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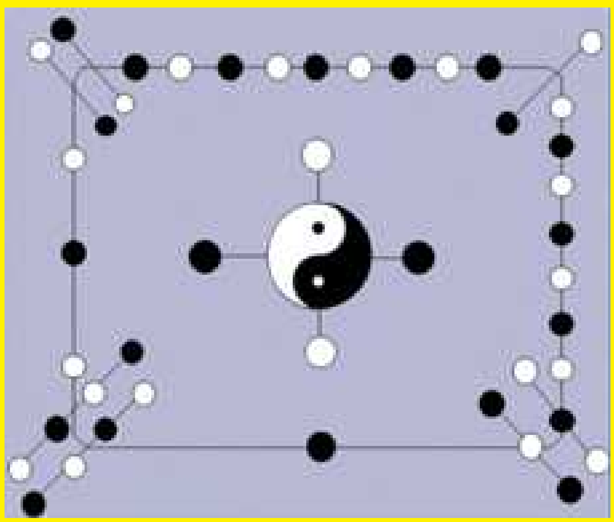
Edited By Linfan MAO



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Academy of Mathematical Combinatorics & Applications

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Aims and Scope: The **Mathematical Combinatorics (International Book Series)** is a fully refereed international book series, quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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Editorial Board (3nd)

Famous Words:

Nothing in life is to be feared. It is only to be understood.

By Marie Curie, a Polish and naturalized-French physicist and chemist.

Math.Combin.Book Ser. Vol.1(2015), 1-13

N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

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Abstract: In this paper, when the unit Darboux vector of the partner curve of Mannheim curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the Mannheim curve. Besides, we illustrate example of our main results.

Key Words: Mannheim curve, Mannheim partner curve, Smarandache Curves, Frenet invariants.

AMS(2010): 53A04

§1. Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve ([12]). Special Smarandache curves have been studied by some authors .

Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache TB_2 curves in the space E_1^4 ([12]). Ahmad T.Ali studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case ([1]). Muhammed Çetin , Yılmaz Tunçer and Kemal Karacan investigated special Smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential goematric properties of Smarandache curves, also they found the centers of the osculating spheres and curvature spheres of Smarandache curves ([5]). Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves ([4]). Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in E^3 ([2]). Nurten Bayrak, Özcan Bektaş and Salim Yüce studied some special Smarandache curves in E_1^3 [3]. Kemal Taşköprü, Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 ([11]).

In this paper, special Smarandache curve belonging to α^* Mannheim partner curve such as N^*C^* drawn by Frenet frame are defined and some related results are given.

[1]Received September 8, 2014, Accepted February 12, 2015.

§2. Preliminaries

The Euclidean 3-space E^3 be inner product given by

$$\langle , \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by ([6], [9])

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

For any unit speed $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = \tau(s)T(s) + \kappa(s) + B(s).$$

If consider the normalization of the Darboux $C = \dfrac{1}{\|W\|}W$, we have

$$\cos\varphi = \frac{\kappa(s)}{\|W\|}, \quad \sin\varphi = \frac{\tau(s)}{\|W\|},$$

$$C = \sin\varphi T(s) + \cos\varphi B(s) \tag{2.2}$$

where $\angle(W, B) = \varphi$. Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector N of the curve α is linearly dependent on the binormal vector B of the curve α^*, then (α) is called a Mannheim curve and (α^*) a Mannheim partner curve of (α). The pair (α, α^*) is said to be Mannheim pair ([7], [8]). The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are as follows:

$$\begin{cases} T^* = \cos\theta T - \sin\theta B \\ N^* = \sin\theta T + \cos\theta B \\ B^* = N \end{cases} \tag{2.3}$$

$$\begin{cases} \cos\theta = \dfrac{ds*}{ds} \\ \sin\theta = \lambda\tau^* \dfrac{ds*}{ds} \end{cases} . \tag{2.4}$$

where $\angle(T, T^*) = \theta$ ([8]).

Theorem 2.1([7]) *The distance between corresponding points of the Mannheim partner curves in $\mathbb{E}^3$ is constant.*

Theorem 2.2 *Let (α, α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the curvatures and the torsions of the Mannheim curve pair (α, α^*) we have,*

$$\begin{cases} \kappa = \tau^* \sin\theta \dfrac{ds*}{ds} \\ \\ \tau = -\tau^* \cos\theta \dfrac{ds*}{ds} \end{cases} \tag{2.5}$$

and

$$\begin{cases} \kappa^* = \dfrac{d\theta}{ds*} = \theta' \dfrac{\kappa}{\lambda\tau\sqrt{\kappa^2+\tau^2}} \\ \\ \tau^* = (\kappa\sin\theta - \tau\cos\theta)\dfrac{ds*}{ds} \end{cases} \tag{2.6}$$

Theorem 2.3 *Let (α, α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the torsions τ^* of the Mannheim partner curve α^* we have*

$$\tau^* = \frac{\kappa}{\lambda\tau}$$

Theorem 2.4([10]) *Let (α, α^*) be a Mannheim pair curves in $\mathbb{E}^3$. For the vector C^* is the direction of the Mannheim partner curve α^* we have*

$$C^* = \frac{1}{\sqrt{1+\left(\dfrac{\theta'}{\|W\|}\right)^2}}C + \frac{\dfrac{\theta'}{\|W\|}}{\sqrt{1+\left(\dfrac{\theta'}{\|W\|}\right)^2}}N \tag{2.7}$$

where the vector C is the direction of the Darboux vector W of the Mannheim curve α.

§3. N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame

Let (α, α^*) be a Mannheim pair curves in E^3 and $\{T^*N^*B^*\}$ be the Frenet frame of the Mannheim partner curve α^* at $\alpha^*(s)$. In this case, N^*C^* - Smarandache curve can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \tag{3.1}$$

Solving the above equation by substitution of N^* and C^* from (2.3) and (2.7), we obtain

$$\beta_1(s) = \frac{\left(\cos\theta\|W\| + \sin\theta\sqrt{\theta'^2+\|W\|^2}\right)T + \theta' N + \left(\cos\theta\sqrt{\theta'^2+\|W\|^2} - \sin\theta\|W\|\right)B}{\sqrt{\theta'^2+\|W\|^2}}. \tag{3.2}$$

The derivative of this equation with respect to s is as follows,

$$T_{\beta_1}(s) = \frac{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\cos\theta - \frac{\theta'\kappa\cos\theta}{\lambda\tau\|W\|}\right]T + \left[\frac{\kappa}{\lambda\tau} - \left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\|W\|}{\theta'}\right]N}{\sqrt{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]}}$$

$$+\frac{\left[\frac{\theta'\kappa\sin\theta}{\lambda\tau\|W\|} - \left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\sin\theta\right]B}{\sqrt{\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]}}. \quad (3.3)$$

In order to determine the first curvature and the principal normal of the curve $\beta_1(s)$, we formalize

$$T'_{\beta_1}(s) = \frac{\sqrt{2}\left[(\bar{r_1}\cos\theta + \bar{r_2}\sin\theta)T + \bar{r_3}N + (-\bar{r_1}\sin\theta + \bar{r_2}\cos\theta)B\right]}{\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2 + \frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|} - 2\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{1}{\theta'}\right]\right)^2}$$

where

$$\begin{aligned}
\bar{r}_1 = {} & 2\left(\frac{\kappa}{\lambda\tau}\right)^2\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]'\right)\left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right) \\
& -\left(\frac{\theta'\kappa}{\lambda\tau\|W\|}\right)\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]' \\
& \left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right)^2 - \left(\frac{\kappa}{\lambda\tau}\right)\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]\left(\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right. \\
& \left.\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]'\right)\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)\left(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\right) - \left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right. \\
& \left.\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^4\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right) - \left(\frac{\kappa}{\lambda\tau}\right)^2\left(\frac{\theta'\kappa}{\lambda\tau\|W\|}\right)' - \left(\frac{\kappa}{\lambda\tau}\right)^2\left[\left(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\right)'\right.
\end{aligned}$$

$$\times\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\kappa}{\lambda\tau}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$-2\kappa^*\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'$$

$$\Big(\frac{\kappa}{\lambda\tau}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\tau^*\Big(\frac{\kappa}{\lambda\tau}\Big)'$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$

$$+\Big(\frac{\kappa}{\lambda\tau}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\bigg)$$

$$+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)'-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)\bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\bigg)$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\kappa}{\lambda\tau}\Big)'\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big),$$

$$\begin{aligned}\bar{r}_2 \quad = \quad & \Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3\\ & \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+3\big(\frac{\kappa}{\lambda\tau}\big)^2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\end{aligned}$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$

$$+3(\frac{\kappa}{\lambda\tau})^3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+(\frac{\kappa}{\lambda\tau})$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-2(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-4\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^4-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$(\frac{\kappa}{\lambda\tau})^2+3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$\bar{r}_3 = 2(\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})'-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)$$

$$(\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$

$$+(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'$$

$$\times\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^4\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$
$$-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-(\frac{\kappa}{\lambda\tau})'$$
$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3$$
$$\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)^2+2(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)$$
$$\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)-(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'-(\frac{\kappa}{\lambda\tau})\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'+(\frac{\kappa}{\lambda\tau})\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big)+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$
$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)\Big(\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big).$$

The first curvature is

$$\kappa_{\beta_1}=\frac{\sqrt{2}(\sqrt{\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2})}{\left(\left[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\right]^2+\frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|}\left[\frac{\kappa}{\lambda\tau\|W\|}-2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{1}{\theta'}\right]\right)^2}.$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_{\beta_1} = \frac{(\bar{r_1}\cos\theta + \bar{r_2}\sin\theta)T + \bar{r_3}N + (-\bar{r_1}\sin\theta + \bar{r_2}\cos\theta)B}{\sqrt{\bar{r_1}^2 + \bar{r_2}^2 + \bar{r_3}^2}}, \tag{3.4}$$

$$B_{\beta_1}(s) = \frac{\xi_1}{\xi_4}T + \frac{\xi_2}{\xi_4}N + \frac{\xi_3}{\xi_4}B, \tag{3.5}$$

where

$$\begin{cases} \xi_1 = \bar{r_2}\cos\theta\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} - \bar{r_2}\cos\theta\frac{\kappa}{\lambda\tau} - \Big[\bar{r_1}\frac{\kappa}{\lambda\tau} - \bar{r_1}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} \\ \quad -\bar{r_3}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' + \bar{r_3}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big]\sin\theta \\ \xi_2 = \bar{r_1}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' - \frac{\theta'\kappa}{\lambda\tau\|W\|}\Big] \\ \xi_3 = \bar{r_2}\sin\theta\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} - \frac{\kappa}{\lambda\tau}\Big] + \Big[\bar{r_1}\frac{\kappa}{\lambda\tau} - \bar{r_1}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'} \\ \quad -\bar{r_3}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' + \bar{r_3}\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big]\cos\theta \\ \xi_4 = \sqrt{\begin{aligned}&\Big((\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2 + (\bar{r_1}^2+\bar{r_2}^2+\bar{r_3}^2)\frac{\kappa(\theta'^2+\|W\|^2)}{\lambda\tau\|W\|} \\ &\Big[\frac{\kappa}{\lambda\tau\|W\|} - 2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{1}{\theta'}\Big]\Big)\end{aligned}}\ . \end{cases}$$

In order to calculate the torsion of the curve β_1, we differentiate

$$\begin{aligned} \beta_1'' \;=\; & \frac{1}{\sqrt{2}}\Bigg(\Bigg[\cos\theta\Bigg(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}} \\ & -\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}} - \Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Bigg) + \\ & +\sin\theta\Bigg(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}} + \Big(\frac{\kappa}{\lambda\tau}\Big) \\ & \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}} - \Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2 - \Big(\frac{\kappa}{\lambda\tau}\Big)^2\Bigg)\Bigg]\mathbf{T} \\ & +\Bigg[\Big(\frac{\kappa}{\lambda\tau}\Big) - \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}} \end{aligned}$$

$$-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}\Big]\mathbf{N}$$

$$\Big[-\sin\theta\Big(\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big)+$$

$$+\cos\theta\Big(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\kappa}{\lambda\tau}\Big)$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2-\Big(\frac{\kappa}{\lambda\tau}\Big)^2\Big)\Big]\mathbf{B}\Big).$$

and thus

$$\beta_1''' = \frac{(t_1\cos\theta + t_2\sin\theta + t_3)T + t_3N + (t_2\cos\theta - t_1\sin\theta + t_3)T}{\sqrt{2}},$$

where

$$\begin{aligned}
t_1 \;=\; & \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]''\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' \\
& \frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}} \\
& -\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'' \\
& -\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big) \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\kappa}{\lambda\tau}\Big)^2
\end{aligned}$$

$$\begin{aligned}
t_2 \;=\; & 2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big) \\
& \Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'
\end{aligned}$$

$$+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\big(\frac{\kappa}{\lambda\tau}\big)'$$
$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+2\big(\frac{\kappa}{\lambda\tau}\big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2$$
$$\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}-3\big(\frac{\kappa}{\lambda\tau}\big)\big(\frac{\kappa}{\lambda\tau}\big)'$$

$$t_3 = \Bigg(\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\big(\frac{\kappa}{\lambda\tau}\big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-3\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$
$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\Bigg)\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Bigg(\big(\frac{\kappa}{\lambda\tau}\big)^2$$
$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]''$$
$$+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\Bigg)\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\big(\frac{\kappa}{\lambda\tau}\big)$$
$$-\big(\frac{\kappa}{\lambda\tau}\big)^3+\big(\frac{\kappa}{\lambda\tau}\big)''$$

The torsion is then given by

$$\tau_{\beta_1}=\frac{det(\beta_1',\beta_1'',\beta_1''')}{\|\beta_1'\wedge\beta_1''\|^2}\quad,$$

$$\tau_{\beta_1}=\sqrt{2}\frac{\Omega_1}{\Omega_2}$$

where

$$\Omega_1=-2t_1\big(\frac{\kappa}{\lambda\tau}\big)^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+t_1\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2\frac{\|W\|^2}{\theta'^2}-t_1\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$
$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_2\big(\frac{\kappa}{\lambda\tau}\big)+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\theta'^2}{\theta'^2+\|W\|^2}t_2$$

$$+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2 t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)-2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2-\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' t_3(\frac{\kappa}{\lambda\tau})^2$$

$$-\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)' t_2(\frac{\kappa}{\lambda\tau})-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)$$

$$t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+t_2\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-t_2\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2 t_3\frac{\kappa}{\lambda\tau}\frac{\|W\|}{\theta'}+t_2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'$$

$$\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_3\frac{\kappa}{\lambda\tau}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}-t_1\frac{\kappa}{\lambda\tau}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$+t_1\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2+t_1\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})+t_2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3\frac{\|W\|^2}{\theta'^2+\|W\|^2}$$

$$-t_2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'(\frac{\kappa}{\lambda\tau})+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)t_3(\frac{\kappa}{\lambda\tau})^2+t_3\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+t_1(\frac{\kappa}{\lambda\tau})^3,$$

$$\Omega_2=\Bigg(\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Big]^2+\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-2(\frac{\kappa}{\lambda\tau})^2\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\|W\|}{\theta'}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2(\frac{\kappa}{\lambda\tau})+(\frac{\kappa}{\lambda\tau})^3\Bigg)^2$$

$$+\Bigg(\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'-(\frac{\kappa}{\lambda\tau})'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'$$

$$\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]$$

$$-\frac{\kappa}{\lambda\tau}\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2-\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]'\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-\frac{\kappa}{\lambda\tau}\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)'+\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})')\Bigg)^2+\Bigg(\Bigg[\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)$$

$$\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+(\frac{\kappa}{\lambda\tau})\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]^2\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}$$

$$-2\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]-(\frac{\kappa}{\lambda\tau})^2\Big[\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\frac{\sqrt{\theta'^2+\|W\|^2}}{\theta'}\Big]\Bigg]$$

$$\frac{\theta'}{\sqrt{\theta'^2+\|W\|^2}}+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)^3+\Big(\frac{\theta'\kappa}{\lambda\tau\|W\|}\Big)(\frac{\kappa}{\lambda\tau})^2-\Big(\frac{\kappa}{\lambda\tau}\Big)\frac{\kappa}{\lambda\tau}\Big(\frac{\|W\|}{\sqrt{\theta'^2+\|W\|^2}}\Big)'\Bigg)^2. \qquad \square$$

Example 3.1 Let us consider the unit speed Mannheim curve and Mannheim partner curve:

$$\alpha(s)=\frac{1}{\sqrt{2}}\big(-\cos s,-\sin s,s\big),\quad \alpha^*(s)=\frac{1}{\sqrt{2}}(-2\cos s,-2\sin s,s)\cdot$$

The Frenet invariants of the partner curve, $\alpha^*(s)$ are given as following

$$T^*(s)\quad=\quad\frac{1}{\sqrt{5}}\big(2\sin s,-2\cos s,1\big),$$

$$
\begin{aligned}
N^*(s) &= \frac{1}{\sqrt{5}}\big(\sin s, \cos s, -2\big)\\
B^*(s) &= \big(\cos s, \sin s, 0\big)\\
C^*(s) &= \big(\frac{2}{5}\sin s + \frac{2}{\sqrt{5}}\cos s, -\frac{2}{5}\cos s + \frac{2}{\sqrt{5}}\sin s, \frac{1}{5}\big)\\
\kappa^*(s) &= \frac{2\sqrt{2}}{5}\\
\tau^*(s) &= \frac{\sqrt{2}}{5}.
\end{aligned}
$$

In terms of definitions, we obtain special Smarandache curve, see Figure 1.

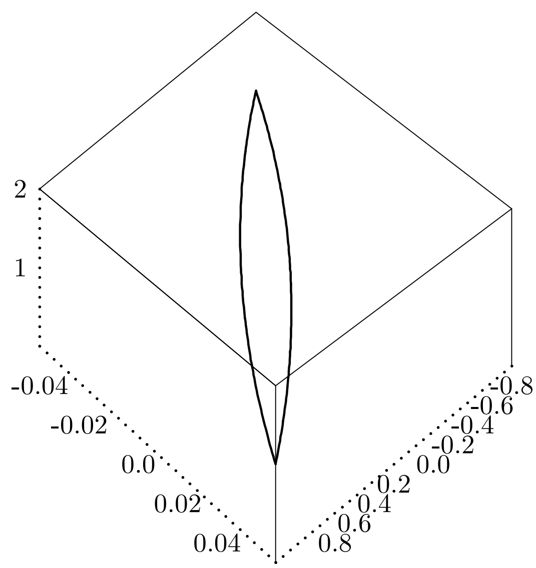

Figure 1 $\beta_1 = \frac{1}{5\sqrt{5}}\big((5+2\sqrt{5})\sin s + 10\cos s, (5-2\sqrt{5})\cos s + 10\sin s, -9\sqrt{5}\big)$

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Fixed Point Theorems of Two-Step Iterations for Generalized Z-Type Condition in CAT(0) Spaces

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Abstract: In this paper, we establish some strong convergence theorems of modified two-step iterations for generalized Z-type condition in the setting of CAT(0) spaces. Our results extend and improve the corresponding results of [3, 6, 28] and many others from the current existing literature.

Key Words: Strong convergence, modified two-step iteration scheme, fixed point, CAT(0) space.

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§1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Fixed point theory in a CAT(0) space was first studied by Kirk (see [19, 20]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2], [9], [11]-[13], [17]-[18], [21]-[22], [24]-[26] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [7].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points

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in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [7]).

1.1 CAT(0) Space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\ d^2(x, y_1) + \frac{1}{2}\ d^2(x, y_2) - \frac{1}{4}\ d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Tits [8]. The above inequality was extended in [12] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) \quad &\leq \quad \alpha d^2(z, x) + (1-\alpha)d^2(z, y) \\ &\quad\ -\alpha(1-\alpha)d^2(x, y) \end{aligned} \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [7, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \big\{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\big\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

We recall the following definitions in a metric space (X, d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a\, d(x, y) \ \text{ for all } x,\ y\ \in X, \tag{1.5}$$

where $a \in (0, 1)$.

The mapping T is called Kannan mapping [15] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \tag{1.6}$$

for all $x, y \in X$.

The mapping T is called Chatterjea mapping [10] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \tag{1.7}$$

for all $x, y \in X$.

In 1972, Zamfirescu [29] proved the following important result.

Theorem Z *Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0,1)$, b, $c \in (0, \frac{1}{2})$ such that for any pair x, $y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T which satisfies at least one of the contractive conditions (z_1), (z_2) and (z_3) is called a *Zamfirescu operator* or a Z-operator.

In 2004, Berinde [5] proved the strong convergence of Ishikawa iterative process defined by: for $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \tag{1.8}$$

to approximate fixed points of Zamfirescu operator in an arbitrary Banach space E. While proving the theorem, he made use of the condition,

$$\|Tx - Ty\| \leq \delta\ \|x - y\| + 2\delta\ \|x - Tx\| \tag{1.9}$$

which holds for any x, $y \in E$ where $0 \leq \delta < 1$.

In 1953, W.R. Mann defined the Mann iteration [23] as

$$u_{n+1} = (1 - a_n)u_n + a_n T u_n, \tag{1.10}$$

where $\{a_n\}$ is a sequence of positive numbers in [0,1].

In 1974, S.Ishikawa defined the Ishikawa iteration [14] as

$$s_{n+1} = (1 - a_n)s_n + a_n T t_n,$$

$$t_n = (1 - b_n)s_n + b_n T s_n, \tag{1.11}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2008, S.Thianwan defined the new two step iteration [27] as

$$\nu_{n+1} = (1-a_n)w_n + a_n T w_n,$$
$$w_n = (1-b_n)\nu_n + b_n T \nu_n, \tag{1.12}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

Recently, Agarwal et al. [1] introduced the S-iteration process defined as

$$x_{n+1} = (1-a_n)Tx_n + a_n T y_n,$$
$$y_n = (1-b_n)x_n + b_n T x_n, \tag{1.13}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in (0,1).

In this paper, inspired and motivated [5, 29], we employ a condition introduced in [6] which is more general than condition (1.9) and establish fixed point theorems of S- iteration scheme in the framework of CAT(0) spaces. The condition is defined as follows:

Let C be a nonempty, closed, convex subset of a CAT(0) space X and $T\colon C \to C$ a self map of C. There exists a constant $L \geq 0$ such that for all x, $y \in C$, we have

$$d(Tx,Ty) \leq e^{L\,d(x,Tx)}\Big[\delta\, d(x,y) + 2\delta\, d(x,Tx)\Big], \tag{1.14}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$. Throughout this paper, we call this condition as generalized Z-type condition.

Remark 1.1 If $L = 0$, in the above condition, we obtain

$$d(Tx,Ty) \leq \delta\, d(x,y) + 2\delta\, d(x,Tx),$$

which is the Zamfirescu condition used by Berinde [5] where

$$\delta = \max\left\{a,\, \frac{b}{1-b},\, \frac{c}{1-c}\right\},\ 0 \leq \delta < 1,$$

while constants a, b and c are as defined in Theorem Z.

Example 1.2 Let X be the real line with the usual norm $\|.\|$ and suppose $C = [0,1]$. Define $T\colon C \to C$ by Tx $= \frac{\text{x+1}}{2}$ for all $x, y \in C$. Obviously T is self-mapping with a unique fixed point 1. Now we check that condition (1.14) is true. If $x, y \in [0,1]$, then $\|Tx - Ty\| \leq e^{L\,\|x-Tx\|}[\delta\ \|x-y\| + 2\delta\ \|x-Tx\|]$ where $0 \leq \delta < 1$. In fact

$$\|Tx - Ty\| \quad = \quad \left\|\frac{x-y}{2}\right\|$$

and

$$e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big] = e^{L\left\|\frac{x-1}{2}\right\|}\Big[\delta\,\|x-y\| + \delta\,\|x-1\|\Big].$$

Clearly, if we chose $x = 0$ and $y = 1$, then contractive condition (??) is satisfied since

$$\|Tx - Ty\| = \left\| \frac{x-y}{2} \right\| = \frac{1}{2},$$

and for $L \geq 0$, we chose $L = 0$, then

$$e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big] = e^{L\left\|\frac{x-1}{2}\right\|}\Big[\delta\,\|x-y\| + \delta\,\|x-1\|\Big]$$

$$= e^{0(1/2)}(2\delta) = 2\delta, \quad \text{where} \quad 0 < \delta < 1.$$

Therefore

$$\|Tx - Ty\| \leq e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big].$$

Hence T is a self mapping with unique fixed point satisfying the contractive condition (1.14).

Example 1.3 Let X be the real line with the usual norm $\|.\|$ and suppose $K = \{0, 1, 2, 3\}$. Define $T\colon K \to K$ by

$$\begin{cases} Tx = 2, & \text{if} \quad x = 0 \\ \quad\;\; = 3, & \text{otherwise.} \end{cases}$$

Let us take $x = 0$, $y = 1$ and $L = 0$. Then from condition (1.14), we have

$$\begin{aligned} 1 &\leq e^{0(2)}[\delta(1) + 2\delta(2)] \\ &\leq 1(5\delta) = 5\delta \end{aligned}$$

which implies $\delta \geq \frac{1}{5}$. Now if we take $0 < \delta < 1$, then condition (1.14) is satisfied and 3 is of course a unique fixed point of T.

1.2 Modified Two-Step Iteration Schemes in CAT(0) Space

Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ be a contractive operator. Then for a given $x_1 = x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme as follows:

$$\begin{aligned} x_{n+1} &= (1-a_n)Tx_n \oplus a_n Ty_n, \\ y_n &= (1-b_n)x_n \oplus b_n Tx_n, \end{aligned} \tag{1.15}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in (0,1). Iteration scheme (1.15) is called modified S-iteration scheme in CAT(0) space.

$$\begin{aligned} \nu_{n+1} &= (1-a_n)w_n \oplus a_n Tw_n, \\ w_n &= (1-b_n)\nu_n \oplus b_n T\nu_n, \end{aligned} \tag{1.16}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1]. Iteration scheme (1.16) is called modified S.Thianwan iteration scheme in CAT(0) space.

$$s_{n+1} = (1-a_n)s_n \oplus a_n T t_n,$$

$$t_n = (1-b_n)s_n \oplus b_n T s_n, \tag{1.17}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1]. Iteration scheme (1.17) is called modified Ishikawa iteration scheme in CAT(0) space.

We need the following useful lemmas to prove our main results in this paper.

Lemma 1.4([24]) *Let X be a CAT(0) space.*

(i) For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that

$$d(x,\, z) = t\, d(x,\, y) \text{ and } d(y,\, z) = (1-t)\, d(x,\, y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\, y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z).$$

Lemma 1.5([4]) *Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \le (1-s_n)p_n + q_n + r_n, \quad \forall\, n \ge 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

§2. Strong Convergence Theorems in CAT(0) Space

In this section, we establish some strong convergence theorems of modified two-step iterations to converge to a fixed point of generalized Z-type condition in the framework of CAT(0) spaces.

Theorem 2.1 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.15). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof From the assumption $F(T) \neq \emptyset$, it follows that T has a fixed point in C, say u. Since T satisfies generalized Z-type condition given by (1.14), then from (1.14), taking $x = u$

and $y = x_n$, we have

$$\begin{aligned} d(Tu, Tx_n) &\leq e^{L\,d(u,Tu)}\Big(\delta\, d(u, x_n) + 2\delta\, d(u, Tu)\Big) \\ &= e^{L\,d(u,u)}\Big(\delta\, d(u, x_n) + 2\delta\, d(u, u)\Big) \\ &= e^{L\,(0)}\Big(\delta\, d(u, x_n) + 2\delta\,(0)\Big), \end{aligned}$$

which implies that

$$d(Tx_n, u) \leq \delta\, d(x_n, u). \tag{2.1}$$

Similarly by taking $x = u$ and $y = y_n$ in (1.14), we have

$$d(Ty_n, u) \leq \delta\, d(y_n, u), \tag{2.2}$$

Now using (1.15), (2.2) and Lemma 1.4(ii), we have

$$\begin{aligned} d(y_n, u) &= d((1-b_n)x_n \oplus b_n Tx_n, u) \\ &\leq (1-b_n)d(x_n, u) + b_n\, d(Tx_n, u) \\ &\leq (1-b_n)d(x_n, u) + b_n\delta\, d(x_n, u) \\ &= (1 - b_n + b_n\,\delta)d(x_n, u). \end{aligned} \tag{2.3}$$

Now using (1.15), (2.1), (2.3) and Lemma 1.4(ii), we have

$$\begin{aligned} d(x_{n+1}, u) &= d((1-a_n)Tx_n \oplus a_n Ty_n, u) \\ &\leq (1-a_n)d(Tx_n, u) + a_n d(Ty_n, u) \\ &\leq (1-a_n)\delta\, d(x_n, u) + a_n\delta\, d(y_n, u) \\ &\leq (1 - a_n + a_n\,\delta)d(x_n, u) + a_n\delta(1 - b_n + b_n\,\delta)d(x_n, u) \\ &= [1-(1-\delta)a_n]d(x_n, u) + a_n\delta[1-(1-\delta)b_n]d(x_n, u) \\ &= [1-(1-\delta)a_n + a_n\delta(1-(1-\delta)b_n)]d(x_n, u) \\ &= [1-\{(1-\delta)a_n + \delta(1-\delta)a_n b_n\}]d(x_n, u) = (1-\mu_n)d(x_n, u) \end{aligned} \tag{2.4}$$

where $\mu_n = (1-\delta)a_n + \delta(1-\delta)a_n b_n$. Since $0 \leq \delta < 1$; a_n, $b_n \in (0,1)$; $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, it follows that $\sum_{n=0}^{\infty} \mu_n = \infty$. Setting $p_n = d(x_n, u)$, $s_n = \mu_n$ and by applying Lemma 1.5, it follows that $\lim_{n\to\infty} d(x_n, u) = 0$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that u_1, $u_2 \in F(T)$ and $u_1 \neq u_2$. Applying generalized Z-type condition given by (1.14) and using the fact that $0 \leq \delta < 1$, we obtain

$$\begin{aligned} d(u_1, u_2) &= d(Tu_1, Tu_2) \\ &\leq e^{L\,d(u_1,Tu_1)}\Big\{\delta\, d(u_1, u_2) + 2\delta\, d(u_1, Tu_1)\Big\} \\ &= e^{L\,d(u_1,u_1)}\Big\{\delta\, d(u_1, u_2) + 2\delta\, d(u_1, u_1)\Big\} \end{aligned}$$

$$\begin{aligned} &= e^{L\,(0)}\Big\{\delta\, d(u_1,u_2) + 2\delta\,(0)\Big\} \\ &= \delta\, d(u_1,u_2) < d(u_1,u_2), \end{aligned}$$

which is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T. □

Theorem 2.2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.16). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof The proof of Theorem 2.2 is similar to that of Theorem 2.1. □

Theorem 2.3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.17). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof The proof of Theorem 2.3 is also similar to that of Theorem 2.1. □

If we take $L = 0$ in condition (1.14), then we obtain the following result as corollary which extends the corresponding result of Berinde [5] to the case of modified S-iteration scheme and from arbitrary Banach space to the setting of CAT(0) spaces.

Corollary 2.4 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ a Zamfirescu operator. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.15). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Remark 2.5 Our results extend and improve upon, among others, the corresponding results proved by Berinde [3], Yildirim et al. [28] and Bosede [6] to the case of generalized Z-type condition, modified S-iteration scheme and from Banach space or normed linear space to the setting of CAT(0) spaces.

§3. Conclusion

The generalized Z-type condition is more general than Zamfirescu operators. Thus the results obtained in this paper are improvement and generalization of several known results in the existing literature (see, e.g., [3, 6, 28] and some others).

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Antidegree Equitable Sets in a Graph

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called a Smarandachely antidegree equitable k-set for any integer k, $0 \leq k \leq \Delta(G)$, if $|deg(u) - deg(v)| \neq k$, for all $u, v \in S$. A Smarandachely antidegree equitable 1-set is usually called an antidegree equitable set. The antidegree equitable number $AD_e(G)$, the lower antidegree equitable number $ad_e(G)$, the independent antidegree equitablenumber $AD_{ie}(G)$ and lower independent antidegree equitable number $ad_{ie}(G)$ are defined as follows:

$AD_e(G) = max\{|S| : \ S \text{ is a maximal antidegree equitable set in } G\}$,
$ad_e(G) = min\{|S| : \ S \text{ is a maximal antidegree equitable set in } G\}$,
$AD_{ie}(G) = max\{|S| : \ S \text{ is a maximal independent and antidegree equitable set in } G\}$,
$ad_{ie}(G) = min\{|S| : \ S \text{ is a maximal independent and antidegree equitable set in } G\}$.

In this paper, we study these four parameters on Smarandachely antidegree equitable 1-sets.

Key Words: Smarandachely antidegree equitable k-set, antidegree equitable set, antidegree equitable number, lower antidegree equitable number, independent antidegree equitable number, lower independent antidegree equitable number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The number of vertices in a graph G is called the order of G and number of edges in G is called the size of G. For standard definitions and terminologies on graphs we refer to the books [2] and [3].

In this paper we introduce four graph theoretic parameters which just depend on the basic concept of vertex degrees. We need the following definitions and theorems, which can be found in [2] or [3].

Definition 1.1 *A graph G_1 is isomorphic to a graph G_2, if there exists a bijection ϕ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if, and only if, $\phi(u)\phi(v) \in E(G_2)$.*

If G_1 is isomorphic to G_2, we write $G_1 \cong G_2$ or sometimes $G_1 = G_2$.

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Definition 1.2 *The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg(v)$ or $deg_G(v)$.*

The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Theorem 1.3 *In any graph G, the number of odd vertices is even.*

Theorem 1.4 *The sum of the degrees of vertices of a graph G is twice the number of edges.*

Definition 1.5 *The corona of two graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.*

Theorem 1.6 *Let G be a simple graph i.e, a undirected graph without loops and multiple edges, with $n \geq 2$. Then G has atleast two vertices of the same degree.*

Definition 1.7 *Any connected graph G having a unique cycle is called a unicyclic graph.*

Definition 1.8 *A graph is called a caterpillar if the deletion of all its pendent vertices produces a path graph.*

Definition 1.9 *A subset S of the vertex set V in a graph G is said to be independent if no two vertices in S are adjacent in G.*

The maximum number of vertices in an independent set of G is called the *independence number* and is denoted by $\beta_0(G)$.

Theorem 1.10 *Let G be a graph and $S \subset V$. S is an independent set of G if, and only if, $V - S$ is a covering of G.*

Definition 1.11 *A clique of a graph is a maximal complete subgraph.*

Definition 1.12 *A clique is said to be maximal if no super set of it is a clique.*

Definition 1.13 *The vertex degrees of a graph G arranged in non-increasing order is called degree sequence of the graph G.*

Definition 1.14 *For any graph G, the set $D(G)$ of all distinct degrees of the vertices of G is called the degree set of G.*

Definition 1.15 *A sequence of non-negative integers is said to be graphical if it is the degree sequence of some simple graph.*

Theorem 1.16([1]) *Let G be any graph. The number of edges in G^{de} the degree equitable graph of G, is given by*

$$\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2},$$

where, $S_i = \{v|v \in V,\ deg(v) = i \text{ or } i+1\}$ and $S_i{}' = \{v|v \in V,\ deg(v) = i\}$.

Theorem 1.17 *The maximum number of edges in G with radius $r \geq 3$ is given by*

$$\frac{n^2 - 4nr + 5n + 4r^2 - 6r}{2}.$$

Definition 1.18 *A vertex cover in a graph G is such a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha(G)$ of G.*

Recently A. Anitha, S. Arumugam and E. Sampathkumar [1] have introduced degree equitable sets in a graph and studied them. "The characterization of degree equitable graphs" is still an open problem. In this paper we give some necessary conditions for a graph to be degree equitable. For this purpose, we introduce another concept "Antidegree equitable sets" in a graph and we study them.

§2. Antidegree Equitable Sets

Definition 2.1 *Let $G = (V, E)$ be a graph. A non-empty subset S of V is called an antidegree equitable set if $|deg(u) - deg(v)| \neq 1$ for all $u, v \in S$.*

Definition 2.2 *An antidegree equitable set is called a maximal antidegree equitable set if for every $v \in V - S$, there exists at least one element $u \in S$ such that $|deg(u) - deg(v)| = 1$.*

Definition 2.3 *The antidegree equitable number $AD_e(G)$ of a graph G is defined as $AD_e(G) = max\{|S| : S$ is a maximal antidegree equitable set$\}$.*

Definition 2.4 *The lower antidegree equitable number $ad_e(G)$ of a graph G is defined as $ad_e(G) = min\{|S| : S$ is a maximal antidegree equitable set$\}$.*

A few $AD_e(G)$ and $ad_e(G)$ of some graphs are listed in the following:

(i) For the complete bipartite graph $K_{m,n}$, we have

$$AD_e(K_{m,n}) = \begin{cases} m+n & \text{if } |m-n| \neq 1, \\ max\{m,n\} & \text{if } |m-n| = 1 \end{cases}$$

and

$$ad_e(K_{m,n}) = \begin{cases} m+n & \text{if } |m-n| \neq 1, \\ min\{m,n\} & \text{if } |m-n| = 1. \end{cases}$$

(ii) For the wheel W_n on n-vertices, we have

$$AD_e(W_n) = \begin{cases} n & \text{if } n \neq 5, \\ 4 & \text{if } n = 5 \end{cases}$$

and

$$ad_e(W_n) = \begin{cases} n & \text{if } n \neq 5, \\ 1 & \text{if } n = 5. \end{cases}$$

(iii) For the complete graph K_n, we have $AD_e(K_n) = ad_e(K_n) = n - 1$.

