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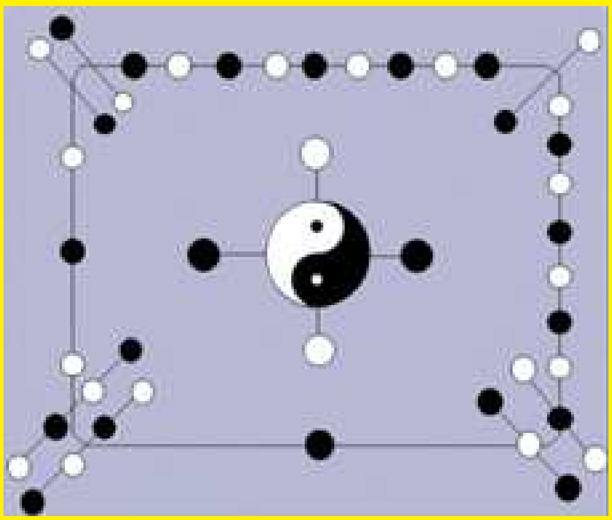
Edited By Linfan MAO



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Special Smarandache Curves According to Bishop Frame in Euclidean Spacetime

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Abstract: In this paper, we introduce some special Smarandache curves according to Bishop frame in Euclidean 3-space E^3. Also, we study Frenet-Serret invariants of a special case in E^3. Finally, we give an example to illustrate these curves.

Key Words: Smarandache curve, Bishop frame, Euclidean spacetime.

AMS(2010): 53A04, 53A05.

§1. Introduction

In the theory of curves in the Euclidean and Minkowski spaces, one of the interesting problems is the characterization of a regular curve. In the solution of the problem, the curvature functions κ and τ of a regular curve have an effective role. It is known that the shape and size of a regular curve can be determined by using its curvatures κ and τ ([7],[8]). For instance, Bertrand curves and Mannheim curves arise from this relationship. Another example is the Smarandache curves. They are the objects of Smarandache geometry, that is, a geometry which has at least one Smarandachely denied axiom [1]. The axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes.

By definition, if the position vector of a curve β is composed by the Frenet frame's vectors of another curve α, then the curve β is called a Smarandache curve [9]. Special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors ([6], [10]). For instance, the special Smarandache curves according to Darboux frame in E^3 are characterized in [5].

In this work, we study special Smarandache curves according to Bishop frame in the Euclidean 3-space E^3. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

[1]Received August 23, 2016, Accepted February 2, 2017.

§2. **Preliminaries**

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $v \in E^3$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. A curve α is called an unit speed curve if velocity vector α' of satisfies $\|\alpha'\| = 1$. For vectors $u, v \in E^3$ it is said to be orthogonal if and only if $\langle u, v\rangle = 0$. Let $\varrho = \varrho(s)$ be a regular curve in E^3. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve α in the space E^3. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulas is given by ([7]).

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1$, $\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$. Then, we write Frenet invariants in this way: $T(s) = \alpha'(s)$, $\kappa(s) = \|T'(s)\|$, $N(s) = T'(s)/\kappa(s)$, $B(s) = T(s) \times N(s)$ and $\tau(s) = -\langle N(s), B'(s)\rangle$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express Bishop of an orthonormal frame along a curve simply by parallel transporting each component of the frame [2]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [3]). The Bishop frame is expressed as ([2], [4]).

$$\begin{pmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \tag{2}$$

Here, we shall call the set $\{T, N_1, N_2\}$ as Bishop trihedra and $k_1(s)$ and $k_2(s)$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta(s) & -\sin\vartheta(s) \\ 0 & \sin\vartheta(s) & \cos\vartheta(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{3}$$

where

$$\begin{cases} \vartheta(s) = arc\tan\left(\dfrac{k_2}{k_1}\right), \ k_1 \neq 0 \\ \tau(s) = -\dfrac{d\vartheta(s)}{ds} \\ \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)} \end{cases} \tag{4}$$

Here, Bishop curvatures are defined by

$$\begin{cases} k_1(s) = \kappa(s)\cos\vartheta(s), \\ k_2(s) = \kappa(s)\sin\vartheta(s). \end{cases} \tag{5}$$

Let $\alpha = \alpha(s)$ be a regular non-null curve parametrized by arc-length in Euclidean 3-space E^3 with its Bishop frame $\{T, N_1, N_2\}$. Then TN_1, TN_2, N_1N_2 and TN_1N_2-Smarandache curve of α are defined, respectively as follows ([9]):

$$\begin{aligned} \mathcal{B} &= \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\Big(T(s) + N_1(s)\Big), \\ \mathcal{B} &= \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\Big(T(s) + N_2(s)\Big), \\ \mathcal{B} &= \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\Big(N_1(s) + N_2(s)\Big), \\ \mathcal{B} &= \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{3}}\Big(T(s) + N_1(s) + N_2(s)\Big). \end{aligned}$$

§3. Special Smarandache Curves According to Bishop Frame in E^3

Definition 3.1 *A regular curve in Euclidean space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

In the light of the above definition, we adapt it to regular curves according to Bishop frame in the Euclidean 3-space E^3 as follows.

Definition 3.2 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_1-Smarandache curves are defined by*

$$\mathcal{B} = \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\left(T(s) + N_1(s)\right). \tag{6}$$

Let us investigate Frenet invariants of TN_1-Smarandache curves according to $\alpha = \alpha(s)$. By differentiating Eqn.(6) with respected to s and using Eqn.(2), we get

$$\mathcal{B}' = \frac{d\mathcal{B}}{d\wp}\frac{d\wp}{ds} = \frac{1}{\sqrt{2}}\Big(-k_1T + k_1N_1 + k_2N_2\Big), \tag{7}$$

and hence

$$T_{\mathcal{B}} = \frac{-k_1 T + k_1 N_1 + k_2 N_2}{\sqrt{2k_1^2 + k_2^2}}, \tag{8}$$

where

$$\frac{d\wp}{ds} = \sqrt{\frac{2k_1^2 + k_2^2}{2}}. \tag{9}$$

In order to determine the first curvature and the principal normal of the curve $\mathcal{B}$, we formalize

$$T'_{\mathcal{B}} = \frac{dT_{\mathcal{B}}}{d\wp}\frac{d\wp}{ds} = \dot{T}_{\mathcal{B}}\frac{d\wp}{ds} = \frac{\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2}{\left(2k_1^2 + k_2^2\right)^{\frac{3}{2}}}, \tag{10}$$

where

$$\begin{cases} \zeta_1 = \left[k_1(2k_1k_1' + k_2k_2') - (2k_1^2 + k_2^2)(k_1' + k_1^2 + k_2^2)\right], \\ \zeta_2 = \left[(2k_1^2 + k_2^2)(k_1' - k_1^2) - k_1(2k_1k_1' + k_2k_2')\right], \\ \zeta_3 = \left[(2k_1^2 + k_2^2)(k_2' - k_1k_2) - k_2(2k_1k_1' + k_2k_2')\right]. \end{cases} \tag{11}$$

Then, we have

$$\dot{T}_{\mathcal{B}} = \frac{\sqrt{2}}{\left(2k_1^2 + k_2^2\right)^2}\Big(\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2\Big). \tag{12}$$

So, the first curvature and the principal normal vector field are respectively given by

$$\kappa_{\mathcal{B}} = \|\dot{T}_{\mathcal{B}}\| = \frac{\sqrt{2}\sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}}{\left(2k_1^2 + k_2^2\right)^2}. \tag{13}$$

and

$$N_{\mathcal{B}} = \frac{\zeta_1 T + \zeta_2 N_1 + \zeta_3 N_2}{\sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}}. \tag{14}$$

On other hand, we express

$$T_{\mathcal{B}} \times N_{\mathcal{B}} = \frac{1}{p\,q}\begin{vmatrix} T & N_1 & N_2 \\ -k_1 & k_1 & k_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix}, \tag{15}$$

where $p = \sqrt{2k_1^2 + k_2^2}$ and $q = \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}$. So, the binormal vector is

$$B_{\mathcal{B}} = \frac{1}{p\,q}\Big\{\left[k_1\zeta_3 - k_2\zeta_2\right]T + \left[k_1\zeta_3 + k_2\zeta_1\right]N_1 + k_1\left[\zeta_1 + \zeta_2\right]N_2\Big\}. \tag{16}$$

In order to calculate the torsion of the curve $\mathcal{B}$, we differentiate Eqn.(7) with respected to s, we have

$$\mathcal{B}'' = \frac{1}{\sqrt{2}}\Big\{ - \left[k_1' + k_1^2 + k_1k_2 + \right]T + \left[k_1' - k_1^2\zeta_1\right]N_1 + \left[k_2' - k_1k_2\right]N_2\Big\}. \tag{17}$$

and thus

$$\mathcal{B}''' = \frac{\nu_1 T + \nu_2 N_1 + \nu_2 N_2}{\sqrt{2}}, \tag{18}$$

where

$$\begin{cases} \nu_1 = -\big[k_1'' + k_1'(3k_1 + k_2) + k_2'(k_1 + k_2) - k_1(k_1^2 + k_2^2)\big], \\ \nu_2 = k_1'' - k_1(k_1^2 + 3k_1' + k_1k_2), \\ \nu_3 = k_2'' - k_1k_2' - k_2(k_1^2 + 2k_1' + k_1k_2). \end{cases} \tag{19}$$

The torsion is then given by:

$$\tau_{\mathcal{B}} = \frac{\sqrt{2}\big[(k_1^2 - k_1')(k_1\nu_3 + k_2\nu_1) + k_1(k_2' - k_1k_2)(\nu_1 + \nu_2) + (k_1^2 + k_1' + k_1k_2)(k_1\nu_3 - k_2\nu_2)\big]}{(k_1k_2' - k_1'k_2)^2 + \big[k_1k_2' + k_2(k_1' + k_1k_2)\big]^2 + k_1^2(2k_1^2 + k_1k_2)^2}. \tag{20}$$

Corollary 3.1 *Let $\alpha = \alpha(s)$ be a curve lying fully in E^3 with the moving frame $\{T, N, B\}$. If α is contained in a plane, then the Bishop curvatures becomes constant and the TN_1-Smarandache curve is also contained in a plane and its curvature satisfying the following equation*

$$\kappa_{\mathcal{B}} = \frac{\sqrt{2\big[k_1^2(k_2^2 + 1) + (k_1^2 + k_2^2)^2\big]}}{2k_1^2 + k_2^2}.$$

.

Definition 3.3 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_2-Smarandache curves are defined by*

$$\mathcal{B} = \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\Big(T(s) + N_2(s)\Big). \tag{21}$$

Remark 3.1 The Frenet invariants of TN_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Definition 3.4 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. N_1N_2-Smarandache curves are defined by*

$$\mathcal{B} = \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{2}}\Big(N_1(s) + N_2(s)\Big). \tag{22}$$

Remark 3.2 The Frenet invariants of N_1N_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Definition 3.5 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N_1, N_2\}$ be its moving Bishop frame. TN_1N_2-Smarandache curves are defined by*

$$\mathcal{B} = \mathcal{B}(\wp(s)) = \frac{1}{\sqrt{3}}\Big(T(s) + N_1(s) + N_2(s)\Big). \tag{23}$$

Remark 3.3 The Frenet invariants of TN_1N_2-Smarandache curves can be easily obtained by the apparatus of the regular curve $\alpha = \alpha(s)$.

Example 3.1 Let $\alpha(s) = \frac{1}{\sqrt{2}}\big(-\cos s, -\sin s, s\big)$ be a curve parametrized by arc length. Then it is easy to show that $T(s) = \frac{1}{\sqrt{2}}\big(\sin s, -\cos s, 1\big)$, $\kappa = \frac{1}{\sqrt{2}} \neq 0$, tanh $= -\frac{1}{\sqrt{2}} \neq 0$ and $\vartheta(s) = \frac{1}{\sqrt{2}}s + c$, $c = constant$. Here, we can take $c = 0$. From Eqn.(4), we get $k_1(s) = \frac{1}{\sqrt{2}}\cos\big(\frac{s}{\sqrt{2}}\big)$,

$k_2(s) = \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)$. From Eqn.(1), we get $N_1(s) = \int k_1(s)T(s)ds$, $N_2(s) = \int k_2(s)T(s)ds$, then we have

$$\begin{aligned} N_1(s) &= \Bigg(\frac{\sqrt{2}}{4(1+\sqrt{2})}\cos\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\cos\Big((1-\sqrt{2})s\Big), \\ &\quad -\frac{\sqrt{2}}{4(1+\sqrt{2})}\sin\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\sin\Big((1-\sqrt{2})s\Big), \frac{\sqrt{2}}{2}\sin\left(\frac{s}{\sqrt{2}}\right)\Bigg) \\ N_2(s) &= \Bigg(\frac{\sqrt{2}}{4(1+\sqrt{2})}\sin\Big((1+\sqrt{2})s\Big) - \frac{\sqrt{2}}{4(1-\sqrt{2})}\sin\Big((1-\sqrt{2})s\Big), \\ &\quad \frac{\sqrt{2}}{4(1+\sqrt{2})}\cos\Big((1+\sqrt{2})s\Big) + \frac{\sqrt{2}}{4(1-\sqrt{2})}\cos\Big((1-\sqrt{2})s\Big), \frac{\sqrt{2}}{2}\cos\left(\frac{s}{\sqrt{2}}\right)\Bigg). \end{aligned}$$

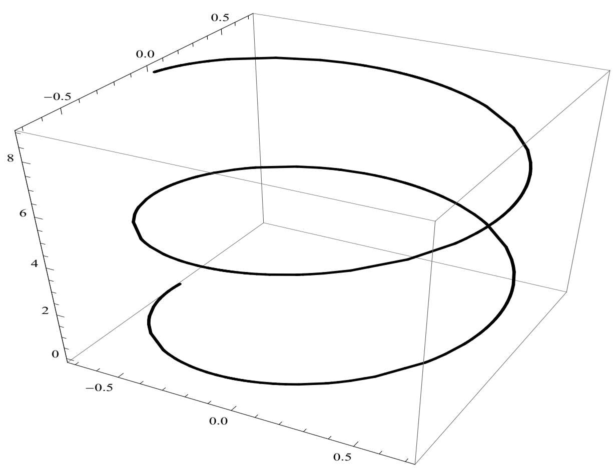

Figure 1 The curve $\alpha = \alpha(s)$.

In terms of definitions, we obtain special Smarandache curves, see Figures 2 - 5.

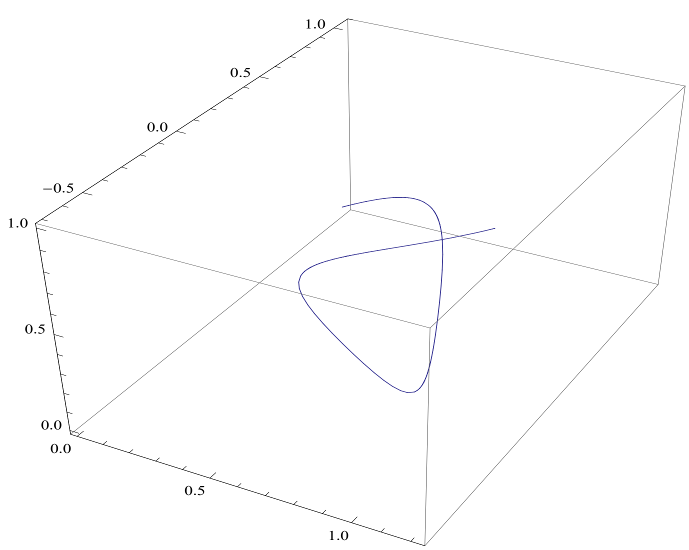

Figure 2 TN_1-Smarandache curve.

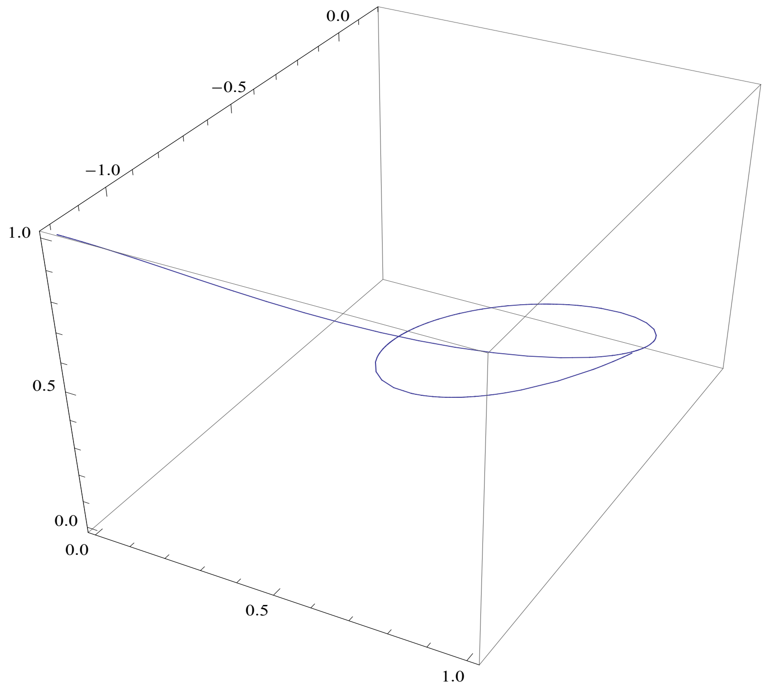

Figure 3 TN_2-Smarandache curve.

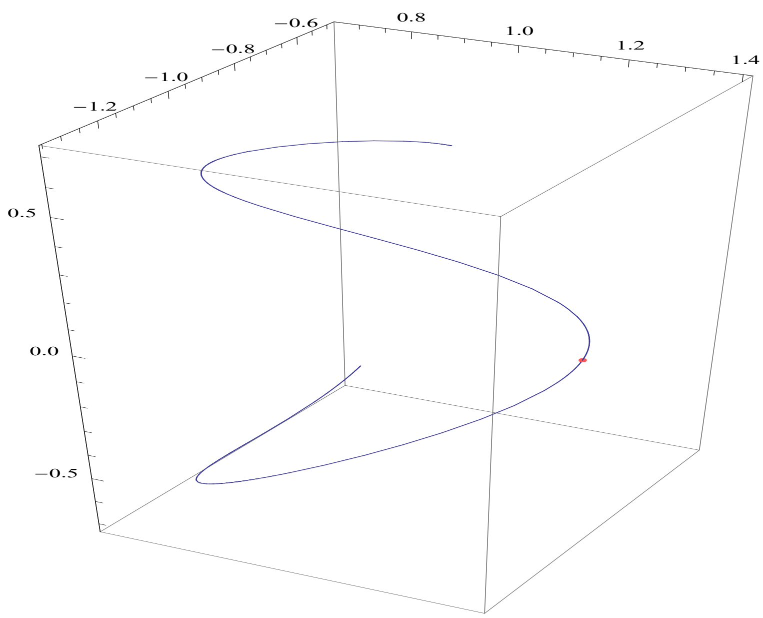

Figure 4 N_1N_2-Smarandache curve.

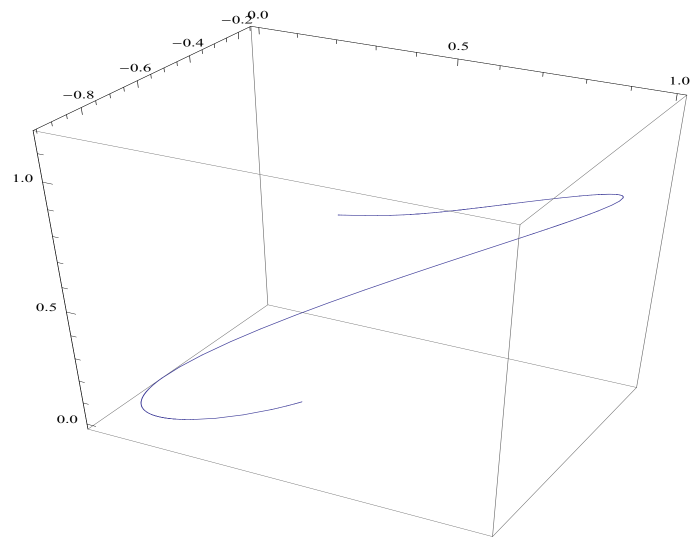

Figure 5 TN_1N_2-Smarandache curve.

§4. Conclusion

Consider a curve $\alpha = \alpha(s)$ parametrized by arc-length in Euclidean 3-space E^3 that the curve $\alpha(s)$ is sufficiently smooth so that the Bishop frame adapted to it is defined. In this paper, we study the problem of constructing Frenet-Serret invariants $\{T_{\mathcal{B}}, N_{\mathcal{B}}, B_{\mathcal{B}}, \kappa_{\mathcal{B}}, \tau_{\mathcal{B}}\}$ from a given some special curve $\mathcal{B}$ according to Bishop frame in Euclidean 3-space E^3 that posses this curve as Smarandache curve. We list an example to illustrate the discussed curves. Finally, we hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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Spectra and Energy of Signed Graphs

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Abstract: The energy of a signed graph Σ is defined as $\varepsilon(\Sigma) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of Σ. In this paper, we study the spectra and energy of a class of signed graphs which satisfy pairing property. We show that it is possible to compare the energies of a pair of bipartite and non-bipartite signed graphs on n vertices by defining quasi-order relation in such a way that the energy is increasing. Further, we extend the notion of extended double cover of graphs to signed graphs to find the spectra of unbalanced signed bipartite graphs and also we construct non-cospectral equienergetic signed bipartite graphs.

Key Words: Signed graph, Smarandachely signed graph, signed energy, extended double cover(EDC) of signed graphs, equienergetic signed bipartite graphs.

AMS(2010): 05C22, 05C50.

§1. Introduction

A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where G is the underlying graph of Σ and $\sigma : E \rightarrow \{+1, -1\}$, called signing (or a signature), is a function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. It is said to be homogeneous if its edges are all positive or negative otherwise heterogeneous, and a Smarandachely signed if $|e_+ - e_-| \geq 1$, where e_+, e_- are numbers of edges signed by $+1$ or -1 in $E(G)$, respectively. Negation of a signed graph is the same graph with all signs reversed. In figure, we denote positive edges with solid lines and negative edges with dotted lines.

The adjacency matrix of a signed graph is the square matrix $A(\Sigma) = (a_{ij})$ where (i, j) entry is $+1$ if $\sigma(v_iv_j) = +1$ and -1 if $\sigma(v_iv_j) = -1$, 0 otherwise. The characteristic polynomial of the signed graph Σ is defined as $\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma))$, where I is an identity matrix of order n. The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$ are called the eigenvalues of signed graph Σ. If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and their multiplicities are $m_1, m_2, \cdots, m_n$ then the spectrum of Σ is

$$Spec(\Sigma) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & ... & \lambda_n \\ m_1 & m_2 & ... & ... & m_n \end{pmatrix}.$$

[1]Received August 16, 2016, Accepted February 3, 2017.

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion for balance in signed graph is given by B. D. Acharya as follows:

Theorem 1.1([1]) *A signed graph is balanced if and only if it is cospectral with the underlying graph. i.e.* $Spec(\Sigma) = Spec(G)$.

The sign of a cycle in a signed graph is the product of the signs of its edges. Thus a cycle is positive if and only if it contains an even number of negative edges. A signed graph is said to be balanced if all of its cycles are positive otherwise unbalanced.

In a signed graph Σ, the degree of a vertex v is defined as $sdeg(v) = d(v) = d^+_\Sigma(v) + d^-_\Sigma(v)$, where $d^+_\Sigma(v)(d^-_\Sigma(v))$ is the number of positive(negative) edges incident with v. It is said to be regular if all its vertices have same degree. The net degree of a vertex v of a signed graph Σ is $d^\pm_\Sigma(v) = d^+_\Sigma(v) - d^-_\Sigma(v)$. It is said to be net-regular of degree k if all its vertices have same net-degree equal to k.

Spectra of graphs is well documented in [5] and signed graphs is discussed in [7, 8, 9, 11]. For standard terminology and notations in graph theory we follow D. B. West [15] and for signed graphs we follow T. Zaslavsky [16].

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of Σ, then $\varepsilon(\Sigma) = \sum_{i=1}^{n} |\lambda_i|$. Two signed graphs Σ_1 and Σ_2 are said to be equienergetic if $\varepsilon(\Sigma_1) = \varepsilon(\Sigma_2)$. Naturally, cospectral signed graphs are equienergetic. Equienergetic signed graphs are constructed in [3, 13].

The cartesian product $\Sigma_1 \times \Sigma_2$ of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is defined as the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the Cartesian product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i, u_k), & \text{if j = l} \\ \sigma_2(v_j, v_l), & \text{if i = k} \end{cases}$$

The Kronecker product of $\Sigma_1 \bigotimes \Sigma_2$ of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the Kronecker product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by $\sigma((u_i, v_j)(u_k, v_l)) = \sigma_1(u_i, u_k)\sigma_2(v_j, v_l)$.

Generally, quasi-order relation is used to compare the energies of bipartite graphs. In this paper, we use quasi-order method to compare the energies of two signed graphs of order n which are bipartite and unbalanced non-bipartite signed graphs. Fundamental question in the energy theory is to find the maximal and minimal energy graphs over a significant class of graphs. It is natural to find for signed graphs also. Here we give maximum energy signed graphs which belong to the class of pairing property. Further, we study the spectra and energy of extended double cover (EDC) of signed graphs and also construct non-cospectral equienergetic signed bipartite graphs.

§2. Energy of Signed Graphs in Δ_n

A graph G is a bipartite graph if and only if $\lambda_i = -\lambda_{n+1-i}$, for $1 \leq i \leq \frac{1}{2}(n-1)$. This result is known as *pairing theorem* by Coulson and Rushbrooke [6]. But non-bipartite signed graphs also satisfy pairing property and examples are given in [3]. The class of signed graphs satisfying pairing property we denote it as Δ_n.

The following result is given by Bhat and Pirzada in [3] which gives the spectral criterion of signed graphs on Δ_n.

Theorem 2.1 *Let Σ be a signed graph of order n which satisfies the pairing property. Then the following statements are equivalent:*

(1) *spectrum of Σ is symmetric about the origin;*

(2) $\Phi_\Sigma(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} \lambda^{n-2k}$, *where b_{2k} are non-negative integers for all $k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$;*

(3) *Σ and $-\Sigma$ are cospectral, where $-\Sigma$ is the signed graph obtained by negating sign of each edge of Σ.*

Now it is possible to define a quasi-order relation over Δ_n in such a way that the energy is increasing. Note that Δ_n consists of signed bipartite as well as unbalanced non-bipartite signed graphs which satisfy pairing property.

Definition 2.2 *Let Σ_1 and Σ_2 be two signed graphs of order n in Δ_n. From Theorem 2.1 we can express*

$$\Phi_\Sigma(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} \lambda^{n-2k}$$

where b_{2k} are non-negative integers for all $k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$. If $b_{2k}(\Sigma_1) \leq b_{2k}(\Sigma_2)$ for all k where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ then we can write $\Sigma_1 \leq \Sigma_2$. Further, if $b_{2k}(\Sigma_1) < b_{2k}(\Sigma_2)$ for all k where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ then we write $\Sigma_1 < \Sigma_2$. Hence

$$\Sigma_1 \preceq \Sigma_2 \Rightarrow \varepsilon(\Sigma_1) \leq \varepsilon(\Sigma_2),$$

$$\Sigma_1 \prec \Sigma_2 \Rightarrow \varepsilon(\Sigma_1) < \varepsilon(\Sigma_2),$$

which implies that the energy is increasing in a quasi order relation over Δ_n.

In [13], it is shown that Coulson's Integral formula remains valid for signed graphs also.

Theorem 2.3([13]) *If Σ is a signed graph then the energy of signed graph Σ is*

$$\varepsilon(\Sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{i\lambda \phi'(i\lambda)}{\phi(i\lambda)} \right] d\lambda.$$

Following result is the consequence of Coulson's Integral formula for signed graphs.

Corollary 2.4 *Let Σ be a signed graph. Then*

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[\lambda^k \phi_\Sigma(\frac{i}{\lambda})\right] d\lambda.$$

Theorem 2.5 *Let $\Sigma \in \Delta_n$. Then the energy of a signed graph can be expressed as*

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right] d\lambda.$$

and if $\Sigma_1, \Sigma_2 \in \Delta_n$ and $\Sigma_1 < \Sigma_2$ then $\varepsilon(\Sigma_1) < \varepsilon(\Sigma_2)$.

Proof Coulson's Integral formula can be expressed as

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[\lambda^k \phi_\Sigma(\frac{i}{\lambda})\right] d\lambda.$$

Since $\Sigma \in \Delta_n$, from Theorem 2.1 we can deduce

$$\phi_\Sigma\left(\frac{i}{\lambda}\right) = \left(\frac{i^n}{\lambda^n}\right)\left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right]$$

and substituting in the above expression, we get

$$\begin{aligned} \varepsilon(\Sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[i^n(1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k})\right] d\lambda \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right] d\lambda. \end{aligned}$$

But $\frac{1}{\pi}$ p.v. $\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln[i^n] d\lambda = 0$ where p.v. is the principal value of Cauchy's integral. Hence $\varepsilon(\Sigma)$ is a monotonically increasing function on the coefficients of $b_{2k}(\Sigma)$. □

Now the question is which signed graphs are having maximum signed energy in Δ_n.

Theorem 2.6([14]) *Let Σ be a signed graph with n vertices and m edges, then*

$$\sqrt{2m + n(n-1)|\det(A(\Sigma))|^{2/n}} \leq \varepsilon(\Sigma) \leq \sqrt{2mn}.$$

Corollary 2.7 $\varepsilon(\Sigma) = \sqrt{2mn} = n\sqrt{r}$ *if and only if* $\Sigma^T\Sigma = (\Sigma)^2 = rI_n$, *where r is the maximum degree of Σ and I_n is the identity matrix of order n.*

Proof Notice that $\varepsilon(\Sigma) = n\sqrt{r}$ if and only if there exists a constant t such that $|\lambda_i|^2 = t$ for all i and Σ is an r-regular signed graph. Hence equality holds if and only if $\Sigma^T\Sigma = (\Sigma)^2 = tI$ and $t = r$. □

The following two examples are given by the present author in [12, 14].

Example 2.8 Following unbalanced signed cycle, we denote it as (C_4^-).

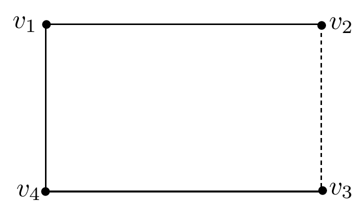

Fig.1 Signed cycle with maximum signed energy

$$A(C_4^-) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is $\phi(C_4^-) = \lambda^4 - 4\lambda^2 + 4$ and Spec (C_4^-)=$\{(\sqrt{2})^2, (-\sqrt{2})^2\}\in \Delta_n$. Hence $\varepsilon(C_4^-) = 4\sqrt{2} = n\sqrt{r}$.

Example 2.9 Following unbalanced signed complete graph, we denote it as (K_6^-).

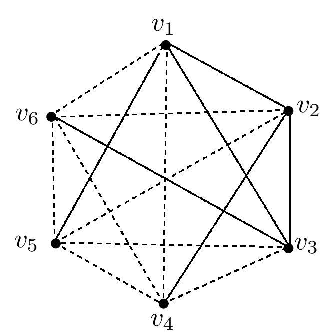

Fig.2 Signed Complete graph with maximum signed energy

$$A(K_6^-) = \begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & -1 & -1 & 0 \end{pmatrix}$$

which is a symmetric conference matrix having the characteristic polynomial $\phi(K_6^-) = \lambda^5 - 15\lambda^3 + 75\lambda - 125$ and Spec $A(K_6^-)$=$\{(\sqrt{5})^3, (-\sqrt{5})^3\}\in \Delta_n$. The signed energy of $\varepsilon(K_6^-) = 6\sqrt{5} = n\sqrt{r}$.

Lemma 2.10([3, 8]) *Let Σ_1 and Σ_2 be two signed graphs with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n_1}$ and $\mu_1, \mu_2, \cdots, \mu_{n_2}$. Then*

(1) *the eigenvalues of $\Sigma_1 \times \Sigma_2 = \lambda_i + \mu_j$, for all $i = 1, 2, \cdots, n_1, j = 1, 2, \cdots, n_2$;*

(2) *the eigenvalues of $\Sigma_1 \bigotimes \Sigma_2 = \lambda_i \mu_j$, for all $i = 1, 2, \cdots, n_1, j = 1, 2, \cdots, n_2$.*

Theorem 2.11 *There exists an infinite family of signed graphs having maximum signed energy in Δ_n.*

Proof Let Σ_1, Σ_2 be two signed graphs in Δ_n with orders n_1 and n_2 having maximum energies $n_1\sqrt{r_1}, n_2\sqrt{r_2}$ respectively. The Kronecker product of $\Sigma_1 \bigotimes \Sigma_2$ is a symmetric matrix of order $n_1 n_2$. From Lemma 2.10, $\Sigma_1 \bigotimes \Sigma_2$ has maximum energy $n_1 n_2 \sqrt{r_1 r_2}$. □

Here we note that maximum energy signed graphs belong to the class of Δ_n.

§3. Spectra of Signed Biparitite Graphs in Δ_n.

In [2], N. Alon introduced the concept of extended double cover of a graph. Here we extend this notion to signed graphs in order to establish the spectrum of various signed bipartite graphs. The ordinary spectrum of EDC of graph is given by Z. Chen in [4].

Lemma 3.1([4]) *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eignenvalues of the graph G. Then the eigenvalues of extended double cover of graph are $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \cdots, \pm(\lambda_n + 1)$.*

Now we define extended double cover of signed graph Σ as follows:

Definition 3.2 *Let Σ be a signed graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. Let Σ^* be a signed bipartite graph with $V(\Sigma^*) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ where,*

1) *v_i is adjacent to u_i and either $\sigma(v_i u_i) = +1$ or $\sigma(v_i u_i) = -1$;*

2) *v_i is adjacent to u_j if v_i is adjacent to v_j in Σ;*

3) *$\sigma(v_i u_j) = +1$ if $\sigma(v_i v_j) = +1$ and $\sigma(v_i u_j) = -1$ if $\sigma(v_i v_j) = -1$.*

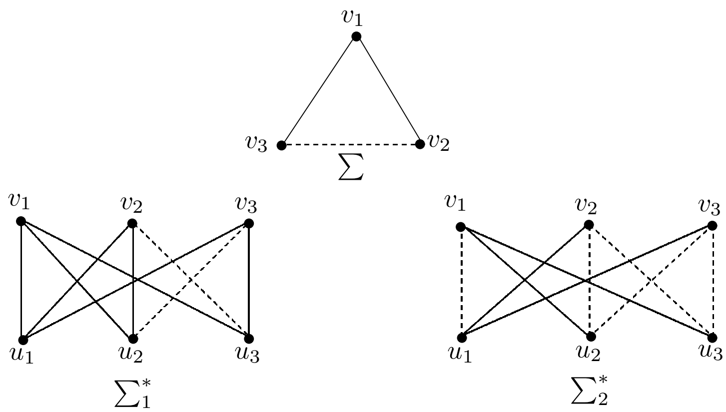

Fig.3 Extended double covers of signed graph Σ.

Then Σ^* is known as extended double cover of signed graph of signed graph Σ and in short we write it as EDC of Σ. Since we get two EDCs of signed graph, we denote it as Σ_1^* if $\sigma(v_iu_i) = +1$ and Σ_2^* if $\sigma(v_iu_i) = -1$.

We need the following Lemma from [10] for further investigation.

Lemma 3.3([10]) *Let* $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ *be a symmetric* 2×2 *block matrix. Then the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

The following Lemma gives the relation between the spectrum of a signed graph and its EDC of signed graph.

Lemma 3.4 *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the eigenvalues of a signed graph then the spectrum of EDCs of signed graph is*

(1) $Spec(\Sigma_1^*) = \left\{ \pm(\lambda_1 + 1), \quad \pm(\lambda_2 + 1), \quad \cdots \quad, \pm(\lambda_n + 1) \right\}$

(2) $Spec(\Sigma_2^*) = \left\{ \pm(\lambda_1 - 1), \quad \pm(\lambda_2 - 1), \quad \cdots \quad, \pm(\lambda_n - 1) \right\}$

Proof Let the adjacency matrix of the signed graph Σ be A. Then the adjacency matrix of EDC of signed graph of Σ is $\begin{pmatrix} 0 & A+I \\ A+I & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & A-I \\ A-I & 0 \end{pmatrix}$, where I is an identity matrix.

From Lemma 3.3, it is clear that the eigenvalues of Σ^* are $\pm(\lambda_i + 1)$ if $\sigma(v_iu_i) = +1$ and $\pm(\lambda_i - 1)$ if $\sigma(v_iu_i) = -1$ for each eigenvalue λ of Σ. □

Theorem 3.5 *Let* Σ *be a connected signed graph. Then* Σ_1^*, Σ_2^* *and* $(\Sigma \times K_2)$ *are co-spectral if and only if* Σ *belongs to the class of* Δ_n.

Proof Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of a signed graph Σ then

(*i*) $Spec(\Sigma_1^*) = \left\{ \pm(\lambda_1 + 1), \quad \pm(\lambda_2 + 1), \quad \cdots \quad, \pm(\lambda_n + 1) \right\}$;

(*ii*) $Spec(\Sigma_2^*) = \left\{ \pm(\lambda_1 - 1), \quad \pm(\lambda_2 - 1), \quad \cdots \quad, \pm(\lambda_n - 1) \right\}$;

(iii) $Spec(\Sigma \times K_2) = \left\{ (\lambda_1 \pm 1), \quad (\lambda_2 \pm 1), \quad ..... \quad , (\lambda_n \pm 1) \right\}$.

So, $Spec(\Sigma_1^*) = Spec(\Sigma_2^*) = Spec(\Sigma \times K_2)$ if and only if $\lambda_i = -\lambda_{n+1-i}$, for $i = 1, 2, \cdots, n$. Hence the proof. □

Now we give spectra of various signed bipartite graphs.

Proposition 3.6 *Let $(P_n)_1^*$ and $(P_n)_2^*$ be the extended double covers of signed path P_n. Then the spectrum is*

$$(1) \quad Spec(P_n)_1^* = \begin{pmatrix} \pm(2cos\frac{\pi i}{n+1} + 1) \\ n \end{pmatrix}, \ i = 1, \cdots, n.$$

$$(2) \quad Spec(P_n)_2^* = \begin{pmatrix} \pm(2cos\frac{\pi i}{n+1} - 1) \\ n \end{pmatrix}, \ i = 1, \cdots, n.$$

Remark 3.7 If Σ is a signed path then EDCs of signed paths are balanced. Hence EDCs of signed paths are having same energy as underlying graph.

Proposition 3.8 *Let C_n^+ (C_n^-) be the positive(negative) signed cycles on C_n. Then the spectrum of EDCs are respectively*

(1) *If n is odd, then $Spec(C_n^+)_1^* = \left[\pm(2cos\frac{2\pi i}{n} + 1), i = 1, 2, \cdots, n\right]$ and $Spec(C_n^+)_2^* = \left[\pm(2cos\frac{2\pi i}{n} - 1), i = 1, 2, \cdots, n\right]$;*

(2) *If n is even, then (i)$Spec(C_n^-)_1^* = \left[\pm(2cos\frac{(2i+1)\pi}{n} + 1), i = 1, 2, \cdots, n\right]$ and $Spec(C_n^-)_2^* = \left[\pm(2cos\frac{(2i+1)\pi}{n} - 1), i = 1, 2, \cdots, n\right]$.*

If the signed graph is $+K_n$ then EDCs of $+K_n$ are $(K_n)_1^* = +K_{n,n}$ and $(K_n)_2^*$. Spec$(K_{n,n})$ $= \{\pm n, 0^{2n-2}\}$. Following result gives the spectrum of $(K_n)_2^*$ which is an unbalanced net-regular signed complete bipartite graph.

