SUKANTO BHATTACHARYA

Utility, Rationality and Beyond –

From Behavioral Finance to Informational Finance

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2005

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I. Summary

This work covers a substantial mosaic of related concepts in utility theory as applied to financial decision-making. It reviews some of the classical notions of Benthamite utility and the normative utility paradigm offered by the von Neumann-Morgenstern expected utility theory; exploring its major pitfalls before moving into what is postulated as an *entropic* notion of utility. *Extrinsic utility* is proposed as a cardinally measurable quantity; measurable in terms of the *expected information content* of a set of alternative choices. The entropic notion of utility is subsequently used to model the financial behavior of individual investors based on their governing risk-return preferences involving *financial structured products* manufactured out of complex, *multi-asset options*. *Evolutionary superiority* of the Black-Scholes function in dynamic hedging scenarios is computationally demonstrated using a *haploid genetic algorithm* model programmed in Borland C. The work explores, both theoretically and computationally, the psycho-cognitive factors governing the financial behavior of individual investors both in the presence as well as absence of downside risk and postulates the concepts of *resolvable* and *irresolvable* risk. A formal theorem of *consistent preference* is proposed and proved. The work also analyzes the utility of an *endogenous capital guarantee* built within a financial structured product. The aspect of *investor empowerment* is discussed in terms of how financial behavior of an investor may be transformed if he or she is allowed a choice of one or more assets that may gain entry into the financial structured product. Finally there is a concluding section wherein the different facets are placed in their proper perspective and a number of interesting future research directions are also proposed.

II. Background and Literature Survey

The assumption of rationality underlies most of the theoretical constructs making up modern economics. The basic assumption is that all human beings, when faced with a decision problem, act in a rational manner under all circumstances so as to maximize the intrinsic utility obtainable from the outcome of such decision. This brings up the problem of establishing a reliable quantitative model for the Benthamite notion of *intrinsic utility* i.e. that inherent, sublime quality in a certain good to satisfy a particular human want (Bentham, 1789).

One of the earliest economic thinkers to have formally modeled utility was the Swiss mathematician Daniel Bernoulli who, ca 1738 A. D., proposed that the utility of money could be best measured by using the *logarithm of the number of units of money*. He hypothesized that the utility of additional sums of money to an individual must be in some way inversely proportional to the amount of money the person already has i.e. his or her initial wealth. It follows that more initial wealth a person has the less is the utility of additional amounts of money.

However it does not require a high degree of analytical prowess to decide that the Bernoullian assumption about diminishing marginal utility of wealth is over-simplistic. Empiricists have attacked the rationalist view on utility on the basis that ultimately being a subjective, psychological notion, it can differ for different persons over the same situation or same person over different situations. Thus there can be no unique, general model of utility that is independent of spatio-temporal and behavioral variables.

The problem of utility is best illustrated by the problem of self-insurance. Individuals faced with the risk of loss due to the occurrence (or non-occurrence) of a chance event can either assume the risk themselves or pay an insurance company to assume the risk for them. Then the pertinent question is when to assume the risk on one's own and when to purchase an insurance policy? By Bernoulli's logarithmic measure of utility, the expected payoff to the individual without insurance is given as follows:

$$\mathbf{E\ (X) = p\ (X)\ log_e X + [1 - p\ (X)]\ log_e\ [X - f\ (X)]} \qquad \mathbf{\ldots\ (1)}$$

In the above equation, X is the total pay-off at stake, f (X) is a pre-determined loss function and p (X) is the probability of receiving the total pay-off. If the individual decides to purchase an insurance policy, the expected pay-off function will be as follows:

$$\mathbf{E\ (X) = log_e\ (X - k)} \qquad \mathbf{\ldots\ (2)}$$

In the above equation, k is the cost of insurance payable by the insured party to the insurance provider. The insurance is cost-effective only if the following inequality holds:

$$\mathbf{log_e\ (X - k) \geq p\ (X)\ log_e X + [1 - p\ (X)]\ log_e\ [X - f\ (X)]} \qquad \mathbf{\ldots\ (3)}$$

$$\text{i. e.} \quad \log_e (X - k) \geq \log_e [X^{p(X)} \{X - f(X)\}^{\{1 - p(X)\}}] \quad \ldots (4)$$

$$\text{i. e.} \quad k \leq X [1 - X^{\{p(X) - 1\}} \{X - f(X)\}^{\{1 - p(X)\}}] \quad \ldots (5)$$

However, it can be intuitively reasoned out that a problem would surface if one is dealing with a situation where one of the probable outcomes is a total loss i.e. bankruptcy whereby $f(X) \geq X$ (Kahnemann and Tversky, 1979). The logarithm of zero is negative infinity, so $f(X) = X$ could still possibly be explained away as "infinite loss" but there is no logically acceptable way to explain the log of a negative number if $f(X) > X$.