Now we study some important basic properties of antidegree equitable sets and independent antidegree equitable sets in a graph.

Theorem 2.5 *Let G be a simple graph on n-vertices. Then*

(i) $1 \leq ad_e(G) \leq AD_e(G) \leq n$;

(ii) $AD_e(G) = 1$ *if, and only if,* $G = K_1$;

(iii) $ad_e(G) = ad_e(\overline{G})$, $AD_e(G) = AD_e(\overline{G})$.

(iv) $ad_e(G) = 1$ *if, and only if, there exists a vertex* $u \in V(G)$ *such that* $|deg(u) - deg(v)| = 1$ *for all* $v \in V - \{u\}$;

(v) *If G is a non-trivial connected graph and $ad_e(G) = 1$, then $AD_e(G) = n - 1$ and n must be odd.*

Proof (i) follows from the definition.

(ii) Suppose $AD_e(G) = 1$ and $G \neq K_1$. Then G is a non-trivial graph and from Theorem 1.6 there exists at least two vertices of same degree and they form an antidegree equitable set in G. So $AD_e(G) \geq 2$ which is a contradiction. The converse is obvious.

(iii) Since $deg_{\overline{G}}(u) = (n-1) - deg_G(u)$, it follows that an antidegree equitable set in G is also an antidegree equitable set in $\overline{G}$.

(iv) If $ad_e(G) = 1$ and there is no such vertex u in G, then $\{u\}$ is not a maximal antidegree equitable set for any $u \in V(G)$ and hence $ad_e(G) \geq 2$ which is a contradiction. The converse is obvious.

(v) Suppose G is a non-trivial connected graph with $ad_e(G) = 1$. Then there exists a vertex $u \in V$ such that $|deg(u) - deg(v)| = 1$, $\forall\ v \in V - \{u\}$. Clearly, $|deg(v) - deg(w)| = 0$ *or* 2, $\forall\ v,\ w \in V - \{u\}$. Hence, $AD_e(G) = |V - \{u\}| = n - 1$. It follows from Theorem 1.4 that $(n-1)$ is even and thus n is odd. □

Theorem 2.6 *Let G be a non-trivial connected graph on n-vertices. Then $2 \leq AD_e(G) \leq n$ and $AD_e(G) = 2$ if, and only if, $G \cong K_2$ or P_2 or P_3 or $L(H)$ or $L^2(H)$ where H is the caterpillar T_5 with spine $P = (v_1v_2)$.*

Proof By Theorem 2.5, for a non-trivial connected graph G on n-vertices, we have $2 \leq AD_e(G) \leq n$. Suppose $AD_e(G) = 2$. Then for each antidegree equitable set S in G, we have $|S| \leq 2$. Let $D(G) = \{d_1, d_2, \ldots, d_k\}$, where $d_1 < d_2 < d_3 < \cdots < d_k$. As there are at least two vertices with same degree, we have $k \leq n - 1$. Since $AD_e(G) = 2$, more than two vertices cannot have the same degree. Let $d_i \in D(G)$ be such that exactly two vertices of G have degree d_i. Since the cardinality of each antidegree equitable set S cannot exceed two, it follows that

$\cdots, d_i-3, d_i-2, d_i+2, d_i+3, d_i+4, \cdots$ do not belong to $D(G)$. Thus $D(G) \subset \{d_i-1, d_i, d_i+1\}$.

Case 1. If $d_i - 1, d_i + 1$ do not belong to $D(G)$ then $D(G) = \{d_i\}$ and the degree sequence $\{d_i, d_i\}$ is clearly graphical. Thus $n = 2$ and $d_i = 1$ which implies $G = K_2$.

Case 2. If $d_i - 1, d_i + 1 \in D(G)$, then the degree sequence $\{d_i - 1, d_i, d_i, d_i + 1\}$ is graphical. Thus $n = 4$ and $d_i = 2$ which implies $G \cong L(H)$, where H is the caterpillar T_5 with spine $P = (v_1v_2)$.

Case 3. If $d_i-1 \in D(G)$ and d_i+1 does not belong to $D(G)$, then d_i-1 may or may not repeat twice in degree sequence. Thus degree sequence is given by $\{d_i-1, d_i, d_i\}$ or $\{d_i-1, d_i-1, d_i, d_i\}$. The first sequence is not graphical but the second sequence is graphical. Thus $n = 4$ and $d_i = 2$ which implies $G \cong P_4$.

Case 4. If $d_i - 1$ does not belong to $D(G)$ and $d_i + 1 \in D(G)$, then the degree sequence is given by $\{d_i, d_i, d_i + 1\}$ or $\{d_i, d_i, d_i + 1, d_i + 1\}$. Both sequences are graphical. In the first case $n = 3$, $d_i = 1$ which implies $G \cong P_2$, and in the second case $n = 4$, $d_i = 1$ or 2 which implies $G \cong P_3$ or $G \cong L^2(H)$ respectively.

The converse is obvious. □

Theorem 2.7 *If a and b are positive integers with $a \leq b$, then there exists a connected simple graph G with $ad_e(G) = a$ and $AD_e(G) = b$ except when $a = 1$ and $b = 2m + 1$, $m \in N$.*

Proof If $a = b$ then for any regular graph of order a, we have $ad_e(G) = AD_e(G) = a$. If $b = a + 1$, then for the complete bipartite graph $G = k_{a,a+1}$ we have $ad_e(G) = a$ and $AD_e(G) = a + 1 = b$. If $b \geq a + 2$, $a \geq 2$, and $b > 4$, then for the graph G consisting of the wheel W_{b-1} and the path $P_a = (v_1v_2v_3 \ldots v_a)$ with an edge joining a pendant vertex of P_a to the center of the wheel W_{b-1}, we have $ad_e(G) = a$, $AD_e(G) = b$. If $a = 1$ and $b = 2m$, $m \in N$, then the graph consisting of two cycles C_m and C_{m+1} along with edges joining i^{th} vertex of C_m to i^{th} vertex of C_{m+1}, we have $ad_e(G) = 1 = a$ and $AD_e(G) = 2m = b$.

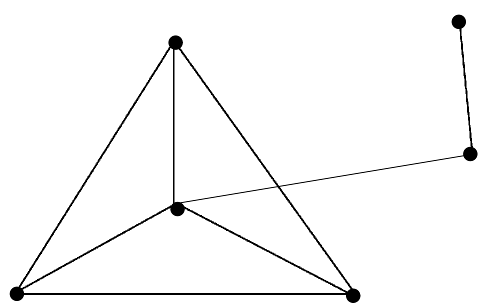

Figure 1

For $a = 2$ and $b = 4$ we consider graph G in Figure 1, for which $ad_e(G) = 2$ and $AD_e(G) = 4$. Also, it follows from Theorem 2.5 that there is no graph G with $ad_e(G) = 1$ and $AD_e(G) = 2m + 1$. □

Theorem 2.8 *Let G be a non-trivial connected graph on n vertices and let S^* be a subset of V such that $|deg(u) - deg(v)| \geq 2$ for all $u, v \in S^*$. Then $1 \leq |S^*| \leq \left[\frac{\Delta - \delta}{2}\right] + 1$ and also, if S^* is a maximal subset of V such that $|deg(u) - deg(v)| \geq 2$ for all u, $v \in S^*$, then $S = \bigcup_{v \in S^*} S_{deg(v)}$ is a maximal antidegree equitable set in G, where $S_{deg(v)} = \{u \in V : deg(u) = deg(v)\}$.*

Proof For any two vertices $u, v \in S^*$, $d(u)$ and $d(v)$ cannot be two successive members of $A = \{\delta, \delta + 1, \delta + 2, \ldots, \delta + k = \Delta\}$ and $D(G) \subset A$. Hence

$$|S^*| \leq \left[\frac{|D(G)| + 1}{2}\right] \leq \left[\frac{|A| + 1}{2}\right] = \left[\frac{\Delta - \delta}{2}\right] + 1.$$

If $a, b \in S = \bigcup_{v \in S^*} S_{deg(v)}$, then it is clear that either $|deg(a) - deg(b)| = 0$ or $|deg(a) - deg(b)| \geq 2$ and hence S is an antidegree equitable set. Suppose $u \in V - S$. Then $deg(u) \neq deg(v)$ for any $v \in S^*$. So, u do not belong to S^* and hence $|deg(u) - deg(v)| = 1$ for all $v \in S$. This implies that S is a maximal antidegree equitable set. □

Theorem 2.9 *Given a positive integer k, there exists graphs G_1 and G_2 such that $ad_e(G_1) - ad_e(G_1 - e) = k$ and $ad_e(G_2 - e) - ad_e(G_2) = k$.*

Proof Let $G_1 = K_{k+2}$. Then $ad_e(G_1) = k + 2$ and $ad_e(G_1 - e) = 2$, where $e \in E(G_1)$. Hence $ad_e(G_1) - ad_e(G_1 - e) = k$. Let G_2 be the graph obtained from C_{k+1} by attaching one leaf e at $(k + 1)^{th}$ vertex of C_{k+1}. Then $ad_e(G_2 - e) - ad_e(G_2) = k$. □

Theorem 2.10 *Given two positive integers n and k with $k \leq n$. Then there exists a graph G of order n with $ad_e(G) = k$.*

Proof If $k < \frac{n}{2}$, then we take G to be the graph obtained from the path $P_k = (v_1 v_2 v_3 \ldots v_k)$ and the complete graph K_{n-k} by joining v_1 and a vertex of K_{n-k} by an edge. Clearly, $ad_e(G) = k$. If $k \geq \frac{n}{2}$, then we take G to be the graph obtained from the cycle C_k by attaching exactly one leaf at $(n - k)$ vertices of C_k. Clearly, $ad_e(G) = k$. □

§3. Independent Antidegree Equitable Sets

In this section, we introduce the concepts of independent antidegree equitable number and lower independent antidegree equitable number and establish important results on these parameters.

Definition 3.1 *The independent antidegree equitable number $AD_{ie}(G) = max\{|S| : S \subset V, S$ is a maximal independent and antidegree equitable set in $G\}$.*

Definition 3.2 *The lower independent antidegree equitable number $ad_{ie}(G) = min\{|S| :$*

S is a maximal independent and antidegree equitable set in $G\}$.

A few AD_{ie} and ad_{ie} of graphs are listed in the following.

(i) For the star graph $K_{1,n}$ we have, $AD_{ie}(K_{1,n}) = n$ and $ad_{ie}(K_{1,n}) = 1$.

(ii) For the complete bipartite graph $K_{m,n}$ we have $AD_{ie}(K_{m,n}) = max\{m,n\}$ and $ad_{ie}(K_{m,n}) = min\{m,n\}$.

(iii) For any regular graph G we have, $AD_{ie}(G) = ad_{ie}(G) = \beta_o(G)$.

The following theorem shows that on removal of an edge in G, $AD_{ie}(G)$ can decrease by at most one and increase by at most 2.

Theorem 3.3 *Let G be a connected graph, $e = uv \in E(G)$. Then*

$$AD_{ie}(G) - 1 \leq AD_{ie}(G - e) \leq AD_{ie}(G) + 2.$$

Proof Let S be an independent antidegree equitable set in G with $|S| = AD_{ie}(G)$. After removing an edge $e = uv$ from the graph G, we shall give an upper and a lower bound for $AD_{ie}(G - e)$.

Case 1. If u, v does not belong to S, then S is a maximal independent antidegree equitable set in $G - e$ as well as in G. Hence, $AD_{ie}(G - e) = AD_{ie}(G)$.

Case 2. If $u \in S$ and v does not belong to S, then $S - \{u\}$ is an independent antidegree equitable set in $G - e$. Hence, $AD_{ie}(G - e) \geq |S - \{u\}| = AD_{ie}(G) - 1$. Thus, $AD_{ie}(G) - 1 \leq AD_{ie}(G - e)$.

Now, Let S be an independent antidegree equitable set in $G - e$ with $|S| = AD_{ie}(G - e)$.

Case 3. If $u, v \in S$, then $S - \{u, v\}$ is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S - \{u, v\}| = AD_{ie}(G - e) - 2$.

Case 4. If $u \in S$ and v does not belong to S, then $S - \{u\}$ is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S - \{u\}| = AD_{ie}(G - e) - 1$.

Case 5. If u, v do not belong to S, then S is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S| = AD_{ie}(G - e)$. It follows that $AD_{ie}(G) \geq AD_{ie}(G - e) - 2$. Hence,

$$AD_{ie}(G) - 1 \leq AD_{ie}(G - e) \leq AD_{ie}(G) + 2. \qquad \square$$

Theorem 3.4 *Let G be a connected graph. $AD_{ie}(G) = 1$ if, and only if, $G \cong K_n$ or for any two non-adjacent vertices $u, v \in V$, $|deg(u) - deg(v)| = 1$.*

Proof Suppose $AD_{ie}(G) = 1$.

Case 1. If $G \cong K_n$, then there is nothing to prove.

Case 2. Let $G \neq K_n$, and u, v be any two non-adjacent vertices in G. Since $AD_{ie}(G) = 1$, $\{u, v\}$ is not an antidegree equitable set and hence $|deg(u) - deg(v)| = 1$. The converse is obvious. □

Theorem 3.5 *Let G be a connected graph. $ad_{ie}(G) = 1$ if, and only if, either $\Delta = n - 1$ or for any two non-adjacent vertices $u, v \in V$, $|deg(u) - deg(v)| = 1$.*

Proof Suppose $ad_{ie}(G) = 1$, then for any two non-adjacent vertices u and v, $\{u, v\}$ is not an antidegree equitable set.

Case 1. If $\Delta = n - 1$, then there is nothing to prove.

Case 2. Let $\Delta < n - 1$, and u, v be any two non-adjacent vertices in G. Then $\{u, v\}$ is not an antidegree equitable set and hence, $|deg(u) - deg(v)| = 1$.

The converse is obvious. □

Remark 3.6 Theorems 3.4 and 3.5 are equivalent.

§4. Degree Equitable and Antidegree Equitable Graphs

After studying the basic properties of antidegree equitable and independent antidegree equitable sets in a graph, in this section we give some conditions for a graph to be degree equitable. We recall the definition of degree equitable graph given by A. Anitha, S. Arumugam, and E. Sampathkumar [1].

Definition 4.1 *Let $G = (V, E)$ be a graph. The degree equitable graph of G, denoted by G^{de} is defined as follows:$V(G^{de}) = V(G)$ and two vertices u and v are adjacent vertices in G^{de} if, and only if, $|deg(u) - deg(v)| \leq 1$.*

Example 4.2 For any regular graph G on n vertices, we have $G^{de} = K_n$.

Definition 4.3 *A graph H is called degree equitable graph if there exists a graph G such that $H \cong G^{de}$.*

Example 4.4 Any complete graph K_n is a degree equitable graph because $K_n = G^{de}$ for any regular graph G on n-vertices.

Theorem 4.5 *Let $G = (V, E)$ be any graph on n vertices with radius $r \geq 3$. Then*

(i) $1 \leq \beta_0(G^{de}) \leq \sqrt{n^2 - 4nr + 5n + 4r^2 - 6r}$.
(ii) $\beta_0(G^{de}) \leq \left[\frac{\Delta - \delta}{2}\right] + 1$, *where $\Delta = \Delta(G)$ and $\delta = \delta(G)$.*

Proof (*i*) Let A be an independent set of G^{de} such that $|A| = \beta_0(G^{de})$. Then A is an antidegree equitable set in G and hence

$$\sum_{v \in V} deg_G(v) \geq \sum_{v \in A} deg_G(v) = \sum_{\ell=1}^{\beta_0(G^{de})} 2\ell - 1 = {\beta_0}^2(G^{de}).$$

By Theorem 1.17 it follows that

$$2\left(\frac{n^2-4nr+5n+4r^2-6r}{2}\right)\geq {\beta_0}^2(G^{de}).$$

Therefore,

$$1\leq \beta_0(G^{de})\leq \sqrt{n^2-4nr+5n+4r^2-6r}.$$

(ii) We know that every independent set A in G^{de} is an antidegree equitable set in G and hence by Theorem 2.8,

$$|A|\leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$

Therefore,

$$\beta_0(G^{de})\leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$

This completes the proof. □

Theorem 4.6 *Let H be any degree equitable graph on n vertices and $H=G^{de}$ for some graph G. Then*

$$\sqrt{\sum_{v\in A} deg_G(v)}\leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1$$

where A is an independent set in G^{de} such that $|A|=\beta_0(G^{de})$.

Proof We know that if A is an independent set in H then it is an antidegree equitable set in G. Hence,

$$\sum_{v\in A} deg_G(v)\leq \sum_{\ell=1}^{\beta_0(H)} 2\ell-1={\beta_0}^2(H).$$

By Theorem 4.5

$$\sum_{v\in A} deg_G(v)\leq \left(\left[\frac{\Delta(G)-\delta(G)}{2}\right]+1\right)^2.$$

Therefore,

$$\sqrt{\sum_{v\in A} deg_G(v)}\leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$ □

We introduce a new concept antidegree equitable graph and present some basic results.

Definition 4.7 *Let $G=(V,E)$ be a graph. The antidegree equitable graph of G, denoted by G^{ade} defined as follows: $V(G^{ade})=V(G)$ and two vertices u and v are adjacent in G^{ade} if, and only if, $|deg(u)-deg(v)|\neq 1$.*

Example 4.8 For a complete bipartite graph $K_{m,n}$, we have

$$G^{ade}=\begin{cases} K_{m+n} & \text{if } |m-n|\geq 2,\text{or }=0\\ K_m\cup K_n & \text{if } |m-n|=1.\end{cases}$$

Definition 4.9 *A graph H is called an antidegree equitable graph if there exists a graph G such that $H \cong G^{ade}$.*

Example 4.10 Any complete graph K_n is an antidegree equitable graph because $K_n = G^{ade}$ for any regular graph G on n-vertices.

Theorem 4.11 *Let G be any graph on n vertices. Then the number of edges in G^{ade} is given by*

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1}\binom{|S_i|}{2} + \binom{|S_\delta{}'|}{2} + 2\sum_{i=\delta+1}^{\Delta}\binom{|S_i{}'|}{2},$$

where $S_i = \{v|\ v \in V\ deg_G(v) = i \text{ or } i+1\}$, $S_i{}' = \{v|\ v \in V\ deg_G(v) = i\}$, $\Delta = \Delta(G)$ and $\delta = \delta(G)$.

Proof By Theorem 1.16, we have the number of edges in G^{ade} with end vertices having the difference degree greater than two in G is

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1}\binom{|S_i|}{2} + \sum_{i=\delta+1}^{\Delta}\binom{|S_i{}'|}{2}.$$

and also, the number of edges in G^{ade} with end vertices having the same degree is

$$\sum_{i=\delta}^{\Delta}\binom{|S_i{}'|}{2}.$$

Hence, the total number of edges in G^{ade} is

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1}\binom{|S_i|}{2} + \sum_{i=\delta+1}^{\Delta}\binom{|S_i{}'|}{2} + \sum_{i=\delta}^{\Delta}\binom{|S_i{}'|}{2}$$

$$= \binom{n}{2} - \sum_{i=\delta}^{\Delta-1}\binom{|S_i|}{2} + \binom{|S_\delta{}'|}{2} + 2\sum_{i=\delta+1}^{\Delta}\binom{|S_i{}'|}{2}. \qquad \square$$

Theorem 4.12 *Let G be any graph on n vertices. Then*

(i) $\alpha(G^{ade}) \leq \sqrt{n(n-1)}$;
(ii) $\alpha(G^{ade}) \leq \left[\frac{\Delta-\delta}{2}\right] + 1$, *where $\Delta = \Delta(G)$ and $\delta = \delta(G)$.*

Proof Let $A \subset V$ be the set of vertices that covers all edges of G^{ade}. Then A is an antidegree equitable set in G. Hence,

$$\sum_{v \in A} deg_G(v) \geq \sum_{\ell=1}^{\alpha(G^{ade})} 2\ell - 1 = \alpha^2(G^{ade}).$$

Therefore,

$$2\left(\frac{n(n-1)}{2}\right) \geq \alpha^2(G^{ade}),$$

$$\alpha(G^{ade}) \leq \sqrt{n(n-1)}.$$

Since, the set A is an antidegree equitable set in G, by Theorem 2.8, we have

$$|A| \leq \left[\frac{\Delta - \delta}{2}\right] + 1.$$

This implies

$$\alpha(G^{ade}) \leq \left[\frac{\Delta - \delta}{2}\right] + 1. \qquad \square$$

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A New Approach to Natural Lift Curves of The Spherical Indicatrices of Timelike Bertrand Mate of a Spacelike Curve in Minkowski 3-Space

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Abstract: In this study, we present a new approach the natural lift curves for the spherical indicatrices of the timelike Bertrand mate of a spacelike curve on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$ in Minkowski 3-space and we give some new characterizations for these curves. Additionally we illustrate an example of our main results.

Key Words: Bertrand curve, natural lift curve, geodesic spray, spherical indicatrix.

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§1. Introduction

Bertrand curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. It is proved in most texts on the subject that the characteristic property of such a curve is the existence of a linear relation between the curvature and the torsion; the discussion appears as an application of the Frenet-Serret formulas. So, a circular helix is a Bertrand curve. Bertrand mates represent particular examples of offset curves [11] which are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). For classical and basic treatments of Bertrand curves, we refer to [3], [6] and [12].

There are recent works about the Bertrand curves. Ekmekçi and İlarslan studied Nonnull Bertrand curves in the n-dimensional Lorentzian space. Straightforward modication of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Izumiya

[1]Received August 28, 2014, Accepted February 19, 2015.

and Takeuchi [16] have shown that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Also, the representation formulae for Bertrand curves were given by [8].

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as $w = \tau t + \kappa b$. In addition, the concepts of the natural lift and the geodesic sprays have first been given by Thorpe (1979). On the other hand, Çalışkan et al. [4] have studied the natural lift curves and the geodesic sprays in Euclidean 3-space $\mathbb{R}^3$. Bilici et al. [7] have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the involute-evolute curve couple in $\mathbb{R}^3$. Recently, Bilici [9] adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space.

Kula and Yaylı [17] have studied spherical images of the tangent indicatrix and binormal indicatrix of a slant helix and they have shown that the spherical images are spherical helices. In [19] Süha et. all investigated tangent and trinormal spherical images of timelike curve lying on the pseudo hyperbolic space H_0^3 in Minkowski space-time. İyigün [20] defined the tangent spherical image of a unit speed timelike curve lying on the on the pseudo hyperbolic space H_0^2 in $\mathbb{R}_1^3$.

Şenyurt and Çalışkan [22] obtained arc-lengths and geodesic curvatures of the spherical indicatrices (T^*) , (N^*) , (B^*) and the fixed pole curve (C^*) which are generated by Frenet trihedron and the unit Darboux vector of the timelike Bertrand mate of a spacelike curve with respect to Minkowski space $\mathbb{R}_1^3$ and Lorentzian sphere S_1^2 or hyperbolic sphere H_0^2. Furthermore, they give some criteria of being integral curve for the geodesic spray of the natural lift curves of this spherical indicatrices.

In this study, the conditions of being integral curve for the geodesic spray of the the natural lift curves of the the spherical indicatrices (T^*) , (N^*) , (B^*) are investigated according to the relations given by [8] on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$ in Minkowski 3-space. Also, we present an example which illustrates these spherical indicatrices (Figs. 1-4). It is seen that the principal normal indicatrix (N^*) is geodesic on S_1^2 and its natural lift curve is an integral curve for the geodesic spray on $T\left(S_1^2\right)$.

§2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves and hypersurfaces in the Minkowski 3-space are briefly presented in this section. A more detailed information can be found in [10].

The Minkowski 3-space $\mathbb{R}_1^3$ is the real vector space $\mathbb{R}^3$endowed with standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. A vector $V = (v_1, v_2, v_3) \in \mathbb{R}_1^3$ is said to be timelike if $g(V,V) < 0$, spacelike if $g(V,V) > 0$ or $V = 0$ and null (lightlike) if

$g(V,V)=0$ and $V\neq 0$ .Similarly, an arbitrary $\Gamma=\Gamma(s)$ curve in $\mathbb{R}_1^3$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors Γ' are respectively timelike, spacelike or null (lightlike), for every t $\in$ I $\subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $V\in\mathbb{R}_1^3$ is given by $\|V\|=\sqrt{|g(V,V)|}$. Γ is called a unit speed curve if the velocity vector V of Γ satisfies $\|V\|=1$. A timelike vector V is said to be positive (resp. negative) if and only if $v_1>0$ (resp.$v_1<0$) .

Let Γ be a unit speed spacelike curve with curvature κ and torsion τ. Denote by $\{t(s),n(s),b(s)\}$ the moving Frenet frame along the curve Γ in the space $\mathbb{R}_1^3$. Then t,n and b are the tangent, the principal normal and the binormal vector of the curve Γ, respectively.

The angle between two vectors in Minkowski 3-space is defined by [21]

Definition 2.1 *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace, then we have $|g(X,Y)|\leq\|X\|\|Y\|$ and hence, there is a unique positive real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\cos\varphi.$$

The real numberφ is called the Lorentzian spacelike angle between X and Y.

Definition 2.2 *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace, then we have $|g(X,Y)|>\|X\|\|Y\|$ and hence, there is a unique positive real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\cosh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.3 *Let X be a spacelike vector and Y a positive timelike vector in $\mathbb{R}_1^3$, then there is a unique non-negative real number φ such that*

$$|g(X,Y)|=\|X\|\|Y\|\sinh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.4 *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}_1^3$, then there is a unique non-negative real number φ such that*

$$g(X,Y)=\|X\|\|Y\|\cosh\varphi.$$

The real number φ is called the Lorentzian timelike angle between X and Y.

Case I. Let Γ be a unit speed spacelike curve with a spacelike binormal. For these Frenet vectors, we can write

$$T\times N=-B,\ N\times B=-T,\ B\times T=N$$

where "$\times$" is the Lorentzian cross product in space $\mathbb{R}_1^3$. Depending on the causal character of the curve Γ, the following Frenet formulae are given in [5].

$$\dot{T}=\kappa N,\ \dot{N}=\kappa T+\tau B, \dot{B}=\tau N$$

The Darboux vector for the spacelike curve with a spacelike binormal is defined by [11]:

$$w = -\tau T + \kappa B$$

If b and w spacelike vectors that span a spacelike vector subspace then by the Definition1. we can write

$$\begin{aligned} \kappa &= \|w\| \cosh \varphi \\ \tau &= \|w\| \sinh \varphi, \end{aligned}$$

where $\|w\|^2 = g(w, w) = \tau^2 + \kappa^2$.

Case II. Let Γ be a unit speed spacelike curve with a timelike binormal. For these Frenet vectors, we can write

$$T \times N = B,\ N \times B = -T,\ B \times T = -N$$

Depending on the causal character of the curve Γ , the following Frenet formulae are given in [5].

$$\dot{T} = \kappa N,\ \dot{N} = -\kappa T + \tau B, \dot{B} = \tau N$$

The Darboux vector for the spacelike curve with a timelike binormal is defined by [11]:

$$w = \tau T - \kappa B$$

There are two cases corresponding to the causal characteristic of Darboux vector w.

(i) If $|\kappa| < |\tau|$, then w is a timelike vector. In this situation, we have

$$\begin{aligned} \kappa &= \|w\| \sinh \varphi \\ \tau &= \|w\| \cosh \varphi, \end{aligned}$$

where $\|w\|^2 = -g(w, w) = \tau^2 - \kappa^2$. So the unit vector c of direction w is

$$c = \frac{1}{\|w\|} w = \sinh \varphi T - \cosh \varphi B.$$

(ii) If $|\kappa| > |\tau|$, then w is a spacelike vector. In this situation, we can write

$$\begin{aligned} \kappa &= \|w\| \cosh \varphi \\ \tau &= \|w\| \sinh \varphi, \end{aligned}$$

where $\|w\|^2 = g(w, w) = \kappa^2 - \tau^2$. So the unit vector c of direction w is

$$c = \frac{1}{\|w\|} w = \sinh \varphi T - \cosh \varphi B.$$

Proposition 2.5([13]) *Let α be a timelike (or spacelike) curve with curvatures κ and τ. The* *curve is a general helix if and only if* $\frac{\tau}{\kappa}$ *is constant.*

Remark 2.6 We can easily see fromequations of the section Case I and Case II that: $\frac{\tau}{\kappa} = \tan\varphi$, $\frac{\tau}{\kappa} = \tanh\varphi$ $\left(or\ \frac{\tau}{\kappa} = \coth\varphi\right)$, if φ =constant then α is a general helix.

Lemma 2.7([9]) *The natural lift* $\overline{\alpha}$ *of the curve* α *is an integral curve of the geodesic spray* X *if and only if* α *is a geodesic on* M.

Definition 2.8 *Let* $\alpha = (\alpha(s); T(s), N(s), B(s))$ *and* $\beta = (\beta(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ *be two regular non-null curves in* $\mathbb{R}^3_1$. $\alpha(s)$ *and* $\beta(s^*)$ *are called Bertrand curves if* $N(s)$ *and* $N^*(s^*)$ *are linearly dependent. In this situation,*(α, β) *is called a Bertrand couple in* $\mathbb{R}^3_1$. (See [1] for the more details in the n-dimensional space).

Lemma 2.9 *Let* α *be a spacelike curve with a timelike binormal. In this situation,* β *is a timelike Bertrand mate of* α. *The relations between the Frenet vectors of the* (α, β) *is as follow*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} \sinh\theta & 0 & \cosh\theta \\ 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, g(T, T^*) = \sinh\theta = constant, [8].$$

Definition 2.11([10]) *Let* S^2_1 *and* H^2_0 *be hypersphere in* $\mathbb{R}^3_1$. *The Lorentzian sphere and hyperbolic sphere of radius 1 in are given by*

$$S^2_1 = \left\{V = (v_1, v_2, v_3) \in \mathbb{R}^3_1 \ : \ g(V, V) = 1\right\}$$

and

$$H^2_0 = \left\{V = (v_1, v_2, v_3) \in \mathbb{R}^3_1 \ : \ g(V, V) = -1\right\}$$

respectively.

Definition 2.12([9]) *Let* M *be a hypersurface in* $\mathbb{R}^3_1$ *equipped with a metric* g. *Let* TM *be the set* $\cup\{T_p(M) : p \in M\}$ *of all tangent vectors to* M. *Then each* $v \in TM$ *is in a unique* $T_p(M)$, *and the projection* $\pi : TM \to M$ *sends* v *to* p. *Thus* $\pi^{-1}(p) = T_p(M)$. *There is a natural way to make*TM *a manifold, called the tangent bundle of* M.

A vector field $X \in \chi(M)$ *is exactly a smooth section of* TM, *that is, a smooth function* $X : M \to TM$ *such that* $\pi \circ X = id_M$.

Definition 2.13([9]) *Let* M *be a hypersurface in* $\mathbb{R}^3_1$. *A curve* $\alpha : I \to TM$ *is an integral curve of* $X \in \chi(M)$ *provided* $\dot{\alpha} = X_\alpha$ *; that is*

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \ \textit{for all } s \in I, [10]. \tag{1}$$

Definition 2.14 *For any parametrized curve* $\alpha : I \to TM$, *the parametrized curve given by*

$\overline{\alpha} : I \to TM$

$$s \to \overline{\alpha}(s) = (\alpha(s), \dot{\alpha}(s)) = \dot{\alpha}(s) \mid_{\alpha(s)} \tag{2}$$

is called the natural lift of α *on* TM. *Thus, we can write*

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s) \mid_{\alpha(s)}\right) = D_{\dot{\alpha}'(s)}\dot{\alpha}(s), \tag{3}$$

where D *is the standard connection on* $\mathbb{R}^3_1$.

Definition 2.15([9]) *For* $v \in TM$ *, the smooth vector field* $X \in \chi(TM)$ *defined by*

$$X(v) = \varepsilon g(v, S(v))\,\xi \mid_{\alpha(s)}, \ \varepsilon = g(\xi, \xi) \tag{4}$$

is called the geodesic spray on the manifold TM, *where* ξ *is the unit normal vector field of* M *and* S *is the shape operator of* M.

§3. Natural Lift Curves for the Spherical Indicatrices of Spacelike-Timelike Bertrand Couple in Minkowski 3-Space

In this section we investigate the natural lift curves of the spherical indicatrices of Bertrand curves (α, β) as in Lemma 2.9. Furthermore, some interesting theorems about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2_1\right)$ or $T\left(H^2_0\right)$.

Note that $\bar{D}$ and $\bar{\bar{D}}$ are Levi-Civita connections on S^2_1 and H^2_0 , respectively. Then Gauss equations are given by the followings

Let D, $\bar{D}$ and $\bar{\bar{D}}$ be connections in $\mathbb{R}^3_1$, S^2_1 and H^2_0 respectively and ξ be a unit normal vector field of S^2_1 and H^2_0. Then Gauss Equations are given by the followings

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)\,\xi, \ D_X Y = \bar{\bar{D}}_X Y + \varepsilon g(S(X), Y)\,\xi, \varepsilon = g(\xi, \xi)$$

where ξ is a unit normal vector field and S is the shape operator of S^2_1 (or H^2_0).

3.1 The natural lift of the spherical indicatrix of the tangent vector of β

Let (α, β) be Bertrand curves as in Lemma 2.9. We will investigate the curve α to satisfy the condition that the natural lift curve of $\bar{\beta}_{T^*}$ is an integral curve of geodesic spray, where β_{T^*} is the tangent indicatrix of β. If the natural lift curve $\bar{\beta}_{T^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.9. we get,

$$\bar{\bar{D}}_{\dot{\beta}_{T^*}}\dot{\beta}_{T^*} = 0, \tag{5}$$

where $\bar{\bar{D}}$ is the connection on the hyperbolic unit sphere H^2_0 and the equation of tangent indicatrix is $\beta_{T^*} = T^*$. Thus from the Gauss equation we can write

$$D_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} = \overline{\overline{D}}_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} + \varepsilon g\left(S\left(\dot{\beta}_{T^*}\right), \dot{\beta}_{T^*}\right) T^*, \varepsilon = g\left(T^*, T^*\right) = -1$$

On the other hand, from the Lemma 2.9. straightforward computation gives

$$\dot{\beta}_{T^*} = t_{T^*} = \frac{d\beta_{T^*}}{ds}\frac{ds}{ds_{T^*}} = \left(\kappa \sinh\theta + \tau\cosh\theta\right) N \frac{ds}{ds_{T^*}}$$

Moreover, we get

$$\frac{ds}{ds_{T^*}} = \frac{1}{\kappa\sinh\theta + \tau\cosh\theta}, t_{T^*} = N,$$

$$D_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} T + \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} B$$

and $g\left(S\left(t_{T^*}\right), t_{T^*}\right) = -1$.

Using these in the Gauss equation, we immediately have

$$\overline{\overline{D}}_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} T + \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} B - T^*.$$

From the Eq. (5) and Lemma 2.9.ii) we get

$$\left(-\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} - \sinh\theta\right) T + \left(\frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} - \cosh\theta\right) B$$

Since T, N, B are linearly independent, we have

$$-\frac{\kappa}{\kappa\sinh\theta + \tau\cosh\theta} - \sinh\theta = 0, \qquad \frac{\tau}{\kappa\sinh\theta + \tau\cosh\theta} - \cosh\theta = 0.$$

It follows that,

$$\kappa\cosh\theta + \tau\sinh\theta = 0 \tag{6}$$

$$\frac{\tau}{\kappa} = -\coth\theta \tag{7}$$

So from the Eq. (7) and Remark 2.6. we can give the following proposition.

Proposition 3.1 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the tangent indicatrix β_{T^*} of β is a geodesic on H_0^2.*

Moreover from Lemma 2.7. and Proposition 3.1 we can give the following theorem to characterize the natural lift of the tangent indicatrix of β without proof.