Proposition 3.9 *Let $(K_n)_2^*$ be the EDC of $+K_n$. Then the spectrum of $(K_n)_2^*$ is*

$$Spec(K_n)_2^* = \begin{pmatrix} -2 & -k & k & 2 \\ n-1 & 1 & 1 & n-1 \end{pmatrix},$$

where $k = d^{\pm}(K_n)_2^* = n - 2$.

Remark 3.10 From above Proposition 3.9, $\varepsilon(K_n)_2^* = 2(3n - 8)$.

Theorem 3.11([13]) *The spectrum of heterogeneous unbalanced signed complete graph (K_n^{net}) is*

$$Spec(K_n^{net}) = \begin{pmatrix} 5-n & 1 + 4\cos(\frac{2\pi i}{n}) \\ 1 & 1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

where (K_n^{net}) is a net regular signed complete graph defined on $+K_n$.

Proposition 3.12 *If* $(K_n^{net})_1^*$ *and* $(K_n^{net})_2^*$ *are the net-regular signed complete bipartite graphs of EDCs of* K_n^{net}*. Then the spectrum is*

(1)

$$Spec(K_n^{net})_1^* = \begin{pmatrix} \pm k & \pm(2 + 4\cos(\frac{2\pi i}{n})) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, (n-1),$$

where $k = (6 - n)$ *gives net regularity of* $(K_n^{net})_1^*$.

(2)

$$Spec(K_n^{net})_2^* = \begin{pmatrix} \pm k & \pm(1 + 4\cos(\frac{2\pi i}{n})) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1,$$

where $k = (4 - n)$ *gives net regularity of* $(K_n^{net})_2^*$.

From the above Propositions, we are having the following result.

Theorem 3.13 *EDCs of signed graphs are net-regular if and only if signed graph* Σ *is net-regular.*

§4. Equienergetic Signed Graphs in Δ_n

Here we construct equienergetic signed bipartite graphs on $4n$ vertices which are non-cospectral and equienergetic.

Theorem 4.1 *There exists a pair of non-cospectral equienergetic signed bipartite graphs on* $4n$ *vertices where* n *is odd and* $n \geq 3$.

Proof Let Σ be a signed cycle of order n and of odd length with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and let the extended double covers of signed graph Σ be Σ_1^* and Σ_2^*.

Case 1. If Σ is balanced then

$$Spec(\Sigma) = \begin{pmatrix} 2 & \lambda_i \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

By Lemma 3.4,

$$Spec(\Sigma_1^*) = \begin{pmatrix} \pm 3 & \pm(\lambda_i + 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

and

$$Spec(\Sigma_2^*) = \begin{pmatrix} \pm 1 & \pm(\lambda_i - 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

Hence Σ_1^* and Σ_2^* are non-cospectral bipartite signed graphs on $2n$ vertices where n is odd and Σ_1^* is balanced and Σ_2^* is unbalanced.

Further, let H_1, H_2 and K_1, K_2 be second iterated extended double cover signed graphs of Σ_1^* and Σ_2^* respectively. By Theorem 3.5, Spec H_1 = Spec H_2 and Spec K_1= Spec K_2. Let Spec S = Spec H_1 = Spec H_2 and Spec T= Spec K_1 = Spec K_2.

$$Spec(S) = \begin{pmatrix} \pm(4) & \pm(2) & \pm(\pm(\lambda_i+1)+1) \\ 1 & 1 & 2(n-1) \end{pmatrix},\ i = 1, \cdots, n-1.$$

and

$$Spec(T) = \begin{pmatrix} \pm(2) & \pm(0) & \pm(\pm(\lambda_i-1)+1) \\ 1 & 1 & 2(n-1) \end{pmatrix},\ i = 1, \cdots, n-1.$$

Hence $S = (\Sigma_1^*)^*$ and $T = (\Sigma_2^*)^*$ are non-cospectral bipartite signed graphs on $4n$ vertices where n is odd.

$$\varepsilon(S) = 2[4 + 2 + \sum_{i=1}^{n-1} | \pm (\lambda_i + 1) + 1|],$$

$$\varepsilon(T) = 2[2 + 0 + \sum_{i=1}^{n-1} | \pm (\lambda_i - 1) + 1|].$$

If $\varepsilon(S) = \varepsilon(T)$ then $4 = \sum_{i=1}^{n-1}(| \pm(\lambda_i - 1) + 1 \mid - \mid \pm(\lambda_i + 1) + 1 |)$, then we know that

$$4 = \sum_{i=1}^{n-1}(|2 - \lambda_i| + |\lambda_i| - |\lambda_i + 2| - |\lambda_i|),$$

$$4 = \sum_{i=1}^{n-1}(|\lambda_i - 2| - |\lambda_i + 2|).$$

Since Σ is a balanced signed cycle $\lambda_i = 2cos\frac{2\pi i}{n},\ i = 1, \cdots, n-1$,

$$4 = \sum_{i=1}^{n-1}(|2cos\theta_i - 2| - |2cos\theta_i + 2|),$$

$$1 = \sum_{i=1}^{n-1}(sin^2(\frac{\theta_i}{2}) - cos^2(\frac{\theta_i}{2})),$$

$$-1 = \frac{1}{2}\sum_{i=1}^{n-1} 2cos\theta_i.$$

Since $\sum_{i=1}^{n-1} \lambda_i = -2$, so $\varepsilon(S) = \varepsilon(T)$.

Case 2. If Σ is unbalanced then

$$Spec(\Sigma) = \begin{pmatrix} -2 & \lambda_i \\ 1 & n-1 \end{pmatrix},\ i = 1, \cdots, n-1.$$

By Lemma 3.4,

$$Spec(\Sigma_1^*) = \begin{pmatrix} \pm 1 & \pm(\lambda_i + 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

and

$$Spec(\Sigma_2^*) = \begin{pmatrix} \pm 3 & \pm(\lambda_i - 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

By a similar argument as in Case 1, we get $\varepsilon(S) = \varepsilon(T)$. Hence the proof. □

Example 4.2 Consider the signed graphs Σ_1^* and Σ_2^* as shown in Fig.3. By Lemma 3.4, the characteristic polynomials of Σ_1^* and Σ_2^* are

$$\phi(\Sigma_1^*) = (\lambda+2)^2(\lambda-2)^2(\lambda+1)(\lambda-1)$$

$$\phi(\Sigma_2^*) = \lambda^4(\lambda+3)(\lambda-3)$$

The characteristic polynomials of $(\Sigma_1^*)^*$and $(\Sigma_2^*)^*$ are

$$\phi(\Sigma_1^*)^* = \lambda^2(\lambda+1)^2(\lambda-1)^2(\lambda+3)^2(\lambda-3)^2(\lambda+2)(\lambda-2),$$

$$\phi(\Sigma_2^*)^* = (\lambda+1)^4(\lambda-1)^4(\lambda+4)(\lambda-4)(\lambda+2)(\lambda-2).$$

Hence $Spec(\Sigma_1^*)^* \neq Spec(\Sigma_2^*)^*$ but $\varepsilon(\Sigma_1^*)^* = \varepsilon(\Sigma_2^*)^* = 20$.

Another example of equienergetic signed bipartite graphs on $4n$ vertices is given below.

Example 4.3 Consider the signed graphs Σ_1^* and Σ_2^* as shown in Fig.3. By Lemma 2.10, the characteristic polynomials of $(\Sigma_1^* \times K_2)$ and $(\Sigma_2^* \times K_2)$ are

$$\phi(\Sigma_1^* \times K_2) = \lambda^2(\lambda+1)^2(\lambda-1)^2(\lambda+3)^2(\lambda-3)^2(\lambda+2)(\lambda-2),$$

$$\phi(\Sigma_2^* \times K_2) = (\lambda+1)^4(\lambda-1)^4(\lambda+4)(\lambda-4)(\lambda+2)(\lambda-2).$$

Hence $Spec(\Sigma_1^* \times K_2) \neq Spec(\Sigma_2^* \times K_2)$ but $\varepsilon(\Sigma_1^* \times K_2) = \varepsilon(\Sigma_2^* \times K_2) = 20$.

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On Transformation and Summation Formulas for Some Basic Hypergeometric Series

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Abstract: In this paper, we give an alternate and simple proofs for Sear's three term ${}_3\phi_2$ transformation formula, Jackson's ${}_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum by using q-exponential operator techniques. We also give an alternate proof for a nonterminating form of the q-Vandermonde sum. We also obtain some interesting special cases of all the three identities, some of which are analogous to the identities stated by Ramanujan in his lost notebook.

Key Words: Transformation formula, q-series, operator identity.

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§1. Introduction

In 1951 Sears [15] has established the following useful three term transformation formula for ${}_3\phi_2$ series.

Theorem 1.1

$$\sum_{n=0}^{\infty}\frac{(a,b,c)_n}{(q,e,f)_n}\left(\frac{ef}{abc}\right)^n=\frac{(b,e/a,f/a,ef/bc)_\infty}{(e,f,b/a,ef/abc)_\infty}\sum_{n=0}^{\infty}\frac{(a,e/b,f/b)_n}{(q,aq/b,ef/bc)_n}q^n$$
$$+\frac{(a,e/b,f/b,ef/ac)_\infty}{(e,f,a/b,ef/abc)_\infty}\sum_{n=0}^{\infty}\frac{(b,e/a,f/a)_n}{(q,bq/a,ef/ac)_n}q^n, \tag{1.1}$$

where $|q|<1$, $\left|\dfrac{ef}{abc}\right|<1$ *and as usual*

$$(a)_\infty:=(a;q)_\infty:=\prod_{n=0}^{\infty}(1-aq^n),$$
$$(a)_n:=(a;q)_n:=\frac{(a)_\infty}{(aq^n)_\infty},\qquad n \text{ is an integer},$$
$$(a_1,a_2,a_3,\cdots,a_m)_n=(a_1)_n(a_2)_n(a_3)_n\cdots(a_m)_n,\qquad n \text{ is an integer or } \infty.$$

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Recently, Liu [9] has established (1.1) by parameter augmentation method. This formula was used by Agarwal [1] to deduce an identity of Andrews [2, Thoerem 1] which was instrumental in deriving sixteen partial theta function identities of Ramanujan found in his lost notebook [4], [11].

The main objective of this paper is to give an alternate proof for (1.1) and to give proofs for Jackson's ${}_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum found in [5] by using q-exponential operator techniques. And also we give a simple proof for a nonterminating form of the q-Vandermonde sum. Also we obtain a number of interesting applications of these formulas.

We first list some definitions and identities that we use in the remainder of this paper. For any function f, the q-difference operator $D_{q,a}$ is defined by

$$D_{q,a}\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The q-shift operator η_a is defined by

$$\eta_a\{f(a)\} = f(aq)$$

and the operator θ_a is given by

$$\theta_a = \eta^{-1} D_{q,a}.$$

The operator identity $T(bD_{q,a})$ [9] is defined by

$$T(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(bD_{q,a})^n}{(q;q)_n} \tag{1.2}$$

and the basic identity for $T(bD_{q,a})$ operator is

$$T(bD_{q,a}) \left\{ \frac{1}{(as, at; q)_\infty} \right\} = \frac{(abst; q)_\infty}{(as, at, bs, bt; q)_\infty}. \tag{1.3}$$

The Cauchy operator $T(a, b; D_{q,c})$ [6] is defined by

$$T(a, b; D_{q,c}) := \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_{q,c})^n. \tag{1.4}$$

The two basic identities for the Cauchy operator (1.4) are

$$T(a, b; D_{q,c}) \left\{ \frac{1}{(ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}, \qquad |bt| < 1, \tag{1.5}$$

$$T(a, b; D_{q,c}) \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, cs, v/t)_n}{(q, cv, abs)_n} (bt)^n. \tag{1.6}$$

The q-exponential operator $R(bD_{q,a})$ [7] is defined by

$$R(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n} D_{q,a}^n. \tag{1.7}$$

The two basic identities for $R(bD_{q,a})$ are

$$R(bD_{q,a})\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{(bt;q)_\infty}{(at;q)_\infty} \tag{1.8}$$

and

$$R(bD_{q,a})\left\{\frac{(av;q)_\infty}{(at,as;q)_\infty}\right\} = \frac{(bs;q)_\infty}{(as;q)_\infty} \sum_{n=0}^{\infty} \frac{(v/t,b/a)_n}{(q,bs)_n}(at)^n. \tag{1.9}$$

The q-binomial theorem [5, equation(II.3), p.354] is given by

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}. \tag{1.10}$$

Heine's transformations for ${}_2\phi_1$-series [5, equation(III.1), (III.2), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\ \beta)_n}{(q,\ \gamma)_n} z^n = \frac{(\beta,\ \alpha z)_\infty}{(\gamma,\ z)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\beta, z)_n}{(q,\ \alpha z)_n} \beta^n. \tag{1.11}$$

The Rogers-Fine identity [12, equation(12), p.576] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^n = \sum_{n=0}^{\infty} \frac{(\alpha, \alpha z q/\beta)_n \beta^n z^n q^{n^2-n}(1-\alpha z q^{2n})}{(\beta)_n (z)_{n+1}}. \tag{1.12}$$

The Sears' transformation for ${}_3\phi_2$-series [5, equation (III.9), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta,\gamma)_n}{(q,\delta,\epsilon)_n} \left(\frac{\delta\epsilon}{\alpha\beta\gamma}\right)^n = \frac{(\epsilon/\alpha, \delta\epsilon/\beta\gamma)_\infty}{(\epsilon, \delta\epsilon/\alpha\beta\gamma)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \delta/\beta, \delta/\gamma)_n}{(q,\delta,\delta\epsilon/\beta\gamma)_n} \left(\frac{\epsilon}{\alpha}\right)^n. \tag{1.13}$$

The three-term ${}_2\phi_1$ transformation formula [5, equation (III.31), p.363] is given by

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\alpha,\beta)_n}{(q,\gamma)_n} z^n &= \frac{(\alpha\beta z/\gamma,\ q/\gamma)_\infty}{(\alpha z/\gamma,\ q/\alpha)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\alpha, \gamma q/\alpha\beta z)_n}{(q, \gamma q/\alpha z)_n} \left(\frac{\beta q}{\gamma}\right)^n \\ &\quad - \frac{(\beta, q/\gamma, \gamma/\alpha, \alpha z/q, q^2/\alpha z)_\infty}{(\gamma/q, \beta q/\gamma, q/\alpha, \alpha z/\gamma, \gamma q/\alpha z)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha q/\gamma, \beta q/\gamma)_n}{(q, q^2/\gamma)_n} z^n.\end{aligned} \tag{1.14}$$

The Jackson's transformation [3, p. 526] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta)_n}{(q,\gamma)_n} z^n = \frac{(\alpha z)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \gamma/\beta)_n (-\beta z)^n}{(\gamma, \alpha z, q)_n} q^{n(n-1)/2}. \tag{1.15}$$

The Ramanujan's [10, Ch. 16] definition of the theta function is

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\qquad |ab|<1. \tag{1.16}$$

The Jacobi's triple product identity [8] is given by

$$\sum_{n=-\infty}^{\infty}q^{n^2}z^n=(-qz,-q/z,q^2;q^2)_\infty,\qquad z\neq 0. \tag{1.17}$$

If we set $qz=a, q/z=b$ in (1.17), we obtain

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.18}$$

which is the Jacobi's triple product identity in Ramanujan's notation [10, Ch.16, entry 19]. It follows from (1.16) and (1.18) that [10, Ch. 16, entry 22]

$$\varphi(q):=f(q,q)=1+2\sum_{n=1}^{\infty}q^{n^2}=\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.19}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.20}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty \tag{1.21}$$

and

$$\chi(q):=(-q;q^2)_\infty. \tag{1.22}$$

The Ramanujan's functions are given by [4], [11]

$$G_6(q):=(q^3;q^6)_\infty^2(q^6;q^6)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2}=\varphi(-q^3), \tag{1.23}$$

$$H_6(q):=(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2-2n}=f(-q,-q^5) \tag{1.24}$$

and

$$J_6(q):=(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty=\sum_{n=0}^{\infty}q^{n(3n+1)/2}=f(q,q^2). \tag{1.25}$$

§2. Main Theorems

In this section, we prove the main results.

Proof of Theorem 1.1. Setting $\alpha=b,\beta=a/c,\gamma=qb/c$ and $z=q$ in (1.14), we obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c)_n}{(q,qb/c)_n}q^n=\frac{(a,c/b)_\infty}{(c,q/b)_\infty}\sum_{n=0}^{\infty}\frac{(q/a)_n}{(q)_n}\left(\frac{a}{b}\right)^n$$
$$-\frac{(a/c,c/b,b)_\infty}{(b/c,a/b,c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b)_n}{(q,qc/b)_n}q^n. \quad (2.1)$$

On using q-binomial theorem for the first series on the right side of (2.1), we obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c)_n}{(q,qb/c)_n}q^n+\frac{(a/c,c/b,b)_\infty}{(b/c,a/b,c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b)_n}{(q,qc/b)_n}q^n=\frac{(a,c/b)_\infty}{(c,a/b)_\infty}. \quad (2.2)$$

Divide the identity (2.2) throughout by $(a/c,c/b,b)_\infty$ to obtain

$$\frac{(a)_\infty}{(b,c,a/b,a/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_nq^n}{(q,qb/c)_n(aq^n/c)_\infty}$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_nq^n}{(q,qc/b)_n(aq^n/b)_\infty}. \quad (2.3)$$

Applying $T(d,e;D_{q,a})$ to both the sides of the identity (2.3) and using (1.5) and (1.6), we obtain

$$\frac{(a,de/b)_\infty}{(b,c,a/b,a/c,e/b)_\infty}\sum_{n=0}^{\infty}\frac{(d,a/b,c)_n}{(q,de/b,a)_n}\left(\frac{e}{c}\right)^n=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(deq^n/c)_\infty q^n}{(q,qb/c)_n(aq^n/c,eq^n/c)_\infty}$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n(deq^n/b)_\infty q^n}{(q,qc/b)_n(aq^n/b,eq^n/b)_\infty}. \quad (2.4)$$

Multiply the identity (2.4) throughout by $(b,c,a/b,a/c,e/b)_\infty/(a,de/b)_\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(d,a/b,c)_n}{(q,de/b,a)_n}\left(\frac{e}{c}\right)^n=\frac{(c,a/b,e/b,de/c)_\infty}{(a,c/b,e/c,de/b)_\infty}\sum_{n=0}^{\infty}\frac{(b,a/c,e/c)_n}{(q,qb/c,de/c)_n}q^n$$
$$+\frac{(b,a/c)_\infty}{(a,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b,e/b)_n}{(q,qc/b,de/b)_n}q^n. \quad (2.5)$$

Change a to A, b to C, c to B, d to A/D and e to E in (2.5) to obtain

$$\sum_{n=0}^{\infty}\frac{(B,A/D,A/C)_n}{(q,A,AE/CD)_n}\left(\frac{E}{B}\right)^n$$
$$=\frac{(B,A/C,E/C,AE/BD)_\infty}{(A,B/C,E/B,AE/CD)_\infty}\sum_{n=0}^{\infty}\frac{(C,A/B,E/B)_n}{(q,Cq/B,AE/BD)_n}q^n$$
$$+\frac{(C,A/B)_\infty}{(A,C/B)_\infty}\sum_{n=0}^{\infty}\frac{(B,A/C,E/C)_n}{(q,Bq/C,AE/CD)_n}q^n. \quad (2.6)$$

Setting $\alpha = B$, $\beta = A/D$, $\gamma = A/C$, $\delta = A$ and $\epsilon = AE/CD$ in (1.13), using the resulting identity on the left side of (2.6) and then multiplying the resulting identity throughout by $(E/B, AE/CD)_\infty/(E, AE/BCD)_\infty$; change A to e, B to b, C to a, D to c and E to f in the resulting identity, we obtain (1.1). □

Remark 1. The identity (2.3) can be used to prove Lemma 2.1 of Somashekara, Narasimha Murthy and Shalini [13], which played a key role in giving a unified approach to the proofs of the reciprocity theorem of Ramanujan and its generalizations.

Remark 2. The identity (2.3) can also be used to prove Theorem 2.2 of Somashekara, Kim, Kwon and Shalini [14], which played a key role in giving proofs for ten identities of Ramanujan found in his lost notebook [4].

Theorem 2.1([5, equation III.5, p. 359]) *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} z^n = \frac{(abz/c)_\infty}{(bz/c)_\infty} \sum_{n=0}^{\infty} \frac{(a,c/b,0)_n}{(q,c,cq/bz)_n} q^n + \frac{(a,bz,c/b)_\infty}{(c,z,c/bz)_\infty} \sum_{n=0}^{\infty} \frac{(z,abz/c,0)_n}{(q,bz,bzq/c)_n} q^n. \tag{2.7}$$

Proof Applying $R(dD_{q,a})$ to both the sides of the identity (2.3) and using (1.8), (1.9), we obtain

$$\frac{(d/c)_\infty}{(b,c,a/c)_\infty} \sum_{n=0}^{\infty} \frac{(b,d/a)_n}{(q,d/c)_n} \left(\frac{a}{b}\right)^n = \frac{1}{(b,c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n (dq^n/c)_\infty}{(q,bq/c)_n (aq^n/c)_\infty} q^n + \frac{1}{(c,b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n (dq^n/b)_\infty}{(q,cq/b)_n (aq^n/b)_\infty} q^n. \tag{2.8}$$

Multiply the identity (2.8) throughout by $(b,c,a/c)_\infty/(d/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty} \frac{(b,d/a)_n}{(q,d/c)_n} \left(\frac{a}{b}\right)^n = \frac{(c)_\infty}{(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b,a/c,0)_n}{(q,bq/c,d/c)_n} q^n + \frac{(b,a/c,d/b)_\infty}{(a/b,b/c,d/c)_\infty} \sum_{n=0}^{\infty} \frac{(c,a/b,0)_n}{(q,cq/b,d/b)_n} q^n. \tag{2.9}$$

Change a to az, b to a, c to abz/c and d to abz in (2.9) to obtain (2.7). □

Theorem 2.2([5, equation II.23, p. 356]) *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} q^n + \frac{(q/c,a,b)_\infty}{(c/q,aq/c,bq/c)_\infty} \sum_{n=0}^{\infty} \frac{(aq/c,bq/c)_n}{(q,q^2/c)_n} q^n = \frac{(q/c,abq/c)_\infty}{(aq/c,bq/c)_\infty}. \tag{2.9'}$$

Proof Change lower case letters to upper case letters in (2.2) and then change B to a, A/C to b and Bq/C to c to obtain (2.9'). □

Theorem 2.3([5, equation II.24, p. 356]) *We have*

$$\sum_{n=0}^{\infty}\frac{(a,b,c)_n}{(q,e,f)_n}q^n+\frac{(q/e,a,b,c,qf/e)_\infty}{(e/q,aq/e,bq/e,cq/e,f)_\infty}\sum_{n=0}^{\infty}\frac{(aq/e,bq/e,cq/e)_n}{(q,q^2/e,qf/e)_n}q^n$$
$$=\frac{(q/e,f/a,f/b,f/c)_\infty}{(aq/e,bq/e,cq/e,f)_\infty}, \tag{2.10}$$

where $ef=abcq$.

Proof Divide (2.3) throughout by $(a)_\infty$ to obtain

$$\frac{1}{(b,c,a/b,a/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_nq^n}{(q,qb/c)_n(aq^n/c,a)_\infty}$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_nq^n}{(q,qc/b)_n(aq^n/b,a)_\infty}. \tag{2.11}$$

Applying $T(dD_{q,a})$ to both the sides of the identity (2.11) and using (1.3), we obtain

$$\frac{(ad/bc)_\infty}{(b,c,a/b,a/c,d/b,d/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(adq^n/c)_\infty}{(q,bq/c)_n(aq^n/c,a,dq^n/c,d)_\infty}q^n$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n(adq^n/b)_\infty}{(q,qc/b)_n(aq^n/b,a,dq^n/b,d)_\infty}q^n. \tag{2.12}$$

Multiply the identity (2.12) throughout by $(a,b,d,a/c,c/b,d/c)_\infty/(ad/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c,d/c)_nq^n}{(q,bq/c,ad/c)_n}+\frac{(c/b,b,a/c,d/c,ad/b)_\infty}{(b/c,c,a/b,d/b,ad/c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b,d/b)_nq^n}{(q,qc/b,ad/b)_n}$$
$$=\frac{(c/b,ad/bc,d,a)_\infty}{(c,a/b,d/b,ad/c)_\infty}. \tag{2.13}$$

Change lower case letters to upper case letters in (2.13) and then change B to a, A/C to b, D/C to c, Bq/C to e and AD/C to f to obtain (2.10). □

§3. Some Applications of Main Results

In this section, we derive some interesting special cases of the main identities. These special cases are found to be analogues to some identities of Ramanujan found in his lost notebook [4], [11].

Setting $a = C, b = B/A, c = D$ and $z = A$ in (2.7), we obtain

$$\sum_{n=0}^{\infty} \frac{(C, B/A)_n}{(q, D)_n} A^n = \frac{(BC/D)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C, AD/B)_n}{(q, D, qD/B)_n} q^n + \frac{(B, C, AD/B)_\infty}{(A, D, D/B)_\infty} \sum_{n=0}^{\infty} \frac{(A, BC/D)_n}{(q, B, qB/D)_n} q^n. \tag{3.1}$$

Change B to β, C to τ, D to τq and then let $A \to 0$ in (3.1) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n q^{n(n-1)/2}}{(q;q)_n (1-\tau q^n)} = \frac{(\beta/q)_\infty}{(\beta/\tau q)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (\tau q^2/\beta)_n (1-\tau q^n)} + \frac{(1-\beta/q)(\beta)_\infty}{(\tau q/\beta)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (\beta/\tau)_n (1-\beta q^{n-1})}. \tag{3.2}$$

Change q to q^2 and set $\tau = -1$ and $\beta = -q^3$ in (3.2) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n (1+q^{2n})} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1} (1-q^{4n})} - \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n} (1-q^{4n+2})}. \tag{3.3}$$

Use (1.22) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_{n-1} (1-q^{4n})} = \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1} (1-q^{4n})} - \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n} (1-q^{4n+2})}.$$

Setting $\alpha = B/A, \beta = C, \gamma = D$ and $z = A$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \frac{(B/A, C)_n}{(q, D)_n} A^n = \frac{(B, C)_\infty}{(A, D)_\infty} \sum_{n=0}^{\infty} \frac{(A, D/C)_n}{(q, B)_n} C^n. \tag{3.4}$$

Using (3.4) in (3.1) and then multiplying the resulting identity throughout by $(A, D)_\infty/(B, C)_\infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{(D/C, A)_n}{(q, B)_n} C^n = \frac{(A, D, BC/D)_\infty}{(B, C, B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C, AD/B)_n}{(q, D, Dq/B)_n} q^n + \frac{(AD/B)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(A, BC/D)_n}{(q, B, qB/D)_n} q^n. \tag{3.5}$$

Change q to q^2 and set $A = t, B = -aq^3, C = -a$ and $D = -aq^2$ in (3.5) and then let

$t \to 0$; divide the resulting identity throughout by $(1+aq)$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(-aq;q^2)_{n+1}} = \frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q;q)_{2n}(1+aq^{2n})} - \frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(q;q)_{2n+1}(1+aq^{2n+1})}. \tag{3.6}$$

In Rogers-Fine identity, change q to q^2, set $\alpha = 0, \beta = -aq^3$ and $z = -a$; multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(-aq;q^2)_{n+1}} &= \sum_{n=0}^{\infty}\frac{a^{2n}q^{2n^2+n}}{(-a;q^2)_{n+1}(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}\frac{a^{2n}q^{2n^2+n}}{(-a;q)_{2n+2}}\\ &= \sum_{n=0}^{\infty}\frac{a^{2n}q^{2n^2+n}((1+aq^{2n+1})-aq^{2n+1})}{(-a;q)_{2n+2}}\\ &= \sum_{n=0}^{\infty}\frac{a^{2n}q^{2n^2+n}}{(-a;q)_{2n+1}} - \sum_{n=0}^{\infty}\frac{a^{2n+1}q^{2n^2+3n+1}}{(-a;q)_{2n+2}} = \sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}}.\end{aligned} \tag{3.7}$$

Use (3.7) in (3.6) and also use (1.21) to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}} = \frac{f(-q^2)}{f(-q)}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q;q)_{2n}(1+aq^{2n})} - \frac{f(-q^2)}{f(-q)}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(q;q)_{2n+1}(1+aq^{2n+1})}. \tag{3.8}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5) and let $t \to 0$ in the resulting identity; multiply the resulting identity throughout by $1/(1-aq)$ and also use (1.21) to obtain on some simplifications

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \frac{f(-q^2)}{f(-q^4)}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^4;q^4)_n(1-a^2q^{4n+2})}. \tag{3.9}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = 0, \beta = aq^3$ and $z = -aq$ and then multiply the resulting identity throughout by $1/(1-aq)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}}. \tag{3.10}$$

Use (3.10) in (3.9) to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}} = \frac{f(-q^2)}{f(-q^4)}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^4;q^4)_n(1-a^2q^{4n+2})}. \tag{3.11}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5) and multiply the resulting identity throughout by $1/(1-aq)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(t;q^2)_n(-aq)^n}{(aq;q^2)_{n+1}}=\frac{(t;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-t;q^2)_nq^{2n}}{(q^4;q^4)_n(1+aq^{2n+1})}+\frac{(-t;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_nq^{2n}}{(q^4;q^4)_n(1-aq^{2n+1})}. \tag{3.12}$$

Set $a=-1$ and $t=q$ in (3.12) to obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q;q^2)_{n+1}}=\frac{(q;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}+\frac{(-q;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \tag{3.13}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha=z=q$ and $\beta=-q^3$; multiply the resulting identity throughout by $1/(1+q)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q;q^2)_{n+1}}=\sum_{n=0}^{\infty}(-1)^nq^{2n(n+1)}. \tag{3.14}$$

Use (3.14) in (3.13) and also use (1.19), (1.20) and (1.21) to obtain

$$2\sum_{n=0}^{\infty}(-1)^nq^{2n(n+1)}=\frac{f(-q)}{f(-q^4)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}+\frac{\varphi(q)}{\psi(q)}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \tag{3.15}$$

In (3.5), set $A=q, B=-aq, C=\tau$ and $D=a^2q$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q/\tau)_n}{(-aq)_n}\tau^n=\frac{(-\tau/a,q,a^2q)_\infty}{(-1/a,-aq,\tau)_\infty}\sum_{n=0}^{\infty}\frac{(\tau)_n}{(q,a^2q)_n}q^n+\frac{(-aq)_\infty}{(-1/a)_\infty}\sum_{n=0}^{\infty}\frac{(-\tau/a)_n}{(-aq,-q/a)_n}q^n. \tag{3.16}$$

In Rogers-Fine identity, set $\alpha=a^2q/\tau, \beta=-aq$ and $z=\tau$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q/\tau)_n}{(-aq)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q/\tau)_na^nq^{n^2}(1-a^2q^{2n+1})}{(\tau)_{n+1}}. \tag{3.17}$$

Use (3.17) in (3.16) and then let $\tau \to 0$ in the resulting identity to obtain

$$\sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) = \frac{(a^2q)_\infty f^2(-q)}{f(aq,1/a)} \sum_{n=0}^{\infty} \frac{q^n}{(q,a^2q)_n} + \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq,-q/a)_n}. \tag{3.18}$$

Set $a = 1$ in (3.18) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) = \frac{f^3(-q)}{f(q,1)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \tag{3.19}$$

In (1.11), set $\gamma = z = q$ and then $\alpha = 0, \beta = 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_n^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{3.20}$$

Use (3.20) in (3.19) and also use (1.20) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) = \frac{f(-q)\psi(-q)}{f(q,1)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}.$$

In (3.16), let $\tau \to 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq)_n} = \frac{(q,a^2q)_\infty}{(-1/a,-aq)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q,a^2q)_n} + \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq,-q/a)_n}. \tag{3.21}$$

Set $a = 1$ in (3.21) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \tag{3.22}$$

The left side of (3.22) yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(-q;q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}((1+q^{2n+1})-q^{2n+1})}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}.\end{aligned} \tag{3.23}$$

Use (3.23) in (3.22) to obtain

$$\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}=\frac{f^3(-q)}{f(1,q)}\sum_{n=0}^{\infty}\frac{q^n}{(q)_n^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^n}{(-q)_n^2}. \tag{3.24}$$

Use (3.20) in (3.24) to obtain

$$\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}=\frac{f^3(-q)}{f(1,q)}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^n}{(-q)_n^2}. \tag{3.25}$$

Use the definition of ψ to obtain

$$\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}=\frac{f(-q)\psi(-q)}{f(1,q)}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^n}{(-q)_n^2}.$$

In (3.5), replace q by q^2, set $A=q^2, B=-aq^3, C=\tau$ and $D=a^2q^2$; multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q^2/\tau;q^2)_n}{(-aq;q^2)_n}\tau^n=\frac{(q^2,a^2q^2,-q\tau/a;q^2)_\infty}{(-aq,\tau,-q/a;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(\tau;q^2)_n}{(q^2,a^2q^2;q^2)_n}q^{2n}$$
$$+\frac{(-aq;q^2)_\infty}{(-q/a;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q\tau/a;q^2)_n}{(-q^3/a;q^2)_n(-aq;q^2)_{n+1}}q^{2n}. \tag{3.26}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha=a^2q^2/\tau, \beta=-aq^3, z=\tau$ and then multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q^2/\tau;q^2)_n}{(-aq;q^2)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q^2/\tau;q^2)_n\tau^n a^n q^{2n(n+1)}(1-a^2q^{4n+2})}{(1+aq^{2n+1})(\tau;q^2)_{n+1}}. \tag{3.27}$$

Use (3.27) in (3.26) and then let $\tau\to 0$ to obtain

$$\sum_{n=0}^{\infty}a^{3n}q^{3n^2+2n}(1-aq^{2n+1})=\frac{(q^2,a^2q^2;q^2)_\infty}{(-q/a,-aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^2,a^2q^2;q^2)_n}$$
$$+\frac{(-aq;q^2)_\infty}{(-q/a;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n}}{(-aq;q^2)_{n+1}(-q^3/a;q^2)_n}. \tag{3.28}$$

In (3.5), replace q to q^2, set $A=q^2, B=-q^3, D=q^2$ and then let $C\to 0$; multiply the resulting identity throughout by $1/(1+q)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}}=\frac{(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n^2}+(1+q)\sum_{n=0}^{\infty}\frac{q^{2n}}{(-q;q^2)_{n+1}^2}. \tag{3.29}$$

In (2.10), replace q by q^6, set $a = q, b = q^4, c = q^2, e = q^3$and $f = q^7$ to obtain

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{(q^3; q^6)_\infty^2 (q^6; q^6)_\infty}{(q; q^6)_\infty^2 (q^4; q^6)_\infty}. \quad (3.30)$$

Use (1.21), (1.23) and (1.24) to obtain on some simplifications

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{f(-q^2)}{H_6(q)} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{G_6^2(q) H_6^2(q) f(-q^2)}{(q; q^6)_\infty^2 f(-q) f^2(-q^6)}. \quad (3.31)$$

In (2.10), replace q by q^3, set $a = c = -q, b = e = -q^2$ and $f = q^3$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2}$$
$$= \frac{(-q; q^3)_\infty^2 (-q^2; q^3)_\infty^2 (q^3; q^3)_\infty^2}{(q^2; q^3)_\infty^2 (q^3; q^3)_\infty^4}. \quad (3.32)$$

Use (1.25) to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2} = \frac{J_6^2(q) (q; q^3)_\infty^2}{(q; q)_\infty^2 (q^3; q^3)_\infty^4}.$$

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Some New Generalizations of the Lucas Sequence

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Abstract: In this paper, we investigate the generalized Lucas, the generalized complex Lucas and the generalized dual Lucas sequence using the Lucas number. Also, we investigate special cases of these sequences. Furthermore, we give recurrence relations, vectors, the golden ratio and Binet's formula for the generalized Lucas and the generalized dual Lucas sequence.

Key Words: Smarandache-Fibonacci triple, Fibonacci number, Lucas number, Lucas sequence, generalized Fibonacci sequence, generalized complex Lucas sequence, generalized dual Lucas sequence.

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§1. Introduction

Let $S(n), n \geq 0$ with $S(n) = S(n-1) + S(n-2)$ be a Smarandache-Fibonacci triple, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Particularly, let $S(n)$ be $F(n)$ or $L(n)$, we get the Fibonacci or Lucas sequence as follows:

A Fibonacci sequence

$$1\,,\ 1\,,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233, \cdots, F_n, \cdots$$

is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\ , \quad (n \geqslant 3),$$

with $F_1 = F_2 = 1$, where F_n is the n-th term of the Fibonacci sequence (F_n) (Leonardo Fibonacci, 1202). The Fibonacci sequence is named after Italian mathematician Leonardo of Pisa, known as Fibonacci. The name "Fibonacci Sequnce" was first used by the $19th$- century number theorist Edouard Lucas. Some recent generalizations for the Fibonacci sequence have

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produced a variety of new and extended results, [1],[5],[6],[9],[13].

A Lucas sequence

$$2\,,\ 1\,,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322, \cdots, L_n, \cdots$$

is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}\,, \quad (n \geqslant 3),$$

with $L_1 = 2\,, L_2 = 1$, where L_n is the n-th term of the Lucas sequence (L_n) (François Edouard Anatole Lucas, 1876). There are a lot of generalizations of the Lucas sequences,[15],[16],[17].

The generalized Fibonacci sequence defined by

$$H_n = H_{n-1} + H_{n-2}\,, \quad (n \geqslant 3) \tag{1.1}$$

with $H_1 = p\,,\ H_2 = p+q$ where p, q are arbitrary integers [3]. That is, the generalized Fibonacci sequence is

$$p\,,\ p+q\,,\ 2p+q,\ 3p+2q,\ 5p+3q,\ 8p+5q, \cdots, (p-q)F_n + qF_{n+1}, \cdots \tag{1.2}$$

Using the equations (1.1) and (1.2) , it was obtained

$$\begin{aligned} H_{n+1} &= q\,F_n + p\,F_{n+1} \\ H_{n+2} &= p\,F_n + (p+q)\,F_{n+1}. \end{aligned} \tag{1.3}$$

For the generalized Fibonacci sequence, it was obtained the following properties:

$$H_{n-1}^2 + H_n^2 = (2p-q)H_{2n-1} - e\,F_{2n-1}\,, \tag{1.4}$$

$$H_{n+1}^2 - H_{n-1}^2 = (2p-q)H_{2n} - e\,F_{2n}\,, \tag{1.5}$$

$$H_{n-1}\,H_{n+1} - H_n^2 = (-1)^n e\,, \tag{1.6}$$

$$H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \quad (n \geqslant 3) \tag{1.7}$$

$$H_{n+1-r}\,H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r} e\ F_r^2\,, \tag{1.8}$$

$$H_{n+1}^2 + e\ F_n^2 = p\,H_{2n+1}\,, \tag{1.9}$$

$$H_n\,H_{n+1+r} - H_{n-s}\,H_{n+r+s+1} = (-1)^{n+s} e\ F_sF_{r+s+1}\,, \tag{1.10}$$

$$\left[2H_{n+1}H_{n+2}\right]^2 + \left[H_nH_{n+3}\right]^2 = \left[2H_{n+1}H_{n+2} + H_n^2\right]^2 \tag{1.11}$$

$$\frac{H_{n+r} + (-1)^r H_{n-r}}{H_n} = F_{r+1} + (-1)^r F_{r-1} \tag{1.12}$$

where e $= p^2 - p\,q - q^2$.