Speculative trading in derivative securities like stock index futures with a highly leveraged position could indeed have unfortunate cases where $f(X) > X$ indicating a finite probability of complete financial ruin for the trader but no meaningful mathematical expression can model this case under the assumption of logarithmic utility!

Additionally, a very small probability of "infinite loss" will not always deter enterprising profit-seekers. As an illustrative example we may cite the case of playing a game whereby a person can walk off with \$5 or can drive up a narrow, winding hill-road for a mile and claim a prize of \$50 at the end of the trail. Of course, no matter how skillful a driver the person is, there will always be a finite chance of a fatal accident on the half-mile stretch along a dangerous road thereby resulting in an expected pay-off function with a positive likelihood of infinite loss. On the other hand, the expected pay-off function for the \$5 represents a "sure thing" with zero likelihood of loss. But there will definitely be enterprising people who will happily set off in their cars without a thought!

The economics of general insurance works on the basic principle of *risk shifting*. This principle is extendible also to the notion of portfolio insurance. When faced with the possibilty of heavy financial losses due to adverse market movements, many if not most investors will prefer to pay an *insurer* a fixed premium in exchange for guaranteeing a floor value to his or her investment portolio. For example in case of a *protective put* portfolio insurance strategy, the investor with a long position in a risky asset pays a fixed sum up front to the option writer as a premium for the put option on the asset (or a representative index in case of a portfolio of assets) which effectively imparts a downside limit without appreciably limiting the upside potential.

For the option buyer (insuree) the decision of an insurance purchase may be statistically conceptualized as a decision problem under risk with a pre-ascertained loss distribution. To reduce the risk the investor pays the premium and is compensated in case loss materializes. If the loss does not materialize i.e. in case of a protective put strategy, and if the asset (or portfolio of assets) does not suffer an erosion of value, then the insurance premium is taken as a normal business overhead expense provided the premium amount is a fair reflection of the expected loss potential. However, in reality the situation is not all that simple and not always explicable using the known rules of statistical probability theory.

Individuals often have very distorted perceptions of risk – people are prepared to pay extraordinarily high amounts for flight insurance (Eisner and Strotz, 1961) but not flood insurance even when the latter is offered at a highly subsidized price (Kunreuther et. al., 1978). Subjective cognitive processes ultimately determine the nature of an investment decision (as possibly that of many other critical decisions in life e.g. decision to get married, decision to change jobs or decision to undergo a cosmetic surgical procedure). Though one may predict with acceptable statistical reliability what decision an *average* human being will take given the circumstances governing the decision environment, no individual fits the profile of an *average individual* all time under all circumstances. Each individual will have some *individuality* – though this may connotate naïvity nevertheless it is a basic fact that is often blatantly ignored when we try to force all our predictive economic theories to snugly fit the crisp profile of the *rational economic person*.

In 1990, a well-known business personality and self-proclaimed meteorologist Iben Browning estimated that there was a 50% chance that a severe earthquake would occur on the notorious New Madrid fault during a two-day period around December 3, 1990. Professional geologists did not agree with this prediction and said that there was no scientific reason to believe that the probability of earthquake would vary significantly on a day to day basis. Indeed December 3, 1990 passed off without any earthquake. But a significant turbulance did occur though it was financial and not geological in nature! As a result of Browning's offhand and somewhat strange doomsday prediction, price and sales of earthquake insurance policies in the region sky-rocketed. According to one insurer, more than 650,000 State Farm policyholders in the eight states near the fault added an earthquake endorsement to their homeowners' insurance policies in the two months preceding December 3, 1990 – the predicted date of the catastrophe! Now it is not difficult to analogically extend this scenario to a situation where, for example, a renowned market guru makes a startling prediction and sends the global markets in frenzy all because such information temporarily unsettles the rational thinking power of a large number of investors and completely distorts their individual risk-return perceptions! The other aspect is that with continual and rapid information dissemination, the amount of media coverage any single piece of information receives tends to disproportionately inflate the importance of that information (Combs and Slovic, 1979). Psycho-cognitive biases can indeed affect the desirability of general insurance (extendible to portfolio insurance) by systematically distorting the probability or magnitude of worst loss (Lichtenstein et. al., 1978).

