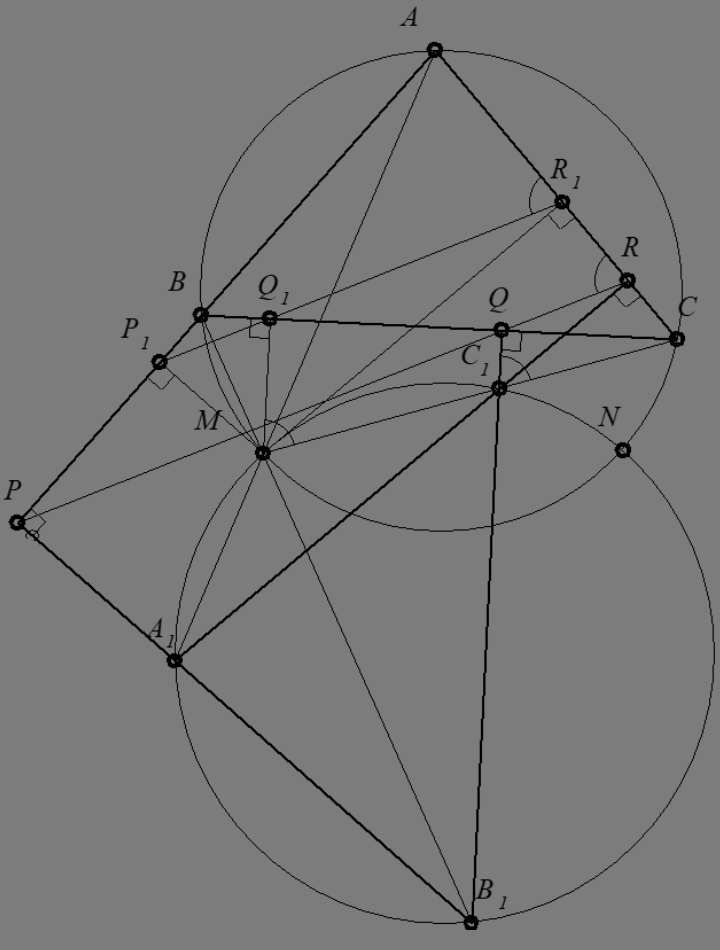
Ion Pătrașcu

Florentin Smarandache

THE GEOMETRY OF THE ORTHOLOGICAL TRIANGLES



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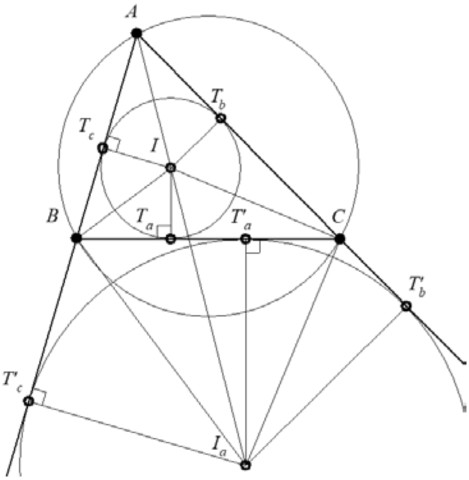

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Ion Pătrașcu
Florentin Smarandache

THE GEOMETRY OF THE ORTHOLOGICAL TRIANGLES

Pons Editions
Brussels, 2020

To my grandchildren LUCAS and EVA-MARIE,
with all my love – ION PĂTRAȘCU

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FOREWORD

Plants and trees grow perpendicular to the plane tangent to the soil surface, at the point of penetration into the soil; in vacuum, the bodies fall perpendicular to the surface of the Earth - in both cases, if the surface is horizontal. Starting from the property of two triangles to be orthological, the two authors have designed this work that seeks to provide an integrative image of elementary geometry by the prism of this "filter".

Basically, the property of orthology is the skeleton of the present work, which establishes many connections of some theorems and geometric properties with it.

The book "The Geometry of The Orthological Triangles" is divided into ten chapters. In the first seven, the topic is introduced and developed by connecting it with other beautiful properties of geometry, such as the homology of triangles. Chapter 8 includes three annexes intended to clarify to the readers some results used in the rest of the chapters. Chapter 9 is a collection of problems where orthological triangles usually appear; it is especially intended for students preparing for participation in different mathematics competitions. The last chapter contains solutions and answers to the problems in chapter 9.

The work ends with a rich bibliography that has been consulted and used by the authors. It is noteworthy that this book highlights the contributions of Romanian mathematicians Traian Lalescu, Gheorghe Ţițeica, Cezar Coşniţa, Alexandru Pantazi et al. to this treasure that constitutes THE GEOMETRY!

We congratulate the authors for the beauty and depth of the chosen topic, which can be explained by the passion of the distinguished teachers, complementary in the complex world of integral culture in the sense of A. Huxley - C. P. Snow:

– The geometer Ion Patrașcu - the classical teacher attracted by exciting topics such as orthogonality - an instrument of duality and lighthouse in knowledge / progress in Mathematics;

– The scientist Florentin Smarandache - renowned and unpredictable innovator in the Philosophy of Science, from Fundamentals to his well-known work "Hadron Models and related New Energy issues" (2007), also targeted in our pop-symphonic composition "LHC - Large Hadron Collider", uploaded on YouTube.

The orthogonality is a universal geometric property, because it represents the local quintessence of the system {point M, geodesic g} in any Riemannn-dimensional space, in particular in an Euclidean space, for $n =$ at least 2. I propose to the readers to try an "orthological reading" on the classical sphere, starting from the system $\{M, g\}$ and advancing, through similarity, towards geodesic triangles. The topic can be taken further, in post-university / doctoral studies, in the Riemannian context, under the empire of the incredible "isometric diving theorems" of brilliant John Forbes Nash, Jr. (MAGIC mechanism - Great ATTENTION! - of "<reverse> Nash teleportation": from the n-Dim Euclidean world to the Riemannian k-Dim world, where k is a polynomial of degree -I think- 2 or 3 inn - *see Professor web*).

Prof. Dr. **Valentin Boju**
Member of the "Cultural Merit" Order of Romania
Officer rank, Category H, Scientific Research

AUTHORS' NOTE

The idea of this book came up when writing our previous book, *The Geometry of Homological Triangles* (2012).

We try to graft on the central theme - the orthology of triangles-, many results from the elementary geometry. In particular, we approach the connection between the orthological and homological triangles; also, we review the "S" triangles, highlighted for the first time by the great Romanian mathematician Traian Lalescu.

The book is addressed to both those who have studied and love geometry, as well as to those who discover it now, through study and training, in order to obtain special results in school competitions. In this regard, we have sought to prove some properties and theorems in several ways: synthetic, vectorial, analytical.

Basically, the book somewhat resembles a quality police novel, in which the pursuits are the orthological triangles, and, in their search, GEOMETRY is actually discovered.

We adress many thanks to the distinguished professor **Mihai Miculiţa** from Oradea for drawing the geometrical figures in this book and contributed with interesting observations and mentions that enriched our work.

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1

INTRODUCTION

1.1 Orthological triangles: definition

Let ABC be a scalene triangle and P – a point in its plane. We build from P the lines a_1, b_1, c_1, respectively perpendicular to BC, CA and AB. On these lines, we consider the points A_1, B_1, C_1, such that they are not collinear (see *Figure 1*). About the triangle $A_1B_1C_1$ we say that it is an orthological triangle in relation to the triangle ABC, and about the point P we say that it is the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC.

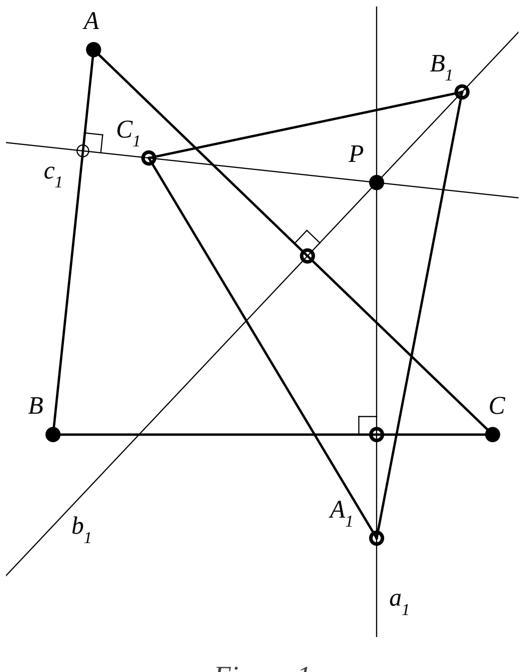

Figure 1

Definition 1

We say that the triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC if the perpendiculars taken from A_1, B_1, C_1 respectively to BC, CA and AB are concurrent.
About the concurrency point of the previously mentioned perpendiculars, we say that it is the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC.

Observation 1

The presented construction leads to the conclusion that, being given a triangle ABC and a point P in its plane, we can build an infinity of triangles $A_nB_nC_n$, such that $A_nB_nC_n$ to be orthological with ABC, $n \in \mathbb{N}$.

Exercise 1

Being given a triangle ABC, build the triangle $A_1B_1C_1$ to be orthological in relation to ABC, such that its orthology center to be the vertex A.

1.2. Characterization of the orthology relation

A triangle ABC can be considered orthological in relation to itself.

Indeed, the perpendiculars taken from A, B, C respectively to BC, CA, AB are also the altitudes of the triangle, therefore they are concurrent lines.

The orthology center is the orthocenter H of the triangle.

We can say that the orthology relation is reflexive in the set of triangles.

We establish in the following the conditions that are necessary and sufficient for two triangles to be in an orthology relation.

The following theorem plays an important role in this approach:

Theorem 1 (L. Carnot, 1803)

If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of a given triangle ABC, the perpendiculars raised at these points to BC, CA respectively AB are concurrent if and only if the following relation takes place:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0. \qquad (1)$$

Proof

We consider that the perpendiculars raised in A_1, B_1, C_1, to BC, CA respectively AB, are concurrent in a point P (see *Figure 2*).

From Pythagoras' Theorem applied in the triangles PA_1B, PA_1C, we have $PB^2 = PA_1{}^2 + A_1B^2$ and $PC^2 = PA_1{}^2 + A_1C^2$.

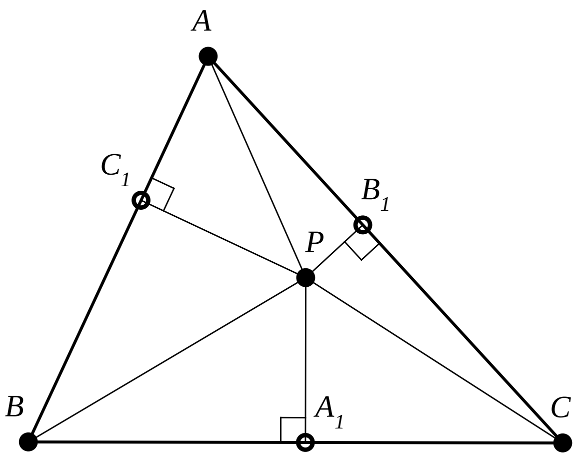

Figure 2

By subtraction member by member, we find:

$PB^2 - PC^2 = A_1B^2 - A_1C^2$. (2)

Similarly, we find the relations:

$PC^2 - PA^2 = B_1C^2 - B_1A^2$, (3)

$PA^2 - PB^2 = C_1A^2 - C_1B^2$. (4)

By addition member by member of the relations (2), (3) and (4), we get the relation (1).

Reciprocally

Let us assume that the relation (1) is true and let us prove the concurrency of the perpendiculars raised in A_1, B_1, C_1, respectively to BC, CA and AB. Let P be the intersection of the perpendicular in A_1 to BC with the perpendicular in B_1 to CA; we denote by C'_1 the projection of P to AB. According to those previously demonstrated, we have the relation:

$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C'_1A^2 - C'_1B^2 = 0$ (5)

This relation and the relation (1) implies that:

$C_1A^2 - C_1B^2 = C'_1A^2 - C'_1B^2$ (6)

We prove that this relation is true if and only if $C_1 = C'_1$.

Indeed, let us suppose that $C_1 \in (AB)$ and $B \in (AC'_1)$ (see *Figure 3*) and that the relation (6) takes place here.

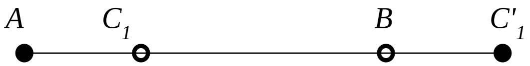

Figure 3

We obtain that:

$(C_1A - C_1B)(C_1A + C_1B) = (C'_1A - C'_1B)(C'_1A + C'_1B)$. (7)

Because $C_1A + C_1B = AB$ and $C'_1A + C'_1B = AB$, from (7) we have that $C_1A - C_1B = C'_1A - C'_1B$ is absurd.

Similarly, we find that the the relation (6) cannot be satisfied in the hypothesis C_1, C'_1 separated by the points A or B. If, for example, $C_1, C'_1 \in (AB)$, then relation (7) leads to $C_1A = C'_1A$, which implies $C_1 = C'_1$ and the implication of the theorem is proved.

Observation 2

a) The points A_1, B_1, C_1 from the theorem's hypothesis can be collinear.

b) If the points A_1, B_1, C_1 from Carnot's theorem statement are noncollinear, then relation (1) expresses a necessary and sufficient condition that the triangle $A_1B_1C_1$ to be orthological in relation to the triangle ABC.

c) From the proof of the Carnot's theorem, the following lemma was inferred:

Lemma 1

The geometric place of the points M in plane with the property $MA^2 - MB^2 = k$, where A and B are two given fixed points and k – a real constant, is a line perpendicular to AB.

d) Carnot's theorem can be used to prove the concurrency of perpendiculars raised on the sides of a triangle.

Exercise 2

Prove with the help of Carnot's theorem that:

a) The mediators of a triangle are concurrent;

b) The altitudes of a triangle are concurrent.

Theorem 2

Let ABC and $A_1B_1C_1$ be two triangles in plane. $A_1B_1C_1$ is orthological in relation to the triangle ABC if and only if the following relation is true:

$$A_1B^2 + B_1C^2 + C_1A^2 = A_1C^2 + B_1A^2 + C_1B^2 \quad (7)$$

Proof

We consider that $A_1B_1C_1$ is orthological in relation to ABC; we denote by P the orthology point; let $\{A'\} = BC \cap PA_1$, $\{B'\} = AC \cap B_1P$, $\{C'\} = AB \cap PC_1$ (see *Figure 4*); we prove the relation (7).

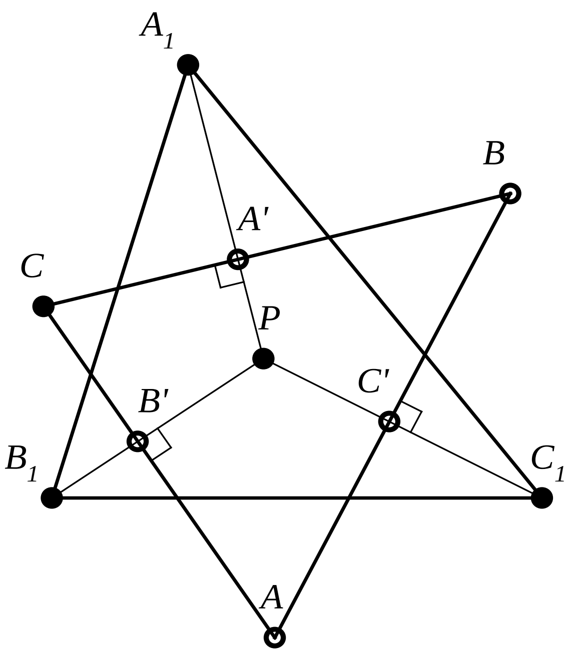

Figure 4

According to *Theorem 1*, we have:

$$A'B^2 - A'C^2 + B'C^2 - B'A^2 + C'A^2 - C'B^2 = 0 \quad (8)$$

But: $A'B^2 - A'C^2 = A_1B^2 - A_1C^2$ (Pythagoras' theorem).

Similarly:

$B'C^2 - B'A^2 = B_1C^2 - B_1A^2$ and $C'A^2 - C'B^2 = C_1A^2 - C_1B^2$.

By adding member by member the last three relations, and taking into account (8), we deduce the relation (7).

Reciprocally

Let us consider the triangles A_1, B_1, C_1 and ABC such that the relation (7) is satisfied; we prove that A_1, B_1, C_1 is orthological in relation to ABC.

Pythagoras' theorem and the relation (7) lead to:

$A'C^2 - A'B^2 + B_1C^2 - B_1A^2 + C'A^2 - C'B^2 = 0$.

According to Theorem 1, the perpendiculars in A', B', C' to BC, CA, AB (which pass respectively through A_1, B_1, C_1) are concurrent, therefore the triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC.

Another way to determine the orthology of two triangles is given by:

Theorem 3

The triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC if and only if the following relation takes place for any point M in their plane: $\overrightarrow{MA_1} \cdot \overrightarrow{BC} + \overrightarrow{MB_1} \cdot \overrightarrow{CA} + \overrightarrow{MC_1} \cdot \overrightarrow{AB} = 0$. (9)

Proof

We denote: $E(M) = \overrightarrow{MA_1} \cdot \overrightarrow{BC} + \overrightarrow{MB_1} \cdot \overrightarrow{CA} + \overrightarrow{MC_1} \cdot \overrightarrow{AB}$, and we prove that $E(M)$ has this value whatever M.

Let $E(N) = \overrightarrow{NA_1} \cdot \overrightarrow{BC} + \overrightarrow{NB_1} \cdot \overrightarrow{CA} + \overrightarrow{NC_1} \cdot \overrightarrow{AB}$, where N is a point in the plane, different from M.

We have:

$E(M) - E(N) = \left(\overrightarrow{MA_1} - \overrightarrow{NA_1}\right) \cdot \overrightarrow{BC} +$

$+\left(\overrightarrow{MB_1} - \overrightarrow{NB_1}\right) \cdot \overrightarrow{CA} + (\overrightarrow{MC_1} - \overrightarrow{NC_1}) \cdot \overrightarrow{AB}$.

Therefore $E(M) - E(N) = \overrightarrow{MN} \cdot (\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB})$.

Since $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$, it follows that $E(M) - E(N) = \overrightarrow{MN} \cdot \vec{0} = 0$.

We proved that, if the relation (9) is true for a point in the plane, then it is true for any other point in the plane. Let us consider now that the triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC, and that P is the orthology center. Obviously, $\overrightarrow{PA_1} \cdot \overrightarrow{BC} = \overrightarrow{PB_1} \cdot \overrightarrow{CA} = \overrightarrow{PC_1} \cdot \overrightarrow{AB} = 0$ and hence the relation (9) is true for the point P, consequently it is also true for any other point M from plane.

Reciprocally

If the relation (9) is satisfied, let us prove that the triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC. Let us denote by M the intersection of the given perpendicular from A_1 to BC with the perpendicular taken from B_1 to CA. The relation (9) becomes in this case: $\overrightarrow{MC_1} \cdot \overrightarrow{AB} = 0$, which shows that MC_1 is perpendicular to AB and consequently the triangle $A_1B_1C_1$ is orthological in relation to ABC, the point M being the orthology center.

Observation 3

From *Theorem 3*, we note that, in order to prove that a triangle $A_1B_1C_1$ is orthological in relation to another triangle ABC, it is sufficient to show that there exists a point M in their plane such that the relation (9) to be satisfied.

1.3. The theorem of orthological triangles

We noticed that the orthology relation, in the set of triangles in plane, is *reflexive*.

The following theorem shows that the orthology relation is *symmetrical*.

Theorem 4 (J. Steiner, 1828 – Theorem of orthological triangles)

If the triangle $A_1B_1C_1$ is orthological in relation to the triangle ABC, then the triangle ABC is orthological as well in relation to $A_1B_1C_1$.

Proof 1

It is based on Theorem 2. The relation (7), being symmetrical, we can write it like this:

$$AB_1^2 + BC_1^2 + CA_1^2 = AC_1^2 + BA_1^2 + CB_1^2 \qquad (10)$$

From Theorem 2, it follows that the triangle ABC is orthological in relation to the triangle $A_1B_1C_1$.

Proof 2

We use Theorem 3. Let the triangle $A_1B_1C_1$ be orthological in relation to the triangle ABC; then the relation (9) takes place, and we consider here $M = A_1$; it follows that: $\overrightarrow{A_1C_1} \cdot \overrightarrow{CA} + \overrightarrow{C_1A_1} + \overrightarrow{AC} \cdot \overrightarrow{A_1B_1} = 0$; now see relation (9), where $M = A$, which shows that the triangle ABC is orthological in relation to the triangle $A_1B_1C_1$.

Proof 3

Let P be the orthology center of the triangle $A_1B_1C_1$, in relation to the triangle ABC, and P_1 be the point of intersection of perpendiculars taken from A and B respectively to B_1C_1 and C_1A_1 (see *Figure 5*).

We denote: $\overrightarrow{PA_1} = \overrightarrow{a_1}$, $\overrightarrow{PB_1} = \overrightarrow{b_1}$, $\overrightarrow{PC_1} = \overrightarrow{c_1}$; $\overrightarrow{P_1A} = \vec{a}$, $\overrightarrow{P_1B} = \vec{b}$, $\overrightarrow{P_1C} = \vec{c}$.

From the hypothesis, we deduce that $\overrightarrow{a_1} \cdot \left(\vec{b} - \vec{c}\right) = 0$, $\overrightarrow{b_1} \cdot (\vec{c} - \vec{a}) = 0$ and $\overrightarrow{a_1} \cdot \left(\overrightarrow{b_1} - \vec{c}_1\right) = \vec{b} \cdot (\vec{c}_1 - \vec{a}_1) = 0$.

Using the obvious identity: $\overrightarrow{a_1} \cdot \left(\vec{b} - \vec{c}\right) + \overrightarrow{b_1} \cdot (\vec{c} - \vec{a}) + \overrightarrow{c_1} \cdot \left(\vec{a} - \vec{b}\right) = \vec{a} \cdot \left(\overrightarrow{b_1} - \vec{c}_1\right) + \vec{b} \cdot (\vec{c}_1 - \vec{a}_1) + \vec{c} \cdot (\vec{a}_1 - \vec{c}_1)$, we deduce that $\vec{c} \cdot (\vec{a}_1 - \vec{c}_1) = 0$, ie. $P_1C \perp AB$, which shows that the triangle ABC is orthological in relation to $A_1B_1C_1$, the orthology center being the point P_1.

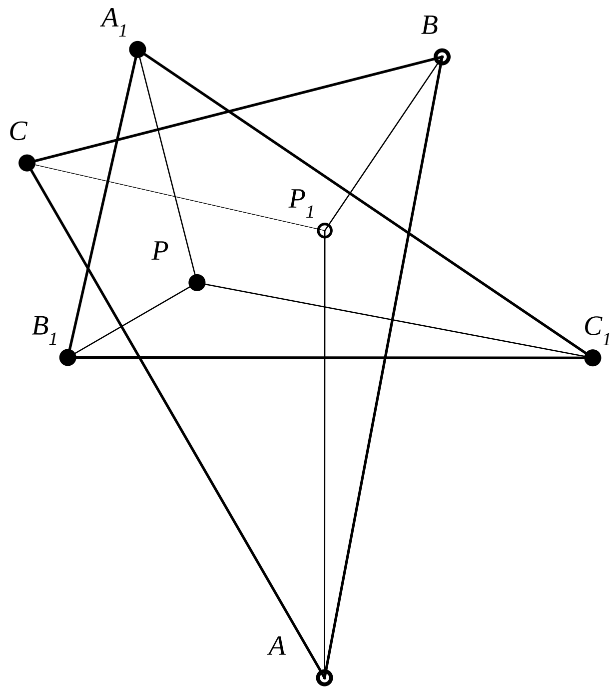

Figure 5

Proof 4

We denote by $A'B'C'$ the pedal triangle of orthology center P of the triangle $A_1B_1C_1$ in relation to the triangle ABC (see *Figure 6*). Also, we denote by A'', B'', C'' the intersections with BC, CA, AB of perpendiculars taken from A, B, C respectively to B_1C_1, C_1A_1 and $A'B'$.

We have: $\Delta A''AB \sim \Delta A'C_1P$ (because $\sphericalangle A''AB \equiv \sphericalangle A'C_1P$ and $\widehat{ABA''} \equiv \widehat{C_1PA'}$ as angles with the sides respectively perpendicular). It follows that:

$$\frac{A''A}{A'C_1} = \frac{A''B}{A'P}. \quad (1)$$

$\Delta A''AC \sim \Delta A'B_1P$ (because $\sphericalangle A''AC \equiv \sphericalangle A'B_1P$ and $\widehat{ACA''} \equiv \widehat{B_1PA'}$ as angles with sides respectively perpendicular). It follows that:

$$\frac{A''A}{A'B_1} = \frac{A''C}{A'P}. \quad (2)$$

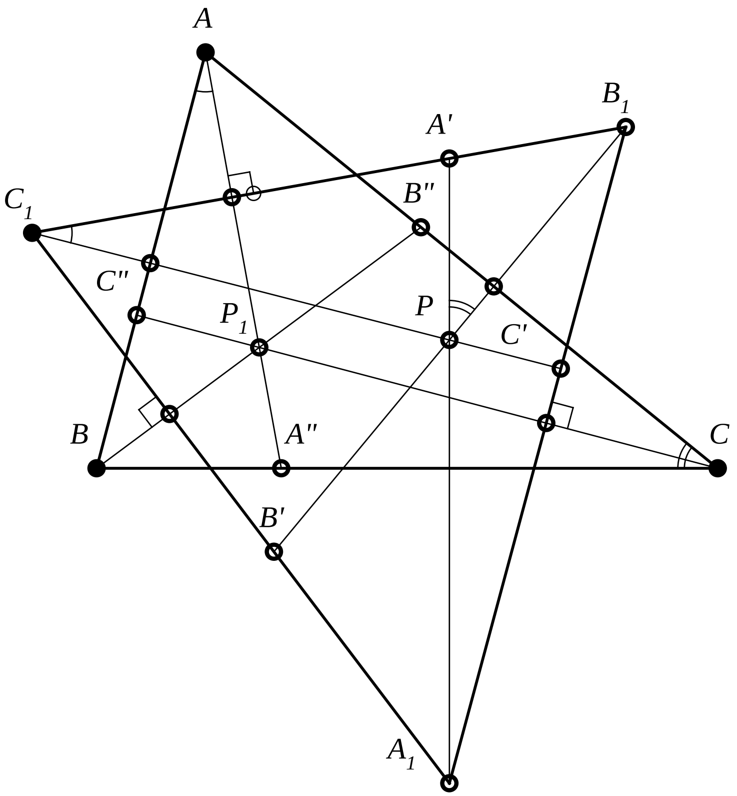

Figure 6

From relations (1) and (2), we obtain that:

$\frac{A'B_1}{A'C_1} = \frac{A''B}{A''C}$ (3).

Similarly, we obtain that:

$\frac{B'C_1}{B'A_1} = \frac{B''C}{B''A}$, (4)

$\frac{C'A_1}{C'B_1} = \frac{C''A}{C''B}$. (5)

Because A_1A', B_1B', C_1C' are concurrent in P, we obtain from Ceva's theorem that:

$\frac{A'B_1}{A'C_1} \cdot \frac{B'C_1}{B'A_1} \cdot \frac{C'A_1}{C'B_1} = 1.$ (6)

The relations (3), (4), (5), (6) and Ceva's theorem show that the cevians AA'', BB'', CC'' are concurrent, hence the triangle ABC is orthological in relation to $A_1B_1C_1$.

Proof 5

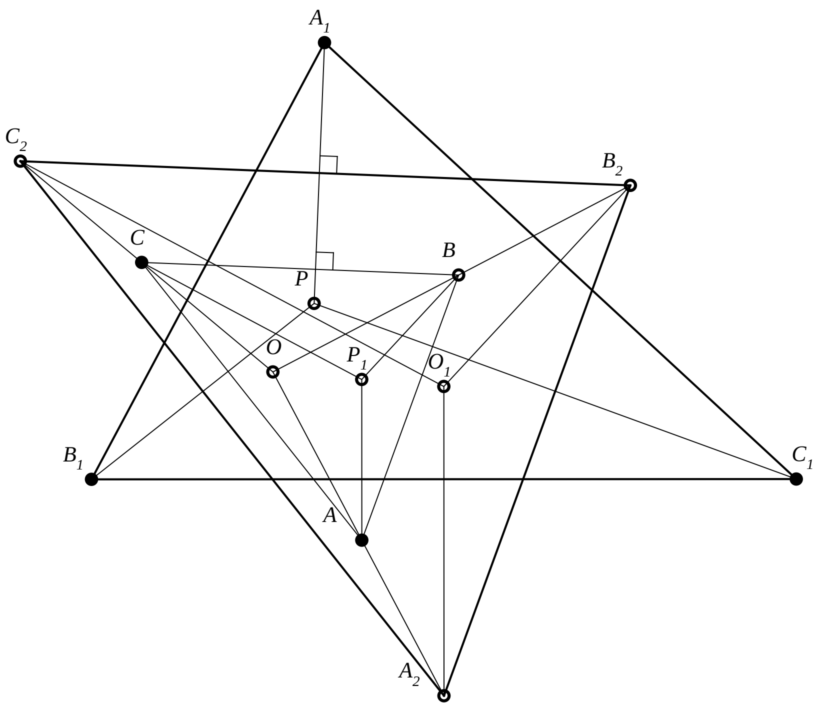

Figure 7

Let P be the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC. We denote by A_2, B_2 and C_2 the centers of the circles circumscribed to the triangles B_1PC_1, C_1PA_1, A_1PB_1 (see *Figure 7*).

The lines of the centers B_2C_2; C_2A_2; A_2B_2 being the mediators of the segments PA_1, PB_1, respectively PC_1, are parallel with the sides of the triangle ABC. The triangles $A_2B_2C_2$ and ABC are homothetic (having parallel sides), and the homothety center was denoted by O.

Perpendiculars from A_2, B_2 and C_2 on the sides of the triangle $A_1B_1C_1$ are its mediators, and hence they are concurrent in the center of the circle circumscribed to the triangle $A_1B_1C_1$, which we denote by O_1.

Because the triangles $A_2B_2C_2$ and ABC are homothetic, it follows that the perpendiculars taken from A; B, C to B_1C_1, C_1A_1, respectively A_1B_1, will also be concurrent (they are parallels with A_2O_1, B_2O_1 and C_2O_1) in a point P_1, which shows that the triangle ABC is orthological in relation to the triangle $A_1B_1C_1$.

Remark 1

The theorem of orthological triangles shows that, in the set of triangles in the plane, the relation of orthology is symmetrical.

Remark 2

Saying that the triangles $A_1B_1C_1$ and ABC are orthological, it is obvious that we must assent that the order of the vertices of the two triangles was put in agreement.

Proof 6 (analytical)

We consider the triangles $A_1B_1C_1$ and ABC such that: $A_1(a_1, a_2)$, $B_1(b_1, b_2)$, $C_1(c_1, c_2)$, $A(0, a)$, $B(b, 0)$, $C(c, 0)$ (see *Figure 8*).

The equations of the sides BC, AB, AC are:

BC: $y = 0$,

AB: $\frac{x}{b} + \frac{y}{a} - 1 = 0$,

AC: $\frac{x}{c} + \frac{y}{a} - 1 = 0$.

The perpendiculars taken from A_1, B_1, C_1 to BC, CA, respectively AB have the equations:

$$x - a_1 = 0, y - b_2 = \frac{c}{a}(x - b_1), y - c_2 = \frac{b}{a}(x - c_1).$$

The fact that these perpendiculars are concurrent in a point P (see *Figure 8*) is expressed by the condition:

$$\begin{vmatrix} 1 & 0 & -a_1 \\ c & -a & ab_2 - cb_1 \\ b & -a & ac_2 - bc_1 \end{vmatrix} = 0 \qquad (11)$$

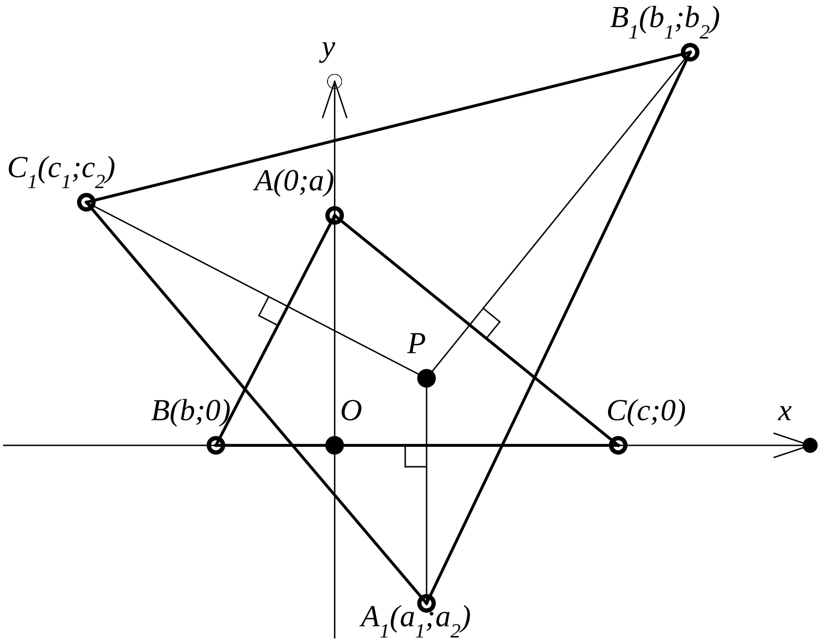

Figure 8

This condition can be written equivalently:

$$a(b_2 - c_2) + b(c_1 - a_1) + c(a_1 - b_1) = 0 \qquad (12)$$

The equations of the sides of the triangle $A_1B_1C_1$ are:

B_1C_1: $\begin{vmatrix} x & y & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$ or

$(b_2 - c_2)x - (b_1 - c_1)y + b_1c_2 - b_2c_1 = 0$;

C_1A_1: $\begin{vmatrix} x & y & 1 \\ a_1 & a_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$ or

$(a_2 - c_2)x - (a_1 - c_1)y + a_1c_2 - a_2c_1 = 0$;

A_1B_1: $\begin{vmatrix} x & y & 1 \\ a & a_2 & 1 \\ b_1 & b_2 & 1 \end{vmatrix} = 0$ or

$(a_2 - b_2)x - (a_1 - b_2)y + a_1b_2 - a_2b_1 = 0$.

The slopes of these lines are:

$m_{B_1C_1} = \frac{b_2-c_2}{b_1-c_1}, m_{C_1A_1} = \frac{a_2-c_2}{a_1-c_1}, m_{A_1B_1} = \frac{a_2-b_2}{a_1-b_2}.$

The perpendiculars taken from A, B, C respectively to B_1C_1, C_1A_1 and A_1B_1 have the equations:

$y - a = \frac{b_1-c_1}{b_2-c_2}x,$

$y = -\frac{a_1-c_1}{a_2-c_2}(x - b),$

$y = -\frac{a_1-b_1}{a_2-c_2}(x - c).$

The concurrency of these lines is expressed by the condition:

$$\begin{vmatrix} b_1 - c_1 & b_2 - c_2 & -a(b_2 - c_2) \\ a_1 - c_1 & a_2 - c_2 & -b(a_1 - c_1) \\ a_1 - b_1 & a_2 - b_2 & -c(a_1 - b_1) \end{vmatrix} = 0 \qquad (13)$$

In this determinant, if from line 1 we subtract line 2 and add line 3 ($L_1 \to L_1 - L_2 + L_3$), in the obtained determinant, taking into account condition (12), we find that the first line is null, therefore the determinant is null, and condition (13) is satisfied.

Definition 2

If two orthological triangles have different orthology centers, we will say that the line determined by them is the orthology axis of the triangles.

Problem 1

Show that the orthology relation in the set of triangles in the plane is not a transitive relation.

Problem 2

Let ABC be a triangle, P – a point in its interior, and O_A, O_B, O_c – the centers of the circles circumscribed to the triangles PBC, PCA respectively PAB. Prove that the triangles ABC and $O_AO_BO_C$ are orthological. Specify the orthology axis.

2

ORTHOLOGICAL REMARKABLE TRIANGLES

2.1 A triangle and its complementary triangle

Definition 3

It is called complementary triangle or median triangle of a given triangle – the triangle determined by the sides of that triangle.

Proposition 1

A given triangle and its complementary triangle are orthological triangles.

The orthology centers are respectively the orthocenter and the center of the circle circumscribed to the given triangle.

Proof

We denote by $A_1B_1C_1$ the complementary triangle of the given triangle ABC (see *Figure 9*).

Because B_1C_1 is a median line in the triangle ABC, the perpendicular from A to B_1C_1 is also the altitude from A of the triangle ABC; similarly, the perpendiculars from B and C to C_1A_1, respectively A_1B_1 are altitudes.

The altitudes of the triangle ABC, being concurrent in H the orthocenter of the triangle, it follows that ABC is orthological in relation to $A_1B_1C_1$.

The perpendiculars from A_1, B_1, C_1 to BC, CA, AB are the mediators of the triangle ABC, consequently O – the center of the circle circumscribed to the triangle ABC, is the orthology center of the complementary triangle in relation to the triangle ABC.

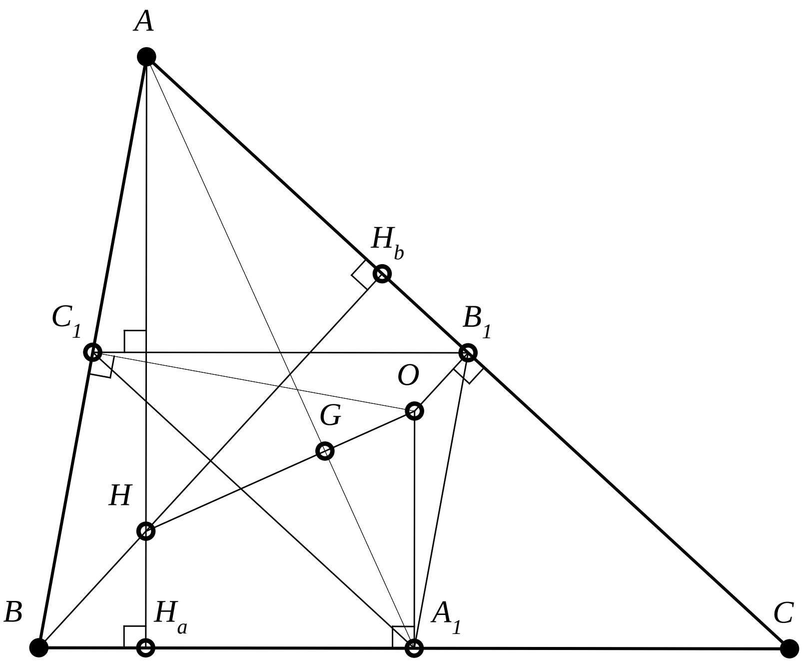

Figure 9

Observation 4

a) The center of the circle circumscribed to the triangle ABC, O, is the orthocenter of the complementary triangle $A_1B_1C_1$.

b) The triangle ABC and its complementary triangle are homothetic by homothety $h\left(G, -\frac{1}{2}\right)$. The homothety center is G – the gravity center of the triangle ABC.

c) The points H, G, O are collinear, and their line is called Euler line.

Problem 3

Let ABC be a given triangle, $A_1B_1C_1$ its complementary triangle and $A_2B_2C_2$ the complementary triangle of the triangle $A_1B_1C_1$. Prove that the triangle ABC and the triangle $A_2B_2C_2$ are orthological. Determine the orthology centers.

2.2 A triangle and its anti-complementary triangle

Definition 4

It is called anticomplementary triangle of a given triangle the triangle determined by the parallels taken through the vertices of the triangle at its opposite sides.

Proposition 2

A given triangle and its anticomplementary triangle are orthological triangles. The orthology centers are the center of the circumscribed circle and the orthocenter of the anticomplementary triangle.

Proof

The proof of Proposition 2 is immediate if we take into account the fact that, for the anticomplementary triangle $A_1B_1C_1$ of the given triangle ABC (see *Figure 10*), the latter is a complementary triangle.

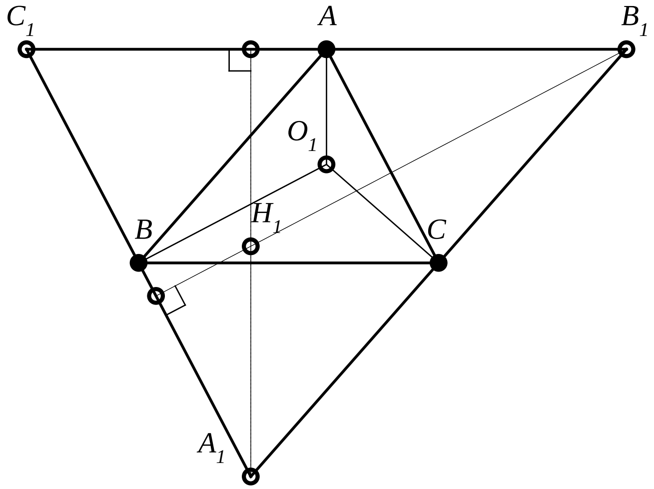

Figure 10

We are thus in the conditions of Proposition 1. However, we notice that the perpendiculars raised in A, B, C, respectively to B_1C_1, C_1A_1 and A_1B_1, are the mediators of the triangle $A_1B_1C_1$, and the perpendiculars taken from A_1, B_1, C_1 to BC, CA and AB are altitudes in the anticomplementary triangle.

Observation 5

a) The center of the circumscribed circle of the anticomplementary triangle of a triangle is the orthocenter of the given triangle.
b) The anticomplementary and complementary triangles of a given triangle are homothetic by homothety of center in the gravity center of the given triangle and ratio 4:1.
c) The anticomplementary and complementary triangles of a given triangle are orthological triangles. The orthology centers are the orthocenter and the center of the circle circumscribed to the given triangle.

2.3 A triangle and its orthic triangle

Definition 5

It is called orthic triangle of a given (non-right) triangle – the triangle created by the feet of the altitudes of the given triangle.

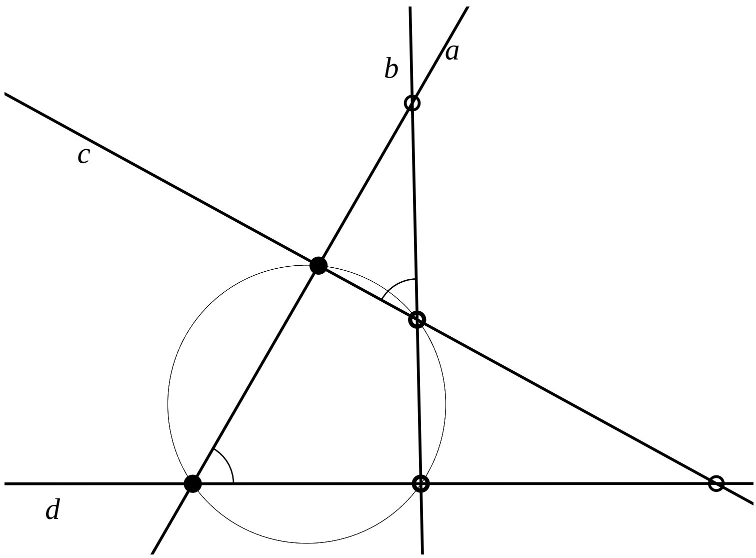

Figure 11

Observation 6

For a right triangle, the notion of orthic triangle is not defined.

Definition 6

We say that the lines c, d are antiparallels in relation to the concurrent lines a and b if the angle created by the lines a and d is congruent with the angle created by the lines b and c.

In *Figure 11*, the lines (c, d) are antiparallels in relation to (a, b); $\sphericalangle(a, d) \equiv \sphericalangle(b, c)$.

Observation 7

a) The lines (c, d), antiparallel with (a, b) form an inscribable quadrilateral with them.
b) If c, d are antiparallels in relation to a, b, then the lines a, b are as well antiparallels in relation with the lines c, d.

Proposition 3

If the lines c, d are antiparallels in relation to the lines a, b and the line c' is parallel with c, then the lines c', d are antiparallels in relation to a, b.

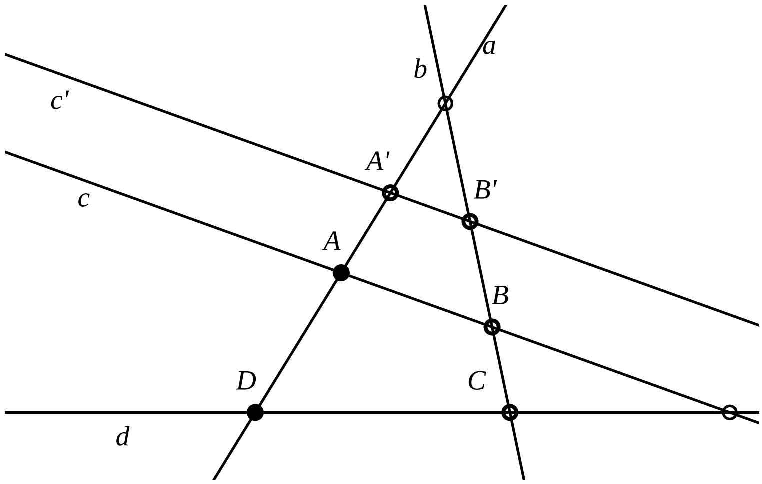

Figure 12

Proof

In *Figure 12*, we denote by A, B, C, D the vertices of the inscribable quadrilateral determined by the antiparallels c, d in relation to the lines a, b.

We denote by A' and B' – the intersections of the parallel c' with a respectively b, and we observe that the quadrilateral $A'B'CD$ is inscribable; consequently, c', d are antiparallels in relation to a, b.

Remark 3

If ABC is a non-isosceles triangle ($AB \neq AC$) and the line a intersects AB respectively AC in A' and B' such that $\sphericalangle AB'A' \equiv \sphericalangle ABC$, we say about a and BC that they are antiparallels or that a is an antiparallel to BC (see *Figure 13*).

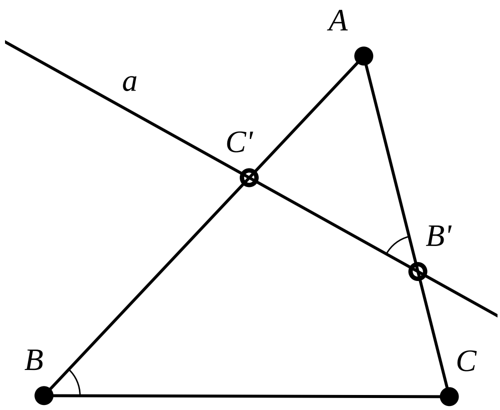

Figure 13

Proposition 4

The orthic triangle of a given non-isosceles triangle has the sides respectively antiparallels with the sides of this triangle.

Proof

In *Figure 14*, we consider ABC an obtuse triangle and $A_1B_1C_1$ its orthic triangle. Because $\sphericalangle BB_1C = \sphericalangle BC_1C = 90^0$, it follows that the quadrilateral BCB_1C_1 is inscribable, and consequently, B_1C_1 is antiparallel with BC. Similarly, it is shown that A_1C_1 and A_1B_1 are antiparallels with AC and respectively AB.

Proposition 5

In a (non-isosceles) triangle, the tangent at a vertex to the circumscribed circle is antiparallel with the opposite side.

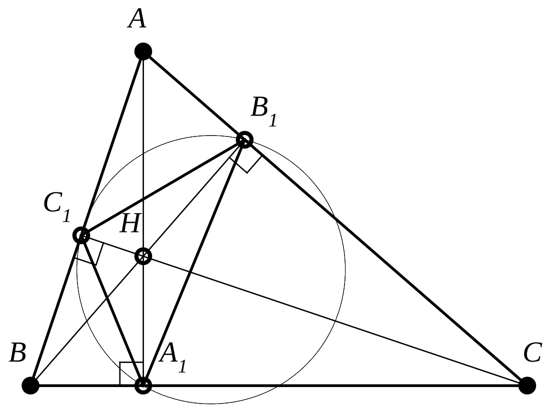

Figure 14

Proof

Let PA be the tangent to the circumscribed circle of the triangle ABC (see *Figure 15*). We have $\sphericalangle PAB \equiv \sphericalangle ACB$ (inscribed angles with the same measure). The previous relation shows that PA and BC are antiparallels in relation to AB and AC.

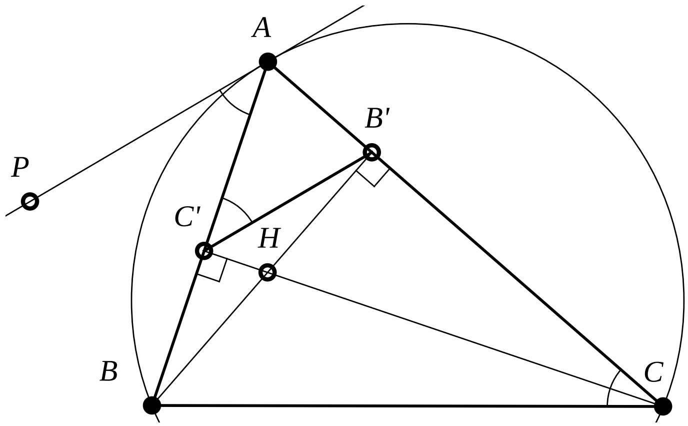

Figure 15

Remark 4

If we take the projections of B and C on AC respectively AB, and we denote them by B' respectively C', we note that:

The tangent in A to the circumscribed circle of the triangle ABC is parallel with the side $B'C'$ of the orthic triangle $A'B'C'$ of this triangle.

Indeed, we established in Proposition 4 that $B'C'$ is antiparallel with BC, since $\sphericalangle A'C'B' \equiv \sphericalangle ACB$; taking into account relation (4), we obtain that $\sphericalangle PAB \equiv \sphericalangle AC'B'$ which shows that $PA \parallel B'C'$.

Proposition 6

A given triangle and its orthic triangle are orthological triangles. The orthology centers are the center of the circumscribed circle and the orthocenter of the given triangle.

Proof

We consider the acute triangle ABC, and let $A'B'C'$ be its orthic triangle (see *Figure 16*).

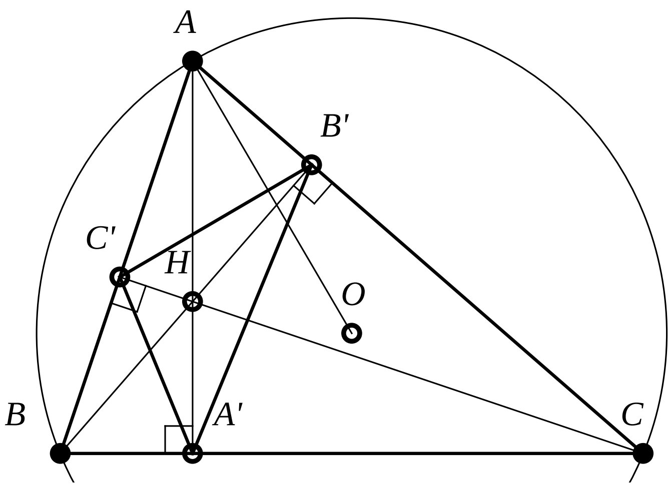

Figure 16

Obviously, the altitudes $A'A$, $B'B$, $C'C$ are concurrent in the orthocenter H, hence $A'B'C'$ is orthological in relation to ABC.

From the theorem of orthological triangles, it follows that ABC is also orthological in relation to $A'B'C'$. Let us prove that the orthology center is O, the center of the circle circumscribed to the triangle ABC.

The perpendicular taken from A to $B'C'$, considering Proposition 4, is also perpendicular to the tangent in A to the circumscribed circle, therefore it passes through O. Similarly, the perpendiculars from B and C to $A'C'$, respectively $A'B'$ pass through the center of the circumscribed circle.

The theorem can similarly be proved if ABC is an obtuse triangle.

Remark 5

The orthology centers H and O are isogonal conjugate points.

Observation 8

The triangles ABC and $A'B'C'$ are homological. The homology center is H, and the homology axis is the orthic axis (see [24]).

2.4 The median triangle and the orthic triangle

Theorem 5

In a given triangle, the median triangle, the orthic triangle and the triangle with the vertices in the midpoints of the segments determined by the orthocenter and the vertices of the given triangle are inscribed in the same circle (the circle of nine points).

Proof

We denote by $A_1B_1C_1$ the orthic triangle, $A_2B_2C_2$ – the median triangle, and A_3, B_3, C_3 – the midpoints of the segments HA, HB, HC (H – the orthocenter of the triangle ABC, see *Figure 17*).

We have that B_2C_3 is midline in the triangle AHC, therefore $B_2C_3 \parallel AH$ and $B_2C_3 = \frac{1}{2}AH$. Similarly, A_2C_3 is midline in the triangle BHC, therefore $A_2C_3 \parallel BH$; since $OB_2 \perp AC$, therefore $OB_2 \parallel BH$; we have that $A_2C_3 \parallel OB_2$; having $OA_2 \parallel B_2C_3$, it follows that the quadrilateral $OA_2C_3B_2$ is parallelogram, hence $B_2C_3 = OA_2$. Because $OA_2 \parallel A_3H$ and $(OA_2) = (A_3H)$, we obtain that the quadrilateral OA_2HA_3 is parallelogram. Denoting by O_9 the midpoint of the segment OH, we have that A_3, O_9, A_2 are collinear and $O_9A_3 = O_9A_2$.

Because O_9A_3 is midline in the triangle AHO, we have that: $O_9A_3 = \frac{1}{2}OA$, therefore $O_9A_3 = \frac{1}{2}R$. In the right triangle $A_3A_1A_2$, A_1O_9 is median, therefore $A_1O_9 = O_9A_3 = O_9A_2 = \frac{1}{2}R$.

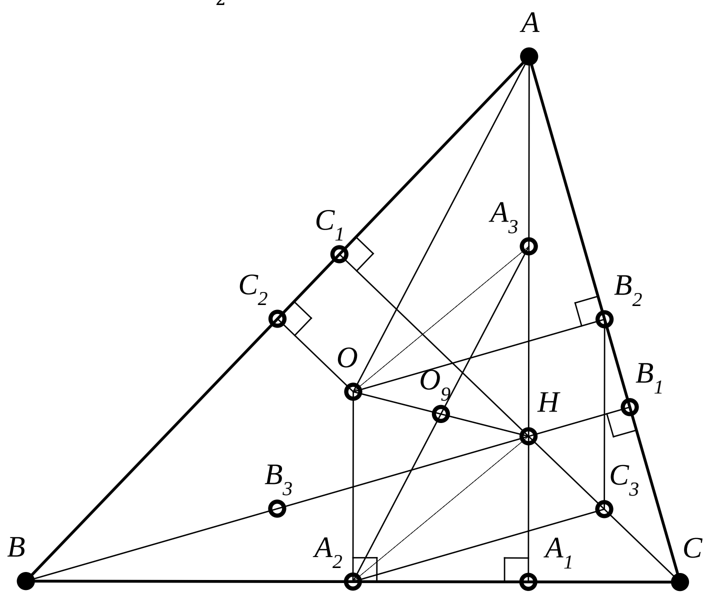

Figure 17

Similarly, it follows that the points B_1, B_2, B_3 are at distance $\frac{1}{2}R$ from O_9 and also from the points C_1, C_2, C_3. The circle of points A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 is also called Euler circle; as we saw, its radius is half the radius of the circle circumscribed to the given triangle.

Observation 9

a) The median triangle and the triangle determined by the midpoints of the segments HA, HB, HC are congruent.

b) The proof of the Theorem 5 can be made in the same way if the triangle ABC is obtuse.

c) The Euler circle is homothetic to the circle circumscribed to the triangle by homothety of center H and ratio $\frac{1}{2}$.

d) From the previous Observation, it derives two useful properties related to the orthocenter of a triangle, namely:

Proposition 7

The symmetric point of the orthocenter of a triangle towards the sides of the triangle belongs to the circumscribed circle.

Proposition 8

The symmetric points of a triangle's orthocenter towards the vertices of the median triangle belong to the circle circumscribed to the given triangle.

Proposition 9

The median triangle and the orthic triangle of a given triangle are orthological triangles. The orthology centers are the center of the circle of the nine points and the orthocenter of the given triangle.

Proof

The segment A_2A_3 is diameter in the circle of the nine points, having $B_1A_2 = C_1A_2 = \frac{1}{2}BC$ (medians in rectangular triangles), and $\sphericalangle A_2B_1A_3 \equiv \sphericalangle A_2C_1A_3 = 90^0$; we have that $\Delta A_2B_1A_3 \equiv \Delta A_2C_1A_3$, therefore also $A_3B_1 \equiv A_3C_1$, and hence A_2A_3 is mediator of the segment B_1C_1. Similarly, we show that the perpendicular from B_2 to A_1C_1 passes through center O_9 of the circle of nine points. The fact that the orthocenter H is the orthology center of the orthic triangle in relation to the median triangle is obvious.

Observation 10

We can prove that the perpendicular from A_2 to B_1C_1 passes through O_9 and, considering that B_1C_1 is antiparallel to BC, therefore B_1C_1 is parallel with the tangent in A_2 to the circle of the nine points, and consequently the perpendicular from A_2 to B_1C_1, being perpendicular to the tangent in A_2 passes through the center O_9 of the circle.

Problem 4

Show that the complementary triangle and the anticomplementary triangle of a given triangle are orthological triangles. Specify the orthology centers.

Problem 5

Show that the orthic triangle and the anticomplementary triangle of a given triangle are orthological triangles. Specify the orthology centers.

2.5 A triangle and its contact triangle

Definition 7

We call a contact triangle of a given triangle – the triangle determined by the tangent (contact) points of the circle inscribed in the triangle with its sides.

Observation 11

In *Figure 18*, the contact triangle of the triangle ABC was denoted by $C_aC_bC_c$. I is the center of the circle inscribed in the triangle ABC (the bisectors' intersection).

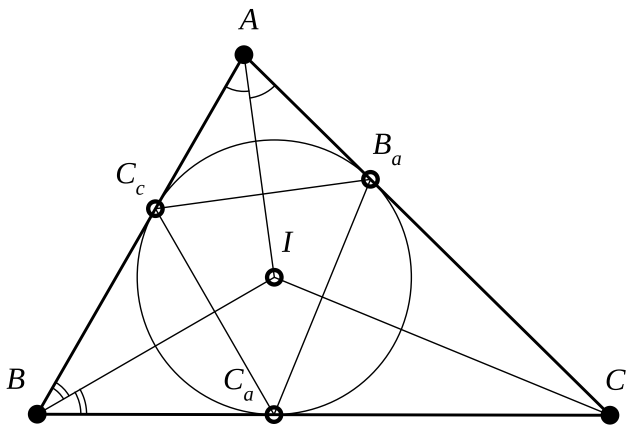

Figure 18

Proposition 10

A given triangle and its contact triangle are orthological triangles. The contact centers of these triangles coincide with the center of the circle inscribed in the given triangle.

Proof

We use *Figure 18*. The tangents AC_b, AC_c taken from A to the inscribed circle are equal, so the perpendicular taken from A to C_bC_c is the bisector of the angle BAC, which, obviously, passes through I. The perpendicular taken from C_a to BC is radius of the inscribed circle; it contains the circle's center I, which is the common center of the two orthologies between the considered triangles.

Observation 12

a) The triangle ABC and its contact triangle $C_aC_bC_c$ are bilogical triangles.
b) The triangles ABC and $C_aC_bC_c$ are homological triangles, the center of homology being Gergonne point, and the axis of homology being Lemoine line (see [24]).

Proposition 11

The contact triangle and the median triangle of a given triangle are orthological triangles. The orthology centers are the centers of the inscribed circles in the given triangle and in the median triangle of the given triangle.

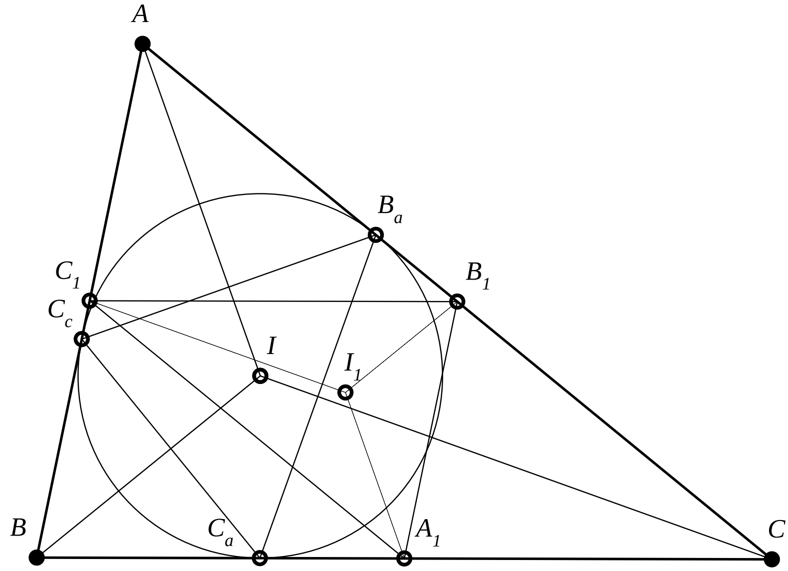

Figure 19

Proof

We denote by $A_1B_1C_1$ the median triangle of the given triangle ABC (see *Figure* 19); because $B_1C_1 \parallel BC$ and $IC_a \perp BC$, it follows that the perpendicular taken from C_a to B_1C_1 passes through I, the center of the circle inscribed in the triangle ABC. The quadrilateral $A_1B_1AC_1$ is parallelogram, the angle bisectors B_1AC_1 and $B_1A_1C_1$ are parallel; since $AI \perp C_bC_c$, it follows that also $A_1I_1 \perp C_bC_c$ (we denoted by I_1 the center of the circle inscribed in the median triangle).

2.6 A triangle and its tangential triangle

Definition 8

The tangential triangle of a given triangle ABC is the triangle formed by the tangents in A, B and C to the circumscribed circle of the triangle ABC.

Observation 13

a) In *Figure 20*, we denoted by $T_aT_bT_c$ the tangential triangle of the triangle ABC.
b) The triangle ABC is the contact triangle for $T_aT_bT_c$, its tangential triangle.
c) The center of the circle circumscribed to the triangle ABC, O, is the center of the circle inscribed in its tangential triangle.

Proposition 12

A given triangle and its tangential triangle are orthological triangles. The common center of orthology is the center of the circle circumscribed to the given triangle.

Observation 14

a) Proof of this property derives from the proof of Proposition 10.
b) From Proposition 11, the tangential triangle of a given triangle and the median triangle of the tangential triangle are orthological.
c) The tangential triangle of a given triangle and this triangle are homological triangle. The center of homology is the symmedian center K (Lemoine point in the given triangle, see [24]).

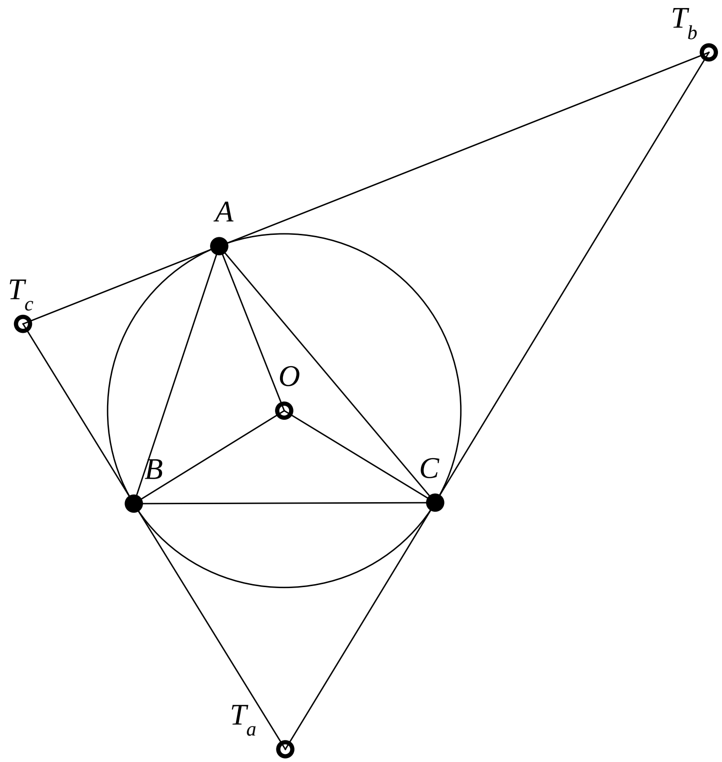

Figure 20

Proposition 13

The tangential triangle and the median triangle of a given triangle are orthological triangles. The orthology centers are the center of the circumscribed circle and the center of the nine points circle of the given triangle.

Proof

Let $T_aT_bT_c$ and $A_1B_1C_1$ the tangential and median triangles of the given triangle ABC (see *Figure 21*). Obviously, T_aA_1 is mediator of segment BC and, since $B_1C_1 \parallel BC$, we have that $T_aA_1 \perp B_1C_1$, and T_aA_1 passes through O, the center of the circle circumscribed to the triangle ABC.

Similarly, T_bB_1 passes through O and T_cC_1 passes through O, therefore O is orthology center.

If we denote by $A_2B_2C_2$ the orthic triangle of the triangle ABC, we have that the perpendicular taken from A_1 to B_2C_2 passes through A_1; since $B_2C_2 \parallel T_bT_c$ (both are antiparallels to BC), we have that the perpendicular from O_9 to T_bT_c passes through O_9. Similarly, it follows that the perpendiculars from B_1 and C_1, respectively to T_aT_c and T_aT_b pass through O_9.

Note 1

In [1], Proposition 13 is presented as Cantor's Theorem.

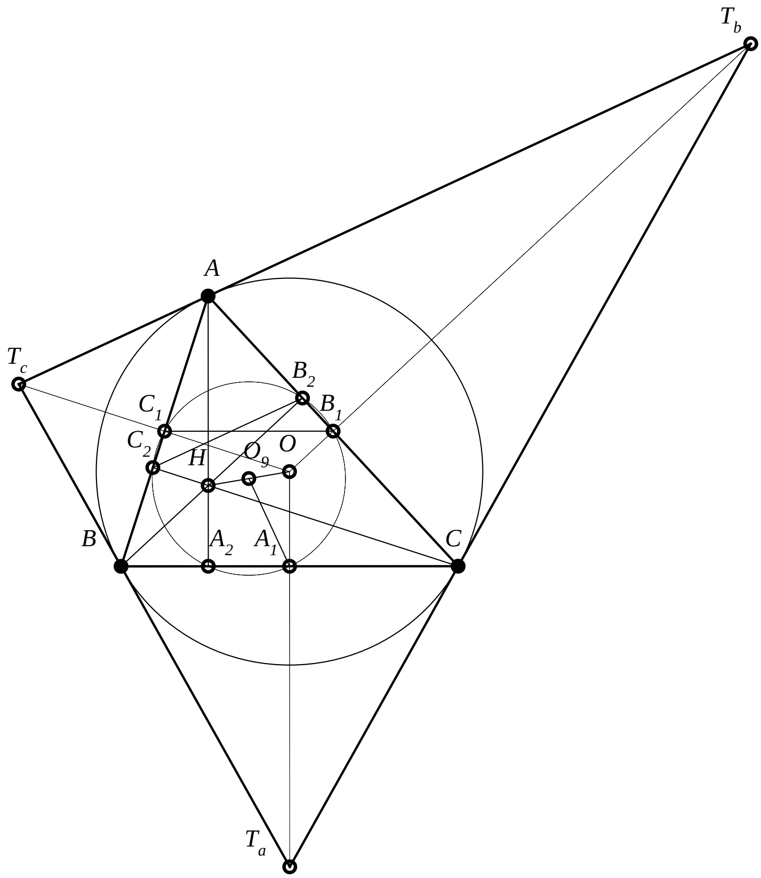

Figure 21

Proposition 14

The tangential triangle $T_aT_bT_c$ of a given triangle ABC and the median triangle of the orthic triangle of the triangle ABC are orthological triangles. The orthology centers are the orthocenter of $T_aT_bT_c$ and the center O_9 of the nine points circle of the triangle ABC.

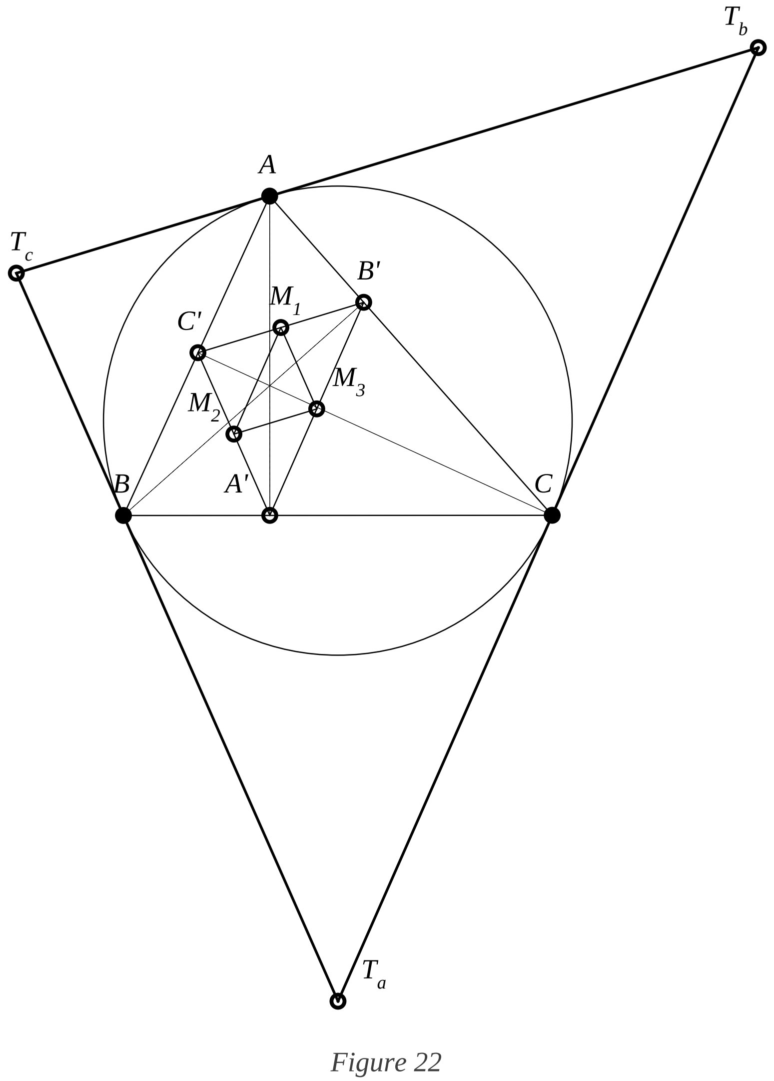

Figure 22

Proof

We denote by $M_1M_2M_3$ the median triangle of the orthic triangle $A'B'C'$ corresponding to the given acute triangle ABC (see *Figure 22*). Because $O_9M_1 \perp B'C'$ and $B'C' \parallel T_bT_c$, it follows that $O_9M_1 \perp T_bT_c$; similarly, $O_9M_2 \perp T_aT_c$ and $O_9M_3 \perp T_aT_b$; hence, the triangles $M_1M_2M_3$ and $T_aT_bT_c$ are orthological, the orthology center being O_9 – the center of the circle of nine points of triangle ABC. The triangles $M_1M_2M_3$ and $T_aT_bT_c$ have the sides respectively parallel. Ee denote by H_T the orthocenter of the tangential triangle; then T_aH_T will be perpendicular to M_2M_3, therefore H_T is orthology center.

2.7 A triangle and its cotangent triangle

Definition 9

It is called cotangent triangle of a given triangle the triangle determined by the contacts of the ex-inscribed circles with the sides of the triangle.

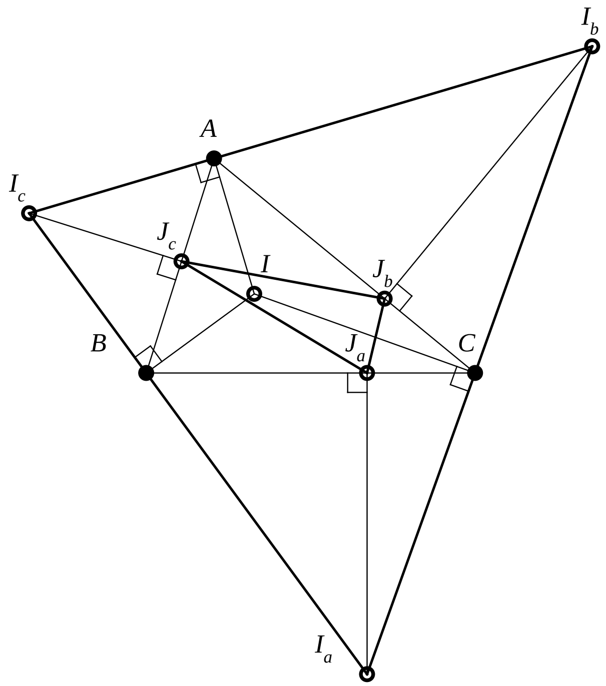

Figure 23

Observation 15

In *Figure 23*, $J_aJ_bJ_c$ is cotangent triangle of the triangle ABC.

Proposition 15

A triangle and its cotangent triangle are orthological triangles.

Proof

It is calculated without difficulty: $AJ_b = p - c$,$AJ_c = p - b$, $BJ_c = p - a$,$BJ_a = p - c$,$CJ_a = p - b$,$CJ_b = p - a$.

We note that:

$AJ_c^2 + BJ_a^2 + CJ_b^2 = J_bA^2 + J_cB^2 + J_aC^2$.

According to the Theorem 2, the triangles ABC and $J_aJ_bJ_c$ are orthological.

Definition 10

It is called Bevan point of the triangle ABC – the intersection of perpendiculars taken from the centers of ex-inscribed circles I_a, I_b, I_c respectively to BC, CA and AB.

Proposition 16

The cotangent triangle of a given triangle and the given triangle have as orthology center the Bevan point.

Proof

From Proposition 15, the perpendiculars taken from J_a, J_b, J_c to BC, CA, AB are concurrent. The points J_a, J_b, J_c being respectively the contacts of the ex-inscribed circles with the sides BC, CA, AB, from the uniqueness of the perpendicular in a point on a line, we have that perpendiculars taken in J_a to BC, in J_b to CA and in J_c to AB pass respectively through I_a, I_b, I_c, therefore the Bevan point is orthology center of the cotangent triangle in relation to the given triangle.

Observation 16

The triangle ABC and its cotangent triangle $J_aJ_bJ_c$ are homological triangles. The homology center is the Nagel point (see [24]).

Problem 6

Let ABC be a triangle and $A_1 \in (BC)$, $B_1 \in (AC)$, $C_1 \in (AB)$, such that $AB_1 = BA_1$, $BC_1 = CB_1$ and $AC_1 = CA_1$. Prove that the triangles ABC and $A_1B_1C_1$ are orthological. What can you say about the triangle $A_1B_1C_1$?

Definition 11

We call adjoint A-cotangent triangle of the triangle ABC the triangle that has as vertices the projections of the center of the A-ex-inscribed circle, I_a, on the sides of the triangle ABC.
It can be defined similarly the adjoint cotangent triangles corresponding to the vertices B and C.

Proof

We denote by $I_aI_b'I_c'$ the adjoint A-cotangent triangle of the triangle ABC (see *Figure 24*). Obviously, the perpendiculars taken in the contacts of the A-ex-inscribed circle on the sides BC, CA, AB pass through I_a.

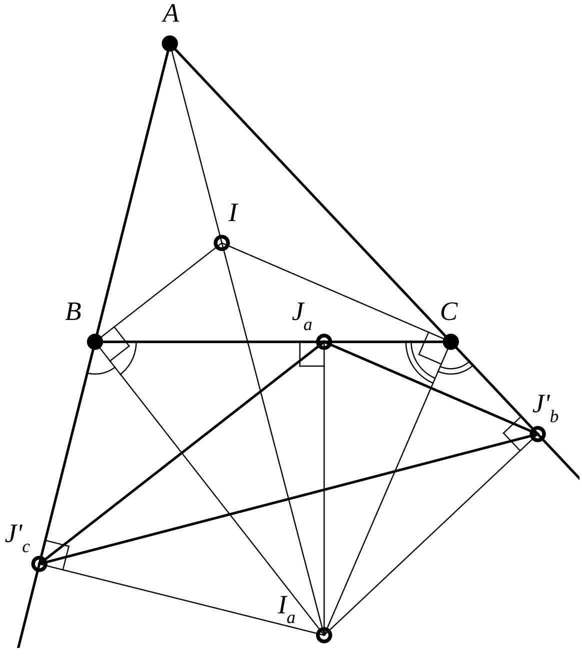

Figure 24

Proposition 17

A given triangle and its adjoint cotangent triangle are orthological triangles. The common center of orthology is the center of the corresponding ex-inscribed circle.

Proof

We denote by $J_aJ'_bJ'_c$ the adjoint A-cotangent triangle of the triangle ABC (see *Figure 25*). Obviously, the perpendiculars taken in the contacts of the A-ex-inscribed circle to the sides BC, CA, AB pass through I_a - the center of the A-ex-inscribed circle; hence, the triangle $J_aJ'_bJ'_c$ is orthological in relation to ABC. Because $AJ'_b = AJ'_c$ (tangents taken from A to the A-ex-inscribed circle), it follows that the the perpendicular from A to $J'_bJ'_c$ is the bisector of the angle BAC, therefore it passes through I_a; similarly, the perpendiculars taken from B and C to $J_aJ'_c$, respectively to $J_aJ'_b$, are exterior bisectors corresponding to the angles B and C of the triangle ABC, thereforethey pass through I_a.

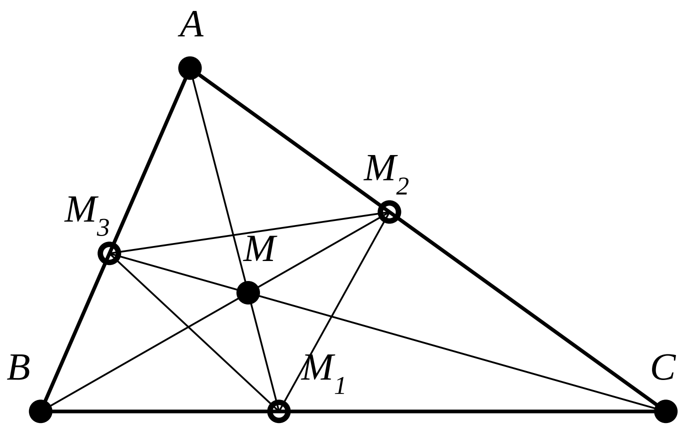

Figure 25

2.8 A triangle and its anti-supplementary triangle

Definition 12

It is called antisupplementary triangle of a given triangle the triangle determined by exterior bisectors of this triangle.

Observation 17

a) The antisupplementary triangle of the triangle ABC is the triangle $I_aI_bI_c$ with the vertices in the centers of the circles ex-inscribed to the triangle ABC.
b) The triangle ABC is the orthic triangle of its triangle $I_aI_bI_c$.

Proposition 18

A triangle and its antisupplementary triangle are orthological triangles. The orthology centers are the center of the inscribed circle and Bevan point of the given triangle.

The proof of this Proposition derives from Propositions 6 and 16.

Observation 18

a) The center of the circle inscribed in the triangle ABC, the point I, is the orthocenter of the antisupplementary triangle $I_aI_bI_c$.
b) The Bevan point of the triangle ABC is the center of the circle circumscribed to the antisupplementary triangle $I_aI_bI_c$.
c) The orthology axis of a triangle and of its antisupplementary triangle is Euler line of the antisupplementary triangle.

Proposition 19

If ABC is a given triangle, I is the center of its inscribed circle, and $I_aI_bI_c$ its antisupplementary triangle, then the pairs of triangles $(I_aI_bI_c, II_bI_c)$, $(I_aI_bI_c, II_cI_a)$, $(I_aI_bI_c, II_aI_b)$ have the same orthology center. Their orthology center is I.

Proof of this property is obvious, because the altitudes of the antisupplementary triangle are the bisectors of the given triangle.

Definition 13

It is called pedal triangle of a point M from the plane of the triangle ABC – the triangle that has as vertices the intersections of the cevians AM, BM, CM, respectively with BC, CA and AB.

