MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO

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Famous Words:

> *God was constructed out of mankind's need for hope, for purpose, for meaning: an invisible protector and conscientious father.*

> By Howards Mel, an American writer.

Some Results in Fuzzy and Anti Fuzzy Group Theory

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Abstract: This paper is to further investigate some properties of an anti fuzzy subgroup of a group in relation to pseudo coset. It also uses isomorphism theorems to establish some results in relation to level subgroups of a fuzzy subgroup μ of a group G.

Key Words: Fuzzy group, level subgroup, Smarandache fuzzy algebra, anti fuzzy group, anti fuzzy subgroup, group homomorphism, group isomorphism.

AMS(2010): 20N25

§1. Introduction

Major part of this work leans on the work of [5]. There are some new results using isomorphism theorems with some results in [5].

§2. Preliminaries

Definition 2.1 *Let X be a non-empty set. A fuzzy subset μ of the set G is a function $\mu : G \to [0,1]$.*

Definition 2.2 *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
(ii) $\mu(x^{-1}) = \mu(x)$;
(iii)μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G.

Definition 2.3 *Let G be a group and μ a fuzzy subset of G. Then μ is called an anti fuzzy subgroup of G if*

(i) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$;
(ii) $\mu(x^{-1}) = \mu(x)$.

Definition 2.4 *Let μ and λ be any two fuzzy subsets of a set X. Then*

(i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X;

(ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X;

(iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$.

Definition 2.5 *Let μ be a fuzzy subset (subgroup) of X. Then, for some t in $[0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset (subgroup) of the fuzzy subset (subgroup) μ.*

Remark 2.5.1 The set μ_t if it is group can be represented as G_μ^t.

Definition 2.6 *Let μ be a fuzzy subgroup of a group G. The set $H = \{x \in G : \mu(x) = \mu(e)\}$ is such that $o(\mu) = o(H)$.*

Definition 2.7 *Let μ be a fuzzy subgroup of a group G. μ is said to be normal if $\sup \mu(x) = 1$ for all x in G. It is said to be normalized if there is an x in G such that $\mu(x) = 1$.*

Definition 2.8 *Let G be a group and μ a fuzzy subset of G. Then μ is called an anti fuzzy subgroup of G if and only if $\mu(xy^{-1}) \leq \max\{\mu(x), \mu(y)\}$, and μ is called an anti fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y.*

Definition 2.9 *Let μ be a fuzzy subset of X. Then, for $t \in [1,0]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level subset of the fuzzy subset μ.*

Definition 2.10 *Let μ be an anti fuzzy subgroup of X. Then, for $t \in [1,0]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level subgroup of μ.*

Definition 2.11 *Let μ be an anti fuzzy subgroup of a group G of finite order. Then, the image of μ is $Im(\mu) = \{t_i \in I : \mu(x) = t_i$ for some x in $G\}$, where $I = [0,1]$.*

Definition 2.12 *Let μ be an anti fuzzy subgroup of a group G. For a in G, the anti fuzzy coset $a\mu$ of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ for all x in G.*

Definition 2.13 *Let μ be an anti fuzzy subgroup of a group G. For a and b in G, the anti fuzzy middle coset $a\mu b$ of G is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for all x in G.*

Definition 2.14 *Let μ be an anti fuzzy subgroup of G and an element a in G. Then pseudo anti fuzzy coset $(a\mu)^p$ is defined by $(a\mu)^p(x) = p(a)\mu(x)$ for all x in G and p in P.*

Definition 2.15 *The Cartesian product $\lambda \times \mu : X \times Y \to [0,1]$ of two anti fuzzy subgroups is defined by $(\lambda \times \mu)(x,y) = \max\{\lambda(x), \mu(y)\}$ for all (x,y) in $X \times Y$ and R_λ is a binary anti fuzzy relation defined by $R_\lambda(x,y) = \max\{\lambda(x), \lambda(y)\}$. The anti fuzzy relation R_λ is said to be a similarity relation if*

(i) $R_\lambda(x,x) = 1$;

(ii) $R_\lambda(x,y) = R_\lambda(y,x)$;

(iii) $\max\{R_\lambda(x,y), R_\lambda(y,z)\} \leq R_\lambda(x,z)$.

Definition 2.16 *Let G be a finite group of order n and μ a fuzzy subgroup of G. Then for t_1, t_2 in $[0,1]$ such that $t_1 \leq t_2, \mu_{t_2} \subseteq \mu_{t_1}$.*

Definition 2.17 *Let G be a finite group of order n and μ an anti fuzzy subgroup of G. Then for $t_1, t_2 \in [0,1]$ such that $t_1 \le t_2, \mu_{t_1} \subseteq \mu_{t_2}$.*

Definition 2.18 *Let f be a group homomorphism from a group G to H. Then there is an isomorphism $\phi : f(G) \to G/Kerf$, where ϕ is the canonical isomorphism associated with f.*

Definition 2.19 *Let G be a group and H, K normal subgroups of G such that $H \le K$. Then there is a natural isomorphism $G/K \cong (G/H)/(K/H)$.*

Proposition 2.20 *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if G_μ^t is a level subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G.*

Proposition 2.21 *H as described in **2.6** can be realized as a level subgroup.*

Theorem 2.22 *G is a Dedekind or Hamiltonian group if and only if every fuzzy subgroup of G is fuzzy normal subgroup.* (A Dedekind and Hamiltonian groups have all the subgroups to be normal).

§3. Briefly on Properties of Anti Fuzzy Subgroup

Proposition 3.1 *Any two pseudo cosets of an anti fuzzy subgroup of a group G are either identical or disjoint.*

Proof Assume that $(a\mu)^p$ and $(b\mu)^p$ are any two identical pseudo anti fuzzy cosets of μ for any a and b in G. Then, $(a\mu)^p(x) = (b\mu)^p(x)$ for all x in G. Assume also on the contrary that they are disjoint. Then, there is no y in G such that $(a\mu)^p(y) = (b\mu)^p(y)$ which implies that $p(a)\mu(y) \ne p(b)\mu(y)$. The consequence is that $p(a) \ne p(b)$. This makes the assumption $(a\mu)^p(x) = (b\mu)^p(x)$ false.

Conversely, assume that $(a\mu)^p$ and $(b\mu)^p$ are disjoint, then $p(a)\mu(y) \ne p(b)\mu(y)$ for every y in G. But if it is assumed that this is also identical, then $p(a)\mu(y) = p(b)\mu(y)$ and that means $p(a) = p(b)$ so that $p(a)\mu(y) \ne p(b)\mu(y)$ cannot be true. $\square$

Proposition 3.2 *Let μ be an anti fuzzy subgroup of any group G. Let $\{\mu_i\}$ be a partition of μ. Then*

    (i) *each μ_i is normal if μ is normalized;*
    (ii) *each μ_i is normal if μ is normal.*

Proof Note that for each i, $\mu_i \subseteq \mu$ which implies that $\mu_i(x) \le \mu(x)$ for all x in G.

(i) Since μ is normalized, there is an x_0 in G such that $\mu_i(x) \le \mu(x) \le \mu(x_0) = 1$ for each i. Whence, $\mu_i(x) \le 1$. Then $\sup \mu_i(x) = 1$.

(ii) Since μ is normal, $\sup \mu(x) = 1$, then $\mu(x) \le 1$. Note that $\mu_i(x) \le \mu(x) \le 1$. Then $\mu_i(x) \le 1$ and $\sup \mu_i(x) = 1$. $\square$

Proposition 3.3 *Let μ be an anti fuzzy subgroup of any group G. Then $\mu(e) \le 1$ even if μ is*

normalized.

　Proof Note that for all x in G, $0 \le \mu(x) \le 1$.

$$\mu(e) = \mu(xx^{-1}) \le \max\{\mu(x), \mu(x^{-1})\} = \mu(x) \text{ since } \mu(x) = \mu(x^{-1}) \text{ for all } x \text{ in } G.$$

　But since μ is normal, there is an x_0 in G such that $\mu(e) \le \mu(x) \le \mu(x_0) = 1$. Hence $\mu(e) \le 1$.   □

Proposition 3.4 *Let μ be an anti fuzzy subgroup of any group G and $R_\mu : G \times G \to [0,1]$ be given by $R_\mu(x, y) = \mu(xy^{-1})$. R_μ is not a similarity relation.*

　Proof The reference [4] has shown that this is a similarity relation when μ is a fuzzy subgroup of G. But
$$R_\mu(x, x) = \mu(xx^{-1}) = \mu(e) \le 1.$$

R_μ is not symmetric, hence not a similarity relation.   □

§4. Application of Isomorphism Theorems of Groups to Fuzzy Subgroups

Proposition 4.1 *Let f be a group homomorphism between G and H. Let μ be a fuzzy subgroup of H. Then G is isomorphic to a level subgroup of H.*

　Proof Since f is a homomorphism, it is defined on G.

$$Ker\ f = \{x \in G : f(x) = e_H\} \Leftrightarrow \{x \in G : \mu f(x) = \mu(e_H) \le 1\}.$$

Hence, $\mu f(x) \le 1$ for all x in G since μ is a fuzzy subgroup of H and $f(x)$ is in H.

$$Ker\ f = G \text{ so that } \mu f(G) \le 1.$$

Also, note that
$$f(G) = \{y = f(x) \in H : \mu f(x) = \mu(y) = \mu(e_H)\}.$$

By 2.21 and 2.6, $f(G)$ is a level subgroup, say H_μ^t of H.

$$G/G = G \cong H_\mu^t$$

by Definition 2.18.   □

Remark 4.2 It can be said then that every group G is isomorphic to a level subgroup of a group H if there is a group homomorphism between G and H and μ a fuzzy subgroup of H exits.

Proposition 4.3 *Let G be a Dedekind or an Hamiltonian group and μ a fuzzy subgroup of G. For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ and $G/G_{1\mu}^t \cong (G/G_{2\mu}^t)/(G_{1\mu}^t/G_{2\mu}^t)$.*

Proof By Proposition 2.20, $G_{1\mu}^t$ and $G_{2\mu}^t$ are subgroups of G and by Theorem 2.22, they are normal subgroups. Also by Definition 2.16,

$$G_{2\mu}^t \leq G_{1\mu}^t.$$

Then, $f : G/G_{2\mu}^t \to G/G_{1\mu}^t$ is a group homomorphism and

$$Im(f) = G/G_{1\mu}^t \ \ \text{if} \ \ f(gG_{2\mu}^t) = gG_{1\mu}^t.$$

Also, it can be shown that $Ker\, f = G_{1\mu}^t/G_{2\mu}^t$.

Then apply Definition 2.19 so that $G/G_{1\mu}^t \cong (G/G_{2\mu}^t)/(t_{1\mu}^t/G_{2\mu}^t)$. $\qquad\square$

Remarks 4.4 It is equally of note that if μ is an anti fuzzy subgroup of a group G,

$$\text{for} \ \ t_1 < t_2, \ G_{1\mu}^t \leq G_{2\mu}^t$$

by Definition 2.17.

Following the same argument as in Proposition 4.3,

$$G/G_{2\mu}^t \cong (G/G_{1\mu}^t)/(G_{2\mu}^t/G_{1\mu}^t). \qquad\square$$

Acknowledgement

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Contributions to Differential Geometry of Partially

Null Curves in Semi-Euclidean Space E_1^4

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Abstract: In this paper, some characterizations of partially null curves of constant breadth and inclined partially null curves in Semi-Rieamannian Space E_1^4 are presented.

Key Words: Semi-Rieamannian Space, partially null curves, curves of constant breadth.

AMS(2010): 53A05, 53B25, 53B30

§1. **Introduction**

The partially null curves, lying fully in the Minkowski space-time are defined in [1] as space-like curves along which respectively the first binormal is null vector and second binormal is null vector. The Frenet equations of a partially null curve, lying fully in the Minkowski space-time are given in [14, 2], using those Frenet equations authors give some characterizations. Another work, in [10], authors define Frenet equations of such curves and study some of characterizations in Semi-Euclidean space.

Recently, a method has been developed by B.Y.Chen to classify curves with the solution of differential equations with constant coefficients, see [3, 4, 11]. Furthermore, classifications all space-like W curves are given in [11].

Curves of constant breadth were introduced by L. Euler,1870. Ö. Köse (1984) wrote some geometric properties of plane curves of constant breadth. And, in another work Ö. Köse (1986) extended these properties to the Euclidian3-space E^3 [6]. Morever, M. Fujivara (1914) obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined "breadth" for space curves and obtained these curves on a surface of constant breadth [5]. A. Mağden and Ö. Köse (1997) studied this kind curves in four dimensional Euclidean space E^4 [7]. S. Yılmaz and M. Turgut extended the notation of curves of constant breadth to null curves in Semi-Rieamannian space E_2^4, see [13].

Inclined curves are well-known concept in the classical differential geometry [8].

---

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [9].) Minkowski space-time E_1^4 is a Euclidean space E_1^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is rectangular coordinate system in E_1^4. Since g is an definite metric, recall that a vector $\vec{\vartheta} \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(\vec{\vartheta}, \vec{\vartheta}) > 0$ or $\vec{\vartheta} = 0$, timelike if $g(\vec{\vartheta}, \vec{\vartheta}) < 0$ and null (ligth-like) if $g(\vec{\vartheta}, \vec{\vartheta}) = 0$ and $\vec{\vartheta} \neq 0$. Similarly, and arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ in E_1^4 can be locally be space-like, time-like or null (ligth-like) if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null. Also recall the norm of a vector $\vec{\vartheta}$ is given by $\left\|\vec{\vartheta}\right\| = \sqrt{\left|g(\vec{\vartheta}, \vec{\vartheta})\right|}$. Therefore, $\vec{\vartheta}$ is a unit vector if $g(\vec{\vartheta}, \vec{\vartheta}) = \pm 1$. Next vectors $\vec{\vartheta}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(\vec{\vartheta}, \vec{w}) = 0$. The velocity of the curve α is given by $\left\|\vec{\alpha}'\right\|$. Thus, a space-like or a time-like curve $\vec{\alpha}$ is said to be parameterized by arc-length function s, if $g(\vec{\alpha}', \vec{\alpha}') = \pm 1$. The Lorentzian hypersphere of center $\vec{m} = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_1^4 defined by

$$S_1^3 = \left\{ \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : g(\vec{\alpha} - \vec{m}, \vec{\alpha} - \vec{m}) = r^2 \right\}.$$

Denoted by $\left\{ \vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s) \right\}$ the moving Frenet frame along the curve $\vec{\alpha}$ in the space E_1^4.

Then $\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2$ are, respectively, the tangent, the principal normal, the first binormal and second binormal vector fields. Recall that a space-like curve with time-like principal normal $\vec{N}$ and null first and second binormal is called a partially null curve in E_1^4 [1]. For a partially null unit speed curve $\vec{\alpha}$ in E_1^4 the following Frenet equations are given in [2, 14]

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & \sigma \end{bmatrix} \cdot \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}$$

where $\vec{T}, \vec{N}, \vec{B}_1$ and $\vec{B}_2$ are mutually orthogonal vectors satisfying equations

$$g(\vec{T}, \vec{T}) = g(\vec{B}_1, \vec{B}_2) = 1, \ g(\vec{N}, \vec{N}) = -1$$

$$g(\vec{B}_1, \vec{B}_1) = g(\vec{B}_2, \vec{B}_2) = 0.$$

And here, $\kappa(s), \tau(s)$ and $\sigma(s)$ are first, second and third curvature of the curve $\vec{\alpha}$, respectively.

In the same space, the authors, in [2], expressed a characterizations of partially null curves with the following theorem.

Theorem 2.1 *A partially null unit speed curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$, in E_1^4, with curvatures $\kappa \neq 0$, $\tau \neq 0$ for each $s \in I \subset \mathbb{R}$ has $\sigma = 0$ for eachs.*

In [13], S. Yılmaz and M. Turgut studied same characterizations of spherical and inclined partially null curves.

§3. Partially Null Curves of Constant Breadth in E_1^4

Let $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ and $\overrightarrow{\alpha}^* = \overrightarrow{\alpha}^*(s)$ be simple closed partially null curves in the space E_1^4. These curves will be denoted by C. Moreover let P and Q at points respectively curves α and α^*. The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a partially null curve in the class Γ as in M. Fujivara (1914) having parallel tangents $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{T}}^*$ in opposite directions at the opposite points α and α^* of the curve. A simple closed curve of constant breadth at opposite points can be represented with respect to Frenet frame by the equation

$$\overrightarrow{\alpha}^* \;\; = \;\; \overrightarrow{\alpha} + m_1\overrightarrow{T} + m_2\overrightarrow{N} + m_3\overrightarrow{B}_1 + m_4\overrightarrow{B}_2 \tag{3.1}$$

where $m_i(s)$, $1 \leq i \leq 4$ arbitrary functions of s, $\overrightarrow{\alpha}$ and $\overrightarrow{\alpha}^*$ are opposite points. The vector $d = \overrightarrow{\alpha}^* - \overrightarrow{\alpha}$ is called "the distance vector" of C. Differentiating both sides of (3.1) and considering Frenet equations, we have

$$\frac{d\alpha^*}{ds} = \overrightarrow{T}^*\frac{ds^*}{ds} \;\; = \;\; (\frac{dm_1}{ds} - m_2\kappa + 1)\overrightarrow{T} + (\frac{dm_2}{ds} - m_4\tau + m_1\kappa)\overrightarrow{N}$$

$$+ (\frac{dm_3}{ds} + m_2\tau + m_3\sigma)\overrightarrow{B}_1 + (\frac{dm_4}{ds} + m_4\sigma)\overrightarrow{B}_2 \tag{3.2}$$

Since $\overrightarrow{T}^* = -\overrightarrow{T}$, rewriting (3.2) we obtain following system of equations,

$$\frac{dm_1}{ds} - m_2\kappa + 1 + \frac{ds^*}{ds} = 0$$

$$\frac{dm_2}{ds} + m_1\kappa - m_4\tau = 0$$

$$\frac{dm_3}{ds} + m_2\tau = 0 \tag{3.3}$$

$$\frac{dm_4}{ds} = 0$$

If we call θ as the angle between the tangent of the curve C at point $\overrightarrow{\alpha}$ with a given fixed

direction and s arc length parameter of $\overrightarrow{\alpha}(s)$, consider $\dfrac{d\theta}{ds} = \kappa$, we have (3.3) as following:

$$\frac{dm_1}{d\theta} = m_2 - f(\theta)$$

$$\frac{dm_2}{d\theta} = -m_1 + m_4\rho\tau$$

$$\frac{dm_3}{d\theta} = -m_2\rho\tau \tag{3.4}$$

$$\frac{dm_4}{d\theta} = 0$$

where $f(\theta) = \rho + \rho^*$, $\rho = \dfrac{1}{\kappa}$ and $\rho^* = \dfrac{1}{\kappa^*}$ denote the radius of curvature at $\overrightarrow{\alpha}$ and $\overrightarrow{\alpha}^*$, respectively. It is not difficult to see that $m_4 = c_4$ =constant. then, using system (3.4) we easily have following differential equations with respect to m_1 and m_2 as

$$\frac{d^2 m_1}{d\theta^2} + m_1 + \frac{df}{d\theta} - c_4\rho\tau = 0$$

$$\frac{d^2 m_2}{d\theta^2} + m_2 - c_4\frac{d}{d\theta}(\rho\tau) - f(\theta) = 0 \tag{3.5}$$

These equations are characterizations for the curve $\overrightarrow{\alpha}^*$. If the distance between opposite points of C and C^* is constant, then, due to null frame vectors, we can write that

$$\|\overrightarrow{\alpha}^* - \overrightarrow{\alpha}\|^2 = m_1^2 + m_2^2 + 2m_3 m_4 = l^2 = \text{constant.} \tag{3.6}$$

Hence, by the differentiation we have

$$m_1\frac{dm_1}{d\theta} + m_2\frac{dm_2}{d\theta} + m_3\frac{dm_4}{d\theta} + m_4\frac{dm_3}{d\theta} = 0 \tag{3.7}$$

Considering system (3.4), we get

$$m_1(\frac{dm_1}{d\theta} - m_2) = 0 \tag{3.8}$$

Since, we arrive $m_1 = 0$ or $\dfrac{dm_1}{d\theta} = m_2$. Therefore, we shall study in the following cases.

Case 1 $m_1 = 0$. Moreover, let us suppose that $c_4 \neq 0$.

In this case $(3.5)_1$ deduce other components, respectively

$$m_2 = f(\theta) = c_4 \int_0^\theta \rho\tau d\theta \tag{3.9}$$

and

$$m_3 = -\int_0^\theta (\rho + \rho^*)\rho\tau d\theta \tag{3.10}$$

If $c_4 = 0$, we have $f(\theta) = c =$ constant. By this way, we know

$$m_2 = c$$

$$m_3 = -c \int_0^\theta \rho\tau d\theta \tag{3.11}$$

$$\rho + \rho^* - c = 0$$

Case 2 $\dfrac{dm_1}{d\theta} = m_2$.

In this case, from (3.4), we know $f(\theta) = m_2 = 0$. And first let us suppose that $c_4 \neq 0$. Thus the equation $(3.5)_1$ has the form

$$\frac{d^2 m_1}{d\theta^2} + m_1 = c_4\rho\tau \tag{3.12}$$

By the method of variation of parameters, the solution of (3.12) yields that

$$m_1 = \cos\theta \left[-\int_0^\theta c_4\rho\tau \sin\theta d\theta + A \right] + \sin\theta \left[\int_0^\theta c_4\rho\tau \cos\theta d\theta + B \right] \tag{3.13}$$

where A, B real numbers. From $(3.4)_3$ and $(3.4)_4$ we get

$$m_3 = c_3 \tag{3.14}$$

and

$$m_4 = c_4 \tag{3.15}$$

And if $c_4 = 0$, we write that

$$\frac{d^2 m_1}{d\theta^2} + m_1 = 0 \tag{3.16}$$

We write the solution of (3.16) as

$$m_1 = l_1 \cos\theta + l_2 \sin\theta \tag{3.17}$$

Considering (3.4), we have other components

$$m_2 = -l_1 \sin\theta + l_2 \cos\theta \tag{3.18}$$

and

$$m_3 = \int_0^\theta (-l_1 \sin\theta + l_2 \cos\theta)\rho\tau d\theta. \tag{3.19}$$

§4. The Inclined Partially Null Curves In E_1^4

Theorem 4.1 *Let $\alpha = \alpha(s)$ be a unit speed partially null curve in E_1^4. α is an inclined curve, if and only if*

$$\frac{\kappa}{\tau} = \text{constant} \tag{4.1}$$

Proof Let $\alpha = \alpha(s)$ be a unit speed partially null curve is E_1^4 and also be an inclined curve from definition of inclined curves, we write that

$$g(\overrightarrow{T}, \overrightarrow{u}) = \cos \Psi \tag{4.2}$$

where $\overrightarrow{u}$ is a constant space-like vector and Ψ is a constant angle. Differentiating (4.2) respect s, we have

$$\kappa g(\overrightarrow{N}, \overrightarrow{u}) = 0 \tag{4.3}$$

which implies that $\overrightarrow{N} \perp \overrightarrow{u}$. And therefore we compose constant vector $\overrightarrow{u}$ as

$$\overrightarrow{u} = u_1 \overrightarrow{T} + u_2 \overrightarrow{B}_1 + u_3 \overrightarrow{B}_2 \tag{4.4}$$

Differentiating (4.4) and considering Frenet equations we have following equation system:

$$\begin{aligned} \frac{du_1}{ds} &= 0 \\ \frac{du_2}{ds} &= 0 \\ \frac{du_3}{ds} &= 0 \\ u_1\kappa - u_3\tau &= 0 \end{aligned} \tag{4.5}$$

Solution of (4.5) yields that

$$\frac{\kappa}{\tau} = \text{constant} \tag{4.6}$$

Conversely, let us consider a vector given by

$$\overrightarrow{u} = \left\{ \overrightarrow{T} + \overrightarrow{B}_1 + \frac{\kappa}{\tau}\overrightarrow{B}_2 \right\} \cos \Psi \tag{4.7}$$

Where Ψ is a constant angle. Differentiating (4.7), we have

$$\frac{d\overrightarrow{u}}{ds} = 0 \tag{4.8}$$

(4.8) implies that $\overrightarrow{u}$ is a constant vector. And then considering a partially null curve $\alpha = \alpha(s)$; using inner product, we get

$$g(\overrightarrow{T}, \overrightarrow{u}) = \cos \Psi, \tag{4.9}$$

which shows that α is a inclined curve in E_1^4.

In the same space, S.Yılmaz gave a formulation about inclined curves with following the-

orem in [12]:

Let $\alpha = \alpha(s)$ be a space-like curve in E_1^4 parametrized by arclength. The curve α is an inclined curve if and only if

$$\frac{\kappa}{\tau} = A\cosh(\int\limits_0^s \sigma ds) + B\sinh(\int\limits_0^s \sigma ds) \tag{4.10}$$

where $\tau \neq 0$ and $\sigma \neq 0$, $A, B \in \mathbb{R}$.

Whence, we know that α is partially null curve, so $\sigma = 0$. Using (4.10) we have

$$\frac{\kappa}{\tau} = \text{constant.} \tag{4.11}$$

This completes the proof.   □

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Existence Results of Unique Fixed Point in 2-Banach Spaces

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Abstract: In this paper, we establish some fixed point theorems in the setup of 2-Banach spaces. The results obtained are the 2-Banach space extension of the result of Zhao [9].

Key Words: Fixed point, 2-Banach space, 2-Banach contraction, 2-Kannan contraction, 2-Chatterjea contraction, 2-Zamfirescu operator.

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§1. Introduction

The concept of 2-Banach space and some basic fixed point results in such spaces are initially given by Gahler ([3], [4]) during $1960's$. Later on some fixed point results have been obtained in such spaces by Iseki [5], Khan et al. [6], Rhoades [7] and many others extending the fixed point results for non expansive mappings from Banach space to 2-Banach space. In 2011, Choudhury and Som [2] (J. Indian Acad. Math. 33(2) (2011), 411-418) have established common fixed point and coincidence fixed point results for a pair of non-linear mappings in 2-Banach space which generalize the results of Som [8], Cho et al. [1] and Zhao [9] in turn. In this paper we establish some fixed point theorems satisfying the contractive type condition in 2-Banach spaces.

§2. Preliminaries

Here we give some preliminary definitions related to 2-Banach spaces which are needed in the sequel.

Definition 2.1 (See [1]) *Let X be a linear space and $\|., .\|$ be a real valued function defined on X satisfying the following conditions:*

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$;
(iii) $\|x, ay\| = |a| \, \|x, y\|$ for all $x, y \in X$ and real a;
(iv) $\|x, y + z\| = \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

Then, $\|., .\|$ is called a 2-norm and the pair $(X, \|., .\|)$ is called a linear 2-normed space.

Some of the basic properties of the 2-norms are that they are non negative and

$$\|x,\, y + ax\| = \|x,\, y\|$$

for all $x, y \in X$ and all real number a.

Definition 2.2(See [1])   *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|., .\|)$ is called a Cauchy sequence if $\lim_{m,\,n\to\infty} \|x_m - x_n,\, y\| = 0$ for all $y \in X$.*

Definition 2.3(See [1])   *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|., .\|)$ is said to be convergent to a point x in X if $\lim_{n\to\infty} \|x_n - x,\, y\| = 0$ for all $y \in X$.*

Definition 2.4(See [1])   *A linear 2-normed space $(X, \|., .\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.*

Definition 2.5(See [1])   *Let X be a 2-Banach space and T be a self mapping of X. T is said to be continuous at x if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$.*

Definition 2.6   *Let $(X, \|., .\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be 2-Banach contraction if there is $a \in [0, 1)$ such that*

$$\|Tx - Ty,\, u\| \le a\, \|x - y,\, u\|$$

for all $x, y, u \in X$.

Definition 2.7   *Let $(X, \|., .\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be 2-Kannan contraction if there is $b \in [0, \frac{1}{2})$ such that*

$$\|Tx - Ty,\, u\| \le b\left[\, \|x - Tx,\, u\| + \|y - Ty,\, u\| \,\right]$$

for all $x, y, u \in X$.

Definition 2.8   *Let $(X, \|., .\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be 2-Chatterjea contraction if there is $c \in [0, \frac{1}{2})$ such that*

$$\|Tx - Ty,\, u\| \le c\left[\, \|x - Ty,\, u\| + \|y - Tx,\, u\| \,\right]$$

for all $x, y, u \in X$.

Definition 2.9   *Let $(X, \|., .\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be 2-Zamfirescu operator if there are real numbers $0 \le a < 1$, $0 \le b < 1/2$, $0 \le c < 1/2$ such that for all $x, y, u \in X$ at least one of the conditions is true:*

(z_1) $\|Tx - Ty,\, u\| \le a\, \|x - y,\, u\|$;
(z_2) $\|Tx - Ty,\, u\| \le b\,(\|x - Tx,\, u\| + \|y - Ty,\, u\|)$;
(z_3) $\|Tx - Ty,\, u\| \le c\,(\|x - Ty,\, u\| + \|y - Tx,\, u\|)$.

Condition 2.1 *Let X be a 2-Banach space (with dim $X \geq 2$) and let T be a self mapping of X such that for all x, y, u in X satisfying the condition:*

$$
\begin{aligned}
\|Tx - Ty, \, u\| \quad \leq \quad & h \max \Big\{ \|x - y, \, u\|, \frac{(\|x - Tx, \, u\| + \|y - Ty, \, u\|)}{2}, \\
& \frac{(\|x - Ty, \, u\| + \|y - Tx, \, u\|)}{2} \Big\}
\end{aligned}
\tag{2.1}
$$

where $0 < h < 1$.

Remark 2.1 It is obvious that each of the conditions $(z_1) - (z_3)$ implies (2.1).

§3. Main Results

In this section we shall prove a fixed point theorem using condition (2.1) in the setting of 2-Banach spaces.

Theorem 3.1 *Let X be a 2-Banach space (with dim $X \geq 2$) and let T be a continuous self mapping of X satisfying the condition (2.1), then T has a unique fixed point in X.*

Proof For given each $x_0 \in X$ and $n \geq 1$, we choose $x_1, x_2 \in X$ such that $x_1 = Tx_0$ and $x_2 = Tx_1$. In general we define sequence of elements of X such that $x_{n+1} = Tx_n = T^{n+1}x_0$. Now for all $u \in X$, using (2.1), we have

$$
\begin{aligned}
\|x_n - x_{n+1}, \, u\| \quad = \quad & \|Tx_{n-1} - Tx_n, \, u\| \\
\leq \quad & h \max \Big\{ \|x_{n-1} - x_n, \, u\|, \frac{(\|x_{n-1} - Tx_{n-1}, \, u\| + \|x_n - Tx_n, \, u\|)}{2}, \\
& \frac{(\|x_{n-1} - Tx_n, \, u\| + \|x_n - Tx_{n-1}, \, u\|)}{2} \Big\} \\
= \quad & h \max \Big\{ \|x_{n-1} - x_n, \, u\|, \frac{(\|x_{n-1} - x_n, \, u\| + \|x_n - x_{n+1}, \, u\|)}{2}, \\
& \frac{(\|x_{n-1} - x_{n+1}, \, u\| + \|x_n - x_n, \, u\|)}{2} \Big\} \\
= \quad & h \max \Big\{ \|x_{n-1} - x_n, \, u\|, \frac{(\|x_{n-1} - x_n, \, u\| + \|x_n - x_{n+1}, \, u\|)}{2}, \\
& \frac{\|x_{n-1} - x_{n+1}, \, u\|}{2} \Big\} \\
\leq \quad & h \max \Big\{ \|x_{n-1} - x_n, \, u\|, \frac{(\|x_{n-1} - x_n, \, u\| + \|x_n - x_{n+1}, \, u\|)}{2}, \\
& \frac{(\|x_{n-1} - x_n, \, u\| + \|x_n - x_{n+1}, \, u\|)}{2} \Big\}.
\end{aligned}
\tag{3.1}
$$

But

$$
\frac{(\|x_{n-1} - x_n, \, u\| + \|x_n - x_{n+1}, \, u\|)}{2} \leq \max \Big\{ \|x_{n-1} - x_n, \, u\|, \|x_n - x_{n+1}, \, u\| \Big\}
\tag{3.2}
$$

From (3.1) and (3.2), we get

$$
\begin{aligned}
\|x_n - x_{n+1},\, u\| \;\leq\; & h \, \max\Big\{\, \|x_{n-1} - x_n,\, u\|,\, \|x_{n-1} - x_n,\, u\|,\, \|x_n - x_{n+1},\, u\|, \\
& \|x_{n-1} - x_n,\, u\|,\, \|x_n - x_{n+1},\, u\| \Big\} \\
\leq\; & h \, \|x_{n-1} - x_n,\, u\|.
\end{aligned}
\tag{3.3}
$$

Similarly, we have

$$
\|x_{n-1} - x_n,\, u\| \leq h \, \|x_{n-2} - x_{n-1},\, u\|.
\tag{3.4}
$$

Hence form (3.3) and (3.4), we have

$$
\|x_n - x_{n+1},\, u\| \leq h^2 \, \|x_{n-2} - x_{n-1},\, u\|.
\tag{3.5}
$$

On continuing in this process, we get

$$
\|x_n - x_{n+1},\, u\| \leq h^n \, \|x_0 - x_1,\, u\|.
\tag{3.6}
$$

Also for $n > m$, we have

$$
\begin{aligned}
\|x_n - x_m,\, u\| \;\leq\; & \|x_n - x_{n-1},\, u\| + \|x_{n-1} - x_{n-2},\, u\| + \ldots \\
& + \|x_{m+1} - x_m,\, u\| \\
\leq\; & \left(h^{n-1} + h^{n-2} + \cdots + h^m \right) \|x_1 - x_0,\, u\| \\
\leq\; & \left(\frac{h^m}{1-h} \right) \|x_1 - x_0,\, u\|.
\end{aligned}
\tag{3.7}
$$

Since $0 < h < 1$ by condition 2.1, $\left(\frac{h^m}{1-h} \right) \to 0$ as $m \to \infty$. Hence $\|x_n - x_m,\, u\| \to 0$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exist a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of T that $Tz = z$. Thus z is a fixed point of T.

For the uniqueness, let $Tv = v$ be another fixed point of the mapping T. Then, we have

$$
\begin{aligned}
\|z - v,\, u\| \;=\; & \|Tz - Tv,\, u\| \\
\leq\; & h \, \max\Big\{\, \|z - v,\, u\|,\, \frac{(\|z - Tz,\, u\| + \|v - Tv,\, u\|)}{2}, \\
& \frac{(\|z - Tv,\, u\| + \|v - Tz,\, u\|)}{2} \Big\} \\
\leq\; & h \, \max\Big\{\, \|z - v,\, u\|,\, 0,\, \|z - v,\, u\| \Big\} \\
\leq\; & h \, \|z - v,\, u\| \\
<\; & \|z - v,\, u\|, \text{ since } 0 < h < 1,
\end{aligned}
\tag{3.8}
$$

a contradiction. Hence $z = v$ and for all $u \in X$. Thus z is a unique fixed point of T. This completes the proof. □

Since Condition 2.1 includes the 2-Banach contraction condition, 2-Kannan contraction condition, 2-Chatterjea contraction condition and 2-Zamfirescu operator. Thus from Theorem 3.1, we obtain the following results as corollaries.