There is reasonably well-established theory of classical utility maximization in the case of deductible general insurance policy on non-financial assets whereby the basic underlying assumption is that cost of insurance is a convex function of the expected indemnification. Such an assumption has been shown to satisfy the sufficiency condition for expected utility maximization when individual preferences exhibit risk aversion (Meyer and Ormiston, 1999). Under the classical utility maximization appoach, the wealth function at the end of the insurance period is given as follows:

$$\mathbf{Z_T = Z_0 + M - x + I\,(x) - C\,(x = D)} \qquad \ldots \textbf{(6)}$$

In the above equation, Z_T is the terminal wealth at time t = T, Z_0 is the initial wealth at time t = 0, x is a random loss variable, I(x) is the indemnification function, C(x) is the cost of insurance and $0 \leq D \leq M$ is the level of the deductible.

However, the extent to which parallels may be drawn between ordinary insurance and portfolio insurance is limited in the sense that an investment portfolio consists of correlated assets which are continuously traded on the floors of organized financial markets. While the form of an insurance contract might look familier for portfolio insurance – an assured value in return for a price – the mechanism of providing such assurance will be quite different. This is because, unlike other tangible assets such as houses or cars, when one portfolio of financial assets collapses, virtually all others are likely to follow suit thereby making "risk pooling", the typical method of insurance, inappropriate (Leland and Rubinstein, 1988).

We have already witnessed some possible pitfalls of the classical expected utility maximization approach which governs the risk-return trade-off in some of the most celebrated *modern portfolio theory* models (e.g. Markowitz, 1952; Sharpe, 1964; Ross, 1976 etc.). Also it may prove somewhat inadequate in measuring the utility emanating from complex portfolio insurance structures involving several underlying assets because the capital guarantee mechanism that could potentially be embedded in such *structured products* (Braddock, 1997; Fabozzi, 1998) impart an additional dimension to investor utility by eliminating downside risk.

Moreover, besides eliminating downside potential, financial structured products also allow the investors a greater element of choice through the availability of a number of different assets that can enter the structure. Some of the more traditional capital guarantee mechanisms like using zero-coupon bonds for example, cannot provide this additional *utility of choice*. A zero-coupon bond capital guarantee scheme involves investment in discounted zero-coupon bonds of α% such that (1 – α)% is left for an *investment account* to be invested in high-risk high-return assets e.g. stock index futures. At the end of the investment horizon the amount invested in the discounted zero-coupon bond yields the original capital while the amount in the investment account provides the scope for an additional return. This is a rather simplistic, straightforward approach with very little element of choice for the investor. The proportion of (1 – α)% spared for the investment account is often inadequate by itself and has to be leveraged. The standard ordinal utility formalisms are quite sufficient for assessing the utility of such simplistic capital guarantee schemes. However, in order to completely explore the utility forms that may evolve out of an endogenously capital guaranteed financial structured product one feels the need to go beyond the traditional utility measures and use one which will appropriately capture this dimension of choice utility. The ultimate objective of our current research is to devise and implement such a utility measure.

Staying within the confines of the classical normative utility theory (von Neumann and Morgenstern, 1947), one may formulate a *maximum entropic measure of utilty* whereby a formal method may be devised to assign utility values under partial information about decison maker's preferences (Abbas, 2002). Abbas proposed a “maximum entropy utility”on the preference behavior of a decison maker by proposing an analogy between utiltiy and probability through the notion of a *utiltiy density function*. The core idea was to define a utility density function as the derivative of some utility function that is normalized to yield values between zero and unity.

Abbas’ works modifies earlier works on interpretation of normalized utility functions as a probability distribution of some hypothetical game of chance that is independent of the problem faced by the decision maker (Castagnoli and LiCalzi, 1996) and rescaling probability distributions with the objective of obtaining convenient expressions for utility functions (Berhold, 1973). According to Abbas’ formulation, when faced with the problem of drawing inference on the basis of partial information, one should use that utiltiy curve (or utility vector) whose utiltiy density function (or utility increment vector) has maximum entropy subject to the limited known preference constraints.

There have also been theoretical advances in the esoteric area of *econophysics* whereby mathematical analogs have been proposed and substantiated between utility theory and classical thermodynamics in so far as that both neoclassical economics and classical thermal physics seek to model natural systems in terms of solutions to constrained optimization problems. Both economic and physical state variables come in intensive/extensive variable pairs and the one such pair that is receiving much current intellectual attention is that of *temperature and entropy* and its purported economical analog – *price and utility* (Foley, 1994; Candeal et. al., 2001).