Observation 19

a) In *Figure 25*, the triangle $M_1M_2M_3$ is the pedal triangle of the point M in relation to the triangle ABC. We say about the triangle $M_1M_2M_3$ that it is the M-pedal triangle of the triangle ABC.

b) The orthic triangle of the triangle ABC is its H-pedal triangle.

Proposition 20

The antisupplementary triangle of the triangle ABC is orthological with the I-pedal triangle of the triangle ABC (I is the center of the circle inscribed in the triangle ABC).

Proof

We denote by $I_1I_2I_3$ the I-pedal triangle of the triangle ABC (see *Figure 26*).

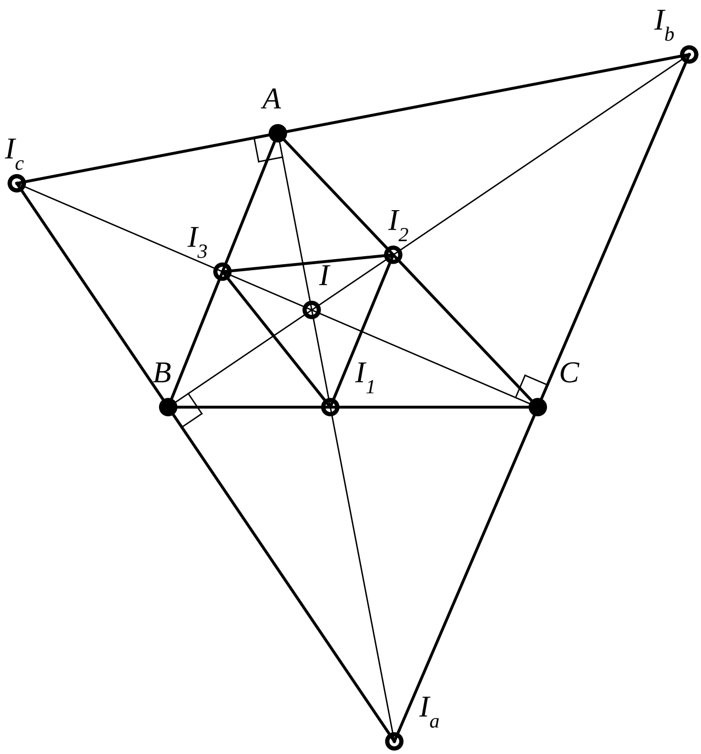

Figure 26

The triangle ABC is the orthic triangle of its antisupplementary triangle $I_aI_bI_c$, therefore the orthology center of the triangle $I_1I_2I_3$ in relation to $I_aI_bI_c$ is the orthocenter of $I_aI_bI_c$, therefore I.

From the orthological triangles theorem, it follows that the perpendiculars taken from I_a, I_b, I_c respectively to I_2I_3, I_1I, I_1I_2 are concurrent as well in the second orthology center.

Problem 7

Let ABC be an isosceles triangle with $AB = AC$ and $I_aI_bI_c$ its antisupplementary triangle. Show that the triangle $I_aI_bI_c$ is orthological in relation to I_a-pedal triangle of the triangle ABC.

2.9 A triangle and its *I*-circumpedal triangle

Definition 14

It is called circumpedal triangle (or metaharmonic triangle) of a point M from the plane of a triangle ABC – the triangle with the vertices in the intersections of the semi-lines $(AM$, $(BM$, $(CM$ with the circumscribed circle of the triangle ABC.

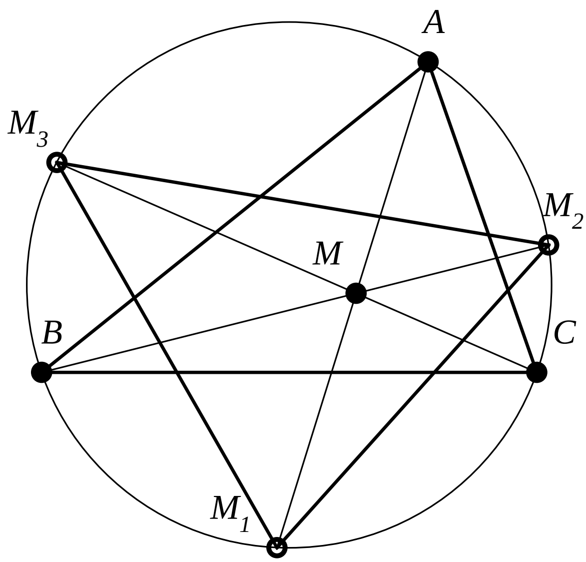

Figure 27

Observation 20

In *Figure 27*, we denoted by $M_1M_2M_3$ the circumpedal triangle of the point M in relation to the triangle ABC. We will say about $M_1M_2M_3$ that it is a M-circumpedal triangle.

Proposition 21

A given triangle ABC and its I-circumpedal triangle are orthological. The orthology centers are I and O – the center of the circle circumcribed to the triangle ABC.

Proof

Let $I_1I_2I_3$ the I-circumpedal triangle of the triangle ABC (see *Figure 28*). We denote $\{A'\} = I_2I_3 \cap AI_1$; we observe that $m(\sphericalangle I_1A'I_2) = \frac{1}{2}m(\overset{\frown}{AI_3}) + \frac{1}{2}m(\overset{\frown}{CI_2}) + \frac{1}{2}m(\overset{\frown}{CI_1}) = \frac{1}{2}[m(\hat{A}) + m(\hat{B}) + m(\hat{C})] = 90^0$.

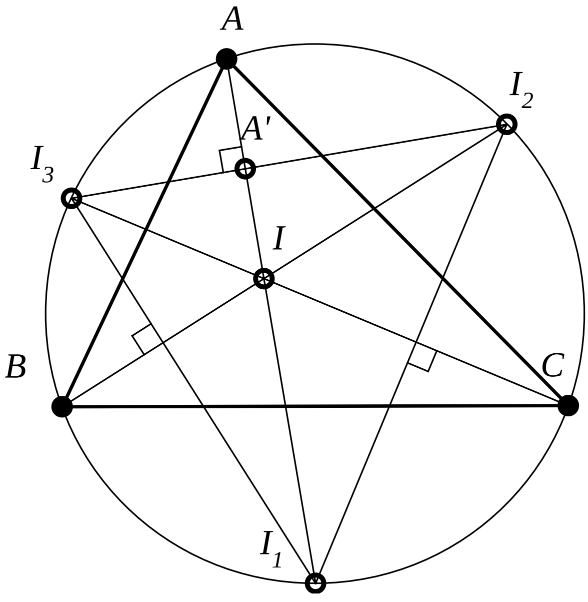

Figure 28

Consequently: $AI_1 \perp I_2I_3$; similarly, it follows that $BI_2 \perp I_1I_3$ and $CI_3 \perp I_1I_2$, therefore the triangles ABC and $I_1I_2I_3$ are orthological, and the orthology center is I.

Because I_1 is the midpoint of the arc BC, it means that the perpendicular from I_1 to BC is the mediator of BC, therefore it passes through O – the center of the circle circumscribed to the triangle ABC. This is the second orthology center of the considered triangles.

Observation 21

a) The center of the circle inscribed in the triangle ABC is the orthocenter of the I-circumpedal triangle of the triangle ABC.

b) The line OI is the Euler line of the I-circumpedal triangle.

Proposition 22

The I-circumpedal triangle of the triangle ABC and the contact triangle $C_aC_bC_c$ of the triangle ABC are orthological. The orthology centers are the points I and H' (the orthocenter of the contact triangle).

Proof

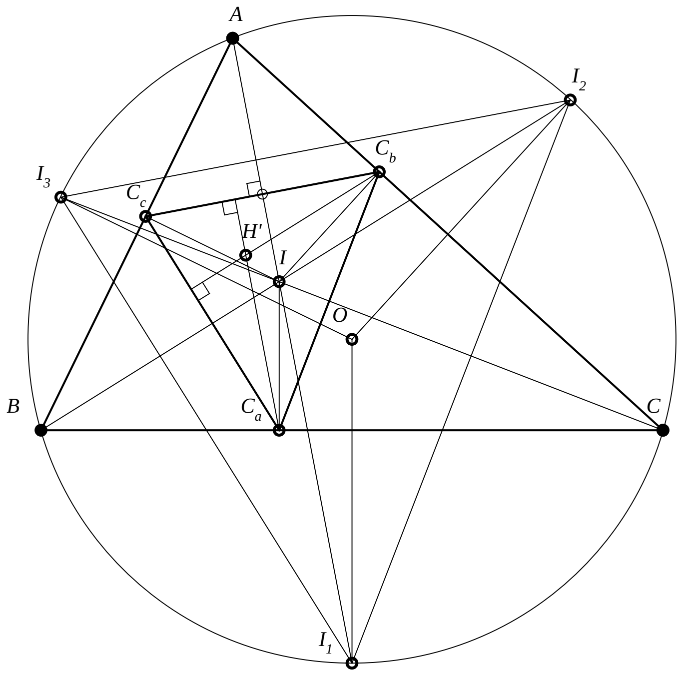

Figure 29

We have $AI_1 \perp C_bC_c$, $BI_2 \perp C_cC_a$ and $CI_3 \perp C_aC_b$, therefore the I-circumpedal triangle $I_1I_2I_3$ and the contact triangle $C_aC_bC_c$ are orthological.

Because the contact triangle and the I-circumpedal triangle are homothetic, it follows that the second orthology center of the triangles $C_aC_bC_c$ and $I_1I_2I_3$ is the orthocenter H' of the contact triangle.

Remark 6

1. The center of homothety of the triangles $I_1I_2I_3$ and $C_aC_bC_c$ is the isogonal N' of the Nagel point of the triangle ABC (see [15], p. 290).
2. The isogonal of the Nagel point of the triangle ABC is to be found on the line OI (see [24] and [15], p. 291).
3. The points N', H', I and O are collinear.

2.10 A triangle and its *H*-circumpedal triangle

Proposition 23

A non-right given triangle ABC and its H-circumpedal triangle are orthological. The orthology centers are H and O (the orthocenter and the center of the circle circumscribed to the triangle ABC).

Proof

The H-circumpedal triangle of the triangle ABC is the homothetic of the orthic triangle of the triangle ABC by homothety of center H and ratio 2 (see *Proposition* 6). Because the orthic triangle of the triangle ABC is orthological with it (see *Proposition* 6), it follows that the the perpendiculars taken from A, B, C on the sides of the H-circumpedal triangle (parallels to the sides of the orthic triangle) will be concurrent in O. The other orthology center is obviously the orthocenter H of the triangle ABC.

2.11 A triangle and its *O*-circumpedal triangle

Definition 15

The symmetric of the orthocenter H of the triangle ABC with respect to the center O of its circumscribed circle is called Longchamps point, L, of the triangle ABC.

Proposition 24

A non-right triangle ABC and its O-circumpedal triangle are orthological triangles. The orthology centers are the orthocenter H and the Longchamps point L of the triangle ABC.

Proof

Let $O_1O_2O_3$ be the O-circumpedal triangle of the acute triangle in *Figure 30*; this is the symmetric of the triangle ABC with respect to O, hence $O_2O_3 \parallel BC$, $O_3O_1 \parallel AC$ and $O_1O_2 \parallel AB$. The perpendiculars taken from A, B, C, respectively on O_2O_3, O_3O_1, O_1O_2 are the altitudes of the triangle ABC, consequently H is the orthology center.

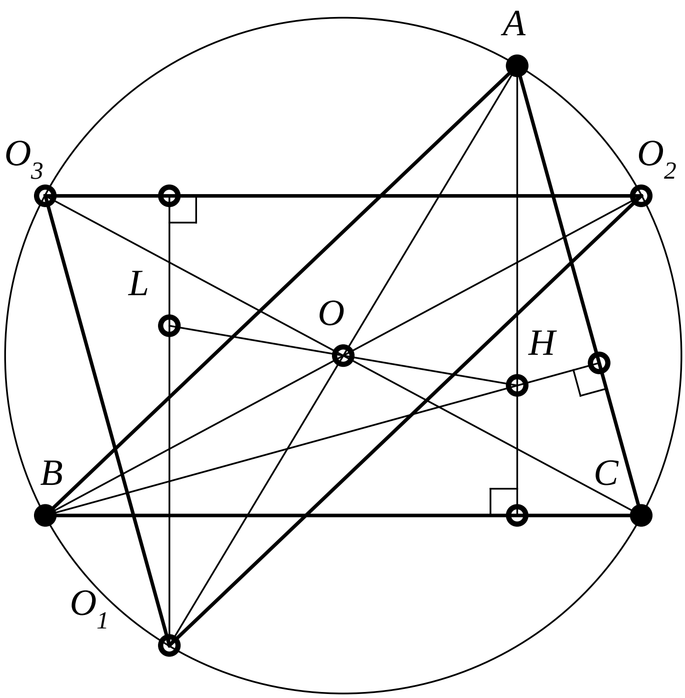

Figure 30

For reasons of symmetry, it follows that the other orthology center will be the symmetric of H with respect to O, ie. the Longchamps point L of the triangle ABC.

2.12 A triangle and its I_a-circumpedal triangle

Proposition 25

The I_a-circumpedal triangle of a given triangle ABC and the triangle ABC are orthological triangles.

Proof

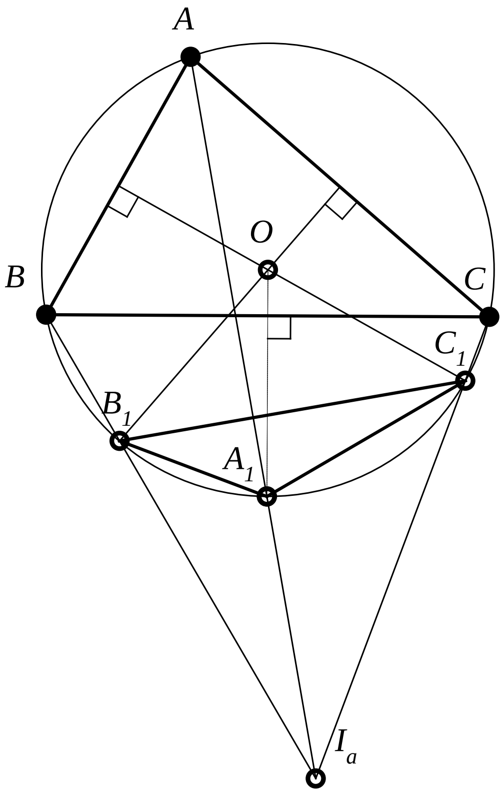

Figure 31

In *Figure 31*, we denoted by $A_1B_1C_1$ the I_a-circumpedal triangle of the center of the A-ex-inscribed circle of the triangle ABC (with $\hat{A} > \hat{C}$). Because A_1 is the midpoint of the arc BC, it follows that the the perpendicular taken from A_1 to BC is mediator of BC, therefore it passes through O, the center of the circle circumscribed to the triangle ABC.

We prove that the perpendiculars taken from B_1 to AC and from C_1 to AB pass through O as well. The quadrilateral B_1BAC is inscribed in the circumscribed circle, therefore:

$m\widehat{B_1AC} = m\widehat{B_1BC} = \frac{1}{2}m(\hat{A}+\hat{C})$; $m\widehat{B_1CA} = m(\hat{C}) + m\widehat{B_1CB}$.

But $m\widehat{B_1CB} = m\widehat{BAB_1} = m(\hat{A}) - m\widehat{B_1AC} = m(\hat{A}) - \frac{1}{2}m(\hat{A}+\hat{C})$.

It follows that $m\widehat{B_1CB} = \frac{1}{2}m(\hat{A}-\hat{C})$ and $\widehat{B_1CA} = m(\hat{C}) + \frac{1}{2}m(\hat{A}-\hat{C}) = \frac{1}{2}m(\hat{A}+\hat{C})$.

Hence: $\widehat{B_1AC} \equiv \widehat{B_1CA}$, therefore $B_1A = B_1C$ and, consequently, the perpendicular taken from B_1 to AC passes through O; similarly, we show that $C_1A = C_1B$, therefore the perpendicular taken from C_1 to AB passes through O as well.

We showed that the triangle $A_1B_1C_1$ is orthological with ABC and the orthology center is O.

Observation 22

In the previous proof, we showed that:

The intersection of the exterior bisector of the angle of a triangle with the circle circumscribed to it, is the midpoint of the high arc subtended by the angle's opposite side.

2.13 A triangle and its ex-tangential triangle

Definition 26

Let ABC a non-right given triangle and its ex-inscribed circles of centers I_a, I_b, I_c. The exterior common tangents to the ex-inscribed circles (which does not contain the sides of the triangle ABC) determine a triangle $E_aE_bE_c$ called ex-tangential triangle of the triangle ABC.

Observation 23

a) The ex-tangential triangle $E_aE_bE_c$ of the triangle ABC is represented in *Figure 32*.
b) If ABC is a right triangle, then the ex-tangential triangle is not defined for this triangle. Indeed, let ABC a triangle with $m(\hat{A}) = 90^0$.

Because the common tangent AB to A-ex-inscribed and B-ex-inscribed circles is perpendicular to the interior common tangent AC taken from the symmetry center with respect to the line I_aI_b, the exterior common tangent to the ex-inscribed circles (I_a), (I_b) will be perpendicular to BC.

Similarly, the exterior common tangent to the circles (I_a), (I_b) will be perpendicular to BC.

Since the exterior common tangent taken through E_b and E_c from the ex-inscribed circles are parallel, the triangle ex-tangential is not defined.

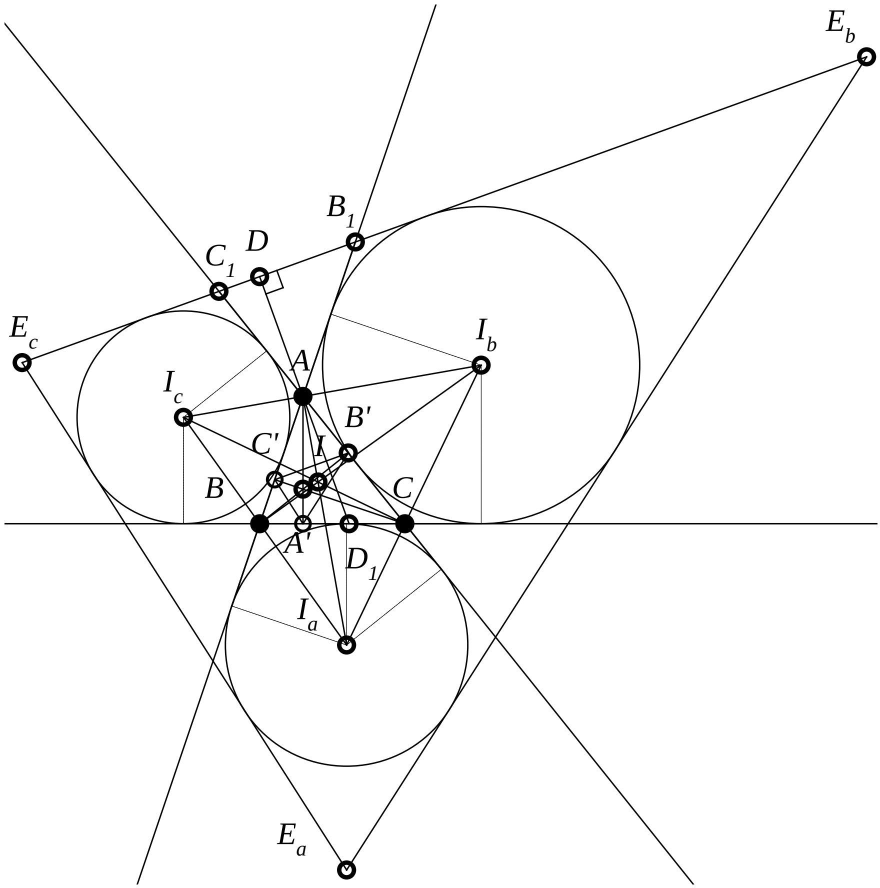

Figure 32

Proposition 26

A non-right given triangle and its ex-tangential triangle are orthological triangles. The orthology center of the given triangle and its ex-tangential triangle is the center of the circle circumscribed to the given triangle.

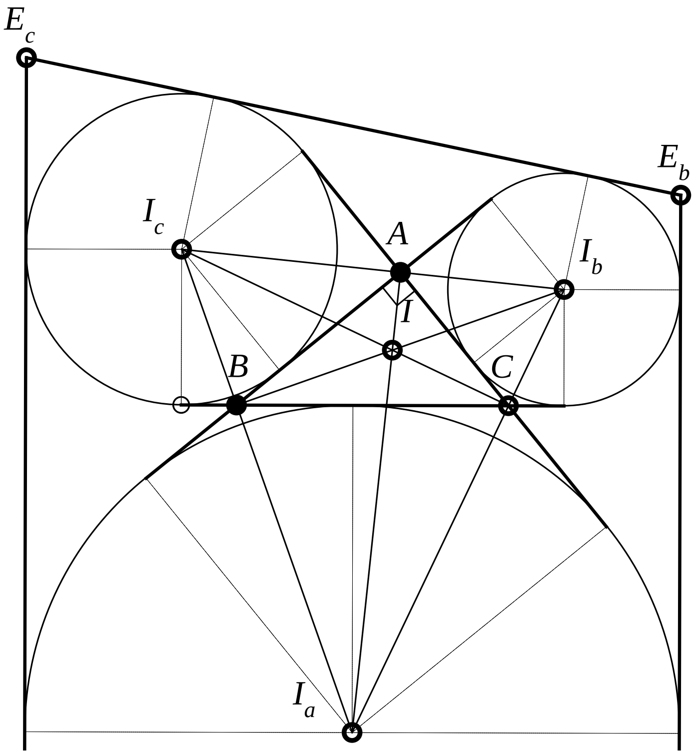

Figure 33

Proof 1

For the proof, we first make the following mentions:

Definition 17

Two cevians of a triangle are called isogonal if they are symmetric in relation to the bisector of the triangle with which they have the common vertex.

Lemma 3

The altitude from a vertex of the triangle and the radius of the circumscribed circle corresponding to that vertex are isogonal cevians.

Proof

Let AD be an altitude in the triangle ABC and O – the center of the circumscribed circle (see *Figure 34*). We have: $m(\widehat{DAC}) = 90^0 - m(\hat{C})$, $m(\widehat{AOB}) = 2m(\hat{C})$.

Therefore $m(\widehat{BAO}) = \frac{1}{2} \cdot [180^0 - 2m(\hat{C})] = m(\widehat{DAC})$.

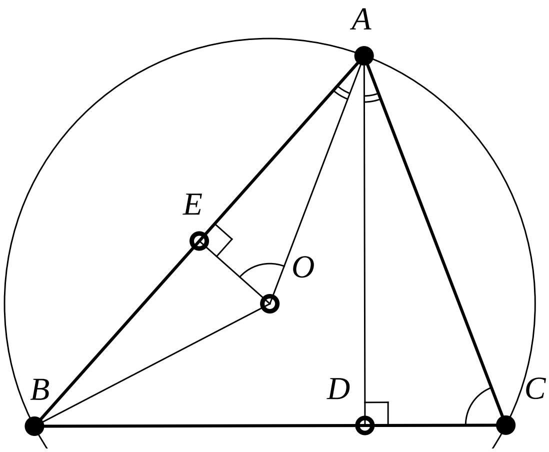

Figure 34

Or (*Solution by Mihai Miculița*):

$$\left.\begin{array}{r} m\left(\widehat{EOA}\right) = \frac{1}{2}.m\left(\widehat{AOB}\right) = m\left(\widehat{ACB}\right) \Rightarrow \widehat{EOA} \equiv \widehat{ACB} \\ \left.\begin{array}{r} OE \perp AB \\ AD \perp BC \end{array}\right\} \Rightarrow \widehat{AEO} \equiv \widehat{ADC} \end{array}\right\} \Rightarrow \widehat{OAE} \equiv \widehat{CAD}$$

(Two triangles which have two angles respectively congruent, have the third congruent angle as well).

Observation 24

a) Obviously: $\sphericalangle AOE = \sphericalangle C$. Writing in ΔAEO (right-angled):
$\sin AOE = \frac{AE}{OA}$, we find $\sin C = \frac{C}{2R}$, therefore $\frac{C}{\sin C} = 2R$ – sinus theorem.

b) Lemma 3 is proved similarly in the case of the obtuse or right triangles.

We prove now Proposition 26. We denote: $\{B_1\} = AB \cap E_bE_c$ and $\{C_1\} = AC \cap E_bE_c$. For reasons of symmetry, the line I_bI_c is axis of symmetry of the figure formed by the B-ex-inscribed and C-exinscribed circles and from their exterior and interior common tangents. It follows that the triangle ABC is congruent with the triangle AC_1B_1. We take $AD \perp B_1C_1$, $D \in B_1C_1$. We denote $\{D_1\} = AD \cap BC$, and we have that $\sphericalangle DAB_1 = \sphericalangle BAD_1$; taking into account Lemma 1 and the indicated congruence of triangles, it follows that AD_1 passes through O, the circumscribed center of the triangle ABC. Similarly, we show that the perpendicular taken from B to E_aE_c passes through O and that the perpendicular from C to E_aE_b passes through O.

Proof 2

We use the following lemma:

Lemma 4

The ex-tangential triangle of the non-right triangle ABC and the orthic triangle $A'B'C'$ of the triangle ABC are homothetic triangles.

Proof of Lemma

We use *Figure 3*2; from the congruence of triangles ABC and AC_1B_1, it follows that $\sphericalangle ABC \equiv \sphericalangle AC_1B_1$. On the other hand, $B'C'$ is antiparallel with BC, therefore $\sphericalangle AB'C' \equiv \sphericalangle ABC$.

The previous relations lead to $\sphericalangle AB'C' \equiv \sphericalangle AC_1B_1$, which implies that $E_bE_c \parallel B'C'$.

Similarly, it is shown that $E_aE_b \parallel A'B'$ and $E_aE_c \parallel B'C'$. The ex-tangential triangle and the orthic triangle, having respectively parallel sides, are hence homothetic.

Observation 25

The homothety center of the orthic triangle is called Clawson point.

We are able now to complete Proof 2 of Proposition 26.

We observed (Proposition 6) that the non-right triangle *ABC* and its orthic triangle are orthological. The perpendiculars taken from A, B, C to the sides of the orthic triangle are concurrent in the center of the circumscribed circle O.

Because the sides of the orthic triangle are parallels with those of the ex-tangential triangle $E_aE_bE_c$, it follows that the triangle ABC and its ex-tangential triangle are orthological, the orthology center being O.

2.14 A triangle and its podal triangle

Definition 18

It is called podal triangle of a point from the plane of a given triangle – the triangle determined by the orthogonal projections of the point on the sides of the triangle.

Observation 26

a) In *Figure 35*, the podal triangle of the point P is $A'B'C'$.
b) The podal triangle of the orthocenter of a non-right triangle is the orthic triangle of that triangle.

Remark 7

For the points M that belongs to the circumscribed circle of a triangle ABC, the podal triangle is not defined because the projections of the point M on sides are collinear points. The line of these projections is called Simson line.

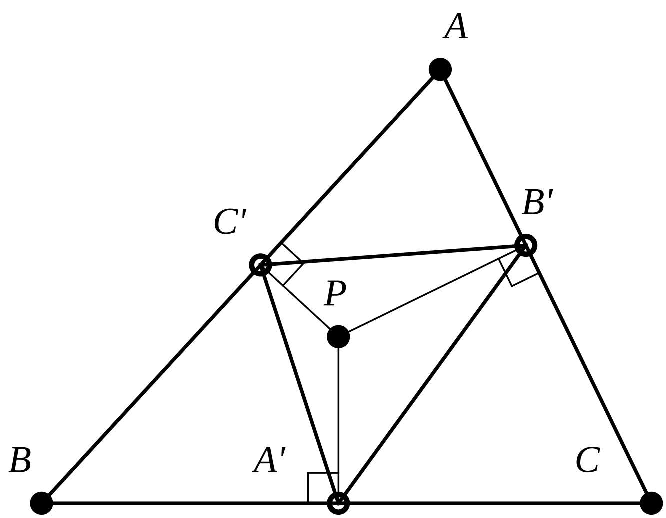

Figure 35

Definition 19

The circumscribed circle of the podal triangle of a point is called podal circle.

Proposition 28

The isogonal cevians of concurrent cevians in a triangle are concurrent.

Definition 2

The concurrency points P and P' of cevians in a triangle and of their isogonal are called isogonal points or isogonal conjugate points.

Remark 8

The center of the circle circumscribed to a triangle and its orthocenter are isogonal points.

Theorem 6

The podal triangle of a point from the interior of a triangle and the given triangle are orthological triangles. The orthology centers are the given point and its isogonal point.

Proof

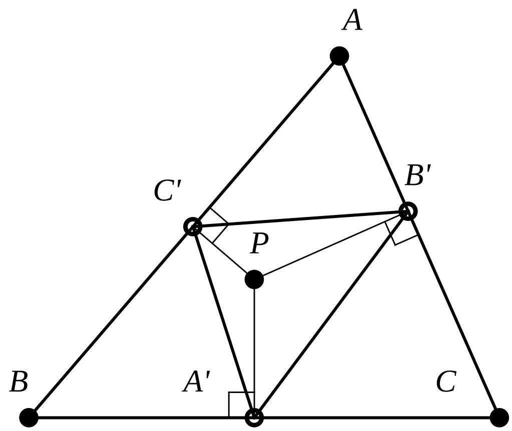

Figure 36

Let $A'B'C'$ be the podal triangle of P in relation to the triangle ABC (see *Figure 36*). Obviously, the triangles $A'B'C'$ and ABC are orthological, and P is orthology center. From the theorem of orthological triangles, it follows that the perpendiculars taken from A, B, C to $B'C'$, $C'A'$ respectively $A'B'$ are concurrent in a point P'. It remains to be shown that the points P and P' are isogonal points.

Because the quadrilateral $AC'PB'$ is inscribable, it follows that $\sphericalangle AP'C' \equiv \sphericalangle AB'C'$. These congruent angles are the complements of the angles PAB respectively $P'AC$. The congruence of these last angles show that the cevians PA and $P'A$ are isogonal.

Similarly, it follows that PB and $P'B$ are isogonal cevians and PC and $P'C$ are isogonal cevians, consequently P and P' – the orthology centers, are isogonal conjugate points.

Observation 28

a) Theorem 6 is true also for the case when the point P is located in the exterior of triangle ABC.
b) Theorem 6 generalizes Propositions 1, 2, 6, 10.
c) From Theorem 6, it follows that:

Proposition 29

A given triangle and the podal of the center of its ex-inscribed circle are orthological triangles. The common orthology center is the center of the ex-inscribed circle.

Proposition 30

The podal triangle of a point P from the interior of the given triangle ABC and the complementary triangle of ABC are orthological triangles.

Proof

Let $A'B'C'$ be the podal of the point P and $A_1B_1C_1$ the triangle of the complementary triangle ABC (see *Figure 37*).

Having $B_1C_1 \parallel BC$, the perpendicular from A' to BC will be also perpendicular to B_1C_1 and will pass through P. The point P is orthology center of the triangle $A'B'C'$ in relation to $A_1B_1C_1$.

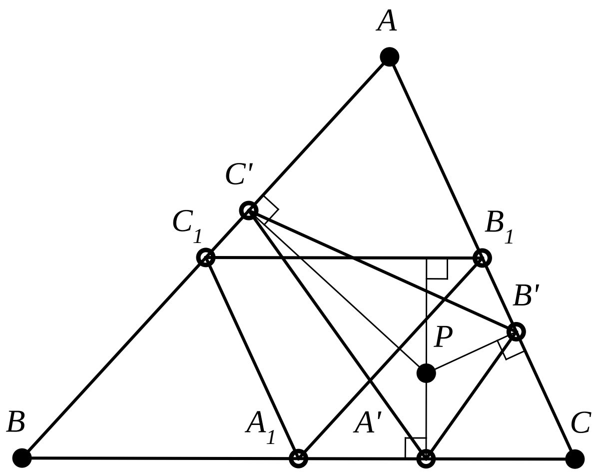

Figure 37

Observation 29

The previous Proposition is true for any point P from the exterior of the triangle ABC which does not belong to its circumscribed circle.

Definition 20

The symmetric of a median of a triangle with respect to the bisector of the triangle with the origin at the same vertex of the triangle is called symmedian.

Observation 30

a) The symmedians of a triangle are concurrent. Their concurrency point is called the symmedian center of the triangle or Lemoine point of the triangle.
b) The symmedian center and the gravity center are isogonal conjugate points.

Proposition 31

The podal triangle of the symmedian center and the median triangle of a median triangle of a given triangle are orthological triangles. The orthology centers are the symmedian center and the gravity center of the given triangle.

Proof

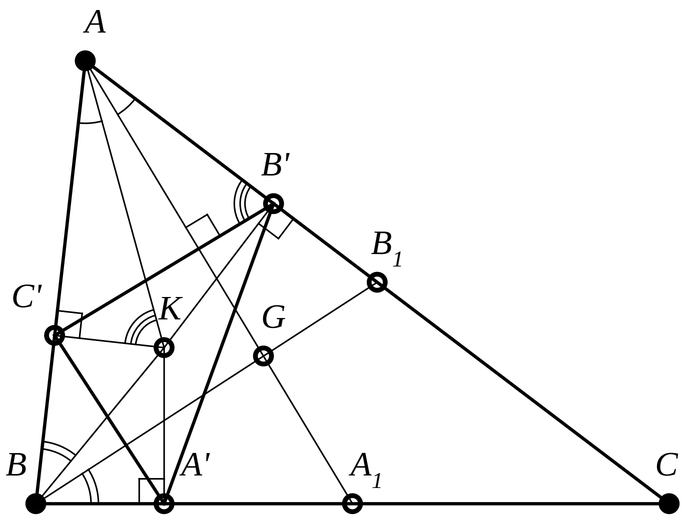

Figure 38

Let $A'B'C'$ be the podal of the symmedian center K and $A_1B_1C_1$ the median triangle of ABC (see *Figure 38*). The cevians AK and AA' being isogonal, we have that:

$\sphericalangle A_1AC \equiv \sphericalangle BAK.$ (1)

The quadrilateral $AC'KB'$este inscribable, therefore:

$\sphericalangle C'KA \equiv \sphericalangle C'B'A.$ (2)

Because $m(\widehat{BAK}) + m(\widehat{C'KA}) = 90^0,$ (3)

from (1) and (2) we obtain that:

$m(\widehat{A_1AC}) + m(\widehat{C'B'A}) = 90^0.$ (4)

This relation shows that $AA_1 \perp B'C'$, therefore the perpendicular from A_1 to $B'C'$ is the median AA_1. Similarly, it follows that $BB_1 \perp A'C'$ and $CC_1 \perp A'B'$, therefore the gravity center $\{G\} = AA_1 \cap BB_1 \cap CC_1$ is the orthology center of the triangle $A_1B_1C_1$ in relation to $A'B'C'$.

Observation 31

Proposition 31 can be proved also using Theorem 6. Indeed, the podal of K and ABC are orthological triangles, therefore the perpendiculars from A, B, C on the sides of the triangle $A'B'C'$ pass through the isogonal of K, videlicet through the gravity center G. These perpendiculars, being the medians of the triangle ABC, pass through A_1, B_1, C_1. The uniqueness of the perpendicular taken from a point to a line show that G is orthology center of the triangle $A_1B_1C_1$ and $A'B'C'$.

Theorem 7 (The circle of six points)

If the points P_1, P_2 are isogonal conjugate points in the interior of the triangle ABC, $A_1B_1C_1$ and $A_2B_2C_2$ their podal triangles, then these triangles have the same podal circle.

Proof

From Theorem 6, it follows that:

$CP_1 \perp A_2B_2$. (1)

If we denote $m(\widehat{P_1B_1A_1}) = x$, since PA_1CB_1 is inscribable, it follows that:

$m(\widehat{P_1CA_1}) = x$. (2)

Taking (1) into consideration, we have that $m(\widehat{B_2A_2C}) = 90^0 - x$. But $m(\widehat{A_1B_1C}) = 90^0 - x$ as well, therefore the points A_1, A_2, B_1, B_2 (3) are concyclic (the lines A_1B_1 and A_2B_2 are antiparallels (see *Figure 39*).

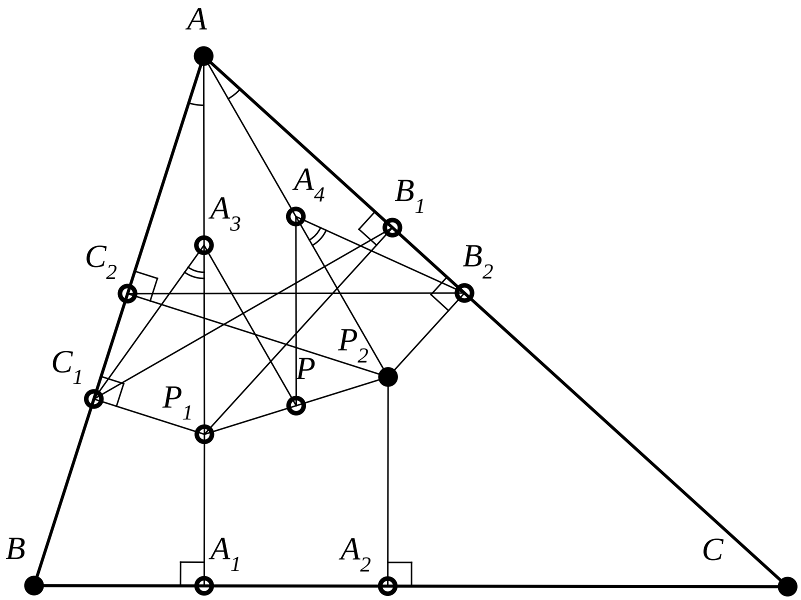

Figure 39

Because the mediators of the segments A_1A_2, B_1B_2 pass through P the midpoint of the segment P_1P_2, it follows that P is the center of the circle on which the points from (3) are found. Similarly, we show that the points B_1, B_2, C_1, C_2 are concyclic, and that P is the center of their circle (4).

From (3) and (4), we have that the points A_1, A_2, B_1, B_2, C_1, C_2 are at the same distance from P, therefore they are concyclic.

Observation 32

The circle of projections on the sides of a triangle of two isogonal conjugate points from its interior is called the circle of six points.

Theorem 8 (The reciprocal of the Theorem 7)

If P_1, P_2 are two distinct points in the interior of the triangle ABC and their podal circles coincide, then P_1, P_2 are isogonal conjugate points.

Proof

We use *Figure 39*; if A_1, A_2, B_1, B_2, C_1, C_2 are concyclic, then the center of this circle will be P – the midpoint of the segment P_1P_2.

Let A_3, A_4 be the midpoints of segment P_1A respectively P_2A. From the conciclicity of the six points, we have that $PC_1 = PB_2$. We observe that: $\Delta PC_1A_3 \equiv \Delta PA_1P_2$ (S.S.S.) - PA_3 is midline in ΔAP_1P_2, therefore $PA_3 = \frac{1}{2}P_2A$, and B_2A_4 is median in the right triangle P_2B_2A, hence $B_2A_4 = \frac{1}{2}P_2A$, so $PA_3 = B_2A_4$; similarly, it follows that $C_1A_3 = PA_4$. The congruence of triangles implies that $\sphericalangle C_1A_3P \equiv \sphericalangle B_2A_4P$, and from here, taking into account that $\sphericalangle P_1A_3P \equiv \sphericalangle P_2A_1P \equiv \sphericalangle P_1AP_2$, we obtain that $\sphericalangle P_1A_3C_1 \equiv \sphericalangle P_2A_4B_2$. These latter angles are exterior angles to the isosceles triangles C_1A_3A, respectively B_2A_4A, hence $\sphericalangle P_1AC_1 \equiv \sphericalangle P_2AB_2$, and thus it follows that the cevians P_1A and P_2A are isogonal. Similarly, it is proved that P_1B and P_2B are isogonal, and that P_1C and P_2C are isogonal cevians, hence P_1 and P_2 are isogonal conjugate points.

Proposition 32

Let P_1 be a point in the interior of the triangle ABC, and $A_1B_1C_1$ its podal triangle; the podal circle of P_1 intersects a second time (BC), (CA), respectively (AB) in the points A_2, B_2, respectively C_2.

Then the triangles $A_2B_2C_2$ and ABC are orthological. The orthology centers are the points P_1 and P_2, where P_2 is the isogonal of P_1.

Proof

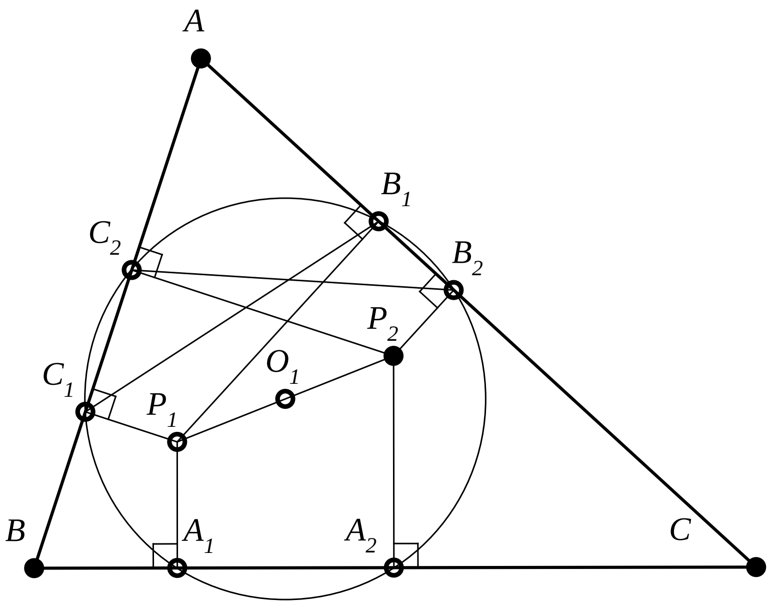

Figure 40

We denote by O_1 the center of the podal circle of the point P_1 (see *Figure 40*). The mediators of the segments A_1A_2, B_1B_2, C_1C_2 are obviously concurrent in O_1. We denote by P_2 the symmetric of the point P_1 with respect to O_1. The symmetric of the line P_1A_1 with respect to O_1 will be P_2A_2 and P_2A_2 is perpendicular to BC; similarly, the symmetric of the line P_1B_1 and P_2C_1 with respect to O_1 will be perpendicular in B_2 and C_2 to AC respectively AB; they also contain the point P_2. The points P_1 and P_2 have the same podal circle; applying Theorem 8, it follows that these points are isogonal.

Applying now Theorem 6, we obtain that the point P_2 – the isogonal of P_1, is orthology center of triangles ABC and $A_2B_2C_2$; from here, we have as well that P_1 is orthology center of the triangle ABC in relation to the triangle $A_2B_2C_2$.

2.15 A triangle and its antipodal triangle

Definition 21

It is called an antipodal triangle of the point P from the plane of the triangle ABC – the triangle formed by the perpendiculars in A, B, C, respectively AP, BP, CP.

Observation 33

a. In *Figure 41*, the antipodal triangle $A'B'C'$ of the point P is represented. This point is called the antipodal point of the triangle $A'B'C'$.

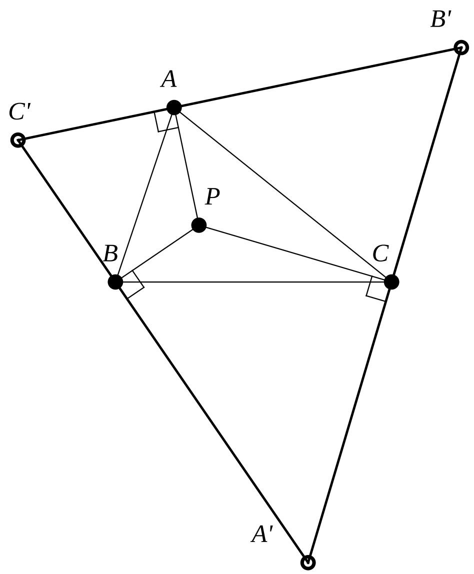

Figure 41

b. The antipodal triangle of the center of the circle circumscribed to a triangle is its tangential triangle.
c. The antipodal triangle of the orthocenter of a triangle is the anticomplementary triangle of this triangle.
d. The antipodal triangle of the center of the circle inscribed in a triangle is the antisupplementary triangle of this triangle.
e. For the points that belong to the sides of the given triangle, the antipodal triangle is not defined.

Proposition 33

A triangle and its antipodal triangle are orthological triangles.

The proof of this Proposition is obvious, because perpendiculars from A, B, C to $B'C'$, $A'C'$ and $C'A'$ are concurrent in P (see *Figure 38*).

Observation 34

The orthology centers of the triangles ABC and its antipodal triangle $A'B'C'$ are the point P and its isogonal conjugate P' in the antipodal triangle, as it follows from Theorem 6.

Proposition 34

The antipodal triangle of a point P is orthological with the P-pedal triangle in the triangle ABC.

Proof

In *Figure 42*, let $A'B'C'$ the antipodal triangle of P and $A_1B_1C_1$ the P-pedal triangle in ABC.

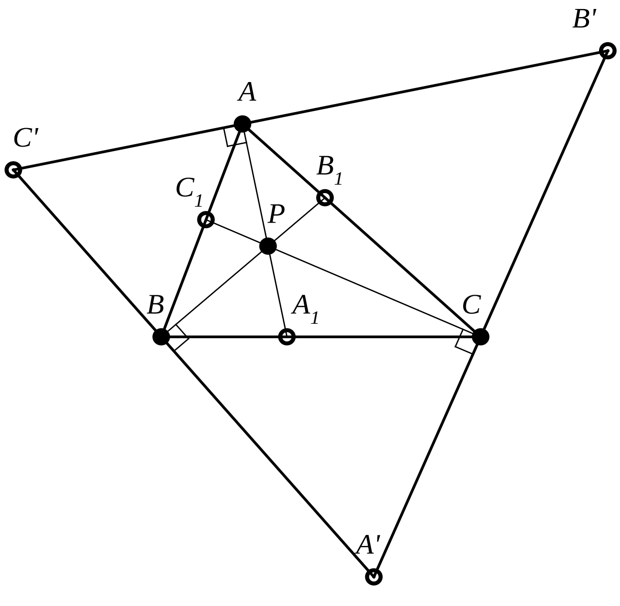

Figure 42

It is obvious that the perpendiculars taken from A_1, B_1, C_1, respectively $B'C'$, $C'A'$ and $A'B'$ are concurrent in P, therefore $A_1B_1C_1$ is orthological with the antipode $A'B'C'$, the orthology center being P.

Observation 35

This Proposition generalizes the Proposition 20.

Proposition 35

The antipodal triangle of the orthocenter of a non-right triangle and its orthic triangle are orthological triangles. The orthology center is the orthocenter of the given triangle, point that coincides with the center of the circle circumscribed to the antipodal triangle.

Proof

The antipodal triangle of the orthocenter H of the triangle ABC is the anticomplementary triangle of ABC; the perpendiculars taken from A_1, B_1, C_1 to $B'C'$, $C'A'$ and $A'B'$ are the mediators of the antipodal triangle $A'B'C'$ and they are concurrent in H – the orthocenter of the triangle ABC.

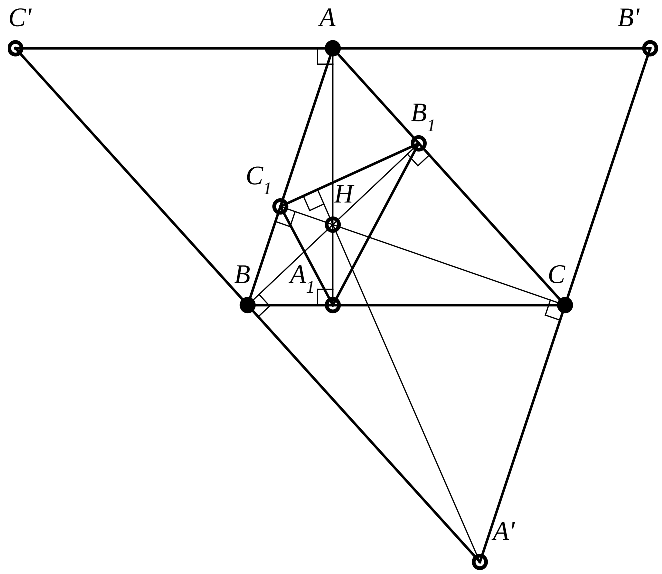

Figure 43

On the other hand, the perpendicular taken from A' to B_1C_1 ($A_1B_1C_1$ is the orthic triangle of ABC, see *Figure 43*) is radius in the circumscribed circle of the triangle $A'B'C'$ (*Proposition 5*), therefore it passes through the center of the circle circumscribed to the triangle $A'B'C'$, which we showed that it is H.

Remark 9

Starting from Theorem 6 and referring to *Figure 33* that we "complete" with the antipodal triangle of the point P', we observe that the antipodal triangle of P' has parallel sides with the podal triangle of P, therefore it is homothetic with it.

We formulate this way:

Proposition 36

The antipodal triangle of a point P from interior of a given triangle is homothetic with the podal triangle of the isogonal conjugate point P' of P.

Observation 36

The antipodal triangle of a point P from the interior of a given triangle and the podal triangle of its isogonal P' being homothetic are orthological triangles.

Proposition 37

The antipodal triangle of a point P from the interior of the triangle ABC is orthological with the O-circumpedal triangle of the triangle ABC.

Proof

In *Figure 44*, we denoted by $A'B'C'$ the antipode of P and by $A_1B_1C_1$ the O-circumpedal triangle of ABC.

Because A_1, B_1, C_1 are symmetrics of A, B, C with respect to O, we will have that $A_1B_1C_1$ has parallel sides to the sides of the triangle ABC. Since ABC and its antipode are orthological, the perpendiculars from A', B', C' to BC, CA, AB are concurrent, but these lines are perpendicular also to B_1C_1, C_1A_1, A_1B_1, therefore $A'B'C'$ and $A_1B_1C_1$ are orthological triangles.

For reasons of symmetry, the orthology center of triangles $A_1B_1C_1$ and $A'B'C'$ will be the symmetric P' of the point P with respect to O.

(Indeed, the triangles APO and $A_1P'O$ are congruent with $AP \parallel A_1P'$, it follows that $A_1P' \perp B'C'$.)

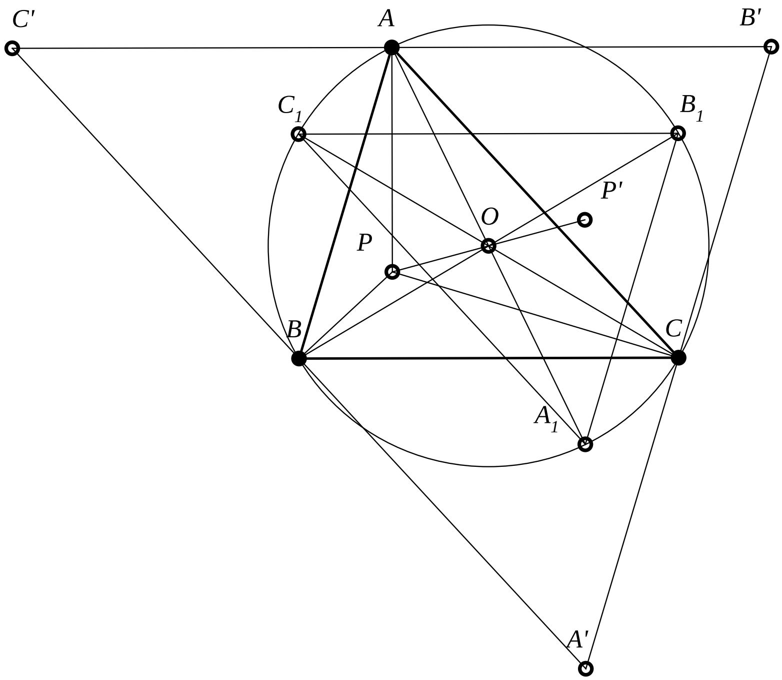

Figure 44

Problem 8

Let ABC be a triangle in which the angles are smaller than 120^0; iin this triangle there is a point T (called isogon center), such that $m\left(\widehat{ATB}\right) = m\left(\widehat{BTC}\right) = m\left(\widehat{CTA}\right) = 120^0$. Prove that the antipodal triangle of the point T is equilateral.

2.16 A triangle and its cyclocevian triangle

Definition 22

Let P be the concurrency point of cevians AA', BB', CC' from triangle ABC. The circumscribed circle of the triangle $A'B'C'$ intersects the second time the sides of the triangle ABC in the points A'', B'', C''. The triangle $A''B''C''$ is called the cyclocevian triangle of the triangle ABC corresponding to the point P.

Observation 37

In *Figure 45*, the triangle $A''B''C''$ is the cyclocevian triangle of the triangle ABC, corresponding to the point P, the intersection of cevians AA', BB', CC'. We can say that the circumscribed circle of the P-pedal triangle of a point intersect the sides of the triangle in the vertices of the cyclocevian triangle of the point P – we call the triangle $A''B''C''$ the P-cyclocevian triangle of the triangle ABC.

Definition 23

The triangles ABC and $A'B'C'$ are called homological triangles if the lines AA', BB', CC' are concurrent. The concurrency point is called the homology center.

Theorem 9 (Terquem – 1892)

The triangle ABC and its P-cyclocevian triangle are homological triangles.

Proof

For proof, we use *Figure 45*. Since AA', BB', CC' are concurrent in P, we have from Ceva's theorem:

$$A'B \cdot B'C \cdot C'A = A'C \cdot B'A \cdot C'B. \qquad (1)$$

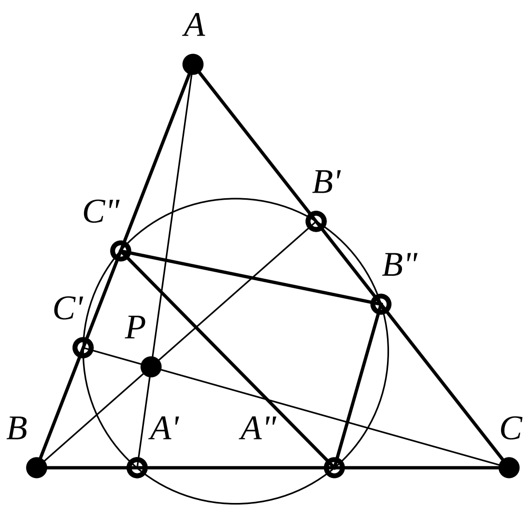

Figure 45

Considering the powers of vertices of triangle ABC over the circumscribed circle of the triangle $A'B'C'$, we have:

$AC' \cdot AC'' = AB' \cdot AB''$, (2)

$BC' \cdot BC'' = BA' \cdot BA''$, (3)

$CA' \cdot CA'' = CB' \cdot CB''$. (4)

Multiplying these last three relations and taking into account the relation (1), we get:

$AC'' \cdot BA'' \cdot CB'' = AB'' \cdot BC'' \cdot CA''$, or equivalent:

$\frac{A''B}{A''C} \cdot \frac{B''C}{B''A} \cdot \frac{C''A}{C''B} = 1$. (5)

The relation (5) and Ceva's theorem show that the cevians AA'', BB'', CC'' are concurrent, consequently the triangles ABC and $A''B''C''$ are homological.

Definition 24

The concurrency point of lines AA'', BB'', CC'' is called the cyclocevian of the point P.

Observation 38

a) The orthocenter H and the gravity center G of a triangle are cyclocevian points because the orthic triangle and the median triangle are inscribed in the circle of nine points.
b) The median triangle is the H-cyclocevian triangle of the triangle ABC.
c) The orthic triangle is the G-cyclocevian triangle.

Theorem 10

In the triangle ABC, let $A_1B_1C_1$ be P-pedal triangle and $A_2B_2C_2$ P-cyclocevian triangle. If the triangles $A_1B_1C_1$ and ABC are orthological, and Q_1, Q_2 are its orthological centers, then:

i. Q_1 and Q_2 are isogonal conjugate points;
ii. The triangles ABC and $A_2B_2C_2$ are orthological;
iii. The orthology centers of the triangles ABC and $A_2B_2C_2$ are the points Q_1 and Q_2.

Proof

i. Let Q_1 be the orthology center of the triangles $A_1B_1C_1$ and ABC (the triangle $A_1B_1C_1$ is the podal triangle of the point Q_1). We denote by Q_2 the second orthology center of triangles $A_1B_1C_1$ and ABC. According to Theorem 6, we have Q_1 and Q_2 – isogonal points.

ii. If we denote by $A_2'B_2'C_2'$ the podal triangle of Q_2 and we take into account Theorem 9, it follows that the points A_1, A_2', B_1, B_2', C_1, C_2' are concyclic. If two circles have three points in common, then they coincide, it follows that $A_2' = A_2$, $B_2' = B_2$, $C_2' = C_2$, therefore the podal triangle of Q_2 is the P-cyclocevian triangle, videlicet $A_2B_2C_2$. This triangle, being podal triangle of ABC, is orthological with ABC (Theorem 6), the orthology center being Q_a.

iii. Applying Theorem 6, we also have that the perpendiculars taken from A, B, C to B_2C_2, C_2A_2, A_2B_2 are concurrent in the isogonal of the point Q_2, therefore in the point Q_1.

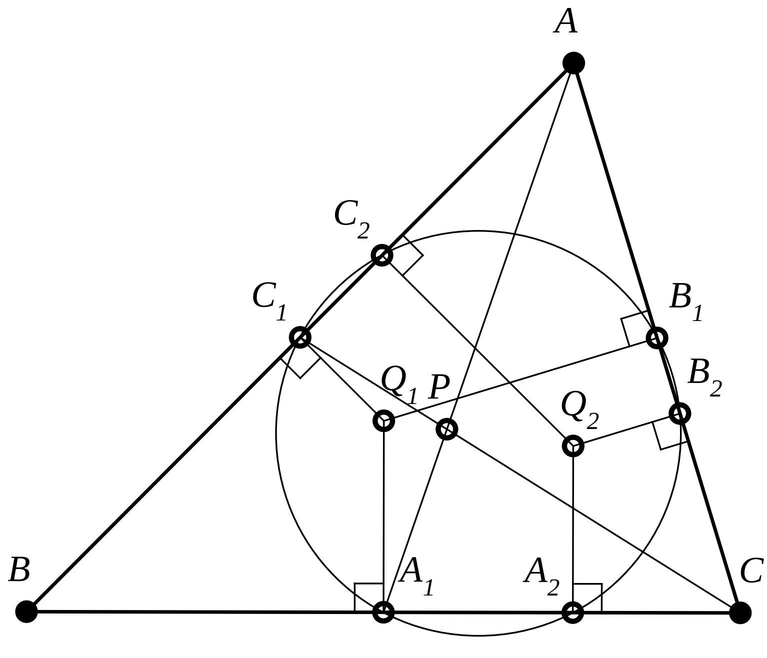

Figure 46

Definition 25

About two triangles that are simultaneously orthological and homological triangles we say that they are bilogical triangles.

Observation 39

The triangles ABC and $A_2B_2C_2$ from the previous theorem are bilogical triangles. The homology derives from Theorem 9.

2.17 A triangle and its three images triangle

Definition 26

The triangle having as vertices the symmetrics of vertices of a given triangle to its opposite sides is called the triangle of the three images of the given triangle.

Observation 40

a) In *Figure 47*, the triangle $A''B''C''$ is the triangle of the three images of the triangle ABC.

b) The triangle of the three images is not possible in the case of a right triangle.

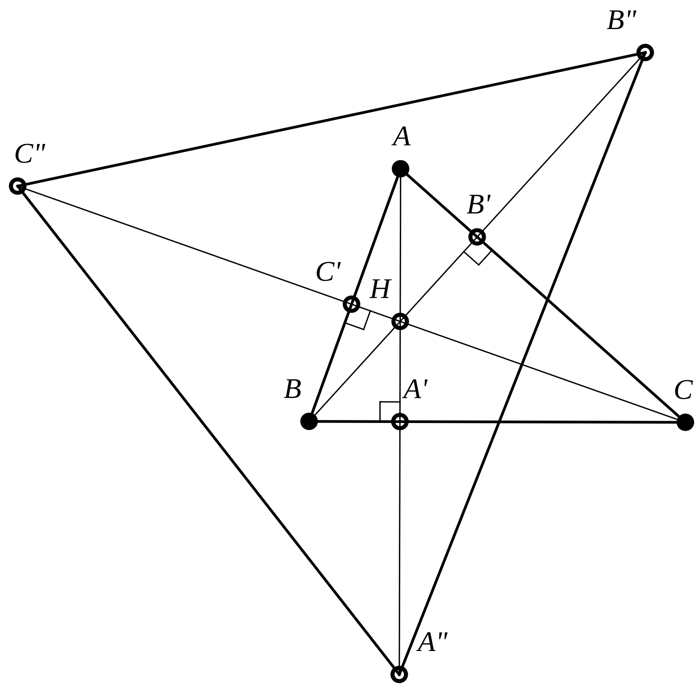

Figure 47

Proposition 28

A given non-right triangle and the triangle of its three images are bilogical triangles.

Proof

Obviously, AA'', BB'', CC'' (see *Figure 43*) are concurrent in H, the orthocenter of the triangle ABC, point which is the homology center of triangles ABC and $A''B''C''$. The perpendiculars taken from A'', B'', C'' to BC, CA, AB are "altitudes" in ABC, therefore the orthocenter is also an orthology center of triangles $A''B''C''$ and ABC.

Theorem 11 (V. Thébault – 1947)

The triangle of the three images of a given triangle is homothetic with the podal triangle of the center of the circle of the nine points corresponding to the given triangle.

Proof

Let H be the orthocenter of triangle ABC, $A_1B_1C_1$ the median triangle of ABC and $A_2B_2C_2$ the O_9-podal triangle in ABC (see *Figure 48*). We denote by H_1 the symmetric of H to BC, videlicet the intersection of semi-line $(HA'$ with the circumscribed circle of the triangle ABC. Because O_9 is the midpoint of the segment OH, it follows that in the right trapeze $HA'A_1O$ we have O_9A_2 midline, therefore $2O_9A_2 = OA_1 + HA'$.

Taking into account that $2OA_1 = AH$, we have: $4O_9A_2 = 2OA_1 + 2HA'$, namely: $4O_9A_2 = HH_1 + A''H_1 = HA''$.

On the other hand, $4O_9G = GH$ (G - the gravity center), we get from the last relation that: $\frac{O_9A_2}{HA''} = \frac{O_9G}{HG} = \frac{1}{4}$. Because H, O_9, G are collinear and $O_9A_2 \parallel HA'$, we have that the triangles O_9GA_2 and HGA'' are similar, therefore the points G, A_2, A'' are collinear.

Consequently, we have that: $\frac{GA_2}{GA''} = \frac{1}{4}$. Similarly, we find that G, B_2, B'' and G, C_2, C'' are collinear and: $\frac{GB_2}{GB''} = \frac{GC_2}{GC''} = \frac{1}{4}$. The obtained relations show that the triangles $A_2B_2C_2$ (the podal of O_9) and $A''B''C''$ the triangle of the three images are homothetic by homothety $h\left(G; \frac{1}{4}\right)$.

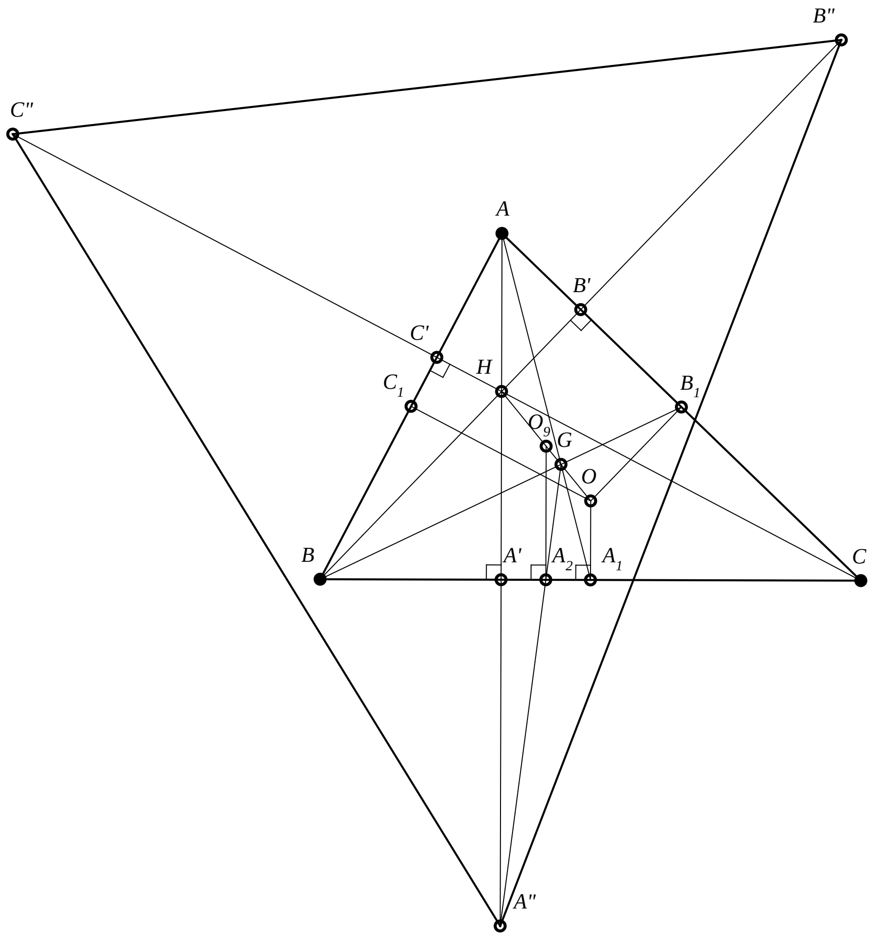

Figure 48

Proposition 39

The triangle of the three images of a given triangle and the podal triangle of the center of the circle of the nine points are orthological triangles.

The proof of this Proposition follows from Theorem 11 and from the fact that if two triangles are homothetic, they are orthological.

2.18 A triangle and its Carnot triangle

Definition 27

We call Carnot circles of the given non-right triangle ABC, of orthocenter H, the circles circumscribed to triangles BHC, CHA, AHB.

Definition 28

The triangle $O_aO_bO_c$ determined by the centers of the Carnot circles of the triangle ABC is called the Carnot triangle of the triangle ABC.

Proposition 40

A given triangle and its Carnot triangle are orthological triangles. The orthology centers are the orthocenter and the center of the circle circumscribed to the given triangle.

Proof

In *Figure 49*, we consider ABC an acute triangle of orthocenter H. Because O_bO_c is mediator of segment AH (common chord in Carnot circles (O_b), (O_c)) and $AH \perp BC$, it follows that $O_bO_c \parallel BC$.

Similarly, $O_aO_b \parallel AB$ and $O_aO_c \parallel AC$, hence the triangles ABC and $O_aO_bO_c$ have respectively parallel sides, therefore are homothetic and, consequently, orthological.

The orthology center of the triangle ABC in relation to $O_aO_bO_c$ is H, because the perpendiculars taken from A, B, C to the sides of the triangle $O_aO_bO_c$ are altitudes of the triangle ABC.

The perpendiculars from O_a, O_b, O_c to BC, CA, AB are the mediators of the triangle ABC, therefore the second orthology center is O, the center of the circle circumscribed to the triangle ABC.

Proposition 41

The Carnot circles of a triangle are congruent with the circle circumscribed to the triangle.

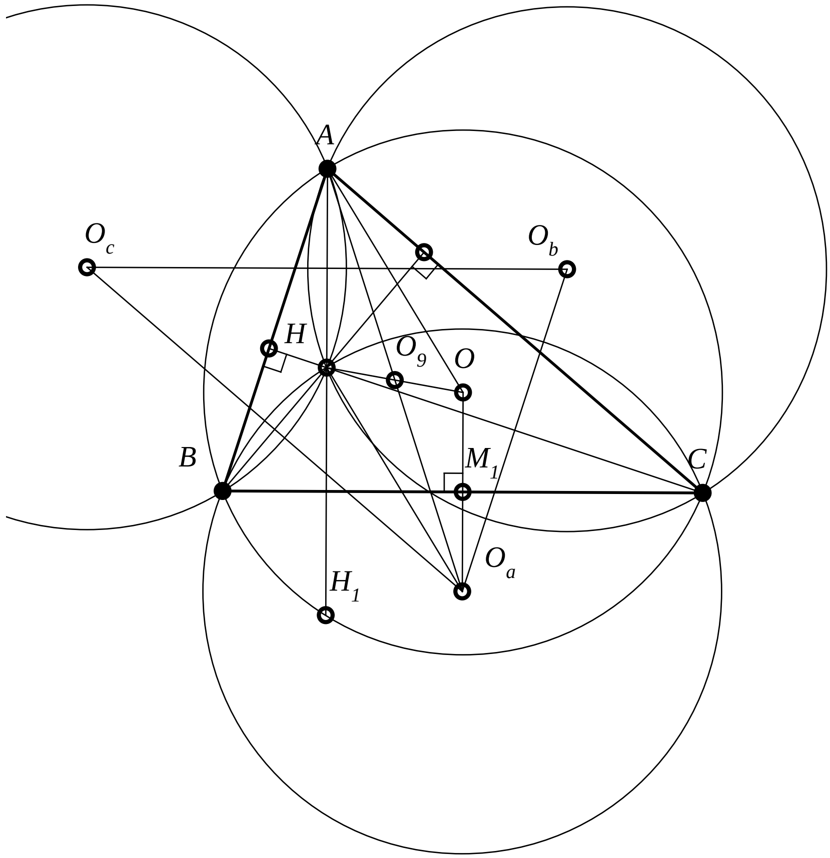

Figure 49

Proof

We denote by H_1 the symmetric of H to BC; according to Proposition 8, this belongs to the circle circumscribed to the triangle ABC, hence the symmetric of the triangle BHC to BC is the triangle BH_1C, and consequently the symmetric of Carnot circle (O_a) to BC is the circumscribed circle of the triangle ABC, therefore these circles are congruent.

Observation 41

Propositions 40, 41 are true also if ABC is an obtuse triangle.

Proposition 42

The Carnot triangle $O_aO_bO_c$ is congruent with the triangle ABC.

Proof

Let M_1 be the midpoint of (BC); we have $OM_1 = \frac{1}{2}AH$; since M_1 is the midpoint of (OO_a), it follows that the quadrilateral AHO_aO is parallelogram. The parallelogram's center is the midpoint O_9 of segment OH, consequently AO_a passes through O_9. We notice that $\Delta OO_aO_b \equiv \Delta HAB$ (S.A.S.), hence $(AB) = (O_aO_b)$, similarly we find that $(BC) = (O_bO_c)$ and $(CA) = (O_aO_c)$. The triangles $O_aO_bO_c$ and ABC are congruent; also we have that its Carnot triangle is the homothetic of triangle ABC by homothety $h(O_9, -1)$ or equivalently the Carnot triangle is symmetric to O_9 of the triangle ABC.

Observation 42

Proposition 42 can be proved similarly in the case of the obtuse triangle.

Definition 29

A quartet of points (a quadruple) such that any of them is the orthocenter of the triangle determined by the other three points is called an orthocentric quadruple.

Remark 10

The quadruple formed by the vertices of a non-right triangle and its orthocenter is a non-right orthocentric quadruple.

Proposition 43

If ABC is a non-right triangle with H – orthocenter, and we denote by O the center of the circumscribed circle, and $O_aO_bO_c$ is Carnot triangle, then the triangles BHC and OO_bO_c, CHA and OO_aO_c, AHB and OO_aO_b are orthological. The proof derives from Proposition 40. The orthology centers of each pair of triangles in the hypothesis are the points O and H.

Observation 43

The triangles in the pairs mentioned in the previous Proposition are symmetrical to O_9 – the center of the circle of the nine points of the triangle ABC.

2.19 A triangle and its Fuhrmann triangle

Definition 30

It is called Fuhrmann triangle of the triangle ABC the triangle whose vertices are the symmetrics of the means of small arcs $\breve{BC}$, $\breve{CA}$, $\breve{AB}$ of the circumscribed circle to the sides BC, CA, respectively AB.

Observation 43

In *Figure 46*, the Fuhrmann triangle is denoted by $F_aF_bF_c$. The circle circumscribed to Fuhrmann triangle is called Fuhrmann circle.

Proposition 44

A given triangle and its Fuhrmann triangle are orthological triangle.

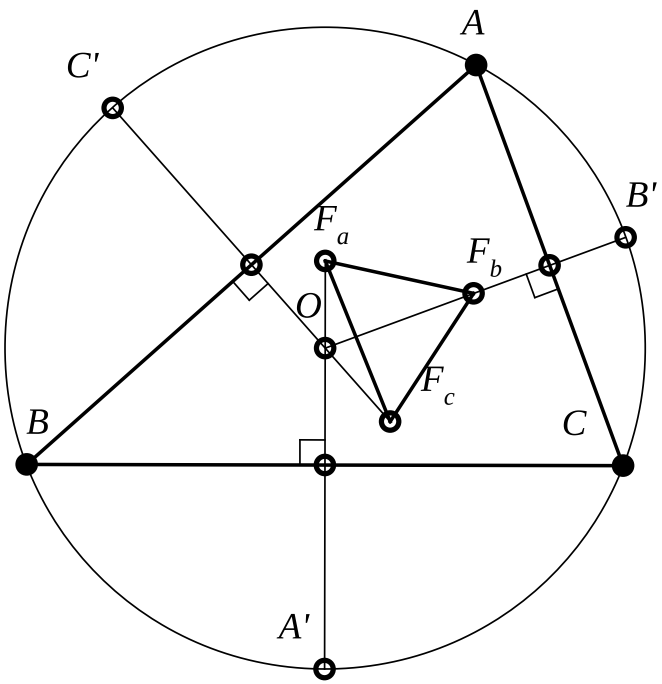

Figure 50

Proof

In *Figure 50*, we denoted by A', B', C' the midpoints of the small arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$. The lines $A'F_a$, $B'F_b$, $C'F_c$ are the mediator of the sides of the triangle ABC, hence they are concurrent in O – the center of the circumscribed circle, therefore the triangles $F_aF_bF_c$ and ABC are orthological, and O is orthology center. We will denote the other orthology center by P. We further prove some properties that will help us define P starting from the Fuhrmann triangle $F_aF_bF_c$ of the triangle ABC.

Proposition 45

In a triangle ABC, the lines determined by the orthocenter H and by the vertices of the Fuhrmann triangle are respectively perpendicular to the bisectors AI, BI, CI.

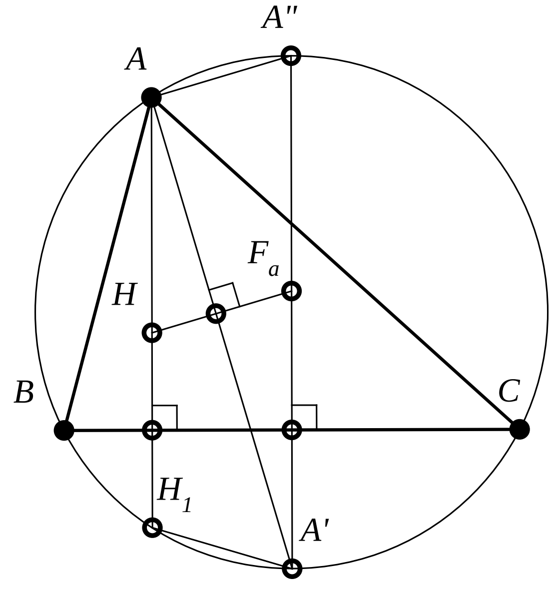

Figure 51

Proof

In *Figure 51*, we considered an acute triangle where the bisector AI intersects the circumscribed circle of the triangle in $A^{'}$, and AH intersects the same circle in H_1. We denote by $A^{''}$ the diameter of $A^{'}$ in the circumscribed circle. We have that $AH \parallel A'A''$ (the latter being mediator of BC), therefore the chords $A'A''$ and H_1A' are congruent. On the other hand, H_1 being the symmetric of H with respect to BC and A' the symmetric of F_a with respect to BC, we have that the quadrilateral H_1AF_aA' is isosceles trapezoid, consequently $HF_a = H_1A'$. From $AA'' = H_1A'$ and from previous relation, we obtain that $AA'' = HF_a$; this equality (together with $AH \parallel A''F_a$) leads to the ascertainment that AHF_aA'' is a parallelogram. Because $A''A \perp AA'$, it follows that $HF_a \perp AI$. Similarly, we prove the other perpendicularities.

Observation 44

The parallelism $AA'' \parallel HF_a$ can be deduced thus: AA'' is antiparallel with H_1A'; HF_a is antiparallel with H_1A', hence, AA'' and HF_a are parallel (see Proposition 3).

Theorem 12 (Housel's line)

In a triangle, the center of the inscribed circle, the gravity center and the Nagel point are collinear and $GN = 2IG$.

Proof 1

We denote by C_a the projection of I on BC and by I'_a the foot of cevian AN. The point I'_a is the projection on BC of the center I_a of the A-ex-inscribed circle, also we denote by A' the foot of the altitude in A (see *Figure 52*).

We prove that the triangles $AA'I'_a$ and IC_aA_1 are similar (CA_1 is the midpoint of BC).

Because the points C_a and I'_a are isotomic, we have: $BC_a = CI'_a = p - b$. We calculate:

$I'_aA' = a - BA' - I'_aC = a - c\cos B - (p - b)$.

Because $c\cos B = \frac{a^2+c^2-b^2}{2a}$ we obtain $I'_aA' = \frac{p(b-c)}{a}$. We have: $C_aA_1 = \frac{1}{2}(b - c)$, $IC_a = r = \frac{S}{p}$, $AA' = \frac{2S}{a}$. It follows that: $\frac{I'_aA'}{A_1C_a} = \frac{AA'}{IC_a} = \frac{2p}{a}$.

The indicated triangles are similar, and consequently: $IA_1 \parallel AH$.

Applying Menelaus's theorem in the triangle AI'_aC for transverses B-N-I'_b we find that $\frac{AN}{AI'_a} = \frac{p}{a}$ (I'_b - the foot of Nagel cevian BN). We denote $\{G'\} = IN \cap AA_1$, we have: $\frac{IG'}{G'N} = \frac{IA_1}{AN} = \frac{1}{2}$, which shows that G' divides the median AA_1 by ratio $\frac{2}{1}$, hence $G' = G$ – the gravity center of the triangle ABC and $2IG = GN$.

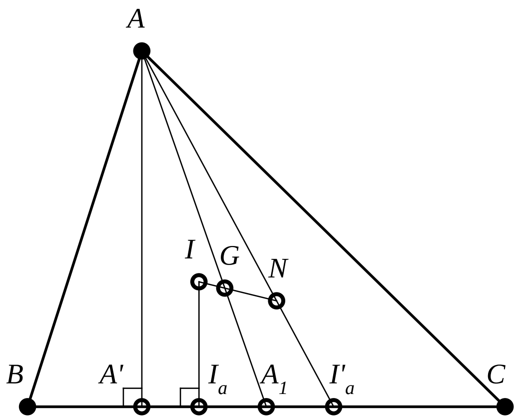

Figure 52

Proof 2

Let P be a point in the plane of the triangle ABC; because $\frac{BI'_a}{I'_aC} = \frac{p-c}{p-b}$, we have: $\overrightarrow{PI'_a} = \frac{\overrightarrow{PB} + \frac{p-c}{p-b} \cdot \overrightarrow{PC}}{1 + \frac{p-c}{p-b}}$. We obtain: $\overrightarrow{PI'_a} = \frac{p-b}{a} \cdot \overrightarrow{PB} + \frac{p-c}{a} \overrightarrow{PC}$. Since $\frac{AN}{NI'_a} = \frac{Q}{p-a}$, it follows that $\overrightarrow{PN} = \frac{\overrightarrow{PA} + \frac{a}{p-a} \cdot \overrightarrow{PI'_a}}{1 + \frac{a}{p-a}}$, therefore the position vector of Nagel point is: $\overrightarrow{PN} = \frac{p-a}{p} \cdot \overrightarrow{PA} + \frac{p-b}{p} \cdot \overrightarrow{PB} + \frac{p-c}{p} \cdot \overrightarrow{PC}$.