Also, for $p=1$, $q=0$, we get the following well-known results:

$$F_{n-1}^2 + F_n^2 = F_{2n-1}\,, \quad \text{(Catalan)}, \tag{1.13}$$

$$F_{n-1}\,F_{n+1} - F_n^2 = (-1)^n\,, \quad \text{(Simpson or Cassini)}, \tag{1.14}$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \quad \text{(Lucas)}. \tag{1.15}$$

In this paper, we will define the generalized Lucas, the generalized complex Lucas and the generalized dual Lucas sequences respectively, denoted by G_n , $\mathbb{C}_n$, $\mathbb{D}_n$.

§2. Generalized Lucas Sequence and Lucas Vectors

In this section, we will define the generalized Lucas sequence denoted by $\mathbb{L}_n$. The generalized Lucas sequence defined by

$$\mathbb{L}_n = \mathbb{L}_{n-1} + \mathbb{L}_{n-2}\,, \quad (n \geqslant 3), \tag{2.1}$$

with $\mathbb{L}_1 = 2\,p - \,q$, $\mathbb{L}_2 = p + 2\,q$ where p, q are arbitrary integers,[3]. That is, the generalized Lucas sequence is

$$2\,p - q\,,\ p + 2\,q\,,\ 3\,p + q\,,\ 4\,p + 3\,q\,,\ 7\,p + 4\,q\,,\ 11\,p + 7\,q\,, \cdots, (p-q)L_n + qL_{n+1}, \cdots \tag{2.2}$$

Using the equations (2.1) and (2.2) , we get

$$\begin{aligned} \mathbb{L}_{n+1} &= q\,L_n + p\,L_{n+1}\,, \\ \mathbb{L}_{n+2} &= p\,L_n + (p+q)\,L_{n+1}. \end{aligned} \tag{2.3}$$

Putting $n=r$ in (2.3) and using (2.1), we find in turn

$$\begin{aligned} \mathbb{L}_{r+3} &= (2\,p + q\,)L_{r+1} + (p + q\,)\,L_r = H_3 L_{r+1} + H_2 L_r \\ \mathbb{L}_{r+4} &= (3\,p + 2\,q)\,L_{r+1} + (2\,p + q)\,L_r = H_4 L_{r+1} + H_3 L_r \end{aligned} \tag{2.4}$$

So, in general, we have obtain relations between generalized Lucas sequence and generalized Fibonacci sequence as follows:

$$\mathbb{L}_{n+r} = H_{n-1} L_r + H_n L_{r+1} \tag{2.5}$$

Also, certain results follow almost immediately from (2.1)

$$\mathbb{L}_{n+2} - 2\,\mathbb{L}_n - \mathbb{L}_{n-1} = 0, \tag{2.6}$$

$$\mathbb{L}_{n+1} - 2\,\mathbb{L}_n + \mathbb{L}_{n-2} = 0, \tag{2.7}$$

$$\sum_{i=0}^{n-1} \mathbb{L}_{2i+1} = \mathbb{L}_{2n} - (2\,p - q), \tag{2.8}$$

$$\sum_{i=1}^{n} \mathbb{L}_{2i} = \mathbb{L}_{2n+1} - (p + 2\,q), \tag{2.9}$$

$$\sum_{i=1}^{n} (\mathbb{L}_{2i-1} - \mathbb{L}_{2i}) = -\,\mathbb{L}_{2n-1} - p + 3\,q. \tag{2.10}$$

For the generalized Lucas sequence, we have the following properties:

$$\mathbb{L}_{n-1}^2 + \mathbb{L}_n^2 = (2p-q)(\mathbb{L}_{2n-2} + \mathbb{L}_{2n}) - e_L\left(L_{2n-2} + L_{2n}\right), \tag{2.11}$$

$$\mathbb{L}_{n+1}^2 - \mathbb{L}_{n-1}^2 = (2p-q)(\mathbb{L}_{2n+2} - \mathbb{L}_{2n-2}) - e_L\left(L_{2n+2} - L_{2n-2}\right), \tag{2.12}$$

$$\mathbb{L}_{n-1}\,\mathbb{L}_{n+1} - \mathbb{L}_n^2 = 5\,(-1)^{n+1}\,e_L\ , \tag{2.13}$$

$$\mathbb{L}_{n+1}^2 + e_L\ L_n^2 = p\left(\mathbb{L}_{2n+2} + \mathbb{L}_{2n}\right), \tag{2.14}$$

$$\frac{\mathbb{L}_{n+r} + (-1)^r \mathbb{L}_{n-r}}{\mathbb{L}_n} = L_r \tag{2.15}$$

where $e_L = p^2 - p\,q - q^2$.

Theorem 2.1 *If $\mathbb{L}_n$ is the generalized Lucas number, then*

$$\lim_{n\to\infty} \frac{\mathbb{L}_{n+1}}{\mathbb{L}_n} = \frac{p\alpha + q}{q\alpha + (p-q)},$$

where $\alpha = (1+\sqrt{5})/2 = 1.618033\cdots$ is the golden ratio.

Proof We have for the Lucas number L_n,

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha,$$

where

$$\alpha = (1+\sqrt{5})\Big/2 = 1.618033\cdots$$

is the golden ratio [12].

Then for the generalized Lucas number $\mathbb{L}_n$, we obtain

$$\lim_{n\to\infty} \frac{\mathbb{L}_{n+1}}{\mathbb{L}_n} = \lim_{n\to\infty} \frac{pL_{n+1} + qL_n}{qL_{n+1} + (p-q)L_n} = \frac{p\alpha + q}{q\alpha + (p-q)}. \tag{2.16}$$

□

Theorem 2.2 *The Binet's formula*[2] *for the generalized Lucas sequence is as follows;*

$$\mathbb{L}_n = (\,\bar{\alpha}\;\alpha^n + \bar{\beta}\;\beta^n) \tag{2.17}$$

where $\bar{\alpha} = \alpha(2\,p - q) - (p + 2\,q)$, $\bar{\beta} = (p + 2\,q) - \beta\,(2\,p - q)$.

Proof The characteristic equation of recurrence relation $\mathbb{L}_{n+2} = \mathbb{L}_{n+1} + \mathbb{L}_n$ is

$$t^2 - t - 1 = 0. \tag{2.18}$$

The roots of this equation are

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}, \tag{2.19}$$

where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$.

Using recurrence relation and initial values $\mathbb{L}_0 = (2\,p - q)$, $\mathbb{L}_1 = (p + 2\,q)$ the Binet's formula for $\mathbb{L}_n$, we get

$$\mathbb{L}_n = A\;\alpha^n + B\,\beta^n = \left[\,\bar{\alpha}\;\alpha^n + \bar{\beta}\;\beta^n\right], \tag{2.20}$$

where

$$A = \frac{\mathbb{L}_1 - \beta\,\mathbb{L}_0}{\alpha - \beta},\;\; B = \frac{\alpha\,\mathbb{L}_0 - \mathbb{L}_1}{\alpha - \beta}$$

and $\bar{\alpha} = \alpha(2\,p - q) - (p + 2\,q)$, $\bar{\beta} = (p + 2\,q) - \beta\,(2\,p - q)$. □

A generalized Lucas vector is defined by

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} = (\mathbb{L}_n\,,\,\mathbb{L}_{n+1},\,\mathbb{L}_{n+2})$$

Also, from equation (2.2) it can be expressed as

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \;\; = (p-q)\,\overrightarrow{\mathrm{L}}_{n} + q\,\overrightarrow{\mathrm{L}}_{\mathrm{n+1}} \tag{2.21}$$

where $\overrightarrow{\mathrm{L}}_{\mathrm{n}} = (\,L_n\,,\,L_{n+1},\,L_{n+2})$ and $\overrightarrow{\mathrm{L}}_{\mathrm{n+1}} = (\,L_{n+1}\,,\,L_{n+2},\,L_{n+3})$ are the Lucas vectors.

The product of $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\lambda \in \mathbb{R}$ is given by

$$\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n}}} = (\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n}}},\,\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n+1}}},\,\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n+2}}})$$

[2]Binet's formula is the explicit formula to obtain the n-th Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet's formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n$$

respectively, where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$ and $\alpha = (1+\sqrt{5})\big/2$, $\beta = (1-\sqrt{5})\big/2$,[7], [8].

and $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$ are equal if and only if

$$\begin{aligned}\mathbb{L}_n &= \mathbb{L}_m\\ \mathbb{L}_{n+1} &= \mathbb{L}_{m+1}\\ \mathbb{L}_{n+2} &= \mathbb{L}_{m+2}.\end{aligned}$$

Theorem 2.3 *Let $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$ be two generalized Lucas vectors. The dot product of $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$ is given by*

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm m}}\right\rangle &= p^2\,(5\,F_{n+m+3} + L_n\,L_m)\\ &\quad + p\,q\,[2\,L_{n+m-1} + 10\,F_{n+m+2}\,]\\ &\quad + q^2(5\,F_{n+m+1} + L_{n-1}L_{m-1})\,. \end{aligned} \tag{2.22}$$

Proof The dot product of $\overrightarrow{\mathbb{L}_{\rm n}} = (\mathbb{L}_n, \mathbb{L}_{n+1}, \mathbb{L}_{n+2})$ and $\overrightarrow{\mathbb{L}_{\rm m}} = (\mathbb{L}_m, \mathbb{L}_{m+1}, \mathbb{L}_{m+2})$ defined by

$$\left\langle\overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm m}}\right\rangle = \mathbb{L}_n\mathbb{L}_m + \mathbb{L}_{n+1}\mathbb{L}_{m+1} + \mathbb{L}_{n+2}\mathbb{L}_{m+2}.$$

Also, using the equations (2.1), (2.2) and (2.3), we obtain

$$\mathbb{L}_n\,\mathbb{L}_m = p^2\,(L_n\,L_m) + p\,q\,[L_nL_{m-1} + L_{n-1}\,L_m] + q^2\,(L_{n-1}\,L_{m-1}), \tag{2.23}$$

$$\mathbb{L}_{n+1}\,\mathbb{L}_{m+1} = p^2\,(L_{n+1}\,L_{m+1}) + p\,q\,[L_{n+1}L_m + L_n\,L_{m+1}] + q^2\,(L_n\,L_m), \tag{2.24}$$

$$\begin{aligned}\mathbb{L}_{n+2}\,\mathbb{L}_{m+2} &= p^2\,(L_{n+2}L_{m+2}) + p\,q\,[L_{n+2}L_{m+1} + L_{n+1}L_{m+2}]\\ &\quad + q^2\,(L_{n+1}L_{m+1}).\end{aligned} \tag{2.25}$$

Then, from the equations (2.23), (2.24) and (2.25), we have

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm m}}\right\rangle &= p^2(L_nL_m + L_{n+1}L_{m+1} + L_{n+2}L_{m+2})\\ &\quad + (p\,q)\,[L_nL_{m-1} + L_{n-1}L_m + L_{n+1}L_m + L_nL_{m+1}\\ &\quad + L_{n+2}L_{m+1} + L_{n+1}L_{m+2}]\\ &\quad + q^2\,(L_{n-1}\,L_{m-1} + L_n\,L_m + L_{n+1}\,L_{m+1})\\ &= p^2(5\,F_{n+m+3} + L_nL_m)\\ &\quad + (p\,q)\,[10F_{n+m+2} + 2\,L_{n+m-1}]\\ &\quad + q^2(5\,F_{n+m+1} + L_{n-1}\,L_{m-1})\,.\end{aligned} \tag{2.26}$$

□

Case 1. For the dot product of the generalized Lucas vectors $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}}_{\rm n+1}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm n+1}} \right\rangle &= \mathbb{L}_n\mathbb{L}_{n+1} + \mathbb{L}_{n+1}\mathbb{L}_{n+2} + \mathbb{L}_{n+2}\mathbb{L}_{n+3} \\ &= p^2 \left[\, 5\, F_{2n+4} + L_n L_{n+1} \right] \\ &\quad +(p\, q) \left[\, 10\, F_{2n+3} + 2\, L_{2n} \,\right] \\ &\quad +q^2 \left[\, 5\, F_{2n+2} + L_{n-1}\, L_n \right] \end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm n}} \right\rangle &= (\mathbb{L}_n)^2 + (\mathbb{L}_{n+1})^2 + (\mathbb{L}_{n+2})^2 \\ &= p^2[\, L_n^2 + L_{n+1}^2 + L_{n+2}^2] \\ &\quad +(p\, q)[\, 2\, L_n L_{n-1} + 2\, L_{n+1} L_n + 2\, L_{n+2} L_{n+1}] \\ &\quad +q^2[\, L_{n-1}^2 + L_n^2 + L_{n+1}^2]. \end{aligned} \tag{2.28}$$

Then for the norm of the generalized Lucas vector, using identities of the Fibonacci numbers

$$\begin{aligned} L_{n+1}^2 + L_n^2 &= 5\, F_{2n+1} \\ L_{n+1}^2 - L_{n-1}^2 &= 5\, F_{2n} \\ L_{n+1}^2 - L_n^2 &= L_{n-1}\, L_{n+2} \\ L_n L_m + L_{n+1} L_{m+1} &= 5\, F_{n+m+1} \end{aligned}$$

we have

$$\begin{aligned}\left\| \overrightarrow{\mathbb{L}_{\rm n}} \right\|^2 &= \left\langle \overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm n}} \right\rangle = {\mathbb{L}_n}^2 + \mathbb{L}_{n+1}^2 + \mathbb{L}_{n+2}^2 \\ &= p^2 \left[5\, F_{2n+3} + L_n^2 \right] \\ &\quad + (p\, q) \left[2\, F_{2n+2} + 2\, L_n\, L_{n-1} \right] \\ &\quad +q^2 \left[5\, F_{2n+1} + L_{n-1}^2 \right]. \end{aligned} \tag{2.29}$$

Case 2. For $p = 1$, $q = 0$, in the equations (2.26), (2.27) and (2.29), we have

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm m}} \right\rangle &= \left[\, 5\, F_{n+m+3} + L_n\, L_m \right], \\ \left\langle \overrightarrow{\mathbb{L}_{\rm n}}, \overrightarrow{\mathbb{L}_{\rm n+1}} \right\rangle &= \left[\, 5\, F_{2n+4} + L_n\, L_{n+1} \right] \end{aligned}$$

and

$$\left\| \overrightarrow{\mathbb{L}_{\rm n}} \right\| = \sqrt{5\, F_{2n+3} + L_n^2}\, .$$

Theorem 2.4 *Let $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$ be two generalized Lucas vectors. The cross product of $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$ is given by*

$$\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}} = 5\, (-1)^n\, F_{m-n}\, (p^2 - p\, q - q^2)\, (i + j - k). \tag{2.30}$$

Proof The cross product of $\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}$ defined by

$$\begin{aligned}\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}} &= \begin{vmatrix} i & j & k \\ \mathbb{L}_n & \mathbb{L}_{n+1} & \mathbb{L}_{n+2} \\ \mathbb{L}_m & \mathbb{L}_{m+1} & \mathbb{L}_{m+2} \end{vmatrix} \\ &= i\left(\mathbb{L}_{n+1}\mathbb{L}_{m+2} - \mathbb{L}_{n+2}\mathbb{L}_{m+1}\right) \\ &\quad + j\left(\mathbb{L}_{n+2}\mathbb{L}_{m} - \mathbb{L}_{n}\mathbb{L}_{m+2}\right) + k\left(\mathbb{L}_{n}\mathbb{L}_{m+1} - \mathbb{L}_{n+1}\mathbb{L}_{m}\right).\end{aligned} \tag{2.31}$$

Now, we calculate the cross products. Using the property $L_n L_{m+1} - L_{n+1} L_m = 5\,(-1)^n\, F_{m-n}$ we get

$$\mathbb{L}_{n+1}\,\mathbb{L}_{m+2} - \mathbb{L}_{n+2}\,\mathbb{L}_{m+1} = 5\,(-1)^n F_{m-n}\,(p^2 - p\,q - q^2) = 5\,(-1)^n F_{m-n}\, e_L, \tag{2.32}$$

$$\mathbb{L}_{n+2}\,\mathbb{L}_{m} - \mathbb{L}_{n}\,\mathbb{L}_{m+2} = 5\,(-1)^n F_{m-n}\,(p^2 - pq - q^2) = 5\,(-1)^n F_{m-n}\, e_L, \tag{2.33}$$

and

$$\mathbb{L}_{n}\,\mathbb{L}_{m+1} - \mathbb{L}_{n+1}\,\mathbb{L}_{m} = 5\,(-1)^{n+1} F_{m-n}\,(p^2 - p\,q - q^2) = 5\,(-1)^{n+1} F_{m-n}\, e_L\,. \tag{2.34}$$

Then from the equations (2.32), (2.33) and (2.34), we obtain the equation (2.30).

Case 3. For $p = 1$, $q = 0$, in the equation (2.30), we have

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}} = 5\,(-1)^n F_{m-n}\,(i + j - k).$$

□

Theorem 2.5 *Let $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$, $\overrightarrow{\mathbb{L}_{\mathrm{m}}}$ and $\overrightarrow{\mathbb{L}_{\mathrm{k}}}$ be the generalized Lucas vectors. The mixed product of these vectors is*

$$\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{k}}} \right\rangle = 0. \tag{2.35}$$

Proof Using $\overrightarrow{\mathbb{L}_{\mathrm{k}}} = (\mathbb{L}_k, \mathbb{L}_{k+1}, \mathbb{L}_{k+2})$, we can write,

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{k}}} \right\rangle &= \begin{vmatrix} \mathbb{L}_n & \mathbb{L}_{n+1} & \mathbb{L}_{n+2} \\ \mathbb{L}_m & \mathbb{L}_{m+1} & \mathbb{L}_{m+2} \\ \mathbb{L}_k & \mathbb{L}_{k+1} & \mathbb{L}_{k+2} \end{vmatrix} \\ &= \mathbb{L}_n\left(\mathbb{L}_{m+1}\,\mathbb{L}_{k+2} - \mathbb{L}_{m+2}\,\mathbb{L}_{k+1}\right) \\ &\quad + \mathbb{L}_{n+1}\left(\mathbb{L}_{m+2}\,\mathbb{L}_{k} - \mathbb{L}_{m}\,\mathbb{L}_{k+2}\right) + \mathbb{L}_{n+2}\left(\mathbb{L}_{m}\,\mathbb{L}_{k+1} - \mathbb{L}_{m+1}\,\mathbb{L}_{k}\right).\end{aligned} \tag{2.36}$$

Also, using the equations (2.32), (2.33) and (2.34), we obtain

$$\begin{aligned} \mathbb{L}_n\,(\mathbb{L}_{m+1}\,\mathbb{L}_{k+2}-\mathbb{L}_{m+2}\,\mathbb{L}_{k+1}) &\quad +\mathbb{L}_{n+1}\,(\mathbb{L}_{m+2}\,\mathbb{L}_{k}-\mathbb{L}_{m}\,\mathbb{L}_{k+2}) \\ &\quad +\mathbb{L}_{n+2}\,(\mathbb{L}_{m}\,\mathbb{L}_{k+1}-\mathbb{L}_{k}\,\mathbb{L}_{m+1}) \\ &= 5\,(-1)^m\,F_{k-m}\,e_L\,(\,\mathbb{L}_n+\mathbb{L}_{n+1}-\mathbb{L}_{n+2}\,) \\ &= 5\,(-1)^m\,F_{k-m}\,e_L\,(\,\mathbb{L}_{n+2}-\mathbb{L}_{n+2}\,)=0. \end{aligned} \tag{2.37}$$

Thus, we have the equation (2.35). □

§3. Generalized Complex Lucas Sequence

In this section, we will define the generalized complex Lucas sequence denoted by $\mathbb{C}_n$. The generalized complex Lucas sequence defined by

$$\mathbb{C}_n=\mathbb{L}_n+i\,\mathbb{L}_{n+1}\,, \tag{3.1}$$

with $\mathbb{C}_0=(2\,p-q)\,+i\,(p+2\,q)$, $\mathbb{C}_1=(p+2\,q)\,+\,i\,(3\,p+q)$, $\mathbb{C}_2=(3\,p+q)\,+\,i\,(4\,p+3\,q)$, where p,q are arbitrary integers. That is, the generalized complex Lucas sequence is

$$\begin{aligned} &(2\,p-q)\,+i\,(p+2\,q)\,,\ (p+2\,q)\,+\,i\,(3\,p+q),\ (3\,p+q)\,+i\,(4\,p+3\,q)\,, \\ &(4\,p+3\,q)\,+\,i\,(7\,p+4\,q)\,,\,\dots\,,\,(\,p-q\,+i\,q\,)L_n+\,(\,q+\,i\,p\,)L_{n+1},\dots \end{aligned} \tag{3.2}$$

Case 1. From the generalized complex Lucas sequence $(\mathbb{C}_n)$ for $p=1,\ q=0$ in the equation (3.2), we obtain complex Lucas sequence (C_n) as follows:

$$(C_n)\ :\ 2\,+\,i\,,\ 1+i\,3\,,\ 3+i\,4\,,\ 4+i\,7,\cdots,\,L_n+\,i\,L_{n+1},\cdots.$$

For the generalized complex Lucas sequence, we have the following properties:

$$\begin{aligned} \mathbb{C}_n^2\,+\,\mathbb{C}_{n-1}^2 &= [\,(2p-\,q)+i\,(p\,+2\,q)\,]\,(\mathbb{C}_{2\,n-2}+\mathbb{C}_{2\,n}\,) \\ &\quad -(\,2\,+i\,)\,e_L\,(L_{2\,n-2}+L_{2\,n}\,)\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathbb{C}_{n+1}^2\,+\,\mathbb{C}_{n-1}^2 &= [\,(2p-\,q)+i\,(p\,+2\,q)\,]\,(\mathbb{C}_{2\,n+2}+\mathbb{C}_{2\,n-2}\,) \\ &\quad -(\,2\,+i\,)\,e_L\,(L_{2\,n+2}+L_{2\,n-2}\,)\,, \end{aligned} \tag{3.4}$$

$$\mathbb{C}_{n-1}\,\mathbb{C}_{n+1}-\mathbb{C}_n^2=5\,(-1)^{n+1}\,(\,2\,+\,i\,)\;e_L\;, \tag{3.5}$$

$$\mathbb{C}_{n+1}^2\,+(\,2\,+i\,)\,e_L\,L_n^2\,=[\,(\,2\,p+q\,)+(\mathbb{C}_{2\,n+2}+\mathbb{C}_{2\,n}\,)\,, \tag{3.6}$$

$$\frac{\mathbb{C}_{n+r}+(-1)^r\mathbb{C}_{n-r}}{C_n}=L_r\ . \tag{3.7}$$

where $e_{\mathbb{C}}=(\,2\,+i\,)\,e_L$.

§4. Generalized Dual Lucas Sequence

In this section, we will define the generalized dual Lucas sequence denoted by $\mathbb{D}_n^L$. The generalized dual Lucas sequence defined by

$$\mathbb{D}_n^L=\mathbb{L}_n+\varepsilon\,\mathbb{L}_{n+1}\ , \tag{4.1}$$

with $\mathbb{D}_0^L=(2\,p-q)+\varepsilon\,(\,p+2\,q)\,,\ \mathbb{D}_1^L=(p+2\,q)+\varepsilon\,(3\,p+q)$ where p,q are arbitrary integers. That is, the generalized dual Lucas sequence is

$$\begin{aligned}&(2\,p-q)\ +\varepsilon\,(3\,p+q)\,,\ (p+2\,q)\ +\ \varepsilon\,(3\,p+q),\ (3\,p+q)\ +\varepsilon\,(4\,p+3\,q),\\&(\,4\,p+3\,q\,)+\varepsilon(7\,p+4\,q),\ (\,7\,p+4\,q\,)+\ \varepsilon(11\,p+7\,q\,),\\&\cdots,\,(\,p-q+\ \varepsilon\,q\,)L_n+\ (\,q+\ \varepsilon p\,)\,L_{n+1},\cdots\end{aligned} \tag{4.2}$$

Using the equations (4.1) and (4.2) , we get

$$\begin{aligned}\mathbb{D}_n^L&=(\,p-q+\ \varepsilon\,q\,)\,L_n+\ (\,q+\varepsilon\,p\,)\,L_{n+1}\,,\\ \mathbb{D}_{n+1}^L&=\ (\,q+\varepsilon\,p\,)\,L_n+[\,p\ +\ \varepsilon\,(\,p+\ q\,)\,]\,L_{n+1}\,,\\ \mathbb{D}_{n+2}^L&=\ [\,p+\varepsilon(\,p+\ q\,)\,]\,L_n+[\,(p+\ q)+\varepsilon\,(\,2\,p+\ q\,)\,]L_{n+1}\,.\end{aligned} \tag{4.3}$$

Case 1. From the generalized dual Lucas sequence $(\mathbb{D}_n^L)$ for $p=1,\ q=0$ in the equation (4.2), we obtain dual Lucas sequence (D_n^L) as follows:

$$(D_n^L)\ :\ 2+\varepsilon\,,\,1+3\,\varepsilon\,,\ 3+4\,\varepsilon\,,\ 4+7\,\varepsilon\,,\ 7+11\,\varepsilon,\ 11+18\,\varepsilon,\cdots,\,L_n+\,\varepsilon\,L_{n+1},\cdots$$

For the generalized dual Lucas sequence, we have the following properties:

$$\begin{aligned}(\mathbb{D}_n^L)^2\ +\ (\mathbb{D}_{n-1}^L)^2\quad&=[\,(2p-q)+\ \varepsilon(p+2\,q)\,]\,\mathbb{D}_{2n-2}^L+\mathbb{D}_{2n}^L\\&\quad-e_{\mathbb{D}}\ (L_{2n-2}+L_{2n})\,,\end{aligned} \tag{4.4}$$

$$\begin{aligned}(\mathbb{D}_{n+1}^L)^2-(\mathbb{D}_n^L)^2\quad&=[\,(2p-q)+\ \varepsilon(p+2\,q)\,]\,\mathbb{D}_{2n+2}^L+\mathbb{D}_{2n-2}^L\\&\quad-e_{\mathbb{D}}\ (L_{2n+2}-L_{2n-2})\,,\end{aligned} \tag{4.5}$$

$$(\mathbb{D}_{n+1}^L)^2+\ e_{\mathbb{D}}\ {L_n}^2=[\,p+\ \varepsilon(p+\ q)\,]\,\mathbb{D}^{\mathbb{L}}{}_{2n+2}+\mathbb{D}_{2n}^L\,, \tag{4.6}$$

$$\mathbb{D}^L_{n-1}\,\mathbb{D}^L_{n+1} - (\mathbb{D}^L_n)^2 = 5\,(-1)^{n+1}\; e_{\mathbb{D}}\,, \tag{4.7}$$

$$\frac{\mathbb{D}^L_{n+r} + (-1)^r\mathbb{D}^L_{n-r}}{\mathbb{D}^L_n} = L_r\ , \tag{4.8}$$

where $e_{\mathbb{D}} = (\,1+\varepsilon)\,e_L$.

Case 2. From properties of the generalized dual Lucas sequence $(\mathbb{D}^L_n)$ for $p=1,\ q=0$ in the equations (4.4) - (4.8), we obtain dual Lucas sequence (D^L_n) as follows:

$$(D^L_n)^2 + \,(D^L_{n-1})^2 = (2+\,\varepsilon)\,D^L_{2n-2} + D^L_{2n} - (1+\,\varepsilon)\;(L_{2n-2} + L_{2n})\,, \tag{4.9}$$

$$(D^L_{n+1})^2 - \,(D^L_n)^2 = (2+\,\varepsilon)\,D^L_{2n+2} + D^L_{2n-2} - (1+\,\varepsilon)\;(L_{2n+2} - L_{2n-2})\,, \tag{4.10}$$

$$(D^L_{n+1})^2 + \,(1+\,\varepsilon)\;{L_n}^2 = (1+\,\varepsilon)\,(D^L_{2n+2} + D^L_{2n})\,, \tag{4.11}$$

$$D^L_{n-1}\,D^L_{n+1} - (D^L_n)^2 = 5\,(-1)^{n+1}\;(1+\,\varepsilon)\,, \tag{4.12}$$

$$\frac{D^L_{n+r} + (-1)^r D^L_{n-r}}{D^L_n} = L_r\ , \tag{4.13}$$

Theorem 4.1 *If $\mathbb{D}^L_n$ is the generalized dual Lucas number, then*

$$\lim_{n\to\infty}\frac{\mathbb{D}^L_{n+1}}{\mathbb{D}^L_n} = \frac{(\,p\,q)\,\alpha^2 + (\,p^2 -\,p\,q +\,q^2\,)\alpha +\,(\,p\,q - q^2)}{q^2\alpha^2 +\,2\,q\,(p-\,q\,)\alpha + (p-q)^2},$$

where $\alpha = 1.618033\cdots$

Proof For the generalized dual Lucas number $\mathbb{D}^L_n$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{D}^L_{n+1}}{\mathbb{D}^L_n} &= \lim_{n\to\infty}\frac{(p-q+\varepsilon q)L_{n+1} + (q+\varepsilon p)L_{n+2}}{(p-q+\varepsilon q)L_n + (q+\varepsilon p)L_{n+1}}\\
&= \lim_{n\to\infty}\frac{(p^2-pq+q^2)L_nL_{n+1} + (pq-q^2)L_n^2 + pqL_{n+1}^2}{q^2L_{n+1}^2 + 2q(p-q)L_nL_{n+1} + (p-q)^2L_n^2}\\
&\quad + \lim_{n\to\infty}\varepsilon\frac{5(-1)^n(p^2-pq-q^2)}{q^2L_{n+1}^2 + 2q(p-q)L_nL_{n+1} + (p-q)^2L_n^2}\\
&= \frac{(pq)\alpha^2 + (p^2-pq+q^2)\alpha + (pq-q^2)}{q^2\alpha^2 + 2q(p-q)\alpha + (p-q)^2},
\end{aligned} \tag{4.14}$$

where $L_{n+2} = L_{n+1} + \,L_n$.

Case 3. For $p=1$, $q=0$ in the equation (4.14), we obtain

$$\lim_{n\to\infty}\frac{\mathbb{D}^L_{n+1}}{\mathbb{D}^L_n}=\lim_{n\to\infty}\frac{D^L_{n+1}}{D^L_n}=\alpha+0=\alpha. \qquad \square$$

Theorem 4.2 *The Binet's formula for the generalized dual Lucas sequence is as follows:*

$$\mathbb{D}^L_n=(\tilde{\alpha}\,\alpha^n+\tilde{\beta}\,\beta^n) \tag{4.15}$$

where $\tilde{\alpha}=(p-q+\varepsilon q)+\alpha(q+\varepsilon p)$ *and* $\tilde{\beta}=(p-q+\varepsilon q)+\beta(q+\varepsilon p)$.

Proof If we use definition of the generalized dual Lucas sequence and substitute first equation in footnote, then we get

$$\begin{aligned}\mathbb{D}^L_n &= (p-q+\varepsilon q)L_n+(q+\varepsilon p)L_{n+1}\\ &= (p-q+\varepsilon q)(\alpha^n+\beta^n)+(q+\varepsilon p)(\alpha^{n+1}+\beta^{n+1})\\ &= \alpha^n(p-q+\varepsilon q+\alpha q+\alpha\varepsilon p)+\beta^n(p-q+\varepsilon q+\beta q+\beta\varepsilon p)\\ &= \tilde{\alpha}\,\alpha^n+\tilde{\beta}\,\beta^n.\end{aligned} \tag{4.16}$$

where $\tilde{\alpha}=(p-q+\varepsilon q)+\alpha(q+\varepsilon p)$ and $\tilde{\beta}=(p-q+\varepsilon q)+\beta(q+\varepsilon p)$. $\square$

§5. Generalized Dual Lucas Vectors

A generalized dual Lucas vector is defined by

$$\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}=(\mathbb{D}^L_n,\mathbb{D}^L_{n+1},\mathbb{D}^L_{n+2})$$

Also, from equations (4.1), (4.2) and (4.3) it can be expressed as

$$\begin{aligned}\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} &= \overrightarrow{\mathbb{L}}_{\mathrm{n}}+\varepsilon\overrightarrow{\mathbb{L}_{\mathrm{n+1}}}\\ &= (p-q+\varepsilon q)\overrightarrow{\mathrm{L}}_n+(q+\varepsilon p)\overrightarrow{\mathrm{L}}_{\mathrm{n+1}}\end{aligned} \tag{5.1}$$

where $\overrightarrow{\mathbb{L}_{\mathrm{n}}}=(\mathbb{L}_n,\mathbb{L}_{n+1},\mathbb{L}_{n+2})$ and $\overrightarrow{\mathrm{L}}_{\mathrm{n}}=(L_n,L_{n+1},L_{n+2})$ are the generalized Lucas vector and the Lucas vector, respectively.

The product of $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\lambda\in\mathbb{R}$ is given by

$$\lambda\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}=\lambda\overrightarrow{\mathbb{L}_{\mathrm{n}}}+\varepsilon\lambda\overrightarrow{\mathbb{L}_{\mathrm{n+1}}}$$

and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}}$ are equal if and only if

$$\begin{aligned}\mathbb{L}_n &= \mathbb{L}_m\\ \mathbb{L}_{n+1} &= \mathbb{L}_{m+1}\\ \mathbb{L}_{n+2} &= \mathbb{L}_{m+2}\end{aligned}$$

Some examples of the generalized dual Lucas vectors can be given easily as:

$$\begin{aligned}\overrightarrow{\mathbb{D}_1^{\mathrm{L}}} &= (\mathbb{D}_1^L\,,\,\mathbb{D}_2^L,\,\mathbb{D}_3^L)\\ &= (\mathbb{L}_1\,,\,\mathbb{L}_2,\,\mathbb{L}_3)+\varepsilon(\mathbb{L}_2\,,\,\mathbb{L}_3,\,\mathbb{L}_4)\\ &= (p+2q)+\varepsilon(3p+q),(3p+q)+\varepsilon(4p+3q),(4p+3q)+\varepsilon(7p+4q))\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathbb{D}_2^{\mathrm{L}}} &= (\mathbb{L}_2\,,\,\mathbb{L}_3,\,\mathbb{L}_4)+\varepsilon(\mathbb{L}_3\,,\,\mathbb{L}_4,\,\mathbb{L}_5)\\ &= ((3p+q)+\varepsilon(4p+3q),(4p+3q)+\varepsilon(7p+4q),(7p+4q)+\varepsilon(11p+18q))\end{aligned}$$

Theorem 5.1 *Let* $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}$ *and* $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}$ *be two generalized dual Lucas vectors. The dot product of* $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}$ *and* $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}$ *is given by*

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}},\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}\right\rangle = \;& p^2[(L_n\,L_m+5\,F_{n+m+3})+\varepsilon(L_n\,L_{m+1}+L_{n+1}\,L_m+10\,F_{n+m+4})\,]\\ &+p\,q\,[(5\,L_{n+m}+10\,F_{n+m+2})\\ &+\varepsilon(L_{n-1}\,L_m+L_n\,L_{m-1}+10\,F_{n+m-1}+20\,F_{n+m+3})\,]\\ &+q^2[(L_{n-1}\,L_{m-1}+5\,F_{n+m+1})\\ &+\varepsilon(L_{n-1}\,L_m+L_n\,L_{m-1}+10\,F_{n+m+2})\,]\end{aligned} \tag{5.2}$$

Proof The dot product of $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}=(\mathbb{D}_n^L\,,\,\mathbb{D}_{n+1}^L,\,\mathbb{D}_{n+2}^L)$ and $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}=(\mathbb{D}_m^L\,,\,\mathbb{D}_{m+1}^L,\,\mathbb{D}_{m+2}^L)$ defined by

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}},\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}\right\rangle &= \mathbb{D}_n^L\mathbb{D}_m^L+\mathbb{D}_{n+1}^L\mathbb{D}_{m+1}^L+\mathbb{D}_{n+2}^L\mathbb{D}_{m+2}^L\\ &= \left\langle\overrightarrow{\mathbb{L}_{\mathrm{n}}},\overrightarrow{\mathbb{L}_{\mathrm{m}}}\right\rangle+\varepsilon\left[\left\langle\overrightarrow{\mathbb{L}_{\mathrm{n}}},\overrightarrow{\mathbb{L}_{\mathrm{m+1}}}\right\rangle+\left\langle\overrightarrow{\mathbb{L}_{\mathrm{n+1}}},\overrightarrow{\mathbb{L}_{\mathrm{m}}}\right\rangle\right]\end{aligned}$$

where $\overrightarrow{\mathbb{L}_{\mathrm{n}}}=(\,\mathbb{L}_n\,,\,\mathbb{L}_{n+1},\,\mathbb{L}_{n+2})$ is the generalized Lucas vector. Also, the equations (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{L}_{\mathrm{n}}},\overrightarrow{\mathbb{L}_{\mathrm{m}}}\right\rangle = \;& p^2(L_n\,L_m+5\,F_{n+m+3})\\ &+p\,q\,(5\,F_{n+m}+10\,F_{n+m+2})\\ &+q^2(L_{n-1}\,L_{m-1}\,+5\,F_{n+m+1})\end{aligned} \tag{5.3}$$

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{L}_{\mathrm{n}}},\overrightarrow{\mathbb{L}_{\mathrm{m+1}}}\right\rangle = \;& p^2(L_n\,L_{m+1}+5\,F_{n+m+4})\\ &+p\,q\,(5\,F_{n+m-1}+10\,F_{n+m+3}+L_{n-1}\,L_m)\\ &+q^2(L_{n-1}\,L_m\,+5\,F_{n+m+2})\,,\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{L}_{\mathrm{n+1}}},\overrightarrow{\mathbb{L}_{\mathrm{m}}}\right\rangle = \;& p^2(L_{n+1}\,L_m+5\,F_{n+m+4})\\ &+p\,q\,(5\,F_{n+m-1}+10\,F_{n+m+3}+L_n\,L_{m-1})\\ &+q^2(L_n\,L_{m-1}\,+5\,F_{n+m+2})\end{aligned} \tag{5.5}$$

Then from equation (5.3), (5.4) and (5.5), we have the equation (5.2). □

Case 1. For the dot product of generalized dual Lucas vectors $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n+1}}}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n+1}}} \right\rangle &= \mathbb{D}^L_n \mathbb{D}^L_{n+1} + \mathbb{D}^L_{n+1}\mathbb{D}^L_{n+2} + \mathbb{D}^L_{n+2}\mathbb{D}^L_{n+3} \\ &= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle + \varepsilon \{ \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+2}}} \right\rangle + \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n+1}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle \} \\ &= p^2[(L_n\, L_{n+1} + 5\, F_{2n+4}) \\ &+\varepsilon (L_n\, L_{n+2} + L_{n+1}\, L_{n+1} + 10\, F_{2n+5})\,] \\ &+p\, q\,[(5\, L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3}) \\ &+\varepsilon (L_{n+1}\, L_{n+2} + 5\, F_{2n} + 10\, F_{2n+4})\,] \\ &+q^2[(L_{n-1}\, L_n + 5\, F_{2n+2}) \\ &+\varepsilon (L_{n-1}\, L_{n+1} + L_n\, L_n + 10\, F_{2n+3})\,]\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\rangle &= (\mathbb{D}^L_n)^2 + (\mathbb{D}^L_{n+1})^2 + (\mathbb{D}^L_{n+2})^2 \\ &= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\rangle + 2\,\varepsilon \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle \\ &= p^2[(L_n\, L_n + 5\, F_{2n+3}) \\ &+2\,\varepsilon (L_n\, L_{n+1} + 5\, F_{2n+4})\,] \\ &+p\, q\,[(5\, F_{2n} + 10\, F_{2n+2}) \\ &+2\,\varepsilon (L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3})\,] \\ &+q^2[(L_{n-1}\, L_{n-1} + 5\, F_{2n+1}) \\ &+2\,\varepsilon (L_{n-1}\, L_n + 5\, F_{2n+2})\,]\,.\end{aligned} \tag{5.7}$$

Then for the norm of the generalized dual Lucas vector [3], we have

$$\begin{aligned}\left\| \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\| &= \sqrt{\left[\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\rangle \right]} = \sqrt{[(\mathbb{D}^L_n)^2 + (\mathbb{D}^L_{n+1})^2 + (\mathbb{D}^L_{n+2})^2]} \\ &= \sqrt{p^2(L_n\, L_n + 5\, F_{2n+3}) + p\, q(5\, F_{2n} + 10\, F_{2n+2})} \\ &+ \sqrt{q^2(L_{n-1}\, L_{n-1} + 5\, F_{2n+1})]} \\ &+ \sqrt{2\varepsilon\{p^2(L_n\, L_{n+1} + 5\, F_{2n+4}) + pq(L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3})} \\ &+ \sqrt{q^2(L_{n-1}\, L_n + 5\, F_{2n+2})}\}.\end{aligned} \tag{5.8}$$

Case 2. For $p = 1$, $q = 0$, in the equations (5.2), (5.6) and (5.8), we have

$$\left\langle \overrightarrow{D^L_{\mathrm{n}}}, \overrightarrow{D^L_{\mathrm{m}}} \right\rangle = [(L_n\, L_m + 5\, F_{n+m+3}) + \varepsilon (L_n\, L_{m+1} + L_{n+1}\, L_m + 10\, F_{n+m+4})\,],$$

---

[3] Norm of dual number as follows ([2], [14]):

$$\left\| \overrightarrow{\mathbb{A}} \right\| = \sqrt{a + \varepsilon\, a^*} = \sqrt{a} + \varepsilon\, a^* \frac{1}{2\sqrt{a}}, A = a + \varepsilon\, a^*$$

$$\left\langle \overrightarrow{D_{\rm n}^L}, \overrightarrow{D_{\rm n+1}^L} \right\rangle = [(L_n\, L_{n+1} + 5\, F_{2n+4}) + \varepsilon(L_n\, L_{n+2} + L_{n+1}\, L_{n+1} + 10\, F_{2n+5})\,]$$

and

$$\begin{aligned} \left\| \overrightarrow{D_{\rm n}^L} \right\| &= \sqrt{(L_n\, L_n + 5\, F_{2n+3}) + 2\,\varepsilon\,(L_n\, L_{n+1} + 5\, F_{2n+4})} \\ &= (L_n\, L_n + 5\, F_{2n+3}) + \varepsilon\, \frac{(L_n\, L_{n+1} + 5\, F_{2n+4})}{\sqrt{(L_n\, L_n + 5\, F_{2n+3})}}. \end{aligned}$$

Theorem 5.2 *Let $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}}$ and $\overrightarrow{\mathbb{D}_{\rm m}^{\rm L}}$ be two generalized dual Lucas vectors. The cross product of $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}}$ and $\overrightarrow{\mathbb{D}_{\rm m}^{\rm L}}$ is given by*

$$\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} \times \overrightarrow{\mathbb{D}_{\rm m}^{\rm L}} = 5\,(-1)^{n+1}\, F_{m-n}\,(1+\varepsilon)\, e_L\ (i+j-k). \tag{5.9}$$

Proof The cross product of $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} = \overrightarrow{\mathbb{L}_{\rm n}} + \varepsilon\, \overrightarrow{\mathbb{L}_{\rm n+1}}$ and $\overrightarrow{\mathbb{D}_{\rm m}^{\rm L}} = \overrightarrow{\mathbb{L}_{\rm m}} + \varepsilon\, \overrightarrow{\mathbb{L}_{\rm m+1}}$ defined by

$$\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} \times \overrightarrow{\mathbb{D}_{\rm m}^{\rm L}} = (\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}) + \varepsilon\,(\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m+1}} + \overrightarrow{\mathbb{L}_{\rm n+1}} \times \overrightarrow{\mathbb{L}_{\rm m}})$$

where $\overrightarrow{\mathbb{L}_{\rm n}}$ is the generalized Lucas vector and $\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}$ is the cross product for the generalized Lucas vectors $\overrightarrow{\mathbb{L}_{\rm n}}$ and $\overrightarrow{\mathbb{L}_{\rm m}}$.