Theorem 3.2 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{T^*}$ of the tangent indicatrix β_{T^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$.*

3.2 The natural lift of the spherical indicatrix of the principal normal vectors of β

Let β_{N^*} be the spherical indicatrix of principal normal vectors of β and $\bar{\beta}_{N^*}$ be the natural lift of the curve . If $\bar{\beta}_{N^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get,

$$\bar{D}_{t_{N^*}} t_{N^*} = 0, \tag{8}$$

that is

$$D_{t_{N^*}} t_{N^*} = \bar{D}_{t_{N^*}} t_{N^*} + \varepsilon g\left(S\left(t_{N^*}\right), t_{N^*}\right) N^*, \varepsilon = g\left(N^*, N^*\right) = 1$$

On the other hand, from Lemma 2.9. and Case II. i) straightforward computation gives

$$\dot{\beta}_{N^*} = t_{N^*} = -\sinh\varphi T + \cosh\varphi B$$

Moreover we get

$$D_{t_{N^*}} t_{N^*} = -\frac{\dot{\varphi}\cosh\varphi}{\|W\|} T + \frac{-\kappa\sinh\varphi + \tau\cosh\varphi}{\|W\|} N + \frac{\dot{\varphi}\sinh\varphi}{\|W\|} B \text{ and } g\left(S\left(t_{N^*}\right), t_{N^*}\right) = 1$$

Using these in the Gauss equation, we immediately have

$$\bar{D}_{t_{N^*}} t_{N^*} = -\frac{\dot{\varphi}\cosh\varphi}{\|W\|} T + \frac{\dot{\varphi}\sinh\varphi}{\|W\|} B.$$

Since T, N, B are linearly independent, we have

$$-\frac{\dot{\varphi}\cosh\varphi}{\|W\|} = 0, \qquad \frac{\dot{\varphi}\sinh\varphi}{\|W\|} = 0.$$

It follows that,

$$\dot{\varphi} = 0, \tag{9}$$

$$\frac{\tau}{\kappa} = cons\tan t. \tag{10}$$

So from the Eq. (10) and Remark 2.6. we can give the following proposition.

Proposition 3.3 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the principal normal indicatrix β_{N^*} of β is a geodesic on S_1^2.*

Moreover from Lemma 2.7. and Proposition 4.3. we can give the following theorem to characterize the natural lift of the principal normal indicatrix of β without proof.

Theorem 3.4 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{N^*}$ of the principal normal indicatrix of β_{N^*} is β an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.*

3.3 The natural lift of the spherical indicatrix of the binormal vectors of β

Let β_{B^*} be the spherical indicatrix of binormal vectors of β and $\bar{\beta}_{B^*}$ be the natural lift of the curve β_{B^*} . If $\bar{\beta}_{B^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get

$$\bar{D}_{t_{B^*}} t_{B^*} = 0, \tag{11}$$

that is

$$D_{t_{B^*}} t_{B^*} = \bar{D}_{t_{B^*}} t_{B^*} + \varepsilon g\left(S\left(t_{B^*}\right), t_{B^*}\right) B^*, \varepsilon = g\left(B^*, B^*\right) = 1$$

On the other hand, from Lemma 2.9.ii) straightforward computation gives

$$t_{B^*} = (\kappa \cosh\theta + \tau \sinh\theta) N \frac{ds}{ds_{B^*}}$$

Moreover we get

$$\frac{ds}{ds_{B^*}} = \frac{1}{\kappa \cosh\theta + \tau \sinh\theta}, t_{B^*} = N,$$

$$D_{t_{B^*}} t_{B^*} = -\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} T + \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} B$$

and $g\left(S\left(t_{B^*}\right), t_{B^*}\right) = -1$.

Using these in the Gauss equation, we immediately have

$$\bar{D}_{t_{B^*}} t_{B^*} = \frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} T - \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} B + B^*$$

From the Eq. (11) and Lemma 2.9.ii) we get

$$\left(-\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} + \cosh\theta\right) T + \left(\frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} + \sinh\theta\right) B = 0.$$

Since T, N, B are linearly independent, we have

$$\begin{aligned} -\frac{\kappa}{\kappa \cosh\theta + \tau \sinh\theta} + \cosh\theta &= 0 \\ \frac{\tau}{\kappa \cosh\theta + \tau \sinh\theta} + \sinh\theta &= 0 \end{aligned}$$

it follows that

$$\kappa \sinh\theta + \tau \cosh\theta = 0 \tag{12}$$

$$\frac{\tau}{\kappa} = -\tanh\theta \tag{13}$$

So from the Eq. (13) and Remark 2.6. we can give the following proposition.

Proposition 3.5 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the binormal indicatrix β_{B^*} of β is a geodesic on S_1^2.*

Moreover from Lemma 2.7. and Proposition 4.5. we can give the following theorem to characterize the natural lift of the binormal indicatrix of β without proof.

Theorem 3.6 *Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\bar{\beta}_{B^*}$ of the binormal indicatrix β_{B^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.*

From the classification of all W-curves (i.e. a curves for which a curvature and a torsion are constants) in (Walrawe, 1995), we have following proposition with relation to curve.

Proposition 3.7 (1) *If the curve* α *with* $\kappa =$*constant* > 0 , $\tau = 0$ *then* α *is a part of a circle;*

(2) *If the curve* α *with* $\kappa =$*constant* > 0 , $\tau =$*constant* $\neq 0$, *and* $|\tau| > \kappa$ *then* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa \sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa \cosh\left(\sqrt{K}s\right)\right), \ K = \tau^2 - \kappa^2;$$

(3) *If the curve* α *with* $\kappa =$*constant* > 0, $\tau =$*constant* $\neq 0$ *and* $|\tau| < \kappa$, *then* α *is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa \cos\left(\sqrt{K}s\right), \kappa \sin\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s,\right), \ K = \kappa^2 - \tau^2;$$

From Lemma 3.1 in Choi et al 2012, we can write the following proposition.

Proposition 3.8 *There is no spacelike general helix of spacelike curve with a timelike binormal in Minkowski 3-space with condition* $|\tau| = |\kappa|$.

Example 3.9 Let $\alpha(s) = \frac{1}{3}\left(\sinh\left(\sqrt{3}s\right), 2\sqrt{3}s, \cosh\left(\sqrt{3}s\right)\right)$be a unit speed spacelike hyperbolic helix with

$$\begin{aligned} T &= \frac{\sqrt{3}}{3}\left(\cosh\left(\sqrt{3}s\right), 2, \sinh\left(\sqrt{3}s\right)\right) \\ N &= \left(\sinh\left(\sqrt{3}s\right), 0, \cosh\left(\sqrt{3}s\right)\right), \qquad \kappa = 1 \text{ and } \tau = 2 \\ B &= \frac{\sqrt{3}}{3}\left(2\cosh\left(\sqrt{3}s\right), 1, 2\sinh\left(\sqrt{3}s\right)\right) \end{aligned}$$

In this situation, spacelike with spacelike binormal Bertrand mate for can be given by the equation

$$\beta(s) = \left(\left(\lambda + \frac{1}{3}\right)\sinh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}s, \left(\lambda + \frac{1}{3}\right)\cosh\left(\sqrt{3}s\right)\right), \lambda \in \mathbb{R}$$

For $\lambda = \frac{\sqrt{7}-1}{3}$, we have

$$\beta(s) = \left(\frac{\sqrt{7}}{3}\sinh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}s, \frac{\sqrt{7}}{3}\cosh\left(\sqrt{3}s\right)\right).$$

The straight forward calculations give the following spherical indicatrices and natural lift curves of spherical indicatrices for β,

$$\begin{aligned}
\beta_{T^*} &= \frac{\sqrt{3}}{3}\left(\sqrt{7}\cosh\left(\sqrt{3}s\right), 2, \sqrt{7}\sinh\left(\sqrt{3}s\right)\right)\\
\beta_{N^*} &= \left(\sinh\left(\sqrt{3}s\right), 0, \cosh\left(\sqrt{3}s\right)\right)\\
\beta_{B^*} &= \frac{\sqrt{3}}{3}\left(-2\cosh\left(\sqrt{3}s\right), \frac{\sqrt{7}}{3}, -2\sinh\left(\sqrt{3}s\right)\right)
\end{aligned}$$

$$\begin{aligned}
\bar{\beta}_{T^*} &= \frac{\sqrt{3}}{3}\left(\sqrt{7}\sinh\left(\sqrt{3}s\right), 0, \sqrt{7}\cosh\left(\sqrt{3}s\right)\right)\\
\bar{\beta}_{N^*} &= \left(\cosh\left(\sqrt{3}s\right), 0, \sinh\left(\sqrt{3}s\right)\right)\\
\bar{\beta}_{B^*} &= -2\left(\sinh\left(\sqrt{3}s\right), 0, \cosh\left(\sqrt{3}s\right)\right)
\end{aligned}$$

respectively, (Figs. 1-4).

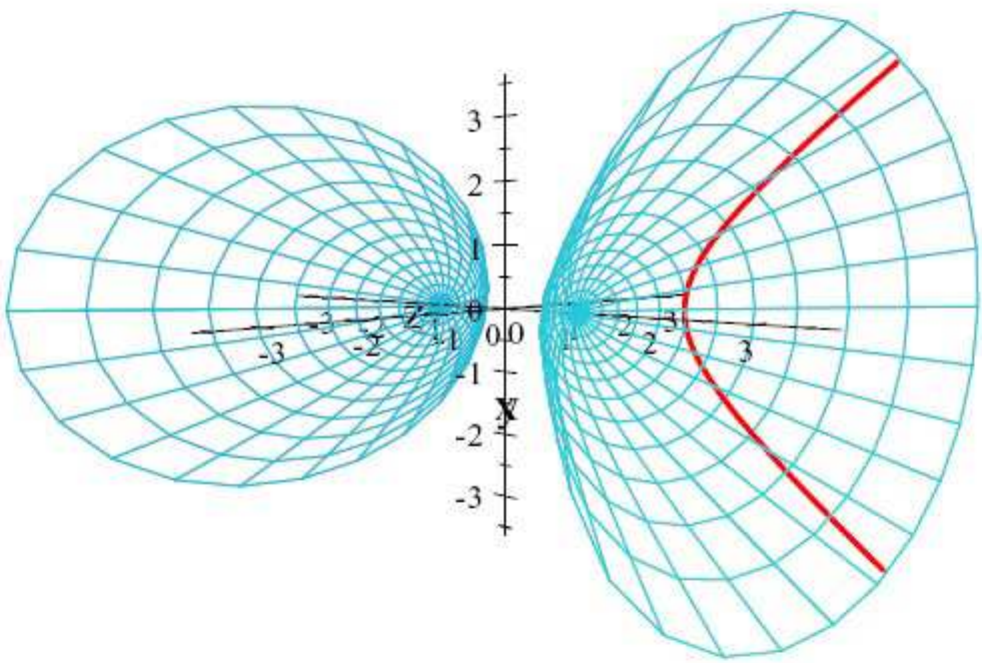

Figure 1. Tangent indicatrix β_{T^*} for Bertrand mate of α on H_0^2

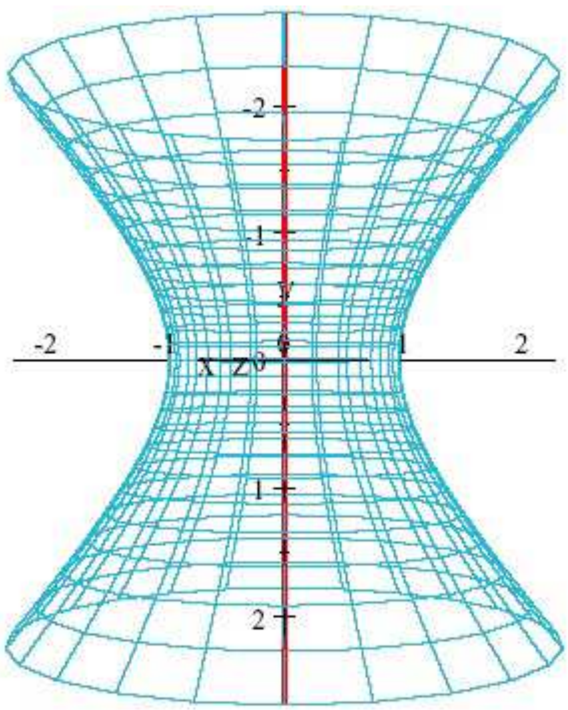

Figure 2. Principal norma indicatrix β_{N^*} for Bertrand mate of α on S_1^2

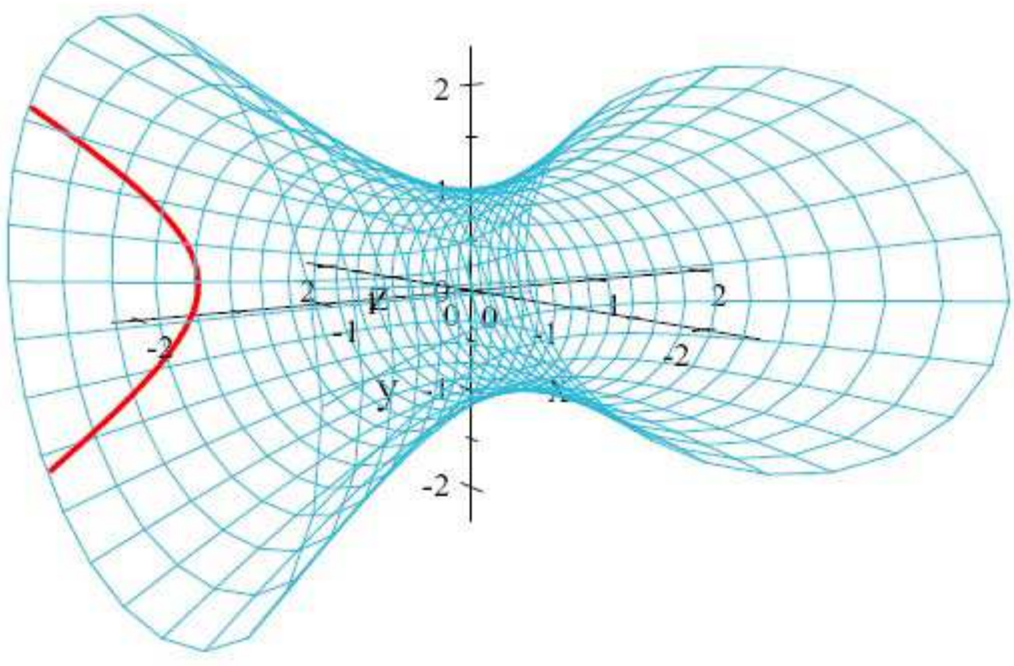

Figure 3. Binormal indicatrix β_{B^*} for Bertrand mate of α on S_1^2

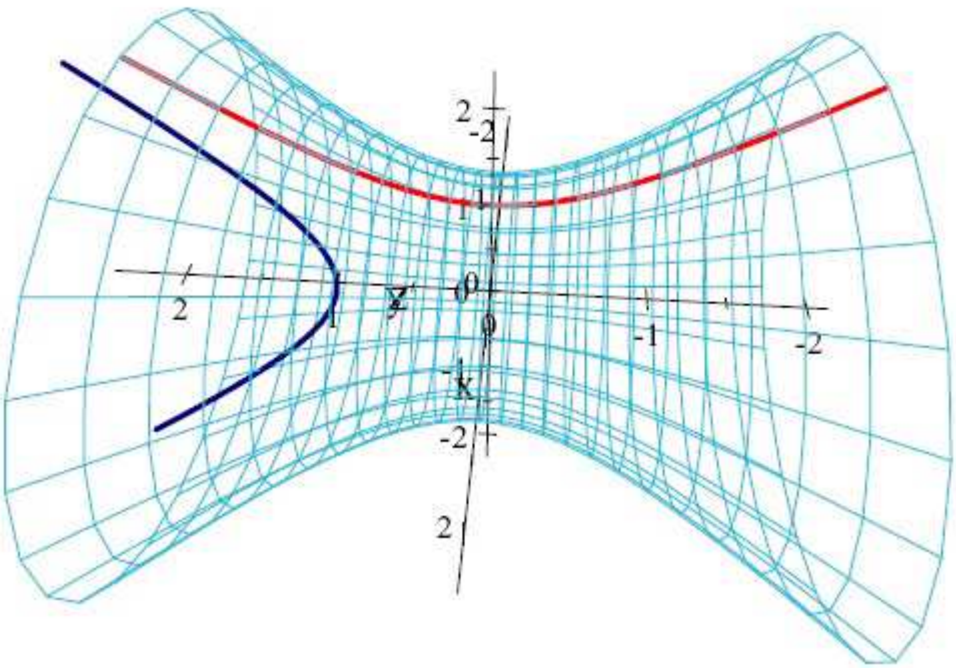

Figure 4. Principal norma indicatrix β_{N^*} and its natural lift curve $\bar{\beta}_{N^*}$ on S_1^2

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Totally Umbilical Hemislant Submanifolds of Lorentzian (α)-Sasakian Manifold

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Abstract: This paper is summarized as follows. In the first section we have given a brief history about slant and hemi-slant submanifold of Lorentzian (α)-Sasakian manifold. This section is followed by some preliminaries about Lorentzian (α)-Sasakian manifold. Finally, we have derived some interesting results on the existence of extrinsic sphere for totally umbilical hemi-slant submanifold of Lorentzian (α)-Sasakian manifold.

Key Words: Totally Umbilical, hemi-slant submanifold, extrinsic sphere.

AMS(2010): 53C25, 53C40, 53C42, 53D15

§1. Introduction

Chen in 1990 [2] initiated the study of slant submanifold of an almost Hermitian manifold as a natural generalization of both holomorphic and totally real submanifolds. After this many research papers on slant submanifolds appeared. The notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by A. Lotta in 1996 [5]. He studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-contact manifold. Further investigation regarding slant submanifolds of a Sasakian manifold [8] was done by Cabrerizo et al. in 2000. Khan et al. in 2010 defined and studied slant submanifolds in Lorentzian almost paracontact manifolds [14].

The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [9]. Recently, in 2009 totally umbilical slant submanifolds of Kaehler manifold was studied by B.Sahin. Later on, in 2011 Siraj Uddin et.al. studied totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold [21].

Our present note deals with a special kind of manifold i.e. Lorentzian (α)-Sasakian manifold. At first we give some introduction about the development of such manifold. An almost contact metric structure (ϕ,ξ,η,g) on $\tilde{M}$ is called a trans-Sasakian structure [17] if (MXR, J, G) belongs to the class W_4 [11], where J is the almost complex structure on (MXR) defined by

$$(J, X\frac{d}{dt}) = (\phi X - f, \eta(X)\frac{d}{dt})$$

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for all vector fields X on M and smooth functions f on $M \times R$, G is the product metric on MXR. This may be expressed by the condition

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + \eta(Y)X] + \beta[g(\phi X, Y) - \eta(Y)\phi X],$$

for some smooth functions α and β on M in [1], and we say that the trans-Sasakian structure is of type (α, β). A trans-Sasakian structure of type (α, β) is α-Sasakian, if $\beta = 0$ and α a nonzero constant [13]. If $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold. Also in 2008 and 2009 many scientists have extended the study to Lorentzian (α)-Sasakian manifold in [22], [18]. In this paper we have studied some special properties of totally umbilical hemislant submanifolds of Lorentzian (α)-Sasakian manifold.

§2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracontact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longmapsto \mathbf{R}$ is a non-degenerate inner product of signature $(-,+,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbf{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike if it satisfies $g_p(v,v) < 0$ [16]. Let $\tilde{M}$ be an n-dimensional differentiable manifold. An almost paracontact structure $(\phi, \xi, \eta, \tilde{g})$, where ϕ is a tensor of type $(1,1)$, ξ is a vector field, η is a 1-form and g is Lorentzian metric, satisfying following properties :

$$\phi^2 X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), g(X,\xi) = \eta(X). \tag{2.2}$$

for all vector fields X, Y on $\tilde{M}$. On $\tilde{M}$ if the following additional condition hold for any $X, Y \in T\tilde{M}$,

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + \eta(Y)X], \tag{2.3}$$

$$\tilde{\nabla}_X \xi = \alpha\phi X, \tag{2.4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, then $\tilde{M}$ is said to be an Lorentzian α-Sasakian manifold (Matsumoto, 1989 [15], [22]).

Let M be a submanifold of $\tilde{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g) with induced metric g and let ∇ is the induced connection on the tangent bundle TM and $\nabla^{\perp}$ is the induced connection on the normal bundle $T^{\perp}M$ of M.

The Gauss and Weingarten formulae are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for any $X, Y \in TM$, $N \in T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.7}$$

For any $X \in \Gamma(TM)$ we can write,

$$\phi X = TX + FX, \tag{2.8}$$

where TX is the tangential component and FX is the normal component of ϕX. Similarly for any $N \in \Gamma(T^{\perp}M)$ we can put

$$\phi V = tV + fV, \tag{2.9}$$

where tV denote the tangential component and fV denote the normal component of ϕV. The covariant derivatives of the tensor fields T and F are defined as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y \qquad \forall\ X, Y \in T\tilde{M}, \tag{2.10}$$

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \qquad \forall\ X, Y \in TM, \tag{2.11}$$

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \quad \forall\ X, Y \in TM. \tag{2.12}$$

From equation (2.3), (2.5), (2.8), (2.9), (2.11) and (2.12) we can calculate

$$(\tilde{\nabla}_X T)Y = \alpha[g(X,Y)\xi + \eta(Y)X] + A_{FY}X + th(X,Y), \tag{2.13}$$

$$(\tilde{\nabla}_X F)Y = -h(X, TY) + fh(X,Y). \tag{2.14}$$

A submanifold M is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if T is identically zero, i.e., $\phi X \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$.

A submanifold M of $\tilde{M}$ is called totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is denoted by $H = \sum_{i=1}^{k} h(e_i, e_i)$, where k is the dimension of M and $\{e_1, e_2, e_3, \cdots, e_k\}$ is the local orthonormal frame on M. A submanifold M is said to be totally geodesic if $h(X,Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

§3. Slant Submanifolds of a Lorentzian (alpha)-Sasakian Manifold

Here, we consider M as a proper slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$. We always consider such submanifold tangent to the structure vector field ξ.

Definition 3.1 *A submanifold M of $\tilde{M}$ is said to be slant submanifold if for any $x \in M$ and $X \in T_xM \setminus \xi$, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\tilde{M}$. If $\theta = 0$ the submanifold is invariant submanifold, if $\theta = \pi/2$*

then it is anti-invariant submanifold and if $\theta \neq 0, \pi/2$ then it is proper slant submanifold.

From [20] we have

Theorem 3.1 *Let M be a submanifold of an Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{3.1}$$

Again, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

From [20], for any X, Y tangent to M, we can easily draw the following results for an Lorentzian (α)-Sasakian manifold $\tilde{M}$,

$$g(TX, TY) = cos^2\theta\{g(X,Y) + \eta(X)\eta(Y)\}, \quad g(FX, FY) = sin^2\theta\{g(X,Y) + \eta(X)\eta(Y)\}.$$

Definition 3.2 *A submanifold M of $\tilde{M}$ is said to be hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ if there exists two orthogonal distribution D_1 and D_2 on M such that*

(*a*) $TM = D_1 \oplus D_2 \oplus < \xi >$;

(*b*) *The distribution D_1 is anti-invariant i.e., $\phi D_1 \subseteq T^\perp M$;*

(*c*) *The distribution D_2 is slant with slant angle $\theta \neq \pi/2$.*

If μ is invariant subspace under ϕ of the normal bundle $T^\perp M$, then in the case of hemi-slant submanifold, the normal bundle $T^\perp M$ decomposes as

$$T^\perp M = <\mu> \oplus \phi D^\perp \oplus FD_\theta.$$

The curvature tensor of an Lorentzian (α)-Sasakian manifold is defined as [4]

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.2}$$

For the curvature tensor we can compute by using the equations (2.10) and (3.2) the relation

$$\begin{aligned} \tilde{R}(X,Y)\phi Z &= \phi\tilde{R}(X,Y)Z + \alpha^2 g(Y,Z)\phi X - \alpha^2 g(X,Z)\phi Y \\ &\quad -\alpha^2 g([X,Y],Z)\phi X + \alpha g(X, \tilde{\nabla}_Y Z)\xi + \alpha\eta(\tilde{\nabla}_Y Z)X \\ &\quad -\alpha g(Y, \tilde{\nabla}_X Z)\xi - \alpha\eta(\tilde{\nabla}_X Z)Y - \alpha\eta(Z)\tilde{\nabla}_X Y \\ &\quad +\alpha\eta(Z)\tilde{\nabla}_Y X - \alpha\eta(Z)[X,Y] + \alpha g(\tilde{\nabla}_X Y, Z)\xi + \alpha g(\tilde{\nabla}_Y X, Z)\xi. \end{aligned} \tag{3.3}$$

Definition 3.3 *A submanifold of an arbitrary Lorentzian (α)-Sasakian manifold which is totally umbilical and has a nonzero parallel mean curvature vector [10] is called an Extrinsic sphere.*

§4. **Main Results**

This section mainly deals with a special class of hemi-slant submanifolds which are totally umbilical. Throughout this section we have considered M as a totally umbilical hemi-slant submanifold of Lorentzian (α)-Sasakian manifold. We derive the following.

Theorem 4.1 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that the mean curvature vector $H \in <\mu>$. Then one of the following is true:*

(i) M is totally geodesic;

(ii) M is semi-invariant submanifold.

Proof For $V \in \phi D^{\perp}$ and $X \in D_{\theta}$, we have from (2.3), (2.5),(2.6) and (2.10)

$$\alpha[g(X,V)\xi + \eta(V)X] = \nabla_X \phi V + g(X, \phi V)H + \phi A_V X - \phi \nabla^{\perp}_X V. \tag{4.1}$$

Since the distributions are orthogonal and from the assumption that $H \in \mu$, above equation can be written as

$$g(\nabla^{\perp}_X V, H) = g(V, \nabla^{\perp}_X H) = 0. \tag{4.2}$$

This implies $\nabla^{\perp}_X H \in \mu \oplus FD_{\theta}$. Now for any $X \in D_{\theta}$, we obtain on using the Gauss and Weingarten equations

$$\alpha[g(X,H)\xi + \eta(H)X] = \nabla^{\perp}_X \phi H - A_{\phi H}X + \phi A_H X - \phi \nabla^{\perp}_X H. \tag{4.3}$$

Now, using the assumption that , M is totally umbilical we have

$$\alpha\eta(H)X = \nabla^{\perp}_X \phi H - Xg(H, \phi H) + \phi X g(H,H) - \phi \nabla^{\perp}_X H. \tag{4.4}$$

On using equation (2.8) we calculate

$$\alpha\eta(H)X = \nabla^{\perp}_X \phi H + TXg(H,H) + FXg(H,H) - \phi \nabla^{\perp}_X H. \tag{4.5}$$

Taking inner product with $FX \in FD_{\theta}$,

$$\alpha\eta(H)g(X,FX) = g(\nabla^{\perp}_X \phi H, FX) + g(FX,FX)g(H,H) - g(\phi \nabla^{\perp}_X H, FX). \tag{4.6}$$

From Theorem 3.1 the equation becomes

$$\alpha\eta(H)g(X,FX) - g(\nabla^{\perp}_X \phi H, FX) - sin^2\theta ||H||^2 ||X||^2 + g(\phi \nabla^{\perp}_X H, FX) = 0. \tag{4.7}$$

If either $H \neq 0$ then $D_{\theta} = \{0\}$, i.e. M is totally real submanifold, and if $D_{\theta} \neq \{0\}$, M is totally geodesic submanifold or M is semi-invariant submanifold. For any $Z \in D^{\perp}$ from (2.13) we get

$$\nabla_Z TZ - T\nabla_Z Z = \alpha[g(Z,Z)\xi + \eta(Z)Z] + A_{FZ}Z + th(Z,Z). \tag{4.8}$$

Taking inner product with $W \in D^{\perp}$ the above equation takes the form

$$g(\nabla_Z TZ, W) - g(T\nabla_Z Z, W) = \alpha[g(Z,Z)g(\xi,W) + \eta(Z)g(Z,W)] + g(A_{FZ}Z, W) + g(th(Z,Z), W). \tag{4.9}$$

As M is totally umbilical hemi-slant submanifold and using (2.7) we can write

$$g(\nabla_Z TZ, W) - g(T\nabla_Z Z, Z) = \alpha g(Z,W)g(H,FZ) + g(tH,W)||Z||^2. \tag{4.10}$$

The above equation has a solution if either $H \in \mu$ or dim $D^{\perp} = 1$. □

If however, H does not belong to μ then we give the next theorem.

Theorem 4.2 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that the dimension of slant distribution $D_\theta \geq 4$ and F is parallel to the submanifold, then M is either extrinsic sphere or anti-invariant submanifold.*

Proof Since the dimension of slant distribution $D_\theta \geq 4$, therefore we can select a set of orthogonal vectors $X, Y \in D_\theta$, such that $g(X,Y) = 0$. Now by replacing Z by TY in (3.4) we have for any $X, Y, Z \in D_\theta$,

$$\begin{aligned}\tilde{R}(X,Y)\phi TY &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad -\alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad +\alpha g(X,\tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad -\alpha g(Y,\tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y.\end{aligned} \tag{4.11}$$

Now using equation (2.3) and (3.1) we obtain on calculation

$$\begin{aligned}\tilde{R}(X,Y)FTY + cos^2\theta\tilde{R}(X,Y)Y &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad -\alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad +\alpha g(X,\tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad -\alpha g(Y,\tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y.\end{aligned} \tag{4.12}$$

Again if F is parallel, then above equation can be written as

$$\begin{aligned}F\tilde{R}(X,Y)TY + cos^2\theta\tilde{R}(X,Y)Y &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad -\alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad +\alpha g(X,\tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad -\alpha g(Y,\tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y.\end{aligned} \tag{4.13}$$

Taking inner product with $N \in T^{\perp}M$, we obtain on using (3.3) and the orthogonality of X and Y vectors,

$$cos^2\theta||Y||^2 g(\nabla^{\perp}_X H, N) = 0$$

The above equation has a solution if either $\theta = \pi/2$ i.e. M is anti-invariant or $\nabla^{\perp}_X H = 0 \; \forall \; X \in D_\theta$. Similarly for any $X \in D^{\perp} \oplus < \xi >$ we can obtain $\nabla^{\perp}_X H = 0$, therefore $\nabla^{\perp}_X H = 0 \; \forall \; X \in TM$ i.e. the mean curvature vector H is parallel to submanifold, i.e., M is extrinsic sphere. Hence the theorem is proved. □

Now we are in a position to draw our main conclusions following.

Theorem 4.3 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian* (α)*-Sasakian manifold $\tilde{M}$. then M is either totally geodesic, or semi-invariant, or dim $D^{\perp} = 1$, or Extrinsic sphere, and the case (iv) holds if F is parallel and dim $M \geq 5$.*

Proof The proof follows immediately from Theorems 4.1 and 4.2. □

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On Translational Hull Of Completely $\mathcal{J}^{*,\sim}$-Simple Semigroups

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Abstract: In this paper, we give a construction theorem about the translational hull of completely $\mathcal{J}^{*,\sim}-$simple semigroups which extends the translational hulls of completely $\mathcal{J}^{*}-$simple semigroups and completely simple semigroups.

Key Words: Translational hull; completely $\mathcal{J}^{*,\sim}-$simple semigroup; construction.

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§1. Introduction

Let S be a semigroup. A mapping λ from a semigroup S to itself is a left translation of S if $\lambda(ab) = (\lambda a)b$ for all elements a, b of S; a mapping ρ from S to itself is a right translation of S if $(ab)\rho = a(b\rho)$ for all elements a, b of S. A left translation λ and right translation ρ are linked if $a(\lambda b) = (a\rho)b$ for all a, b of S, in this case, the pair (λ, ρ) is a bitranslation of S. The set $\Lambda(S)$ of all left translations of S and the set $P(S)$ of all right translations of S are semigroups under the composition of mappings. The translational hull of S is the subsemigroup $\Omega(S)$ of $\Lambda(S) \times P(S)$ of all bitranslations of S. A left translation λ is inner if $\lambda = \lambda_a$ for some $a \in S$, where $\lambda_a x = ax$ for all $x \in S$; an inner right translation ρ_a is defined dually; the pair $\pi_a = (\lambda_a, \rho_a)$ is an inner bitranslation and the set $\Pi(S)$ of all inner bitranslations is the inner part of $\Omega(S)$ (actually an ideal of $\Omega(S)$).

The translation hull of semigroups plays an important role in the algebraic theory of semigroups. It is an important tool in the study of ideal extensions. For more related details of translational hulls, the reader is referred to [1], [4], [5], [7],[14],[15]..

In order to generalize regular semigroups, new Green's relations, namely, the Green's $*$-relations on a semigroup have been introduced as follows ([11], [12]):

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb\},$$

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$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*,$$

$$(a,b) \in \mathcal{J}^* \Leftrightarrow J^*(a) = J^*(b),$$

where $J^*(a)$ and $J^*(b)$ are the principal $*-$ ideals generated by a and b respectively.

In [3], Fountain investigated a class of semigroups called abundant semigroups in which each $\mathcal{L}^*-$class and each $\mathcal{R}^*-$class of S contain at least an idempotent. And from which, we know that, the class of regular semigroups are properly contained in the one of abundant semigroups.

According to [3], a semigroup in which every idempotent is primitive is said to be a primitive semigroup, and an abundant semigroup S is called a completely $\mathcal{J}^*-$simple semigroup if S itself is primitive and the idempotents of S generate a regular subsemigroup of S. Clearly, completely $\mathcal{J}^*-$simple semigroups extend completely simple semigroups studied by Clifford and Petrich in [2].

Later on, Ren and Shum [16] investigated the structure of superabundant semigroups, and generalized the corresponding results of completely regular semigroups in [15].

On the other hand, in order to further generalize completely regular semigroups [superabundant semigroups] in the class of rpp semigroups, Guo, Shum and Gong [10] introduced the so-called $(*,\sim)$-Green's relations on a semigroup S. The relations $\mathcal{L}^{*,\sim}$ and $\mathcal{R}^{*,\sim}$ are respectively defined as $\mathcal{L}^*$ and $\mathcal{R}$. The intersection and the join of $\mathcal{L}^{*,\sim}$ and $\mathcal{R}^{*,\sim}$ are respectively denoted by $\mathcal{H}^{*,\sim}$ and $\mathcal{D}^{*,\sim}$. The relation $\mathcal{J}^{*,\sim}$ is defined by the rule that $a\mathcal{J}^{*,\sim}b$ if and only if $J^{*,\sim}(a) = J^{*,\sim}(b)$, Where, for any $a, b \in S$, $a\mathcal{R}b$ if and only if for all $e \in E(S), ea = a$ if and only if $eb = b$, and $J^{*,\sim}(a)$ is the smallest ideal containing a and saturated by $\mathcal{L}^{*,\sim}$ and $\mathcal{R}^{*,\sim}$.

According to [10], a semigroup S is called an r-ample semigroup if S is $\mathcal{L}^{*,\sim}$-abundant and $\mathcal{R}^{*,\sim}$-abundant, here we call that S is σ-abundant, if every equivalence σ-class of S contains idempotents of S. An r-ample semigroup is called a super-r-ample semigroup, if S is $\mathcal{H}^{*,\sim}$-abundant. The class of super-r-ample semigroups forms a proper extension class of the class of superabundant semigroups. It was shown in [10] that $\mathcal{R}^{*,\sim}$ usually is not a left congruence on S even if S is an$\mathcal{R}^{*,\sim}$-abundant semigroup, but in a super-r-ample semigroup S, the relation $\mathcal{R}^{*,\sim}$ is a left congruence on S.

In [9], the authors defined a class of completely $\mathcal{J}^{*,\sim}$-simple semigroups, and give the structure of such semigroups which extended the celebrated Rees theorem for completely simple semigroups. According to [9], a super-r-ample semigroup S is called a completely $\mathcal{J}^{*,\sim}$-simple semigroup if S is $\mathcal{J}^{*,\sim}$-simple. Clearly, a completely $\mathcal{J}^*$-simple semigroup must be completely $\mathcal{J}^{*,\sim}$-simple.

Note from [15] and [1] that, the translational hulls of completely simple semigroups and completely $\mathcal{J}^*-$simple semigroups have been solved, so naturally, we will quote such a question: what is the translational hull of completely $\mathcal{J}^{*,\sim}-$simple semigroups, do we have some similar results with the ones of completely $\mathcal{J}^{*,\sim}-$simple semigroups or completely simple semigroups?

In this paper, we will set out to discuss the above question, and finally establish a construction theorem about the translational hull of completely $\mathcal{J}^{*,\sim}-$simple semigroups which extend the translational hulls of completely $\mathcal{J}^*-$simple semigroups and completely simple semigroups.

For notations and terminologies not mentioned in this paper, the readers are referred to [5],[9] or [10].

§2. Main Results

Definition 2.1 ([6], Definition 1) *Let $\mathcal{M}[T; I, \Lambda; P]$ be a Rees matrix semigroup and P the $\Lambda \times I$ matrix over a left cancellative monoid T. Then P is said to be normalized at 1 if there is an element $1 \in I \cap \Lambda$ such that $p_{1i} = p_{\lambda 1} = e$ for all $i \in I$, $\lambda \in \Lambda$, where e is the identity of the left cancellative monoid T. Furthermore, the Rees matrix semigroup $\mathcal{M}[T; I, \Lambda; P]$ is called normalized if P is normalized.*

Lemma 2.2 ([6], Theorem 1) *Let T be a left cancellative monoid with an identity element e and I, Λ be nonempty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix where each entry in P is a unit of T. Suppose that P is normalized at $1 \in I \cap \Lambda$. Then the normalized Rees matrix semigroup $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ is completely $\mathcal{J}^{*,\sim}-$simple semigroup.*

Conversely, every completely $\mathcal{J}^{*,\sim}-$simple semigroup is isomorphic to a normalized Rees matrix semigroup $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ over a left cancellative monoid T.