Corollary 3.1 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a self mapping of X satisfying the condition:*

$$\|Tx - Ty, u\| \leq a \, \|x - y, u\|$$

for all $x, y, u \in X$, where a is a constant in $(0, 1)$. Then T has a unique fixed point in X.

Corollary 3.2 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying the condition:*

$$\|Tx - Ty, u\| \leq b \left[\|x - Tx, u\| + \|y - Ty, u\|\right]$$

for all $x, y, u \in X$, where b is a constant in $(0, \frac{1}{2})$. Then T has a unique fixed point in X.

Corollary 3.3 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying the condition:*

$$\|Tx - Ty, u\| \leq c \left[\|x - Ty, u\| + \|y - Tx, u\|\right]$$

for all $x, y, u \in X$, where c is a constant in $(0, \frac{1}{2})$. Then T has a unique fixed point in X.

Corollary 3.4 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying 2-Zamfirescu operator, that is, satisfying at least one of the conditions in $(z_1) - (z_3)$. Then T has a unique fixed point in X.*

Remark 3.1 Our results extend the corresponding result of Zhao [9] (Acta Math. Sinica 22(1979), 459-470), Cho et al. [1] (Far East Jour. Math. Sci. 3(2)(1995), 125-133) and many others from the existing literature.

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Total Domination in Lict Graph

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Abstract: For any graph $G = (V, E)$, lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contains any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of the graph G is a minimum cardinality of total lict dominating set of graph G. In this paper many bounds on $\gamma_t(\eta(G))$ are obtained and its exact values for some standard graphs are found in terms of parameters of G. Also its relationship with other domination parameters is investigated.

Key Words: Smarandachely k-dominating set, total lict domination number, lict graph, edge domination number, total edge domination number, split domination number, non-split domination number.

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§1. Introduction

The graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual $'p'$ and $'q'$ denote the number of vertices and edges of a graph G. For any undefined term or notation in this paper can be found in Harary [1].

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number $\gamma_k(G)$* of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

The lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating

set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of G is a minimum cardinality of total lict dominating set of G.

The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of G. $L(G)$ is the line graph of G, $\gamma_e^{'}(G)$ is the complementary edge domination number, $\gamma_s(G)$ is the split dominating number, $\gamma_t^{'}(G)$ is the total edge dominating number , $\gamma_{ns}(G)$ is the non-split dominating number, $\chi(G)$ is the chromatic number and $\omega(G)$ is the clique number of a graph G. The degree of an edge $e = uv$ of G is $deg(e) = deg(u) + deg(v) - 2$. The minimum (maximum) degree of an edge in G is denoted by $\delta^{'}(\Delta^{'})$. A subdivision of an edge $e = uv$ of a graph G is the replacement of an edge e by a path (u, v, w) where $w \ni E(G)$. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$. For any real number $X, \lceil X \rceil$ denotes the smallest integer not less than X and $\lfloor X \rfloor$ denotes the greatest integer not greater than X.

In this paper we established the relationship of this concept with the other domination parameters. We use the following theorems for our later results.

Theorem A([2]) *For any graph* $G, \gamma_e(G) \geq \left\lceil \dfrac{q}{\Delta^{'} + 1} \right\rceil$.

Theorem B([2]) *For any graph G of order $p \geq 3$,*

  (i) $\beta_1(G) + \beta_1(\bar{G}) \leqslant 2 \left\lceil \dfrac{p}{2} \right\rceil$.

  (ii) $\beta_1(G) * \beta_1(\bar{G}) \leqslant \left\lceil \dfrac{p}{2} \right\rceil^2$.

Theorem C([3]) *For any graph G,*

  (i) $\gamma_t^{'}(S(K_p)) = 2 \left\lceil \dfrac{p}{2} \right\rceil$.

  (ii) $\gamma_t^{'}(S(K_{p,q})) = 2q(p \leq q)$.

  (iii) $\gamma_t^{'}(S(G)) = 2(p - \beta_1)$.

Theorem D([4]) *For every graph G of order p,*

  (i) $\chi(G) \geq \omega(G)$.

  (ii) $\chi(G) \geq \dfrac{q}{\beta_0}(G)$.

Theorem E([5]) *For any connected graph G with $p \geq 3$ vertices, $\gamma_t^{'}(G) \leq \left\lceil \dfrac{2p}{3} \right\rceil$.*

Theorem F([5]) *If G is a connected graph G with $p \geq 4$ vertices and q edges then $\dfrac{q}{\Delta^{'}} \leq \gamma_t^{'}(G)$, further equality holds for every cycle C_p where $p = 4n, n \geq 1$.*

§2. Main Results

Theorem 1 *First list out the exact values of $\gamma_t(\eta(G))$ for some standard graphs:*

(i) *For any cycle C_p with $p \geq 3$ vertices,*

$$\gamma_t(\eta(C_p)) = \begin{cases} p/2 & \text{if } p \equiv 0 (mod 4). \\ \left\lfloor \dfrac{p}{2} \right\rfloor + 1 & otherwise. \end{cases}$$

(ii) *For any path P_p with $p \geq 4$ vertices,* $\gamma_t(\eta(P_p)) = \left\lfloor \dfrac{2q}{3} \right\rfloor$.

(iii) *For any star graph $K_{1,p}$ with $p \geq 3$ vertices,* $\gamma_t(\eta(K_{1,p})) = 2$.

(iv) *For any wheel graph W_p with $p \geq 4$ vertices,* $\gamma_t(\eta(W_p)) = \left\lfloor \dfrac{p}{2} \right\rfloor$.

(v) *For any complete graph K_p with $p \geq 3$ vertices,* $\gamma_t(\eta(K_p)) = \left\lfloor \dfrac{2p}{3} \right\rfloor$.

(vi) *For any friendship graph F_p with k blocks,* $\gamma_t(\eta(F_p)) = k$.

Initially we obtain a lower bound of total lict domination number with edge and total edge domination number.

Theorem 2 *For any graph G, $\gamma_t(\eta(G)) \geq \gamma_e(G)$.*

Proof Let D be a γ_e set of graph G, if D is a total lict dominating set of a graph G, then for every edge $e_1 \in D$ there exists an edge $e_2 \in D, e_1 \neq e_2$ such that e_1 is adjacent to e_2. Hence $\gamma_t(\eta(G)) = \gamma_e(G)$. Otherwise for each isolated edge $e_i \in D$, choose an edge $e_j \in N(e_i)$. Let $E_1 = \{e_j / e_j \in N(e_i)\}$, then $D \cup E_1$ is a total lict dominating set of G and $|D \cup E_1| \geq |D|$. Hence, $\gamma_t(\eta(G)) \geq \gamma_e(G)$. $\qquad\square$

Theorem 3 *For any graph G $\gamma_t(\eta(G)) \geq \gamma_t'(G)$, equality holds if G is non-separable.*

Proof Let D be a γ_t' set of G, if all the cut vertices of G are incident with at least one edge of D, then $\gamma_t(\eta(G)) = \gamma_t'(G)$. Otherwise there exists at least one cut vertex v_c of graph G which is not incident with any edge of D, then $\gamma_t(\eta(G)) \geq |D \cup e| \geq \gamma_t'(G) + 1$, where e is an edge incident with v_c and $e \in N(D)$. Thus, $\gamma_t(\eta(G)) \geq \gamma_t'(G)$.

For the equality, note that if the graph G is non-separable, then $\eta(G) = L(G)$. Thus $\gamma_t(\eta(G)) = \gamma_t(L(G)) = \gamma_t'(G)$. $\qquad\square$

Next we obtain an inequality of total lict domination in terms of number of vertices, number of edges and maximum edge degree of graph G.

Theorem 4 *For any connected graph G with $p \geq 3$ vertices, then $\gamma_t(\eta(G)) \leq 2\left\lceil \dfrac{q}{3} \right\rceil$.*

Proof Let $E(G) = \{e_1, e_2, e_3, \cdots, e_l\}$ and let $D = \{e_l / 1 \leq i \leq l \text{ and } i \neq 0 (mod 3)\} \cup \{e_{l-1}\}$. Then D is total lict dominating set of G and $|D| = 2\left\lceil \dfrac{q}{3} \right\rceil$. Hence, $\gamma_t(\eta(G)) \leq 2\left\lceil \dfrac{q}{3} \right\rceil$. $\qquad\square$

Theorem 5 *For any non-separable graph G,*

(i) $\gamma_t(\eta(G)) \leq \left\lceil \dfrac{2p}{3} \right\rceil$, $p \geq 3$.

(ii) $\dfrac{q}{\Delta'} \leq \gamma_t(\gamma(G))$, $p \geq 4$ vertices, equality holds for every cycle C_p, where $p = 4n, n \geq 1$.

Proof Let G be a non-separable graph,then $\gamma_t(\eta(G)) = \gamma_t'(G)$. Using Theorems E and F, the result follows.   □

Theorem 6 *For any connected graph* G,$\gamma_t(\eta(G)) \leq q - \Delta'(G) + 1$, *where* Δ' *is a maximum degree of an edge.*

Proof Let e be an edge with degree Δ' and let S be a set of edges adjacent to e in G. Then $E(G) - S$ is the lict dominating set of graph G. We consider the following two cases.

Case 1 If $\langle E(G) - S \rangle$ contains at least one isolate in $\eta(G)$ other than the vertex corresponding to e in $\eta(G)$.

Let E_1 be the set of all such isolates, then for each isolate $e_i \in E_1$, let $E_2 = \{e_j/e_j \in (N(e_i) \cap N(e))\}$, then $F = [\{(E(G) - S) - E_1\} \cup E_2]$ is a total lict dominating set of graph G. Thus, $\gamma_t(\eta(G)) \leq q - \Delta'(G)$.

Case 2 If $\langle E(G) - S \rangle$ contains only e as an isolate in $\eta(G)$.

Then for an edge $e_i \in N(e)$, $\{(E(G) - S) \cup e_i\}$ is a total lict dominating set of a graph G. Thus, $\gamma_t(\eta(G)) \leq |(E(G) - S) \cup e_i| = q - \Delta'(G) + 1$.

From Cases 1 and 2, the result follows.   □

Theorem 7 *For any connected graph* G,$\gamma_t(\eta(G)) \geq \left\lceil \dfrac{q}{\Delta' + 1} \right\rceil$.

Proof Using Theorem 2 and Theorem A, the result follows.   □

Theorem 8 *For any connected graph* G,$\gamma_t(\eta(G)) \leq p - 1$.

Proof Let T be a spanning tree of a graph G. Let $A = \{e_1, e_2, e_3, \cdots, e_k\}$ be the set of edges of spanning tree T, A covers all the vertices and cut vertices of a graph $\eta(G)$. Hence,$\gamma_t(\eta(G)) \leq A = p - 1$.   □

Now we obtain the relationship between total lict domination and total domination of a line graph.

Theorem 9 *For any graph* G,*with* k *number of cut vertices,*

$$\gamma_t(\eta(G)) \leq \gamma_t(L(G)) + k.$$

Proof We consider the following two cases.

Case 1 $k = 0$.

Then the graph G is non-separable, and in that case $\eta(G) = L(G)$. Hence, $\gamma_t(\eta(G)) = \gamma_t(L(G))$.

Case 2 $k \neq 0$.

Let D be a total dominating set of $L(G)$ and let S be the set of cut vertices which is not incident with any edge of D, then for each cut vertex $v_c \in S$, choose exactly one edge in

E_1, where $E_1 = \{e_j \in E(G)/e_j \text{ is incident with } v_c \text{ and } e_j \in N(D)\}$ with $|E_1| = |v_c|$. Hence, $\gamma_t(\eta(G)) \leq \gamma_t(L(G)) + |E_1| = \gamma_t(L(G)) + |v_c| = \gamma_t(L(G)) + k$.

From Cases 1 and 2, the result follows. □

In the following theorems we obtain total lict domination of any tree in terms of different parameters of G.

Theorem 10 *For any tree T with k number of cut vertices,$\gamma_t(\eta(G)) \leq k+1$, further equality holds if $T = K_{1,p}$, $p \geq 3$.*

Proof Let $A = \{v_1, v_2, v_3, \cdots, v_k\} \subset V(G)$ be the set of all cut vertices of a tree T with $|A| = k$. Since every edge in T is incident with at least one element of A, A covers all the edges and cut vertices of $\eta(G)$, if for every cut vertex $v \in A$ there exists a vertex $u \in A, u \neq v$, such that v is adjacent to u. Otherwise let $e_1 \in E(G)$ such that e_1 is incident with A, so that $\gamma_t(\eta(G)) \leq \{A \cup e_1\} = |A| + 1 = k+1$.

To prove the equality, let $K_{1,p}$ be a star and C be the cut vertex and e be any edge of $K_{1,p}$. Then $D = \{C \cup e\}$ is the γ_t set of $\eta(G)$ with cardinality $k+1$. □

Theorem 11 *For any tree T,$\gamma_t(\eta(T)) \geq \chi(T)$ and equality holds for all star graph $K_{1,p}$.*

Proof $\chi(T) = 2$ and $2 \leq \gamma_t(T) \leq p$. Hence, $\gamma_t(\eta(T)) \geq \chi(T)$. For $T = K_{1,p}$, clearly $\chi(T) = 2$. Using Theorem 1(iii), the equality follows. □

Theorem 12 *For any tree T ,$\gamma_t(\eta(T)) \geq \omega(T)$.*

proof The result follows from Theorem 11 and Theorem D. □

Theorem 13 *For any tree T, $\gamma_t(\eta(T)) \geq \dfrac{q}{\beta_0(T)}$.*

Proof The result follows from Theorem 11 and Theorem D. □

Theorem 14 *For any tree T, $\gamma_t(\eta(T)) \leq \gamma_t(T)$.*

Proof Let T be a tree and D be γ_t of T. Let E_1 denotes the edge set of the induced graph $\langle D \rangle$. Let F be the set of cut vertices which are not incident with any edge of E_1. we consider the following two cases.

Case 1 If $F = \Phi$, and in $\eta(T)$ if E_1 does not contains any isolates then E_1 is a total lict dominating set of T. Otherwise for each isolated edge $e_i \in E_1$, choose exactly one edge in E_2, where $E_2 = \{e_j \in E(T)/e_j \in N(e_i)\}$. Then $D^* = E_1 \cup E_2$ is a total lict dominating set of tree T. Hence, $\gamma_t(\eta(T)) \leq |D^*| \leq |D| = \gamma_t(T)$.

Case 2 If $F \neq \Phi$, then for each cut vertex $v_c \in F$. Let $E_2 = \{e_j \in E(T)/e_j \in N(e_i)$ and incident with $v_c\}$.Then $D^* = E_1 \cup E_2$ is a total lict dominating set of tree T. Hence, $\gamma_t(\eta(T)) \leq |D^*| \leq |D| = \gamma_t(T)$.

From Cases 1 and 2, the result follows. □

Theorem 15 *For any tree T with $p \geq 3$, in which every non-end vertex is incident with an end vertex, then $\gamma_t(\eta(T)) \leq \beta_0(T)$.*

Proof We consider the following two cases.

Case 1 $T = K_{1,p}$.

Noticing that $\beta_0(T) = p - 1 \geq 2$ for $p \geq 3$, and using Theorem 1(iii), the result follows. Hence,$\gamma_t(\eta(T)) \leq \beta_0(T)$.

Case 2 $T \neq K_{1,p}$.

Let $B = \{v_1, v_2, v_3, \cdots, v_m\} \subset V(G)$ such that $|B| = \beta_0(T)$. Let $S \subseteq B$ be the set of k end vertices of T and $N \subseteq B$ be the set of l non-end vertices of T such that $S \cup N = B$. In T, for each vertex $v_i \in S$ there exists cut vertex $C_i \in N(v_i)$. Then in $\eta(T)$ the cut vertex C_i covers the edges incident with cut vertex C_i of T where $i = 1, 2, 3, 4, 5, ...k$ and for each vertex $v_i \in N$ in T, a vertex $v_j \in \eta(T)$ which is a cut vertex of T covers all the edges incident with v_j where $j = 1, 2, 3, 4, 5.......l$. Thus $\{C_i\}_{i=1}^{k} \cup \{v_j\}_{j=1}^{l}$ forms a total lict dominating set of T. Hence $\gamma_t(\eta(T)) \leq |S \cup N| \leq |B| = \beta_0(T)$.

From case(1) and case(2) the result follows.   □

Theorem 16 *Let T be any order $p \geq 3$ and n be the number of pendent edges of T, then $n \leq \gamma_t(\eta(S(T))) \leq 2(p-1) - n$ and equality holds for all $K_{1,p}$.*

Proof Let $u_1 v_1, u_2 v_2, u_3 v_3, u_4 v_4, \cdots, u_n v_n$ be the pendent edges of T. Let w_i be the vertex set of $S(T)$ that subdivides the edges $u_i v_i, i = 1, 2, 3, 4, \cdots, n$. Any total lict dominating set of $S(T)$ contains the edges $u_i w_i, i = 1, 2, 3, 4, \cdots, n$ and hence $\gamma_t(\eta(S(T))) \geq n$. Further $E(S(T)) - S$, where S is the set of all pendent edges of $S(T)$ forms a total lict dominating set of $S(T)$. Hence,$\gamma_t(\eta(S(T))) \leq 2(p-1) - n$.

Notice that the edges of $D = \{u_i w_i\}, i = 1, 2, 3, \cdots, n$ will forms a γ_t of $\eta(S(T))$ for $K_{1,p}$. Thus, the equality $\gamma_t(\eta(S(T))) = n$. Similarly, the set $\{E(S(T) - S\}$ will forms a γ_t of $\eta(S(T))$ for $K_{1,p}$. So $\gamma_t(\eta(S(T))) = 2(p-1) - n$.   □

Now we obtain the relation between total lict domination in terms of complimentary edge domination, total domination and split domination and non-split domination.

Theorem 17 *For any graph G if $\gamma_e(G) = \gamma_e'(G)$, then $\gamma_t(\eta(G)) \geq \gamma_e'(G)$.*

Proof Let us consider the graph G, with $\gamma_e(G) = \gamma_e'(G)$ and using Theorem 2.2, the result follows.   □

Corollary 1 *Let D be the γ_e set of a non-separable graph G then, $\gamma_t(\eta(G)) \geq \gamma_e'(G)$.*

Proof Since every complementary edge dominating set is an edge dominating set, the follows from Theorem 2.   □

Theorem 18 *For any non-separable graph G with $p \geq 3$, then $\gamma_t(G) \leq \gamma_t(\eta(G))$, equality holds for all cycle C_p.*

Proof Let $D = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_t set of a graph G. Let $E^* = \{e_i \in E(G)/e_i$ is incident with $v_i\}, i = 1, 2, 3, 4, \cdots, k$. Then every edge in $\langle E(G) - E^* \rangle$ is adjacent to at least one edge in E^*. Clearly E^* covers all the vertices in $\eta(G)$, and $\langle E^* \rangle$ does not contain any isolates, E^* is a total lict dominating set of graph G and $|D| \leq |E^*|$. Hence, $\gamma_t(G) \leq \gamma_t(\eta(G))$.

For any cycle C_p, $\eta(G) = L(G), \gamma_t(L(G)) = \gamma_t(G)$. Hence, $\gamma_t(G) = \gamma_t(\eta(G))$. $\qquad\square$

Theorem 19 *For any cycle C_p $p \geq 3$, $\gamma_s(C_p) \leq \gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.*

Proof We consider the following two cases.

Case 1 $\gamma_s(C_p) \leq \gamma_t(\eta(C_p))$.

Let $A = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_s dominating set of cycle C_p. For any cycle C_p, $\eta(G) = L(G)$, the corresponding edges $B = \{e_1, e_2, e_3, \cdots, e_k\}$ will be a split dominating set of $\eta(G)$. Since $\langle B \rangle$ is disconnected, $\gamma_t(\eta(C_p)) \leq \gamma_s(C_p) + 1$. Hence, $\gamma_s(C_p) \leq \gamma_t(\eta(C_p))$.

Case 2 $\gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.

Let $A = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_{ns} dominating set of cycle C_p. For any cycle C_p, $\eta(G) = L(G)$, the corresponding edges $B = \{e_1, e_2, e_3, \cdots, e_k\}$ will be a split dominating set of $\eta(G)$. Since $\langle B \rangle$ is connected. Hence, $\gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.

The result follows from Cases 1 and 2. $\qquad\square$

Now we obtain the total lict dominating number in terms of independence number and edge covering number.

Theorem 20 *For any graph $G, \gamma_t(\eta(G)) \leq 2\beta_1(G)$.*

Proof Let S be a maximum independent edge set in a graph G. Then every edge in $E(G) - S$ is adjacent to at least one edge in S. Let D be the set of cut vertices that is not incident with any edge of S and let $E_1 = \{e_i \in E(G) - S/e_i \in N(S)\}$. We consider the following two cases.

Case 1 If $D = \phi$, then for each edge $e_j \in S$, pick exactly one edge $e_i \in E_1$, such that $e_i \in N(e_j)$. Let D_1 be the set of all such edges with $|D_1| \leq |S|$. Then $F = S \cup D_1$ is a total lict dominating set of G. Hence, $\gamma_t(\eta(G)) \leq |S \cup D_1| = |S| + |D_1| \leq |S| + |S| = 2\beta 1(G)$.

Case 2 If $D \neq \phi$, then for each cut vertex $v_c \in D$. Let $E_2 = \{e_i \in E(G) - S/e_j \in N(S)$ and incident with $v_c\}$, $E_3 = \{e_k \in S/e_k \in N(E_2)\}$ and $D_2 = S - E_3$. Now for each edge $e_l \in D_2$, pick exactly one edge in $e_i \in E_1$, such that e_l is adjacent to e_i. Let D_3 be the set of all such edges. Then $F = D_2 \cup D_3 \cup E_2 \cup E_3$ is a total lict dominating set of G. Hence,

$$
\begin{aligned}
\gamma_t(\eta(G)) &\leq |F| = |D_2 \cup E_3 \cup D_3 \cup E_2| \\
&\leq |D_2 \cup E_3| + |D_3 \cup E_2| \\
&= |S| + |S| = 2|S| = 2\beta_0(G)
\end{aligned}
$$

From Cases 1 and 2, the result follows. $\qquad\square$

Theorem 21 *For any graph* $G, \gamma_t(\eta(G)) \leq 2\alpha_0(G)$.

Proof Let $S = \{v_1, v_2, v_3, v_4, \cdots, v_k\} \subset V(G)$ such that $|S| = \alpha_0(G)$. Then for each vertex v_i, choose exactly one edge in E_1 where $E_1 = \{e_i \in E(G)/e_i$ is incident with $v_i\}$ such that $|E_1| \leq |S|$. Let D be the set of cut vertices that is not incident with any edge of E_1 and let $E_2 = \{e_j \in E(G) - E_1/e_j \in N(E_1)\}$. We consider the following two cases.

Case 1 If $D = \phi$, then for each edge $e_i \in E_1$, pick exactly one edge $e_j \in E_2$, such that $e_j \in N(e_i)$. Let D_1 be the set of all such edges with $|D_1| \leq |E_1| = |S|$. Then $F = E_1 \cup D_1$ is a total lict dominating set of G. Hence, $\gamma_t(\eta(G)) \leq |E_1 \cup D_1| = |E_1| + |D_1| \leq |S| + |S| = 2\alpha_0(G)$.

Case 2 If $D \neq \phi$, then for each cut vertex $v_c \in D$. Let $E_3 = \{e_l \in E(G) - E_1/e_l \in N(E_1)$ and incident with $v_c\}$,

$E_4 = \{e_k \in E_1/e_k \in N(E_3)\}$ and $D_3 = E_1 - E_4$. Now for each edge $e_r \in D_2$, pick exactly one edge in $e_j \in E_2$, such that e_r is adjacent to e_j. Let D_3 be the set of all such edges. Then $F = D_2 \cup D_3 \cup E_3 \cup E_4$ is a total lict dominating set of G. Hence,

$$\begin{aligned}
\gamma_t(\eta(G)) &\leq |F| = |D_2 \cup E_4 \cup D_3 \cup E_3| \\
&\leq |D_2 \cup E_4| + |D_2 \cup E_4| \\
&= |E_1| + |E_1| = |S| = 2\alpha_0(G)
\end{aligned}$$

From Cases 1 and 2, the result follows.   □

Now we obtain the total lict dominating number of a subdivision graph of a graph G in terms of edge independence number and number of vertices of a graph G.

Theorem 22 *For any graph* G, $\gamma_t(\eta(S(G))) \leq 2q - 2\beta_1 + p_0$, *where* p_0 *is the number of vertices that subdivides* β_1.

Proof Let $A = \{u_i v_i/1 \leq i \leq n\}$ be the edge set of a graph G. Let $X = \{u_i v_i/1 \leq i \leq n\}$ be a maximum independent edge set of graph G. Then X is edge dominating set of a graph G. Let w_i be the vertex set of $S(G)$ and let $p_0 \in w_i$ be the set of vertices that subdivides X. Then for each vertex p_0, choose exactly one edge in E_1, where $E_1 = \{u_i w_i$ or $w_i v_i \in S(G)/u_i w_i$ or $w_i v_i$ is incident with p_0 and adjacent to $A - X\}$. Let $F = \{\{\{A - \{X\}\} - \{E_1\}\}$ covers all the edges and cut vertices of $S(G)$. Hence, $\gamma_t(\eta(S(G))) \leq F = |A - X - E_1| = 2q - 2\beta_1 + p_0$.□

Theorem 23 *For any non-separable graph* G,

(i) $\gamma_t(\eta(S(K_p)) = 2\lceil \frac{p}{2} \rceil$.
(ii) $\gamma_t(\eta(S(K_{p,q})) = 2q(p \leq q)$.
(iii) $\gamma_t(\eta(S(G)) = 2(p - \beta_1)$.

Proof Using the definitions of total lict dominating set and total edge dominating set of a graph, the result follows from Theorem C.   □

Next, we obtain the Nordhus-Gaddam results for a total domination number of a lict graph.

Theorem 24 *For any connected graph G of order $p \geq 3$ vertices,*

$(i)\ \ \gamma_t(\eta(G)) + \gamma_t(\eta(\bar{G})) \leq 4\lceil \frac{p}{2} \rceil.$

$(ii)\ \ \gamma_t(\eta(G)) * \gamma_t(\eta(\bar{G})) \leq 4\lceil \frac{p}{2} \rceil^2.$

Proof The result follows from Theorem B and Theorem 20. $\qquad\square$

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The Genus of the Folded Hypercube

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Abstract: The folded hypercube FQ_n is a variance of the hypercube network and is superior to Q_n in some properties[IEEE Trans. Parallel Distrib. Syst. 2 (1991) 31-42]. The genus of n-dimensional hypercube Q_n were given by G. Ringel. In this paper, the genus $\gamma(FQ_n)$ of FQ_n is discussed. That is, $\gamma(FQ_n) = (n-3)2^{n-3} + 1$ if n is odd and $(n-3)2^{n-3} + 1 \leq \gamma(FQ_n) \leq (n-2)2^{n-3} + 1$ if n is even.

Key Words: n-Dimensional hypercube, folded hypercube, genus, surface, Smarandache λ^S-drawing,.

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§1. Introduction

Let $G = (V(G), E(G))$ be a *graph*, where $V(G)$ is a finite *vertex set* and $E(G)$ is the *edge set* which is the subset of $\{(u,v)|(u,v)$ is an unordered pair of $V(G)\}$. Two vertices u and v are *adjacent* if $(u,v) \in E(G)$. A *path*, written as $\langle v_0, v_1, v_2, \cdots, v_m \rangle$, is a sequence of adjacent vertices, in which all the vertices $v_0, v_1, v_2, \cdots, v_m$ are distinct except possibly $v_0 = v_m$, the path with $v_0 = v_m$ is a *cycle*. The *girth* of a graph G is the length of the shortest cycle of G.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E(G)$ of fewer then l edges, then G is called *l-edge-connected*. The greatest integer l such that G is l-edge-connected is the *edge-connectivity* $\lambda(G)$ of G.

A *surface* is a compact connected orientable 2-manifold which could be thought of as a sphere on which has been placed a number of handles. The number of handles is referred to as the *genus* of the surface. A drawing of graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S if existing, is called an embedding of G on S.

A *region* of a graph G embedded on a surface is the connected sections of the surface bounded by a set of edges of G. This set of edges is called the *boundary* of the region, and

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the number of edges is the length of the region. We will use $(v_0, v_1, v_2, \cdots, v_m)$, called a *facial cycle*, to denote the region bounded by edges $(v_0, v_1), (v_1, v_2), \cdots$, and (v_m, v_0). so a facial cycle of a graph is a region of the graph. A region is a *k-cycle* if its length is k. A region is a *2-cell* if any simple closed curve within the region can be collapsed to a single point. An embedding of a graph G on a surface S is a *2-cell embedding* if all embedded regions are 2-cells.

An embedding of G into an oriented surface S induce a rotation system as follows: The *local rotation* at a vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around v. A *rotation system* of the given embedding of G in S is the collection of local rotations at all vertices of G. It is proved [19] that every 2-cell embedding of a graph G in an orientable surface is uniquely determined, up to homeomorphism, by its rotation system.

Let G be a graph and π be an embedding of G, the corresponding rotation system is denoted by ρ_π. For any $v \in V$, the local rotation at v determined by ρ_π is denoted by $\rho_\pi(v)$. In the following, we consider 2-cell embedding of simple undirected graphs on orientable surfaces, the rotation at a vertex is clockwise. The readers are referred to [1] for undefined notations.

The *genus* $\gamma(G)$ of a graph G is meant the minimum genus of all possible surfaces on which G can be embedded with no edge crossings, similarly, the $\gamma_M(G)$ is the *maximum genus*. As a measure of the complexity of a network, the genus gives an indication of how efficiently the network can be laid out. The smaller the genus, the more efficient the layout. The planer graphs have genus zero since no handles are needed to prevent edge intersections.

Let G be a connected graph with a 2-cell embedding on an orientable surface of genus g, having m vertices, q edges and r regions, then the well known Euler's formula [16] is: $m - q + r = 2 - 2g$. For embedding, Duke's interpolation theorem [5] is that a connected graph G has a 2-cell embedding on surface S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$, where k is the genus of surface S_k.

Graph embeddings have been studied by many authors over years. Especially the study of the maximum and minimum orientable genus $\gamma_M(G)$ and $\gamma(G)$ of a graph G, they have been proved polynomial [7] and NP-complete [22], respectively. The embedding properties of a graph and some results about surfaces are extensively treated in the books [3,4,8,19]. More results about genera and embedding genus distributions are referred to see [9-11,13-15,17-18,20,23-25,27] etc.. Although there are much results about maximal genera, but minimum genera for most kinds of graphs are not known. The folded hypercube FQ_n is a variance of the hypercube network and is superior to Q_n in some properties such as diameters [6]. The genus $\gamma(Q_n)$ of n-dimensional hypercube Q_n were given by G. Ringel [21], the genus of n-cube is discussed by Beineke and Harary [2].

In this paper, the genus $\gamma(FQ_n)$ of FQ_n is discussed. That is, $\gamma(FQ_n) = (n-3)2^{n-3} + 1$ for n is odd and $(n-3)2^{n-3} + 1 \leq \gamma(FQ_n) \leq (n-2)2^{n-3} + 1$ for n is even.

§2. Main Results

The n-dimensional hypercube, denoted by Q_n, is a bipartite graph with 2^n vertices, its any vertex v is denoted by an n-bit binary string $v = x_n x_{n-1} \cdots x_2 x_1$ or $(x_n x_{n-1} \cdots x_2 x_1)$, where

$x_i \in \{0,1\}$ for all i, $1 \leq i \leq n$. Two vertices of Q_n are adjacent if and only if their binary strings differ in exactly one bit position. So Q_n is an n-regular graph.

If $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ are two vertices in Q_n such that $y_i = 1 - x_i$ for $1 \leq i \leq n$, then we denote $y = \overline{x}$, and we say that x and $\overline{x}$ have complementary addresses. As a variance of the Q_n, the n-dimensional folded hypercube, denoted by FQ_n, proposed first by El-Amawy and Latifi[?], is defined as follows: FQ_n is an $(n+1)$-regular graph, its vertex set is exactly $V(Q_n)$, and its edge set is $E(Q_n) \bigcup E_0$, where $E_0 = \{x\overline{x} | x \in V(Q_n)\}$. In other words, FQ_n is a graph obtained from Q_n by adding edges, called complementary edges, between any pair of vertices with complementary addresses. FQ_2 and FQ_3 are shown in Fig.1.
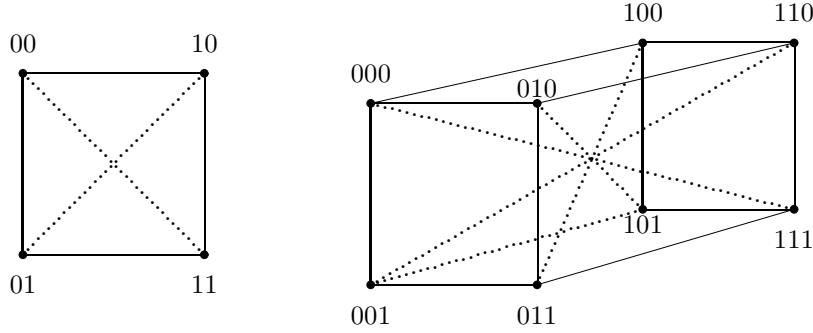
Fig.1 FQ_2 and FQ_3

Lemma 2.1([2, 21]) *Let Q_n be an n-hypercube, then $\gamma(Q_n) = (n-4)2^{n-3} + 1$.*

Lemma 2.1([6])   *The edge-connectivity of n-folded hypercube $\lambda(FQ_n) \geq n+1$.*

Lemma 2.3(Jungerman [12], Xuong [25]) *If G is a 4-edge-connected graph with m vertices and q edges, then $\gamma_M(G) = \lfloor \dfrac{q - m + 1}{2} \rfloor$.*

Lemma 2.4 *Let Q_n be an n-dimensional hypercube. Then there exists an embedding π_n of Q_n for $n \geq 3$ on the surface S of genus $(n-4)2^{n-3} + 1$, such that each of the following three kinds of cycles for $x_i \in \{0,1\}, 3 \leq i \leq n$,*

$((x_n \cdots x_3 10), (x_n \cdots x_3 00), (x_n \cdots x_3 01), (x_n \cdots x_3 11))$;

$((x_n \cdots x_3 10), (x_n \cdots x_3 00), (\overline{x_n} x_{n-1} \cdots x_3 00), (\overline{x_n} x_{n-1} \cdots x_3 10))$ *and*

$((x_n \cdots x_3 01), (x_n \cdots x_3 11), (\overline{x_n} x_{n-1} \cdots x_3 11), (\overline{x_n} x_{n-1} \cdots x_3 01))$

is a facial 4-cycle of π_n.