Now, we calculate the cross products $\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}$, $\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m+1}}$ and $\overrightarrow{\mathbb{L}_{\rm n+1}} \times \overrightarrow{\mathbb{L}_{\rm m}}$:
Using the property $L_nL_{m+1} - L_{n+1}L_m = 5\,(-1)^n\, F_{m-n}$, we get

$$\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}} = 5\,(-1)^{n+1}\, F_{m-n}\,(i+j-k)\, e_L, \tag{5.10}$$

$$\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m+1}} = 5\,(-1)^{n+1}\, F_{m-n+1}(i+j-k)\, e_L, \tag{5.11}$$

and

$$\overrightarrow{\mathbb{L}_{\rm n+1}} \times \overrightarrow{\mathbb{L}_{\rm m}} = 5\,(-1)^{n+2}\, F_{m-n-1}(i+j-k)\, e_L. \tag{5.12}$$

Then from the equations (5.10), (5.11) and (5.12), we obtain the equation (5.9). □

Case 3. For $p = 1$, $q = 0$ in the equations (5.9), we have

$$\overrightarrow{D_{\rm n}^L} \times \overrightarrow{D_{\rm m}^L} = 5\,(-1)^{n+1}\, F_{m-n}\,(1+\varepsilon)\,(i+j-k).$$

Theorem 5.3 *Let $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}}$, $\overrightarrow{\mathbb{D}_{\rm m}^{\rm L}}$ and $\overrightarrow{\mathbb{D}_{\rm k}^{\rm L}}$ be the generalized dual Lucas vectors. The mixed product of these vectors is*

$$\left\langle \overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} \times \overrightarrow{\mathbb{D}_{\rm m}^{\rm L}}\ ,\ \overrightarrow{\mathbb{D}_{\rm k}^{\rm L}} \right\rangle = 0. \tag{5.13}$$

Proof Using the properties

$$\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} \times \overrightarrow{\mathbb{D}_{\rm m}^{\rm L}} = (\,\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}\,) + \varepsilon\,(\,\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m+1}} + \overrightarrow{\mathbb{L}_{\rm n+1}} \times \overrightarrow{\mathbb{L}_{\rm m}}\,)$$

and

$$\overrightarrow{\mathbb{D}^{L}_{k}} = \overrightarrow{\mathbb{L}_{k}} + \varepsilon\, \overrightarrow{\mathbb{L}_{k+1}},$$

we can write,

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{L}_{n}} \times \overrightarrow{\mathbb{D}^{L}_{m}},\ \overrightarrow{\mathbb{D}^{L}_{k}} \right\rangle &= \left\langle \overrightarrow{\mathbb{L}_{n}} \times \overrightarrow{\mathbb{L}_{m}}, \overrightarrow{\mathbb{L}_{k}} \right\rangle + \varepsilon \left[\left\langle \overrightarrow{\mathbb{L}_{n}} \times \overrightarrow{\mathbb{L}_{m}}, \overrightarrow{\mathbb{L}_{k+1}} \right\rangle \right. \\ &\quad \left. + \left\langle \overrightarrow{\mathbb{L}_{n}} \times \overrightarrow{\mathbb{L}_{m+1}}, \overrightarrow{\mathbb{L}_{k}} \right\rangle + \left\langle \overrightarrow{\mathbb{L}_{n+1}} \times \overrightarrow{\mathbb{L}_{m}}, \overrightarrow{\mathbb{L}_{k+1}} \right\rangle \right].\end{aligned}$$

Then using equations (5.10), (5.11) and (5.12), we obtain

$$\left\langle (i + j - k),\ \overrightarrow{\mathbb{L}_{k}} \right\rangle = \mathbb{L}_{k} + \mathbb{L}_{k+1} - \mathbb{L}_{k+2} = 0,$$

$$\left\langle (i + j - k),\ \overrightarrow{\mathbb{L}_{k+1}} \right\rangle = \mathbb{L}_{k+1} + \mathbb{L}_{k+2} - \mathbb{L}_{k+3} = 0\,.$$

Thus, we have the equation (5.13). □

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Fixed Point Results Under Generalized Contraction Involving Rational Expression in Complex Valued Metric Spaces

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Abstract: The aim of this paper is to study common fixed point under generalized contraction involving rational expression in the setting of complex valued metric spaces. The results presented in this paper extend and generalize several results from the existing literature.

Key Words: Common fixed point, generalized contraction involving rational expression, complex valued metric space.

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§1. Introduction

Fixed point theory plays a very crucial role in the development of nonlinear analysis. The Banach [2] fixed point theorem for contraction mapping has been generalized and extended in many directions. This famous theorem can be stated as follows.

Theorem 1.1([2]) *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq \alpha\, d(x, y),\ \forall\, x, y \in X, \tag{1.1}$$

where α is a constant in $[0, 1)$. Then T has a fixed point $p \in X$.

The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [4, 5].

Recently, Azam et al. [1] introduced the concept of complex valued metric space and established some fixed point results for mappings satisfying a rational inequality. Complex-valued metric space is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering and electrical engineering, for more details, see, [7, 8].

In this paper, we establish common fixed point results for generalized contraction involving rational expression in the framework of complex valued metric spaces.

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§2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$, $Im(z_1) \le Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(*i*) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(*ii*) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(*iii*) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(*iv*) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), or (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition was introduced by Azam et al. in 2011 (see, [1]).

Definition 2.1([1]) *Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{C}$ satisfies:*

(C_1) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *with* $x \neq y$ *and* $d(x, y) = 0 \Leftrightarrow x = y$*;*
(C_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(C_3) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.2 Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{it}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $t \in [0, \frac{\pi}{2}]$. Then (X, d) is a complex valued metric space.

Example 2.3([1]) Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{3i}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then (X, d) is a complex valued metric space.

Example 2.4 Let $X = \mathbb{C}$. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ia}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and a is any real constant. Then (X, d) is a complex valued metric space.

Definition 2.5 (*i*) *A point $x \in X$ is called an interior point of a subset $G \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that*

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq G.$$

(ii) A point $x \in X$ is called a limit of G whenever for every $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) \cap \Big(G - \{x\}\Big) \neq \emptyset.$$

(iii) The set $G \subseteq X$ is called open whenever each element of G is an interior point of G. A subset $H \subseteq X$ is called closed whenever each limit point of H belongs to H.

The family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.6([1]) *Let (X,d) be a complex valued metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(i) $\{x_n\}$ is called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.

(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$. If every Cauchy sequence converges in X, then X is called a complete complex valued metric space.

Definition 2.7([6]) *Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if*

(i) $T_iT_j = T_jT_i$, $i,j \in \{1,2,\cdots,m\}$;
(ii) $S_kS_l = S_lS_k$, $k,l \in \{1,2,\cdots,n\}$;
(iii) $T_iS_k = S_kT_i$, $i \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\cdots,n\}$.

Lemma 2.8([1]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n, x)| = 0$.*

Lemma 2.9([1]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.*

§3. Main Results

In this section we shall prove some common fixed point results under generalized contraction involving rational expression in the framework of complex valued metric spaces.

Theorem 3.1 *Let (X,d) be a complete complex valued metric space. Suppose that the mappings $S, T\colon X \to X$ satisfy:*

$$\begin{aligned} d(Sx,Ty) \quad \precsim \quad & \alpha\, d(x,y) + \beta\Big[\frac{d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)}{d(x,Ty) + d(y,Sx)}\Big] \\ & +\gamma\, d(x,Sx) + \delta\, d(y,Ty) \\ & +\lambda\,[d(x,Ty) + d(y,Sx)] \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha + \beta + \gamma + \delta + 2\lambda < 1$. Then S

and T have a unique common fixed point in X.

Proof Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k},\ x_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \cdots .$$

Then from (3.1), we have

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\
&\precsim \alpha\, d(x_{2k}, x_{2k+1}) \\
&\quad +\beta \Big[\frac{d(x_{2k}, Sx_{2k}) d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Tx_{2k+1}) d(x_{2k+1}, Sx_{2k})}{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})}\Big] \\
&\quad +\gamma\, d(x_{2k}, Sx_{2k}) + \delta\, d(x_{2k+1}, Tx_{2k+1}) \\
&\quad +\lambda\, [d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})] \\
&= \alpha\, d(x_{2k}, x_{2k+1}) \\
&\quad +\beta \Big[\frac{d(x_{2k}, x_{2k+1}) d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+2}) d(x_{2k+1}, x_{2k+1})}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})}\Big] \\
&\quad +\gamma\, d(x_{2k}, x_{2k+1}) + \delta\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\lambda\, [d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})] \\
&\precsim (\alpha + \beta + \gamma) d(x_{2k}, x_{2k+1}) + \delta\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\lambda [d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\
&= (\alpha + \beta + \gamma + \lambda) d(x_{2k}, x_{2k+1}) + (\delta + \lambda) d(x_{2k+1}, x_{2k+2}). \qquad (3.2)
\end{aligned}$$

This implies that

$$d(x_{2k+1}, x_{2k+2}) \precsim \Big(\frac{\alpha + \beta + \gamma + \lambda}{1 - \delta - \lambda}\Big)\, d(x_{2k}, x_{2k+1}). \qquad (3.3)$$

Similarly, we have

$$\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+1}, Tx_{2k+2}) \\
&\precsim \alpha\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\beta \Big[\frac{d(x_{2k+1}, Sx_{2k+1}) d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Tx_{2k+2}) d(x_{2k+2}, Sx_{2k+1})}{d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})}\Big] \\
&\quad +\gamma\, d(x_{2k+1}, Sx_{2k+1}) + \delta\, d(x_{2k+2}, Tx_{2k+2}) \\
&\quad +\lambda\, [d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})] \\
&= \alpha\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\beta \Big[\frac{d(x_{2k+1}, x_{2k+2}) d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+3}) d(x_{2k+2}, x_{2k+2})}{d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})}\Big] \\
&\quad +\gamma\, d(x_{2k+1}, x_{2k+2}) + \delta\, d(x_{2k+2}, x_{2k+3}) \\
&\quad +\lambda\, [d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})] \\
&\precsim (\alpha + \beta + \gamma) d(x_{2k+1}, x_{2k+2}) + \delta\, d(x_{2k+2}, x_{2k+3}) \\
&\quad +\lambda [d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\
&= (\alpha + \beta + \gamma + \lambda) d(x_{2k+1}, x_{2k+2}) + (\delta + \lambda) d(x_{2k+2}, x_{2k+3}). \qquad (3.4)
\end{aligned}$$

This implies that

$$d(x_{2k+2}, x_{2k+3}) \precsim \Big(\frac{\alpha+\beta+\gamma+\lambda}{1-\delta-\lambda}\Big)\, d(x_{2k+1}, x_{2k+2}). \tag{3.5}$$

Putting

$$h = \Big(\frac{\alpha+\beta+\gamma+\lambda}{1-\delta-\lambda}\Big).$$

As $\alpha+\beta+\gamma+\delta+2\lambda < 1$, it follows that $0 < h < 1$, we have

$$d(x_{n+1}, x_{n+2}) \precsim h\, d(x_n, x_{n+1}) \precsim \cdots \precsim h^{n+1} d(x_0, x_1). \tag{3.6}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + d(x_{n+m-1}, x_m) \\ &\precsim [h^n + h^{n+1} + h^{n+2} + \cdots + h^{n+m-1}] d(x_1, x_0) \\ &\precsim \Big[\frac{h^n}{1-h}\Big] d(x_1, x_0) \end{aligned}$$

and so

$$|d(x_n, x_m)| \leq \Big[\frac{h^n}{1-h}\Big] |d(x_1, x_0)| \to 0 \text{ as } m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $w \in X$ such that $x_n \to w$ as $n \to \infty$. It follows that $w = Sw$, otherwise $d(w, Sw) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(w, x_{2n+2}) + d(x_{2n+2}, Sw) \precsim d(w, x_{2n+2}) + d(Sw, Tx_{2n+1}) \\ &\precsim d(w, x_{2n+2}) + \alpha\, d(w, x_{2n+1}) \\ &\quad +\beta \Big[\frac{d(w, Sw) d(w, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1}) d(x_{2n+1}, Sw)}{d(w, Tx_{2n+1}) + d(x_{2n+1}, Sw)}\Big] \\ &\quad +\gamma\, d(w, Sw) + \delta\, d(x_{2n+1}, Tx_{2n+1}) + \lambda\, [d(w, Tx_{2n+1}) + d(x_{2n+1}, Sw)] \\ &= d(w, x_{2n+2}) + \alpha\, d(w, x_{2n+1}) \\ &\quad +\beta \Big[\frac{d(w, Sw) d(w, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) d(x_{2n+1}, Sw)}{d(w, x_{2n+2}) + d(x_{2n+1}, Sw)}\Big] \\ &\quad +\gamma\, d(w, Sw) + \delta\, d(x_{2n+1}, x_{2n+2}) + \lambda\, [d(w, x_{2n+2}) + d(x_{2n+1}, Sw)]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &\leq |d(w, x_{2n+2})| + \alpha\, |d(w, x_{2n+1})| \\ &\quad +\beta \Big[\frac{|z||d(w, x_{2n+2})| + |d(x_{2n+1}, x_{2n+2})||d(x_{2n+1}, Sw)|}{|d(w, x_{2n+2})| + |d(x_{2n+1}, Sw)|}\Big] \\ &\quad +\gamma\, |z| + \delta\, |d(x_{2n+1}, x_{2n+2})| + \lambda\, [|d(w, x_{2n+2})| + |d(x_{2n+1}, Sw)|]. \end{aligned}$$

Letting $n \to \infty$, it follows that

$$|z| \leq (\gamma + \lambda)|z| \leq (\alpha + \beta + \gamma + \delta + 2\lambda)|z| < |z|$$

which is a contradiction and so $|z| = 0$, that is, $w = Sw$.

In an exactly the same way, we can prove that $w = Tw$. Hence $Sw = Tw = w$. This shows that w is a common fixed point of S and T.

We now show that S and T have a unique common fixed point. For this, assume that w^* is another common fixed point of S and T, that is, $Sw^* = Tw^* = w^*$ such that $w \neq w^*$. Then

$$\begin{aligned} d(w, w^*) &= d(Sw, Tw^*) \\ &\precsim \alpha\, d(w, w^*) + \beta \Big[\frac{d(w, Sw)d(w, Tw^*) + d(w^*, Tw^*)d(w^*, Sw)}{d(w, Tw^*) + d(w^*, Sw)}\Big] \\ &\quad + \gamma\, d(w, Sw) + \delta\, d(w^*, Tw^*) + \lambda\, [d(w, Tw^*) + d(w^*, Sw)] \\ &= \alpha\, d(w, w^*) + \beta \Big[\frac{d(w, w)d(w, w^*) + d(w^*, w^*)d(w^*, w)}{d(w, w^*) + d(w^*, w)}\Big] \\ &\quad + \gamma\, d(w, w) + \delta\, d(w^*, w^*) + \lambda\, [d(w, w^*) + d(w^*, w)] \\ &= (\alpha + 2\lambda)(w, w^*) \end{aligned}$$

So that $|d(w, w^*)| \leq (\alpha + 2\lambda)(w, w^*) < |d(w, w^*)|$, since $0 < (\alpha + 2\lambda) < 1$, which is a contradiction and hence $d(w, w^*) = 0$. Thus $w = w^*$. This shows that S and T have a unique common fixed point in X. This completes the proof. □

Putting $S = T$ in Theorem 3.1, we have the following result.

Corollary 3.2 *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\begin{aligned} d(Tx, Ty) &\precsim \alpha\, d(x, y) + \beta \Big[\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}\Big] \\ &\quad + \gamma\, d(x, Tx) + \delta\, d(y, Ty) + \lambda\, [d(x, Ty) + d(y, Tx)] \end{aligned} \tag{3.7}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha + \beta + \gamma + \delta + 2\lambda < 1$. Then T has a unique fixed point in X.

Corollary 3.3 *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$\begin{aligned} d(T^n x, T^n y) &\precsim \alpha\, d(x, y) + \beta \Big[\frac{d(x, T^n x)d(x, T^n y) + d(y, T^n y)d(y, T^n x)}{d(x, T^n y) + d(y, T^n x)}\Big] \\ &\quad + \gamma\, d(x, T^n x) + \delta\, d(y, T^n y) + \lambda\, [d(x, T^n y) + d(y, T^n x)] \end{aligned} \tag{3.8}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha + \beta + \gamma + \delta + 2\lambda < 1$. Then T has a unique fixed point in X.

Proof By Corollary 3.2, there exists $q \in X$ such that $T^n q = q$. Then

$$\begin{aligned}
d(Tq, q) &= d(TT^n q, T^n q) = d(T^n Tq, T^n q) \\
&\precsim \alpha\, d(Tq, q) \\
&\quad + \beta \Big[\frac{d(Tq, T^n Tq) d(Tq, T^n q) + d(q, T^n q) d(q, T^n Tq)}{d(Tq, T^n q) + d(q, T^n Tq)}\Big] \\
&\quad + \gamma\, d(Tq, T^n Tq) + \delta\, d(q, T^n q) + \lambda\, [d(Tq, T^n q) + d(q, T^n Tq)] \\
&= \alpha\, d(Tq, q) \\
&\quad + \beta \Big[\frac{d(Tq, TT^n q) d(Tq, T^n q) + d(q, T^n q) d(q, TT^n q)}{d(Tq, T^n q) + d(q, TT^n q)}\Big] \\
&\quad + \gamma\, d(Tq, TT^n q) + \delta\, d(q, T^n q) + \lambda\, [d(Tq, T^n q) + d(q, TT^n q)] \\
&= \alpha\, d(Tq, q) \\
&\quad + \beta \Big[\frac{d(Tq, Tq) d(Tq, q) + d(q, q) d(q, Tq)}{d(Tq, q) + d(q, Tq)}\Big] \\
&\quad + \gamma\, d(Tq, Tq) + \delta\, d(q, q) + \lambda\, [d(Tq, q) + d(q, Tq)] \\
&= (\alpha + 2\lambda)\, d(Tq, q).
\end{aligned}$$

So that $|d(Tq, q)| \leq (\alpha + 2\lambda)\, |d(Tq, q)| < |d(Tq, q)|$, since $0 < (\alpha + 2\lambda) < 1$, which is a contradiction and hence $d(Tq, q) = 0$. Thus $Tq = q$. This shows that T has a unique fixed point in X. This completes the proof. □

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 3.4 *If $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are two finite pairwise commuting finite families of self-mappings defined on a complete complex valued metric space (X, d) such that S and T (with $T = T_1 T_2 \cdots T_m$ and $S = S_1 S_2 \cdots S_n$) satisfy the condition* (3.1)*, then the component maps of the two families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ have a unique common fixed point.*

Proof In view of Theorem 3.1 one can conclude that T and S have a unique common fixed point g, that is, $T(g) = S(g) = g$. Now we are required to show that g is a common fixed point of all the components maps of both the families. In view of pairwise commutativity of the families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$, (for every $1 \leq k \leq m$) we can write

$$T_k(g) = T_k S(g) = S T_k(g) \quad \text{and} \quad T_k(g) = T_k T(g) = T T_k(g)$$

which show that $T_k(g)$ (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_k(g) = g$ (for every k) which shows that g is a common fixed point of the family $\{T_i\}_{i=1}^m$. Using the same arguments as above, one can also show that (for every $1 \leq k \leq n$) $S_k(g) = g$. This completes the proof. □

By taking $T_1 = T_2 = \cdots = T_m = G$ and $S_1 = S_2 = \cdots = S_n = F$, in Theorem 3.4, we derive the following result involving iterates of mappings.

Corollary 3.5 *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X,d) satisfying the condition*

$$\begin{aligned} d(F^n x, G^m y) &\precsim \alpha\, d(x,y) + \beta \Big[\frac{d(x,F^n x)d(x,G^m y) + d(y,G^m y)d(y,F^n x)}{d(x,G^m y) + d(y,F^n x)}\Big] \\ &\quad + \gamma\, d(x,F^n x) + \delta\, d(y,G^m y) + \lambda\, [d(x,G^m y) + d(y,F^n x)] \end{aligned} \tag{3.9}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then F and G have a unique common fixed point in X.

By setting $m = n$ and $F = G = T$ in Corollary 3.5, we deduce the following result.

Corollary 3.6 *Let (X,d) be a complete complex valued metric space and let the mapping $T\colon X \to X$ satisfies (for fixed n)*

$$\begin{aligned} d(T^n x, T^n y) &\precsim \alpha\, d(x,y) + \beta \Big[\frac{d(x,T^n x)d(x,T^n y) + d(y,T^n y)d(y,T^n x)}{d(x,T^n y) + d(y,T^n x)}\Big] \\ &\quad + \gamma\, d(x,T^n x) + \delta\, d(y,T^n y) + \lambda\, [d(x,T^n y) + d(y,T^n x)] \end{aligned} \tag{3.10}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then T has a unique fixed point in X.

Proof By Corollary 3.2, we obtain $p \in X$ such that $T^n p = p$. The rest of the proof is same as that of Corollary 3.3. This completes the proof. □

By taking $\alpha = h$ and $\beta = \gamma = \delta = \lambda = 0$ in Corollary 3.3, we draw following corollary which can be viewed as an extension of Bryant (see, [4]) theorem to complex valued metric space.

Corollary 3.7 *Let (X,d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfying the condition*

$$d(T^n x, T^n y) \precsim h\, d(x,y)$$

for all $x, y \in X$ and $h \in [0,1)$ is a constant. Then T has a unique fixed point in X.

The following example demonstrates the superiority of Bryant (see, [3]) theorem over Banach contraction theorem.

Example 3.8 Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ by $d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$ where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $(\mathbb{C}, d)$ is a complex valued metric space. Define $T\colon \mathbb{C} \to \mathbb{C}$ as

$$T(x+iy) = \begin{cases} 0, & \text{if } x, y \in Q, \\ i, & \text{if } x, y \in Q^c, \\ 1, & \text{if } x \in Q^c, y \in Q, \\ 1+i, & \text{if } x \in Q, y \in Q^c. \end{cases}$$

Now for $x = \frac{1}{\sqrt{2}}$ and $y = 0$, we get

$$d(T(\frac{1}{\sqrt{2}}), T(0)) = d(1,0) \precsim \lambda d(\frac{1}{\sqrt{2}}, 0) = \frac{\lambda}{\sqrt{2}}.$$

Thus $\lambda \geq \sqrt{2}$ which is a contradiction that $0 \leq \lambda < 1$. However, we notice that $T^2(z) = 0$, so that

$$0 = d(T^2(z_1), T^2(z_2)) \precsim \lambda d(z_1, z_2),$$

which shows that T^2 satisfies the requirement of Bryant theorem and $z = 0$ is a unique fixed point of T.

Finally, we conclude this paper with an illustrative example which satisfied all the conditions of Corollary 3.2.

Example 3.9 Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $' \precsim '$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued metric d be given as

$$d(x,y) = |x-y|\sqrt{2}e^{i\frac{\pi}{4}} = |x-y|(1+i) \text{ for } x, y \in X.$$

Let $T\colon X \to X$ be defined as follows:

$$T(0) = 0, T(\frac{1}{2}) = 0, T(2) = \frac{1}{2}.$$

Case 1. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$d(Tx, Ty) = 0 \leq \Big(\frac{1+i}{2}\Big)(\alpha + \beta + \gamma + \lambda).$$

This implies that $\alpha = \beta = \gamma = 0$ and $\delta = \lambda = \frac{1}{2}$ or $\alpha = \beta = \gamma = \frac{1}{9}$ and $\delta = \lambda = \frac{1}{6}$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

Case 2. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1+i}{2} \leq \alpha.\Big(\frac{3(1+i)}{2}\Big) + \beta.\Big(\frac{3(1+i)}{2}\Big) + \gamma.\Big(\frac{3(1+i)}{2}\Big) \\ &\quad + \delta.\frac{1+i}{2} + \lambda.2(1+i). \end{aligned}$$

This implies that $\alpha = \beta = \gamma = \delta = \lambda = \frac{1}{13}$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

Case 3. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Corollary 3.2, then we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1+i}{2} \leq \alpha.2(1+i) + \beta.\Big(\frac{3(1+i)}{2}\Big) + \gamma.\Big(\frac{3(1+i)}{2}\Big) \\ &\quad +\lambda.\frac{5(1+i)}{2}. \end{aligned}$$

This implies that $\alpha = \beta = \gamma = \lambda = \frac{1}{14}$ and $\delta = 0$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

§4. Conclusion

In this paper, we establish common fixed point theorems using generalized contraction involving rational expression in the setting of complex-valued metric spaces and give an example in support of our result. Our results extend and generalize several results from the current existing literature.

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A Study on Cayley Graphs over Dihedral Groups

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Abstract: Let G be the dihedral group D_n and $Cay(G,S)$ is the Cayley graph of G with respect to S , and let $C_G(x)$ is the centralizer of an element x in G and $\bar{x}$ is the orbit of x in G. In this paper, we prove that if G act on G by conjugation, the vertex induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\bar{x})$ is either Hamiltonian or it contain Hamiltonian decompositions. But if n is prime, it is always Hamiltonian.

Key Words: Smarandache-Cayley graph, Cayley graph, dihedral group, Hamiltonian cycle, complete graph.

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§1. Introduction

Let $(G;\cdot)$ be a finite group. A *Smarandache-Cayley graph* of G respect to a pair $\{S,T\}$ of non-empty subsets $S\subset G$, $T\subset G\setminus S$ is the graph with vertex set G and edge set consisting of pairs (x,y) such that $s\cdot x=t\cdot y$, where $s\in S$ and $t\in T$. Particularly, let $T=\{1_G\}$. Then such a Smarandache-Cayley graph is the usual Cayley graph $Cay(G,S)$, whose vertex set is G and edges are the pairs (x,y) such that $s\cdot x=y$ for some $s\in S$ and $x\neq y$. Arthur Cayley (1878) introduced the Cayley graphs of groups and it has received much attention in the literature. Brian Alspach et al. (2010) proved that every connected Cayley graphs of valency at least three on a generalized dihedral group, whose order is divisible by four is Hamilton-connected, unless it is bipartite. Recently Adrian Pastine and Daniel Jaume (2012) proved that given a dihedral group D_H and a generating subset S, if $S\cap H\neq\phi$, then the Cayley digraph $Cay(D_H,S)$ is Hamiltonian. In this paper, we denote a group $(G;\cdot)$ by G for convenience.

§2. Main Results

In this section we deals with some basic definitions and terminologies of group theory and graph theory which are needed in sequel. For details see Fraleigh (2003), Gallian (2009) and Diestel (2010).

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Definition 2.1 *Let G be a group. The orbit of an element x under G is usually denoted as $\bar{x}$ and is defined as $\bar{x} = \{gx/g \in G\}$.*

Definition 2.2 *Let x be a fixed element in a group G . The centralizer of an element x in G , $C_G(x)$ is the set of all element in G that commute with x. In symbols, $C_G(x) = \{g \in G/gx = xg\}$.*

Definition 2.3 *A group G act on G by conjugation means $gx = gxg^{-1}$ for all $x \in G$.*

Definition 2.4 *An element x in a group G is called an involution if $x^2 = e$.*

Definition 2.5 *The n^{th} dihedral group D_n is the group of symmetries of the regular n-gon and $D_n \subset S_n$, where S_n is the symmetric group of n letters for $n \geq 3$ with $|D_n| = 2n$.*

The structure of D_n is $\{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$, where g denote rotation by $\frac{2\pi}{n}$ and y be any one of reflections (reflections along perpendicular bisector of sides or along diagonal flips). D_n can be represented as $G_1 \cup G_2$ where $G_1 =< g >$ and $G_2 = \{y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$. We say g and y are generators of D_n, and the equations $g^n = y^2 = e$, the identity and $yg = g^{n-1}y$ are relations for these generators. Generally all reflections are involutions and rotations may or may not. If n is odd, e is the only involution in G_1 and G_2 consist of reflections along perpendicular bisector of sides only. Except for e , generally G_1 and G_2 never commute and G_2 is non-abelian , but if n is even, $g^{\frac{n}{2}}$ is the only involution in G_1 which commute G_2.

Definition 2.6 *A subgraph (U, F) of a graph (V, E) is said to be vertex induced subgraph if F consist of all the edges of (V, E) joining pairs of vertices of U.*

Definition 2.7 *A Hamiltonian path is a path in (V, E) which goes through all the vertices in (V, E) exactly ones. A Hamiltonian cycle is a closed Hamiltonian path. A graph (V, E) is said to be Hamiltonian, if it contains a Hamiltonian cycle.*

Theorem 2.8 *Let G be the dihedral group D_p, p is prime and G act on G by conjugation. Then for every element $x \in G_1$ with $x \neq e$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian.*

Proof Given $G = D_p$, so $G = \{g, g^2, g^3, \cdots, g^p, y, yg, yg^2, \cdots, yg^{p-1}\}$. Since $x \in G_1$, we have $C_G(x) = \{x, x^2, x^3, \cdots, x^p\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. $\overline{x}$ is the orbit of $x \in G$ with $x^2 \neq e$ and G act on G by conjugation, we have $\overline{x} = \{x, x^{p-1}\}$, since $C_G(x)$ is abelian and $yx = x^{n-1}y$. We can choose an element $s \in \overline{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e)e = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x) = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Continuing in this way, we get a finite path $u \to ux \to ux^2 \to ux^3 \to \cdots \to ux^p = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$, which is Hamiltonian. In particular for $u = e$, we get a Hamiltonian cycle $e \to x \to x^2 \to x^3 \to \cdots \to x^p = e$. □

Definition 2.9 *A graph (V, E) is said to be complete if for eah pair of arbitrary vertices in*

(V,E) can be joined by an edge. A complete graph of n vertices is denoted as K_n.

Theorem 2.10 *Let G be the dihedral group D_{2n+1} and G act on G by conjugation. Then for every element $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is K_2.*

Proof Given $G = D_{2n+1}$, so we have $G = \{g, g^2, g^3, \cdots, g^{2n+1}, y, yg, yg^2, \cdots, yg^{2n}\}$. Since $x \in G_2$, which is non-abelian, we have $C_G(x) = \{x, e\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\overline{x}$ is the orbit of $x \in G_2$ and G act on G by conjugation, we have $x \in \overline{x}$. We can choose the element $s = x \in \overline{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2$, so there exist an edge from ux to ux^2 and consequently a path from u to ux^2. Since $x^2 = e$, we get a Hamiltonian cycle $u \to ux \to ux^2 = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$, which is K_2. □

Corollary 2.11 *Let G be the dihedral group D_p, where p is prime and G act on G by conjugation. Then for every element $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is Hamiltonian.*

Theorem 2.12 *Let G be the dihedral group D_p ,where p is prime and G act on G by conjugation. Then for $x \in G$ with $x \neq e$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is Hamiltonian.*

Proof Since $|G| = 2p$, we have an element $x \in G$ such that either $x^p = e$ or $x^2 = e$. So there exists a Hamiltonian cycle $u \to ux \to ux^2 \to ux^3 \cdots \to ux^p = u$ by Theorem 2.8 or a Hamiltonian cycle $u \to ux \to ux^2 = ue = u$ by Theorem 2.10 in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$. □

Definition 2.13 *A graph (V,E) is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of (V,E) is of the form $\{a,b\}$ with $a \in V_1$ and $b \in V_2$.*

Theorem 2.14 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G_1$ with $x \neq e$ and $C_G(x) = G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is a bipartite graph on n vertices.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$. Since $x \in G_1$ with $C_G(x) = G$, we have either $x = e$ or $x = g^{\frac{n}{2}}$. But $x \neq e$. Let $u \in C_G(x)$. Then $ux = xu$ for all $u \in G$. $\bar{x}$ is the orbit of $x \in G_1$ and G act on G by conjugation ,we have $\bar{x} = \{x\}$. We can choose the element $s = x \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Since $x = g^{\frac{n}{2}}$, we have $x^2 = e$. Thus we get a complete graph $u \to ux \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$.

Let us consider the following cases.

Case 1. If $u = g^i$, $i = 1, 2, 3, \cdots, n$, we get $\frac{n}{2}$ distinct complete graph of two vertices with one end vertex in $\{g, g^2, g^3, \cdots, g^{\frac{n}{2}}\}$ and other in $\{g^{\frac{n}{2}+1}, g^{\frac{n}{2}+2}, \cdots, g^n\}$ as shown below.

$g \to g^{\frac{n}{2}+1} \to g$, $g^2 \to g^{\frac{n}{2}+2} \to g^2$ $,\cdots,$ $g^{\frac{n}{2}} \to g^n \to g^{\frac{n}{2}}$, $g^{\frac{n}{2}+1} \to g \to g^{\frac{n}{2}+1}$ $,\cdots,$ $g^n \to g^{\frac{n}{2}} \to g^n$.

Case 2. If $u = yg^i$,$i = 1, 2, 3, \cdots, n$, we get another $\frac{n}{2}$ distinct complete graph of two vertices with one end vertex in $\{yg, yg^2, yg^3, \cdots, yg^{\frac{n}{2}}\}$ and other in $\{yg^{\frac{n}{2}+1}, yg^{\frac{n}{2}+2}, \cdots, yg^n\}$ as shown below.

$yg \to yg^{\frac{n}{2}+1} \to yg$, $yg^2 \to yg^{\frac{n}{2}+2} \to yg^2$, $\cdots$, $yg^{\frac{n}{2}} \to y \to yg^{\frac{n}{2}}$, $yg^{\frac{n}{2}+1} \to yg \to yg^{\frac{n}{2}+1}$,$\cdots$, $yg^n \to yg^{\frac{n}{2}} \to yg^n$.

Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices. □

Remark 2.15 By Theorem 2.14 ,the graphs in case.2 have been completely characterized. If $n = pq$ with p and q are distinct primes, the induced subgraph with vertex set $C_G(x)$ of the Caylay graph $Cay(G, \overline{x})$ has $\frac{n}{2}$ distinct complete graphs on two vertices with one end vertex in $\{y, yg^2, \cdots, yg^{n-2}\}$ and other in $\{yg, yg^3, \cdots, yg^{n-1}\}$.

If $n \neq pq$, we get $\frac{n}{2}$ distinct complete graph on two vertices. Out of which $\frac{n}{4}$ graphs have one end vertex in $\{y, yg^2, yg^4, \cdots, yg^{\frac{n}{2}-2}\}$ and other in $\{yg^{\frac{n}{2}}, yg^{\frac{n}{2}+2}, \cdots, yg^{n-2}\}$ and the remaining $\frac{n}{4}$ graphs have one end vertex in $\{yg, yg^3, yg^5, \cdots, yg^{\frac{n}{2}-1}\}$ and others in $\{yg^{\frac{n}{2}+1}, yg^{\frac{n}{2}+3}, \cdots, yg^{n-1}\}$.

