich, and S. Klavar, Product graphs, *Structure and Recognition*, John Wiley and Sons. Incorporated, New York. (2000).
- [7] D. Kornhauser, G. Miller, and P. Spirakis. Coordinating pebble motion on graphs, the diameter of permutation groups, and applications, *Foundations of Computer Science*, Annual IEEE Symposium on, 0:241-250, 1984.
- [8] I. Parberry, A real-time algorithm for the  $(n^2 - 1)$ -puzzle, *Inform. Process. Lett.*, 56, 23-28, 1995.
- [9] C. H. Papadimitriou, P. Raghavan, M. Sudan and H. Tamaki, Motion planning on a graph, in: *FOCS'94*, 511-520, 1994.
- [10] D. Ratner and M. K. Warmuth, The  $(n^2 - 1)$ -puzzle and related relocation problems, *J. Symbolic Comput.*, 10(1990) 111-137.
- [11] R. M. Wilson, Graph puzzles, homotopy and the alternating group, *J. Combin. Theory (B)*, 16, 86-96, 1974.

*To know what people really think, pay regard to what they do rather than what they say.*

By Rene Descartes, a French philosopher.

## Author Information

**Submission:** Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration both in **International Journal of Mathematical Combinatorics** and **Mathematical Combinatorics (International Book Series)**. An effort is made to publish a paper duly recommended by a referee within a period of 3 – 4 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

**Abstract:** Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

## Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

## Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

**Figures:** Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

**Copyright:** It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

**Proofs:** One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.





March 2018

## Contents

### Some Fixed Point Results for Contractive Type Conditions

in Cone  $b$ -Metric Spaces and Applications G.S.Saluja ..... 01

### A Note on Slant and Hemislant Submanifolds of an $(\epsilon)$ -Para Sasakian Manifold

Barnali Laha and Arindam Bhattacharyya ..... 18

A Study on Set-Valuations of Signed Graphs P.K.Ashraf, K.A.Germina and N.K.Sudev 34

### Extended Quasi Conformal Curvature Tensor on $N(k)$ -Contact

Metric Manifold Venkatesha, H.Aruna Kumara and R.T.Naveen Kumar ..... 41

### The Sturm-Liouville Fuzzy Problem According to the Boundary Conditions

Hülya GÜLTEKİN ÇİTİL and Nihat ALTINIŞIK ..... 51

### Nonholonomic Frames for Finsler Space with Deformed Special $(\alpha, \beta)$ Metric

Brijesh Kumar Tripathi ..... 61

### Homology of a Type of Octahedron

Shouvik Datta Choudhury, Santu Dey and Arindam Bhattacharyya ..... 68

### Even Modular Edge Irregularity Strength of Graphs

K.Muthugurupackiam and S.Ramya ..... 75

### Finslerian Hypersurfaces and Quartic Change of Finsler Metric

Gauree Shanker and Ramdayal Singh Kushwaha ..... 83

### Adjacency Matrices of Some Directional Paths and Stars

Rajesh Kumar T.J and Mathew Varkey T.K ..... 90

### Minimum Equitable Dominating Randić Energy of a Graph

Rajendra P. and R.Rangarajan ..... 97

### Linear Cyclic Snakes as Super Vertex Mean Graphs

A.Lourdusamy and Sherry George ..... 109

Equal Degree Graphs of Simple Graphs T.Chalapathi and R.V M S S Kiran Kumar .... 127

4-Remainder Cordial Labeling of Some Graphs R.Ponraj, K.Annathurai and R.Kala ... 138

### Motion Planning in Certain Lexicographic Product Graphs

A. D Akwu and O. Oyewumi ..... 146

ISBN 978-1-59973-558-0



Mathematical Combinatorics (Book Series)