Proof It is true for Q_3, shown in Fig.2. Assume it is true for $Q_{n-1}, n \geq 4$. There exists an embedding π_{n-1} of Q_{n-1} on the surface S' of genus $(n-5)2^{n-4} + 1$, such that each of three kinds of cycles $((x_{n-1} \cdots x_3 10), (x_{n-1} \cdots x_3 00), (x_{n-1} \cdots x_3 01), (x_{n-1} \cdots x_3 11))$; $((x_{n-1} \cdots x_3 10), (x_{n-1} \cdots x_3 00), (\overline{x_{n-1}} x_{n-2} \cdots x_3 00), (\overline{x_{n-1}} x_{n-2} \cdots x_3 10))$ and $((x_{n-1} \cdots x_3 01),$ $(x_{n-1} \cdots x_3 11), (\overline{x_{n-1}} x_{n-2} \cdots x_3 11), (\overline{x_{n-1}} x_{n-2} \cdots x_3 01))$ for $x_i \in \{0,1\}, 3 \leq i \leq n-1$, is a facial cycle on embedding π_{n-1} of Q_{n-1}. So the rotations of π_{n-1} are as follows:

$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 10) = (A'(x_{n-1}\cdots x_3 11)(x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 00) = (B'(x_{n-1}\cdots x_3 10)(x_{n-1}\cdots x_3 01)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 01) = (C'(x_{n-1}\cdots x_3 00)(x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 11) = (D'(x_{n-1}\cdots x_3 01)(x_{n-1}\cdots x_3 10))$$

because of $((x_{n-1}\cdots x_3 10),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 01),(x_{n-1}\cdots x_3 11))$ being facial cycles along counter-clockwise; or

$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 10) = (A'(x_{n-1}\cdots x_3 00)(x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 00) = (B'(x_{n-1}\cdots x_3 01)(x_{n-1}\cdots x_3 10)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 01) = (C'(x_{n-1}\cdots x_3 11)(x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 11) = (D'(x_{n-1}\cdots x_3 10)(x_{n-1}\cdots x_3 01))$$

because of $((x_{n-1}\cdots x_3 11),(x_{n-1}\cdots x_3 01),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 10))$ being facial cycles along counter-clockwise, where A',B',C',D' are the ordered subsequences of vertices which incident with $(x_{n-1}\cdots x_3 10),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 01)$ and $(x_{n-1}\cdots x_3 11)$, respectively.

By Euler's formula, the boundary of every region in π_{n-1} of Q_{n-1} on S' is a 4-cycle. Let Q_{n-1} embed on another copy surface S'' of genus $(n-5)2^{n-4}+1$ such that the embedding of Q_{n-1} on S'' is a "mirror image" of the embedding of Q_{n-1} on S'. As a subgraph of Q_n, the vertices in embedding of Q_{n-1} on S' and on S'' are labeled by $(0x_{n-1}\cdots x_3 x_2 x_1)$ and $(1x_{n-1}\cdots x_3 x_2 x_1)$ respectively, where $x_i \in \{0,1\}, 1 \le i \le n-1$. For simplification, we also use the signals of A',B',C' and D' in the following.

Based on π_{n-1}, the rotation system of π_n is given as follows:

$$\rho_{\pi_n}(x_n\cdots x_3 10) = ((\overline{x_n}x_{n-1}\cdots x_3 10)A'(x_n\cdots x_3 11)(x_n\cdots x_3 00)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 00) = (B'(\overline{x_n}x_{n-1}\cdots x_3 00)(x_n\cdots x_3 10)(x_n\cdots x_3 01)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 01) = ((\overline{x_n}x_{n-1}\cdots x_3 01)C'(x_n\cdots x_3 00)(x_n\cdots x_3 11)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 11) = (D'(\overline{x_n}x_{n-1}\cdots x_3 11)(x_n\cdots x_3 01)(x_n\cdots x_3 10)); \text{ or}$$

$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 10) = (A'(x_n\cdots x_3 10)(\overline{x_n}x_{n-1}\cdots x_3 00)(\overline{x_n}x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 00) = ((x_n\cdots x_3 00)B'(\overline{x_n}x_{n-1}\cdots x_3 01)(\overline{x_n}x_{n-1}\cdots x_3 10)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 01) = (C'(x_n\cdots x_3 01)(\overline{x_n}x_{n-1}\cdots x_3 11)(\overline{x_n}x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 11) = ((x_n\cdots x_3 11)D'(\overline{x_n}x_{n-1}\cdots x_3 10)(\overline{x_n}x_{n-1}\cdots x_3 01)),$$

where $\overline{x_i} = 1 - x_i$.

By using the method of researching regions of embedding from rotation system in [19], the following four kinds of facial cycles on S' or S''

$$((00x_{n-2}\cdots x_3 10),(00x_{n-2}\cdots x_3 00),(01x_{n-2}\cdots x_3 00),(01x_{n-2}\cdots x_3 10));$$
$$((11x_{n-2}\cdots x_3 10),(11x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 10));$$
$$((00x_{n-2}\cdots x_3 11),(00x_{n-2}\cdots x_3 01),(01x_{n-2}\cdots x_3 01),(01x_{n-2}\cdots x_3 11));$$
$$((11x_{n-2}\cdots x_3 11),(11x_{n-2}\cdots x_3 01),(10x_{n-2}\cdots x_3 01),(10x_{n-2}\cdots x_3 11));$$

are replaced in π_n by the following eight facial 4-cycles:

$$((00x_{n-2}\cdots x_3 10),(00x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 10));$$

$$((00x_{n-2}\cdots x_300),(01x_{n-2}\cdots x_300),(11x_{n-2}\cdots x_300),(10x_{n-2}\cdots x_300));$$

$$((01x_{n-2}\cdots x_300),(01x_{n-2}\cdots x_310),(11x_{n-2}\cdots x_310),(11x_{n-2}\cdots x_300));$$

$$((01x_{n-2}\cdots x_310),(00x_{n-2}\cdots x_310),(10x_{n-2}\cdots x_310),(11x_{n-2}\cdots x_310));$$

$$((00x_{n-2}\cdots x_311),(00x_{n-2}\cdots x_301),(10x_{n-2}\cdots x_301),(10x_{n-2}\cdots x_311));$$

$$((00x_{n-2}\cdots x_301),(01x_{n-2}\cdots x_301),(11x_{n-2}\cdots x_301),(10x_{n-2}\cdots x_301));$$

$$((01x_{n-2}\cdots x_301),(01x_{n-2}\cdots x_311),(11x_{n-2}\cdots x_311),(11x_{n-2}\cdots x_301));$$

$$((01x_{n-2}\cdots x_311),(00x_{n-2}\cdots x_311),(10x_{n-2}\cdots x_311),(11x_{n-2}\cdots x_311))$$

and the other regions are not changed. As a result, each region of π_n is a 4-cycle. By the Euler's formula, the genus of embedding π_n of Q_n is exactly $2((n-5)2^{n-4}+1)+2^{n-3}-1 = (n-4)2^{n-3}+1$.

Further more, it could be found that the following three kinds of 4-cycles

$$((x_n\cdots x_310),(x_n\cdots x_300),(x_n\cdots x_301),(x_n\cdots x_311));$$

$$((x_n\cdots x_310),(x_n\cdots x_300),(\overline{x_n}x_{n-1}\cdots x_300),(\overline{x_n}x_{n-1}\cdots x_310)) \text{ and}$$

$$((x_n\cdots x_301),(x_n\cdots x_311),(\overline{x_n}x_{n-1}\cdots x_311),(\overline{x_n}x_{n-1}\cdots x_301))$$

for $x_i \in \{0,1\}$ and $3 \le i \le n$ are facial 4-cycles on π_n. $\square$
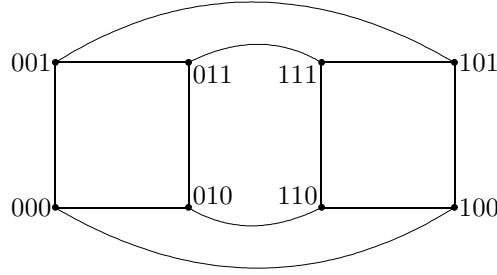
Fig.2 Q_3

Lemma 2.5([26])

(1) FQ_n is a bipartite graph if and only if n is odd.

(2)   If n is even, then the length of any shortest odd cycle in FQ_n is $n+1$.

Theorem 2.6   *The genus of $FQ_n(n \ge 3)$ is given as $\gamma(FQ_n) = (n-3)2^{n-3} + 1$ for n is odd and $(n-3)2^{n-3} + 1 \le \gamma(FQ_n) \le (n-2)2^{n-3} + 1$ for n is even.*

Proof FQ_n is embedded on the surface of genus $\gamma(FQ_n)$ with m vertices, q edges and r regions, where $m = 2^n$ and $q = (n+1)2^{n-1}$. From Lemma 2.5, the girth of FQ_n is 4 for $n \ge 3$. By Euler's formula, $4r \le 2q$, $m - q + r = 2 - 2\gamma(FQ_n) \le m - \frac{q}{2}$, so $2\gamma(FQ_n) - 2 \ge \frac{q}{2} - m$. That implies $\gamma(FQ_n) \ge (n-3)2^{n-3} + 1$.

To finish the proving, we only need to give an embedding of FQ_n such that the genus of embedded surface is $(n-3)2^{n-3}+1$ if n is odd, and is $(n-2)2^{n-3}+1$ if n is even, respectively.

First, Q_n is embedded on the surface with rotation system σ which is the same as the embedding π_n in Lemma 2.4, then we have the following rotations:

$$\rho_\sigma(x_n \cdots x_3 10) = (A(x_n \cdots x_3 11)(x_n \cdots x_3 00)),$$
$$\rho_\sigma(x_n \cdots x_3 00) = (B(x_n \cdots x_3 10)(x_n \cdots x_3 01)),$$
$$\rho_\sigma(x_n \cdots x_3 01) = (C(x_n \cdots x_3 00)(x_n \cdots x_3 11)),$$
$$\rho_\sigma(x_n \cdots x_3 11) = (D(x_n \cdots x_3 01)(x_n \cdots x_3 10)).$$

(2.1)

Or

$$\rho_\sigma(x_n \cdots x_3 10) = (A(x_n \cdots x_3 00)(x_n \cdots x_3 11)),$$
$$\rho_\sigma(x_n \cdots x_3 00) = (B(x_n \cdots x_3 01)(x_n \cdots x_3 10)),$$
$$\rho_\sigma(x_n \cdots x_3 01) = (C(x_n \cdots x_3 11)(x_n \cdots x_3 00)),$$
$$\rho_\sigma(x_n \cdots x_3 11) = (D(x_n \cdots x_3 10)(x_n \cdots x_3 01)),$$

(2.2)

where A, B, C, D are the ordered sequences of vertices which is incident with $(x_n \cdots x_3 10)$, $(x_n \cdots x_3 00), (x_n \cdots x_3 01), (x_n \cdots x_3 11)$, respectively.

According to ρ_σ of formulae (2.1) and (2.2) respectively and the fact that graph FQ_n is obtained from Q_n by adding complementary edges, the rotation system, denoted by θ, of FQ_n is gotten from rotation system σ as followings:

$$\rho_\theta(x_n \cdots x_3 10) = (A(x_n \cdots x_3 11)(\overline{x_n} \cdots \overline{x_3} 01)(x_n \cdots x_3 00)),$$
$$\rho_\theta(x_n \cdots x_3 00) = (B(x_n \cdots x_3 10)(\overline{x_n} \cdots \overline{x_3} 11)(x_n \cdots x_3 01)),$$
$$\rho_\theta(x_n \cdots x_3 01) = (C(x_n \cdots x_3 00)(\overline{x_n} \cdots \overline{x_3} 10)(x_n \cdots x_3 11)),$$
$$\rho_\theta(x_n \cdots x_3 11) = (D(x_n \cdots x_3 01)(\overline{x_n} \cdots \overline{x_3} 00)(x_n \cdots x_3 10)).$$

(2.3)

Or

$$\rho_\theta(\overline{x_n} \cdots \overline{x_3} 10) = (A(\overline{x_n} \cdots \overline{x_3} 00)(x_n \cdots x_3 01)(\overline{x_n} \cdots \overline{x_3} 11)),$$
$$\rho_\theta(\overline{x_n} \cdots \overline{x_3} 00) = (B(\overline{x_n} \cdots \overline{x_3} 01)(x_n \cdots x_3 11)(\overline{x_n} \cdots \overline{x_3} 10)),$$
$$\rho_\theta(\overline{x_n} \cdots \overline{x_3} 01) = (C(\overline{x_n} \cdots \overline{x_3} 11)(x_n \cdots x_3 10)(\overline{x_n} \cdots \overline{x_3} 00)),$$
$$\rho_\theta(\overline{x_n} \cdots \overline{x_3} 11) = (D(\overline{x_n} \cdots \overline{x_3} 10)(x_n \cdots x_3 00)(\overline{x_n} \cdots \overline{x_3} 01)),$$

(2.4)

where $\overline{x_i} = 1 - x_i$.

If n is odd, by the embedding σ of Q_n, the two kinds of 4-cycles

$$((x_n \cdots x_3 10), (x_n \cdots x_3 00), (x_n \cdots x_3 01), (x_n \cdots x_3 11));$$
$$((\overline{x_n} \cdots \overline{x_3} 10), (\overline{x_n} \cdots \overline{x_3} 11), (\overline{x_n} \cdots \overline{x_3} 01), (\overline{x_n} \cdots \overline{x_3} 00))$$

(2.5)

are facial cycles of this embedding of Q_n on the clockwise direction (or counter-clockwise direction). From the definition of θ of FQ_n, the following four kinds of complementary edges are

added in the facial cycles (2.5) shown in (a)(b) of Fig.3.

$$((x_n \cdots x_3 10), (\overline{x_n} \cdots \overline{x_3} 01)); \ (x_n \cdots x_3 00), \overline{x_n} \cdots \overline{x_3} 11));$$
$$((x_n \cdots x_3 01), (\overline{x_n} \cdots \overline{x_3} 10)); \ (x_n \cdots x_3 11), \overline{x_n} \cdots \overline{x_3} 00)). \tag{2.6}$$

(a)

(b)

(c)

(d)
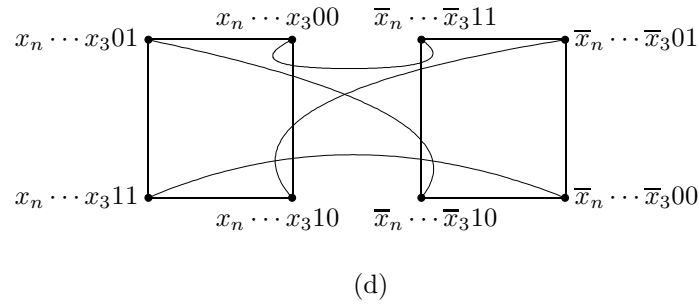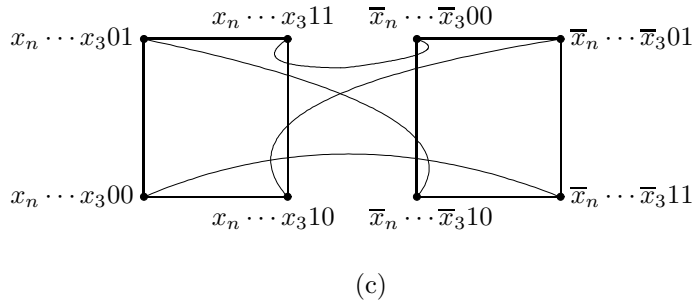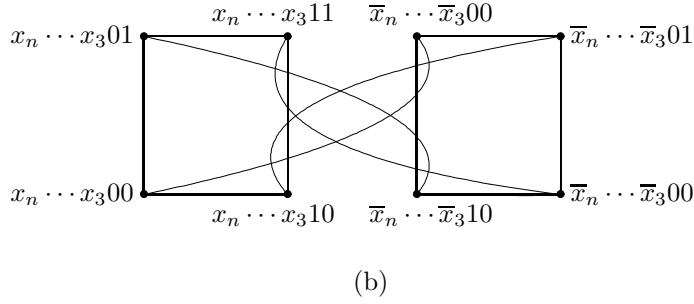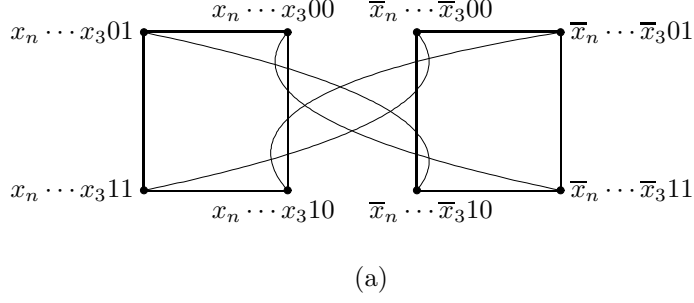
Fig.3 Two kinds of embedding depending on n being odd or even

As a result, the regions (2.5) of σ are replaced by the following four kinds of 4-regions in θ of FQ_n:

$$
\begin{aligned}
&((x_n \cdots x_3 11), (x_n \cdots x_3 10), (\overline{x_n} \cdots \overline{x_3} 01), (\overline{x_n} \cdots \overline{x_3} 00)); \\
&((x_n \cdots x_3 01), (x_n \cdots x_3 11), (\overline{x_n} \cdots \overline{x_3} 00), (\overline{x_n} \cdots \overline{x_3} 10)); \\
&((x_n \cdots x_3 00), (x_n \cdots x_3 01), (\overline{x_n} \cdots \overline{x_3} 10), (\overline{x_n} \cdots \overline{x_3} 11)); \\
&((x_n \cdots x_3 10), (x_n \cdots x_3 00), (\overline{x_n} \cdots \overline{x_3} 11), (\overline{x_n} \cdots \overline{x_3} 01)).
\end{aligned}
\tag{2.7}
$$

The other regions are not changed, thus all regions of embedding θ of FQ_n are all 4-cycles, and the number of regions is $2^{n-2}(n+1)$. Recalled that FQ_n have 2^n vertices, $2^{n-1}(n+1)$ edges. By Euler's formula, the total genus of θ of FQ_n for n being odd is $(n-3)2^{n-3}+1$.

If n is even, by the embedding σ of Q_n, the two kinds of 4-cycles

$$
\begin{aligned}
&((x_n \cdots x_3 10), (x_n \cdots x_3 00), (x_n \cdots x_3 01), (x_n \cdots x_3 11)); \\
&((\overline{x_n} \cdots \overline{x_3} 10), (\overline{x_n} \cdots \overline{x_3} 00), (\overline{x_n} \cdots \overline{x_3} 01), (\overline{x_n} \cdots \overline{x_3} 11))
\end{aligned}
\tag{2.8}
$$

are facial cycles of this embedding of Q_n on the clockwise direction (or counter-clockwise direction). From the definition of θ of FQ_n, By adding four kinds of complementary edges of (2.6) in facial cycles (2.8) shown in (c) and (d) of Fig.3, the regions in (2.8) of σ are replaced by the following two kinds of 8-cycles in θ of FQ_n:

$$
\begin{aligned}
&((x_n \cdots x_3 10), (x_n \cdots x_3 11), (\overline{x_n} \cdots \overline{x_3} 00), (\overline{x_n} \cdots \overline{x_3} 10), \\
&\quad (x_n \cdots x_3 01), (x_n \cdots x_3 00), (\overline{x_n} \cdots \overline{x_3} 11), (\overline{x_n} \cdots \overline{x_3} 01)); \\
&((x_n \cdots x_3 10), (x_n \cdots x_3 00), (\overline{x_n} \cdots \overline{x_3} 11), (\overline{x_n} \cdots \overline{x_3} 10), \\
&\quad (x_n \cdots x_3 01), (x_n \cdots x_3 11), (\overline{x_n} \cdots \overline{x_3} 00), (\overline{x_n} \cdots \overline{x_3} 01)).
\end{aligned}
$$

As a result, the number of regions in θ is less 2^{n-1} than regions in σ. By Euler's formula $2^n - 2^{n-1}(n+1) + (2^{n-2}n - 2^{n-1}) = 2 - 2h$, the genus h of embedding θ of FQ_n for n being even is $(n-2)2^{n-3}+1$. $\qquad \square$

From Lemmas 2.2 and 2.3, the following theorem is immediately obtained.

Theorem 2.7 *The maximum genus of FQ_n is given by $\gamma_M(FQ_n) = (n-1)2^{n-2}$ for $n \geq 3$. Furthermore, $\gamma(FQ_2) = 0, \gamma_M(FQ_2) = 1$.*

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Characteristic Polynomial & Domination Energy of

Some Special Class of Graphs

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Abstract: Representation of a set of vertices in a graph by means of a matrix was introduced by Sampath Kumar. Let $G(V, E)$ be a graph and $S \subseteq V$ be a set of vertices, we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. In this paper we define set energy and find its properties and also study the special case of set S being a dominating set and corresponding domination energy of some special class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, domination number, eigenvalues, energy of graph.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities, as the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is representation of molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent, if there is a bond connecting them.

Eigenvalues and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. From the pioneering work of Coulson [2] there exists a continuous interest towards the general Mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [7-10].

The importance of eigenvalues is not only used in theoretical chemistry but also in analyze structures, car designers analyze eigenvalues in order to damp out the noise to reduce the vibration of the car due to music, eigenvalues can be used to test for cracks or deformities in a solid, oil companies frequently use eigenvalue analysis to explore land for oil, eigenvalues are also used to discover new and better designs for the future [23].

Representation of a set of vertices in a graph by means of a matrix was introduced by Sampath Kumar [5]. Let $G(V, E)$ be a graph and $S \subseteq V$ be a set of vertices we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S, denoted by $A_S(G)$ and energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the special case of a set S being a dominating set and the corresponding matrix is domination matrix denoted by $A_\gamma(G)$ of G and energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$. For any undefined terms or notation in this paper, we refer Harary [6]. In this paper we define set energy and find its properties and also study the special case of set S being a dominating set and corresponding domination energy of some special class of graphs.

Let the graph G be connected and let its vertices be labelled as $v_1, v_2, v_3, \ldots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The eigenvalues of the dominating matrix are denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be A_γ eigenvalues of G. Since the A_γ matrix is symmetric, its eigenvalues are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$.

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [7-9]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ are the ordinary graph eigenvalues [7] that is, the eigenvalues of the adjacency matrix $A(G)$. Recall that in the few years, the graph energy $E(G)$ has been extensively studied in the mathematics [11-15] and mathematic-chemical literature [16-25].

§2. **Main Results**

Let $G(V, E)$ be a graph, $S \subseteq V$ and $A(G)$ be the adjacency matrix of G. Replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S.

Definition 2.1(Minimal dominating energy) *A dominating set D in G is minimal dominating set, if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for*

minimal dominating set is called Minimal Dominating Energy denoted by $E_{\gamma-\min}(G)$.

Definition 2.2(Minimal dominating energy) *A dominating set D in G is maximal dominating set, if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for maximal dominating set is called Maximal Dominating Energy denoted by $E_{\gamma-\max}(G)$.*

Observation 2.3 If $A(G)$ is the adjacency matrix corresponds to the graph $G(V, E)$, $A_{\gamma-\min}(G)$ is the adjacency matrix corresponding to the minimal dominating set $S_{\min}$ and $A_{\gamma-\max}(G)$ is the adjacency matrix corresponding to the maximum dominating set $S_{\max}$. Cardinality $|S_{\min}| \leqslant |S| \leqslant |S_{\max}|$ where set S is the dominating set whose cardinality is in between minimal and maximal dominating set. A graph $G(V, E) \neq K_n$, $n \geqslant 3$ then $E_{\gamma-\min}(G) \pm \varepsilon \leqslant E_\gamma(G) \leqslant E_{\gamma-\max}(G) \pm \varepsilon$, where ε is the error factor such that $|\varepsilon| \leqslant 1$.

Corollary 2.4 *A graph $G(V, E) \neq K_n$, $n \geqslant 3$ then $E(G) \leqslant E_{\gamma-\min}(G)$.*

Observation 2.5 A graph $G(V, E) = K_n$, $n \geqslant 3$ then $E_{\gamma-\min}(G) \pm \varepsilon \geqslant E_\gamma(G) \geqslant E_{\gamma-\max}(G) \pm \varepsilon$, where ε is the error factor such that $|\varepsilon| \leqslant 1$.

Corollary 2.6 *If graph $G(V, E) = K_n$, $n \geqslant 3$ then $E(G) \geqslant E_{\gamma-\min}(G)$. $E(K_n) = 2(n-1) \geqslant E_{\gamma-\min}(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$ (Theorem 4.3).*

Observation 2.7[Set Energy] Domination energy is the energy calculated w.r.t. the dominating set, but in order to understand the spectra of dominating set we generalize the concept as set energy. That is w.r.t. the set of different cardinality the energy were found. Energy for the $|S| = 0$ is the energy of the Graph $E(G)$. Similarly we find the energy for $|S| = 1$ to n. The particular case of set energy is the domination energy.

1. P_2 and C_6 are the only graphs with set energy of $|\varphi| = |2|$ and $|\varphi| = |6|$, $|1| = |5|$, $|2| = |4|$ respectively. Spectra are different but energy is same.

2. In energy of graph $\sum_{i=1}^n \lambda_i^2 = 2m$, m is the number of edges where as for Set energy $\sum_{i=1}^n \kappa_i^2 = 2m + |S|$, $|S|$ is the cardinality of set for which energy is calculated.

3. Set energies are symmetry in nature i.e., w.r.t. the shape of the graph (molecule). This can be proved by showing the matrix for the respected set will be same with the corresponding operation $R_i \leftrightarrow R_j$, $C_i \leftrightarrow C_j$. Example in a cycle of order n, label the vertices as $v_1, v_2, v_3, \ldots, v_n$ clockwise then $E_S(v_i, v_{i+1}) = E_S(v_{i+1}, v_{i+2})$ $E_S(v_i, v_{i+2}) = E_S(v_{i+1}, v_{i+3})$ etc. where $i = 1$ to $n-1$ and $j = 1$ to n.

4. In energy of graph $\sum_{i<j} \lambda_i \lambda_j = -m$, m is the number of edges where as for Set energy, $\sum_{i<j} \kappa_i \kappa_j \geqslant -m$, for $|S| \neq 1$, $\sum_{i<j} \kappa_i \kappa_j > -m$.

5. In energy of graph $\sum_{i=1}^n \lambda_i = 0$ where as for the set energy $\sum_{i=1}^n \lambda_i = |S|$, $i = 1$ to n.

6. It was found that there are same spectra for different sets of same cardinality (symmetry w.r.t. shape). Different spectra for different sets of same cardinality. Different spectra with set energy being same for the set with different cardinality.

7. If λ_1 is the highest eigenvalue w.r.t. energy of graph then $\sqrt{\Delta} \leqslant \lambda_1 \leqslant \Delta$. If κ_1 is the highest eigenvalue w.r.t. set energy of graph then $\sqrt{\Delta + 1} \leqslant \kappa_1 \leqslant \Delta + 1$.

8. In energy of a graph, characteristic polynomial is given by $\varphi(G : \lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}+\ldots+a_n$, $a_1 = 0$, $-a_2$ is number of edges, $-a_3$ is twice the number of triangles in G. For set energy the characteristic polynomial is given by $\varphi(G : \kappa) = \kappa^n + a_1\kappa^{n-1} + a_2\kappa^{n-2} + \ldots + a_n$, a_1 and a_2 are same for all characteristic polynomial with same cardinality of the set S, where $a_1 = -|S|$, but a_2 varies w.r.t. the cardinality of the set, i.e., when $|S| = 1$, $a_2 = -e = u_1$, e is the number of edges for a given graph. When $|S| = 2$, $a_2 = u_1 + 1 = u_2$, when $|S| = 3$, $a_2 = u_2 + 2 = u_3,\ldots$. When $|S| = n$, $a_2 = u_{n-1} + (n-1)$. Finding a_i for $i > 2$ is difficult for different cardinality of the set for the same graph.

§3. **Preliminary Results**

The following results comes from [22].

1. A graph $G(V, E)$ with $n \geqslant 3$ and $G \neq K_n$ then

$$\sqrt{2m + n(n-1)\left(\det A\right)^{2/n}} \leqslant E_{\gamma-\min}(G) \leqslant \sqrt{2mn},$$

where m is the number of edges and n is the number of vertices in G.

2. A graph G with n vertices without isolated vertices, with $n \geqslant 3$ and $G \neq K_n$ then $E_{\gamma-\min}(G) \geqslant 2\sqrt{n} + \varepsilon$.

3. $K_{n,n}$ is a Complete regular bipartite graph with $n \geqslant 3$, then

$$E_{\gamma-\min}(K_{n,n}) \leqslant 2|V| - 2,$$

where $|V|$ is the cardinality of vertices in G.

4. A graph $G(V, E)$ with $n \geqslant 3$ then $E_{\gamma-\min}(G) \leqslant \dfrac{n}{2}\left(\sqrt{n+1}\right) + \varepsilon$ where n is the number of vertices in G.

5. A graph $G(V, E)$ is a complete graph with $n \geqslant 3$ then $E_{\gamma}(K_n) \leqslant \sqrt{mn}$ where m is the number of edges and n is the number of vertices in G.

6. A graph $G(V, E)$ with $n \geqslant 3$ then $E_{\gamma-\min}(K_{1,n-1}) \leqslant E_{\gamma-\min}(T_n) \leqslant E_{\gamma-\min}(P_n)$ where $K_{1,n-1}$ is star graph with n vertices, T_n is tree with n vertices and P_n is path with n vertices.

§4. **Characterizing Graphs w.r.t. to the Unique Dominating Set**

Case 1 $\gamma(G) = 1$.

The characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also a modified method of Leverrier's method.

Theorem 4.1 *For any given star $K_{1,n-1}$ with $n \geqslant 3$, the characteristic polynomial is given by*

$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} = 0$ *with*

$$E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}$$
$$E(K_{1,n-1}) = 2\sqrt{n-1} \leqslant E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}.$$

Proof Consider a star, $K_{1,n-1}$. Label the vertices $v_1, v_2, v_3, \ldots, v_n$ such that v_1 has the maximum degree, hence in the domination matrix $a_{11} = 1$ and all other $a_{ii} = 0$, the characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method. That is, the characteristic polynomial is given by $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0$ where,

$$A_1 = A, \quad q_1 = -\text{Trace}A_1, \quad B_1 = A_1 + q_1 I_n,$$

where I_n is the unit matrix of order n.

$$A_2 = AB_1, \quad q_2 = -\frac{1}{2}\text{Trace}A_2, \ B_2 = A_2 + q_2 I_n,$$
$$A_3 = AB_2, \quad q_3 = -\frac{1}{3}\text{Trace}A_3, \ B_3 = A_3 + q_3 I_n$$
$$\ldots\ldots$$
$$A_n = AB_{n-1}, \quad q_n = -\frac{1}{n}\text{Trace}A_n, \quad B_n = A_n + q_n I_n.$$

Now consider an domination matrix of $K_{1,n-1}$, whose $\gamma(K_{1,n-1}) = 1$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & 0 \end{bmatrix}, \quad A_1 = A, \quad q_1 = -\text{Trace}A_1 = -1.$$

$B_1 = A_1 + q_1 I_n = A_1 - I_n$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & - & - & 1 \\ 1 & -1 & 0 & 0 & - & - & 0 \\ 1 & 0 & -1 & 0 & - & - & 0 \\ 1 & 0 & 0 & -1 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & -1 \end{bmatrix}$$

$A_2 = AB_1$

$$
= \begin{bmatrix}
1 & 1 & 1 & 1 & - & - & 1 \\
1 & 0 & 0 & 0 & - & - & 0 \\
1 & 0 & 0 & 0 & - & - & 0 \\
1 & 0 & 0 & 0 & - & - & 0 \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & - \\
1 & 0 & 0 & 0 & - & - & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 1 & - & - & 1 \\
1 & -1 & 0 & 0 & - & - & 0 \\
1 & 0 & -1 & 0 & - & - & 0 \\
1 & 0 & 0 & -1 & - & - & 0 \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & - \\
1 & 0 & 0 & 0 & - & - & -1
\end{bmatrix}
$$

$$
= \begin{bmatrix}
n-1 & 0 & 0 & 0 & - & - & 0 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & - & - & - & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1
\end{bmatrix}
$$

$$
q_2 = -\frac{1}{2}\text{Trace}A_2 = -\frac{1}{2}\left(n-1+n-1\right) = -\frac{1}{2}\left(2n-2\right) = -\left(n-1\right).
$$

$B_2 = A_2 + q_2 I_n = A_2 - (n-1)I_n.$

$$
= \begin{bmatrix}
n-1 & 0 & 0 & 0 & - & - & 0 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & - & - & - & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1
\end{bmatrix}
-
\begin{bmatrix}
n-1 & 0 & 0 & 0 & - & - & 0 \\
0 & n-1 & 0 & 0 & 0 & -0 & 0 \\
0 & 0 & n-1 & 0 & - & - & 0 \\
0 & 0 & 0 & n-1 & 0 & - & 0 \\
- & 0 & - & - & - & - & 0 \\
- & - & - & - & - & - & 0 \\
0 & 0 & 0 & 0 & - & - & n-1
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & 0 & - & - & 0 \\
0 & -n+2 & 1 & 1 & - & - & 1 \\
0 & 1 & -n+2 & 1 & - & - & 1 \\
0 & 1 & 1 & -n+2 & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & 1 \\
0 & - & - & - & - & - & 1 \\
0 & 1 & 1 & 1 & - & - & -n+2
\end{bmatrix}
$$

$A_3 = AB_2$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & -n+2 & 1 & 1 & - & - & 1 \\ 0 & 1 & -n+2 & 1 & - & - & 1 \\ 0 & 1 & 1 & -n+2 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & - & - & - & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & -n+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 \end{bmatrix},$$

$q_3 = -\dfrac{1}{3}\text{Trace}A_3 = 0$

$B_3 = A_3 + q_3 I_n = A_3 + 0I_n = A_3.$

$$A_4 = AB_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 \end{bmatrix},$$

$q_4 = -\dfrac{1}{4}\text{Trace}A_4 = 0$

Similarly all the further A_i will have $a_{ii} = 0$ hence $q_i = 0$, for $i = 3$ to n.

Hence the resultant characteristic polynomial is $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} = 0$ implies $\kappa^n - \kappa^{n-1} - (n-1)\kappa^{n-2} = 0$. Solving this equation we get roots (eigenvalues). $\kappa^{n-2}\left(\kappa^2 - \kappa - (n-1)\right) = 0. Whence, \kappa^{n-2} = 0$ or $\left(\kappa^2 - \kappa - (n-1)\right) = 0$.