Corollary 2.16 *Let G be the dihedral group D_n, where n is even and G act on G by conjugation. Then for the element $x = g^{\frac{n}{2}} \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices.*

Theorem 2.17 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is Hamiltonian.*

Proof Let $G = D_{4n}$. So we have $G = \{g, g^2, \cdots, g^{4n}, y, yg, yg^2, \cdots, yg^{4n-1}\}$. Since $x \in G$ with $x^2 = e$ and $C_G(x) \neq G$, we have $x \neq e$ and $x \neq g^{\frac{n}{2}}$. Thus $x \in G_2$ and $C_G(x) = \{x, e, xg^{2n}, g^{2n}\}$. We decompose G_1 as $G_1^{'} \cup G_1^{''}$ where $G_1^{'} = \{g^2, g^4, \cdots, g^{4n}\}$ and $G_1^{''} = \{g, g^3, \cdots, g^{4n-1}\}$. Similarly G_2 an be decomposed as $G_2^{'} \cup G_2^{''}$, where $G_2^{'} = \{y, yg^2, yg^4, \cdots, yg^{4n-2}\}$ and $G_2^{''} = \{yg, yg^3, \cdots, yg^{4n-1}\}$. Since $x \in G_2$, we have either $x \in G_2^{'}$ or $x \in G_2^{''}$. If $x \in G_2^{'}$, it implies that $xg^{2n} \in G_2^{'}$. From the composition table and also from the relation $yg = g^{4n-1}y$, we get $G_1^{'}G_2^{'}(G_1^{'})^{-1} = G_1^{''}G_2^{'}(G_1^{''})^{-1} = G_2^{'}G_2^{'}(G_2^{'})^{-1} = G_2^{''}G_2^{'}(G_2^{''})^{-1} = G_2^{'}$. Thus $\bar{x} = G_2^{'}$. Similarly if $x \in G_2^{''}$,implies that $xg^{2n} \in G_2^{''}$. From the composition table it follows that $G_1^{'}G_2^{''}(G_1^{'})^{-1} = G_1^{''}G_2^{''}(G_1^{''})^{-1} = G_2^{'}G_2^{''}(G_2^{'})^{-1} = G_2^{''}G_2^{''}(G_2^{''})^{-1} = G_2^{''}$ and hence $\bar{x} = G_2^{''}$.

Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. We can choose two involutions s_1 and s_2 in $\bar{x}$ such that $s_1 = (ux)x(ux)^{-1}$ and $s_2 = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}$. Now $s_1u = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux.

Again $s_2(ux) = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}(ux) = (uxg^{2n})xg^{2n}(g^{2n})^{-1}x^{-1}u^{-1}ux = (uxg^{2n})x = (ux)g^{2n}x = u(xg^{2n})x = u(g^{2n}x)x = (ug^{2n})x^2 = ug^{2n}$, then there is an edge from ux to ug^{2n} and consequently a path from u to ug^{2n}. Again $s_1(ug^{2n}) = (ux)x(ux)^{-1}(ug^{2n}) = (ux)xx^{-1}u^{-1}ug^{2n} = uxg^{2n}$, then there is an edge from ug^{2n} to uxg^{2n} and consequently a path from u to uxg^{2n}. Again $s_2(uxg^{2n}) = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}(uxg^{2n}) = (uxg^{2n})xg^{2n} = (uxg^{2n})g^{2n}x = uxg^{4n}x = ux^2 = ue = u$. Thus we get a Hamiltonian cycle $u \to ux \to ug^{2n} \to uxg^{2n} \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. In particular for $u = e$, we get a Hamiltonian cycle $e \to x \to g^{2n} \to xg^{2n} \to e$. □

Corollary 2.18 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is Hamiltonian.*

Theorem 2.19 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4.*

Proof Since $x \in D_{4n}$ with $x^2 = e$ and $C_G(x) \neq G$ by Theorem 2.17, we get a Hamiltonian cycle $u \to ux \to ug^{2n} \to uxg^{2n} \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. To prove that this graph is K_4, it is enough to show that there exist edges from $u \to ug^{2n}$ and $ux \to uxg^{2n}$. We can choose $s = g^{2n}$ as $ug^{2n}u^{-1}$. Now $su = (ug^{2n}u^{-1})u = ug^{2n}$, then there is an edge from u to ug^{2n}. Similarly we get an edge from ux to uxg^{2n}, since $s(ux) = (ug^{2n}u^{-1})ux = ug^{2n}x = uxg^{2n}$. □

Corollary 2.20 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4.*

Theorem 2.21 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices.*

Proof Given $G = D_{4n+2}$, so we have $G = \{g, g^2, g^3, \cdots, g^{4n+2}, y, yg, yg^2, \cdots, yg^{4n+1}\}$. Since $x \in G$ with $x^2 = e$ and $C_G(x) \neq G$, clearly $x \in G_2$ and hence $C_G(x) = \{x, e, g^{2n+1}, xg^{2n+1}\}$. Since $x \in G_2$, either $x \in G_2^{'}$ or $x \in G_2^{''}$, where $G_2^{'} = \{y, yg^2, \cdots, yg^{4n}\}$ and $G_2^{''} = \{yg, yg^3, \cdots, yg^{4n+1}\}$. If $x \in G_2'$, then $xg^{2n+1} \in G_2''$. Since $\bar{x}$ is the orbit of an element x in $G_2^{'}$ and G act on G by conjugation, we get $\bar{x} = G_2^{'}$. Similarly if $x \in G_2''$, we have $xg^{2n+1} \in G_2'$ and $\bar{x} = G_2^{''}$. Thus there exist exactly one involution in $\bar{x} \cap C_G(x)$. We can choose that $s \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$.

Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2 = ue = u$. Thus we get a Hamiltonian cycle $u \to ux \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. Since $|C_G(x)| = 4$, there exist an element other than u and ux in $C_G(x)$. Since ug^{2n+1} commute with all reflections, we have $ug^{2n+1} \in C_G(x)$. Again $s(ug^{2n+1}) = (ux)x(ux)^{-1}(ug^{2n+1}) = (ux)(xx^{-1})(u^{-1}u)g^{2n+1} = uxg^{2n+1}$ and on the other hand $s(uxg^{2n+1}) = (ux)x(ux)^{-1}(uxg^{2n+1}) = (ux)(xx^{-1})(u^{-1}u)xg^{2n+1} = ux^2g^{2n+1} = ug^{2n+1}$. Thus we get another cycle $ug^{2n+1} \to uxg^{2n+1} \to ug^{2n+1}$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices. □

Corollary 2.22 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices.*

Theorem 2.23 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian.*

Proof Since $G = D_{4n+2}$ and G act on G by conjugation, by Theorem 2.21, for every $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on 4 vertices with one cycle $u \to ux \to u$ and another cycle $ug^{2n+1} \to uxg^{2n+1} \to ug^{2n+1}$. If we add an element g^{2n+1} in $\bar{x}$, then we get an edge from u to ug^{2n+1} and ux to uxg^{2n+1} , since $g^{2n+1}u = ug^{2n+1}$ and $g^{2n+1}(ux) = (ux)g^{2n+1}$. Thus we get a Hamiltonian cycle $ug^{2n+1} \to u \to ux \to uxg^{2n+1} \to ug^{2n+1}$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$. □

Corollary 2.24 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian.*

Theorem 2.25 *Let G be the dihedral group D_n, n is even and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{\frac{n}{2}})$ is Hamiltonian.*

Proof Suppose $G = D_{4n}$ and G act on G by conjugation. Then by Corollary 2.20, for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4. Also we have if $G = D_{4n+2}$ and G act on G by conjugation, by Corollary 2.24, for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian. Thus if $G = D_n$, n is even ,we get for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{\frac{n}{2}})$ is Hamiltonian. □

Theorem 2.26 *Let G be the dihedral group D_n and G act on G by conjugation. Then for $x \in G$ with $x = g^m$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian if $gcd(m, n) = 1$.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, \cdots, yg^{n-1}\}$. Since $x \in G$ with $x = g^m$ and $gcd(m, n) = 1$, we get $C_G(x) = \{x, x^2, x^3, \cdots, x^n\}$ and $\bar{x} = \{x, x^{n-1}\}$. As in the proof Theorem 2.8, we get a Hamiltonian cycle $u \to ux \to ux^2 \to ux^3 \to \cdots \to ux^n = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. □

Theorem 2.27 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G$ with $x = g^m$ with $C_G(x) \neq G$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, \cdots, yg^{n-1}\}$. Since $x \in G$ with $x = g^m$ and $C_G(x) \neq G$, we have $m \neq \frac{n}{2}$ and n. Thus $x \in G_1$ other than $g^{\frac{n}{2}}$ and g^n and hence $C_G(x) = \{x, x^2, x^3, \cdots, x^{m-1}, x^m, x^{m+1}, \cdots, x^n\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we have $\bar{x} = \{x, x^{n-1}\}$. Choose an element $s = x \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Continuing in this way, we get a cycle $u \to ux \to ux^2 \to \cdots \to ux^{\frac{n}{d}} = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. In particular, for $u = g^i$,$i = 1, 2, \cdots, n$, we get d Hamiltonian decompositions on $\frac{n}{d}$ vertices as $g \to g^{1+m} \to g^{1+2m} \to \cdots \to g^{1+\frac{mn}{d}} = g$, $g^2 \to g^{2+m} \to g^{2+2m} \to \cdots \to g^{2+\frac{mn}{d}} = g^2, \cdots, g^d \to g^{d+m} \to g^{d+2m} \to \cdots \to g^{d+\frac{mn}{d}} = g^d$, $g^{d+1} \to g^{d+1+m} \to g^{d+1+2m} \to \cdots \to g^{d+1}, \ldots, g^n \to g^m \to \cdots \to g^n$ of which the decompositions when $u = g^i$ and $u = g^{i+d}$ are same. □

Theorem 2.28 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G_1$ with $x = g^m$ and $x \neq e$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$.*

Proof Given $G = D_n$ and G act on G by conjugation.Then by Theorem 2.27, for every element $x \in G$ with $x = g^m$ with $C_G(x) \neq G$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$. Also we have ,by Theorem 2.14, for every $x \in G_1$ with $x \neq e$ and $C_G(x) = G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices. Thus if $G = D_n$ and G act on G by conjugation, for every element $x \in G_1$ with $x = g^m$ and $x \neq e$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$. □

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On the Second Order Mannheim Partner Curve in E^3

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Abstract: In this study first we worked on the Mannheim curve pair $\{\alpha, \alpha_1\}$ and Mannheim curve pair $\{\alpha_1, \alpha_2\}$ We called α_2 as the second order Mannheim partner curve of the Mannheim curve α. We examined the Frenet apparatus of second order Mannheim partner curve in terms of, Frenet apparatus of Mannheim curve α, with the offset property of second order Mannheim partner α_2. Further we examined third order Mannheim partner α_3 where $\{\alpha_2, \alpha_3\}$ are Mannheim curve pair.

Key Words: Mannheim curve, Frenet apparatus, second Mannheim curve, modified Darboux vector.

AMS(2010): 53A40, 53B30.

§1. Introduction

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\dfrac{\kappa}{\kappa^2+\tau^2}$ is a nonzero constant, κ is the curvature and τ is the torsion. Mannheim curve was redefined in [6], if the principal normal vector N of first curve and binormal vector B_1 of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim partner curves. For more detail see in [6]. Frenet-Serret apparatus of the curve $\alpha : I \to E^3$ are $\{T, N, B, \kappa, \tau\}$. For any unit speed curve α, the Darboux and modified Darboux vectors are, respectively ([2],[4])

$$D(s) = \tau(s)T(s) + \kappa(s)B(s), \tag{1.1}$$

$$\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s). \tag{1.2}$$

In [7] Mannheim curves are studied and Mannheim partner curve of α can be represented

$$\alpha(s_1) = \alpha_1(s_1) + \lambda(s_1)B_1(s_1) \tag{1.3}$$

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for some function λ, since N and B_1 are linearly dependent, equation can be rewritten as

$$\alpha_1(s) = \alpha(s) - \lambda(s)N(s), \tag{1.4}$$

where

$$\lambda(s) = \frac{-\kappa(s)}{\big(\kappa(s)\big)^2 + \big(\tau(s)\big)^2}. \tag{1.5}$$

Frenet-Serret apparatus of Mannheim partner curve α_1 are $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$. The relationship α and α_1 Frenet vectors are as follows

$$\begin{aligned} T_1 &= \cos\theta\ T - \sin\theta\ B \\ N_1 &= \sin\theta\ T + \cos\theta\ B \\ B_1 &= N. \end{aligned} \tag{1.6}$$

where $\angle(T, T_1) = \cos\theta$. The first curvature and the second curvature (torsion) are

$$\kappa_1 = -\frac{d\theta}{ds_1} = \frac{\theta'}{\cos\theta}, \qquad \tau_1 = \frac{\kappa}{\lambda\tau}. \tag{1.7}$$

We use dot $\cdot$ to denote the derivative with respect to the arc length parameter of the curve α. Also

$$\frac{ds}{ds_1} = \frac{1}{\cos\theta} = \frac{-\lambda\tau_1}{\sin\theta}, \tag{1.8}$$

for more detail see in [7], or we can write

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{1+\lambda\tau}}. \tag{1.9}$$

§2. Second Order Mannheim Partner and Frenet Apparatus

Definition 2.1 *Let $\{\alpha\ , \alpha_1\}$ and $\{\alpha_1, \alpha_2\}$ be the Mannheim pairs of α and α_1 respectively. We called as α_2 is a second order Mannheim partner of the curve α. which has the following parametrization ,*

$$\alpha_2 = \alpha + \lambda_1 \sin\theta T - \lambda N + \lambda_1 \cos\theta B, \tag{2.1}$$

where

$$\alpha_1 = \alpha(s) - \lambda N(s) \quad \textit{and} \quad \alpha_2 = \alpha_1(s) - \lambda_1 N_1(s). \tag{2.2}$$

Theorem 2.1 *The Frenet vectors of second order Mannheim partner α_2 of a Mannheim curve*

α, based on the Frenet apparatus of Mannheim curve α are

$$\begin{cases} T_2 = \cos\theta_1 \cos\theta \ T - \sin\theta_1 \ N - \cos\theta_1 \sin\theta \ B \\ N_2 = \sin\theta_1 \cos\theta \ T + \cos\theta_1 \ N - \sin\theta_1 \sin\theta \ B \\ B_2 = \sin\theta \ T + \cos\theta \ B. \end{cases} \tag{2.3}$$

Proof Let α_2 be second order Mannheim partner of a Mannheim curve α. Also α_2 be the Mannheim partner of Mannheim partner α_1. The Frenet vector fields T_1, N_1, B_1 and T_2, N_2, B_2 which are belong to the curves α_1 and α_2, respectively. It is easy to say that Frenet vectors of second order Mannheim partner α_2, based on the Frenet vectors of Mannheim curve α_1 are

$$\begin{cases} T_2 = \cos\theta_1 \ T_1 - \sin\theta_1 \ B_1 \\ N_2 = \sin\theta_1 \ T_1 + \cos\theta_1 \ B_1 \\ B_2 = N_1 \end{cases}$$

where $\angle(T_1, T_2) = \theta_1$. By substituting T_1, N_1, B_1 we have the equalities in terms of the curve α.

$$T_2 = \cos\theta_1 \ (\cos\theta \ T - \sin\theta \ B) - \sin\theta_1 \ N$$
$$N_2 = \sin\theta_1 \ (\cos\theta \ T - \sin\theta \ B) + \cos\theta_1 \ N$$
$$B_2 = \sin\theta \ T + \cos\theta \ B$$

This completes the proof. Also the following product give us the same equalities;

$$\begin{bmatrix} T_2 \\ N_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \qquad \square$$

Theorem 2.2 *Let α_2 be second order Mannheim partner of a Mannheim curve α. The curvature and torsion of the second order Mannheim partner α_2 are*

$$\kappa_2 = \frac{-\theta_1'}{\cos\theta} \frac{1}{\cos\theta_1}, \quad \tau_2 = \frac{-\theta'}{\cos\theta} \frac{\lambda\tau}{\lambda_1 \kappa}. \tag{2.4}$$

Proof Since $\kappa_1 = \dfrac{-\theta'}{\cos\theta}$ and $\tau_1 = \dfrac{\kappa}{\lambda\tau}$, we have the curvature as in the following way

$$\kappa_2 = -\frac{d\theta_1}{ds_2} = \frac{-\theta_1'}{\cos\theta\cos\theta_1}.$$

Also as in the following way we have the torsion

$$\tau_2 = \frac{\kappa_1}{\lambda_1 \tau_1} = \frac{-\theta'}{\cos\theta} \frac{\lambda}{\lambda_1} \frac{\tau}{\kappa}.$$

We use mark to denote the derivative with respect to the parameter of the curve α. Due to this theorem we also get

$$\frac{ds}{ds_2} = \frac{1}{\cos\theta\cos\theta_1}. \tag{2.5}$$

□

Theorem 2.3 *The modified Darboux vector of Mannheim partner α_1 of a Mannheim curve α, is*

$$\tilde{D}_1(s) = \frac{\kappa}{\lambda\tau}\frac{\cos^2\theta}{\theta'}\ T + N - \frac{\kappa}{\lambda\tau}\frac{\cos\theta\sin\theta}{\theta'}\ B \tag{2.6}$$

Proof Similarly from the equation (1.2)

$$\tilde{D}_1(s) = \frac{\tau_1}{\kappa_1}T_1(s) + B_1(s). \tag{2.7}$$

Substituting the equation (2.7) into equation (1.6)and (1.7), the proof is complete. □

Theorem 2.4 *The modified Darboux vector of second order Mannheim partner α_2 of a Mannheim curve α, is*

$$\begin{aligned}\tilde{D}_2 &= \left(\frac{\lambda\tau\cos^2\theta_1\cos\theta}{\lambda_1\kappa} + \sin\theta\right)T - \frac{\lambda\tau\cos\theta_1\sin\theta_1}{\lambda_1\kappa}\ N \\ &\quad - \left(\frac{\lambda\tau\cos^2\theta_1\sin\theta}{\lambda_1\kappa} - \cos\theta\right)B.\end{aligned} \tag{2.8}$$

Proof Since

$$\tilde{D}_2(s) = \frac{\tau_2}{\kappa_2}T_2(s) + B_2(s). \tag{2.9}$$

Substituting the equation (2.9) into equation (2.3)and (2.4), the proof is complete. □

Theorem 2.5 *The offset property of second order Mannheim partner α_2 can be given if and only if the curvature κ and the torsion τ of α satisfy the following equation*

$$\lambda_1 = \frac{-\theta'\tau\cos\theta}{\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta}, \tag{2.10}$$

where $\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta \neq 0$.

Proof Notice that $\kappa_1 = \frac{-\theta'}{\cos\theta}$,$\tau_1 = \frac{\kappa}{\lambda\tau}$ with the offset property $-\kappa_1 = \lambda_1\left(\kappa_1^2+\tau_1^2\right)$ and

$$\begin{aligned}\left(\kappa_1^2+\tau_1^2\right) &= \frac{-\kappa_1}{\lambda_1} \\ \lambda_1 &= \frac{-\theta'}{\cos\theta}\ \frac{1}{\dfrac{\theta'^2\tau + \cos^2\theta\left(\kappa^2+\tau^2\right)^2}{\tau\cos^2\theta}} \\ \lambda_1 &= \frac{-\theta'\tau\cos\theta}{\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta}.\end{aligned}$$

This completes the proof. □

Theorem 2.6 *The second order Mannheim partner α_2 is not a Mannheim partner curve α.*

Proof Since the definition of Mannheim partner curve,

$$\langle B_2(s), N(s)\rangle = \langle \sin\theta\ T + \cos\theta\ B, N\rangle = 0,$$

hence $N(s)$ and $B_2(s)$ are linear independent. □

Definition 2.2 *Let $\{\alpha\ ,\alpha_1\}$, $\{\alpha_1,\alpha_2\}$ and $\{\alpha_2,\alpha_3\}$ be the Mannheim pairs of α , α_1 and α_2 respectively. We called as α_3 is a third order Mannheim partner of the curve α, which has the following parametrizations,*

$$\begin{aligned} \alpha_3(s) &= \alpha_2(s) - \lambda_2 N_2(s) \\ &= \alpha + (\lambda_1 \sin\theta\ +\ \lambda_2 \sin\theta_1 \cos\theta)\ T - (\lambda - \lambda_2 \cos\theta_1)\ N \\ &\quad + (\lambda_1 \cos\theta - \lambda_2 \sin\theta_1 \sin\theta)\ B, \end{aligned} \tag{2.11}$$

where

$$\alpha_2 = \alpha + \lambda_1 \sin\theta T - \lambda N + \lambda_1 \cos\theta B \tag{2.12}$$

and

$$|\lambda + \lambda_1 + \lambda_2|$$

is the distance between the arclengthed curves α and α_3.

Theorem 2.7 *The Frenet vectors of third order Mannheim partner α_3 of a Mannheim curve α, based on the Frenet apparatus of Mannheim curve α are*

$$\begin{cases} T_3 = (\cos\theta_2 \cos\theta_1 \cos\theta - \sin\theta_2 \sin\theta)\ T\ - \cos\theta_2 \sin\theta_1\ N \\ \qquad - (\sin\theta_2 \cos\theta + \cos\theta_2 \cos\theta_1 \sin\theta)\ B \\ \\ N_3 = (\sin\theta_2 \cos\theta_1 \cos\theta + \cos\theta_2 \sin\theta\ ) T - \sin\theta_2 \sin\theta_1\ N \\ \qquad + (\cos\theta_2 \cos\theta - \sin\theta_2 \cos\theta_1 \sin\theta\ ) B \\ \\ B_3 = \sin\theta_1 \cos\theta\ T + \cos\theta_1\ N - \sin\theta_1 \sin\theta\ B \end{cases} \tag{2.13}$$

where $\angle(T_2, T_3) = \cos\theta_2$.

Proof Since

$$\begin{bmatrix} T_3 \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

we have the proof. □

Corollary 2.1 *The product of Frenet vector fields of third order Mannheim partner α_3 and Mannheim curve α, has the following matrix form*

$$[\mathbf{V}_3]^{\mathbf{T}}[\mathbf{V}] = \begin{bmatrix} \cos\theta_2\cos\theta_1\cos\theta - \sin\theta_2\sin\theta & -\cos\theta_2\sin\theta_1 & -\sin\theta_2\cos\theta - \cos\theta_2\cos\theta_1\sin\theta \\ \sin\theta_2\cos\theta_1\cos\theta + \cos\theta_2\sin\theta & -\sin\theta_2\sin\theta_1 & \cos\theta_2\cos\theta - \sin\theta_2\cos\theta_1\sin\theta \\ \sin\theta_1\cos\theta & \cos\theta_1 & -\sin\theta_1\sin\theta \end{bmatrix} \quad (2.14)$$

where $[\mathbf{V}_3] = [T_3, N_3, B_3]$ *and* $[\mathbf{V}] = [T, N, B]$.

Corollary 2.2 *Let α_3 be third order Mannheim partner of a Mannheim curve α. The curvature and torsion of the third order Mannheim partner α_3 are*

$$\kappa_3 = -\frac{\theta_2'}{\cos\theta\cos\theta_1\cos\theta_2}, \qquad \tau_3 = \frac{\theta_1'\lambda_1\kappa}{\theta'\cos\theta_1\lambda_2\lambda\tau}. \quad (2.15)$$

Proof We can write

$$\kappa_3 = -\frac{d\theta_2}{ds_3} = \frac{-\theta_2'}{\cos\theta\cos\theta_1\cos\theta_2}$$

and

$$\tau_3 = \frac{\kappa_2}{\lambda_2\tau_2} = \frac{\theta_1'\lambda_1\kappa}{\theta'\cos\theta_1\lambda_2\lambda\tau}$$

or also since

$$\cos\theta\cos\theta_1\cos\theta_2 = \frac{-\theta_2'}{\kappa_3}$$

and

$$\cos\theta\cos\theta_1 = \frac{-\theta_1'}{\kappa_2}.$$

□

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Math.Combin.Book Ser. Vol.1(2017), 78-87

The $\beta-$Change of Special Finsler Spaces

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Abstract: We have considered the $\beta-$change of Finsler metric L given by $L^* = f(L, \beta)$, where f is any positively homogeneous function of degree one in L and β. We have obtained the $\beta-$change of $C-$reducible Finsler spaces, $S3-$like Finsler spaces and $T-$tensor. Particular case when b_i in β is concurrent vector field has been studied.

Key Words: $\beta-$change, Finsler metric, $T-$tensor, $C-$reducible, $S3-$like Finsler spaces.

AMS(2010): 53B20, 53B28, 53B40, 53B18, 53C60.

§1. Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space on the differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. B. N. Prasad and Bindu Kumari [1] and C. Shibata [2] considered the $\beta-$change of Finsler metric given by

$$L^*(x, y) = f(L, \beta), \tag{1.1}$$

where f is positively homogeneous function of degree one in L and β and β given by $\beta(x, y) = b_i(x)\, y^i$ is a one-form on M^n. The Finsler space (M^n, L^*) obtained from F^n by the $\beta-$change (1.1) will be denoted by F^{*n}. The Homogeneity of f in (1.1) gives

$$Lf_1 + \beta f_2 = f, \tag{1.2}$$

where the subscripts '1' and '2' denote the partial derivatives with respect to L and β respectively.

Differentiating (1.2) with respect to L and β respectively, we get

$$Lf_{11} + \beta f_{12} = 0 \quad \text{and} \quad Lf_{12} + \beta f_{22} = 0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{L\beta} = \frac{f_{22}}{L^2},$$

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which gives

$$f_{11} = \beta^2\omega, \quad f_{22} = L^2\omega, \quad f_{12} = -\beta L\omega,$$

where the Weierstrass function ω is positively homogeneous function of degree -3 in L and β. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \tag{1.3}$$

Again ω_2 is positively homogeneous of degree -4 in L and β, so

$$L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0. \tag{1.4}$$

Throughout the paper we frequently use above equations (1.2) to (1.4) without quoting them. The concept of concurrent vector field has been given by Matsumoto and K. Eguchi [6] and S. Tachibana [7], which is defined as follows:

The vector field b_i is said to be a concurrent vector field if

$$(i) \qquad b_{i|j} = -\,g_{ij}, \qquad (ii) \qquad b_i|_j = 0, \tag{1.5}$$

where small and long solidus denote the $h-$ and $v-$covariant derivatives respectively.

It has been proved by by Matsumoto that b_i and its contravariant components b^i are functions of coordinates alone. Therefore from (1.5)(ii), we have

$$C_{ijk}\, b^i = 0.$$

§2. Fundamental Quantities of F^{*n}

To find the relation between fundamental quantities of F^n and F^{*n}, we use the following results

$$\dot\partial_i\beta = b_i, \qquad \dot\partial_i L = l_i, \qquad \dot\partial_j l_i = L^{-1}h_{ij}, \tag{2.1}$$

where $\dot\partial_i$ stands for $\dfrac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of F^n given by $h_{ij} = g_{ij} - l_i\, l_j = L\dot\partial_j\dot\partial_j L$.

The successive differentiation of (1.1) with respect to y^i and y^j gives:

$$l_i^* = f_1 l_i + f_2 b_i, \tag{2.2}$$

$$h_{ij}^* = \frac{ff_1}{L}h_{ij} + fL^2\omega m_i m_j, \tag{2.3}$$

where $m_i = b_i - \frac{\beta}{L}\, l_i$. The quantities corresponding to F^{*n} will be denoted by putting star on the top of those quantities.

From (2.2) and (2.3) we get the following relations between metric tensors of F^n and F^{*n}

$$g_{ij}^* = \frac{ff_1}{L}g_{ij} - \frac{p\beta}{L}l_i l_j + (fL^2\omega + f_2^2)b_i b_j + p(l_i b_j + l_j b_i), \tag{2.4}$$

where $p = (f_1 f_2 - f\beta L\omega)$.

The contravariant components of the metric tensor of F^{*n} will be dertived from (2.4) as follows:

$$g^{*ij} = \frac{L}{ff_1}g^{ij} + \frac{pL^3}{f^3 f_1 t}\left(\frac{f\beta}{L^2} - \triangle f_2\right)l^i l^j - \frac{L^4\omega}{ff_1 t}b^i b^j - \frac{pL^2}{f^2 f_1 t}(l^i b^j + l^j b^i), \tag{2.5}$$

where we put $b^i = g^{ij}b_j, \quad l^i = g^{ij}l_j, \quad b^2 = g^{ij}b_i b_j$ and

$$t = f_1 + L^3\omega\triangle, \qquad \triangle = b^2 - \frac{\beta^2}{L^2}. \tag{2.6}$$

Putting $q = 3f_2\omega + f_2\omega$, we find that

$$\begin{aligned}
&(a)\ \ \dot\partial_i f = \frac{f}{L}l_i + f_2 m_i,\\
&(b)\ \dot\partial_i f_1 = -\beta L\omega m_i,\\
&(c)\ \dot\partial_i f_2 = L^2\omega m_i,\\
&(d)\ \dot\partial_i\omega = -\frac{3\omega}{L}l_i + \omega_2 m_i,\\
&(e)\ \dot\partial_i b^2 = -2C_{..i},\\
&(f)\ \dot\partial_i\triangle = -2C_{..i} - \frac{2\beta}{L^2}m_i
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&(a)\ \dot\partial_i p = -\beta L q m_i,\\
&(b)\ \dot\partial_i t = -2L^3\omega C_{..i} + (L^3\triangle\omega_2 - 3\beta L\omega)m_i,\\
&(c)\ \dot\partial_i q = -\frac{3q}{L}l_i + (4f_2\omega_2 + 3\omega^2 L^2 + f\omega_{22})m_i,
\end{aligned} \tag{2.8}$$

where '.' denotes the contraction with b^i, viz. $C_{..i} = C_{jki}b^j b^k$.

Differentiating (2.4) with respect to y^k, using (2.1) and (2.7), we get the following relation between the Cartan's $C-$tensors ($C^*_{ijk} = \frac{1}{2}\dot\partial_k\, g^*_{ij}$ and $C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij}$):

$$C^*_{ijk} = \frac{ff_1}{L}C_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2}m_i m_j m_k. \tag{2.9}$$

It is to be noted that

$$m_i l^i = 0, \quad m_i m^i = \triangle = m_i b^i, \quad h_{ij}l^j = 0, \quad h_{ij}m^j = h_{ij}b^j = m_i, \tag{2.10}$$

where $m^i = g^{ij}m_j = b^i - \frac{\beta}{L}l^i$.

To find $C^{*i}_{jk} = g^{*ih}C^*_{jhk}$ we use (2.5), (2.9), (2.10), we get

$$\begin{aligned}
C^{*i}_{jk} = C^i_{jk} + \frac{p}{2ff_1}(h_{jk}m^i + h^i_j m_k + h^i_k m_j) + \frac{qL^3}{2ff_1}m_j m_k m^i\\
- \frac{L}{ft}C_{.jk}n^i - \frac{pL\triangle}{2f^2 f_1 t}h_{jk}n^i - \frac{2pL + L^4\triangle q}{2f^2 f_1 t}m_j m_k n^i,
\end{aligned} \tag{2.11}$$

where $n^i = fL^2\omega b^i + pl^i$.

We have the following relations corresponding to the vectors with components n^i and m^i:

$$C_{ijk}m^i = C_{.jk}, \quad C_{ijk}n^i = fL^2\omega C_{.jk}, \quad m_i m^i = fL^2\omega\triangle. \tag{2.12}$$

§3. The $\beta-$Change of $C-$reducible Finsler Space

Let F^n be a $C-$reducible Finsler space. Then [5]

$$C_{hjk} = \frac{1}{n+1}(h_{hj}C_k + h_{hk}C_j + h_{jk}C_h), \tag{3.1}$$

where $C_k = C_{hjk}g^{hj}$.

Using equation (3.1) in equation (2.9), we get

$$C^*_{hjk} = (p_k h_{hj} + p_j h_{hk} + p_h h_{jk}) + \frac{qL^2}{2} m_h m_j m_k, \tag{3.2}$$

where

$$p_k = \frac{ff_1}{L(n+1)}C_k + \frac{p}{2L}m_k. \tag{3.3}$$

Using equation (2.3) in equation (3.2), we get

$$C^*_{hjk} = \frac{L}{ff_1}(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}) + q_h\, m_j m_k + q_j\, m_h m_k + q_k\, m_j m_h, \tag{3.4}$$

where

$$q_h = \frac{qL^2}{6}m_h - \frac{L^3\omega}{f_1}p_h. \tag{3.5}$$

Now suppose that the transformation (1.1) is such that $(n+1)(f_1\omega_2+3\beta L\omega^2)m_h = 6f_1\omega C_h$, then $q_h = 0$. So equation (3.4) reduces to

$$C^*_{hjk} = \frac{L}{ff_1}(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}) \tag{3.6}$$

which will give $\dfrac{C^*_k}{n+1} = \dfrac{L}{ff_1}\, p_k$, so that

$$C^*_{hjk} = \frac{1}{n+1}(C^*_k h^*_{hj} + C^*_j h^*_{hk} + C^*_h h^*_{jk}) \tag{3.7}$$

Hence F^{*n} is also a $C-$reducible. Therefore we have the following result.

Theorem 3.1 *Under the $\beta-$change of Finsler metric with the condition $(n+1)(f_1\omega_2 + 3\beta L\omega^2)m_h = 6f_1\omega C_h$, the $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space.*

In the theorem (3.1) we have assumed that $(n+1)(f_1\omega_2 + 3\beta L\omega^2)m_h = 6f_1\omega C_h$. However if this condition is not satisfied then a $C-$reducible Finsler space may not be transformed to a $C-$reducible Finsler space. In the following we discuss under what condition a $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by $\beta-$change of Finsler metric.

In both the spaces F^n and F^{*n} are $C-$reducible then from (3.1) and its corresponding equation for F^{*n} we find, on using (2.9), that

$$\begin{aligned}
&\frac{fL^2\omega}{t}[(Q_h m_j m_k + Q_j m_h m_k + Q_k m_j m_h) - f_1(C_{..h}h_{jk} + C_{..j}h_{hk} \\
&+ C_{..k}h_{jh})] = \left(\frac{p}{2L} - \frac{ff_1 r}{L(n+1)}\right)(h_{jk}m_h + h_{hj}m_k + h_{hk}m_j) \\
&+ \left(\frac{qL^2}{2} - 3fL^2\omega r\right) m_h m_j m_k,
\end{aligned} \tag{3.8}$$

where $Q_h = tC_h - L^3\omega C_{..h}$ and $r = (n-2)pt + f_1(3p + L^3 q\triangle)$. Thus, we have the following result.

Theorem 3.2 *A $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by a $\beta-$change of Finsler metric if and only if (3.8) holds.*

The condition (3.8) of theorem (3.2) is too complicated to study any geometrical concept of Finsler space. So we consider that our β in $\beta-$change of Finsler metric is such that b_i is a concurrent vector field [6] so that $C_{.i} = 0$, $C_{..i} = 0$. Hence equation (3.8) reduces to

$$\begin{aligned}
&fL^2\omega(C_h m_j m_k + C_j m_h m_k + C_k m_j m_h) = \left(\frac{p}{2L} - \frac{ff_1 r}{2L}\right)(h_{jk}m_h + h_{hj}m_k \\
&+ h_{hk}m_j) + \left(\frac{qL^2}{2} - 3f^2\omega r\right) m_h m_j m_k.
\end{aligned}$$

Contracting this equation with g^{jk}, we find

$$2fL^3\omega\triangle C_h = \{(n+1)(p - ff_1 r) + (qL^3 - 6f^2L\omega r)\triangle\}\, m_h.$$

Hence we have the following result.

Theorem 3.3 *If a $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by a concurrent $\beta-$change of Finsler metric, then the vector C_h is along the direction of the vector m_h.*

§4. The $\beta-$Change of $v-$Curvature Tensor

To find the $v-$curvature tensor of F^{*n} with respect to Cartan's connection, we use the following:

$$C^h_{ij}h_{hk} = C_{ijk}, \quad h^k_j h^i_k = h^i_j, \quad h_{ij}n^i = fL^2\omega m_j. \tag{4.1}$$

The $v-$curvature tensors S^*_{hijk} of F^{*n} [4] is defined as

$$S^*_{hijk} = C^{*r}_{hk}C^*_{rij} - C^{*r}_{hj}C^*_{ikr}. \tag{4.2}$$

From (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14) we get the following relation between $v-$curvature tensors of F^n and F^{*n} [1]:

$$S^*_{hijk} = \frac{ff_1}{L}S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{4.3}$$

where

$$d_{ij} = L\sqrt{\frac{s}{t}}C_{.ij} - \frac{pf_1}{2L^2\sqrt{ts}}h_{ij} + \frac{2\omega p - qf_1}{2\sqrt{ts}}\,Lm_im_j, \tag{4.4}$$

$$E_{ij} = \frac{p}{2L^2\sqrt{f\omega}}\,h_{ij} - \frac{p\omega - qf_1}{2f_1\sqrt{f\omega}}\,Lm_im_j \tag{4.5}$$

and $s = ff_1\omega$.

Now suppose that b_i is a concurrent vector field and F^n is an $S3-$like Finsler space [4], then $C_{.ij} = 0$,

$$S_{hijk} = \frac{S}{L^2}\,(h_{hk}h_{ij} - h_{hj}h_{ik}),$$

where S is any scalar function of x and y.

In view of these equations, we have from (4.3)

$$\begin{aligned} S^*_{hijk} &= \left(\frac{ff_1S}{L^3} + \frac{p^2f_1^2}{4L^4ts} - \frac{p^2}{4L^4f\omega}\right)(h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ \left\{\frac{p(p\omega - qf_1)}{4L^2ff_1\omega} - \frac{pf_1(2\omega p - qf_1)}{4Lts}\right\}(h_{hj}m_im_k + h_{ik}m_hm_j \\ &- h_{hk}m_im_j - h_{ij}m_hm_k). \end{aligned} \tag{4.6}$$

Now suppose that the transformed Finsler space F^{*n} is also $S3-$like. Then

$$S^*_{hijk} = \frac{S^*}{L^{*2}}\,(h^*_{hk}h^*_{ij} - h^*_{hj}h^*_{ik}). \tag{4.7}$$

Now from (2.3), it follows that

$$\begin{aligned} (h^*_{hk}h^*_{ij} - h^*_{hj}h^*_{ik}) &= \left(\frac{ff_1}{L}\right)^2(h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ f^2f_1L\omega(h_{hk}m_im_j + h_{ij}m_hm_k - h_{hj}m_km_i - h_{ik}m_hm_j). \end{aligned} \tag{4.8}$$

In view of (4.6), (4.7) and (4.8), we have

$$\begin{aligned} &\left(\frac{ff_1S}{L^3} + \frac{p^2f_1^2}{4L^4ts} - \frac{p^2}{4L^4f\omega} - \frac{S^*f_1^2}{L^2}\right)(h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ \left\{\frac{p(p\omega - qf_1)}{4L^2ff_1\omega} - \frac{pf_1(2\omega p - qf_1)}{4Lts} - S^*f_1L\omega\right\}(h_{hk}m_im_j \\ &+ h_{ij}m_hm_k - h_{hj}m_im_k - h_{ik}m_hm_j) = 0. \end{aligned} \tag{4.9}$$

Contracting (4.9) by $g^{ij}g^{hk}$, we get

$$\begin{aligned}&\left(\frac{ff_1S}{L^3}+\frac{p^2f_1^2}{4L^4ts}-\frac{p^2}{4L^4f\omega}-\frac{S^*f_1^2}{L^2}\right)(n-1)(n-2)\\&+\ 2\left\{\frac{p(p\omega-qf_1)}{4L^2ff_1\omega}-\frac{pf_1(2\omega p-qf_1)}{4Lts}-S^*f_1L\omega\right\}\triangle=0.\end{aligned}\tag{4.10}$$

Hence, we have the following result.