Considering in this relation $P = G$ (the gravity center) and taking into account that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$, we get:

$$\overrightarrow{GN} = -\frac{1}{p}\left(a\overrightarrow{GA} + b\overrightarrow{GB} + c\overrightarrow{GC}\right). \tag{1}$$

It is known that the position vector of the center of the inscribed circle I is:

$\overrightarrow{PI} = \frac{1}{2r}\left(a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right)$, taking $P = G$, we have:

$$\overrightarrow{GI} = \frac{1}{2p}\left(a\overrightarrow{GA} + b\overrightarrow{GB} + c\overrightarrow{GC}\right). \tag{2}$$

The relations (1) and (2) show that I, G, H are collinear, and that $2IG = GH$.

Proposition 46

The Fuhrmann circle of the triangle ABC has as diameter the segment HN determined by the orthocenter and by Nagel point.

Proof

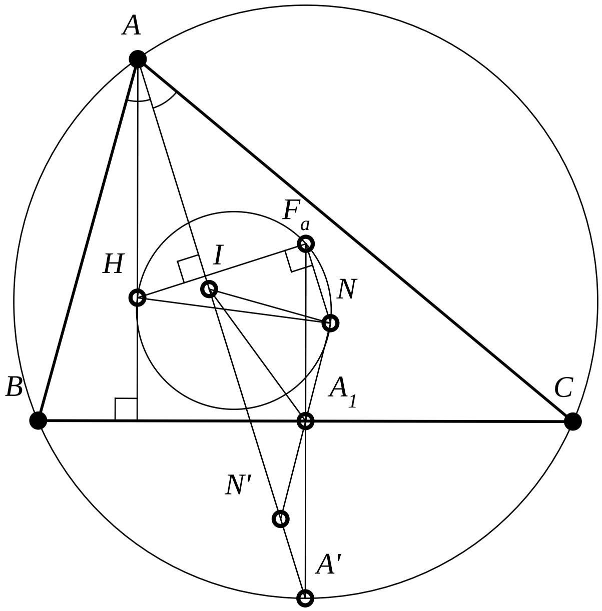

Figure 53

In *Figure 53*, we considered an acute triangle of orthocenter H. From Theorem 11, it follows that $IA_1 \parallel AN$ and $2IA_1 = AN$ (A_1, the midpoint of BC). We build N' – the intersection of the line NA_1 with AI; because in the triangle $N'AN$ we have $IA_1 \parallel AN$ and $2IA_1 = AN$, it follows that IA_1 is midline in the triangle $N'NA$, therefore $N'A_1 = A_1N$; having $A'A_1 = A_1F_a$, we obtain that the quadrilateral $NF_aN'A'$ is parallelogram, consequently $NF_a \parallel AI$. We proved that $HF_a \perp AI$, it follows therefore that $m\left(\widehat{HF_aN}\right) = 90^0$, ie. F_a belongs to the circle of diameter HN. Similarly, it is shown that F_b, F_c are on the circle of diameter HN, and similarly the Proposition can be proved in the case of the obtuse triangle.

Proposition 47

The measures of the angles of the Fuhrmann triangle of the triangle ABC are $90^0 - \frac{A}{2}$, $90^0 - \frac{B}{2}$, $90^0 - \frac{C}{2}$.

Proof

HF_a is perpendicular to the bisector AI, and HF_b is perpendicular to the bisector BI. Because $m(\widehat{AIB}) = 90^0 + \frac{C}{2}$, and the angle $\widehat{F_aNF_b}$ is its supplement; it has the measure $90^0 - \frac{C}{2}$. On the other hand, $\widehat{F_aHF_b} \equiv \widehat{F_aF_cF_b}$, therefore $m(\widehat{F_aF_cF_b}) = 90^0 - \frac{C}{2}$. Similarly, it follows that $m(\widehat{F_bF_aF_b}) = 90^0 - \frac{A}{2}$ and $m(\widehat{F_cF_bF_a}) = 90^0 - \frac{B}{2}$.

Proposition 48

The second orthology center, P, of the triangle ABC and of the Fuhrmann triangle, $F_aF_bF_c$, is the intersection of the circles: $\mathcal{C}(F_a; F_aB)$, $\mathcal{C}(F_b; F_bC)$, $\mathcal{C}(F_c; F_cA)$.

Proof

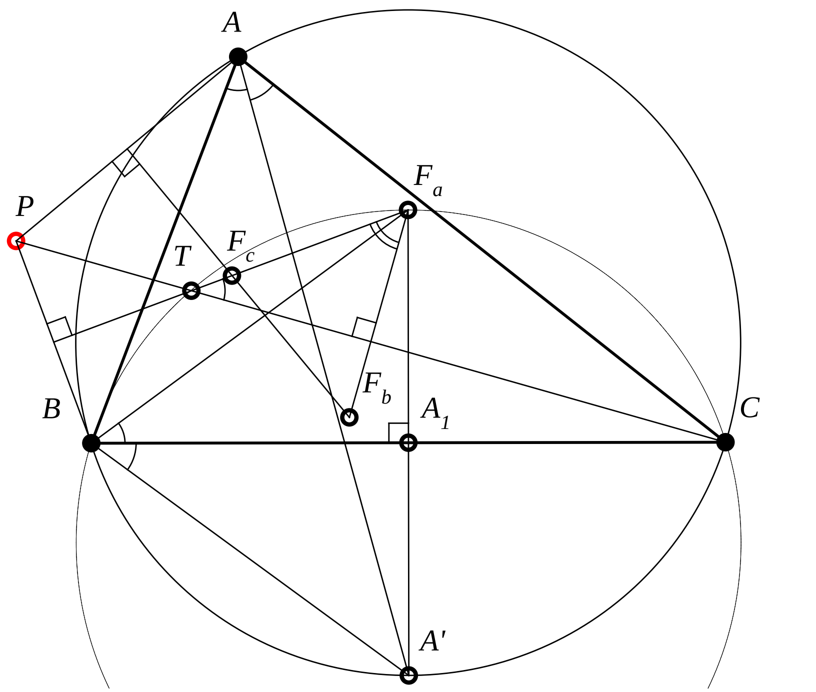

Figure 54

We denote by T the intersection of the perpendicular taken from C to F_aF_b with the line F_aF_c (see *Figure 54*). Since $m(\widehat{F_bF_aF_c}) = 90^0 - \frac{A}{2}$, it follows that $m(\widehat{F_cBC}) = \frac{A}{2}$, also, $m(\widehat{F_aBC}) = \frac{A}{2}$; we obtain that the point T is on the circumscribed circle of the triangle ABC.

If P is the center of orthology of triangles ABC and $F_aF_bF_a$, we note that $\sphericalangle BPC \equiv \sphericalangle TF_aF_b$ (angles with sides respectively perpendicular), also, we have $\sphericalangle TF_aB \equiv \sphericalangle BCT$ (the quadrilateral BCF_aT is inscribable). The angles BCT and $F_bF_aA_1$ are also congruent (sides respectively perpendicular); we obtain that: $\sphericalangle TF_aB \equiv \sphericalangle F_bF_aA_1$, and then that $\sphericalangle BPC \equiv \sphericalangle BF_aA_1$ or that: $\sphericalangle BPC = \frac{1}{2}\sphericalangle BF_aC$. This last relation shows that the point P is on the circle of center F_a which passes through B and C. In the same way, we prove that P belongs to the circles: $\mathcal{C}(F_b, F_bC)$, $\mathcal{C}(F_c, F_cA)$.

Proposition 49

The symmetrics of vertices of an acute given triangle with respect to the sides of its H-circumpedal triangle are the vertices of Fuhrmann triangle of its H-circumpedal triangle.

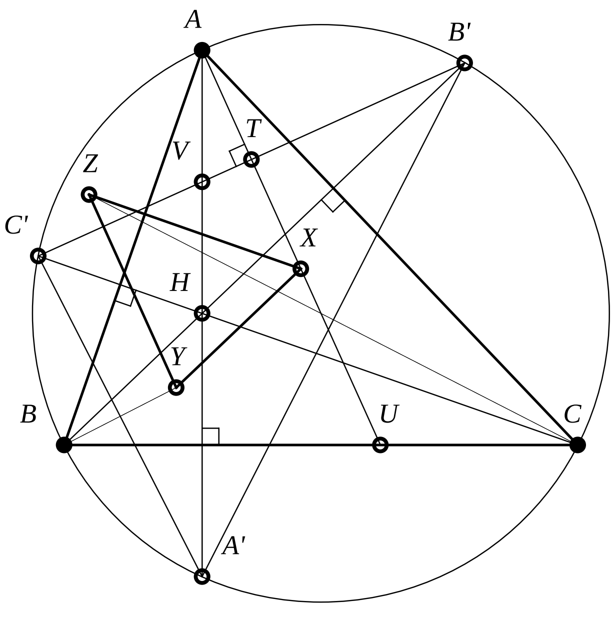

Figure 55

Proof

In *Figure 55*, we denote by $A'B'C'$ the H-circumpedal triangle of ABC and by X, Y, Z the symmetrics of the points A, B, C to $B'C'$, $C'A'$, respectively $A'B'$. We noticed that the H-circumpedal triangle of ABC is homothetic with the orthic triangle of ABC. Also, the triangle ABC and its orthic triangle are orthological, and O, the center of the circumscribed circle, is orthology center.

Because the perpendicular taken from A to $B'C'$ passes through O, it means that it is also the mediator of the side $B'C'$; hence, A is the midpoint of the arc $\overset{\frown}{B'C'}$ and X is vertex of the Fuhrmann triangle of the triangle $A'B'C'$.

Similar proof for the other vertices.

Theorem 13 (M. Stevanovic – 2002)

In a triangle acute, the orthocenter of Fuhrmann triangle coincides with the center of the circle inscribed in the given triangle.

Proof

We use *Figure 51*, where we note that XYZ is Fuhrmann triangle of the triangle $A'B'C'$ (H-circumpedal of ABC). It is sufficient to prove that H, the orthocenter of ABC and the center of the circle inscribed in $A'B'C'$, is the orthocenter of XYZ. We will prove that $XH \perp YZ$, showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.

We have:

$$\overrightarrow{XH} = \overrightarrow{AH} - \overrightarrow{AX}, \tag{1}$$

$$\overrightarrow{YZ} = \overrightarrow{YB} + \overrightarrow{BC} + \overrightarrow{CZ}. \tag{2}$$

Because Y and Z are the symmetrics of B, respectively C, with respect to $A'C'$, respectively $A'B'$, with the parallelogram rule, we have: $\overrightarrow{BY} = \overrightarrow{BA'} + \overrightarrow{BC'}$, $\overrightarrow{CZ} = \overrightarrow{CA'} + \overrightarrow{CB'}$; replacing these relations in (2), it results:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{CA'} + \overrightarrow{CB'} - \overrightarrow{BA'} - \overrightarrow{BC'}. \tag{3}$$

But $\overrightarrow{BC} + \overrightarrow{CA'} + \overrightarrow{A'B} = 0$, therefore:

$$\overrightarrow{YZ} = \overrightarrow{CB'} + \overrightarrow{C'B}. \tag{4}$$

Because:

$$\overrightarrow{BC} + \overrightarrow{CB'} + \overrightarrow{B'C'} + \overrightarrow{C'B} = \vec{0}, \tag{5}$$

we get: $\overrightarrow{YZ} = \overrightarrow{CB} + \overrightarrow{C'B'}$. We evaluate: $\overrightarrow{XH} \cdot \overrightarrow{YZ} = \left(\overrightarrow{AH} - \overrightarrow{AX}\right) \cdot \left(\overrightarrow{CB} - \overrightarrow{C'B'}\right)$, it follows that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{CB} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{CB} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$.

But $AH \perp CB$, so $\overrightarrow{AH} \cdot \overrightarrow{CB} = 0$ and $AX \perp C'B'$.

Therefore $\overrightarrow{AX} \cdot \overrightarrow{C'B'} = 0$; so:

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} + \overrightarrow{AX} \cdot \overrightarrow{BC}. \quad (6)$$

We denote by $\{U\} = AX \cap BC$ and $\{V\} = AH \cap B'C'$, we have:

$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot \cos \widehat{AX, BC} = AX \cdot BC \cdot \cos \widehat{AUC}$,

$\overrightarrow{AN} \cdot \overrightarrow{C'B'} = AN \cdot C'B' \cdot \cos \widehat{AH, B'C'} = AH \cdot C'B' \cdot \cos(\widehat{AYC'})$.

We observe that $\sphericalangle AUC \equiv \sphericalangle AYC'$ (sides respectively perpendicular). The point B' is the symmetric of H with respect to AC, therefore $\sphericalangle HAC \equiv \sphericalangle CAB'$; similarly, it is obtained that $\sphericalangle HAB \equiv \sphericalangle BAC'$ and from these two relations: $\sphericalangle B'AC' = 2\hat{A}$.

Sinus theorem in the triangles $AB'C'$ and ABC provides the relations $B'C' = 2R \sin 2A$, $BC = 2R \sin 2A$ (R – the radius of the circumscribed circle). We show that $AX \cdot BC = AH \cdot B'C'$ is equivalent to $AX \cdot 2R \sin A = AH \cdot 2R \sin 2A$, videlicet with $AX = 2AH \cdot \cos A$. From $\sphericalangle B'AC' = 2\hat{A}$ and $AX \perp B'C'$, it follows that $\sphericalangle TAC' = \hat{A}$. On the other hand, $AC' = AH$ (because $AH = AB'$ and $AB' = AC'$). Since $AT = \frac{1}{2} AX$ and $AT = AC' \cos A = AH \cdot \cos A$, it follows that $AX = 2AH \cos A$. The angles of lines $\sphericalangle(AH, C'B')$ and $\sphericalangle(AX, BC)$ are supplementar, therefore we get from (6) that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$

Similarly, we prove that $YH \perp XZ$, so H is the orthocenter of the Fuhrmann triangle XYZ.

Proposition 50

The Fuhrmann triangle and *the Carnot triangle* corresponding to a given triangle are orthological.

Proof of this property is immediate if we observe that O_a and F_a belong to the mediator of the side BC. The orthology center of the Carnot triangle $O_a O_b O_c$ in relation to the Fuhrmann triangle $F_a F_b F_c$ is O – the center of the circle circumscribed to the triangle ABC.

3

ORTHOLOGICAL DEGENERATE TRIANGLES

In this section, we will define the concept of orthopole of a line in relation to a triangle, and we will establish connections between this notion and the orthological triangles.

3.1 Degenerate triangles; the orthopole of a line

Definition 31

We say that a triplet of distinct collinear points, joined together by segments, is a degenerate triangle of sides $(\boldsymbol{AB}), (\boldsymbol{BC}), (\boldsymbol{CA})$ and of vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$.

We will admit that any two parallel lines are "concurrent" in a point "thrown to infinity"; consequently, we can formulate:

Proposition 51

Two degenerate triangles ABC and $A_1B_1C_1$ are orthological.

An important case is when we consider a triangle ABC – a scalene triangle, and a triangle $A_1B_1C_1$ – a degenerate triangle.

Theorem 14 (The Orthopole Theorem; Soons – 1886)

If ABC is a given triangle, d is a certain line, and A_1, B_1, C_1 are the orthogonal projections of vertices A, B, C on d, then the perpendiculars taken from A_1, B_1, C_1 respectively to BC, CA and AB are concurrent in a point called the orthopole of the line d in relation to the triangle ABC.

Proof 1 (Niculae Blaha, 1949)

We consider A_1, B_1, C_1 – the vertices of a degenerate triangle. The lines AA_1, BB_1, CC_1, being perpendicular to d, are concurrent to infinity; consequently, the triangles ABC and $A_1B_1C_1$ are orthological. By the theorem of orthological triangles, we have that the perpendiculars taken from A_1, B_1, C_1 respectively to BC, CA and AB are concurrent as well.

Proof 2

We denote by A', B', C' the orthogonal projections of the points A_1, B_1, C_1 respectively on BC, CA and AB (see *Figure 56*).

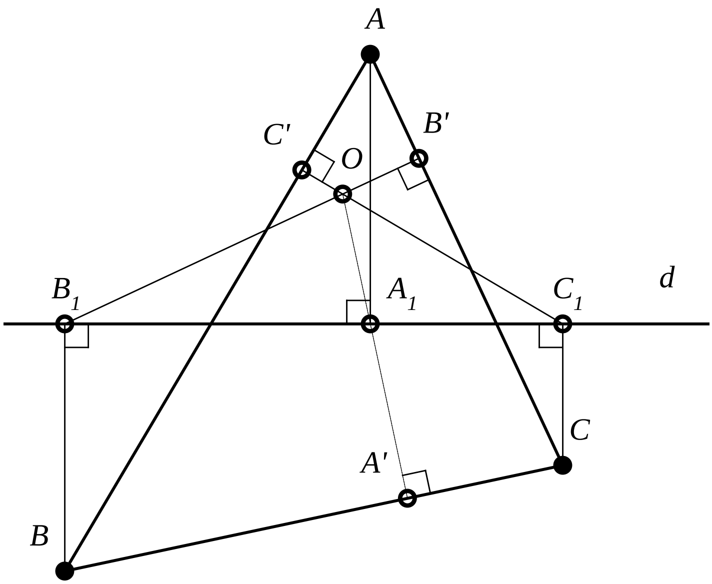

Figure 56

We have that:

$$A'B^2 - A'C^2 = A_1B^2 - A_1C^2 = BB_1^2 + B_1A_1^2 - CC_1^2 - C_1A_1^2$$
$$B'C^2 - B'A^2 = B_1C^2 - B_1A^2 = CC_1^2 + B_1C_1^2 - AA_1^2 - B_1A_1^2$$
$$C'A^2 - C'B^2 = C_1A^2 - C_1B^2 = AA_1^2 + A_1C_1^2 - BB_1^2 - B_1C_1^2$$

From the three previous relations, we obtain that:

$A'B^2 - A'C^2 + B'C^2 - B'A^2 + C'A^2 - C'B^2 = 0$.

According to Carnot's theorem, it follows that the lines $A'A_1$, $B'B_1$, $C'C_1$ are concurrent.

Proof 3 (Traian Lalescu – 1915)

The points B_1 and C_1 are equally distant from the midpoint M_a of the side BC because the perpendicular taken from M_a to d is mediator of the side $B_1C_{i_0}$.

Similarly, A_1 and C_1 are equally distant from M_b – the midpoint of AC, and A_1 and B_1 are equally distant from M_c (see *Figure 57*).

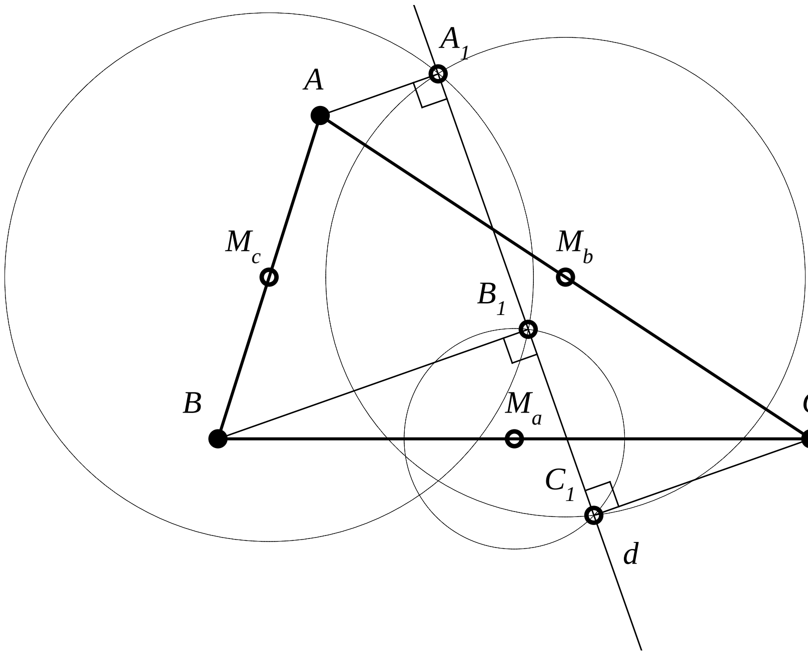

Figure 57

We consider the circles $\mathcal{C}(M_a, M_aB_1)$, $\mathcal{C}(M_b, M_bC_1)$, $\mathcal{C}(M_c, M_cA_1)$. The first two circles, having in common the point C_1, still have another common point, and their common chord is perpendicular to M_aM_b (the center line), since M_aM_b is parallel with AB, which means that the common chord is perpendicular to AB.

Similarly, the circles of centers M_b and M_c have in common the chord that have the extremity A_1 which, being perpendicular to M_bM_c, is also perpendicular to BC.

Finally, the circles of centers M_a and M_c have in common a chord that has the extremity B_1, and it is perpendicular to AB.

The three circles, having two by two a common chord and their centers being noncollinear – according to a theorem, it follows that these chords are concurrent.

Their point of concurrency, namely the radical center of the considered circles, is the orthopole of the line d in relation to the triangle ABC.

Definition 32

If ABC is a triangle; A_1, B_1, C_1 are the projections of its vertices on a line d; and O is the orthopole of the line d, – then the circumscribed circles of triangles OA_1B_1, OB_1C_1, OC_1A_1 are called orthopolar circles of the triangle ABC.

Observation 45

From this Definition and from Proof 3 of the previous theorem, it follows that the centers of the orthopolar circles are midpoints of the sides of the given triangle.

3.2 Simson line

Theorem 15 (Wallace, 1799)

The projections of a point belonging to the circumscribed circle of a triangle on the sides of that triangle are collinear points.

Proof

Let M be a point on the circumscribed circle of the triangle ABC and $A_1B_1C_1$ the orthogonal projections of BC, CA, respectively AB (see *Figure 58*).

From the inscribable quadrilateral $ABMC$, it follows that: $\sphericalangle MCB \equiv \sphericalangle MAC_1$, and from this relation we obtain that:

$\sphericalangle CMA_1 \equiv \sphericalangle C_1MA$, (1)

being the complement of the previous angles.

The quadrilateral MB_1A_1C and MB_1AC_1 are inscribable, hence:

$\sphericalangle A_1B_1C \equiv \sphericalangle A_1MC$, (2)

$\sphericalangle C_1BA \equiv \sphericalangle C_1MA$. (3)

The relations (1), (2) and (3) implies $\sphericalangle A_1B_1C \equiv \sphericalangle AB_1C_1$, consequently the points A_1, B_1 and C_1 are collinear.

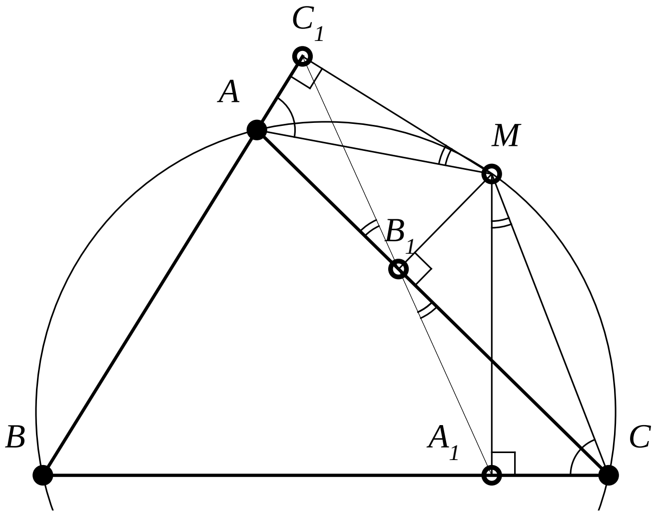

Figure 58

Observation 46

a) It is called the Simson line of a point M to the triangle ABC the line of the points A_1, B_1, C_1.

b) If $A_1B_1C_1$ is the Simson line of the triangle ABC in relation to the point M, we can consider $A_1B_1C_1$ as a degenerate triangle. This triangle is orthological with the triangle ABC, the point M being an orthology center, and the other orthology center being "thrown to infinity".

Theorem 16 (The reciprocal of the Simson-Wallace theorem)

Let ABC be a given triangle and $A_1B_1C_1$ a degenerate triangle, with $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$. The orthology center of the triangle $A_1B_1C_1$ in relation to ABC is a point that belongs to the circle circumscribed to the triangle ABC.

Proof

Because ABC is orthological in relation to $A_1B_1C_1$ (the orthology center is "thrown to infinity"), it follows that $A_1B_1C_1$ is orthological in relation to ABC. Let M be the orthology center (see *Figure 59*).

Because $A_1B_1C_1$ is a degenerate triangle, we have:

$\sphericalangle A_1B_1C \equiv \sphericalangle C_1B_1A$. (1)

On the other hand, from the inscribable quadrilaterals MB_1AC_1 and MB_1A_1C, we note that:

$\sphericalangle A_1B_1C \equiv \sphericalangle A_1MC,$ (2)

$\sphericalangle C_1B_1A \equiv \sphericalangle AMC_1.$ (3)

From relations (1), (2) and (3), we get:

$\sphericalangle A_1MC \equiv \sphericalangle C_1MA.$ (4)

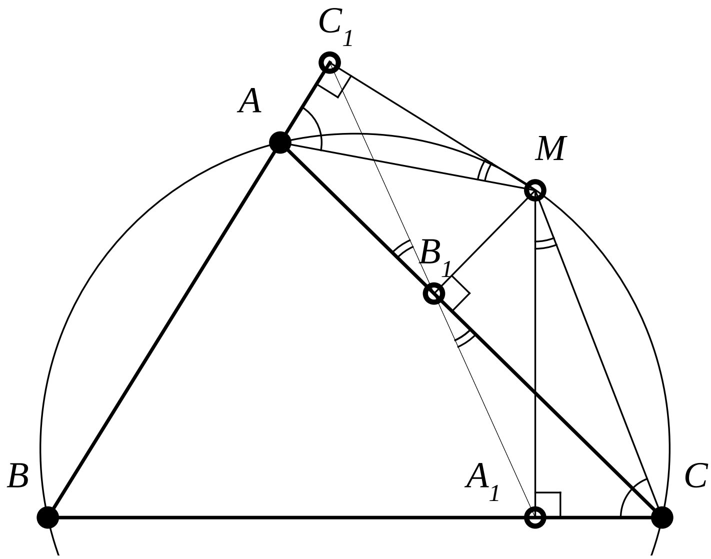

Figure 59

The angles from the relation (4) are complements of the angles $\sphericalangle MCA_1$ and $\sphericalangle MAC_1$, and consequently:

$\sphericalangle MCA_1 \equiv \sphericalangle MAC_1.$ (5)

The relation (5) shows that the quadrilateral $MABC$ is inscribable, therefore M belongs to the circle circumscribed to the triangle ABC.

Proposition 52

In the triangle ABC, the point M belongs to the circle circumscribed to the triangle. We denote by M' the second intersection of the perpendicular MA_1 taken from M to BC on the circle ($A_1 \in BC$). Then, the Simson line of the point M is parallel with AM'.

Proof

The quadrilateral MB_1A_1C is inscribable (B_1 and C_1 are feet of perpendiculars taken from M to AC respectively AB, see *Figure 60*); it follows that $\sphericalangle B_1A_1M \equiv \sphericalangle B_1CM$ (1).

But $\sphericalangle B_1A_1M \equiv \sphericalangle AM'M$ (2) (having the same measure $\frac{1}{2}m(\widehat{AM'})$.

It follows that $\sphericalangle AM'M \equiv \sphericalangle B_1A_1M$ and hence $A_1B_1 \parallel AM'$.

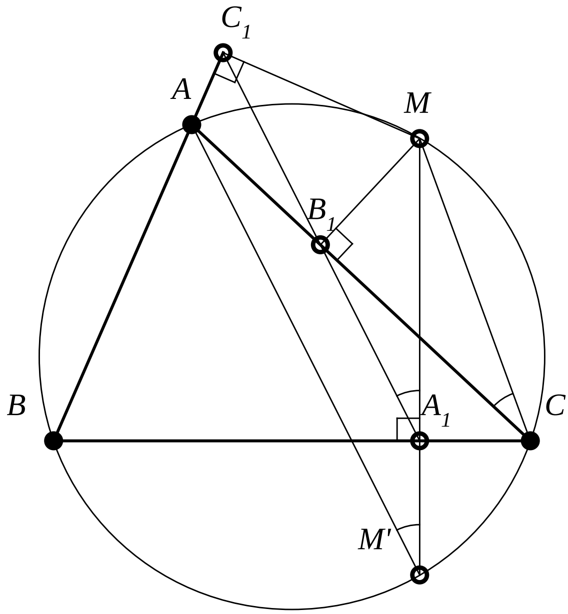

Figure 60

Observation 47

Similarly, it is proved that the Simson line of the point M' is parallel with AM.

Proposition 53

The Simson lines of two diametrically opposed points in the circumscribed circle of the triangle ABC are perpendicular.

Proof

Let M and M' two diametrically opposed points in the circumscribed circle of the triangle ABC (see *Figure 61*).

We denote by M_1 the second intersection with the circle of the perpendicular MA_1 taken to BC, and with M_1' the second intersection with the circle of the perpendicular $M'A_1'$ taken to BC.

From Proposition 52, it follows that the Simson line of the point M is parallel with the line AM_1, and the Simson line of M' is parallel with the line MA_1', because the lines AM_1 and AM_1' are perpendicular, we obtain that the mentioned Simson lines are perpendicular.

Indeed, the quadrilateral $MM_1M'M_1'$ is a rectangle because the points M, M' are diametrically opposed; it follows that A_1 and A_1', their orthogonal projections, are equally distanced from the center O of the circumscribed circle and hence the chords MM_1 and $M_1'M'$ are parallel; being equally distant from O, they are congruent. The points M_1, O, M_1' are collinear and consequently $M_1A \perp M_1'A$.

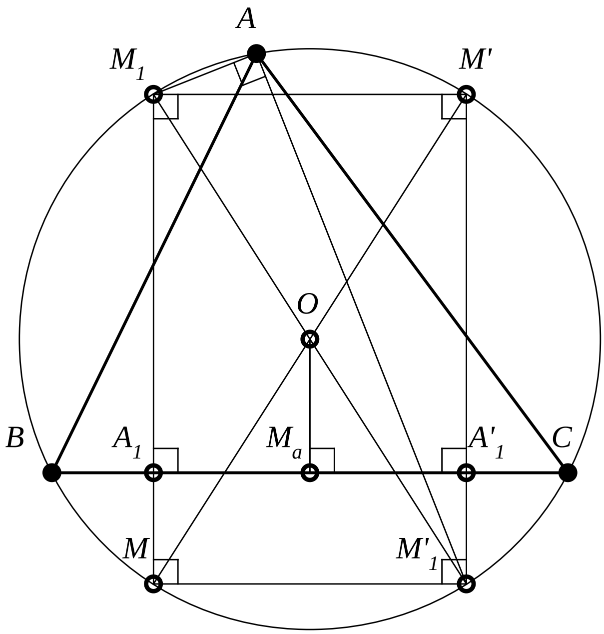

Figure 61

Observation 48

The Simson lines of the the extremities of a diameter are isotomic transverse. Indeed, the points A_1 and A'_1 and analogs are isotomic points.

Theorem 17 (J. Steiner)

The Simson line of a point M that belongs to the circumscribed circle of the triangle ABC contains the midpoint of the segment determined by M and by the orthocenter H of the triangle ABC.

Proof

We build the symmetric of the circle circumscribed to the triangle ABC with respect to the side BC. We denote by A_1 the projection of the point M on BC and with M' the intersection of MA_1 with the circumscribed circle of the triangle ABC; also, we denote by M'' the intersection of chord (MM') with the circle symmetric with the circle circumscribed to the triangle ABC (see *Figure 62*).

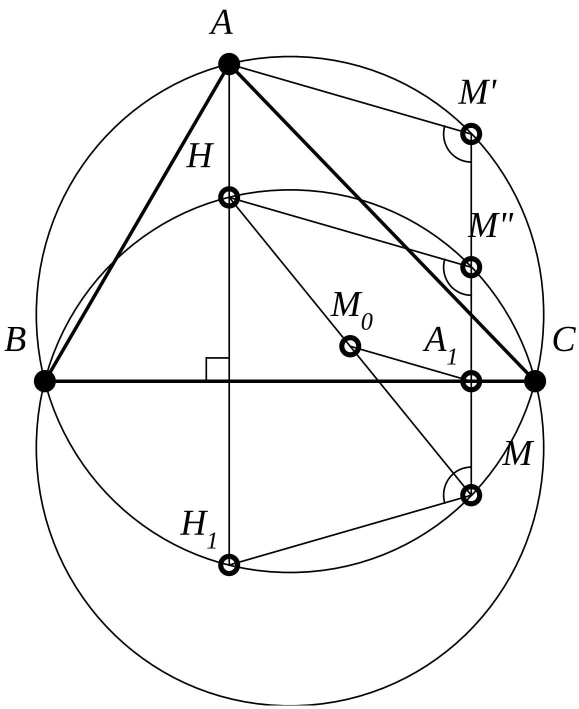

Figure 62

Also, we denote by H_1 the second intersection of the altitude AH with the circumscribed circle of the triangle ABC. From Proposition 52, we have that the Simson line of the point M is the parallel taken through A_1 with AM'.

On the other hand, we have that the quadrilaterals AH_1MM' and $H_1MM''H$ are isosceles trapezoids; the first – because $AH_1 \parallel MM'$; and the second – because $HH_1 \parallel MM''$, and H_1 is the symmetric of H with respect to BC (see Proposition 7), and, also, M'' is the symmetric of M with respect to BC due to the performed construction.

From the considered isosceles trapezoids, we obtain that $MM'' \parallel AM'$, and then the Simson line of the point M is parallel with HM'', because A_1 is the midpoint of the segment $[MM'']$; it follows that the Simson line of the point M is middle line in the triangle $MM''H$ and consequently passes through the midpoint of the segment MH.

Proposition 54

The midpoint of the segment determined by the point M that belongs to the circle circumscribed to the triangle ABC and by the orthocenter H of the triangle belongs to the circle of nine points of the triangle ABC.

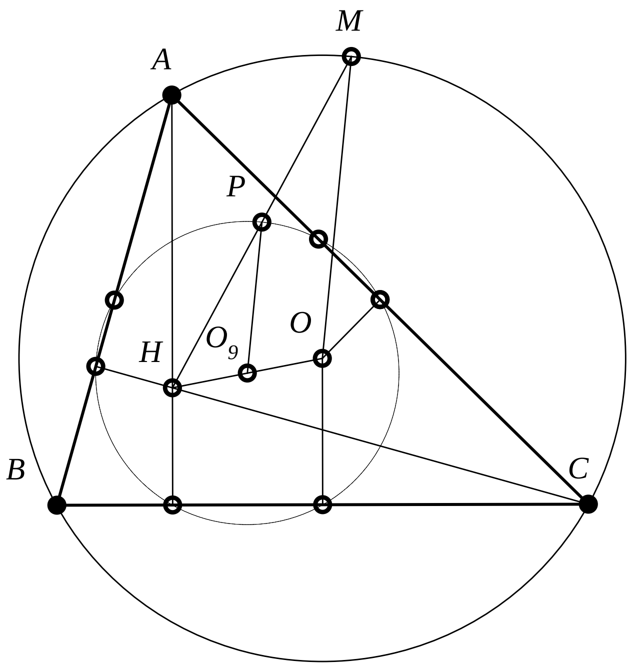

Figure 63

Proof

Let O be the center of the circumscribed circle, and let O_9 – the center of the circle of nine points (the midpoint of the segment (OH), see *Figure 63*). If P is the midpoint of the segment $[HM]$, then in the triangle HOM, PO_9 is middle line, therefore $PO_9 = \frac{OM}{2}$; consequently, $PO_9 = \frac{R}{2}$, which shows that the point P belongs to the circle of nine points.

Remark 11

This Proposition shows that the circle of nine points is homothetic to the circle circumscribed to the triangle ABC by homothety of center H and ratio $\frac{1}{2}$.

Theorem 18

Let ABC be a given triangle, and $A_1B_1C_1$, $A_2B_2C_2$ – degenerate triangles, with $A_i \in BC$, $B_i \in CA$, $C_i \in AB$, $i \in \{1, 2\}$; the orthology centers of the latter in relation to ABC are M_1, M_2. We denote by $\{Q\} = A_1B_1 \cap A_2B_2$; then Q is the orthopole of the line M_1M_2 in relation to the triangle ABC.

Proof

We denote by B' and C' the projections of points B and C on the line M_1M_2 (see *Figure 64*). We prove that the quadrilateral $A_1B'C'A_2$ is inscribable.

Indeed, from the inscribable quadrilateral M_2A_2CC', we note that:

$\sphericalangle M_2C'A_2 \equiv \sphericalangle M_2CA_2$. (1)

From the inscribable quadrilateral BM_1M_2C, we have:

$\sphericalangle M_2CB \equiv \sphericalangle BM_1B'$. (2)

The inscribable quadrilateral $B'BA_1M_1$ leads to:

$\sphericalangle BM_1B' \equiv \sphericalangle B'A_1B$. (3)

From the relations (1), (2) and (3), we obtain that: $\sphericalangle M_2C'A_2 \equiv \sphericalangle B'A_1B$, therefore the points B', A_1, A_2, C' are concyclic.

We denote by Q' the orthopole of the line M_1M_2 in relation to the triangle ABC. Having $B'Q' \perp AC$ and $C'Q' \perp AB$, it follows that $m\left(\widehat{B'Q'C'}\right) = 180^0 - A$. We prove that Q' is on the circle of the points B', A_1, A_2, C'. It is sufficient to show that $m\widehat{B'A_2C'} = 180^0 - A$.

We have: $m\widehat{B'A_2C'} = 180^0 - \left[m\widehat{B'A_2B} + m\widehat{C'A_2C}\right]$.

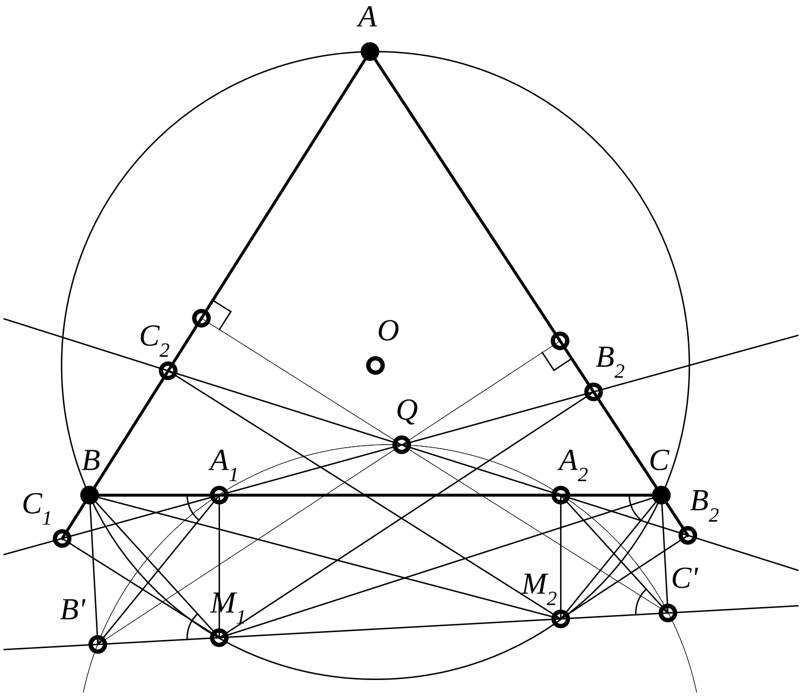

Figure 64

From the inscribable quadrilateral BA_2M_2B', we note that $\widehat{B'A_2B} \equiv \widehat{B'M_2B}$. On the other hand, M_1 and M_2 are on the circumscribed circle of the triangle ABC, so we have: $\sphericalangle B'M_2B \equiv \sphericalangle BAM_1$, therefore $\sphericalangle B'A_2B \equiv \sphericalangle BAM_1$. Similarly, we find that $\widehat{C'A_2C} \equiv \widehat{M_1AC}$. We obtain that $\sphericalangle B'A_2B + \sphericalangle C'A_2C = \sphericalangle A$ and, consequently, $m(\sphericalangle B'A_2C') = 180^0 - A$.

Let us show that $Q' \equiv Q$. It is sufficient to prove that $Q' \in A_1C_1$ and $Q' \in A_2B_2$. Because $Q' \in A_1C_1$, it is necessary to prove that $\sphericalangle Q'A_1A_2 \equiv \sphericalangle C_1A_1B$. But $\sphericalangle Q'A_1A_2 \equiv \sphericalangle C_2C'Q'$. Also, $BM_1 \parallel A_2C'$ (because $\sphericalangle BM_1B' \equiv \sphericalangle BCM_2 \equiv \sphericalangle A_2C'M_2$), $M_1C_1 \parallel C'Q'$ (being perpendicular to AB). It follows that $\sphericalangle A_2C'Q' \equiv \sphericalangle C_2M_1B$.

Because the quadrilateral $BC_1M_1A_1$ is inscribable, $\sphericalangle C_1MB \equiv \sphericalangle C_1A_1B$. Thus, we have that $\sphericalangle Q'A_1A_2 \equiv \sphericalangle BA_1C_1$, as such Q' is on the Simson line of the point M_1.

Similarly, we show that $Q' \in A_2B_2$, therefore $Q' \equiv Q$.

Remark 12

1. The theorem shows that the orthopole of the line M_1M_2 (which intersect the circumscribed circle of the triangle ABC in M_1 and M_2) is the intersection of Simson lines of the points M_1 and M_2 in relation to ABC.

2. From the theorem, it follows that, if a line d "rotates" around a point M from the circle to the circle circumscribed to the triangle ABC, then the orthopole of the line d belongs to the Simson line of the point M in relation to the triangle ABC.

3. Also, from this theorem, we obtain that:

The orthopole of a diameter of the circumscribed circle of a triangle in relation to this triangle belongs to the circle of nine points of the triangle.

Proposition 55

The orthopolar circles of a triangle relative to a tangent taken to the circumscribed circle of the triangle are tangent to the sides of the triangle. The tangent points are collinear, and their lines contain the orthopole of the tangent.

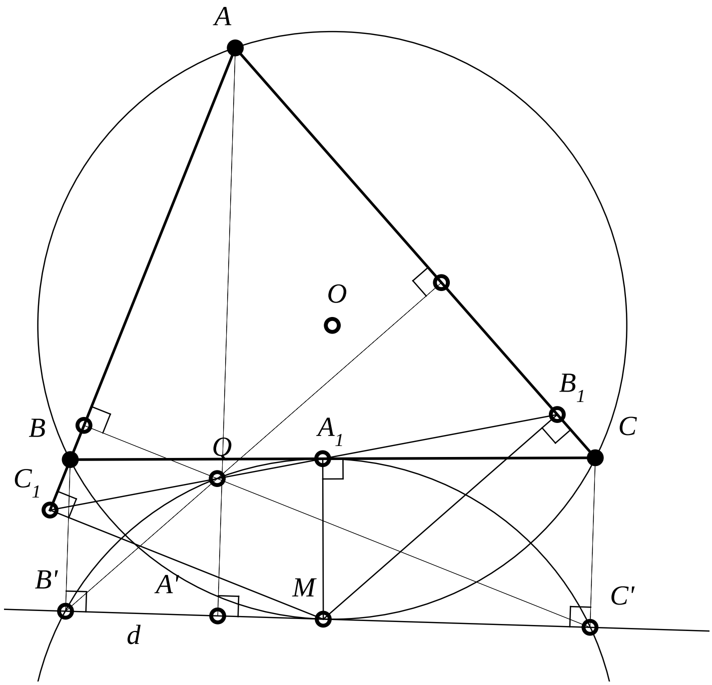

Figure 65

Proof

Let M the tangent point with the circle of tangent d (see *Figure 65*). We denote by A', B', C' the projections of vertices of the triangle on the line d and by A_1, B_1, C_1 the projections of M on the sides of the triangle ABC. Also, we denote by Q the orthopole of tangent d, and by Q_a – the center of the orthopolar circle circumscribed to the triangle $B'QC'$.We prove that A_1 belongs to this circle.

We have $m\left(\widehat{B'QC'}\right) = 180^0 - A$. (1)

We prove as well that $m\left(\widehat{B'A_1C'}\right) = 180^0 - A$. (2)

The quadrilateral $A_1MB'B$ and $A_1MC'C$ are inscribable; from here, we note that:

$\sphericalangle B'A_1B \equiv \sphericalangle B'MB$, (3)

$\sphericalangle C'A_1C \equiv \sphericalangle C'MC$. (4)

Since d is tangent to the circle circumscribed to the triangle ABC, we have:

$\sphericalangle B'MB \equiv \sphericalangle BAM$, (5)

$\sphericalangle C'MB \equiv \sphericalangle CAM$. (6)

But: $\sphericalangle BAM + \sphericalangle CAM = \sphericalangle A$. (7)

Also: $m(\sphericalangle B'A_1C') = 180^0 - m(\sphericalangle B'A_1B) + m(\sphericalangle C'A_1C)$, therefore we obtain the relation (2), which, together with (1), prove the concyclicity of points B', Q, A_1, C'.

We prove that the orthopolar circle (Q_a) is tangent in A_1 to the BC.

From the inscribable quadrilateral $A_1MC'C$, we have the relation (4), which, together with $\sphericalangle CMC' \equiv \sphericalangle$MBC (consequence of the fact that $B'C'$ is tangent to the circumscribed circle) and with $\sphericalangle MBA_1 \equiv \sphericalangle A_1B'$M (the quadrilateral $A_1MB'B$ is inscribable) lead to $\sphericalangle A_1B'M \equiv \sphericalangle C'A_1C$, which shows that BC is tangent to the orthopolar circle (Q_a). Similarly, we show that B_1 and C_1 are contact points with AC respectively AB of the orthopolar circles (Q_b) and (Q_c). The points A_1, B_1, C_1 belong to the Simson line of the point M. We prove that Q belongs to this Simson line. It is satisfactory to prove that:

$\sphericalangle B_1A_1C \equiv \sphericalangle$Q$A_1B$. (8)

From the inscribable quadrilateral A_1MCB_1, we note that:

$\sphericalangle B_1A_1C \equiv \sphericalangle B_1MC$. (9)

The line d is tangent to the the circumscribed circle, therefore:

$\sphericalangle MBC \equiv \sphericalangle$M$B'A_1$. (10)

From (8) and (9), it follows that: $B'A_1 \parallel MC$. (11)

Since $B'Q$ and MB_1 are parallel, we find that:

$\sphericalangle QB'A_1 \equiv \sphericalangle B_1MC$. (12)

The line BC is tangent in A_1 to the orthopolar circle (Q_c), consequently:
$\sphericalangle QB'A_1 \equiv \sphericalangle QA_1B$. (13)
The relations (9), (12) and (13) lead to the relation (8).

Remark 13

This Proposition can be considered a particular case of Theorem 18. Indeed, if we consider the tangent in M to the "limit" position of a secant M_1M_2 with M_1 tending to be confused with M_2, then also the projections A_1, A_2 on BC are to be confused, and the circle circumscribed to the quadrilateral $B'A_1A_2C'$ becomes tangent in A_1 to BC; also it has been observed that Q belongs to this circle and, since Q is found at the intersection of Simson lines of the points M_1, M_2, it will be situated on the Simson line corresponding to the point of tangent M.

Proposition 56

A given triangle and the triangle formed by the centers of the orthopolar circles corresponding to the orthopole of a secant to the circle are orthological triangles.

Proof

The center of the circle (Q_a) is the center of the circle circumscribed to the quadrilateral $B'A_1A_2C'$ (see *Figure* 64). The perpendicular from Q_a to $B'C'$ passes through the midpoint of $(B'C')$. This perpendicular is parallel with BB' and with CC', and hence passes through the midpoint M_a of the side (BC). If we denote by P the midpoint of the chord M_1M_2 to the circle circumscribed to the triangle ABC, we have $QP \perp M_1M_2$, so $QP \perp Q_aM_a$. The mediator of (A_1A_2) passes through P and through Q_a (it is parallel with M_1A_1 and it is middle line in the trapezoid $M_1A_1A_2M_2$), hence Q_aP is parallel with OM_a. The quadrilateral Q_aPOM_a is parallelogram. Since $OM_a \perp BC$, it follows that the perpendicular taken from Q_a to BC passes through P.

Similarly, we show that the perpendiculars from Q_b and from Q_c to AC, respectively AB, pass through the midpoint P of the chord $[M_1M_2]$, point that is orthology center of the triangle $Q_aQ_bQ_c$ in relation to ABC.

We show that Q_aM_a is parallel and congruent with OP; similarly, it follows that Q_bM_b and Q_cM_c are parallel and congruent with OP, and it is obtained that the triangle $Q_aQ_bQ_c$ is congruent with $M_aM_bM_c$ and have sides parallel to its sides, so actually the triangle $Q_aQ_bQ_c$ is the translation of the median triangle $M_aM_bM_c$ by vector translation $\overrightarrow{OP}$.

The triangles ABC and $M_aM_bM_c$ are orthological, and the orthology center is the orthocenter H of the triangle ABC; it follows that H is also the orthology center of the triangle ABC in relation to the triangle $Q_aQ_bQ_c$.

Observation 48

The orthology centers of the triangles $Q_aQ_bQ_c$ and ABC are the orthocenters of these triangles. Indeed, the perpendiculars taken from Q_a, Q_b, Q_c to BC, CA respectively AB are concurrent in P, but BC is parallel with M_bM_c, and M_bM_c is parallel with Q_bQ_c, hence the perpendicular from Q_a to BC is perpendicular to Q_bQ_c, so P belongs to the altitude from Q_a of the triangle $Q_aQ_bQ_c$; similarly, we obtain that P belongs also to the altitude from Q_b of the same triangle, therefore P is the orthocenter of the triangle $Q_aQ_bQ_c$

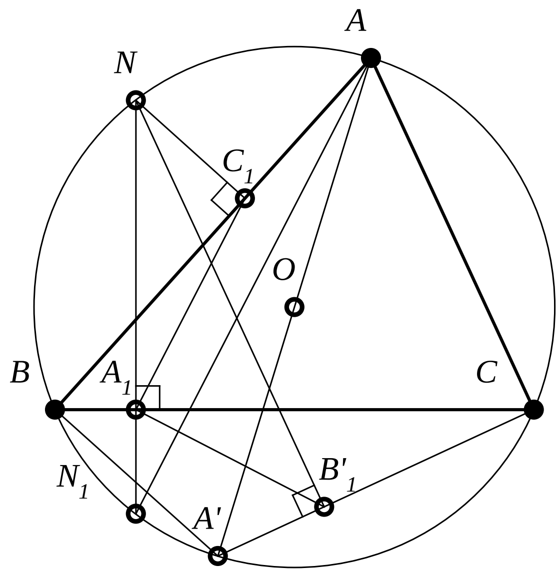

Figure 66

Proposition 57

Let ABC and $A'BC$ two triangles inscribed in the same circle such that the points A and A' are diametrically-opposed. The Simson lines of a point N that belongs to the circle in relation to the triangles ABC and $A'BC$ are orthogonal, and their intersection point is the orthogonal projection on BC of the point N.

Proof

The Simson line of the point N, denoted A_1C_1 in *Figure 66*, in relation to the triangle ABC, is parallel with AN_1 (Proposition 52). By N_1 we denoted the intersection of the perpendicular taken from N to BC with the circle. Also, the Simson line of N in relation to the triangle $A'BC$, denoted $A_1B_1{}'$, is parallel with $A'N_1$. Because the angle AN_1A' is right, it follows that the Simson lines are also perpendicular. They obviously pass through the projection A_1 of N to BC.

Proposition 58

In a triangle, the orthopole of the diameter of the circumscribed circle is the symmetric of projection of a vertex of the triangle on this diameter, in relation to the side of the median triangle opposite to that vertex.

Proof

Let A' be the projection of vertex A on ABC on the diameter d (see *Figure 67*). The point A' belongs obviously to the circle circumscribed to the triangle AM_bM_c (we denoted by $M_aM_bM_c$ the median triangle of ABC).

This circle has as diameter the radius AO of the circumscribed circle and it is the symmetric with respect to M_bM_c of the circle of nine points of the triangle ABC.

We know that the orthopole Q of d belongs to the latter circle, on the other hand Q belongs to the perpendicular taken from A' to M_bM_c, therefore Q is the symmetric of A' with respect to M_bM_c.

Observation 49

This Proposition can be as well formulated this way:

The projections of vertices of a triangle on a diameter of the circumscribed circle are the vertices of a degenerate orthological triangle with the median triangle of the given triangle and having as orthology center the orthopole of diameter in relation to the given triangle.

Proposition 59

Let ABC be a triangle inscribed in the circle of center O and d a line that passes through O. We denote by A_1, B_1, C_1 the symmetrics of vertices of the triangle ABC to d, and by A_2, B_2, C_2 – the symmetrics of points A_1, B_1, C_1 respectively to BC, CA and AB.

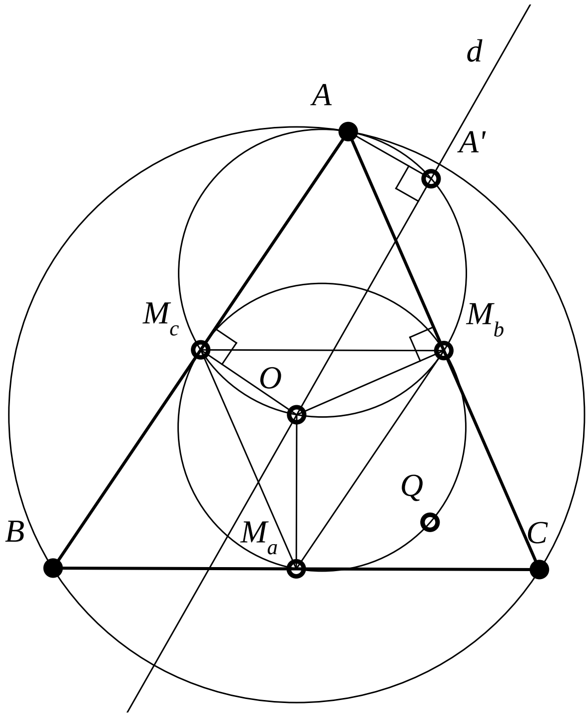

Figure 67

Then:

i. The triangle ABC and the triangle $A_2B_2C_2$ are symmetrical to the orthopole Q of the line d in relation to the triangle ABC.

ii. The triangles ABC and $A_2B_2C_2$ are orthological. Their orthology centers are the orthocenters of these triangles, H and H_2, and the line HH_2 passes through the orthopole Q of the line d.

Proof

Let A' be the projection of A to the line d that passes through O. This point is on the circle with center in O_1, the midpoint of OA (see *Figure 68*). This circle is the homothetic of the circle circumscribed to the triangle ABC by homothety of center A and ratio $\frac{1}{2}$. The symmetric of A with respect to the line d is A_1, and the symmetric of A' with respect to M_bM_c is Q, as we proved in the previous Proposition. The symmetric of A_1 with respect to BC is A_2, and $A'Q \parallel A_1A_2$, hence A, Q, A_2 are collinear, and A_1, A_2 are symmetrical with respect to Q.

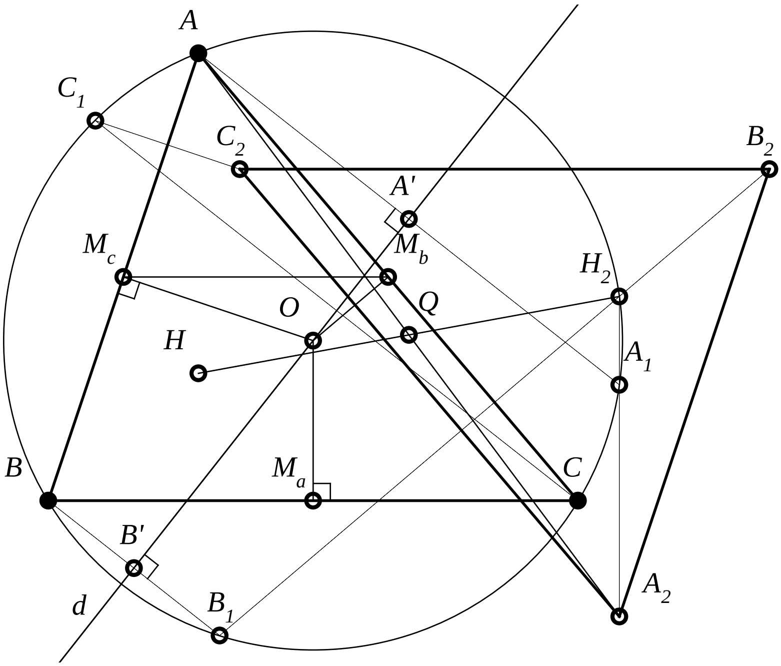

Figure 68

Similarly, we prove that B_1, B_2 and C_1, C_2 are symmetrical with respect to Q. The triangles ABC and $A_2B_2C_2$ have parallel and congruent homologous sides. It is obvious that the perpendiculars taken from A, B, C to BC, CA respectively AB (altitudes of the triangle) will be perpendicular to B_2C_2, C_2A_2 respectively A_2B_2 and concurrent in H.

Similarly, H_2 is the orthocenter of the triangle $A_2B_2C_2$ in relation to ABC; moreover, H and H_2 are symmetric in relation to Q, the orthopole of the line d in relation to the triangle ABC.

4

$\mathcal{S}$ TRIANGLES OR ORTHOPOLAR TRIANGLES

The $\mathcal{S}$ triangles were introduced in geometry by the illustrious Romanian mathematician Traian Lalescu. In this chapter, we will present this notion and some theorems related to it, and establish connections with the orthological triangles.

4.1 $\mathcal{S}$ triangles: definition, construction, properties

Definition 33

We say that the triangle $A_1B_1C_1$ is a $\mathcal{S}$ triangle in relation to the triangle ABC if these triangles are inscribed in the same circle and if the Simson line of the vertex of the triangle $A_1B_1C_1$ (in relation to the triangle ABC) is perpendicular to the opposite side of that vertex of the triangle $A_1B_1C_1$.

Construction of $\mathcal{S}$ triangles

Being given a triangle ABC inscribed in a circle (O), we show how another triangle $A_1B_1C_1$ can be built in order to be $\mathcal{S}$ triangle in relation to ABC.

We present two ways of accomplishing the construction (see *Figure 69*).

I. 1. We fix a point A_1 on the circle (O).

2. We build the Simson line A'-B'-C'.

3. We build the chord (B_1C_1) in the circle (O), perpendicular to the Simson line $A'B'$.

The triangle $A_1B_1C_1$ is $\mathcal{S}$ triangle in relation to the triangle ABC.

II. 1. Let us fix the points B_1 and C_1 on the circumscribed circle of the triangle ABC.

2. We build the chord (AA''), perpendicular to B_1C_1.
3. We build the chord $(A''A_1)$, perpendicular to BC.

The triangle $A_1B_1C_1$ is $\mathcal{S}$ triangle in relation to the triangle ABC.

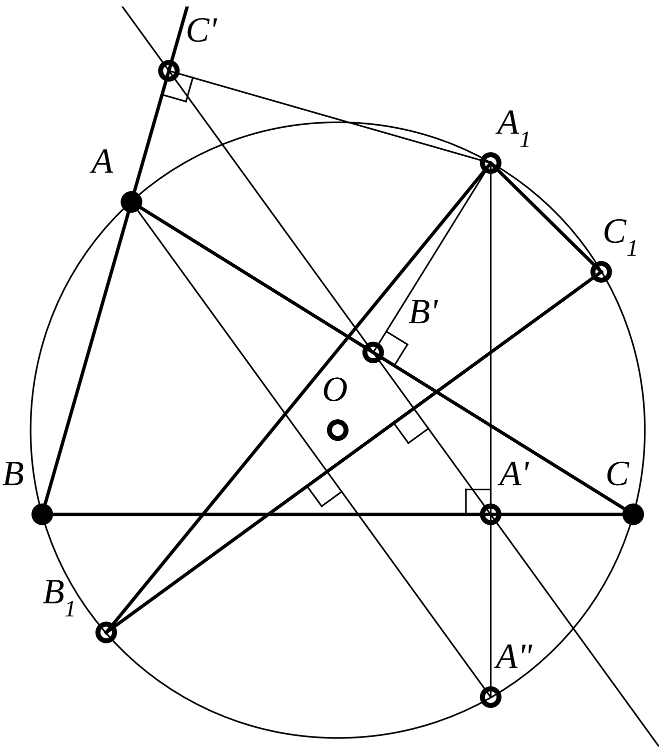

Figure 69

Figure 69 was made to illustrate both constructions above.

The construction II is based on the result of Proposition 52.

Observation 50

1. From the construction I, it follows that, being fixed the point A on the circumscribed circle of the triangle ABC, we can build an infinity of triangles $A_1B_1C_1$ to be $\mathcal{S}$ triangles in relation to ABC. These triangles have the side B_1C_1 of fixed direction (that of the perpendicular to the Simson line of the point A_1).

We can formulate:

Proposition 60

If the triangle $A_1B_1C_1$ is $\mathcal{S}$ triangle in relation to the triangle ABC, then any triangle $A'_1B'_1C'_1$, where $B'_1C'_1$ is a chord parallel with BC in the circumscribed circle of the triangle ABC, is $\mathcal{S}$ triangle in relation to ABC.

2. Both constructions provide an infinity of $\mathcal{S}$ triangles in relation to *ABC*.

Theorem 19 (Traian Lalescu, 1915)

If the triangle $A_1B_1C_1$ is $\mathcal{S}$ triangle in relation to the triangle ABC, then:

1. The algebraic sum of measures of the arcs $\overset{\frown}{AA_1}$, $\overset{\frown}{BB_1}$, $\overset{\frown}{CC_1}$ considered on the circumscribed circle of the triangle ABC, on which a positive sense of traversing was set, equals zero.
2. The Simson lines of the vertices of triangle $A_1B_1C_1$ in relation to the triangle ABC are respectively perpendiculars on the opposite sides of the triangle $A_1B_1C_1$.
3. The Simson lines of vertices of the triangle $A_1B_1C_1$ in relation to the triangle ABC are concurrent.
4. The triangle ABC is a $\mathcal{S}$ triangle in relation to the triangle $A_1B_1C_1$.
5. The six Simson lines of vertices of the triangles $A_1B_1C_1$ and ABC are concurrent in the midpoint of the segment determined by the orthocenters of these triangles.

Proof

1. We refer to *Figure 70.* Suppose that the trigonometric direction of traversing the arcs was fixed on the circle circumscribed to the triangle ABC. The Simson line of point A_1 is perpendicular to B_1C_1. We denote: $\{X\} = BC \cap B_1C_1$; we have: $\widehat{CXC_1} \equiv \widehat{AA''A_1}$ (as angles with perpendicular sides), where A'' is the intersection with the circle of the perpendicular A_1A' to BC. We have:

$m(\widehat{CXC_1}) = \frac{1}{2}[m(\overset{\frown}{BB_1}) + m(\overset{\frown}{C_1C})]$,

$m(\widehat{AA''A_1}) = \frac{1}{2}m(\overset{\frown}{A_1A})$.

From $m(\overset{\frown}{A_1A}) = m(\overset{\frown}{BB_1}) + m(\overset{\frown}{CC_1})$, it follows that:

$m(\overset{\frown}{A_1A}) + m(\overset{\frown}{B_1B}) + m(\overset{\frown}{CC_1}) = 0.$

2. We prove the Simson line of vertex B_1 in relation to the triangle ABC is perpendicular to A_1C_1.

We take the perpendicular from B to A_1C_1 and we denote by B'' its intersection with the circle. We join B'' with B_1 and we denote by $\{Y\} = AC \cap A_1C_1$. We have:

$m(\widehat{AYA_1}) = \frac{1}{2}[m(\breve{A_1A}) - m(\breve{CC_1})]$,

$m(\widehat{BB''B_1}) = \frac{1}{2}m(\breve{B_1B})$.

Since $m(\breve{A_1A}) + m(\breve{B_1B}) + m(\breve{C_1C}) = 0$, it derives that:

$\sphericalangle AYA_1 \equiv \sphericalangle BB''B_1$.

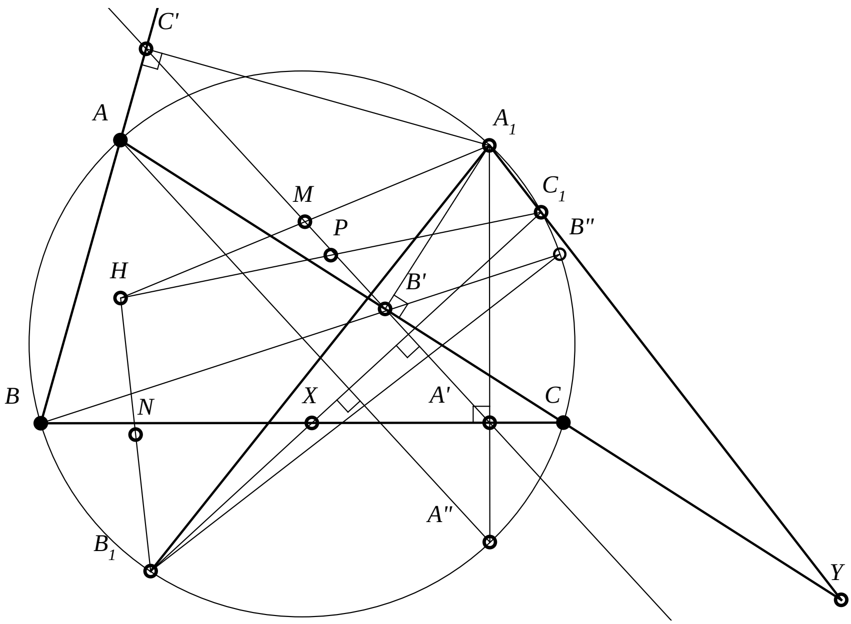

Figure 70

These acute angles, having $BB'' \perp A_1Y$, they also have $B_1B'' \perp AC$, therefore the Simson line of vertex B_1 is parallel with BB'' and as such it is perpendicular to A_1C_1.

Similarly, it is proved that the Simson line of the vertex C_1 is perpendicular to A_1B_1.

Remark 14

Basically the condition $m(\overset{\frown}{AA_1}) + m(\overset{\frown}{BB_1}) + m(\overset{\frown}{CC_1}) = 0(\text{mod } 360^0)$ is necessary and sufficient as the triangle $A_1B_1C_1$ to be $\mathcal{S}$ triangle in relation to the triangle ABC.

3. We denote by M, N, P respectively the midpoints of segments HA_1, HB_1, HC_1, where H is the orthocenter of ABC. From Steiner's theorem, it follows that the Simson lines of vertices A_1, B_1, C_1 pass respectively through M, N respectively P, and, on the other hand, these Simson lines are perpendicular to B_1C_1, C_1A_1 respectively A_1B_1, lines respectively parallel to NP, PM and MN. Consequently, the Simson lines of vertices A_1, B_1, C_1 are altitudes in the triangle MNP, therefore they are concurrent.

4. It derives from $m(\overset{\frown}{A_1A}) + m(\overset{\frown}{B_1B}) + m(\overset{\frown}{C_1C}) = 0(\text{mod } 360^0)$.

5. The triangle MNP is the homothetic of the triangle $A_1B_1C_1$ by homothety of center H and ratio $\frac{1}{2}$. It follows that its orthocenter is the midpoint of the segment determined by the homothety center H and the homologue point H_1, the orthocenter of the triangle $A_1B_1C_1$. We saw that the Simson lines of the vertices of triangle $A_1B_1C_1$ pass through this point. The relation between the triangles $A_1B_1C_1$ and ABC being symmetrical, we have that the Simson lines of the vertices of the triangle ABC in relation to $A_1B_1C_1$ are concurrent in the same point, the midpoint of the segment $[HH_1]$.

We write:

1. $\Delta A_1B_1C_1\mathcal{S}\Delta ABC$ and read: the triangle $A_1B_1C_1$ is in "$\mathcal{S}$" relation with the triangle ABC.
2. $\mathcal{T}_{\boxed{\Delta}}$ - the set of triangles inscribed in a given circle.

4.2 The relation of $\mathcal{S}$ equivalence in the set of triangles inscribed in the same circle

Definition 34

We say that the triangle $A_1B_1C_1$ is in $\mathcal{S}$ relation with the triangle ABC if the triangle $A_1B_1C_1$ is $\mathcal{S}$ triangle in relation to the triangle ABC.

Proposition 61

The $\mathcal{S}$ relation in the set $\mathcal{T}_{\boxed{\Delta}}$ is an equivalence relation.

Proof

We must prove that the $\mathcal{S}$ relation has the properties: reflexivity, symmetry and transitivity.

Reflexivity

Whatever the triangle ABC from the set $\mathcal{T}_{\boxed{\Delta}}$, we have: $\Delta A_1B_1C_1\mathcal{S}\Delta ABC$. Indeed, the Simson line of vertex A in relation to ABC is the altitude from A of the triangle ABC, this being perpendicular to BC; we obtain that ABC is $\mathcal{S}$ triangle in relation to itself.

Another proof: $m(\breve{AA}) + m(\breve{BB}) + m(\breve{CC}) = 0$ (mod 360^0).

Therefore: $\Delta ABC\mathcal{S}\Delta ABC$.

Symmetry

If $\Delta A_1B_1C_1\mathcal{S}\Delta ABC$, it has been shown in T. Lalescu's theorem that $\Delta ABC\mathcal{S}\Delta A_1B_1C_1$, therefore the "$\mathcal{S}$" relation is symmetrical.

Transitivity

In the set $\mathcal{T}_{\boxed{\Delta}}$, we consider the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ such that: $\Delta ABC\mathcal{S}\Delta A_1B_1C_1$ and $A_1B_1C_1\mathcal{S}\Delta A_2B_2C_2$.

Let us prove that: $\Delta ABC\mathcal{S}\Delta A_2B_2C_2$.

From $\Delta ABC\mathcal{S}\Delta A_1B_1C_1$, we have that:

$m(\breve{AA_1}) + m(\breve{BB_1}) + m(\breve{CC_1}) = 0$ (mod 360^0).

From $A_1B_1C_1\mathcal{S}\Delta A_2B_2C_2$, we have that:

$m(\breve{A_1A_2}) + m(\breve{B_1B_2}) + m(\breve{C_1C_2}) = 0$ (mod 360^0).

Adding member by member the previous relations, we obtain that:

$m(\breve{AA_2}) + m(\breve{BB_2}) + m(\breve{CC_2}) = 0$ (mod 360^0),

hence: $\Delta ABC\mathcal{S}\Delta A_2B_2C_2$.

If ABC is a fixed triangle from $\mathcal{T}_{\boxed{\Delta}}$, we define the set:

$\mathcal{S}^{\Delta ABC} = \{\Delta A'B'C' \in \mathcal{T}_{\boxed{\Delta}} \,/\, \Delta ABC\mathcal{S}\Delta A'B'C'\}$.

The set $\mathcal{S}^{\Delta ABC}$ is a modulo "$\mathcal{S}$" equivalence class of set $\mathcal{T}_{\boxed{\Delta}}$.

Proposition 62

The Simson line of a point that belongs to the circle circumscribed to the triangles of the same equivalence class has the fixed direction in relation to these triangles.

Proof

Let M be a point on the circumscribed circle of the triangle ABC, and d – a line determining together with M a $\mathcal{S}$ triangle in relation to ABC. Because the Simson line of M in relation to ABC, or any other triangle from $\mathcal{S}\Delta ABC$, is perpendicular to d, it will have a fixed direction.

Remark 15

The denomination of $\mathcal{S}$ triangles of the orthopolar triangles is justified by the fact that two $\mathcal{S}$ triangles in relation to a line have the same orthopole.

Indeed, let ABC be a triangle and d – a line. The projections of vertices of triangle ABC on d are A_1, B_1, C_1 ; and the perpendiculars taken from A_1, B_1, C_1 to BC, CA respectively AB are concurrent with O, the orthopole of line d in relation to ABC. This point O is the intersection of the Simson lines of the intersection points with the circle (of line d) circumscribed to the triangle ABC. Basically, these perpendiculars are Simson lines of the $\mathcal{S}$ triangle in relation to the $\mathcal{S}$ triangle that has the line *d* as one of its sides.

Proposition 63

In two $\mathcal{S}$ triangles, the orthopoles of one's sides in relation to the other coincide with the midpoint of the segment determined by its orthocenters.

Proof

The orthopole O of a line d in relation to a scalene triangle is the intersection point of Simson lines, of the intersection points of line d with the circle. From T. Lalescu's theorem, the six Simson lines of the vertices of a triangle in relation to the other triangle are concurrent lines in the midpoint of the segment determined by its orthocenters.

4.3 Simultaneously orthological and orthopolar triangles

Lemma 5

If the triangles ABC and $A_1B_1C_1$ are inscribed in the same circle, with $AA_1 \parallel BC$, $BB_1 \parallel CA$, $CC_1 \parallel AB$, then: B_1C_1 and BC are antiparallels in relation to AB and AC; B_1A_1, and BA – antiparallels in relation to CB and CA; A_1C_1 and AC – antiparallels in relation to BC and BA.

Proof

B_1C_1 and BC form the inscribed quadrilateral BB_1CC_1, therefore they are antiparallels in relation to BB_1 and CC_1; since $BB_1 \parallel CA$ and $CC_1 \parallel AB$, we have that B_1C_1 and BC are antiparallels in relation to AB and AC. Similarly, the other requirements are proved.

Theorem 19

If the triangles ABC and $A_1B_1C_1$ inscribed in the same circle have $AA_1 \parallel BC$, $BB_1 \parallel CA$, $CC_1 \parallel AB$, then the triangles are simultaneously orthological and orthopolar.

Proof

First of all, we prove that the triangles ABC and $A_1B_1C_1$ are orthological. Indeed, the perpendicular from A to B_1C_1, which is antiparallel with B, passes through O – the center of the circle circumscribed to the triangle ABC (Proposition 4). Similarly, the perpendiculars from B and C to C_1A_1 respectively A_1B_1 pass through O, therefore O is an orthology center. We prove that the triangles ABC and $A_1B_1C_1$ are $\mathcal{S}$ triangles. We show that:

$$m(\widetilde{AA_1}) + m(\widetilde{BB_1}) + m(\widetilde{CC_1}) = 0 \ (\text{mod}360^0). \qquad (1)$$

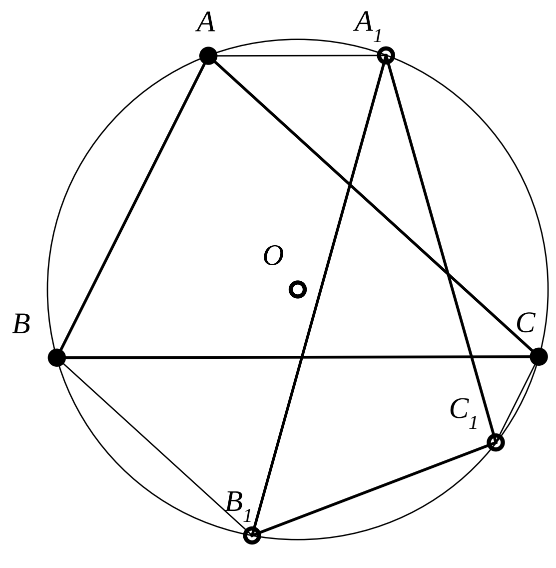

Figure 71

We use *Figure 71*; we have that the quadrilateral AA_1CB is an isosceles trapezoid, therefore:

$m(\overset{\frown}{AA_1}) = 360^0 - 4m(\hat{C}) - 2m(\hat{A})$. (2)

The quadrilateral BB_1CA is also an isosceles trapezoid; we have:

$m(\overset{\frown}{BB_1}) = 360^0 - 4m(\hat{C}) - 2m(\hat{B})$. (3)

The quadrilateral CC_1AB is an isosceles trapezoid; it follows that:

$m(\overset{\frown}{CC_1}) = 360^0 - 4m(\hat{A}) - 2m(\hat{C})$. (4)

Taking into account that, in the relation (1), the arcs must be transversed in the same direction, we have that: $m(\overset{\frown}{AA_1}) = 4m(\hat{C}) + 2m(\hat{A})$.

Then:

$$m(\overset{\frown}{AA_1}) + m(\overset{\frown}{BB_1}) + m(\overset{\frown}{CC_1}) = 4m(\hat{C}) + 2m(\hat{A}) + 360^0 - 4m(\hat{C}) - 2m(\hat{B}) + 360^0 - 4m(\hat{A}) - 2m(\hat{C}).$$

It follows that:

$$m(\overset{\frown}{AA_1}) + m(\overset{\frown}{BB_1}) + m(\overset{\frown}{CC_1}) = 720^0 - 2[m(\hat{A}) + m(\hat{B}) + m(\hat{C})].$$

Therefore:

$m(\overset{\frown}{AA_1}) + m(\overset{\frown}{BB_1}) + m(\overset{\frown}{CC_1}) = 360^0$,

and, consequently, the relation (1) is true; therefore the triangles ABC and $A_1B_1C_1$ are $\mathcal{S}$ triangles.

Proposition 64

The median triangle and the orthic triangle of a given non-right triangle are simultaneously orthopolar and orthological triangles. Proof of this property follows as a consequence of Theorem 5. The fact that the median triangle and the orthic triangle are orthological was established in Proposition 9.

Proposition 65

A given triangle ABC and the triangle $A_0B_0C_0$ determined by the intesections of the exterior bisectors of the angles A, B, C with the circumscribed circle of the triangle ABC are simultaneously orthopolar and orthological triangles. Proof of this property derives from the fact that the triangle ABC and the triangle $A_0B_0C_0$ are respectively the orthic triangle and the median triangle in the antisupplementary triangle $I_aI_bI_c$ of the triangle ABC.

5

ORTHOLOGICAL TRIANGLES WITH THE SAME ORTHOLOGY CENTER

In this chapter, we will prove some important theorems regarding the orthological triangles with common orthology center, and we will address some issues related to the biorthological triangles.

5.1 Theorems regarding the orthological triangles with the same orthology center

Theorem 20

Two orthological triangles with common orthology center are homological triangles.

In the proof of this theorem, we will use:

Theorem 21 (N. Dergiades, 2003)

Let $\mathcal{C}_1(O_1, R_1)$, $\mathcal{C}_2(O_2, R_2)$, $\mathcal{C}_3(O_3, R_3)$ be three circles that pass respectively through vertices B and C, C and A, A and B of a triangle ABC. We denote by D, E, F the second intersection point respectively between $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_1)$, $(\mathcal{C}_1)$ and $(\mathcal{C}_1)$. The perpendiculars taken from the points D, E, F to AD, BE respectively CF intersect the sides BC, CA, AB in the points X, Y, Z. The points X, Y and Z are collinear.

Proof (Ion Pătrașcu)

Let $A_1B_2C_3$ be the median triangle of the triangle ABC (see *Figure 72*). In the proof, we use Menelaus's reciprocal theorem.

We have:

$\frac{XB}{XC} = \frac{Aria(XDB)}{Aria(XDC)} = \frac{DB \cdot \sin(XDB)}{DC \cdot \sin(XDC)} = \frac{DB \cdot \cos(ADB)}{DC \cdot \cos(ADC)}$.

Similarly, we find:

$\frac{YC}{YA} = \frac{EC}{EA} \cdot \frac{\cos(BEC)}{\cos(BEA)}, \frac{ZA}{ZB} = \frac{FA}{FB} \cdot \frac{\cos(CFA)}{\cos(CFB)}$.

From the inscribed quadrilaterals $ADEB$, $BEFC$ and $CFDA$, we note that:

$\sphericalangle ADB = \sphericalangle BEA, \sphericalangle BEC = \sphericalangle CFB, \sphericalangle CFA = \sphericalangle ADC$.