Notice that $\kappa^{n-2} = 0$ implies $n-2$ roots are zero and solving $\kappa^2 - \kappa - (n-1) = 0$ enables one knowing that

$$\kappa = \frac{1 \pm \sqrt{1 - 4(1)(-n+1)}}{2} = \frac{1 \pm \sqrt{4n-3}}{2},$$

where $n \geqslant 3$. Hence the roots are

$$\kappa_1 = \frac{1 + \sqrt{4n-3}}{2} \quad \text{and} \quad \kappa_2 = -\left(\frac{\sqrt{4n-3}-1}{2}\right).$$

Thus, $$E_{\gamma-\min}(K_{1,n-1}) = \sum_{i=1}^{n} |\kappa_i| = \frac{1 + \sqrt{4n-3} + \sqrt{4n-3}-1}{2} = \sqrt{4n-3}$$

$$\text{i.e., } E(K_{1,n-1}) = 2\sqrt{n-1} \leqslant E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}. \qquad \square$$

Corollary 4.2 *For any given thorn star $S_{k,t}$ for $k = 1$, $S_{k,1}$ is a star with t vertices.*

Theorem 4.3 *For any given Complete Graph K_n with $n \geqslant 3$, the characteristic polynomial is given by $(\kappa-1)^{n-2}\left(\kappa^2 - (n-1)\kappa - 1\right) = 0$ and $E_{\gamma-\min}(K_n) = \sqrt{n^2 - 2n + 5} + (n-2)$.*

Proof Label the vertices $v_1, v_2, v_3, \ldots, v_n$ such that v_1 is the dominating set, hence in the domination matrix $a_{11} = 1$ and all other $a_{ii} = 0$, the characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method similar to Theorem 1. That is, the characteristic polynomial is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \ldots + q_{n-1}\kappa + q_n = 0.$$

It can be shown that the characteristic polynomial of complete graph is given by

$$(\kappa - 1)^{n-2}\left(\kappa^2 - (n-1)\kappa - 1\right) = 0.$$

On solving the equation we get

$$(\kappa-1)^{n-2} = 0 \text{ or } \left(\kappa^2 - (n-1)\kappa - 1\right) = 0.$$

Notice that $(\kappa - 1)^{n-2} = 0$ implies $\kappa = -1, -1, -1, \ldots, -1(n-2)$ times and

$$\kappa^2 - (n-1)\kappa - 1 = 0.$$
$$\kappa = \frac{n-1 \pm \sqrt{(n-1)^2 - 4(1)(-1)}}{2} = \frac{n-1 \pm \sqrt{n^2 - 2n + 5}}{2}$$

where $n \geqslant 3$. Hence the roots are

$$\kappa_1 = \frac{n-1 + \sqrt{n^2 - 2n + 5}}{2} \quad \text{and} \quad \kappa_2 = -\left(\frac{\sqrt{n^2 - 2n + 5} - (n-1)}{2}\right).$$

Thus, $$E_{\gamma-\min}(K_n) = \sum_{i=1}^{n} |\kappa_i|$$

$$= \frac{n-1 + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 - 2n + 5} - (n-1)}{2} + n - 2$$

$$E_{\gamma-\min}(K_n) = \sqrt{n^2 - 2n + 5} + (n-2). \qquad \square$$

Case 2 $\gamma(G) = 2$.

During the study of chemical graphs and its Weiner number, the Yugoslavian Chemist Ivan Gutman introduced the concept of Thorn graphs. This idea was further extended to the broader concept of generalized thorny graphs by Danail Bonchev and Douglas J Klein of USA. This class of graphs gain importance in spectral theory as it represents the structural formula of aliphatic and aromatic hydrocarbons [3].

Definition 4.4(Thorn Rod) *A Thorn rod is a graph $P_{p,t}$ which includes a linear chain (termed as a rod) of p vertices and degree t terminal vertices at each of the two rod ends.*

Definition 4.5(Thorn Star) *A Thorn Stars are the graphs obtained from a k arm star by attaching $t - 1$ terminal vertices to each of the star arms and are denoted as $S_{k,t}$.*

Definition 4.6(Thorn Ring) *A t Thorny Ring has a simple cycle as the parent, and $t - 2$ thorns at each cycle vertex.*

C_n^+ consists of 2n vertices where n vertices on the cycle are degree three and remaining n vertices are pendant vertices.

$C_n^\vee$ consists of $n(t-1)$ vertices of which n vertices are in cycle each of degree t and $n(t-2)$ pendant vertices.

Theorem 4.7 *For any given thorn rod $P_{2,t}$, the characteristic polynomial is given by $\kappa^{2t-4}(\kappa^2 - (t-1))(\kappa^2 - 2\kappa - (t-1))$, where n being the order of G given by $n = 2t$ and $E_{\gamma-\min}(P_{2,t}) = 2\sqrt{t-1} + 2\sqrt{t}$.*

Proof The characteristic polynomial can be found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method. Instead we generalize the result obtained for few thorn rods. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices. The characteristic polynomial of $P_{2,t}$ for $t > 2$ is given by, for $P_{2,3}$, $\kappa^2(\kappa^2 - 2)(\kappa^2 - 2\kappa - 2)$, $P_{2,4}$, $\kappa^4(\kappa^2 - 3)(\kappa^2 - 2\kappa - 3)$, $P_{2,5}$, $\kappa^6(\kappa^2 - 4)(\kappa^2 - 2\kappa - 4)$, $P_{2,6}$, $\kappa^8(\kappa^2 - 5)(\kappa^2 - 2\kappa - 5)$, hence for $P_{2,t}$ the characteristic polynomial is given by, $\kappa^{2k-4}(\kappa^2 - (t-1))(\kappa^2 - 2\kappa - (t-1))$, $n = 2t$. Solving the two quadratic equations and summing their absolute eigenvalues we obtain $E_{\gamma-\min}(P_{2,t}) = 2\sqrt{t-1} + 2\sqrt{t}$. □

Theorem 4.8 *For any given thorn rod $P_{3,t}$, the characteristic polynomial is given by $\kappa^{2t-3}(\kappa^2 - \kappa - (t-1))(\kappa^2 - \kappa - (t+1))$, $n = 2t + 1$ and $E_{\gamma-\min}(P_{3,t}) = \sqrt{4t-3} + \sqrt{4t+5}$.*

Proof The proof is similar to the above theorem. For $t = 1$, $P_{3,1}$ is a path with 3 vertices, $t = 2 P_{3,2}$ is a path with 5 vertices. For $t > 2$ the characteristic polynomial is given by, for $P_{3,3}$, $\kappa^3(\kappa^2 - \kappa - 2)(\kappa^2 - \kappa - 4)$, $P_{3,4}$, $\kappa^5(\kappa^2 - \kappa - 3)(\kappa^2 - \kappa - 5)$, $P_{3,5}$, $\kappa^7(\kappa^2 - \kappa - 4)(\kappa^2 - \kappa - 6)$, hence for $P_{3,t}$, $\kappa^{2k-3}(\kappa^2 - \kappa - (t-1))(\kappa^2 - \kappa - (t+1))$, $n = 2t+1$. The corresponding minimal domination energy is $\sqrt{4t-3} + \sqrt{4t+5}$. □

Theorem 4.9 *For any given thorn rod $P_{4,t}$, the characteristic polynomial is given by $\kappa^{2t-4}(\kappa^3 - (t+1)\kappa - (t-1))(\kappa^3 - 2\kappa^2 - (t-1)\kappa + (t-1))$, $n = 2t + 2$.*

Proof For $t = 1$, $P_{4,1}$ is a path with 4 vertices, $t = 2$, $P_{4,2}$ is a path with 6 vertices. For $t > 2$ the characteristic polynomial is given by, for $P_{4,3}$, $\kappa^2(\kappa^3 - 4\kappa - 2)(\kappa^3 - 2\kappa^2 - 2\kappa + 2)$, $P_{4,4}$, $\kappa^4(\kappa^3 - 5\kappa - 3)(\kappa^3 - 2\kappa^2 - 3\kappa + 3)$, $P_{4,5}$, $\kappa^6(\kappa^3 - 6\kappa - 4)(\kappa^3 - 2\kappa^2 - 4\kappa + 4)$, hence for $P_{4,t}$, $\kappa^{2k-4}(\kappa^3 - (t+1)\kappa - (t-1))(\kappa^3 - 2\kappa^2 - (t-1)\kappa + (t-1))$, $n = 2t + 2$. Solving a cubic equation is quite difficult. □

Corollary 4.10 *For any given thorn star $S_{k,t}$ for $k = 2$, $S_{2,t}$ is a $P_{3,t}$.*

Case 3 $\gamma(G) = 3$.

Theorem 4.11   *For any given thorn star $S_{3,t}$, the characteristic polynomial is given by,* $\kappa^{3t-5}(\kappa^2 - \kappa - (t+2))(\kappa^2 - \kappa - (t-1))^2$, $n = 3t + 1$ *and* $E_{\gamma-\min}(S_{3,t}) = \sqrt{4t+9} + 2\sqrt{4t-3}$.

Proof Thorny star $S_{3,t}$ has 3 arm and $t - 1$ terminal vertices, hence $\gamma(G) = 3$. The characteristic polynomial of $S_{3,2}$, $\kappa^1(\kappa^2 - \kappa - 4)(\kappa^2 - \kappa - 1)^2$, $S_{3,3}$, $\kappa^4(\kappa^2 - \kappa - 5)(\kappa^2 - \kappa - 2)^2$, $S_{3,4}$, $\kappa^7(\kappa^2 - \kappa - 6)(\kappa^2 - \kappa - 3)^2$, hence for $S_{3,t}$, $\kappa^{3k-5}(\kappa^2 - \kappa - (t+2))(\kappa^2 - \kappa - (t-1))^2$, $n = 3t + 1$. The corresponding minimal domination energy is $\sqrt{4t+9} + 2\sqrt{4t-3}$. □

Theorem 4.12   *For a given thorn ring C_3^t, with three vertices on the cycle of degree t and $n(t-1)$ vertices, the characteristic polynomial is given by,* $\kappa^{3t-9}(\kappa^2 - (t-2))^2(\kappa^2 - 3\kappa - (t-2))$ *and* $E_{\gamma-\min}(C_3^t) = 4\sqrt{t-2} + \sqrt{4t+1}$.

Proof Thorn ring C_n^t, has $n(t-1)$ vertices, n vertices on the cycle and n (t-2) pendant vertices, t is the degree of each vertex on the cycle. The characteristic polynomial of C_3^3, $\kappa^0(\kappa^2 - 1)^2(\kappa^2 - 3\kappa - 1)$, C_3^4, $\kappa^3(\kappa^2 - 2)^2(\kappa^2 - 3\kappa - 2)$, C_3^5, $\kappa^6(\kappa^2 - 3)^2(\kappa^2 - 3\kappa - 3)$, C_3^6, $\kappa^9(\kappa^2 - 4)^2(\kappa^2 - 3\kappa - 4)$. Hence for C_3^t, $\kappa^{3t-9}(\kappa^2 - (t-2))^2(\kappa^2 - 3\kappa - (t-2))$. The corresponding minimal domination energy is $4\sqrt{t-2} + \sqrt{4t+1}$. □

Case 4 $\gamma(G) = 4$.

Theorem 4.13   *For any given thorn star $S_{4,t}$, the characteristic polynomial is given by,* $\kappa^{4t-7}(\kappa^2 - \kappa - (t+3))(\kappa^2 - \kappa - (t-1))^3$, $n = 4t + 1$ *and* $E_{\gamma-\min}(S_{4,t}) = \sqrt{4t+13} + 3\sqrt{4t-3}$.

Proof Thorny Star $S_{4,t}$ has 4 arm and $t - 1$ terminal vertices, hence $\gamma(G) = 4$. The characteristic polynomial of $S_{4,2}$, $\kappa^1(\kappa^2 - \kappa - 5)(\kappa^2 - \kappa - 1)^3$, $S_{4,3}$, $\kappa^5(\kappa^2 - \kappa - 6)(\kappa^2 - \kappa - 2)^3$, $S_{4,4}$, $\kappa^9(\kappa^2 - \kappa - 7)(\kappa^2 - \kappa - 3)^3$, hence for $S_{4,t}$, $\kappa^{4k-7}(\kappa^2 - \kappa - (t+3))(\kappa^2 - \kappa - (t-1))^3$, $n = 4t + 1$. The corresponding minimal domination energy is $\sqrt{4t+13} + 3\sqrt{4t-3}$. □

Theorem 4.14 *For any given thorn rod $P_{5,t}$, the characteristic polynomial is given by,* $\kappa^{2t-3}(\kappa^2 - \kappa - t)(\kappa^4 - 2\kappa^3 - (t+1)\kappa^2 + (t+2)\kappa + (2t-2))$, $n = 2t + 3$.

Proof For $P_{5,t}$, $\gamma(G) = 3$, for $t = 1$, $P_{5,1}$ is a path with 5 vertices, $t = 2$, $P_{5,2}$ is a path with 7 vertices. For $t > 2$ the characteristic polynomial is given by, for $P_{5,3}$, $\kappa^3(\kappa^2 - \kappa - 3)(\kappa^4 - 2\kappa^3 - 4\kappa^2 + 5\kappa + 4)$, $P_{5,4}$, $\kappa^5(\kappa^2 - \kappa - 4)(\kappa^4 - 2\kappa^3 - 5\kappa^2 + 6\kappa + 6)$, $P_{5,5}$, $\kappa^7(\kappa^2 - \kappa - 5)(\kappa^4 - 2\kappa^3 - 6\kappa^2 + 7\kappa + 8)$, $P_{5,6}$, $\kappa^9(\kappa^2 - \kappa - 6)(\kappa^4 - 2\kappa^3 - 7\kappa^2 + 8\kappa + 10)$, hence for $P_{5,t}$, $\kappa^{2k-3}(\kappa^2 - \kappa - t)(\kappa^4 - 2\kappa^3 - (t+1)\kappa^2 + (t+2)\kappa + (2t-2))$, n = 2t + 3. These result can

be extended to $P_{7,t}$ which has a unique minimal dominating set where as $P_{6,t}$ has two minimal dominating set. □

Theorem 4.15 *For any given thorn ring C_4^t, with four vertices on the cycle of degree t and $n(t-1)$ vertices, the characteristic polynomial is given by, $\kappa^{4t-12}(\kappa^2 + \kappa - (t-2))(\kappa^2 - \kappa - (t-2))^2(\kappa^2 - 3\kappa - (t-2))$ and $E_{\gamma-\min}(C_4^t) = \sqrt{4t+1} + 3\sqrt{4t-7}$.*

Proof The characteristic polynomial of C_4^3, $\kappa^0(\kappa^2 + \kappa - 1)(\kappa^2 - \kappa - 1)^2(\kappa^2 - 3\kappa - 1)$, C_4^4, $\kappa^4(\kappa^2 + \kappa - 2)(\kappa^2 - \kappa - 2)^2(\kappa^2 - 3\kappa - 2)$, C_4^5, $\kappa^8(\kappa^2 + \kappa - 3)(\kappa^2 - \kappa - 3)^2(\kappa^2 - 3\kappa - 3)$, hence for C_4^t $\kappa^{4t-12}(\kappa^2 + \kappa - (t-2))(\kappa^2 - \kappa - (t-2))^2(\kappa^2 - 3\kappa - (t-2))$. The corresponding minimal domination energy is $\sqrt{4t+1} + 3\sqrt{4t-7}$. □

§5. Open Problems

1. Domination energy for other standard graphs can be explored.

2. The relation between these parameters can be extended to other classes of graphs and other types of domination.

3. Application of Set and domination energy has to Explored.

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On Variation of Edge Bimagic Total Labeling

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Abstract: An edge magic total labeling of a graph $G(V, E)$ with p vertices and q edges is a bijection $f : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ such that $f(u) + f(v) + f(uv)$ is a constant k for any edge $uv \in E(G)$. If there exist two constants k_1 and k_2 such that the above sum is either k_1 or k_2, it is said to be an edge bimagic total labeling. A total edge magic (edge bimagic) graph is called a super edge magic (super edge bimagic) if $f(V(G)) = \{1, 2, \cdots, p\}$ and it is called superior edge magic(bimagic) if $f(E(G)) = \{1, 2, \cdots, q\}$. In this paper, we investigate and exhibit super and superior edge bimagic labeling for some classes of graphs.

Key Words: Graph, labeling, bimagic labeling, bijective function.

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§1. Introduction

All graphs considered in this article are finite, simple and undirected. A labeling of a graph G is an assignment of labels to either the vertices or the edges, or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of mathematical models from a broad range of applications such as coding theory, X-ray, Crystallography, radar, astronomy, circuit design, communication networks and data base management. Graph labeling was first introduced in the late 1960s. A useful survey on graph labeling by Gallian (2012) can be found in [4]. We follow the notation and terminology of [5].

A graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection $f : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ such that $f(u) + f(v) + f(uv) = k$ for any edge $uv \in E(G)$. The original concept of total edge-magic graph is due to Kotzig and Rosa [6] who called it magic graph. A total edge-magic graph is called a super edge-magic if $f(V(G)) = \{1, 2, \cdots, p\}$.

Wallis [7] called super edge-magic as strongly edge-magic. The notion of edge bimagic labeling was introduced by Baskar Babujee [1]. A graph G with p vertices and q edges is called total edge bimagic if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ and two

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constants k_1 and k_2 such that $f(u) + f(v) + f(uv)$ is either k_1 or k_2 for any edge $uv \in E(G)$. A total edge-bimagic graph is called super edge-bimagic if $f(V(G)) = \{1, 2, \cdots, p\}$ and it is called superior edge bimagic if $f(E(G)) = \{1, 2, \cdots, q\}$. In this article, $C_n \hat{o} C_n^+$, $C_n \hat{e} C_n^+$, $C_n \hat{o} C_n$, $C_n \hat{e} C_n$, $(K_{1,m} + K_1) \hat{o} C_n^+$ and $G \hat{o} (P_2 + mK_1)$ are shown to admit super and superior edge bimagic labeling.

Definition 1.1([2]) *A bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p + q\}$ is said to be super edge bimagic total labeling of G if there exist two constants k_1 and k_2 such that $f(u) + f(v) + f(uv)$ is either k_1 or k_2 for any edge $uv \in E(G)$ and $f(V(G)) = \{1, 2, \cdots, p\}$.*

Definition 1.2([8]) *A graph G with p vertices and q edges is called superior edge magic if there is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p + q\}$ such that $f(u) + f(v) + f(uv)$ is a constant for any edge $uv \in E(G)$, where $f(E(G) = \{1, 2, \cdots, q\}$. If $f(u) + f(v) + f(uv)$ are all distinct for all $uv \in E(G)$, then the graph is called superior edge antimagic total labeling.*

Definition 1.3([3]) *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two connected graphs then the graph obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1 is denoted by $G_1 \hat{o} G_2$. The resultant graph $G = G_1 \hat{o} G_2$ contains $p_1 + p_2$ - 1 vertices and $q_1 + q_2$ edges. In general, there are $p_1 p_2$ possibilities of getting from G_1 and G_2.*

Definition 1.4 *$G_1 \hat{e} G_2$ is obtained from G_1 and G_2 by introducing an edge between an arbitrary vertex of G_1 and an arbitrary vertex of G_2. If $G_1(p_1, q_1)$ has p_1 vertices and q_1 edges and $G_2(p_2, q_2)$ has p_2 vertices and q_2 edges then $G_1 \hat{e} G_2$ will have $(p_1 + p_2)$ vertices and $(q_1 + q_2 + 1)$ edges. If $G_1 = K_{1,m}$, $G_2 = P_n$.*

Interesting graph structures $K_{1,m} \hat{e} P_n$ is obtained respectively using our operation defined above and we prove the following results.

§2. **Main Results**

In this section, we obtain super and superior edge bimagic labeling from connected magic graphs.

Theorem 2.1 *There exists at least one graph G from the class $C_n \hat{o} C_n^+$, $(n \geq 3)$ when n is odd that admits super edge bimagic labeling.*

Proof Let the graph G is obtained by superimposing a vertex of C_n on a pendant vertex of C_n^+ is denoted by $C_n \hat{o} C_n^+$. We define that the vertex set $V(G) = \{v_1^j, v_2^j; 1 \leq j \leq n\} \cup \{u_1^k; 1 \leq k \leq n-1\}$ and edge set $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{v_1^j v_2^j; 1 \leq j \leq n\}$, $E_2 = \{v_2^j v_2^{j+1}; 1 \leq j \leq n-1\}$, $E_3 = \{u_1^k u_1^{k+1}; 1 \leq k \leq n-2\}$, $E_4 = \{v_2^1 v_2^n, v_1^1 u_1^1, v_1^1 u_1^{n-1}\}$. Define a bijective function $f : V(G) \bigcup E(G) \to \{1, 2, 3, \cdots, 6n - 1\}$ is as follows:

For $j = 1$ to n, let $f(v_1^j) = n - 1 + j$; For $j = 1$ to n, when $j \equiv 1 \pmod 2$, let $f(v_2^j) = \dfrac{5n - j}{2}$, $f(v_1^j v_2^j) = \dfrac{10n - j - 1}{2}$ and when $j \equiv 0 \pmod 2$, let $f(v_2^j) = \dfrac{6n - j}{2}$, $f(v_2^j v_2^{j+1}) = 3n + j - 1$.

For $k = 1$ to $n - 2$, let $f(u_1^k u_1^{k+1}) = 6n - k$; For k $= 1$ to n-1 ; when $k \equiv 1(mod\ 2)$, let $f(u_1^k) = \dfrac{n+k}{2}$ and when $k \equiv 0(mod\ 2)$, let $f(u_1^k) = \dfrac{k}{2}$.

Let $f(u_1^1 v_1^1) = 5n$, $f(v_2^1 v_2^n) = 4n - 1$, $f(v_1^1 u_1^{n-1}) = 5n + 1$.

In the following cases, it is justified that the above assignment results in the required labeling.

Case 1 For edges in E_1, when $j \equiv 1(mod\ 2)$, we have

$$
\begin{aligned}
f(v_1^j) + f(v_2^j) + f(v_1^j v_2^j) &= n - 1 + j + \frac{5n - j}{2} + \frac{10n - j - 1}{2} \\
&= \frac{17n - 3}{2} = k_1
\end{aligned}
$$

and when $j \equiv 0(mod\ 2)$, we have

$$
\begin{aligned}
f(v_1^j) + f(v_2^j) + f(v_1^j v_2^j) &= n - 1 + j + \frac{6n - j}{2} + \frac{9n - j - 1}{2} \\
&= \frac{17n - 3}{2} = k_1.
\end{aligned}
$$

Case 2 For edges in E_2, when $j \equiv 1(mod\ 2)$, we have

$$
\begin{aligned}
f(v_2^j) + f(v_2^{j+1}) + f(v_2^j v_2^{j+1}) &= \frac{5n - j}{2} + \frac{6n - j - 1}{2} + 3n - 1 + j \\
&= \frac{17n - 3}{2} = k_1
\end{aligned}
$$

and when $j \equiv 0(mod\ 2)$, we have

$$
\begin{aligned}
f(v_2^j) + f(v_2^{j+1}) + f(v_2^j v_2^{j+1}) &= \frac{6n - j}{2} + \frac{5n - j - 1}{2} + 3n - 1 + j \\
&= \frac{17n - 3}{2} = k_1.
\end{aligned}
$$

Case 3 For edges in E_3, when $k \equiv 1(mod\ 2)$, we have

$$
\begin{aligned}
f(u_1^k) + f(u_1^{k+1}) + f(u_1^k u_1^{k+1}) &= \frac{n+k}{2} + \frac{k+1}{2} + 6n - k \\
&= \frac{13n + 1}{2} = k_2
\end{aligned}
$$

and when $k \equiv 0(mod\ 2)$, we have

$$
\begin{aligned}
f(u_1^k) + f(u_1^{k+1}) + f(u_1^k u_1^{k+1}) &= \frac{k}{2} + \frac{n+k+1}{2} + 6n - k \\
&= \frac{13n + 1}{2} = k_2.
\end{aligned}
$$

Case 4 For the edges in E_4, we have

$$f(v_2^n) + f(v_2^1) + f(v_2^1 v_2^n) = 2n + \frac{5n-1}{2} + 4n - 1 = \frac{17n-3}{2} = k_1,$$

$$f(u_1^1) + f(v_1^1) + f(u_1^1 v_1^1) = \frac{n+1}{2} + n + 5n = \frac{13n+1}{2} = k_2,$$

$$f(v_1^1) + f(v_1^{n-1}) + f(v_1^1 u_1^{n-1}) = n + \frac{n-1}{2} + 5n + 1 = \frac{13n+1}{2} = k_2.$$

We observe that there are two constants k_1 and k_2 such that for each edge $uv \in E(G)$, $f(u)+f(v)+f(uv)$ is either k_1 or k_2. From the above cases we have two constants $k_1 = \dfrac{17n-3}{2}$ and $k_2 = \dfrac{13n+1}{2}$. Hence the resultant graph admits super edge bimagic labeling. □

Illustration 1 The graph $C_9 \hat{o} C_9^+$ is given in Figure 1. It is super edge bimagic labelling is also indicated in the same figure.
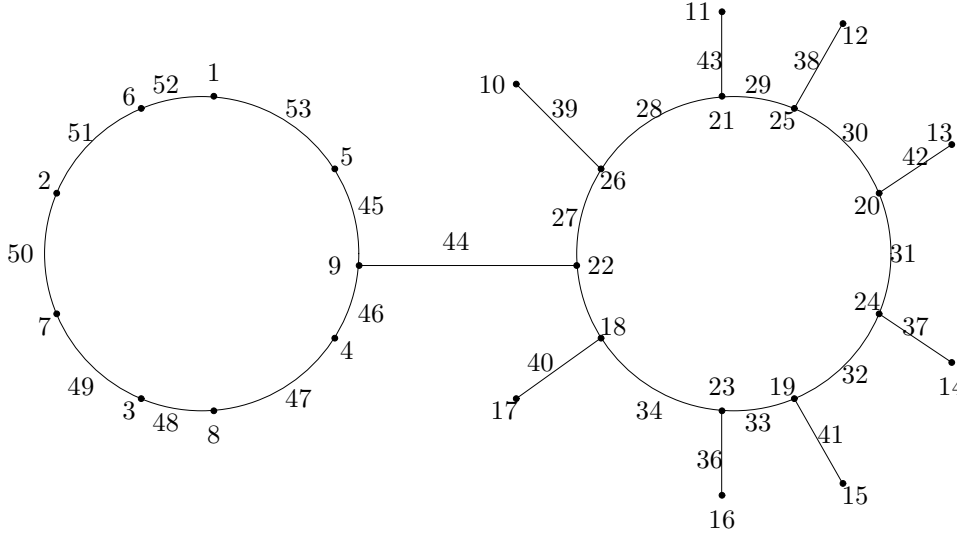
Figure 1 $k_1 = 75$ $k_2 = 59$

Theorem 2.2 *There exists at least one graph G from the class $C_n \hat{e} C_n^+$, $(n \geq 3)$, when n is odd that admits super edge bimagic total labeling.*

Proof Let the graph G is obtained by introducing an edge between a vertex of C_n and a pendant vertex of C_n^+ is denoted by $C_n \hat{e} C_n^+$. We define that the vertex set $V(G) = \{u_i, v_j, w_j; 1 \leq i, j \leq n\}$ and edge set $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{u_i u_{i+1}; 1 \leq i \leq n-1\}$, $E_2 = \{v_j w_j; 1 \leq j \leq n\}$, $E_3 = \{w_j w_{j+1}; 1 \leq j \leq n-1\}$, $E_4 = \{u_1 u_n, v_1 u_n, w_1 w_n\}$. Define a bijective function $f : V(G) \bigcup E(G) \to \{1, 2, 3, \cdots, 6n-1\}$ is as follows:

For $i = 1$ to $n-1$, let $f(u_i u_{i+1}) = 6n - i + 1$; For $i = 1$ to n, when $i \equiv 1 (mod\ 2)$, let $f(u_i) = \dfrac{i+1}{2}$ and when $i \equiv 0 (mod\ 2)$, let $f(u_i) = \dfrac{n+i+1}{2}$. For $j = 1$ to $n-1$, let

$f(w_j w_{j+1}) = 3n + j$; For $j = 1$ to n, when $j \equiv 1 (mod\ 2)$, let $f(w_j) = \dfrac{5n - j + 2}{2}$ and when $j \equiv 0 (mod\ 2)$, let $f(w_j) = \dfrac{6n - j + 2}{2}$. For $j = 1$ to n, when $j \equiv 1 (mod\ 2)$, let $f(v_j) = n + j$, $f(w_j v_j) = \dfrac{10n - j + 1}{2}$ and when $j \equiv 0 (mod\ 2)$, let $f(v_j) = n + j$, $f(w_j v_j) = \dfrac{19n - j + 1}{2}$. Let $f(u_1 u_n) = 6n + 1$, $f(v_1 u_n) = 5n + 1$ and $f(w_1 w_n) = 4n$.

In the following cases, it is justified that the above assignment results in the required labeling.

Case 1 For edges in E_1, when $i \equiv 1 (mod\ 2)$, we obtain

$$
\begin{aligned}
f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) &= \frac{i+1}{2} + \frac{n+i+2}{2} + 6n - i + 1 \\
&= \frac{13n + 5}{2} = k_1
\end{aligned}
$$

and when $i \equiv 0 (mod\ 2)$, we have

$$
\begin{aligned}
f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) &= \frac{n+i+1}{2} + \frac{i+2}{2} + 6n - i + 1 \\
&= \frac{13n + 5}{2} = k_1.
\end{aligned}
$$

Case 2 For edges in E_2, when $j \equiv 1 (mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_j) + f(w_j) + f(v_j w_j) &= n + j + \frac{5n - j + 2}{2} + \frac{10n - j + 2}{2} + \frac{10n - j + 1}{2} \\
&= \frac{17n + 3}{2} = k_2
\end{aligned}
$$

and when $j \equiv 0 (mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_j) + f(w_j) + f(v_j w_j) &= n + j + \frac{6n - j + 2}{2} + \frac{9n - j + 1}{2} \\
&= \frac{17n + 3}{2} = k_2.
\end{aligned}
$$

Case 3 For edges in E_3, when $j \equiv 1 (mod\ 2)$, we obtain

$$
\begin{aligned}
f(w_j) + f(w_{j+1}) + f(w_j w_{j+1}) &= \frac{5n - j + 2}{2} + \frac{6n - j + 1}{2} + 3n + j \\
&= \frac{17n + 3}{2} = k_2
\end{aligned}
$$

and when $j \equiv 0 (mod\ 2)$, we have

$$
\begin{aligned}
f(w_j) + f(w_{j+1}) + f(w_j w_{j+1}) &= \frac{6n - j + 2}{2} + \frac{5n - j + 1}{2} + 3n + j \\
&= \frac{17n + 3}{2} = k_2.
\end{aligned}
$$

**Case** 4 For the edges in E_4, we have,

$$f(u_1) + f(u_n) + f(u_1 u_n) = 1 + \frac{n+i}{2} + 6n + 1 = \frac{13n+5}{2} = k_1,$$

$$f(v_1) + f(u_n) + f(v_1 u_n) = n + 1 + \frac{n+1}{2} + 5n + 1 = \frac{13n+5}{2} = k_1,$$

$$f(w_1) + f(w_n) + f(w_1 w_n) = 2n + 1 + \frac{5n+1}{2} + 4n = \frac{17n+3}{2} = k_2.$$

We observe that there are two constants k_1 and k_2 such that for each edge $uv \in E(G)$, $f(u)+f(v)+f(uv)$ is either k_1 or k_2. From the above cases we have two constants $k_1 = \dfrac{13n+5}{2}$ and $k_2 = \dfrac{17n+3}{2}$. Hence the graph $C_n \hat{e} C_n^+$, $(n \geq 3)$ admits super edge bimagic labeling. □

Illustration 2 The graph $C_{11} \hat{e} C_{11}^+$ is given in Figure 2. It is super edge bimagic labelling is also indicated in the same figure.
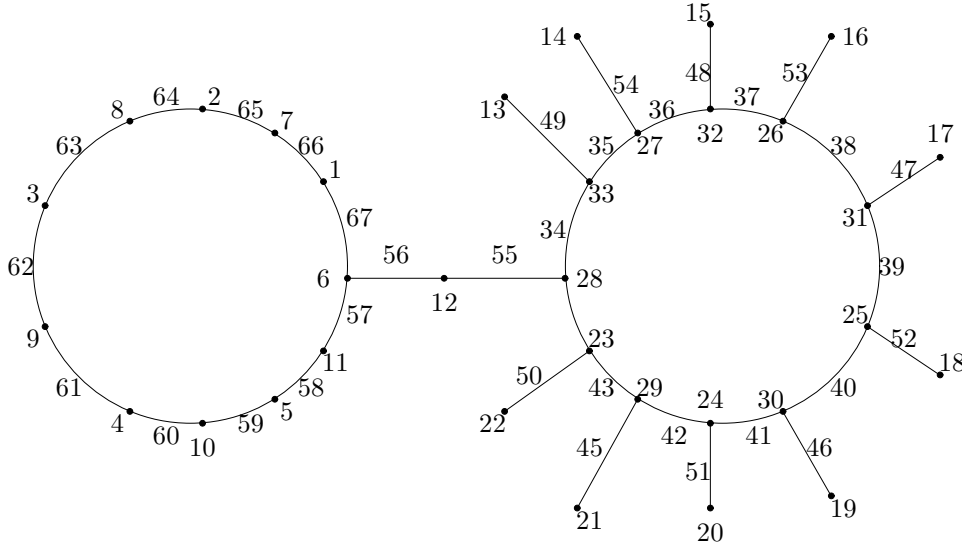
Figure 2 $k_1 = 74$ $k_2 = 95$

Theorem 2.3 *There exists at least one graph G from the class $C_n \hat{o} C_n$, $(n \geq 3)$ when n is odd that admits superior edge bimagic total labeling.*

Proof Let the graph G is obtained by superimposing a vertex of C_n on a vertex of the same copy denoted by $C_n \hat{o} C_n$. Now, we define that the vertex set $V(G) = \{u_i, v_j; 1 \leq i \leq n\}, 1 \leq j \leq n-1\}$ and edge set $E(G) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{u_i u_{i+1}; 1 \leq i \leq n-1\}$, $E_2 = \{v_j v_{j+1}; 1 \leq j \leq n-2\}$, $E_3 = \{u_1 u_n, v_1 u_n, u_n v_{n-1}\}$. A bijective function $f : V(G) \bigcup E(G) \to \{1, 2, 3, \cdots, 4n-1\}$ is given bellow:

For $i = 1$ to $n-1$, let $f(u_i u_{i+1}) = 6n - i + 1$; For $i = 1$ to n, when $i \equiv 1(mod\ 2)$, let $f(u_i) = \dfrac{i+1}{2}$ and when $i \equiv 0(mod\ 2)$, let $f(u_i) = \dfrac{n+i+1}{2}$. For $i = 1$ to $n-1$, let

$f(u_iu_{i+1}) = i$; For $i = 1$ to n, when $i \equiv 1(mod\ 2)$, let $f(u_i) = \dfrac{7n - i}{2}$ and when $i \equiv 0(mod\ 2)$,

let $f(u_i) = \dfrac{8n - i}{2}$. For $j = 1$ to $n - 2$, let $f(v_jv_{j+1}) = n + 2 + j$; For $j = 1$ to $n - 1$,

when $j \equiv 1(mod\ 2)$, let $f(v_j) = \dfrac{5n - j}{2}$, when $j \equiv 0(mod\ 2)$, let $f(v_j) = \dfrac{6n - j}{2}$. Let

$f(v_{n-1}) = \dfrac{5n + 1}{2}$, $f(u_nv_{n-1}) = n + 1$, $f(v_1u_n) = n + 2$, $f(u_1u_n) = n$.

The above assigned labels are justified in the following cases.