Theorem 4.1 *If a $S3-$like Finsler space is transformed to a $S3-$like Finsler space under the concurrent $\beta-$change, then equation (4.10) holds.*

§5. The $\beta-$Change of $T-$Tensor

The $T-$tensor of Finsler space F^n is defined by [3]:

$$T_{hijk}=LC_{hij}|_k+l_hC_{ijk}+l_iC_{hjk}+l_jC_{hik}+l_kC_{hij},\tag{5.1}$$

where

$$C_{hijk}|_k=\frac{\partial C_{hij}}{\partial y^k}-C_{rij}C^r_{hk}-C_{hrj}C^r_{ik}-C_{hir}C^r_{jk}.\tag{5.2}$$

To find the $T-$tensor of F^{*n}, first of all we find

$$C^*_{hij}||_k=\frac{\partial C^*_{hij}}{\partial y^k}-C^*_{rij}C^{*r}_{hk}-C^*_{hrj}C^{*r}_{ik}-C^*_{hir}C^{*r}_{jk},$$

where $||$ denotes $v-$covariant derivative in F^{*n}. The derivatives of m_i and h_{ij} with respect to y^k are given by

$$\begin{aligned}\dot\partial_k m_i&=-\frac{\beta}{L^2}h_{ik}-\frac{1}{L}l_im_k,\\\dot\partial_k(h_{ij})&=2C_{ijk}-\frac{1}{L}(l_ih_{jk}+l_jh_{ki}).\end{aligned}\tag{5.3}$$

From (2.7), (2.8), (2.9) and (5.3), we get

$$\begin{aligned}\frac{\partial C^*_{hij}}{\partial y^k}&=\frac{ff_1}{L}\frac{\partial C_{hij}}{\partial y^k}+\frac{p}{L}(C_{hij}m_k+C_{ijk}m_h+C_{jhk}m_i+C_{ihk}m_j)\\&-\frac{p\beta}{2L^3}(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk})+\frac{p}{2L^2}(h_{jk}l_hm_i+h_{hk}l_jm_i\\&+h_{hk}l_im_j+h_{ik}l_hm_j+h_{jk}l_im_h+h_{ik}l_jm_h+h_{ij}l_hm_k+h_{hj}l_im_k\\&+h_{ih}l_jm_k+h_{ij}l_km_h+h_{jh}l_km_i+h_{hi}l_km_j)-\frac{\beta q}{2}(h_{ij}m_hm_k\\&+h_{jh}m_im_k+h_{hi}m_jm_k+h_{ik}m_jm_h+h_{jk}m_im_h+h_{hk}m_im_j)\\&-\frac{qL}{2}(l_im_jm_hm_k+l_jm_hm_im_k+l_hm_im_jm_k+l_km_im_jm_h)\\&+\frac{L^2}{2}(4f_2\omega_2+3L^2\omega^2+f\omega_{22})\,m_im_jm_hm_k.\end{aligned}\tag{5.4}$$

From equation (2.9), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
C^{*}_{rij}\,C^{*r}_{hk} &= \frac{ff_1}{L}C_{rij}C^{r}_{hk} + \frac{p}{2L}(C_{hjk}m_i + C_{hik}m_j + C_{hij}m_k \\
&+ C_{ijk}m_h) + \frac{f_1 p}{2Lt}(C_{.ij}h_{hk} + C_{.hk}h_{ij}) - \frac{ff_1L^2\omega}{t}C_{.ij}C_{.hk} \\
&+ \frac{p^2\triangle}{4fLt}h_{ij}h_{hk} + \frac{L^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_hm_k + C_{.hk}m_im_j) \\
&+ \frac{p(p + L^3q\triangle)}{4Lft}(h_{ij}m_km_h + h_{hk}m_im_j) + \frac{p^2}{4Lff_1}(h_{ij}m_km_h \\
&+ h_{hk}m_im_j + h_{jk}m_im_k + h_{jk}m_im_h + h_{ih}m_jm_k + h_{ik}m_jm_h) \\
&+ \frac{L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3q\triangle)\}}{4ff_1t}m_im_jm_hm_k.
\end{aligned} \tag{5.5}$$

From equation (5.4) and (5.5), we get

$$\begin{aligned}
C^{*}_{hij}||_k &= \frac{ff_1}{L}C_{hij}|_k - \frac{p}{2L}(C_{hij}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{ihk}m_j) \\
&- \frac{p(2f\beta t + L^2p\triangle)}{4fL^3t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) - \Big(\frac{\beta q}{2} \\
&+ \frac{f_1p^2 + f_1L^3pq\triangle + 3p^2}{4Lff_1t}\Big)(h_{ij}m_km_h + h_{hk}m_im_j + h_{jh}m_im_k \\
&+ h_{ik}m_jm_h + h_{hi}m_jm_k + h_{jk}m_im_h) - \frac{p}{2L^2}\{l_h(h_{jk}m_i \\
&+ h_{ij}m_k + h_{ik}m_j) + l_j(h_{hk}m_i + h_{ik}m_h + h_{ih}m_k) + l_i(h_{hk}m_j \\
&+ h_{jk}m_h + h_{hj}m_k) + l_k(h_{ij}m_h + h_{jh}m_i + h_{hi}m_j)\} - \frac{qL}{2}(l_im_jm_hm_k \\
&+ l_jm_hm_im_k + l_hm_im_jm_k + l_km_im_jm_h) - \frac{f_1p}{2Lt}(C_{.ij}h_{hk} + C_{.hj}h_{ik} \\
&+ C_{.hk}h_{ij} + C_{.ik}h_{hj} + C_{.hi}h_{jk} + C_{.jk}h_{hi}) + \frac{ff_1L^2\omega}{t}(C_{.ij}C_{.hk} \\
&+ C_{.hj}C_{.ik} + C_{.hi}C_{.jk}) - \frac{L^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_km_h + C_{.hk}m_im_j \\
&+ C_{.hj}m_im_k + C_{.ik}m_jm_h + C_{.hi}m_jm_k + C_{.jk}m_hm_i) \\
&+ \Big[\frac{L^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22}) \\
&- \frac{3L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3q\triangle)\}}{4ff_1t}\Big]m_im_jm_hm_k.
\end{aligned} \tag{5.6}$$

Using equations (2.2), (2.9) and (5.6), we get the following relation between $T-$tensors of

Finsler spaces F^n and F^{*n}:

$$\begin{aligned}
T^*_{hijk} &= \frac{f^2 f_1}{L^2} T_{hijk} + \frac{f(f_1 f_2 + f\beta L\omega)}{2L}(C_{hij} m_k + C_{ijk} m_h + C_{hjk} m_i \\
&+ C_{ihk} m_j) + \frac{f^2 L^2 f_1 \omega}{t}(C_{.ij} C_{.hk} + C_{.hj} C_{.ik} + C_{.hi} C_{.jk}) - \frac{f f_1 p}{2Lt} \\
&(C_{.ij} h_{hk} + C_{.hk} h_{ij} + C_{.hj} h_{ik} + C_{.ik} h_{hj} + C_{.hi} h_{jk} + C_{.jk} h_{hi}) \\
&- \frac{f L^2 (q f_1 - 2p\omega)}{2t}(C_{.ij} m_k m_h + C_{.hk} m_i m_j + C_{.hj} m_i m_k \\
&+ C_{.ik} m_j m_h + C_{.hi} m_j m_k + C_{.jk} m_h m_i) - \frac{p(2f\beta t + L^2 p \triangle)}{4L^3 t}(h_{ij} h_{hk} \\
&+ h_{hj} h_{ik} + h_{ih} h_{jk}) - \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4L f_1 t} - \frac{p f_2}{L}\right) \\
&(h_{ij} m_k m_h + h_{hk} m_i m_j + h_{jh} m_i m_k + h_{ik} m_j m_h + h_{hi} m_j m_k \\
&+ h_{jk} m_i m_h) + \left[\frac{f L^2}{2}(4 f_2 \omega_2 + 3L^2 \omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2} \right. \\
&\left. - \frac{3L^2\{2pqt + (q f_1 - 2p\omega)(2p + L^3 q \triangle)\}}{4 f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.7}$$

If b_i is a concurrent vector field in F^n, then $C_{.ij} = 0$. Therefore from (5.7), we have

$$\begin{aligned}
T^*_{hijk} &= \frac{f^2 f_1}{L^2} T_{hijk} - \frac{p(2f\beta t + L^2 p \triangle)}{4L^3 t}(h_{ij} h_{hk} + h_{hj} h_{ik} + h_{ih} h_{jk}) \\
&- \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4L f_1 t} - \frac{p f_2}{L}\right) \\
&(h_{ij} m_k m_h + h_{hk} m_i m_j + h_{jh} m_i m_k + h_{ik} m_j m_h + h_{hi} m_j m_k \\
&+ h_{jk} m_i m_h) + \left[\frac{f L^2}{2}(4 f_2 \omega_2 + 3L^2 \omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2} \right. \\
&\left. - \frac{3L^2\{2pqt + (q f_1 - 2p\omega)(2p + L^3 q \triangle)\}}{4 f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.8}$$

If b_i is a concurrent vector field in F^n, with vanishing $T-$tensor then $T-$tensor of F^{*n} is given by

$$\begin{aligned}
T^*_{hijk} &= -\frac{p(2f\beta t + L^2 p \triangle)}{4L^3 t}(h_{ij} h_{hk} + h_{hj} h_{ik} + h_{ih} h_{jk}) \\
&- \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4L f_1 t} - \frac{p f_2}{L}\right) \\
&(h_{ij} m_k m_h + h_{hk} m_i m_j + h_{jh} m_i m_k + h_{ik} m_j m_h + h_{hi} m_j m_k \\
&+ h_{jk} m_i m_h) + \left[\frac{f L^2}{2}(4 f_2 \omega_2 + 3L^2 \omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2} \right. \\
&\left. - \frac{3L^2\{2pqt + (q f_1 - 2p\omega)(2p + L^3 q \triangle)\}}{4 f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.9}$$

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Peripheral Distance Energy of Graphs

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Abstract: The peripheral distance matrix of a graph G of order n with k peripheral vertices is a square symmetric matrix of order $k \times k$, denoted as D_p-matrix of G and is defined as $D_p(G) = [d_{ij}]$, where d_{ij} is the distance between two peripheral vertices v_i and v_j in G. The peripheral distance energy of a graph G is the sum of the absolute values of the eigenvalues of D_p-matrix of G. The sum of the distances between all pairs of peripheral vertices is a peripheral Wiener index of a graph G. In this paper, we study some preliminary facts of D_p-matrix of G and give some bounds for peripheral distance energy of a graph G. Specially the bounds are presented for a graph of diameter less than 3. Bounds of peripheral distance energy in terms of peripheral Wiener index are also obtained for graphs of $diam(G) \leq 2$.

Key Words: Distance, peripheral Wiener index, peripheral distance matrix, peripheral distance energy.

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§1. Introduction

Let G be a connected, nontrivial graph with vertex set $V(G)$ and edge set $E(G)$ and let $|V(G)| = n$ and $|E(G)| = m$. Let u and v be two vertices of a graph G. The *distance* $d(u, v|G)$ between the vertices u and v is the length of a shortest path connecting u and v. If $u = v$ then $d(u, v|G) = 0$. The *eccentricity* $e(v)$ of a vertex v in a graph G is the distance between v and a vertex farthest from v in G. The *diameter* $diam(G)$ of G is the maximum eccentricity of G, while the *radius* $rad(G)$ is the smallest eccentricity of G. A vertex v with $e(v) = diam(G)$ is called a *peripheral* vertex of G. The set of peripheral vertices of G is called as periphery and is denoted as $P(G)$.

We claim that the adjacency matrix of a graph is the distance based matrix such that the entries of adjacency matrix are 1 if the distance between two vertices is 1 and 0 otherwise.

The *distance matrix* of a graph G is defined as a square matrix $D = D(G) = [d_{ij}]$, where d_{ij} is the distance between v_i and v_j in G. For the application and the background of the distance matrix on the chemistry, one can refer to [1, 32].

Peripheral distance matrix or D_p-matrix, D_p of a graph G is defined as, $D_p = D_p(G) =$

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$[d_{ij}]$, where d_{ij} is the distance between two peripheral vertices v_i and v_j in G. The eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$ of the D_p-matrix are said to be D_p-eigenvalues of G denoted by $D_p - spec(G)$. Since D_p-matrix of G is symmetric, all of its eigenvalues are real and can be arranged in a non-increasing order as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. Recalling the definition of peripheral distance matrix, a graph G of order n with k peripheral vertices, the peripheral distance matrix of G is a $(k \times k)$ matrix, whose entries are as follows:

$$D_p(G) = [d_{ij}] = \ [d(v_i, v_j)]; \quad where \ \ v_i, v_j \ \in P(G).$$

The peripheral distance energy (D_p-energy (in short)) of a graph G is defined as the sum of the absolute values of D_p- eigenvalues of D_p-matrix of G. i.e,

$$E_{D_P}(G) = \sum_{i=1}^{k} |\mu_i|. \tag{1}$$

The form of (1) is chosen so as to be fully analogous to the definition of graph energy [5, 6, 9].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the ordinary eigenvalues [3], i.e the eigenvalues of the adjacency matrix $A(G)$. Observe that the graph energy $E(G)$ in past a few years has been extensively studied and surveyed in Mathematics and Chemistry [8, 11, 14, 18, 19, 20, 21, 22, 25, 26, 27, 29, 30, 31, 33]. Through out the paper $|P(G)| = k$ with labellings $v_1, v_2, \cdots, v_k$, where $2 \leq k \leq n$.

The *characteristic polynomial* of $D_p(G)$ is the $\det(\mu I - D_p(G))$, it is referred to as a characteristic polynomial of G and is denoted by $\psi(G;\mu) = c_0\mu^k + c_1\mu^{k-1} + c_2\mu^{k-2} + \cdots + c_k$. The roots $\mu_1, \mu_2, \cdots, \mu_k$ of the polynomial $\psi(G;\mu)$ are called the *eigenvalues* of $D_p(G)$. The eigenvalues of $D_p(G)$ are said to be the *peripheral distance eigenvalues* (or D_p-eigenvalues (in short)) of G. Since $D_p(G)$ is a real symmetric matrix, the D_p-eigenvalues are real and can be ordered in non-increasing order, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. Then the D_p*-spectrum* of a graph G is the set of eigenvalues of $D_p(G)$, together with the multiplicities of D_p-eigenvalues of $D_p(G)$. If the D_p-eigenvalues of $D_p(G)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ and their multiplicities are $m(\mu_1), m(\mu_2), \cdots, m(\mu_k)$, then we shall write

$$D_p - spec(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ m(\mu_1) & m(\mu_2) & \cdots & m(\mu_k) \end{pmatrix}.$$

For example, let G be a graph as shown in Fig.1. Then

$$D_p(G) = \begin{bmatrix} . & v_1 & v_2 & v_3 \\ v_1 & 0 & 3 & 3 \\ v_2 & 3 & 0 & 2 \\ v_3 & 3 & 2 & 0 \end{bmatrix}$$

Clearly, the characteristic polynomial of G is $\psi(G;\mu) = -\mu^3 + 22\mu + 36$, whose D_p- eigenvalues are $1+\sqrt{19},\ 1-\sqrt{19}$ and -2. Hence E_{D_p}-energy of G is 10.7178.

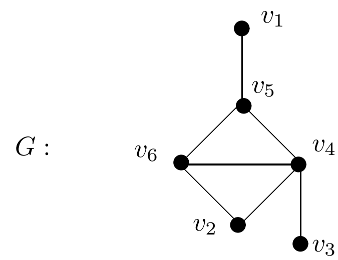

Fig.1 G is a graph of order $n = 6$ with $k = 3$ peripheral vertices.

This paper is organized as follows: In the forthcoming section some preliminary facts of peripheral distance matrix $D_p(G)$ of G are obtained. In section 3 bounds of peripheral distance energy in terms peripheral Wiener index are deduced. In section 4 bounds for the peripheral distance energy are established. In the last section the smallest peripheral distance energy of a graph is obtained thereby posing an open problem for the maximum peripheral distance energy.

§2. Preliminary Results

Lemma 2.1 *Let G be a graph of order n with k peripheral vertices and let $\mu_1, \mu_2, \cdots, \mu_k$ be its peripheral distance eigenvalues. Then,*

(1) $\sum_{i=1}^{k} \mu_i = 0$;

(2) $\sum_{i=1}^{k} {\mu_i}^2 = 2 \sum_{1\le i<j\le k} (d_{ij})^2$.

Proof Since, $\sum_{i=1}^{k} \mu_i = trace[D_p(G)]$ but $d_{ii} = 0$ in $D_p(G)$, therefore, $\sum_{i=1}^{k} \mu_i = 0$.

For $i = 1, 2, \cdots, k$, the $(i,i)^{th}$ entry of $[D_p(G)]^2$ is equal to

$$\sum_{i=1}^{k} d_{ij}, d_{ji} = \sum_{j=1}^{k} (d_{ij})^2$$

since $D_p(G)$ is symmetric. Therefore,

$$\begin{aligned}
\sum_{i=1}^{k} {\mu_i}^2 &= trace[D_p(G)]^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} (d_{ij})^2 = 2\cdot\sum_{i<j} (d_{ij})^2 \\
\Longrightarrow \sum_{i=1}^{k} {\mu_i}^2 &= 2\sum_{i<j} (d_{ij})^2. \qquad (3)
\end{aligned}$$

□

Lemma 2.2 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sum_{i=1}^{k} {\mu_i}^2 = 6\binom{n}{2} + 2\binom{k}{2} - 6m.$$

Proof In the peripheral distance matrix D_p of G there are $x = 2m - 2\{(n-k)k + \frac{(n-k)(n-k-1)}{2}\}$ elements equal to unity, and $y = k(k-1) - x$ elements equal to two. Therefore,

$$\begin{aligned}
\sum_{i=1}^{k} {\mu_i}^2 &= trace[D_p(G)]^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} (d_{ij})^2 = 2\sum_{i<j} (d_{ij})^2 \\
\Longrightarrow \sum_{i=1}^{k} {\mu_i}^2 &= (x).1^2 + (y).2^2 \\
&= (x).1^2 + (k(k-1) - x).2^2 \\
&= 4.k(k-1) - 3x \\
&= 4.k(k-1) - 3\{2m + k(k-1) - n(n-1)\} \\
&= k(k-1) + 3.n(n-1) - 6m \\
\sum_{i=1}^{k} {\mu_i}^2 &= 6\binom{n}{2} + 2\binom{k}{2} - 6m.
\end{aligned}$$

§3. Preliminary Results with Respect to Peripheral Wiener Index

Definition 3.1([4, 7]) *The thorn graph of the graph G , with parameters $t_1, t_2, \cdots, t_n$ is obtained by attaching t_i new vertices of degree one to the vertex v_i of the graph $G; i = 1, 2, \cdots, n$. The thorn graph of the graph G will be denoted by G^*, or if the respective parameters need to be specified, by $G^*(t_1, t_2, \cdots, t_n)$.*

Definition 3.2([7, 28]) *The thorn graph of the graph G obtained by attaching t new vertices of* *degree one to all the vertices v_i of the graph G is denoted by G^{+t}.*

If we partition the vertex set $V(G)$ of a graph into two sets, with peripheral vertices in one set and non-peripheral vertices in other. Then the sum of the distances between all pairs of peripheral vertices is the peripheral Wiener index of a graph G. More formally

$$PWI(G) = \sum_{1 \le i < j \le k} d(v_i, v_j|G), \tag{4}$$

where G is an (n, m)-graph with k peripheral vertices and $v_i, v_j \in P(G)$.

Theorem 3.3([17]) *Suppose G is a graph of order n and size m with k peripheral vertices having $diam(G) \le 2$. Then,*

$$PWI(G) = \binom{n}{2} + \binom{k}{2} - m. \tag{5}$$

Theorem 3.4 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \le 2$. Then, for G^{+t}*

$$\sum_{i=1}^{tk} {\mu_i}^2 = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$$

Proof In the peripheral distance matrix $D_p(G^{+t})$ there are $x_1 = kt$ elements equal to 0, $x_2 = k(t^2 - t)$ elements equal to 2, $x_3 = t^2\{2m + 2\binom{k}{2} - 2\binom{n}{2}\}$ elements equal to 3 and $x_4 = t^2\{2\binom{n}{2} - 2m\}$ elements equal to 4. Therefore,

$$\begin{aligned}
\sum_{i=1}^{kt} {\mu_i}^2 &= trace[D_p(G^{+t})]^2 \\
&= \sum_{i=1}^{kt}\sum_{j=1}^{kt} (d_{ij})^2 = 2\sum_{i<j} (d_{ij})^2 \\
\Longrightarrow \sum_{i=1}^{kt} {\mu_i}^2 &= (x_1).0^2 + (x_2).2^2 + (x_3)\dot{3}^2 + (x_4).4^2 \\
&= \{k(t^2 - t)\}.2^2 + \left\{t^2\left\{2m + 2\binom{k}{2} - 2\binom{n}{2}\right\}\right\}.3^2 + \left\{t^2\left\{2\binom{n}{2} - 2m\right\}\right\}.4^2 \\
&= \{4k(t^2 - t)\} + \left\{9t^2\left\{2m + 2\binom{k}{2} - 2\binom{n}{2}\right\}\right\} + \left\{16t^2\left\{2\binom{n}{2} - 2m\right\}\right\} \\
&= (4kt^2 - 4kt) + \left\{t^2\left\{18m + 18\binom{k}{2} - 18\binom{n}{2}\right\}\right\} + \left\{t^2\left\{32\binom{n}{2} - 32m\right\}\right\} \\
&= 4kt^2 - 4kt + 18mt^2 + 18\binom{k}{2}t^2 - 18\binom{n}{2}t^2 + 32\binom{n}{2}t^2 - 32mt^2 \\
\sum_{i=1}^{kt} {\mu_i}^2 &= \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt. \qquad (6)
\end{aligned}$$

□

Corollary 3.5 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then, for G^{+t}*

$$\sum_{i=1}^{tk} {\mu_i}^2 = \left\{4k + 4\binom{k}{2} + 14PWI(G)\right\} t^2 - 4kt.$$

Proof The proof follows directly from Theorems 3.3 and 3.4. □

Proposition 3.6 *Suppose $G(n,m)$ is a graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sum_{i=1}^{k} {\mu_i}^2 = 6PWI(G) - 4\binom{k}{2},$$

where PWI(G) is the peripheral Wiener index of G.

Proof From the Lemma 2.2 we have,

$$\begin{aligned}\sum_{i=1}^{k} {\mu_i}^2 &= 6\binom{n}{2} + 2\binom{k}{2} - 6m \\ &= 6\binom{n}{2} + 2\binom{k}{2} - 6m + 4\binom{k}{2} - 4\binom{k}{2} \\ &= 6\left\{\binom{n}{2} + \binom{k}{2} - m\right\} - 4\binom{k}{2} \\ &= 6\left\{PWI(G)\right\} - 4\binom{k}{2}\end{aligned}$$

from Theorem 3.3. □

§4. Bounds for the Peripheral Distance Energy

Theorem 4.1 *Suppose G is a graph with k peripheral vertices. Then*

$$\sqrt{2\sum_{i<j}(d_{ij})^2} \leq E_{D_P}(G) \leq \sqrt{2.k.\sum_{i<j}(d_{ij})^2}. \tag{7}$$

Proof We have from Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} {a_i}^2\right)\left(\sum_{i=1}^{k} {b_i}^2\right)$$

Put $a_i = 1$ and $b_i = |\mu_i|$ then

$$\begin{aligned}[E_{D_P}(G)]^2 = \left(\sum_{i=1}^{k} |\mu_i|\right)^2 &\leq \left(\sum_{i=1}^{k} 1\right) \left(\sum_{i=1}^{k} {\mu_i}^2\right) \\ &= k \left(\sum_{i=1}^{k} {\mu_i}^2\right) \\ &= k \left(2\sum_{i<j} (d_{ij})^2\right)\end{aligned}$$

from Eq.(3) and

$$[E_{D_P}(G)] \leq \sqrt{2\ k\ \sum_{i<j} (d_{ij})^2}. \tag{8}$$

We have from the definition

$$\begin{aligned}[E_{D_P}(G)]^2 &= \left(\sum_{i=1}^{k} |\mu_i|\right)^2 = \sum_{i=1}^{k} {\mu_i}^2 + 2\sum_{i<j} |\mu_i||\mu_j| \\ &= 2\sum_{i<j} (d_{ij})^2 + 2\sum_{i<j} |\mu_i||\mu_j|,\end{aligned}$$

$$[E_{D_P}(G)]^2 = 2\sum_{i<j} (d_{ij})^2 + \sum_{i\neq j} |\mu_i||\mu_j|, \tag{9}$$

$$[E_{D_P}(G)]^2 - 2\sum_{i<j} (d_{ij})^2 = \sum_{i\neq j} |\mu_i||\mu_j|. \tag{10}$$

Also, we know that

$$\begin{aligned}[E_{D_P}(G)]^2 &= \left(\sum_{i=1}^{k} |\mu_i|\right)^2 \geq \sum_{i=1}^{k} |\mu_i|^2 = 2\sum_{i<j} (d_{ij})^2 \\ \Longrightarrow [E_{D_P}(G)]^2 &\geq 2\sum_{i<j} (d_{ij})^2 \\ \Longrightarrow [E_{D_P}(G)] &\geq \sqrt{2\sum_{i<j} (d_{ij})^2}. \end{aligned} \tag{11}$$

Inequations (8) and (11) complete the proof. □

Corollary 4.2 *Suppose G is any graph with k peripheral vertices and $diam(G) = d$. Then,*

$$\sqrt{k(k-1)} \leq E_{D_P}(G) \leq d.k.\sqrt{k-1}.$$

Proof Since $d(v_i, v_j) = d_{ij} \geq 1$, for $i \neq j$ and totally $\binom{k}{2}$ pairs of peripheral vertices in G

form lower bound of Corollary 4.2.

$$\begin{aligned} E_{D_P}(G) &\geq \sqrt{2.\sum_{i<j}(d_{ij})^2} \geq \sqrt{2.[1]^2\binom{k}{2}} \\ &= \sqrt{2.1.\frac{k(k-1)}{2}}, \\ E_{D_P}(G) &\geq \sqrt{k(k-1)}. \end{aligned} \qquad (12)$$

Also, $d(v_j, v_j) = d_{ij} \leq d$, for $i \neq j$ and totally $\binom{k}{2}$ pair of peripheral vertices in G form upper bound of Corollary 4.2.

$$\begin{aligned} E_{D_P}(G) &\leq \sqrt{2.k.\sum_{i<j}(d_{ij})^2} \leq \sqrt{2.k.[d]^2\binom{k}{2}} \\ &= \sqrt{2.k.[d]^2\frac{k(k-1)}{2}} \\ E_{D_P}(G) &\leq d.k.\sqrt{k-1}. \end{aligned} \qquad (13)$$

Inequations (12) and (13) complete the proof. □

Theorem 4.3 *Suppose G is any graph with k peripheral vertices. Then,*

(1) $\sqrt{2\sum_{i<j}(d_{ij})^2 + k(k-1)\Delta^{2/k}} \leq E_{D_P}(G)$;

(2) $E_{D_P}(G) \leq \frac{2}{k}\sum_{i<j}(d_{ij})^2 + \sqrt{(k-1)[2\sum_{i<j}(d_{ij})^2 - (\frac{2}{k}\sum_{i<j}(d_{ij})^2)^2]}$,

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$.

Proof We know that, for non-negative numbers the arithmetic mean is not smaller than the geometric mean.

$$\begin{aligned} \frac{1}{k(k-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{k(k-1)}} = \left(\prod_{i=1}^{k}|\mu_i|^{2(k-1)}\right)^{\frac{1}{k(k-1)}} \\ &= \left(\prod_{i=1}^{k}|\mu_i|\right)^{2/k} = |\det(D_p(G))|^{2/k} = (\Delta)^{2/k} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \sum_{i\neq j}|\mu_i||\mu_j| &\geq k(k-1).(\Delta)^{2/k} \\ \Longrightarrow [E_{D_P}(G)]^2 - 2\sum_{i<j}(d_{ij})^2 &\geq k(k-1).(\Delta)^{2/k} \\ [E_{D_P}(G)]^2 &\geq k(k-1).(\Delta)^{2/k} + 2\sum_{i<j}(d_{ij})^2, \end{aligned}$$

$$[E_{D_P}(G)] \geq \sqrt{k(k-1).\,(\Delta)^{2/k} + 2\sum_{i<j}(d_{ij})^2}. \tag{14}$$

Therefore, the equation (14) proves lower bound.

To prove the upper bound we follow the ideas of Koolen and Moulton [18, 19], who obtained an analogous upper bound for ordinary graph energy $E(G)$. By applying the Cauchy-Schwartze inequality to the two $(k-1)$ vectors $(1,1,\cdots,1)$ and $(|\mu_1|,|\mu_2|,\cdots,|\mu_k|)$ we get.

$$\begin{aligned}\left(\sum_{i=2}^{k}|\mu_i|\right)^2 &\leq (k-1)\left(\sum_{i=2}^{k}\mu_i^2\right)\\ \left(E_{D_p}(G)-\mu_1\right)^2 &\leq (k-1)\left(2\sum_{i<j}(d_{ij})^2-\mu_1^2\right)\\ E_{D_p}(G) &\leq \mu_1+\sqrt{(k-1)\left(2\sum_{i<j}(d_{ij})^2-\mu_1^2\right)}\end{aligned}$$

Define the function

$$f(x) = x + \sqrt{(k-1)\left(2\sum_{i<j}(d_{ij})^2 - x^2\right)}$$

we set $x=\mu_1$ and bear in mind that $\mu_1 \geq 1$.

From Equation (3) we get $x^2 = \mu_1^2 \leq 2\sum_{i<j}(d_{ij})^2 \implies x \leq \sqrt{2\sum_{i<j}(d_{ij})^2}$.

Now $f^{'}(x) = 0$ implies, $x = \sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2}$. Therefore $f(x)$ is a decreasing function in the interval

$$\sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2} \leq x \leq 2\sqrt{\sum_{i<j}(d_{ij})^2}.$$

and

$$\sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2} \leq \frac{2}{k}\sum_{i<j}(d_{ij})^2 \leq \mu_1.$$

Hence

$$f(\mu_1) \leq f\left(\frac{2}{k}\sum_{i<j}(d_{ij})^2\right).$$

Hence the proof. □

Theorem 4.4 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{6\binom{n}{2}+2\binom{k}{2}-6m} \leq E_{D_P}(G) \leq \sqrt{k\left\{6\binom{n}{2}+2\binom{k}{2}-6m\right\}}.$$

Proof From Theorem 4.1 we have

$$\sqrt{2\sum_{i<j}(d_{ij})^2} \leq E_{D_P}(G) \leq \sqrt{2.k.\sum_{i<j}(d_{ij})^2}$$

and next from Lemma 2.2,

$$2\sum_{i<j}(d_{ij})^2 = \sum_{i=1}^{k}{\mu_i}^2 = 6\binom{n}{2} + 2\binom{k}{2} - 6m.$$

By replacing the $2\sum\limits_{i<j}(d_{ij})^2$ by $6\binom{n}{2}+2\binom{k}{2}-6m.$ in Ineq.7 gives the proof. □

Corollary 4.5 *Suppose G is a graph with $diam(G) \leq 2$. having k peripheral vertices. Then,*

$$\sqrt{6PWI(G) - 4\binom{k}{2}} \leq E_{D_P}(G) \leq \sqrt{k.\left\{6PWI(G) - 4\binom{k}{2}\right\}},$$

where $PWI(G)$ is the peripheral Wiener index of a graph G.

Proof The proof follows from Theorem 4.1 and Proposition 3.6. □

Theorem 4.6 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{S} + 2\binom{k}{2}\Delta^{2/k}} \leq E_{D_P}(G) \leq \frac{1}{k}\{\mathbb{S}\} + \sqrt{(k-1)[\mathbb{S} - (\frac{1}{k}\{\mathbb{S}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$ and $\mathbb{S} = 6\binom{n}{2} + 2\binom{k}{2} - 6m$.

Proof The proof follows from Theorem 4.3 and Lemma 3.4. □

Corollary 4.7 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{S} + 2\binom{k}{2}\Delta^{2/k}} \leq E_{D_P}(G) \leq \frac{1}{k}\{\mathbb{S}\} + \sqrt{(k-1)[\mathbb{S} - (\frac{1}{k}\{\mathbb{S}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$, $\mathbb{S} = 6PWI(G) - 4\binom{k}{2}$ and $PWI(G)$ is the peripheral Wiener index of a graph G.

Proof The proof follows from Theorem 4.3 and Proposition 3.6. □

Theorem 4.8 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T}} \leq E_{D_P}(G^{+t}) \leq \sqrt{kt\,\{\mathbb{T}\}},$$

where $\mathbb{T} = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$

Proof The proof follows from Theorem 4.1 and Lemma 2.2. □

Corollary 4.9 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T}} \leq E_{D_P}(G^{+t}) \leq \sqrt{kt\,\{\mathbb{T}\}},$$

where $\mathbb{T} = \left\{4k + 4\binom{k}{2} + 14PWI(G)\right\} t^2 - 4kt$ *and $PWI(G)$ is the peripheral Wiener index of a graph G.*

proof The proof follows from Theorem 4.1 and Corollary 3.5. □

Theorem 4.10 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T} + 2\binom{kt}{2}\Delta^{2/kt}} \leq E_{D_P}(G^{+t}) \leq \frac{1}{kt}\{\mathbb{T}\} + \sqrt{(kt-1)[\mathbb{T} - (\frac{1}{kt}\{\mathbb{T}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G^{+t})$ and $\mathbb{T} = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$

proof The proof follows from Theorem 4.3 and Lemma 3.4. □

§5. The Smallest Peripheral Distance Energy of a Graph

By studying the bounds for peripheral distance energy, there arise a common question that, which n vertex graphs with k peripheral vertices have the smallest and greatest peripheral distance energy. Among all n-vertex connected graphs with k peripheral vertices the complete graph is the unique graph with the smallest peripheral distance energy.

Theorem 5.1 *The complete graph $K_{n=k}$ with k peripheral vertices is the graph with smallest peripheral distance energy, which is equal to $2(k-1)$.*

Proof Let G be a graph with k peripheral vertices and K_k be a complete graph on k peripheral vertices. Let A be a peripheral distance matrix of K_k. B be a peripheral distance matrix of G with the D_p-eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$. Clearly A and B are non-negative matrices and obviously $0 \leq A \leq B$. Now, from the fact that if $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$. And for the complete graph, $\rho(A) = n - 1$ and $E_D(K_k) = 2(k-1)$ hence,

$$\begin{aligned} 2(k-1) &= 2\rho(A) \leq 2\rho(B) \\ &\leq \rho(B) + \sum_{i=2}^{k} |\mu_i|. \end{aligned}$$

By using Perron Frobenius theorem, it implies that $\rho(B)$ is a positive eigenvalues. Hence,

$$2(k-1) \quad \leq \quad \sum_{i=1}^{k} |\mu_i| = E_{D_p}(G).$$

But

$$2(k-1) = E_{D_p}(K_k) \leq E_{D_p}(G).$$

Hence, we conclude that the peripheral distance energy of a graph with k peripheral vertices is greater than the peripheral distance energy of a complete graph on k vertices. This proves that among k peripheral vertices graphs complete graph has the smallest peripheral distance energy $= 2(k-1)$. □

Since, distance matrix D of a complete graph is equal to peripheral distance matrix D_p of a complete graph, also distance energy E_D of a complete graph is equal to peripheral distance matrix E_{D_p} of a complete graph, therefore this also settles the conjecture posed by Ramane et al. in [24]. However, in [2], the authors have given the direct reason for the proof of the conjecturer in [24]. Since, we do not have a sufficient stuff to prove graph with greatest peripheral distance energy, but the graph with k peripheral vertices such that all the peripheral vertices are at the distance d $(= diam(G))$ from each other is certainly deserve to be seriously considered graph. In this connection it looks plausible to pose an open problem:

Open Problem *The graph G with k peripheral vertices such that all of its peripheral vertices are at the same distance d $(= diam(G))$ from each other has maximum peripheral distance energy.*

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Some Properties of a h-Randers Finsler Space

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Abstract: The purpose of the present paper is to obtain the relation between imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, L^*) where L^*(x, y) is obtained from the transformation of L(x, y) is given by

$$L^*(x,y) \to L(x,y) + b_i(x,y)y^i$$

Key Words: Riemannian metric, h-vector, imbedding class.

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§1. Introduction

In 1971 Matsumoto [5] introduced the transformation of Finsler metric

$$\bar{L}(x,y) \to L(x,y) + b_i y^i \tag{1.1}$$

and obtain the relation between the imbedding class numbers of a tangent Riemannian spaces to (M^n, L) and a Finsler space $(M^n, \bar{L})$ which is obtained by the transformation of the Finsler metric L by the relation given by in the equation (1.1). Since a concurrent vector field is a function of (x) i.e., position only, assuming $b_i(x)$ as a concurrent vector field, Matsumoto [6] studied the R3-likeness of Finsler spaces (M^n, L) and $(M^n, \bar{L})$. Singh and Prasad [14,11] generalized the concept of concurrent vector field and introduced the semi-parallel and concircular vector fields which are functions of (x) only. Assuming $b_i(x)$ as a concicular vector field, Prasad, Singh and Singh [11] studied the R3-likeness of (M^n, L) and $(M^n, \bar{L})$.

If $L(x, y)$ is a metric function of Riemannian space then $\bar{L}(x, y)$ reduces to the metric function of Rander's space. Such a Finsler metric was first introduced by G. Randers [13]from the standpoint of general theory of relativity and applied to the theory of the electron microscope by R. S. Ingarden [3] who first named it us Randers space. The geometrical properties of this space have been studied by various workers [2, 7, 9, 12, 15].. In 1970 Numata [10] has studied the properties of $(M^n, \bar{L})$ which is obtained from Minkowski space (M^n, L) by transformation

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(1.1). In all those works the function $b_i(x)$ are assumed to be functions of (x) only.

In 1980, Izumi [4] while studying the conformal transformation of Finsler spaces, introduced the h-vector b_i which is v-covariantly constant with respect to Cartan's connection $C\Gamma$ and satisfies the relation

$$LC^h_{ij}b_h = \rho h_{ij}$$

Thus the h-vector b_i is not only a function of (x) but it is also a function of directional arguments satisfying $L\dot{\partial}_j b_i = \rho h_{ij}$. The purpose of the present paper is to obtain the relation between imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, L^*) where L^*(x, y) is obtained from the transformation of L(x, y) is given by

$$L^*(x,y) \rightarrow L(x,y) + \beta(x,y), \tag{1.2}$$

where $\beta(x,y) = b_i(x,y)y^i$, i.e. $b_i(x,y)$ is the function of position and direction both.

§2. **An h-Vector in** (M^n, L)

Let b_i be a vector field in the Finsler space (M^n, L). If $b_i(x,y)$ satisfies the conditions

$$b_i|_j = 0, \tag{2.1}$$

$$LC^h_{ij}b_h = \rho h_{ij}, \tag{2.2}$$

then the vector field b_i is called an h-vector [4]. Here $|_i$ denotes the v-covariant derivative with respect to y^i in the case of Cartan's connection $C\Gamma$, C^h_{ij} is the cartan's C-tensor, h_{ij} is the angular metric tensor and ρ is given by

$$\rho = \frac{LC^i b_i}{(n-1)}, \tag{2.3}$$

where C^i is the torsion tensor given by $C^i_{jk}g^{jk}$.

Lemma 2.1([4]) *If b_i is an h-vector then the function ρ and are independent of y.*

Since The v-covariant derivation of $b^2 = g^{ij}b_ib_j$ and the fact that g^{ij} is v-covariantly constant yield

$$b\dot{\partial}_k b = g^{ij}b_ib_j|_k.$$

In the view of (2.1) we have

$$\dot{\partial}_k b = 0.$$

Thus we have

Lemma 2.2 *The magnitude b of an h-vector is independent of y.*

From (2.1), Ricci identity [8] and the fact that $S_{ihjk} = g_{hr}S^r_{ijk}$ is skew-symmetric in h and

i we have

$$b_i|_j|_k - b_i|_k|_j = -S^h_{ijk} b_h = 0.$$

Thus we have

Lemma 2.3 *For an h-vector b_i we have $S_{hijk} b^h = 0$, where S_{hijk} are components of v-curvature tensor of Cartan's connection $C\Gamma$.*

The concept of concurrent vector field in (M^n, L) has been introduced by Tachibana [16] and its properties have been studied by Matsumoto [6]. A vector field b_i in (M^n, L) is said to be concurrent if it satisfies the condition (2.1) and

$$b_{i|j} = -g_{ij}, \tag{2.4}$$

where $_{|j}$ denotes h-covariant differentiation with respect to x^i in the sense of Cartan's connection $C\Gamma$.