By Lemma 2.2, we know that if S is a completely $\mathcal{J}^{*,\sim}-$ simple semigroup, then it can be isomorphic to a normalized Rees matrix semigroup $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ over a left cancellative monoid T. Hence, to discuss the translational hulls of completely$\mathcal{J}^{*,\sim}-$simple semigroups, we can also consider the cases of normalized Rees matrix semigroups $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ over a left cancellative monoid T for convenience.

In the following, we will establish the translational hull of a normalized Rees matrix semigroup $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ over a left cancellative monoid T. Before we give our main result, it will be useful to make use of the following notation.

Notation We set $\mathcal{M}[T; I, \Lambda; P]$ with P normalized at $1 \in I \times \Lambda$ and denoted by e the identity of T. Let $T(S) = \{(F, t, \Phi) \in \mathcal{T}^{'}(I) \times T \times \mathcal{T}(\Lambda) |$ for all $i \in I$, $\mu \in \Lambda$, $p_{\mu,Fi}tp_{1\Phi,i} = p_{\mu,F1}tp_{\mu\Phi,i}\}$ with multiplication $(F, t, \Phi)(F^{'}, t^{'}, \Phi^{'}) = (FF^{'}, tp_{1\Phi,F^{'}1}t^{'}, \Phi\Phi^{'})$ for all $i \in I$ and $\lambda \in \Lambda$, where $\mathcal{T}^{'}(I)$ ($\mathcal{T}(\Lambda)$) means the semigroup of all full transformations on I (Λ) and all of the transformations are written on the left (right).

Theorem 2.3 *Let $S = \mathcal{M}[T; I, \Lambda; P]$ with P normalized, each entry in P is a unit of T,and let e be the identity of T. Define a mapping σ from $\Omega(S)$ to $T(S)$ by*

$$\sigma : (\lambda, \rho) \to (F, t, \Phi) \quad ((\lambda, \rho) \in \Omega(S))$$

where F, t and Φ are defined by the requirements

$$\lambda(i, e, 1) = (Fi, \cdots, \cdots) \in S, \qquad (1)$$

$$(1, e, 1)\rho = (\cdots, t, \cdots) \in S, \qquad (2)$$

$$(1, e, \mu)\rho = (\cdots, \cdots, \mu\Phi) \in S. \qquad (3)$$

Further define a mapping τ *from* $T(S)$ *to* $\Omega(S)$ *by*

$$\tau : (F, t, \Phi) \to (\lambda, \rho) \quad ((F, t, \Phi) \in T(S)),$$

where λ *and* ρ *are defined by the formulas*

$$\lambda(i, h, \mu) = (Fi, tp_{1\Phi,i}h, \mu) \quad ((i, h, \mu) \in S), \quad (4)$$

$$(i, h, \mu)\rho = (i, hp_{\mu,F1}t, \mu\Phi) \quad ((i, h, \mu) \in S). \quad (5)$$

Then σ *and* τ *are mutually inverse isomorphisms between* $\Omega(S)$ *and* $T(S)$.

Proof We will show the theorem by the following steps.

(i) σ is a mapping.

Let $(\lambda, \rho) \in \Omega(S)$. For any $(i, h, \mu) \in S$, we have

$$\lambda(i, h, \mu) = \lambda[(i, h, 1)(1, e, \mu)] = [\lambda(i, h, 1)](1, e, \mu),$$

so that $\lambda(i, h, \mu) = (j, h', \mu)$ for some $j \in I$ and $h' \in T$. Similarly, we have $(i, h, \mu)\rho = (i, h'', \nu)$ for some $h'' \in T$ and $\nu \in \Lambda$. In the following, we will use the above statements repeatedly. In particular, we may define s_i and r_μ by

$$\lambda(i, e, 1) = (Fi, s_i, 1) \quad (i \in I),$$

$$(1, e, \mu)\rho = (1, r_\mu, \mu\Phi) \quad (\lambda \in \Lambda).$$

By the definition of t in this theorem, we have $t = r_1$. Also, notice that

$$[(1, e, 1)\rho](i, e, 1) = (1, t, 1\Phi)(i, e, 1) = (1, tp_{1\Phi,i}, 1),$$

$$(1, e, 1)[\lambda(i, e, 1)] = (1, e, 1)(Fi, s_i, 1) = (1, s_i, 1),$$

we have $s_i = tp_{1\Phi,i}$. Thus,

$$\begin{aligned} \lambda(i, h, \mu) &= \lambda[(i, e, 1)(1, h, \mu)] = [\lambda(i, e, 1)](1, h, \mu) \\ &= (Fi, s_i, 1)(1, h, \mu) = (Fi, tp_{1\Phi,i}, 1)(1, h, \mu) \\ &= (Fi, tp_{1\Phi,i}h, \mu). \end{aligned}$$

This proves (4). With a similar argument, we can establish (5). Hence,

$$(1, e, \mu)[\lambda(i, e, 1)] = (1, e, \mu)(Fi, tp_{1\Phi,i}, 1) = (1, p_{\mu,Fi}tp_{1\Phi,i}, 1),$$

$$[(1, e, \mu)\rho](i, e, 1) = (1, p_{\mu,F1}, \mu\Phi)(i, e, 1) = (1, p_{\mu,F1}tp_{\mu\Phi,i}, 1).$$

Since $(\lambda, \rho) \in \Omega(S)$, we have $p_{\mu,Fi}tp_{1\Phi,i} = p_{\mu,F1}tp_{\mu\Phi,i}$, and then $(F,t,\Phi) \in T(S)$, and (i) holds.

(ii) τ is a mapping.

Let $(F,t,\Phi) \in T(S)$, and let λ and ρ be defined as (4) and (5) respectively. Then for $(i,h,\mu),(j,k,\nu) \in S$, we have

$$\begin{aligned}[\lambda(i,h,\mu)](j,k,\nu) &= (Fi, tp_{1\Phi,i}h, \mu)(j,k,\nu) = (Fi, tp_{1\Phi,i}hp_{\mu j}k, \nu) \\ &= \lambda(i, hp_{\mu j}k, \nu) = \lambda[(i,h,\mu)(j,k,\nu)].\end{aligned}$$

Hence, λ is a left translation. Similarly, we can show that ρ is a right translation. Further, on the one hand,

$$(i,h,\mu)[\lambda(j,k,\nu)] = (i,h,\mu)(Fj, tp_{1\Phi,j}k, \nu) = (i, hp_{\mu,Fj}tp_{1\Phi,j}k, \nu), \quad (6)$$

on the other hand,

$$[(i,h,\mu)\rho](j,k,\nu) = (i, hp_{\mu,F1}t, \mu\Phi)(j,k,\nu) = (i, hp_{\mu,F1}tp_{\mu\Phi,j}k, \nu), \quad (7)$$

and notice that $(F,t,\Phi) \in T(S)$, we can immediately obtain that (6) and (7) are equal. And then $(\lambda,\rho) \in \Omega(S)$. Thus, τ is a mapping from $T(S)$ to $\Omega(S)$, and (ii) holds.

(iii) $\sigma\tau$ is an identity mapping on $\Omega(S)$.

Let $(\lambda,\rho) \in \Omega(S)$, and let $(\lambda,\rho)\sigma\tau = (F,t,\Phi)\tau = (\lambda^{'}, \rho^{'})$ so that $\lambda^{'}(i,h,\mu) = (Fi, tp_{1\Phi,i}h, \mu)$. By the proof of (i), we know (4) holds for λ,thus, we have $\lambda = \lambda^{'}$. Similarly, we have $\rho = \rho^{'}$. Hence, (iii) holds.

(iv) $\tau\sigma$ is an identity mapping on $T(S)$.

Let $(F,t,\Phi) \in T(S)$, and let $(F,t,\Phi)\tau\sigma = (\lambda,\rho)\sigma = (F^{'}, t^{'}, \Phi^{'})$. Then (4) and (5) are satisfied, and thus $\lambda(i,e,1) = (Fi, tp_{1\Phi,1}, 1)$, $(1,e,\mu)\rho = (1, p_{\mu,F1}t, \mu\Phi)$. By (1),(2) and (3), we immediately obtain that $F = F^{'}, t = t^{'}$, and $\Phi = \Phi^{'}$. Consequently, $\tau\sigma$ is the identity mapping on $T(S)$.

(v) τ is a homomorphism.

Let $(F,t,\Phi)\tau = (\lambda,\rho)$, $(F^{'}, t^{'}, \Phi^{'})\tau = (\lambda^{'}, \rho^{'})$, and $(FF^{'}, tp_{1\Phi,F^{'}1}t^{'}, \Phi\Phi^{'})\tau = (\xi,\eta)$. On the one hand, we have

$$\lambda\lambda^{'}(i,h,\mu) = \lambda(F^{'}i, t^{'}p_{1\Phi^{'},i}h, \mu) = (FF^{'}i, tp_{1\Phi,F^{'}i}t^{'}p_{1\Phi^{'},i}h, \mu). \quad (8)$$

On the other hand ,

$$\xi(i,h,\mu) = (FF^{'}i, tp_{1\Phi,F^{'}1}t^{'}p_{1\Phi\Phi^{'},i}h, \mu). \quad (9)$$

Since $(F^{'}, t^{'}, \Phi^{'}) \in T(S)$, we have $p_{1\Phi,F^{'}i}t^{'}p_{1\Phi^{'},i} = p_{1\Phi,F^{'}1}t^{'}p_{1\Phi\Phi^{'},i}$, and then (8) and (9) are equal. That is, $\lambda\lambda^{'} = \xi$. Similarly, we can prove that $\rho\rho^{'} = \eta$. Therefore, τ is a homomorphism.

(vi) Analogous with the proof of (v), we can prove that σ is a homomorphism.

Summing up the six steps above, we have shown that both σ and τ are isomorphisms. □

Remark 2.4 From Theorem 2.3, we know that, under the isomorphism, the translational hull of a normalized Rees matrix semigroup $\mathcal{M} = \mathcal{M}[T; I, \Lambda; P]$ over a left cancellative monoid T can regard as the semigroup $T(S)$, whose elements and multiplications are defined in the Notation. And then by Lemma 2.2, the translational hull of a completely $\mathcal{J}^{*,\sim}$-simple semigroup can be also regarded as this form up to isomorphism.

Further, from Remark 1 in [9], we know that if S is an abundant semigroup, then $\mathcal{R}^{*,\sim}=\mathcal{R}^*$. Hence S is a completely $\mathcal{J}^{*,\sim}$ -simple semigroup if and only if S is a completely $\mathcal{J}^*$-simple semigroup; S is a left cancellative monoid if and only if S is a cancellative monoid. If S is a regular semigroup, then $\mathcal{R}^{*,\sim}=\mathcal{R}^*$, $\mathcal{L}^{*,\sim}=\mathcal{L}^*$. Hence S is a completely $\mathcal{J}^{*,\sim}$-simple semigroup if and only if S is a completely simple semigroup; S is a left cancellative monoid if and only if S is a group.

Now, if we let left cancellative monoid T be a cancellative monoid in Theorem 2.3, then we can immediately get the translational hull of a completely $\mathcal{J}^*$–simple semigroup which is the main theorem in [1].

Corollary 2.5 *Let $S = \mathcal{M}[T; I, \Lambda; P]$ with P normalized, each entry in P is a unit of cancellative monoid T,and let e be the identity of T. Define a mapping σ by*

$$\sigma : (\lambda, \rho) \to (F, t, \Phi) \quad ((\lambda, \rho) \in \Omega(S))$$

where F, t and Φ are defined by the requirements

$$\lambda(i, e, 1) = (Fi, \cdots, \cdots) \in S, \qquad (1)$$

$$(1, e, 1)\rho = (\cdots, t, \cdots) \in S, \qquad (2)$$

$$(1, e, \mu)\rho = (\cdots, \cdots, \mu\Phi) \in S. \qquad (3)$$

Further define a mapping τ by

$$\tau : (F, t, \Phi) \to (\lambda, \rho) \quad ((F, t, \Phi) \in T(S)),$$

where λ and ρ are defined by the formulas

$$\lambda(i, h, \mu) = (Fi, tp_{1\Phi,i}h, \mu) \quad ((i, h, \mu) \in S), \qquad (4)$$

$$(i, h, \mu)\rho = (i, hp_{\mu,F1}t, \mu\Phi) \quad ((i, h, \mu) \in S). \qquad (5)$$

Then σ and τ are mutually inverse isomorphisms between $\Omega(S)$ and $T(S)$.

Also, if we let T be a group G in Theorem 2.3, then we can immediately get the translational hull of a completely simple semigroup which is the Theorem III.7.2 in [15].

Corollary 2.6 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ with P normalized, and let e be the identity of group G. Define a mapping σ by*

$$\sigma : (\lambda, \rho) \to (F, g, \Phi) \quad (\lambda, \rho) \in \Omega(S))$$

where F, g and Φ are defined by the requirements

$$\lambda(i, e, 1) = (Fi, \cdots, \cdots) \qquad (1)$$

$$(1, e, 1)\rho = (\cdots, g, \cdots) \qquad (2)$$

$$(1, e, \mu)\rho = (\cdots, \cdots, \mu\Phi) \qquad (3)$$

Further define a mapping τ by

$$\tau : (F, g, \Phi) \to (\lambda, \rho) \quad ((F, g, \Phi) \in T(S)),$$

where λ and ρ are defined by the formulas

$$\lambda(i, h, \mu) = (Fi, gp_{1\Phi,i}h, \mu) \quad (i, h, \mu) \in S), \quad (4)$$

$$(i, h, \mu)\rho = (i, hp_{\mu,F1}g, \mu\Phi), \quad (i, h, \mu) \in S). \quad (5)$$

Then σ and τ are mutually inverse isomorphisms between $\Omega(S)$ and $T(S)$.

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Some Minimal $(r, 2, k)$-Regular Graphs Containing a Given Graph and its Complement

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Abstract: A graph G is called $(r, 2, k)$-regular graph if each vertex of G is at a distance 1 away from r vertices of G and each vertex of G is at a distance 2 away from k vertices of G [9]. This paper suggest a method to construct a $((m+2(n-1)), 2, (m-1)(2n-1)))$-regular graph H_4 of smallest order $2mn$ containing a given graph G of order $n \geq 2$, and its complement G^c as induced subgraphs, for any $m > 1$. Also, in this paper we calculate the topological indices Wiener index W, hyper Wiener index WW, degree distance DD, variance of degrees, first, second and third Zagreb indexes of the graphs H_4 which we constructed in this paper.

Key Words: Induced subgraph; clique number; independent number; (d, k)-regular graphs; $(2, k)$-regular graphs;$(r, 2, k)$-regular graphs; semiregular.

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§1. Introduction

In this paper, we consider only finite, simple, connected graphs. For basic definitions and terminologies we refer Harary [7] and J.A.Bondy and U.S.R.Murty [4]. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively. The degree of a vertex v is the number of edges incident at v. A graph G is regular if all its vertices have the same degree.

For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest (u, v) path. Therefore, the degree of a vertex v is the number of vertices at a distance 1 from v, and it is denoted by $d(v)$. This observation suggests a generalization of degree. That is, $d_d(v)$ is defined as the number of vertices at a distance d from v. Hence $d_1(v) = d(v)$ and $N_d(v)$ denote the set of all vertices that are at a distance d away from v in a graph G. That is, $N_1(v) = N(v)$ and $N_2(v)$ denotes the set of all vertices that are at a distance 2 away from v in a graph G and closed neighbourhood $N[v] = N(v) \cup \{v\}$.

The concept of distance d-regular graph was introduced and studied by G.S. Bloom, J.K. Kennedy and L.V.Quintas [3]. A graph G is said to be distance d-regular if every vertex of G has the same number of vertices at a distance d from it. A graph G is said to be (d, k)-regular

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graph if $d_d(v) = k$, for all v in $V(G)$. A graph G is $(2, k)$ regular if $d_2(v) = k$, for all v in $V(G)$. The concept of the semiregular graph was introduced and studied by Alison Northup [2]. We observe that $(2, k)$ - regular graph and k - semiregular graph are the same. A graph G is said to be $(r, 2, k)$-regular if $d(v) = r$ and $d_2(v) = k$, for all $v \in V(G)$.

An induced subgraph of G is a subgraph H of G such that $E(H)$ consists of all edges of G whose end points belong to $V(H)$.In 1936, Konig [8] proved that if G is any graph, whose largest degree is r, then there is an r-regular graph H containing G as an induced subgraph. In 1963, Paul Erdos and Paul Kelly [6] determined the smallest number of new vertices which must be added to a given graph G to obtain such a graph. We now suggest a method that may be considered an analogue to Konig's theorem for $(r, 2, k)$-regular graph.

With this motivation, already we have constructed a $(m+n-2), 2, (m-1)(n-1))$-regular graph S of order mn containing a given graph G of order $n \geq 2$ as an induced subgraph, for any $m > 1$ [12]. In this paper, our main objective is to construct a $((m+2(n-1)), 2, (m-1)(2n-1))$-regular graph of smallest order $2mn$ containing the given graph G of order $n \geq 2$, and its complement G^c as induced subgraphs, for any $m > 1$.

§2. $(r, 2, k)$-Regular Graph

Definition 2.1 *A graph G is called $(r, 2, k)$-regular if each vertex in graph G is at a distance one from exactly r-vertices and at a distance two from exactly k vertices. That is, $d(v) = r$ and $d_2(v) = k$, for all v in G.*

Example 2.2 A few $(r, 2, k)$-regular graphs are listed following.

(1) The Peterson graph is a (3, 2, 6)-regular graph .

(2) A complete bipartite graph $K_{n,n}$ is a $(n, 2, (n-1))$-regular graph.

Observation 2.3 For any $n \geq 1$, the smallest order of $(n, 2, (n-1))$- regular graph containing the complete bipartite graph $K_{n,n}$ of order $2n$ is $K_{n,n}$ itself.

The following facts can be verified easily.

Observation 2.4([9]) If G is $(r, 2, k)$-regular graph, then $0 \leq k \leq r(r-1)$.

Observation 2.5([10]) For any $r > 1$, a graph G is $(r, 2, r(r-1))$-regular if G is r-regular with girth at least five.

Observation 2.6([11]) For any odd $r \geq 3$, there is no $(r, 2, 1)$-regular graph.

Observation 2.7([11]) Any $(r, 2, k)$-regular graph has at least $k + r + 1$ vertices.

Observation 2.8([11]) If r and k are odd, then $(r, 2, k)$-regular graph has at least $k + r + 2$ vertices.

Observation 2.9([12]) For any $m \geq 1$, every graph G of order $n \geq 2$ is an induced subgraph of $(n + m - 1, 2, (mn - 1))$-regular graph H of order $2mn$.

Observation 2.10([13]) For any $m > 1$, every graph G of order $n \geq 2$ is an induced subgraph of $(n + m - 2, 2, (m-1)(n-1))$-regular graph H of order mn.

§3. Minimal $(r,2,k)$-Regular Graphs Containing Given Graph and Its Complement as an Induced Subgraph

In this section we construct a smallest $(r,2,k)$-regular graphs containing given graph and its complement as an induced subgraph.

Theorem 3.1 *For a graph G of order $n \geq 2$, there exists a $(m+2(n-1),2,(m-1)(2n-1))$-regular graph H_4 of order $2mn$ such that G and G^c are the induced subgraphs of H_4.*

Proof Let G be a graph of order $n \geq 2$, G and G^c has the same vertex set $\{v_i^1 : 1 \leq i \leq n\}$. Take a graph G' which is isomorphic to G^c. The vertex set of G' is denoted as $\{u_i^1 : 1 \leq i \leq n\}$ and u_i^1 corresponds to $v_i^1 (1 \leq i \leq n)$. Let $G_1 = G \cup G'$. Then $V(G_1) = \{v_i^1, u_i^1 : 1 \leq i \leq n\}$. Let $G_t (2 \leq t \leq m)$ be the $(m-1)$ copies of G_1 with the vertex set $V(G_t) = \{v_i^t, u_i^t : 1 \leq i \leq n\}$, for $(2 \leq t \leq m)$ and $v_i^t, u_i^t (2 \leq t \leq m)$ correspond to $v_i^1, u_i^1 (1 \leq i \leq n)$ respectively. The desired graph H_4 has the vertex set $V(H_4) = \bigcup\limits_{t=1}^{m} V(G_t)$, and edge set $E(H_4) = \bigcup\limits_{t=1}^{m} E(G_t) \bigcup E_1 \bigcup E_2 \bigcup E_3 \bigcup E_4 \bigcup E_5$, where,

$$E_1 = \bigcup_{t=1}^{m-1} \{v_j^t v_i^{t+1}, v_j^m v_i^1 : v_j^1 v_i^1 \notin E(G_1)(1 \leq j \leq n), (j+1 \leq i \leq n)\},$$

$$E_2 = \bigcup_{k=1}^{n} \{v_k^i v_k^{i+j} : (1 \leq i \leq m-1), (1 \leq j \leq m-i)\},$$

$$E_3 = \bigcup_{t=1}^{m-1} \{u_j^t u_i^{t+1}, u_j^m u_i^1 : u_j^1 u_i^1 \notin E(G_1)(1 \leq j \leq n), (j+1 \leq i \leq n)\},$$

$$E_4 = \bigcup_{k=1}^{n} \{u_k^i u_k^{i+j} : (1 \leq i \leq m-1), (1 \leq j \leq m-i)\},$$

$$E_5 = \bigcup_{t=1}^{m-1} \{v_j^t u_i^{t+1}, v_j^m u_i^1 : (1 \leq i, j \leq n)\}.$$

The resulting graph H_4 contains G_1 as an induced subgraph. More over in $H_4, (1 \leq t \leq m), d(v_i^t) = m + 2(n-1)$, for $(1 \leq i \leq n)$. Then H_4 is $m + 2(n-1)$, regular graph with $2mn$ vertices. Hence H_4 contains G and G^c as induced subgraphs. In $H_4, d(v_i) = d(v_i^1) = d(u_i) = d(u_i^1) = m + 2n - 2$, $1 \leq i \leq n$. To find the d_2 degree of each vertex in H_4, the following cases are examined.

Case 1. $t = 1$. If $v \in V(G_1)$, then $v \in V(G)$ (or) $v \in V(G')$.

Subcase 1.1 If $v \in V(G)$, then $v = v_j^1$, for some j. Let $v_j^1 \in V(H_4) - N[v_i^1]$. Then v_j^1 and v_i^1 are non-adjacent vertices in H_4. By our construction, v_j^1 is adjacent to v_i^2 and v_i^2 is adjacent to v_i^1. Then $d(v_j^1, v_i^1) = 2$. Hence $v_j^1 \in N_2(v_i^1)$. This implies that $V(H_4) - N[v_i^1] \subseteq N_2(v_i^1)$. If $v_j^1 \in N_2(v_i^1)$, then v_j^1 is non-adjacent with v_i^1. This implies that $v_j^1 \in V(H_4) - N[v_i^1]$. Hence $N_2(v_i^1) = V(H_4) - N[v_i^1]$, $(1 \leq i \leq n)$ and $d_2(v_i^1) = (m-1)(2n-1), (1 \leq i \leq n)$.

Subcase 1.2 If $v \in V(G')$, then $v = u_j^1$, for some j.

Let $u_j^1 \in V(H_4) - N[u_i^1]$. Then, u_j^1 and u_i^1 are non-adjacent vertices in H_4. By our construction, u_j^1 is adjacent to u_i^2 and u_i^2 is adjacent to u_i^1. Then $d(u_j^1, u_i^1) = 2$. Hence $u_j^1 \in N_2(u_i^1)$. This implies that $V(H_4) - N[u_i^1] \subseteq N_2(u_i^1)$. if $u_j^1 \in N_2(u_i^1)$, then u_j^1 is non-adjacent with u_i^1. Hence $u_j^1 \in V(H_4) - N[u_i^1]$. This implies that $N_2(u_i^1) = V(H_4) - N[u_i^1]$, $(1 \leq i \leq n)$ and

$d_2(u_i^1) = (m-1)(2n-1), (1 \le i \le n)$.

Case 2. $2 \le t \le m-1$. If $v \in V(G_t)$, then $v = v_j^t$ (or) $v = u_j^t$, for some j.

Subcase 2.1 If $v = v_j^t$ and if $v_j^t \in V(H_4) - N[v_i^1]$, then v_j^t and v_i^1 are non-adjacent vertices in H_4. By our construction, v_j^t is adjacent to v_i^t and v_i^t is adjacent to v_i^1. Then $d(v_j^t, v_i^1) = 2$. Hence $v_j^t \in N_2(v_i^1)$. This implies that $V(H_4) - N[v_i^1] \subseteq N_2(v_i^1)$. If $v_j^t \in N_2(v_i^1)$, then, v_j^t is non-adjacent with v_i^1. This implies that $v_j^t \in V(H_4) - N[v_i^1]$. Hence $N_2(v_i^1) = V(H_4) - N[v_i^1]$, $(1 \le i \le n)$ and $d_2(v_i^1) = (m-1)(2n-1), (1 \le i \le n)$.

Subcase 2.2 If $v = u_j^t$ and if $u_j^t \in V(H_4) - N[u_i^1]$, then u_j^t and u_i^1 are non-adjacent vertices in H_4. By our construction, u_j^t is adjacent to u_i^{t+1} and u_i^{t+1} is adjacent to u_i^1. Then $d(u_j^t, u_i^1) = 2$. Hence $u_j^t \in N_2(u_i^1)$. This implies that $V(H_4) - N[u_i^1] \subseteq N_2(u_i^1)$. If $u_j^t \in N_2(u_i^1)$, then, u_j^t is non-adjacent with u_i^1. Hence $u_j^t \in V(H_4) - N[u_i^1]$. This implies that $N_2(u_i^1) = V(H_4) - N[u_i^1]$, $(1 \le i \le n)$ and $d_2(u_i^1) = (m-1)(2n-1), (1 \le i \le n)$.

Case 3. $t = m$. If $v \in V(G_m)$, then $v = v_j^m$ (or) $v = u_j^m$ for some j.

Subcase 3.1 If $v = v_j^m$ and if $v_j^m \in V(H_4) - N[v_i^1]$, then v_j^m and v_i^1 are non-adjacent vertices in H_4. By our construction, v_j^m is adjacent to v_i^m and v_i^m is adjacent to v_i^1. Then $d(v_j^m, v_i^1) = 2$. Hence $v_j^m \in N_2(v_i^1)$. This implies that $V(H_4) - N[v_i^1] \subseteq N_2(v_i^1)$. If $v_j^m \in N_2(v_i^1)$, then v_j^m is non-adjacent with v_i^1. Hence $v_i^m \in V(H_4) - N[v_i^1]$. This implies that $N_2(v_i^1) = V(H_4) - N[v_i^1]$, $1 \le i \le n$ and $d_2(v_i^1) = (m-1)(2n-1), (1 \le i \le n)$.

Subcase 3.2 If $v = u_j^m$ and if $u_j^m \in V(H_4) - N[u_i^1]$, then u_j^m and u_i^1 are non-adjacent vertices in H_4. By our construction, u_j^m is adjacent to u_i^m and u_i^m is adjacent to u_i^1. Then $d(u_j^m, u_i^1) = 2$. Hence $u_j^m \in N_2(v_i)$. This implies that $V(H_4) - N[u_i] \subseteq N_2(u_i)$. If $u_j^m \in N_2(u_i^1)$, then u_j^m is non-adjacent with u_i^1. Hence $u_j^m \in V(H_4) - N[u_i^1]$. This implies that $N_2(u_i^1) = V(H_4) - N[u_i^1]$, $1 \le i \le n$ and $d_2(u_i^1) = (m-1)(2n-1), (1 \le i \le n)$. Similarly for $(1 \le t \le m) d_2(v_i^t) = d_2(u_i^t) = (m-1)(2n-1), (1 \le i \le n)$. H_4 is $(m+2(n-1), 2, (m-1)(2n-1))$-regular graph of order $2mn$ containing a given graph G of order $n \ge 2$ and its complement as induced subgraphs. □

Corollary 3.2 *For any $m \ge 1$, the smallest order of $(m+2(n-1), 2, (m-1)(2n-1))$-regular graph containing a given graph of order $n \ge 2$ and its complement is $2mn$.*

Proof For the graph H_4 constructed in Theorem 3.1 is $(m+2(n-1), 2, (m-1)(2n-1))$-regular graph of order $2mn$. Suppose H_4 is $(m+2(n-1), 2, (m-1)(2n-1))$-regular graph of order $2mn-1$. Then, for each $v_i \in H_4$, $d_2(v_i) = (m-1)(2n-1)$ and $d(v_i) = m+2(n-1)$. Hence H_4 has at least $((m-1)(2n-1) + (m+2(n-1)+1) = 2mn$ vertices, a contradiction.□

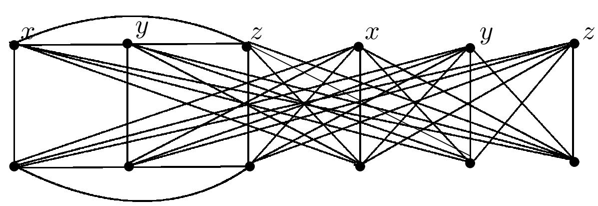

Figure 1

Corollary 3.3 *Every graph G of order $n \ge 2$, and its complement G^c are the induced sub-graphs* *of* $(2n, 2, (2n-1))$*-regular graph of smallest order* $4n$.

In Figure 1, Corollary 3.3 is illustrated for $G = K_3$, in which the graph G is induced by the vertices x, y, z.

Corollary 3.4 *Every graph* G *of order* $n \geq 2$, *and its complement* G^c *are the induced subgraphs of* $(2n+1, 2, 2(2n-1))$*-regular graph of smallest order* $6n$.

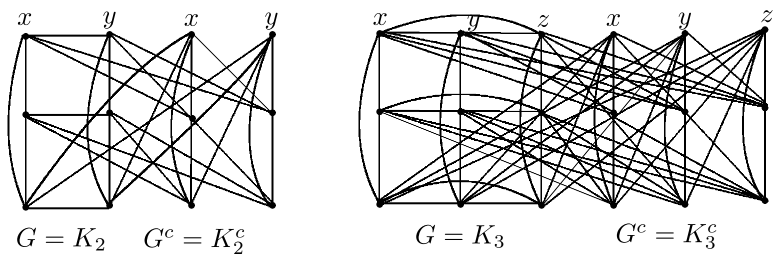

Figure 2

In Figure 2, Corollary 3.4 is illustrated for $G = K_2$ and $G = K_3$, in which the graph G and $G^c$is induced by the vertices x, y for $G = K_2$. In the second graph, the graph G and $G^c$is induced by the vertices x, y, z for $G = K_3$.

Corollary 3.5 *Every graph* G *of order* $n \geq 2$, *and its complement* G^c *are the induced subgraphs of* $(2n+2, 2, 3(2n-1))$*-regular graph of smallest order* $8n$.

Corollary 3.6 *Every graph* G *of order* $n \geq 2$, *and its complement* G^c *are the induced subgraph of* $(2n+3, 2, 4(2n-1))$*-regular graph of smallest order* $10n$.

Remark 3.7 There are at least as many $(m+2(n-1), 2, (m-1)(2n-1))$-regular of order $2mn$ as there are graphs G of order $n \geq 2$. If $m = 2, 3, 4, 5, \ldots$, then there are $(2n, 2, (2n-1)), (2n+1, 2, 2(2n-1)), (2n+2, 2, 3(2n-1)), (2n+3, 2, 4(2n-1)), \ldots$ regular graphs of smallest order $4n, 6n, 8n, 10n, 12n \ldots$ respectively containing any graph G of order $n \geq 2$ and its complement as induced subgraphs.

§4. Topological Indices of the Graph H_4

The topological indices Wiener Index W, Hyper Wiener Index WW, Degree Distance DD, Variance of degrees, The first Zagreb index, The second Zagreb Index and the third Zagreb Index of the graph H_4, which was constructed in Theorem 3.1 are calculated in this section.

Topological index $Top(G)$ of a graph G is a number with this property that for every graph H isomorphic to G, $Top(G) = Top(H)$. For historical background, computational techniques and mathematical properties of Zagreb indices and Wiener, Hyper Wiener one can refer to [21,22,23,24,25].

The graph H_4 is $(m+2n-2, 2, (m-1)(2n-1))$-regular graph having $2mn$ vertices and $mn(m+2n-2))$ edges with diameter 2. Also, for each $v \in H_4$, $d_2(v) = (m-1)(2n-1)$ and

$d(v) = m + 2n - 2$.

Computation of W, WW and DD for H_4 is done by using the following theorem [14]:

Let G be a graph with n vertices, m edges and with diameter 2, then

(1) $W(G) = n(n-1) - m$;

(2) $WW(G) = 3/2(n(n-1)) - 2m$;

(3) $DD(G) = 4(n-1)m - M_1(G)$.

The Wiener index W is the first and important topological index in chemistry which was introduced by H. Wiener in 1947 to study the boiling points of parafins. This index is useful to describe molecular structures and also crystal lattice that depends on its W value.

Definition 4.1 *The Wiener index, $W(G)$ of a finite, connected graph G is defined to be $W(G) = \frac{1}{2}\sum d(u,v)$, where $d(u,v)$ denotes the distance between u and v in G.*

$$\begin{aligned}\text{Wiener Index of a graph } H_4 &= W(H_4) = 2mn(2mn-1) - ((mn)(m+2(n-1)) \\ &= mn(4mn - 2 - m - 2n + 2) = (mn)(4mn - (m+2n))\end{aligned}$$

The Hyper Wiener index WW was introduced by Randic. The Hyper Wiener Index WW is used as a structure descriptor for predicting physicochemical properties of organic compounds.

Definition 4.2 *The Hyper Wiener index $WW(G)$ of a finite, connected graph G is defined to be $WW(G) = \frac{1}{2}(W_1(G) + W_2(G))$, where $W_1(G) = W(G)$ and $W_\lambda(G) = \sum d_G(k)(k^\lambda)$ is called the Wiener-type invariant of G associated to a real number.*

$$\begin{aligned}\text{Hyper Wiener Index of a graph } H_4 &= WW(H_4) \\ &= (3/2)(2mn(2mn-1) - 2mn(m+2(n-1))) \\ &= (mn)(6mn - 3 - 2m - 4n + 4) \\ &= (mn)(6mn - (2m+4n) + 1)\end{aligned}$$

The Zagreb indices were introduced by Gutman and Trinajestic [7,10,14].

Definition 4.3 *The oldest and most investigated topological graph indices are defined as: First Zagreb index $M_1(G) = \sum_{v\in V(G)}(d_G(v))^2$, second Zagreb index $M_2(G) = \sum_{uv\in E(G)}(d_G(u)d_G(v))$ and third Zagreb index $M_3(G) = \sum|d(u) - d(v)|, uv \in E(G)$.*

The Zagreb Indices of graph H_4 are

$$
\begin{aligned}
1.\ M_1(H_4) &= \sum d(u)d(u) = \sum d(u)^2 = 2mn((m+2n-2)^2) \\
2.\ M_2(H_4) &= \sum d(u)d(v), uv \in E(H_4) \\
&= (mn)(m+2(n-1)(m+2(n-1)(m+2(n-1) \\
&= (mn)((m+2(n-1))^3) \\
3.\ M_3(H_4) &= \sum |d(u)-d(v)||, uv \in E(H_4) \\
&= \sum |(m+2(n-1))-(m+2(n-1))| = 0.
\end{aligned}
$$

Definition 4.4([4]) *The degree distance (Schultz index) of G was introduced by Dobrynin and Kochetova and Gutman as a weighted version of the Wiener index defined as $DD(G) = \sum(d(u)+d(v))d(u,v)$. It is to be noted that $DD(G)$ and $W(G)$ are closely mutually related for certain classes of molecular graphs.*

The degree distance of graph H_4 is

$$
\begin{aligned}
DD(H_4) &= 4(2mn-1)(mn)(m+2(n-1)) - M_1(H_4) \\
&= 2mn(m+2n-2)[2(2mn-1)-(m+2n-2)) \\
&= 2mn(m+2n-2)[4mn-(m+2n)]
\end{aligned}
$$

Definition 4.5([13]) *The status, or distance sum of a vertex v in a graph is defined by $s(v) = \sum d(u,v)$, where $d(u,v)$ is the distance between the vertices u and v and $u \neq v$. The status sequence of a graph consists of a list of the stati of all the vertices.*

Since diameter of H_4 is two, the status of a vertex v in H_4 is

$$
\begin{aligned}
s(v) &= (m+2(n-1)+2(m-1)(2n-1) \\
&= m+2n-2+4mn-2m-4n+2 = 4mn-2(m+n)
\end{aligned}
$$

Definition 4.6 *A graph is said to be self-median, or SM, if the stati of its vertices are all equal.*

Every vertex in H_4 has the same status $4mn-2(m+n)$. Whence, H_4 is a self-median graph.