Case 1 For edges in E_1, when $i \equiv 1(mod\ 2)$, we obtain

$$
\begin{aligned}
f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) &= \frac{7n - i}{2} + \frac{8n - i - 1}{2} + i \\
&= \frac{15n - 1}{2} = k_1
\end{aligned}
$$

and when $i \equiv 0(mod\ 2)$, we obtain

$$
\begin{aligned}
f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) &= \frac{8n - i}{2} + \frac{7n - i - 1}{2} + i \\
&= \frac{15n - 1}{2} = k_1.
\end{aligned}
$$

Case 2 For edges in E_2, when $j \equiv 1(mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_j) + f(v_{j+1}) + f(v_jv_{j+1}) &= \frac{5n - j}{2} + \frac{6n - j - 1}{2} + n + 2 + j \\
&= \frac{13n + 3}{2} = k_2;
\end{aligned}
$$

when $j \equiv 0(mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_j) + f(v_{j+1}) + f(v_jv_{j+1}) &= \frac{6n - j}{2} + \frac{5n - j - 1}{2} + n + 2 + j \\
&= \frac{13n + 3}{2} = k_2.
\end{aligned}
$$

Case 3 For the edges in E_3, we have

$$
\begin{aligned}
f(u_1) + f(u_n) + f(u_1u_n) &= \frac{7n - 1}{2} + 3n + n = \frac{15n - 1}{2} = k_1, \\
f(v_1) + f(u_n) + f(v_1u_n) &= \frac{5n - 1}{2} + 3n + n + 2 = \frac{13n + 3}{2} = k_2, \\
f(u_n) + f(v_{n-1}) + f(u_uu_{n-1}) &= 3n + \frac{5n + 1}{2} + n + 1 = \frac{13n + 3}{2} = k_2.
\end{aligned}
$$

Therefore, when we observe from the above cases, we have the constant $k_1 = \dfrac{15n - 1}{2}$ and

$k_2 = \dfrac{13n + 3}{2}$. Hence the graph $G = C_n \ \hat{o} \ C_n$, $(n \geq 3)$ admits superior edge bimagic total labeling. $\square$

Theorem 2.4 *There exists at least one graph G from the class $C_n \ \hat{e} \ C_n$, $(n \geq 3)$ when n is*

odd that admits super edge bimagic total labeling.

Proof Let the graph G is obtained by introducing an edge between a vertex of C_n and a vertex of the same copy denoted by $C_n \, \hat{e} \, C_n$. Now, we define that the vertex set $V(G) = \{u_i, v_j; 1 \leq i \leq n\}, 1 \leq j \leq n\}$ and edge set $E(G) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{u_i u_{i+1}; 1 \leq i \leq n-1\}$, $E_2 = \{v_j v_{j+1}; 1 \leq j \leq n-1\}$, $E_3 = \{u_1 v_n, u_1 u_n, v_1 v_n\}$. A bijective function $f : V(G) \bigcup E(G) \rightarrow \{1, 2, 3, \cdots, 4n+1\}$ is given bellow:

For $i = 1$ to $n-1$, let $f(u_i u_{i+1}) = 3n - i$; For $i = 1$ to n, when $i \equiv 1 (mod \ 2)$, let $f(u_i) = \dfrac{2n + i + 1}{2}$; when $i \equiv 0 (mod \ 2)$, let $f(u_i) = \dfrac{3n + i + 1}{2}$. For $j = 1$ to $n-1$, let $f(v_j v_{j+1}) = 4n + 1 + j$; For $j = 1$ to n, when $j \equiv 1 (mod \ 2)$, let $f(v_j) = \dfrac{j+1}{2}$, when $j \equiv 0 (mod \ 2)$, let $f(v_j) = \dfrac{n + j + 1}{2}$. Let $f(u_1 u_n) = 3n$, $f(u_1 v_n = 3n + 1)$, $f(v_1 v_n = 4n + 1)$.

The above assigned labels are justified in the following cases.

Case 1 For edges in E_1, when $i \equiv 1 (mod 2)$, we obtain

$$\begin{aligned} f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) &= \frac{2n+i+1}{2} + \frac{3n+i+2}{2} + 3n - i \\ &= \frac{11n+3}{2} = k_1 \end{aligned}$$

and when $i \equiv 0 (mod \ 2)$, we obtain

$$\begin{aligned} f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) &= \frac{3n+i+1}{2} + \frac{2n+i+2}{2} + 3n - i \\ &= \frac{11n+3}{2} = k_1. \end{aligned}$$

Case 2 For edges in E_2, when $j \equiv 1 (mod \ 2)$, we obtain

$$\begin{aligned} f(v_j) + f(v_{j+1}) + f(v_j v_{j+1}) &= \frac{j+1}{2} + \frac{n+j+2}{2} + 4n + 1 - j \\ &= \frac{9n+5}{2} = k_2. \end{aligned}$$

When $j \equiv 0 (mod \ 2)$, we obtain

$$\begin{aligned} f(v_j) + f(v_{j+1}) + f(v_j v_{j+1}) &= \frac{n+j+1}{2} + \frac{j+2}{2} + 4n + 1 - j \\ &= \frac{9n+5}{2} = k_2. \end{aligned}$$

Case 3 For the dges in E_3, we have

$$f(u_1) + f(u_n) + f(u_1 u_n) = n + 1 + \frac{3n+1}{2} + 3n = \frac{11n+3}{2} = k_1,$$

$$f(v_1) + f(u_n) + f(v_1 u_n) = 1 + \frac{n+1}{2} + 4n + 1 = \frac{9n+5}{2} = k_2,$$

$$f(u_1) + f(v_n) + f(u_1 v_n) = n + 1 + \frac{n+1}{2} + 3n + 1 = \frac{9n+5}{2} = k_2.$$

Therefore, $k_1 = \dfrac{11n+3}{2}$ and $k_2 = \dfrac{9n+5}{2}$. Hence the graph $G = C_n \hat{e} C_n$, $(n \geq 3)$ admits super edge bimagic total labeling. $\qquad\square$

Theorem 2.5 *There exists at least one graph G' from the class $G\hat{o}C_n^+$, $(n \geq 3)$, (when n is odd) that admits super edge bimagic total labeling, where G is any graph from $K_{1,m} + K_1, (m \geq 2)$.*

Proof Let the graph G' is obtained by merging of two graphs with a vertex of above degree 2 in G and a pendant vertex of C_n^+. We define the graph $G\hat{o}C_n^+$ with vertex set $V(G) = \{u_i, v_i; 1 \leq i \leq n\} \cup \{w_1\} \cup \{w_2^j; 1 \leq j \leq m\}$ and edge set $E(G) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{u_i v_i; 1 \leq i \leq n\}$, $E_2 = \{v_i v_{i+1}; 1 \leq i \leq n-1\}$, $E_3 = \{u_1 w_2^j; 1 \leq j \leq m\} \cup \{w_1 w_2^j; 1 \leq j \leq m\} \cup \{u_1 w_1, v_1 v_n\}$. A bijective function $f : V(G) \bigcup E(G) \to \{1, 2, 3, \cdots, 4n + 3m + 2\}$ is given bellow:

For $i = 1$ to $n-1$, let $f(v_i v_{i+1}) = 2n + 1 + m + i$; For $i = 1$ to n, when $i \equiv 1(mod\ 2)$, let $f(v_i) = \dfrac{3n + 4 - i}{2} + m$. When $i \equiv 0(mod\ 2)$, let $f(v_i) = \dfrac{4n - i + 4}{2} + m$. For $i = 1$ to n, let $f(u_i) = m + 1 + i$; For $i = 1$ to n, when $i \equiv 1(mod\ 2)$, let $f(u_i v_i) = \dfrac{8n - i + 3}{2}$. When $i \equiv 0(mod\ 2)$, let $f(u_i v_i) = \dfrac{7n + 3 - i}{2}$. For $i = 1$ to n, let $f(v_1 v_n) = 3n + m + 1$ and for $j = 1$ to m, let $f(w_2^j) = 1 + j$, $f(u_1 w_2^j) = \dfrac{7n + 1}{2} + 3m - j$, $f(w_1 w_2^j) = \dfrac{7n + 3}{2} + 4m - j$. Let $f(w_1) = 1$, $f(u_1 w_1) = \dfrac{7n + 1}{2} + 3m$.

In the following cases, it is justified that the above assignment results in the required labeling.

Case 1 For any edge $u_i v_i \in E_1$, when $i \equiv 1(mod\ 2)$, we obtain

$$
\begin{aligned}
f(u_i) + f(v_i) + f(u_i v_i) &= m + 1 + i + \dfrac{3n + 4 - i}{2} + m + \dfrac{8n - i + 3}{2} + m \\
&= \dfrac{6m + 11n + 9}{2} = k_1.
\end{aligned}
$$

When $i \equiv 0(mod\ 2)$, we obtain

$$
\begin{aligned}
f(u_i) + f(v_i) + f(u_i v_i) &= m + 1 + i + \dfrac{n - i + 3}{2} + m + 2n + m + 1 + i \\
&= \dfrac{6m + 11n + 9}{2} = k_1.
\end{aligned}
$$

Case 2 For any edge $v_i v_{i+1} \in E_2$, when $i \equiv 1(mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_i) + f(v_{i+1}) + f(v_i v_{i+1}) &= \dfrac{3n + 4 - i}{2} + m + \dfrac{4n - i - 3}{2} + m + 2n + m + 1 + i \\
&= \dfrac{6m + 11n + 9}{2} = k_1.
\end{aligned}
$$

When $i \equiv 0(mod\ 2)$, we obtain

$$
\begin{aligned}
f(v_i) + f(v_{i+1}) + f(v_iv_{i+1}) &= \frac{4n+4-i}{2} + m + \frac{3n-i-3}{2} + m + 2n + m + 1 + i \\
&= \frac{6m+11n+9}{2} = k_1.
\end{aligned}
$$

Case 3 For the edges in E_3, we have

$$
\begin{aligned}
f(u_1) + f(w_2^j) + f(u_1w_2^j) &= m + 2 + 1 + j + \frac{7n+1}{2} + 3m - j = \frac{8m+7n+7}{2} = k_2, \\
f(w_1) + f(w_2^j) + f(w_1w_2^j) &= 1 + 1 + j + \frac{7n+3}{2} + 4m - j = \frac{8m+7n+7}{2} = k_2, \\
f(v_1) + f(v_n) + f(v_1v_n) &= \frac{3n+3}{2} + m + n + 2 + m + 3n + m + 1 = \frac{6m+11n+9}{2} = k_1, \\
f(u_1) + f(w_1) + f(u_1w_1) &= m + 2 + 1 + \frac{7n+1}{2} + 3m = \frac{8m+7n+7}{2} = k_2.
\end{aligned}
$$

We observe that there are two common counts k_1 and k_2 such that for each edge $uv \in E(G)$, $f(u) + f(v) + f(uv)$ is either k_1 or k_2. From the above cases we have two constants $k_1 = \frac{6m+11n+9}{2}$ and $k_2 = \frac{8m+7n+7}{2}$. Hence as per our construction G' admits super edge bimagic labeling.

Illustration 3 The graph $(K_{1,6}+K_1)\hat{o}C_9^+$ is given in figure 3. It is super edge bimagic labelling is also indicated in the same figure.
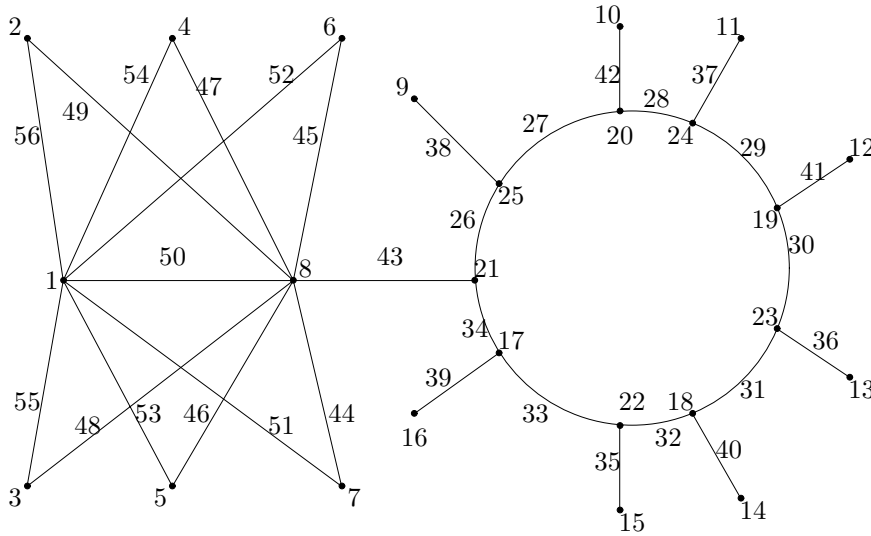
Figure 3 $k_1 = 72$ $k_2 = 59$

Theorem 2.6 *If G is an arbitrary graph that admits total edge magic labeling then there exists at least one graph from the class $G\hat{o}(P_2 + mK_1)$ admits edge bimagic total labeling.*

Proof Let G(p, q) be total edge magic graph with the bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \cdots, p+q\}$ such that $f(u) + f(v) + f(uv) = k_1$. Let $w \in V(G)$ must be vertex whose label $f(w) = p + q$ is the maximum value. Consider the graph $(P_2 + mK_1)$ with vertex set $\{u_0, v_0, u_i : 1 \leq i \leq m\}$ and edge set $E(G) = \{u_0 u_i, v_0 u_i : 1 \leq i \leq m\} \cup \{u_0 v_0\}$. We superimpose the vertex v_0 is degree more than two of the $(P_2 + mK_1)$ graph on the vertex $w \in V(G)$ of G. Now we define the new graph $G' = G \, \hat{o} \, (P_2 + mK_1) : 1 \leq i \leq m$ and edge set $E'(G') = E \cup E_1 \cup E_2 \cup E_3$ where $E_1 = \{u_0 u_i : 1 \leq i \leq m\}$, $E_2 = \{w u_i : 1 \leq i \leq m\}$, $E_3 = \{u_0 w\}$. Consider the bijection $g : V'(G') \cup E'(G') = \{1, 2, 3, \cdots, p+q+3m+2\}$ defined by $g(v) = f(v)$ for all $v \in V(G)$ and $g(uv) = f(uv)$ for all $uv \in E(G)$.

From our construction of new graph G', the labels are defined as follows:

$f(w) = g(v_0) = g(w) = p + q$, $g(u_i) = p + q + i$, for $1 \leq i \leq m$;

$g(w u_i) = p + q + 3m + 3 - i$, for $1 \leq i \leq m$;

$g(u_0 u_i) = p + q + 2m + 2 - i$, for $1 \leq i \leq m$;

$g(u_0) = p + q + m + 1$ and $g(u_0 w) = p + q + 2m + 2$.

Since the graph G is total edge magic with constant k_1 and implies that $g(u) + g(uv) + g(v) = k_2$ for all $uv \in E'(G')$.

Next, we have to prove that the remaining edges w and u_0 joining with $\{u_i : 1 \leq i \leq m\}$ have the constant k_2.

For the edges in $E_1 \cup E_2 \cup E_3$,

$$
\begin{aligned}
g(u_0) + g(u_0 u_i) + g(u_i) &= p + q + m + 1 + p + q + 2m + 2 - i + p + q + i \\
&= 3(p + q + m + 1) = k_2, \\
g(w) + g(u_i) + g(w u_i) &= p + q + p + q + i + p + q + 3m + 3 - i \\
&= 3(p + q + m + 1) = k_2 \text{ and} \\
g(u_0) + g(u_0 w) + g(w) &= p + q + m + 1 + p + q + p + q + 2m + 2 \\
&= 3(p + q + m + 1) = k_2.
\end{aligned}
$$

Therefore, the resultant graph $G \hat{o} (P_2 + mK_1)$ has two common counts k_1 and k_2. Hence the graph admits edge bimagic total labeling. $\square$

Conclusion In our present study, we have investigated super and superior edge bimagic labeling for some special graphs. Investigating super and superior edge bimagic total labeling for the graph from the class $G_1 \hat{o} G_2$ and $G_1 \hat{e} G_2$ for some arbitrary graph G_1 and G_2 with this conditions. This is our future plan.

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Characterization of Pathos Adjacency Blict Graph of a Tree

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Abstract: In this paper we introduce the concept of pathos adjacency blict graph $PBn(T)$ of a tree T and present the characterization of graphs whose pathos adjacency blict graphs are planar, outerplanar, minimally non-outerplanar and Eulerian.

Key Words: Pathos, outerplanar, Smarandachely blict graph, crossing number cr(G), inner vertex number $i(G)$, minimally non-outerplanar.

AMS(2010): 05C05, 05C45

§1. Introduction

All graphs considered in this paper are finite and simple. For standard terminology and notation in graph theory, not specifically defined in this paper, the reader is referred to Harary [3]. The operation of forming a graph valued function of a graph G is probably the most interesting operation by which one graph is obtained from another. The concept of *pathos* of a graph G was introduced by Harary [4], as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T is equal to k, where $2k$ is the number of odd degree vertices of T. Also, the end vertices of every path of any pathos of a tree T are of odd degree [2]. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

A *pathos vertex* is a vertex corresponding to a path P in any pathos and a *block vertex* is a vertex corresponding to a block(or an edge) of a tree T. The *edge degree* of an edge pq of a tree T is the sum of the degrees of p and q.

The *lict graph* (Here "*lict*" indicates "*line cut vertex*") of a graph G [6], written $n(G)$, is the graph whose vertices are the edges and cut vertices of G, with two vertices of $n(G)$ adjacent whenever the corresponding edges of G are adjacent or the corresponding members of G are incident, where the edges and cut vertices of G are called its members. Let C be a block set of G. A *Smarandachely blict graph* $B^C(G)$ is the graph whose vertices are the edges, cut vertices

---

and blocks in C, with two vertices of $Bn(G)$ adjacent whenever the corresponding members of G are adjacent or incident, where the edges, cut vertices and blocks in C are called its *members*. Particularly, if C is all blocks of G, such a $B^C(G)$ is called a *blict graph* (Here "*blict*" indicates "*block line cut vertex*") of a graph G [1], written by $Bn(G)$.

The *pathos line graph* of a tree T [1], written $PL(T)$, is the graph whose vertices are the edges and paths of pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent and the edges that lie on the corresponding path P_i of pathos of T. The *pathos lict graph* of a tree T [1], written $Pn(T)$, is the graph whose vertices are the edges, cut vertices and paths of pathos of T, with two vertices of $Pn(T)$ adjacent whenever the corresponding edges of T are adjacent, edges that lie on the corresponding path P_i of pathos and the edges incident to the cut vertex of T.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be an *outerplanar*, *i.e.* $i(G) = 0$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing its outer planarity. A graph G is said to be *minimally non-outerplanar* if $i(G)=1$ [5]. The least number of edge-crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by cr(G).
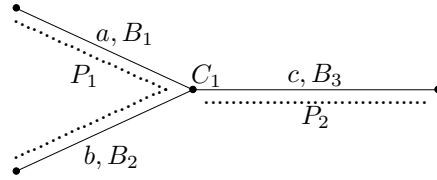
Figure 1 Tree T
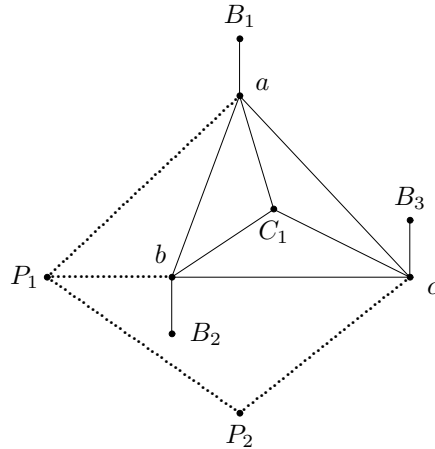
Figure 2 Pathos adjacency blict graph $PBn(T)$ of T

Definition 1.1 *The pathos adjacency blict graph of a tree T, written $PBn(T)$, is the graph whose vertices are the edges, paths of pathos, cut vertices and blocks of T, with two vertices of $PBn(T)$ adjacent whenever the corresponding edges of T are adjacent, edges that lie on the corresponding path P_i of pathos, edges incident to cut vertex and edges that lie on the blocks of T. Two distinct pathos vertices P_m and P_n are adjacent in $PBn(T)$ whenever the corresponding paths of pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex, say v_c in T.*

Since the pattern of pathos for a tree is not unique, the corresponding pathos adjacency blict graph is also not unique. Figure 1 shows a tree T and Figure 2 is its corresponding $PBn(T)$.

The following existing results are required to prove further results.

Theorem A([1]) *The pathos line graph $PL(T)$ of a tree T is planar if and only if $\Delta(T) \leq 4$.*

Theorem B([1]) *Let T be a tree on p vertices and $q = p - 1$ edges such that d_i and C_j are the degrees of vertices and cut vertices C of T, respectively. Then the pathos lict graph $Pn(T)$ has $(q + k + C)$ vertices and $\dfrac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j$ edges, where k is the path number of T.*

Theorem C([1]) *The blict graph $Bn(G)$ of a graph G is planar if and only if $\Delta(T) \leq 3$ and every vertex of degree three is a cut vertex.*

Theorem D([3]) *If G is a graph on p vertices and q edges, then $L(G)$ has q vertices and $-q + \dfrac{1}{2}\sum_{i=1}^{p} d_i^2$ edges, where d_i is the degree of vertices of G.*

Theorem E([6]) *The lict graph $n(G)$ of a graph G is planar if and only if G is planar and the degree of each vertex is at most three.*

§2. Preliminary Results

Remark 2.1 For any tree T with $p \geq 3$ vertices, $L(T) \subseteq PL(T) \subseteq PBn(T)$, $L(T) \subseteq Bn(T) \subseteq PBn(T)$ and $L(T) \subseteq Pn(T) \subseteq PBn(T)$. Here $\subseteq$ is the subgraph notation.

Remark 2.2 If the edge degree of an edge pq in a tree T is even(odd) and p and q are the cut vertices, then the degree of the corresponding vertex pq in $PBn(T)$ is even(odd).

Remark 2.3 If the degree of an end edge(or pendant edge) in a tree T is even(odd), then the degree of the corresponding vertex in $PBn(T)$ is odd(even).

Remark 2.4 For any tree T (except star graph), the number of edges in $PBn(T)$ whose end vertices are the pathos vertices is given by $(k - 1)$, where k is the path number of T.

Remark 2.5 If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges in $PBn(T)$ whose end vertices are the pathos vertices is given by $\dfrac{k(k-1)}{2}$, where k is the path number of T. For example, edge $P_1 P_2$ in Figure 2.

Remark 2.6 Since every block vertex of $PBn(T)$ is an end vertex (For example, the block vertices B_1, B_2 and B_3 in *Figure 2*), $PBn(T)$ does not contain a spanning cycle. Hence it is always non-Hamiltonian.

§3. Lemmas

Here we present two simple lemmas on the graph $PBn(T)$.

Lemma 3.1 *Let T be a tree (except star graph) on p vertices and q edges such that d_i and C_j are the degrees of vertices and cut vertices C of T, respectively. Then $PBn(T)$ contains $(2q + k + C)$ vertices and*

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j + q + (k-1)$$

edges, where k is the path number of T.

Proof Let T be a tree (except star graph) on p vertices and q edges. By definition, the number of vertices in $PBn(T)$ equals the sum of number of edges, paths of pathos, cut vertices and the blocks of T. Since every edge of T is a block, $PBn(T)$ contains $(2q + k + C)$ vertices.

By Theorem B, the number of edges in $Pn(T)$ is $\dfrac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j$. The number of edges in $PBn(T)$ equals the sum of edges of $Pn(T)$, edges that lie on the corresponding path P_i of pathos of T and the number of edges whose end vertices are the pathos vertices. By Remark 2.4, the number of edges in $PBn(T)$ is given by

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j + q + (k-1). \qquad \square$$

Lemma 3.2 *Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges such that d_i and C_j are the degrees of vertices and cut vertex C of T, respectively. Then $PBn(T)$ contains $(2m + k + 1)$ vertices and $\dfrac{1}{2}\sum_{i=1}^{n} d_i^2 + 2m + \dfrac{k(k-1)}{2}$ edges, where k is the path number of T.*

Proof Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges. Since T has exactly one cut vertex C, $PBn(T)$ contains $(2m + k + 1)$ vertices. For a star graph T, the number of edges in $PBn(T)$ equals the sum of number of edges of $L(T)$, thrice the number of edges of T and the number of edges whose end vertices are the pathos vertices.

By Theorem D and Remark 2.5, we know that

$$-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 3m + \frac{k(k-1)}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 2m + \frac{k(k-1)}{2}.$$

Whence, we get the conclusion. $\square$

§4. Main Results

Theorem 4.1 *The pathos adjacency blict graph $PBn(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$, for every vertex $v \in T$.*

Proof Suppose $PBn(T)$ is planar. Assume that $\Delta(T) > 3$. If there exists a vertex p of degree 4 in T, by Theorem[A], $PL(T)$ is planar and contains K_4 as an induced subgraph. In $Pn(T)$, the vertex p is adjacent to every vertex of K_4. This gives K_5 as subgraph in $PBn(T)$. Clearly, $PBn(T)$ is nonplanar, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_n on $n \geq 3$ vertices, then each block of $n(T)$ is K_3 and it has exactly $(n-2)$ blocks. The path number of T is exactly one and the corresponding pathos vertex is adjacent to at most two vertices of each block of $n(T)$. The pathos vertex together with each block of of $n(T)$ gives $(n-2)$ number of $\langle K_4 - e \rangle$ subgraphs in $Pn(T)$. Furthermore, every edge of T is a block. Hence the adjacency of block vertices and the vertices of $L(T)$ gives $(n-2)$ number of $\langle K_4 - e \rangle$ subgraphs in $PBn(T)$. Clearly, the crossing number of $PBn(T)$ is zero, *i.e.* cr($PBn(T)$)=0. Hence $PBn(T)$ is planar.

Case 2 Suppose that T is not a path such that $\Delta(T) \leq 3$. By Theorem E, $n(T)$ is planar. Moreover, each block of $n(T)$ is either K_3 or K_4. The path number of T is at least one and the corresponding pathos vertices are adjacent to at most two vertices of each block of $n(T)$. Hence $Pn(T)$ contains at least one copy of K_3 and K_4 as its subgraphs. Finally, on embedding $PBn(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the crossing number of $PBn(T)$ becomes zero, *i.e.*, cr($PBn(T)$)=0. Hence $PBn(T)$ is planar. This completes the proof. □

Theorem 4.2 *The pathos adjacency blict graph $PBn(T)$ of a tree T is an outerplanar if and only if T is a path on P_n on $n \geq 3$ vertices.*

Proof Suppose $PBn(T)$ is an outerplanar. Assume that T has a vertex p of degree three. The edges incident to p and the cut vertex p gives K_4 as subgraph in $Pn(T)$. By Remark 2.1, the inner vertex number of $PBn(T)$ is non-zero, *i.e.* $i(PBn(T)) \neq 0$, a contradiction.

Conversely, suppose that T is a path P_n on $n \geq 3$ vertices. By Case 1 of Theorem 4.1, $PBn(T)$ contains $(n-2)$ number of $\langle K_4 - e \rangle$ as its subgraphs. Clearly, *i.e.* $i(PBn(T))$=0. Hence $PBn(T)$ is an outerplanar. This completes the proof. □

Theorem 4.3 *For any tree T, $PBn(T)$ is not maximal outerplanar.*

Proof By Theorem 4.2, $PBn(T)$ is an outerplanar if and only if T is a path P_n on $n \geq 3$ vertices. Suppose that T is a path P_n on $n \geq 3$ vertices with the edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. By Case 1 of Theorem 4.1, $Pn(T)$ contains $(n-2)$ number of $\langle K_4 - e \rangle$ as its subgraphs. Moreover, each edge of T is a block. Hence by definition, block vertices and the vertices of $L(T)$ are adjacent in $PBn(T)$, which in turn forms $(n-1)$ number of end edges in $PBn(T)$. Finally, since the addition of an edge between the block vertices increases the inner

vertex number of $PBn(T)$ by at least one, $PBn(T)$ is not maximal outerplanar. This completes the proof.   □

Theorem 4.4 *For any tree T, $PBn(T)$ is not minimally non-outerplanar.*

Proof Proof by contradiction. Suppose that $PBn(T)$ of a tree T is minimally non-outerplanar. We consider the following cases.

Case 1 Suppose that $\Delta(T) \leq 2$. By Theorem 4.2, $PBn(T)$ is an outerplanar, a contradiction.

Case 2 Suppose that $\Delta(T) \geq 3$.

We consider the following subcases of Case 2.

Subcase 2.1 Suppose that $\Delta(T) > 3$. By Theorem 4.1, $PBn(T)$ is nonplanar, a contradiction.

Subcase 2.2 Suppose that $\Delta(T) = 3$. Let p be a vertex of degree 3 in T. By Case 2 of Theorem 4.1, cr($PBn(T)$)=0, but it is easy to observe that (For example, the graph $PBn(T)$ in Figure 2) on embedding $PBn(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the inner vertex number of $PBn(T)$ is at least two, *i.e.* $i(PBn(T)) \geq 2$. Hence $PBn(T)$ is not minimally non-outerplanar. This completes the proof.   □

Theorem 4.5 *For any tree T with $p \geq 3$ vertices, $PBn(T)$ is non-Eulerian.*

Proof Suppose that T is a tree with $p \geq 3$ vertices. Then there exists at least one cut vertex C of T which is incident to at least one end edge q or at least one block B. We consider the following two cases.

Case 1 If the degree of cut vertex C is odd, then the edge degree of q in T is even. By Remark 2.3, $PBn(T)$ contains odd degree vertex. Hence $PBn(T)$ is non-Eulerian.

Case 2 If the degree of cutvertex C is even, then the edge degree of q in T is odd. By Remark 2.3, $PBn(T)$ contains even degree vertex. But, since every edge of T is a block, degree of the corresponding block vertex in $PBn(T)$ is exactly one. Hence $PBn(T)$ is non-Eulerian. This completes the proof.   □

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Regularization and Energy Estimation of Pentahedra (Pyramids) Using Geometric Element Transformation Method

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Abstract: By geometric element transformation method (GETMe) always we get a new element. It is based on geometric transformations, which, if applied iteratively, lead to the regularization of a pyramid (under conditions). Energy function is a cost function for pentahedra which is applicable also for hexahedra, octahedra, decahera etc. is defined by a particular process, which we call as base diagonal apex method (BDAMe). Here, we try to investigate the characterization of different cost functions using BDAMe when we transform a pyramid by GETMe.

Key Words: Mesh quality, iterative element regularization, finite element mesh, objective function, cost function.

AMS(2010): 65M50, 65N50

§1. Introduction

In many finite element applications unstructured tessellations of the geometry under consideration play a fundamental role. Therefore, the generation of quality meshes are essential steps of the simulation process, since mesh quality has an impact on solution accuracy and the efficiency of the computational methods involved [1,2].

In [4] the geometric element transformation method (GETMe) has been introduced as new type of geometry-based mesh smoothing for triangular surface meshes. Based on a simple geometric element transformation, which iteratively transforms low quality elements to regular, hence perfect elements, mesh improvement is accomplished by sequentially improving the worst element of the mesh. In [5] this approach has been generalized to a simultaneous approach for triangular or quadrilateral mixed surface meshes in which all mesh elements are transformed simultaneously and node updates are obtained by transformed node averaging. As has been shown in [6,7] such regularizing transformations exist for polygons with an arbitrary number of nodes. Furthermore, the sequential as well as the simultaneous GETMe approach naturally extend to tetrahedral meshes [8].

In finite element simulation the mesh quality is a crucial aspect for good numerical be-

haviour of the method. In a first stage, some automatic 3-D mesh generators construct meshes with poor quality and in special cases, for example when node movement is required, inverted elements may appear. So, it is necessary to develop a procedure that optimises the pre-existing mesh. This process must be able to smooth and untangle the mesh.

The most usual technique to improve the quality of a valid mesh, that is, one that does not have inverted elements, are based upon local smoothing. In short, these technique consists of finding the new positions that the mesh nodes must hold, in such a way that they optimize an objective function. Usually, objective functions are appropriate to improve the quality of a valid mesh, but they do not work properly when there are inverted elements. To avoid this problem we can proceed as Freitag et al in [9,10,11].

In this paper, we have defined the characterization of energy function of a pyramid using base diagonal apex method (BDAMe). Then we have proved that the energy function always lies between 0 and 1 and discussed regularization properties of a pyramid and tried to regularize by using geometric element transformation method (GETMe). Finally we have studied the characterization of energy function of a particular type of pentahedron using GETMe and BDAMe.

§2. Characterization of Energy Function of a Pyramid

For 3-simplex the cost function which referred to as energy function, which is also discussed in [3]. But we can not estimate energy function for all 3-D shapes. In this paper, we shall try to estimate the energy function of a pentahedron by a particular method, which we have defined as base diagonal apex method (BDAMe).

2.1 Base Diagonal Apex Method (BDAMe)

In this method, we add the two diagonal of the base of the pyramid and then add between the intersection point of the diagonal and the apex of the pyramid. This line (from apex to the intersection point of the diagonals) may be the height of the pyramid or may not be the height of the pyramid, totally depend upon the type of pyramid we choose. If we follow this method, we get four 3-simplex, that is, four tetrahedra. Now each tetrahedron has a cost function or energy function. Therefore, we get four cost functions and then we can easily define the cost function of a pyramid, and to define cost function of 3-D shapes except 3-simplex, we introduce the function $h(v_i, \sigma^n)$, the signed distance from $c(\sigma^n)$ to $\text{aff}(\sigma_i^{n-1})$ with the convention that $h(v_i, \sigma^n) \geq 0$ when $c(\sigma^n)$ and v_i are on the same side of $\text{aff}(\sigma_i^{n-1})$. Here σ^n are the n-simplex, $c(\sigma^n)$ the circumcenter of the n-simplex, facet $\text{aff}(\sigma_i^{n-1})$ and vertex v_i. We work on basically all 3-D figures, so in that case $n = 3$. The magnitude of $h(v_i, \sigma^n)$ can be computed as the distance between $c(\sigma^n)$ and $c(\sigma_i^{n-1})$, and its sign can be computed by testing whether $c(\sigma^n)$ and v_i have the same orientation with respect to $\text{aff}(\sigma_i^{n-1})$. Now by BDAMe, the pentahedron is the sum of the maximum number of four 3-simplexes. Here we divide the quantity $h(v_i, \sigma^n)$ by the circumradius $R(\sigma^n)$ to get a quantity called cost function or energy function. Note that $-1 < h(v_i, \sigma^n)/R(\sigma^n) < 1$ for finite σ^n, because $R(\sigma^n)^2 = h(v_i, \sigma^n)^2 + R(\sigma_i^{n-1})^2$.

We consider the energy function

$$f_p(\sigma^n) = \frac{1}{4}\Sigma_{j=1}^4 max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}| \tag{1}$$

Now we prove the following theorem.

Theorem 2.1 *The energy function (using BDAMe)* $f_p(\sigma^n) = \frac{1}{N}\Sigma_{j=1}^N max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|$, *where N is maximum total number of tetrahedron, of a 3D-figure (pentahedron, hexahedron, decahedron, octahedron,$\cdots$, etc) always lies between 0 and 1 that is,* $0 < f_p(\sigma^n) < 1$.