However it is the entropic formulation advanced by Abbas that serves as a major theoretical cornerstone of our present work. What we have aimed to accomplish is to devise an entropic measure of *extrinsic utility* of choice; as an additional dimension over and above the *intrinsic utility* which may be measured by the known methods of von Neumann-Morgenstern expected utility; to completely account for the utility derived by an individual investor from an endogenously capital-guaranteed financial structured product enveloping multiple assets.

The strength of our approach in breaking down total utility into intrinsic and extrinsic components is that while one may choose whatever appropriate paradigm to model intrinsic utility of a financial structured product, the additional dimension of choice utility can always be adequately captured using the analytical framework we've proposed here.

III. Structure and Methodology

Alongside exploring existing, cutting-edge developments in the area, this work also aims to cover new grounds in behavioral finance modeling. Our approach is primarily one of building intuitive logic based on behavioral reasoning, supported in most parts by rigorous mathematical exposition or numerical simulation and sensitivity analysis. The major part of the work attempts to build on existing, established theoretical models by incorporating additional mechanisms within the existing framework. Though based extensively on intuitive-cognitive models, our approach is predominantly numerate and substantiated by rigorous mathematics rather than being merely normative or speculative.

The main body of the work is divided into four relevant chapters. The first chapter takes up the notion of *resolvable risk* i.e. systematic investment risk which may be attributed to actual market movements as against *irresolvable risk* which is primarily born out of the inherent imprecision associated with the information gleaned out of market data such as price, volume, open interest etc. A *neutrosophic* model of risk classification is proposed – neutrosophic logic being a new branch of mathematical logic which allows for a three-way generalization of binary fuzzy logic by considering a third, *neutral* state in between the *high* and *low* states associated with binary logic circuits.

A plausible application of the postulated model is proposed in reconciliation of price discrepancies in the long-term options market where the only source of resolvable risk is the long-term implied volatility. The chapter postulates that inherent imprecision in the way market information is subjectively processed by psycho-cognitive factors governing human decision-making actually contributes to the creation of heightened risk appraisals. Such heightened notions of *perceived risk* make investors predisposed in favour of *safe* investments even when pure economic reasoning may not entirely warrant such a choice.

The second chapter explores one of the simplest of such *safe* investment strategies which are now in vogue – *options-based portfolio insurance*. In this chapter we propose and mathematically prove a theorem which we have named the "*theorem of consistent preference*" whereby we show that an option buyer will always show consistency in revealing his or her preference for a particular strategy within a pre-defined probability space. In Appendix (i), we use a *Monte Carlo* simulation technique to numerically derive the utility structures which evolve out of an options-based portfolio insurance strategy.

We explored utility structures which evolve out of an options-based portfolio insurance strategy that uses exchange-traded put options in the second chapter. However an alternative scenario could be that instead of buying a real put option in the options market, the payoff from a long put and long underlying asset (the combination for portfolio insurance) is synthetically replicated by a position in the underlying asset and cash using the standard *Black-Scholes* analytical framework. The Black-Scholes framework is the most efficient framework for engineering this type of synthetic portfolio insurance as it has ingrained *evolutionary optimality*. This is exactly what we have computationally demonstrated in the third chapter by using a haploid *Genetic Algorithm* model programmed in Borland C. The detailed Genetic Algorithm structure and output is given in Appendix (ii).

Having computationally demonstrated the evolutionary optimality of a Black-Scholes type expected payoff (utility) maximization function in a dynamic portfolio insurance scenario, we proceeded to explore the utility afforded to individual investor profiles by an *endogenously capital guaranteed financial structured product* based on the fundamental principle of multi-asset, dynamic hedging strategy in the fourth and last chapter.

It is in this chapter that we have postulated an *Information Theoretic model of utility of choice* evolving out of multi-asset financial structured products and also the additional utility afforded by such financial products due to the element of *investor empowerment* it can bring about by allowing the investor a say in the type of assets that are to be included within the product keeping the financial structured product as a whole endogenously capital-guaranteed.

These financial structured products are the type of *safe* investment strategies which are fast catching the fancy of investors on a global scale as they can, besides affording protection against untoward downside market movements (i.e. the *resolvable risk*), also provide assurance against the *irresolvable risk* by creating a sense of protection in the mind of the individual investor whereby he or she does not feel exposed to worst-case drawdown scenarios or risks of ruin even when, in reality, the actual statistical expectation of such ultra-negative scenarios are extremely small. For example, the *risk of* *ruin* i.e. the probability of losing the entire investment account in a single trade; is something that may be statistically controlled using fractional money management techniques and brought down to very low levels. But even if this probability is made to become very small (say 0.001; i.e. one-in-a-thousand chance of going bankrupt in a subsequent trade), there still remains a finite risk of this extreme event which can assume a larger than life image in the eyes of a small-time trader. However, if he or she knows that there is a floor below which the size of the investment account cannot fall *under any circumstances* then this can surely impart an additional sense of security and assurance.