§5. Open Problems

For further investigation, the following open problem is suggested:

(1) Construct (r, m, k)-regular graphs containing a given graph G and its complement of order $n \geq 2$, as induced subgraph, for $m \geq 3$.

(2) Construct (r, m, k)-regular graphs containing a given graph G and its complement of order $n \geq 2$, as induced subgraph, for all values of k.

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On Signed Graphs Whose Two Path Signed Graphs are Switching Equivalent to Their Jump Signed Graphs

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Abstract: In this paper, we obtained a characterization of signed graphs whose jump signed graphs are switching equivalent to their two path signed graphs.

Key Words: Smarandachely k-signed graph, t-Path graphs, jump graphs, signed graphs, balance, switching, t-path signed graphs and jump signed graphs.

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§1. Introduction

For standard terminology and notation in graph theory we refer Harary [6] and Zaslavsky [20] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Given a graph Γ and a positive integer t, the *t-path graph* $(\Gamma)_t$ of Γ is formed by taking a copy of the vertex set $V(\Gamma)$ of Γ, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in Γ. The notion of *t-path graphs* was introduced by Escalante et al. [4]. A graph G for which

$$(\Gamma)_t \cong \Gamma \tag{1}$$

has been termed as *t-path invariant graph* by Esclante et al. in [4], Escalante & Montejano [5] where the explicit solution to (1) has been determined for $t = 2, 3$. The structure of t-path invariant graphs are still remains uninvestigated in literature for all $t \geq 4$.

The *line graph* $L(\Gamma)$ of a graph $\Gamma = (V, E)$ is that graph whose vertices can be put in one-to-one correspondence with the edges of Γ so that two vertices of $L(\Gamma)$ are adjacent if, and only if, the corresponding edges of Γ are adjacent.

The *jump graph* $J(\Gamma)$ of a graph $\Gamma = (V, E)$ is $\overline{L(\Gamma)}$, the complement of the line graph $L(\Gamma)$ of Γ (see [6]).

A *Smarandachely k-signed graph* is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ ($\mu : V \to$

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$(\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph is called abbreviated to *signed graph*, where $\Gamma = (V, E)$ is a graph called *underlying graph of* Σ and $\sigma : E \to \{+, -\}$ is a function. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced*, if every cycle in Σ has an even number of negative edges (See [7]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of Σ is positive.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [8] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [3] provided a mathematical model for balance through graphs. For more new notions on signed graphs refer the papers (see [12-17, 20]).

A *marking* of Σ is a function $\zeta : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

A switching function for Σ is a function $\zeta : V \to \{+, -\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^\zeta := (\Sigma|\sigma^\zeta)$. We say that Σ switched by ζ. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cut set $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [18]) or *cycle isomorphic* (see [19]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [19]).

Theorem 1.(T. Zaslavsky, [19]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [11], the authors introduced the switching and cycle isomorphism for signed digraphs. The notion of t-path graph of a given graph was extended to the class of signed graphs by Mishra [9] as follows:

*Given a signed graph Σ and a positive integer t, the t-*path signed graph *$(\Sigma)_t$ of Σ is formed by taking a copy of the vertex set $V(\Sigma)$ of Σ, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in S and then by defining its sign to be $-$ whenever in every $u - v$ path of length t in Σ all the edges are negative.*

In [13], P. S. K. Reddy introduced a variation of the concept of t-path signed graphs studied above. The motivation stems naturally from one's mathematically inquisitiveness as to ask why not define the sign of an edge $e = uv$ in $(\Sigma)_t$ as the product of the signs of the vertices u and v in Σ. The *t-path signed graph* $(\Sigma)_t = ((\Gamma)_t, \sigma')$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $(\Gamma)_t$ called *t-path graph* and sign of any edge $e = uv$ in $(\Sigma)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called *t-path signed graph*, if $\Sigma \cong (\Sigma')_t$, for some signed graph Σ'. In this paper, we follow the notion of t-path signed graphs defined by P. S. K. Reddy as above.

Theorem 2.(P. S. K. Reddy, [13]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its t-path signed graph $(\Sigma)_t$ is balanced.*

Corollary 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its 2-path signed graph $(\Sigma)_2$ is balanced.*

The *jump signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $J(S) = (J(G), \sigma')$, where for any edge ee' in $J(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by M. Acharya and D. Sinha [2] (See also E. Sampathkumar et al. [10]).

Theorem 4.(M. Acharya and D. Sinha, [2]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its jump signed graph $J(\Sigma)$ is balanced.*

§2. Switching Equivalence of Two Path Signed Graphs and Jump Signed Graphs

The main aim of this paper is to prove the following signed graph equation

$$(\Sigma)_2 \sim J(\Sigma).$$

We first characterize graphs whose two path graphs are isomorphic to their jump graphs.

Theorem 5. *A graph $\Gamma = (V, E)$ satisfies $(\Gamma)_2 \cong J(\Gamma)$ if, and only if, $G = C_4$ or C_5.*

Proof Suppose Γ is a graph such that $(\Gamma)_2 \cong J(\Gamma)$. Hence number of vertices and number of edges of Γ are equal and so Γ must be unicyclic. Let $C = C_m$ be the cycle of length $m \geq 3$ in Γ.

Case 1. $m = 3$.

Let $V(C) = \{u_1, u_2, u_3\}$. Then C is also a cycle in $(\Gamma)_2$, where as in $J(\Gamma)$, the vertices corresponds the edges of C are mutually non-adjacent. Since $(\Gamma)_2 \cong J(\Gamma)$, $J(\Gamma)$ must also contain a C_3 and hence Γ must contain $3K_2$, disjoint union of 3 copies of K_2. Whence Γ must contain either Γ_1, Γ_2 or Γ_3 as shown in Figure 1 as induced subgraph.

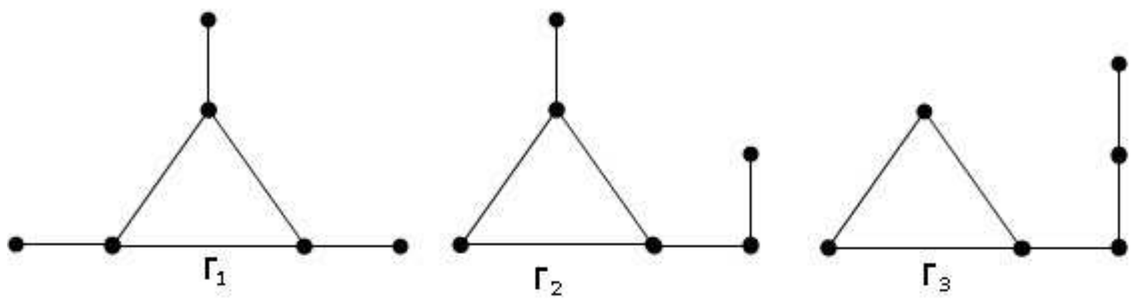

Figure 1.

Subcase 1.1 If Γ contains Γ_2 or Γ_3. Let v be the vertex satisfying $d(v, C) = 2$. Since $d(v, C) = 2$ there exists a vertex u in Γ adjacent to v and a vertex in C say u_1. Now, in $(\Gamma)_2$, the vertex v is not adjacent to u. Since C is also a cycle in $(\Gamma)_2$ and w is adjacent to u_1 in Γ, the vertices w u_2 and u_3 forms a cycle C' in $(\Gamma)_2$. Further, the vertex v is not adjacent to C'. Hence, Γ contains $H = C_3 \cup K_1$ as a induced subgraph. But since $\overline{\Gamma'} = K_{1,3}$ which is a forbidden induced subgraph for line graph $L(\Gamma)$ and $J(\Gamma) = \overline{L(\Gamma)}$, we must have $(\Gamma)_2 \ncong J(\Gamma)$.

Subcase 1.2 If Γ contains Γ_1. Then by subcase(i), $\Gamma = \Gamma_1$. But clearly, $(\Gamma)_2 \ncong J(\Gamma)$.

Case 2. $m \geq 4$.

Suppose that $m \geq 4$ and there exists vertex v in Γ which is not on the cycle C. Let $C = (v_1, v_2, v_3, v_4, ..., v_m, v_1)$. Since Γ is connected v is adjacent to a vertex say, v_i in C. Then the subgraph induced by the vertices v_{i-1}, v, v_{i+1} and v_i in $(\Gamma)_2$ is $K_3 \cup K_1$. Now the graph $\overline{K_3 o K_1}$ is $K_{1,3}$ which is a forbidden induced subgraph for $L(\Gamma) = \overline{J(\Gamma)}$. Hence Γ is not a jump graph. Hence Γ must be a cycle. Clearly $(\Gamma)_2(C_4) = 2K_2 = J(C_4)$ and $J(C_5) = (\Gamma)_2(C_5) = C_5$ it remains to show that for $\Gamma = C_m$ with $m \geq 6$ does not satisfy $J(C_m) = (\Gamma)_2(C_m)$, for $m \geq 6$.

Suppose that $m \geq 6$. Then clearly every vertex in $J(C_m)$ is adjacent to at least $m - 2 \geq 4$ vertices where as in $(\Gamma)_2$, degree of every vertex is 2. This proves the necessary part. The converse part is obvious. □

We now give a characterization of signed graphs whose two path signed graphs are switching equivalent to their jump signed graphs.

Theorem 6. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $(\Sigma)_2 \sim J(\Sigma)$ if, and only if, the underlying graph Γ is either C_5 or C_4.*

Proof Suppose that $(\Sigma)_2 \sim J(\Sigma)$. Then clearly, $(\Gamma)_2 \cong J(\Gamma)$. Hence by Theorem 5, Γ must be either C_4 or C_5.

Conversely, suppose that Σ is a signed graph on C_4 or C_5. Then by Theorem 5, $(\Gamma)_2 \cong J(\Gamma)$.

Since for any signed graph Σ, by Corollary 3 and Theorem 4, $(\Sigma)_2$ and $J(\Sigma)$ are balanced, the result follows by Theorem 1. □

Remark The only possible cases for which $(\Sigma)_2 \cong J(\Sigma)$ are shown in Figure 2.

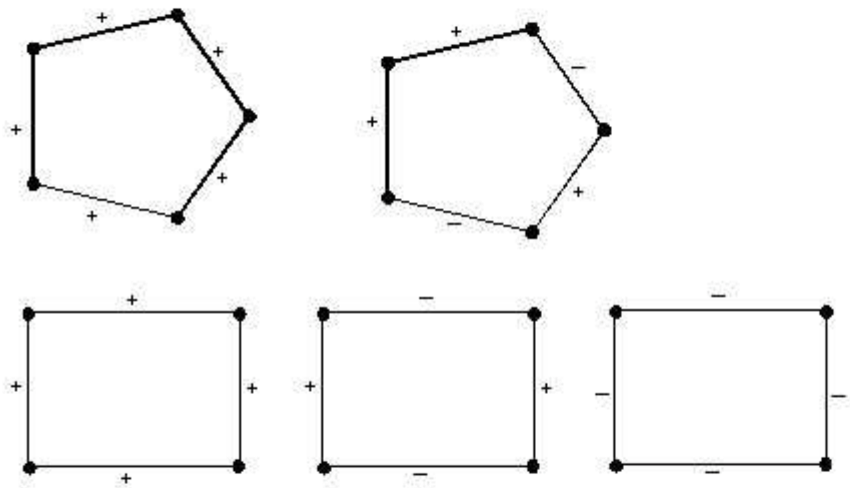

Figure 2.

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A Note on Prime and Sequential Labelings of Finite Graphs

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Abstract: A labeling or valuation of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. In this paper, we study some classes of graphs and their corresponding labelings.

Key Words: Labeling, sequential graph, harmonious graph, prime graph, Smarandache common k-factor labeling.

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§1. Introduction

Unless mentioned or otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices. For all terminology and notations in Graph Theory, we follow [5] and all terminology regarding to sequential labeling, we follow [3]. Graph labelings where the vertices are assigned values subject to certain conditions have been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolutional codes with optimal autoconvolutional properties. They facilitate the optimal nonstandard encodings of integers.

Labeled graphs have also been applied in determining ambiguities in X-ray crystallographic analysis, to design a communication network addressing system, in determining optimal circuit layouts and radio astronomy problems etc. A systematic presentation of diverse applications of graph labelings is presented in [1].

Let G be a (p, q)-graph. Let $V(G), E(G)$ denote respectively the vertex set and edge set of G. Consider an injective function $g : V(G) \to X$, where $X = \{0, 1, 2, \cdots, q\}$ if G is a tree and $X = \{0, 1, 2, \cdots, q-1\}$ otherwise. Define the function $f^* : E(G) \to N$, the set of natural numbers such that $f^*(uv) = f(u) + f(v)$ for all edges uv. If $f^*(E(G))$ is a sequence of distinct consecutive integers, say $\{k, k+1, \cdots, k+q-1\}$ for some k, then the function f is said to be *sequential labeling* and the graph which admits such a labeling is called a *sequential graph.*

Another labeling has been suggested by Graham and S Loane [4] named as harmonious

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labeling which is a function $h : V(G) \to Z_q$, q is the number of edges of G such that the induced edge labeling given by $g^*(uv) = [g(u) + g(v)] \pmod q$ for any edge uv is injective.

The notion of prime labeling of graphs, was defined in [6]. A graph G with n-vertices is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \cdots, n$ such that for each edge uv the labels assigned to u and v are relatively prime. Such a graph admitting a prime labeling is known as a *prime graph*. Generally, a *Smarandache common k-factor labeling* is such a labeling with distinct integers $1, 2, \cdots, n$ such that the greatest common factor of labels assigned to u and v is k for $\forall uv \in E(G)$. Clearly, a prime labeling is nothing else but a Smarandache common 1-factor labeling. A graph admitting a Smarandache common k-factor labeling is called a *Smarandache common k-factor graph*. Particularly, a graph admitting a prime labeling is known as a *prime graph* in references.

Notation 1.1 $(a, b) = 1$ means that a and b are relatively prime.

§2. Cycle Related Graphs

In [2], showed that every cycle with a chord is graceful. In [9] proved that a cycle C_n with a chord joining two vertices at a distance 3 is sequential for all odd $n, n \geqslant 5$. Now, we have the following theorems.

Theorem 2.1 *Every cycle C_n, with a chord is prime, for all $n \geqslant 4$.*

Proof Let G be a graph such that $G = C_n$ with a chord joining two non- adjacent vertices of C_n, for all n greater than or equal to 4. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of G. Let the number of vertices of G be n and the edges be $n + 1$. Define a function $f : V(G) \to \{1, 2, \cdots n\}$ such that $f(v_i) = i, i = 1, 2, \cdots, n$. It is obvious that $(f(v_i), f(v_{i+1})) = 1$ for all $i = 1, 2, \cdots, (n-1)$. Also $(1, n) = 1$ for all n greater than 1. Now select the vertex v_1 and join this to any vertex of C_n, which is not adjacent to v_1, G admits a prime labeling. □

Theorem 2.2 *Every cycle C_n, with $\left[\dfrac{n-1}{2}\right] - 1$ chords from a vertex is prime, for all n greater than or equal to 5.*

Proof Let G be a graph such that $G = C_n$, n greater than or equal to 5. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Label the vertices of C_n as in Theorem 2.1. Next select the vertex v_2. By our labeling $f(v_2) = 2$. Now join v_2 to all the vertices of C_n whose f-values are odd. Then it is clear that there exists exactly $[\frac{n-1}{2}] - 1$ chords, and G admits a prime labeling. □

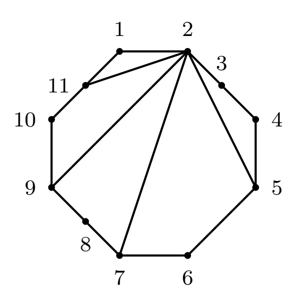

Figure 1

Remark 2.1 From Theorem 2.1, it is clear that there is possible to get $n-3$ chords and Theorem 2.2 tells us there are $[\frac{n-1}{2}]-1$ chords. Thus the bound $n-3$ is best possible and all other possible chords of less than these two bounds.

Example 2.1 Figure 1 gives the prime labeling of C_{11} with 4-chords.

Theorem 2.3 *The graph $C_n+\bar{K}_{1,t}$ is sequential for all odd $n, n\geqslant 3$.*

Proof Let $v_1, v_2, \cdots, v_n$ (n is odd) be the set of vertices of C_n and $u, u_1, u_2, \cdots, u_t$ be the $t+1$ isolated vertices of $\bar{K}_{1,t}$. Let $G=C_n+\bar{K}_{1,t}$ and note that, G has $n+t+1$ vertices and $n(2+t)$ edges.

Define a function $f: V(G)\to\{0,1,2,\cdots,\frac{n-1}{2}+tn\}$ such that

$$\begin{aligned} f(v_{2i-1}) &= i-1,\ \text{for } i=1,2,\cdots,\frac{n+1}{2}\\ f(v_{2i}) &= \frac{n-1}{2}+i,\ \text{for } i=1,2,\cdots,\frac{n-1}{2}\\ f(u_1) &= \frac{3}{2}n-1 \end{aligned}$$

and

$$f(u_i) = \frac{n-1}{2}+ni, i=2,3,\cdots,t$$

We can easily observe that the above defined f is injective. Hence f becomes a sequential labeling of $C_n+\bar{K}_{1,t}$. Thus $C_n+\bar{K}_{1,t}$ is sequential for all odd n, $n\geqslant 3$. □

Corollary 2.4 *The graph $C_n+\bar{K}_{1,t}$ is harmonious for all odd $n\geqslant 3$.*

Proof Any sequential is harmonius implies that $C_n+\bar{K}_{1,t}$ is harmonius, $n\geqslant 3$. □

Theorem 2.5 *The graph $C_n+\bar{K}_{1,1,t}$ is sequential and harmonius for all odd $n, n\geqslant 3$.*

Theorem 2.6 *The graph $C_n+\bar{K}_{1,1,1,t}$ is sequential and harmonius for all odd n, $n\geqslant 3$.*

Example 2.2 Figure 2 gives the sequential labeling of the graph $C_5+\bar{K}_{1,1,1,3}$.

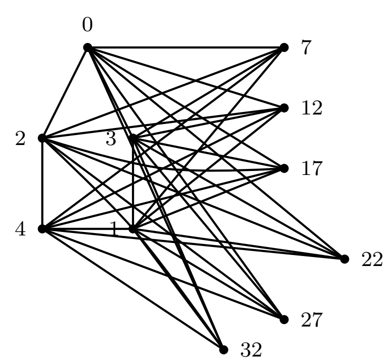

Figure 2

Theorem 2.7 *The graph $C_n + \bar{K}_{1,1,\dots,1,t}$ is sequential as harmonius for odd n, $n \geqslant 3$.*

Theorem 2.8 *The graph $C_n + \bar{K}_{1,m,n}$ is sequential and harmonius for all odd n, $n \geqslant 3, m \geqslant 1$.*

§3. On Join of Complete Graphs

In [7], it is shown that $L_n + K_1$ and $B_n + K_1$ are prime and join of any two connected graphs are not odd sequential. Now, we have the following.

Theorem 3.1 *The graph $K_{1,n} + K_2$ is prime for $n \geqslant 4$.*

Proof Let $G = K_{1,n} + K_2$. We can notice that G has $(n+3)$-vertices and $(3n+2)$-edges. Let $\{w, v_1, v_2, \cdots, v_n\}$ be the vertices of $K_{1,n}$ and $\{u_1, u_2\}$ be the two vertices of K_2. Assign the first two largest primes less than or equal to $n+3$ to the two vertices of K_2. Assign 1 to w and remaining n values to the n vertices arbitrarily, we can obtain a prime numbering of $K_{1,n} + K_2$. □

Corollary 3.1 *The graph $K_{1,n} + \bar{K}_2$ is prime for all $n \geqslant 4$.*

§4. Product Related Graphs

Definition 4.1 *Let G and H be graphs with $V(G) = V_1$ and $V(H) = V_2$. The cartesian product of G and H is the graph $G \Box H$ whose vertex set is $V_1 \times V_2$ such that two vertices $u = (x, y)$ and $v = (x', y')$ are adjacent if and only if either $x = x'$ and y is adjacent to y' in H or $y = y'$ and x is adjacent to x' in G. That is, u adj v in $G \Box H$ whenever $[x = x'$ and y adj $y']$ or $[y = y'$ and x adj $x']$.*

In [8] A.Nagarajan, A.Nellai Murugan and A.Subramanian proved that $P_n \Box K_2$, $P_n \Box P_n$ are near mean graphs.

Definition 4.2 *Let P_n be a path on n vertices and K_4 be a complete graph on 4 vertices. The Cartesian product P_n and K_4 is denoted as $P_n \Box K_4$ with $4n$ vertices and $10n - 4$ edges.*

Theorem 4.1 *The graph $P_n \Box K_4$ is sequential, for all $n \geqslant 1$.*

Proof Let $G = P_n \Box K_4$. Let $\{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} / i = 1, 2, \cdots, n\}$ be the vertex set of G.

Define a function $f : V(G) \rightarrow \{0, 1, 2, \cdots, 5n-1\}$ such that

$$f(v_{2i-1,1}) = 10i - 6 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,2}) = 10(i-1) \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,3}) = 10i - 9 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i-1,4}) = 10i - 8 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n+1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,1}) = 10i - 4 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,2}) = 10i - 1 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$
$$f(v_{2i,3}) = 10i - 3 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$

and

$$f(v_{2i,4}) = 10i - 5 \quad ; \quad 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even or } 1 \leqslant i \leqslant \frac{n-1}{2} \quad \text{if } n \text{ is odd.}$$

(a) Clearly we can see that f is injective.

(b) Also, $\max_{v\in V} f(v) = \max\{\max_i 10i-6; \max_i 10(i-1); \max_i 10i-9; \max_i 10i-8; \max_i 10i-4; \max_i 10i-1; \max_i 10i-3; \max_i 10i-5\} = 5n-1$. Thus, $f(v) = \{0, 1, 2, \ldots, 5n-1\}$. Finally, it can be easily verified that the labels of the edge values are distinct positive integers in the interval $[1, 10n-4]$. Thus, f is a sequential numbering. Hence, the graph G is sequential. □

Example 4.1 Figure 4 gives the sequential labeling of the graph $P_4\Box K_4$.

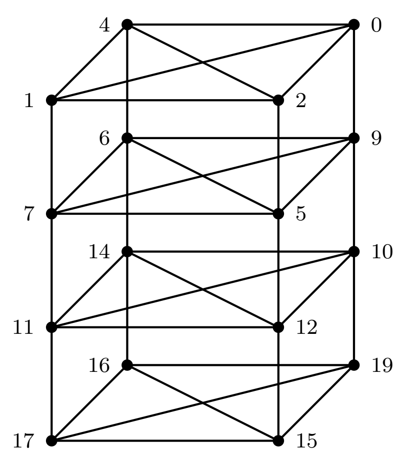

Figure 3

Corollary 4.1 *The graph $P_n\Box K_4$ is harmonius, for $n \geqslant 2$.*

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The Forcing Vertex Monophonic Number of a Graph

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Abstract: For any vertex x in a connected graph G of order $p \geq 2$, a set $S_x \subseteq V(G)$ is an *x-monophonic set* of G if each vertex $v \in V(G)$ lies on an $x-y$ monophonic path for some element y in S_x. The minimum cardinality of an x-monophonic set of G is the *x-monophonic number* of G and is denoted by $m_x(G)$. A subset T_x of a minimum x-monophonic set S_x of G is an *x-forcing subset* for S_x if S_x is the unique minimum x-monophonic set containing T_x. An x-forcing subset for S_x of minimum cardinality is a *minimum x-forcing subset* of S_x. The *forcing x-monophonic number* of S_x, denoted by $f_{m_x}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The *forcing x-monophonic number* of G is $f_{m_x}(G) = min\{f_{m_x}(S_x)\}$, where the minimum is taken over all minimum x-monophonic sets S_x in G. We determine bounds for it and find the forcing vertex monophonic number for some special classes of graphs. It is shown that for any three positive integers a, b and c with $2 \leq a \leq b < c$, there exists a connected graph G such that $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G, where $cm_x(G)$ is the connected x-monophonic number of G.

Key Words: monophonic path, vertex monophonic number, forcing vertex monophonic number, connected vertex monophonic number, Smarandachely geodetic k-set, Smarandachely hull k-set.

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§1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an $x-y$ *geodesic*. The *neibourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neibourhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete.

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The *closed interval* $I[x,y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x,y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. The geodetic number of a graph was introduced in [1,8] and further studied in [2,5]. A geodetic set of cardinality $g(G)$ is called a $g-set$ of G. Generally, for an integer $k \geq 0$, a subset $S \subseteq V$ is called a *Smarandachely geodetic k-set* if $I[S \bigcup S^+] = V$ and a *Smarandachely hull k-set* if $I_h(S \bigcup S^+) = V$ for a subset $S^+ \subset V$ with $|S^+| \leq k$. Let $k = 0$. Then a Smarandachely geodetic 0-set and Smarandachely hull 0-set are nothing else but the geodetic set and hull set, respectively.

The concept of vertex geodomination number was introduced in [9] and further studied in [10]. For any vertex x in a connected graph G, a set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x-y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a $g_x - set$.

A chord of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of a graph G is a *monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path in G for some $x, y \in S$. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$.

The concept of vertex monophonic number was introduced in [11]. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S_x \subseteq V(G)$ is an *x-monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path for some element y in S_x. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a $m_x - set$ of G. The concept of upper vertex monophonic number was introduced in [13]. An x-monophonic set S_x is called a *minimal x-monophonic set* if no proper subset of S_x is an x-monophonic set. The *upper x-monophonic number*, denoted by $m_x^+(G)$, is defined as the minimum cardinality of a minimal x-monophonic set of G. The connected x-monophonic number was introduced and studied in [12]. A *connected x-monophonic set* of G is an x-monophonic set S_x such that the subgraph $G[S_x]$ induced by S_x is connected. The minimum cardinality of a connected x-monophonic set of G is the *connected x-monophonic number* of G and is denoted by $cm_x(G)$. A connected x-monophonic set of cardinality $cm_x(G)$ is called a $cm_x - set$ of G.

The following theorems will be used in the sequel.

Theorem 1.1([11]) *Let x be a vertex of a connected graph G.*

(1) *Every simplicial vertex of G other than the vertex x (whether x is simplicial vertex or not) belongs to every $m_x - set$;*

(2) *No cut vertex of G belongs to any $m_x - set$.*

Theorem 1.2([11]) (1) *For any vertex x in a cycle $C_p (p \geq 4)$, $m_x(C_p) = 1$;*

(2) *For the wheel $W_p = K_1 + C_{p-1} (p \geq 5)$, $m_x(W_p) = p - 1$ or 1 according as x is K_1 or x is in C_{p-1}.*

Theorem 1.3([11]) *For $n \geq 2$, $m_x(Q_n) = 1$ for every vertex x in Q_n.*

Throughout this paper G denotes a connected graph with at least two vertices.

§2. Vertex Forcing Subsets in Vertex Monophonic Sets of a Graph

Let x be any vertex of a connected graph G. Although G contains a minimum x-monophonic set there are connected graphs which may contain more than one minimum x-monophonic set. For example, the graph G given in Figure 2.1 contains more than one minimum x-monophonic set. For each minimum x-monophonic set S_x in a connected graph G there is always some subset T of S_x that uniquely determines S_x as the minimum x-monophonic set containing T. Such sets are called "vertex forcing subsets" and we discuss these sets in this section. Also, forcing concepts have been studied for such diverse parameters in graphs as the geodetic number [3], the domination number [4] and the graph reconstruction number [7].

Definition 2.1 *Let x be any vertex of a connected graph G and let S_x be a minimum x-monophonic set of G. A subset T of S_x is called an x-forcing subset for S_x if S_x is the unique minimum x-monophonic set containing T. An x-forcing subset for S_x of minimum cardinality is a minimum x-forcing subset of S_x. The forcing x-monophonic number of S_x, denoted by $f_{m_x}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The forcing x-monophonic number of G is $f_{m_x}(G) = min\{f_{m_x}(S_x)\}$, where the minimum is taken over all minimum x-monophonic sets S_x in G.*

Example 2.2 For the graph G given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum forcing vertex monophonic sets and the forcing vertex monophonic numbers are given in Table 2.1.

Vertex x	Minimum x-monophonic sets	$m_x(G)$	Minimum forcing x-monophonic sets	$f_{m_x}(G)$
u	$\{r, y\}, \{r, z\}, \{r, s\}$	2	$\{y\}, \{z\}, \{s\}$	1
v	$\{u, r, y\}, \{u, r, z\}, \{u, r, s\}$	3	$\{y\}, \{z\}, \{s\}$	1
w	$\{u, r\}$	2	$\emptyset$	0
y	$\{u, r\}$	2	$\emptyset$	0
z	$\{u, r\}$	2	$\emptyset$	0
s	$\{u, r\}$	2	$\emptyset$	0
t	$\{u, r, w\}, \{u, r, y\}, \{u, r, z\}$	3	$\{w\}, \{y\}, \{z\}$	1
r	$\{u, w\}, \{u, y\}, \{u, z\}$	2	$\{w\}, \{y\}, \{z\}$	1

Table 2.1

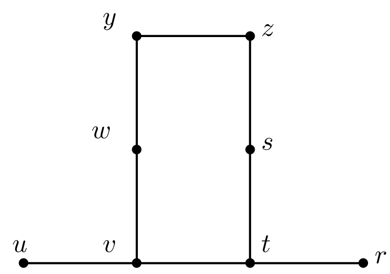

Figure 2.1

Theorem 2.3 *For any vertex x in a connected graph G, $0 \leq f_{m_x}(G) \leq m_x(G)$.*

Proof Let x be any vertex of G. It is clear from the definition of $f_{m_x}(G)$ that $f_{m_x}(G) \geq 0$. Let S_x be a minimum x-monophonic set of G. Since $f_{m_x}(S_x) \leq m_x(G)$ and since $f_{m_x}(G) = min\{f_{m_x}(S_x) : S_x$ is a minimum x-monophonic set in $G\}$, it follows that $f_{m_x}(G) \leq m_x(G)$. Thus $0 \leq f_{m_x}(G) \leq m_x(G)$. □

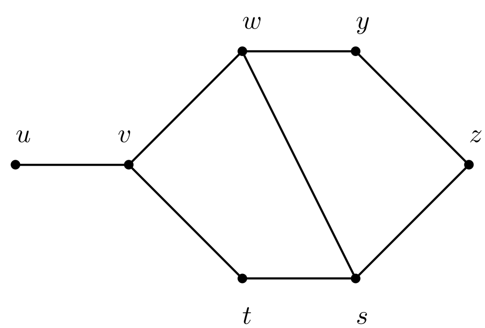

Figure 2.2

Remark 2.4 The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2.2, $S = \{u, z, t\}$ is the unique minimum w-monophonic set of G and the empty set ϕ is the unique minimum w-forcing subset for S. Hence $f_{m_w}(G) = 0$. Also, for the graph G given in Figure 2.2, $S_1 = \{y\}$ and $S_2 = \{z\}$ are the minimum u-monophonic sets of G and so $m_u(G) = 1$. It is clear that no minimum u-monophonic set is the unique minimum u-monophonic set containing any of its proper subsets. It follows that $f_{m_u}(G) = 1$ and hence $f_{m_u}(G) = m_u(G) = 1$. The inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $m_u(G) = 2$ and $f_{m_u}(G) = 1$. Thus $0 < f_{m_u}(G) < m_u(G)$.

In the following theorem we characterize graphs G for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{m_x}(G) = 1$.

Theorem 2.5 *Let x be any vertex of a connected graph G. Then*

(1) $f_{m_x}(G) = 0$ *if and only if* G *has a unique minimum* x*-monophonic set;*

(2) $f_{m_x}(G) = 1$ *if and only if* G *has at least two minimum* x*-monophonic sets, one of which is a unique minimum* x*-monophonic set containing one of its elements, and*

(3) $f_{m_x}(G) = m_x(G)$ *if and only if no minimum* x*-monophonic set of* G *is the unique minimum* x*-monophonic set containing any of its proper subsets.*

Definition 2.6 *A vertex* u *in a connected graph* G *is said to be an* x*-monophonic vertex if* u *belongs to every minimum* x*-monophonic set of* G*.*

For the graph G in Figure 2.1, $S_1 = \{u, r, y\}$, $S_2 = \{u, r, z\}$ and $S_3 = \{u, r, s\}$ are the minimum v-monophonic sets and so u and r are the v-monophonic vertices of G. In particular, every simplicial vertex of G other than x is an x-monophonic vertex of G.

Next theorem follows immediately from the definitions of an x-monophonic vertex and forcing x-monophonic subset of G.

Theorem 2.7 *Let* x *be any vertex of a connected graph* G *and let* $\mathcal{F}_{m_x}$ *be the set of relative complements of the minimum* x*-forcing subsets in their respective minimum* x*-monophonic sets in* G*. Then* $\bigcap_{F\in\mathcal{F}_{m_x}} F$ *is the set of* x*-monophonic vertices of* G*.*

Theorem 2.8 *Let* x *be any vertex of a connected graph* G *and let* M_x *be the set of all* x*-monophonic vertices of* G*. Then* $0 \leq f_{m_x}(G) \leq m_x(G) - |M_x|$*.*

Proof Let S_x be any minimum x-monophonic set of G. Then $m_x(G) = |S_x|$, $M_x \subseteq S_x$ and S_x is the unique minimum x-monophonic set containing $S_x - M_x$ and so $f_{m_x}(G) \leq |S_x - M_x| = m_x(G) - |M_x|$. □

Theorem 2.9 *Let* x *be any vertex of a connected graph* G *and let* S_x *be any minimum* x*-monophonic set of* G*. Then*

(1) *no cut vertex of* G *belongs to any minimum* x*-forcing subset of* S_x*;*

(2) *no* x*-monophonic vertex of* G *belongs to any minimum* x*-forcing subset of* S_x*.*

Proof (1) Since any minimum x-forcing subset of S_x is a subset of S_x, the result follows from Theorem 1.1(2).

(2) Let v be an x-monophonic vertex of G. Then v belongs to every minimum x-monophonic set of G. Let $T \subseteq S_x$ be any minimum x-forcing subset for any minimum x-monophonic set S_x of G. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S_x is the unique minimum x-monophonic set containing T' so that T' is an x-forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum x-forcing subset for S_x. Hence $v \notin T$. □

Corollary 2.10 *Let* x *be any vertex of a connected graph* G*. If* G *contains* k *simplicial vertices, then* $f_{m_x}(G) \leq m_x(G) - k + 1$*.*

Proof This follows from Theorem 1.1(1) and Theorem 2.9(2). □

Remark 2.11 The bound for $f_{m_x}(G)$ in Corollary 2.10 is sharp. For a non-trivial tree T with

k end-vertices, $f_{m_x}(T) = 0 = m_x(T) - k + 1$ for any end-vertex x in T.

Theorem 2.12 (1) *If T is a non-trivial tree, then $f_{m_x}(T) = 0$ for every vertex x in T;*

(2) *If G is the complete graph, then $f_{m_x}(G) = 0$ for every vertex x in G.*

Proof This follows from Theorem 2.9. □

Theorem 2.13 *For every vertex x in the cycle $C_p (p \geq 3), f_{m_x}(C_p) = \begin{cases} 0 & \text{if } p = 3, 4 \\ 1 & \text{if } p \geq 5 \end{cases}$.*

Proof Let $C_p : u_1, u_2, \cdots, u_p, u_1$ be a cycle of order $p \geq 3$. Let x be any vertex in C_p, say $x = u_1$. If $p = 3$ or 4, then C_p has unique minimum x-monophonic set. Then by Theorem 2.5(1), $f_{m_x}(C_p) = 0$. Now, assume that $p \geq 5$. Let y be a non-adjacent vertex of x in C_p. Then $S_x = \{y\}$ is a minimum x-monophonic set of C_p. Hence C_p has more than one minimum x-monophonic set and it follows from Theorem 2.5(1) that $f_{m_x}(C_p) \neq 0$. Now it follows from Theorems 1.2(1) and 2.3 that $f_{m_x}(G) = m_x(G) = 1$. □

Theorem 2.14 *For any vertex x in a complete bipartite graph $K_{m,n} (m, n \geq 2)$, $f_{m_x}(K_{m,n}) = 0$.*

Proof Let (V_1, V_2) be the bipartition of $K_{m,n}$. If $x \in V_1$, then $S_x = V_1 - \{x\}$ is the unique minimum x-monophonic set of G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. If $x \in V_2$, then $S_x = V_2 - \{x\}$ is the unique minimum x-monophonic set of G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. □

Theorem 2.15 (1) *If G is the wheel $W_p = K_1 + C_{p-1} (p = 4, 5)$, then $f_{m_x}(G) = 0$ for any vertex x in W_p;*

(2) *If G is the wheel $W_p = K_1 + C_{p-1} (p \geq 6)$, then $f_{m_x}(G) = 0$ or 1 according as x is K_1 or x is in C_{p-1}.*

Proof Let $C_{p-1} : u_1, u_2, \cdots, u_{p-1}, u_1$ be a cycle of order $p - 1$ and let u be the vertex of K_1.