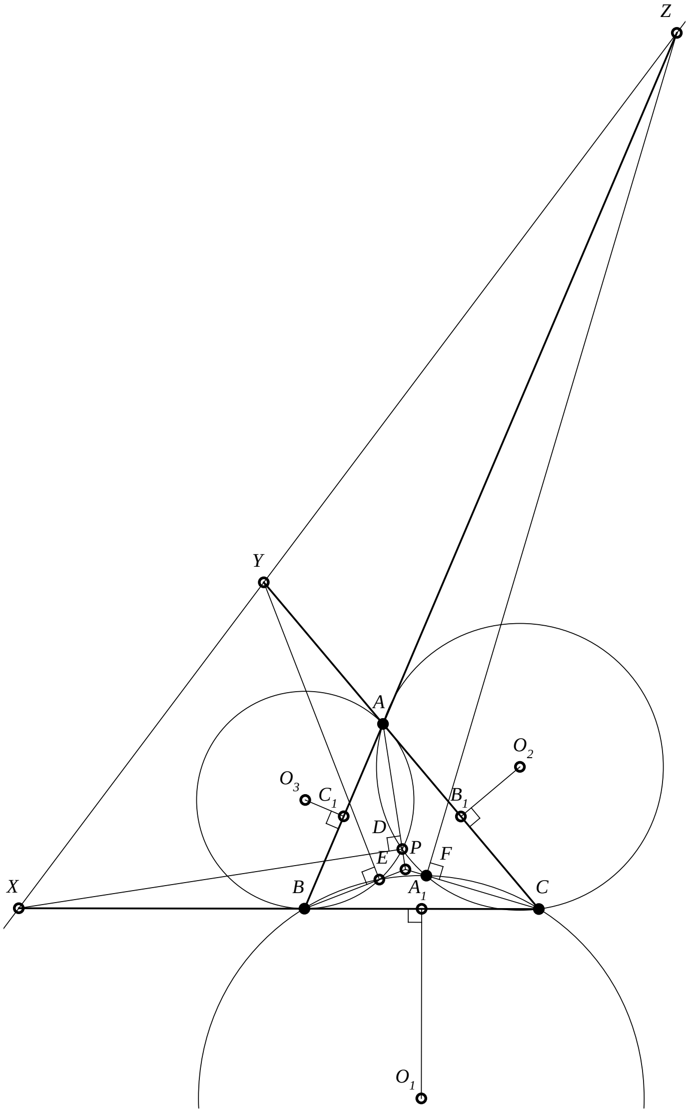

Figure 72

Consequently:

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB}. \quad (1)$$

On the other hand:

$DB = 2R_3\sin(BAD)$,
$EA = 2R_3\sin(ABE)$,
$DC = 2R_2\sin(CAD)$,
$FA = 2R_2\sin(ACF)$,
$FB = 2R_1\sin(BCF)$,
$EC = 2R_1\sin(CBE)$.

Coming back to the relation (1), we obtain:

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{\sin(BAD)}{\sin(CAD)}\cdot\frac{\sin(CBE)}{\sin(ABE)}\cdot\frac{\sin(ACF)}{\sin(BCF)}. \quad (2)$$

According to Carnot's theorem, the common chords of circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are concurrent, which means that $AD \cap BE \cap CF = \{P\}$ (P is the radical center of these circles).

The cevians AD, BE, CF being concurrent, we can write the trigonometric form of Ceva's theorem, from where we find:

$$\frac{\sin(BAD)}{\sin(CAD)}\cdot\frac{\sin(CBE)}{\sin(ABE)}\cdot\frac{\sin(ACF)}{\sin(BCF)}=1.$$

By this relation, from (2) we obtain that:

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=1,$$

which shows the collinearity of the points X, Y, Z.

To prove Theorem 20, we need:

Lemma 6

Let ABC and $A'B'C'$ be two orthological triangles with the same orthology center, O.

If E and F are the orthogonal projections of vertices B and C on the support lines of the sides $[A'C']$ and respectively $[A'B']$, then the points B, C, E and F are four concyclic points.

Proof 1

Let O be the common orthology center of the given triangle; we denote by E, F the projections of B and C on $A'C'$ respectively on $A'B'$; also, we denote: $\{B''\} = EO \cap A'B'$, $\{C''\} = FO \cap A'C'$ (see *Figure 73*).

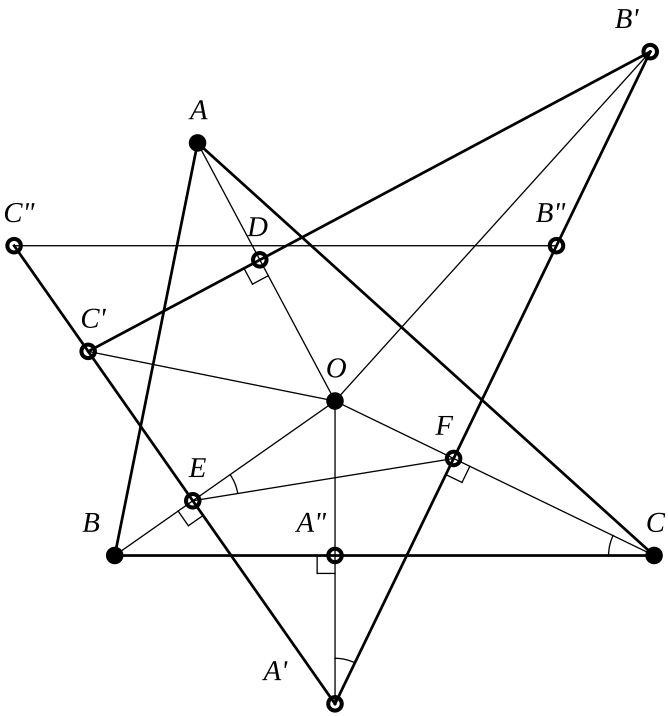

Figure 73

O is the orthocenter in the triangle $A'B''C''$. It consequently follows that $A'O \perp B''C''$. Because EF is antiparallel with $B''C''$, it follows from Proposition 4 that EF is antiparallel with BC, which shows that the quadrilateral $BCFE$ is inscribable.

Observation 51

Similarly, if D is the projection of A on $B'C'$, it follows that the points A, D, F, C and A, D, E, B are concyclic.

Proof 2 (Ion Pătrașcu)

We denote: $\{A''\} = A'O \cap BC$. The quadrilaterals $BEA''A'$, $A'A''FC$ are inscribable. The power of the point O over the circles circumscribed to these quadrilaterals provides the relations: $OA'' \cdot OA' = OF \cdot OC$ and $OA'' \cdot OA' = OE \cdot OB$. We get from here that: $OE \cdot OB = OF \cdot OC$, accordingly the points B, C, F, E are concyclic.

Proof 3 (Mihai Miculița)

Denoting $\{A''\} = A'O \cap BC = pr_{BC}(A')$ (see *Figure 73*), we have:

$$\left.\begin{array}{r} BO \perp A'C'' \\ A'O \perp BC \\ CO \perp A'B' \end{array}\right\} \Rightarrow \begin{cases} OEAF - \text{inscribable} \Rightarrow \sphericalangle \mathrm{OEF} \equiv \sphericalangle \mathrm{O}A'\mathrm{F} \\ A''A'CO - \text{inscribable} \Rightarrow \sphericalangle \mathrm{O}A'\mathrm{F} \equiv \sphericalangle \mathrm{BCO} \end{cases}$$

$\Rightarrow \sphericalangle \mathrm{OEF} \equiv \sphericalangle BCO \Rightarrow BCFE -$ inscribable.

Proof of *Theorem 20*

We use the configuration from *Figure 73*. The quadrilaterals $BCFE$, $CFDA$, $ADEB$ being inscribable, we observe that their circumscribed circles satisfy the hypotheses in Theorem 21.

Applying this theorem, it follows that the lines $B'C'$ and BC, $C'A'$ and CA, $A'B'$ and AB intersect respectively in the points X, Y, Z, which are collinear. Using Desargues's theorem (see [24]), it follows that AA', BB' and CC' are concurrent and, hence, the triangles ABC and $A'B'C'$ are homological.

Remark 16

a) The triangles $O_1O_2O_3$ (formed by the centers of the circles circumscribed to the quadrilaterals $BCFE$, $CFDA$ respectively $ADFB$) and ABC are orthological. The orthology centers are the points P – the radical center of the circles of centers O_1, O_2, O_3; and O – the center of the circle circumscribed to ABC.

b) The triangles $O_1O_2O_3$ and DEF (formed by the projections of vertices of triangles ABC on the sides of $A'B'C'$) are orthological. The orthology centers are: the center of the circle circumscribed to the triangle DEF; and P – the center of the radical circle of the circles of centers O_1, O_2, O_3. Indeed, the perpendiculars taken from O_1, O_2, O_3 to EF, FD respectively DE are mediators of these segments, therefore they are concurrent in the center of the circle circumscribed to the triangle DEF, and the perpendiculars taken from D, E, F to the sides of the triangle $O_1O_2O_3$, being the common chords AD, BE, CF, they will be concurrent in the point P.

Theorem 22

If O is a point in the interior of the given triangle ABC, $A_1B_1C_1$ is its podal triangle, and the points A', B', C' are such that:

i. $\overrightarrow{OA_1}$, $\overrightarrow{OA'}$ are collinear vectors; $\overrightarrow{OB_1}$, $\overrightarrow{OB'}$ are collinear vectors; $\overrightarrow{OC_1}$, $\overrightarrow{OC'}$ are collinear vectors;

ii. $\overrightarrow{OA_1} \cdot \overrightarrow{OA'} = \overrightarrow{OB_1} \cdot \overrightarrow{OB'} = \overrightarrow{OC_1} \cdot \overrightarrow{OC'}$, so the triangles ABC and $A'B'C'$ are orthological with O – common orthology center.

Proof

We build the circle of diameter AC' and we denote by D its second point of intersection with AO. We obtain (see *Figure 74*) that $C'D \perp AO$ (1) and $OC_1 \cdot OC' = OD \cdot OA$ (2). Because $OC_1 \cdot OC' = OB_1 \cdot OB'$ (3), from relations *i)* and *ii)*, we have $OD \cdot OA = OB_1 \cdot OB'$. This relation shows that the points B_1, B', A, D are concyclic.

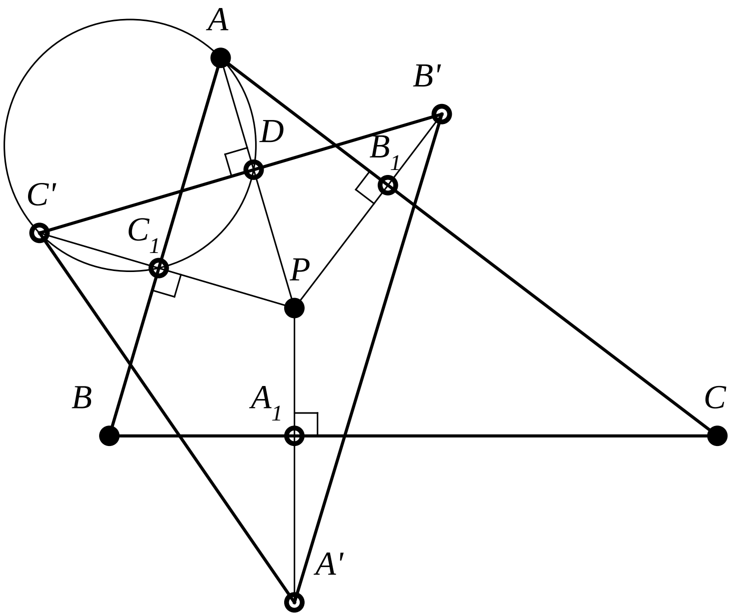

Figure 74

The angle AB_1B' is right; it follows that $\sphericalangle ADB'$ is also right, hence $B'D \perp AO$ (4). The relations (1) and (4) show that the points B', D, C' are collinear, and $AO \perp B'C'$. Similarly, we deduce that $BO \perp A'C'$ and $CO \perp A'B'$, therefore the triangles ABC and $A'B'C'$ are orthological, of common center O.

Observation 52

The reciprocal of Theorem 22 is also true, and its proof is made without difficulty.

Proposition 66

If ABC and $A'B'C'$ are two orthological triangles of common orthology center O, and of homology with the homology axis XYZ, then OX, OY, OZ are respectively perpendiculars to AA', BB', CC'.

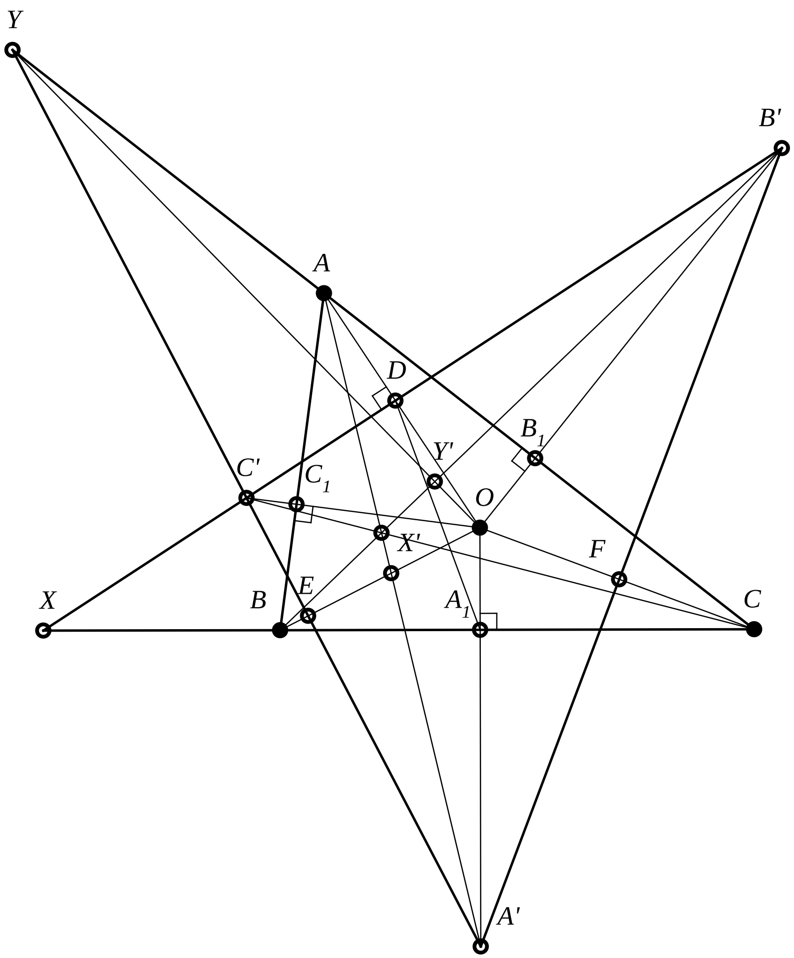

Figure 75

Proof

We denote by D', E', F' the orthogonal projections of O on the sides BC, CA respectively AB, and by D, E, F – the orthogonal projections of O on $B'C'$, $C'A'$, respectively $A'B'$.

Applying *Lemma 6*, it follows that the points A', B', E', D' are concyclic (see *Figure 75*).

Considering the power of the point O to the circle of the previous points, we write:

$OD' \cdot OA' = OE' \cdot OB'$. (1)

The points F and E' are on the circle of diameter CB'; considering the power of the point O over this circle, it follows that:

$OF \cdot OC = OE' \cdot OB'$. (2)

Also from *Lemma 6*, we note that the points A, C, F, D are concyclic; the power of O over the circle of these points implies that:

$OF \cdot OC = OD \cdot OA$. (3)

From the relations (1), (2), (3), we obtain:

$OD' \cdot OA' = OD \cdot OA$. (4)

The relation (4) shows that the points A, A', D', D are concyclic, therefore:

$\sphericalangle DD'O \equiv \sphericalangle A'AO$. (5)

The points O, D, X, D' belong to the circle of diameter (OX), therefore we have:

$\sphericalangle DXO \equiv \sphericalangle D'DO$. (6)

The relations (5) and (6) lead to:

$\sphericalangle D'DO \equiv \sphericalangle A'AO$. (7)

This relation, together with $OA \perp B'C'$ and with the reciprocal theorem of the angles with the perpendicular sides, we obtain that $AA' \perp OX$. Similarly, we prove that $BB' \perp OY$ and $CC' \perp OZ$.

Proposition 67

If the triangles ABC and $A'B'C'$ are orthological, of common center O, and homological with P and XY – respectively their center of homology and the axis of homology; then: $OP \perp XY$.

Proof

From the previous Proposition, we remember that $OX \perp AA'$, $OY \perp BB'$, $OZ \perp CC'$.

We denote $\{X'\} = OX \cap AA'$, $\{Y'\} = OY \cap BB'$, $\{Z'\} = OZ \cap CC'$ (see *Figure 75*). The quadrilateral $XX'O'A'$ is inscriptable; considering the power of point O over its circumscribed circle, we write:

$OX' \cdot OX = OD' \cdot OA'$. (1)

On the other hand:

$OA_1 \cdot OA' = OB_1 \cdot OB'$.

The points B', B_1, Y', Y are located on the circle of diameter YB'; writing the power of O over this circle, it turns out that:

$OB_1 \cdot OB' = OY' \cdot OY$. (2)

The relations (1), (2), (3) lead to:

$OX' \cdot OX = OY' \cdot OY$, (3)

relation showing that the points X', X, Y, Y' are concyclic, hence $X'Y'$ is antiparallel with XY in relation to OX and OY. Also, $X'Y'$ is antiparallel with the tangent taken in O to the circle of diameter OP. Consequently, the tangent to the circle is parallel with XY, and since OP is perpendicular to the tangent, we have that $OP \perp XY$.

5.2 Reciprocal polar triangles

Definition 34

Two triangles are called reciprocal polar in relation to a given circle if the sides of one are the other's vertices polars with respect to a circle.
The circle against which the two triangles are reciprocal polars is called director circle.

Theorem 23

Two reciprocal polar triangles in relation to a given circle are orthological triangles, having the center of the circle as common orthology center.

Proof

Let ABC and $A'B'C'$ be two reciprocal polar triangles in relation to the director circle of center O (see *Figure 76*). Because the polar of A, videlicet $B'C'$, is perpendicular to the line determined by the point A and by the circle's center O, we have that $OA \perp B'C'$, similarly $OB \perp A'C'$ and $OC \perp A'B'$.

Also, the polar of A' with respect to the circle, videlicet BC, is perpendicular to $A'O$, and similarly $B'O \perp AC$ and $C'O \perp AB$, therefore ABC and $A'B'C'$ are orthological triangles with the point O – common orthology center.

The points A_0, B_0, C_0 are the inverses of points A, B, C – with respect to the inversion circle, with O as center.

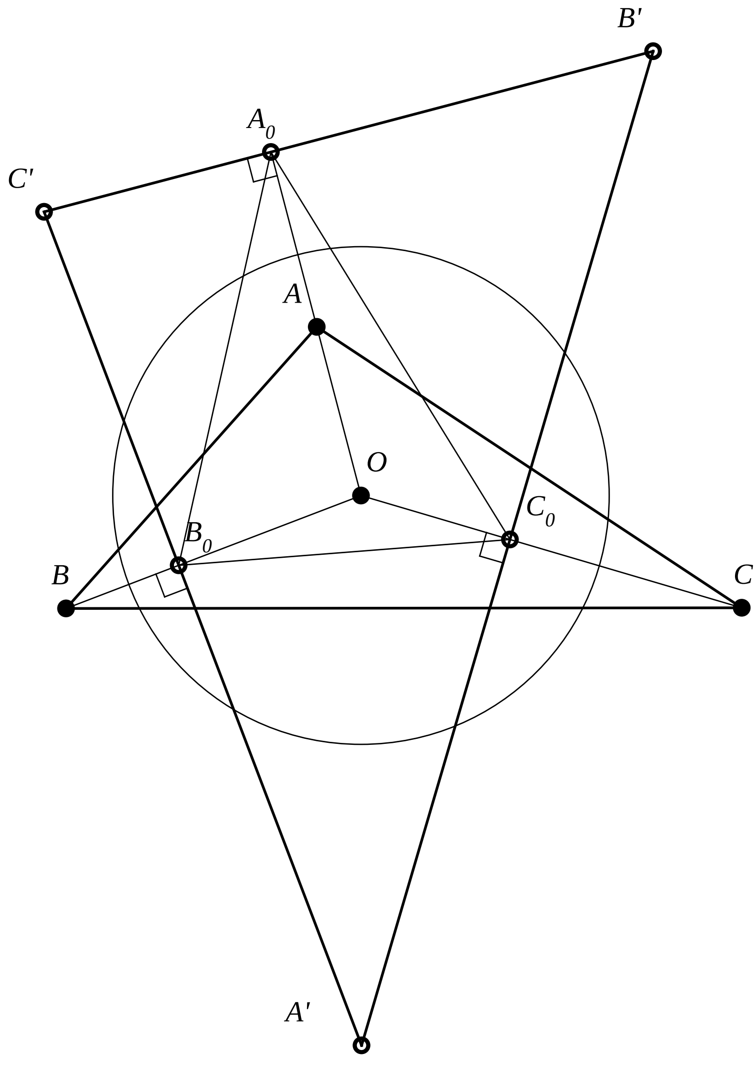

Figure 76

Remark 17

A triangle ABC and its tangential triangle are reciprocal polar triangles in relation to the circumscribed circle of the triangle ABC, therefore they are orthological triangles with the common orthology center in the circle circumscribed to the triangle ABC.

5.3 Other remarkable orthological triangles with the same orthology center

In the Second Chapter, we presented a few pairs of orthological triangles with only one orthology center, namely a given triangle and its contact triangle; the given triangle and its tangent triangle.

In the following, we will investigate another pair of orthological triangles that have the same orthology center.

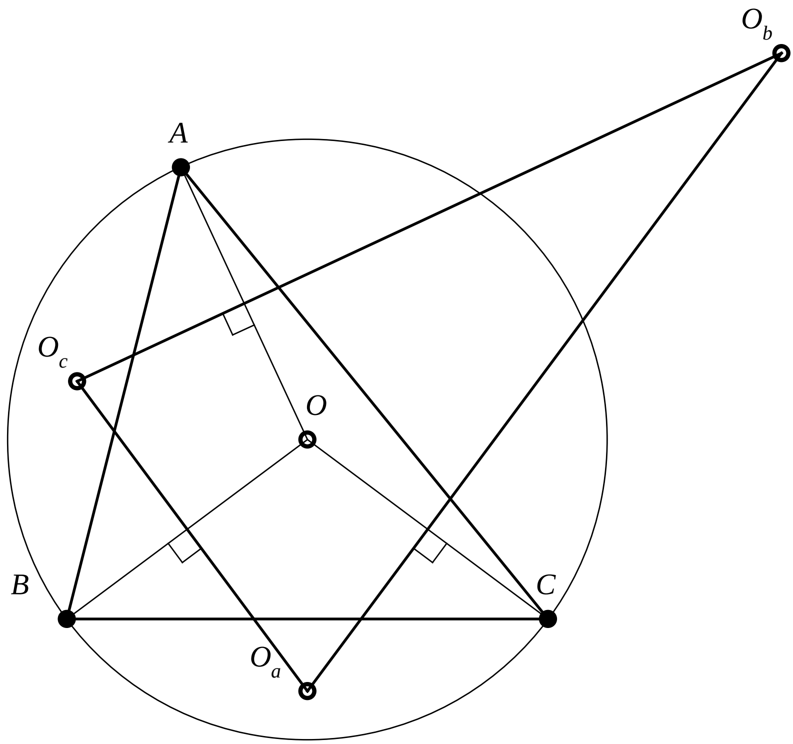

Figure 77

Definition 35

Being given a non-right triangle ABC, the triangle with the vertices in the centers of the circles circumscribed to the triangles OBC, OCA, OAB, where O is the center of the circle circumscribed to the triangle ABC – is called Coșniță triangle.

Observation 53

In *Figure 77*, the Coșniță triangle of the triangle ABC was denoted $O_aO_bO_c$.

Proposition 68

A non-right given triangle and the Coșniță triangle are orthological triangles with the common orthology center O – the center of the circle circumscribed to the given triangle.

Proof

Obviously, the perpendiculars taken from O_a, O_b, O_c to BC, CA, AB are the mediators of the triangle ABC, therefore are concurrent in O. On the other hand, AO is a common chord in the circles circumscribed to triangles AOB and AOC, hence $AO \perp O_bO_c$. Similarly, it follows that the perpendiculars from B and C respectively to O_aO_c and O_aO_b pass through the point O.

Remark 18

1. In [24], we proved that the triangles ABC and $O_aO_bO_c$ are homological (with the help of Ceva's theorem); this result can now be obtained using Theorem 20. We recall that the homology center is called Coșniță point and it is the isogonal conjugate of the center of the circle of nine points; also, the homology axis is called Coșniță line.
2. If the triangle ABC is acute; $A_1B_1C_1$ is the podal triangle of the center of the circumscribed circle O; and $O_aO_bO_c$ is *Coșniță triangle*, – it can be shown that:

   $OA_1 \cdot OO_a = OB_1 \cdot OO_b = OC_1 \cdot OO_c = \frac{1}{2}R^2.$

If we consider A', B', C' belonging respectively to the lines A_1O_a, B_1O_b, C_1O_c, such that $\overrightarrow{OA_1} \cdot \overrightarrow{OA'} = \overrightarrow{OB_1} \cdot \overrightarrow{OB'} = \overrightarrow{OC_1} \cdot \overrightarrow{OC'}$, then, from Theorem 22, we obtain that the triangles ABC and $A'B'C'$ are orthological. From Theorem 20, we see that these triangles are homological.

In [21], we called the homology center of these triangles – the generalized Coșniță point, and the homology axis of these triangles is called – the generalized Coșniță line. We call the triangle *A'B'C'* – the generalized Coșniță triangle.

Considering Proposition 68, we can state:

> The line determined by the center of the circumscribed circle of a triangle and the generalized Coșniță point is perpendicular to the generalized Coșniță line.

5.4. Biorthological triangle

Definition 36

If the triangle ABC is orthological with the triangle $A_1B_1C_1$ and with the triangle $B_1C_1A_1$, we say that the triangles ABC and $A_1B_1C_1$ are biorthological.

Observation 54

In *Figure 78*, the triangles ABC and $A_1B_1C_1$ are biorthological. We denoted by O_1 the orthology center of the triangle ABC in relation to the triangle $A_1B_1C_1$ and with O_2 – the orthology center of the triangle ABC in relation to the triangle $B_1C_1A_1$.

Theorem 24 (A. Pantazi, 1896 – 1948)

If the triangle ABC is simultaneously orthological in relation to the triangles $A_1B_1C_1$ and $B_1C_1A_1$, then the triangle ABC is orthological in the triangle $C_1A_1B_1$.

Proof 1

The triangles ABC and $A_1B_1C_1$ being orthological, we have:

$$AB_1^2 - AC_1^2 + BC_1^2 - BA_1^2 + CA_1^2 - CB_1^2 = 0. \quad (1)$$

The triangles ABC and $B_1C_1A_1$ being orthological, we can write:

$$AC_1^2 - AA_1^2 + BA_1^2 - BB_1^2 + CB_1^2 - CC_1^2 = 0. \quad (2)$$

Adding member by member the relations (1) and (2), we obtain:

$$AB_1^2 - AA_1^2 + BC_1^2 - BB_1^2 + CA_1^2 - CC_1^2 = 0. \quad (3)$$

The relation (3) expresses the necessary and sufficient condition for the triangles ABC and $C_1A_1B_1$ to be orthological.

Proof 2

Let the triangle ABC simultaneously orthological with the triangles $A_1B_1C_1$ and $B_1C_1A_1$.

Using Theorem 3, we have:

$$\overrightarrow{MA}\cdot\overrightarrow{B_1C_1}+\overrightarrow{MB}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MC}\cdot\overrightarrow{A_1B_1}=0, \quad (4)$$

$$\overrightarrow{MA}\cdot\overrightarrow{C_1A_1}+\overrightarrow{MB}\cdot\overrightarrow{AB_1}+\overrightarrow{MC}\cdot\overrightarrow{B_1C_1}=0, \quad (5)$$

Adding member by member the relations (4) and (5), we obtain:

$$\overrightarrow{MA}\cdot\left(\overrightarrow{B_1C_1}+\overrightarrow{C_1A_1}\right)+\overrightarrow{MB}\cdot\left(\overrightarrow{C_1A_1}+\overrightarrow{A_1B_1}\right)+$$
$$+\overrightarrow{MC}\cdot\left(\overrightarrow{A_1B_1}+\overrightarrow{B_1C_1}\right)=0, \quad (6)$$

whatever M – a point in plane.

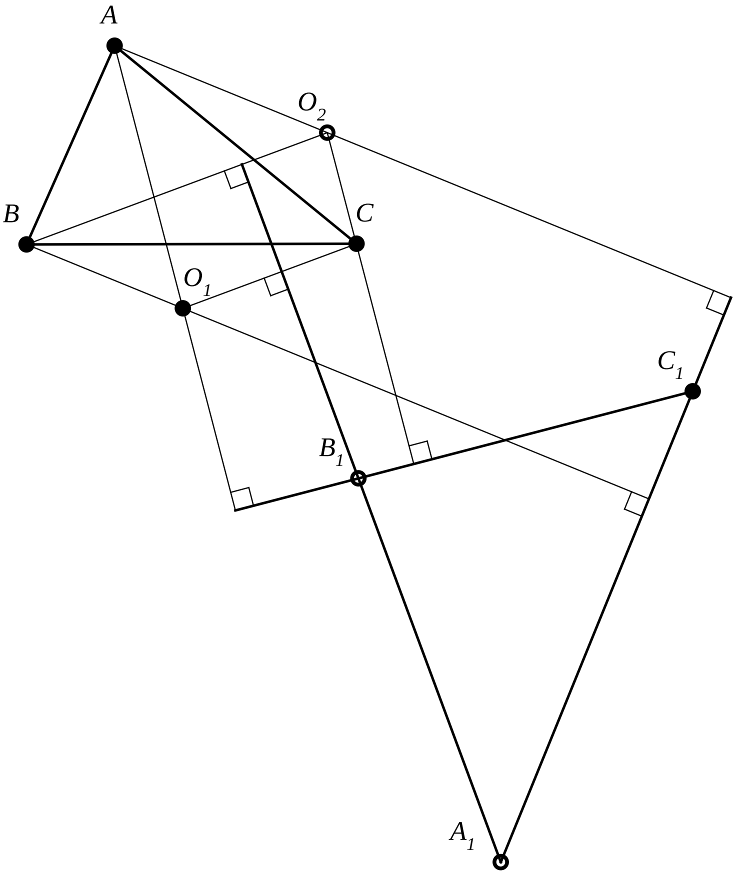

Figure 78

Because $\overrightarrow{B_1C_1} + \overrightarrow{C_1A_1} = \overrightarrow{B_1A_1}$, $\overrightarrow{C_1A_1} + \overrightarrow{A_1B_1} = \overrightarrow{C_1B_1}$ and $\overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} = \overrightarrow{A_1C_1}$ (Chasles relation), we have:

$$\overrightarrow{MA} \cdot \overrightarrow{B_1A_1} + \overrightarrow{MB} \cdot \overrightarrow{C_1B_1} + \overrightarrow{MC} \cdot \overrightarrow{A_1C_1} = 0, \quad (7)$$

whatever M – a point in plane.

This relation shows that the triangles ABC and $C_1A_1B_1$ are orthological.

Remark 18

The Pantazi's theorem can be formulated as follows:

If two triangles are biorthological, then they are triorthological.

Theorem 25 (C. Cocea, 1992)

(i) Two inversely oriented equilateral triangles ABC and $A_1B_1C_1$ are three times orthological and namely in the orders:

$$\begin{pmatrix} A & B & C \\ A_1 & B_1 & C_1 \end{pmatrix}; \begin{pmatrix} A & B & C \\ B_1 & C_1 & A_1 \end{pmatrix}; \begin{pmatrix} A & B & C \\ C_1 & A_1 & B_1 \end{pmatrix}.$$

(ii) If we denote by O'_1, O'_2, O'_3 the orthology centers corresponding to the terns above, then the triangle $O'_1O'_2O'_3$ is equilateral and congruent with the triangle ABC.

Proof

i) Denoting by $A' = pr_{B_1C_1}(A)$, $B' = pr_{A_1C_1}(B)$ and by $\{O_1\} = AA' \cap BB'$, to prove that ΔABC is orthological with $\Delta A_1B_1C_1$, it is sufficient to show that:

$\boxed{O_1C \perp A_1B_1}$ (see *Figure 79*).

We have:

$$\left.\begin{matrix} AA' \perp B_1C_1 \\ BB' \perp A_1C_1 \end{matrix}\right\} \Rightarrow O_1A'C_1B' \text{- inscribable}$$

$$\Rightarrow \left.\begin{matrix} \widehat{BCA} \equiv \widehat{A_1C_1B_1} \\ \widehat{A_1C_1B_1} \equiv \widehat{BO_1A'} \\ [AA'] \cap [BB'] = \{O_1\} \Rightarrow \widehat{BO_1A'} \equiv \widehat{BO_1A} \end{matrix}\right\} \Rightarrow \widehat{BCA} \equiv \widehat{BO_1A} \quad (1)$$

$$\Rightarrow ABO_1C\text{- inscribable} \Rightarrow \begin{cases} O_1 \in ABC \\ AO_1C \equiv ABC \end{cases} \quad (2, 3)$$

Finally, denoting: $\{C'\} = O_1C \cap A_1B_1$, we obtain that:

$$\left.\begin{array}{l} m\left(\widehat{BO_1A}\right)^{(1)} = m\left(\widehat{BCA}\right) = 60^0 \\ m\left(\widehat{AO_1C}\right)^{(3)} = m\left(\widehat{ABC}\right) = 60^0 \end{array}\right\} \Rightarrow m(C'O_1B') =$$

$$= 180^0 - \left[m\left(\widehat{BO_1A}\right) + m\left(\widehat{AO_1C}\right)\right] =$$

$$= 180^0 - (60^0 + 60^0) = 60^0 = m\left(\widehat{B_1A_1C_1}\right) \Longrightarrow$$

$$\Longrightarrow \widehat{C'O_1B'} \equiv \widehat{B_1A_1C_1} \Longrightarrow \begin{array}{c} O_1B'A_1C' - \text{inscribable} \\ BB' \perp A_1C_1 \end{array} \Longrightarrow$$

$$\Longrightarrow \boxed{CC'(O_1C) \perp A_1B_1}.$$

Similarly, it is shown that the triangles ABC and $B_1C_1A_1$ are orthological, of center O_2 (O_2 belongs to the circle circumscribed to the triangle ABC), and that the triangles ABC and $C_1A_1B_1$ are orthological, of orthology center O_3, a point that belongs to the circle circumscribed to the triangle ABC.

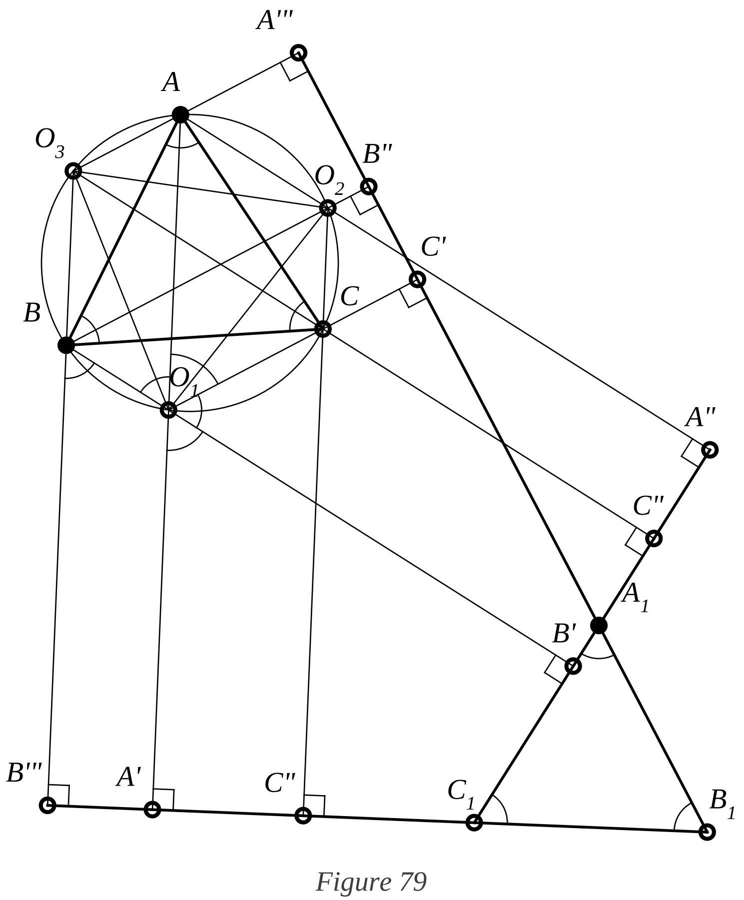

Figure 79

ii) Because $CO_3 \parallel AO_2$ (they are perpendicular to A_1C_1) and $A_1O_2C_1O_3$ is inscribable quadrilateral (these points belong to the circle circumscribed to the triangle ABC), we have that AO_2CO_3 is an isosceles trapezoid, therefore $O_2O_3 = AC$. Similarly, AO_1BO_3 is an isosceles trapezoid, hence $O_1O_3 = AB$; and BO_2CO_1 is an isosceles trapezoid, hence $O_1O_2 = BC$. But the triangle ABC is equilateral, thus $O_1O_2O_3$ is an equilateral triangle, congruent with ABC.

Observation 55

Similarly, it can be shown that, if we denote by O'_1, O'_2, O'_3 the orthology centers corresponding to the terns below:

$\begin{pmatrix} A_1B_1C_1 \\ ABC \end{pmatrix}; \begin{pmatrix} B_1C_1A_1 \\ ABC \end{pmatrix}; \begin{pmatrix} C_1A_1B_1 \\ ABC \end{pmatrix}$,

then the triangle $O'_1O'_2O'_3$ is equilateral and congruent with the triangle $A_1B_1C_1$.

It can be proved without difficulty that the triangle $O_1O_2O_3$ is biorthological in relation to the triangle $O'_1O'_2O'_3$.

Theorem 26 (Mihai Miculița)

Two certain similar triangles ABC and $A_1B_1C_1$, inversely oriented, are orthological.

Consequences

1). During the proof, we showed that the quadrilateral $PB_1A_1C_1$ is inscribable, it follows that $P \in A_1B_1C_1$. Therefore, the orthology center P of the triangle $A_1B_1C_1$ with respect to the triangle ABC is to be found on the circumscribed circle of the triangle $A_1B_1C_1$.

2). If ABC and $A_1B_1C_1$ are two inversely oriented equilateral triangles, then we have: $\Delta ABC \sim \Delta A_1B_1C_1 \sim \Delta B_1C_1A_1 \sim \Delta C_1A_1B_1$, so the point *i*) of Theorem 25 is a particular case of the above Theorem.

Proof

Denoting by $A' = pr_{BC}(A_1)$, $C' = pr_{AB}(C_1)$ and $\{P\} = AA' \cap CC'$, and by $\{B'\} = AC \cap B_1P$, the proof is now reduced to show only:

$B' = pr_{AC}(B_1)$ (see *Figure 80*).

We have:

$$\left.\begin{array}{l} A' = pr_{BC}(A_1) \Leftrightarrow A_1A' \perp BC \\ C' = pr_{AN}(A_1) \Leftrightarrow C_1C' \perp AB \end{array}\right\} \Rightarrow \widehat{CA'P} \equiv \widehat{BC'P} \Rightarrow$$

$$\left.\begin{array}{c}\{P\} = AA' \cap CC' \Rightarrow \widehat{A_1PC_1} \Longrightarrow \widehat{C'PA'} \\ \Longrightarrow A'BC'P - \text{inscribable} \Longrightarrow \widehat{C'PA'} = \widehat{ABC} \\ \Delta ABC \sim \Delta A_1B_1C_1 \Longrightarrow \widehat{ABC} = \widehat{A_1B_1C_1}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \widehat{A_1PC_1} = \widehat{A_1B_1C_1} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{c}\widehat{A_1PC_1} = \widehat{A_1B_1C_1} \Longrightarrow PB_1A_1C_1 - \text{inscribable} \Longrightarrow \widehat{A_1C_1B_1} = \widehat{A_1PB_1} \\ [AA'] \cap [BB'] = \{P\} \Longrightarrow \widehat{A_1PB_1} = \widehat{A'PB'} \\ \Delta ABC \sim \Delta A_1B_1C_1 \Longrightarrow \widehat{ACB} = \widehat{A_1C_1B_1}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \widehat{A'PB'} = \widehat{ACB} \Longrightarrow$$

$$\left.\begin{array}{c}\Longrightarrow A'PB'C - \text{inscribable} \Longrightarrow \widehat{PB'C} = \widehat{PA'B} \\ PA' \perp BC \Longrightarrow m\left(\widehat{PA'B}\right) = 90^0\end{array}\right\} \Longrightarrow m\left(\widehat{PB'C}\right) = 90^0 \Longrightarrow$$

$$\Longrightarrow B' = pr_{AC}(B_1)$$

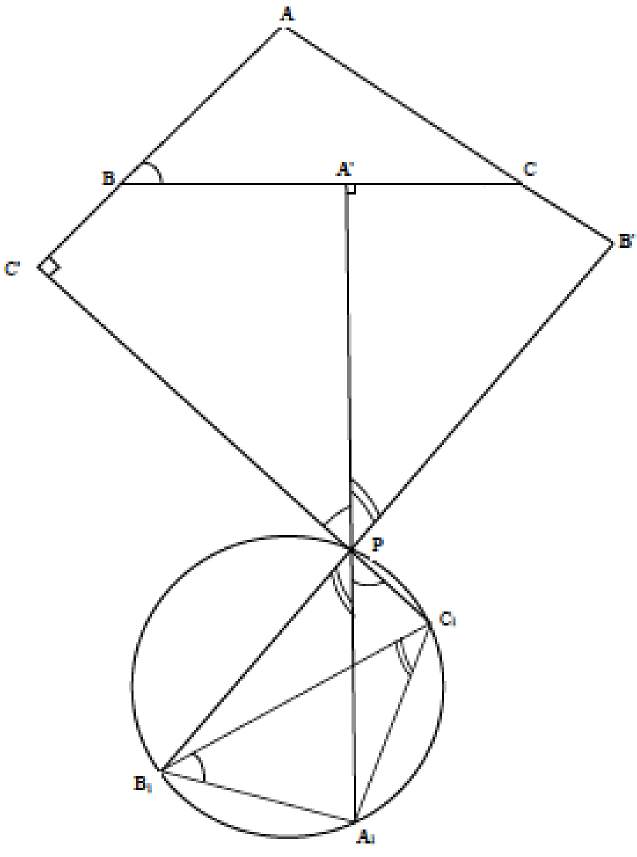

Figure 80

Theorem 27 (Lemoine)

The geometric place of the points M in the plane of a triangle ABC, whose podal triangles in relation to it are triorthological with ABC, is Lemoine line of the triangle ABC.

Proof

We consider the triangle ABC such that in Cartesian frame of reference xOY we have $A(0,a)$, $B(b,0)$, $C(c,0)$.

It is known that the equation of a circle determined by three noncollinear points, A_1, A_2, A_3, $A_i(x_i,y_i)$, $i \in \{1,2,3\}$, is given by:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

For the points A, B, C with the above coordinates, this equation is the following, after development of the determinant:

$$a(x^2+y^2) - a(b+c)x - (a^2+bc)y + abc = 0.$$

We write the Lemoine equation of the triangle ABC. The equation is determined by the intersection of tangents to the circle circumscribed with the opposite sides of the triangle.

The equation of the tangent in $A(0,a)$ to the circumscribed circle is:

$$a(b+c)x - y(a^2-bc) + a(a^2-bc) = 0.$$

Intersecting this tangent with $0x$, we find the point D, with $D\left(\frac{bc-a^2}{b+c};0\right)$.

The equation of the tangent in $B(b,0)$ to the circumscribed circle of the triangle ABC is:

$$a(b-c)x - (a^2+bc)y - ab(b-c) = 0.$$

We intersect this tangent with the line AC:

$$ax + cy - ac = 0.$$

We obtain:

$$E\left(\frac{c(a^2+b^2)}{a^2-c^2+2bc};\frac{-a(b-c)^2}{a^2-c^2+2bc}\right).$$

The slope of Lemoine line of the triangle ABC is:

$$m_{DE} = \frac{a(b+c)(b-c)^2}{-a^4-2a^2bc-bc^3-b^3c-b^2c^2}.$$

The equation of Lemoine line of the triangle ABC is:

$$\boxed{a(b+c)(b-c)^2x + (a^4+2a^2bc+bc^3+b^3c-b^2c^2)y - a(bc-a^2)(b-c)^2 = 0}.$$

Let us consider now a point $M(x_0, y_0)$ that has the property from statement, namely its podal triangle, which we denote $A_1B_1C_1$, is triorthological with the triangle ABC. We have that $A_1(x_0, 0)$. The equation of the perpendicular taken from M to AB is: $y - y_0 = \frac{c}{a}(x - x_0)$.

Its intersection with AB: $ax + by - ab = 0$ is:

$C_1\left(\frac{b(a^2+bx_0-ay_0)}{a^2+b^2}; \frac{a(b^2-bx_0+ay_0)}{a^2+b^2}\right)$.

The perpendicular taken from M to AC has the equation:

$cx - ay - cx_0 + ay_0 = 0$.

Intersecting this line with AC: $ax + cy - ac = 0$, we obtain:

$B_1\left(\frac{c(a^2+cx_0-ay_0)}{a^2+c^2}; \frac{a(c^2-cx_0+ay_0)}{a^2+c^2}\right)$.

The equation of the perpendicular taken from A_1 to AC is:

$cx - ay - cx_0 = 0$.

The equation of the perpendicular taken from B_1 to AB is:

$$b(a^2+c^2)x - a(a^2+c^2)y - x_0(bc^2+a^2c) + y_0 \cdot a(a^2+bc) - a^2(c^2-bc) = 0.$$

The equation of the perpendicular taken from C_1 to BC is:

$(a^2+c^2)x - b(a^2+cx_0-ay_0) = 0$.

These perpendiculars are concurrent if and only if:

$$\begin{vmatrix} c & -a & -cx_0 \\ b(a^2+c^2) & -a(a^2+c^2) & -x_0(bc^2+ac^2)+y_0(a^3+abc+a^2c^2-a^2bc) \\ a^2+b^2 & 0 & -b(a^2+bx_0-ay_0) \end{vmatrix} = 0.$$

Developing this determinant, making reductions, we obtain:

$$a(b+c)(b-c)^2x_0 + (a^4+2a^2bc+bc^3+b^3c-b^2c^2)y_0 - a(bc-a^2)(b-c)^2 = 0$$

which shows that the point M belongs to the Lemoine line of the triangle ABC.

6

BILOGICAL TRIANGLES

In 1922, J. Neuberg proposed the name of bilogical triangles for the triangles that are simultaneously homological and orthological.

6.1 Sondat's theorem. Proofs

Theorem 28 (P. Sondat, 1894)

If two triangles ABC and $A_1B_1C_1$ are bilogical with the homology center P and the orthology centers Q_1 and Q, then P, Q and Q_1 are on the same line, which is perpendicular to the axis of homology.

Proof 1 (V. Thébault, 1952)

We denote by d the axis of homology of the given triangle. On this line, there exist the points: $\{A'\} = BC \cap B_1C_1$, $\{B'\} = AC \cap A_1C_1$, $\{C'\} = AB \cap A_1B_1$.

The orthology center Q_1 is the intersection of the perpendiculars taken from A, B, C respectively on B_1C_1; A_1C_1 and A_1B_1. The idea of proving the collinearity of points P, Q, Q_1 is to show that $PQ \perp d$ and that $PQ_1 \perp d$.

To prove that $PQ \perp d$, it is necessary and sufficient to prove the relation:

$$B'P^2 - B'Q^2 = A'P^2 - A'Q^2. \qquad (1)$$

We employ Stewart's theorem in the triangle PAC; $B' \in AC$; we have:

$$B'P^2 \cdot AC + CP^2 \cdot B'A - PA^2 \cdot B'C = B'A \cdot AC \cdot B'C. \qquad (2)$$

The point P being the homology center, we have:

$$\overrightarrow{PA_1} = \alpha \cdot \overrightarrow{AA_1}, \overrightarrow{PB_1} = \beta \cdot \overrightarrow{BB_1}, \overrightarrow{PC_1} = \gamma \cdot \overrightarrow{CC_1}, \qquad (3)$$

α, β, γ real numbers (in the case of *Figure 81*, α, β, γ are strictly positive).

Menelaus's theorem applied in the triangle PAC for the transverse $B' - C_1 - A_1$ implies:

$$\frac{B'C}{B'A} \cdot \frac{A_1A}{A_1P} \cdot \frac{C_1P}{C_1C} = 1. \qquad (4)$$

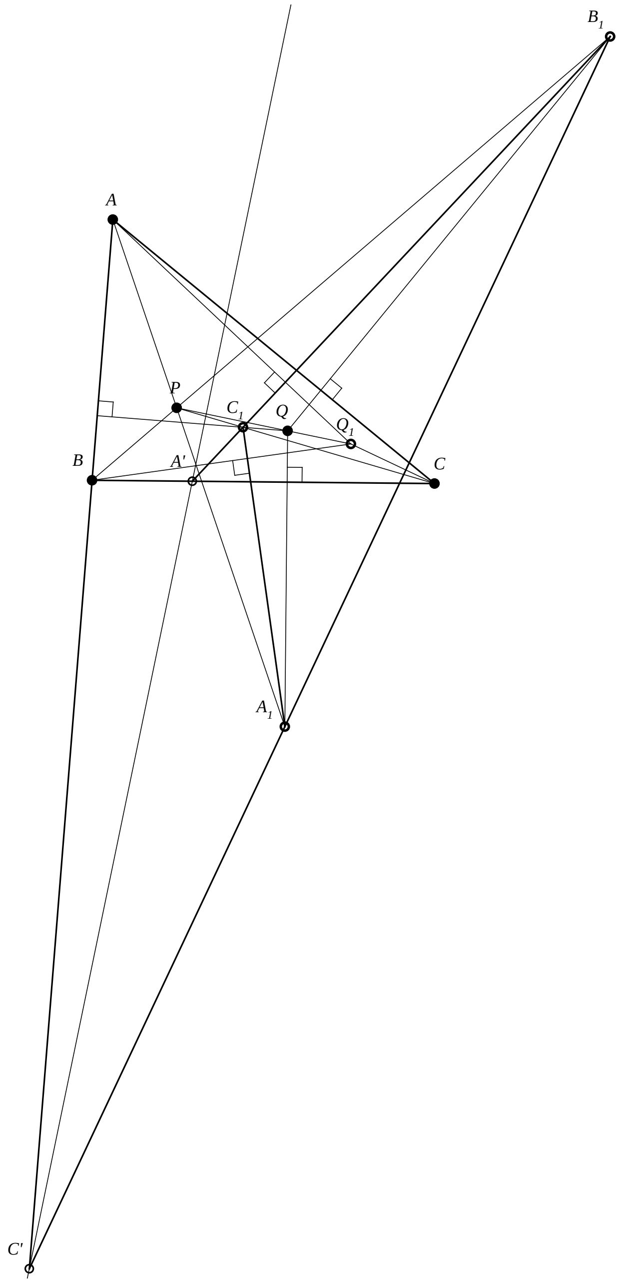

Figure 81

Taking into account (3), we get from (4) that:

$\frac{B'C}{B'A} = \frac{\alpha}{\gamma}$. (5)

Stewart's theorem applied in the triangle QAC, $B' \in AC$, leads to:

$B'Q^2 \cdot AC + CQ^2 \cdot B'A - QA^2 \cdot B'C = B'A \cdot AC \cdot B'C$. (6)

We equal the relations (2) and (6); we get:

$$B'P^2 \cdot AC + CP^2 \cdot B'A - PA^2 \cdot B'C = B'Q^2 \cdot AC + CQ^2 \cdot B'A - QA^2 \cdot B'C$$

or:

$(B'P^2 - B'Q^2) \cdot AC = (CQ^2 - CP^2) \cdot B'A + (PA^2 - QA^2) \cdot B'C$. (7)

We denote:

$PA^2 - QA^2 = u$, $PB^2 - QB^2 = v$, $PC^2 - QC^2 = t$. (8)

We find:

$B'P^2 - B'Q^2 = \frac{\alpha u - \gamma t}{\alpha - \gamma}$. (9)

We apply Stewart's theorem in the triangles PBC and QBC, $A' \in BC$:

$A'P^2 \cdot BC - PB^2 \cdot A'C + PC^2 \cdot A'B = A'B \cdot BC \cdot A'C$, (10)

$A'Q^2 \cdot BC - QB^2 \cdot A'C + QC^2 \cdot A'B = A'B \cdot BC \cdot A'C$. (11)

Menelaus's theorem in the triangle PBC for the transverse $A' - C_1 - B_1$, leads to:

$\frac{A'B}{A'C} = \frac{\gamma}{\beta}$. (12)

We equal the relations (10) and (11), taking into account (8) and (12):

$A'P^2 - A'Q^2 = \frac{v\beta - t\gamma}{\beta - \gamma}$. (13)

The relation (1) is equivalent to:

$\alpha\beta(u - v) + \beta\gamma(v - t) + \gamma\alpha(t - u) = 0$. (14)

To prove (14), we apply Stewart's theorem in the triangles PAC and PAB, $A_1 \in AP$; we get:

$CA^2 \cdot PA_1 - CP^2 \cdot AA_1 + CA_1^2 \cdot PA = PA \cdot PA_1 \cdot AA_1$, (15)

$BA^2 \cdot PA_1 - BP^2 \cdot AA_1 + BA_1^2 \cdot PA = PA \cdot PA_1 \cdot AA_1$. (16)

We equal these relations and we get:

$(BA^2 - CA^2)PA_1 + (PC^2 - PB^2)AA_1 + (BA_1^2 - CA_1^2)PA = 0$. (17)

Because $A_1Q \perp BC$, we have that: $BA_1^2 - CA_1^2 = QB^2 - QC^2$.

Substituting this relation into (17); taking into account that $\frac{PA_1}{AA_1} = \alpha$; and the relations (8), – we obtain:

$BA^2 - CA^2 + QC^2 - QB^2 = \frac{v - t}{\alpha}$. (18)

Similarly, we get the relations:

$$CB^2 - AB^2 + QA^2 - QC^2 = \frac{t-u}{\beta}, \quad (19)$$

$$AC^2 - BC^2 + QB^2 - QA^2 = \frac{u-v}{\gamma}. \quad (20)$$

Adding the last three member-to-member relations, we get:

$$\frac{v-t}{\alpha} + \frac{t-u}{\beta} + \frac{u-v}{\gamma} = 0. \quad (21)$$

The relations (14) and (21) are equivalent because $PQ \perp d$. Similarly, it is proved that $PQ_1 \perp d$, which ends the proof of P. Sondat's theorem.

Proof 2 (adapted after the proof given by Jean-Louis Aymé)

Let A_2 be the intersection of the perpendicular taken from Q_1 to BC with AA_1; B_2 – the intersection of the parallel taken through A_2 to A_1B_1 with BB_1; and C_2 – the intersection of the parallel taken through B_2 to B_1C_1 with CC_1 (see *Figure 82*).

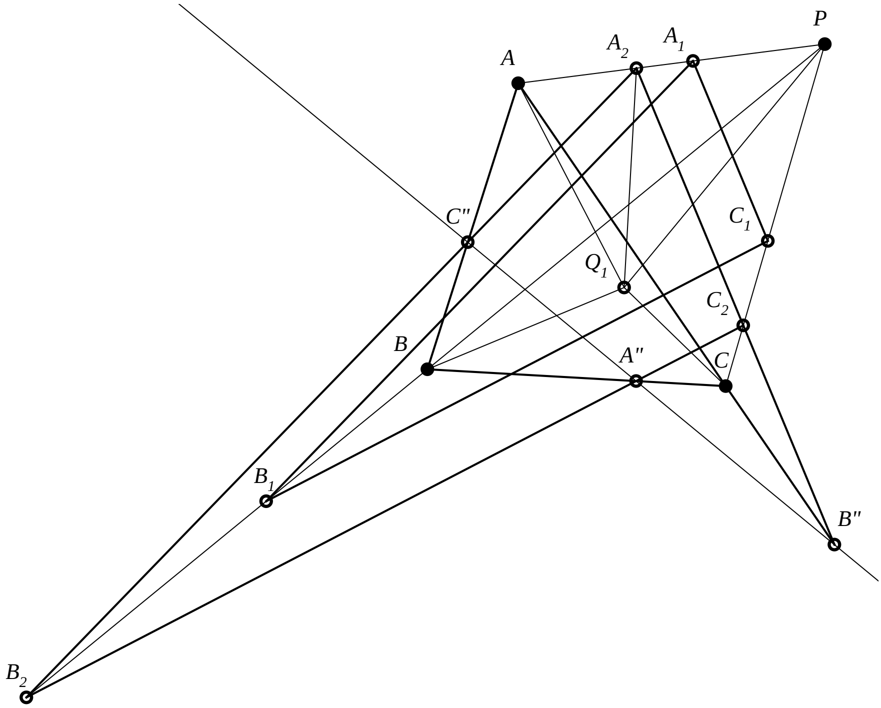

Figure 82

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic by a homothety of center P (they have respectively parallel sides). Because $AQ_1 \perp B_1C_1$ and $B_1C_1 \perp B_2C_2$, it follows that $AQ_1 \perp B_2C_2$; similarly, $BQ_1 \perp A_2C_2$ and $CQ_1 \perp A_2B_2$, therefore Q_1 is common orthology center for the triangles $A_2B_2C_2$ and ABC.

The triangle $A_2B_2C_2$ being homothetic with $A_1B_1C_1$, and $A_2B_2C_2$ being homological with ABC (of center P), we denote by $A''B''C''$ the homology axis of triangles $A_2B_2C_2$ and ABC, we have that $A''B'' \parallel A'B'$ ($A'B'$ is the homology axis of triangles ABC and $A_1B_1C_1$). Applying now Proposition 67, it follows that $PQ_1 \perp A''B''$, therefore:

$PQ_1 \perp A'B'$. (1)

We denote by A_3 the intersection of the perpendiculars taken from Q to B_1C_1 with AA_1; let B_3 – the intersection of the parallel taken from A_3 to AB_1 with BB_1; and let C_3 – the intersection of the parallel taken from B_3 to BC with CC_1. The triangles ABC and $A_3B_3C_3$ are homothetic of center P (they have respectively parallel sides). Having $B_1Q \perp AC$ and $A_3C_3 \parallel AC$, it follows that $B_1Q \perp A_3C_3$; from $A_1Q \perp BC$ and $B_3C_3 \parallel BC$, we obtain that $A_1Q \perp B_3C_3$. In conclusion, the triangles $A_1B_1C_1$ and $A_3B_3C_3$ are orthological, with the common orthology center the point Q, and homological, of center P. Applying the Proposition 67, it follows that $PQ \parallel A'''B'''$ (where $A'''B'''$ is the homology axis of triangles $A_1B_1C_1$ and $A_3B_3C_3$. Since $A_3B_3C_3$ is homothetic with ABC, it follows that $A'''B'''$ is parallel with $A'B'$; consequently:

$PQ \perp A'B'$. (2)

The relations (1) and (2) lead to the conclusion.

6.2 Remarkable bilogical triangles

6.2.1 A triangle and its first Brocard triangle

Definition 37

We call the first Brocard triangle of a given triangle – the triangle determined by the projections of the symmedian center on the mediators of the given triangle.

Observation 56

In *Figure 83*, K is the intersection of symmedians of triangles ABC; OA', OB', OC' are its mediators; and $A_1B_1C_1$ is the first Brocard triangle.

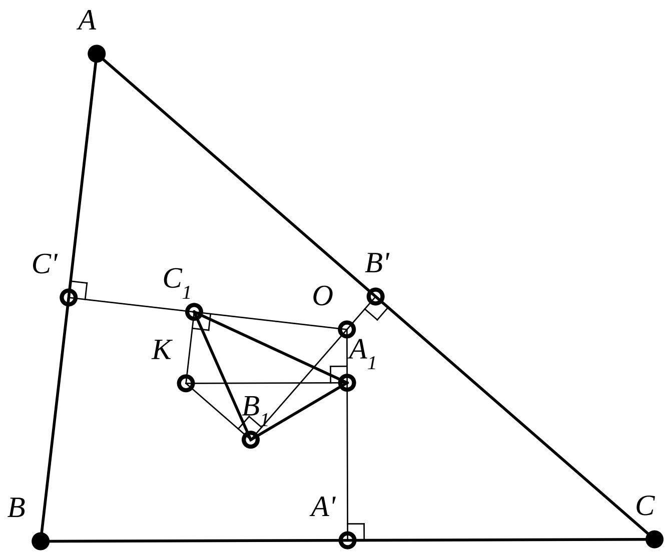

Figure 83

Proposition 69

The first Brocard triangle is similar with the given triangle.

Proof

We observe that $m(\widehat{KA_1O}) = 90^0$, therefore A_1 belongs to the circle of diameter OK; similarly B_1, C_1 belong to this circle. We have: $\widehat{B_1A_1C_1} \equiv \widehat{B_1OC_1}$ (subtending the same arc in the circle circumscribed to the first Brocard triangle). On the other hand, $\sphericalangle B_1OC_1 \equiv \sphericalangle BAC$ (they have sides respectively perpendicular). We obtain that $\sphericalangle B_1A_1C_1 \equiv \sphericalangle BAC$. Similarly, it follows that $\sphericalangle A_1B_1C_1 \equiv \sphericalangle ABC$, and, consequently, the given triangle ABC and its first Brocard triangle $A_1B_1C_1$ are similar.

Observation 57

1. The circle circumscribed to the first Brocard triangle is called Brocard circle.
2. The previous Proposition is proved in the same way in the case of the obtuse (or right) triangle.

Theorem 29

The triangle ABC and its first Brocard triangle, $A_1B_1C_1$, are bilogical triangles.

We will prove this theorem in two steps:

I. We prove that the triangles $A_1B_1C_1$ and ABC are orthological.

Indeed, the perpendiculars taken from A_1, B_1, C_1 to BC, CA, AB are mediators of the triangle ABC and, consequently, O – the center of the circle circumscribed to the triangle ABC, is the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC. According to the theorem of orthological triangles, the perpendiculars taken from A_1, B_1, C_1 respectively to the sides of the first Brocard triangle $A_1B_1C_1$ are concurrent.

Because this point is important in the geometry of the triangle, we will define it and prove the concurrency of previous lines.

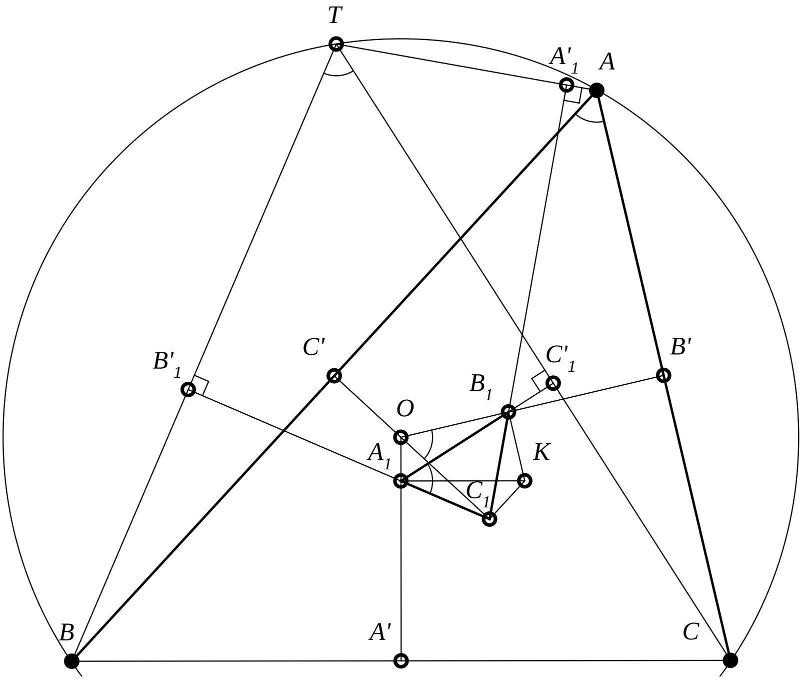

Figure 84

Definition 38

It is called a Tarry point of the triangle ABC the orthology center of the triangle ABC in relation to the first Brocard triangle.

Proposition 70

The orthology center of the triangle ABC in relation to $A_1B_1C_1$ – its first Brocard triangle, is the Tarry point, T, and this point belongs to the circle circumscribed to the triangle ABC.

Proof

We denote: $\{B_1'\} = BT \cap A_1C_1$, $\{C_1'\} = CT \cap A_1B_1$ (see *Figure 84*). We have: $\sphericalangle C_1A_1B_1 \equiv \sphericalangle A$; it follows that $m\left(\widehat{B_1'A_1C_1'}\right) = 180^0 - A$, consequently $\sphericalangle BTC \equiv \sphericalangle A$, which shows that T belongs to the circle circumscribed to the triangle ABC.

II. To prove that the triangles ABC and $A_1B_1C_1$ are homological, we need some helpful results:

Definition 39

The points $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega'}$ from the interior of triangle $\boldsymbol{ABC}$, with the properties:

$\boldsymbol{m(\sphericalangle \Omega AB) = m(\sphericalangle \Omega BC) = m(\sphericalangle \Omega CA) = \omega}$**,**

$\boldsymbol{m(\sphericalangle \Omega' BA) = m(\sphericalangle \Omega' AC) = m(\sphericalangle \Omega' CB) = \omega}$**,**

are called Brocard points, and $\boldsymbol{\omega}$ is called Brocard angle.

Lemma 7

In the triangle ABC, where Ω is the first Brocard point and $A\Omega \cap BC = \{A''\}$, we have $\frac{BA''}{CA''} = \frac{c^2}{a^2}$.

Proof

$$\text{Area}\Delta ABA'' = \frac{1}{2}AB \cdot AA'' \cdot sin\omega, \quad (1)$$

$$\text{Area}\Delta ACA'' = \frac{1}{2}AC \cdot AA'' \cdot sin(A-\omega). \quad (2)$$

From (1) and (2), it follows that:

$$\frac{\text{Area}\Delta ABA''}{\text{Area}\Delta ACA''} = \frac{c \cdot sin\omega}{b \cdot sin(A-\omega)}. \quad (3)$$

On the other hand, the mentioned triangles have the same altitude taken from A, hence:

$$\frac{\text{Area}\Delta ABA''}{\text{Area}\Delta ACA''} = \frac{BA''}{CA''}. \quad (4)$$

The relations (3) and (4) lead to:

$$\frac{BA''}{CA''} = \frac{c}{a} \cdot \frac{sin\omega}{sin(A-\omega)}. \quad (5)$$

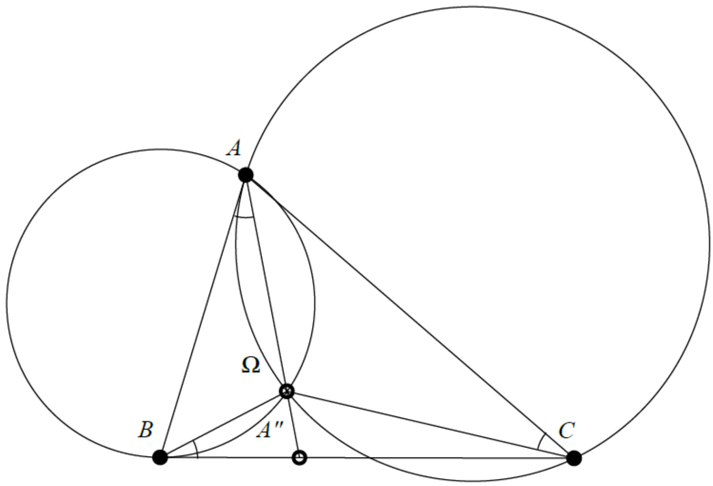

Figure 85

Applying the sinus theorem in the triangles $A\Omega C$ and $B\Omega C$, we get:

$$\frac{C\Omega}{sin(A-\omega)} = \frac{AC}{\sin(A\Omega C)}, \quad (6)$$

$$\frac{C\Omega}{sin\omega} = \frac{BC}{\sin(B\Omega C)}. \quad (7)$$

Because $m\left(\widehat{A\Omega C}\right) = 180^0 - A$ and $m\left(\widehat{B\Omega C}\right) = 180^0 - C$, from relations (6) and (7) – it follows that:

$$\frac{sin\omega}{sin(A-\omega)} = \frac{b}{a} \cdot \frac{sinC}{sinA}. \quad (8)$$

The sinus theorem in the triangle ABC provides:

$$\frac{sinC}{sinA} = \frac{c}{a}. \quad (9)$$

The relations (5), (8) and (9) lead to:

$$\frac{BA''}{CA''} = \frac{c^2}{a^2}.$$

Observation 58

1. Denoting $\{B''\} = B\Omega \cap AC$ and $\{C''\} = C\Omega \cap AB$, we obtain similarly the relations:
   $\frac{CB''}{B''A} = \frac{a^2}{b^2}, \frac{AC''}{C''B} = \frac{b^2}{c^2}$.
2. Denoting $\{A'''\} = A\Omega' \cap BC$,$\{B'''\} = B\Omega' \cap AC$, $\{C'''\} = C\Omega' \cap AB$, and proceeding similarly, we find:
   $\frac{BA'''}{CA'''} = \frac{a^2}{b^2}; \frac{CB'''}{AB'''} = \frac{b^2}{c^2}; \frac{AC'''}{BC'''} = \frac{c^2}{a^2}$.

Lemma 8

In a triangle ABC, the following relation is true:

$$\cot\omega = \cot A + \cot B + \cot C. \qquad (10)$$

Proof

From relation (8), it follows that:

$$\sin(A-\omega) = \frac{a}{b} \cdot \frac{\sin A}{\sin C} \cdot \sin\omega. \qquad (11)$$

But $\frac{\sin A}{\sin B} = \frac{a}{b}$; replacing in (11), we find that:

$$\sin(A-\omega) = \frac{\sin^2 A \cdot \sin\omega}{\sin B \cdot \sin C}.$$

We develop: $\sin(A-\omega) = \sin A \cdot \sin\omega - \sin\omega \cdot \cos A$.

We have: $\sin A \cdot \cos\omega - \sin\omega \cdot \cos A = \frac{\sin^2 A \cdot \sin\omega}{\sin B \cdot \sin C}$. (12)

We divide the relation (12) by $\sin A \cdot \sin\omega$, and considering that $\sin A = \sin(B+C)$, and $\sin(B+C) = \sin B \cdot \cos C + \sin C \cdot \cos B$, we get the relation (10).

Observation 59

From (10), we obtain that:

$$\tan\omega = \frac{4S}{a^2+b^2+c^2} \qquad (13)$$

Lemma 9

If, in the triangle ABC, K is symmedian center, and K_1 is the projection of K on BC, then:

$$KK_1 = \frac{1}{2}a \cdot \tan\omega.$$

Proof

If AA_2 and CC_2 are symmedians, applying Menelaus's theorem in the triangle AA_2B and taking into account that $\frac{BA_2}{CA_2} = \frac{c^2}{b^2}$ and $\frac{C_2A}{C_2B} = \frac{b^2}{a^2}$, we get that $\frac{AK}{KA_2} = \frac{b^2+c^2}{a^2}$, and from here $\frac{AA_2}{KA_2} = \frac{a^2+b^2+c^2}{a^2}$.

But $\frac{AA_2}{KA_2} = \frac{h_a}{KK_1}$ (h_a is the altitude from A of the triangle ABC).

From $\frac{h_a}{KK_1} = \frac{a^2+b^2+c^2}{a^2}$, since $h_a = \frac{2S}{a}$, it follows that $KK_1 = \frac{1}{2}a \cdot \tan\omega$.

Observation 60

If K_2, K_3 are the projections of K on AC and AB, then the following relation takes place:

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} = \frac{1}{2}\tan\omega.$$

Lemma 10

In the triangle ABC the cevians of Brocard $B\Omega$ and $C\Omega'$ intersect in the point A_1 (vertex of the first Brocard triangle).

Proof

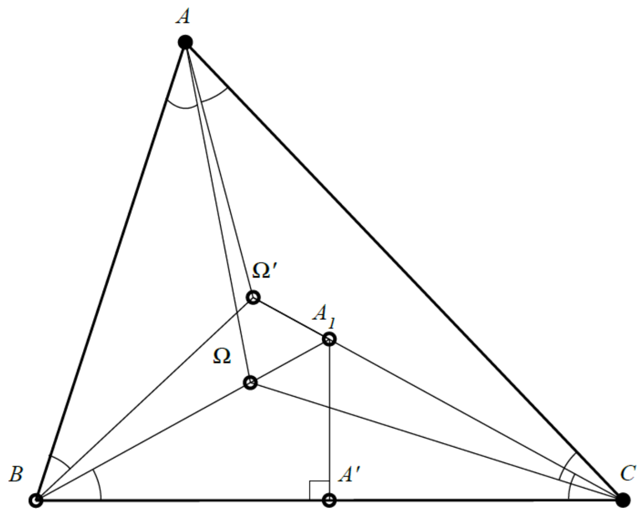

Figure 86

Let $\{A_1'\} = B\Omega \cap C\Omega'$ (see *Figure 86*). Because $\sphericalangle A_1'BC \equiv \sphericalangle A_1'CB = \omega$, it follows that the triangle $BA_1'C$ is isosceles if $A_1'A' \perp BC$ (A' is the midpoint of BC). Moreover, having: $A_1'A' = \frac{1}{2} \cdot a \cdot \tan\omega$, therefore $A_1'A' = KK_1$, it follows that $A_1' = A_1$.

Lemma 11

Let ABC – a triangle, and $A_1B_1C_1$ – its first Brocard triangle. The lines AA_1, BB_1, CC_1 are the isotomics of the symmedians AA_2, BB_2, CC_2.

Proof

We denote by $B\Omega \cap AC = \{B''\}$, $C\Omega' \cap AB = \{C'''\}$ and $AA_1 = \{A_2'\}$.

Since $B\Omega$, $C\Omega'$ and AA_1 are concurrent, applying Ceva's theorem, we have that:

$\frac{CB''}{B''A} \cdot \frac{C'''A}{C'''B} \cdot \frac{A_2'B}{A_2'C} = 1$, but $\frac{CB''}{B''A} = \frac{a^2}{b^2}$ and $\frac{C'''A}{C'''B} = \frac{c^2}{a^2}$, we obtain that: $\frac{A_2'B}{A_2'C} = \frac{b^2}{c^2}$. Because AA_2 is symmedian and $\frac{A_2B}{A_2C} = \frac{c^2}{b^2}$, it follows that the points A_2 and A_2' are symmetric with respect to midpoint A' of BC, therefore AA_2' is the isotomic of symmedian AA_2; similarly, we have that BB_2' and CC_2' are the isotomic of symmedian BB_2; respectively of symmedian CC_2.

We are now completing the second step of the proof. It is known that the isotomics of concurrent cevians in a triangle are concurrent cevians and, since AA_1, BB_1, CC_1 are isotomics of symmedian, it follows that they are concurrent and, consequently, the triangle ABC and its first Brocard triangle $A_1B_1C_1$ are homological triangles. The center of these homologies is denoted, by some authors, Ω'', and it is called the third Brocard point.

Remark 20

Sondat's theorem implies the collinearity of points: O – the center of the circle circumscribed to the triangle ABC; T – Tary point; and Ω'' – third Brocard point of the triangle ABC.

6.2.2 A triangle and its Neuberg triangle

Definition 40

Two triangles that have the same Brocard angle are called equibrocardian triangles.

Observation 61

a) Two similar triangles are equibrocardian triangles.
b) A given triangle and its first Brocard triangle are equibrocardian triangles.

Theorem 30

The geometric place of the points M in plane located on the same section of the side BC of a given triangle ABC, that has the property that the triangle MBC is equibrocardian with ABC, is a circle of center N_a located on the mediator of the side BC, such that $m(\sphericalangle BN_aC) = 2\omega$, and of radius $n_a = \frac{1}{2} \cdot a \cdot \sqrt{\cot^2 \omega - 3}$ (Neuberg circle – 1882).

Proof

Let ABC be a given triangle (see *Figure 87*). We start the proof by building some points of the geometric place in the idea of "identifying" the shape of the place. It is clear that the Brocard point of a triangle that is equibrocardian with ABC can be considered on the semi-line $(B\Omega$. We choose Ω' – the intersection between $(B\Omega$ and the mediator of the side BC.

We build the geometric place of the points M in the boundary plane BC, containing the point A, of which the segment $B\Omega'$ "is seen" from an angle of measure ω. This geometric place is the circle of center O' – the intersection of mediator of the segment $B\Omega'$ with the perpendicular in B to BC, having $O'B$ as radius.

We build now a secant to this circle, CM_1, such that $m\left(\widehat{\Omega' CM_1}\right) = \omega$; let M_2 be the second intersection point of this secant with the circle $\mathcal{C}(O'; O'B)$. The triangles M_1BC and M_2BC have the same Brocard point Ω' and the same angle ω, therefore they are equibrocardian with ABC; therefore the points M_1, M_2 belong to the sought geometric place. We have now three points of the sought geometric place, A, M_1, M_2.

Obviously, we can build the symmetric of this figure with respect to mediator of the segment BC, and we get other three points, A', M'_1, M'_2. It is possible that the sought geometric place to be a circle and its center be, for reasons of symmetry, located on the mediator of the segment BC. We denote by N_a the intersection of this mediator with the circle $\mathcal{C}(O'; O'B)$. We prove that N_a is the center of the circle – geometric place – the Neuberg circle. We have $\sphericalangle BN_a\Omega' = \omega$, also $\sphericalangle O'BN_a = \omega$ ($O'B \parallel N_a\Omega'$). The triangle $O'BN_a$ is isosceles, hence $\sphericalangle O'N_aB = \omega$.

For reasons of symmetry, we have that $\sphericalangle CN_a\Omega' = \omega$, consequently N_a is a fixed point located on the mediator $N\Omega'$ of BC, because $m\left(\widehat{BN_aC}\right) = 2\omega$. We show that $N_aM_1 = N_aM_2$.

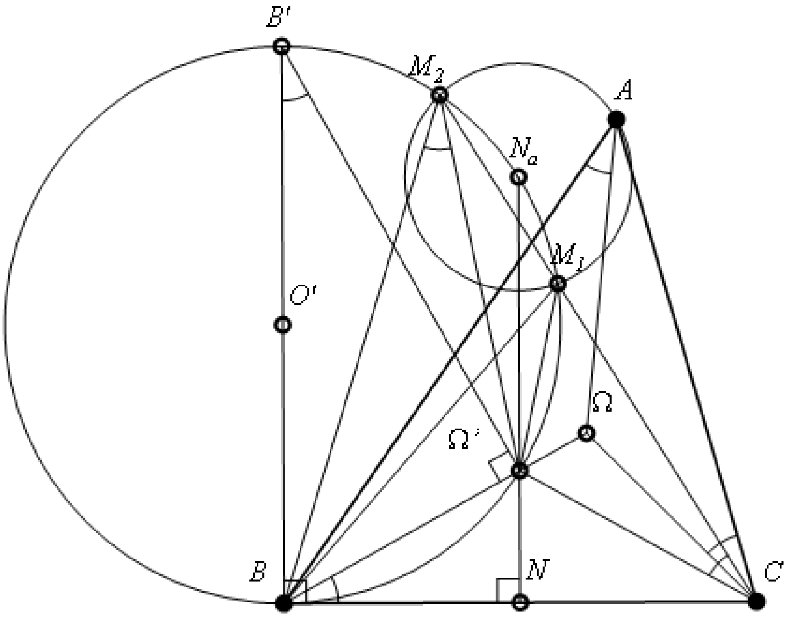

Figure 87

We denote by I the intersection between CM_1 and $\Omega'N_a$, and by J the intersection between CM_1 and $O'N_a$; because $m\left(\widehat{NCI}\right) = 2\omega$, $\sphericalangle NIC \equiv \sphericalangle N_aIJ$, and $m\left(\widehat{JN_aN}\right) = 2\omega$, it follows that $N_aO' \perp M_1M_2$; consequently: $N_aM_1 = N_aM_2$.

We prove now that $N_aA = \frac{1}{2}a \cdot \sqrt{\cot^2\omega - 3}$. We denote by P the projection of A on N_aN, and we have:

$N_aA^2 = AN^2 + NN_a{}^2 - 2NN_a \cdot PN$ (generalized Pythagorean theorem).

From the median theorem in the triangle ABC, we have:

$4AN^2 = 2(b^2 + c^2) - a^2$, apoi$NN_a = \frac{1}{2} \cdot a \cdot \cot\omega$.