The Information Theoretic (entropic) model of utility of choice we have developed in the last chapter is proposed as a seminal contribution to the collective body of knowledge that is referred to as *Computational Finance*. In Appendix (iii) we have used actual market data to construct a capital-guaranteed financial structured product and measured the investor's extrinsic utilities. Our model allows for a seamless integration of quantitative behavioral finance theory with the binary structure of information science.

The concluding section attempts to weave the different facets of the work and put them in proper perspective besides proposing a few interesting future research directions.

1. *Neutrosophic* Notion of Irresolvable Risk – A Proposed Behavioral Explanation of Investor Preference for Downside-protected Investments

The *efficient market hypothesis* based primarily on the statistical principle of *Bayesian inference* has been proved to be only a special-case scenario. The generalized financial market, modeled as a *binary, stochastic system* capable of attaining one of two possible states (High $\rightarrow$ 1, Low $\rightarrow$ 0) with finite probabilities, is shown to reach *efficient equilibrium* with p . M = p if and only if the transition probability matrix M_{2x2} obeys the additionally imposed condition $\{m_{11} = m_{22}, m_{12} = m_{21}\}$, where m_{ij} is an element of M (Bhattacharya, 2001).

Efficient equilibrium is defined as the stationery condition p = [0.50, 0.50] i.e. the state in t + 1 is equi-probable between the two possible states given the market vector in time t. However, if this restriction $\{m_{11} = m_{22}, m_{12} = m_{21}\}$ is removed, we arrive at inefficient *equilibrium* $\rho = [m_{21}/(1\text{-}v), m_{12}/(1\text{-}v)]$, where $v = m_{11} - m_{21}$ may be derived as the *eigenvalue of* M and ρ is a generalized version of p whereby the elements of the market vector are no longer restricted to their efficient equilibrium values. Though this proves that the generalized financial market cannot possibly be reduced to *pure random walk* if we do away with the *assumption of normality*, it does not necessarily rule out the possibility of *mean reversion* as M itself undergoes transition over time implying a probable re-establishment of the condition $\{m_{11} = m_{22}, m_{12} = m_{21}\}$ at some point of time in the foreseeable future. The temporal drift rate may be viewed as the *mean reversion parameter* k such that $k^j M_t$ tends to M_{t+j}. In particular, the options market demonstrates a perplexing departure from efficiency. In a *Black-Scholes type world*, if stock price volatility is known *a priori*, the option prices are completely determined and any deviations are quickly arbitraged away.

Therefore, statistically significant mispricings in the options market are somewhat unique as the only non-deterministic variable in option pricing theory is volatility. Moreover, given the knowledge of implied volatility on the short-term options, the miscalibration in implied volatility on the longer term options seem odd as the parameters of the process driving volatility over time can simply be estimated by an AR(1) model (Stein, 1993).

Clearly, the process is not quite as straightforward as a simple parameter estimation routine from an autoregressive process. Something does seem to affect the market players' collective pricing of longer term options, which clearly overshadows the straightforward considerations of implied volatility on the short-term options. One clear reason for inefficiencies to exist is through *overreaction* of the market players to new information. Some inefficiency however may also be attributed to purely *random white noise* unrelated to any coherent market information. If the process driving volatility is indeed mean reverting then a low implied volatility on an option with a shorter time to expiration will be indicative of a higher implied volatility on an option with a longer time to expiration. Again, a high implied volatility on an option with a shorter time to expiration will be indicative of a lower implied volatility on an option with a longer time to expiration. However statistical evidence often contradicts this *rational expectations hypothesis* for the *implied volatility term structure*.