(1) If $p = 4$ or 5, then G has unique minimum x-monophonic set for any vertex x in G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$.

(2) Let $p \geq 6$. If $x = u$, then $S_x = \{u_1, u_2, \cdots, u_{p-1}\}$ is the unique minimum x-monophonic set and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. If $x \in V(C_{p-1})$, say $x = u_1$, then $S_i = \{u_i\} (3 \leq i \leq p-2)$ is a minimum x-monophonic set of G. Since $p \geq 6$, there is more than one minimum x-monophonic set of G. Hence it follows from Theorem 2.5(1) that $f_{m_x}(G) \neq 0$. Now it follows from Theorems 1.2(2) and 2.3 that $f_{m_x}(G) = m_x(G) = 1$. □

Theorem 2.16 *For any vertex x in the n-cube Q_n $(n \geq 2)$, then $f_{m_x}(Q_n) = \begin{cases} 0 & \text{if } n = 2 \\ 1 & \text{if } n \geq 3 \end{cases}$.*

Proof If $n = 2$, then Q_n has unique minimum x-monophonic set for any vertex x in Q_n and so by Theorem 2.5(1), $f_{m_x}(Q_n) = 0$. If $n \geq 3$, then it is easily seen that there is more than one minimum x-monophonic set for any vertex x in Q_n. Hence it follows from Theorem 2.5(1) that $f_{m_x}(Q_n) \neq 0$. Now it follows from Theorems 1.3 and 2.3 that $f_{m_x}(Q_n) = m_x(Q_n) = 1$. □

The following theorem gives a realization result for the parameters $f_{m_x}(G)$, $m_x(G)$ and $m_x^+(G)$.

Theorem 2.17 *For any three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $m_x^+(G) = c$ for some vertex x in G.*

Proof For each integer i with $1 \leq i \leq a-1$, let $F_i : u_{0,i}, u_{1,i}, u_{2,i}, u_{3,i}$ be a path of order 4. Let $C_6 : t, u, v, w, x, y, t$ be a cycle of order 6. Let H be a graph obtained from F_i and C_6 by joining the vertex x of C_6 to the vertices $u_{0,i}$ and $u_{3,i}$ of $F_i (1 \leq i \leq a-1)$. Let G be the graph obtained from H by adding $c-a$ new vertices $y_1, y_2, \cdots, y_{c-b}, v_1, v_2, \cdots, v_{b-a}$ and joining each $y_i (1 \leq i \leq c-b)$ to both u and y, and joining each $v_j (1 \leq j \leq b-a)$ with x. The graph G is shown in Figure 2.3.

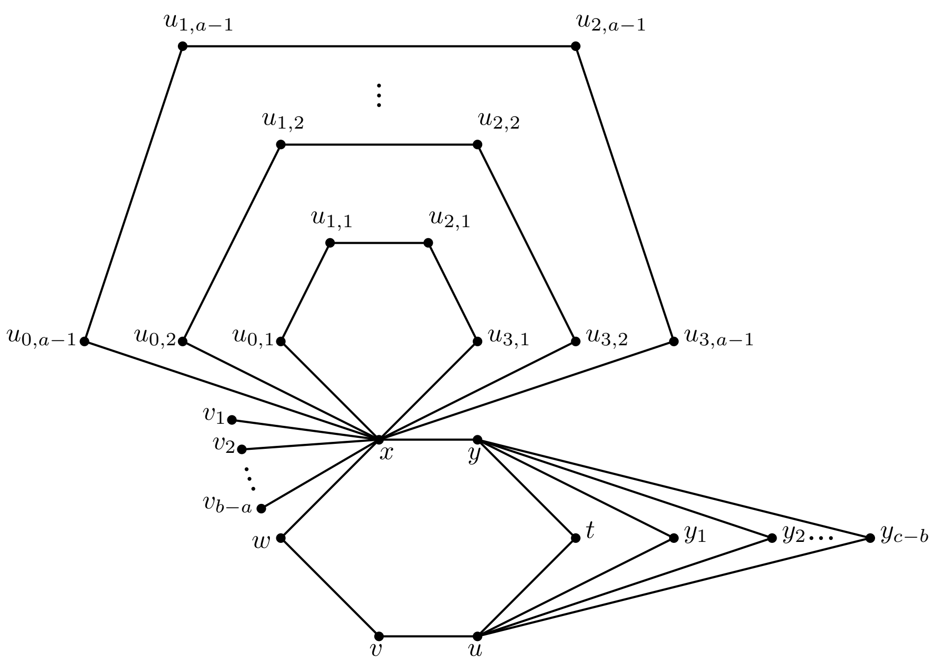

Figure 2.3

Let $S = \{v_1, v_2, \cdots, v_{b-a}\}$ be the set of all simplicial vertices of G. For $1 \leq j \leq a-1$, let $S_j = \{u_{1,j}, u_{2,j}\}$. If $b = c$, then let $S_a = \{u, v, t\}$. Otherwise, let $S_a = \{u, v\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains S and exactly one vertex from each set $S_j (1 \leq j \leq a)$ so that $m_x(G) \geq b$. Since $S_x^{'} = S \bigcup \{u, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G, we have $m_x(G) = b$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = S \bigcup \{u, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Since S is the set of all x-monophonic vertices of G and by Theorem 2.8, $f_{m_x}(G) \leq m_x(G) - |S| = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x^{'} = (S_x - \{u_{1,1}\}) \bigcup \{u_{2,1}\}$. Then $S_x^{'} \neq S_x$ and $S_x^{'}$ is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x a minimum x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Next, we show that $m_x^+(G) = c$. Let $U_x = S \bigcup \{u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}, t, y_1, y_2, \cdots, y_{c-b}\}$. Clearly U_x is a minimal x-monophonic set of G and so $m_x^+(G) \geq c$. Also, it is clear that every minimal x-monophonic set of G contains at most c elements and hence $m_x^+(G) \leq c$. Therefore, $m_x^+(G) = c$. □

The following theorem gives a realization for the parameters $f_{m_x}(G)$, $m_x(G)$ and $cm_x(G)$.

Theorem 2.18 *For any three positive integers a, b and c with $2 \leq a \leq b < c$, there exists a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G.*

Proof We prove this theorem by considering three cases.

Case 1. $2 \leq a < b < c$.

For each integer i with $1 \leq i \leq a-1$, let $F_i : y_1, u_{1,i}, u_{2,i}, y_3$ be a path of order 4. Let $P_{c-b+2} : y_1, y_2, y_3, \cdots, y_{c-b+2}$ be a path of order $c-b+2$ and let $P : v_1, v_2, v_3$ be a path of order 3. Let H_1 be a graph obtained from $F_i (1 \leq i \leq a-1)$ and P_{c-b+2} by identifying the vertices y_1 and y_3 of all $F_i (1 \leq i \leq a-1)$ and P_{c-b+2}. Let H_2 be the graph obtained from H_1 and P by joining the vertex v_1 of P to the vertex y_2 of H_1 and joining the vertex v_3 of P to the vertex y_3 of H_1. Let G be the graph obtained from H_2 by adding $b-a$ new vertices $z_1, z_2, ..., z_{b-a}$ and joining each $z_i (1 \leq i \leq b-a)$ with the vertex y_{c-b+2}. The graph G is shown in Figure 2.4.

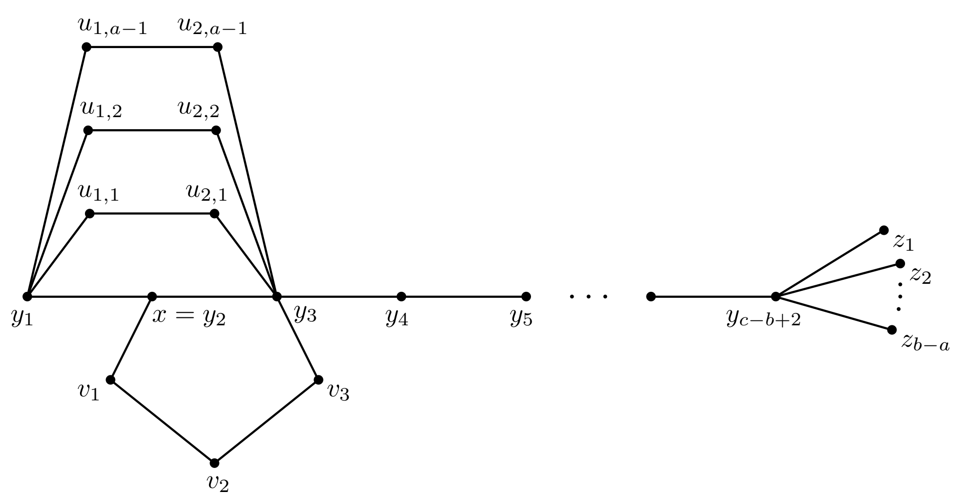

Figure 2.4

Let $x = y_2$ and let $S = \{z_1, z_2, \cdots, z_{b-a}\}$ be the set of all simplicial vertices of G. For $1 \leq j \leq a-1$, let $S_j = \{u_{1,j}, u_{2,j}\}$ and let $S_a = \{v_2, v_3\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains S and exactly one vertex from each set $S_j (1 \le j \le a)$. Hence $m_x(G) \ge b$. Since $S_x^{'} = S \bigcup \{v_2, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G with $|S_x^{'}| = b$, it follows that $m_x(G) = b$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = S \bigcup \{v_2, u_{1,1}, u_{1,2}, ..., u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Since S is the set of all x-monophonic vertices of G and by Theorem 2.8, $f_{m_x}(G) \le m_x(G) - |S| = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x^{'} = (S_x - \{u_{1,1}\}) \bigcup \{u_{2,1}\}$. Then $S_x^{'} \ne S_x$ and $S_x^{'}$ is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x an x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Clearly, $S \bigcup \{v_3, u_{2,1}, u_{2,2}, \cdots, u_{2,a-1}, y_3, y_4, \cdots, y_{c-b+2}\}$ is the unique minimum connected x-monophonic set of G, we have $cm_x(G) = c$.

Case 2. $2 \le a = b < c$ and $c = b + 1$.

Construct the graph H_2 in Case 1. Then $G = H_2$ has the desired properties (S is the empty set).

Case 3. $2 \le a = b < c$ and $c \ge b + 2$. For each i with $1 \le i \le a - 1$, let $F_i : y_1, u_{i,1}, u_{i,2}, y_3$ be a path of order 4. Let $P_{c-a+1} : y_1, y_2, y_3, \cdots, y_{c-a+1}$ be a path of order $c - a + 1$ and let $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of order 5. Let H be a graph obtained from F_i and P_{c-a+1} by identifying the vertices y_1 and y_3 of all $F_i (1 \le i \le a - 1)$ and P_{c-a+1}. Let G be the graph obtained from H by identifying the vertex y_{c-a+1} of P_{c-a+1} and v_1 of C_5. The graph G is shown in Figure 2.5. Let $x = y_2$.

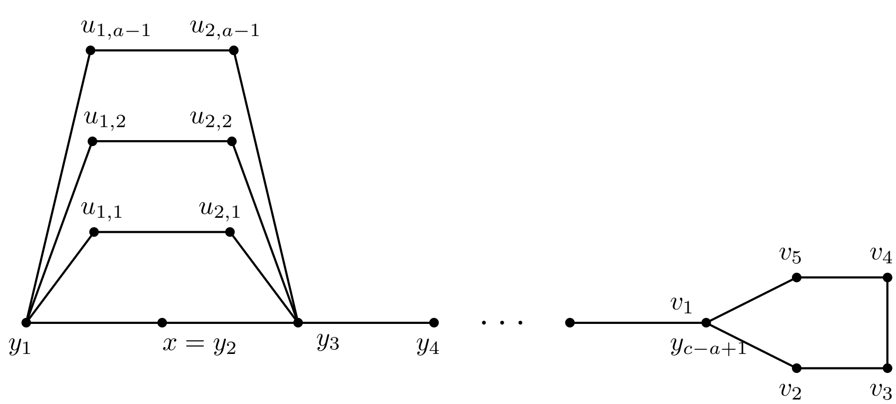

Figure 2.5

For $1 \le j \le a - 1$, let $S_j = \{u_{1,j}, u_{2,j}\}$ and let $S_a = \{v_3, v_4\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains exactly one vertex from each set $S_j (1 \le j \le a)$ so that $m_x(G) \ge a$. Since $S_x^{'} = \{v_3, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G with $|S_x^{'}| = a$, we have $m_x(G) = a$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = \{v_3, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Then by Theorem 2.3, $f_{m_x}(G) \leq m_x(G) = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x^{'} = (S_x - \{u_{1,1}\}) \bigcup \{u_{1,2}\}$. Then $S_x^{'} \neq S_x$ and $S_x^{'}$ is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x an x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Let $S = \{v_2, v_3, u_{2,1}, u_{2,2}, \cdots, u_{2,a-1}, y_3, y_4, \cdots, y_{c-a+1}\}$. It is easily verified that S is a minimum connected x-monophonic set of G and so $cm_x(G) = c$. □

Problem 2.19 *For any three positive integers a, b and c with $2 \leq a \leq b = c$, does there exist a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G?*

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Skolem Difference Odd Mean Labeling of H-Graphs

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Abstract: A graph G with p vertices and q edges is said to have a skolem difference odd mean labeling if there exists an injective function $f : V(G) \to \{1, 2, 3, \cdots, 4q-1\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ is a bijection. A graph that admits skolem difference odd mean labeling is called a skolem difference odd mean graph. In this paper, we investigate skolem difference odd mean labeling of some H-graphs.

Key Words: Skolem difference odd mean labeling, skolem difference odd mean graph.

AMS(2010): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Path on n vertices is denoted by P_n. $K_{1,m}$ is called a star and is denoted by S_m. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. The H-graph of a path P_n, denoted by H_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even. The corona of a graph G on p vertices $v_1, v_2, \cdots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \ldots, u_p$ and the new edges u_iv_i for $1 \leq i \leq p$. The corona of G is denoted by $G \odot K_1$. The 2-corona of a graph G, denoted by $G \odot S_2$ is a graph obtained from G by identifying the center vertex of the star S_2 at each vertex of G. The disjoint union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

The concept of mean labeling was introduced and studied by S. Somasundaram and R. Ponraj [5]. Some new families of mean graphs are studied by S.K. Vaidya et al. [6]. Further some more results on mean graphs are discussed in [4,7,8]. A graph G is said to be a mean graph if there exists an injective function f from $V(G)$ to $\{0, 1, 2, \cdots, q\}$ such that the induced map f^* from $E(G)$ to $\{1, 2, 3, \cdots, q\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection.

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In [2], K. Manickam and M. Marudai introduced odd mean labeling of a graph. A graph G is said to be odd mean if there exists an injective function f from $V(G)$ to $\{0, 1, 2, 3, \cdots, 2q-1\}$ such that the induced map f^* from $E(G)$ to $\{1, 3, 5, \cdots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection. Some more results on odd mean graphs are discussed in [9,10].

The concept of skolem difference mean labeling was introduced and studied by K. Murugan and A. Subramanian [3]. A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $1, 2, 3, \cdots, p+q$ in such a way that for each edge $e = uv$, let $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the resulting labels of the edges are distinct and are from $1, 2, 3, \cdots, q$. A graph that admits a skolem difference mean labeling is called a skolem difference mean graph.

The concept of skolem difference odd mean labeling was introduced in [11]. A graph with p vertices and q edges is said to have a skolem difference odd mean labeling if there exists an injective function $f : V(G) \to \{1, 2, 3, \cdots, 4q-1\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ is a bijection. A graph that admits a skolem difference odd mean labeling is called a skolem difference odd mean graph.

A skolem difference odd mean labeling of $B_{4,7}$ is shown in Figure 1.

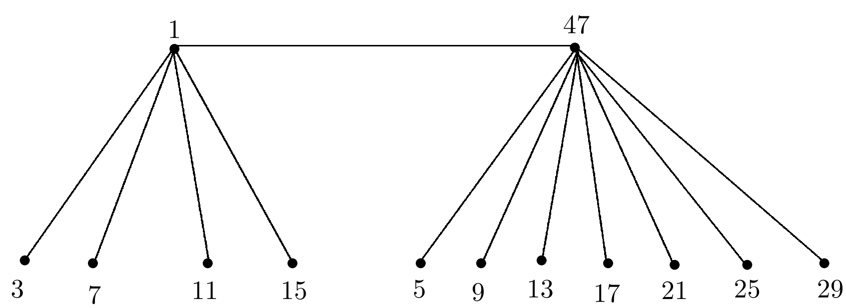

Figure 1

In this paper, we prove that the H-graph, corona of a H-graph, 2-corona of a H-graph are skolem difference odd mean graph. Also we prove that union of any two skolem difference odd mean H-graphs is also a skolem difference odd mean graph.

§2. Skolem Difference Odd Mean Graphs

Theorem 2.1 *The H-graph G is a skolem difference odd mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ be the vertices of the H-graph G. The graph G has $2n$ vertices and $2n-1$ edges.

Define $f : V(G) \to \{1, 2, 3, \cdots, 4q-1 = 8n-5\}$ as follows:

$$f(v_i) = \begin{cases} 2i-1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 8n-2i-1, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(u_i) = \begin{cases} 6n-2i-1, & \text{if } n \text{ is odd, } 1 \le i \le n \text{ and } i \text{ is odd} \\ 2n+2i-1, & \text{if } n \text{ is odd, } 1 \le i \le n \text{ and } i \text{ is even} \\ 2n+2i-1, & \text{if } n \text{ is even, } 1 \le i \le n \text{ and } i \text{ is odd} \\ 6n-2i-1, & \text{if } n \text{ is even, } 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$$f^*(v_iv_{i+1}) = 4n-2i-1, \quad 1 \le i \le n-1$$
$$f^*(u_iu_{i+1}) = 2n-2i-1, \quad 1 \le i \le n-1$$

$$f^*\left(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}\right) = 2n-1 \quad \text{if n is odd and}$$
$$f^*\left(v_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) = 2n-1 \quad \text{if n is even.}$$

Thus, f is a skolem difference odd mean labeling and hence the H-graph G is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of H-graphs G_1 and G_2 are shown in Figure 2.

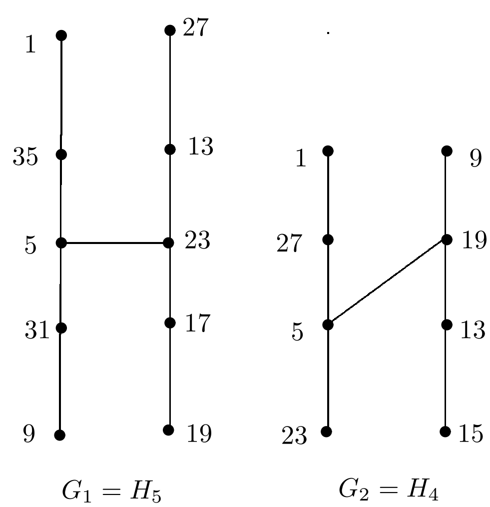

Figure 2

Theorem 2.2 *For a H-graph G, $G \odot K_1$ is a skolem difference odd mean graph.*

Proof By Theorem 2.1, there exists a skolem difference odd mean labeling f for G. Let

$v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ be the vertices of G.

$$\text{Let } V(G \odot K_1) = V(G) \cup \{v'_1, v'_2, \ldots, v'_n\} \cup \{u'_1, u'_2, \ldots, u'_n\}$$
$$\text{and } E(G \odot K_1) = E(G) \cup \{v_i v'_i, u_i u'_i : 1 \le i \le n\}.$$

Case 1. n is odd.

Define $g : V(G \odot K_1) \to \{1, 2, \cdots, 16n-5\}$ as follows:

$$\begin{aligned}
g(v_{2i-1}) &= f(v_{2i-1}), \quad 1 \le i \le \frac{n+1}{2} \\
g(v_{2i}) &= f(v_{2i}) + 8n, \quad 1 \le i \le \frac{n-1}{2} \\
g(u_{2i-1}) &= f(u_{2i-1}) + 8n, \quad 1 \le i \le \frac{n+1}{2} \\
g(u_{2i}) &= f(u_{2i}) \\
g(v'_{2i-1}) &= g(u_n) - 4n - 4(i-1), \quad 1 \le i \le \frac{n+1}{2} \\
g(v'_{2i}) &= g(u_{n-1}) + 4n + 4i, \quad 1 \le i \le \frac{n-1}{2} \\
g(u'_{2i-1}) &= g(u_n) - 2n + 4(i-1), \quad 1 \le i \le \frac{n+1}{2} \\
g(u'_{2i}) &= g(u_{n-1}) + 2n - 4(i-1), \quad 1 \le i \le \frac{n-1}{2}.
\end{aligned}$$

For the vertex labeling g, the induced edge labeling g^* is given as follows:

$$\begin{aligned}
g^*(v_i v_{i+1}) &= f^*(v_i v_{i+1}) + 4n, \quad 1 \le i \le n-1 \\
g^*(u_i u_{i+1}) &= f^*(u_i u_{i+1}) + 4n, \quad 1 \le i \le n-1 \\
g^*(v_i v'_i) &= 4n + 1 - 2i, \quad 1 \le i \le n \\
g^*(u_i u'_i) &= 2n + 1 - 2i, \quad 1 \le i \le n \\
g^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) &= 3f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) + 2.
\end{aligned}$$

Case 2. n is even.

Define $g : V(G \odot K_1) \to \{1, 2, 3, \cdots, 16n-5\}$ as follows:

$$\begin{aligned}
g(v_{2i-1}) &= f(v_{2i-1}), \quad 1 \le i \le \frac{n}{2} \\
g(v_{2i}) &= f(v_{2i}) + 8n, \quad 1 \le i \le \frac{n}{2} \\
g(u_{2i-1}) &= f(u_{2i-1}), \quad 1 \le i \le \frac{n}{2} \\
g(u_{2i}) &= f(u_{2i}) + 8n, \quad 1 \le i \le \frac{n}{2} \\
g(v'_{2i-1}) &= g(u_{n-1}) + 4n + 6 - 4i, \quad 1 \le i \le \frac{n}{2}
\end{aligned}$$

$$g(v'_{2i}) = g(u_n) - 4n - 2 + 4i, \quad 1 \leq i \leq \frac{n}{2}$$
$$g(u'_{2i-1}) = g(u_{n-1}) + 2(n+1) - 4(i-1), \quad 1 \leq i \leq \frac{n}{2}$$
$$g(u'_{2i}) = g(u_n) - 2(n+1) + 4i, \quad 1 \leq i \leq \frac{n}{2}.$$

For the vertex labeling g, the induced edge labeling g^* is obtained as follows:

$$g^*(v_i v_{i+1}) = f^*(v_i v_{i+1}) + 4n$$
$$g^*(u_i u_{i+1}) = f^*(u_i u_{i+1}) + 4n$$
$$g^*(v_i v'_i) = 4n + 1 - 2i$$
$$g^*(u_i u'_i) = 2n + 1 - 2i$$
$$g^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) = 3f^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) + 2.$$

Thus, g is a skolem difference odd mean labeling and hence $G \odot K_1$ is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of H-graphs $G_1, G_2, G_1 \odot K_1$ and $G_2 \odot K_1$ are shown in Figure 3.

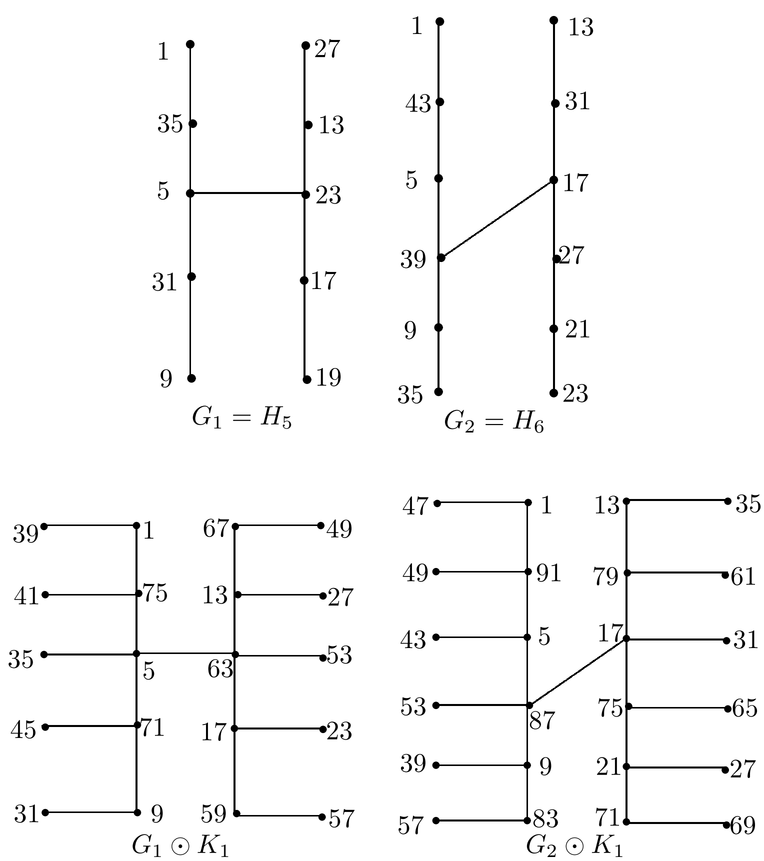

Figure 3

Theorem 2.3 *For a H-graph G, $G \odot S_2$ is a skolem difference odd mean graph.*

Proof By Theorem 2.1, there exists a skolem difference odd mean labeling f for G. Let $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ be the vertices of G. Let $V(G)$ together with $v'_1, v'_2, \cdots, v'_n, v''_1, v''_2, \cdots, v''_n, u'_1, u'_2, \cdots, u'_n$ and $u''_1, u''_2, \cdots, u''_n$ form the vertex set of $G \odot S_2$ and the edge set is $E(G)$ together with $\{v_iv'_i, v_iv''_i, u_iu'_i, u_iu''_i : 1 \leq i \leq n\}$.

Case 1. n is odd.

Define $g : V(G \odot S_2) \to \{1, 2, 3, \cdots, 24n - 5\}$ as follows:

$$g(v_{2i-1}) = f(v_{2i-1}), \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(v_{2i}) = f(v_{2i}) + 16n, \quad 1 \leq i \leq \frac{n-1}{2}$$
$$g(u_{2i-1}) = f(u_{2i-1}) + 16n, \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(u_{2i}) = f(u_{2i}), \quad 1 \leq i \leq \frac{n-1}{2}$$
$$g(v'_{2i-1}) = g(u_n) - 4n - 12(i-1), \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(v'_{2i}) = g(u_{n-1}) + 4n - 4 + 12i, \quad 1 \leq i \leq \frac{n-1}{2}$$
$$g(v''_{2i-1}) = g(v'_{2i-1}) - 4, \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(v''_{2i}) = g(v'_{2i}) + 4, \quad 1 \leq i \leq \frac{n-1}{2}$$
$$g(u'_{2i-1}) = g(u_n) - 6n + 12(i-1), \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(u'_{2i}) = g(u_{n-1}) + 4n + 18 - 12i, \quad 1 \leq i \leq \frac{n-1}{2}$$
$$g(u''_{2i-1}) = g(u'_{2i-1}) + 4, \quad 1 \leq i \leq \frac{n+1}{2}$$
$$g(u''_{2i}) = g(u'_{2i}) - 4, \quad 1 \leq i \leq \frac{n-1}{2}.$$

For the vertex labeling g, the induced edge labeling g^* is given as follows:

$$g^*(v_iv_{i+1}) = f^*(v_iv_{i+1}) + 8n, \quad 1 \leq i \leq n-1$$
$$g^*(u_iu_{i+1}) = f^*(u_iu_{i+1}) + 8n, \quad 1 \leq i \leq n-1$$
$$g^*(v_iv'_i) = 8n + 3 - 4i, \quad 1 \leq i \leq n$$
$$g^*(v_iv''_i) = 8n + 1 - 4i, \quad 1 \leq i \leq n$$
$$g^*(u_iu'_i) = 4n + 3 - 4i, \quad 1 \leq i \leq n$$
$$g^*(u_iu''_i) = 4n + 1 - 4i, \quad 1 \leq i \leq n$$
$$g^*\left(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}\right) = 5f^*\left(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}\right) + 4.$$

Case 2. n is even.

Define $g : V(G \odot S_2) \to \{1, 2, 3, \cdots, 24n-5\}$ as follows:

$$g(v_{2i-1}) = f(v_{2i-1}), \quad 1 \le i \le \frac{n}{2}$$
$$g(v_{2i}) = f(v_{2i}) + 16n, \quad 1 \le i \le \frac{n}{2}$$
$$g(u_{2i-1}) = f(u_{2i-1}), \quad 1 \le i \le \frac{n}{2}$$
$$g(u_{2i}) = f(u_{2i}) + 16n, \quad 1 \le i \le \frac{n}{2}$$
$$g(v'_{2i-1}) = g(u_{n-1}) + 12n + 14 - 12i, \quad 1 \le i \le \frac{n}{2}$$
$$g(v'_{2i}) = g(u_n) - 12n - 6 + 12i, \quad 1 \le i \le \frac{n}{2}$$
$$g(v''_{2i-1}) = g(v'_{2i-1}) - 4, \quad 1 \le i \le \frac{n}{2}$$
$$g(v''_{2i}) = g(v'_{2i}) + 4, \quad 1 \le i \le \frac{n}{2}$$
$$g(u'_{2i-1}) = g(u_{n-1}) + 6n + 14 - 12i, \quad 1 \le i \le \frac{n}{2}$$
$$g(u'_{2i}) = g(u_n) - 6n - 6 + 12i, \quad 1 \le i \le \frac{n}{2}$$
$$g(u''_{2i-1}) = g(u'_{2i-1}) - 4, \quad 1 \le i \le \frac{n}{2}$$
$$g(u''_{2i}) = g(u'_{2i}) + 4, \quad 1 \le i \le \frac{n}{2}.$$

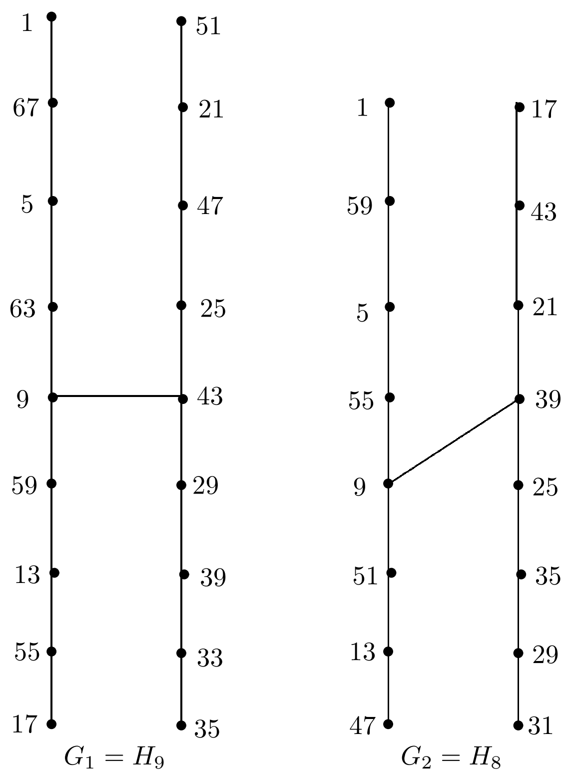

Figure 4

For the vertex labeling g, the induced edge labeling g^* is obtained as follows:

$$g^*(v_iv_{i+1}) = f^*(v_iv_{i+1}) + 8n, \quad 1 \le i \le n-1$$
$$g^*(u_iu_{i+1}) = f^*(u_iu_{i+1}) + 8n, \quad 1 \le i \le n-1$$
$$g^*(v_iv_i') = 8n + 3 - 4i, \quad 1 \le i \le n$$
$$g^*(v_iv_i'') = 8n + 1 - 4i, \quad 1 \le i \le n$$
$$g^*(u_iu_i') = 4n + 3 - 4i, \quad 1 \le i \le n$$
$$g^*(u_iu_i'') = 4n + 1 - 4i, \quad 1 \le i \le n$$
$$g^*\left(v_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) = 5f^*\left(v_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) + 4.$$

Thus, f is a skolem difference odd mean labeling and hence the graph $G \odot S_2$ is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of H-graphs $G_1, G_2, G_1 \odot S_2$ and $G_2 \odot S_2$ are shown in Figures 4 and 5.

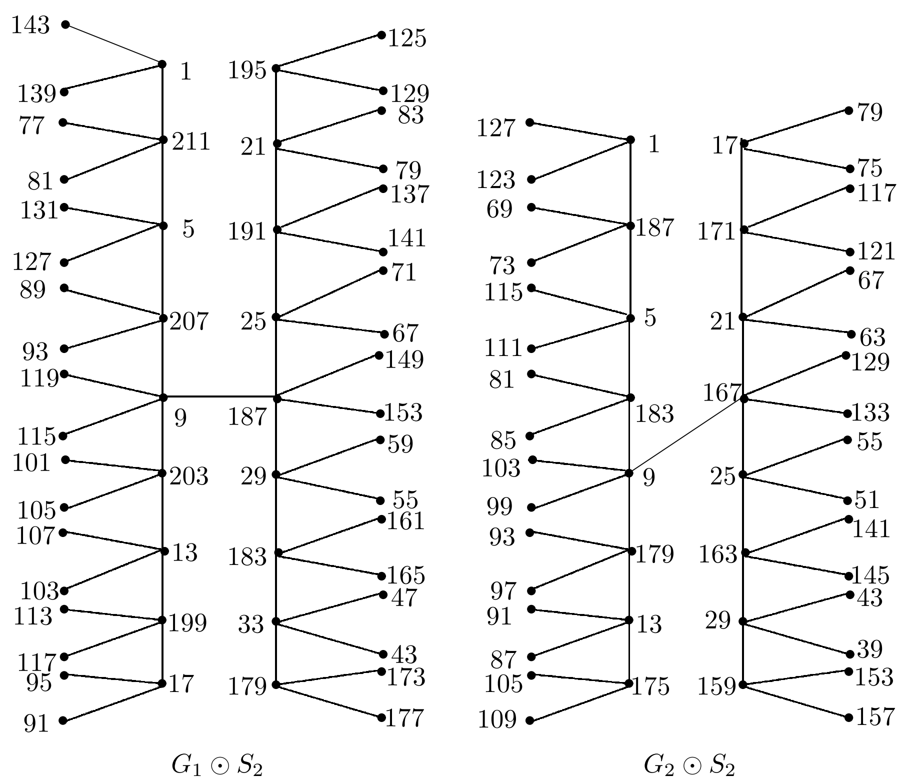

Figure 5

Theorem 2.4 *If G_1 and G_2 are skolem difference odd mean H-graphs, then $G_1 \cup G_2$ is also a skolem difference odd mean graph.*

Proof Let $V(G_1) = \{u_i, v_i : 1 \le i \le n\}$ and $V(G_2) = \{s_j, t_j : 1 \le j \le m\}$ be the vertices of the H-graphs G_1 and G_2 respectively. Then the graph $G_1 \cup G_2$ has $2(n+m)$ vertices and $2(n+m-1)$ edges. Let $f : V(G_1) \to \{1, 2, 3, \cdots, 8n-5\}$ and $g : V(G_2) \to \{1, 2, 3, \cdots, 8m-5\}$ be a skolem difference odd mean labeling of G_1 and G_2 respectively.

Define $h : V(G_1 \cup G_2) \to \{1, 2, 3, \cdots, 4q-1 = 8(n+m)-9\}$ as follows:

For $1 \le i \le n$ and $n \ge 1$,

$$h(u_i) = \begin{cases} f(u_i) & \text{if } i \text{ is odd} \\ f(u_i) + 8m - 4 & \text{if } i \text{ is even} \end{cases}$$

$$h(v_i) = \begin{cases} f(v_i) + 8m - 4 & \text{if } n \text{ is odd and } i \text{ is odd} \\ f(v_i) & \text{if } n \text{ is odd and } i \text{ is even} \\ f(v_i) & \text{if } n \text{ is even and } i \text{ is odd} \\ f(v_i) + 8m - 4 & \text{if } n \text{ is even and } i \text{ is even.} \end{cases}$$

For $1 \le j \le m$ and $m \ge 1$,

$$h(s_j) = g(s_j) + 2$$
$$h(t_j) = g(t_j) + 2.$$

For the vertex labeling h, the induced edge labeling h^* is given as follows:

For $1 \le i \le n-1$ and $n \ge 1$,

$$h^*(u_i u_{i+1}) = f^*(u_i u_{i+1}) + 4m - 2$$
$$h^*(v_i v_{i+1}) = f^*(v_i v_{i+1}) + 4m - 2$$
$$h^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) + 4m - 2 \quad \text{if n is odd and}$$
$$h^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) + 4m - 2 \quad \text{if n is even.}$$

For $1 \le j \le m-1$ and $m \ge 1$,

$$h^*(s_j s_{j+1}) = g^*(s_j s_{j+1})$$
$$h^*(t_j t_{j+1}) = g^*(t_j t_{j+1})$$

$$h^*\left(s_{\frac{m+1}{2}} t_{\frac{m+1}{2}}\right) = g^*\left(s_{\frac{m+1}{2}} t_{\frac{m+1}{2}}\right) \quad \text{if m is odd}$$
$$h^*\left(s_{\frac{m}{2}+1} t_{\frac{m}{2}}\right) = g^*\left(s_{\frac{m}{2}+1} t_{\frac{m}{2}}\right) \quad \text{if m is even.}$$

Thus, h is a skolem difference odd mean labeling of $G_1 \cup G_2$ and hence the graph $G_1 \cup G_2$ is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of $G_1 \cup G_2$ where $G_1 = H_3$; $G_2 =$

$H_5,\ G_1 = H_5;\ G_2 = H_6,\ G_1 = H_4;\ G_2 = H_6$ and $G_1 = H_4;\ G_2 = H_4$ are shown in Figures 6-9 following.