We established that: $\cot\omega = \frac{a^2+b^2+c^2}{4S}$.

Area$\Delta MBC = \frac{1}{2} \cdot a \cdot MN \cdot \cos(\widehat{MNN_a})$.

From cosine theorem applied in the triangle MNN_a, we have:

$M_a^2 = MN^2 + N_aN^2 - 2MNNN_a \cdot \cos(\widehat{MNN_a})$.

We established that: $N_aN = \frac{1}{2}a \cdot \cot\omega$.

By replacing in the preceding formula, we have:

$n_a^2 = NM^2 + \frac{1}{4}a^2 \cdot \cot^2\omega - a \cdot \cot\omega \cdot \frac{2MN^2+\frac{3}{2}a^2}{2a\cdot\cot\omega}$.

The radius of Neuberg circle: $n_a = \frac{1}{2} \cdot a \cdot \sqrt{\cot^2\omega - 3}$.

We have:

$\frac{1}{4}a^2 \cdot \cot^2\omega - \frac{3}{4}a^2 = MN^2 + \frac{1}{4}a^2 \cdot \cot^2\omega - \frac{3}{4}a^2 \cdot \frac{\cot\omega}{\cot\omega'} - \frac{\cot\omega}{\cot\omega'} \cdot MN^2$.

It follows that:

$\frac{\cot\omega'-\cot\omega}{\cot\omega}\left(MN^2 + \frac{3}{4}a^2\right) = 0$.

From this relation, we get $\cot\omega' = \cot\omega$, which implies $\omega' = \omega$.

Remark 21

Let us reformulate the statement of Theorem 30 as follows:

Find the geometric place of the points M in the plane of the triangle ABC with the property that the triangles with a vertex in M and the other two to be vertices of the given triangle and have the same Brocard angle as ABC; the answer will be given in the same way as before, but there are six Neuberg circles (two symmetrical with respect to each side of the given triangle).

We observe that $2S = a \cdot PN$, therefore denoting $AN = m_a$ we have: $\cot\omega = \frac{3a^2+4m_a^2}{4a\cdot PN}$.

$N_aA^2 = m_a^2 + \frac{1}{4}a^2 \cdot \cot^2\omega - a \cdot \cot\omega \cdot \frac{3a^2+4m_a^2}{4a\cdot\cot\omega}$.

We get: $N_aA = \frac{1}{2}a \cdot \sqrt{\cdot\cot^2\omega - 3}$.

We proved that any vertex of a triangle that is equibrocardian with ABC and has the common side BC with ABC belongs to the Neuberg circle $\mathcal{C}(N_a;\ n_a)$.

We prove the reciprocal, namely that any point M that belongs to the circle $\mathcal{C}(N_a;\ n_a)$ is the vertex of the triangle MBC that is equibrocardian with the given triangle ABC.

We denote by ω' – Brocard angle of the triangle MBC (see *Figure 88*), $M \in \mathcal{C}(N_a;\ n_a)$, then $\cot\omega' = \frac{MB^2+MC^2+BC^2}{4\cdot\text{Area }\Delta MBC}$.

From the median theorem applied in the triangle MBC, we note that:

$MB^2 + MC^2 = 2MN^2 + \frac{a^2}{4}$.

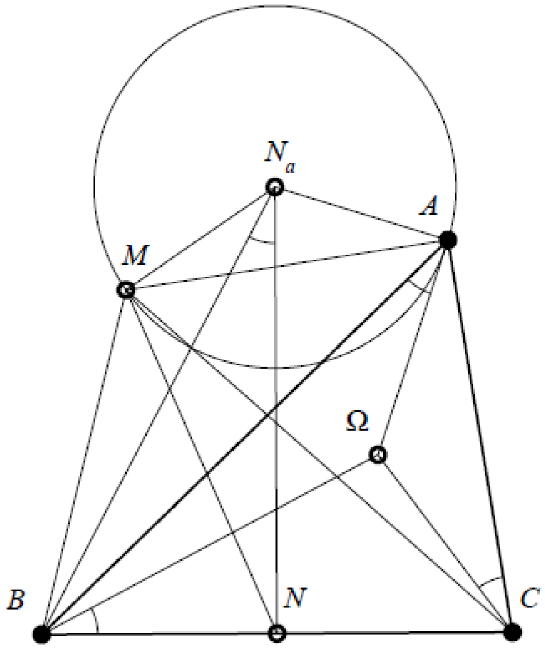

Figure 88

Definition 41

We call Neuberg triangle of a given triangle ABC – the triangle $N_aN_bN_c$ formed by the centers of Neuberg circles (located in semi-planes determined by a side and by a vertex of the triangle ABC).

Theorem 31

The triangle ABC and its Neuberg triangle $N_aN_bN_c$ are bilogical triangles. The homology center is the Tarry point of the triangle ABC, and an orthology center is O – the center of the circle circumscribed to the triangle ABC.

Proof

The perpendiculars taken from N_a, N_b, N_c to the sides of the triangle ABC are its mediators, hence O – the center of the circle circumscribed to the triangle ABC, is orthology center of Neuberg triangle in relation to the triangle ABC.

In Proposition 70, we proved that the perpendiculars taken from A, B, C to the sides of the first Brocard triangle are concurrent in the Tarry point, T, of the triangle ABC. To prove that T is the homology center of triangles $N_aN_bN_c$ and ABC, it is sufficient to prove that the points N_a, A, T; N_b, B, T; N_c, C, T are collinear.

The condition N_a, A, T to be collinear is equivalent to $N_aA \perp B_1C_1$, namely to $\overrightarrow{AN_a} \cdot \overrightarrow{B_1C_1} = 0$; $\overrightarrow{AN_a} = \overrightarrow{AA'} + \overrightarrow{A'N_a}$;$\overrightarrow{B_1C_1} = \overrightarrow{B_1B'} + \overrightarrow{B'C'} + \overrightarrow{C'C_1}$; $\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$; $\overrightarrow{B'C'} = -\frac{1}{2}\overrightarrow{BC}$; $\overrightarrow{AN_a} \cdot \overrightarrow{B_1C_1} = \left(\overrightarrow{AA'} + \overrightarrow{A'N_a}\right)\left(\overrightarrow{B_1B'} + \overrightarrow{B'C'} + \overrightarrow{C'C_1}\right) = \frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{B_1B'} + \frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{B'C'} + \frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{C'C_1} + \frac{1}{2}\overrightarrow{AC} \cdot \overrightarrow{B_1B'} + \frac{1}{2}\overrightarrow{AC} \cdot \overrightarrow{B'C'} + \frac{1}{2}\overrightarrow{AC} \cdot \overrightarrow{C'C_1} + \overrightarrow{A'N_a} \cdot \overrightarrow{B_1B'} + \overrightarrow{A'N_a} \cdot \overrightarrow{B'C'} + \overrightarrow{A'N_a} \cdot \overrightarrow{C'C_1}$.

But $\overrightarrow{AB} \cdot \overrightarrow{C'C_1} = 0$; $\overrightarrow{AC} \cdot \overrightarrow{B_1B'} = 0$; $\overrightarrow{A'N_a} \cdot \overrightarrow{B'C'} = 0$.

$\frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{B_1B'} = -\frac{1}{4}bc \cdot \tan\omega \cdot \sin A$,

$\frac{1}{2}\overrightarrow{AC} \cdot \overrightarrow{C'C_1} = \frac{1}{4}bc \cdot \tan\omega \cdot \sin A$,

$\frac{1}{2}\overrightarrow{AB} \cdot \overrightarrow{B'C'} = \frac{1}{4}ac \cdot \cos B$,

$\frac{1}{2}\overrightarrow{AC} \cdot \overrightarrow{B'C'} = -\frac{1}{4}ab \cdot \cos C$,

$\overrightarrow{A'N_a} \cdot \overrightarrow{B_1B'} = \frac{1}{4}ab \cdot \cos C$,

$\overrightarrow{A'N_a} \cdot \overrightarrow{C'C_1} = \frac{1}{4}ac \cdot \cos B$.

We consequently obtain that $\overrightarrow{AN_a} \cdot \overrightarrow{B_1C_1} = 0$.

Similarly, we show that the points N_b, B, T;N_c, C, T are collinear.

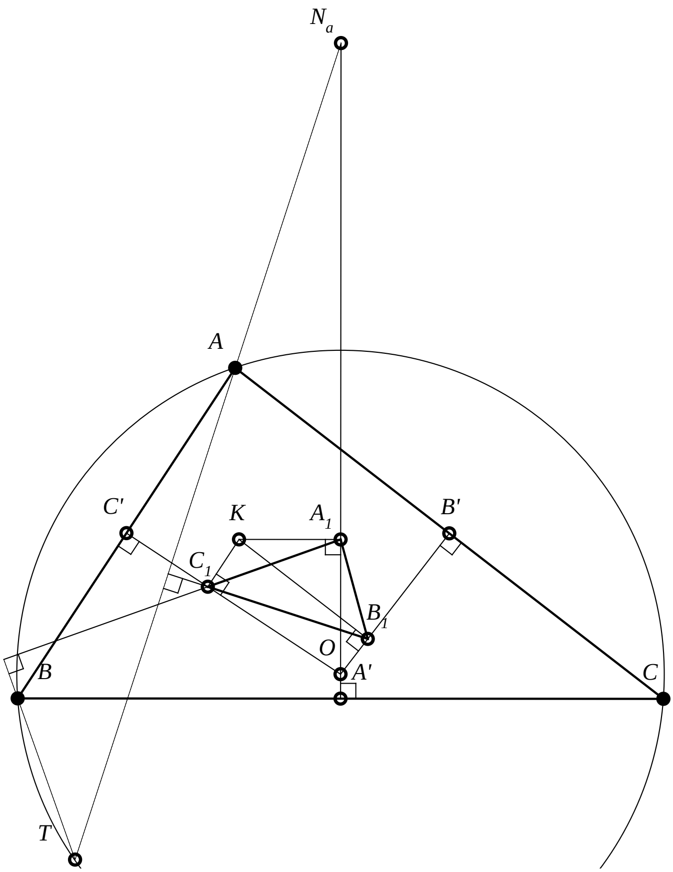

Figure 89

Remark 22

The Neuberg triangle and the first Brocard triangle of a given triangle are bilogical triangles. The homology center is the center of the circle circumscribed to the given triangle, and one of the orthology centers is the Tarry point of the triangle.

Theorem 32

If ABC is a given triangle, $N_aN_bN_c$ is its Neuberg triangle, and $A_1B_1C_1$ is its first Brocard triangle, then: these triangles are bilogical two by two, have the same axis of orthology and the same axis of homology.

Proof

We noticed that the points O, T, Ω'' are collinear; applying Theorem 19 from [24], it follows that the triangles ABC, $A_1B_1C_1$, $N_aN_bN_c$ have two by two the same axis of homology. If we denote by U the orthology center of the triangle ABC in relation to $N_aN_bN_c$ and by V – the orthology center of the triangle $A_1B_1C_1$ in relation to $N_aN_bN_c$, then, according to Sondat's theorem, it follows that the orthology axis of the triangles ABC and $A_1B_1C_1$, namely OT, is perpendicular to their homology axis. Because O is orthology center of the triangles $N_aN_bN_c$ and ABC, it means that their axis of orthology is the perpendicular from O to their axis of homology, hence it coincides with OT, and the orthology axis of the triangles $N_aN_bN_c$ and $A_1B_1C_1$, passing through T and being perpendicular to the axis of homology, is OT. Sondat's theorem implies the collinearity of points T, O, Ω'', U and V.

6.2.3 A triangle and the triangle that determines on its sides three congruent antiparallels

Theorem 33 (R. Tucker)

Three congruent antiparallels in relation to the sides of a triangle determines on these sides six concyclic points.

Proof

Let (A_1A_2), (B_1B_2), (C_1C_2) be the three antiparallels respectively to the congruent sides BC, CA, AB (see *Figure 90*).

We denote by A_3, B_3, C_3 the intersection of the pairs of antiparallels $(B_1B_2;\ C_1C_2)$, $(C_1C_2;\ A_1A_2)$ and $(A_1A_2;\ B_1B_2)$. The triangles $A_3B_1C_2$; $B_3C_1A_2$; $C_3B_2A_1$ are isosceles. Indeed, $\sphericalangle BB_1B_2 \equiv \sphericalangle A$ and $\sphericalangle C_1C_2C \equiv \sphericalangle A$, therefore $\sphericalangle C_1C_2C \equiv \sphericalangle BB_1B_2$.

These angles being opposite at vertex with $\widehat{A_3C_2B_1}$ and $\widehat{A_3B_1C_2}$, we obtain that the triangle $A_3B_1C_2$ is isosceles. Similarly, it is shown that the triangles $B_3C_1A_2$ and $C_3B_2A_1$ are isosceles. We obtain that the bisectors of triangle $A_3B_3C_3$ are mediators of the segments (B_1C_2); (C_1A_2); (A_1B_2).

Let T be the intersection of these bisectors (the center of the circle inscribed in the triangle $A_3B_3C_3$), we have the relations $TB_1 = TC_2$; $TC_1 = TA_2$; $TB_2 = TA_1$. The triangles TB_1A_3 and TC_2A_3 are congruent (S.S.S.), it follows that $\sphericalangle TB_1A_3 \equiv \sphericalangle TC_2A_3$, with the consequence: $\sphericalangle TB_1B_2 \equiv \sphericalangle TC_2C_1$.

This relation (together with $B_1B_2 = C_1C_2$ and $TB_1 = TC_2$) leads to $\Delta TB_1B_2 \equiv \Delta TC_2C_1$, hence $TB_2 = TC_1$.

Similarly, it follows that $\Delta TA_2A_1 \equiv \Delta TC_1C_2$, with the consequence: $TA_1 \equiv TC_2$.

Thus, we obtain that: $TA_1 = TA_2 = TB_1 = TB_2 = TC_1 = TC_2$, which shows that the points A_1, A_2, B_1, B_2, C_1, C_2 are on a circle with the center T. This circle is called Tucker circle.

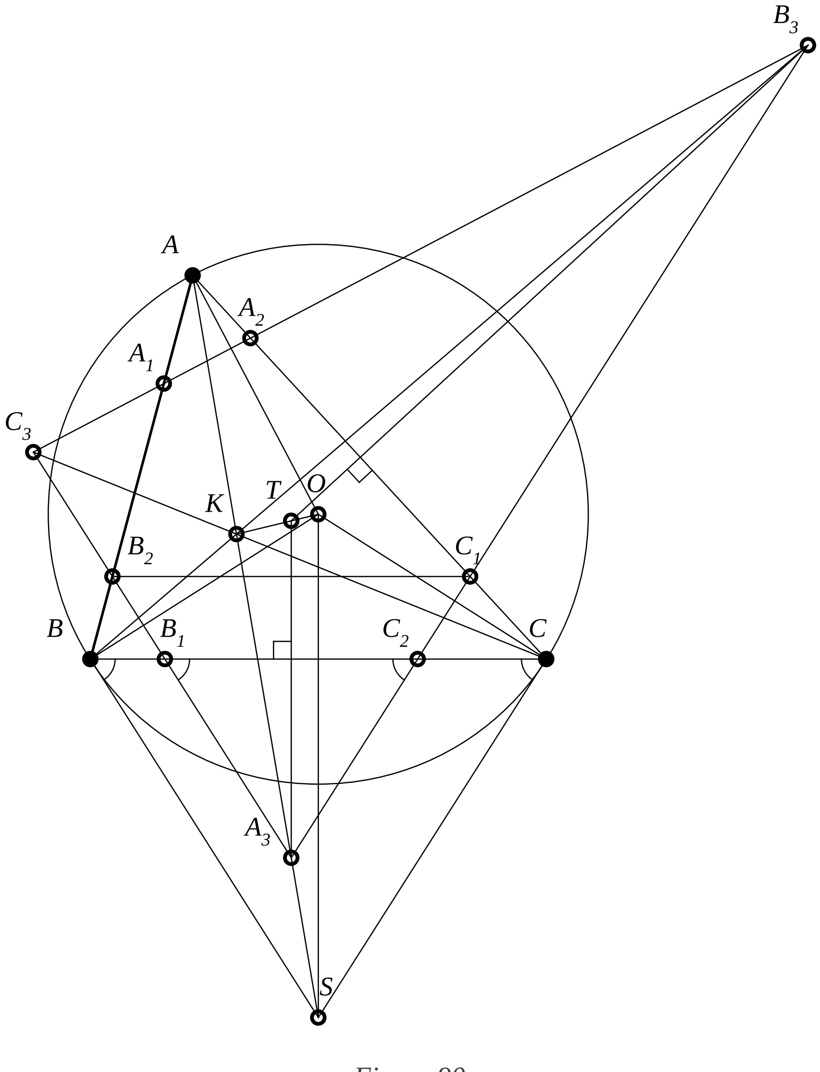

Figure 90

Theorem 34

The triangles ABC and $A_3B_3C_3$ are bilogical. The homology center is the symmedian center K of the triangle ABC, and the orthology centers are the center of the Tucker circle, T, and the center O of the circle circumscribed to the triangle ABC.

Proof

We build the tangents in B and C to the circumscribed circle of the triangle ABC, and we denote by S their intersection. It is known that AS is symmedian in the triangle ABC.

We prove that the points A, A_3, S are collinear (see *Figure 91*).

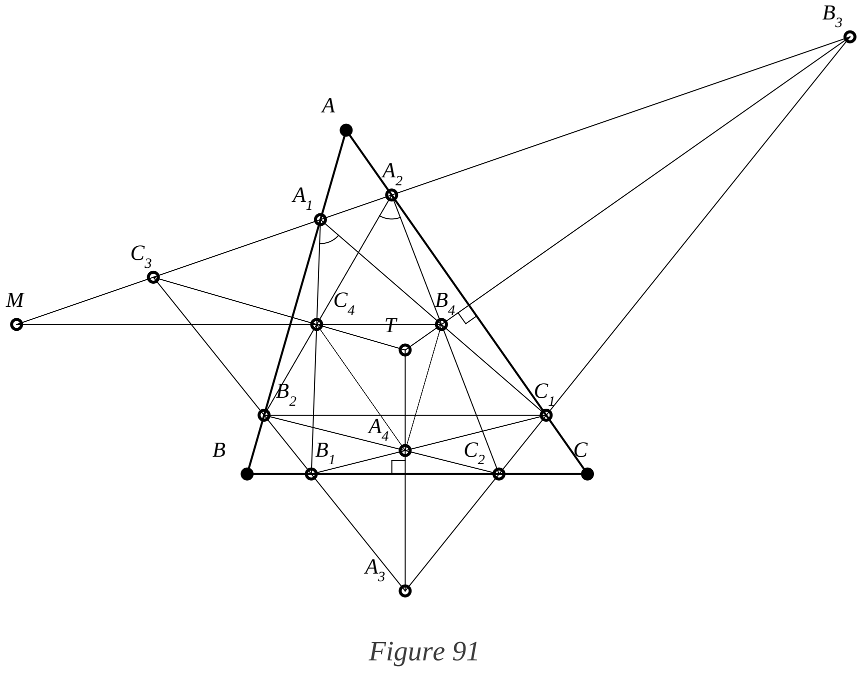

Figure 91

Indeed, because $B_1C_2C_1B_2$ is isosceles trapezoid, we have that $B_2C_1 \parallel BC$; on the other hand, BS is antiparallel with AC, therefore $BS \parallel B_2A_3$; similarly, $CS \parallel C_1A_3$. The triangles $A_3C_1B_2$ and SCB have respectively parallel sides, hence they are homothetic, because $\{A\} = BB_2 \cap CC_1$, it follows that the homothety center is A, consequently the points A, A_3, S are collinear, so AA_3 is symmedian in the triangle ABC; similarly, it follows that BB_3 and CC_3 are the other symmedians, therefore they are concurrent in K – the symmedian center. This point is the homology center of the triangles ABC and $A_3B_3C_3$. We noticed that the perpendiculars from A_3, B_3, C_3 to BC, CA, AB are bisectors of the triangle $A_3B_3C_3$, therefore they are concurrent in T – the center of Tucker circle. This point is orthology center in the triangle $A_3B_3C_3$ in relation to ABC. The perpendiculars from A, B, C to the antiparallels B_3C_3, C_3A_3, A_3B_3 are also concurrent. Because these antiparallels are parallel with the tangents taken from A, B respectively C to the circle circumscribed, it means that the perpendiculars taken in A, B, C to tangents pass through O – the center of the circumscribed circle, and this point is consequently the orthology center of the triangle ABC in relation to the triangle $A_3B_3C_3$.

Remark 23

1. The homology of the triangles ABC and $A_3B_3C_3$ can be proved with the help of Pascal's theorem relative to an inscribed hexagon (see [24]). Indeed, applying this theorem in the inscribed hexagon $A_1A_2C_1C_2B_1B_2$, we obtain that its opposite sides, videlicet A_1A_2 and BC; B_1B_2 and AC; C_1C_2 and AB, intersect respectively in the points M, N, P, and these are collinear points. They determine the homology axis of triangles ABC and $A_3B_3C_3$; according to Desargues's theorem, the lines AA_3, BB_3, CC_3 are concurrent.
2. From Sondat's theorem, we obtain that the points K, O, T are collinear. Also from this theorem, we obtain that OK is perpendicular to the homology axis of the triangles ABC and $A_3B_3C_3$.

Proposition 71

The triangles $A_3B_3C_3$ and $A_4B_4C_4$, formed by the intersections to the diagonals of the trapezoids $B_1C_2C_1B_2$; $C_2C_1A_2A_1$; $A_1B_2B_1A_2$, are homological. Their homology center is T, the center of Tucker circle, and the line determined by T and by the center of the circle circumscribed to the triangle $A_4B_4C_4$ – is perpendicular to the homology axis of the triangles.

Proof

The quadrilateral $A_1A_2B_1B_2$ is isosceles trapezoid; the triangle $C_3A_1A_2$ is isosceles; it follows that C_3C_4 is mediator of the segment A_1B_2, hence it passes through T, the center of Tucker circle of the triangle ABC (see *Figure 92*).

Similarly, A_3A_4 and B_3B_4 pass through T, hence the center of Tucker circle of the triangle ABC is homology center of triangles $A_3B_3C_3$ and $A_4B_4C_4$.

We denote by:

$\{L\} = A_3B_3 \cap A_4B_4$,

$\{M\} = B_3C_3 \cap B_4C_4$,

$\{N\} = A_3C_3 \cap A_4C_4$.

According to Desargues's theorem, the points M, N, P are collinear and belong to the homology axis of the previously indicated triangles.

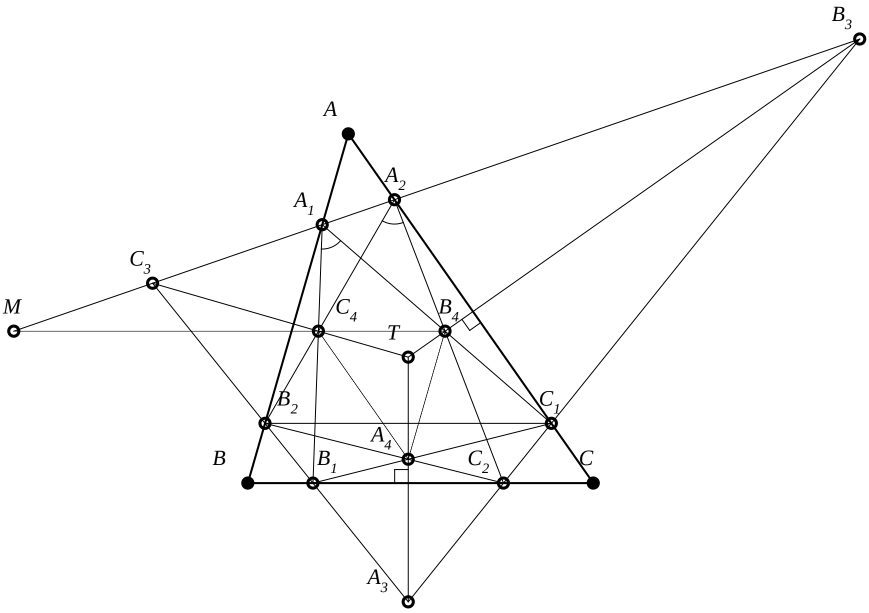

Figure 92

Because $\widehat{C_4A_1B_4} \equiv \widehat{C_4A_2B_4}$ (subtending congruent arcs in Tucker circle), it follows that the quadrilateral $A_1A_2B_4C_4$ is inscribable, we have $MA_1 \cdot MA_2 = MC_4 \cdot MB_4$, therefore the point M has equal powers over Tucker circle and over the circumscribed circle of the triangle $A_4B_4C_4$, so it belongs to the radical axis of these circles.

Similarly, it is shown that the points L and N belong to the same radical axis.

6.2.4 A triangle and the triangle of the projections of the center of the circle inscribed on its mediators

Theorem 35

Let ABC be a scalene triangle and let $A'B'C'$ be the triangle determined by the projections of the center of the inscribed circle I in the triangle ABC on its mediators; then, the triangles ABC and $A'B'C'$are bilogical.

Proof

We prove by vector that the homology center of the triangles ABC and $A'B'C'$ is the Nagel point, N. We have:

$\overrightarrow{AA'} = \overrightarrow{AI} + \overrightarrow{IA'} = \frac{b}{2p} \cdot \overrightarrow{AB} + \frac{c}{2p} \cdot \overrightarrow{BC} - \frac{b-c}{2a} \cdot \overrightarrow{AB} + \frac{b-c}{2a} \cdot \overrightarrow{AC}$.

We considered the triangle ABC, with $AB \leq AC$, and took into account that: $AI = \frac{b}{2p} \cdot \overrightarrow{AB} + \frac{c}{2p} \cdot \overrightarrow{AC}$, $\overrightarrow{IA'} = \alpha \overrightarrow{BC}$, $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$, and C_a is the projection of I on BC.

M_a the midpoint of (BC), having $BC_a = p - b$ and $BM_a = \frac{a}{2}$, we find $C_aM_a = \frac{b-c}{2a}$, therefore $\alpha = \frac{b-c}{2a}$.

$\overrightarrow{AA'} = \left(\frac{b}{2p} - \frac{b-c}{2a}\right)\overrightarrow{AB} + \left(\frac{c}{2p} + \frac{b-c}{2a}\right)\overrightarrow{AC}$.

Let $\{D_a\} = AA' \cap BC$, we have:

$\overrightarrow{AD_a} = \lambda \frac{ab-p(b-c)}{2ap}\overrightarrow{AB} + \lambda \frac{ac+p(b-c)}{2ap}\overrightarrow{AC}$.

On the other hand, $\overrightarrow{D_aC} = \mu\left(\overrightarrow{AC} - \overrightarrow{AB}\right)$.

The scalars λ and μ are such that:

$\overrightarrow{D_aC} = \overrightarrow{AC} - \overrightarrow{AD_a}$, therefore:

$\mu\left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \overrightarrow{AC} - \lambda \frac{ab-p(b-c)}{2ap}\overrightarrow{AB} - \lambda \frac{ac+p(b-c)}{2ap}\overrightarrow{AC}$.

It derives that:

$\left(\mu - 1 + \lambda \frac{ac+p(b-c)}{2ap}\right)\overrightarrow{AC} + \left(\lambda \frac{ab-p(b-c)}{2ap} - \mu\right)\overrightarrow{AB} = 0$.

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are non-collinear; it follows that:

$\mu - 1 + \lambda \frac{ac+p(b-c)}{2ap} = 0$,

$-\mu + \lambda \frac{ab-p(b-c)}{2ap} = 0$.

We find: $\lambda = \frac{p}{b+c}$ and $\mu = \frac{p-b}{a}$, consequently $\overrightarrow{D_aC} = \frac{p-b}{a} \cdot \overrightarrow{BC}$, therefore $D_aC = p - b$, and the point D_a is the isotomic of C_a (contact of the inscribed circle with BC), hence AD_a pass through the Nagel point, N.

Similarly, it is shown that $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ pass through N. Obviously, the triangles $A'B'C'$ and ABC are orthological, and the orthology center is O – the center of the circle circumscribed to the triangle ABC.

According to the the orthological triangles theorem, it follows that the perpendiculars taken from A, B, C respectively to $B'C'$, $C'A'$ and $A'B'$ are concurrent.

We denote this orthology center by Φ and we show that Φ belongs to the circle circumscribed to the triangle ABC.

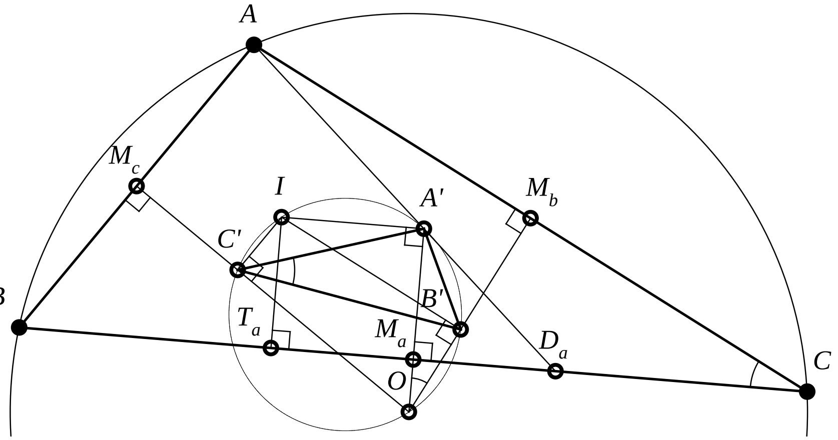

Figure 93

We have $\sphericalangle B\Phi A = 180^0 - \sphericalangle A'C'B'$. On the other hand, the points A', B', C' belong to the circle of diameter OI, therefore: $\sphericalangle A'C'B' \equiv -\sphericalangle A'IB'$; the latter has the sides perpendicular to the mediators of the sides BC and AC; hence, it is congruent with $\sphericalangle BCA$.

Having $m\left(\widehat{B\Phi A} = 180^0 - m(\hat{C})\right)$, it means that the quadrilateral $A\Phi BC$ is inscribable, therefore Φ belongs to the circle circumscribed to the triangle ABC.

Sondat's theorem implies the collinearity of points N, O, Φ.

Remark 24

In the anticomplementary triangle of the triangle ABC, the circle circumscribed to the triangle ABC is the circle of the nine points, and N – Nagel point, is the center of the circle inscribed in the anticomplementary triangle.

From Feuerbach's theorem (see [15]), these circles are tangent, the point of tangency being the point Φ (the orthology center of the triangle ABC in relation to the triangle $A'B'C'$) – called Feuerbach point.

The collinearity of points N, O, Φ indicated above, and the fact that the mentioned circles are interior tangents lead to $ON = R - 2r$.

6.2.5 A triangle and the triangle of the projections of the centers of ex-inscribed circles on its mediators

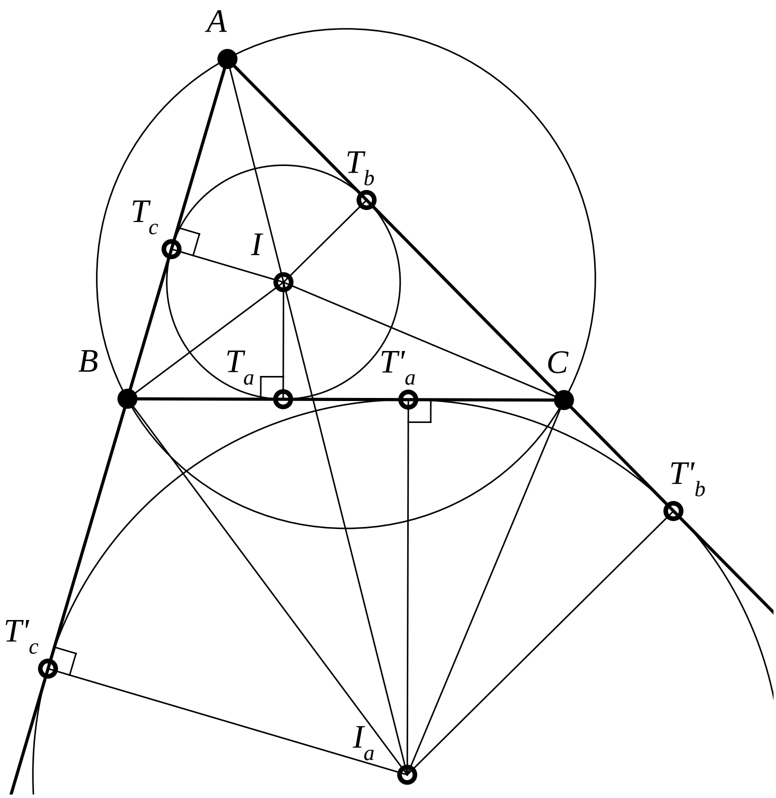

Figure 94

Proposition 72

Let ABC be a given triangle and $A'B'C'$ the triangle whose vertices are the projections of the centers I_a, I_b, I_c of the ex-inscribed circles respectively on the sides mediators (BC), (CA) and (AB).

Then the triangles ABC and $A'B'C'$ are bilogical. The orthology center of the triangle ABC in relation to $A'B'C'$ belongs to the line ΓO (Γ is Gergonne point of the triangle ABC and O – the center of its circumscribed circle).

Proof

Let C_a and D_a the projections of centers I and I_a on BC (see *Figure 94*).

These points are isotomic. The inscribed circle and the A-ex-inscribed circle are homothetic by homothety of center A and by ratio $\frac{IC_a}{ID_a}$. We denote by D'' the diameter of D_a in the circle A-ex-inscribed; we have that C_a and D'' are homotetical points, therefore A, C_a, D'' are collinear. The point A', the projection of I_a on the mediator of the side BC, is the midpoint of the segment C_aD'' because OA' contains the midline of the rectangle C_aD_aD'', hence A' belongs to Gergonne cevian AC_a, similarly B' and C' belong to Gergonne cevian BB_a, CC_a. As these cevians are concurrent in Γ (Gergonne point), it follows that this point is homology center of the triangles ABC and $A'B'C'$.

Obviously, the triangle $A'B'C'$ is orthological in relation to ABC and the orthology center is O.

From Sondat's theorem, it follows that the orthology center of the triangle ABC in relation to $A'B'C'$ belongs to the line ΓO.

6.2.6 A triangle and its Napoleon triangle

Definition 42

If ABC is a triangle and we build in its exterior the equilateral triangles BCA_1', CAB_1', ABC_1', we say about the triangle $O_1O_2O_3$ which has the centers of the circles circumscribed to the equilateral triangles that we just built – that it is a Napoleon exterior triangle corresponding to the triangle ABC.

Observation 62

In *Figure 95*, the Napoleon exterior triangle is $O_1O_2O_3$.

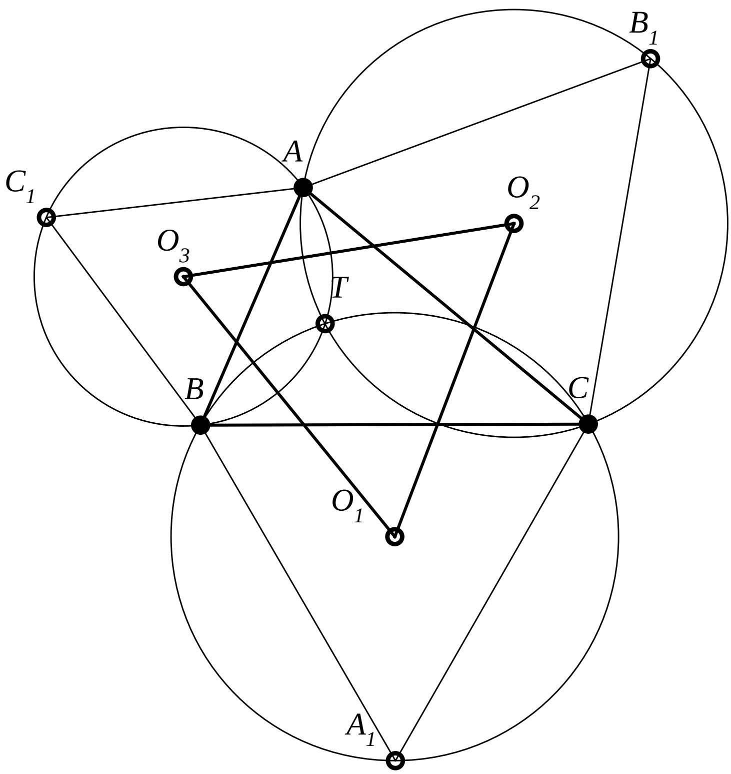

Figure 95

Definition 43

The circles circumscribed to the equilateral triangles BCA_1, CAB_1, ABC_1 – are called Toricelli circles.

Definition 44

The triangle $O_1'O_2'O_3'$ determined by the centers of the circles circumscribed to the equilateral triangles BCA_1', CAB_1', ABC_1' built on the sides of the triangle ABC to its interior – is called a Napoleon interior triangle.

Theorem 36

On the sides of the triangle ABC, we build in its exterior the equilateral triangles BCA_1, CAB_1, ABC_1; then:

i) The Toricelli circles intersect in a point T;
ii) The Napoleon exterior triangle is equilateral;
iii) $AA_1 = BB_1 = CC_1$;
iv) The triangles ABC and $A_1B_1C_1$ are bilogical.

Proof

i) We denote by T the second point of intersection of Toricelli circles circumscribed to the triangles ABC_1 and ACB_1. We have $m(\widehat{ATB}) = m(\widehat{ATC}) = 120^0$; assuming that T is located in the interior of triangle ABC, it follows that $m(\widehat{BTC}) = 120^0$, consequently the quadrilateral $BTCA_1$ is inscribable, and, consequently, T belongs to the circles circumscribed to the equilateral triangle BCA_1.

Observation 63

a) If $m(\widehat{BAC}) > 120^0$, then T is in the exterior of the triangle and $m(\widehat{BTA}) = m(\widehat{CTA}) = 60^0$; it follows that $m(\widehat{BTC}) = 120^0$, therefore T belongs to Toricelli circle circumscribed to BCA_1.
b) In case that $m(\widehat{BAC}) = 120^0$, the point T coincides with the vertex A.
c) The point T is called Toricelli-Fermat point of the triangle ABC.

ii) If the measures of the angles of the triangle ABC are smaller than 120^0, then: $m(\widehat{B_1AC_1}) = 120^0 + A$, $m(\widehat{A_1BC_1}) = 120^0 + B$, $m(\widehat{B_1CA_1}) = 120^0 + C$.

Also: $m(\widehat{O_2AO_3}) = 60^0 + A$, $m(\widehat{O_1BO_3}) = 60^0 + B$ and $m(\widehat{O_1CO_2}) = 60^0 + C$.

We calculate the sides of Napoleon triangle with the help of the cosine theorem.

We have: ${O_2O_3}^2 = O_3A^2 + O_2A^2 - 2O_3A \cdot O_2A \cdot \cos(60^0 + A)$.

Because $O_3A = c \cdot \frac{\sqrt{3}}{3}$, $O_2A = b \cdot \frac{\sqrt{3}}{3}$ and $\cos(60^0 + A) = \frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A$, we have ${O_2O_3}^2 = \frac{b^2}{3} + \frac{c^2}{3} - \frac{bc}{3}\cos A + \frac{bc}{3} \cdot \sqrt{3}\sin A$.

But $2bc\cos A = b^2 + c^2 - a^2$ and $bc \cdot \sin A = 2S$.

We obtain: ${O_2O_3}^2 = \frac{a^2+b^2+c^2+4S\sqrt{3}}{6}$.

Similarly, ${O_3O_2}^2$ and ${O_2O_1}^2$ are given by the same expression, hence the triangle $O_1O_2O_3$ is equilateral.

iii) We consider T in the interior of the triangle ABC, therefore no angle of the triangle ABC has a measure greater than or equal to 120^0. We have $m(\widehat{ATB}) = m(\widehat{BTC}) = m(\widehat{CTA}) = 120^0$ (this property makes that the point T in this case to be called isogon center of the triangle ABC). On the other hand, $m(\widehat{BTA_1}) = 60^0$, hence $m(\widehat{ATB_1}) = 120^0 + 60^0 = 180^0$, therefore the points A, T, A_1 are collinear (similarly B, T, A_1 and C, T, A_1 are collinear). From Van Schooten relation, we have that $TB + TC = TA_1$, since A, T, A_1 are collinear, it follows that $AA_1 = TA + TB + TC$, similarly $BB_1 = CC_1 = TA + TB + TC$.

Observation 64

It can be shown that $AA_1 = BB_1 = CC_1$ by direct calculation with cosine theorem; it is found that ${AA_1}^2 = \frac{a^2+b^2+c^2+4S\sqrt{3}}{2}$.

iv) We have shown before that A, T, A_1; B, T, B_1; C, T, C_1 are collinear; hence, the triangles ABC and $A_1B_1C_1$ are homological, and the homology center is the Toricelli-Fermat point, T. The perpendiculars taken from A_1, B_1, C_1 to BC, CA respectively AB are the mediators of these sides, consequently O – the center of the circle circumscribed to the triangle ABC is the orthology center of the triangle $A_1B_1C_1$ in relation to ABC.

Theorem 37

The triangle ABC where no angle is greater than 120^0 and its Napoleon exterior triangle – are bilogical triangles.

i) The orthology centers are the center O of the circle circumscribed to the triangle ABC and the isogonal center T of the triangle ABC;

ii) The homology center belongs to the axis of orthology and the axis of orthology is perpendicular to the axis of homology.

Proof

i) The perpendiculars taken from O_1, O_2, O_3 to BC, CA respectively AB are the mediators of these sides; hence, O is the orthology center of the Napoleon triangle $O_1O_2O_3$ in relation to ABC. The segments TA, TB, TC are common chords in the Toricelli circles and, consequently, O_2O_3 is mediator of $[TA]$, O_3O_1 is mediator of $[TB]$ and O_1O_2 is mediator of $[TC]$.

It follows that T is the second orthology center of the triangles indicated in the statement.

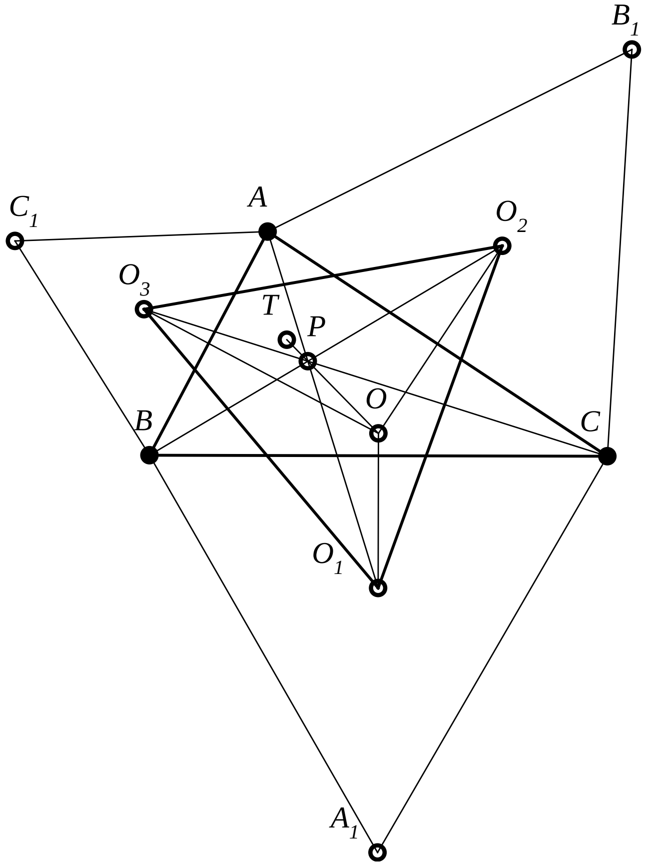

Figure 96

ii) Let A', B', C' be the intersections of the cevians AO_1, BO_2, CO_3 respectively with BC, CA, AB. We have:

$\frac{BA'}{CA'} = \frac{\text{Area}\Delta(ABA')}{\text{Area}\Delta(ACA')} = \frac{\text{Area}\Delta(BO_1A')}{\text{Area}\Delta(CO_1A')} = \frac{\text{Area}\Delta(ABO_1)}{\text{Area}\Delta(ACO_1)}$.

We obtain:

$\frac{BA'}{CA'} = \frac{AB \cdot BO_1 \cdot \sin(ABO_1)}{AC \cdot CO_1 \cdot \sin(ACO_1)} = \frac{c}{b} \cdot \frac{\sin(B+30^0)}{\sin(C+30^0)}$.

Similarly:

$\frac{CB'}{B'A} = \frac{c}{a} \cdot \frac{\sin(C+30^0)}{\sin(A+30^0)}$,

$\frac{C'A}{C'B} = \frac{b}{a} \cdot \frac{\sin(A+30^0)}{\sin(B+30^0)}$.

From $\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1$ and Ceva's reciprocal theorem, we obtain that AO_1, BO_2, CO_3 are concurrent, hence the triangle ABC and the Napoleon exterior triangle, $O_1O_2O_3$, are homological. We denote by P the center of this homology.

From Sondat's theorem, it follows that the points T, O, P are collinear and that the axis of orthology OT is perpendicular to the axis of homology of bilogical triangles ABC and $O_1O_2O_3$.

Theorem 38

On the sides of the given triangle ABC we build the equilateral triangles BCA_2, CAB_2, ABC_2 (whose interiors intersect the interior of the triangle ABC). Then:

i) The circumscribed circles of these equilateral triangles have a common point T'.

ii) The Napoleon interior triangle, $O_1'O_2'O_3'$, is equilateral.

iii) $AA_2 = BB_2 = CC_2$.

iv) The triangles ABC and $A_2B_2C_2$ are bilogical triangles.

Proof

i) Let T' be the second intersection point of the circles circumscribed to the triangles ACB_2 and ABC_2 (see *Figure 97*).

We have: $m\left(\widehat{BT'C}\right) = 60^0$ and $m\left(\widehat{AT'C}\right) = m\left(\widehat{AT'C_2}\right) = 60^0$.

From the last relation, it follows the collinearity of the points T', C_2, C.

Because $m\left(\widehat{BT'C}\right) = m\left(\widehat{BA_2C}\right) = 60^0$, we obtain that T' belongs to the circle circumscribed to the equilateral triangle BCA_2.

ii) We calculate the length of sides using the cosine theorem, bearing in mind that, in general, the angles $\sphericalangle O_1'AO_3'$, $\sphericalangle O_2'BO_3'$, $\sphericalangle O_2'CO_1'$ have measures equal to: $A - 60^0$, $B - 60^0$ or $C - 60^0$ (or $60^0 - A$, $60^0 - B$, $60^0 - C$) in the case that $m(\hat{A}) < 30^0$ or $m(\hat{B}) < 30^0$ or $m(\hat{C}) < 60^0$.

It is obtained that:

$O_1'O_3'^2 = O_2'O_3'^2 = O_1'O_2'^2 = \frac{a^2+b^2+c^2-4S\sqrt{3}}{6}$.

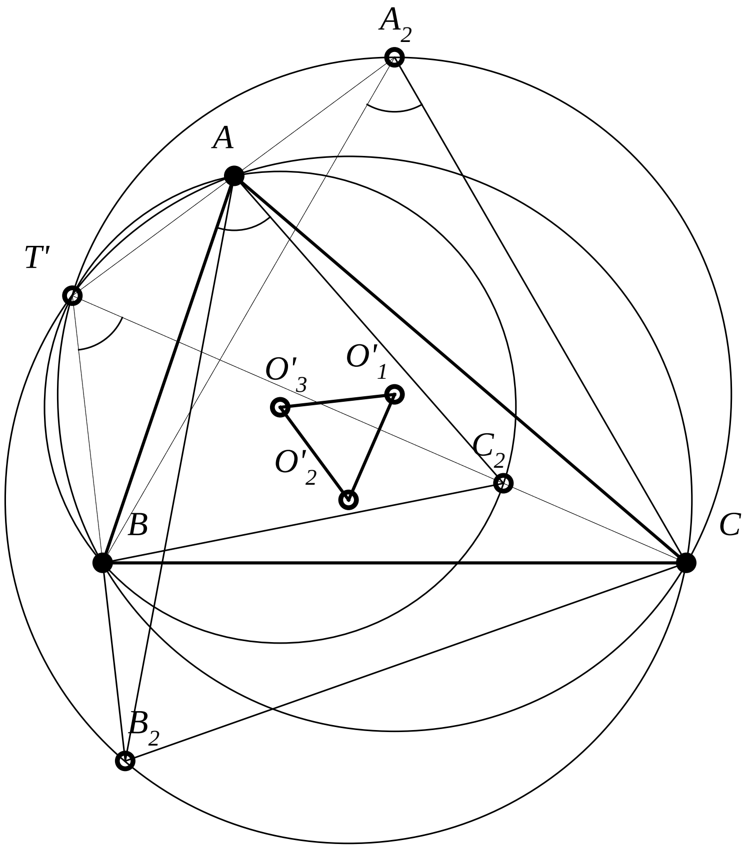

Figure 97

Remark 25

From the preceding expression, it is obtained that, in a triangle ABC, the inequality $a^2 + b^2 + c^2 \geq 4S\sqrt{3}$ is true. The equality takes place if and only if ABC is equilateral.

iii) $\Delta BAB_2 \equiv \Delta C_2AC$ (*SAS*), $BA = C_2A$, $AB_2 = AC$ and $m\left(\widehat{BAB_2}\right) = m\left(\widehat{C_2AC}\right) = A - 60^0$ (in the case of *Figure 97*), it follows that $BB_2 = CC_2$. Similarly, $\Delta ABA_2 \equiv \Delta C_2BC$, it follows that $AA_2 = CC_2$.

iv). Similarly, since the colinearity T', A, A_2 was proved, the colinearity of the points T', B, B_2 and T', C, C_2 is also proved. This collinearity implies that ABC and $A_2B_2C_2$ are homological. The homology center is the point T', which is called the second Torricelli-Fermat point. The perpendiculars taken from A_2, B_2, C_2 respectively to BC, CA and AB are the mediators of these sides; consequently, the center of the circle circumscribed to the triangle ABC, O, is the orthology center of these triangles.

Theorem 39

The given triangle ABC and its Napoleon interior triangle $O_1'O_2'O_3'$ are bilogical.

Proof

The perpendiculars taken from O_1', O_2', O_3' to BC, CA respectively AB are mediators of these three sides, hence they are concurrent in O – the center of the circle circumscribed to the triangle ABC, point that is the orthology center of the triangles $O_1'O_2'O_3'$ and ABC.

Because $T'A$ is a common chord in the Toricelli circle of centers O_3' and O_2', it follows that $O_3'O_2'$ is mediator of the segment $T'A$, therefore the perpendicular from A to $O_3'O_2'$ passes through T'; similarly, it follows that the perpendiculars taken from B to $O_1'O_3'$ and from C to $O_1'O_2'$ pass through the second Toricelli-Fermat point, T' - point that is the orthology center of the triangles ABC and $O_1'O_2'O_3'$. Let A', B', C' be the intersections of cevians AO_1', BO_2', CO_3' respectively with BC, CA and AB.

We have:

$$\frac{BA'}{CA'} = \frac{\text{Arie}\Delta ABA'}{\text{Arie}\Delta ACA'} = \frac{\text{Arie}\Delta BO_1'A'}{\text{Arie}\Delta CO_1'A'} = \frac{\text{Arie}\Delta ABO_1'}{\text{Arie}\Delta ACO_1'}.$$

We obtain:

$$\frac{BA'}{CA'} = \frac{AB \cdot BO_1' \cdot \sin \widehat{ABO_1'}}{AC \cdot CO_1' \cdot \sin \widehat{ACO_1'}} = \frac{c \cdot \sin(B - 30^0)}{b \cdot \sin(C - 30^0)}.$$

Similarly, it follows that:

$$\frac{CB'}{B'A} = \frac{a}{c} \cdot \frac{\sin(C - 30^0)}{\sin(A - 30^0)},$$

$$\frac{C'A}{C'B} = \frac{b}{a} \cdot \frac{\sin(A - 30^0)}{\sin(B - 30^0)}.$$

Because $\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1$, the Ceva's reciprocal theorem implies the concurrency of lines AO_1', BO_2', CO_3' and, consequently, the homology of triangles ABC and $O_1'O_2'O_3'$. We denote by P' the homology center.

Sondat's theorem shows that the points T', O, P' are collinear and OT' is perpendicular to the homology axis of bilogical triangles ABC and $O_1'O_2'O_3'$.

7

ORTHOHOMOLOGICAL TRIANGLES

7.1. Orthogonal triangles

Definition 42

Two triangles ABC and $A_1B_1C_1$ are called orthogonal if they have the sides respectively perpendicular.

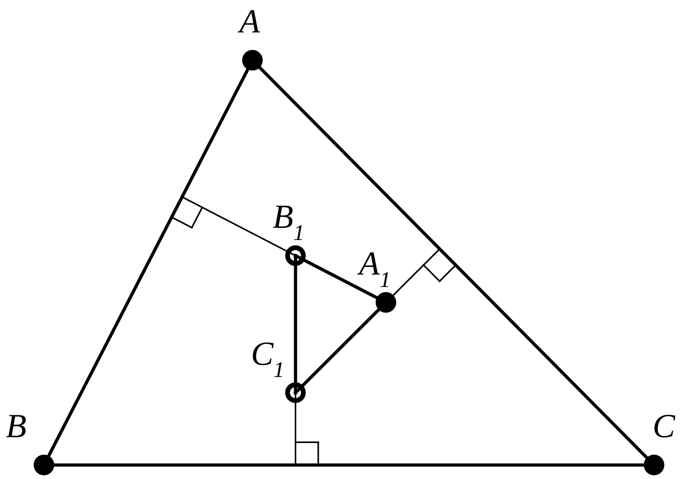

Figure 98

In *Figure 98*, the triangles ABC and $A_1B_1C_1$ are orthogonal. We have: $AB \perp A_1B_1$, $BC \perp B_1C_1$ and $CA \perp C_1A_1$.

Observation 62

If the triangles ABC and $A_1B_1C_1$ are orthogonal, and the vertices of the triangle $A_1B_1C_1$ are respectively on the sides of the triangle ABC, we say that the triangles are orthogonal, and $A_1B_1C_1$ is inscribed in ABC.

Problem 9

Being given a triangle $A_1B_1C_1$, build a triangle ABC such that ABC and $A_1B_1C_1$ to be orthogonal triangles and $A_1B_1C_1$ to be inscribed in the triangle ABC.

Solution

If we build the perpendicular d_1 in A_1 to A_1C_1, the perpendicular d_2 in B_1 to B_1A_1 and the perpendicular d_3 in C_1 to C_1B_1, then, denoting $\{A\} = d_2 \cap d_3$, $\{B\} = d_1 \cap d_3$ and $\{C\} = d_1 \cap d_2$, the triangle ABC is orthogonal with $A_1B_1C_1$, and the latter is inscribed in ABC (see *Figure 99*).

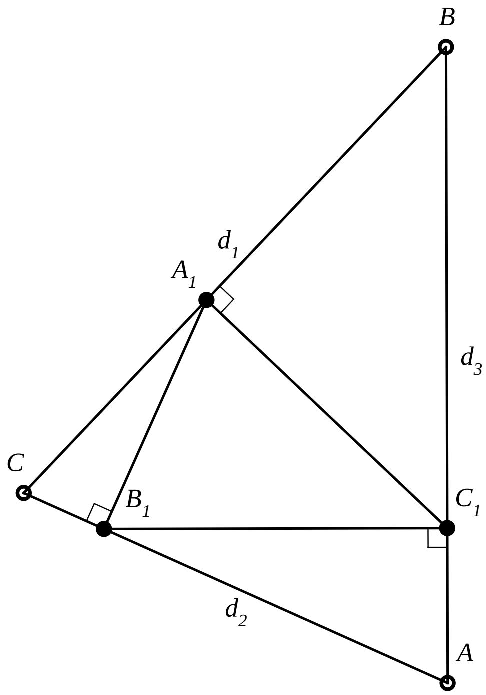

Figure 99

If we build the perpendicular d_1 in A_1 to A_1B_1, the perpendicular d_2 in B_1 to B_1C_1 and the perpendicular d_3 in C_1 to C_1A_1, and we denote by $\{A\} = d_1 \cap d_3$, $\{B\} = d_2 \cap d_1$ and $\{C\} = d_2 \cap d_3$, the triangle ABC is also a solution for the proposed problem.

Problem 10

Build the triangle $A_1B_1C_1$ inscribed in the given triangle ABC such as to be orthogonal with it.

Solution

We suppose the problem is solved and we ratiocinate about the configuration in *Figure 100*, where the triangle $A_1B_1C_1$ is inscribed in the given triangle ABC, and it is orthogonal with it.

We build the perpendiculars in A, B, C respectively to AC, AB and BC; consequently, we obtain the triangle $A_2B_2C_2$ orthogonal with ABC. Next, we build the perpendiculars in A_2, B_2, C_2 respectively to A_2C_2, B_2A_2 and C_2B_2, obtaining at their intersections the triangle $A_3B_3C_3$ orthogonal with $A_2B_2C_2$.

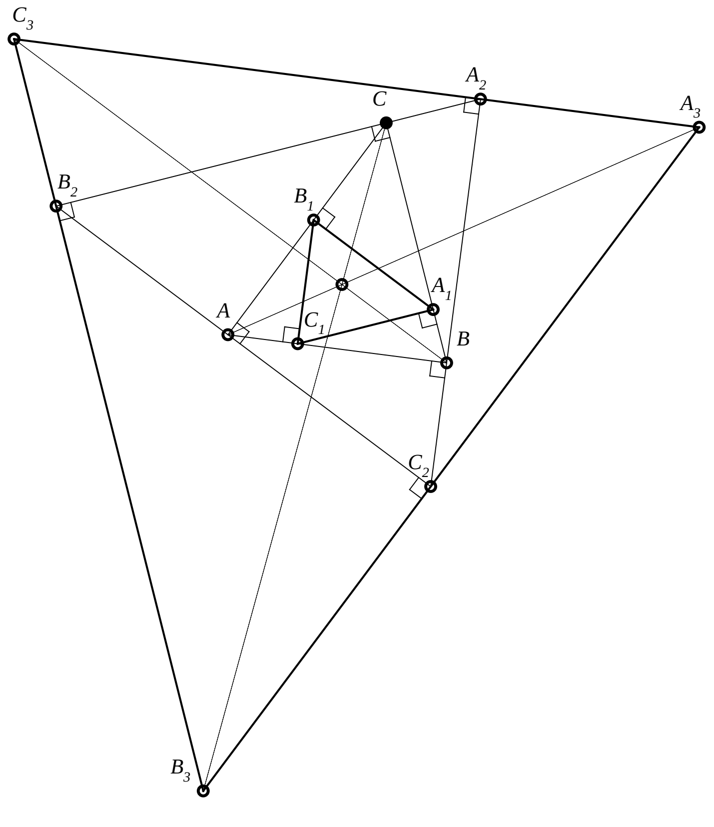

Figure 100

We note that $AC \parallel A_3C_3$, $BC \parallel B_3C_3$, $AB \parallel A_3B_3$; hence, the triangles ABC and $A_3B_3C_3$ are homothetic. The center of homothety is $\{O\} = AA_3 \cap BB_3$. We also observe that $A_1B_1 \parallel A_2B_2$, $A_1C_1 \parallel A_2C_2$ and $B_1C_1 \parallel B_2C_2$, therefore the triangles $A_2B_2C_2$ and $A_1B_1C_1$ are homothetic as well.

Because the homothetic of the side A_3C_3 is the side AC by homothety of center O and ratio $\frac{OA_3}{OA}$, since $A_2 \in A_3C_3$, if we take A_2O and we denote by A_1' the intersection with AC, we have:

$\frac{OA_2}{OA_1{}'} = \frac{OA_3}{OA}$.

Similarly we find that:

$\frac{OB_2}{OB_1{}'} = \frac{OB_3}{OB}$ and $\frac{OC_2}{OC_1{}'} = \frac{OC_3}{OC}$, hence the triangle $A_1{}'B_1{}'C_1{}'$ is the homothetic of the triange $A_2B_2C_2$ by homothety of center O and ratio $\frac{OA_3}{OA}$. Because the homothety is a transformation that preserves the measures of angles and transforms the lines into lines, and the triangle $A_2B_2C_2$ is orthogonal with the triangle $A_3B_3C_3$, it means that the triangle $A_1{}'B_1{}'C_1{}'$ is also orthogonal with ABC, hence $A_1{}' = A_1$, $B_1{}' = B_1$, $C_1{}' = C_1$.

We can build the triangle $A_1B_1C_1$ in the following way:

1. We build the triangle $A_2B_2C_2$ orthogonal with the given triangle ABC, and ABC inscribed in $A_2B_2C_2$ (we build effectively the perpendiculars to A, B, C, respectively to AC, AB and BC).
2. We build the triangle $A_3B_3C_3$ orthogonal with $A_2B_2C_2$, such that $A_2B_2C_2$ to be inscribed in $A_3B_3C_3$.
3. We unite A with A_3, B with B_3 and we denote $\{O\} = AA_3 \cap BB_3$.
4. We unite A_2 with O, B_2 with O, C_2 with O. At the intersection of these lines with AC, AB, BC we find the points A_1, B_1, C_1 – the vertices of the requested triangle.

Observation 63

Because the triangle $A_2B_2C_2$ can be build in two ways, it follows that we can obtain at least two solutions for the proposed problem.

Problem 11

Being given a triangle ABC, build a triangle $A_1B_1C_1$ such that the triangles to be orthogonal.

Solution

We consider a point C_1 in the plane of the triangle ABC (see *Figure 101*). We take the orthogonal projections of C_1 on BC, CA, AB, denoted A', B', C'. We consider $B_1 \in (A'C_1)$; we take from B_1 the perpendicular to AB and we denote by A_1 its intersection with (C_1B'). The triangles $A_1B_1C_1$ and ABC are orthogonal.

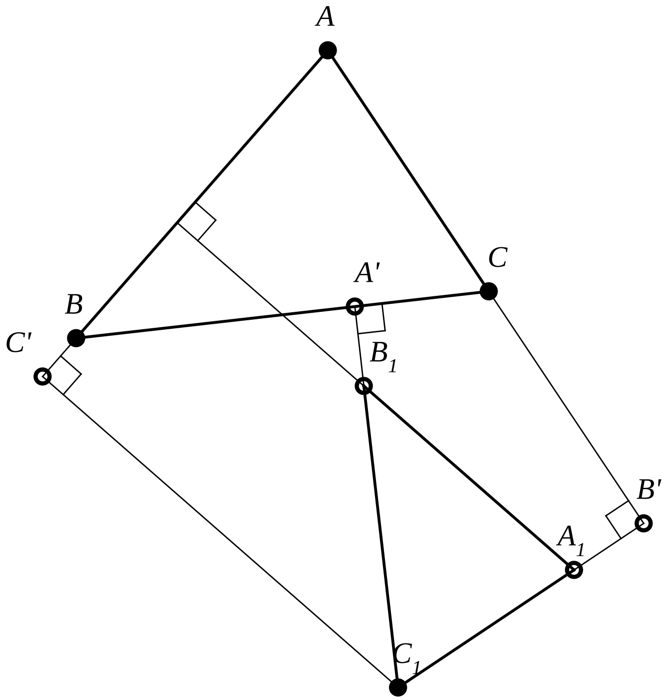

Figure 101

7.2. Simultaneously orthogonal and orthological triangles

Proposition 73 (Ion Pătrașcu)

If the orthogonal triangles ABC and $A_1B_1C_1$ are given, and they are orthological as well, then the orthology is in the sense that ABC is orthological with $B_1A_1C_1$, the orthology centers are the vertices C and C_1, and the triangles ABC and $A_1B_1C_1$ are similar.

Proof

Because ABC and $A_1B_1C_1$ are orthogonal, we have $AB \perp A_1B_1$, $BC \perp B_1C_1$, $CA \perp C_1A_1$ (see *Figure 102*).

Let us consider that ABC and $A_1B_1C_1$ are orthological in the sense that the perpendicular taken from A to B_1C_1, the perpendicular taken from B to A_1C_1, and the perpendicular taken from C to A_1B_1 are concurrent in a point O. Then, the perpendicular from B to A_1C_1 will be parallel with AC, the perpendicular from A to B_1C_1 will be parallel with BC; which leads to the conclusion that the point O is such that the quadrilateral $ACBO$ is parallelogram. On the other hand, the perpendicular taken from C to A_1B_1 must be parallel with AB and it must pass through O, which is absurd, because it is not possible that in the parallelogram $ACBO$ the diagonal CO to be parallel with the diagonal AB.

Let us consider that ABC and $A_1B_1C_1$ are orthological in the sense that the perpendicular from A to A_1B_1, the perpendicular taken from B to B_1C_1, the perpendicular taken from C to C_1A_1 are concurrent in a point O. Then, the perpendicular from A to A_1B_1 is AB, the perpendicular from B to B_1C_1 is BC; in this moment, the point O coincides with B; the perpendicular from C to A_1C_1, ie. AC, should pass though B, which is impossible.

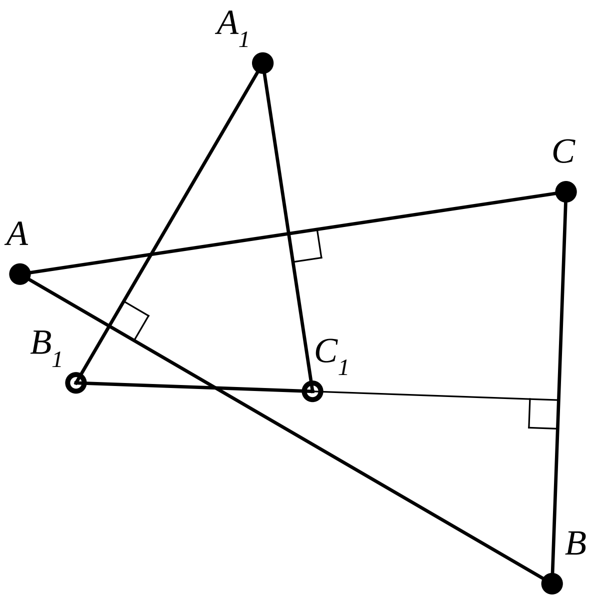

Figure 102

Finally, let us consider that ABC and $A_1B_1C_1$ are orthological triangles in the sense that the perpendicular taken from A to C_1A_1, the perpendicular taken from B to B_1C_1 and the perpendicular taken from C to A_1B_1 are concurrent in a point O. Because the perpendicular from A to A_1C_1 must be parallel with AC, it follows that this perpendicular is actually AC. The perpendicular from B to C_1B_1 is actually BC, hence the point O coincides with C. The perpendicular from C to A_1B_1 will be parallel with AB; which is possible, and hence, the vertex C is the orthology center of the triangle ABC in relation to the triangle $B_1A_1C_1$. According to the theorem of orthological triangles, the triangle $B_1A_1C_1$ is orthological as well in relation to ABC. We find that the orthology center is the vertex C_1.

In *Figure 102*, it is observed that the angles ACB and AC_1B_1 have the sides respectively perpendicular, hence they are congruent. So does the angle BAC and the angle $B_1A_1C_1$ (they also have the sides respectively perpendicular), therefore they are congruent. It consequently follows that: $\Delta ABC \sim \Delta A_1B_1C_1$.

Proposition 74 (Ion Pătrașcu)

If ABC is a right triangle in A and $A_1B_1C_1$ is an orthogonal triangle with it, then:

i) The triangle $A_1B_1C_1$ is a right triangle in A_1;

ii) The triangles ABC and $A_1B_1C_1$ are triorthological.

Proof

i) From $A_1C_1 \perp AC$, $A_1B_1 \perp AB$ and $m(\hat{A}) = 90^0$, it follows that $m(\widehat{A_1}) = 90^0$ (see *Figure 103*).

ii) The perpendicular taken from A to A_1C_1 is AC, and the perpendicular taken from B to C_1B_1 is CB. These are concurrent in the point C, through which passes, obviously, the perpendicular taken from C to A_1B_1. Consequently, the triangles ABC and $B_1A_1C_1$ are orthological, and the orthology center is the vertex C. The perpendicular taken from A to A_1B_1 is AB, and the perpendicular taken from C to B_1C_1 is CB. These perpendiculars intersect in the point B; the perpendicular taken from B to A_1C_1 passes, obviously, through B, hence the point B is the orthology center of the triangle ABC in relation to the triangle $C_1B_1A_1$. We can affirm that the triangle ABC is biorthological with the triangle $A_1B_1C_1$ and, applying Pantazi's theorem, we have that ABC and $A_1B_1C_1$ are triorthological. The fact that the triangles ABC and $A_1C_1B_1$ are orthological can be proved as above. The orthology center is A.

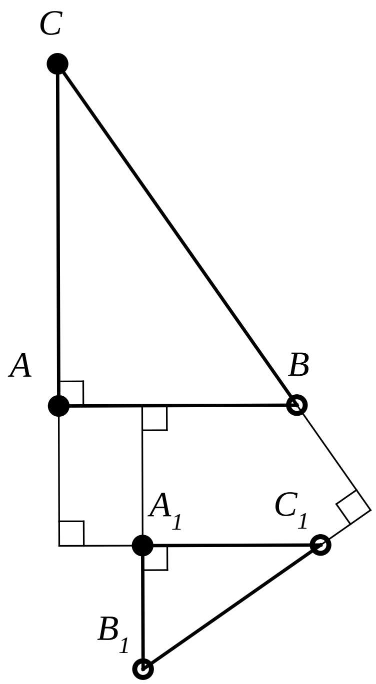

Figure 103

Observation 64

Obviously, the triangle $A_1B_1C_1$ is also triorthological in relation to the triangle ABC; the orthology centers are the vertices of the triangle $A_1B_1C_1$.

7.3. Orthohomological triangles

Definition 43

Two triangles that are simultaneously orthological and homological are called orthohomological triangles (J. Neuberg).

Problem 12

The triangle ABC being given, build the triangle $A_1B_1C_1$, such that ABC and $A_1B_1C_1$ to be orthohomological triangles.

To solve this problem, we prove:

Lemma 12

Let $\mathcal{C}(O,r)$ and $\mathcal{C}(O_1,r_1)$ be two secant circles with the common points M and N. We take through M the secants A, M, A_1 and B, M, B_1. The angle of the chords AB and A_1B_1 is congruent with the angle of the given circles.

Definition 44

The angle of two secant circles is the angle created by the tangents taken to the circles in one of the common points.

Proof

We denote: $\{P\} = AB \cap A_1B_1$, and MT, MT_1 – the tangents taken in M to the two circles (see *Figure 104*). We have: $\sphericalangle PAM \equiv \sphericalangle BMT$, $\sphericalangle PA_1M \equiv \sphericalangle B_1MT_1$.

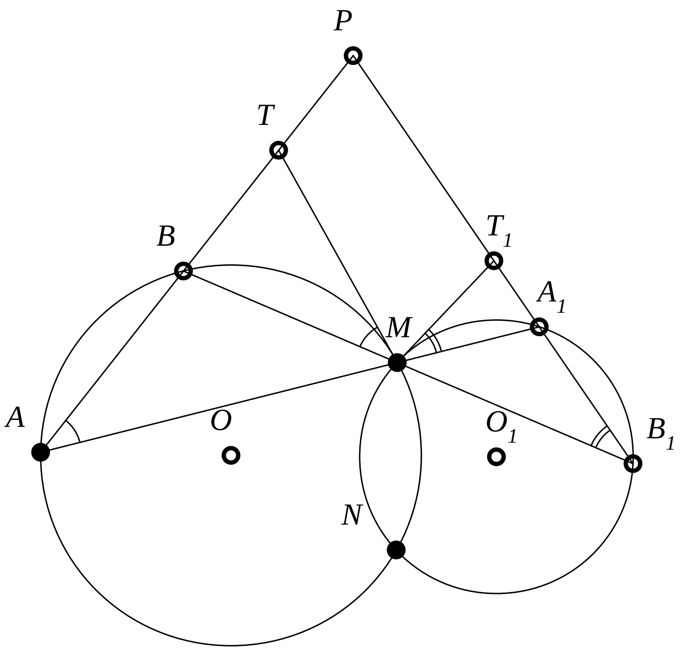

Figure 104

Adding these relations, and considering that supplements of the measures found in this addition are equal, we obtain: $\sphericalangle APA_1 \equiv \sphericalangle TMT_1$.

Definition 45

Two circles are called orthogonal if the angle they create is a right angle.

Observation 65

If we consider two orthogonal circles, $\mathcal{C}(O,r)$ and $\mathcal{C}(O_1,r_1)$, and two chords, AB and A_1B_1, in these circles, such that A, M, A_1 and B, M, B_1 to be collinear (M is a common point for the given circles), then, according to *Lemma 12*, it follows that $AB \perp A_1B_1$.

Solution of *Problem 12*

We build the circumscribed circle of the triangle ABC and then we build an orthogonal circle of this circle. We denote by M one of their common points (see *Figure 105*).

We take the lines AM, BM, CM and we denote by A_1, B_1, C_1 their second point of intersection with the orthogonal circle to the circle circumscribed to the triangle ABC.

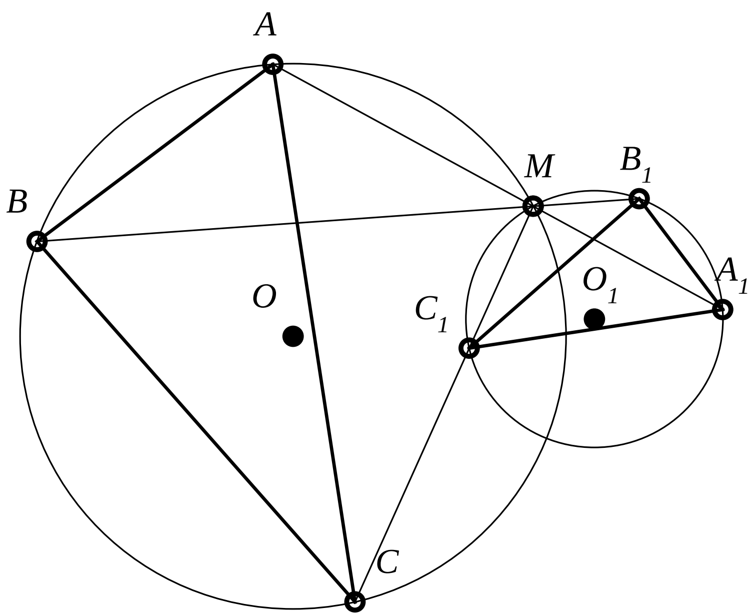

Figure 105

According to Lemma 12, it follows that the triangles ABC and $A_1B_1C_1$ have their sides respectively perpendiculars, therefore they are orthogonal triangles. On the other hand, AA_1, BB_1, CC_1 are concurrent in M, therefore the triangles are homological as well.

Remark 24

In order to build two orthogonal circles $\mathcal{C}(O,r)$ and $\mathcal{C}(O_1,r_1)$, we take into account:

Theorem 40

Two circles $\mathcal{C}(O,r)$ and $\mathcal{C}(O_1,r_1)$ are orthogonal if and only if $r^2+r_1^2=OO_1^2$.

Proof

If the circles are orthogonal and M is one of their common points, then MO and MO_1 are tangent to the circles, the triangle OMO_1 is a right triangle and, consequently, $r^2+r_1^2=OO_1^2$. Reciprocally, if the circles are such that $r^2+r_1^2=OO_1^2$, it follows that the angle OMO_1 is a right angle and, also, the angle TMM_1, created by the tangents in M to the circles, is also a right angle, hence, the circles are orthogonal.

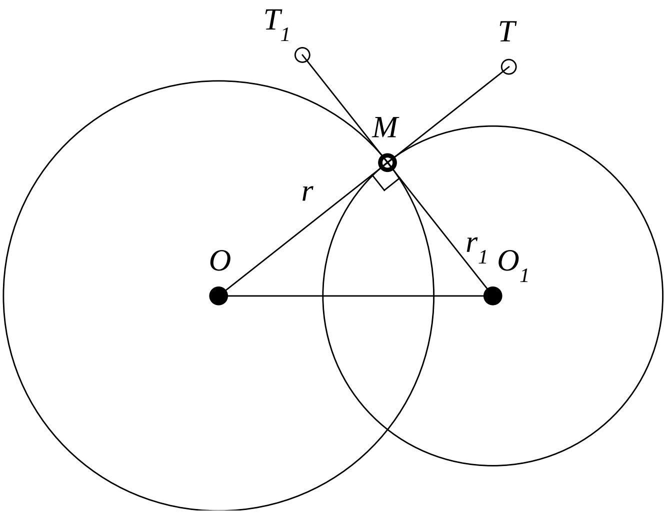

Figure 106

Theorem 41

If ABC and $A_1B_1C_1$ are two orthohomological triangles, then:

i. Their circumscribed circles are secant, and one of their common points is the center of homology;
ii. The circles circumscribed to the given triangles are orthogonal;
iii. The other common point of the circles circumscribed to the given triangles is the similarity center of these triangles;
iv. The Simson lines of the homology center in relation to the given triangles are parallel to their homology axis;
v. The Simson lines of the similarity center of the triangles in relation to the given triangles are orthogonal in a point belonging to their homology axis.

Proof

i) We denote by M the homology center of the given triangles, therefore $\{M\} = AA_1 \cap BB_1 \cap CC_1$; also, we denote by P, Q, R the homology axis (see *Figure 107*):

$\{P\} = A_1B_1 \cap AB$,
$\{Q\} = B_1C_1 \cap BC$,
$\{R\} = C_1A_1 \cap AC$.

P, Q, R are collinear and due to the orthogonality of the given triangles, we have that the angles from P, Q and R are right angles.

ii) Because the chords AB and A_1B_1 in the two circles are perpendicular, taking into account Lemma 12, it follows that the circles circumscribed to the triangles ABC and $A_1B_1C_1$ are orthogonal.

iii) It follows from *The Theory of Similar Figures*, see *Annex no. 2*.

iv) (Mihai Miculița; see *Figure 108*)

$$\left.\begin{array}{c}\left.\begin{array}{r} MR_1 \perp AC \\ MQ_1 \perp BC \end{array}\right\} \Rightarrow MCR_1Q_1 - \text{inscribable} \Rightarrow \widehat{Q_1R_1A} \equiv \widehat{Q_1MC} \\ MQ_1, C_1Q \perp BC \Rightarrow MQ_1 \parallel C_1Q \Rightarrow \widehat{Q_1MC} \equiv \widehat{QC_1C} \\ \left.\begin{array}{r} C_1Q \perp BC \\ C_1R \perp AC \end{array}\right\} \Rightarrow C_1CRQ - \text{inscribable} \Rightarrow \widehat{QC_1C} \equiv \widehat{QRA} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \widehat{Q_1R_1A} \equiv \widehat{QRA} \Rightarrow \boxed{Q_1R_1 \parallel QR}.$$

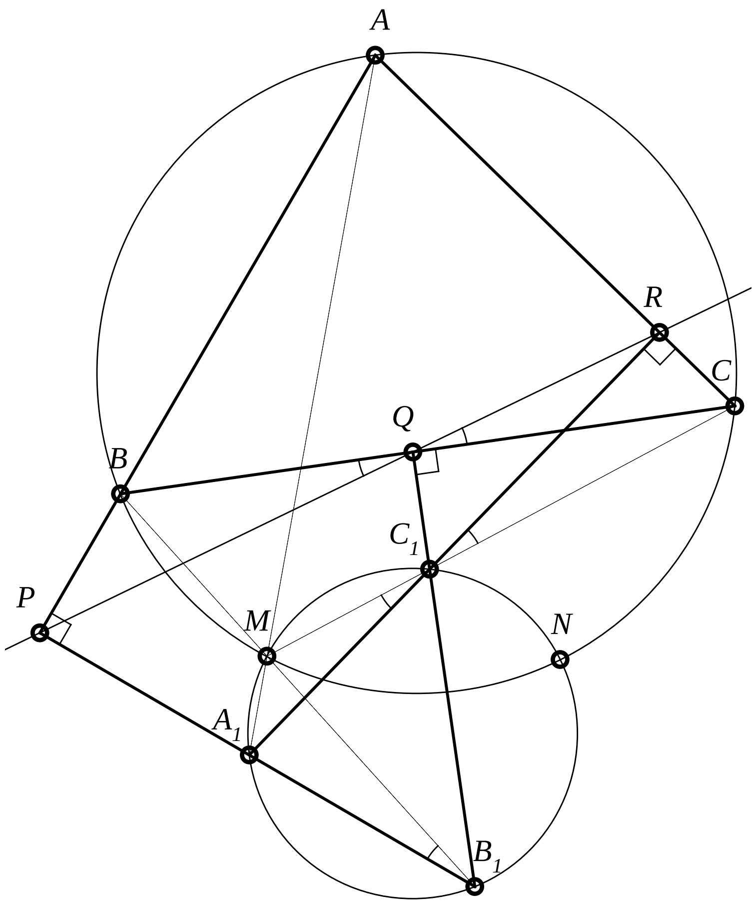

Figure 107

v) We denote by N the second point of intersection of the circles circumscribed to the given triangles, the quadrilateral $ARNA_1$ is inscribable (from R, AA_1 is seen from a right angle, and from N, also, AA_1 is seen from a right angle, N being its own homologous point). From the same considerations, the quadrilateral APA_1N is inscribable, we obtain that the points A, P, A_1, N, R are on the circle of diameter AA_1. Considering the triangles APR and A_1PR, and applying Proposition 53, we obtain that the Simson lines of the point N in relation to the triangles ABC and $A_1B_1C_1$ are perpendicular and intersect in a point situated on the line PQ.