Proof First we break (using BDAMe) the 3D-figure with maximum number of tetrahedra (pairwise disjoint) which can cover the hole 3D-figure. If it is not possible, then we have to break it in maximum number of pentahedra (pairwise disjoint) which can cover the hole 3D-figure and then use BDAMe in pentahedron. Therefore, we can get the cost function (using BDAMe) of any regular 3D-figure. Therefore, overall the total number of tetrahedra gives the value of N. Note that $-1 < h(v_i,\sigma^n)/R(\sigma^n) < 1$ for finite σ^n, because $R(\sigma^n)^2 = h(v_i,\sigma^n)^2 + R(\sigma_i^{n-1})^2$ and when we consider energy function, we take the maximum ratio (h/R) with positive value of each tetrahedron. Therefore, the value $f_p(\sigma^n)$ is always greater than zero and we divide $\Sigma_{j=1}^N max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|$ by N (total number of tetrahedra), hence it is always less than one. Therefore we can write, $0 < f_p(\sigma^n) < 1$.   □

For instance, if we consider a hexahedron, which is six pentahedra. After using BDAMe we get 24 tetrahedra. Therefore the energy function

$$f_H(\sigma^n) = \frac{1}{24}\Sigma_{j=1}^{24} max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|,$$

where H for hexahedron.

§3. Methods of Transformation

Here we use several methods of transformation to regularize the 3-D figure, like pentahedra. By this, regularizing means that if the transformation is applied iteratively to a single element, it becomes regular. Consequently, this section focuses on the properties of the transformations applied to a single pentahedron.

3.1 Transformation of a pentahedron using GETMe

Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with five pairwise disjoint nodes $p_i \in R^3$, $i \in \{1, \cdots, 5\}$, which are positively oriented. Let

$$n_1 := (p_5 - p_2) \times (p_3 - p_2),$$

$$n_2 := (p_5 - p_3) \times (p_4 - p_3),$$

$$n_3 := (p_4 - p_5) \times (p_1 - p_5),$$

$$n_4 := (p_1 - p_5) \times (p_2 - p_5),$$

$$n_5 := (p_4 - p_3) \times (p_2 - p_3)$$

denote the inside oriented face normal of P. A new pentahedron P' with nodes p'_i is derived from P by constructing on each node p_i the opposing face normal n_i scaled by $\sigma/\sqrt{|n_i|}$, where $\sigma \in R_0^+$. That is

$$P' = \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \\ p'_5 \end{pmatrix} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} + \sigma \begin{pmatrix} \frac{1}{\sqrt{|n_1|}} n_1 \\ \frac{1}{\sqrt{|n_2|}} n_2 \\ \frac{1}{\sqrt{|n_3|}} n_3 \\ \frac{1}{\sqrt{|n_4|}} n_4 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{2}$$

It is clear that if $\sigma = 0$ then P' and P are same.

3.2 Apex transformation of a pentahedron using GETMe

Apex transformation means, we transform the apex (top vertex) of the pentahedron (pyramid) using geometric element transformation method (GETMe) as discussed in the article (3.1). So, we transform p_5 (apex) to p'_5 using only the inside oriented face normal n_5, $n_5 := (p_4 - p_3) \times (p_2 - p_3)$ of P. In that case, a new pentahedron P' with nodes p'_i is derived from P by constructing the node p_5 the opposing face normal n_5 scaled by $\sigma/\sqrt{|n_5|}$, where $\sigma \in R_0^+$. That is

$$P' = \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \\ p'_5 \end{pmatrix} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{3}$$

It is also clear that if $\sigma = 0$ then P' and P are same.

3.3 New pentahedron derived from centroid transformation of a pentahedron using GETMe

Let P denote a pentahedron with nodes p_k and $\sigma \in R^+$ an arbitrary scaling factor. The nodes p'_k of the transformed pentahedron P' are given by

$$p'_k := c_k + \frac{\sigma}{\sqrt{|n_k|}} n_k, k \in \{1, ..., 5\}. \tag{4}$$

That is p'_k is obtained by adding the centroid c_k of the kth pentahedron face with the associated

normal n_k scaled by $\sigma/\sqrt{|n_k|}$.

3.4 Apex transformation (one step) of the pentahedron using centroid transformation

In this case, we only transform the apex (top vertex) of the pentahedron using method (4). So, we only transform p_5 to p_5' and the transformed pentahedron is given by

$$
P' = \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \end{pmatrix} := \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{5}
$$

Here c_k is the centroid of kth pentahedron face where the associated normal n_k scaled by $\sigma/\sqrt{|n_k|}$, $k \in \{1, ..., 5\}$ and $n_5 := (p_4 - p_3) \times (p_2 - p_3)$ of P.

§4. Procedure of Transformations of a Pentahedron

Now it is very important to note the procedure of transformations of a pentahedron. When we consider the 3-D figure like pentahedron, the three points of each four faces must be coplanar. In this paper the construction of pentahedron is in such a way that the four vertices of the pentahedron lie on a plane which means they are coplanar and this plane forms the base of the pentahedron. That means in Fig 1, Fig 2, Fig 3 and Fig 4, one must verify that the points p_1, p_2, p_3, p_4 are coplanar or not. From [15], we can check whether those points are coplanar or not.

Next we discuss about the procedure of first method given by (2). In [8] when the authors transformed a tetrahedron, they took the face normals n_1, n_2, n_3, n_4 opposite to the points p_1, p_2, p_3, p_4 respectively but our case is not exactly the same as in [8]. So, in this case, if we consider Fig 1, let p_1 be any vertices of the pentahedron. Here we find that there are two opposite faces namely face$\{p_2, p_3, p_5\}$ and face$\{p_3, p_4, p_5\}$. In this method we consider the face which is first when we start from p_1 in the anti-clockwise sense. In Fig 1, face$\{p_2, p_3, p_5\}$ is the opposite face of the point p_1. But when we transform p_5 (apex of the pyramid), for this case, the opposite face always forms the base of the pyramid. In Fig.1, the base is face$\{p_1, p_2, p_3, p_4\}$ of the pyramid $P := (p_1, p_2, p_3, p_4, p_5)^t$ which is the opposite face of the p_5. We follow the same procedure in method (3). Now it is important to note that when we use method (3) then it will be necessary to check that after the transformation, the base points of the pentahedron are coplanar or not.

Now for method (4), the procedure is not the same as in [12]. For this case, we consider the face which is first when we start from p_1 in the anti clockwise sense and take the centroid of the face. In Fig.3 let p_1 be any vertices of the pentahedron, here we see that there are two opposite faces namely face$\{p_2, p_3, p_5\}$ and face$\{p_3, p_4, p_5\}$, but in this case we take face$\{p_2, p_3, p_5\}$ and

then consider the centroid of the face $\{p_2, p_3, p_5\}$. But when we transform p_5, for this case the opposite face is always base of the pyramid. particularly, for Fig.3 $c_1 = \{p_2 + p_3 + p_5\}/3$, $c_2 = \{p_3 + p_4 + p_5\}/3$, $c_3 = \{p_4 + p_1 + p_5\}/3$, $c_4 = \{p_2 + p_1 + p_5\}/3$, $c_5 = \{p_1 + p_2 + p_3 + p_4\}/4$. So, in this way we get new element (pyramid) using method (4). But here, in that case, the new elements are not linearly formed that means, when new elements form after transformation, the base of this new element is opposite the original element. It happens according to the iteration. For this procedure it will be necessary to check that after the transformation, the base points of the pentahedron are coplanar or not.
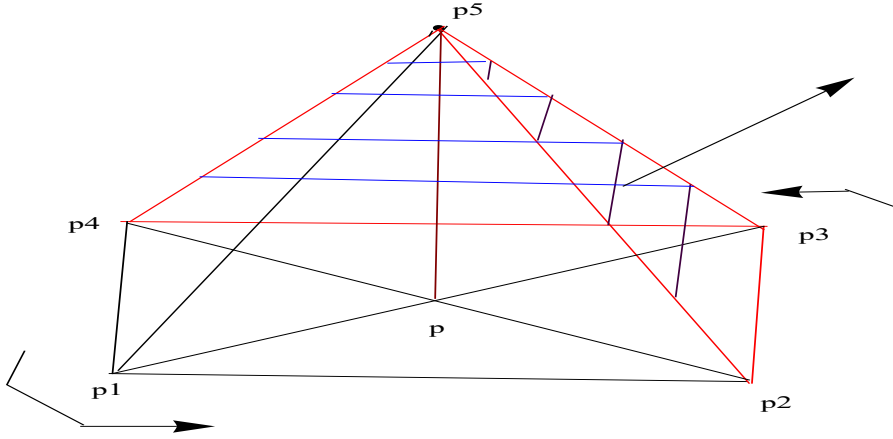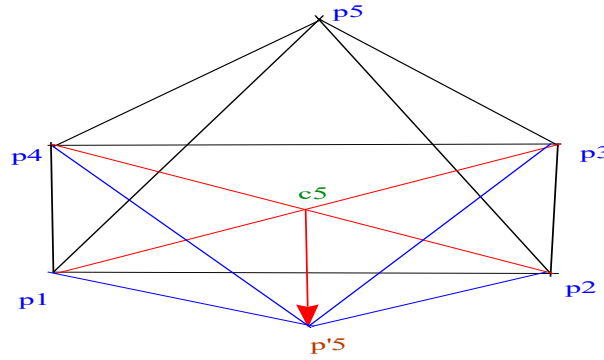

Fig.1 Transformation of a pentahedron using method (2)
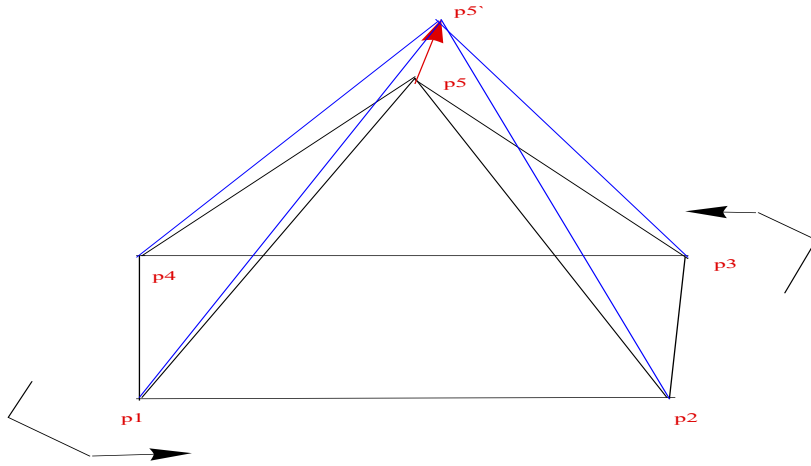
Fig.2 Transformation of a pentahedron using method (3)
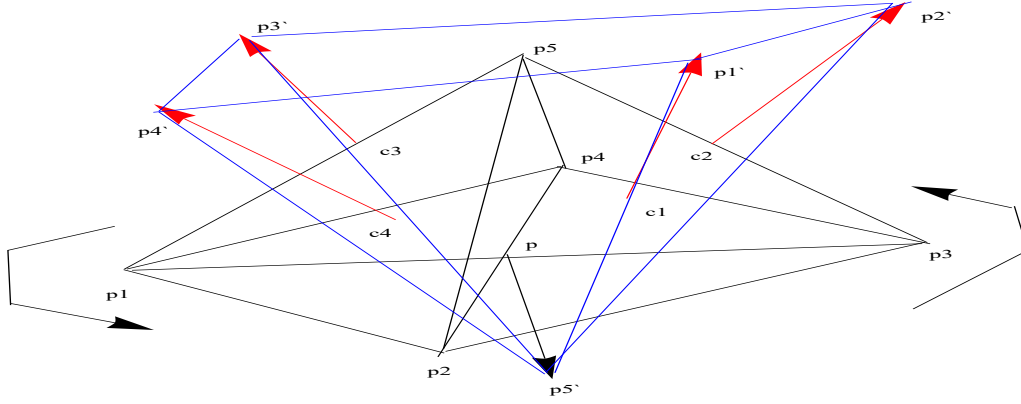
Fig.3 Transformation of a pentahedron using method (4)

Fig.4 Transformation of a pentahedron using method (5)

§5. Properties of the Transformations

In this section, we discuss the basic properties of the above three transformations.

5.1 The transformations are scale invariant

The transformations given by (2), (3), (4) and (5) are scale invariant that means for $s > 0$,

$(sP)' = sP'$. Since the normals n_i are scaled by $1/\sqrt{|n_i|}$, therefore the transformations are scale invariant. To check this property we shall consider some examples where we choose $\sigma = 0.1$.

Example 1 In this example, we use the transformation (2) and then investigate the property of $(sP)' = sP'$. Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (0.8, 0, 0.6)$, $p_2 \equiv (0, -0.8, 0.6)$, $p_3 \equiv (-0.8, 0, 0.6)$, $p_4 \equiv (0, 0.8, 0.6)$ and $p_5 \equiv (0, 0, 1)$. Let $s = 0.5$ and applying the transformation (1) both on $(sP)'$ and sP' after that we get the following table.

	Vertex Coordinates of $(sP)'$					Vertex Coordinates of sP'		
	x	y	z			x	y	z
$(sp_1)'$	0.38	-0.02	0.34		sp_1'	0.38	-0.02	0.34
$(sp_2)'$	-0.02	-0.38	0.34		sp_2'	-0.02	-0.38	0.34
$(sp_3)'$	-0.42	-0.02	0.26		sp_3'	-0.42	-0.02	0.26
$(sp_4)'$	-0.02	0.42	0.26		sp_4'	-0.02	0.42	0.26
$(sp_5)'$	0	0	0.44		sp_5'	0	0	0.45

Hence for this pentahedron the transformation is scale invariant. Now, if we use the method of apex transformation (3) on a pentahedron then one can verify from the above table that the transformation is scale invariant. Next we give an example using the method of centroid transformation of a pentahedron.

Example 2 In this example, we use the transformation (4) and then try to investigate the property of $(sP)' = sP'$. We use the same pentahedron which is used in example (1) with $s = 0.5$ and then applying the transformation (4) both on $(sP)'$ and sP' and after calculations we get the following table.

	Vertex Coordinates of $(sP)'$					Vertex Coordinates of sP'		
	x	y	z			x	y	z
$(sp_1)'$	-0.15	-0.15	0.403		sp_1'	-0.15	-0.15	0.403
$(sp_2)'$	-0.15	0.15	0.403		sp_2'	-0.15	0.15	0.403
$(sp_3)'$	0.12	0.12	0.33		sp_3'	0.12	0.12	0.33
$(sp_4)'$	0.12	-0.12	0.33		sp_4'	0.12	-0.12	0.33
$(sp_5)'$	0	0	0.357		sp_5'	0	0	0.356

One can also show that the transformation (5) is scale invariant.

5.2 Transformations (2), (3), (4) and (5) do not preserve the centroid of the pentahedron

It should be noted, that the transformations given by (2), (3), (4) and (5) do not preserve the centroid of the pentahedron, that is $\frac{1}{5}\Sigma_{i=1}^{5} p_i \neq \frac{1}{5}\Sigma_{i=1}^{5} p_i'$, where p_1, p_2, p_3, p_4, p_5 are the vertex coordinates of original pentahedron and p_1', p_2', p_3', p_4', p_5' are the vertex coordinates of the transformed pentahedron. As the scale normals $n_i/\sqrt{|n_i|}$ have been used to ensure the

scale invariance of the transformation, so the transformations (2) and (3) does not preserve the centroid of the pentahedron. We verify it by an example.

Example 3 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (1, 0, 0)$, $p_2 \equiv (1, 1, 0)$, $p_3 \equiv (0, 1, 0)$, $p_4 \equiv (0, 0, 0)$ and $p_5 \equiv (0, 0, 2)$. Then using the transformation (3) we get $p_1' \equiv (1, 0, 0)$, $p_2' \equiv (1, 1, 0)$, $p_3' \equiv (0, 1, 0)$, $p_4' \equiv (0, 0, 0)$ and $p_5' \equiv (0, 0, 2.10)$.

Centroid of the pentahedron	
Before transformation ($\frac{1}{5}\Sigma_{i=1}^{5}p_i$)	After transformation ($\frac{1}{5}\Sigma_{i=1}^{5}p_i'$)
(0.4,0.4,0.4)	(0.4,0.4,0.42)

Hence from the above we can say that the transformations (3) does not preserve the centroid of the pentahedron. Now, if we use transformation (2), then we can show that it also does not preserve the centroid of the pentahedron, provided after transformation the base of the pentahedron must also be coplanar.

Example 4 In this example, we have shown that after using transformation (4) on a pentahedron, it does not satisfy the preserving property of centroid of the pentahedron. Although in [12], the transformation given by (4) preserve the centroid of the initial hexahedron. In this case, we use the same pentahedron as used in the above example 1. Then using the transformation (4) we get $p_1' \equiv (0.33, 0.53, 0.60)$, $p_2' \equiv (0.14, 0.33, 0.67)$, $p_3' \equiv (0.33, -0.14, 0.67)$, $p_4' \equiv (0.80, 0.33, 0.73)$ and $p_5' \equiv (0.50, 0.50, 0.10)$.

Centroid of the pentahedron	
Before transformation ($\frac{1}{5}\Sigma_{i=1}^{5}p_i$)	After transformation ($\frac{1}{5}\Sigma_{i=1}^{5}p_i'$)
(0.4,0.4,0.4)	(0.42,0.31,0.55)

Hence from the above we can say that the transformations (4) does not preserve the centroid of the pentahedron. One can also easily see that the transformation (5) also does not preserve the centroid of the pentahedron.

5.3 Characterization of mean ratio quality of a pentahedron

To define mean ratio quality for pentahedron, first we use BDAMe to get four tetrahedra and then choose any tetrahedron. Let $T := (p_1, p_2, p_3, p_4)$ denote a tetrahedron with the four pairwise disjoint nodes $p_i \in R^3$, $i \in \{1, \cdots, 4\}$, which is positively oriented. That is $det(A) > 0$ with $A := (p_2 - p_1, p_3 - p_1, p_4 - p_1)$ representing the (3×3) Jacobian matrix of the difference vectors, which span the tetrahedron. In [8,11,12,13] authors have discussed how to get mean ratio quality of a tetrahedron and using this procedure we define the mean ratio quality for pentahedron, as

$$q(P) := \frac{1}{4}\sum_{k=1}^{4} \frac{3det(S_k)^{2/3}}{\|S\|_F},$$

with $\|S\| := \sqrt{trace(S^t S)}$ denoting the Frobenius norm of the matrix $S_k := A_k W^{-1}$ where

$$W = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/6 \\ 0 & 0 & \sqrt{2}/\sqrt{3} \end{pmatrix}$$

denotes the difference matrix of a regular reference tetrahedron. Now in the case of pentahedron, the criterion of $q(P)$ is not same as in [8,12]. In that case, if P is regular then $q(P) \in [0,1]$, where very small values indicate nearly degenerated elements and large values element good quality. Now, if the transformation is applied iteratively, the resulting pentahedron became more and more regular. In order to assess the regularity of a pentahedron P numerically, the mean ratio quality criterion will be used. Now, next we give an example of a pentahedron which is regular square pyramid but $q(P) \neq 1$.

Example 5 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (0,0,0)$, $p_2 \equiv (1,0,0)$, $p_3 \equiv (1,1,0)$, $p_4 \equiv (0,1,0)$ and $p_5 \equiv (0.5, 0.5, 1)$. Here, $q(p) = 0.797$.

5.4 Significance of the scaling factor σ

The resulting iteration numbers are totally depended upon the scaling factor σ. This can be used in order to control the regularization speed by a quality depended choice of the scaling factor. For a given pentahedron, there is most important thing to choice the scaling factor when we try to regularize it. For this, if the transformation is applied iteratively, the resulting pentahedron more and more regular. Now, depending upon the choice of the scaling factor σ, the size of the pentahedron might also change significantly.

Also the important fact is that there is no specific choice of σ, for which the transformation given exactly once to any arbitrary pentahedron results a regular one. To show this, we give an example.

Example 6 Let us choose the pentahedron with the same coordinate as given in article (5.2) Example 1. According to (2), the nodes of the transformed pentahedron P' are given by $p_1' \equiv (1+\sigma(0), 0+\sigma(-1.3), 0+\sigma(-0.7))$, $p_2' \equiv (1+\sigma(1.4), 1+\sigma(0), 0+\sigma(0))$, $p_3' \equiv (0+\sigma(0), 1+\sigma(-1.4), 0+\sigma(0))$, $p_4' \equiv (0+\sigma(1.3), 0+\sigma(0), 0+\sigma(0.7))$ and $p_5' \equiv (0+\sigma(0), 0+\sigma(0), 0+\sigma(-1))$ using an arbitrary scaling factor $\sigma \in R_0^+$. In order to be regular, all edge lengths of the base of the transformed pentahedron have to be equal if the base is square and in that case our example is square base pyramid but not regular. Since the equation $|p_2' - p_3'| = |p_3' - p_4'|$ has only valid solution $\sigma = 25.45$ and on the other way $|p_1' - p_2'| = |p_2' - p_3'|$ has only valid solution if $\sigma = 0.91$. Therefore there is a contradiction that there is no $\sigma \in R_0^+$ for which the pentahedron P' obtained by one step of the transformation is regular.

5.5 Uniqueness of the circumsphere and volume of the pentahedron

Now for any 3-simplex, we can always draw a sphere through the four vertices of the 3-simplex, but for pentahedron and other 3D-figures we always do not get a sphere through the all vertices of the 3D-figure except tetrahedron. But if we choose a pentahedron so that its all vertices

satisfy some particular sphere equation and then we use the transformation (2), (3), (4) and (5) we can show that after transformation the transformed pentahedron may not satisfy some particular sphere equation. Let us give an example using transformation (2) and (3) and one can also verify show using transformation (4) and (5).

Example 7 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (-0.8, 0, 0.6)$, $p_2 \equiv (0, -0.8, 0.6)$, $p_3 \equiv (-0.8, 0, 0.6)$, $p_4 \equiv (0, 0.8, 0.6)$ and $p_5 \equiv (0, 0, 1)$. Here all vertices of the pentahedron P satisfy the sphere equation $x^2 + y^2 + z^2 = 1$. After transformation using formula (2) the transformed pentahedron does not satisfy any sphere equation. If we use apex transformation then after transformation always we get a pentahedron and this pentahedron does not satisfy any particular sphere equation. For the given example after one step (using method (3)) we get, $p'_1 \equiv (-0.8, 0, 0.6)$, $p'_2 \equiv (0, -0.8, 0.6)$, $p'_3 \equiv (-0.8, 0, 0.6)$, $p'_4 \equiv (0, 0.8, 0.6)$ and $p'_5 \equiv (0, 0, 2.13)$ and p'_i's do not satisfy any sphere equation.

On the other hand the volume of the pentahedron will also be changed and does not depend on whether the pentahedron is regular or not. The volume of the transformed pentahedron will decrease or increase that depends upon the choice of the pentahedron.

Example 8 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (4, 3, 0)$, $p_2 \equiv (4, 7, 0)$, $p_3 \equiv (0, 7, 0)$, $p_4 \equiv (0, 3, 0)$ and $p_5 \equiv (3, 6, 10)$. Now the volume of the pentahedron is 73.34 cube unit and using the formula (3) to the pentahedron P we get the volume of the transformed pentahedron 76.26 cube unit.

§6. Regularization of a Pyramid (Pentahedron)

Here we shall consider a process to regularize a pentahedron. In the case of pentahedron regular means that the base of the pentahedron is regular (square, rectangle, $\cdots$, etc.) and the upper all edge lengths are equal. So, when we consider an arbitrary pentahedron it is quite difficult to regularize the pentahedron, but if we take the base of the pentahedron is regular and upper portion of the pentahedron is irregular then we can regularize the pentahedron using apex transformations (3) and (5). One can also use transformations (2) and (4) provided after transformations the base points are coplanar. Here we furnish an example where transformation (3) and transformation (5) are used to regularize the pentahedron whose base is regular.

Example 9 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (4, 3, 4)$, $p_2 \equiv (4, 7, 4)$, $p_3 \equiv (0, 7, 4)$, $p_4 \equiv (0, 3, 4)$ and $p_5 \equiv (3, 8, 10)$. This is square based pyramid but not regular because the base edges length $p_1p_2 = p_2p_3 = p_3p_4 = p_4p_1 = 4$ and for the upper portion length of the edges are $p_1p_5 = 7.87$, $p_2p_5 = 6.16$, $p_3p_5 = 6.78$, $p_4p_5 = 8.37$ which all are not equal. Now, if we use the transformation (3) on the given pentahedron, then in first step, length of the edges (upper portion) are $p_1p'_5 = 8.18$, $p_2p'_5 = 6.44$, $p_3p'_5 = 7.14$, $p_4p'_5 = 8.66$ and in third step, length of the edges are $p_1p''_5 = 8.50$, $p_2p''_5 = 6.95$, $p_3p''_5 = 7.50$, $p_4p''_5 = 8.96$. Here we observe that the pyramid tends to regularize but slow. The speed of the regularization depends upon the choice of the scaling factor σ. In this case we take the scaling factor $\sigma = 0.1$.

Next we use the apex transformation (5) to the given pentahedron and we get $p'_5 \equiv$

$(2, 5, 4.40)$. After calculations we see that length of the edges (upper portion) are $p_1 p_5' = p_2 p_5' = p_3 p_5' = p_4 p_5' = 2.86$. Hence the given pyramid converge to regularize and it turns to a regular square pyramid.

§7. Characterization of Energy Function of a Particular Type of Pentahedron Using GETMe and BDAMe

The changing cost function, after transforming the pentahedron by GETMe, is given by

$$f(\sigma_p) = |f_{p_k}(\sigma^n) \sim f_{p_{k+1}}(\sigma^n)|.$$

Now using BDAMe we find the numerical values of changing cost function. We can calculate the changing cost function of the pyramid provided after transformation the base points are coplanar. Let us consider $p_1(4, 3, 0)$, $p_2(4, 7, 0)$, $p_3(0, 7, 0)$, and $p_4(0, 3, 0)$ are the base points and $p_5(2, 5, 10)$ be the apex of the pyramid and $p(2, 5, 0)$ be the intersection point of the base diagonals. In this case, it is regular square base pyramid.

After calculation, we get

	Apex Transformations			
	Initial Step	2nd Step	3rd Step	4th step
$f_p(\sigma^n)$	0.944	0.934	0.939	0.941

Now, this example is similar to the example of section (6) and it can be regularized by using the method (3). Here we see that the values of the changing cost function $f(\sigma_p)$ are 0.01, 0.005 and 0.002. Therefore for this example we see that when it converges to regularize the changing cost function also decreases. One can also calculate the changing cost function for any arbitrary pyramid (using the method (2) and (4)) provided after transformations the base points are coplanar.

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Algorithmic and NP-Completeness
Aspects of a Total Lict Domination Number of a Graph

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Abstract: A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of G is a minimum cardinality of total lict dominating set of G. The current paper studies total lict domination in graph from an algorithmic point of view. In particular we had obtained the algorithm for a total lict domination number of any graph. Also we had obtained the time complexity of a proposed algorithm. Further we discuss the NP-Completeness of a total lict domination number of the split graph, bipartite graph and chordal graph.

Key Words: Smarandachely k-dominating set, total lict dominating number, lict graph, vertex independence number, bipartite graph, split graph, chordal graph.

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§1. Introduction

All graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual 'p' and 'q' denotes the number of vertices and edges of a graph G.

The concept of domination in graph theory is a natural model for many location problems in operations research. In a graph G, a vertex is said to dominate itself and all of its neighbors.

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

A dominating set D of a graph G is a total dominating set if the dominating set D does not contain any isolates. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of total dominating set.

The lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set

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of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G)$ of G is the minimum cardinality of total lict dominating set of G.

A vertex cover C of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that for every edge $uv \in E$, we have $u \in C$ or $v \in C$. A cut-vertex of a connected graph G is a vertex v such that $G - \{v\}$ is disconnected.

A stable set in a graph G is a pair-wise non-adjacent vertices subset of $V(G)$, and a clique is a pairwise adjacent vertices subset of $V(G)$. A graph is split if its vertex set can be partitioned into a stable set and a clique. A graph is bipartite if its vertex set can be partitioned into two stable sets. A graph is chordal if every cycle of length at least 4 has at least one chord, which is an edge joining two non-consecutive vertices in the cycle.

In this paper, we obtain the algorithm for a total lict domination number of any graph. Also, we had obtained the time complexity of a proposed algorithm. Further we discuss the NP-Completeness of a total lict domination number of a graph with respect to split graph, bipartite graph and chordal graph.

§2. Algorithm

To find the algorithm for the minimum total lict domination set of a graph we use initially, the DFS algorithm to the find the cut vertices of a given graph [1], the VSA algorithm [2] to find the minimum vertex cover of a graph and shortest path algorithm [3] to find the shortest path in a graph. The edges in the shortest path gives a total lict domination set of graph G. Then we reduce this to a minimum set which gives the minimum total lict domination set of any graph G.

Algorithm to find the minimum total lict domination set of a given graph:

Input: A graph $G = (V, E)$.

Output: A minimum total lict domination set D of a graph $G = (V, E)$.

Step 1: Initialize $D = \phi$.

Step 2: Label the vertices of a graph G as $\{v_i / i = 1, 2, 3, 4, 5, \cdots, n\}$ and label the edges of a graph G as $\{e_j / j = 1, 2, 3, 4, 5, \cdots, m\}$.

Step 3: Let A=$\{v_i / v_i$ is a cut vertex of a graph $G(V, E)\}$.

Step 4: Compute the set C of all minimal vertex covers in G, such that C does not contain vertex of degree one.

Step 5: FOR the minimal vertex cover set $c \in C$, DO

Step 6: IF $|V(c)| = 1$.
GOTO Step 7.
ELSE

IF $|V(c)| = 2$ and they are adjacent
GOTO Step 8.
ELSE
GOTO Step 9.
END IF.

Step 7: $D = D \cup \{$ any two adjacent edges of $E(G)\}$.
GOTO Step 13.

Step 8: $D = D \cup \{(e_i, e_j), e_i$ is a common edge incident with $V(c)$ and $e_j \in N(e_i)\}$
GOTO Step 13.

Step 9: Let $E_1 = \{e_q/e_q \in E(G)$, where e_q is the set of edges in the shortest path
connecting all the vertices of $V(c)$ and $\langle E_1 \rangle \neq K_{1,n}$ if there is any other shortest
path $\}$.
$K = \{e_l/e_l$ is an end edge $\in E_1\}$.
$R = \{e_j/e_j \in E(G) - E_1/e_j$ is adjacent to $K\}$
FOR $|E_1| \neq 1$ or 0 DO,
Let two edges $E_2 = (e_i, e_j) \in E_1$ such that $e_j \in N(e_i)$.
IF $e_i \in N(e_j)$ and $e_i \in N(e_k)$, where e_k or e_j is and end edge.
Then $E_2 = (e_i,$an end edge$)$
ELSE IF $e_i \in N(e_j, e_k)$ and $e_j \in N(e_l, e_m), (e_l, e_m) \neq e_i$
Then $E_2 = (e_i, e_k)$
END IF
END IF
$D = D \cup E_2$.
$B = \{e_p/e_p \in N(e_i, e_j)$ in $E_1\}$.
$C_1 = \{e_r/e_r \in N(B) \cap E_1 - (D \cup B), e_r$ is not incident with $A, e_r \neq (v_i, v_j), v_i, v_j \in C\}$.
$E_1 = E_1 - (B \cup C_1)$.
END FOR.

Step 10: IF $|E(E_1)| = 0$ then
GOTO Step 11.
ELSE
$D = D \cup \{E_1 \cup e_i, e_i \in E_1$ and $e_i \in N(D)\}$.
GOTO Step 11.
END IF.

Step 11: FOR $R \neq \phi$ DO,
Let any edge in R
$D = D \cup \{e_k, e_k \in E_1$ and $e_k \in N(e_i)\}$.
$R = R - \{e_i\} \cup \{e_s/e_s \in N(D)\}$.
END FOR

Step 12: END FOR (from Step 4)

Step 13: RETURN D, a minimum total lict domination set of a graph G.

Step 14: STOP.

§3. **Time Complexity**

The worst case time complexity of finding the solution of the minimum total lict domination problem of a graph using the proposed algorithm can be obtained as follows:

Assume that there are n vertices and m edges in the proposed algorithm.

(i) DFS algorithm [1] to find the cut vertices of a given graph which requires a running time of O(mn).

(ii) VSA algorithm [2] to find the minimum vertex cover of a given graph which requires the running time of $0(mn^2)$.

(iii) Shortest path algorithm [3] to find the shortest path connecting the vertices of $V(c)$ which requires the worst case of running time of O($m + n$).

(iv) For a FOR loop in step 9 requires the worst case running time of $0\left(\frac{m-1}{3}\right)$.

(v) For a FOR loop in step 11 requires the worst case running time of $0\left(\frac{2n}{3} - 2\right)$.

(vi) So the overall time is

$$O(mn) + 0(mn^2) + O(m+n) + 0\left(\frac{m-1}{3}\right) + 0\left(\frac{2n}{3} - 2\right) = 0(mn^2).$$

§4. **NP-Completeness of total lict domination number of a graph**

This section establishes NP-Complete results for the total lict domination problem in bipartite graph, split graph and in chrodal graph. The transformation is from the vertex cover problem, which is known to be NP-Complete.
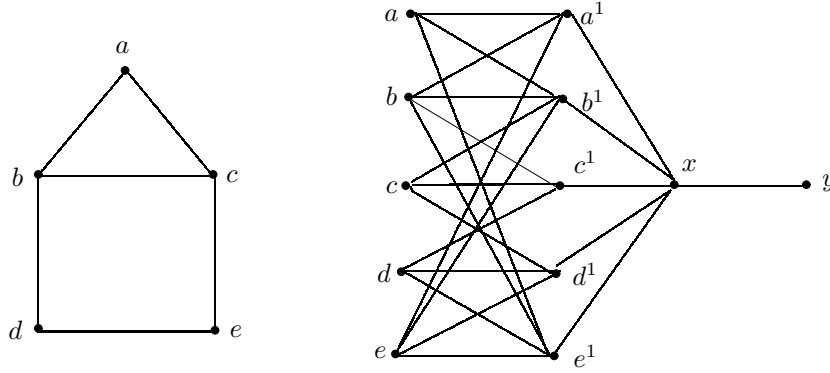
Fig.1 A constructed bipartite graph G' from the graph G

Theorem 4.1 *The total lict domination number problem is NP-Complete for bipartite graph.*

Proof The total lict domination number problem for bipartite graph is NP-Complete as we can transform the vertex cover problem to it as follows. Given a non-trivial graph $G = (V, E)$,

construct the graph $G' = (V', E')$ with the vertex set V' consists of two copies of V denoted by V and V', together with two special vertices x and y and whose edges E' consists of

(i) edges uv' and $u'v$ for each edge $uv \in E(G)$.

(ii) edges of the form uu' for each vertex $u \in V$.