Applying Ricci Identity [8]

$$b_{i|j}|_k - b_i|_{k|j} = -b_h P^h_{ijk} - b_{i|h} C^h_{jk} - b_i|_h P^h_{jk}$$

and using (2.1) and (2.4) we have

$$P^h_{ijk} b_h + C_{ijk} = 0.$$

Since $P_{imjk} = g_{mh} P^h_{ijk}$ is skew-symmetric in i and m, contraction of above equation with $b^i = g^{ij} b_j$ gives $C_{ijk} b^i = 0$. Hence we have the following

Lemma 2.4 *An h-vector b_i with $\rho \neq 0$ is not a concurrent vector field.*

§3. Properties of the h-Randers Finsler Space

Let b_i be an h-vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental function $L^*(x, y)$ is given by (1.2).

Since b_i is an h-vector, from (2.1) and (2.2), we get

$$\dot{\partial}_j b_i = L^{-1} \rho h_{ij}, \tag{3.1}$$

which after using the indicatory property of h_{ij} yields $\dot{\partial}_j \beta = b_j$.

Definition 3.1 *Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot{x}^i)$ of M^n. A change in the fundamental function L by the equation (1.2) on the same manifold M^n is called h-Randers change. A space equipped with fundamental metric L^* is called h-Randers changed Finsler space F^{*n}.*

Now differentiating (1.2) with respect to y^i we have

$$l_i^* = l_i + b_i, \tag{3.2}$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in (M^n, L) and $l_i^* = \dot{\partial}_i L^*$ is the normalized element of support in (M^n, L^*). The quantities of (M^n, L^*) will be denoted by starred letter. Now differentiating (3.2) with respect to y^j then the angular metric tensor $h_{ij}^* = \dot{\partial}_j l_i^*$ is given by

$$h_{ij}^* = \sigma h_{ij}, \tag{3.3}$$

where $\sigma = LL^{-1}(1+\rho)$. Hence we have

$$g_{ij}^* = \sigma g_{ij} + (1-\sigma) l_i l_j + (l_i b_j + l_j b_i) + b_i b_j. \tag{3.4}$$

From (3.4) the relation between the contravariant components of the fundamental tensors can be derived as follows

$$g^{*ij} = \sigma^{-1} g^{ij} - (1+\rho^2)\sigma^{-3}(1 - b^2 - \sigma) l^i l^j - (1+\rho)\sigma^{-2}(l^i b^j + l^j b^i), \tag{3.5}$$

where b is the magnitude of the vector b_i.

From the lemma (2.1) and (3.2) we have

$$\dot{\partial}_i \sigma = \frac{(1+\rho)}{L} m_i, \tag{3.6}$$

$$m_i = b_i - \frac{\beta}{L} l_i. \tag{3.7}$$

Now differentiating (3.3) with respect to y^k (3.2), (3.6), (3.3) and the fact

$$\dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1}(h_{ik} l_j + h_{jk} l_i),$$

we have

$$C_{ijk}^* = \sigma C_{ijk} + (1+\rho) \frac{h_{ij} m_k + h_{jk} m_i + h_{ki} m_j}{2L}. \tag{3.8}$$

From the definition of m_i, it is evident that

$$\begin{array}{ll} (a)\ m_i l^i, & (b)\ m_i b^i = b^2 - \frac{\beta^2}{L^2} = m^i m_i, \\ (c)\ h_{ij} m^i = h_{ij} b^i = m_j, & (d)\ C_{ij}^h m_h = L^{-1} \rho h_{ij}. \end{array} \tag{3.9}$$

From (2.1), (3.5), (3.8) and (3.9) we have

$$\begin{aligned} C_{ij}^{*r} &= C_{ij}^r + \frac{(h_{ij} m^r + h_j^r m_i + h_i^r m_j)}{2L^*} - \frac{1}{L^*}[\{\rho \\ &\quad + \frac{L}{2L^*}(b^2 - \frac{\beta^2}{L^2})\} h_{ij} + \frac{L}{L^*} m_i m_j] l^r. \end{aligned} \tag{3.10}$$

Proposition 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element l_i^*, angular metric tensor h_{ij}^*, fundamental metric tensor g_{ij}^* and (h)hv-torsion tensor C_{ijk}^* of F^{*n} are given by (3.2), (3.3), (3.4) and (3.8) respectively.*

Proposition 3.2 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the reciprocal of the fundamental metric tensor g_{ij}^* is given by (3.5).*

The curvature tensor S_{hijk} of (M^n, L^*) is given by

$$S_{hijk}^* = C_{hkm}^* C_{ij}^{*m} - C_{hjm}^* C_{ik}^{*m}. \tag{3.11}$$

From the equation (3.8) and (3.10), we have

$$\begin{aligned} C_{hkm}^* C_{ij}^{*m} \quad = \quad & \sigma C_{hkm} C_{ij}^m + \alpha h_{ij} h_{hk} + \frac{(1+\rho)}{2L}\{C_{ijk} m_h + C_{hjk} m_i \\ & + C_{hik} m_j + C_{hij} m_k\} + \frac{(1+\rho)}{4LL^*}\{2h_{ij} m_k m_h \\ & + 2h_{hk} m_i m_j + h_{ik} m_j m_h + h_{ih} m_j m_k + h_{jk} m_i m_h \\ & + h_{jh} m_i m_k\}, \end{aligned} \tag{3.12}$$

where $\alpha = \frac{(1+\rho)\rho}{4L^2} + \frac{1+\rho}{4LL^*}(b^2 - \frac{\beta^2}{L^2})$. Thus from (3.11) we have

$$S_{hijk}^* = \sigma S_{hijk} + h_{ij} d_{hk} + h_{hk} d_{ij} - h_{ik} d_{jh} - h_{hj} d_{ik}, \tag{3.13}$$

where $d_{ij} = \frac{\sigma}{2} h_{ij} + \frac{1+\rho}{4LL^*} m_i m_j$.

If we define the tensor A_{ij} and B_{ij} as

$$A_{ij} = \frac{h_{ij} + d_{ij}}{\sqrt{2}}, \qquad B_{ij} = \frac{h_{ij} - d_{ij}}{\sqrt{2}}, \tag{3.14}$$

then S_{hijk}^* is written as

$$S_{hijk}^* = \sigma S_{hijk} - (A_{hj} A_{ik} - A_{hk} A_{ij}) + (B_{hj} B_{ik} - B_{hk} B_{ij}). \tag{3.15}$$

Thus we have

Proposition 3.3 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the curvature tensor S_{hijk}^* is given by (3.15).*

If $|_j$ denotes v-covariant differentiation with respect to y^j in (M^n, L^*) then we have

$$h_{ij}|_k - h_{ik}|_j = \frac{(h_{ij} l_k - h_{ik} l_j)}{L}, \tag{3.16}$$

$$m_i|_j - m_j|_i = \frac{(m_i l_j - m_j l_i)}{L}, \tag{3.17}$$

$$d_{ij}|_k - d_{ik}|_j = \frac{(d_{ij} l_k - d_{ik} l_j)}{L}. \tag{3.18}$$

Hence from (3.14), (3.16) and (3.18), we get

$$A_{ij}|_k - A_{ik}|_j = \frac{(B_{ij} l_k - B_{ik} l_j)}{L}, \tag{3.19}$$

$$B_{ij}|_k - B_{ik}|_j = \frac{(A_{ij} l_k - A_{ik} l_j)}{L}. \tag{3.20}$$

§4. Imbedding Class Numbers of Tangent Riemannian Space to (M^n, L) and (M^n, L^*)

The tangent vector space M^n_x to M^n at every point x is regarded as n-dimensional Riemannian space (M^n_x, g_x) with Riemannian metric $g_x = g_{ij}(x, y)dy^i dy^j$. Thus the component C^i_{jk} of Cartan's C-tensor are the Christoffel symbols associated with g_x, i.e.

$$C^i_{jk} = \frac{1}{2} g^{ih}(\cdot\partial_k g_{jh} + \dot{\partial}_j g_{hk} + \dot{\partial}_h g_{jk}).$$

Hence C^i_{jk} defines the Riemannian connection on M^n_x. It is observed from the definition if S_{hijk} that the curvature tensor of the Riemannian space (M^n_x, g_x) at a point x. The space (M^n_x, g_x) equipped with such a Riemannian connection will be called the tangent Riemannian space.

In the theory of Riemannian space, we know that any n-dimensional Riemannian space V^n, can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n-1)}{2}$. If $n + r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically then the integer r is called imbedding class number of V^n. The fundamental theorem of isometric imbedding [1] states that the tangent Riemannian n-space (M^n_x, g_x) is locally imbedded isometrically in an Euclidean $n + r$ space if and only if there exist r numbers, and $\lambda = \pm 1$, r symmetric tensor $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(PQ)i} = H_{(QP)i}$, $\quad Q = 1, 2, 3, \cdots, r$ satisfying the Gauss equations,

$$S_{hijk} = Sigma\lambda_{(P)}\{H_{(P)hj}H_{(P)ik} - H_{(P)hk}H_{(P)ij}\},$$

where summation is given over P.

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \Sigma\lambda_{(Q)}\{H_{(Q)ij}H_{(QP)k} - H_{(Q)ik}H_{(QP)j}\},$$

where summation is given over Q and Ricci-Kuhne equations

$$H_{(PQ)i}|_j - H_{(PQ)j}|_i + \Sigma\lambda_{(R)}\{H_{(RP)i}H_{(RQ)j}$$
$$-H_{(RP)j}H_{(PQ)i}\} + g^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} = 0.$$

For a special case when (M_x^n, g_x) is of imbedding class 1, the above equations reduce to

$$S_{hijk} = \lambda(H_{hj}H_{ik} - H_{hk}H_{ij}), \tag{4.1}$$

$$H_{ij}|_k - H_{ik}|_j = 0. \tag{4.2}$$

Since $S_{hijk}y^k = 0$, from (3.21), we have

$$H_{hj}H_{i0} - H_{h0}H_{ij} = 0$$

contracting above equation by y^i, we have

$$H_{hj}H_{00} - H_{h0}H_{0j} = 0,$$

which implies that $H_{0j} = 0$ or $H_{ij} = H_{00}^{-1}H_{h0}H_{0j}$. In the latter case we get $S_{hijk} = 0$. In the theory of spaces of imbedding class 1, [17] introduced the concept of type number t, which is the rank of matrix $\| H_{ij} \|$ provided to the rank is more than 1. If the rank is 0 or 1, then S vanishes. Therefore if (M_x^n, g_x) is of imbedding class 1, the second fundamental tensor H_{ij} satisfies $H_{ij}y^j = 0$ and thus the type number t is less than n.

Again by virtue of Lemma 2.3 and equation (4.1), we get

$$H_{hj}H_{ik} - H_{hk}H_{ij}b^h = 0.$$

From this equation we have

$$H_{hj}b^h b^j H_{ik} - H_{hk}b^h H_{ij}b^j = 0.$$

This gives

$$H_{hk}b^h = 0, \quad or \quad H_{ik} = \frac{H_{hk}b^h H_{ij}b^j}{H_{hj}b^h b^j}.$$

In the latter case $S_{hijk} = 0$. Thus for an imbedding class 1, $H_{hk}b^k = 0$. Now we shall put

$$H^*_{(1)ij} = \sqrt{\sigma}H_{ij}, \qquad \varepsilon^*_1 = \varepsilon, \tag{4.3}$$

$$H^*_{(2)ij} = A_{ij}, \qquad \varepsilon^*_2 = -1, \qquad 4.4$$

$$H^*_{(3)ij} = B_{ij}, \qquad \varepsilon^*_3 = 1, \qquad 4.5$$

then from (3.15) and (4.1), we get

$$S^*_{hijk} = \Sigma\lambda^*_P\{H^*_{(P)hj}H^*_{(P)ik} - H^*_{(P)hh}H^*_{(P)ij}\},$$

where summation is varies from $P = 1, 2, 3$. Thus the above equation is noting but Gauss equation of (M_x^n, g_x^*).

Now we put

$$H^*_{(21)i} = -H^*_{(12)i} = 0, \tag{4.6}$$

$$H^*_{(31)i} = -H^*_{(13)i} = 0, \tag{4.7}$$

$$H^*_{(32)i} = -H^*_{(23)i} = \frac{1}{L} l_i \tag{4.8}$$

and using (4.2), (4.3), (3.3), Lemma 2.1 and the fat that $H_{i0} = 0$, we get

$$H^*_{(1)ij}|_k - H^*_{(1)ik}|_j = 0. \tag{4.9}$$

Again in view of (4.4), (4.5), (4.6), (4.7) and (4.8), equations (3.19) and (3.20) reduce to

$$H^*_{(2)ij}|_k - H^*_{(2)ik}|_j = \Sigma\lambda^*_Q\{H^*_{(Q)ij}H^*_{(Q2)k} - H^*_{(Q)ik}H^*_{(Q2)j}\}, \quad 4.10$$

$$H^*_{(3)ij}|_k - H^*_{(3)ik}|_j = \Sigma\lambda^*_Q\{H^*_{(Q)ij}H^*_{(Q3)k} - H^*_{(Q)ik}H^*_{(Q3)j}\}, \quad 4.11$$

where summation is varies from $Q = 1, 2, 3$.

The equations (4.9), (4.10) and (4.11) are the Codazzi equations of (M^n_x, g^*_x). Now we have to verify Ricci-Kuhne equations, we have from (3.10),

$$l_i|_j = L^{-1}h_{ij+L^{*-1}}[\{\rho + (2L^*)^{-1}(v^2 - \frac{\beta^2}{L^2})\}h_{ij} + L^{*-1}m_i m_j]$$

from which we get $l_i|_j - l_j|_i = 0$. Hence from (4.10), we get

$$H^*_{(32)i}|_j - H^*_{(23)j}|_i = 0,$$

which are the Ricci-Kuhne equations of (M^n_x, g^*_x) as

$$M^*_{(12)} - M^*_{(21)} = 0, \quad and \quad M^*_{(13)} - M^*_{(31)} = 0.$$

Thus from above we have

Theorem 4.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then if the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class 1, then the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L^*) is at most of imbedding class 3.*

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Pure Edge-Neighbor-Integrity of Graphs

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Abstract: In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. This study introduces a new vulnerability parameter, pure edge-neighbor-integrity of graphs. The pure edge-neighbor-integrity of a graph G is defined to be $PENI(G) = \min_{\Re \subseteq E(G)}\{|\Re| + \varpi_e(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G and $\varpi_e(G/\Re)$ is the number of edges in the largest component of $G/\Re$. A set $\Re \subseteq E(G)$, is said to be a $PENI$-set of G if $PENI(G) = |\Re| + \varpi_e(G/\Re)$. In this paper, several properties and bounds on the $PENI$ are presented over here and the relation between $PENI$ with other parameters is investigated. The $PENI$ of some classes of graphs is also computed.

Key Words: Vulnerability, integrity, neighbor-integrity, edge-neighbor-integrity.

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§1. Introduction

Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks. In all applications, vulnerability and reliability are crucial and important features. Network designers often build a network configuration around specific processing, performance and cost requirements. But there is little consideration given to the stability of the networks communication structure when under the pressure of link or node loses. This lack of consideration makes the networks have low survivability. Therefore, network design process must identify the critical points of failure and be able to modify the design to eliminate them [18].

A network can be modeled by a graph whose vertices represent the stations and whose edges represent the communication lines. Vulnerability measures the resistivity of the network to the disruption of its operation due to the failure of certain stations or communication links. Losing links or nodes eventually lead to a loss of the effectiveness of the network. Communication networks must be constructed so as to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph

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theoretical parameters have been used in the past to describe the stability of communication networks, including connectivity, integrity, toughness and binding number. However, these parameters do not take into account the effect that the removal of a vertex has on the neighbors of that vertex. If a station is destroyed, the adjacent stations are betrayed and become useless to the network as a whole. The neighbor integrity is a measure of the vulnerability of graphs to the disruption caused by the consecutive removal of a vertex and all of its adjacent vertices [8, 9, 10, 15] a probabilistic basis. However, sometimes it is important to take subjective reliability estimates into consideration. Among the relevant issue of importance, we are particularly interested in one of the vulnerabilities. That is, in an unfriendly external environment, how vulnerable is such a distributed system to certain external destruction and how much computing power can be sustained in the face of destruction.

The concept of network vulnerability is motivated by the design and analysis of networks under a hostile environment. Several graph theoretic models under various assumptions have been proposed for the study and assessment of network vulnerability. Graph integrity, introduced by Barefoot et al. [4, 5], is one of these models that has received wide attention [2, 11].

In 1994, Margaret B. Cozzens and Wu [7] introduced a new graph parameter called the edge-neighbor-integrity. They consider the edge analogue of (vertex)neighbor-integrity a measure of the vulnerability of graphs to disruption caused by the removal of edges, their incident vertices, and all of their incident edges. The integrity of a graph $G = (V, E)$, which was introduced as a useful measure of the vulnerability of the graph, is defined as follows: $I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of the largest component. Barefoot, Entringer and Swart defined the edge-integrity of a graph G with edge set $E(G)$ by $I'(G) = \min\{|S| + m(G - S) : S \subseteq E(G)\}$. The weak integrity was introduced by Kirlangic [14] and is defined as $I_w(G) = \min\{|S| + m_e(G - S) : S \subseteq V(G)\}$, where $m_e(G - S)$ denotes the number of edges in a largest component of $G - S$. Let u be a vertex in G. $N(u) = \{v \in V(G)|u \neq v, v \textit{ and } u \textit{ are adjacent}\}$ is the open neighbourhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood $N(u)$ is deleted from G. A set of vertices $S = \{u_1, u_2, \cdots, u_n\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. Let G/S be the survival-subgraph when S has been a vertex subversion strategy of G. The closed neighborhood of a vertex subset $S, N[S]$, is $\cup_{u\in S} N[u]$. Hence $G/S = G - N[S] = G - (\cup_{u\in S} N[u])$. The vertex-neighbor-integrity of a graph $G, VNI(G)$, is defined to be $VNI(G) = min_{S\subseteq V(G)}\{|S| + \omega(G/S)\}$, where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. The edge $e = (v, w)$ in G is said to be subverted if the edge e, all of its incident edges, and the two ends of e, namely v and w, are removed from G. (For simplicity, an edge $e = (v, w)$ is subverted if the two ends of the edge e, namely v and w, are deleted from G.) A set of edges $\Re = \{e_1, e_2, \cdots, e_n\}$ is called an edge subversion strategy of G if each of the edges in $\Re$ has been subverted from G. Let $G/\Re$ be the survival-subgraph when $\Re$ has been an edge subversion strategy of G. The edge-neighbor-integrity of a graph G, is defined to be $ENI(G) = min_{\Re\subseteq E(G)}\{|\Re| + \varpi(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G, and $\varpi(G/\Re)$ is the maximum order of the components of $G/\Re$. We now introduce a new measure of stability of a graph G in this sense and it is called pure edge-neighbor-integrity. Formally, the pure edge-neighbor-integrity $PENI(G)$ of a graph G is defined as $PENI(G) = \min_{\Re \subseteq E(G)}\{|\Re| + \varpi_e(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G and $\varpi_e(G/\Re)$ is the number of edges of a largest component of $G/\Re$. Any set $\Re$ with property that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ is called a $PENI-$ set of G. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges, with vertex set $V(G) = \{v_1, v_2, \cdots, v_p\}$. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest v_iv_j path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. The order and size of G are denoted by p and q, respectively. We use Bondy and Murty [6, 12] for terminology and notations not defined here. In general, the degree of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by $degv$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G), (\delta(G))$. We denote the minimum number of edges in edge cover of G (i.e., edge cover number) by $\alpha_1(G)$ and the minimum number of edges in independent set of edges of G (i.e., edge independence number) by $\beta_1(G)$. A vertex of degree one is called a pendant vertex. The symbols $\alpha(G)$, $\kappa(G)$, $\lambda(G)$, and $\beta(G)$ denote the vertex cover number, the connectivity, the edge-connectivity, and the independence number of G, respectively.

A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G [16].

The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [12]. In the present work, the basic properties of pure edge-neighbor-integrity and of $PENI-$sets are explored, bounds and relationship between pure edge-neighbor-integrity and other graphical parameters are considered. Finally, the pure edge-neighbor-integrity of binary operations of some graphs are determined. We need the following to prove main results.

Lemma 1.1([13]) *If $D \subseteq E(G)$, then $L(G - D) = L(G) - D$.*

Theorem 1.1([14]) *If a graph G of order n is isomorphic to a cycle graph or a tree, then $I_w(G) = I(G) - 1$.*

Theorem 1.2([12]) *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Lemma 1.2 *For any graph G, $\beta_1(G) \leq \alpha(G)$.*

Theorem 1.3([1]) *For any connected graph G of even order p, $\gamma' = \frac{p}{2}$ if and only if G is isomorphic to K_p or $K_{\frac{p}{2},\frac{p}{2}}$.*

Theorem 1.4([2]) *The integrity of*

(a) the complete graph K_p is p;

(b) the complete bipartite graph $K_{m,n}$ is $1 + min\{m, n\}$.

§2. Main results

Proposition 2.1 *(a) For any complete graph K_p, $PENI(K_p) = \lfloor \frac{p}{2} \rfloor$;*

(b) For any path P_p with $p \geq 3$, $PENI(P_p) = \lceil 2\sqrt{p+2} \rceil - 4$;

(c) For any cycle C_p,

$$PENI(C_p) = \begin{cases} 1, & \text{if } p = 3\ ; \\ 2, & \text{if } p = 4\ ; \\ \lceil 2\sqrt{p} \rceil - 3, & \text{if } p \geq 5. \end{cases}$$

(d) For the star $K_{1,p-1}$, $PENI(K_{1,p-1}) = 1$;

(e) For the double star $S_{n,m}$, $PENI(S_{n,m}) = 1$;

(f) For the complete bipartite graph $K_{n,m}$, $PENI(K_{n,m}) = \min\{n, m\}$;

(g) For the wheel graph $W_{1,p-1}$, $p \geq 5$, $PENI(W_{1,p-1}) = \lceil 2\sqrt{p} \rceil - 3$.

Remark 2.1 (1) If H is a subgraph of G, then $PENI(H) \leq PENI(G)$;

(2) Pure edge-neighbor integrity of a connected graph for $p \geq 2$, takes its minimum value at $K_{1,p-1}$ and its maximum value at K_p complete graph;

(3) $0 \leq PENI(G) \leq q$.

Lemma 2.1 *If G is a non-trivial graph, then for all $v \in V(G)$, $PENI(G-v) \geq PENI(G)-1$, the bound is sharp for $G = K_4$.*

Proposition 2.2 *(a) If G has enough components close in size to the largest one, then $PENI(G) = \varpi_e(G)$. In particular, if $G = pH$ with $p \geq \varpi_e(H)$, then $PENI(G) = \varpi_e(H)$;*

(b) Suppose that G is disconnected and $m(G) = k$, if G has at least $k-1$ components of order k, then empty set is an $PENI(G)$-set of G.

Lemma 2.2 *If $\Re$ is $PENI$-set of G, then $\varpi_e(G/\Re) = PENI(G/\Re)$ and ϕ is $PENI$-set of $G/\Re$.*

Proof Let $\Re$ is $PENI$-set of G and $\Re^\star$ be $PENI$-set of $G/\Re$. Thus

$$\begin{aligned} |\Re| + \varpi_e(G/\Re) &= PENI(G) \\ &\leq \varpi_e(G/(\Re \cup \Re^\star)) + |\Re \cup \Re^\star| \\ &= |\Re| + \varpi_e[(G/\Re)/\Re^\star] + |\Re^\star| \\ &= |\Re| + PENI(G/\Re). \end{aligned}$$

So, $\varpi_e(G/\Re) \leq PENI(G/\Re)$, but $\varpi_e(G/\Re) \geq PENI(G/\Re)$. This completes the proof. □

Lemma 2.3 *If $D \subseteq E(G)$,$PENI(L(G-D)) = PENI(L(G)-D))$.*

Proof The proof follows by Lemma 1.1. □

Theorem 2.1 *If G is a simple graph such that $\overline{G} \cong L(G)$, then $PENI(G) = PENI(L(G)) =$*

$PENI(\overline{G})$ if and only if $G = C_5$ or G is the graph shown in the Figure 1.

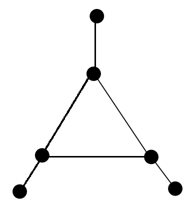

Figure 1 G

Proposition 2.3 *If a connected graph G is isomorphic to its line graph, then $PENI(G) = PENI(L(G))$. But the converse is not true, for example the graph G is given in the following Figure 2.*

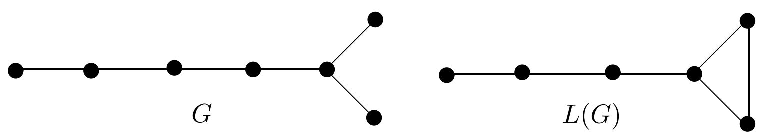

Figure 2 G and $L(G)$

Notice that $PENI(G) = 2 = PENI(L(G))$, but G and $L(G)$ are not isomorphic.

Lemma 2.4 *Let G be a connected graph of order at least 3. If $PENI(G) = 1$, then the diameter of G is ≤ 3.*

Proof The diameter of G is ≥ 4 is Supposed, then G contains a path P_5. Hence for any edge e in $G, \varpi_e(G/e) \geq 1$, and for any two edges e_1 and e_2 in $G, \varpi_e(G/\{e_1, e_2\}) \geq 0$. Thus $EENI(G) \geq 2$, a contradiction. Hence, the diameter of G is ≤ 3. □

Lemma 2.5 *For any a graph G, $PENI(G) = VNI(L(G))$.*

Proof Since every edge dominating set in G is a dominating set in the line graph of G, the set of edges S that satisfies $PENI(G)$ equal to the set of vertices S that satisfies $VNI(L(G))$, this completes the proof. □

Lemma 2.6 *For any (p, q) graph G, $\lceil \frac{q}{\Delta'+1} \rceil \leq PENI(G) \leq q - \beta_1$, where Δ' denotes the maximum degree of an edge in G.*

Observation 2.1 For any connected graph G, $PENI(G) = q - \beta_1$ if and only if $G \cong P_p, 3 \leq p \leq 6$, $G \cong p_8$, $G \cong C_4$ or G in the Figure 3.

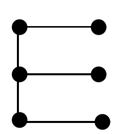

Figure 3 G

Corollary 2.1 *For any connected (p,q) graph, $PENI(G)=p-q$ if and only if G is isomorphic to $K_{1,p-1}$ or $S_{n,m}$.*

Observation 2.2 Let G be a graph, and let $\Re$ be $PENI$-set of G such that $|\Re|=1$, then the following hold

(a) $PENI(G)=1$;
(b) $|E-\Re|=\sum_{e\in\Re} deg(e)$;
(c) $\Delta^{'}(G)=q-1$.

Corollary 2.2 *For any connected graph G of even order p, $PENI(G)=\frac{p}{2}$ if and only if G is isomorphic to K_p or $K_{\frac{p}{2},\frac{p}{2}}$.*

Theorem 2.2 *For any integer $n\geq 1$, there does not exist any graph G satisfy $PENI(G)=I(G)=\gamma'(G)=n$.*

Proof Let G be a graph of order p. By Theorem 1.3 and Corollary 2.2, $PENI(G)=\frac{p}{2}=\gamma'(G)$ if p is even and $G\cong K_p$ or $G\cong K_{\frac{p}{2},\frac{p}{2}}$, but from Theorem 1.4, $I(K_p)=p$, and $I(K_{\frac{p}{2},\frac{p}{2}})=\frac{p}{2}+1$. Hence the result. □

Theorem 2.3 *For any integer $k\geq 1$, there exists a graph G of size $q\geq k$ with $PENI(G)=\gamma(G)=k$, where $\gamma(G)$ is domination number.*

Proof The result is true for $k=1,2$, since $G_1=K_2, G_2=K_3$ have the desired property. For $k\geq 3$, consider the graph G_k which is obtained from k disjoint copies of the complete graph K_3 and joining the vertex v_i in the i^{th} copy with the vertex v_{i+1} in the $(i+1)^{th}$ copy, and joining the vertex u_i in the i^{th} copy with the vertex w_i in the $(i+1)^{th}$ copy. The graph G_3 shown in Figure 4.

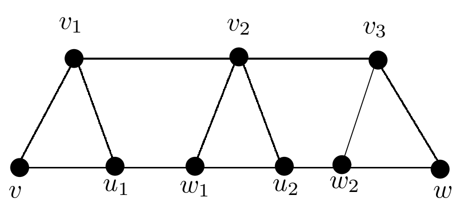

Figure 4 G_3

Consider $D=\{v_1,v_2,v_3,\cdots,v_k\}$ be a dominating set for G_k, and $|D|=k$. Let's claim that set D is a minimum dominating set. Since each $v_i, 2\leq i\leq k-1$, is adjacent to w_{i-1} and u_i. If v_i is removed from set D, then w_{i-1} and u_i will not be dominated by any vertex. Hence D is a minimum domination set. Therefore, $\gamma(G_k)=k$. Consider $\Re=\{(v_1,v),(v_2,w_1),(v_3,w_2),\cdots,(v_i,w_{i-1}),1\leq i\leq k\}$. Then $|\Re|=k$, and $\varpi_e(G_k/\Re)=0$. Therefore, $PENI(G_k)\leq|\Re|+\varpi_e(G_k/\Re)=k$. Consider $\Re_1=\{(v_1,v),(v_2,w_1),(v_3,w_2),...,(v_{i-1},w_{i-2}),1\leq i\leq k\}$. Then $|\Re_1|=k-1$, and $\varpi_e(G_k/\Re_1)=4$, this implies that $|\Re_1|+\varpi_e(G_k/\Re_1)>|\Re|+\varpi_e(G_k/\Re)$. If $\varpi_e(G_k/\Re)=1$, then $|\Re|\geq k$. Thus, $PENI(G_k)\geq k+1$. Therefore, $PENI(G_k)=k$. □

Corollary 2.3 *For every integer $n\geq 1$, there exists graph G with $PENI(G)=n$.*

Lemma 2.7 *Let G be a graph of order p, $PENI(G) = 0$ if and only if $G \cong \overline{K}_p$.*

Theorem 2.4 *For any graph G of order p, $PENI(G) \leq I_w(G) \leq I'(G)$.*

Proof Clearly, $I_w(G) \leq I'(G)$. If G is complete, then $PENI(G) = \lfloor \frac{p}{2} \rfloor \leq |p-1| = I_w(G)$. G is non-complete is supposed and $S' = \{u_1, u_2, \cdots, u_p\}$ be an I_w-set of G. Then S' is a vertex cut-set of G, and u_i, where $1 \leq i \leq p$, is not an isolated vertex of G. Let $\Re = \{(u_i, v_i) \in E(G)/$ for some vertex $v_i \in V, u_i \in S',\ where\ i = 1, 2, \cdots, p\}$ thus $|\Re| = |S'| = p$. Therefore,

$$\begin{aligned} G/\Re &= G - \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_p\} \\ &= G - (S' \cup \{v_i \in V(G)/(u_i, v_i) \in \Re, u_i \in S'\}) \subseteq G - S', \end{aligned}$$

it follows that $\varpi_e(G/\Re) \leq m_e(G - S')$, then

$$\begin{aligned} PENI(G) &\leq |\Re| + \varpi_e(G/\Re) \\ &\leq |S'| + m_e(G - S') = I_w(G). \end{aligned}$$

□

Observation 2.3 If $PENI(G) = I_w(G)$, then the induced subgraph of G, $< S >$ must be a null graph, where S is an I_w-set of G. But the converse is not true, for example in the graph in Figure 5. $S = \{u_1, u_2, u_3\}$ is an I_w-set of G is noted. Therefore, $I_w(G) = 4$ and $\Re = \{e_1, e_2\}$ is a $PENI$- set of G. Thus $PENI(G) = 2$. $< S >$ is a null graph, but $I_w(G) \neq PENI(G)$.

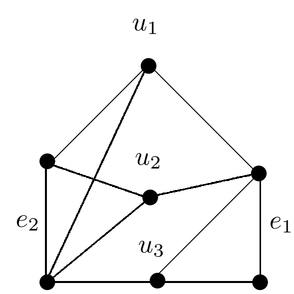

Figure 5

Lemma 2.8 *If diam$(L(G)) = 1$, then $PENI(G) = 1$.*

Proof Since diam$(L(G)) = 1$, then G is either K_3 or $K_{1,p-1}$. Hence the result. □

Remark 2.2 If G is a graph with $\alpha(G) = 1$, $PENI(L(G)) = \lfloor \frac{p}{2} \rfloor$.

Theorem 2.5 *For any graph G, $VNI(G) \leq PENI(G) + 1$.*

Proof Let $\Re = \{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\}$ be a $PENI$-set of G, let S be a set of one end vertex of each edge in $\Re$. Thus $|S| \leq |\Re|$ and $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\} \subseteq N[S]$. Therefore $G/S = G - N[S] \subseteq G - \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\} = G/\Re$, and $|S| + \omega(G/S) \leq |\Re| + \omega(G/\Re) \leq |\Re| + \varpi_e(G/\Re) + 1 = PENI(G) + 1$. So $VNI(G) = min_{S \subseteq V(G)}\{|S| + \omega(G/S)\} \leq |S'| + \omega(G/S') \leq PENI(G) + 1$. □

Theorem 2.6 *For any graph G, $PENI(G) \geq ENI(G) - 1$.*

Proof Let $\Re$ be a $PENI$-set of G. Since $ENI(G) \leq |\Re| + \varpi(G/\Re)$ and $\varpi(G/\Re) \leq \varpi_e(G/\Re) + 1$, for every $\Re \subseteq E(G)$, hence the result. □

Theorem 2.7 *For any graph G and $e \in E(G)$, $PENI(G - e) \geq PENI(G) - 1$.*

Proof Let $\Re^\star$ be a $PENI$-set of $G - e$, and $PENI(G - e) = |\Re^\star| + \varpi_e((G - e)/\Re^\star)$, let $\Re^{\star\star} = \Re^\star \cup \{e\}$. Then $|\Re^{\star\star|} = |\Re^\star| + 1$. Then $\Re^{\star\star}$ is $PENI$-set of G and $\varpi_e(G/\Re^{\star\star}) = \varpi_e(G/e)/\Re^\star)$ Therefore,

$$\begin{aligned} PENI(G) &\leq |\Re^{\star\star}| + \varpi_e(G/\Re^{\star\star}) \\ &\leq |\Re^\star| + \varpi_e[(G/e)/\Re^\star] + 1 \\ &= PENI(G - e) + 1. \end{aligned}$$

Then $PENI(G - e) \geq PENI(G) - 1$. □

Theorem 2.8 *For any graph G, $PENI(G) \leq \alpha_1(G)$.*

Proof Let $\Re$ be $PENI$-set of G such that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ and let C be a minimum edge covering of G. Since each vertex of G is an end vertex of some edge in C, we have $G/C = \phi$ and $\varpi_e(G/C) = 0$.

Thus

$$\begin{aligned} PENI(G) &= |\Re| + \varpi_e(G/\Re) \\ &\leq |C| + \varpi_e(G/C) = |C| = \alpha_1(G). \end{aligned}$$ □

Theorem 2.9 *For any graph G, $PENI(G) \leq \beta_1(G)$.*

Proof Let $\Re$ be $PENI$-set of G such that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ and let M be a maximum matching in G. It is clear $G/M = \phi$ or a set of isolated vertices, hence $\varpi_e(G/M) = 0$. Then

$$\begin{aligned} PENI(G) &= |\Re| + \varpi_e(G/\Re) \\ &\leq |M| + \varpi_e(G/M) = |M| = \beta_1(G). \end{aligned}$$ □

Theorem 2.10 *For any graph G, $PENI(G) \leq \alpha(G)$.*

Proof The proof follows from Lemma 1.2 and Theorem 2.9. □

Theorem 2.11 *For any tree T, $PENI(T) \geq \delta(T)$.*

Proof Let $\Re$ be a $PENI$-set of T such that $PENI(T) = |\Re| + \varpi_e(T/\Re)$. Then $\varpi_e(T/\Re) \geq \delta(T/\Re) \geq \delta(T) - |\Re|$, So, $PENI(T) = |\Re| + \varpi_e(T/\Re) \geq |\Re| + \delta(T) - |\Re| = \delta(T)$. □

Lemma 2.9 *For any tree T, $PENI(T) \geq \lambda(T)$.*

Proof The proof follows from Theorems 2.11 and 1.2. □

Lemma 2.10 *For any tree T, $PENI(T) \geq \kappa(T)$.*

Proof The proof follows from Lemma 2.9 and Theorem 1.2. □

Notice that $\alpha_1(G), \beta_1(G)$ and $\alpha(G)$ are upper bounds of $PENI(G)$, while $\delta(G), \lambda(G)$ and $\kappa(G)$ are lower bounds of $PENI(G)$.
However, the independence number β, has no such relationship with $PENI(G)$. For example,

(1) $PENI(K_{1,n}) < \beta(K_{1,n})$;
(2) $PENI(K_p) > \beta(K_p)$;
(3) $PENI(K_{n,m}) = \begin{cases} n = m = \beta(K_{n,m}), \text{ if } n = m\ ; \\ min\{n, m\} < \beta(K_{n,m}), \text{ if } n \neq m. \end{cases}$

Corollary 2.4 *For any graph G, $PENI(G) \leq \lfloor \frac{p}{2} \rfloor$.*

Proof Let M be a maximum matching of G. Then $|M| = \beta_1(G) \leq \lfloor \frac{p}{2} \rfloor$. Two cases are discussed.

Case 1. If $\beta_1(G) = \lfloor \frac{p}{2} \rfloor$, then $G/M = \phi$ (if p is even) or a single vertex (if p is odd), hence $PENI(G) \leq |M| + \varpi_e(G/M) = \lfloor \frac{p}{2} \rfloor$.

Case 2. If $\beta_1(G) < \lfloor \frac{p}{2} \rfloor$, then by Theorem 2.9, we have $PENI(G) \leq \beta_1(G) < \lfloor \frac{p}{2} \rfloor$. □

Theorem 2.12 *For any graph G, $PENI(G) \geq \lceil \frac{I(G)}{2} \rceil - 1$.*

Proof Let $\Re = \{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\}$ be a $PENI$-set of G. So $PENI(G) = |\Re| + \varpi_e(G/\Re) = n + \varpi_e(G/\Re)$.

Let $S^\star = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Since $\Re$ may not be edge independent in G, $|S^\star| \leq 2n$. Then

$$\begin{aligned} I(G) &= min_{S \subseteq V(G)}\{|S| + m(G - S)\} \\ &\leq |S^\star| + m(G - S^\star) \leq 2n + \varpi_e(G/\Re) + 1 \\ &\leq 2(n + \varpi_e(G/\Re)) + 1 = 2PENI(G) + 1. \end{aligned}$$

Therefore, $PENI(G) \geq \lceil \frac{I(G)}{2} \rceil - 1$. □

Corollary 2.5 *For any graph G, $PENI(G) \geq \lceil \frac{I_w(G)}{2} \rceil$.*

§3. Pure-Edge Neighbor Integrity of Some Graph Operators

Definition 3.1([12]) *The (Cartesian)product $G \times H$ of graphs G and H has $V(G) \times V(H)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Theorem 3.1 *For a graph $K_2 \times P_p$,*

$$PENI(K_2 \times P_p) = \begin{cases} \frac{p}{2}+1, & p \text{ is even;} \\ \frac{p-1}{2}+1, & p \text{ is odd.} \end{cases}$$

Proof The number of vertices of graph $K_2 \times P_p$ is $2p$ and the number of edges is $3p-2$. The graph $K_2 \times P_p$ is shown in Figure 6, we have two cases.