Denoted by $\sigma'_t(t)$, (where the symbol ' indicates first derivative) the implied volatility at time t of an option expiring at time T is given in a Black-Scholes type world as follows:

$$\boldsymbol{\sigma'_t (t) = {}_{j=0}\int^{T} [\{\sigma_M + k^j (\sigma_t - \sigma_M)\}/T]\, dj}$$

$$\boldsymbol{\sigma'_t (t) = \sigma_M + (k^T - 1)(\sigma_t - \sigma_M)/(T \ln k)} \qquad \textbf{... (1.1)}$$

Here σ_t evolves according to a *continuous-time, first-order Wiener process* as follows:

$$\boldsymbol{d\sigma_t = - \beta_0 (\sigma_t - \sigma_M)\, dt + \beta_1 \sigma_t\, \varepsilon \sqrt{dt}} \qquad \textbf{... (1.2)}$$

$\beta_0 = - \ln k$, where k is the *mean reversion parameter*. Viewing this as a *mean reverting AR(1) process* yields the expectation at time t, $E_t(\sigma_{t+j})$, of the instantaneous volatility at time t+j, in the required form as it appears under the integral sign in equation (1.1).

This theorizes that volatility is rationally expected to gravitate geometrically back towards its long-term mean level of σ_M. That is, when instantaneous volatility is above its mean level ($\sigma_t > \sigma_M$), the implied volatility on an option should be decreasing as $t \rightarrow T$. Again, when instantaneous volatility is below the long-term mean, it should be rationally expected to be increasing as $t \rightarrow T$. That this theorization does not satisfactorily reflect reality is attributable to some kind of combined effect of overreaction of the market players to *excursions in implied volatility of short-term options* and their *corresponding underreaction to the historical propensity of these excursions to be short-lived*.

1.1 A Cognitive Dissonance Model of Behavioral Market Dynamics

Whenever a group of people starts acting in unison guided by their hearts rather than their heads, two things are seen to happen. Their individual suggestibilities decrease rapidly while the suggestibility of the group as a whole increases even more rapidly. The 'leader', who may be no more than just the most vociferous agitator, then primarily shapes the "groupthink". He or she ultimately becomes the focus of group opinion. In any financial market, it is the gurus and the experts who often play this role. The crowd hangs on their every word and makes them the uncontested Oracles of the marketplace.

If figures and formulae continue to speak against the prevailing groupthink, this could result into a *mass cognitive dissonance* calling for reinforcing self-rationalizations to be strenuously developed to suppress this dissonance. As individual suggestibilities are at a lower level compared to the group suggestibility, these self-rationalizations can actually further fuel the prevailing groupthink. This groupthink can even crystallize into something stronger if there is also a simultaneous "vigilance depression effect" caused by a tendency to filter out the dissonance-causing information. (Bem, 1967). The non-linear feedback process could in effect be the force which keeps blowing up the bubble until a critical point is reached and the bubble bursts ending the prevailing groupthink with a recalibration of the position by the experts.

That reasoning that we advance has two basic components – a *linear feedback process* containing no looping and a *non-linear feedback process* fuelled by an "unstable rationalization loop". Our conjecture is that it is due to this unstable rationalization loop that perceived true value of an option might be pushed away from its theoretical true value. The market price of an option will follow its *perceived true value* rather than its *theoretical true value* and hence inefficiencies arise. This does not mean that the market as a whole has to be inefficient – the market can very well be close to strong efficiency! Only it is the perceived true value that determines the actual price-path meaning that all market information (as well as some of the random white noise) could become automatically *anchored* to this perceived true value. This theoretical model would also explain why excursions in short-term implied volatilities tend to dominate the historical considerations of mean reversion – the perceived term structure simply becomes anchored to the prevailing groupthink about the nature of the implied volatility.

Our conceptual model is based on two primary assumptions:

- The unstable rationalization loop comes into effect if and only if the group is a reasonably well-bonded one i.e. if the initial group suggestibility has already attained a certain minimum level as, for example, in cases of strong cartel formations and;

- The unstable rationalization loop stays in force until some critical point in time t* is reached in the life of the option. Obviously t* will tend to be quite close to T – the time of expiration. At that critical point any further divergence becomes unsustainable due to the extreme pressure exerted by real economic forces 'gone out of sync' and the gap between perceived and theoretical values closes rapidly.

1.2 The Classical Cognitive Dissonance Paradigm

Since Leon Festinger presented it well over four decades ago, cognitive dissonance theory has continued to generate much interest as well as controversy (Festinger, 1957). This was mainly due to the fact that the theory was originally stated in much generalized, abstract terms. As a consequence, it presented possible areas of application covering a number of psychological issues involving the interaction of cognitive, motivational, and emotional factors. Festinger's dissonance theory began by postulating that pairs of cognitions (elements of knowledge), given that they are relevant to one another, can either be in agreement with each other or otherwise. If they are in agreement they are said to be *consonant*, otherwise they are termed *dissonant*. The mental condition that forms out of a pair of dissonant cognitions is what Festinger calls *cognitive dissonance*.