(iii) edges of the form $u'x$ for every vertex $u \in V$.
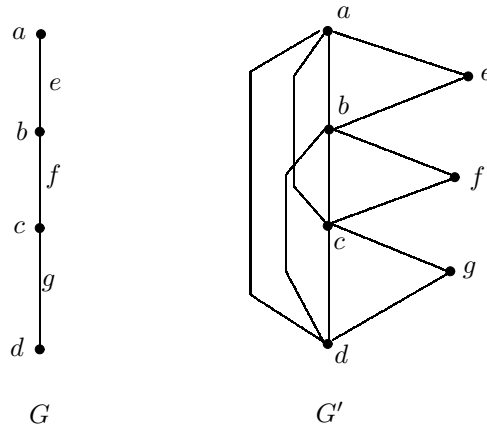
(iv) the one additional edge xy.

We claim that $G = (V, E)$ has a vertex cover of size k if and only if $G' = (V', E')$ has a minimal total lict domination set of size $k + (p - k)$. Let C be the vertex cover of G of size k. Let $B = \{u'x/u \in V\}$ such that $|B| = k$. Let $D = B \cup R$, where $R = \{u'x/u \in V - C\}$ with $|R| = p - k$. Then it is clear that, D is a total lict dominating number of a bipartite graph with cardinality $k + (p - k)$.

On the other hand suppose D is a minimal total lict domination set of the graph G' with cardinality $k + (p - k)$. Let $A = \{v_i/v_i \in V', v_i$ is incident with $e_i \in D\}$ with $|A| = |D|$. The vertex set A in G' is $V(G)$, such that A consists of copies of V and $V - C$ and whose vertices are adjacent to atleast one vertex of C. So, the graph G has a vertex cover of size k. $\square$

Theorem 4.2 *The total lict domination number problem is NP-Complete for split graph.*

Proof The total lict domination number problem for split graph is NP-Complete as we can transform the vertex cover problem to it as follows.

Given a non-trivial graph $G = (V, E)$ construct the graph $G' = (V', E')$ with the vertex set $V' = V \cup E$ and $E' = \{uv : u \neq v, u, v \in V\} \cup \{ve : v \in V, e \in E, v \in e\}$.

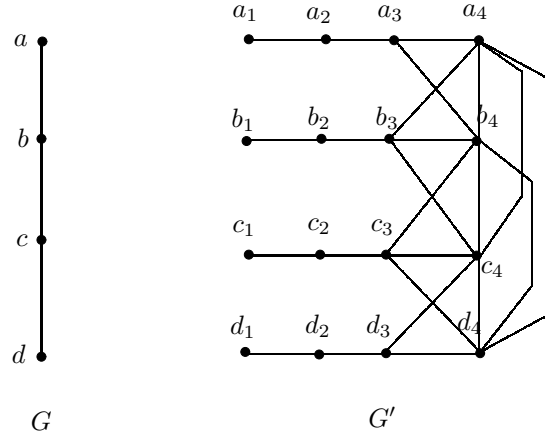**Fig.**2 A constructed split graph G^1 from a graph G

We claim that $G = (V, E)$ has a vertex cover of size k if and only if $G' = (V', E')$ has a total lict domination set of size $k + (p - k) - 1$. Let C be the vertex cover of G of size k. Let $B = \{e_i/e_i \in E(G') \cap E(G), e_i$ is incident with $V' \in C$ and $V' \in V - C$ in $G\}$. Then it is clear that B is a total lict dominating set of a split graph with cardinality $k + (p - k) - 1$.

On the other hand, suppose D is the total lict domination number of the graph G' with

cardinality $k + (p - k) - 1$. Let $A = \{v_i/v_i \in V', v_i$ is incident with $e_i \in D \cap E(G)\}$ with cardinality equal to $|D| + 1 = k + (p - k)$. The vertex set A in G' is $V(G)$ such that A consists copies of V and $V - C$ whose vertices are adjacent to at least to one vertex of C. So, the graph Ghas a vertex cover of size k. $\qquad \square$

Theorem 4.3 *The total lict domination number problem is NP-Complete for chordal graph.*

Proof we shall transform the vertex cover problem in general graph to the total lict domination in chordal graph. Therefore, the NP-Completeness of the total lict domination problem in chordal graph follows from that of the vertex cover problem in general graph. For any graph G consider the chordal graph $G' = (V', E')$ with vertex set $V' = \{v_1, v_2, v_3, v_4/v \in V\}$ and the edge set $E' = \{v_1v_2, v_2v_3, v_3v_4/v \in V\} \cup \{u_3v_4/uv \in E\} \cup \{u_4v_4/uv \in V, u \neq v\}$.

Fig.3 A constructed chordal graph G^1 from a graph G

We claim that $G = (V, E)$ has vertex cover of size k if and only if $G' = (V', E')$ has a minimal total lict domination set of size $2(k + (p - k))$. Let C be the vertex cover of G of size k. Let $B = \{v_2v_3, v_3v_4/v \in V\}$. Then it is clear that B is a minimal total lict dominating set of a chordal graph with cardinality $2(k + (p - k))$.

On the other hand suppose D is the minimal total lict domination number of the graph G' with cardinality $2(k + (p - k))$. Let $A = \{v_3/v_3 \in V', v_3$ is incident with $v_2v_3, v_3v_4 \in D\}$ with $|A| = \frac{D}{2} = k + (p - k)$. The vertex set A in G' is $V(G)$ such that A consists copies of V and $V - C$ whose vertices are adjacent to at least to one vertex of C. So, the graph G has a vertex cover of size k. $\qquad \square$

§4. **Conclusion**

The main purpose of this paper is to establish an algorithm for the total lict domination problem in general graph. NP-Complete results for the problem are also shown for split graph, chordal graph and for bipartite graphs.

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Upper Singed Domination Number of Graphs

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Abstract: A function $f : V(G) \to \{-1, 1\}$ defined on the vertices of a graph G is a signed dominating function (SDF) if $f(N[v]) \geq 1$, $\forall\ v \in V$, where $N[v]$ is the closed neighborhood of v. A SDF f is minimal if there does not exists signed dominating function g, $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The signed domination number of a graph G is the minimum weight of a minimal SDF on G and upper signed domination number of G is the maximum weight of a minimal SDF on G. In this paper, we obtain the upper signed domination number of path, cycle and complete bipartite graph.

Key Words: Signed (minus) dominating function, signed (minus) dominating function.

AMS(2010): 05C22

§1. Introduction

For all terminology and notation in graph theory we refer the reader to [2]. However, unless mentioned otherwise, we shall consider here only connected simple graphs.

Let $G = (V, E)$ be a simple graph, the *open neighborhood* of a vertex v is $N(v) = \{u : uv \in E(G)\}$ and *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For any real valued function $f : V \to \mathbb{R}$ and $S \subseteq V$, let $f(S) = \sum_{\forall u \in S} f(u)$ and then the *weight* of f is defined as $wt(f) := f(V)$.

A function $f : V \to \{-1, 0, 1\}$, is said to be a *minus dominating function* (MDF) if $f(N[v]) \geq 1$, $\forall\ v \in V$ and the function $f : V \to \{-1, 1\}$ is called a *signed dominating function* (SDF) of G if $f(N[v]) \geq 1$, $\forall\ v \in V$. A SDF (MDF) f on a graph G is minimal if there does not exist an SDF (MDF) g ($g \neq f$) for which $g(v) \leq f(v)$ for every $v \in V$.

The minus domination number for a graph G, denoted by $\gamma^-(G)$ and defined as $\gamma^-(G) = min\{wt(f) :\ f$ is a minus dominating function on $G\}$. Likewise, the upper minus domination

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number for a graph G, denoted by $\Gamma^-(G)$ and defined as

$$\Gamma^-(G) = max\{wt(f): \ f \text{ is a minimal minus dominating functionon } G\}.$$

The sign domination number for a graph G, denoted by $\gamma_s(G)$ and defined as $\gamma_s(G) = min\{wt(f): \ f$ is a sign dominating function on $G\}$. Likewise, the upper sign domination number for a graph G, denoted by $\Gamma_s(G)$ and defined as

$$\Gamma_s(G) = max\{wt(f): \ f \text{ is a minimal minus dominating function on} G\}.$$

In [4], Dunbar et.al. characterized the minimal signed dominating function which is as follows:

Proposition 1.1(Dunbar et.al.[4]) *A SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$.*

In [5], Henning and Slater posed an open problem to find the good bound for upper signed domination number. Towards solving this problem Favaron [6] found the following sharp bound for the regular graphs.

Theorem 1.2(Favaron [6]) *If G is a k-regular graph, $k \geq 1$ of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \dfrac{n(k+1)}{k+3} & if \ k \ is \ even; \\ \dfrac{n(k+1)^2}{k^2+4k-1} & if \ k \ is \ odd. \end{cases}$$

In 2001, Wang and Mao [1] gave upper bound for nearly regular graphs.
Theorem 1.3(Wang and Mao [1]) *If G is a nealy $(k+1)$-regular graph of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \dfrac{n(k+2)^2}{k^2+6k+4} & if \ k \ is \ even; \\ \dfrac{n(k^2+3k+4)}{k^2+5k+2} & if \ k \ is \ odd \end{cases}$$

and this bound is sharp.

The next result which was stated in [3] provides the best possible bound for a graph in terms of minimum degree δ and maximum degree Δ of the graph.

Theorem 1.4(Tang and Chen [3]) *If G is a graph of order n, then $\Gamma_s(G) \leq \dfrac{(\delta\Delta + 4\Delta - \delta)n}{\delta\Delta + 4\Delta + \delta}$ for δ even and $\Gamma_s(G) \leq \dfrac{(\delta\Delta + 3\Delta - \delta + 1)n}{\delta\Delta + 3\Delta + \delta - 1}$ for δ odd. Furthermore, if G is an Eulerian graph then $\Gamma_s(G) \leq \dfrac{(\delta\Delta + 2\Delta - \delta)n}{\delta\Delta + 2\Delta + \delta}$.*

It is easy to observe that if a graph has a pendent vertex then by Theorems 1.3 and 1.4, $\Gamma_s \leq n$ which is not a good bound, however, from a survey of literature and to the best of our

knowledge, the upper signed domination number of basic graphs like path, cycle, caterpillar and bipartite graphs are not known. Thus, in this paper we have find the upper signed domination number of path, cycle and complete bipartite graph.

§2. Upper Singed Domination Number of Path and Cycle

In this section we give the upper signed domination number of path and cycle.

Theorem 2.1 *For every path P_n of order n, $\Gamma_s(P_n) = n - 2\left\lfloor \frac{n}{5} \right\rfloor$.*

Proof If f is a minimal SDF of P_n with weight Γ_s, then

$$\Gamma_s = |P_f| - |M_f|,$$

where $P_f = \{u \in V(P_n) : f(u) = +1\}$ and $M_f = \{u \in V(P_n) : f(u) = -1\}$. Therefore,

$$\Gamma_s = n - 2|M_f|.$$

In order to prove the result it is suffices to show that $|M_f| = \left\lfloor \frac{n}{5} \right\rfloor$. Let $n = 5k + l$ for some non negative integers k and l. Let $g : V(P_n) \to \{-1, 1\}$ be a function such that,

$$M_g = \begin{cases} \{v_{5i}\} \cup \{v_{n-2}\} & \forall, 1 \le i \le k-1 & if \ k = 0, 1; \\ \{v_{5i}\} & \forall, 1 \le i \le k & if \ k = 2, 3, 4, \end{cases}$$

and $P_f = V(P_n) \backslash M_g$. One can check that given function G is a minimal SDF with $|M_g| = \left\lfloor \frac{n}{5} \right\rfloor$. Therefore,

$$|M_f| \le |M_g| = \left\lfloor \frac{n}{5} \right\rfloor. \tag{1}$$

If v_i and v_j are two vertices in M_f such that there is no other vertices between v_i and v_j in M_f. Now, suppose that the distance between v_i and v_j is less than or equal to two i.e., $d(v_i, v_j) \le 2$. Then there exists a vertex v_x adjacent to v_i and v_j. Since $f(v_i) = f(v_j) = -1$,

$$f(N[v_x]) = f(v_i) + f(v_x) + f(v_j) = -1 + 1 - 1 < 0,$$

which is a contradiction to the assumption that f is an SDF. Therefore, $d(v_i, v_j) \ge 3$.

On the other hand, if the distance between v_i and v_j is greater than or equal to six i.e., $d(v_i, v_j) \ge 6$ then there exist a sub path $P_i = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, \cdots, v_{i+t}, v_j\}$ for every $t \ge 5$, such that all the vertices $\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, \cdots, v_{i+t}\}$ are positive and $f(v_i) = f(v_j) = -1$. By Proposition 1.1 a SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$, but $f(v_{i+3}) = 1$ and $f(N[v_{i+2}]) = f(N[v_{i+3}]) = f(N[v_{i+4}]) = 3$ (see Figure 1) therefore Proposition 1.1 implies that f can not be minimal SDF, this contradicts the assumption that f is a minimal SDF. Therefore $d(v_i, v_j) \le 5$.
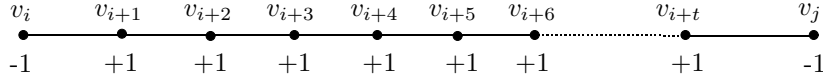
Figure 1

Hence

$$3 \leq d(v_i, v_j) \leq 5,$$

from this one can conclude that

$$|M_f| \geq \left\lfloor \frac{n}{5} \right\rfloor. \tag{2}$$

From (1) and (2)

$$|M_f| = \left\lfloor \frac{n}{5} \right\rfloor.$$

Hence,

$$\Gamma_s(P_n) = n - 2|M_f| = n - 2\left\lfloor \frac{n}{5} \right\rfloor. \qquad \square$$

Corollary 2.2   *For every cycle C_n of order n, $\Gamma_s(C_n) = n - 2\lceil \frac{n}{5} \rceil$.*

     Proof The proof of this corollary can be given by the arguments analogues to those used in the above Theorem 2.1. $\square$

§3. Upper Signed Domination Number of Complete Bipartite Graphs

Theorem 3.1   *If $K_{m,n}$, the complete bipartite graph with $m \leq n$, then $\Gamma_s = (m+n) - 2\left\lfloor \frac{m}{2} \right\rfloor$.*

     Proof Consider $K_{m,n} = (U, W)$ the complete bipartite graph with partite sets U and W having $|U| = m \leq n = |W|$ $(m, n \geq 2)$ and f be a minimal SDF with weight $\Gamma_s(K_{m,n})$, then

$$\Gamma_s(K_{m,n}) = |P_f| - |M_f| = (m+n) - 2|M_f|.$$

Where P_f and M_f are as defined in Theorem 2.1. In order to establish the desired result, it is sufficient to show that $|M_f| = \left\lfloor \frac{m}{2} \right\rfloor$.

     Let $|U \cap M_f| = m^-$ and $|W \cap M_f| = n^-$. Since $K_{m,n}$ is a complete bipartite graph with $m \leq n$, then $m^- \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $n^- \leq \lfloor \frac{n}{2} \rfloor$. This gives,

$$|M_f| \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Suppose $|M_f| < \left\lfloor \frac{m}{2} \right\rfloor$, then there exists a positive integer k such that

$$|M_f| = \left\lfloor \frac{m}{2} \right\rfloor - k$$

$$m^- + n^- = \left\lfloor \frac{m}{2} \right\rfloor - k. \tag{3}$$

Consider,

$$
\begin{aligned}
f(N[w_i]) &= \sum_{u_i \in U} f(u_i) + f(w_i) \\
&= \sum_{u_i \notin M_f} f(u_i) + \sum_{u_i \in M_f} f(u_i) + f(w_i) \\
&= m - m^- - m^- + f(w_i) \\
&= m - 2m^- + f(w_i) \\
&= m - 2\left\lfloor \frac{m}{2} \right\rfloor + 2k + 2n^- + f(w_i) \qquad \text{by equation (3),} \\
&\geq 3 \qquad\qquad\qquad\qquad\qquad\qquad \forall, w_i \in W.
\end{aligned}
$$

Following the above procedure, we calculate the value of $f(N[u_i])$

$$
\begin{aligned}
f(N[u_i]) &= \sum_{w_i \in W} f(w_i) + f(u_i) \\
&= \sum_{w_i \notin M_f} f(w_i) + \sum_{w_i \in M_f} f(w_i) + f(u_i) \\
&= n - n^- - n^- + f(u_i) \\
&= n - 2n^- + f(u_i) \\
&= n - 2\left\lfloor \frac{m}{2} \right\rfloor + 2k + 2m^- + f(u_i) \qquad \text{by equation (3),} \\
&\geq 3 \qquad\qquad\qquad\qquad\qquad\qquad \forall, u_i \in U.
\end{aligned}
$$

This implies that, if $|M_f| < \left\lfloor \frac{m}{2} \right\rfloor$ then $f(N[v]) \geq 3$ for all $v \in V(K_{m,n})$ and by Proposition 1.1 a SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$, but $f(N[v]) \geq 3$ for all $v \in V(K_{m,n})$ hence f is not a minimal SDF, which is a contradiction to the assumption that f is an minimal SDF. Therefore,

$$
|M_f| = \left\lfloor \frac{m}{2} \right\rfloor.
$$

This implies

$$
\begin{aligned}
\Gamma_s(K_{m,n}) &= (m+n) - 2|M_f| \\
&= (m+n) - 2\left\lfloor \frac{m}{2} \right\rfloor.
\end{aligned}
$$

Hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Acknowledgment

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Star Chromatic and Defining Number of Graphs

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Abstract: Let u and v be adjacent vertices in G. If we assign colors to $N[v]$ and $N[u]$ such that the assignment colors to $N[v]$ are different with the assignment colors to $N[u]$, then this colorings is said to be vertex star colorings. In this paper we initiate the study of the star chromatic number and star defining number.

Key Words: Star coloring, star chromatic number, star defining number, Smarandachely Λ-coloring.

AMS(2010): 05C15

§1. Introduction

In the whole paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. We use [9] for terminology and notation which are not defined here.

Let Λ be a subgraph of a graph G. A Smarandachely Λ-coloring $\varphi_\Lambda|_{V(G)} : \mathscr{C} \to V(G)$ of a graph G by colors in $\mathscr{C}$ is a mapping $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G)$ such that $\varphi(u) \neq \varphi(v)$ if u and v are vertices of a subgraph isomorphic to Λ in G. Particularly, if $\Lambda = G$, such a coloring is called a k-coloring of G. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable. Let $\chi(G) \leq k \leq |V(G)|$. A set $S \subseteq V(G)$ with an assignment of colors to them is called a defining set of the vertex coloring of G if there exists a unique extension of S to a k-coloring of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, k)$, for more see [1, 3, 4, 5, 6, 7].

In this note we introduce vertex star coloring of graphs as follows:

If u and v are arbitrary adjacent vertices in G, then the set of colors that we assign to $N[v]$ is different with the set of colors that assign to $N[u]$. We call this vertex coloring as *vertex star coloring*. It is obvious that vertex star coloring does not include the family of graphs with

following property:

$\exists u, v \in V(G)$ with $N[v] = N[u]$, for which $uv \in E(G)$.

The chromatic number and defining number of vertex star coloring are called *the star chromatic number* (χ^*) and *star defining number* (d^*), respectively.

We make the following observations:

Observation 1 *For every connected graph G of order $n \geqslant 3$, $\chi^*(G) \geq 3$.*

Observation 2 *If $\chi^*(G) = 3$, then $|f(N[v])| = 2, |f(N[u])| = 3$ for every two adjacent vertices u, $v \in V(G)$ for which f is a star coloring function.*

Our purpose in this paper is to initiate the study of the star chromatic number and the star defining number (d^*) of cycles, paths and complete bipartite, hyper cube and Cartesian product $P_n \times P_m$ graphs.

§2. **Star Chromatic Numbers**

In this section the star chromatic number of cycle, path, complete bipartite and Cartesian product $P_n \times P_m$ graphs are studied.

First, we present a general result as follows:

Proposition 3 *Let G be a graph. Then $\chi^*(G) > \chi(G)$.*

Proof On the one hand, $\chi^*(G) \geq \chi(G)$. On the other hand, it is enough to show that $\chi^*(G) \neq \chi(G)$. Suppose to the contrary. First, we increasingly order vertices of G and color the vertex with the least index by 1. Now, we color the remaining vertices by this manner, i.e: for the next uncolored vertex, we assign an unused color on its neighbors or a new color if be necessary (Greedy algorithm). Hence, a vertex color by $\chi(G)$ such that its neighbors colored by $\{1, 2, \cdots, \chi(G) - 1\}$. And a vertex color by $\chi(G) - 1$ such that its neighbors colored by $\{1, 2, \cdots, \chi(G) - 2\}$. Without loss of generality, we may assume that u and v are two vertices which colored by $\chi(G) - 1$ and $\chi(G)$. It follows that the set $\{1, 2, \ldots, \chi(G)\}$ is the used colors on u and its neighbors, and on the vertex v and its neighbors, a contradiction. $\square$

Proposition 4 *(i)* $\chi^*(C_n) = 3$ *where $n = 4m$.*
(ii) $\chi^*(C_n) = 4$ *where $n = 4m + 2$.*

Proof *(i)* Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & i \text{ is odd,} \\ 1 & i = 4t + 2, \\ 3 & i = 4t. \end{cases}$$

It implies that $\chi^*(G) \leq 3$. Hence, by Proposition 3 the desired result follows.

(ii) Define the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & i \text{ is odd and } i \neq 4m+1, \\ 3 & i = 4t+2 \text{ and } i \leq 4m, \\ 1 & i = 4t, 4m+2, \\ 4 & i = 4m+1. \end{cases}$$

It follows that $\chi^*(G) \leq 4$. Now, we show that $\chi^*(G) \geq 4$. It is easy to check that for any four consecutive vertices in C_n, namely $v_i, v_{i+1}, v_{i+2}, v_{i+3}$, we have $f(v_i) \neq f(v_{i+3})$. Otherwise, a contradiction. Moreover, we must use 3 different colors on any four consecutive vertices. Using the star coloring function f in the proof of Part (i), which implies that the vertex v_{n-1} cannot be colored by 2. The set of the colors of v_{4m+1} and its neighbors will be the same as the ones of v_{4m+2} and its neighbors. Thus, it can be colored by 4. Hence the desired result follows. □

Now, we continue the study of the star chromatic numbers on odd cycle.

Proposition 5 $\chi^*(C_n) = 4$ where $n(\neq 5, 7)$ is an odd integer.

Proof For $n = 5$, the star coloring function of C_5 can be defined as follows: $f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 2$, $f(v_4) = 4$, $f(v_5) = 5$.

For $n = 7$, the star coloring function of C_7 can be defined as follows: $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 1$, $f(v_4) = 3$, $f(v_5) = 4$, $f(v_6) = 3$, $f(v_7) = 5$.

Let $n - 1 = 6t + 4$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, t \geq 1 \text{ and } i = 1, 3, \\ 4 & i = 6t+4, \\ 2 & i = 6t, 2, \\ 1 & i = n \text{ and } i \text{ is odd and } i \neq 1, 3. \end{cases}$$

Let $n - 1 = 6t$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, n, \\ 4 & i = 6t+4, n-1, \\ 2 & i = 6t \text{ and } i = 1, n-3, \\ 1 & i \text{ is odd and } i \neq 1, n. \end{cases}$$

Let $n - 1 = 6t + 2, n > 9$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, t \geq 1 \text{ and } i = 1, 3 \\ 4 & i = 6t+4, n-1, \\ 2 & i = 6t \text{ and } i = 6t, 2, \\ 1 & i \text{ is odd and } i \neq 1, 3. \end{cases}$$

Hence, by Proposition 3 and the fact that $\chi(C_n) = 3$ for which n is an odd integer, we get that $\chi^*(G) = 4$. □

Proposition 6 (i) $\chi^*(P_n) = 3$ where n is an odd integer.

(ii) $\chi^*(P_n) = 4$ where $n \geqslant 4$ is an even integer.

Proof (i) Define the the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & i = 2t, \\ 1 & i = 4t+1, \\ 3 & i = 4t+3. \end{cases}$$

This completes the proof.

(ii) Using a same fashion star coloring function f in Part (i), but $f(v_{n=2m}) = 4$. It follows that $\chi^*(P_{n=2m}) \leq 4$. Now, we consider two cases as follows.

Case 1 If $m = 2t$, then, according to the star coloring function f, let $f(v_{2m-1}) = 3$. It follows that the vertex v_{2m} cannot be colored by 2 or 3. Color the vertex v_{n-1} by 3, so the vertex v_n cannot be colored by 1, 2 and 3. Thus, it can be colored by 4. Hence the result holds.

Case 2 If $m = 2t + 1$, In the same manner in Case 1 settle this case.   □

Proposition 7 $\chi^*(K_{m,n}) = 3$.

Proof Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{m,n}$. On the one hand, we may define the star coloring function f as follows: $f(v_i) = 1 (1 \leq i \leq m)$, $f(u_j) = 2$ $(1 \leq j \leq n-1)$, $f(u_n) = 3$. Thus $\chi^*(K_{m,n}) \leq 3$. On the other hand, if we use two colors on vertices of complete bipartite graphs, we imply that $N[u] = N[v]$ for every vertex $u \in X$ and $v \in Y$. So $\chi^*(K_{m,n}) \geq 3$. Hence the result holds.   □

Theorem 8 $\chi^*(P_n \times P_m) = 3$.

Proof Let v_{ij} be the vertex in ith row and jth column. Define the star coloring function c^* as follows:

$$c^*(v_{ij}) = \begin{cases} 2 & j \equiv 2 \ (mod\ 4) \text{ and } i \text{ is odd or } j \equiv 3 \ (mod\ 4) \text{ and } i \text{ is even}, \\ 3 & j \equiv 0 \ (mod\ 4) \text{ and } i \text{ is odd } or \ j \equiv 1 \ (mod\ 4) \text{ and } i \text{ is even}, \\ 1 & o.w. \end{cases}$$

Hence the result holds.   □

The following observation has straightforward proof.

Observation 9 $\chi^*(Q_k) = 3$.

§3. **Star Defining Numbers**

Proposition 10 $d^*(C_n, \chi^*) = 2$ where $n = 4m$.

Proof Let $S = \{v_1, v_3\}$ and define the star coloring function f on S as follows: $f(v_1) = 1$, $f(v_3) = 3$. It is easy to check that the remaining vertices are forced to get one color which implies that $d^*(C_{n=4m}, \chi^*) \leq 2$.

On the other side, it is well-known that $d^*(C_{n=4k}, \chi^*) \geq \chi^*(G) - 1 = 2$. This completes the proof. $\qquad \square$

Now, the star defining numbers of odd paths are studied.

Proposition 11 *(i)* $d^*(P_n, \chi^*) \leq m - 1$ *where* $n = 2m$.
$\quad$ *(ii)* $d^*(P_n, \chi^*) = 2$ *where* $n = 2m + 1$.

Proof *(i)* We define $S = \{v_i | i = 3t + 1 \text{ and } t(> 0) \ t \text{ is even}\} \cup \{v_i | i = 3t, \ t = 1 \text{ and } t(\geqslant 3) \text{ is odd}\} \cup \{v_i | i = 3t + 2 \text{ and } t \text{ is odd}\}$ with

$$
f(v_i) = \begin{cases}
2 & i = 3t \text{ and } t = 1 \text{ and } t \geq 3 \text{ and } t \text{ is odd}, \\
4 & i = 3t + 1 \text{ and } t > 0 \text{ and } t \text{ is even}, \\
3 & i = 3t + 2 \text{ and } t \text{ is odd}.
\end{cases}
$$

$\quad$ *(ii)* Define $S = \{v_1, v_2\}$ with $f(v_1) = 1$, $f(v_2) = 2$. The rest of vertices orderly get colors from $v_3, \ v_4, \ \cdots, \ v_{2n+1}$. We know that for every graph G, $d^*(G, \chi^*) \geq \chi^* - 1$. Therefore $d^*(P_n, \chi^*) = 2$ where $n = 2m + 1$. $\qquad \square$

Proposition 12 $d^*(K_{1,n}, \chi^*) = n$.

Proof Let $X = \{x_1\}$ and $Y = \{y_1, \cdots, y_n\}$ be partite sets of $K_{1,n}$. Define $S = Y$ with $f(y_i) = 2 \ (1 \leq i \leq n - 1)$, $f(y_n) = 3$. So $f(x_1) = 1$. Thus, $d^*(K_{1,n}, \chi^*) \leq n$.
$\quad$ Now, we show that $d^*(K_{1,n}, \chi^*) \geq n$. It is easy to check that if we use two colors on $n - 1$ vertices of Y, thus one can obtain two different colorings. Hence, $d^*(K_{1,n}, \chi^*) = n$. $\qquad \square$

Proposition 13 $d^*(K_{m,n}, \chi^*) = m$ *where* $1 < m \leq n$.

Proof Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{m,n}$. We define $S = \{x_1, x_2, \ldots, x_m\}$ with $f(x_i) = 2 \ (1 \leq i \leq m - 1)$, $f(x_m) = 3$ and get the result $f(y_j) = 1 \ (1 \leq j \leq n)$.
$\quad$ Now, we show that $d^*(K_{m,n}, \chi^* = 3) \geq m$. Suppose that we color $m - 1$ vertices of X by two colors, then the remaining vertex of X can be colored by two different colors, a contradiction. Hence the result. $\qquad \square$

Proposition 14 *If* $G = K_{m,n}$, $m \leq n$ *and* $m > 1$ *then*

$$
d^*(K_{m,n}, c \geq \chi^* + 1) = \begin{cases}
m & c \leq m, \\
m + n & c > max\{m, n\}, \\
n & m < c \leq n.
\end{cases}
$$

Proof The same used manner in Propositions 12 and 13 settles the stated result. $\qquad \square$

Proposition 15 *(i)* $d^*(P_3 \times P_3) = d^*(P_3 \times P_4) = d^*(P_3 \times P_5) = 2$.
$\quad$ *(ii)* $d^*(P_2 \times P_3) = d^*(P_2 \times P_4) = d^*(P_2 \times P_5) = 2$.

Proof We know that $d^*(P_n \times P_m) \geq \chi^*(P_n \times P_m) - 1 = 3 - 1 = 2$. It is enough to

present a star defining set of size 2 for each of these graphs. Define the star defining sets of $P_2 \times P_3, P_2 \times P_4, P_2 \times P_5, P_3 \times P_3, P_3 \times P_4, P_3 \times P_5$, as follows:

$$\begin{bmatrix} * & 2 & * \\ 3 & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ 2 & * & 3 & * \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & 2 & * & 3 \end{bmatrix},$$

$$\begin{bmatrix} * & 2 & * \\ 3 & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & 3 & * & 2 \\ * & * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ 3 & * & 2 & * & * \\ * & * & * & * & * \end{bmatrix}.$$

$\square$

Theorem 16 *If n is an even integer and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) \leq n/2 \times \lfloor m/2 \rfloor$.*

Proof In the following table, a star defining set of size $n/2 \times \lfloor m/2 \rfloor$ is presented.

$$\begin{bmatrix} * & 2 & * & 2 & * & \ldots \\ * & * & * & * & * & \ldots \\ * & 3 & * & 3 & * & \ldots \\ * & * & * & * & * & \ldots \\ * & 2 & * & 2 & * & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & a & * & a & * & \ldots \\ * & * & * & * & * & \ldots \end{bmatrix}$$

if $n = 4k + 2$, then $a = 2$, and if $n = 4k$, then $a = 3$. $\square$

Conjecture 17 *If n is an even number and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) = n/2 \times \lfloor m/2 \rfloor$.*

Theorem 18 *If $m(k + 1) \geq 4$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) \leq m(k + 1) - 2$.*

Proof In the following table, a star defining set of size $m(k + 1) - 2$ is shown.

$$\begin{bmatrix} * & 2 & * & 2 & * & \ldots & 2 & * & 2 & * \\ * & * & * & * & * & \ldots & * & * & * & * \\ * & 3 & * & 3 & * & \ldots & 3 & * & 3 & * \\ * & * & * & * & * & \ldots & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & 3 & * & \ldots & 3 & * & * & * \end{bmatrix}$$
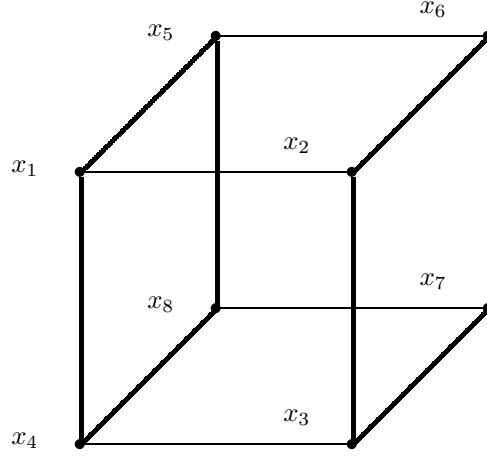
So, the star defining number is less or equal to this value. $\square$

Conjecture 19 *If $m(k + 1) \geq 4$ and $k \leq m$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) = m(k + 1) - 2$.*

Theorem 20 *If $k \geq 2$, then $d^*(Q_k, 3) = 2^{k-2} + 1$.*

Proof First, we show that $d^*(Q_k, \chi^*) \leq 2^{k-2} + 1$. It is well-known that each Q_k is 2^{k-3} copies of Q_3. We label the vertices of Q_3 as the following figure:

We define the star defining set as the following matrix for which ith row is dependent to the vertices of ith copy of Q_3 in Q_k. Note that at the defining set of Q_k, just one vertex gets color i and the remaining vertices get color j.

$$\text{For} \qquad Q_3 : \begin{bmatrix} i & * & j & * & * & .j & * & * \end{bmatrix}.$$

$$\text{For} \qquad Q_4 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \end{bmatrix}.$$

$$\text{For} \qquad Q_5 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \end{bmatrix}.$$

$$\text{For} \qquad Q_6 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \\ * & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \end{bmatrix}.$$

We know that Q_k is constructed by two copies of Q_{k-1}. Therefore, we may give a star defining set in general form for the graph as follows: We assign for the first copy as above. For the next copy; if in a row of the first copy we define $*\ *\ j\ *\ *\ j\ *\ *$, we may define in the symmetric row of the new copy as $*\ j\ *\ *\ *\ *\ j\ *$, and if in the first copy we define $*\ j\ *\ *\ *\ *\ j\ *$, we may define in the symmetric row of the next copy we define $*\ *\ j\ *\ *\ j\ *\ *$. Note that in the first row we have $i\ *\ j\ *\ *\ j\ *\ *$ but for the its symmetric row in the new copy we define as $*\ j\ *\ *\ *\ *\ j\ *$.