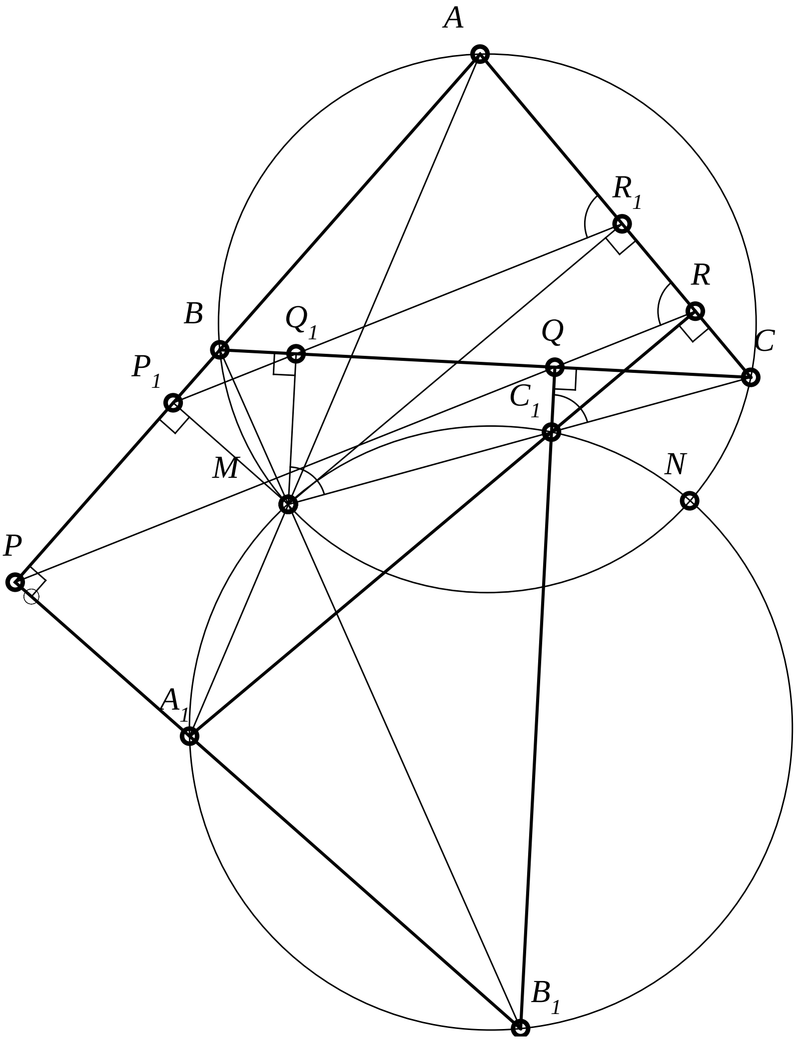

Figure 108

Theorem 42 (P. Sondat)

The homology axis of two orthohomological triangles ABC and $A_1B_1C_1$ passes through the midpoint of the segment HH_1 determined by the ortocenters of these triangles.

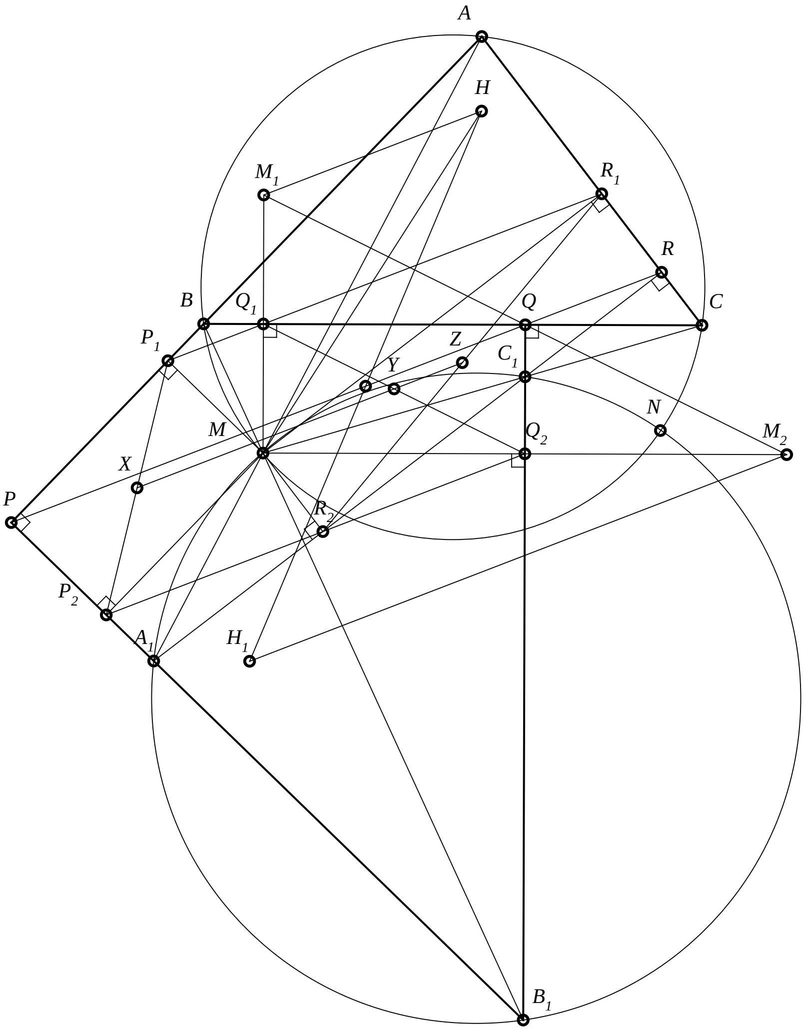

Figure 109

Proof

Let $P_1Q_1R_1$, $P_2Q_2R_2$ the Simson lines of homology center M of the triangles ABC and $A_1B_1C_1$, and PQR their homology axis (see *Figure 109*). We denote by M_1 respectively M_2 the symmetrics of M towards Q_1 respectively Q_2; because the Simson line $P_1Q_1R_1$ passes through the middle of the segment MH

(*Theorem 17*), we have that M_1H is parallel with P_1Q_1, therefore with PQ, similarly M_2H_1 is parallel with PQ. The quadrilateral $MQ_1Q_2Q_3$ is rectangle, if we denote by X its center; we obviously have $Q_1 - X - Q_2$ collinear and $M - X - Q$ collinear. The line M_1M_2 is the homothety of the line Q_1Q_2 by homothety of center M and ratio 2, consequently the point Q is the midpoint of the segment M_1M_2. The quadrilateral $M_1HM_2H_1$ is trapeze (its bases are parallel to the axis of homology PQ), because Q is the midpoint of M_1M_2 and the parallel taken through Q to M_1H is the axis of homology PQ, according to a theorem in trapeze, PQ will also contain the midpoint of the diagonal HH_1.

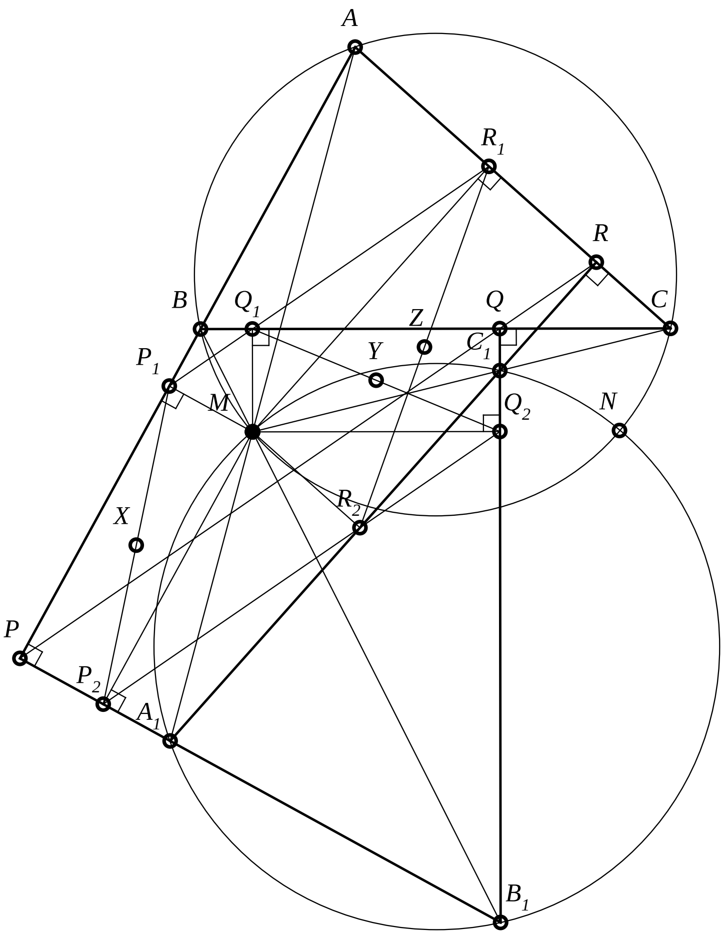

Figure 110

Proposition 75 (Ion Pătrașcu)

Let $A_1B_1C_1$ and $A_2B_2C_2$ be two orthohomological triangles. The Simson lines of the homology center of triangles to the circumscribed circles are P_1, Q_1, R_1, respectively P_2, Q_2, R_2. Then the midpoints of the segments P_1P_2, Q_1Q_2, R_1R_2 are collinear.

Proof

Let M be the homology center and $P-Q-R$ the homology axis of the given triangles (see *Figure 110*).

We denote by $P_1-Q_1-R_1$ and $P_2-Q_2-R_2$ the Simson lines of M in relation to the triangles $A_1B_1C_1$respectively$A_2B_2C_2$.

The quadrilateral MP_1PP_2 is rectangle, therefore the midpoint of P_1P_2 is the center of this rectangle; we denote it by X; similarly, let Y the midpoint of Q_1Q_2 (therefore the center of this rectangle MQ_1QQ_2) and Z the midpoint of R_1R_2.

Because X, Y, Z are the midpoints of the segments MP, MQ, respectively MQ and P, Q, R are collinear points, it follows that X, Y, Z are collinear as well, they belong to the homothetic of the line PQ by the homothety of center M and ratio $\frac{1}{2}$.

Remark 25

We can prove in the same way that the midpoints of the segments determined by the feet of altitudes of the orthohomological triangles $A_1B_1C_1$ and $A_2B_2C_2$ are collinear points.

Theorem 43

Let ABC and $A_1B_1C_1$ two orthohomological triangles having as its axis of homology the line d. Then:

i) The complete quadrilaterals (ABC, d) and $(A_1B_1C_1, d)$ have the same Miquel's point and the same Miquel's circle;

ii) The centers of the circles circumscribed to the triangles ABC and $A_1B_1C_1$ are the extremities of a diameter of the Miquel's circle, and the homology center of these triangles belongs to this Miquel's circle.

Proof

i) Let P, Q, R be the points in which the axis of homology d intersects the sides AB, BC, respectively CA (see *Figure 111*). The triangles ABC and $A_1B_1C_1$ being orthogonal, we have that $\sphericalangle APA_1 \equiv \sphericalangle ARA_1 = 90^0$, therefore the triangles APR and A_1PR have the same circumscribed circle with center O_2.

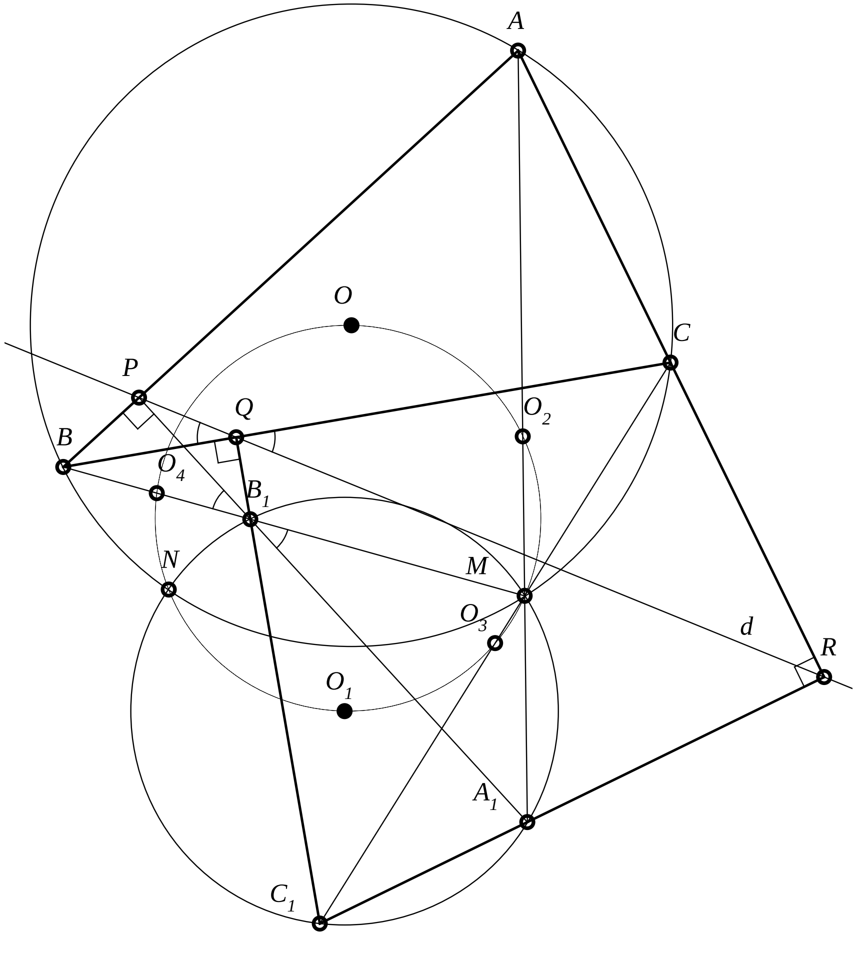

Figure 111

The quadrilaterals $PBQB_1$ and $QRCC_1$ are inscribable, consequently:
$\sphericalangle BQP \equiv \sphericalangle BB_1A_1$, (1)

$\sphericalangle CQR \equiv \sphericalangle RC_1C.$ (2)

We also have:

$\sphericalangle BQP \equiv \sphericalangle CQR$ (opposite vertex), (3)

$\sphericalangle MC_1A_1 \equiv \sphericalangle RC_1C$ (opposite vertex). (4)

From these relations, we obtain that:

$\sphericalangle MB_1A_1 \equiv \sphericalangle MC_1A_1.$ (5)

This relations show that the circumscribed circle of the triangle $A_1B_1C_1$ contains the point M. The condition of orthogonality of the triangles ABC and $A_1B_1C_1$ implies their direct similarity (the angles of the triangles have the sides respectively perpendicular). From the conciclicity of the points M, A_1, B_1, C_1, it derives that:

$\sphericalangle A_1MB_1 \equiv \sphericalangle A_1C_1B_1.$ (6)

But: $\sphericalangle A_1C_1B_1 \equiv \sphericalangle ACB.$ (7)

On the other hand:

$\sphericalangle A_1MB_1 \equiv \sphericalangle BMA$ (opuse la vârf). (8)

We get in this way that $\sphericalangle BMA \equiv \sphericalangle ACB$, condition showing that the point M, the homology center of triangles to the circle circumscribed to the triangle ABC.

Similarly, the triangles CQR and C_1QR have the same circumscribed circle of center O_3, and the triangles BPQ and B_1PQ have the same circumscribed circle of center O_4 – the midpoint of the segment BB_1. If we denote by M and N the points of intersection of the circles circumscribed to the triangles ABC and $A_1B_1C_1$ (of centers O, respectively O_1), then, due to the fact that the circles circumscribed to the triangles APR and CQR coincide with the circles circumscribed to the triangles A_1PR and C_1QR, it means that their second point of intersection is N, the common point of the circles of the complete quadrilaterals (ABC, d), $(A_1B_1C_1, d)$, hence N is the Miquel's point of these quadrilaterals (see *Annex no. 3*). Also, the Miquel's circle of the complete quadrilateral (ABC, d), ie. the circle that contains the points O_1, O_2, O_3, O_4 and the point N coincides with Miquel's circle of the quadrilateral $(A_1B_1C_1, d)$, which contains the points N, O_1, O_2, O_3, O_4.

ii) The circles circumscribed to the triangles ABC and $A_1B_1C_1$ are orthogonal; because N is one of their common points, we have that $m(\widehat{ONO_1}) = 90^0$, and because O, N and O_1 are on the Miquel's circle, it means that O and O_1 are diametrically opposed in this circle.

If we denote by M the homology center of the triangles ABC and $A_1B_1C_1$, then M will be the second point of intersection of these circles; because N is situated on the Miquel's circle and M is its symmetric towards the diameter OO_1, it follows that M also belongs to the common Miquel's circle – complete quadrilaterals $(ABC; d)$, $(A_1B_1C_1;\ d_1)$.

Definition 46

If ABC is a triangle and $P - Q - R$ a transverse ($P \in AB$, $Q \in BC$, $R \in AC$), and the perpendiculars raised in P, Q, R respectively to AB, AC and CA determine a triangle $A_1B_1C_1$, which is called paralogical triangle of ABC.

Observation 66

In *Figure 112*, $A_1B_1C_1$ is the paralogical triangle of ABC.

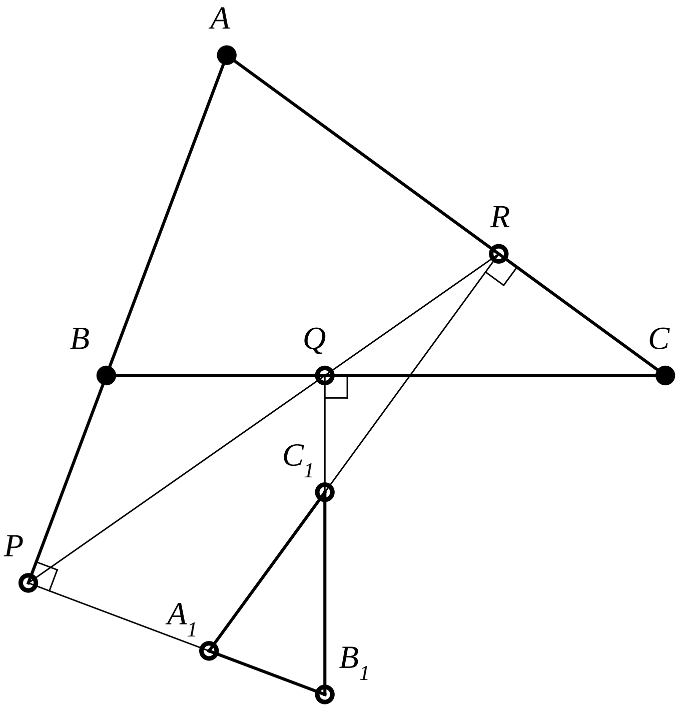

Figure 112

7.4. Metaparallel triangles or parallelologic triangles

Definition 47

Two triangles ABC and $A'B'C'$, with the property that the parallels taken through A, B, C respectively to $B'C'$, $C'A'$, $A'B'$ are concurrent in a point P, are called metaparallel triangles or parallelologic triangles. The point P is called center of parallelogy.

Theorem 44

If the triangles ABC and $A'B'C'$ are parallelologic, then the triangle $A'B'C'$ is also parallelologic in relation to ABC (parallels taken through the vertices A', B', C' respectively to BC, CA, AB are concurrent in a point P' - center of parallelology of the triangle $A'B'C'$ in relation to the triangle ABC).

Proof

Let ABC and $A'B'C'$ be two parallelologic triangles and P the parallelology center of the triangle ABC in relation to $A'B'C'$ (see *Figure 113*). We denote by A_1, B_1, C_1 the intersections of the parallels taken from A, B, C to $B'C'$, $C'A'$, $A'B'$, respectively with BC, CA, AB. Therefore $AA_1 \cap BB_1 \cap CC_1 = \{P\}$.

According to Ceva's theorem, we have that:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1. \quad (1)$$

We denote by A'_1, B'_1, C'_1 the intersections of the parallels taken through A', B', C', respectively to BC, CA, AB with the sides $B'C'$, $C'A'$ and $A'B'$; we note that $\Delta A'_1A'B' \sim \Delta A_1CP$ (they have parallel sides respectively); it follows that:

$$\frac{A'_1A'}{A_1C} = \frac{A'_1B'}{A_1P}. \quad (2)$$

Also, we have $\Delta A'_1A'C' \sim \Delta A_1BP$, from where:

$$\frac{A'_1A'}{A_1B} = \frac{A'_1C'}{A_1P}. \quad (3)$$

From relations (2) and (3), we get:

$$\frac{A'_1B'}{A'_1C'} = \frac{A_1B}{A_1C}. \quad (4)$$

Similarly, the following relations are obtained:

$$\frac{B'_1C'}{B'_1A'} = \frac{B_1C}{B_1A}, \quad (5)$$

$$\frac{C'_1A'}{C'_1B'} = \frac{C_1A}{C_1B}. \quad (6)$$

The relations (4), (5), (6) and (1) show, together with Ceva's theorem, that $A'A'_1$, $B'B_1$, $C'C'_1$ are concurrent in the parallelology center P' of the triangle $A'B'C'$ in relation to the triangle ABC.

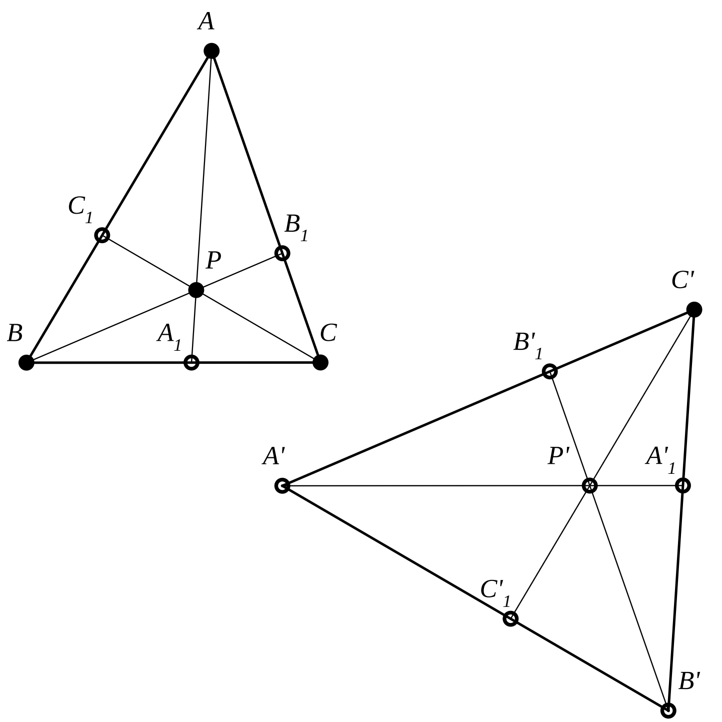

Figure 113

Observation 67

Two orthogonal triangles are parallelologic. Their parallelology centers are the orthocenters.

Remark 26

If ABC and $A_1B_1C_1$ are parallelologic triangles, then, obviously, they are orthogonal triangles.

From the reciprocal of Desargues's theorem (see [24]), it follows that ABC and $A_1B_1C_1$ are also homological triangles; hence, two parallelologic triangles are orthohomological triangles.

The theorem 40 can be formulated for the parallelologic triangles. In the same way, it can be demonstrated:

Theorem 45

If ABC and $A'B'C'$ are two triangles in the same plane, and through the vertices A, B, C some lines that create with $B'C'$, $C'A'$ and $A'B'$ angles of measure φ are taken, and these lines are concurrent, then the lines that pass through A', B', $C'$and create with BC, CA, AB angles of measure $180^0 - \varphi$ are concurrent. (The triangles ABC and $A'B'C'$ are called isologic).

This theorem generalizes the theorem of the orthological triangles.

8

ANNEXES

8.1 Annex 1: Barycentric Coordinates

8.1.1 Barycentric coordinates of a point in a plane

Let us consider a scalene triangle ABC which we will call the reference triangle, and an arbitrary point M in the plane of the triangle. We denote by $A'B'C'$ the intersections of the lines AM, BM, CM with the sides BC, CA, AB (see *Figure 113*).

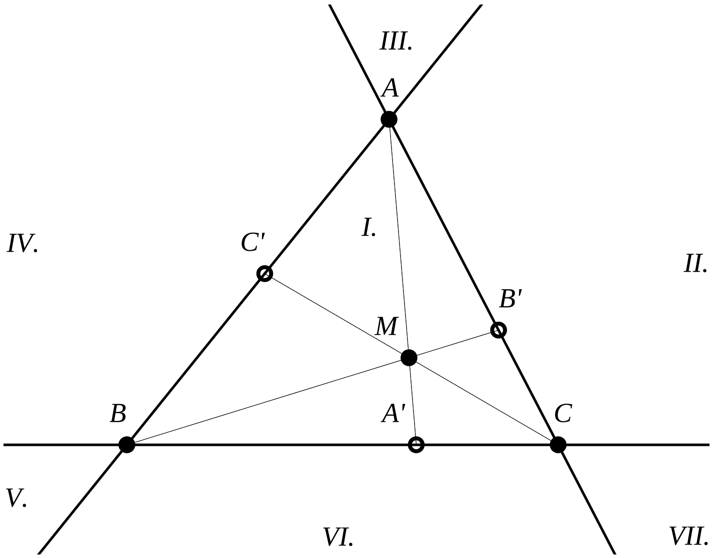

Figure 113

The point M determines with two points of the triangle, generally three triangles MBC, MCA, MAB. The areas of these triangles are considered positive or negative according to the following rule:

If a triangle has a common side with the reference triangle and its other vertex is on the same section of the common side with the “remaining” vertex of the reference triangle, then its area is positive, and if the common side separates its vertex with the other vertex of the reference triangle, the area is negative.

If the point M is situated on a side of the reference triangle, then the area of the "degenerate" triangle determined by it with the vertices of the reference triangle determining the respective side is zero. Denoting by S_a, S_b, S_c the areas of the three triangle MBC, MAC, MAB, it is observed that the signs of these areas correspond to those in the following table:

Region	S_a	S_b	S_c
I	+	+	+
II	+	-	+
III	+	-	-
IV	+	+	-
V	-	+	-
VI	-	+	+
VII	-	-	+

Definition 48

Three real numbers α, β, γ proportional to the three algebraic considered areas S_a, S_b, S_c are called barycentric coordinates of the point M in relation to the triangle ABC.

We denote $M(\alpha, \beta, \gamma)$. If α, β, γ are such that $\alpha + \beta + \gamma = 1$, then α, β, γ are barycentric absolute coordinates of the point M.

If we denote by S the area of the triangle ABC, then the barycentric absolute coordinates of the point M are $\frac{S_a}{S}, \frac{S_b}{S}, \frac{S_c}{S}$.

For example, if G is the center of gravity of the triangle ABC, then the absolute barycentric coordinates of G are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Theorem 46

If ABC is a given triangle and M is a point in its plane, then there exists and it is unique the ordered triplet $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, $\alpha + \beta + \gamma = 1$, such that $\alpha\overrightarrow{MA} + \beta\overrightarrow{MB} + \gamma\overrightarrow{MC} = \vec{0}$.

Reciprocally

For any triplet $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, $\alpha + \beta + \gamma = 1$, there exists and it is unique a point M in the plane of the triangle ABC, such that $\alpha\overrightarrow{MA} + \beta\overrightarrow{MB} + \gamma\overrightarrow{MC} = \vec{0}$.

Observation 68

From the previous theorem, it follows that the point M satisfies the condition $\frac{\overrightarrow{MA'}}{\overrightarrow{AM}} = \frac{\alpha}{\beta+\gamma}$ or $\frac{\overrightarrow{MA'}}{\overrightarrow{AA'}} = \alpha$.

We note that α is negative when BC separates the points A and M, and negative if A and M are on the same section of BC.

On the other hand, $\frac{\overrightarrow{MA'}}{\overrightarrow{AA'}} = \frac{S_a}{S}$, in the sign convention for S_a; in conclusion, the triplet (α, β, γ) from the theorem constitutes the barycentric absolute coordinates of the point M.

Theorem 47 (The position vector of a point)

Let O be a certain point in the plane of the triangle ABC and $M(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 1$. Then: $\overrightarrow{OM} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC}$.

Observation 69

1. We can denote $\overrightarrow{r_M} = \alpha\overrightarrow{r_A} + \beta\overrightarrow{r_B} + \gamma\overrightarrow{r_C}$.
2. From the previous theorem, it follows that $\overrightarrow{AM} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC}$, $\overrightarrow{BM} = \alpha\overrightarrow{BA} + \gamma\overrightarrow{BC}$, $\overrightarrow{CM} = \alpha\overrightarrow{CA} + \beta\overrightarrow{CB}$.

Theorem 48

If $Q_1(\alpha_1, \beta_1, \gamma_1)$, $Q_2(\alpha_2, \beta_2, \gamma_2)$, with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$ two given points in the plane of the triangle ABC, then $\overrightarrow{Q_1Q_2} = (\alpha_2 - \alpha_1)\overrightarrow{r_A} + (\beta_2 - \beta_1)\overrightarrow{r_B} + (\gamma_2 - \gamma_1)\overrightarrow{r_C}$.

Theorem 49 (Barycentric Coordinates of a vector)

Let ABC be a given triangle and O a point in its plane considered to be the origin of the plane. We denote by $\overrightarrow{r_A}, \overrightarrow{r_B}, \overrightarrow{r_C}$ the position vectors of the points A, B, C and with $\vec{u}$ a vector in the plane.
Then there exist and they are unique three real numbers α, β, γ with $\alpha + \beta + \gamma = 0$, such that $\vec{u} = \alpha\overrightarrow{r_A} + \beta\overrightarrow{r_B} + \gamma\overrightarrow{r_c}$.

Reciprocally

For any triplet $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ with $\alpha + \beta + \gamma - 0$, there exists and it is unique a vector $\vec{u}$ that verifies the relation $\vec{u} = \alpha\overrightarrow{r_A} + \beta\overrightarrow{r_B} + \gamma\overrightarrow{r_c}$.

Definition 49

The triplet $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathbb{R}^3$, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} = \mathbf{0}$, with the property that $\boldsymbol{\alpha} \cdot \overrightarrow{\boldsymbol{r_A}} + \boldsymbol{\beta} \cdot \overrightarrow{\boldsymbol{r_B}} + \boldsymbol{\gamma} \cdot \overrightarrow{\boldsymbol{r_c}} = \vec{\boldsymbol{u}}$ constitutes the barycentric coordinates of the vector $\vec{\boldsymbol{u}}$. We denote it by $\vec{\boldsymbol{u}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$.

Observation 70

The barycentric coordinates of vector $\vec{u}$ do not depend on choice of origin O.

Consequence

If $\vec{u}(\alpha, \beta, \gamma)$, then:
$\vec{u} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC}$;
$\vec{u} = \alpha\overrightarrow{BA} + \gamma\overrightarrow{BC}$;
$\vec{u} = \alpha\overrightarrow{CA} + \beta\overrightarrow{CB}$.

Theorem 50 (The position vector of a point that divides a segment into a given ratio)

Let $Q_1(\alpha_1, \beta_1, \gamma_1)$, $Q_2(\alpha_2, \beta_2, \gamma_2)$, $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$ and the point P that divides the segment Q_1Q_2 thus: $\frac{\overrightarrow{PQ_1}}{\overrightarrow{PQ_2}} = k$.

Then $P\left(\frac{\alpha_1 - k\alpha_2}{1-k}, \frac{\beta_1 - k\beta_2}{1-k}, \frac{\gamma_1 - k\gamma_2}{1-k}\right)$.

Consequences

1. The barycentric coordinates of the midpoint of the segment $[Q_1, Q_2]$, $Q_i(\alpha_i, \beta_i, \gamma_i)$ with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$ are given by $M\left(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}, \frac{\gamma_1+\gamma_2}{2}\right)$.
2. If $Q_i(\alpha_i, \beta_i, \gamma_i)$, $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,3}$ are the vertices of a triangle, then the center of gravity G of the triangle has the barycentric coordinates:
$$G\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}, \frac{\beta_1 + \beta_2 + \beta_3}{3}, \frac{\gamma_1 + \gamma_2 + \gamma_3}{3}\right).$$

Theorem 51 (The collinearity condition of two vectors)

Let us have the vectors $\overrightarrow{u_1}(\alpha_1, \beta_1, \gamma_1)$, $\overrightarrow{u_2}(\alpha_2, \beta_2, \gamma_2)$, $\alpha_1 + \beta_1 + \gamma_1 = 0$, $\alpha_2 + \beta_2 + \gamma_2 = 0$.
The vectors $\overrightarrow{u_1}, \overrightarrow{u_2}$ are collinear if and only if $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = \frac{\gamma_1}{\gamma_2}$.

Theorem 52 (The condition of perpendicularity of two vectors)

Let ABC be a given triangle, $BC = a$, $AC = b$, $AB = c$ and $\overrightarrow{u_1}(\alpha_1, \beta_1, \gamma_1)$, $\overrightarrow{u_2}(\alpha_2, \beta_2, \gamma_2)$, with $\alpha_i, \beta_i, \gamma_i = 0$, $i = 1, 2$.
Then: $\overrightarrow{u_1} \perp \overrightarrow{u_2} \Leftrightarrow (\beta_1\gamma_2 + \beta_2\gamma_1)a^2 + (\alpha_1\gamma_2 + \alpha_2\gamma_1)b^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)c^2 = 0$.

Consequence

If $Q_i(\alpha_i, \beta_i, \gamma_i)$, with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$ and $Q_0(\alpha_0, \beta_0, \gamma_0)$, with $\alpha_0 + \beta_0 + \gamma_0 = 0$, then:
$m\left(\widehat{Q_1Q_0Q_2}\right) = 90^0 \Leftrightarrow [(\beta_1 - \beta_0)(\gamma_1 - \gamma_0) + (\beta_2 - \beta_0)(\gamma_1 - \gamma_0)]a^2 + [(\alpha_1 - \alpha_0)(\gamma_2 - \gamma_0) + (\alpha_2 - \alpha_0)(\gamma_1 - \gamma_0)]b^2 + [(\alpha_1 - \alpha_0)(\beta_2 - \beta_0) + (\alpha_2 - \alpha_0)(\beta_1 - \beta_0)]c^2 = 0$.

Theorem 53

If $\vec{u}(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 0$, then $|\vec{u}|^2 = -(\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2)$.

Consequence

Let $Q_i(\alpha_i, \beta_i, \gamma_i)$, with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$.

The distance between Q_1 and Q_2 is given by ${Q_1Q_2}^2 = -\left[[(\beta_2 - \beta_1)(\gamma_2 - \gamma_1)]a^2 + [(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1)]b^2 + [(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)]c^2\right]$.

Theorem 54

Let $\vec{u}_i(\alpha_i, \beta_i, \gamma_i)$ with $\alpha_i + \beta_i + \gamma_i = 0$, $i = \overline{1,2}$; then:
$\vec{u}_1 \cdot \vec{u}_2 - -\frac{1}{2}[(\beta_1\gamma_2 + \beta_2\gamma_1)a^2 + (\gamma_1\alpha_2 + \alpha_2\gamma_2)b^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)c^2]$.

Theorem 55

The points $Q_i(\alpha_i, \beta_i, \gamma_i)$ with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,3}$ are collinear if and only if:
$\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1} = \frac{\beta_2-\beta_1}{\beta_3-\beta_1} = \frac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}$.
If a denominator is 0, then it is agreed that the appropriate numerator to be zero.

Theorem 56 (The three-point collinearity condition)

The points Q_1, Q_2, Q_3 with $Q_i(\alpha_i, \beta_i, \gamma_i)$, $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,3}$ are collinear, if and only if:

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0.$$

Consequence

The point $P(x, y, z)$, $x + y + z = 1$ is located on the line Q_1Q_2, $Q_i(\alpha_i, \beta_i, \gamma_i)$ with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$, if and only if

$$\begin{vmatrix} x & y & z \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

Observation 71

From those previously established, it follows that the equation of a line in barycentric coordinates is $mx + ny + pz = 0, m, n, p \in \mathbb{R}$.

The director vector of the line $d: mx + ny + p = 0$ is given by $\overrightarrow{u_d} = (n - p)\overrightarrow{r_A} + (p - m)\overrightarrow{r_B} + (m - n)\overrightarrow{r_C}$.

Observation 72

The barycentric coordinates of the director vector of the line $d: mx + ny + pz = 0$ are: $(n - p, p - m, m - n)$.

Theorem 57 (The condition of parallelism of two lines)

The lines $d_1: m_1x + n_1y + p_1z = 0$, $d_2: m_2x + n_2y + p_2z = 0$ are parallels if and only if:

$\frac{m_1-n_1}{m_2-n_2} = \frac{n_1-p_1}{n_2-p_2} = \frac{p_1-m_1}{p_2-m_2}$,

or:

$$d_1 \parallel d_2 \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{vmatrix} = 0.$$

Theorem 58 (The condition of perpendicularity of two lines)

The lines $d_1: m_1x + n_1y + p_1z = 0$ and $d_2: m_2x + n_2y + p_2z = 0$ are perpendicular if and only if:

$$[(p_1 - m_1)(m_2 - n_2) + (m_1 - n_1)(p_2 - m_2)]a^2 + [(n_1 - p_1)(m_2 - n_2) + (m_1 - n_1)(m_2 - p_2]\, b^2 + [(n_1 - p_1)\,(p_2 - m_2) + (p_1 - m_1)\,(n_2 - p_2)]c^2 = 0.$$

Theorem 59

The equation of the line determined by the point $P(x_0, y_0, z_0)$, $x_0 + y_0 + z_0 = 1$ and by the director vector $\vec{u}(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 0$ is:

$$\begin{vmatrix} x & y & z \\ x_0 & y_0 & z_0 \\ \alpha & \beta & \gamma \end{vmatrix} = 0.$$

Consequence

The equation of the line that passes through $P(x_0, y_0, z_0)$, $x_0 + y_0 + z_0 = 1$ and is parallel with the line $d \div mx + ny + pz = 0$ is:

$$\begin{vmatrix} x & y & z \\ x_0 & y_0 & z_0 \\ n - p & p - m & m - n \end{vmatrix} = 0.$$

Theorem 60 (Barycentric Coordinates of a vector perpendicular to a given vector)

If $\vec{u}(\alpha,\beta,\gamma)$, $\alpha+\beta+\gamma=0$ is a given vector, and u_1 is the vector perpendicular to $\vec{u}$, then:

$$u_{\perp}\big((\gamma-\beta)a^2-\alpha b^2+\alpha c^2, \beta a^2-\beta c^2+(\alpha-\gamma)b^2, \gamma b^2-\gamma a^2 + (\beta-d)c^2\big).$$

Theorem 61

In the plane of the triangle ABC, we consider the point $Q(\alpha,\beta,\gamma)$, $\alpha+\beta+\gamma=1$. We denote by $\{M\}=AQ\cap BC$, $\{N\}=BQ\cap CA$, $\{P\}=CQ\cap AB$. The barycentric coordinates of the points M,N,P are:

$$M\left(0,\frac{\beta}{\beta+\gamma},\frac{\gamma}{\beta+\gamma}\right);\ N\left(\frac{\alpha}{\alpha+\gamma},0,\frac{\gamma}{\alpha+\gamma}\right);\ P\left(\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta},0\right).$$

Remark

If the coordinates of the point $Q(\alpha,\beta,\gamma)$ are not absolute, then the barycentric (non-absolute) coordinates of the points M,N,P are $M(0,\beta,\gamma)$, $N(\alpha,0,\gamma)$, $P(\alpha,\beta,0)$.

Theorem 62 (The condition of concurrency of three lines)

Let the line d_i of equations:

$m_i x+n_i x+p_i z=0$, $i=\overline{1,3}$.

The lines d_1,d_2,d_3 are concurrent, if and only if:

$$\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}=0.$$

Consequence (The condition of concurrency of three cevians)

If the points Q_1, Q_2, Q_3 situated in the plane of the triangle ABC have the coordinates $Q_i(\alpha_i,\beta_i,\gamma_i)$, $i=\overline{1,3}$, then the lines AQ_1, BQ_2, CQ_3 are concurrent if and only if: $\alpha_3\beta_1\gamma_2=\alpha_2\beta_3\gamma_1$.

Theorem 63

Let Q_1, Q_2 in the plane of the triangle ABC such that:

$\alpha_1\overrightarrow{Q_1A}+\beta_1\overrightarrow{Q_1B}+\gamma_1\overrightarrow{Q_1C}=0$,

$\alpha_2\overrightarrow{Q_2A} + \beta_2\overrightarrow{Q_2B} + \gamma_2\overrightarrow{Q_2C} = 0;$
then the lines Q_1, Q_2 intersect the sides of the triangle ABC in the points $M \in BC$, $N \in CA$, $P \in BA$ that verifies the relations:
$\frac{\overrightarrow{MB}}{\overrightarrow{MC}} = -\frac{\begin{vmatrix}\gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2\end{vmatrix}}{\begin{vmatrix}\beta_1 & \alpha_1 \\ \beta_2 & \alpha_2\end{vmatrix}}, \frac{\overrightarrow{NC}}{\overrightarrow{NA}} = -\frac{\begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{vmatrix}}{\begin{vmatrix}\gamma_1 & \beta_1 \\ \gamma_2 & \beta_2\end{vmatrix}}, \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = -\frac{\begin{vmatrix}\beta_1 & \gamma_1 \\ \beta_2 & \gamma_2\end{vmatrix}}{\begin{vmatrix}\alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2\end{vmatrix}}.$

Theorem 64

Let ABC a triangle and Q a point in its plane such that $\alpha\overrightarrow{QA} + \beta\overrightarrow{QB} + \gamma\overrightarrow{QC} = \vec{0}$, where $\alpha, \beta, \gamma \neq 0$ and $\alpha + \beta + \gamma \neq 0$. We denote $AQ \cap BC = \{M\}$, $BQ \cap AC = \{N\}$, $CQ \cap AB = \{P\}$.
Then: $\frac{\overrightarrow{MB}}{\overrightarrow{MC}} = -\frac{\gamma}{\beta}, \frac{\overrightarrow{NC}}{\overrightarrow{NA}} = -\frac{\alpha}{\gamma}, \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = -\frac{\beta}{\alpha}$.

Theorem 65

Let the triangle ABC and $M \in BC$, $N \in AC$, $P \in AB$, such that: $\frac{\overrightarrow{MB}}{\overrightarrow{MC}} = \frac{-\gamma}{\beta}$, $\frac{\overrightarrow{NC}}{\overrightarrow{NA}} = \frac{-\alpha}{\gamma}$, $\frac{\overrightarrow{PA}}{\overrightarrow{PB}} = \frac{-\beta}{\alpha}$. Then the lines AM, BM, CP are concurrent in the point Q whose barycentric coordinates are $Q(\alpha, \beta, \gamma)$.

Consequence

If AA', BB', CC' are three concurrent cevians in the point X and $\frac{A'B}{A'C} = \alpha$, $\frac{B'C}{B'A} = \beta$, $\frac{C'A}{C'B} = \gamma$, then $X\left(\frac{1}{1-\gamma+\gamma\alpha}, \frac{1}{1-\alpha+\alpha\beta}, \frac{1}{1-\beta+\beta\gamma}\right)$.

Theorem 66

Let $P(\alpha\beta\gamma)$, $P'(\alpha'\beta'\gamma')$ be two isotomic points in the triangle ABC; then $\alpha\alpha' = \beta\beta' = \gamma\gamma'$.

Theorem 67

Let $P(\alpha, \beta, \gamma)$, $P'(\alpha'\beta'\gamma')$ be two isogonal points in the triangle ABC, with $BC = a$, $CA = b$, $AB = c$; then: $\frac{\alpha\alpha'}{a^2} = \frac{\beta\beta'}{b^2} = \frac{\gamma\gamma'}{c^2}$.

8.1.2 The Barycentric Coordinates of some important points in the triangle geometry

– *THE CENTER OF GRAVITY*

$$G:\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

– *THE CENTER OF THE INSCRIBED CIRCLE*

$$I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)$$

– *THE ORTHOCENTER*

$$H(\cot B\cot C,\cot A\cot C,\cot A\cot B)$$

– *THE CENTER OF THE CIRCUMSCRIBED CIRCLE*

$$O\left(\frac{R^2\sin 2A}{2S},\frac{R^2\sin 2B}{2S},\frac{R^2\sin 2C}{2S}\right)$$

– *THE CENTER OF THE A-EX-INSCRIBED CIRCLE*

$$I_a\left(\frac{-a}{2(p-a)},\frac{b}{2(p-a)},\frac{c}{2(p-a)}\right)$$

– *THE NAGEL'S POINT*

$$N\left(\frac{p-a}{p},\frac{p-b}{p},\frac{p-c}{p}\right)$$

– *THE GERGONNE'S POINT*

$$\Gamma\left(\frac{(p-b)(p-c)}{r(4R+r)},\frac{(p-a)(p-c)}{r(4R+r)},\frac{(p-a)(p-b)}{r(4R+r)}\right)$$

– *THE LEMOINE'S POINT*

$$K\left(\frac{a^2}{a^2+b^2+c^2},\frac{b^2}{a^2+b^2+c^2},\frac{c^2}{a^2+b^2+c^2}\right)$$

Observation 73

The above barycentric coordinates are absolute, and the relative barycentric coordinates are: $G(1,1,1)$, $I(a,b,c)$, $O(\sin 2A,\sin 2B,\sin 2C)$, $I_a(-a,b,c)$, $N(p-a,p-b,p-c)$, $\Gamma\left(\frac{1}{p-a},\frac{1}{p-b},\frac{1}{p-c}\right)$, $K(a^2,b^2,c^2)$.

8.1.3 Other Barycentric Coordinates and useful equations

– The coordinates of the vertices of the reference triangle ABC are:

$$A(1,0,0),B(0,1,0),C(0,0,1)$$

– The coordinates of the midpoints M, N, P of the sides of the reference triangle ABC are:

$$M\left(0,\frac{1}{2},\frac{1}{2}\right),N\left(\frac{1}{2},0,\frac{1}{2}\right),P\left(\frac{1}{2},\frac{1}{2},0\right)$$

– The coordinates of the point $M\in BC, \frac{\overline{MB}}{\overline{uC}}=k$ are:

$$M\left(0,\frac{1}{1-k},\frac{-k}{1-k}\right)$$

– The coordinates of an arbitrary point $M\in BC$ are $M(0,b,c)$
– The coordinates of an arbitrary point $N\in CA$ are $N(a,0,c)$
– The coordinates of an arbitrary point $P\in AB$ are $P(a,b,0)$
– The equation of the line BC is $x=0$
– The equation of the line AC is $y=0$
– The equation of the line AB is $z=0$
– The equation of a line that passes through A is $ny+pz=0$
– The equation of a line that passes through B is $mx+pz=0$
– The equation of a line that passes through C is $mx+ny=0$
– The coordinates of a direction vector BC are:

$$\vec{u}_{\overrightarrow{BC}}(0,-1,1)$$

– The coordinates of a direction vector CA are:

$$\vec{u}_{\overrightarrow{CA}}(1,0,-1)$$

– The coordinates of a direction vector AB are:

$$\vec{u}_{\overrightarrow{AB}}(-1,1,0)$$

– The coordinates of a vector perpendicular to BC are:

$$\vec{u}_{\perp BC}(2a^2,-a^2-b^2+c^2,-a^2+b^2-c^2)$$

– The coordinates of a vector perpendicular to CA are:

$$\vec{u}_{\perp CA}(-a^2-b^2+c^2,2b^2,a^2-b^2-c^2)$$

– The coordinates of a vector perpendicular to AB are:

$$\vec{u}_{\perp AB}(-a^2+b^2-c^2,a^2-b^2-c^2,2c^2)$$

– The equation of the mediator of the side BC is:

$$(b^2-c^2)x+a^2y-a^2z=0$$

– The equation of the mediator of the side CA is:

$$-b^2x+(c^2-a^2)y+b^2z=0$$

– The equation of the mediator of the side AB is:

$$c^2x-c^2y+(a^2-b^2)z=0$$

8.1.4 Aplications

1. In a triangle ABC, G – the center of gravity, I – the center of the inscribed circle and the Nagel's point, N, are collinear points.

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \mathrm{p-a} & \mathrm{p-b} & \mathrm{p-c} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ p & p & p \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a & b & c \end{vmatrix} = 0.$$

2. Prove that in a triangle ABC the Gergonne's point, Γ, the Nagel's point, N, and the point R – the isotomic of the orthocenter H, are collinear points.

Solution

The barycentric coordinates of H are:
$(\cot B \cot C\,, \cot C \cot A\,, \cot A \cot B)$.
The barycentric coordinates of its isotomic H' are:
$\left(\frac{1}{\cot B \cot C}, \frac{1}{\cot C \cot A}, \frac{1}{\cot A \cot B}\right)$,
therefore $H'(\tan B \tan C\,, \tan C \tan A\,, A \tan B)$.
We have $\Gamma\left(\frac{1}{p-a}, \frac{1}{p-b}, \frac{1}{p-c}\right)$ and $N(p-a, p-b, p-c)$.
The collinearity of points H', Γ and N is equivalent to:

$$\begin{vmatrix} \tan B \tan C & \tan C \tan A & \tan A \tan B \\ p-a & p-b & p-c \\ \frac{1}{p-a} & \frac{1}{p-b} & \frac{1}{p-c} \end{vmatrix} = 0.$$

The preceding condition is equivalent to:

$$D = \begin{vmatrix} \cot A & \cot B & \cot C \\ p-a & p-b & p-c \\ (p-a)^{-1} & (p-b)^{-1} & (p-c)^{-1} \end{vmatrix} = 0.$$

We know that $\cot A = \frac{b^2+c^2-a^2}{2S} = \frac{P(p-a)-(p-b)(p-c)}{2S}$ and the analogs.
$S = \text{Area}\Delta ABC$.

$$D = \frac{P}{2S}\begin{vmatrix} p-a & p-b & p-c \\ p-a & p-b & p-c \\ (p-a)^{-1} & (p-b)^{-1} & (p-c)^{-1} \end{vmatrix} -$$

$$-\frac{(p-a)(p-b)(p-c)}{2S}\begin{vmatrix} (p-a)^{-1} & (p-b)^{-1} & (p-c)^{-1} \\ p-a & p-b & p-c \\ (p-a)^{-1} & (p-b)^{-1} & (p-c)^{-1} \end{vmatrix} = 0.$$

Observation 74

The point R, the isotomic of the orthocenter of a triangle, is also called the retrocenter of a triangle.

3. If Q_1, Q_2 are points in the plane of the triangle ABC, $Q_i(\alpha_i, \beta_i, \gamma_i)$ with $\alpha_i + \beta_i + \gamma_i = 1$, $i = \overline{1,2}$, deduce the formula:

$$Q_1Q_2^2 = \alpha_2 Q_1A^2 + \beta_2 Q_1B^2 + \gamma_2 Q_1C^2 - \sum a^2\beta_2\gamma_2.$$

Solution

Let $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ be the barycentric coordinates of vertices. We have:

$$Q_1Q_2^2 = -a^2(\beta_2 - \beta_1)(\gamma_2 - \gamma_1) - b^2(\gamma_2 - \gamma_1)(\alpha_2 - \alpha_1) - c^2(\alpha_2 - \alpha_1)(\beta_2 - \beta_1).$$

By doing the calculations, we get:

$$Q_1Q_2^2 = -\sum a^2\beta_2\gamma_2 - \sum a^2\beta_1\gamma_1 + \sum a^2\beta_1\gamma_2 + \sum a^2\beta_2\gamma_1 \qquad (1)$$

We calculate:

$$Q_1A^2 = -a^2\beta_1\gamma_1 + b^2\gamma_1(1-\alpha_1) + c^2\beta_1(1-\alpha_1) = a^2\beta_1\gamma_1 - b^2\gamma_1\alpha_1 - c^2\alpha_1\beta_1 + b^2\gamma_1 + c^2\beta_1.$$

Therefore:

$$Q_1A^2 = -\sum a^2\beta_1\gamma_1 + b^2\gamma_1 + c^2\beta_1.$$

Similarly:

$$Q_1B^2 = -\sum a^2\beta_1\gamma_1 + c^2\alpha_1 + a^2\gamma_1.$$

$$Q_1C^2 = -\sum a^2\beta_1\gamma_1 + a^2\beta_1 + b^2\gamma_1.$$

We evaluate:

$$\alpha_2 Q_1A^2 + \beta_2 Q_1B^2 + \gamma_2 Q_1C^2.$$

We have:

$$\begin{aligned}\alpha_2 Q_1A^2 + \beta_2 Q_1B^2 + \gamma_2 Q_1C^2 &= a^2\left(-\sum a^2\beta_1\gamma_1 + b^2\gamma_1 + c^2\beta_1\right)\\ &+ \beta_2\left(-\sum a^2\beta_1\gamma_1 + c^2\alpha_1 + a^2\gamma_1\right)\\ &+ \gamma_2\left(-\sum a^2\beta_1\gamma_1 + a^2\beta_1 + b^2\gamma_1\right)\\ &= -\sum a^2\beta_1\gamma_1 + \sum a^2\beta_1\gamma_2 + \sum a^2\beta_2\gamma_1. \quad (2)\end{aligned}$$

Comparing the relations (1) and (2), we find that:

$$\boxed{Q_1Q_2^2 = \alpha_2 Q_1A^2 + \beta_2 Q_1B^2 + \gamma_2 Q_1C^2 - \sum a^2\beta_2\gamma_2}. \qquad (3)$$

4. If ABC is a given scalene triangle, O and I are respectively the centers of its circumscribed and inscribed circles; prove the relation: $OI^2 = R^2 - 2Rr$.

Solution

We use the formula (3) from the previous aplication, where $Q_1 = Q$ and $Q_2 = I$. The barycentric coordinates of I are $\frac{a}{2p}, \frac{b}{2p}, \frac{c}{2p}$.

We have $OI^2 = R^2 - \sum a^2 \frac{bc}{4p^2} = R^2 - \frac{abc}{4p^2} \sum a = R^2 - \frac{abc}{2p}$.

Taking into account the known formulas $S = pr$ and $abc = 4RS$, we get:

$OI^2 = R^2 - \frac{4RS}{2p} = R^2 - 2Rr$.

Observation 75

The resulting relation is called Euler's relation. From here, it follows that, in a triangle, $R \geq 2r$ (Euler's inequality).

5. Let ABC be a scalene triangle, O – its circumscribed center, and N – the Nagel's point. Show that $ON = R - 2r$ (R – radius of the circumscribed circle, r – radius of the circle inscribed in the triangle ABC).

Solution

We employ the formula (3) from Application 3, where $Q_1 = Q$ and $Q_2 = N$. The barycentric coordinates of the Nagel's point are $N\left(\frac{p-a}{p}, \frac{p-b}{p}, \frac{p-c}{p}\right)$. We have:

$ON^2 = R^2 - \sum a^2 \frac{(p-b)(p-c)}{p^2}$.

$$ON^2 = R^2 - \frac{1}{4p^2}\sum a^2(a-b+c)(a+b-c) = R^2 - \frac{1}{4p^2}\sum a^2[a^2-(b-c)^2]$$

$$= R^2 - \frac{1}{4p^2}\sum a^2(a^2-b^2-c^2+2bc) = R^2$$

$$+ \frac{1}{4p^2}\left[2\sum b^2c^2 - \sum a^4 - 2abc(a+b+c)\right]$$

$$= R^2 + \frac{1}{4p^2}(a6S^2 - 4pabc) = R^2 + \frac{1}{4p^2}(16p^2r^2 - 16p^2R \cdot r)$$

$$= R^2 + 4r^2 - 4pr - (R-2r)^2$$

The following formulas were used:

$16S^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4$,

$abc = 4R \cdot S$ and $S = p \cdot r$.

8.2 Annex 2: The similarity of two figures

The study of the geometrical properties of the plane figures often calls for plane transformations that increase (or decrease) the distance between points, but retain the shape of the figures.

8.2.1 Properties of the similarity on a plane

Definition 50

An application $a_k: \mathcal{P} \to \mathcal{P}$, where $k \in \mathbb{R}_+^*$, is called a similarity ratio k if $a_k(A)a_k(B) = k \cdot AB$, $(\forall)A, B \in \mathcal{P}$.
We denoted $a_k(A) = A'$, $a_k(B) = B'$ and we say about A' and B' that are the homologues or the similars of the points A, B.
The constant ratio k is called a similarity or a similitude ratio.
From definition, it follows that, if ABC is a given triangle and $A'B'C'$ is the triangle obtained from ABC, applying to it the similarity a_k, having $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{C'C'} = k$, the triangle $A'B'C'$ is similar to the triangle ABC.
We denote $\Delta A'B'C' \sim \Delta ABC$.
Furthermore, if we consider the triangle ABC, oriented such as the vertices A, B, C are read in the trigonometric sense, and if A', B', C' have the same orientation, we say that the similarity is direct.
If the similarity is inverse, ABC and $A'B'C'$ are inversely-orientated.
During this presentation, we will say that two figures are similar instead of directly-similar and we will make the special mention in the case of the inversely-similar figures.

Proposition 76

A similarity transforms three collinear points in three other collinear points keeping the points in order.

Proof

Let A, B, C be three collinear points and let $A' = a_k(A)$, $B^{'} = a_k(B)$ and $C^{'} = a_k(C)$ be their similars.

The points A, B, C are in the order in which they were written, therefore $AB + BC = AC$, having $A'B' = a_k(A)a_k(B) = KAB$, $B'C' = a_k(B)a_k(C) = KBC$ and $A'C' = a_k(A)a_k(C) = KAC$; it follows that $A'B' + B'C' = A'C'$ if A', B', C' are collinear in the order A' - B' - C'.

Remark 27

From the previous property, it follows that a similarity transforms a segment into another segment, a semi-line into another semi-line, a line into another line (keeping the points in order).

Property 77

A similarity transforms two parallel lines into two other parallel lines.

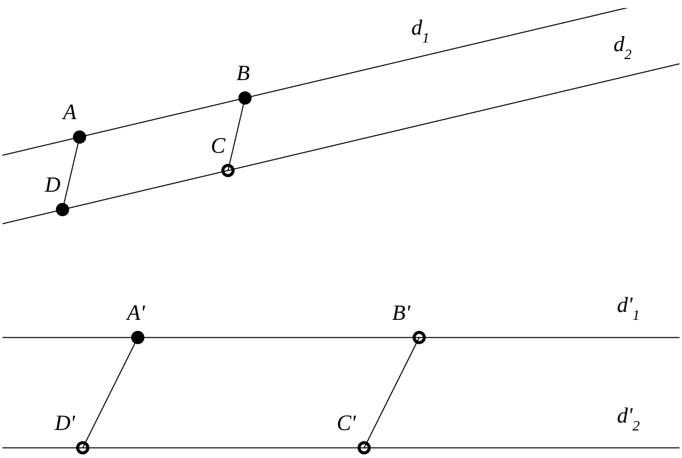

Figure 114

Proof

Let $d_1 \parallel d_2$ and $d_1 = a_k(d_1)$, $d_2 = a_k(d_2)$ (see *Figure 114*). If $A, B \in d_1$, $C, D \in d_2$ such that $ABCD$ is parallelogram, then we have $A' = a_k(A)$, $B' = a_k(B)$, $C' = a_k(C)$, $D' = a_k(D)$. Since $A'B' = k \cdot AB$, $C'D' = k \cdot CD$ and $AB = CD$, it follows that $A'B' = C'D'$ (1).

Also, $A'D' = k \cdot AD$, $B'C' = k \cdot BC$ and $AD = BC$, it follows that $A'D' = B'C'$ (2). The relations (1) and (2) show that $A'B'C'D'$ is a parallelogram, therefore $d_1{}' \parallel d_2{}'$.

Remark 28

The image of an angle $\widehat{AOB}$ by a similarity a_k is an angle $\widehat{A'O'B'}$ and $\widehat{AOB} \equiv \widehat{A'O'B'}$.

Definition 51

Two figures $\mathcal{F}$ and $\mathcal{F}'$ of the plane $\mathcal{P}$ are called similar figures with the similarity ratio k if there exists a similarity $a_k : \mathcal{P} \to \mathcal{P}$ such that $a_k(\mathcal{F}) = \mathcal{F}'$.
We denote $\mathcal{F} \sim \mathcal{F}'$ and read $\mathcal{F}$ is similar to $\mathcal{F}'$.

Observation 76

If $\mathcal{F} \sim \mathcal{F}'$, then any triangle ABC with the vertices belonging to the figure $\mathcal{F}$ are as image a similar triangle $A'B'C'$ with the vertices belonging to the figure $\mathcal{F}'$.

In two similar figures, the homologous segments are proportional to the similarity ratio, and the homologous angles are congruent.

Definition 52

It is called center of similarity (or double point) of two similar figures the point of a figure that coincides with its homologue (similar) in the other figure.

Remark 29

From previous Definition, it follows that, if two figures $\mathcal{F}$ and $\mathcal{F}'$ are similar and if O is their similarity center, AB and $A'B'$ are two homologous segments from the figures $\mathcal{F}$ respectively $\mathcal{F}'$, then the triangles OAB and $OA'B'$ are similar.

Proposition 78

If AB and $A'B'$ two homologous segments from the similar figures $\mathcal{F}$ and $\mathcal{F}'$, such that the lines AB and $A'B'$ intersect in a point M, then the intersection of the circles circumscribed to triangles $AA'M$ and $BB'M$ contain the center of similarity.

Proof

We denote by O the second common point of the circles circumscribed to the triangles $AA'M$ and $BB'M$ (see *Figure 115*).

The quadrilateral $MBB'O$ is inscribed, therefore $\sphericalangle MBO \equiv \sphericalangle MB'O$ (1). Also, we have that $\sphericalangle BMB' \equiv \sphericalangle BOB'$ (1). The quadrilateral $MAA'O$ is inscribed, hence $\sphericalangle AMA' \equiv \sphericalangle AOA'$ (2). From relations (2) and (3), we note that $\sphericalangle BOB' \equiv \sphericalangle AOA'$ (4).

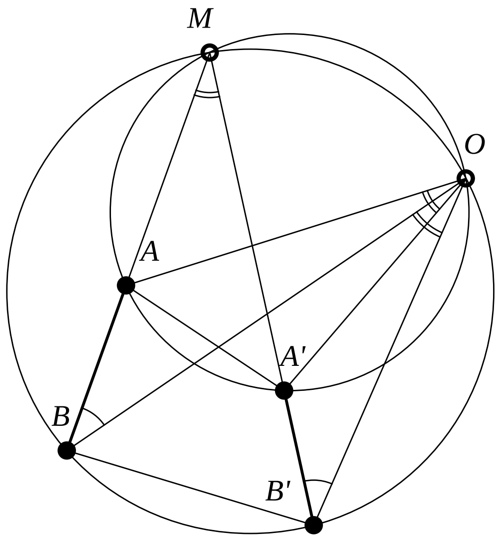

Figure 115

Because $\widehat{AOA'} = \widehat{AOB} + \widehat{BOA'}$ and $\widehat{BOB'} = \widehat{BOA'} + \widehat{A'OB'}$, we obtain that $\sphericalangle AOB \equiv \sphericalangle A'OB'$ (5).

The relations (1) and (5) show that $\Delta OAB \sim \Delta OA'B'$ and, consequently, O is the double point of similarity.

Remark 30

i) The homothety is a particular case of similarity.
ii) In the case of homothety, the similarity center is the homothety center.

Teorema68

Two similar figures can become homothetic by a rotation around their similarity center.

Proof

If the figures $\mathcal{F}$ and $\mathcal{F}'$ are similar and have their similarity center O, their similarity ratio being k, we rotate the figure $\mathcal{F}'$ around the point O with an angle $\varphi = m(\widehat{AOA'})$, where A and A' are homologous points belonging to the figures $\mathcal{F}$ and $\mathcal{F}'$. Then the point A' occupies the position A'' on the radius OA, also because $\Delta AOB \sim \Delta A'OB'$, the point B' occupies the position B'' on the radius OB, and we have $\frac{OB''}{OB} = k$. The reasoning applied to B' is valid for any point $X' \in \mathcal{F}'$; this will pass after rotation in X'' on the homologous radius OX and we have $\frac{OX''}{OX} = k$. The figure $\mathcal{F}'$ occupies a new position after rotation, $\mathcal{F}''$, and $\mathcal{F}''$ is homothetic to the figure $\mathcal{F}$ by homothety of center O and ratio k.

Remark 31

i) Two homologous lines AB and $A'B'$ form between them an angle of measure φ equal to the measure of the angle of rotation which transforms the similar figures into homothetic figures.

ii) The ratio of the distances of the similarity center of two homologous lines AB and $A'B'$ is constant and equal with the similarity ratio.

Theorem 69

The geometric place of the similarity centers of two non-concentric and non-congruent given circles is the circle with the diameter determined by the center of homothety of the two circles.

Proof

Let $\mathcal{C}(O_1, r_1)$ and $\mathcal{C}(O_2, r_2)$, $r_1 < r_2$, be the given circles (see *Figure 116*).

We denote by A and B their center of direct and inverse homothety. If M is a similarity center of the two circles, then $\frac{MO_1}{MO_2} = \frac{r_1}{r_2}$ (obviously the points A and B belong to the geometric place because they are similarity centers).

From $\frac{BO_1}{BO_2} = \frac{r_1}{r_2} = \frac{MO_1}{MO_2}$, it follows that MB is an internal bisector in the triangle MO_1O_2, and from $\frac{AO_1}{AO_2} = \frac{MO_1}{AM}$, we obtain that MA is an external bisector in the triangle MO_1O_2.

Because the internal and external bisectors corresponding to the same angle of a triangle are perpendicular, we have that $m(\widehat{AMB}) = 90^0$ and consequently the point M belongs to the circle of diameter $[AB]$.

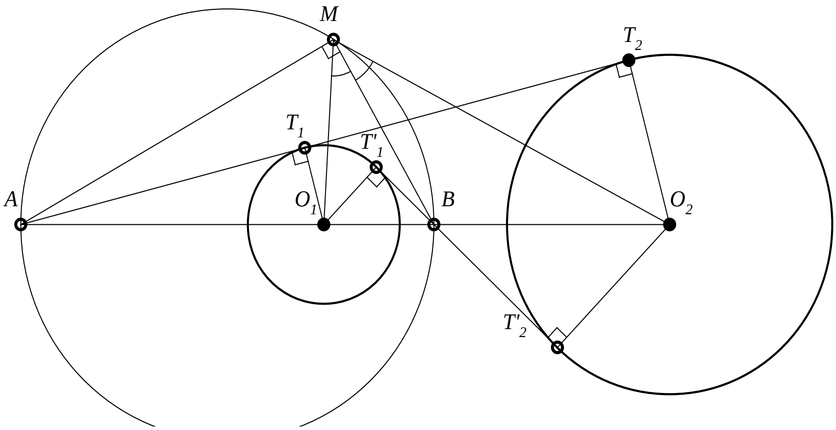

Figure 116

Remark 32

i) If the circles $\mathcal{C}(O_1, r_1)$ and $\mathcal{C}(O_2, r_2)$ are secants in the points M and N, then obviously these points are similarity centers for the given circles.

ii) If ABC is a given scalene triangle, the geometric place of the points M in the plane of the triangle ABC for which $\frac{MB}{MC} = \frac{AB}{AC}$ is a circle, called the A-Apollonius circle of the triangle ABC.

8.2.2 Applications

1. Let $OA_1B_1C_1$ and $OA_2B_2C_2$ two squares in plane with the same orientation. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent.

Solution

$\Delta OA_1B_1 \sim \Delta OA_2B_2$. Because $[A_1B_1]$ and $[A_2B_2]$ are analogous segments in the directly-similar given squares and O is their similarity point, it follows that A_1A_2 and B_1B_2 intersect in the second point of intersection of the circles circumscribed to the squares, point that was denoted by M in *Figure 117*.

The same reasoning applies also to the homologous segments B_1C_1 and B_2C_2, therefore B_1B_2 intersects with C_1C_2 in M as well.

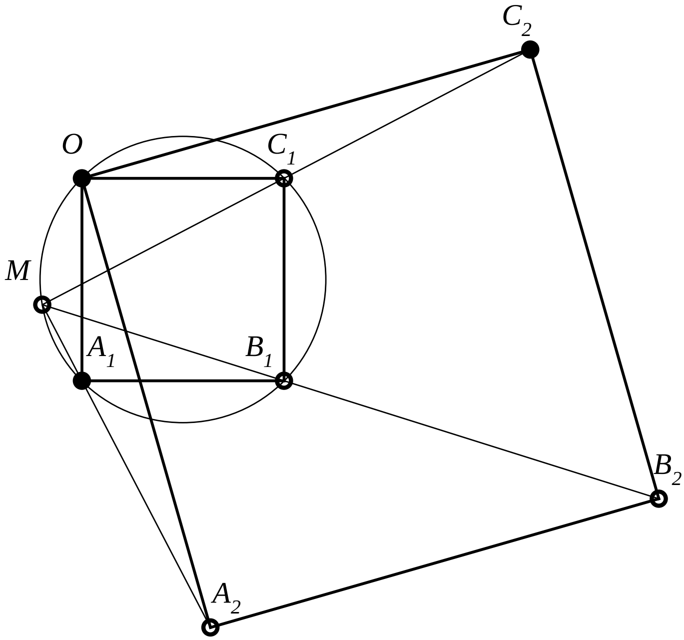

Figure 117

2. Let ABC be a triangle and A_1 - B_1 - C_1 a transverse ($A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$). Prove that the circles circumscribed to the triangles AB_1C_1, ABC, BA_1C_1 and A_1CB_1 have a common point (the Miquel's circles).

Solution

Let O be the second common point of the circles circumscribed to the triangles AB_1C_1 and ABC. This point is the similarity center of the homologous segments (C_1B) and (B_1C) (see *Figure 118*).

Therefore there exists a similarity a such that $a(O) = O$, $a(C_1) = B_1$, $a(B) = C$; there exists also a similarity a' such that $a'(O) = O$, $a'(C_1) = B_1$, $a'(B) = C$; then the circles circumscribed to the triangles B_1A_1C and C_1A_1B pass through the point O.

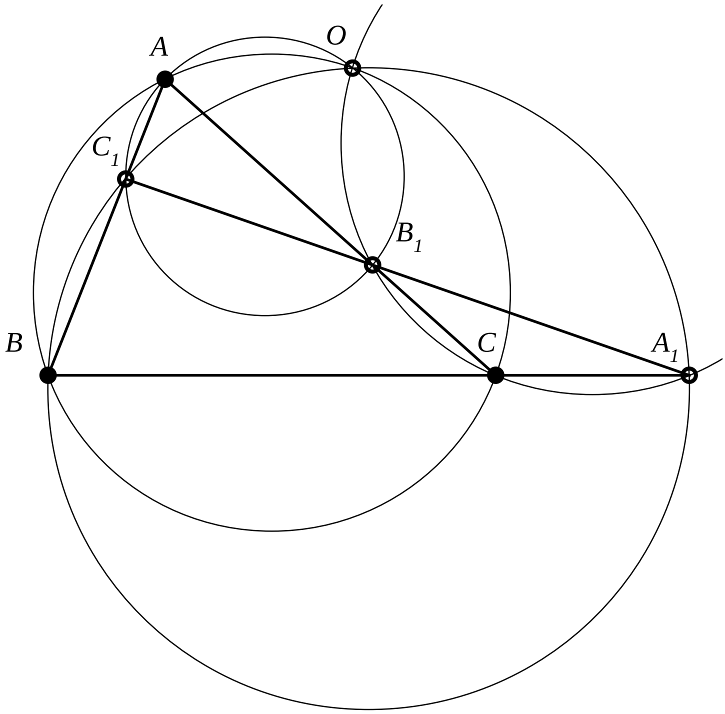

Figure 118

8.3 Annex 3: The point, the triangle and the Miquel's circles

8.3.1 Definitions and theorems

Theorem 70 (J. Steiner, 1827)

The four circles circumscribed to the triangles formed by four lines intersecting two by two pass through the same point (Miquel's point).

Proof

Let A, B, C and A_1, B_1,C_1 be the intersection points of the four lines in the statement (the quadrilateral $BCB_1C_1A_1A$ is a complete quadrilateral, see *Figure 119*). We denote by M the second point of intersection of the circles circumscribed to the triangles AB_1C_1 and CB_1A_1.

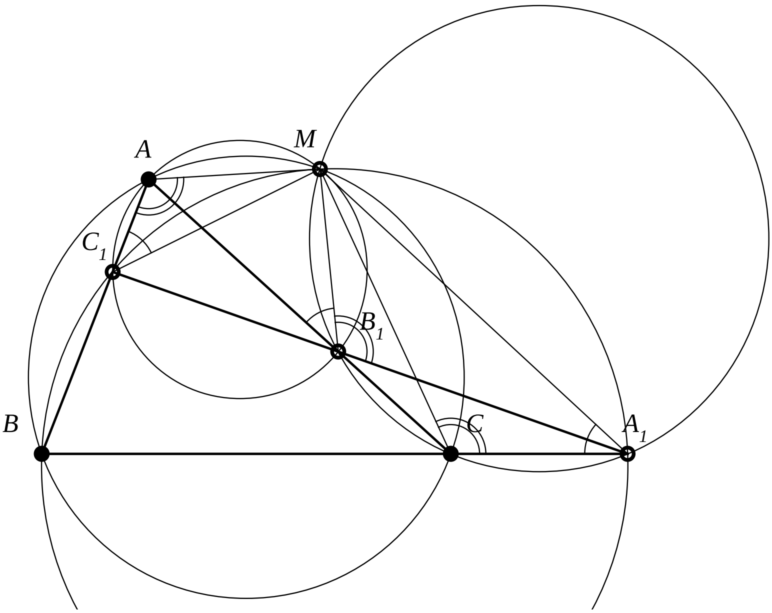

Figure 119

The quadrilaterals MAC_1B_1 and MB_1CA_1 being inscribable, we obtain the relation:

$\sphericalangle MC_1A \equiv \sphericalangle MB_1A \equiv \sphericalangle MA_1C = \varphi.$ (1)

From (1), we note that:

$\sphericalangle MC_1A \equiv \sphericalangle MA_1B,$ (2)

a relation showing that the quadrilateral MC_1BA_1 is inscribable, hence the circle circumscribed to BA_1C_1 passes through the point M. Also from the inscribability of the previous quadrilaterals, we obtain that:

$\sphericalangle MAC_1 \equiv \sphericalangle MB_1A_1 \equiv \sphericalangle MCA_1.$ (3)

Noting from this relation that: $\sphericalangle MAC_1 \equiv \sphericalangle MCA_1$, we conclude that the quadrilateral $MABC$ is inscribable, consequently the circumscribed circle of the triangle ABC passes through M; the theorem is proved.

Observation 77

a) The concurrency point M of the four circles is called the Miquel's point of the complete quadrilateral $BCB_1C_1A_1A$.
b) The preceding theorem can be reformulated; thus: The circles circumscribed to the four triangles of a complete quadrilateral have a common point.

Theorem 71 (J. Steiner, 1827)

The centers of the circles circumscribed to the four triangles of a complete quadrilateral and the Miquel's point of this quadrilateral have five concyclic points (Miquel's circle).

Proof

Let O, O_1, O_2, O_3 respectively be the centers of the circles circumscribed to the triangles ABC, AB_1C_1, BC_1A_1, CB_1A_1 (see *Figure 119*). We denoted $m\left(\widehat{MC_1A}\right) = \varphi$, then $m\left(\widehat{MO_1A}\right) = 2\varphi$.

Because $OO_1 \perp AM$, it follows that $m\left(\widehat{MO_1O}\right) = 180^0 - \varphi$. On the other hand, $m\left(\widehat{MA_1B}\right) = \varphi$ and $m\left(\widehat{MO_2B}\right) = 2\varphi$; also $O_2O \perp MM$, leading to $m\left(\widehat{MO_2O}\right) = \varphi$. From $m\left(\widehat{MO_1O}\right) = 180^0 - \varphi$ and $m\left(\widehat{MO_2O}\right) = \varphi$, it appears that the quadrilateral MO_1OO_2 is inscribable if the points M, O_1, O, O_2 are concyclic.

A similar reasoning leads to $m(\widehat{MO_3O}) = \varphi$, and since $m(\widehat{MO_1O}) = 180^0 - \varphi$, we have the points M, O_1, O, O_3. We note that the points M, O, O_1, O_2, O_3 are concyclic.

Their circle is called Miquel's circle.

Remark 33

a) From Theorem 70, we can observe that the collinear points A_1 - B_1 - C_1 are the projections of the point M (that belongs to the circle circumscribed to the triangle ABC), under the angle of measurement φ and in the same direction (measured) on the sides of the triangle ABC. Thus occurs the generalized Simson's theorem (of angle φ).

The generalized Simson's theorem has been proved by L. Carnot and it is thus stated:

The projections of a point of the circle circumscribed under the same angle and direction on the sides of the triangle are collinear.

In the case of *Figure 119*, we have:

$m(MA_1, BC) = m(MA_1, CA) = m(MC_1, AB) = \varphi$.

It can be shown that the Theorem 70 and the generalized Simson's theorem are equivalent.

b) The equivalence of the previous theorems leads to:

Theorem 72

The circles described on the chords MA, MB, MC of the circle circumscribed to the given triangle ABC capable of the same angle φ intersect two by two in the collinear points A_1, B_1, C_1 situated on the sides of the triangle ABC.

This theorem for the right angle case is owed to the Irish mathematician G. Salmon (1819-1904).

In Theorem 70, the points A_1 - B_1 - C_1 that belong to the sides of the triangle ABC are collinear; in the following, we show that we can define the Miquel's point even if A_1, B_1, C_1 are not collinear.

Theorem 73

If the points A_1, B_1, C_1 repectively belong to the sides BC, CA and AB of the triangle ABC, then the circles circumscribed to the triangles AB_1C_1, BC_1A_1 and CA_1B_1 pass through the same point (Miquel's point).

Proof

We consider A_1, B_1, C_1 on the sides (BC), (CA), (AB) (see *Figure 120*). We denote by M the second point of intersection of the circles circumscribed to the triangles AC_1B_1 and BC_1A_1. The quadrilateral AC_1MB_1 and BC_1MA_1 are inscribable, hence:

$\sphericalangle MB_1A \equiv \sphericalangle MC_1B,$ (1)

$\sphericalangle MC_1B \equiv \sphericalangle MA_1C.$ (2)

The preceding relations imply that:

$\sphericalangle MB_1A \equiv \sphericalangle MA_1C,$ (3)

and this relations shows that the points M, A_1, C, B_1 are situated on a circle.