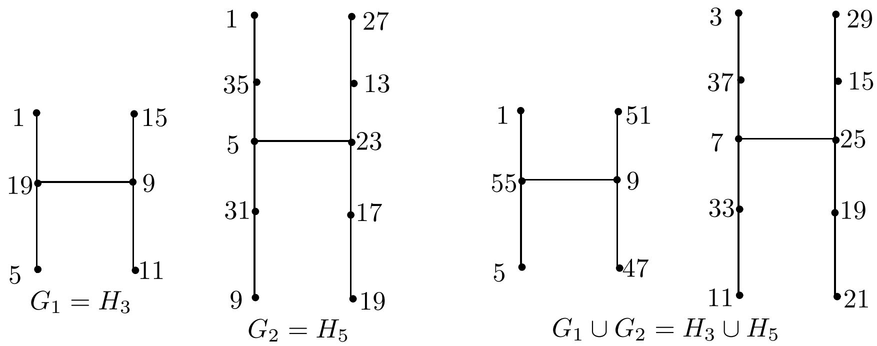

Figure 6

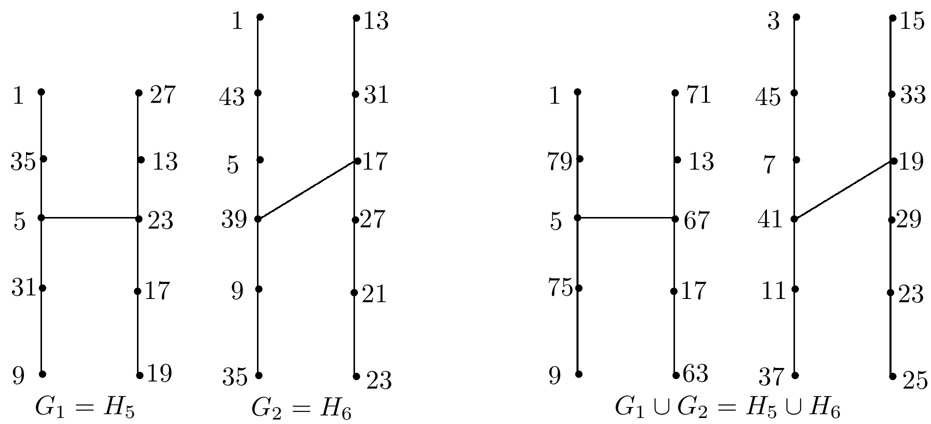

Figure 7

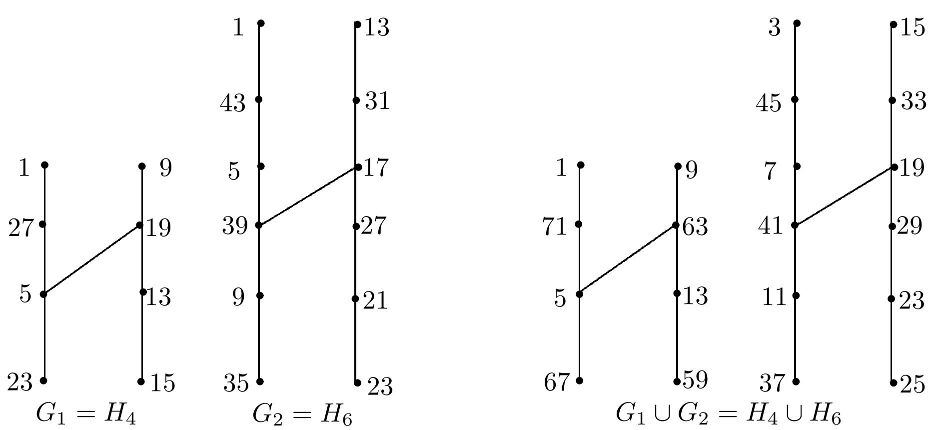

Figure 8

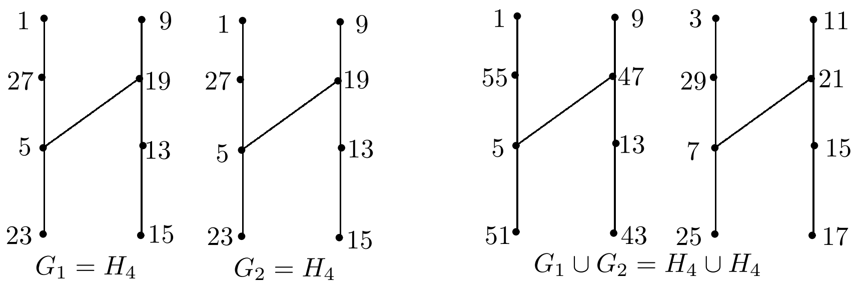

Figure 9

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Equitable Total Coloring of Some Graphs

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Abstract: In this paper we determine the equitable total chromatic number χ_{et} for the double star graph $K_{1,n,n}$ and the fan graph $F_{m,n}$.

Key Words: Equitable total coloring, double star graph, fan graph.

AMS(2010): 05C15, 05C69

§1. Introduction

In this paper, we consider only finite simple graphs without loops or multiple edges. Let $G(V, E)$ be a graph with the set of vertices V and the edge set E. Total coloring $\chi_t(G)$ was introduced by Vizing [7] and Behzad [1]. They both conjectured that for any graph G the following inequality holds: $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$. It is obvious that $\Delta(G) + 1$ is the best possible lower bound. This conjecture is proved so far for some specific classes of graphs. In general the equitable total coloring problem is more difficult than the total coloring problem. In 1994, Fu [4] gave the concepts of an equitable total coloring and the equitable total chromatic number of a graph. For a simple graph $G(V, E)$, let f be a proper $k-$total coloring of G

$$||T_i| - |T_j|| \leq 1, \ i, j = 1, 2, \ldots, k.$$

The partition $\{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\}$ is called a $k-$equitable total coloring ($k-$ETC of G in brief), and

$$\chi_{et}(G) = \min\{k|k - \text{ETC of } G\}$$

is called the equitable total chromatic number [2-6, 10] of G, where $\forall x \in T_i = V_i \cup E_i$, $f(x) = i$, $i = 1, 2, \cdots, k$. It is obvious that $\chi_{et}(G) \geq \Delta + 1$. Furthermore Fu presented a conjecture concerning the equitable total chromatic number (simply denoted by ETCC)

Conjecture 1.1([4]) *For any simple graph $G(V, E)$,*

$$\chi_{et}(G) \leq \Delta(G) + 2.$$

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These Researchers in [2, 3, 5, 6, 8-10] have concentrated in providing the equitable total chromatic number for specific families of graphs.

Lemma 1.2([10]) *For any simple graph* $G\,(V,E)$,

$$\chi_{et}\,(G) \geq \chi_t(G) \geq \Delta\,(G) + 1.$$

Lemma 1.3([4]) *For complete graph* K_p *with order* p,

$$\chi_{et}\,(K_p) = \begin{cases} p, & p \equiv 1 \bmod 2 \\ p+1, & p \equiv 0 \bmod 2. \end{cases}$$

Lemma 1.4([3]) *For* $n \geq 13$ *the total equitable chromatic number of Hypo-Mycielski Graph,* $\chi_{et}(HM(W_n)) = n + 2$.

In [9], equitable total chromatic numbers of some join graphs were given. Gong Kun et.al [2] proved some results on the equitable total chromatic number of $W_m \vee K_n$, $F_m \vee K_n$ and $S_m \vee K_n$. In 2012, Ma Gang and Ma Ming [6] proved some results concerning the equtiable total chromatic number of $P_m \vee S_n$, $P_m \vee F_n$ and $P_m \vee W_n$.

In the present paper, we find the equitable total chromatic number χ_{et} for the double star graph $K_{1,n,n}$ and the fan graph $F_{m,n}$.

§2. Preliminaries

Definition 2.1 *A double star* $K_{1,n,n}$ *is a tree obtained from the star* $K_{1,n}$ *by adding a new pendant edge of the existing* n *pendant vertices. It has* $2n+1$ *vertices and* $2n$ *edges. Let* $V\,(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ *and* $E\,(K_{1,n,n}) = \{e_1, e_2, \ldots, e_n\} \cup \{s_1, s_2, \ldots, s_n\}$.

Definition 2.2 *A Fan graph* $\overline{K_m} + P_n$ *where* P_n *is path on* n *vertices. All the vertices of the fan corresponding to the path* P_n *are labeled from* m *to* n *consecutively. The vertices in the fan corresponding* $\overline{K_m}$ *is labeled* $m+n$.

§3. Equitable Total Coloring of Double Star Graphs

Theorem 3.1 *For any positive integer* n,

$$\chi_{et}\,(K_{1,n,n}) = n + 1.$$

Proof Let $V\,(K_{1,n,n}) = \{u_0\} \cup \{u_i : 1 \leq i \ \leq n\} \cup \{v_i : 1 \leq i \leq n\}$ and $E\,(K_{1,n,n}) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$, where $e_i\,(1 \leq i \leq n)$ is the edge $u_0 u_i\,(1 \leq i \leq n)$ and $s_i(1 \leq i \leq n)$ is the edge $u_i v_i\,(1 \leq i \leq n)$. Now we partition the edge and vertex sets in $K_{1,n,n}$ as follows.

$$\begin{aligned}
T_1 &= \{e_1, u_n, s_{n-1}, v_{n-2}\}\\
T_2 &= \{e_2, u_1, s_n, v_{n-1}\}\\
T_3 &= \{e_3, u_2, v_n\}\\
T_4 &= \{e_4, u_3, s_1\}\\
T_k &= \{e_k, u_{k-1}, s_{k-3}, v_{k-4}\}\,(5 \le k \le n)\\
T_{n+1} &= \{u_0, s_{n-2}, v_{n-3}\}
\end{aligned}$$

Clearly T_1, T_2, T_3, T_4, T_k and T_{n+1} are independent sets of $K_{1,n,n}$. Also $|T_1| = |T_2| = |T_k| = 4\,(5 \le k \le n)$ and $|T_3| = |T_4| = |T_{n+1}| = 3$, it holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i, j). This implies $\chi_{et}(K_{1,n,n}) \le n+1$. Since the set of edges $\{e_1, e_2, \ldots e_n\}$ and u_0 receives distinct color, $\chi_{et}(K_{1,n,n}) \ge \chi_t(K_{1,n,n}) \ge n+1$. Hence $\chi_{et}(K_{1,n,n}) \ge n+1$. Therefore $\chi_{et}(K_{1,n,n}) = n+1$. □

§4. Equitable Total Coloring of Fan Graphs

Theorem 4.1 *For any positive integer n, m $(n \ge m)$, then*

$$\chi_{et}(F_{m,n}) = \begin{cases} \Delta + 2 \text{ if } n = m\\ \Delta + 2 \text{ if } n - m = 2\\ \Delta + 1 \text{ if } n - m = 1\\ n + 2 \text{ if } n - m \ge 3 \end{cases}$$

Proof Let $V(F_{m,n}) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\}$ and $E(F_{m,n}) = \bigcup_{i=1}^{m} \{e_{i,j} : 1 \le j \le n\} \cup \{e_j : 1 \le j \le n-1\}$, where $e_j\,(1 \le j \le n-1)$ is the edge $v_j v_{j+1}\,(1 \le j \le n-1)$ and $e_{i,j}$ is the edge $u_i v_j\,(1 \le i \le m, 1 \le j \le n)$. Now we partition the edge and vertex sets of $F_{m,n}$ in the following cases.

Case 1. $n = m$

$$\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,4}\} \cup \{v_3\}\\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{5,n}, e_{6,n-1}, e_{7,n-2}, \ldots, e_{m,5}\} \cup \{v_4\}\\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{6,n}, e_{7,n-1}, e_{8,n-2}, \ldots, e_{m,6}\} \cup \{v_5\}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
T_{\Delta-4} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m-2,1}\} \cup \{v_n\}\\
T_{\Delta-3} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m-1,1}\}
\end{aligned}$$

$$
\begin{aligned}
T_{\Delta-2} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,1}\}\\
T_{\Delta-1} &= \{e_{2,n}, e_{3,n-1}, \ldots, e_{m,2}\} \cup \{v_1\}\\
T_{\Delta} &= \{e_{3,n}, e_{4,n-1}, \ldots, e_{m,3}\} \cup \{v_2\}\\
T_{\Delta+1} &= \left\{e_{2i} : 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{u_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\right\}\\
T_{\Delta+2} &= \left\{e_{2i-1} : 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil\right\} \cup \left\{u_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \leq 1$ for every pair (i, j). This implies $\chi_{et}(F_{m,n}) \leq \Delta + 2$. Each edge $v_j v_{j+1}$ $(1 \leq j \leq n-1)$ is adjacent with $2m + 2$ edges and incident with two vertices,it forms a triangle with atleast one vertex of $\{u_i : 1 \leq i \leq m\}$. Therefore equitable total coloring needs $\Delta + 2$ colors.$\chi_{et}(F_{m,n}) \geq \chi_t(F_{m,n}) \geq \Delta + 2$. Hence $\chi_{et}(F_{m,n}) \geq \Delta + 2$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 2$.

Case 2. $n - m = 2$

$$
\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,4}\} \cup \{v_3\}\\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,5}\} \cup \{v_4\}\\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,6}\} \cup \{v_5\}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
T_{\Delta-2} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m,1}\} \cup \{v_n\}\\
T_{\Delta-1} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,2}\} \cup \{v_1\}\\
T_{\Delta} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,3}\} \cup \{v_2\}\\
T_{\Delta+1} &= \left\{u_{2i-1} : 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{e_{2i} : 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right\}\\
T_{\Delta+2} &= \left\{u_{2i} : 1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor\right\} \cup \left\{e_{2i-1} : 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \leq 1$ for every pair (i, j). This implies $\chi_{et}(F_{m,n}) \leq \Delta + 2$. Since each edge $v_j v_{j+1}$ $(1 \leq j \leq n-1)$ is adjacent with $2m+2$ edges and incident with two vertices,it forms a triangle with atleast one vertex of $\{u_i : 1 \leq i \leq m\}$. Therefore equitable total coloring needs $\Delta + 2$ colors. $\chi_{et}(F_{m,n}) \geq \chi_t(F_{m,n}) \geq \Delta + 2$. Hence $\chi_{et}(F_{m,n}) \geq \Delta + 2$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 2$.

Case 3. $n - m = 1$

$$
\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,3}\} \cup \{v_2\}\\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,4}\} \cup \{v_3\}\\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,5}\} \cup \{v_4\}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

$$
\begin{aligned}
T_{\Delta-2} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,1}\} \cup \{v_n\} \\
T_{\Delta-1} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_{\Delta} &= \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{u_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\} \\
T_{\Delta+1} &= \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\} \cup \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+1}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i,j). This implies $\chi_{et}(F_{m,n}) \le \Delta + 1$. Since at each vertex v_j $(2 \le j \le n-1)$ there exist Δ mutually adjacent edges and v_j $(2 \le j \le n-1)$ needs one more color. $\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge \Delta + 1$. Hence $\chi_{et}(F_{m,n}) \ge \Delta + 1$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 1$.

Case 4. $n - m \ge 3$

$$
\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,4}\} \cup \{v_3\} \\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,5}\} \cup \{v_4\} \\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,6}\} \cup \{v_5\} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
T_{n-2} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m,1}\} \cup \{v_n\} \\
T_{n-1} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_n &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,3}\} \cup \{v_2\} \\
T_{n+1} &= \left\{u_{2i-1} : 1 \le i \le \left\lceil \frac{m}{2} \right\rceil\right\} \cup \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \\
T_{n+2} &= \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor\right\} \cup \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{n+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i, j). This implies $\chi_{et}(F_{m,n}) \le n + 2$. Since at each vertex u_i $(1 \le i \le m)$ there exist n mutually adjacent edges and u_i $(1 \le i \le m)$ needs one more color. $\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge n + 2$. Hence $\chi_{et}(F_{m,n}) \ge n + 2$. Therefore $\chi_{et}(F_{m,n}) = n + 2$. □

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Some Characterizations for the Involute Curves in Dual Space

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Abstract: In this paper, we investigate some characterizations of involute – evolute curves in dual space. Then the relationships between dual Frenet frame and Darboux vectors of these curves are found.

Key Words: Dual curve, involute, evolute, dual space.

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§1. Introduction

Involute-evolute curve couple was originally defined by Christian Huygens in 1668. In the theory of curves in Euclidean space, one of the important and interesting problems is the characterizations of a regular curve. In particular, the involute of a given curve is a well known concept in the classical differential geometry (for the details see [7]). For classical and basic treatments of Involute-evolute curve couple, we refer to [1], [5], [7-9] and [13].

The relationships between the Frenet frames of the involute-evolute curve couple have been found as depend on the angle between binormal vector B and Darboux vector W of evolute curve, [1]. In the light of the existing literature, similar studies have been constructed on Lorentz and Dual Lorentz space,[2-4, 10-12].

In this paper, The relationships between dual Frenet frame and Darboux vectors of these curves have been found. Additionally, some important results concerning these curves are given.

§2. Preliminaries

Dual numbers were introduced by W.K. Clifford (1849-79) as a tool for his geometrical investi-

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gations. The set $ID = \left\{ A = a + \varepsilon a^* \mid a, a^* \epsilon IR,\ \varepsilon^2 = 0 \right\}$ is called dual numbers set. On this set product and addition operations are described as

$$
\begin{aligned}
(a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon (a^* + b^*), \\
(a + \varepsilon a^*) . (b + \varepsilon b^*) &= ab + \varepsilon (ab^* + a^* b),
\end{aligned}
$$

respectively. The elements of the set $ID^3 = \left\{ \vec{A} \mid \vec{A} = \vec{a} + \varepsilon \vec{a^*},\ \vec{a}, \vec{a^*} \epsilon IR^3 \right\}$ are called dual vectors. On this set, addition and scalar product operations are described as

$$
\begin{aligned}
\oplus &: ID^3 \times ID^3 \to ID^3 ,\ \vec{A} \oplus \vec{B} = \left(\vec{a} + \vec{b} \right) + \epsilon \left(\vec{a^*} + \vec{b^*} \right), \\
\odot &: ID \times ID^3 \to ID^3 ,\quad \widetilde{\lambda} \odot \vec{A} = \lambda \vec{a} + \epsilon \left(\lambda \vec{a^*} + \lambda^* \vec{a} \right),
\end{aligned}
$$

respectively. Algebraic construction $\left(ID^3, \oplus, ID, +, ., \odot \right)$ is a modul. This modul is called *ID-Modul*.

The inner product and vectorel product of dual vectors $\vec{A}, \vec{B} \epsilon ID^3$ are defined by respectively,

$$
\begin{aligned}
\langle\ ,\ \rangle &: ID^3 \times ID^3 \to ID ,\ \left\langle \vec{A}, \vec{B} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b^*} \right\rangle + \left\langle \vec{a^*}, \vec{b} \right\rangle \right) \\
\wedge &: ID^3 \times ID^3 \to ID^3 ,\ \vec{A} \wedge \vec{B} = \left(\vec{a} \wedge \vec{b} \right) + \varepsilon \left(\vec{a} \wedge \vec{b^*} + \vec{a^*} \wedge \vec{b} \right).
\end{aligned}
$$

For $\vec{A} \neq 0$, the norm $\left\| \vec{A} \right\|$of $\vec{A} = \vec{a} + \varepsilon \vec{a^*}$ is defined by

$$
\left\| \vec{A} \right\| = \sqrt{\left\langle \vec{A}, \vec{A} \right\rangle} = \left\| \vec{a} \right\| + \varepsilon \frac{\left\langle \vec{a}, \vec{a^*} \right\rangle}{\left\| \vec{a} \right\|} \ ,\ \left\| \vec{a} \right\| \neq 0.
$$

The angle between unit dual vectors $\vec{A}$ and $\vec{B}$ $\Phi = \varphi + \varepsilon \varphi^*$ is called dual angle and this angle is denoted by ([6])

$$
\left\langle \vec{A}, \vec{B} \right\rangle = \cos\Phi = \cos\varphi - \varepsilon \varphi^* \sin\varphi
$$

Let

$$
\begin{aligned}
\widetilde{\alpha} &: I \subset IR \to ID^3 \\
s &\to \widetilde{\alpha}(s) = \alpha(s) + \varepsilon \alpha^*(s)
\end{aligned}
$$

be differential unit speed dual curve in dual space ID^3. Denote by $\{T, N, B\}$ the moving dual Frenet frame along the dual space curve $\widetilde{\alpha}(s)$ in the dual space ID^3. Then T, N and B are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. The function $\kappa(s) = k_1 + \varepsilon k_1^*$ and $\tau(s) = k_2 + \varepsilon k_2^*$ are called dual curvature and dual torsion of $\widetilde{\alpha}$, respectively. Then for the dual curve $\widetilde{\alpha}$ the Frenet formulae are given by,

$$\begin{aligned} T^{'}(s) &= \kappa(s)N(s) \\ N^{'}(s) &= -\kappa(s)T(s)+\tau(s)B(s) \\ B^{'}(s) &= -\tau(s)N(s) \end{aligned} \qquad (2.1)$$

The formulae (2.1) are called the Frenet formulae of dual curve. In this palace curvature and torsion are calculated by,

$$\kappa(s)=\sqrt{\langle T^{'},T^{'}\rangle}\ ,\ \tau(s)=\frac{\det\left(T,T^{'},T^{''}\right)}{\langle T^{'},T^{'}\rangle} \qquad (2.2)$$

If α is not unit speed curve, then curvature and torsion are calculated by

$$\kappa(s)=\frac{\left\|\alpha^{'}(s)\wedge\alpha^{''}(s)\right\|}{\left\|\alpha^{'}(s)\right\|^{3}}\ ,\ \tau(s)=\frac{\det\left(\alpha^{'}(s),\alpha^{''}(s),\alpha^{'''}(s)\right)}{\left\|\alpha^{'}(s)\wedge\alpha^{''}(s)\right\|^{2}} \qquad (2.3)$$

By separating formulas (2.1) into real and dual part, we obtain

$$\begin{aligned} t^{'}(s) &= k_1 n \\ n^{'}(s) &= -k_1 t+k_2 b \\ b^{'}(s) &= -k_2 n \end{aligned} \qquad (2.4)$$

$$\begin{aligned} t^{*'}(s) &= k_1 n^{*}+k_1^{*}n \\ n^{*'}(s) &= -k_1 t^{*}-k_1^{*}t+k_2 b^{*}+k_2^{*}b \\ b^{*'}(s) &= -k_2 n^{*}-k_2^{*}n \end{aligned} \qquad (2.5)$$

§3. Some Characterizations Involute of Dual Curves

Definition 3.1 *Let $\widetilde{\alpha}: I\rightarrow ID^3$ and $\widetilde{\beta}: I\rightarrow ID^3$ be dual unit speed curves. If the tangent lines of the dual curve $\widetilde{\alpha}$ is orthogonal to the tangent lines of the dual curve $\widetilde{\beta}$, the dual curve $\widetilde{\beta}$ is called involute of the dual curve $\widetilde{\alpha}$ or the dual curve $\widetilde{\alpha}$ is called evolute of the dual curve $\widetilde{\beta}$ (see Fig.1). According to this definition, if the tangent of the dual curve $\widetilde{\alpha}$ is denoted by T and the tangent of the dual curve $\widetilde{\beta}$ is denoted by $\bar{T}$, we can write*

$$\left\langle T,\bar{T}\right\rangle=0 \qquad (3.1)$$

Theorem 3.1 *Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be dual curves. If the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$, we can write*

$$\widetilde{\beta}(s)=\widetilde{\alpha}(s)+[(c_1-s)+\varepsilon c_2]\,T(s)\ \ ,\ \ c_1,c_2\epsilon IR.$$

Proof Then by the definition we can assume that

$$\widetilde{\beta}(s) = \widetilde{\alpha}(s) + \lambda T(s) \quad , \quad \lambda(s) = \mu(s) + \varepsilon \mu^*(s) \tag{3.2}$$

for some function $\lambda(s)$. By taking derivative of the equation (3.2) with respect to s and applying the Frenet formulae (2.1) we have

$$\frac{d\widetilde{\beta}}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N$$

where s and s^* are arc parameter of the dual curves $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. It follows that

$$\bar{T}\frac{ds^*}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N \tag{3.3}$$

the inner product of (3.3) with T is

$$\frac{ds^*}{ds}\left\langle T, \bar{T}\right\rangle = \left(1 + \frac{d\lambda}{ds}\right)\langle T, T\rangle + \lambda \langle T, N\rangle \tag{3.4}$$

From the definition of the involute-evolute curve couple, we can write

$$\left\langle T, \bar{T}\right\rangle = 0$$

By substituting the last equation in (3.4) we get

$$1 + \frac{d\lambda}{ds} = 0 \quad \text{and} \quad \frac{d}{ds}\left(\mu(s) + \varepsilon \mu^*(s)\right) = -1 \tag{3.5}$$

Straightforward computation gives

$$\mu^{'}(s) = -1 \quad \text{and} \quad \mu^{*'}(s) = 0$$

integrating last equation, we get

$$\mu(s) = c_1 - s \quad \text{and} \quad \mu^*(s) = c_2 \tag{3.6}$$

By substituting (3.6) in (3.2), we get

$$\widetilde{\beta}(s) - \widetilde{\alpha}(s) = \left[(c_1 - s) + \varepsilon c_2\right] T(s). \tag{3.7}$$

This completes the proof. □

Corollary 3.1 *The distance between the dual curves* $\widetilde{\beta}$ *and* $\widetilde{\alpha}$ *is* $|c_1 - s| \mp \varepsilon c_2$.

Proof By taking the norm of the equation (3.7) we get

$$d\left(\widetilde{\alpha}(s), \widetilde{\beta}(s)\right) = |c_1 - s| \mp \varepsilon c_2 \tag{3.8}$$

This completes the proof. □

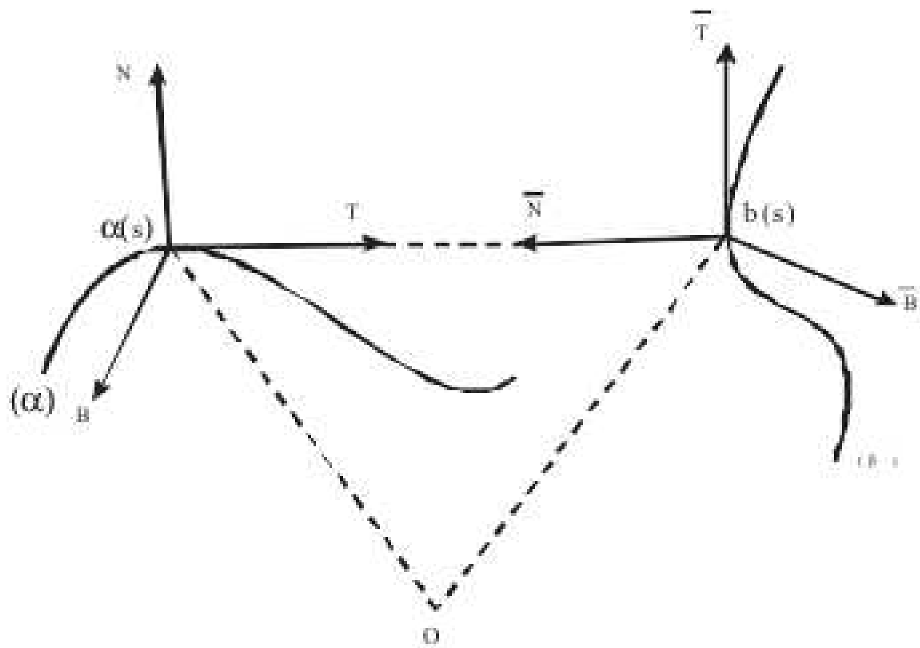

Fig.1

Theorem 3.2 *Let $\widetilde{\alpha},\widetilde{\beta}$ be dual curves. If the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$, then the relationships between the dual Frenet vectors of the dual curves $\widetilde{\alpha}$ and $\widetilde{\beta}$*

$$\begin{aligned}\bar{T} &= N\\ \bar{N} &= -\cos\Phi T+\sin\Phi B\\ \bar{B} &= \sin\Phi T+\cos\Phi B\end{aligned}$$

Proof By differentiating the equation (3.2) with respect to s we obtain

$$\widetilde{\beta}'(s)=\lambda\kappa(s)N(s)\ ,\ \lambda=(c_1-s)+\varepsilon c_2 \tag{3.9}$$

and

$$\left\|\widetilde{\beta}'(s)\right\|=\lambda\kappa(s)$$

Thus, the tangent vector of $\widetilde{\beta}$ is found

$$\bar{T}=\frac{\widetilde{\beta}'(s)}{\left\|\widetilde{\beta}'(s)\right\|}=\frac{\lambda\kappa(s)N(s)}{\lambda\kappa(s)}$$

If we arrange the last equation we obtain

$$\bar{T}=N(s) \tag{3.10}$$

By differentiating the equation (3.9) with respect to s we obtain

$$\widetilde{\beta}'' = -\lambda\kappa^2 T + \left(\lambda\kappa' - \kappa\right) N + \lambda\kappa\tau B$$

If the cross product $\widetilde{\beta}' \wedge \widetilde{\beta}''$ is calculated we have

$$\widetilde{\beta}' \wedge \widetilde{\beta}'' = \lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B \tag{3.11}$$

The norm of vector $\widetilde{\beta}' \wedge \widetilde{\beta}''$ is found

$$\left\| \widetilde{\beta}' \wedge \widetilde{\beta}'' \right\| = \lambda^2\kappa^2\sqrt{\kappa^2 + \tau^2} \tag{3.12}$$

For the dual binormal vector of the dual curve $\widetilde{\beta}$ we can write

$$\bar{B} = \frac{\widetilde{\beta}' \wedge \widetilde{\beta}''}{\left\| \widetilde{\beta}' \wedge \widetilde{\beta}'' \right\|}$$

By substituting (3.11) and (3.12) in the last equation we get

$$\bar{B} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \tag{3.13}$$

For the dual principal normal vector of the dual curve $\widetilde{\beta}$ we can write

$$\bar{N} = \bar{B} \wedge \bar{T}$$

and

$$\bar{N} = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \tag{3.14}$$

Let Φ $\left(\Phi = \varphi + \varepsilon\varphi^* \ , \ \varepsilon^2 = 0\right)$ be dual angle between the dual Darboux vector W of $\widetilde{\alpha}$ and dual unit binormal vector B in this situation we can write

$$\sin\Phi = \frac{\tau}{\kappa^2 + \tau^2} \ , \quad \cos\Phi = \frac{\kappa}{\kappa^2 + \tau^2}. \tag{3.15}$$

By substituting (3.15) in (3.12) and (3.13) the proof is completed. □

The real and dual parts of $\bar{T}$, $\bar{N}$, $\bar{B}$ are

$$\begin{aligned} \bar{T} &= N \\ \bar{N} &= -\cos\Phi T + \sin\Phi B \\ \bar{B} &= \sin\Phi T + \cos\Phi B \end{aligned}$$

is separated into the real and dual part, we can obtain

$$\begin{aligned}\bar{t} &= n,\\ \bar{n} &= -\cos\varphi t + \sin\varphi b,\\ \bar{b} &= \sin\varphi t + \cos\varphi b\end{aligned}$$

$$\begin{aligned}\bar{t}^{*} &= n^{*}\\ \bar{n}^{*} &= -\cos\varphi t^{*} + \sin\varphi b^{*} + \varphi^{*}\left(\sin\varphi t + \cos\varphi b\right)\\ \bar{b}^{*} &= \sin\varphi t^{*} + \cos\varphi b^{*} + \varphi^{*}\left(\cos\varphi t - \sin\varphi b\right)\end{aligned}$$

On the way

$$\begin{cases}\sin\Phi = \sin\left(\varphi + \varepsilon\varphi^{*}\right) = \sin\varphi + \varepsilon\varphi^{*}\cos\varphi\\ \cos\Phi = \cos\left(\varphi + \varepsilon\varphi^{*}\right) = \cos\varphi - \varepsilon\varphi^{*}\sin\varphi\end{cases}$$

If the equation

$$\sin\Phi = \frac{\tau}{\kappa^2+\tau^2}$$

is separated into the real and dual part, we can obtain

$$\begin{aligned}\sin\varphi &= \frac{k_2}{k_1^2+k_2^2}\\ \cos\varphi &= \frac{k_1^2+k_2^{*}-2k_1k_2k_1^{*}-2k_2^2k_2^{*}}{\varphi\left(k_1^2+k_2^2\right)^2}\end{aligned}$$

If the equation

$$\cos\Phi = \frac{\kappa}{\kappa^2+\tau^2}$$

is separated into the real and dual part, we can obtain

$$\begin{aligned}\cos\varphi &= \frac{k_1}{k_1^2+k_2^2}\\ \sin\varphi &= \frac{2k_1^2+k_1^{*}+2k_1k_2k_2^{*}-k_1^2k_1^{*}-k_2^2k_1^{*}}{\varphi\left(k_1^2+k_2^2\right)^2}\end{aligned}$$

Theorem 3.3 *Let $\widetilde{\alpha}, \widetilde{\beta}$ be dual curves. If the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$, curvature and torsion of the dual curve $\widetilde{\beta}$ are*

$$\bar{\kappa}^{2}\left(s\right) = \frac{\kappa^2\left(s\right)+\tau^2\left(s\right)}{\lambda^2\left(s\right)\kappa^2\left(s\right)}\ ,\ \bar{\tau}\left(s\right) = \frac{\kappa\left(s\right)\tau'\left(s\right)-\kappa'\left(s\right)\tau\left(s\right)}{\lambda\left(s\right)\kappa\left(s\right)\left(\kappa^2\left(s\right)+\tau^2\left(s\right)\right)} \qquad (3.16)$$

Proof By the definition of involute we can write

$$\widetilde{\beta}\left(s\right) = \widetilde{\alpha}\left(s\right) + \left|\lambda\right| T\left(s\right) \qquad (3.17)$$

By differentiating the equation (3.17) with respect to s we obtain

$$\frac{d\widetilde{\beta}}{ds^*}\frac{ds^*}{ds} = T(s) + |\lambda|^{'} T(s) + |\lambda|\kappa(s) N(s),$$
$$\frac{d\widetilde{\beta}}{ds^*}\frac{ds^*}{ds} = T(s) - T(s) + |\lambda|\kappa(s) N(s),$$

$$\bar{T}(s)\frac{ds^*}{ds} = |\lambda|\kappa(s) N(s). \quad (3.18)$$

Since the direction of $\bar{T}(s)$ is coincident with $N(s)$ we have

$$\bar{T}(s) = N(s). \quad (3.19)$$

Taking the inner product of (3.18) with T and necessary operation are made we get

$$\frac{ds^*}{ds} = |\lambda(s)|\kappa(s). \quad (3.20)$$

By taking derivative of (3.19) and applying the Frenet formulae (2.1) we have

$$\bar{T}(s) = N(s) \Rightarrow \bar{T}^{'}(s)\frac{ds^*}{ds} = -\kappa T + \tau B. \quad (3.21)$$

From (3.20) and (3.21), we have

$$\bar{T}^{'}(s) = \frac{-\kappa T + \tau B}{|\lambda(s)|\kappa(s)}.$$

From the last equation we can write

$$\bar{\kappa}(s)\bar{N}(s) = \frac{-\kappa T + \tau B}{|\lambda(s)|\kappa(s)}.$$

Taking the inner product the last equation with each other we have

$$\left\langle \bar{\kappa}(s)\bar{N}(s), \bar{\kappa}(s)\bar{N}(s)\right\rangle = \left\langle \frac{-\kappa T + \tau B}{|\lambda(s)|\kappa(s)}, \frac{-\kappa T + \tau B}{|\lambda(s)|\kappa(s)}\right\rangle.$$

Thus, we find

$$\bar{\kappa}^2(s) = \frac{\kappa^2(s) + \tau^2(s)}{\lambda^2(s)\kappa^2(s)}.$$

We know that

$$\widetilde{\beta}' \wedge \widetilde{\beta}^{''} = \lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B.$$