Now, we show that $d^*(Q_k, \chi^*) \geq 2^{k-2} + 1$. If $k = 2$, it is obvious. For completing of the proof, first we show that in each Q_3 of Q_k which colored by three colors i, j, k. Then we have just one way to color of each Q_3. Let $c(i)$ be the set of vertices with color i. It is easy to check that $|c(i)| = 1$, $|c(j)| = 1$ or $|c(k)| = 1$ is not possible. Because, we cannot find a proper star coloring for Q_k. Now, let $|c(i)| = 2$. We have two cases: (a): $|c(j)| = |c(k)| = 3$. By simple verification one can see that this cases also cannot be holden. (b): $|c(j)| = 2$ and $|c(k)| = 4$ (or symmetrically $|c(k)| = 2$ and $|c(j)| = 4$). Hence, we may color the graphs Q_3, Q_4, Q_5 and Q_6 as follows, respectively.

$$Q_3 : \begin{bmatrix} i & k & j & k & k & j & k & i \end{bmatrix}.$$

$$Q_4 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \end{bmatrix}.$$

$$Q_5 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \end{bmatrix}.$$

$$Q_6 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \\ i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \end{bmatrix}.$$

To color of the graph Q_k with $k \geqslant 5$, we should color it by the above method, otherwise we cannot find a proper star coloring for the graph. We may also replace color 2 with 3, and conversely to find a new proper star coloring of Q_k. Let S be a defining set of Q_k. It is so easy that $|S| \geqslant 3$ for Q_3. It is well-known that the graph Q_k with $k \geqslant 3$ containing of 2^{k-3} copies of Q_3. Simple verification shows that there exist no copy Q_3 of Q_k such that $S \cap V(Q_3) = 1$. Because, it is possible to assign at least two star coloring functions. It follows that $S \cap V(Q_3^i) \geqslant 2$ where $2 \leqslant i \leqslant 2^{k-3}$. Hence, the desired result follows. $\square$

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Bounds for the Harmonious Coloring of Myceilskians

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Abstract: In this paper, we find the harmonious chromatic number on Mycielskian graph of cycle, path, complete graph and complete bipartite graph.

Key Words: Harmonious coloring, harmonious chromatic number, Myceilskian graph.

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§1. Introduction

The first paper on harmonious graph coloring was published in 1982 by Frank Harary and M.J.Plantholt [2]. However, the proper definition of this notion is due to J.E.Hopcroft and M.S. Krishnamoorthy [5] in 1983. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard.

A harmonious coloring [1, 2, 3, 5, 6, 9] of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

The concept of harmonious coloring of graphs has been studied extensively by several authors; see [8, 11] for surveys. If G has m edges and G has a harmonious coloring with k colors, then clearly, $\binom{k}{2} \geq m$. Let $k(G)$ be the smallest integer satisfying the inequality. This number can be expressed as a function of m, namely

$$k(G) = \left\lceil \frac{1 + \sqrt{8m + 1}}{2} \right\rceil.$$

Paths are among the first graphs whose harmonious chromatic numbers have been established. Let P_n denote the path of order n. The following fact has been proved [2].

If $k(P_n)$ is odd or if $k(P_n)$ is even and $n - 1 = k(k - 1)/2 - j, j = k/2 - 1, k/2, \cdots, k - 2$, where $k = k(P_n)$, then $\chi_H(P_n) = k(P_n)$. Otherwise, $\chi_H(P_n) = k(P_n) + 1$.

In this present paper, we find the harmonious chromatic number on Mycielskian graph of cycle, path, complete graph and complete bipartite graph.

---

§2. **Mycielskian Graph**

We consider only finite, loopless graphs without multiple edges. For a given graph G on the vertex set $V(G) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(G)$ [4, 7, 10] as follows:

The vertex set of $\mu(G)$ is $V(\mu(G)) = \{X, Y, z\} = \{x_1, \cdots, x_n, y_1, \cdots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,
- $x_i \sim y_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,
- and $y_i \sim z$ in $\mu(G)$ for all $i \in \{1, 2, \cdots, n\}$.

§3. **Harmonious Coloring on Myceilskian Graph of Cycles**

Theorem 3.1 *Let n be a positive integer, then*

$$\chi_H(\mu(C_n)) = 2n + 1.$$

Proof For any cycle C_n with the vertex set $V(C_n) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(C_n)$ as follows. The vertex set of $\mu(C_n)$ is $V(\mu(C_n)) = \{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(C_n)$ if and only if $v_i \sim v_j$ in C_n,
- $x_i \sim y_j$ in $\mu(C_n)$ if and only if $v_i \sim v_j$ in C_n,
- and $y_i \sim z$ in $\mu(C_n)$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(C_n)$ is $4n$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = 4$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(C_n)) \geq 2n + 1$.

Now consider the vertex set $V(\mu(C_n))$ and assign a proper harmonious coloring to $V(\mu(C_n))$ as follows:

For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq n)$, assign the color c_{n+1+i} for x_i. Therefore, $\chi_H(\mu(C_n)) \leq 2n + 1$. Hence, $\chi_H(\mu(C_n)) = 2n + 1$. $\square$
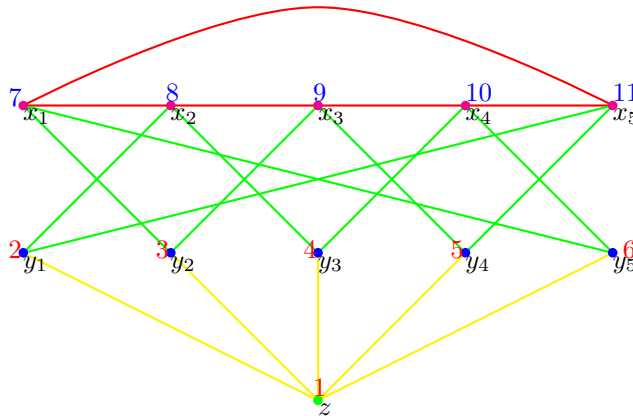
Figure 1 Mycielskian graph of C_5 with $\chi_H(\mu(C_5)) = 11$

§4. **Harmonious Coloring on Myceilskian Graph of Paths**

Theorem 4.1 *Let n be a positive integer, then*

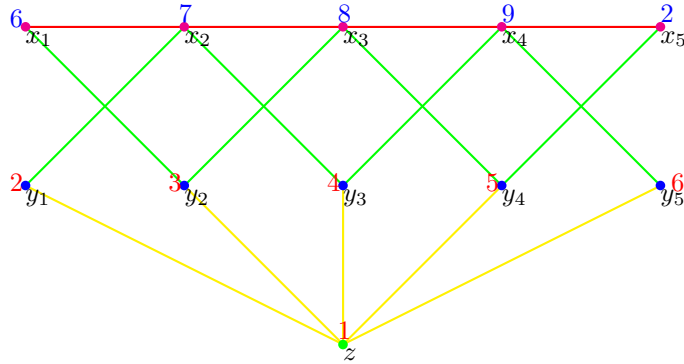$$\chi_H(\mu(P_n)) = 2n - 1, \forall n > 2.$$

Proof For any path P_n with the vertex set $V(P_n) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(P_n)$ as follows. The vertex set of $\mu(P_n)$ is $V(\mu(P_n)) = \{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(P_n)$ if and only if $v_i \sim v_j$ in P_n,
- $x_i \sim y_j$ in $\mu(P_n)$ if and only if $v_i \sim v_j$ in P_n,
- and $y_i \sim z$ in $\mu(P_n)$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(P_n)$ is $4n - 3$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $2 \le deg(x_i) \le 4$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(P_n)) \ge 2n - 1, \forall n > 2$.

Now consider the vertex set $V(\mu(P_n))$ and assign a proper harmonious coloring to $V(\mu(P_n))$ as follows:

For $(1 \le i \le n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \le i \le n)$, assign the color c_{n+i} for x_i. Therefore, $\chi_H(\mu(P_n)) \le 2n - 1, \forall n > 2$. Hence, $\chi_H(\mu(P_n)) = 2n - 1, \forall n > 2$. □

Figure 2 Mycielskian graph of P_5 with $\chi_H(\mu(P_5)) = 9$

§5. **Harmonious Coloring on Myceilskian Graph of Complete Graphs**

Theorem 5.1 *Let n be a positive integer, then*
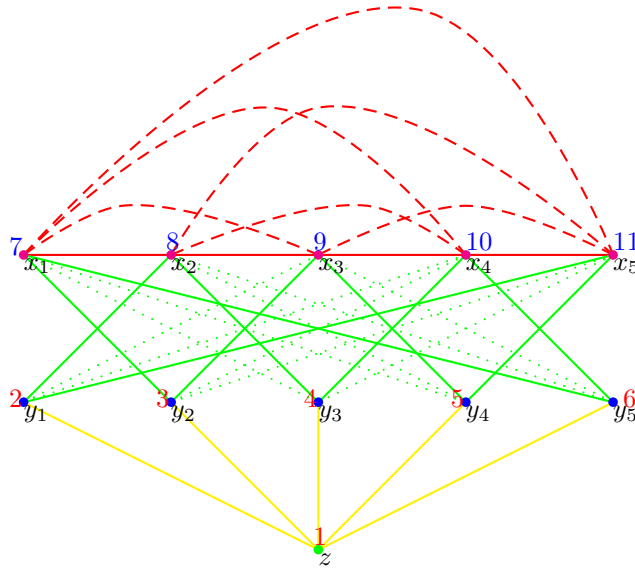
$$\chi_H(\mu(K_n)) = 2n + 1 \ \text{for} \ n \neq 2.$$

Proof For any complete graph K_n with the vertex set $V(K_n) = \{v_1, \dots, v_n\}$, we define its Mycielskian $\mu(K_n)$ as follows. The vertex set of $\mu(K_n)$ is $V(\mu(K_n)) = \{X, Y, z\} = \{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(K_n)$ if and only if $v_i \sim v_j$ in K_n,
- $x_i \sim y_j$ in $\mu(K_n)$ if and only if $v_i \sim v_j$ in K_n,
- and $y_i \sim z$ in $\mu(K_n)$ for all $i \in \{1, 2, \dots, n\}$.

The number of edges in $\mu(K_n)$ is $\dfrac{3n^2 - n}{2}$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = n + 1$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(K_n)) \geq 2n + 1$, for $n \neq 2$.

Now consider the vertex set $V(\mu(K_n))$ and assign a proper harmonious coloring to $V(\mu(K_n))$ as follows:

For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq n)$, assign the color c_{n+1+i} for x_i. Therefore, $\chi_H(\mu(K_n)) \leq 2n+1$, for $n \neq 2$. Hence, $\chi_H(\mu(K_n)) = 2n + 1$, for $n \neq 2$. $\qquad\square$

Figure 3 Mycielskian graph of K_5 with $\chi_H(\mu(K_5)) = 11$

§6. Harmonious Coloring on Myceilskian Graph of Complete Bipartite Graphs

Theorem 6.1 *Let n and m be a positive integers, then*

$$\chi_H(\mu(K_{m,n})) = 2(m + n) + 1.$$

Proof For any complete bipartite graph $K_{m,n}$ with the vertex set $V(K_{m,n}) = \{v_1, \dots, v_n\} \cup$

$\{u_1, \cdots, u_n\}$, we define its Mycielskian $\mu(K_{m,n})$ as follows. The vertex set of $\mu(K_{m,n})$ is

$$V(\mu(K_{m,n})) = \{X, X', Y, Y', z\} = \{x_1, \ldots, x_n, x'_1, \cdots, x'_m, y_1, \ldots, y_n, y'_1, \cdots, y'_m, z\}$$

for a total of $2n + 2m + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(K_{m,n})$ if and only if $v_i \sim v_j$ in $K_{m,n}$,
- $x'_i \sim x'_j$ in $\mu(K_{m,n})$ if and only if $u_i \sim u_j$ in $K_{m,n}$,
- $x_i \sim y_j$ in $\mu(K_{m,n})$ if and only if $v_i \sim v_j$ in $K_{m,n}$,
- $x'_i \sim y'_j$ in $\mu(K_{m,n})$ if and only if $u_i \sim u_j$ in $K_{m,n}$,
- and $y_i \sim z$ in $\mu(K_{m,n})$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(K_{m,n})$ is $m^2 + n^2 + mn + m + n$ and all the vertices z, x_i, x'_i, y_i, y'_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = 2m$, $deg(x'_i) = 2n$, $deg(y_i) = 4$, $deg(y'_i) = 4$ and so all must have distinct colors. Thus we have $\chi_H(\mu(K_{m,n})) \geq 2(m+n) + 1$.

Now consider the vertex set $V(\mu(K_{m,n}))$ and assign a proper harmonious coloring to $V(\mu(K_{m,n}))$ as follows: For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq m)$, assign the color c_{n+1+i} for y'_i. For $(1 \leq i \leq m)$, assign the color $c_{n+m+1+i}$ for y'_i. For $(1 \leq i \leq n)$, assign the color $c_{2m+n+1+i}$ for x_i Therefore, $\chi_H(\mu(K_{m,n})) \leq 2(m+n) + 1$. Hence, $\chi_H(\mu(K_{m,n})) = 2(m+n) + 1$.   □

Case 1
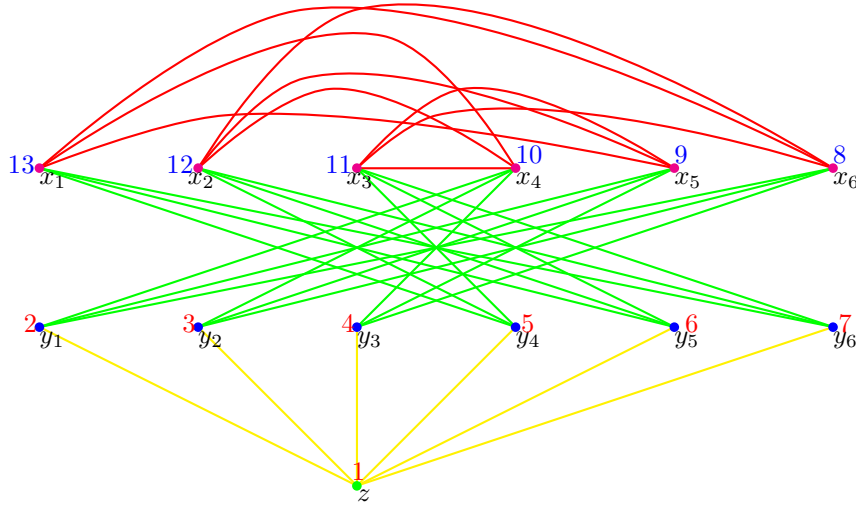
Figure 4 Mycielskian Graph of $K_{3,3}$ with $\chi_H(\mu(K_{3,3})) = 13$

Case 2
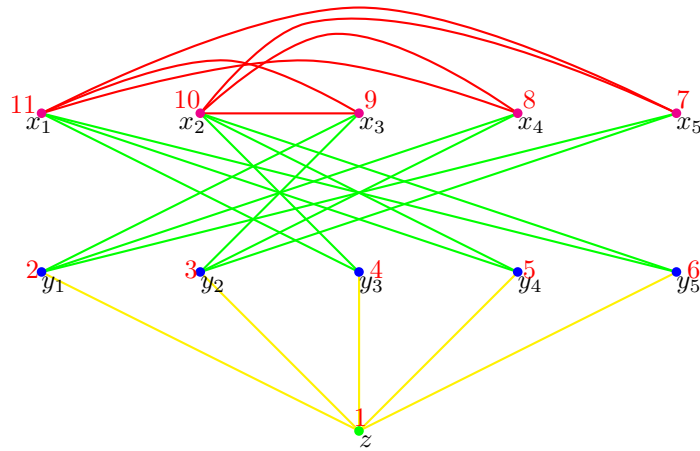
Figure 5 Mycielskian Graph of $K_{2,3}$ with $\chi_H(\mu(K_{2,3})) = 11$

§7. Main Theorem

Theorem 7.1 *Let G be any graph without pendant vertices, then*

$$\chi_H(\mu(G)) = 2(V(\mu(G))) + 1.$$

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A Topological Model for Ecologically Industrial Systems

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Abstract: An ecologically industrial system is such an industrial system in harmony with its environment, especially the natural environment. The main purpose of this paper is to show how to establish a mathematical model for such systems by combinatorics, and find its topological characteristics, which are useful in industrial ecology and the environment protection.

Key Words: Industrial system, ecology, Smarandache multi-system, combinatorial model, input-output analysis, circulating economy.

AMS(2010): 91B60

§1. Introduction

Usually, the entirely life cycle of a product consists of mining, smelting, production, storage, transporting, use and then finally go to the waste, $\cdots$, etc.. In this process, a lot of waste gas, water or solid waste are produced. Such as those shown in Fig.1 for a producing cell following.
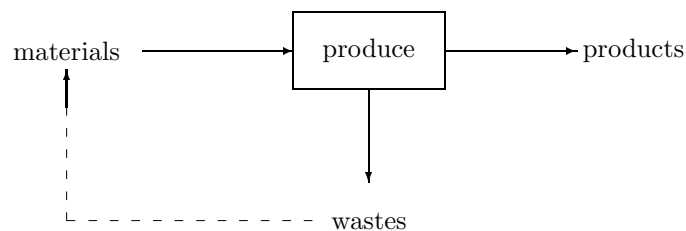

Fig.1

In old times, these wastes produced in industry are directly discarded to the nature without disposal, which brings about an serious problem to human beings, i.e., environment pollution and harmful to our survival. For minimizing the effects of these waste to our survival, the growth of industry should be in coordinated with the nature and the $3R$ *rule*: reduces its amounts, reuses it and furthermore, into recycling, i.e., use these waste into produce again after disposal, or let

---

them be the materials of other products and then reduce the total amounts of waste to our life environment. An *ecologically industrial system* is such a system consisting of industrial cells in accordance with the $3R$ rule by setting up one or more waste disposal centers. Such a system is opened. Certainly, it can be transferred to a closed one by letting the environment as an additional cell. For example, series produces such as those shown in Fig.2 following.
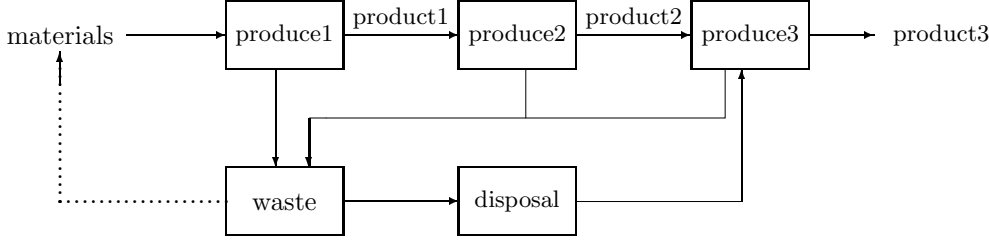
Fig.2

Generally, we can assume that there are $P_1, P_2, \cdots, P_m$ products (including by-products) and $W_1, W_2, \cdots, W_s$ wastes after a produce process. Some of them will be used, and some will be the materials of another produce process. In view of cyclic economy, such an ecologically industrial system is nothing else but a Smarandachely multi-system. Furthermore, it is a combinatorial system defined following.

Definition 1.1([1],[2] and [9]) *A rule in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ, *i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandachely system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$.

Definition 1.2([1],[2] and [9]) *For an integer* $m \geq 2$, *let* $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *be m mathematical systems different two by two. A Smarandache multi-space is a pair* $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ *with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Definition 1.3([1],[2] and [9]) *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1; \mathcal{R}_1),(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *for an integer m, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \ \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{ \ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

The main purpose of this paper is to show how to establish a mathematical model for such systems by combinatorics, and find its topological characteristics with label equations. In fact, such a system of equations is non-solvable. As we discussed in references [3]-[8], such a non-solvable system of equations has significance also for things in our world and its global behavior can be handed by its G-solutions, where G is a topological graph inherited by this non-solvable system.

§2. A Generalization of Input-Output Analysis

The 3R rule on an ecologically industrial system implies that such a system is optimal both in its economical and environmental results.

2.1 An Input-Output Model

The input-output model is a linear model in macro-economic analysis, established by a economist Leontief as follows, who won the Nobel economic prize in 1973.

Assume these are n departments $D_1, D_2, \cdots, D_n$ in a macro-economic system $\mathscr{L}$ satisfy conditions following:

(1) The total output value of department D_i is x_i. Among them, there are x_{ij} output values for the department D_j and d_i for the social demand, such as those shown in Fig.1.

(2) A unit output value of department D_j consumes t_{ij} input values coming from department D_i. Such numbers t_{ij}, $1 \leq i, j \leq n$ are called *consuming coefficients*.
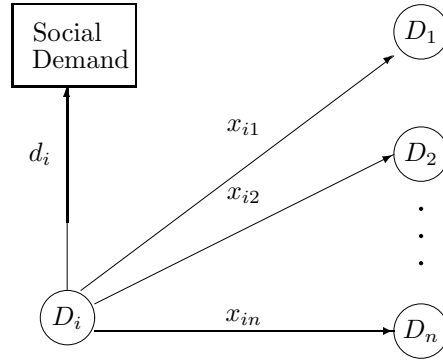

Fig.2

Therefore, such an overall balance macro-economic system $\mathscr{L}$ satisfies n linear equations

$$x_i = \sum_{j=1}^{n} x_{ij} + d_i \tag{1}$$

for integers $1 \leq i \leq n$. Furthermore, substitute $t_{ij} = x_{ij}/x_j$ into equation (10-1), we get that

$$x_i = \sum_{j=1}^{n} t_{ij}x_j + d_i \tag{2}$$

for any integer i. Let $\mathbf{T} = [t_{ij}]_{n \times n}$, $\mathbf{A} = I_{n \times n} - T$. Then

$$\mathbf{A}\overline{x} = \overline{d}, \tag{3}$$

from (2), where $\overline{x} = (x_1, x_2, \cdots, x_n)^T$, $\overline{d} = (d_1, d_2, \cdots, d_n)^T$ are the output vector or demand vectors, respectively.

For example, let $\mathscr{L}$ consists of 3 departments D_1, D_2, D_3, where D_1=agriculture, D_2= manufacture industry, D_3=service with an input-output data in Table 1.

Department	D_1	D_2	D_3	Social demand	Total value
D_1	15	20	30	35	100
D_2	30	10	45	115	200
D_3	20	60	/	70	150

Table 1

This table can be turned to a consuming coefficient table by $t_{ij} = x_{ij}/x_j$ following.

Department	D_1	D_2	D_3
D_1	0.15	0.10	0.20
D_2	0.30	0.05	0.30
D_3	0.20	0.30	0.00

Table 2

Thus

$$\mathbf{T} = \begin{bmatrix} 0.15 & 0.10 & 0.20 \\ 0.30 & 0.05 & 0.30 \\ 0.20 & 0.30 & 0.00 \end{bmatrix}, \quad \mathbf{A} = I_{3\times3} - \mathbf{T} = \begin{bmatrix} 0.85 & -0.10 & -0.20 \\ -0.30 & 0.95 & -0.30 \\ -0.20 & -0.30 & 1.00 \end{bmatrix}$$
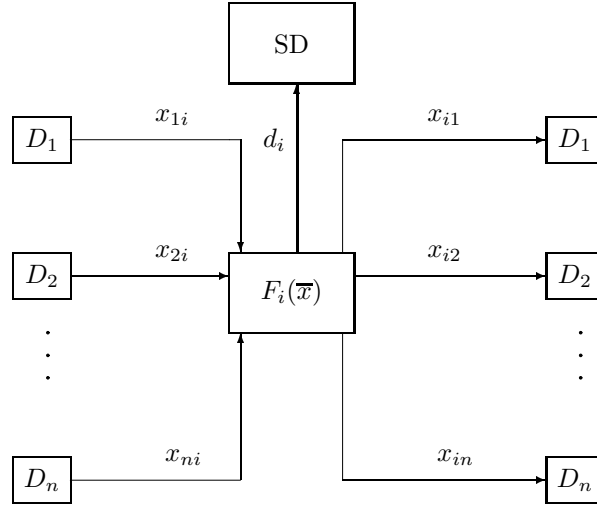
and the input-output equation system is

$$\begin{cases} 0.85x_1 - 0.10x_2 - 0.20x_3 = d_1 \\ -0.30x_1 + 0.95x_2 - 0.30x_3 = d_2 \\ -0.20x_1 - 0.30x_2 + x - 3 = d_3 \end{cases}$$

Solving this linear system of equations enables one to find the input and output data for economy management.

2.2 A Generalized Input-Output Model

Notice that our WORLD is not linear in general, i.e., the assumption $t_{ij} = x_{ij}/x_j$ does not hold in general. A non-linear input-output model is shown in Fig.3, where $\overline{x} = (x_{1i}, x_{2i}, \cdots, x_{ni})$, $D_1, D_2, \cdots, D_n$ are n departments, SD=social demand. Usually, the function $F(\overline{x})$ is called the *producing function*.

Fig.3

In this case, an overall balance input-output model is characterized by equations

$$F_i(\overline{x}) = \sum_{j=1}^{n} x_{ij} + d_i \qquad (4)$$

for integers $1 \le i \le n$, where $F_i(\overline{x})$ may be linear or non-linear and determined by a system of equations such as those of ordinary differential equations

$$1 \le i \le n \begin{cases} F_i^{(n)} + a_1 F_i^{(n-1)} + \cdots + a_{n-1} F_i + a_n = 0 \\ F_i|_{t=0} = \varphi_0, \; F_i^{(1)}\Big|_{t=0} = \varphi_1, \cdots, F_i^{(n-1)}\Big|_{t=0} = \varphi_{n-1} \end{cases} \qquad (OES^n)$$

or

$$1 \le i \le n \begin{cases} \dfrac{\partial F_i}{\partial t} = H_1(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ F_i|_{t=t_0} = \varphi_0(x_1, x_2, \cdots, x_{n-1}) \end{cases}, \qquad (PES^1)$$

which can be solved by classical mathematics. However, the input-output model with its generalized only consider the consuming and producing, neglected the waste and its affection to our environment. So it can be not immediately applied to ecologically industrial systems. However, we can generalize such a system for this objective by introducing environment factors, which are discussed in the next section.

$3. **A Topological Model for Ecologically Industrial Systems**

The essence of input-output model is in the output is equal to the input, i.e., a simple case of the *law of conservation of mass*: *a matter can be changed from one form into another, mixtures can be separated or made, and pure substances can be decomposed, but the total amount of mass remains constant.* Applying this law, it needs the environment as an additional cell for ecologically industrial systems and replaces the departments D_i, $1 \le i \le n$ by input materials M_i, $1 \le i \le n$ or products P_k, $1 \le k \le m$, and SD by W_i, $1 \le i \le s =$ wastes, such as those shown in Fig.4 following.
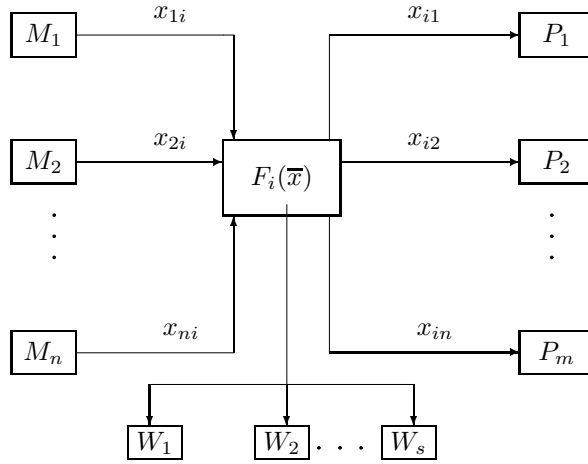
Fig.4

In this case, the balance input-output model is characterized by equations

$$F_i(\overline{x}) = \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{s} W_i \tag{5}$$

for integers $1 \le i \le n$. We construct a topological graphs following.

Construction 3.1 *Let $\mathscr{J}(t)$ be an ecologically industrial system consisting of cells $C_1(t), C_2(t),$ $\cdots, C_l(t)$, R the environment of $\mathscr{J}$. Define a topological graph $G[\mathscr{J}]$ of $\mathscr{J}$ following:*

$$
\begin{aligned}
V(G[\mathscr{J}]) \;&=\; \{C_1(t), C_2(t), \cdots, C_l(t), R\}; \\
E(G[\mathscr{J}]) \;&=\; \{(C_i(t), C_j(t)) \; if \; there \; is \; an \; input \; from \; C_i(t) \; to \; C_j(t), \; 1 \le i, j \le l\} \\
&\quad \bigcup \{(C_i(t), R) \; if \; there \; are \; wastes \; from \; C_i(t) \; to \; R, \; 1 \le i \le l\}.
\end{aligned}
$$

Clearly, $G[\mathscr{J}]$ is an inherited graph for an ecologically industrial system $\mathscr{J}$. By the 3R rule, any producing process X_{i_1} of an ecologically industrial system is on a directed cycle $\overrightarrow{C} = (X_{i_1}, X_{i_2}, \cdots, X_{i_k})$, where $X_{i_j} \in \{C_i, \; 1 \le j \le l; R\}$, such as those shown in Fig.5.
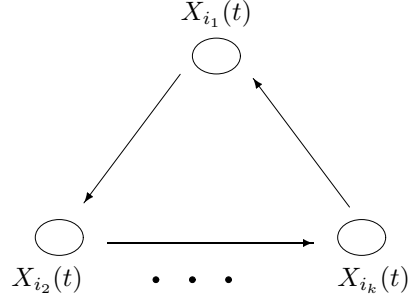
Fig.5

Such structure of cycles naturally determined the topological structure of an ecologically industrial system following.

Theorem 3.2　*Let $\mathscr{J}(t)$ be an ecologically industrial system consisting of producing cells $C_1(t), C_2(t), \cdots, C_l(t)$ underlying a graph $G\left[\mathscr{J}(t)\right]$. Then there is a cycle-decomposition*

$$G\left[\mathscr{J}(t)\right] = \bigcup_{i=1}^{t} \overrightarrow{C}_{k_i}$$

for the directed graph $G\left[\mathscr{J}(t)\right]$ such that each producing process $C_i(t)$, $1 \leq i \leq l$ is on a directed circuit $\overrightarrow{C}_{k_i}$ for an integer $1 \leq i \leq t$. Particularly, $G\left[\mathscr{J}(t)\right]$ is 2-edge connectness.

　　Proof By definition, each producing process $C_i(t)$ is on a directed cycle, which enables us to get a cycle-decomposition

$$GG\left[\mathscr{J}(t)\right] = \bigcup_{i=1}^{t} \overrightarrow{C}_{k_i}. \qquad\qquad \square$$

　　Thus, any ecologically industrial system underlying a topological 2-edge connect graph with vertices consisting of these producing process. Whence, we can always call *G-system* for an ecologically industrial system. Clearly, the global effects of G_1-system and G_2-system are different if $G_1 \not\simeq G_2$ by definition. Certainly, we can also characterize these G-systems with graphs by equations (5) following.

**Theorem** 3.3　*Let consisting of producing cells $C_1(t), C_2(t), \cdots, C_l(t)$ underlying a graph $G\left[\mathscr{J}(t)\right]$. Then*

$$F_v\left(x_{uv}, u \in N^-_{G\left[\mathscr{J}(t)\right]}(v)\right) = \sum_{w \in N^+_{G\left[\mathscr{J}(t)\right]}(v)} (-1)^{\delta(v,w)} x_{vw}$$

with $\delta(v, w) = 1$ if x_{vw}=product, and -1 if x_{vw}=waste, where $N^-_{G\left[\mathscr{J}(t)\right]}, N^+_{G\left[\mathscr{J}(t)\right]}$ are the in or our-neighborhoods of vertex v in $G\left[\mathscr{J}(t)\right]$.

　　Notice that the system of equations in Theorem 3.3 is non-solvable in $\mathbb{R}^{\Delta+1}$ with Δ the maximum valency of vertices in $G\left[\mathscr{J}(t)\right]$. However, we can also find its $G\left[\mathscr{J}(t)\right]$-solution in

$\mathbb{R}^{\Delta+1}$ (See [4]-[6] for details), which can be also applied for holding the global behavior of such G-systems. Such a $G[\mathscr{J}(t)]$-solution is not constant for $\forall e \in E(G[\mathscr{J}(t)])$. For example, let a G-system with G=circuit be shown in Fig.4.

$$C_6$$
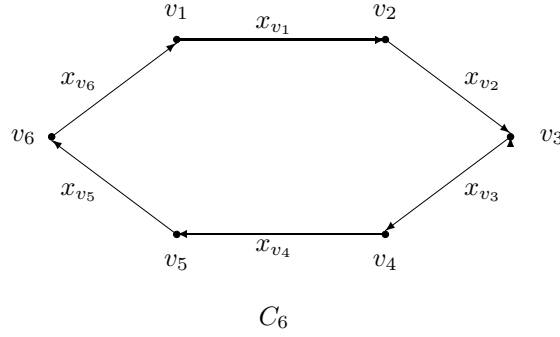
Fig.5

Then there are no wastes to environment with equations

$$F_v(x_{v_i}) = x_{v_{i+1}}, \quad 1 \le i \le 6, \quad \text{where } i \bmod 6, \text{ i.e.,}$$
$$F_{v_i} F_{v_{i+1}} \cdots F_{v_{i+6}} = 1 \text{ for any integer } 1 \le i \le 6.$$

If F_{v_i} is given, then solutions x_{v_i}, $1 \le i \le 6$ dependent on an initial value, for example, $x_{v_1}|_{t=0}$, i.e., one needs the choice criterions for determining the initial values $x_{v_i}|_{t=0}$. Notice that an industrial system should harmonizes with its environment. The only criterion for its choice must be

  optimal in economy with minimum affection to the environment, or approximately, maximum output with minimum input.

According to this criterion, there are 2 types of G-systems approximating to an ecologically industrial system:

  (1) Optimal in economy with all inputs (wastes) $W_{r_1}, W_{r_2}, \cdots, W_{r_s}$ licenced to R;

  (2) Minimal wastes to the environment, i.e., minimal used materials but supporting the survival of human beings.

  For a G-system, let

$$c_v^- = \sum_{u_1 N_{G[\mathscr{J}(t)]}^-(v)} c(x_{uv}) \text{ and } c_v^+ = \sum_{w \in N_{G[\mathscr{J}(t)]}^+(v)} (-1)^{\delta(v,w)} c(x_{vw})$$

be respectively the producing costs and product income at vertex $v \in V(G)$. Then the optimal function is

$$
\begin{aligned}
\Lambda(G) \;\; &= \;\; \sum_{v \in V(G)} \left(c_v^+ - c_v^- \right) \\[2mm]
&= \;\; \sum_{v \in V(G)} \left(\sum_{w \in N_{G[\mathscr{I}(t)]}^+(v)} (-1)^{\delta(v,w)} c(x_{vw}) - \sum_{u_1 N_{G[\mathscr{I}(t)]}^-(v)} c(x_{uv}) \right).
\end{aligned}
$$

Then, a G-system of Types 1 is a mathematical programming

$$
\max \sum_{v \in V(G)} \Lambda(G) \;\; \text{but} \;\; \sum_{v \in V(G)} W_{ri} \le W_{ri}^U,
$$

where W_{ri}^U is the permitted value for waste W_{ri} to the nature for integers $1 \le i \le s$. Similarly, a G-system of Types 2 is a mathematical programming

$$
\min \sum_{v \in V(G)} W_{ri} \;\; \text{but} \;\; \text{all} \;\; \text{prodcuts} \;\; \mathrm{P} \ge \mathrm{P}^{\mathrm{L}},
$$

where P^L is the minimum needs of product P in an area or a country. Particularly, if $W_{ri}^U = 0$, i.e., an ecologically industrial system, such a system can be also characterized by a non-solvable system of equations

$$
F_v(x_{uv}, u \in N_{G[\mathscr{I}(t)]}^-(v)) = \sum_{w \in N_{G[\mathscr{I}(t)]}^+(v)} x_{vw} \;\; \text{for} \;\; \forall v \in V(G).
$$

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I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

Author Information

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Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

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Research papers

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