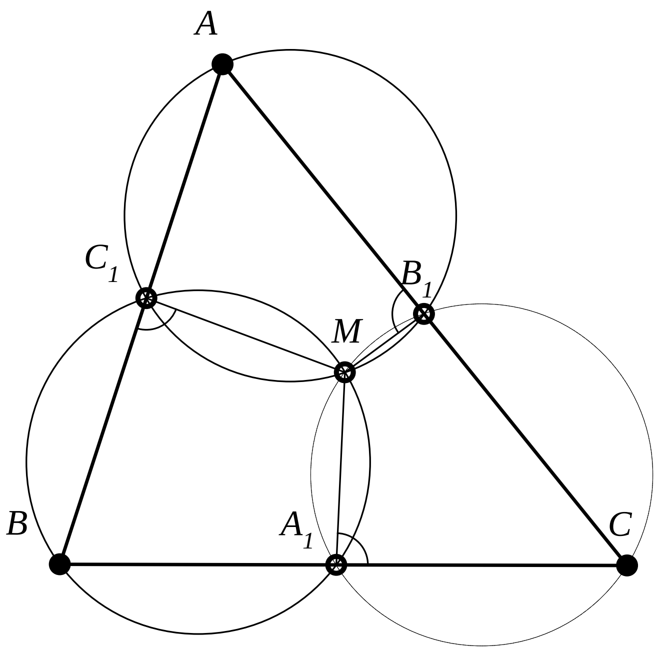

Figure 120

Observation 78

a) The theorem can be proved in the same way even if one of the points A_1, B_1, C_1 are situated on a side, and the other two points are situated on the extensions of the other two sides.

b) The triangle $A_1B_1C_1$ was called the Miquel's triangle, corresponding to the Miquel's point *M*, and the circles circumscribed to the triangles AC_1B_1, BC_1A_1 and CA_1B_1 were called the Miquel's circles.

Proposition 79

If M is a fixed point in the plane of the triangle ABC, then we can build any Miquel's triangles we want, corresponding to the point M.

Indeed, we can build from M the lines MA_1, MB_1, MC_1 that create with BC, CA, AB the same angle measured in the same direction, or we can proceed in this way:

We take through M and A a certain circle that cuts AB and AC in C_1 and B_1. We build the circumscribed circle of the triangle BMC_1; this will intersect BC the second time in the point A_1. The triangle $A_1B_1C_1$ is a Miquel's triangle corresponding to the point M.

Theorem 71

All the Miquel's triangles corresponding to a given point M in the plane of the triangle ABC are directly-similar, and the point M is the double point (their similarity center):

Proof

Let M be a Miquel's point for the triangle ABC and $A_1B_1C_1$ the Miquel's triangle corresponding to this point *M*, supposedly fixed. Then we know the angular coordinates of M, therefore the angles $\widehat{BMC}$, $\widehat{CMA}$, $\widehat{AMB}$ (see *Figure 121*). It is not difficult to establish that:

$\sphericalangle B_1A_1C_1 = \sphericalangle BMC - \sphericalangle BAC$;

$\sphericalangle A_1B_1C_1 = \sphericalangle AMC - \sphericalangle ABC$;

$\sphericalangle B_1C_1A_1 = \sphericalangle AMB - \sphericalangle ACB$.

As it can be seen, the measures of these angles are well determined when the point *M* is fixed. If we will consider the triangle $A_2B_2C_2$ – Miquel's triangle corresponding to the point M, the angles $\sphericalangle B_1A_2C_2$, $\sphericalangle A_2B_2C_2$ and $\sphericalangle B_2C_2A_2$ will be given by the same formulas, hence $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$.

Then, $\sphericalangle A_1MB_1 \equiv \sphericalangle A_2MB_2 = 180^0 - m(\hat{C})$, $\sphericalangle MA_1B_1 \equiv \sphericalangle MCA$, $\sphericalangle MA_2B_2 \equiv \sphericalangle MCA$, consequently $\Delta MA_1B_1 \sim \Delta MA_2B_2$, which shows that the point M is the similarity center of the two Miquel's triangles.

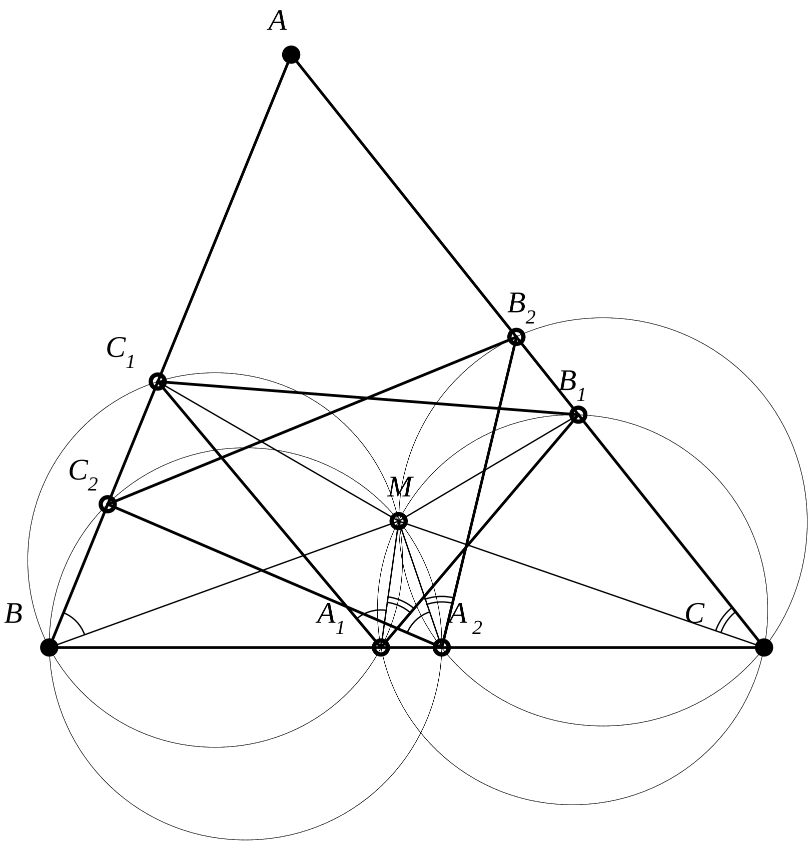

Figure 121

8.3.2 Applications

1. Let M be a Miquel's point in the interior of the triangle ABC and let $A_1B_1C_1$ be its corresponding Miquel's triangle. We denote by A_2, B_2, C_2 the intersections of semi-lines $(AM$,$(BM$, $(CM$ with the circumscribed circle of the triangle ABC. Prove that $\Delta A_2B_2C_2 \sim \Delta A_1B_1C_1$.

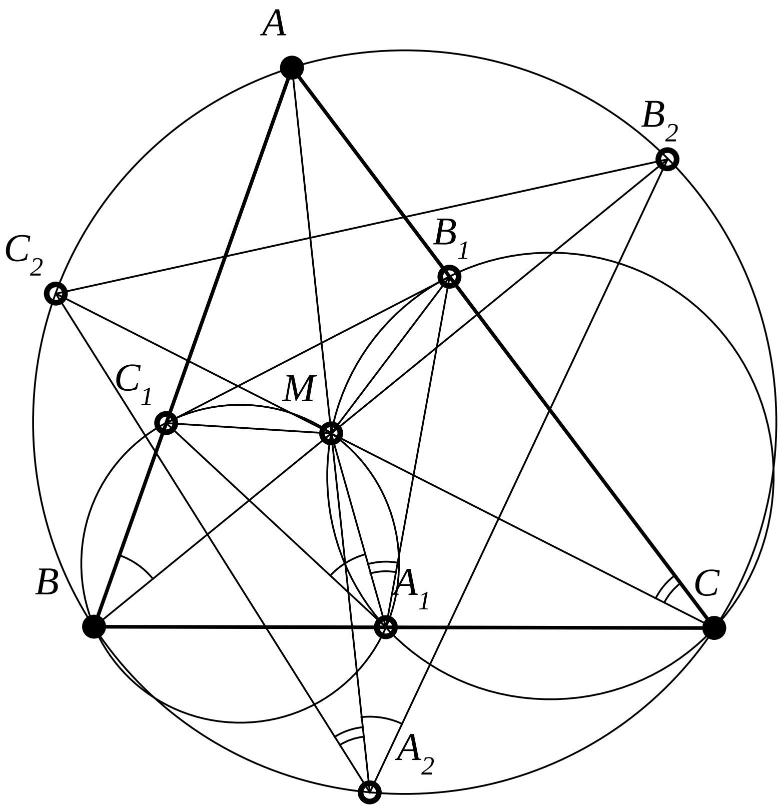

Figure 122

Solution

$\sphericalangle B_2A_2C_2 = \sphericalangle B_2A_2A + \sphericalangle AA_2C_2$ (see *Figure 122*). But $\sphericalangle B_2A_2A = \sphericalangle B_2BA$ and $\sphericalangle B_2BA \equiv \sphericalangle MA_1C_1$, $\sphericalangle AA_2C_2 \equiv \sphericalangle ACC_2$ and $\sphericalangle ACC_2 \equiv \sphericalangle MA_1B_1$. We obtain that $\sphericalangle B_2A_2C_2 \equiv \sphericalangle B_1A_1C_1$. Similarly, we show that $\sphericalangle A_2B_2C_2 \equiv \sphericalangle A_1B_1C_1$.

2. Let M be the Miquel's point corresponding to the Miquel's triangle $A_1B_1C_1$, with the vertices on the sides of the triangle ABC. Three cevians, concurrent in the point P from the interior of the triangle ABC, intersect a second time *the Miquel's circles* in the points A_2, B_2, C_2. Prove that the points A_2, B_2, C_2, M are concyclic.

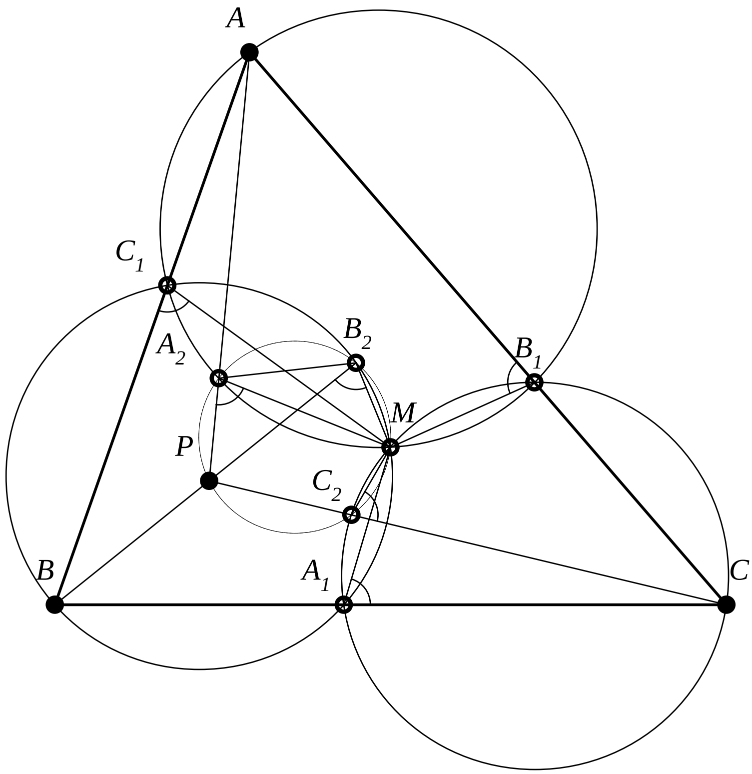

Figure 123

Solution

In *Figure 123*, let A_2 the intersection of cevian AP with the Miquel's circle (AB_1C_1). We denote $\widehat{MA_1C} = \widehat{MB_1A} = \widehat{MC_1B} = \varphi$. We have $\sphericalangle MA_2P = 180^0 - \varphi$, $\sphericalangle MB_2P = 180^0 - \sphericalangle BB_2M = 180^0 - \varphi$. It follows that $\sphericalangle MC_2P = \varphi$, therefore the points M, A_2, B_2, C_2, P are concyclic.

3. Three circles passing respectively through the vertices A, B, C of the triangle ABC intersect in a point S in the interior of the triangle and a second time in the points D, E, F belonging to the sides (BC), (CA), (AB). We take through A, B, C three parallels with a given direction that intersect the second time each circle respectively in the points P, Q and R. Prove that the points P, Q, R are collinear.

Van Khea – Peru, Geometrico 2017

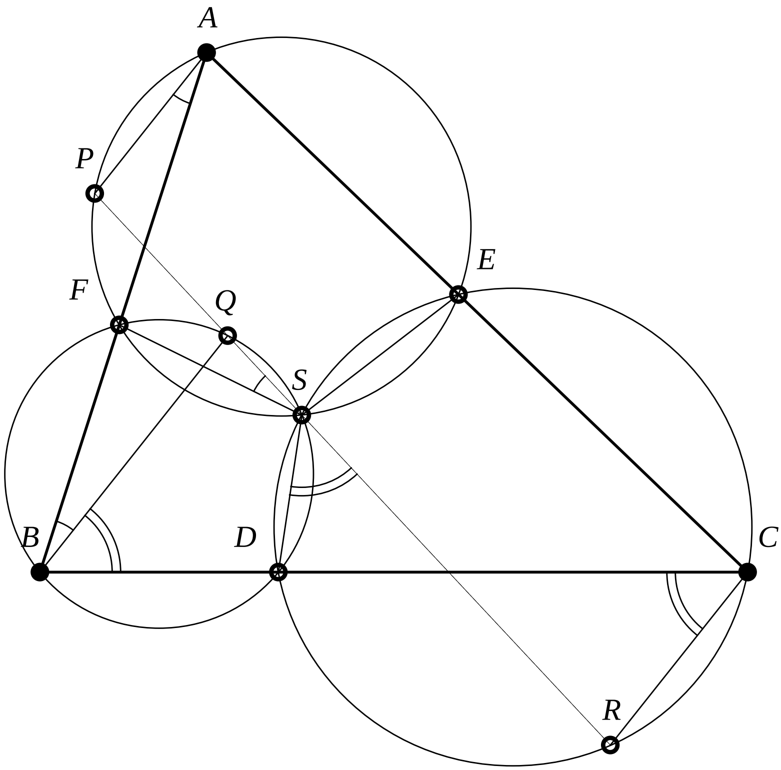

Figure 124

Solution

The quadrilateral $APFS$ is inscribed, therefore:

$\sphericalangle PAF \equiv \sphericalangle FSP$. (1)

From the parallelism of lines AP and BQ, we note that:

$\sphericalangle PAF \equiv \sphericalangle FBQ$. (2)

The quadrilateral $FBSQ$ is inscribed, hence: $\sphericalangle FBQ \equiv \sphericalangle FSQ$. (3)

The relations (1) – (3) lead to $\sphericalangle PSF \equiv \sphericalangle FSQ$. (4)

This relation shows that the points P, Q, S are collinear. Because the quadrilateral $BQSD$ is inscribed, we have that $\sphericalangle QBD \equiv \sphericalangle DSR'$. (5)

We denoted by R' the intersection of the line QS with the circle that passes through C and S (see *Figure 124*).

The quadrilateral $DSCR'$ is inscribed, hence $\sphericalangle DSR' \equiv \sphericalangle DCR'$. (6)

The relations (5) and (6) lead to $\sphericalangle QBD \equiv \sphericalangle DCR'$. (7)

On the other hand, BQ is parallel with CR, therefore:

$\sphericalangle QBD \equiv \sphericalangle DCR$. (8)

The relations (7) and (8) show that $R' = R$. Having P, Q, S collinear and Q, S, R also collinear, it follows that P, Q, R are collinear as well.

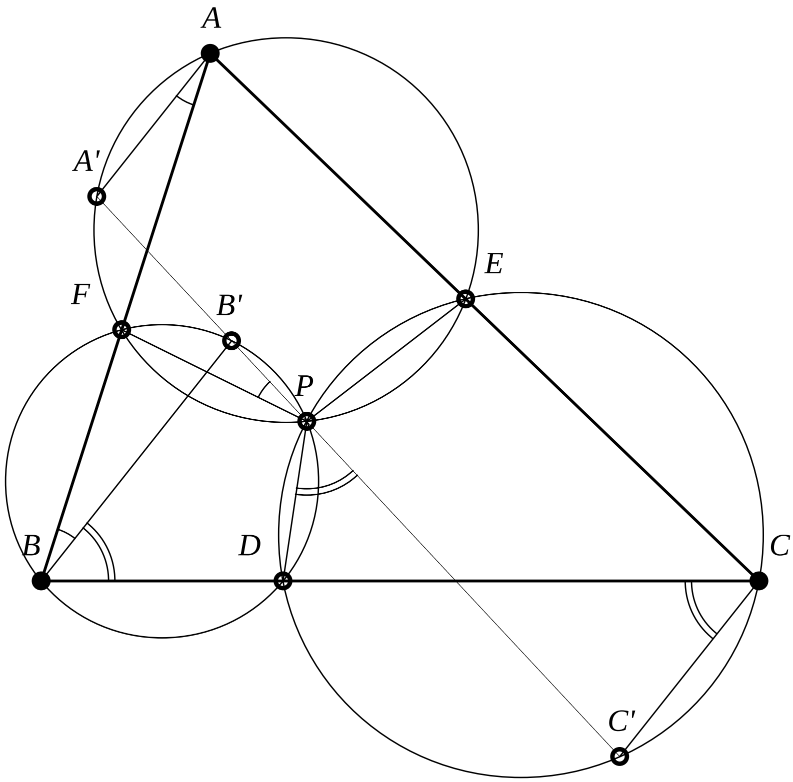

Figure 125

4. Let ABC be a scalene triangle and the points $D \in (BC)$, $E \in (AC)$, $F \in (AB)$. The circles circumscribed to the triangles AEF, BDF and CDE intersect in a point P. An arbitrary line which passes through the point P intersects the second time the circles circumscribed to the triangles AEF, BDF and CDE respectively in the points A', B' and C'. Show that the lines AA', BB', CC' are parallel.

Mihai Miculița – A reciprocal of a problem by Van Khea

Solution (Mihai Miculița)

The quadrilateral $AA'FP$ is inscribed, then $\sphericalangle FAA' \equiv \sphericalangle FPA'$. (1)

The quadrilateral $BPB'F$ is inscribed, therefore:

$\sphericalangle FPA' \equiv \sphericalangle FBB'$. (2)

From relations (1) and (2), we obtain that $\sphericalangle FAA' \equiv \sphericalangle FBB'$. This implies consequently that $AA' \parallel BB'$ (see *Figure 12*5). (3)

The quadrilateral $PB'BD$ is inscribed, therefore:

$\sphericalangle B'BD \equiv \sphericalangle DPC$. (4)

The inscribed quadrilateral $PDC'C$ leads to $\sphericalangle DPC \equiv \sphericalangle DCC'$. (5)

The relations (4) and (5) imply that $\sphericalangle B'BD \equiv \sphericalangle DCC'$.

The consequence is that $BB' \parallel CC'$. (6)

Finally, the relations (3) and (6) lead to the requested conclusion:

$AA' \parallel BB' \parallel CC'$.

9

PROBLEMS CONCERNING ORTHOLOGICAL TRIANGLES

9.1 Proposed Problems

1. The triangles ABC and $A_1B_1C_1$ are symmetrical to the line d. Prove that ABC and $A_1B_1C_1$ are orthological triangles.

2. In the triangle ABC, denote by E and F the contacts of the inscribed circle (of center I) with AC, respectively AB. Let M, N, P be the midpoints of segments BC, CE and respectively BF. Prove that the perpendiculars taken from the points I, B and C respectively to NP, PM and MN are concurrent.

Ion Pătrașcu

3. Let O_1, O_2, O_3 be respectively the centers of the circles circumscribed to the triangles MBC, MAC, MAB, where M is a certain point in the interior of the triangle ABC. Prove that the triangles ABC and $O_1O_2O_3$ are orthological, and specify the orthology centers.

4. Let ABC be a non-right triangle, H – its orthocenter and P – a point located on AH. The perpendiculars taken from H to BP and to CP intersect AC and AB in B_1, respectively C_1. Prove that the lines B_1C_1 and BC are parallel.

Ion Pătrașcu

5. If: P and P' are points in the interior of the triangle ABC; $A'B'C'$ and $A''B''C''$ are their pedal triangles; the set of the orthology centers of the triangles ABC, $A'B'C'$ and ABC with $A''B''C''$ is formed only by the points P, P'; – then show that the points P and P' are isogonal conjugate points.

6. Let ABC be a right triangle in A, and AD its altitude, $D \in (BC)$. Denote by K the midpoint of AD, and by P the projection of K on the mediator of the side BC. Let Q be the intersection of the semi-line $(AP$ with the circumscribed circle of the triangle ABC (whose center is the point O), and S the symmetric of Q to BC. Prove that the Simson line of the point S is parallel with OK.

Ion Pătrașcu

7. Let ABC be an equilateral triangle. Find the positions of the points A_1, B_1, C_1 on the sides BC, CA respectively AB, such that the lines AA_1, BB_1, CC_1 to be concurrent, and the perpendiculars raised on the sides respectively in the points A_1, B_1, C_1 also to be concurrent.

8. In the triangle ABC of orthocenter H, let A_1 be the diametral of A in the circumscribed circle, A_2 – the projection of A_1 on BC, and B_2, C_2 – analogous points. Let A_b, A_c be the intersections of the parallels taken through B and C to AH with AC, respectively AB, and B_c, B_a, C_a, C_b – analogous points. Show that the perpendiculars from A_2, B_2, C_2 respectively to A_bA_c, B_CB_Q, C_aC_b are concurrent in H.

Nicolae Mihăileanu – The Correlative of a Proposition –
Victor Thébault, *G.M.* vol. 41, 1936

9. Let AA_1, BB_1, CC_1 be the concurrent cevians in the point P in the interior of the triangle ABC and let Q be the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC. Denote $\{A'\} = B_1C_1 \cap AA_1$, $\{B'\} = C_1A_1 \cap BB_1$, $\{C'\} = A_1B_1 \cap CC_1$; denote by R the orthology center of the triangle $A'B'C'$ in relation to the triangle ABC. Prove that:

i) The points R, P, Q are collinear if and only if P is the gravity center of the triangle $A_1B_1C_1$;

ii) If P is the gravity center of the triangle $A_1B_1C_1$, then the points R, P, Q, S are collinear (S is the orthology center of the triangle ABC in relation to triangle $A'B'C'$).

Vincențiu Pașol – Ion Pătrașcu

10. Let ABC be a scalene triangle, H – its orthocenter and P – an arbitrary point on AH. Denote by B' and C' the midpoints of the sides AC

and AB, and by Q the point of intersection of the perpendicular taken from B' to the line CP with the perpendicular taken from C' to the line BP. Show that the point Q is found on the mediator of the side BC.

Cities Tour – Russia, 2010

11. Let ABC be a right triangle. Build the rectangle $BCDE$ on the hypotenuse BC, in triangle's exterior. Denote by I the intersection of the perpendicular taken from D to AB with the perpendicular taken from E to AC. Prove that the triangles ABC and IDE are orthological.

Ion Pătrașcu

12. Let ABC be an acute triangle, O – the center of its circumscribed circle and A_1, B_1, C_1 – the symmetrics of O with respect to the sides BC, CA respectively AB.

a) Prove that the triangle ABC and $A_1B_1C_1$ are bilogical;
b) Prove that the homology center of the triangles ABC and $A_1B_1C_1$ belongs to the Euler line of the triangle ABC;
c) If O_1 is the homology center of triangles ABC and $A_1B_1C_1$, calculate $\frac{O_1H}{O_1O}$, where H is the orthocenter of the triangle ABC.

13. Let ABC be a right triangle in A and O – the center of its circumscribed circle. Build the points B' respectively C', such that $BB' = CC' = BO$, on the semi-lines $(BA$ and $(CA$. On the semi-line $(AA_1$, where A_1 is the foot of the altitude from A, build A' such that $AA' = AO$. Prove that the perpendiculars taken from A, B, C, respectively $B'C'$, $C'A'$ and $A'B'$ are concurrent.

14. In a certain triangle ABC, let B_1C_1 be parallel with BC, with $B_1 \in (AB)$, $C_1 \in (AC)$, and A_1 – the orthogonal projection of A on BC. Prove that the perpendicular taken from B to A_1C_1, the perpendicular taken from C to A_1B_1, and AA_1 are concurrent.

Ion Pătrașcu

15. Let ABC be an equilateral triangle and M – a point in its plane. Denote by A', B', C' the symmetrics of M with respect to BC, CA,

respectively AB, and note that $AA' = BB' = CC'$. Prove that the triangles ABC, $A'B'C'$ are orthological and have a common orthology center.

16. The triangles ABC and $A_1B_1C_1$ are orthological, and the orthology centers are O respectively O_1. Let $A'_1B'_1C'_1$ be the translation of a triangle $A_1B_1C_1$ by vector translation $\overrightarrow{O_1O}$. Prove that the triangles ABC and $A'_1B'_1C'_1$ are reciprocal polar to a circle of center O.

17. Let ABC be an acute triangle, H – its orthocenter and $A'B'C'$ – its orthic triangle. Denote: $\{P\} = B'C' \cap BC, \{Q\} = A'B' \cap AB, \{R\} = A'C' \cap AC$, $\{U\} = AP \cap CQ$, $\{V\} = BR \cap CQ$, $\{W\} = AP \cap BR$. Prove that the median triangle $M_aM_bM_c$ of the triangle ABC and the triangle UVW are orthological.

Ion Pătrașcu

18. Let ABC be an acute triangle and let $A_1B_1C_1$ be its orthic triangle. Denote by A_2, B_2 respectively C_2 the projections of vertices A, B, C respectively on B_1C_1, C_1A_1 and A_1B_1. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are bilogical triangles.

19. Let ABC be a scalene triangle. Denote by B_1 and C_1 the intersections of a circle taken through the points B and C with the sides (AC) respectively (AB). Denote by M_a, M_b, M_c the midpoints of the segments B_1C_1, B_1C respectively BC_1. Prove that the triangles $M_aM_bM_c$ and ABC are orthological.

20. Let $A_1B_1C_1$ and $A_2B_2C_2$ be two homological triangles located in different planes, and let O be their center of homology. Denote by A_0, B_0, C_0 the projections of the points A_1, B_1, C_1 on the plane $(A_2B_2C_2)$. Prove that, if the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthological, then the triangles $A_0B_0C_0$ and $A_2B_2C_2$ are bilogical triangles.

Ion Pătrașcu

21. Let ABC be a triangle and $A' \in (BC)$, $B' \in (AC)$, $C' \in (AB)$.

Prove that:

a) The perpendiculars from A', B', C' respectively to BC, CA, AB are concurrent if and only if:
$BC \cdot BA' + CA \cdot CB' + AB \cdot AC' = \frac{1}{2}(AB^2 + BC^2 + CA^2)$.

b) In the case of *a)*, the following inequality takes place:
${BA'}^2 + {CB'}^2 + {AC'}^2 \geq \frac{1}{4}(AB^2 + BC^2 + CA^2)$.

a) If the points A', B', C' are mobile and the hypothesis from *a)* is true, the sum ${BA'}^2 + {CB'}^2 + {AC'}^2$ is minimal if and only if the concurrency point of the three perpendiculars is the center of the circle circumscribed to the triangle ABC.

Ovidiu Pop, professor, Satu Mare
(The annual contest of *G.M.* resolvers, 1990)

22. Let the triangle ABC and the points A_1, $A_2 \in (BC)$, B_1, $B_2 \in (CA)$, C_1, $C_2 \in (AB)$ be such that: $BA_1 = CA_2$; $CB_1 = AB_2$; $AC_1 = BC_2$. Prove that the orthology center of the triangle ABC in relation to the triangle determined by the intersections of the lines of the centers of the circles circumscribed to the triangles: ABA_1 and ACA_2; BCB_1 and BAB_2; CAC_1 and CBC_2 – is the gravity center of the triangle ABC.

Ion Pătrașcu

23. Let $ABCD$ be a convex quadrilateral and A_1, B_1, C_1 the orthocenters of the triangles BCD; ACD and ABD. Prove that the perpendiculars taken from A, B and C respectively to A_1C_1, C_1A_1 and A_1B_1 are concurrent.

24. Show that, if ABC and $A_1B_1C_1$ are two orthological equilateral triangles, and the triangle $A_1B_1C_1$ is inscribed in the triangle ABC $\left(A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)\right)$, then $A_1B_1C_1$ is the median triangle of the triangle ABC.

Ion Pătrașcu

25. Let $A'B'C'$ be the pedal triangle of the center of the inscribed circle I in the triangle ABC. Prove that the triangles ABC and $A'B'C'$ are orthological if and only if the triangle ABC is isosceles.

Reformulation of the problem O: 695, *G.M.* nr. 710-11-12/1992.
Autor: M. Bârsan, Iași

26. Let ABC be an equilateral triangle of side a, and M – a point in its interior. Denote by A_1, B_1, C_1 the projections of M on the sides. Show that:

a) $A_1B + B_1C + C_1A = \frac{3}{2}a$.

b) The lines AA_1, BB_1, CC_1 are concurrent if and only if M is located on one of the altitudes of the triangles.

Laurențiu Panaitopol, County Olympics 1990.

27. In the triangle ABC, denote by A_1, B_1, C_1 the feet of symmedians from A, B, C. Prove that the perpendiculars in the points A_1, B_1, C_1 respectively to the lines BC, CA, AB are concurrent if and only if the triangle is isosceles.

F. Enescu, student, Bucharest – Problem C. 1125, *G.M.* 5/1991.
The annual contest of the resolvers, grades 7-8

28. Let ABC be an equilateral triangle and $A_1B_1C_1$ – a triangle inscribed in ABC, such that $\overrightarrow{AA_1} \cdot \overrightarrow{BC} + \overrightarrow{BB_1} \cdot \overrightarrow{CA} + \overrightarrow{CC_1} \cdot \overrightarrow{AB} = 0$. Show that:

a) The triangles ABC and $A_1B_1C_1$ are orthological.

b) If P is the orthology center of the triangle $A_1B_1C_1$ in relation to ABC and O is the center of the circle circumscribed to ABC, then: $\overrightarrow{PA_1} + \overrightarrow{PB_1} + \overrightarrow{PC_1} = \frac{3}{2}\overrightarrow{PO}$.

29. Let ABC be a given equilateral triangle and M – a point in its interior. Show that there is an infinity of equilateral triangles $A'B'C'$ – orthological with ABC, and having the point M as orthology center.

Ion Pătrașcu

30. Two triangles ABC, $A'B'C'$ are homological (the lines AA', BB', CC' meet in a point I). The perpendiculars in A to AB, AC meet $A'B'$, $A'C'$ respectively in A_c, A_b. Similarly, the perpendiculars taken in B and C on the lines (BA, BC), (CB, CA) determine on $(B'A', B'C')$, $(C'B', C'A')$ the points B_c, B_a, C_a, C_b. Show that the perpendiculars descending from A, B, C on the lines A_bA_c, B_cB_a, C_aC_b are concurrent.

Gh. Țițeica

31. Let $A_1A_2A_3$ be a right triangle in A_1 and D the foot of the perpendicular from A_1. Denote by K the midpoint of A_1D, and – $\{N\} = A_2K \cap A_1A_3$, $\{M\} = A_3K \cap A_1A_2$. Also, $\{B_1\} = MN \cap A_2A_3$ and B_2, B_3 – the projections of B_1 on A_1A_3 respectively A_1A_2. Prove that $A_1A_2A_3$ and $B_1B_2B_3$ are triorthological.

Ion Pătrașcu

32. Denote by A', B', C' the orthogonal projections of the point P from the interior of triangle ABC on the sides BC, CA respectively AB. The circumscribed circle of the triangle $A'B'C'$ intersect the second time the sides BC, CA, AB in the points A_1, B_1, respectively C_1. Prove that the perpendiculars taken from A, B and C respectively to B_1C_1, C_1A_1 and A_1B_1 are concurrent.

33. Take the triangle ABC; the equilateral triangles ABC_1, BCA_1, CAB_1 are built on its sides, in the exterior. Let α, β, γ be the midpoints of the segments B_1C_1; C_1A_1; A_1B_1. Show that the perpendiculars taken from α, β, γ on the sides BC, CA respectively AB are concurrent.

Dan Voiculescu

34. Let ABC be a scalene triangle inscribed in a circle of center O. The parallels taken through A, B, C with BC, CA respectively AB intersect the circle the second time in the points A', B', C'. Prove that the perpendiculars taken from A', B', C' to the sides BC, CA, AB are concurrent.

35. Let ABC be a scalene triangle and A_1, B_1, C_1 be the feet of its altitudes. Denote by A_2, B_2, C_2 the feet of the altitudes of the triangle $A_1B_1C_1$. Show that the circles circumscribed to the triangles AA_1A_2, AB_1B_2, AC_1C_2 still have a common point.

Ion Pătrașcu

36. Let ABC be a triangle and $M \in (AC)$, $N \in (AB)$, $P \in (BC)$ such that $MN \perp AC$, $NP \perp AB$ and $MP \perp BC$. Show that, if the Lemoine point of the triangle ABC coincides with the gravity center of the triangle MNP, then the triangle ABC is equilateral.

Ciprian Manolescu, Problem O: 830, *G.M.* no. 10/1996.

37. Let $ABCD$ be a rectangle of center O. Denote by E and F the intersections of mediator of the diagonal BD with AB and BC. Let M, N, P be respectively the midpoints of the sides AB, AD, DC and let L – the intersection with AB of the perpendicular taken from D to PF. Prove that the triangles DLB and NEM are orthological.

Ion Pătrașcu

38. Let $ABCD$ be a quadrilateral inscribed in the circle of diameter AC. It is known that there exist the point E on (CD) and the point F on (BC) such that the line AE is perpendicular to DF and the line AF is perpendicular to BE. Show that $AB = AD$.

Problem no. 4, National Mathematical Olympiad, grade 9, 2014

39. Let ABC be a right isosceles triangle, $AB = AC$. Denote by M the midpoint of AB. The point Q is defined by $4 \cdot \overrightarrow{AQ} = \overrightarrow{AC}$, $R \in BC$, such that $\overrightarrow{QR}$ is collinear with $\overrightarrow{AB}$, and P is the midpoint of CM. Prove that the triangles PRQ and ABC are orthological and specify the orthology centers.

Ion Pătrașcu

40. Let AA_1, BB_1, CC_1 be concurrent cevians in triangle ABC, $A_1 \in (BC)$, $B_1 \in (AC)$, $C_1 \in (AB)$, such that AA_1 is median and the triangles ABC and $A_1B_1C_1$ are orthological. Prove that the triangle ABC is isosceles or that BB_1 and CC_1 are medians.

Florentin Smarandache

41. Let ABC be an acute triangle; prove that there exists a triangle $A_1B_1C_1$ inscribed in ABC, with $C_1 \in (AB)$, $B_1 \in (AC)$, $A_1 \in (BC)$, such that $A_1B_1 \perp BC$, $C_1B_1 \perp AC$, $A_1C_1 \perp AB$. Denote by O_1, O_2, O_3 respectively the midpoints of segments BA_1, CB_1, AC_1. Prove that the triangles $B_1C_1A_1$ and $O_1O_2O_3$ are orthological.

Ion Pătrașcu

42. Let $\mathcal{C}(O_1)$, $\mathcal{C}(O_2)$, $\mathcal{C}(O_3)$ be three circles having their centers in noncollinear points, exterior two by two. Denote by A the point located on O_2O_3 which has equal powers over the circles $\mathcal{C}(O_2)$ and $\mathcal{C}(O_3)$; by B the point located on O_3O_1 which has equal powers over the circles $\mathcal{C}(O_3)$ and

$\mathcal{C}(O_1)$; and by C the point belonging to the line O_1O_2 which has equal powers over the circles $\mathcal{C}(O_1)$ and $\mathcal{C}(O_2)$. Show that the triangles ABC and $O_1O_2O_3$ are orthological. Specify the orthology centers.

43. Let ABC be a scalene triangle and $C_aC_bC_c$ – its contact triangle. Denote by H_a, H_b, H_c respectively the orthocenters of triangles AC_bC_c, BC_aC_c, CC_aC_b. Prove that the triangles $H_aH_bH_c$ and $C_aC_bC_c$ are orthological. Specify their orthology centers.

Ion Pătrașcu

44. Show that the triangle determined by the points of tangent with the sides of the triangle ABC of its A-ex-inscribed circle, and the triangle ABC are orthological, and they have the same orthology center.

45. The triangles ABC and $A_1B_1C_1$ are orthological; $A_1B_1C_1$ is inscribed in ABC, $A_1 \in (BC)$, $B_1 \in (CA)$, $C_1 \in (AB)$, $B_1C_1 \parallel BC$, $B_1A_1 \parallel AB$. Prove that $A_1B_1C_1$ is the complementary (median) triangle of the triangle ABC.

46. Let ABC be a non-right triangle, and O the center of its circumscribed circle. The mediators of segments AO, BO and CO determine the triangle $A_1B_1C_1$ (B_1, C_1 belong to the mediator of the segment AO). Prove that the triangles ABC and $A_1B_1C_1$ are orthological, and have a common orthology center.

47. Let AA', BB', CC' be three cevians, concurrent in the point P inside the triangle ABC. Build the mediators of segments AP, BP, CP, and denote $A_1B_1C_1$ the triangle determined by them (B_1 and C_1 belong to mediator of the segment AP). Show that $A_1B_1C_1$ is orthological in relation to the triangle ABC, and specify their orthology centers.

48. The triangles ABC and $A_1B_1C_1$ are orthological of center M (M is the orthology center of the triangle $A_1B_1C_1$ in relation to ABC). Let $A'B'C'$ be the contact triangle of the triangle ABC. Denote by A_1', B_1', C_1' respectively the intersection points with the circle inscribed of the perpendiculars taken from A, B, C to B_1C_1, C_1A_1 respectively A_1B_1. Denote by X the intersection between the tangent taken in A_1' to the inscribed circle

with the parallel taken through I (the center of the inscribed circle) with B_1C_1; let Y be the intersection between the tangent taken through B_1' to the inscribed circle with the parallel taken through I with C_1A_1; and let Z be the intersection of the tangent taken through C_1' to the inscribed circle with the parallel taken through I with A_1B_1. Prove that the points X, Y, Z are collinear.

Ion Pătrașcu

49. Let ABC be a scalene triangle, and M – the midpoint of the side BC. The perpendiculars taken from M to AB and AC intersect respectively in P and Q the perpendiculars raised in B' and C' to BC. The points B' and C' are symmetric with respect to M, and are located on BC. Denote: $\{R\} = AB \cap PQ$ and $\{S\} = AC \cap PQ$. Prove that the triangles ARS and AQP are orthological.

50. Let $ABCD$ be a right trapeze, $\hat{A} = \hat{B} = 90^0$. Consider the point E on the side CD and let M – the midpoint of the side AB. Build $MP \perp AE$, $P \in AD$ and $MQ \perp BE$, $Q \in BC$. Denote by R and S the intersections of the line PQ with AE respectively BE. Prove that the triangle ESR and EPQ are orthological.

Ion Pătrașcu

51. Let ABC be a triangle, U – a point in its interior, and V – the isogonal conjugate of U. If the U-circumpedal triangle of the triangle ABC and the triangle ABC are orthological, then the V-circumpedal triangle of the triangle ABC and the triangle ABC are also orthological.

52. Let ABC be a scalene triangle. The following squares are built in the triangle's exterior, on its sides: $BCMN$, $ACPQ$ and $ABRS$. Denote: $\{A_1\} = MP \cap NR$, $\{B_1\} = MP \cap SQ$ and $\{C_1\} = NR \cap SQ$. Prove that the triangles ABC and $A_1B_1C_1$ are orthological. What important point in the triangle ABC is its orthology center in relation to the triangle $A_1B_1C_1$?

53. If the segments AA', BB', CC' that join the vertices of the orthological triangles ABC and $A'B'C'$ are divided by the points A'', B'', C''

and A''', B''', C''' in proportional segments, then the triangles $A''B''C''$ and $A'''B'''C'''$ are also orthological.

J. Neuberg

54. Let H be the orthocenter of an acute triangle ABC. Consider the points A', B', C', with $HA' \equiv BC$, $HB' \equiv CA$ and $HC' \equiv AB$, on the semi-lines $(HA$, $(HB$, $(HC$. Prove that:

a) The orthology center of the triangle ABC in relation to the triangle $A'B'C'$ is the gravity center G of the triangle ABC.
b) If $\{A''\} = BC \cap B'C'$, $\{B''\} = CA \cap C'A'$, then $HG \perp A''B''$.

55. Build the squares $BCMN$, $ACPQ$ and $ABRS$ in the exterior of the triangle ABC. Let C_1 – the intersection of lines SQ and RN, B_1 – the intersection of lines SQ and MP, and A_1 – the intersection of lines MP and RN. Prove that the perpendiculars from B_1, A_1 and C_1 to the lines AC, BC respectively AB are concurrent.

Petru Braica, Satu Mare, Problem 27089, *G.M.* nr. 1/2015

56. Let ABC be a scalene triangle. Build with A, B, C as centers three congruent circles that cut the sides AB and AC in A', A''; the sides BA and BC in B', B''; the sides CB, CA in C', C''. Denote: $B'B'' \cap C'C'' = \{A_1\}$, $A'A'' \cap C'C'' = \{B_1\}$ and $A'A'' \cap B'B'' = \{C_1\}$. Prove that the triangles ABC and $A_1B_1C_1$ are orthological, and specify their orthology centers.

57. Let $ABCDEF$ be a convex hexagon with: $AB = BC$, $CD = DE$, $EF = FA$. Show that the perpendiculars taken from A, C and E respectively to the lines FB, BD and DF are concurrent.

58. Let A', B' and C' be the feet of altitudes of a triangle ABC. Consider the points $M \in AA'$, $N \in BB'$, $P \in CC'$ and denote $MN \cap AB = \{K\}$, $MP \cap AC = \{L\}$. If the points N and P are fixed, and M mobile, it is required to:

a) Prove that ML rotates around a fixed point.
b) Find the geometric place of the intersections of the perpendiculars descending from B and C respectively to MP and MN.

V. Sergiescu, student, Bucharest, Problem 8794, *G.M.* nr. 1/1969

59. Let ABC be a scalene triangle, and AM – its cevian. Denote by A', B' and C' the projections of vertex A on BC and the projections of B and C on AM. Prove that the triangle ABC is orthological in relation to the triangle $A'B'C'$.

Ion Pătrașcu

60. Let ABC be an acute triangle, H – its orthocenter and P – a point on AH. The perpendiculars taken from H to BP and to CP intersect AC and AB in B, respectively C. Prove that the lines B_1C_1 and BC are parallel.

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61. Let ABC be a given triangle and $A_1B_1C_1$ – the triangle formed by the intersections of parallels taken through A, B, C to the interior bisectors of the triangle ABC (the parallel taken through A to the bisector BB' intersects with the parallel taken from B to the bisector CC' in C_1, …). Prove that the triangle $A_1B_1C_1$ is orthological in relation to Fuhrmann triangle of the triangle ABC.

Ion Pătrașcu

62. Let K be the midpoint of the side AB of the triangle ABC, and $L \in (AC)$, $M \in (BC)$ – two points such that $\sphericalangle CLK \equiv \sphericalangle CMK$. Show that the perpendiculars raised in the points K, L and M respectively to AB, AC and BC are concurrent in a point P.

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63. Let ABC be a given triangle and $A_1B_1C_1$ – the podal triangle of center I_a of the A-ex-inscribed circle to the triangle. Denote by Γ_a the intersections of cevians AA_1, BB_1, CC_1, and – $\{X\} = B_1C_1 \cap BC$, $\{Y\} = A_1C_1 \cap AC$. Prove that $I_a\Gamma_a \perp XY$.

64. Let ABC be an acute triangle inscribed in the circle of center O. Denote by A_1, B_1, C_1 respectively the midpoints of the circle's high arcs, supported by the chords BC, CA and AB. Prove that the triangles ABC and $A_1B_1C_1$ are orthological.

65. In a certain triangle ABC, let A_1, B_1,C_1 be the projections of centers of ex-inscribed circles, I_a, I_b, I_c respectively on mediators of the sides BC, CA and AB. Prove that the triangles ABC and $A_1B_1C_1$ are bilogical triangles.

66. Let ABC be a scalene triangle and K – the midpoint of the side AB. Build congruent circles which pass through A and K, and through B and K, and which have the centers on the same section of the line AB as the point C. These circles cut the second time the sides AC and BC in L respectively M. Prove that the triangles MLK and ABC are orthological. Determine the geometric place of the orthology center P of these triangles.

Ion Pătrașcu, Mihai Miculița

67. Let $ABCD$ be a rhombus of center O. Denote by E the projection of O on AD and by F the symmetric of O with respect to the midpoint of the segment AD. The perpendicular taken from F to AD intersects the perpendicular taken from D to EB in H. Prove that $AH \perp CE$.

Ion Pătrașcu, Problem S: L.17.299 – *G.M.* nr. 11/2017

68. Let $ABCD$ be a rectangle with $AB = 2$ and $BC = \sqrt{3}$. Denote by M the midpoint of the side AB, by P – the midpoint of the segment DM, and by S – the symmetric of P with respect to AB. Denote the intersection of the lines AC and DS by T. V be the intersection of perpendicular from D to DM with the parallel taken through P with AB. Let Q be the intersection of bisector of the angle AMD with the perpendicular taken from D to CV. Prove that the points P, Q, T are collinear.

Ion Pătrașcu

69. Let $A_1B_1C_1$ and ABC be two orthological triangles of center P. Denote by A_2, B_2 and C_2 the symmetrics of P with respect to the midpoints of the sides of the triangle $A_1B_1C_1$. Prove that the triangle $A_1B_1C_1$ is orthological with the triangle $A_2B_2C_2$.

70. The following are required:

a) Find the condition which an acute triangle must satisfy in order that the orthic triangle of its orthic triangle does not exist;

b) Find a triangle such that it does not exist an orthic triangle of the orthic triangle of the orthic triangle of the given triangle;
c) Let ABC be a triangle, $A'B'C'$ – its othic triangle, and $A'B'C'$ – the orthic triangle of the triangle $A''B''C''$. What can you say about the relation of orthology relative to the triangles ABC and $A''B''C''$?

71. Let ABC be an acute triangle. Denote by D the projection of B on AC, and by E – the projection of C on AB, and by K, L, M respectively the midpoints of the segments BE, CD and DE. Prove that:
a) The triangles MKL and ABC are orthological;
b) The axis of orthology is perpendicular to KL.

72. Let ABC be a scalene triangle and $A'B'C'$ – its I-circumpedal triangle (I – the center of the circle inscribed in the triangle ABC). Prove that:
a) The triangle ABC is orthological in relation to the triangle $A'B'C'$;
b) The circles $\mathcal{C}(A';\ A'B)$, $\mathcal{C}(B';\ B'C)$, $\mathcal{C}(C';\ C'A)$ intersect in the orthology center of the triangle from the point *a)*;
c) Let – $\{X\} = B'C' \cap BC$, $\{Y\} = A'B' \cap AB$, and O – the center of the circle circumscribed to the triangle ABC; then: $OI \perp XY$.

73. Let ABC be an acute triangle with $AB < AC$, and of orthocenter H. The mediator of the side BC intersects the sides BC, CA and AB respectively on the points M, Q and P. Denote by N the midpoint of the segment PQ. Prove that the triangles BHM and QAN are orthogonal.

74. The circle ω intersect the sides (BC), (CA) and (AB) of the triangle ABC in A_1A_2; B_1B_2, C_1, C_2. Prove that, if the triangles $A_1B_1C_1$ and ABC are orthological, then the triangles $A_2B_2C_2$ and ABC are also orthological.

Reformulation of a problem proposed at the Hungarian Competition, 1914.

75. Let ABC and $A_1B_1C_1$ be two triangles located in distinct planes such that the perpendiculars taken from A, B, C respectively to B_1C_1, C_1A_1 and A_1B_1 are concurrent in a point H. Prove that the triangle $A'B'C'$ (the projection of ABC on the plane $A_1B_1C_1$) and the triangle $A_1B_1C_1$ are orthological.

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76. Let ABC be an isosceles triangle, $AB = AC$; H is its orthocenter, and $A_1B_1C_1$ is its orthic triangle. Show that the triangle HBC is orthological in relation to $A_1B_1C_1$.

77. Let O be the center of the circle circumscribed to a non-isosceles triangle ABC. The circumscribed circle of the triangle OBC intersects the second time the lines AB and AC in the points A_c respectively A_b; the circumscribed circle of the triangle OAC intersects the second time the lines AB and BC in the points B_c respectively B_a; and the circumscribed circle of the triangle OAB intersects the second time the lines BC and AC in the points C_a respectively C_b. Show that A_bB_a, A_cC_a and B_cC_b are three concurent lines.

National Mathematics Olympiad, Brazil, 2009

78. Let A_1, B_1, C_1 be the feet of altitudes of the acute triangle ABC, and $X \in (B_1C_1)$, $Y \in (C_1A_1)$, $Z \in (A_1B_1)$, such that:

$\frac{C_1X}{XB_1} = \frac{b\cos C}{c\cos B}$, $\frac{A_1Y}{YC_1} = \frac{c\cos A}{a\cos C}$ and $\frac{B_1Z}{ZA_1} = \frac{a\cos B}{b\cos A}$.

Show that the lines AX, BY and CZ are concurrent.

Petru Braica, Problem 27309, *G.M.* nr. 12/2016

79. Let ABC be a given triangle, and $A_1B_1C_1$ – a triangle inscribed in ABC and orthologic with it. Denote by O the orthology center of the triangle $A_1B_1C_1$ in relation to ABC. Consider the point P on the perpendicular raised in O in the plane ABC; and the points A_2, B_2, C_2 on the segments PA_1, PB_1, PC_1. Prove that the triangles $A_2B_2C_2$ and ABC are orthological with a single orthology center.

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80. Let E and F be the feet of altitudes from the vertices B and C of the acute triangle ABC, and M – the midpoint of the side BC. Denote: $\{N\} = AM \cap EF$, $P = Pr_{BC}^{(N)}$, $R = Pr_{AC}^{(P)}$, $S = Pr_{AB}^{(P)}$. Show that N is the orthocenter of the triangle ARS.

Nguyễn Minh Hả

81. On the sides of the triangle ABC, consider the points $M \in (BC)$, $N \in (CA)$ and $P \in (AB)$, such that:

$$\frac{MB}{MC} = \frac{NC}{NA} = \frac{PA}{PB} = k.$$

Let a be the perpendicular from M to BC. Define similarly the lines b and c. Then: a, b, c are concurrent if and only if $k = 1$.

M. Monea, Problem 4, The National Mathematical Olympiad, local stage, 2003

82. Let (T_a), (T_b), (T_c) be tangents in the vertices A, B, C of the triangle ABC to the circumscribed circle of the triangle. Prove that the perpendiculars taken from the midpoints of the sides opposed to (T_a), (T_b), (T_c) are concurrent, and determine their concurrency point.

83. An equilateral triangle ABC is given, and D – an arbitrary point in its plane. Denote by A_1, B_1 and C_1 the centers of the circles inscribed in the triangles BCD, CAD and ABD. Prove that the perpendiculars taken from the vertices A, B, C respectively on the sides B_1C_1, C_1A_1 and A_1B_1 are concurrent.

I. Shariguin, Collection of problems, Problem II.17

84. Let d be a given line and d_1, d_2, d_3 – three lines perpendicular to d. Consider A, B, C – points on d, such that:

$d(A, d_2) = a_1$, $d(A, d_3) = a_2$,

$d(B, d_3) = b_1$, $d(B, d_1) = b_2$,

$d(C, d_1) = c_1$, $d(C, d_2) = c_2$.

Find the condition that needs to be met by a_1, a_2, b_1, b_2, c_1, c_2 such that whatever the points A_1, B_1, C_1 on d_1, d_2 respectively d_3, the perpendiculars in A, B, C to B_1C_1, C_1A_1, A_1B_1 to be concurrent.

85. Let $M_aM_bM_c$ be the median triangle of the triangle ABC. If M is a point in the plane of the triangle ABC, and $A_1B_1C_1$ is the triangle formed by the orthogonal projections of the point M on the sides BC, CA respectively AB, then show that the triangles $M_aM_bM_c$ and $A_1B_1C_1$ are orthological triangles.

86. Let ABC and $A_1B_1C_1$ be orthological triangles, and O a certain point in their plane. Denote by $A'B'C'$ the symmetrical triangle with respect to O

of the triangle ABC. Show that the triangles $A'B'C'$ and $A_1B_1C_1$ are orthological.

87. Let $ABCD$ be an orthodiagonal trapezoid of bases BC and AD; denote by O the intersection of diagonals, by E – the projection of O on AD, and by F – the symmetric of O with respect to the midpoint of the segment AD. The perpendicular from F to AD intersects with the perpendicular taken from D to EB in H. Prove that $AH \perp CE$.

Ion Pătrașcu, Problem 27447, *G.M.* no. 11/2017

88. Show that the orthology center of a triangle ABC in relation to the podal triangle of symmedian center is the gravity center of the triangle ABC, and the orthology center of the podal triangle of the symmedian center in relation to the triangle ABC is the gravity center of the podal triangle of the symmedian center.

89. Show that:

a) Two equilateral triangles ABC and $A'B'C'$, inversely oriented, are three times parallelogic, namely with the orders: $\binom{ABC}{A'B'C'}$; $\binom{ABC}{B'C'A'}$; $\binom{ABC}{C'A'B'}$.

b) Denoting by P_1, P_2, P_3 the points of parallelogy corresponding to the terns above, then the triangle $P_1P_2P_3$ is equilateral, with the vertices on the circumscribed circle of the triangle ABC.

Constantin Cocea

90. Draw the triangle ABC, and $A_1B_1C_1$ – its orthic triangle. A_2, B_2, C_2 are the projections of vertices of the triangle ABC respectively on B_1C_1, C_1A_1 and A_1B_1. Prove that the triangles $A_2B_2C_2$ and ABC are orthological.

I. Shariguin, Collection of problems

91. On the sides of the acute triangle ABC, the equilateral triangles BCK, CAL, ABM are being built on the exterior. Show that the median triangle of the triangle KLM and the triangle ABC are orthological.

92. Let ABC be a scalene triangle, and let $A_1B_1C_1$ be the podal triangle of the symmedian center K of the triangle ABC. Denote by A_2, B_2, C_2 the

symmetrics of the points A_1, B_1, C_1 in relation to K. Prove that the triangles $A_2B_2C_2$ and $A_1B_1C_1$ are orthological.

93. Let $BCDE$ be a convex quadrilateral inscribed in the circle of center O, where BE is not parallel with DC. Denote by Q and R the midpoints of the sides CD and BE, and by A the intersection of the lines BE and DC. Prove that the perpendiculars taken from A, B, C respectively to RQ, CE and BD are concurrent.

Ion Pătrașcu

94. Let $ABCDEF$ be a regular hexagon. Show that the triangles BFD and ECA are triorthological. Specify the orthology centers.

Ion Pătrașcu

95. Let ABC be a right triangle in B with $m(\hat{A}) = 60^0$ and $BC = \sqrt{7}$. Draw parallels with BC, AB and CA located at distances $\frac{\sqrt{21}}{7}$, $\frac{2\sqrt{7}}{7}$ and respectively $\frac{\sqrt{7}}{7}$ which intersects the interior of the triangle.

Prove that:

a) The parallels are concurrent in a point M;
b) The podal triangle of the point M, denoted by $A_1B_1C_1$, is equilateral;
c) The triangle ABC and $A_1B_1C_1$ are not bilogical.

Ion Pătrașcu

96. Let ABC be an acute triangle, O – the center of its circumscribed circle, and M, D – the intersections of the semi-line $(AO$ with BC respectively with the circumscribed circle. The tangent in D to the circumscribed circle intersects AB in K and AC in L. The circles circumscribed to triangles DMC and DMB intersect the second time AC and AB respectively in F and E. Prove that the triangle DEF is orthological with the triangle AKL, and the orthology center is the symmetric of the point D to M.

97. Let ABC be an acute triangle, $A_1B_1C_1$ – its orthic triangle, and MNP – its median triangle. Denote by A_2, B_2, C_2 the midpoints of medians

$(AM), (BN), (CP)$. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthological.

Ion Pătrașcu

98. Let ABC and $A_1B_1C_1$ be two orthological triangles. Denote by P – the orthology center of the triangle ABC in relation to the triangle $A_1B_1C_1$; and by P_1 – the orthology center of the triangle $A_1B_1C_1$ in relation to ABC. Then, the barycentric coordinates of P in relation to ABC are equal with the barycentric coordinates of P_1 in relation to $A_1B_1C_1$.

99. Let ABC a triangle inscribed in the circle of center O. The bisectors AD, BE, CF are concurrent in I. The perpendiculars taken from I to BC, CA and AD intersect EF, ED and DE respectively in M, N, P. Show that AM, BN, CP are concurrent in a point situated on OI.

Nguyễn Minh Hả

100. Let ABC be a triangle and P be a point in its interior. Denote by D, E, F the feet of perpendiculars taken from P to BC, CA, respectively AB. Suppose that:

$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PE^2$.

Denote by I_a, I_b, I_c the centers of the ex-inscribed circles to the triangle ABC. Show that P is the center of the circle circumscribed to the triangle $I_aI_bI_c$.

Problem G3 – Short-listed – 44th International Mathematical Olimpiad, Tokyo, Japan, 2003

9.2 Open Problems

1. The pedal triangle of the isogonal of Gergonne point in the triangle ABC is orthological with the triangle ABC if and only if the triangle ABC is an isosceles triangle.

2. The pedal triangle of the isogonal of Nagel point in the triangle ABC is orthological with the triangle ABC if and only if ABC is an isosceles triangle.

3. If $A'B'C'$ is the M-pedal triangle of the point M from the interior of triangle ABC, and the triangles ABC and $A'B'C'$ are orthological, and $A''B''C''$ is the M'-pedal triangle of the isogonal M' of the point M in relation to the triangle ABC, the triangles ABC and $A''B''C''$ are orthological.

4. Let $A_1B_1C_1$ be the G-circumpedal triangle of the triangle ABC (G – the gravity center in the triangle ABC). Is it true that ABC and $A_1B_1C_1$ are orthological triangles if and only if ABC is an equilateral triangle?

5. Let ABC be an isosceles triangle, with $AB = AC$, and G – its gravity center. If U and V are the orthology centers of the orthological and G - circumpedal triangles ABC, and these points are symmetrical with respect to G, find the measures of angles of the triangle ABC.

6. The podal triangle of the orthocenter H of the triangle ABC is orthological with the H-circumpedal triangle. What conditions must the point M from the interior of triangle ABC meet in order that its podal and its M-circumpedal triangles to be orthological triangles?

7. Let ABC and $A_1B_1C_1$ be two equilateral inversely similar triangles. Denote by O_1, O_2, O_3 the orthology centers of triangle ABC in relation to the triangles $A_1B_1C_1$, $B_1C_1A_1$ and $C_1A_1B_1$. Let O_1', O_2', O_3' be the orthology centers of the triangle $A_1B_1C_1$ in relation to the triangles ABC, BCA and CAB. It is known that $O_1O_2O_3$ and $O_1'O_2'O_3'$ are equilateral, inversely similar and triorthological triangles. If one continues for these triangles the construction made for ABC and $A_1B_1C_1$, and then for those determined by their orthology centers, etc., will the process continue endlessly or stop?

8. Let $A'B'C'$ be the pedal triangle of the center of the circumscribed circle O of the acute triangle ABC. Prove that the triangles ABC and $A'B'C'$ are orthological if and only if ABC is an isosceles triangle.

Ion Pătraşcu, Mihai Dinu

10

SOLUTIONS, INDICATIONS, ANSWERS TO THE PROPOSED ORTHOLOGY PROBLEMS

1. *Solution 1*. The triangles ABC and $A_1B_1C_1$ are orthological if and only if $AB_1^2 + BC_1^2 + CA_1^2 = AC_1^2 + BA_1^2 + CB_1^2$. Being symmetrical with respect to the line d, we have that $AB_1 = BA_1$, $BC_1 = CB_1$ and $CA_1 = AC_1$, hence the above relation is verified.

Solution 2. The triangles ABC and $A_1B_1C_1$ are similar and inversely oriented. We apply now *Theorem 26*.

2. We prove that the triangles BIC and MNP are orthological. The perpendicular from M to BC is mediator of BC. The perpendicular from N to CI is the radical axis of the inscribed circle and the null circle $\mathcal{C}$, and the perpendicular from P to BI is the radical axis of the inscribed circle and the null circle $\mathcal{B}$. The radical axis of the null circles $\mathcal{B}$, $\mathcal{C}$ is mediator of BC. The radical axes of three circles are concurrent in their radical center Ω. Because perpendiculars from M, N, P to BC, CI and BI are concurrent, it means that the triangles MBP and BIC are orthological, therefore the perpendiculars taken from I, B, C respectively to NP, MP and MN will be concurrent.

Observation

The problem remains valid even if instead of the inscribed circle (I) we consider the A-ex-inscribed circle (I_a).

3. The perpendiculars taken from O_1, O_2, O_3 respectively to BC, CA, AB are the mediators of these sides, therefore they are concurrent in the center O of the circle circumscribed to the triangle ABC, point that is the orthology center of the triangles $O_1O_2O_3$ and ABC.

The perpendiculars taken from A, B, C to O_2O_3, O_3O_1 respectively O_1O_2 are the chords AM, BM, CM, therefore M is the second orthology center.

4. The triangles B_1C_1A and BPC are orthological, because the perpendicular taken from A to BC, the perpendicular from B_1 to BP and the perpendicular taken from C_1 to CP are concurrent in H. According to the theorem of orthological triangles, the property is also true vice versa: the perpendicular taken from C to C_1A and the perpendicular taken from P to B_1C_1 are concurrent. Because the first two perpendiculars are altitudes in the triangle ABC, they are concurrent in H, and it follows that the third perpendicular passes through H, therefore PH is perpendicular in B_1C_1. On the other hand, PH is perpendicular to BC, consequently BC and B_1C_1 are parallel.

5. The triangles ABC and $A'B'C'$ are orthological triangles. One of the orthology centers is P, and let P'' be the second orthology center. It is known that P'' is the isogonal conjugate of the point P. If P''' is the second orthology center of the triangle ABC (the first is P'), then P''' is the isogonal conjugate of the point P'. Because the set of orthology centers consists only of P and P', it means that P'' and P''' must coincide with P', respectively P, then P and P' are isogonal conjugate points.

6. If A', B', C' is the Simson line of the point S, we have $\sphericalangle SBA \equiv \sphericalangle SA'C'$ (the quadrilateral $SA'BC'$ is inscribable). Because $\sphericalangle SBA \equiv \sphericalangle SQA$, it follows that $\sphericalangle SA'C' \equiv \sphericalangle SQA$, therefore $A'C' \parallel AQ$. The quadrilateral $AKOP$ is parallelogram because $AK \parallel PO$ and $AK \equiv PO$; it follows that $OK \parallel AQ$. $A'C' \parallel AQ$ and $AQ \parallel OK$ lead to $A'C' \parallel OK$.

7. We consider the side of an equilateral triangle of length 1 and let $AC_1 = x$, $BA_1 = y$, $CB_1 = z$. From Ceva's theorem and Carnot's theorem, it follows that:

$xyz = (1-x)(1-y)(1-z)$ and

$x^2 + y^2 + z^2 = (1-x)^2 + (1-y)^2 + (1-z)^2$.

From the second, we note that: $x + y + z = \frac{3}{2}$, therefore $(1-x) + (1-y) + (1-z) = \frac{3}{2}$.

We also have:

$xy + yz + zx = \frac{1}{2}\left(\frac{9}{4} - (x^2 + y^2 + z^2)\right).$

Analogously:

$$(1-x)(1-y) + (1-y)(1-z) + (1-z)(1-x) =$$
$$= \frac{1}{2}\left(\frac{9}{4} - (x^2 + y^2 + z^2)\right).$$

If $x = y \neq z$, then we have the solutions $\{x, z, 1-x, 1-z\}$, from where it follows that $x = 1 - z$, and in any situation we obtain again that $x = y = z = \frac{1}{2}$.

If x, y, z are all different, then $\{x, y, z\} = \{1-x, 1-y, 1-z\}$. If $x = 1 - x$, then $x = \frac{1}{2}$, and if $x = 1 - y$, then $y = 1 - x$ and $z = 1 - z$ lead to $z = \frac{1}{2}$. Hence, the solutions of the equation are $\left\{x, 1-x, \frac{1}{2}\right\}$.

In conclusion, all the positions of the points A_1, B_1, C_1 are given by the triplets $\left(x, 1-x, \frac{1}{2}\right)$ and their permutations, with $x \in (0,1)$, hence one of the points is the midpoint of one side, and the other is equally spaced from the vertices of the respective side.

8. Let $\{X\} = A_bA_c \cap BC$; we suppose that the triangle ABC is acute and $\hat{B} > \hat{C}$. We have $A_cC = a\tan B$, $A_bB = a\tan C$, $\tan\left(\widehat{A_bXC}\right) = \tan B - \tan C$. It is observed that A_2 is the isotomic of A' – the feet of the altitude from A; since $BA' = c \cdot \cos B$, it follows that:

$$A'A_2 = a - 2c \cdot \cos B = 2R\sin A - 4R\sin C\cos B$$
$$= 2R(\sin A - 2\sin C\cos B).$$

$HA' = \cot C \cdot BA' = \frac{C \cdot \cos B \cdot \cos C}{\sin C} = 2R\cos B\cos C.$

$$\tan\left(\widehat{HA_2A'}\right) = \frac{HA'}{A'A_2} = \frac{\cos B \cdot \cos C}{\sin A - 2\sin C \cdot \cos B}$$
$$= \frac{\cos B \cdot \cos C}{\sin(B+C) - 2\sin C \cdot \cos B}$$
$$= \frac{\cos B \cdot \cos C}{\sin B \cdot \cos C + \sin C \cdot \cos B - 2\sin C \cdot \cos B}$$
$$= \frac{\cos B \cdot \cos C}{\sin B \cdot \cos C - \sin C \cdot \cos B} = \frac{1}{\tan B - \tan C}.$$

Because $\tan(\widehat{A_b X C}) = \cot(\widehat{H A_2 A'})$, it follows that $A_2 H \perp A_b A_c$. Similarly, we prove that $B_2 H \perp B_a B_c$ and that $C_2 H \perp C_b C_a$.

Observation

The problem states that the triangles $A_2 B_2 C_2$ and the one with the sides determined by the lines $A_b A_c$, $B_a B_c$, $C_a C_b$ – are orthological, and H is their center of orthology.

9. i) We suppose that R, P, Q are collinear; let $\frac{RP}{PQ} = \lambda$,

$\Delta A'RP \sim \Delta A_1 QP \Longrightarrow \frac{A'R}{A_1 Q} = \frac{A'P}{A_1 P} = \lambda$ (1)

Analogously,

$\Delta B'RP \sim \Delta B_1 QP \Longrightarrow \frac{B'R}{B_1 Q} = \frac{B'P}{B_1 P} = \lambda,$ (2)

$\Delta C'RP \sim \Delta C_1 QP \Longrightarrow \frac{C'R}{C_1 Q} = \frac{C'P}{C_1 P} = \lambda.$ (3)

The relations (1), (2) and (3) lead to $A'B' \parallel A_1 B_1$, $B'C' \parallel B_1 C_1$, $A'C' \parallel A_1 C_1$. The triangles $A_1 B_1 C_1$ and $A'B'C'$ have respectively parallel sides; it follows that the quadrilaterals $A'C'B'C_1$ and $A'C'A_1 B'$ are parallelograms, therefore $A'C' = C_1 B'$ and $A'C' = A_1 B'$, consequently B' is the midpoint of the side $A_1 C_1$, analogously A' is the midpoint of the side $B_1 C_1$, and C' is the midpoint of the side $A_1 B_1$; therefore P is the gravity center of the triangle $A_1 B_1 C_1$.

Let now P be the gravity center of the triangle $A_1 B_1 C_1$ and Q – the orthology center of the triangle $A_1 B_1 C_1$ in relation to ABC. If R is the orthology center of the triangle $A'B'C'$ in relation to ABC, then, because the triangles $A_1 B_1 C_1$ and ABC are homological of center P, according to Sondat's theorem, we obtain that the points Q, P and S are collinear (S is the orthology center of the triangle ABC in relation to $A_1 B_1 C_1$). On the other hand, the triangle $A'B'C'$ is the median triangle of the triangle $A_1 B_1 C_1$, therefore the perpendiculars taken from A, B, C to the sides of $A_1 B_1 C_1$ are perpendicular to the sides of the triangle $A'B'C'$ as well, consequently S is the orthology center of the triangle ABC in relation to $A'B'C'$. According to Sondat's theorem, taking into account that $A'B'C'$ and ABC are homothetic with P the homothety center, we have that the points S, R, P are collinear. From Q, P, S and S, R, P – collinear, it follows that Q, R, S, P are collinear.

ii) It follows from i).

10. The triangle PBC is orthological with the triangle $A'B'C'$ (we denoted by A' the midpoint of BC). The orthology centers are H and Q.

11. The triangles ABC and IED are congruent (A.S.A.). From $AB \parallel IE$ and $AB = IE$, it follows that the quadrilateral $BEIA$ is parallelogram, hence $AI \parallel BE$. Because $BE \perp BC$, it follows that $AI \perp BC$, therefore AI is the altitude from A of the triangle ABC. We obtained that the perpendiculars taken from I, D, E respectively to BC, AB and AE are concurrent, hence the triangles IDE and ABC are orthological.

12. a) Obviously, the triangles $A_1B_1C_1$ and ABC are orthological, the orthology center being O. If we denote by M, N, P the midpoints of the sides BC, CA, AB, we observe that NP is parallel with B_1C_1; and since NP is parallel with BC, it follows that the perpendicular taken from A to B_1C_1 is the altitude AA'. Reasoning analogously, we obtain that the second orthology center of the triangles ABC and $A_1B_1C_1$ is the orthocenter H of the triangle ABC.

b) The quadrilateral AHA_1O is parallelogram because $AH = 2OM$ and $AH \parallel OA_1$; it follows that AA_1 passes through the midpoint O_1 of the segment OH. Analogously, we obtain that BB_1 and CC_1 pass through O_1, therefore the homology center is O_1, and it belongs to Euler line OH of the triangle ABC.

c) From b), we have that $\frac{O_1H}{O_1O} = 1$.

13. The triangle $A'B'C'$ is orthological in relation to ABC, and the orthology center is A. Then ABC is also orthological in relation to $A'B'C'$.

14. The triangles $A_1B_1C_1$ and ABC are orthological. Indeed, the perpendicular taken from A_1 to BC, the perpendicular taken from B_1 to AC (the altitude of the triangle AB_1C_1), and the perpendicular taken from C_1 to AB (the altitude of the triangle AB_1C_1) are concurrent. The orthology center is the the orthocenter H_1 of the triangle AB_1C_1.

15. It is clear that M must be in the interior of the triangle ABC. We have: $\Delta ABB' \equiv \Delta ACC'$ (S.S.S.); it follows that $\sphericalangle B'AC \equiv \sphericalangle C'AB$, therefore also $\widehat{BAM} \equiv \widehat{CAM}$; hence AM is bisector, therefore an altitude in ABC. Similarly, we show that BM is an altitude in the triangle ABC, hence M must be the orthocenter of this triangle. Moreover, AA', BB', CC' are altitudes in ABC. The common orthology center is H.

16. We denote by a, b, c the orthogonal projections of the point O on the sides of the triangle ABC, and by a_1, b_1, c_1 – the projections of O on the sides of the triangle A_1', B_1',C_1'. The quadrilaterals $aA_1'b_1B$, b_1BcC_1', $cC_1'a_1A$, a_1AbB_1', $bB_1'c_1C$, c_1CaA_1' are inscribable, having two opposite right angles.

It follows that:

$$\overrightarrow{O_a} \cdot \overrightarrow{OA_1'} = \overrightarrow{O_b} \cdot \overrightarrow{OB_1'} =$$
$$= \overrightarrow{O_c} \cdot \overrightarrow{OC_1'} = \overrightarrow{O_{a_1}} \cdot \overrightarrow{OA} =$$
$$= \overrightarrow{O_{b_1}} \cdot \overrightarrow{OB} = \overrightarrow{O_{c_1}} \cdot \overrightarrow{OC} = k^2.$$

This shows that A_1', B_1', C_1' – the poles of the sides BC, CA respectively AB with respect to the circle $\mathcal{C}(O;k)$, videlicet the triangles $A_1'B_1'C_1'$ and ABC are reciprocal polar to this circle. *(G.M. XXII)*

17. We consider the circle circumscribed to the quadrilateral $CB'C'B$; let M_a be the center of this circle. The polar of P with respect to this circle is AH, and the polar of A with respect to this circle is PH; it follows that H is the pole of the line AP and, hence, $M_aH \perp AP$ or $M_aH \perp UW$. Analogously, it is shown that $M_bH \perp VW$ and that $M_cH \perp UV$. Consequently, the triangles $M_aM_bM_c$ and UVW are orthological, with the orthology center being H.

18. It is obvious that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological with the help of Ceva's theorem; $\frac{A_2C_1}{A_2B_1} = \frac{b\cdot\cos C}{c\cdot\cos B}$. The orthology derives from the fact that AA_2 – being perpendicular to B_1C_1 (which is antiparallel with BC) – passes through the center O of the circle circumscribed to the triangle ABC.

19. We denote by M_d the midpoint of the side BC. The quadrilateral $M_aM_bM_dM_c$ is parallelogram. The perpendiculars raised in the points M_a, M_b, M_d, M_c to B_1C_1, B_1C, BC respectively AB are concurrent in the center of the circle.

20. Let A_1', B_1', C_1' be the projections of points A_1, B_1, C_1 on B_2C_2; C_2A_2 respectively A_2B_2. We denote by $\{Q\} = A_1A_1' \cap B_1B_1' \cap C_1C_1'$. From the reciprocal of the three perpendicular theorem, it follows that $A_0A_1' \perp B_2C_2$, $B_0B_1' \perp A_2C_2$ and $C_0C_1' \perp A_2B_2$. On the other hand, the planes $(A_0A_1A_1')$, $(B_0B_1B_1')$, $(C_0C_1C_1')$ have in common the point Q and contain respectively the parallel lines A_1A_0, B_1B_0, C_1C_0; it means that they have a line in common with them, which passes through Q and intersects the plane $A_2B_2C_2$ in Q'. This point Q' is on each of the lines A_0A_1', B_0B_1', C_0C_1', hence these lines are concurrent in Q', and this point is the orthology center of the triangle $A_0B_0C_0$ with respect to $A_2B_2C_2$. We denote by M, N, P the homology axis of triangles $A_1B_1C_1$ and $A_2B_2C_2$ (the line of intersection of their planes). Because the projection of the line B_1C_1 on the plane $(A_2B_2C_2)$ is B_0C_0 and $B_1C_1 \cap (A_2B_2C_2) = \{M\}$, it follows that $M \in B_0C_0$, therefore $\{M\} = B_0C_0 \cap B_2C_2$, analogously $\{N\} = A_0C_0 \cap A_2C_2$ and $\{P\} = A_0B_0 \cap A_2B_2$.

21. a) The perpendiculars in A', B', C' respectively to BC, AC, AB are concurrent if and only if:

$A'B^2 - A'C^2 + B'C^2 - B'A^2 + C'A^2 - C'B^2 = 0.$

This relation is equivalent to:

$(A'B - A'C) \cdot BC + (B'C - B'A) \cdot AC + (C'A - C'B) \cdot AB = 0.$

But: $A'C = BC - A'B$; $B'A = AC - B'C$; $C'B = AB - C'A$.

Replacing these relations in the preceding relation, we get the required relation.

b) It is known that, if $a, b, c \in \mathbb{R}$ and $x, y, z \in \mathbb{R}$, the Cauchy-Buniakovski-Schwarz inequality takes place:

$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq (ax + by + cz)^2.$

Taking $BC = a$, $CA = b$, $AB = c$ and $A'B = x$, $B'C = y$, $C'A = z$, the relation *b)* is obtained taking into account the Cauchy-Buniakovski-Schwarz inequality and the relation from *a)*.

c) ${BA'}^2 + {CB'}^2 + {AC'}^2$ is minimal if and only if the equality holds in the inequality from *b)*, ie. if and only if $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. We denote:

$\frac{y}{a} = \frac{y}{b} = \frac{z}{c} = k.$

From *a)*, it derives that:

$ax + by + cz = \frac{1}{2}(a^2 + b^2 + c^2);$

$a^2k + b^2k + c^2k = \frac{1}{2}(a^2 + b^2 + c^2)$, therefore $k = \frac{1}{2}$, which shows that A', B', C' are the midpoints of the sides BC, CA, AB; hence, the intersection point of the perpendiculars in A', B', C' to BC, CA, AB is the center of the circle circumscribed to the triangle ABC.

22. It is sufficient to prove that the medians of the triangle ABC are perpendicular to the lines of the centers of the circles circumscribed to the respective triangles. We prove that AM (the median din A) is perpendicular to O_1O_2 (O_1 is the center of the circle circumscribed to the triangle ABA_1, and O_2 is the center of the circle circumscribed to the triangle ABA_2).

We denote: $AO_1 = \mathrm{r}_1$, $AO_2 = \mathrm{r}_2$, $d(O_1, BC) = d_1$, $d(O_2, BC) = d_2$, $\frac{1}{2}BA_1 = x$ and $MB = \frac{1}{2}a$.

$AM \perp O_1O_2 \Leftrightarrow AO_1^2 - AO_2^2 = MO_1^2 - MO_2^2.$

$MO_1^2 = d_1^2 + \left(\frac{a}{2} - x\right)^2;$

$MO_2^2 = d_2^2 + \left(\frac{a}{2} - x\right)^2.$

On the other hand, $d_1^2 = \mathrm{r}_1^2 - x^2$, $d_2^2 = \mathrm{r}_2^2 - x^2$.

It follows that $MO_1^2 - MO_2^2 = r_1^2 - r_2^2 = O_1A^2 - O_2A^2$.

23. The perpendicular from A_1 to BC, the perpendicular from B_1 to CA and the perpendicular from C_1 to AB intersect in D; in other words, the triangle $A_1B_1C_1$ is orthological with the triangle ABC, and the orthology center is the point D.

24. *Solution 1*. We denote: $BC = a$ and $BA_1 = x$, $CB_1 = y$, $AC_1 = z$.

From $A_1B_1 = B_1C_1$, we get:

$(a - x)^2 + y^2 - (a - x)y = (a - y)^2 + z^2 - (a - y) \cdot z,$

equivalent to:

$(x-z)(x+y+z-a)=a(x-y).$ (1)

Analogously, from $B_1C_1=A_1C_1$, it follows that:

$(y-x)(x+y+z-a)=a(y-z).$ (2)

The triangles ABC and $A_1B_1C_1$ being orthological, we have that:

$$x^2-(a-x)^2+y^2-(a-y)^2+z^2-(a-z)^2=0 \Leftrightarrow$$
$$2ax+2ay+2az=3a^2 \Leftrightarrow$$

$x+y+z=\frac{3}{2}a.$ (3)

Substituting $x+y+z$ from (3) in (1) and (2), we obtain that $x=y=z=\frac{a}{2}$, which shows that $A_1B_1C_1$ is the median triangle of the triangle ABC.

Solution 2. Keeping the previous notations, we prove that $x=y=z$, showing that $\Delta AB_1C_1\equiv\Delta BC_1A_1\equiv\Delta CA_1B_1$.

Indeed, if we denote $m\widehat{AB_1C_1}=\alpha$, then $m\widehat{AC_1B_1}=120^0-\alpha$, and since $m\widehat{A_1C_1B_1}=60^0$, we obtain that: $m\widehat{BC_1A_1}=\alpha$, therefore $m\widehat{BA_1C_1}=120^0-\alpha$. Analogously, we get $m\widehat{B_1A_1C}=\alpha$ and $m\widehat{A_1B_1C}=120^0-\alpha$.

The congruence of the triangles derives now from A.S.A. Then the continuation is as in *Solution 1*.

25. If the triangle ABC is isosceles, for example $AB=AC$, we prove that ABC and $A'B'C'$ are orthological, showing that the relation:

$A'B^2-A'C^2+B'C^2-B'A^2+C'A^2-C'B^2=0$ (1)

is true.

Because $A'B=A'C$, it remains to show that $B'C^2-B'A^2+C'A^2-C'B^2=0$. BB' and CC' are bisectors and $AB=AC$; we obtain without difficulty that $B'C=BC'$ and $B'A=C'A$, therefore (1) is verified.