The existence of dissonance, being psychologically uncomfortable, motivates the person to reduce the dissonance by a process of *filtering out* information that is likely to increase the dissonance. The greater the degree of the dissonance, the greater is the pressure to reduce dissonance and change a particular cognition. The likelihood that a particular cognition will change is determined by the *resistance to change* of the cognition. Again, resistance to change is based on the *responsiveness* of the cognition to reality and on the extent to which the particular cognition is in line with various other cognitions. Resistance to change of cognition depends on the extent of loss or suffering that must be endured and the satisfaction or pleasure obtained from the behavior (Aronson et. al., 1968).

We propose the conjecture that cognitive dissonance is one possible (indeed highly likely) ***critical behavioral trigger*** (Allen and Bhattacharya, 2002) that triggers the rationalization loop and subsequently feeds it.

1.3 Non-linear Feedback Processes Generating a Rationalization Loop

In a linear autoregressive model of order R, a time series y_n is modeled as a linear combination of N earlier values in the time series, with an added correction term x_n:

$$\mathbf{y_n = x_n - \Sigma a_j \, y_{n-j}} \qquad \text{... (1.3)}$$

The autoregressive coefficients a_j (j = 1 ... N) are fitted by minimizing the mean-squared difference between the modeled time series y_n and the observed time series y_n. The minimization process results in a system of linear equations for the coefficients a_n, known as the *Yule-Walker equations*. Conceptually, the time series y_n is considered to be the output of a discrete linear feedback circuit driven by a noise x_n, in which delay loops of lag j have *feedback strength* a_j. For Gaussian signals, an autoregressive model often provides a concise description of the time series y_n, and calculation of the coefficients a_j provides an indirect but highly efficient method of spectral estimation. In a full nonlinear autoregressive model, quadratic (or higher-order) terms are added to the linear autoregressive model. A constant term is also added, to counteract any net offset due to the quadratic terms:

$$\mathbf{y_n = x_n - a_0 - \Sigma a_j\, y_{n-j} - \Sigma b_{j,k}\, y_{n-j} y_{n-k}} \qquad \textbf{... (1.4)}$$

The autoregressive coefficients a_j (j = 0 ... N) and $b_{j,\,k}$ (j, k = 1 ... N) are fit by minimizing the mean-squared difference between the modeled time series y_n and the observed time series y_n^*. The minimization process also results in a system of linear equations, which are generalizations of the *Yule-Walker equations* for the linear autoregressive model.

Unfortunately, there is no straightforward method to distinguish linear time series models (H_0) from non-linear alternatives (H_A). The approach generally taken is to test the H_0 of linearity against a pre-chosen particular non-linear H_A. Using the classical theory of statistical hypothesis testing, several test statistics have been developed for this purpose. They can be classified as Lagrange Multiplier (LM) tests, likelihood ratio (LR) tests and Wald (W) tests. The LR test requires estimation of the model parameters both under H_0 and H_A , whereas the LM test requires estimation only under H_0. Hence in case of a complicated, non-linear H_A containing many more parameters as compared to the model under H_0, the LM test is far more convenient to use. On the other hand, the LM test is designed to reveal specific types of non-linearities. The test may also have some power against inappropriate alternatives. However, there may at the same time exist alternative non-linear models against which an LM test is not powerful. Thus rejecting H_0 on the basis of such a test does not permit robust conclusions about the nature of the non-linearity. One possible solution to this problem is using a W test which estimates the model parameters under a well-specified non-linear H_A (De Gooijer and Kumar, 1992).

In a nonlinear feedback process, the time series y_n is conceptualized as the output of a circuit with nonlinear feedback, driven by a noise x_n. In principle, the coefficients $b_{j,k}$ describes dynamical features that are not evident in the power spectrum or related measures. Although the equations for the autoregressive coefficients a_j and $b_{j,k}$ are linear, the estimates of these parameters are often unstable, essentially because a large number of them must be estimated often resulting in significant estimation errors. This means that all *linear predictive systems* tend to break down once a rationalization loop has been generated. As parameters of the volatility driving process, which are used to extricate the implied volatility on the longer term options from the implied volatility on the short-term ones, are estimated by an AR (1) model, which belongs to the class of regression models collectively referred to as the *GLIM (General Linear Model)*, the parameter estimates go 'out of sync' with those predicted by a theoretical pricing model.