Taking the norm the last equation, we get

$$\left\|\widetilde{\beta}' \wedge \widetilde{\beta}^{''}\right\| = \kappa^4\lambda^4\left(\kappa^2 + \tau^2\right).$$

By substituting these equations in (2.3), we get

$$\bar{\tau}=\frac{\begin{vmatrix} 0 & \kappa\lambda & 0 \\ -\kappa^2\lambda & (\kappa\lambda)' & \kappa\tau\lambda \\ \left(-\kappa^2\lambda\right)'-\kappa(\kappa\lambda)' & -\kappa^3\lambda+(\kappa\lambda)''-\kappa\tau^2\lambda & (\kappa\lambda)'\tau+(\kappa\tau\lambda)' \end{vmatrix}}{\left\|\widetilde{\beta}'\wedge\widetilde{\beta}''\right\|^2},$$

$$\bar{\tau}=\frac{\kappa\tau'-\kappa'\tau}{\kappa|\lambda|\left(\kappa^2+\tau^2\right)}.$$

This completes the proof. □

If the equation (3.16) is separated into the real and dual part, we can obtain

$$\begin{aligned}
\bar{k}_1 &= \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1}, \\
\bar{k}_1^{*} &= \frac{\left(\mu^2k_1^2\right)\left(2k_1k_1^*+2k_2k_2^*\right)-\left(2k_1k_1^*\mu^2\right)\left(k_1^2+k_2^2\right)}{2\mu^3k_1^3\sqrt{k_1^2+k_2^2}}, \\
\bar{k}_2 &= \frac{k_1k_2'-k_2k_1'}{\mu k_1\left(k_1^2+k_2^2\right)}, \\
\bar{k}_2^{*} &= \frac{\left(k_1k_2^{'*}+k_2'k_1^*-k_1'k_2^*-k_2k_1^{'*}\right)}{\left(\mu k_1^3+k_1k_2^2\mu\right)} \\
&- \frac{\left[2\left(k_1k_1^*+k_2k_2^*\right)k_1\mu+\left(k_1^2+k_2^2\right)\left(k_1^*\mu+k_1\mu^*\right)\right]\left(k_1k_2'-k_2k_1'\right)}{\left(\mu k_1^3+k_1k_2^2\mu\right)^2}.
\end{aligned}$$

Theorem 3.4 *Let $\widetilde{\alpha},\widetilde{\beta}$ be dual curves and the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$. If W and $\bar{W}$ are Darboux vectors of the dual curves $\widetilde{\alpha}$ and $\widetilde{\beta}$ we can write*

$$\bar{W}=\frac{1}{\lambda\kappa}\left(W+\Phi' N\right) \tag{3.22}$$

Proof Since $\bar{W}$ is Darboux vector of $\widetilde{\beta}(s)$ we can write

$$\bar{W}(s)=\bar{\tau}(s)\bar{T}(s)+\bar{\kappa}(s)\bar{B}(s) \tag{3.23}$$

By substituting $\bar{\tau},\bar{T},\bar{\kappa},\bar{B}$ in the last equation, we get

$$\bar{W}(s)=\frac{\kappa\tau'-\kappa'\tau}{\kappa|\lambda|\left(\kappa^2+\tau^2\right)}N(s)+\frac{\sqrt{\kappa^2+\tau^2}}{\kappa|\lambda|}\left(\sin\Phi T+\cos\Phi B\right). \tag{3.24}$$

By substituting (3.15) in (3.24), we get

$$\bar{W}(s) = \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2+\tau^2)}N(s) + \frac{\sqrt{\kappa^2+\tau^2}}{\kappa|\lambda|}\left(\frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}\right).$$

The necessary operation are made, we get

$$\bar{W}(s) = \frac{\tau T + \kappa B}{\kappa|\lambda|} + \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2+\tau^2)}N(s),$$

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|}\left(\tau T + \kappa B + \frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2}N\right)$$

and

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|}\left(W + \frac{\left(\frac{\tau}{\kappa}\right)'\kappa^2}{\kappa^2+\tau^2}N\right).$$

Furthermore, Since

$$\frac{\sin\Phi}{\cos\Phi} = \frac{\tau/\sqrt{\kappa^2+\tau^2}}{\kappa/\sqrt{\kappa^2+\tau^2}},$$

$$\frac{\tau}{\kappa} = \tan\Phi.$$

By taking derivative of the last equation, we have

$$\Phi'\sec^2\Phi = \left(\frac{\tau}{\kappa}\right)'.$$

By a straightforward calculation, we get

$$\Phi' = \left(\frac{\tau}{\kappa}\right)'\frac{\kappa}{\kappa^2+\tau^2},$$

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|}\left(W + \Phi' N\right),$$

which completes the proof. □

If the equation (3.22) is separated into the real and dual part, we can obtain

$$\bar{w} = \frac{w + \varphi' n}{\mu k_1},$$

$$\bar{w}^* = \frac{\mu k_1\left(w^* + \varphi' n + \varphi'^* n\right) - (\mu k_1^* + \mu^* k_1)\left(w + \varphi' n\right)}{\mu^2 k_1^2}.$$

If the equation (3.24) is separated into the real and dual part, we can obtain

$$\bar{w} = \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1}(\sin\varphi t + \cos\varphi b),$$

$$\begin{aligned}\bar{w}^{*} &= \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1}\left(\sin\varphi t^{*}+\cos\varphi b^{*}+\varphi^{*}\left(\cos\varphi t-\sin\varphi b\right)\right)\\ &+ \frac{\mu k_1\left(k_1k_1^{*}+k_2k_2^{*}\right)-\left(k_1^2+k_2^2\right)\left(\mu k_1^{*}+\mu^{*}k_1\right)}{\sqrt{k_1^2+k_2^2}\mu^2k_1^2}\left(\sin\varphi t+\cos\varphi b\right)\cdot\end{aligned}$$

Theorem 3.5 *Let $\widetilde{\alpha},\widetilde{\beta}$ be dual curves and the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$. If C and $\bar{C}$ are unit vectors of the direction of W and $\bar{W}$, respectively*

$$\bar{C}=\frac{\Phi^{'}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}N+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}C. \tag{3.25}$$

Proof Since $\widetilde{\beta}$ the dual angle between $\bar{W}$ and $\bar{B}$ we can write

$$\bar{C}\left(s\right)=\sin\widetilde{\beta}\bar{T}\left(s\right)+\cos\widetilde{\beta}\bar{B}\left(s\right).$$

In here, we want to find the statements $\sin\widetilde{\beta}$ and $\cos\widetilde{\beta}$, we know that

$$\sin\widetilde{\beta}=\frac{\bar{\tau}}{\left\|\bar{W}\right\|}=\frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2+\bar{\tau}^2}}.$$

By substituting $\bar{\tau}$ and $\bar{\kappa}$ in the last equation and necessary operations are made, we get

$$\sin\widetilde{\beta}=\frac{\Phi^{'}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}. \tag{3.26}$$

Similarly,

$$\cos\widetilde{\beta}=\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}. \tag{3.27}$$

Thus we find

$$\bar{C}=\frac{\Phi^{'}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}\bar{T}+\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{\Phi'^2+\kappa^2+\tau^2}}C,$$

which completes the proof. □

If the equation (3.25) is separated into the real and dual part, we can obtain

$$\begin{aligned}\bar{c} &= \frac{\varphi^{'}n+\sqrt{k_1^2+k_2^2}c}{\sqrt{\varphi^{'}+k_1^2+k_2^2}},\\ \bar{c}^{*} &= \frac{\varphi^{'}n^{*}+\varphi^{'^{*}}n+\sqrt{k_1^2+k_2^2}c^{*}+\frac{k_1k_1^{*}+k_2k_2^{*}}{\sqrt{k_1^2+k_2^2}}c-\frac{\varphi^{'}n\left(\sqrt{k_1^2+k_2^2}\right)c\left(\varphi^{'}\varphi^{'^{*}}+k_1k_1^{*}+k_2k_2^{*}\right)}{\sqrt{\varphi^{'}+k_1^2+k_2^2}}}{\sqrt{\varphi^{'}+k_1^2+k_2^2}}.\end{aligned}$$

If the equation (3.26) and (3.27) are separated into the real and dual part, we can obtain

$$\begin{aligned} \sin\bar{\varphi} &= \frac{\varphi^{'}}{\sqrt{\varphi^{'} + k_1^2 + k_2^2}}, \\ \cos\bar{\varphi} &= \frac{\left(\Phi^{'2} + \kappa^2 + \tau^2\right)\Phi^{'*} - \varphi^{'}\varphi^{*} + k_1 k_1^* + k_2 k_2^* \varphi^{'}}{\bar{\varphi}^{*}\left(\Phi^{'2} + \kappa^2 + \tau^2\right)^{\frac{3}{2}}}, \end{aligned}$$

$$\begin{aligned} \cos\bar{\varphi} &= \sqrt{\frac{k_1^2 + k_2^2}{\varphi^{'^2} + k_1^2 + k_2^2}}, \\ \sin\bar{\varphi} &= \frac{\left(\varphi^{'}\varphi^{'^*} + k_1 k_1^* + k_2 k_2^*\right)\sqrt{k_1^2 + k_2^2} - \left(\varphi^{'^2} + k_1^2 + k_2^2\right)\left(k_1 k_1^* + k_2 k_2^*\right)}{\bar{\varphi}^{*}\left(\Phi^{'2} + \kappa^2 + \tau^2\right)^{\frac{3}{2}}\sqrt{k_1^2 + k_2^2}}. \end{aligned}$$

Corollary 3.2 *Let $\widetilde{\alpha}, \widetilde{\beta}$ be dual curves and the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$. If evolute curve $\widetilde{\alpha}$ is helix,*

(1) *The vectors $\bar{W}$ and $\bar{B}$ of the involute curve $\widetilde{\beta}$ are linearly dependent;*

(2) $C = \bar{C}$*;*

(3) $\widetilde{\beta}$ *is planar.*

Proof (1) If the evolute curve $\widetilde{\alpha}$ is helix, then we have

$$\frac{\tau}{\kappa} = \tan\Phi = cons \text{ or } \Phi^{'} = 0$$

and then we have

$$\begin{aligned} \sin\bar{\Phi} &= 0, \\ \cos\bar{\Phi} &= 1. \end{aligned}$$

Thus, we get

$$\bar{\Phi} = 0. \tag{3.28}$$

(2) Substituting by the equation (3.28) into the equation (3.25), we have

$$C = \bar{C}.$$

(3) For being is a helix , then we have

$$\frac{\tau}{\kappa} = cons, \quad \left(\frac{\tau}{\kappa}\right)^{'} = 0. \tag{3.29}$$

On the other hand, from the equation (3.16), we can write

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}}{\frac{\left(\kappa^2+\tau^2\right)^{\frac{1}{2}}}{\lambda\kappa}} = \frac{\left(\frac{\tau}{\kappa}\right)'\kappa^2}{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}. \tag{3.30}$$

Substituting by the equation (3.29) into the equation (3.30), then we find

$$\bar{\tau} = 0,$$

which completes the proof. □

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One Modulo N Gracefulness of Some Arbitrary Supersubdivision and Removal Graphs

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Abstract: A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. In this paper we prove that arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful for all positive integer N. Also we prove that the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.

Key Words: Graceful, modulo N graceful, disconnected graphs, arbitrary supersubdivision graphs, $P_n \bigcup C_n$ and $P_n^+ - v_k^{(1)}$.

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§1. Introduction

S. W. Golomb [3] introduced graceful labelling. Odd gracefulness was introduced by R. B. Gnanajothi [4]. C. Sekar [11] introduced one modulo three graceful labelling. In [8,9], we introduced the concept of one modulo N graceful where N is any positive integer. In the case $N=2$, the labelling is odd graceful and in the case $N=1$ the labelling is graceful. Joseph A. Gallian [2] surveyed numerous graph labelling methods. Recently G. Sethuraman and P. Selvaraju [5] have introduced a new method of construction called supersubdivision of a graph. Let G be a graph with n vertices and t edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from G. A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph.

G. Sethuraman and P. Selvaraju [6] proved that every connected graph has some supersub-

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division that is graceful. They pose the question as to whether some supersubdivision is valid for disconnected graphs. [10] We proved that an arbitrary supersubdivision of disconnected paths are graceful. Barrientos and Barrientos [1] proved that any disconnected graph has a supersubdivision that admits an α-labeling. They also proved that every supersubdivision of a connected graph admits an α-labeling.

In this paper we prove that arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful for all positive integer N. When $N = 1$ we get an affirmative answer for their question. Also we prove that the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.

§2. Main Results

Definition 2.1 *A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv)=|\phi(u)-\phi(v)|$.*

Definition 2.2 *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e are merged with the two vertices part of $K_{2,m}$ after removing the edge e from G. H is denoted by $SS(G)$.*

Definition 2.3 *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). H is denoted by $ASS(G)$.*

Definition 2.4 *Let $v_1, v_2, \ldots, v_n$ be the vertices of a path of length n and $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ be the pendant vertices attached with $v_1, v_2, \ldots, v_n$ respectively. The removal of a pendant vertex $v_k^{(1)}$ where $1 \leq k \leq n$ from P_n^+ yields the graph $P_n^+ - v_k^{(1)}$.*

Theorem 2.5 *Arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geqslant 2$.*

Proof Let P_n be a path with successive vertices $v_1, v_2, \cdots, v_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge v_iv_{i+1} of P_n. Let C_r be a cycle with successive vertices $v_{n+1}, v_{n+2}, \cdots, v_{n+r}$ and let $e_i (n+1 \leq i \leq n+r)$ denote the edge v_iv_{i+1}.

Let H be an arbitrary supersubdivision of the disconnected graph $P_n \cup C_r$ where each edge e_i of $P_n \cup C_r$ is replaced by a complete bipartite graph K_{2,m_i} with $m_i \geqslant 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r$. Here the edge $v_{n+r}v_{n+1}$ is replaced by $k_{2,r-1}$. We observe that H has $M = 2(m_1 + m_2 + \cdots + m_{n-1} + m_{n+1} + \cdots + m_{n+r})$ edges.

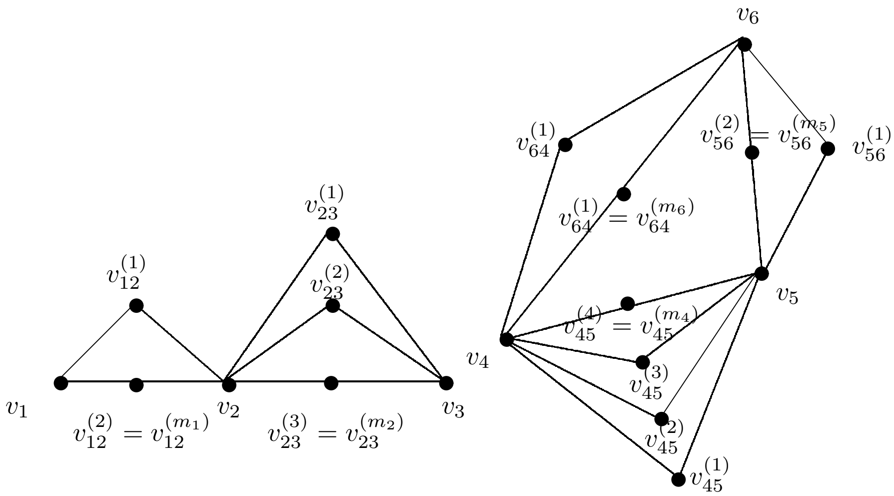

Figure 1 Supersubdivision of $P_3 \cup C_3$

Define

$\phi(v_i) = N(i-1),\ i = 1, 2, 3, \cdots, n,$

$\phi(v_i) = N(i),\ i = n+1, n+2, n+3, \cdots, n+r$, and for $k = 1, 2, 3, \ldots, m_i$,

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1 \\ N(M-2+i)+1-2N(m_1+m_2+\cdots+m_{i-1}+k-1) & \text{if } i = 2,3,\ldots n-1 \\ N(M-1+i)+1-2N(m_1+m_2+\cdots+m_{n-1}+k-1) & \text{if } i = n+1 \\ N(M-1+i)+1-2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{i-1})+k-1] & \text{if } i = n+2, n+3, \ldots n+r-1 \end{cases}$$

and for $k = 1, 2, 3, \cdots, m_{n+r}$, $\phi(v_{n+r,n+1}^{(k)}) = N(n+r-k+m_{n+r})+1$

From the definition of ϕ it is clear that

$$\{\phi(v_i), i = 1, 2, \cdots, n+r\} \bigcup \{\phi(v_{i,i+1}^{(k)}), i = 1, 2, \cdots, n+r-1 \text{ and}$$
$$k = 1, 2, 3, \cdots, m_i\} \bigcup \{\phi(v_{n+r,n+1}^{(k)}), k = 1, 2, 3, \cdots, m_i\}$$
$$= \{0, N, 2N, \cdots, N(n-1)\} \bigcup \{N(n+1), N(n+2), \cdots, N(n+r)\}$$
$$\bigcup \{N[M-2k+1]+1, N[M-2m_1]+1, N[M-2m_1-2]+1 \cdots,$$
$$N[M-2(m_1+m_2)+2]+1, N[M-2(m_1+m_2)+1]+1,$$
$$N[M-2(m_1+m_2)-1]+1, \ldots, N[M-2(m_1+m_2+m_3)+3]+1,$$
$$\ldots, N[M-3+n-2(m_1+m_2+\cdots+m_{n-2})]+1,$$
$$N[M-5+n-2(m_1+m_2+\cdots+m_{n-2})]+1, \ldots,$$
$$N[M-1+n-2(m_1+m_2+\cdots+m_{n-1})]+1,$$
$$N[M+n-2(m_1+m_2+\cdots+m_{n-1})]+1,$$
$$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+1)]+1, \ldots,$$
$$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}-1)]+1,$$
$$N[M+1+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1})]+1,$$

$N[M-1+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1})]+1,\cdots,$
$N[M+3+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M+2+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,\ldots,$
$N[M+4+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2}+m_{n+3})]+1,$
$N[M-2+n+r-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M-2+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-2})]]+1,$
$N[M-4+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-2})]]+1,$
$\cdots,N[M+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-1})]]+1\}$
$\bigcup\{N(n+r-1+m_{n+r})+1,N(n+r-2+m_{n+r})+1,\cdots,N(n+r)+1\}$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

For $k=1,2,\cdots,m_1$, $\phi^*(v_{1,2}^{(k)}v_1)=|\ \phi(v_{1,2}^{(k)})-\phi(v_1)\ |=N(M-2k+1)+1$, $\phi^*(v_{1,2}^{(k)}v_2)=|\ \phi(v_{1,2}^{(k)})-\phi(v_2)\ |=N(M-2k)+1$.

For $k=1,2,\cdots,m_i$ and $i=2,3,\cdots,n-1$, $\phi^*(v_{i,i+1}^{(k)}v_i)=|\ \phi(v_{i,i+1}^{(k)})-\phi(v_i)\ |=N(M-2k+1)-2N(m_1+m_2+\cdots+m_{i-1})+1$, $\phi^*(v_{i,i+1}^{(k)}v_{i+1})=|\ \phi(v_{i,i+1}^{(k)})-\phi(v_{i+1})\ |=N(M-2k)-2N(m_1+m_2+\cdots+m_{i-1})+1$.

For $k=1,2,\cdots,m_{n+1}$, $\phi^*(v_{n+1,n+2}^{(k)}v_{n+1})=|\ \phi(v_{n+1,n+2}^{(k)})-\phi(v_{n+1})\ |=N(M-2k+1)-2N(m_1+m_2+\cdots+m_{n-1})+1$, $\phi^*(v_{n+1,n+2}^{(k)}v_{n+2})=|\ \phi(v_{n+1,n+2}^{(k)})-\phi(v_{n+2})\ |=N(M-2k)-2N(m_1+m_2+\cdots+m_{n-1})+1$.

For $k=1,2,\ldots,m_i$ and $j=n+2,n+3,\cdots,n+r$, $\phi^*(v_{i,i+1}^{(k)}v_i)=|\ \phi(v_{i,i+1}^{(k)})-\phi(v_i)\ |=N(M-2k+1)-2N\{(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{i-1})\}+1$, $\phi^*(v_{i,i+1}^{(k)}v_{i+1})=|\ \phi(v_{i,i+1}^{(k)})-\phi(v_{i+1})\ |=N(M-2k)-2N\{(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{i-1})\}+1$.

For $k=1,2,\cdots,m_{n+r}$, $\phi^*(v_{n+r,n+1}^{(k)}v_{n+r})=|\ \phi(v_{n+r,n+1}^{(k)})-\phi(v_{n+r})\ |=N(m_{n+r}-k)+1$, $\phi^*(v_{n+r,n+1}^{(k)}v_{n+1})=|\ \phi(v_{n+r,n+1}^{(k)})-\phi(v_{n+1})\ |=N(m_{n+r}+r-k-1)+1$.

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1\leq i\leq n-1$ and $n+1\leq i\leq n+r-1$. Therefore the vertices of each $K_{2,m_i},1\leq i\leq n-1$ and $n+1\leq i\leq n+r-1$ in the arbitrary supersubdivision H of $P_n\cup C_r$ have distinct labels and also the edges of each $K_{2,m_i},1\leq i\leq n-1$ and $n+1\leq i\leq n+r-1$ in the arbitrary supersubdivision graph H of $P_n\cup C_r$ have distinct labels. Clearly H is one modulo N graceful. Hence arbitrary supersubdivisions of disconnected path and cycle $P_n\cup C_r$ is one modulo N graceful, for every positive integer N.

Consequently, every disconnected graph has some supersubdivision that is one modulo N graceful. □

Example 2.6 A odd graceful labelling of $ASS(P_3\cup C_4)$ is shown in Figure 2.

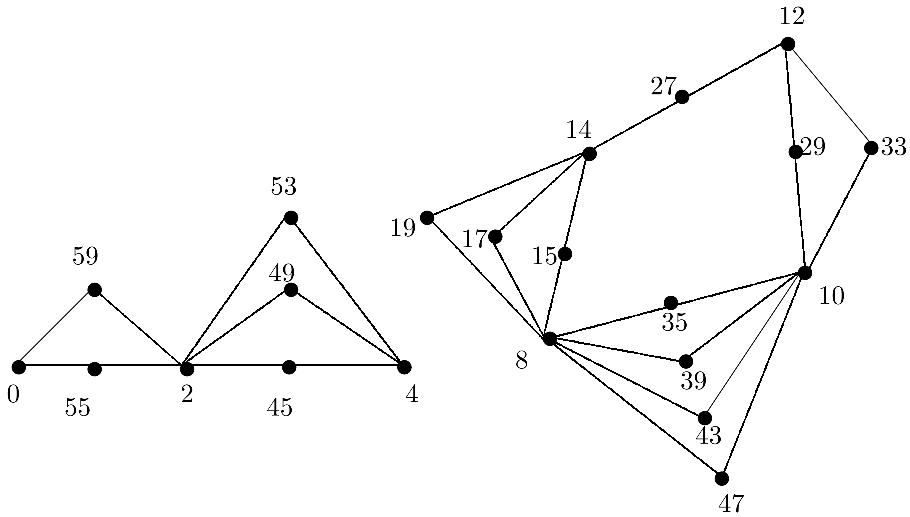

Figure 2

Example 2.7 A graceful labelling of $ASS(P_3 \cup C_3)$ is shown in Figure 3.

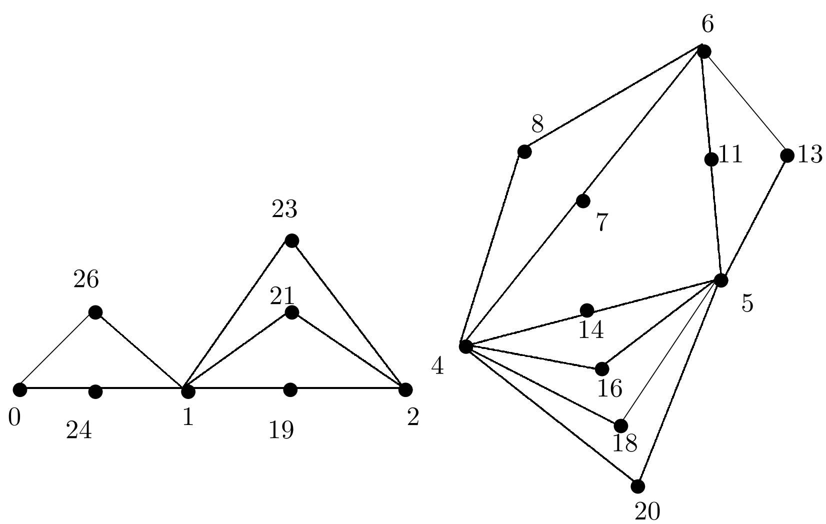

Figure 3

Theorem 2.8 *For any pendant vertex $v_k^{(1)} \in V(P_n^+)$, the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of a path of length n and $v_1^{(1)}, v_2^{(1)}, \cdots, v_n^{(1)}$ the pendant vertices attached with $v_1, v_2, \cdots, v_n$ respectively. Consider the graph $P_n^+ - v_k^{(1)}$, where $1 \leq k \leq n$. It has $2n-1$ vertices and $2n-2$ edges.

Case 1. n is even and k is even

Define

$$\phi(v_{2i-1}) = \begin{cases} N(2n-3)+1-2N(i-1) & \text{for i} = 1,2,\cdots,\frac{k}{2} \\ N(2n-3)+1-2N(\frac{k}{2}-1)-N-2N(i-(\frac{k}{2}+1)) & \text{for i} = \frac{k}{2}+1,\cdots,\frac{n}{2} \end{cases},$$

$\phi(v_{2i}) = N(2i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$,

$$\phi(v_{2i}^{(1)}) = \begin{cases} 2N(n-2)+1-2N(i-1) & \text{for i} = 1,2,\cdots,\frac{k}{2}-1 \\ 2N(n-2)+1-2N(\frac{k}{2}-2)-3N-2N(i-(\frac{k}{2}+1)) & \text{fori} = \frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n}{2} \end{cases},$$

$\phi(v_{2i-1}^{(1)}) = 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$.

From the definition of ϕ it is clear that

$$\begin{aligned}
&\{\phi(v_{2i-1}),\ i = 1,2,\cdots,\frac{n}{2}\} \bigcup \{\phi(v_{2i}),\ i = 1,2,\cdots,\frac{n}{2}\} \\
&\bigcup \{\phi(v_{2i}^{(1)}), i = 1,2,\cdots,\frac{k}{2}-1,\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n}{2}\} \\
&\bigcup \{\phi(v_{2i-1}^{(1)}),\ i = 1,2,\cdots,\frac{n}{2}\} \\
&= \{N(2n-3)+1, N(2n-5)+1,\cdots,N(2n-k-1)+1, N(2n-k-2)+1, \\
&\quad N(2n-k-4)+1,\ldots,Nn+1\} \bigcup \{N, 3N,\cdots,N(n-1)\} \\
&\quad \bigcup \{2N(n-2)+1, 2N(n-3)+1,\cdots,N(2n-k)+1, N(2n-k-3)+1, \\
&\quad N(2n-k-5)+1,\cdots,N(n-1)+1\} \bigcup \{0, 2N,\ldots,N(n-2)\}
\end{aligned}$$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows.

For $i = 1, 2, \cdots, \frac{k}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+1)+1$.

For $i = 1, 2, \cdots, \frac{k}{2}-1$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i+1)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n}{2}-1$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

This show that the edges have the distinct labels $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $q = 2n-2$. Hence for every positive integer N, $P_n^+ - v_k^{(1)}$ is one modulo N graceful if n is even and k is even.

Example 2.9 A one modulo 10 graceful labelling of $P_{10}^{+} - v_6^{(1)}$ is shown in Figure 4.

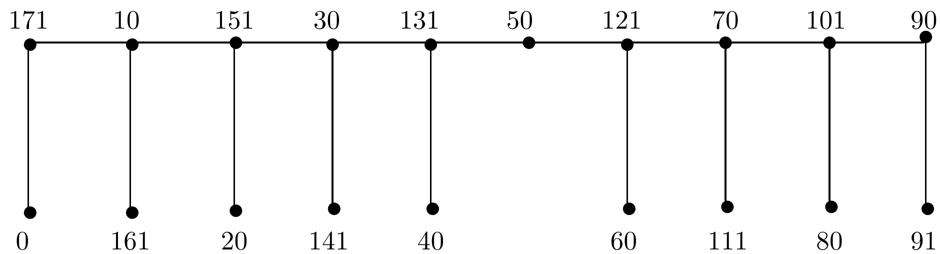

Figure 4

Case 2. n is even and k is odd

Define

$$\phi(v_{2i}) = \begin{cases} N(2i-1) & \text{for i} = 1, 2, \cdots, \frac{k-1}{2} \\ N(k-2) + N + 2N(i - (\frac{(k+1)}{2})) & for\ i = \frac{k+1}{2}, \frac{k+3}{2}, \cdots, \frac{n}{2} \end{cases},$$

$\phi(v_{2i-1}) = N(2n-3) + 1 - 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$,

$$\phi(v_{2i-1}^{(1)}) = \begin{cases} 2N(i-1) & \text{for i} = 1, 2, \cdots, \frac{k-1}{2} \\ 2N(\frac{k-1}{2} - 1) + 3N + 2N(i - (\frac{k+3}{2})) & \text{fori} = \frac{k+3}{2}, \frac{k+5}{2}, \cdots, \frac{n}{2} \end{cases},$$

$\phi(v_{2i}^{(1)}) = 2N(n-2) + 1 - 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$.

The proof is similar to that of Case 1. Hence for every positive integer N, $P_n^{+} - v_k^{(1)}$ is one modulo N graceful if n is even and k is odd.

Example 2.10 A one modulo 4 graceful labelling of $P_{12}^{+} - v_9^{(1)}$ is shown in Figure 5.

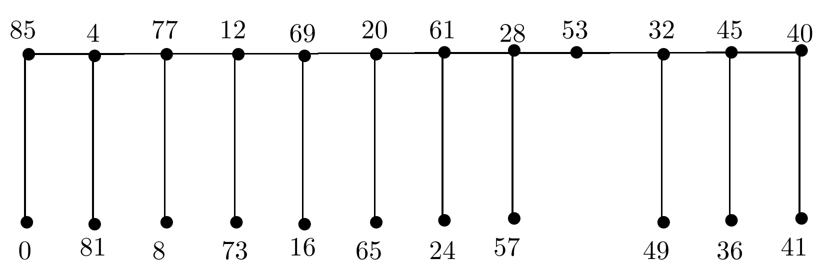

Figure 5

Case 3. n is odd and k is even

Define

$$\phi(v_{2i-1}) = \begin{cases} N(2n-3)+1-2N(i-1) & \text{for i} = 1,2,\cdots,\frac{k}{2} \\ N(2n-3)+1-2N(\frac{k}{2}-1)-N-2N(i-(\frac{k}{2}+1)) & \text{for} i = \frac{k}{2}+1,\cdots,\frac{n-1}{2} \end{cases},$$

$\phi(v_{2i}) = N(2i-1)$ for $i = 1,2,\ldots,\frac{n-1}{2}$,

$$\phi(v_{2i}^{(1)}) = \begin{cases} 2N(n-2)+1-2N(i-1) & \text{for i} = 1,2,\ldots,\frac{k}{2}-1 \\ 2N(n-2)+1-2N(\frac{k}{2}-2)-3N-2N(i-(\frac{k}{2}+1)) & \text{for} i = \frac{k}{2}+1,\cdots,\frac{n-1}{2} \end{cases},$$

$\phi(v_{2i-1}^{(1)}) = 2N(i-1)$ for $i = 1,2,\ldots,\frac{n-1}{2}$.

From the definition of ϕ it is clear that

$$\begin{aligned}
&\{\phi(v_{2i-1}),\ i=1,2,\cdots,\frac{n-1}{2}\}\bigcup\{\phi(v_{2i}),\ i=1,2,\cdots,\frac{n-1}{2}\}\\
&\bigcup\{\phi(v_{2i}^{(1)}),\ i=1,2,\cdots,\frac{k}{2}-1,\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n-1}{2}\}\bigcup\{\phi(v_{2i-1}^{(1)}),\ i=1,2,\cdots,\frac{n-1}{2}\}\\
&=\{N(2n-3)+1,N(2n-5)+1,\ldots,N(2n-k-1)+1,N(2n-k-2)+1,\\
&\quad N(2n-k-4)+1,\ldots,N(n-1)+1\}\ \bigcup\{N,3N,\ldots,N(n-2)\}\\
&\bigcup\{2N(n-2)+1,2N(n-3)+1,\ldots,N(2n-k)+1,N(2n-k-3)+1,\\
&\quad N(2n-k-5)+1,\ldots,Nn+1\}\ \bigcup\{0,2N,\ldots,N(n-1)\}
\end{aligned}$$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

For $i = 1,2,\cdots,\frac{k}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+1)+1$.

For $i = 1,2,\cdots,\frac{k}{2}-1$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n-1}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i+1)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n-1}{2}$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n+1}{2}$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+2)+1$.

This show that the edges have the distinct labels $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $q = 2n-2$. Hence for every positive integer N, $P_n^+ - v_k^{(1)}$ is one modulo N graceful if n is odd and k is even.

Example 2.11 A one modulo 3 graceful labelling of $P_{13}^+ - v_2^{(1)}$ is shown in Figure 6.

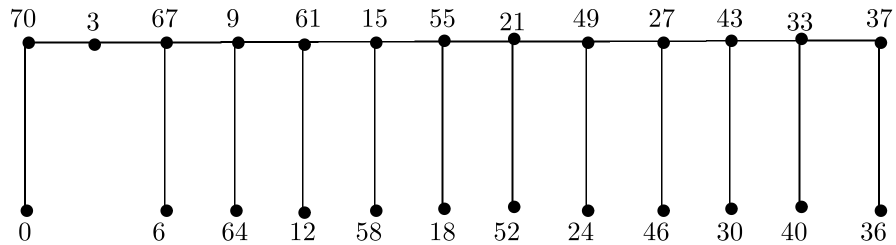

Figure 6

Case 4. n is odd and k is odd

Define

$$\phi(v_{2i}) = \begin{cases} N(2i-1) \ \text{ for i} = 1, 2, \cdots, \frac{\text{k}-1}{2} \\ N(k-2) + N + 2N(i - (\frac{(k+1)}{2})) \ \text{ fori} = \frac{\text{k}+1}{2}, \frac{\text{k}+3}{2}, \cdots, \frac{\text{n}-1}{2} \end{cases},$$

$\phi(v_{2i-1}) = N(2n-3) + 1 - 2N(i-1)$ for $i = 1, 2, \ldots, \frac{n-1}{2}$,

$$\phi(v_{2i-1}^{(1)}) = \begin{cases} 2N(i-1) \ \text{ for i} = 1, 2, \cdots, \frac{\text{k}-1}{2} \\ 2N(\frac{k-1}{2} - 1) + 3N + 2N(i - (\frac{k+3}{2})) \ \text{ fori} = \frac{\text{k}+3}{2}, \frac{\text{k}+5}{2}, \cdots, \frac{\text{n}-1}{2} \end{cases},$$

$\phi(v_{2i}^{(1)}) = 2N(n-2) + 1 - 2N(i-1)$ for $i = 1, 2, \ldots, \frac{n-1}{2}$.

The proof is similar to that of Case 3. Hence for every positive integer N, $P_n^+ - v_k^{(1)}$ is one modulo N graceful if n is odd and k is odd. □

Example 2.12 A one modulo 5 graceful labelling of $P_{11}^+ - v_5^{(1)}$ is shown in Figure 7.

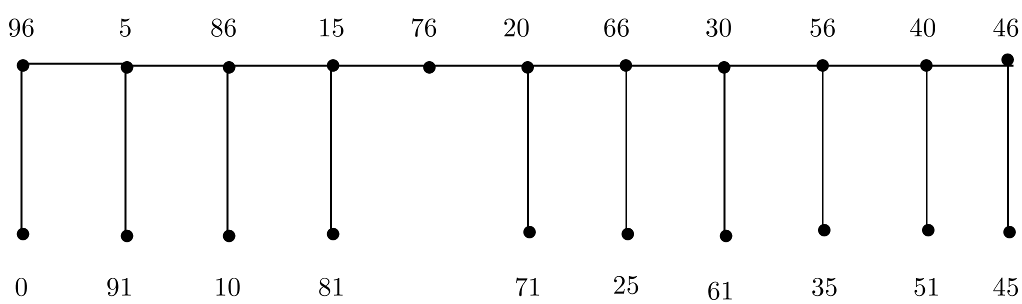

Figure 7

§3. Conclusion

Subdivision or supersubdivision or arbitrary supersubdivision of certain graphs which are not graceful may be graceful. The method adopted in making a graph one modulo N graceful will provide a new approach to have graceful labelling of graphs and it will be helpful to attack standard conjectures and unsolved open problems.

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