Observation

In this hypothesis, we can also prove the concurrency of perpendiculars taken in A', B', C' to BC, CA, AB in this way:

We denote by O_1 the intersection of the perpendicular in B' to AC with AA'. From the congruence of triangles $AB'O$ and $AC'O$, it follows that $\sphericalangle AC'O_1=90^0$, therefore also the perpendicular in C' to AB passes through O_1.

Reciprocally, if the triangles ABC and $A'B'C'$ are orthological, the relation (1) takes place, and let us prove that the triangle ABC is isosceles.

Using the bisector theorem, we find: $A'B = \frac{ac}{b+c}$, $A'C = \frac{ab}{b+c}$, $B'C = \frac{ab}{a+c}$, $B'A = \frac{bc}{a+c}$, $C'A = \frac{bc}{a+b}$, $C'B = \frac{ac}{a+b}$.

We get:

$\frac{a^2(c-b)}{b+c} + \frac{b^2(a-c)}{a+c} + \frac{c^2(b-a)}{b+a} = 0.$

By doing the calculations, we obtain:

$(a-c)(b-a)(b-c)(a+b+c)^2 = 0,$

from where it follows that $a = b$ sau $b = c$ sau $c = a$, therefore the triangle ABC is isosceles.

26. a) The condition of concurrency of lines MA_1, MB_1, MC_1 is equivalent to:

$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0$, or:

$$(A_1B - A_1C)(A_1B + A_1C) + (B_1C - B_1A)(B_1C + B_1A) + (C_1A - C_1B)(C_1A + C_1B) = 0.$$

We get:

$A_1B + B_1C + C_1A = A_1C + B_1A + C_1B = \frac{3a}{2}.$

b) We denote: $A_1B = x$, $B_1C = y$, $C_1A = z$. The condition of concurrency of lines AA_1, BB_1, CC_1 is equivalent to $\frac{a-x}{x} \cdot \frac{a-y}{y} \cdot \frac{a-z}{z} = 1$ or:

$a^3 - (x+y+z)a^3 + (xy+yz+zx)a - 2xyz = 0.$

Because: $x + y + z = \frac{3a}{2}$, it follows that:

$-a^3 + 2(xy+yz+zx)a - 4xyz = 0.$ (1)

On the other hand:

$(a-2x)(a-2y)(a-2z) = a^3 - 2(x+y+z)a^2 + 4(xy+yz+zx)a - 8xyz = a^3 - 3a^3 + 4(xy+yz+zx)a - 8xyz = -2a^3 + 4(xy+yz+zx)a - 8xyz.$ (2)

From relations (1) and (2) we have that:

$(a-2x)(a-2y)(a-2z) = 0,$

from where $x = \frac{a}{2}$ or $y = \frac{a}{2}$ or $z = \frac{a}{2}$, therefore M is on one of the altitudes.

27. Let K be the Lemoine point in the triangle ABC; it is known that: $\frac{A'B}{A'C} = \frac{c^2}{b^2}$; $\frac{B'C}{B'A} = \frac{a^2}{c^2}$; $\frac{C'A}{C'B} = \frac{b^2}{a^2}$.

We obtain:

$A'B = \frac{ac^2}{b^2+c^2}$; $A'C = \frac{ab^2}{b^2+c^2}$; $B'C = \frac{ba^2}{a^2+c^2}$; $B'A = \frac{bc^2}{a^2+c^2}$;

$C'A = \frac{cb^2}{a^2+b^2}$; $C'B = \frac{ca^2}{a^2+b^2}$.

The triangles ABC and $A'B'C'$ are orthological if and only if:

$$A'B^2 - A'C^2 + B'C^2 - B'A^2 + C'A^2 - C'B^2 = 0. \quad (1)$$

If we suppose that the triangle ABC is isosceles, $AB = AC$, then AA' is median, therefore $A'B = A'C$; also, we find that $B'A = C'A$ and $B'C = C'B$, and the relation (1) is satisfied.

If the relation (1) takes place, then we obtain that:

$$\frac{a^2(c^2-b^2)}{b^2+c^2} + \frac{b^2(a^2-c^2)}{a^2+c^2} + \frac{c^2(b^2-a^2)}{a^2+b^2} = 0.$$

This relation is equivalent to:

$$a^2(c^2-b^2)(a^2+c^2)(a^2+b^2) + b^2(a^2-c^2)(b^2+c^2)(a^2+b^2) + c^2(b^2-a^2)(a^2+c^2)(b^2+c^2) = 0.$$

$$a^2(c^2-b^2)(a^2+c^2)(a^2+b^2) + (b^2+c^2)[b^2(a^4+a^2b^2-a^2c^2-b^2c^2) + c^2(a^2b^2-a^4+b^2c^2-a^2c^2)] = 0.$$

$$a^2(c^2-b^2)(a^2+c^2)(a^2+b^2) + (b^2+c^2)[-a^4(c^2-b^2) - a^2(c^4-b^4) + b^2c^2(c^2-b^2)] = 0$$

$$(c^2-b^2)\{a^2(a^2+c^2)(a^2+b^2) + (b^2+c^2)[-a^4-a^2b^2-a^2c^2+b^2c^2]\} = 0$$

$$(c^2-b^2)[a^6-a^4b^2+a^4c^2+a^2b^2c^2-a^4b^2-a^2b^4-a^2b^2c^2+b^4c^2 - a^4c^2-a^2b^2c^2-a^2c^4+b^2c^4] = 0$$

We obtain that:

$(a^2-b^2)(c^2-b^2)(a^2-c^2)(a^2+b^2+c^2) = 0$, therefore:

$(a-b)(c-b)(a-c)(a+b)(b+c)(a+c)(a^2+b^2+c^2) = 0$.

Hence, $a = b$ sau $b = c$ or $a = c$, consequently the triangle ABC is isosceles.

28. a) The given condition is equivalent to the orthology of triangles.

b) We take through the point P the parallels MN, RS, UV with BC, CA respectively AB. Obviously, the triangles PVR, PNU and PSM are equilateral.

We have:

$\overrightarrow{PA_1} = \frac{1}{2}\left(\overrightarrow{PV} + \overrightarrow{PR}\right)$;

$\overrightarrow{PB_1} = \frac{1}{2}\left(\overrightarrow{PN} + \overrightarrow{PU}\right)$;

$\overrightarrow{PC_1} = \frac{1}{2}\left(\overrightarrow{PS} + \overrightarrow{PM}\right)$.

$\overrightarrow{PA_1} + \overrightarrow{PB_1} + \overrightarrow{PC_1} = \frac{1}{2}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right)$.

We took into account that the quadrilaterals $PUAS$; $PMBV$; $PRCN$ are parallelograms. For any point P from the plane of the triangle the relation ABC $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = 3\overrightarrow{PG}$ (where G is the gravity center) takes place. In our case, $G = 0$, because ABC is equilateral, hence $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \frac{3}{2}\overrightarrow{PG}$.

29. Let $A_1B_1C_1$ be the podal triangle of M. We have $m(\sphericalangle A_1MB_1) = m(\sphericalangle B_1MC_1) = m(\sphericalangle C_1MA_1) = 120^0$. If, on the semi-lines $(MA_1$, $(MB_1$, $(MC_1$ we consider the points A', B', C', such as $MA' = MB' = MC'$, then $A'B'C'$ and ABC are orthological, and the orthology center is the point M.

30. The perpendiculars taken from I to BC, CA, AB meet respectively $B'C'$, $C'A'$, $A'B'$ in T_1, T_2, T_3. The triangles ABC and $T_1T_2T_3$ are orthological (I is one of their orthology centers). The lines A_bA_c, B_cB_a, C_aC_b are parallel with the sides of the triangle $T_1T_2T_3$, therefore the perpendiculars taken to them from A, B, respectively C are concurrent in the second orthology center of the triangles ABC and $T_1T_2T_3$.

31. We observe that the triangle $B_1B_2B_3$ is the podal triangle of the point B_1 in relation to the triangle $A_1A_2A_3$, consequently the triangles $B_1B_2B_3$ and $A_1A_2A_3$ are orthological, their orthology center being B_1. Also, the triangle $A_1A_2A_3$ is orthological with the triangle $B_1B_2B_3$, their orthology center being the point A_1. Let us prove now that the perpendicular from A_1 to B_3B_1, the perpendicular from A_2 to B_1B_2, and the perpendicular from A_3 to B_2B_3 are concurrent. We observe that the first two perpendiculars are concurrent in A_2; we prove that the perpendicular from A_3 to B_2B_3 also passes through A_2; basically, we prove that $B_2B_3 \perp A_2A_3$.

The quadrilateral $B_1B_2A_1B_3$ is a rectangle; it follows that $\sphericalangle B_1B_2B_3 \equiv \sphericalangle B_1A_1B_3$. From $\sphericalangle A_1A_3A_2 \equiv \sphericalangle A_2A_1D$, we obtain that $\sphericalangle B_1B_2B_3 \equiv \sphericalangle A_2A_1D$.

These angles have $B_1B_2 \parallel A_2A_1$; it follows that $B_2B_3 \parallel A_1D$; since $A_1D \perp A_2A_3$, it follows that $B_2B_3 \perp A_2A_3$.

We obtained that the second orthology center of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ is A_2. It follows that the triangle $B_1B_2B_3$ is two times orthological with the triangle $A_1A_2A_3$. We show that the second orthology center is B_2. Indeed, the perpendicular from B_1 to A_3A_1 is B_1B_2, therefore it passes through B_2. The perpendicular from B_2 to A_1A_2 passes through B_2, and we have previously shown that the perpendicular from B_3 to A_2A_3 is B_2B_3. It is not difficult to show that the perpendicular from A_1 to B_1B_2, the perpendicular from A_2 to B_2B_3 and the perpendicular from A_3 to B_3B_1 are concurrent in A_3, therefore A_3 is the third orthology center of the triangle $A_1A_2A_3$ in relation to $B_1B_2B_3$. The third orthology center of the triangle $B_1B_2B_3$ in relation to $A_1A_2A_3$ is B_3.

32. We show that the triangle $A_1B_1C_1$ is orthological with ABC. We denote by O' the center of the circle circumscribed to the triangle $A'B'C'$; obviously the mediators of the segments $A'A_1$, $B'B_1$, $C'C_1$ pass through O'. The perpendicular in A_1 to BC passes through P', the symmetric of P with respect to O'. (Indeed, the quadrilateral $A_1A'PP'$ is rectangular trapeze and the mediator of A_1A' contains the midline of this trapeze.) Analogously, it follows that P' belongs also to the perpendiculars in B_1 to AC and in C_1 to AB_1. Consequently, the triangles $A_1B_1C_1$ and ABC are orthological.

33. We show that:

$\alpha B^2 - \alpha C^2 + \beta C^2 - \beta A^2 + \gamma A^2 - \gamma B^2 = 0.$

We calculate αB and αC as medians in the triangles BB_1C_1 and CC_1B_1. It is easy to note the congruence of triangles BAB_1 and CAC_1, since $BB_1 = CC_1$. We have:

$\alpha B^2 = \frac{1}{4}[2(c^2 + BB_1^2) - B_1C_1^2],$

$\alpha C^2 = \frac{1}{4}[2(b^2 + CC_1^2) - B_1C_1^2].$

Therefore $\alpha B^2 - \alpha C^2 = \frac{1}{2}(c^2 - b^2).$

Analogously, we find:

$\beta C^2 - \beta A^2 = \frac{1}{2}(a^2 - c^2),$

$\gamma A^2 - \gamma B^2 = \frac{1}{2}(b^2 - a^2).$

34. The quadrilaterals $AA'CB$; $BB'CA$; $CC'AB$ are isosceles trapezoids. We have $A'C = AB$; $A'B = AC$ etc. We show that $A'B'C'$ is orthological with ABC; we calculate:

$A'B^2 - A'C^2 + B'C^2 - B'A^2 + C'A^2 - C'B^2$;

it is obtained zero, therefore $A'B'C'$ is orthological with ABC. The orthology center of the triangle ABC in relation to $A'B'C'$ is O.

35. Let H_1 be the orthocenter of the triangle $A_1B_1C_1$ and X_1 – the intersection of semi-line $(AH_1$ with the circumscribed circle of the triangle AA_1A_2. We have: $AH_1 \cdot H_1X_1 = A_1H_1 \cdot H_1A_2$ (1) (the power of the point H_1 over the circle AA_1A_2). We denote by X_2 the intersection of semi-line $(AH_1$ with the circle AB_1B_2. We have: $AH_1 \cdot H_1X_2 = B_1H_1 \cdot H_1B_2$ (2). Analogously, if X_3 is the intersection of semi-line $(AH_1$ with the circle AC_1C_2, we have: $AH_1 \cdot H_1X_3 = C_1H_1 \cdot H_1C_2$ (3). From relations (1), (2), (3), because of the fact that $A_1H_1 \cdot H_1A_2 = B_1H_1 \cdot H_1B_2 = C_1H_1 \cdot H_1C_2$ (4), it follows that $X_1 = X_2 = X_3$, consequently the circles still have a common point. The relation (4) derives from the property of the symmetrics of an orthocenter of a triangle to be on its circumscribed circle, and from the power of orthocenter over the circumscribed circle of the triangle.

The reasoning is the same in the case of the obtuse triangle.

36. Let K be the Lemoine point of the triangle ABC and A_1, B_1, C_1 – the projections of K on BC, CA respectively AB. We use the following result: “the Lemoine point K of the triangle ABC is the gravity center of its podal triangle and reciprocally”.

We know about K that it is the gravity center of the triangle $A_1B_1C_1$ and of the triangle MNP; we have:

$\overrightarrow{KA_1} + \overrightarrow{KB_1} + \overrightarrow{KC_1} = \vec{0}$,

$\overrightarrow{KP} + \overrightarrow{KM} + \overrightarrow{KN} = \vec{0}$.

Because $\overrightarrow{KP} = \overrightarrow{KA_1} + \overrightarrow{A_1P}$; $\overrightarrow{KM} = \overrightarrow{KB_1} + \overrightarrow{B_1M}$ and $\overrightarrow{KN} = \overrightarrow{KC_1} + \overrightarrow{C_1N}$, we obtain that: $\overrightarrow{A_1P} + \overrightarrow{B_1M} + \overrightarrow{C_1N} = 0$.

If we denote by M', N', P' the projections of K on the sides of the triangle MNP, it follows that the relation $\overrightarrow{KP'} + \overrightarrow{KM'} + \overrightarrow{KN'} = 0$, which expresses that the point K is the gravity center of its podal triangle $M'N'P'$

in relation to the triangle MNP, hence K is the symmedian center of the triangle MNP.

Because in the triangle MNP the symmedian center coincides with the gravity center, we obtain that this triangle is equilateral. On the other hand, the triangle MNP built as in the statement is similar with the triangle ABC. The triangle MNP being equilateral, it follows that ABC is also equilateral.

37. Let $\{H\} = EF \cap AD$; since MN is middle line in ABD, it follows that $NM \parallel BD$, also $EH \perp NM$. Since $NA \perp EM$, it follows that H is the orthocenter of the triangle NEM, hence $NM \perp EH$. The quadrilateral $PHMF$ is parallelogram of center O, hence $PF \parallel HM$; and since $DL \perp NE$, it follows that $MH \perp DL$. The perpendicular taken from N to LB is NA, the perpendicular taken from E to DB is EO, and the perpendicular taken from M to LD is MH; these perpendiculars are altitudes in the triangle NEM, and are concurrent in H. The orthology center of the triangle NEM in relation to DLB is H – the orthocenter of the triangle NEM, and the orthology center of the triangle DLB in relation to the triangle NEM is the orthocenter of the triangle DLB.

38. We observe that the triangles ACB and AFD are orthological. Indeed, the perpendicular from A to FD, the perpendicular from C to AD, and the perpendicular from B to AF are concurrent in E. The point E is the orthology center of the triangle ACB in relation to AFD. According to the theorem of orthological triangles, we have that the triangle AFD is also orthological in relation to ACB, hence the perpendicular from A to CB, the perpendicular from F to AB, the perpendicular from D to AC are concurrent. Because the perpendicular from A to BC and the perpendicular from F to AB are concurrent in B, it follows that the perpendicular from D to AC must also pass through B; and since AC is diameter in the circle, it follows that B must be the symmetric of D with respect to AC, hence $AD = AB$.

39. From $QR \parallel AB$, M – the midpoint of AB, and $\{N\} = QR \cap CM$, we obtain that N is the midpoint of QR. Also, $AQ = \frac{1}{4} \cdot AC \Longrightarrow MN = \frac{1}{4} \cdot CM$, therefore $PN = \frac{1}{2} \cdot CP$, videlicet P is the gravity center of the triangle CQR (isosceles rectangle), consequently $QP \perp CB$. The perpendicular from R to

AC is RQ, and the perpendicular from Q to AB is QA. From the above, it follows that the triangle PRQ is orthological with the triangle ABC, the orthology center being the point Q. The theorem of orthological triangles shows that the triangle ABC is also orthological in relation to PRQ, the orthology center being the vertex C.

40. We denote $BA_1 = CA_1 = \frac{a}{2}$, $B_1C = y$, $B_1A = b - y$, $C_1A = z$, $C_1B = c - z$. From Ceva's theorem, we find:

$$y = \frac{b}{c}(c - z). \quad (1)$$

The triangle ABC being orthological in relation to $A_1B_1C_1$, we have:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0.$$

We obtain:

$$2by - b^2 + 2cz - c^2 = 0. \quad (2)$$

Replacing y in (2), it follows that:

$$2z(c^2 - b^2) - c(c^2 - b^2) = 0 \Leftrightarrow (c - b)(c + b)(2z - c) = 0.$$

If $z = \frac{c}{2}$, then CC_1 is median, and BB_1 as well. If $b = c$, then the triangle ABC is isosceles.

41. a) Assuming the problem solved, given conditions, we find that $\sphericalangle A_1 \equiv \sphericalangle B$, $\sphericalangle B_1 \equiv \sphericalangle C$, hence $\Delta A_1B_1C_1 \sim \Delta BCA$. We firstly build a triangle $A'B'C'$ with $A' \in (BC)$, $B' \in (CA)$, $C' \in (AB)$ and $\Delta A'B'C' \sim \Delta BCA$. We fix $A' \in (BC)$ and we build $B' \in (AC)$, such that $A'B' \perp BC$. We then build the perpendicular in B' to AC. We build on this perpendicular the point C', such that $\sphericalangle C'A'B' \equiv \sphericalangle B$. Now we draw the line CC' and we denote by C_1 its intersection with (AB). We build $C_1B_1 \perp AC$ with $B_1 \in (AC)$; we build $B_1A_1 \perp BC$ with $A_1 \in (BC)$. The triangle $A_1B_1C_1$ is the required triangle. The proof of the construction results from the fact that $A'B'C'$ is similar with BCA ($\sphericalangle B' = \sphericalangle C$ and $\sphericalangle A' \equiv \sphericalangle B$). The triangle $A'B'C'$ and the triangle $A_1B_1C_1$ are homothetic, therefore $\Delta A_1B_1C_1 \sim \Delta BCA$. Also, it follows that $A_1C_1 \perp AB$.

b) The perpendiculars taken from O_1, O_2, O_3 respectively to A_1C_1, A_1B_1 and B_1C_1 are mediators of the triangle $A_1B_1C_1$.

42. The perpendiculars raised in A, B, C respectively to O_2O_3, O_3O_1 and O_1O_2 are the radical axes of the pairs of circles: $\left(\mathcal{C}(O_2), \mathcal{C}(O_3)\right)$ and $\left(\mathcal{C}(O_3), \mathcal{C}(O_1)\right)$ respectively $\left(\mathcal{C}(O_1), \mathcal{C}(O_2)\right)$. As it is known, these radical axes are concurrent in the radical center Ω of the given circles. Consequently, the triangle ABC and the triangle $O_1O_2O_3$ are orthological, and the orthology center is Ω. The orthology center of the triangle $O_1O_2O_3$ in relation to ABC will be Ω' – the isogonal conjugate of Ω (considering that ABC is the podal triangle of Ω in relation to $O_1O_2O_3$).

43. The triangle AC_bC_c is isosceles. Its altitude from A is the bisector from A of the triangle ABC, hence the perpendicular from A to C_bC_c passes through I – the center of the inscribed circle. Analogously, the perpendiculars taken from H_b respectively H_c to C_aC_c pass through I, consequently the triangles $H_aH_bH_c$ and $C_aC_bC_c$ are orthological, and the orthology center is I. The quadrilaterals $H_aC_cIC_b$, $H_bC_aIC_c$ and $C_aH_cC_bI$ are rhombuses. In the triangle IH_bH_c, the line MN, where M is the midpoint of C_aC_c and N is the midpoint of C_aC_b, is midline, therefore $MN \parallel H_bH_c$. Then, the perpendicular taken from C_a to H_bH_c is perpendicular to MN; hence it is the altitude from C_a of the contact triangle. The orthology center of the triangle $H_aH_bH_c$ in relation to $C_aC_bC_c$ is the orthocenter of the contact triangle.

44. Let D, E, F the contacts with BC, AB respectively AC of the A-ex-inscribed circle. The triangle DEF is orthological in relation to ABC because the perpendiculars taken in D, E, F to BC, AB and AC are concurrent in the center I_a of the A-ex-inscribed circle. Because $AE = AF$ (tangents taken from a point exterior to the circle), it means that the perpendicular from A to EF is a bisector in the triangle ABC; hence it contains the point I_a; analogously, the perpendiculars taken from B and from C to ED respectively DF pass through I_a.

45. From $C_1B_1 \parallel BC$ it follows that $\frac{AC_1}{C_1B} = \frac{AB_1}{B_1C}$ (1). From $B_1A_1 \parallel AB$, it follows that $\frac{B_1C}{B_1A} = \frac{A_1C}{A_1B}$ (2). The triangle ABC being orthological with $A_1B_1C_1$ and $B_1C_1 \parallel BC$, $A_1B_1 \parallel AB$, it means that the orthology center is H,

the orthocenter of ABC; this would mean that A_1C_1 must be parallel with AC. From (1) and (2) we get: $\frac{AC_1}{C_1B} = \frac{A_1B}{A_1C}$ (3). Because $A_1C_1 \parallel AC$, we have: $\frac{AC_1}{C_1B} = \frac{CA_1}{BA_1}$ (4). The relations (3) and (4) lead to $\frac{A_1B}{A_1C} = \frac{A_1C}{A_1B}$, therefore $A_1B = A_1C$, hence A_1 is the midpoint of BC. Having $A_1B_1 \parallel AB$, we obtain that B_1 is the midpoint of AC, and $B_1C_1 \parallel BC$ leads to the conclusion that C_1 is the midpoint of AB. Consequently, $A_1B_1C_1$ is the complementary (median) triangle of the triangle ABC.

46. Obviously, O is the orthology center of the triangle ABC in relation to the triangle $A_1B_1C_1$. Because B_1C_1 is the mediator of AO, we have that $B_1A = BO_1$; since B_1A_1 is the mediator of CO, we have that $B_1O = B_1C$. From the above equalities, we note that $B_1A = B_1C$, therefore B_1 is on the mediator of the side AC of the triangle ABC; it follows analogously that the perpendicular from A_1 to BC is the mediator of BC, and the perpendicular from C_1 to AB is the mediator of AB. Consequently, O – the center of the circumscribed circle, is also the orthology center of the triangle $A_1B_1C_1$ in relation to ABC.

47. The triangles ABC and $A_1B_1C_1$ are obviously orthological, the orthology center being P. From the theorem of the orthological triangles, we have that $A_1B_1C_1$ is also orthological in relation to ABC. Because A_1 belongs to the mediators of the segments BP and CP, it follows that A_1 is the center of the circle circumscribed to the triangle BPC, therefore the perpendicular taken from A_1 to BC is the mediator of BC, consequently the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC is O – the center of the circle circumscribed to the triangle ABC.

48. The triangle ABC being orthological in relation to $A_1B_1C_1$, it means that the perpendiculars taken from A, B, C to B_1C_1, C_1A_1 respectively A_1B_1 are concurrent in the orthology center N. The pole of the perpendicular taken from A to B_1C_1 is the point X, the pole of the perpendicular taken from B to C_1A_1 is the point Y, and the pole of the perpendicular taken from C to A_1B_1 is the point Z. The previous perpendiculars being concurrent in N, it

means that their poles in relation to the duality relative to the inscribed circle are collinear points. In conclusion, X, Y, Z are collinear points.

49. Because the perpendicular from Q to AS and the perpendicular from P to AR are concurrent in M, the triangles AQP and ARS, in order to be orthological, it is necessary that the perpendicular taken from A to RS to pass as well through M. We build the circumscribed circle of the triangle $PB'B$ and the circumscribed circle of the triangle $QC'C$. The point M has equal powers over these circles because:

$MB \cdot MB' = MC \cdot MC'$. (1)

We build the circumscribed circle of the triangle APP' and the circumscribed circle of the triangleAQQ', where $\{P'\} = MP \cap AB$, $\{Q'\} = MQ \cap AC$; let O_1 and O_2 be their centers, and let T – their second point of intersection. From $MP \cdot MP' = MB \cdot MB'$and $MQ \cdot MQ' = MC \cdot MC'$ (2) and from the relation (1), it follows that the point M has equal powers over these circles, therefore it belongs to the radical axis AT. Since $AT \perp O_1O_2$ and $O_1O_2 \parallel PQ$ (midline in the triangle APQ), it follows that $AM \perp RS$.

50. Because the perpendicular taken from P to ER and the perpendicular taken from Q to E are concurrent in M, the triangles EPQ and ESR must have the perpendicular taken from E to RS passing through M in order to be orthological. Therefore, we need to prove that $ME \perp PQ$. We build the circles circumscribed to triangles MAP and MBQ; we denote by T their second point of intersection, and we denote by A' respectively by B' their second point of intersection with AE respectively BE. From $m(\widehat{MTP}) = m(\widehat{MTQ}) = 90^0$, it follows that T belongs to the line PQ. The point A' is the symmetric of A with respect to MP, and B' is the symmetric of B with respect to MQ. The quadrilateral $AA'B'B$ is inscribable ($MA = MA' = MB' = MB$), then $EA' \cdot EA = EB' \cdot EB$, hence E has equal powers over the circles APM and BMQ, consequently E belongs to the radical axis of these circles, namely to the common chord MT. This is perpendicular to the line of the centers of the circles that is parallel to PQ, consequently $ME \perp PQ$.

51. We denote: $A_1B_1C_1$ – the U-circumpedal triangle of ABC and $\alpha = \sphericalangle BAA_1$, $\beta = \sphericalangle CAA_1$, $\delta = \sphericalangle B_1BA$, $\gamma = \sphericalangle B_1BC$, $\varphi = \sphericalangle C_1CB$, $\varepsilon = \sphericalangle C_1CA$. The triangles ABC and $A_1B_1C_1$ are orthological, hence:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0. \quad (1)$$

From the sinus theorem, it follows that $BA_1 = 2R\sin\alpha$, $CA_1 = 2R\sin\beta$ and analogs. R is the radius in the circle circumscribed to the triangle ABC. With these substitutions, the relation (1) becomes:

$$\sin^2\alpha - \sin^2\beta + \sin^2\gamma - \sin^2\delta + \sin^2\varepsilon - \sin^2\varphi = 0. \quad (2)$$

Because V is the isogonal conjugate of U, we have (denoting by $A_2B_2C_2$ the circumpedal of V): $\sphericalangle A_2AB = \beta$, $\sphericalangle A_2AC = \alpha$, $\sphericalangle B_2BA = \gamma$, $\sphericalangle B_2BC = \delta$, $\sphericalangle C_2CA = \varphi$, $\sphericalangle C_2CB = \varepsilon$.

The relation (2) can be written in this way:

$$\sin^2\beta - \sin^2\alpha + \sin^2\delta - \sin^2\gamma + \sin^2\varphi - \sin^2\varepsilon = 0.$$

This relation is equivalent to:

$$A_2B^2 - A_2C^2 + B_2C^2 - B_2A^2 + C_2A^2 - C_2B^2 = 0,$$

which expresses the orthology of the V-circumpedal triangle ABC with the triangle ABC.

52. We show that the perpendicular taken from A to SQ is also a median in the triangle ABC. Let T be the fourth vertex of the parallelogram $AQTS$. It is noticed that the triangle SAT is obtained from the triangle ABC if to the latter a vector translation $\overrightarrow{AS}$ is applied, then a rotation of center S and of right angle. By these transformations, the median SO of the triangle AST ($\{O\} = SQ \cap AT$) is the transformation of the median from A of the triangle ABC. These lines are perpendicular, hence the median from A of triangle ABC is perpendicular to SQ. Analogously, the median from B is perpendicular to A_1C_1, and that from C is perpendicular to A_1B_1. The orthology center is G – the gravity center in the triangle ABC.

53. We have that $A(a)$, $B(b)$, $C(c)$, $A'(a')$, $B'(b')$, $C'(c')$; ABC and $A'B'C'$ are orthological triangles; we know that:

$$a(c' - b') + b(a' - c') + c(b' - a') = 0. \quad (1)$$

Because $\frac{AA''}{A''A'} = \frac{BB''}{B''B'} = \frac{CC''}{C''C'} = \lambda$, we have that:

$$A''\left(\frac{a+\lambda a'}{1+\lambda}\right),\ B''\left(\frac{b+\lambda b'}{1+\lambda}\right),\ C''\left(\frac{c+\lambda c'}{1+\lambda}\right).$$

$\overrightarrow{A''B''}\left(\frac{b-a+\lambda(b'-a')}{1+\lambda}\right)$;
$\overrightarrow{A''B''}\left(\frac{c-b+\lambda(c'-b')}{1+\lambda}\right)$;
$\overrightarrow{C''A''}\left(\frac{a-c+\lambda(c'-a')}{1+\lambda}\right)$.

We evaluate:

$$\frac{a[(c-b)+\lambda(c'-b')]}{1+\lambda}+\frac{b[(a-c)+\lambda(a'-c')]}{1+\lambda}+\frac{c[(b-a)+\lambda(b'-a')]}{1+\lambda}.$$

Taking into account (1) and the fact that $a(c-b)+b(a-c)+c(b-a)=0$, we obtain that the previous sum is zero; hence ABC and $A''B''C''$ are orthological triangles.

Analogously, it is proved that $A'B'C'$ and $A''B''C''$ are orthological triangles. Now $A'B'C'$ is taking over the role of ABC, $A''B''C''$ is taking over the role of $A'B'C'$, and $A'''B'''C'''$ is taking over the role of $A''B''C''$; from previous proof, we find that the triangles $A''B''C''$ and $A'''B'''C'''$ are orthological.

54. a) Let M be the midpoint of the side BC. We show that $AM \perp B'C'$.

We have:

$$\overrightarrow{AM}\cdot\overrightarrow{B'C'}=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{AC}\right)\cdot\left(\overrightarrow{HC'}-\overrightarrow{HB'}\right)$$
$$=\frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{HC'}-\frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{HB'}+\frac{1}{2}\overrightarrow{AC}\cdot\overrightarrow{HC'}-\frac{1}{2}\overrightarrow{AC}\cdot\overrightarrow{HB'}.$$

Because $\overrightarrow{AB}\cdot\overrightarrow{HC'}=\overrightarrow{AC}\cdot\overrightarrow{HB'}=0$, we have:

$\overrightarrow{AM}\cdot\overrightarrow{B'C'}=\frac{1}{2}\left[b\cdot c\cdot\cos\left(\widehat{C'CA}\right)-c\cdot b\cdot\cos\left(\widehat{B'BA}\right)\right]=$
$\frac{1}{2}bc\left[\cos\left(\frac{\pi}{2}+A\right)-cos\left(\frac{\pi}{2}+A\right)\right]=0,$

hence $AM\perp B'C'$, $AG\perp B'C'$.

Analogously, it follows that $BG\perp C'A'$ and $CG\perp A'B'$, in conclusion the triangle ABC is orthological in relation to $A'B'C'$ and the orthology center is the gravity center G of the triangle ABC.

b) The orthology center of the triangle $A'B'C'$ in relation to the triangle ABC is obviously H. The triangles ABC and $A'B'C'$ are homological, of center H. From Sondat's theorem, it follows that the orthology axis HG is perpendicular to the homology axis $A''B''$.

55. We prove that the triangle ABC is orthological in relation to $A_1B_1C_1$, and respectively that the orthology center is the gravity center of the triangle ABC. We denote by A' and A'' the intersections of the perpendicular taken from A to B_1C_1 with B_1C_1 respectively with BC.

Then:

$\frac{BA''}{\sin(\widehat{BAA''})} = \frac{AB}{\sin(\widehat{BA''A})}$ and $\frac{A''C}{\sin(\widehat{CAA''})} = \frac{AC}{\sin(\widehat{CA''A})}$.

It follows that:

$$\frac{BA''}{CA''} = \frac{AB}{AC} \cdot \frac{\sin(\widehat{BAA''})}{\sin(\widehat{CAA''})} = \frac{AB}{AC} \cdot \frac{\cos(\widehat{SAA'})}{\cos(\widehat{QAA'})} = \frac{AB}{AC} \cdot \frac{AQ}{AS} = \frac{AB}{AC} \cdot \frac{AC}{AB} = 1.$$

Consequently, AA'' is a median in the triangle ABC. Analogously, it follows that the perpendiculars from B and C are also medians.

56. The perpendiculars taken from A, B, C to B_1C_1, C_1A_1 and A_1B_1 are bisectors in triangle ABC, therefore I – the center of the inscribed circle, is orthology center. The orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC is O_1 – the center of the center circumscribed to the triangle $A_1B_1C_1$.

57. The triangles DFB and ACE are orthological. The orthology center is the center of the circle circumscribed to the triangle ACE.

58. a) The triangles ABC and MNP are obviously homological. We denote $\{I\} = NP \cap BC$; since NP and BC are fixed, it follows that I is fixed. The homology axis of triangles ABC and MNP is $I - K - L$, therefore KL passes through I.

b) The triangles ABC and MNP are orthological, the orthology center is H – the orthocenter of ABC. Also, the perpendiculars taken from B to MP, from C to MN and from A to NP are concurrent. The points of the geometric place belong to the perpendicular taken from A to NP (fixed line).

59. The perpendicular from A' to BC is the altitude from A of the triangle ABC; the perpendicular from B' to AC and the perpendicular from C' to AB intersect in a point P on the altitude AA'; this point is the orthopole of the line AM in relation to the triangle ABC and is the orthology center of the

triangle $A'B'C'$ in relation to ABC. The theorem of the orthological triangle implies the conclusion that ABC is orthological with $A'B'C'$.

60. The triangle B_1C_1A and the triangle CBP are orthological: indeed, the perpendicular taken from A to BC, the perpendicular taken from B_1 to BP and the perpendicular taken from C_1 to CP are concurrent in H. From the theorem of orthological triangles, it follows that the triangle CBP is also orthological in relation to B_1C_1A. Therefore the perpendicular taken from C to C_1A, from B to B_1A and from P to B_1C_1 are concurrent; since the perpendiculars taken from C to C_1A and from B to B_1A are concurrent in H; it follows that the perpendicular from P to B_1C_1 passes through H, so $PH \perp B_1C_1$; but $PH \perp BC$, therefore $B_1C_1 \parallel BC$.

61. We use the property: the perpendiculars taken from the vertices of Fuhrmann triangle on the interior bisectors are concurrent in the orthocenter of the triangle. The orthology center of the Fuhrmann triangle $F_aF_bF_c$ in relation to $A_1B_1C_1$ is H – the orthocenter of the triangle ABC.

62. *Solution 1 (Mihai Miculița).*We denote by P the intersection point of the perpendicular raised in L to AC with the perpendicular raised in M to the side BC, and by K the projection of P on the side AB. To solve the problem, it must be shown that the point K is the midpoint of the side (AB).

From $\left.\begin{matrix} PK \perp AB \\ PL \perp AC \end{matrix}\right\} \Longrightarrow$ the quadrilateral $AKPL$ is inscribable, therefore:

$\sphericalangle KLA \equiv \sphericalangle KPA.$ (1)

From $\left.\begin{matrix} PK \perp AB \\ PM \perp BC \end{matrix}\right\} \Longrightarrow$ the quadrilateral $BMPK$ is inscribable, therefore:

$\sphericalangle KPB \equiv \sphericalangle KMB.$ (2)

From the hypothesis, $\sphericalangle KLC \equiv \sphericalangle KMC$, and from here it follows that their supplements are congruent, therefore:

$\sphericalangle KLA \equiv \sphericalangle KMB.$ (3)

The relations (1), (2) and (3) lead to $\sphericalangle KPA \equiv \sphericalangle KPB$. This relation, (together with $KP \perp AB$) implies that $[AK] \equiv [BK]$.

Solution 2 (Ion Pătrașcu). The relations from the hypothesis, $[AK] \equiv [BK]$ and $\sphericalangle ALK \equiv \sphericalangle BMK$, and the sinus theorem applied in the triangle AKL and BKM lead to the conclusion that the circles circumscribed to these triangles are congruent. Let O_1 respectively O_2 be the centers of these circles; they are located on the same side of the line AB as the vertex C, and we have that $O_1O_2 \parallel AB$. We denote by P the second intersection point of previous circles. We proved that P is the sought point. The point P being the symmetric of K with respect to O_1O_2, and $O_1O_2 \parallel AB$, we have that $PK \perp AB$. From $PK \perp KA$, it follows that (AP) is diameter in the circle of center O_1. Joining P with L, we have that $PL \perp AC$ (the angle ALP is inscribed in semicircle). Reasoning analogously in the circle of center O_2, we have that (BP) is diameter, and $PM \perp AB$.

Notes and comments

The problem in discussion required basically, in the respective hypotheses, to prove that the triangle MLK is orthological in relation to the triangle ABC, and that the orthology center of MLK in relation to ABC is the point P.

It is known that, if ABC is an orthological triangle in relation to $A'B'C'$, and the orthology center is P, then $A'B'C'$ is orthological in relation to BC, the orthology center being P', the isogonal conjugate of P (see [2]). Thus, it follows that the perpendiculars taken from A to LK, from B to KM and from C to ML are concurrent in a point P' – the isogonal conjugate of P in the triangle ABC. Indeed, if we take the perpendicular from A to KL and we denote by A' its foot, we have that $\sphericalangle KAA' \equiv \sphericalangle PAL$, because their complements, $\sphericalangle AKA'$ respectively $\sphericalangle APL$, are congruent, consequently AA' is the isogonal of AP.

In this regard, we propose to the reader to solve this problem, proving that the perpendicular from A to KL, the perpendicular from B to KM, and the perpendicular from C to ML are concurrent.

63. I_a is the orthology center of bilogical triangles ABC and $A_1B_1C_1$. The homology center of these triangles is Γ_a, and XY is axis of homology. According to Sondat's theorem, we have $I_a\Gamma_a \perp XY$.

64. The perpendicular from A_1 to BC is the mediator of BC, therefore O – the center of the circle circumscribed to the triangle ABC is the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC. From the theorem of orthological triangles, it follows that the triangle $A_1B_1C_1$ is also orthological in relation to the triangle ABC.

65. We first prove that AA_1, BB_1, CC_1 are concurrent. Let D_a be the contact of the A-ex-inscribed circle with BC and let D'_a be the diametral of D_a in the A-exinscribed circle. We denote by C_a and C'_a the contact of the inscribed circle with BC and its diametral in the inscribed circle.

The inscribed and A-ex-inscribed circles are homothetic by homothety of center A and ratio $\frac{r_a}{r}$; by the same homothety, the point I_a corresponds to the point I, the point D_a corresponds to the point C'_a, and the point D'_a corresponds to the point C_a. The projection of I on the mediator of the side BC, denoted by A', is such that AA' contains the Nagel point of the triangle ABC. Because C_a and D_a are isotomic points, it follows that A_1 belongs to AC_a, therefore AA_1 passes through Γ (the Gergonne point). Analogously, it is shown that BB_1 and CC_1 contain Γ.

The triangle $A_1B_1C_1$ is obviously orthological in relation to ABC because the perpendiculars taken from A_1, B_1, C_1 to BC, CA and AB are mediators in the triangle ABC, hence O is the orthology center.

66. We denote by O_1 and O_2 the centers of the circles; the circles being congruent, it follows that $O_1O_2 \parallel AB$ (the chords AK and BK being equal, they are equally spaced from the centers). We denote by P the symmetric of K with respect to O_1O_2; obviously, P is the second intersection point of circles of centers O_1 and O_2. Since $O_1O_2 \parallel AB$ and $O_1O_2 \perp PK$, we obtain that A, O_1, P are collinear and also B, O_2, P are collinear. Because AP is diameter, we have that $\sphericalangle PLA = 90^0$; also BP is diameter, so $\sphericalangle PMB = 90^0$. Having $PM \perp BC$, $PL \perp AC$ and $PK \perp AB$, it means that P is the orthology center of the triangle MLK in relation to ABC. The geometric place of the point P (the orthology center of the triangle MLK in relation to ABC) is the semi-line of K origin, perpendicular to AB, and located in the boundary semi-plane AB containing the point C.

67. The triangle EBC is orthological in relation to the triangle FAD. Indeed, the perpendicular from E to AD is EO, the perpendicular from B to FD is BO (because $ODFA$ is rectangle), and the perpendicular from C to FA is CO, hence O is the orthology center. The orthology relation being symmetrical, it means that the triangle FAD is also orthological in relation to the triangle EBC, therefore the perpendicular from F to BC is FH (it is

perpendicular to AD), the perpendicular from D to EB and the perpendicular from A to EC are concurrent. Because the perpendicular from F to AD and the perpendicular from D to EB intersect in H, it follows that the perpendicular from A to EC passes through H.

68. The triangles DAM and SCV are homological, and T is the homology center (T is the Toricelli point of the triangle DAM). The triangle SCV is orthological in relation to DAM, and the orthology center is the point P (indeed, the perpendicular from S to AM, the perpendicular from C to DM and the perpendicular from V to AD are concurrent in P – the midpoint of DM). From the theorem of orthological triangles, it follows that the triangle DAM is orthological in relation to SCV, and the orthology center is Q. Sondat's theorem implies the collinearity of points T, P, Q.

69. Let $A'B'C'$ be the median triangle of $A_1B_1C_1$. Then the triangle $A_2B_2C_2$ is the homothetic of the triangle $A'B'C'$ by homothety of center P and by ratio 2. The triangle $A_1B_1C_1$ is orthological in relation to $A'B'C'$; the orthology center is the orthocenter of $A_1B_1C_1$. The triangle $A_2B_2C_2$ having the sides parallel to that of the triangle $A'B'C'$, it means that the orthocenter of $A_1B_1C_1$ is orthology center of the triangle $A_1B_1C_1$ in relation to $A_2B_2C_2$.

70. i) If ABC is an acute triangle and $A'B'C'$ is its orthic triangle, then $A'B'C'$ is acute and $m(\widehat{A'}) = 180^0 - 2\hat{A}$,$m(\widehat{B'}) = 180^0 - 2\hat{B}$, $m(\widehat{C'}) = 180^0 - 2\hat{C}$. If $A''B''C''$ is the orthic triangle of the triangle $A'B'C'$, then: $m(\widehat{A''}) = 4\hat{A} - 180^0$, $m(\widehat{B''}) = 4\hat{B} - 180^0$, $m(\widehat{C''}) = 4\hat{C} - 180^0$. We observed that a right triangle does not have an orthic triangle, therefore it would be necessary for the triangle ABC to have an angle of measure 45^0.

ii) If the triangle ABC has, for example, $m(\hat{A}) = 112^0 30'$, then its orthic triangle $A'B'C'$ has $m(\widehat{A'}) = 45^0$, $m(\widehat{B'}) = 2m(\hat{B})$, $m(\widehat{C'}) = 2m(\hat{C})$. The triangle $A''B''C''$ – the orthic triangle of $A'B'C'$, has $m(\widehat{A''}) = 90^0$, $m(\widehat{B''}) = 180^0 - 4m(\hat{B})$, $m(\widehat{C''}) = 180^0 - 4m(\hat{C})$. The triangle $A''B''C''$ being rectangular, it does not have an orthic triangle.

iii) If ABC is an equilateral triangle, then $A'B'C'$ and $A''B''C''$ are equilateral triangles, and ABC is orthological in relation to $A''B''C''$. In general, ABC and $A''B''C''$ are not orthological.

71. a) The perpendicular taken from K to AC is KM because, being midline in the triangle ABD, we have $KM \parallel BD$, therefore $KM \perp AC$. Analogously, the perpendicular from L to AB is LM; obviously, the perpendicular from M to BC passes through M, hence M is the orthology center of the triangle MKL in relation to ABC.

b) The other orthology center is A; indeed, the perpendicular from B to ML is BA, the perpendicular from C to KM is CA and the perpendicular from A to KL passes obviously through A. Since in the triangle AKL the point M is orthocenter (KL and LM are altitudes), it follows that AM is an altitude, hence $AM \perp KL$.

72. a) Because AA' is bisector, we have the arcs $\breve{BA'} \equiv \breve{CA'}$, therefore $BA' = CA'$, and the perpendicular from A' to BC is the mediator of BC. Thus, we obtain that the I-circumpedal triangle is orthological in relation to ABC, and the orthology center is O – the center of the circumscribed circle. It can be shown directly that ABC is orthological in relation to $A'B'C'$, and the orthology center is I.

b) We noticed that $A'B = A'C$; the triangle $A'BI$ is isosceles ($A'B = A'I$) because $\sphericalangle A'BI \equiv \sphericalangle A'IB$, therefore the circle $\mathcal{C}(A';\, A'B)$ passes through I; analogously, it follows that the other circles pass through I as well.

c) The triangles ABC and $A'B'C'$ are homological; the axis of homology is XY; the axis of orthology is OI; according to Sondat's theorem, $OI \perp XY$.

73. Obviously, $BH \perp AQ$, $BM \perp QN$. We prove that we also have $MH \perp AN$. We observe that $m(\widehat{AQM}) = m(\widehat{MQC}) = 90^0 - \hat{C} = \widehat{HBC}$. Also, $m(\widehat{BHC}) = m(\widehat{PAQ}) = 180^0 - \hat{A}$, therefore: $\Delta BHC \sim \Delta QAP$. It also follows from here that $\Delta BHM \sim \Delta QAN$. We note that $\sphericalangle ANQ \equiv \sphericalangle HMB$; because $NM \perp BC$, we have that $m(\widehat{HMN}) + m(\widehat{ANQ}) = 90^0$, consequently $MH \perp NA$.

Observation

It can be seen that the point $\{S\} = MN \cap NA$ belongs to the circle circumscribed to the triangle ABC.

74. If P is the orthology center of the triangle $A_1B_1C_1$ in relation to the triangle ABC, and O is the center of the circle ω, ie. the intersection of mediators of the segment (A_1A_2), (B_1B_2), (C_1C_2), and the perpendicular in A_2 on BC intersects PO in Q, the point Q will be the symmetric of P towards O. Similarly, the perpendicular from B_2 on AC passes through the symmetric of P with respect to O, therefore through Q, and in the same way Q belongs to the perpendicular in C_2 to AB. Consequently: the point Q is the orthology center of the triangle $A_2B_2C_2$ in relation to the triangle ABC.

75. We denote by H' the projection of H on the plane $A_1B_1C_1$ and by A'', B'', C'' the projections of points A, B, C respectively on B_1C_1, C_1A_1 and A_1B_1. From $AA' \perp (A_1B_1C_1)$ and $AA'' \perp B_1C_1$, we obtain that $A'A'' \perp B_1C_1$ (the reciprocal theorem of the three perpendicular). On the other hand, being perpendicular to $(A_1B_1C_1)$, the plane $(AA'A'')$ contains HH', and even more: $H' \in A'A''$. Similarly, we show that $H' \in B'B''$ and $H' \in C'C''$. The point H' is the orthology center of the triangle $A'B'C'$ in relation to $A_1B_1C_1$.

76. The perpendicular from H to B_1C_1 is the mediator of BC, the perpendicular from B to A_1C_1 passes through O – the center of the circle circumscribed to the triangle ABC, and also, the perpendicular from C to A_1B_1, which is antiparallel with AB, passes through O. Therefore O – the center of the circle circumscribed to the triangle ABC, is the orthology center of the triangle HBC in relation to $A_1B_1C_1$.

77. *Solution* (Mihai Miculița). We prove that the line A_bB_a is mediator of the side AB. We have: $OA = OB \Rightarrow \sphericalangle BAO \equiv \sphericalangle ABO$.

$$\left.\begin{array}{r} AOB_aC - \text{inscribable} \Rightarrow \sphericalangle OAB_a \equiv \sphericalangle OCB \\ OB = OC \Rightarrow \sphericalangle OCB \equiv \sphericalangle OBC \end{array}\right\} \Rightarrow \sphericalangle OAB_a \equiv \sphericalangle OBC$$

$$\Rightarrow m(\widehat{BAB_a}) = m(\widehat{BAO}) + m(\widehat{OAB_a}) = m(\widehat{ABO}) + m(\widehat{OBC}) = m(\widehat{ABB_a}) \Rightarrow B_aA = B_aB. \quad (1)$$

On the other hand, from:

$OA = OB \Rightarrow \sphericalangle BAO \equiv \sphericalangle ABO$,

$$\left.\begin{array}{r} OA = OC \Rightarrow \sphericalangle OAC \equiv \sphericalangle OCA \\ COBA_b - \text{inscribable} \Rightarrow \sphericalangle OCA \equiv \sphericalangle OBA_b \end{array}\right\} \Rightarrow \sphericalangle OAC \equiv \sphericalangle OBA_b$$

$\Rightarrow\ m(\widehat{BAA_b}) = m(\widehat{BAO}) + m(\widehat{OAC}) = m(\widehat{ABO}) + m(\widehat{OBA_b}) = m(\widehat{ABA_b}) \Rightarrow \widehat{BAA_b} \equiv \widehat{ABA_b} \Rightarrow A_bA = A_bB.$ (2)

The relations (1) and (2) show that the line A_bB_a is mediator of the side AB, so we have: $A_bB_a \cap A_cC_a \cap B_cC_b = \{O\}$.

Observation

The triangles $A_bA_cB_c$ and CBA are orthological; the orthology center is O. Also, the triangles $B_aC_bC_a$ and CAB are orthological of center O.

78. We have $\frac{A_1C}{B_1C} = \frac{b\cos C}{c\cos B} = \frac{C_1X}{XB_1}$. We take $AX' \perp B_1C_1$, $X' \in (B_1C_1)$.

$B_1X' = AB_1 \cdot \cos B$, $C_1X' = AC_1 \cdot \cos C$.

$AB_1 = c \cdot \cos A$, $AC_1 = b \cdot \cos A$.

$\frac{C_1X'}{B_1X'} = \frac{b\cos C}{c\cos B} = \frac{C_1X}{XB_1}$, it follows that $X' = X$.

The triangles ABC and its orthic triangle $A_1B_1C_1$ are orthological, therefore AX, BY and CZ are concurrent. The concurrency point is the orthology center of the triangle ABC in relation to $A_1B_1C_1$, therefore O – the center of the circle circumscribed to the triangle ABC.

79. Obviously, the triangle $A_2B_2C_2$ is orthological in relation to ABC, because $PA_1 \perp BC$ and $A_2 \in (PA_1)$; from the theorem of the three perpendiculars, it follows that $A_1A_2 \perp BC$; similarly $B_1B_2 \perp AC$ and $C_2C_1 \perp AB$, and $A_1A_2 \cap B_1B_2 \cap C_1C_2 = \{P\}$, hence P is orthology center. Let $BB' \perp A_1C_1$ and let O_1 be the orthology center of the triangle ABC in relation to $A_1B_1C_1$, $B' \in (A_1C_1)$. From $B'A_1^2 - B'C_1^2 = PA_1^2 - PC_1^2$.

80. *Solution* (Mihai Miculița).

We denote the midpoint of the segment EF by Q; we show that $Q \in [AP]$.

$$\left.\begin{array}{l}\left.\begin{array}{r}BE \perp AC\\ MB = MC\end{array}\right\} \Rightarrow ME = \frac{1}{2}BC\\ \left.\begin{array}{r}CF \perp AB\\ MB = MC\end{array}\right\} \Rightarrow MF = \frac{1}{2}BC\end{array}\right\} \Longrightarrow \left.\begin{array}{c}ME = MF\\ QE = QF\end{array}\right\} \Longrightarrow MQ \perp EF. \qquad (1)$$

On the other hand, whereas:

$$\left.\begin{array}{l}BE \perp AC\\ CF \perp AB\end{array}\right\} \Longrightarrow BCEF - \text{inscribable} \Longrightarrow \begin{cases}\sphericalangle AEF \equiv \sphericalangle ABC\\ \sphericalangle QAF \equiv \sphericalangle MAC\end{cases}. \qquad (2, 3)$$

Taking the relations (1) and (3) into consideration, we obtain that:

$$\left.\begin{array}{l}MQ \perp EF\\ NP \perp BC\end{array}\right\} \Longrightarrow \left.\begin{array}{c}MNQP \text{ inscribable}\\ \sphericalangle AQE \equiv \sphericalangle AMB\end{array}\right\} \Rightarrow Q \in AP \Rightarrow (AQ = (AP. \quad (4)$$

From relations (3) and (4), it follows that the points N and P are homological points of the similar triangles $\Delta AEF \sim \Delta ABC$. Hence we have:

$$\frac{NE}{NF} = \frac{PB}{PC}. \quad (5)$$

From relations (4) and (5), it follows now that:

$$\frac{NE}{NF} = \frac{RE}{RC} \Longrightarrow \left.\begin{array}{l}RN \parallel CF\\ CF \perp AB\end{array}\right\} \Longrightarrow RN \perp AC. \quad (6)$$

Similarly, it is shown that $SN \perp AC$. (7)

Finally, the relations (6) and (7) shows that the lines AN and SN are altitudes in the triangle ARS, therefore N is the orthocenter of this triangle.

Observation

During the solution, we solved the following problem:

Let E and F – be the feet of the altitudes taken from the vertices B and C of an acute triangle ABC, and M – the midpoint of the side BC. We denote by $\{N\} = AM \cap EF$ and by P the projection of N on BC. Show that the semi-line $(AP$ is a symmedian of the triangle ABC.

81. Obviously, if $k = 1$, the lines a, b, c are concurrent being the mediators of the triangle ABC. Reciprocally, let $a \cap b \cap c = \{S\}$. Then $\overrightarrow{MS} \cdot \overrightarrow{BC} = 0$, ie. $(\overrightarrow{r_S} - \overrightarrow{r_M})(\overrightarrow{r_C} - \overrightarrow{r_B}) = 0$.

From here, $\overrightarrow{r_S} \cdot (\overrightarrow{r_C} - \overrightarrow{r_B}) = \frac{\overrightarrow{r_B} + k\overrightarrow{r_C}}{1+k}(\overrightarrow{r_C} - \overrightarrow{r_B})$.

We write the analogous relations, and we add them. We obtain:

$$\sum(\overrightarrow{r_B} + k\overrightarrow{r_C})(\overrightarrow{r_C} - \overrightarrow{r_B}) = 0. \quad (1)$$

We consider a system of axes with the origin in the center of the circle circumscribed to the triangle ABC. Suppose this circle has the radius 1. Then: $\overrightarrow{r_B} \cdot \overrightarrow{r_B} = 1$.

The equality (1) becomes:

$\sum(\overrightarrow{r_B} \cdot \overrightarrow{r_C} + k - 1 - k \cdot \overrightarrow{r_B} \cdot \overrightarrow{r_C}) = 0$, namely:

$(k-1)(3 - \overrightarrow{r_A} \cdot \overrightarrow{r_B} - \overrightarrow{r_B} \cdot \overrightarrow{r_C} - \overrightarrow{r_C} \cdot \overrightarrow{r_A}) = 0$.

But:

$|\overrightarrow{r_A} \cdot \overrightarrow{r_B} + \overrightarrow{r_B} \cdot \overrightarrow{r_C} + \overrightarrow{r_C} \cdot \overrightarrow{r_A}| < |\overrightarrow{r_A}||\overrightarrow{r_B}| + |\overrightarrow{r_B}||\overrightarrow{r_C}| + |\overrightarrow{r_C}||\overrightarrow{r_A}| = 3$, consequently $k = 1$.

Observation

The problem expresses that only the triangle MNP – inscribed in the triangle ABC, with the property $\frac{MB}{MC}=\frac{NC}{NA}=\frac{PA}{PB}$, and orthological with ABC, is the median triangle.

82. We denote: $M_aM_bM_c$ the median triangle of the triangle ABC, and $T_aT_bT_c$ – the tangential triangle of the triangle ABC. The triangles $T_aT_bT_c$ and ABC are orthological, and O is their orthology center. Indeed, the perpendiculars taken from T_a, T_b, T_c to BC, CA, AB are bisectors in $T_aT_bT_c$ and consequently pass through O, which is the center of the circle inscribed in the triangle $T_aT_bT_c$. Moreover, T_aO is mediator of (BC), and it consequently passes through M_a; T_aM_a being mediator, it is perpendicular to BC, but also to M_bM_c, which is midline. From the theorem of orthological triangles, it follows that the perpendiculars taken from M_a, M_b, M_c to T_bT_c, T_cT_a respectively T_aT_b are concurrent as well. The point of concurrency is O_9 – the center of the Euler circle of the triangle ABC. We prove this fact. We take $M_aM_1 \perp T_bT_c$; we denote $\{H_1\}=M_aM_1\cap AH$, where H is the orthocenter of the triangle ABC. It is known that $AH = 2OM_a$. We join O with A; we have that $OA \perp T_bT_c$; since $M_aM_1 \perp T_bT_c$ and $AH \parallel OM_a$, it follows that the quadrilateral OM_aH_1A is parallelogram. From $AH_1 = OM_a$ and $AH = 2OM_a$, we obtain that H_1 is the midpoint of (AH), therefore H_1 is on the circle of the nine points of the triangle ABC. On this circle, we find also the points A' (the feet of the altitude from A) and M_a. Since $\sphericalangle AA'M_a = 90^0$, it follows that M_aH_1 is diameter in Euler circle, therefore the midpoint of M_aH_1 is the center of Euler circle, which we denote by O_9. We observe that the quadrilateral H_1HM_aO is parallelogram as well, and it follows that O_9 is the midpoint of OH.

Observation

The triangles $M_aM_bM_c$ and $T_aT_bT_c$ are bilogical.

83. We denote: $AD = x$, $BD = y$, $CD = z$ and $AB = BC = CA = a$; let A_2, B_2, C_2 be the contacts of the circles inscribed in the triangles BDC, CDA and ABD with CB, CA respectively AB.

It is shown without difficulty that:

$BA_2 = \frac{y+a-z}{2}$; $CA_2 = \frac{a+z-y}{2}$; $CB_2 = \frac{z+a-x}{2}$;
$AB_2 = \frac{a+x-z}{2}$; $AC_2 = \frac{x+a-y}{2}$; $BC_2 = \frac{y+a-x}{2}$.

We show that the triangle $A_1B_1C_1$ is orthological in relation to ABC proving the equality:

$BA_2^2 + CB_2^2 + AC_2^2 = CA_1^2 + AB_2^2 + BC_1^2$.

Consequently, it also follows that ABC is orthological in relation to $A_1B_1C_1$.

84. We denote by x, y, z the distances of the points A_1, B_1, C_1 from the line d. In order that the perpendiculars taken from A, B, C respectively to B_1C_1, C_1A_1, A_1B_1 to be concurrent, the following condition must be satisfied:

$AB_1^2 - AC_1^2 + BC_1^2 - BA_1^2 + CA_1^2 - CB_1^2 = 0$.

This condition is equivalent to:

$$(a_1^2 + y^2) - (a_2^2 + z^2) + (b_1^2 + z^2) - (b_2^2 + x^2) + (c_1^2 + x^2) - (c_2^2 + y^2) = 0.$$

From where we find:

$a_1^2 - a_2^2 + b_1^2 - b_2^2 + c_1^2 - c_2^2 = 0$ (it does not depend of x, y, z).

Remark

If the points A, B, C belong to the lines d_1, d_2, d_3, the preceding condition is obviously fulfilled, and the concurrency point of the perpendiculars taken from A, B, C to B_1C_1, C_1A_1 respectively A_1B_1 is the orthopole of the line d in relation to the triangle $A_1B_1C_1$.

85. The perpendicular from A_1 to M_bM_c is A_1M, the perpendicular from B_1 to M_cM_a is B_1M, and the perpendicular from C_1 to M_aM_b is C_1M, hence M is the orthology center of the triangle $A_1B_1C_1$ in relation to the median triangle $M_aM_bM_c$.

86. The triangle $A_1B_1C_1$ is orthological in relation to ABC, therefore the perpendiculars taken from A_1, B_1, C_1 to BC, CA, AB are concurrent in a point P. The triangle $A'B'C'$, the symmetric with respect to O of the triangle ABC, has the sides respectively parallel with its sides. The perpendicular from A_1 to BC will be perpendicular to $B'C'$ as well. The orthology center of the triangle $A_1B_1C_1$ in relation to $A'B'C'$ will be the point P.

87. The triangle EBC is orthological in relation to the triangle FAD, the orthology center being O. Indeed, the perpendicular taken from E to AD, the perpendicular taken from B to FD, and the perpendicular taken from C to FA intersect in O. The relation of orthology being symmetrical, it follows that the triangle FAD is orthological in relation to the triangle EBC as well. Then the perpendicular from F to BC, the perpendicular from D to EB and the perpendicular from A to EC are concurrent. Because the perpendicular from D to EB and the perpendicular from F to AD intersect in H, it follows that the perpendicular from A to EC passes through H as well, therefore $AH \perp EC$.

88. We denote by $A'B'C'$ the podal triangle of the symmedian center K of the triangle ABC; also, we denote by A_1 the intersection of perpendicular from A to $B'C'$ with $B'C'$. We have that $\sphericalangle C'AA_1 \equiv \sphericalangle KC'B'$ (1) (they have the same complement), $\sphericalangle KC'B' \equiv \sphericalangle KAB'$ (2) (the quadrilateral $KC'AB'$is inscribable). From relations (1) and (2), it follows that $\sphericalangle C'AA_1 \equiv \sphericalangle KAB'$, therefore AA_1 is the isogonal of symmedian AK, hence AA_1 passes through G – the gravity center of the triangle ABC. It is obvious that the symmedian center K is orthology center. To prove that K is the gravity center of the triangle $A'B'C'$, it is shown that:

$\text{Area}(\Delta KB'C') = \text{Area}(\Delta KC'A') = \text{Area}(\Delta KA'B')$.

We use the relation:

$\frac{KA'}{BC} = \frac{KB'}{AC} = \frac{KC'}{AB}$.

89. a) We denote by P_1 the intersection of the parallel taken through A with $B'C'$, with the parallel taken through C with $A'B'$. We have that $\sphericalangle AP_1C \equiv \sphericalangle B'C'A'$, therefore $m(AP_1C) = 60^0$, which shows that P_1 belongs to the circle circumscribed to the triangle ABC. From angle measurement, BP_1C is 60^0; and $CP_1 \parallel A'B'$ from the reciprocal theorem of the angle with parallel sides; we have that $BP_1 \parallel A'C'$, hence P_1 is parallelogy center of the triangle ABC in relation to the triangle $A'B'C'$. Similarly reasoning, we find that P_2 and P_3, parallelogy centers of the triangle ABC in relation to $B'C'A'$ and $C'A'B'$, are located on the circumscribed circle of the triangle ABC.

b) AP_2 is parallel with $A'C'$, and BP_1 is also parallel with $A'C'$; it follows that $AP_2 \parallel BP_1$. The trapeze AP_2BP_1, being inscribed, is isosceles, therefore the diagonals AB and P_1P_2 are congruent. Similarly, it is shown that $P_2P_3 = BC$ and $P_1P_3 = AC$, consequently the triangle $P_1P_2P_3$ is equilateral.

90. To prove that the triangle $A_2B_2C_2$ is orthological in relation to ABC, the following relation must be verified:

$A_2B^2 - A_2C^2 + B_2C^2 - B_2A^2 + C_2A^2 - C_2B^2 = 0.$ (1)

Because AA_1, BB_1, CC_1 are concurrent, we have that:

$C_1A^2 - C_1B^2 + A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 = 0.$ (2)

The perpendicular AA_2 to B_1C_1 – which is antiparallel to BC – passes through O – the center of the circle circumscribed to the triangle ABC. Also, BB_2 and CC_2 pass through O. The cevians AA_2, BB_2 and CC_2 being concurrent, we have that:

$A_2C_1^2 - A_2B_1^2 + B_2C_1^2 - B_2A_1^2 + C_2A_1^2 - C_2B_1^2 = 0.$ (3)

We calculate $A_2B^2 - B_2A^2$. We will apply the cosine theorem in the triangles A_2BC_1 respectively B_2AC_1. The lines B_1C_1 and C_1A_1 being antiparallels with BC respectively AC, it follows that $\sphericalangle AC_1B_1 \equiv \sphericalangle BC_1A_1$. The triangles AC_1A_2 and BC_1B_2 are similar; we obtain that:

$AC_1 \cdot C_1B_2 = BC_1 \cdot C_1A_2.$ (4)

$A_2B^2 = C_1B^2 + C_1A_2^2 - 2C_1B \cdot C_1A_2 \cdot \cos \sphericalangle BC_1A_2,$

$B_2A^2 = C_1A^2 + C_1B_2^2 - 2C_1A \cdot C_1B_2 \cdot \cos \sphericalangle AC_1B_2.$

Since $\sphericalangle BC_1A_2 \equiv \sphericalangle AC_1B_2$, taking (4) into consideration, we obtain that:

$A_2B^2 - B_2A^2 = C_1B^2 - C_1A^2 + C_1A^2 - C_1B_2^2.$ (5)

Similarly, we calculate: $B_2C^2 - C_2B^2$ and $C_2A^2 - A_2C^2$ and, considering the relations (5), (2) and (3), we obtain the relation (1).

91. Let M_1 the midpoint of LM, M_2 the midpoint of MK, and M_3 the midpoint of KL. It is shown that:

$M_1B^2 - M_1C^2 + M_2C^2 - M_2A^2 + M_3A^2 - M_3B^2 = 0.$ (1)

We calculate M_1B^2 with the median theorem applied in the triangle MBL, and we have:

$M_1B^2 = \frac{2(AB^2+BL^2)-ML^2}{4},$

$M_1C^2 = \frac{2(AC^2+CM^2)-ML^2}{4}.$

Because $\Delta ABL \equiv \Delta AMC$ (S.A.S), it follows that $BL = MC$ and $M_1B^2 - M_1C^2 = \frac{AB^2 - AC^2}{2}$.

Similarly, we calculate: $M_2C^2 - M_2A^2$ and $M_3A^2 - M_3B^2$.

Replacing in (1), this is verified.

92. The symmedian center K of the triangle ABC is the gravity center of the triangle $A_1B_1C_1$. The triangle $A_2B_2C_2$ is the symmetric, in relation to K, of the triangle $A_1B_1C_1$; we have $B_1C_1 \parallel B_2C_2$; $A_1B_1 \parallel A_2B_2$ and $A_1C_1 \parallel A_2C_2$. The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthological, and their common orthology center is the orthocenter of the triangle $A_1B_1C_1$.

93. We denote by P the midpoint of DE. The perpendiculars taken from P, Q, R to BC, BE and CD are concurrent in the point M – the symmetric of O with respect to the center of the parallelogram $PQSR$ (S is the midpoint of BC). The point M is called the Mathot point of the inscribable quadrilateral $BCDE$. In other words, the triangle PRQ is orthological in relation to ABC. It follows that ABC is orthological as well in relation to PRQ, therefore the perpendicular taken from A to RQ, the perpendicular taken from B to PQ, and the perpendicular taken from C to PR are concurrent. Because $PQ \parallel CE$ and $RP \parallel BD$, we get to the required conclusion.

94. The perpendiculars taken from B, F, D respectively to CA, EA and EC are BE, FC and DA, and they intersect in O – the center of the circle circumscribed to the hexagon, the point which is the common center of orthology of triangles BFD and ECA. The triangle BFD is orthological in relation to the triangle CEA, and the orthology center is the point A. The orthology center of the triangle CEA in relation to BFD is the point D. The triangle BFD is orthological in relation to the triangle ACE, and the orthology center is the point C. The orthology center of the triangle ACE in relation to BFD is the point F.

95. a) Let M be the intersection of the parallel with BC, with the parallel to AC. The quadrilateral MC_1BA_1 is a rectangle; $MA_1 = \frac{\sqrt{21}}{7}$, $MC_1 = \frac{2\sqrt{7}}{7}$,

$A_1C = \frac{5\sqrt{7}}{7}$. The quadrilateral MA_1CB_1 is inscribable; from sinus theorem, we have that: $A_1B_1 = MC \cdot \sin 30^0$, $A_1B_1 = \frac{1}{2}MC$.

From the triangle MA_1C, it follows that $MC = 2$; therefore $A_1B_1 = 1$.

$A_1C_1{}^2 = \left(\frac{\sqrt{21}}{7}\right)^2 + \left(\frac{2\sqrt{7}}{7}\right)^2 = 1$, therefore $A_1C_1 = 1$.

We denote $MB_1 = x$; applying cosine theorem in triangle A_1MB_1, the equation $7x^2 + 3\sqrt{7}x - 4 = 0$ is obtained, with the appropriate solution $x = \frac{\sqrt{7}}{7}$, which shows that M also belongs to the parallel with AC.

b) The quadrilateral MB_1AC_1 is inscribable; $\sphericalangle B_1MC_1 = 120^0$; we calculate B_1C_1 with cosine theorem; it follows that $B_1C_1 = 1$, therefore the triangle $A_1B_1C_1$ is equilateral.

c) In the triangles ABC and $A_1B_1C_1$, the cevians are obviously orthological. They are not bilogical because they are not homological; AA_1, BB_1, CC_1 are not concurrent. Indeed, it is verified that: $\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} \neq 1$.

96. Obviously, $AD \perp KL$ (the radius is perpendicular to the tangent). Since $m\left(\widehat{DCF}\right) = 90^0$, it follows that $m\left(\widehat{DMF}\right) = 90^0$ as well (1). Because $m\left(\widehat{DBA}\right) = 90^0$ (AD diameter), it follows that $m\left(\widehat{EMD}\right) = 90^0$ as well (2). The relations (1) and (2) lead to E, M, F collinear. The triangles DEF and AKL have $EF \parallel KL$; it is obvious that the orthocenter of the triangle AEF is their orthology center; we denote it by P. The quadrilateral $ABDC$ is inscribable (3), $\sphericalangle BAM \equiv \sphericalangle PFM$ (4) (sides respectively perpendicular). From (3) and (4), it follows that $\sphericalangle PFM \equiv \sphericalangle MFD$, therefore $\sphericalangle MFD \equiv \sphericalangle MCD$ (5). In the triangle PFD, MF is an altitude and a bisector, therefore PFD is isosceles, hence $MD = PM$.

97. It is obvious that the triangle MNP is orthological in relation to ABC, and the orthology center is I_a. We prove that AM, BN, CP are concurrent using Ceva's theorem – trigonometric variant. We take $FF' \perp BC$; $MM' \perp BC$; $EE' \perp BC$, we have $\frac{FM}{ME} = \frac{F'M'}{E'M'}$; $F'M' = BF' + BM'$, $BF' = BF \cdot \cos B$; from the theorem of the exterior bisector, it follows that $\frac{FB}{FA} = \frac{a}{b}$; we find that $FB = \frac{ac}{b-a}$; $BM' = p - c$. Then: $F'M' = \frac{c(p-a)}{b-a}$.

$E'M' = CE' + CM' = CE\cos C + p - b.$

But $CE = \frac{a}{c-a}$, it follows that $E'M' = \frac{b(p-c)}{c-a}$, $FA = \frac{bc}{b-a}$, $EA = \frac{bc}{c-a}$.

We get:

$$\frac{\sin BAM}{\sin MAC} = \frac{c(p-b)}{b(p-c)}. \quad (2)$$

Calculating in a similar way, we find:

$$\frac{\sin CBN}{\sin NBF} = \frac{Qp}{c(p-b)}, \quad (3)$$

$$\frac{\sin ECP}{\sin PCB} = \frac{b(p-c)}{ap}. \quad (4)$$

Replacing (2), (3), (4) in relation (1), the equality implying the concurrency of lines AM, BN, CP is verified.

98. The barycentric coordinates of a point P in relation to the triangle ABC are three numbers x, y, z proportional to the areas of the triangles PBC, PCA and PAB. Because $P_1A_1 \perp BC$, $P_1B_1 \perp CA$ and $P_1C_1 \perp AB$, we have that $\sin B_1P_1C_1 = \sin A$, $\sin P_1B_1C_1 = \sin PAC$ and $\sin P_1C_1B_1 = \sin PAB$.

We have:

$$\frac{\text{Area}PAC}{\text{Area}PAB} = \frac{PA\cdot b\cdot \sin PAC}{PA\cdot c\cdot \sin PAB} = \frac{b\cdot \sin P_1B_1C_1}{c\cdot \sin P_1C_1B_1} = \frac{b}{c}\cdot\frac{P_1C_1}{P_1B_1}.$$

Similarly, we get:

$$\frac{\text{Area}PBC}{\text{Area}PCA} = \frac{a}{c} \div \frac{P_1C_1}{P_1A_1}.$$

On the other hand, we find:

$$\frac{\text{Area}(P_1A_1C_1)}{\text{Area}(P_1A_1B_1)} = \frac{P_1A_1\cdot P_1C_1\cdot \sin B}{P_1A_1\cdot P_1B_1\cdot \sin C} = \frac{b}{c}\cdot\frac{P_1C_1}{P_1B_1}.$$

Therefore:

$$\frac{\text{Area}(PAC)}{\text{Area}(PAB)} = \frac{\text{Area}(P_1A_1C_1)}{\text{Area}(P_1A_1B_1)}.$$

Similarly:

$$\frac{\text{Area}(PBC)}{\text{Area}(PCA)} = \frac{\text{Area}(P_1B_1C_1)}{\text{Area}(P_1C_1A_1)},$$

$$\frac{\text{Area}(PAB)}{\text{Area}(PBC)} = \frac{\text{Area}(P_1A_1B_1)}{\text{Area}(P_1B_1C_1)}$$

99. $\frac{FM}{ME} = \frac{\text{Area}(FAM)}{\text{Area}(EAM)} = \frac{FA}{EA}\cdot\frac{\sin\alpha}{\sin(A-\alpha)}$.

We denoted $\alpha = \sphericalangle BAM$.

We take FF', EE' and MM' perpendicular to BC; we have $BM' = p - b$, $\frac{FM}{ME} = \frac{F'M'}{E'M'}$, $BF' = FB \cdot \cos B$. From bisector's theorem, we find:

$FB = \frac{ac}{a+b}$; $FB \cdot \cos B = \frac{a^2+c^2-b^2}{2(a+b)}$;

$F'M' = \frac{a+b+c}{2} - b - \frac{a^2+c^2-b^2}{2(a+b)} = \frac{c(p-c)}{a+b}$;

$E'M' = CM' - CE' = p - c - EC \cdot \cos C$;

$EC = \frac{ab}{a+c}$; we find that $E'M' = \frac{b(p-b)}{a+c}$;

$\frac{FM}{ME} = \frac{F'M'}{E'M'} = \frac{c(p-c)(a+c)}{b(p-b)(a+b)}$;

$$\frac{\sin\alpha}{\sin(A-\alpha)} = \frac{F'M'}{E'M'} \cdot \frac{EA}{FA} = \frac{c(p-c)}{b(p-b)}. \quad (1)$$

Similarly, we find that $\frac{\sin\beta}{\sin(B-\beta)} = \frac{a(p-a)}{c(p-c)}$, (2)

$$\frac{\sin\gamma}{\sin(C-\gamma)} = \frac{b(p-b)}{a(p-a)}. \quad (3)$$

We denoted $\beta = \sphericalangle NBC$ and $\gamma = \sphericalangle PCA$. The relations (1), (2), (3) and Ceva's theorem – trigonometric variant, lead to the concurrency of the cevians AM, BN, CP. If we denote $\{X\} = AM \cap BC$, $\{Y\} = BN \cap AC$, $\{Z\} = CP \cap AB$ and by L – the concurrency point of cevians AM, BN, CP, we find:

$$\frac{XB}{XC} = \frac{c^2(p-c)}{b^2(p-b)}. \quad (4)$$

The relation (4) and the Steiner relation relative to the isogonal cevians show that the cevian AX is the isogonal of Nagel cevian relative to BC. Thus, we find that the point L is the Nagel isogonal point of the triangle ABC. The barycentric coordinates of the points I, J, O, where J is the Nagel point, are:

$I\left(\frac{a}{2p}, \frac{b}{2p}, \frac{c}{2p}\right)$;

$J\left(\frac{p-a}{p}, \frac{p-b}{p}, \frac{p-c}{p}\right)$;

$O(\sin 2A, \sin 2B, \sin 2C)$.

The coordinates of L – the isogonal of J – are:

$L\left(\frac{a^2p}{p-a}, \frac{b^2p}{p-b}, \frac{c^2p}{p-c}\right)$.

To show that I, L, O are collinear, the following condition must be proved:

$$\begin{vmatrix} \frac{a}{2p} & \frac{b}{2p} & \frac{c}{2p} \\ \frac{a^2p}{p-a} & \frac{b^2p}{p-b} & \frac{c^2p}{p-c} \\ \sin 2A & \sin 2B & \sin 2C \end{vmatrix} = 0. \quad (5)$$

Because $\sin 2A = 2\sin A \cos A$ and $\sin A = \frac{a}{2R}$, condition (5) is equivalent to:

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{a}{p-a} & \frac{b}{p-b} & \frac{c}{p-c} \\ \cos A & \cos B & \cos C \end{vmatrix} = 0. \quad (6)$$

Condition (6) implies that:

$$\begin{vmatrix} 1 & 0 & 0 \\ \frac{a}{p-a} & \frac{b}{p-b} - \frac{a}{p-a} & \frac{c}{p-c} - \frac{a}{p-a} \\ \cos A & \cos B - \cos A & \cos C - \cos A \end{vmatrix} = 0. \quad (7)$$

Therefore:

$$\begin{vmatrix} \frac{b}{p-b} - \frac{a}{p-a} & \frac{c}{p-c} - \frac{a}{p-a} \\ \cos B - \cos A & \cos C - \cos A \end{vmatrix} = 0. \quad (8)$$

Considering that $\cos A = \frac{b^2+c^2-a^2}{2bc}$, $\cos B = \frac{a^2+c^2-b^2}{2ac}$, $\cos C = \frac{a^2+b^2-c^2}{2ab}$, it is found that the relation (8) is verified.

100. From the given relation, it follows that $AP^2 - PE^2 = BP^2 - PD^2$, therefore $AE = BD$; we denote: $AE = BD = z$, and we find similarly that $CD = AF = y$ and $BF = CE = x$. We show that all the points D, E, F belong to the sides of the triangle ABC. Indeed, if, for example, B, C and D are in this order, then we have: $AB + BC = (x + y) + (z - y) = x + z = AC$ – contradiction. Denoting by a, b, c the lengths of the sides BC, CA, AB and $p = \frac{a+b+c}{2}$, we find that $x = p - a$, $y = p - b$, $z = p - c$. These relations show that the points D, E, F are contacts of the ex-inscribed circles with BC, CA, AB, hence I_a, D, P are collinear, and also I_b, E, P and I_c, F, P. Thus, we find that P is orthology center of the antisupplementary triangle $I_aI_bI_c$ in relation to the given triangle ABC. We noticed that this triangle and ABC have as orthology center the center of the circle circumscribed to the triangle $I_aI_bI_c$, therefore the point P and the center of the circle inscribed in the triangle ABC.

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The idea of this book came up when writing our previous book, *The Geometry of Homological Triangles* (2012).

As there, we try to graft on the central theme, of the orthological triangles, many results from the elementary geometry. In particular, we approach the connection between the orthological and homological triangles; also, we review the "S" triangles, highlighted for the first time by the great Romanian mathematician Traian Lalescu.

The book is addressed to both those who have studied and love geometry, as well as to those who discover it now, through study and training, in order to obtain special results in school competitions. In this regard, we have sought to prove some properties and theorems in several ways: synthetic, vectorial, analytical.

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