Case 1. p is even. Consider $\Re = \{e_{2+2j}, 0 \le j < \frac{p}{2}\}$, $|\Re| = \frac{p}{2}$, and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times P_p) \le |\Re| + \varpi_e((K_2 \times P_p)/\Re) = \frac{p}{2} + 1. \tag{1}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) = 0$, then $|\Re| \ge p-1$. So

$$PENI(K_2 \times P_p) \ge p - 1. \tag{2}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) \ge 2$, then $|\Re| \ge \frac{p}{2}, p > 2$. Thus

$$PENI(K_2 \times P_p) \ge \frac{p}{2} + 2. \tag{3}$$

Therefore, the inequalities (1), (2) and (3) lead to $PENI(K_2 \times P_p) = \frac{p}{2} + 1$.

Case 2. p is odd. Consider $\Re = \{e_{2+2j}, 0 \le j < \frac{p-1}{2}\}$, $|\Re| = \frac{p-1}{2}$, and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times P_p) \le |\Re| + \varpi_e((K_2 \times P_p)/\Re) = \frac{p-1}{2} + 1. \tag{4}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) = 0$, then $|\Re| \ge p-1$. So

$$PENI(K_2 \times P_p) \ge p - 1. \tag{5}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) \ge 2$, then $|\Re| \ge \frac{p-1}{2} + 1$. Thus

$$PENI(K_2 \times P_p) > \frac{p-1}{2} + 1. \tag{6}$$

Therefore, these inequalities (4), (5) and (6) lead to $PENI(K_2 \times P_p) = \frac{p-1}{2} + 1$. □

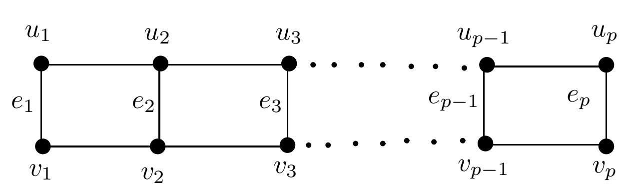

Figure 6 $K_2 \times P_p$

Theorem 3.2 *For a graph* $K_2 \times C_p$,

$$PENI(K_2 \times C_p) = \begin{cases} \frac{p}{2}+1, & p \text{ is even and } p > 2; \\ \frac{p+1}{2}+1, & p \text{ is odd and } p \geq 3. \end{cases}$$

Proof The number of vertices of graph $K_2 \times C_p$ is $2p$ and the number of edges is $3p$. The graph $K_2 \times C_p$ is shown in Figure 7, two cases are considered.

Case 1. p is even. Consider $\Re = \{e_{1+2j}, 0 \leq j < \frac{p}{2}\}$, $|\Re| = \frac{p}{2}$, and $\varpi_e((K_2 \times C_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = \frac{p}{2} + 1. \quad (7)$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) = 0$, then $|\Re| \geq p - 1$. So

$$PENI(K_2 \times C_p) \geq p - 1. \quad (8)$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) \geq 2$, then $|\Re| \geq \frac{p}{2} + 1$. Thus

$$PENI(K_2 \times C_p) \geq \frac{p}{2} + 3. \quad (9)$$

Therefore, these inequalities (7), (8) and (9) lead to

$$PENI(K_2 \times C_p) = \frac{p}{2} + 1.$$

Case 2. (i) p is odd, $p = 3$. Consider $S = \{e_1, e_2\}, |S| = 2$, and $\varpi_e((K_2 \times C_p)/\Re)) = 1$. Thus,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = 3.$$

(ii) $p > 3$, Consider $\Re = \{e_{1+2j}, 0 \leq j < \frac{p+1}{2}\}$, $|\Re| = \frac{p+1}{2}$ and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = \frac{p+1}{2} + 1. \quad (10)$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) = 0$, then $|\Re| \geq p - 1$. So

$$PENI(K_2 \times C_p) \geq p - 1. \quad (11)$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) \geq 2$, then $|\Re| \geq \frac{p+1}{2}$. Thus

$$PENI(K_2 \times C_p) \geq \frac{p+1}{2} + 2. \quad (12)$$

Therefore, these inequalities (10), (11) and (12) lead to

$$PENI(K_2 \times C_p) = \frac{p+1}{2} + 1. \qquad \square$$

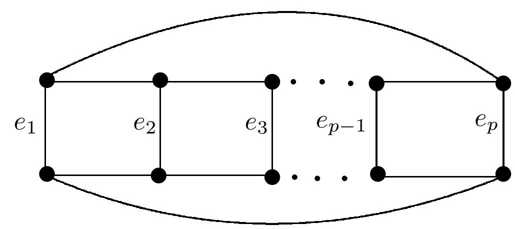

Figure 7 $K_2 \times C_p$

Theorem 3.3 *$PENI(K_2 \times K_{1,p-1}) = 2$.*

Proof The number of vertices of graph $K_2 \times K_{1,p-1}$ is $2p$. The set $\Re = \{e\}$ as shown in Figure 8 is chosen. If we remove the edge e, $p-1$ components such that $\varpi_e((K_2 \times K_{1,p-1})/\Re) = 1$, thus $|\Re| + \varpi_e((K_2 \times K_{1,p-1})/\Re) = 2$. Therefore, $PENI(K_2 \times K_{1,p-1}) = 2$. If $\Re$ is set of any edges such that $\varpi_e((K_2 \times K_{1,p-1})/\Re) = 0$, then $|\Re| \geq p-1$. So $PENI(K_2 \times K_{1,p-1}) \geq p-1$.

If $\varpi((K_2 \times K_{1,p-1})/\Re) \geq 2$, then trivially $|\Re| + \varpi_e((K_2 \times K_{1,p-1})/\Re) > 2$. Thus $HI(K_2 \times K_{1,p-1}) = 2$. □

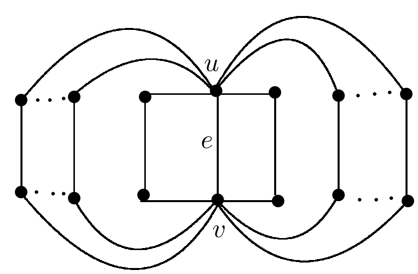

Figure 8 $K_2 \times K_{1,p-1}$

Definition 3.2([12]) *For a simple connected graph G the square of G denoted by G^2, is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 in G.*

Theorem 3.4 *For a graph P_p^2,*

$$PENI(P_p^2) = \begin{cases} \frac{p}{3}, & \text{if } p \equiv 0 (mod\ 3), \\ \frac{p-1}{3}, & \text{if } p \equiv 1 (mod\ 3), \\ \frac{p-2}{3} + 1, & \text{if } p \equiv 2 (mod\ 3). \end{cases}$$

Proof Let $V(P_p) = \{v_1, v_2, \cdots, v_p\}$. Then, $|V(P_p^2)| = p$ and $|E(P_p^2)| = 2p - 3$. The graph P_5^2 is shown in Figure 9.

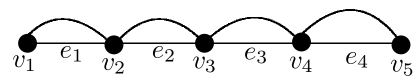

Figure 9 P_5^2

An edge set $\Re$ of P_p^2 as below is considered.

(1) If $p \equiv 0(mod\ 3)$, then $p = 3k$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{2+3i}/0 \leq i \leq k-1\}\ and\ |\Re| = k.$$

We have, $|\Re| = \frac{p}{3}$, and $\varpi_e(P_p^2/\Re) = 0$;

(2) If $p \equiv 1(mod\ 3)$, then $p = 3k+1$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{2+3i}/0 \leq i \leq k-1\}\ and\ |\Re| = k.$$

We have, $|\Re| = \frac{p-1}{3}$, and $\varpi_e(P_p^2/\Re) = 0$;

(3) If $p \equiv 2(mod\ 3)$ then, $p = 3k-1$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{1+3i}/0 \leq i \leq k-1\}\ and\ |\Re| = k.$$

We have, $|\Re| = \frac{p-2}{3} + 1$ and $\varpi_e(P_p^2/\Re) = 0$.

To discuss the minimality of $|\Re| + \varpi_e(P_p^2/\Re)$. Consider any edge set $\Re_1$ of P_p^2 such that, $|\Re_1| \leq |\Re|$, then due to the construction of P_p^2 (i.e., to convert $P_p^2/\Re_1$ into disconnected graph, include at least one edge in $\Re_1$) must be included. It generates a large value of $\varpi_e(P_p^2/\Re_1)$ such that,

$$|\Re| + \varpi_e(P_p^2/\Re) \leq |\Re_1| + \varpi_e(P_p^2/\Re_1) \tag{13}$$

Let $\Re_2$ be any edge set of P_p^2 such that $\varpi_e(P_p^2/\Re_2) \geq 1$. Then

$$|\Re| + \varpi_e(P_p^2/\Re) \leq |\Re_2| + \varpi_e(P_p^2/\Re_2). \tag{14}$$

Therefore, these inequalities (13) and (14) lead to

$$|\Re| + \varpi_e(P_p^2/\Re) = min\{|X| + \varpi_e(G/X) : X \subseteq E(G)\} = PENI(P_p^2).$$ □

Definition 3.3([17]) *The lollipop graph $L_{p,d}$ is obtained from a complete graph K_{p-d} and a path P_d, by joining one of the end vertices of P_d to all the vertices of K_{p-d}.*

Theorem 3.5 *For a lollipop graph $L_{p,d}$,*

$$PENI(L_{p,d}) = \lfloor \frac{p-d+1}{2} \rfloor + \lceil 2\sqrt{d+1} \rceil - 4.$$

Proof The number of the vertices of $L_{p,d}$ is p and the number of edges is $d-1+\frac{(p-d+1)(p-d)}{2}$.

The graph $L_{p,d}$ consists of a complete graph of order $p-d+1$ and a path of order $d-1$. By Proposition 2.1, it follows that

$$PENI(L_{p,d}) = PENI(P_{d-1}) + PENI(K_{p-d+1}) = \lfloor\frac{p-d+1}{2}\rfloor + \lceil 2\sqrt{d+1}\rceil - 4. \qquad \Box$$

Definition 3.4([17]) *A broom graph $B_{p,d}$ consists of a path P_d, together with $(p-d)$ end vertices all adjacent to the same end vertex of P_d.*

Theorem 3.6 *For a broom graph $B_{p,d}$,*

$$PENI(B_{p,d}) = \lceil 2\sqrt{d}\rceil - 3.$$

Proof Let $V(B_{p,d}) = \{u_1, u_2, \cdots, u_d, v_1, v_2, \cdots, v_{p-d}\}$ such that $u_1, u_2, \cdots, u_d$ is a path on d vertices and $v_1, v_2, \cdots, v_{p-d}$ are end vertices that are adjacent to u_d. An edge e as shown in Figure 10 is chosen,

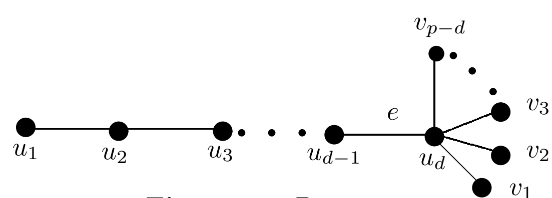

Figure 10 $B_{p,d}$

and e is deleted, we get $p-d+1$ components, namely $(p-d)$ isolated vertices and a path of order $(d-2)$. By Proposition 2.1, it follows that

$$PENI(B_{p,d}) = 1 + PENI(P_{d-2}) = 1 + \lceil 2\sqrt{d}\rceil - 4.$$

Thus

$$PENI(B_{p,d}) = \lceil 2\sqrt{d}\rceil - 3. \qquad \Box$$

Corollary 3.1 *For any broom graph, if $p-d=2$, then*

$$PENI(B_{p,d}) = PENI(L(B_{p,d})).$$

Definition 3.5([12]) *The join of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 + G_2$ consists of vertex set $V = V_1 \cup V_2$, and edge set $E = E_1 \cup E_2$ and all edges joining V_1 with V_2.*

Theorem 3.7 *For a joint graph $K_2 + P_p$,*

$$PENI(K_2 + P_p) = \lceil 2\sqrt{p+2}\rceil - 3.$$

Proof Let K_2 be a complete graph with vertices u_1, u_2 and P_p, a path with vertices $v_1, v_2, \cdots, v_p$. Let G be the graph $K_2 + P_p$. Then, $V(G) = \{u_1, u_2, v_1, \cdots, v_p\}, |V(G)| = p+2$, and $|E(G)| = 3p$.

The graph $K_2 + P_p$ is shown in Figure 11.

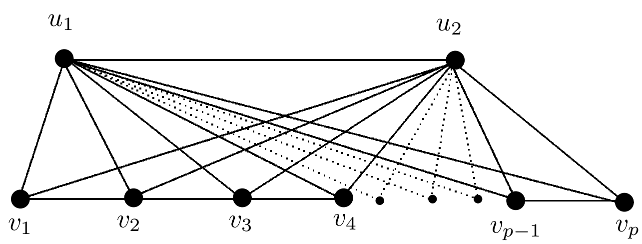

Figure 11 $K_2 + P_p$

Consider $\Re_1 = \{(u_1, u_2)\}$, $|\Re_1| = 1$. Then, $G/\Re_1 = P_p$, so that $\varpi_e(G/\Re_1) = p-1$. Let $\Re_2 = \{e_k = (v_k, v_{k+1}), 1 \le k \le p-1 / e_k \in PENI - set\ of\ P_p\}$. Take $E_1 = \{e_k/e_k \in PENI - set\ of\ P_p\}$ so that $|\Re_2| = |E_1|$. Consider $\Re = \Re_1 \cup \Re_2$. Thus, $|\Re| = |\Re_1| + |\Re_2| = |\Re_1| + |E_1|$ and $G/\Re = P_p/E_1$. So $\varpi_e(G/\Re) = \varpi_e(P_p/E_1)$. By Proposition 2.1, we have

$$\begin{aligned} |\Re| + \varpi_e(G/\Re) &= |\Re_1| + |E_1| + \varpi_e(P_p/E_1) \\ &= |\Re_1| + PENI(P_p) = 1 + \lceil 2\sqrt{p+2} \rceil - 4 \\ &= \lceil 2\sqrt{p+2} \rceil - 3. \end{aligned} \tag{15}$$

To claim that $|\Re| + \varpi_e(G/\Re)$ is minimum. Suppose $\Re_3$ is any edge set of G such that $\Re_3 = \Re_1 \cup \{e\}$ and $|\Re_3| = 2$. Then $|\Re_3| + \varpi_e(G/\Re_3) \ge |\Re| + \varpi_e(G/\Re)$. Let $\Re_5$ be edge set of G such that $\Re_5 = \Re_2$. Then, $\varpi_e(G/\Re_5) \ge p$. Hence, $|\Re_5| + \varpi_e(G/\Re_5) \ge |\Re_2| + p > |\Re| + \varpi_e(G/\Re)$. Therefore, from the above discussion, it follows that $|\Re| + \varpi_e(G/\Re)$ is minimum. Hence, from equation (15) and the minimality of $|\Re| + \varpi_e(G/\Re)$, we have

$$PENI(K_2 + P_p) = \lceil 2\sqrt{p+2} \rceil - 3.$$ □

Theorem 3.8 *For a joint graph $K_2 + C_p$,*

$$PENI(K_2 + C_p) = \begin{cases} p-1, & p=3,\ 4; \\ \lceil 2\sqrt{p} \rceil - 2, & p \ge 5. \end{cases}$$

Proof The proof is similar to that of the Theorem 3.7. □

Theorem 3.9 *For a joint graph $K_2 + K_p$,*

$$PENI(K_2 + K_p) = \lfloor \frac{p+2}{2} \rfloor.$$

Proof Since $K_2 + K_p = K_{p+2}$ is a complete graph of order $p+2$, by Proposition 2.1,

$$PENI(K_2 + K_p) = PENI(K_{(p+2)}) = \lfloor \frac{p+2}{2} \rfloor. \qquad \square$$

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Bounds for the Largest Color Eigenvalue and the Color Energy

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Abstract: In [1] Chandrashekar Adiga et al. introduced the matrix of a vertex colored graph and studied their eigenvalues called color eigenvalues. Further, defined the color energy of the graph and obtained some results. In this paper, we obtain bounds for the largest color eigenvalue and the color energy.

Key Words: Smarandachely vertex coloring, color eigenvalues, color spectral radius, color energy of a graph.

AMS(2010): 05C15, 05C50

§1. Introduction

Let $G = (V, E)$ be finite simple graph with n vertices and m edges. The adjacency matrix of G is the $n \times n$ matrix $A = A(G)$, whose entries a_{ij} are given by $a_{ij} = 1$ if v_i and v_j are adjacent, $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are the eigenvalues of G. The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of $A(G)$ [6]. A survey of development of energy of a graph before 2001 can be found in [7].

Let H be a subgraph of graph G. A *Smarandachely vertex coloring respect to* H of a graph G by colors in $\mathscr{C}$ is a mapping $\varphi_H : \mathscr{C} \to E(G)$ such that $\varphi_H(e_1) \neq \varphi_H(e_2)$ if e_1 and e_2 are edges of a subgraph isomorphic to H in G. Particularly, if $H = G$, such a Smarandachely vertex coloring is the usual *vertex coloring* of a graph G, i.e., a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called *chromatic number* and denoted by $\chi(G)$.

Recently in [1], Chandrashekar Adiga et al. have introduced the $n \times n$ matrix $A = A_c(G)$ of a vertex colored graph G, which is defined as follows: If $c(v_i)$ is the color of v_i, then

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

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The eigenvalues of $A_c(G)$ are called color eigenvalues of G. The color energy $E_c(G)$ is defined to be the sum of the absolute values of the color eigenvalues of G. In [1]C.Adiga et al. have computed the color energy $E_\chi(G)$ of few families of graphs with minimum number of colors. In [2] they have also derived explicit formulas for the color energies of the unitary Cayley graph, the complement of the colored unitary Cayley graph and the gcd-graphs.

The main purpose of this paper is to establish some bounds for largest color eigenvalue and color energy. In literature there are several upper bounds for the spectral radius λ_1 of a graph G. For more details see [3], [4], [5] and [8].

§2. Bounds for the Largest Color Eigenvalue

First we prove the following theorem which is useful to obtain bounds for the largest color eigenvalue of a graph G.

Theorem 2.1 *Let G be a colored graph with n vertices and m edges and H be a (n, m_1)-graph. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are color eigenvalues of G and $\lambda_1{}' \geq \lambda_2{}' \geq \cdots \geq \lambda_n{}'$ are eigenvalues of H, then*

$$\sum_{i=1}^{n} \lambda_i \lambda_i' \leq 2\sqrt{(m + m_c{}')m_1},$$

where $m_c{}'$ is the number of pairs of non-adjacent vertices receiving the same color in G.

Proof By Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} \lambda_i \lambda_i'\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} {\lambda_i'}^2\right) \tag{2.1}$$

In [1], it has been proved that $\sum_{i=1}^{n} \lambda_i^2 = 2(m + m_c{}')$. It is well-known that $\sum_{i=1}^{n} {\lambda_i'}^2 = m_1$. Using these in (2.1) we obtain

$$\left(\sum_{i=1}^{n} \lambda_i \lambda_i'\right) \leq 2\sqrt{(m + m_c{}')m_1}. \qquad \square$$

If we know the spectrum of a graph H with n vertices and m_1 edges, then we can find an upper bound for the largest color eigenvalue of the colored graph G with n vertices.

Using the above theorem we establish bounds for the largest color eigenvalue.

Proposition 2.2 *If G is a colored (n, m)-graph and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are color eigenvalues of G, then*

$$\lambda_1 \leq \frac{1}{p-1}\left[\sqrt{2(m + m_c{}')p(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}\right],$$

where p is any integer $1 < p \leq n$.

Proof Let $H = K_p \bigcup \overline{K_{n-p}}$. Then the spectrum of H is

$$\begin{pmatrix} (p-1) & 0 & -1 \\ 1 & n-p & p-1 \end{pmatrix}.$$

Then by Theorem 2.1 we have

$$\lambda_1(p-1) + \lambda_2(0) + \lambda_3(0) + \cdots + \lambda_{n-p+1}(0) + \lambda_{n-p+2}(-1) + \cdots + \lambda_n(-1)$$
$$\leq 2\sqrt{\frac{(m+m_c{}')p(p-1)}{2}}.$$

Thus,

$$(p-1)\lambda_1 \leq \sqrt{2(m+m_c{}')p(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}$$

Hence,

$$\lambda_1 \leq \frac{1}{p-1}\left[\sqrt{2(m+m_c{}')p(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}\right]. \qquad \square$$

Remark 2.3 If $p = n$ in the above proposition, then

$$\lambda_1 \leq \sqrt{\frac{2(m+m_c{}')(n-1)}{n}}.$$

Remark 2.4 If $p = 2$ in the above proposition, then

$$\lambda_1 - \lambda_n \leq 2\sqrt{(m+m_c{}')}.$$

Proposition 2.5 *If G is a colored (n,m)-graph and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are color eigenvalues of G, then*

$$\sum_{i=1}^{k} \lambda_i \leq \sqrt{\frac{2(m+m_c{}')k(p-1)}{p}},$$

where p is any integer $1 \leq p \leq n$ and $k = \frac{n}{p}$.

Proof Let H be a graph with n vertices and k components each is a complete graph K_p. Then $n = pk$ and H has $\frac{kp(p-1)}{2}$ edges. Thus spectrum of H is

$$\begin{pmatrix} (p-1) & -1 \\ k & k(p-1) \end{pmatrix}.$$

Then by Theorem 2.1 we have

$$(p-1)\lambda_1 + (p-1)\lambda_2 + \cdots + (p-1)\lambda_k + (-1)\lambda_{k+1} + \cdots + (-1)\lambda_n$$
$$\leq 2\sqrt{\frac{(m+m_c{}')kp(p-1)}{2}}.$$

Therefore,

$$p\sum_{i=1}^{k}\lambda_i-\sum_{i=1}^{n}\lambda_i\leq\sqrt{2(m+m_c{}')kp(p-1)}$$

and

$$\sum_{i=1}^{k}\lambda_i\leq\sqrt{\frac{2(m+m_c{}')k(p-1)}{p}}.\qquad\square$$

Remark 2.6 If $k=1$ in the above proposition, then

$$\lambda_1\leq\sqrt{\frac{2(m+m_c{}')(p-1)}{p}}.$$

Proposition 2.7 *If G is a colored (n,m)-graph and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ are color eigenvalues of G, then*

$$\left[\sum_{i=1}^{k}\lambda_i-\sum_{i=1}^{k}\lambda_{n-k+i}\right]\leq 2\sqrt{(m+m_c{}')k},$$

where $1\leq k<n$ and $k|n$.

Proof Let H be a graph with n vertices and k components each is a complete bipartite graph $K_{p,q}$. Then $n=k(p+q)$ and H has kpq edges. Thus, the spectrum of H is

$$\begin{pmatrix}\sqrt{pq} & 0 & -\sqrt{pq}\\ k & k(p+q-2) & k\end{pmatrix}.$$

Then, by Theorem 2.1 we have

$$\sqrt{pq}\lambda_1+\cdots+\sqrt{pq}\lambda_k+(0)\lambda_{k+1}+\cdots+(0)\lambda_{k+k(p+q-2)}+(-\sqrt{pq})\lambda_{k(p+q-1)+1}+\cdots$$
$$+(-\sqrt{pq})\lambda_n\leq 2\sqrt{(m+m_c{}')kpq}.$$

Therefore,

$$\sqrt{pq}\left[\sum_{i=1}^{k}\lambda_i-\sum_{i=1}^{k}\lambda_{n-k+i}\right]\leq 2\sqrt{(m+m_c{}')kpq}$$

and

$$\left[\sum_{i=1}^{k}\lambda_i-\sum_{i=1}^{k}\lambda_{n-k+i}\right]\leq 2\sqrt{(m+m_c{}')k}.\qquad\square$$

Remark 2.8 If $k=1$ in the above proposition, then

$$\lambda_1-\lambda_n\leq 2\sqrt{(m+m_c{}')}.$$

§3. Bounds for Color Energy of a Graph

In [1] Adiga et al. have proved the following results.

Proposition 3.1 *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are color eigenvalues of* $A_c(G)$*, then* $\sum_{i=1}^{n} \lambda_i^2 = 2(m + m_c')$*, where* m_c' *is the number of pairs of non-adjacent vertices receiving the same color in* G*.*

Theorem 3.2 *Let* G *be a connected colored graph with* n *vertices,* m *edges, and* m_c' *be number of pairs of non-adjacent vertices receiving the same color. Then* $E_c(G) \leq \sqrt{2n(m + m_c')}$*.*

Using Proposition 3.1 and the Theorem 3.2 we prove the following result.

Theorem 3.3 *Let* G *be a connected colored graph with* n *vertices and* m *edges. Then*

$$2\sqrt{m + m_c'} \leq E_c(G) \leq 2\sqrt{m(m + m_c')}.$$

Proof Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the color eigenvalues of G. Since

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} {\lambda_i}^2 = 2(m + m_c')$$

we have

$$\sum_{i<j} \lambda_i \lambda_j = -(m + m_c'). \tag{3.1}$$

Now consider

$$\begin{aligned}
[E_c(G)]^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\lambda_j| \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i||\lambda_j| \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 + 2 \,|\sum_{i<j} \lambda_i \lambda_j| \geq 2(m + m_c') + 2(m + m_c')
\end{aligned}$$

by using Proposition 3.1 and equation (3.1). Hence

$$E_c(G) \geq 2\sqrt{m + m_c'}.$$

From Theorem 3.2, we have $E_c(G) \leq \sqrt{2n(m + m_c')}$. Since $n \leq 2m$, we have $E_c(G) \leq 2\sqrt{m(m + m_c')}$. Thus,

$$2\sqrt{m + m_c'} \leq E_c(G) \leq 2\sqrt{m(m + m_c')}. \qquad \square$$

§4. Bounds for Color Spectral Radius and Color Energy

We now establish a lower bound and an upper bound for color spectral radius. Also using these bounds we establish bounds for color energy.

Proposition 4.1 *Let G be a colored (n,m)-graph and $\rho_c(G)=\max_{1\le i\le n}\{|\lambda_i|\}$ be the color spectral radius of G. Then*

$$\sqrt{\frac{2(m+m_c{}')}{n}}\le \rho_c(G)\le \sqrt{2(m+m_c{}')}.$$

Proof Consider

$$\begin{aligned}\rho^2{}_c(G) &= \max_{1\le i\le n}\{|\lambda_i|^2\}\\ &\le \sum_{j=1}^{n}\lambda_j{}^2=2(m+m_c{}').\end{aligned}$$

So,

$$\rho_c(G) \le \sqrt{2(m+m_c{}')}.$$

Next consider

$$\begin{aligned}n\ \rho^2{}_c(G) &\ge \sum_{i=1}^{n}\lambda_i{}^2\\ &\ge 2(m+m_c{}').\end{aligned}$$

we have,

$$\rho_c(G) \ge \sqrt{\frac{2(m+m_c{}')}{n}}.$$

Therefore,

$$\sqrt{\frac{2(m+m_c{}')}{n}}\le \rho_c(G)\le \sqrt{2(m+m_c{}')}.$$

□

Theorem 4.2 *Let G be a colored graph and $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the color eigenvalues of G. If $n\le 2(m+m_c{}')$ and $\lambda_1\ge\frac{2(m+m_c{}')}{n}$, then*

$$E_c(G)\le\frac{2(m+m_c{}')}{n}+\sqrt{(n-1)\left[2(m+m_c{}')-\left(\frac{2(m+m_c{}')}{n}\right)^2\right]}.$$

Proof We have

$$\sum_{i=2}^{n}\lambda_i{}^2=2(m+m_c{}')-\lambda_1{}^2. \tag{4.1}$$

By a special case of Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \leq n \sum_{i=1}^{n}|\lambda_i|^2.$$

Thus,

$$\left(\sum_{i=2}^{n}|\lambda_i|\right)^2 \leq (n-1) \sum_{i=2}^{n}|\lambda_i|^2$$

and hence

$$\left(\sum_{i=2}^{n}|\lambda_i|\right) \leq \sqrt{(n-1) \sum_{i=2}^{n}|\lambda_i|^2}. \tag{4.2}$$

Employing (4.1) in (4.2), we obtain

$$E_c(G) - \lambda_1 \quad \leq \quad \sqrt{(n-1)[2(m+m_c{}') - {\lambda_1}^2]}.$$

i.e,

$$E_c(G) \quad \leq \quad \lambda_1 + \sqrt{(n-1)[2(m+m_c{}') - {\lambda_1}^2]}.$$

Consider the function

$$F(x) = x + \sqrt{(n-1)[2(m+m_c{}') - x^2]}.$$

Then

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{2(m+m_c{}') - x^2}}.$$

Observe that $F(x)$ is decreasing in $\left(\sqrt{\frac{2(m+m_c{}')}{n}}, \sqrt{2(m+m_c{}')}\right)$.

Since $n \leq 2(m+m_c{}')$ and $\frac{2(m+m_c{}')}{n} \leq \lambda_1$, we have

$$\sqrt{\frac{2(m+m_c{}')}{n}} < \frac{2(m+m_c{}')}{n} \leq \lambda_1 \leq \sqrt{2(m+m_c{}')}.$$

Last inequality follows from Proposition 4.1.

Hence

$$E_c(G) \leq \frac{2(m+m_c{}')}{n} + \sqrt{(n-1)\left[2(m+m_c{}') - \left(\frac{2(m+m_c{}')}{n}\right)^2\right]}. \qquad \square$$

As the proof of the following theorem is similar to that of Theorem 4.2 we omit the proof.

Theorem 4.3 *If* $n \leq 2(m + m_c')$ *and* $\sqrt{\frac{2(m+m_c')}{n}} \leq \rho_c(G) \leq \frac{2(m+m_c')}{n}$, *then*

$$E_c(G) \geq \frac{2(m + m_c')}{n} + \sqrt{(n-1)\left[2(m + m_c') - \left(\frac{2(m + m_c')}{n}\right)^2\right]}.$$

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A Note on Acyclic Coloring of Sunlet Graph Families

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Abstract: In this paper, we find the acyclic chromatic number χ_a for the central graph of sunlet graph $C(S_n)$, line graph of sunlet graph $L(S_n)$, middle graph of sunlet graph $M(S_n)$ and the total graph of sunlet graph $T(S_n)$ for all $n \geqslant 3$.

Key Words: Smarandachely vertex coloring, acyclic coloring, sunlet graph, central graph, line graph, middle graph and total graph.

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§1. Introduction

Let G be a finite graph and let $H \prec G$ be a subgraph of G. A Smarandachely vertex coloring respect to a subgraph $H \prec G$ by colors in $\mathscr{C}$ is a mapping $\varphi_H : \mathscr{C} \to E(G)$ such that $\varphi_H(e_1) \neq \varphi_H(e_2)$ if e_1 and e_2 are edges of a subgraph isomorphic to H in G. Particularly, let $H = G$. Then, such a Smarandachely vertex coloring is clearly the usual proper vertex coloring (or proper coloring) of G, i.e., a coloring ϕ: $V \to N^+$ on G such that if v and u are adjacent vertices, then $\phi(v) \neq \phi(u)$. The chromatic number of a graph G is the minimum number of colors required in any proper coloring of G. Generally, The notion of acyclic coloring was introduced by Branko Grünbaum in 1973. An acyclic coloring of a graph G is a proper vertex coloring such that the induced subgraph of any two color classes is acyclic, i.e., disjoint collection of trees. The minimum number of colors needed to acyclically color the vertices of a graph G is called as acyclic chromatic number and is denoted by $\chi_a(G)$.

§2. Preliminaries

A *sunlet graph* on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and denoted by S_n.

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For a given graph $G = (V, E)$ we do an operation on G by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called *central graph* [5] of G denoted by $C(G)$.

A *line graph* [1, 4] of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

A *middle graph* [3] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

The *total graph* [1 ,3, 4] of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

Additional graph theory terminology used in this paper can be found in [1, 4].

In the following sections we find the acyclic chromatic number for the central graph of sunlet graph $C(S_n)$, line graph of sunlet graph $L(S_n)$, middle graph of sunlet graph $M(S_n)$ and the total graph of sunlet graph $T(S_n)$.

Definition 2.1([2]) *An ayclic coloring of a graph G is a proper coloring such that the union of any two color classes induces a forest.*

§3. Acyclic Coloring on Central Graph of Sunlet Graph

Theorem 3.1 *Let S_n be a sunlet graph with $2n$ vertices, then*

$$\chi_a(C(S_n)) = n, \forall\, n \geq 3.$$

Proof Let $V(S_n) = \{u_1, u_2, \cdots, u_n\} \bigcup \{v_1, v_2, \cdots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\}$, where e_i is the edge $v_iv_{i+1}(1 \leq i \leq n-1)$, e_n is the edge v_nv_1 and e'_i is the edge $v_iu_i(1 \leq i \leq n)$. For $1 \leq i \leq n$, u_i is the pendant vertex and v_i is the adjacent vertex to u_i. By the definition of central graph $V(C(S_n)) = V(S_n) \bigcup E(S_n) = \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$, where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.

Assign the following coloring for $C(S_n)$ as acyclic:

(1) For $1 \leq i \leq n$ assign the color c_i to u_i, v_i;

(2) For $1 \leq i \leq n-1$ assign the color c_{i+1} to u'_i and c_1 to u'_n;

(3) For $1 \leq i \leq n-1$ assign the color c_i to v'_i and c_n to v'_1.

Thus, $\chi_a(C(S_n)) = n$, for if $\chi_a(C(S_n)) < n$, say $n-1$. A contradiction to proper coloring since, $\forall n$, $\{u_i : 1 \leq i \leq n\}$ forms a clique of order n. Hence, $\chi_a(C(S_n)) = n, \forall n \geq 3$. □

§4. Acyclic Coloring on Line Graph of Sunlet Graph

Theorem 4.1 *Let $n \geq 3$ be a positive integer, then $\chi_a(L(S_n)) = 3$.*

Proof Let $V(S_n) = \{u_1, u_2, \cdots, u_n\} \bigcup \{v_1, v_2, \cdots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\}$, where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By the definition of line graph $V(L(S_n)) = E(S_n) = \{u'_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n-1\} \bigcup \{v'_n\}$ where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.

Assign the coloring σ as acyclic as follows:

(1) For $1 \leq i \leq 3$, assign the vertices v'_i as $\sigma(v'_i) = c_i$, and for $4 \leq i \leq n$, let $\sigma(v'_i) = c_1$ if $i \equiv 1 \bmod 3$, $\sigma(v'_i) = c_2$ if $i \equiv 2 \bmod 3$, $\sigma(v'_i) = c_3$ if $i \equiv 0 \bmod 3$;

(2) For $1 \leq i \leq n$, assign the vertices u'_i with colors c_1, c_2, c_3 such that $\sigma(u'_i) \neq \sigma(v'_{i-1})$ and $\sigma(u'_i) \neq \sigma(v'_i)$, where $v'_0 = v'_n$.

Thus, $\chi_a(L(S_n)) = 3$, $\forall n \geq 3$.

To the contrary, let $\chi_a(L(S_n)) < 3$, say 2. A contradiction to proper coloring, since, for $\{1 \leq i \leq n-1\}$, $\left\{v'_i, u'_{i+1}, v'_{i+1}\right\}$ is a complete graph K_3. Hence, $\chi_a(L(S_n)) = 3, \forall n \geq 3$. □

§5. Acyclic Coloring on Middle Graph of Sunlet Graph

Theorem 5.1 *Let $n \geq 3$ be a positive integer, then $\chi_a(M(S_n)) = 4$.*

Proof Let $V(S_n) = \{u_1, u_2, \cdots, u_n\} \bigcup \{v_1, v_2, \cdots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\}$, where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By definition of middle graph $V(M(S_n)) = V(S_n) \bigcup E(S_n) = \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{v'_i : 1 \leq i \leq n\} \bigcup \{u'_i : 1 \leq i \leq n\}$, where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.

Define the mapping σ such that $\sigma(V(M(S_n))) \rightarrow c_i$ for $1 \leq i \leq 4$ as follows:

(1) For $1 \leq i \leq n$, assign the vertices v'_i as

$\sigma(v'_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \cdots c_1c_2c_3\}$ if $n \equiv 0 \bmod 3$,
$\sigma(v'_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \cdots c_1c_2c_3 \quad c_2\}$ if $n \equiv 1 \bmod 3$,
$\sigma(v'_i) = \{c_1c_2c_3 \quad c_1c_2c_3 \cdots c_1c_2c_3 \quad c_1c_2\}$ if $n \equiv 2 \bmod 3$;

(2) Assign $\sigma(u_i) = \sigma(v_i) = c_4$ for $1 \leq i \leq n$;

(3) Assign the vertices u'_i with c_1, c_2, c_3 such that $\sigma(u'_i) \neq \sigma(v'_{i-1})$ and $\sigma(u'_i) \neq \sigma(v'_i)$ for $1 \leq i \leq n$ where $v'_0 = v'_n$.

Thus, $\chi_a(M(S_n)) = 4$, $\forall n \geq 3$.

To the contrary, let $\chi_a(M(S_n)) < 4$, say 3. A contradiction to proper coloring, since $\forall n$, $\left\{v'_{i-1}, v'_i, v_i, u'_i\right\}$, where $v'_0 = v'_n$, forms a clique of order 4. Thus σ is a proper acyclic coloring and hence $\chi_a(M(S_n)) = 3, \forall n \geq 3$. □

§6. Acyclic Coloring on Total Graph of Sunlet Graph

Theorem 6.1 *Let S_n be a sunlet graph with $2n$ vertices then for $n \geq 3$, $\chi_a(T(S_n)) = 6$.*

Proof Let $V(S_n) = \{u_1, u_2, \cdots, u_n\} \bigcup \{v_1, v_2, \cdots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \bigcup \{e'_i : 1 \leq i \leq n\}$, where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By the definition of total graph $V(T(S_n)) = V(S_n) \bigcup E(S_n) = \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{v'_i : 1 \leq i \leq n\} \bigcup \{u'_i : 1 \leq i \leq n\}$, where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.

Define the mapping σ such that $\sigma(V(T(S_n))) \to c_i$ for $1 \leq i \leq 6$ as follows:

(1) For $1 \leq i \leq 3$, let $\sigma(v'_i) = c_i$ and for $4 \leq i \leq n$, let

$\sigma(v'_i) = c_1$ if $i \equiv 1 \bmod 3$,
$\sigma(v'_i) = c_2$ if $i \equiv 2 \bmod 3$,
$\sigma(v'_i) = c_3$ if $i \equiv 0 \bmod 3$;

(2) For $1 \leq i \leq 3$, let $\sigma(v_i) = c_{i+3}$ and for $4 \leq i \leq n$, let

$\sigma(v_i) = c_4$ if $i \equiv 1 \bmod 3$,
$\sigma(v_i) = c_5$ if $i \equiv 2 \bmod 3$,
$\sigma(v_i) = c_6$ if $i \equiv 0 \bmod 3$;

(3) Let $\sigma(u'_i) = \sigma(v'_i) + 1$ for $1 \leq i \leq n$;

(4) For $1 \leq i \leq n$, let $\sigma(u_i) = \sigma(v_i) + 1$ and $\sigma(u_i) = c_1$ if $\sigma(v_i) = c_6$.

Thus, $\chi_a(T(S_n)) = 6$ for $n \geq 3$.

For $1 \leq i \leq 6$, the union of any two color classes c_{i-1} and c_i induces subgraph whose components are trees, hence by Definition 1.1, σ is a proper acyclic coloring and $\chi_a(T(S_n)) = 6$ for $n \geq 3$. □

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In almost every face and every person, they may discover fine feathers and defects, good and bad qualities.

By Benjamin Franklin, an American polymath and a leading author, printer, political theorist, politician, freemason, postmaster, scientist, inventor, civic activist, statesman, and diplomat.

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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