1.4 The Zadeh Argument Revisited

In the face of non-linear feedback processes generated by *dissonant information sources*, even mathematically sound rule-based reasoning schemes often tend to break down. As a pertinent illustration, we take Zadeh's argument against the well-known Dempster's rule (Zadeh, 1979). Let $\Theta = \{\theta_1, \theta_2 \ldots \theta_n\}$ stand for a set of n mutually exhaustive, elementary events that cannot be precisely defined and classified making it impossible to construct a larger set Θ_{ref} of disjoint elementary hypotheses.

The assumption of exhaustiveness is not a strong one because whenever θ_j, j = 1, 2 … n does not constitute an exhaustive set of elementary events, one can always add an extra element θ_0 such that θ_j, j = 0, 1 … n describes an exhaustive set. Then, if Θ is considered to be a *general frame of discernment* of the problem of a map m (.): $D^{\Theta} \rightarrow [0, 1]$ may be defined associated with a given body of evidence *B* that can support paradoxical information as follows:

$$\mathbf{m(\phi) = 0} \qquad \textbf{... (1.5)}$$

$$\mathbf{\Sigma_{A \in D^{\Theta}} m(A) = 1} \qquad \textbf{... (1.6)}$$

Then m (A) is called A's *basic probability number*. In line with the *Dempster-Shafer Theory* (Dempster, 1967; Shafer, 1976) the *belief* and *plausibility functions* are defined as follows:

$$\mathbf{Bel(A) = \Sigma_{B \in D^{\Theta}, B \subseteq A}\, m(B)} \qquad \text{... (1.7)}$$

$$\mathbf{Pl(A) = \Sigma_{B \in D^{\Theta}, B \cap A \neq \phi}\, m(B)} \qquad \text{... (1.8)}$$

Now let $Bel_1(.)$ and $Bel_2(.)$ be two belief functions over the same frame of discernment Θ and their corresponding *information granules* $m_1(.)$ and $m_2(.)$. Then the combined *global belief function* is obtained as $Bel_1(.) = Bel_1(.) \oplus Bel_2(.)$ by combining the information granules $m_1(.)$ and $m_2(.)$ as follows for $m(\phi) = 0$ and for $C \neq 0$ and $C \subseteq \Theta$;

$$\mathbf{[m_1 \oplus m_2] (C) = [\Sigma_{A \cap B = C}\, m_1 (A)\, m_2 (B)] / [1 - \Sigma_{A \cap B = \phi}\, m_1 (A)\, m_2 (B)]} \qquad \text{... (1.9)}$$

The summation notation $\Sigma_{A \cap B = C}$ is necessarily interpreted as the sum over all $A, B \subseteq \Theta$ such that $A \cap B = C$. The orthogonal sum m (.) is considered a basic probability assignment if and only if the denominator in equation (5) is non-zero. Otherwise the orthogonal sum m (.) does not exist and the bodies of evidences B_1 and B_2 are said to be in *full contradiction*.

Such a case can arise when there exists $A \subset \Theta$ such that $Bel_1 (A) = 1$ and $Bel_2 (A^c) = 1$ which is a problem associated with *optimal Bayesian information fusion rule* (Dezert, 2001). Extending Zadeh's argument to option market anomalies, if we now assume that under conditions of *asymmetric market information*, two market players with *homogeneous expectations* view implied volatility on the long-term options, then one of them sees it as either arising out of (A) current excursion in implied volatility on short-term options with probability 0.99 or out of (C) random white noise with probability of

0.01. The other sees it as either arising out of (B) historical pattern of implied volatility on short-run options with probability 0.99 or out of (C) white noise with probability of 0.01.

Using Dempster's rule of combination, the unexpected final conclusion boils down to the expression $m(C) = [m_1 \oplus m_2](C) = 0.0001/(1 - 0.0099 - 0.0099 - 0.9801) = 1$ i.e. the determinant of implied volatility on long-run options is white noise with 100% certainty!

To deal with this information fusion problem a new combination rule has been proposed under the name of *Dezert-Smarandache combination rule of paradoxical sources of evidence*, which looks for the optimal combination i.e. the basic probability assignment denoted as $\mathbf{m(.) = m1(.) \oplus m2(.)}$ that *maximizes the joint entropy* of the two information sources (Smarandache 2000; Dezert, 2001).

The Zadeh illustration originally sought to bring out the fallacy of automated reasoning based on the Dempster's rule and showed that some form of the *degree of conflict* between the sources must be considered before applying the rule. However, in the context of financial markets this assumes a great amount of practical significance in terms of how it might explain some of the recurrent anomalies in rule-based information processing by inter-related market players in the face of apparently conflicting knowledge sources. The traditional conflict between the *fundamental analysts* and the *technical analysts* over the credibility of their respective knowledge sources is of course all too well known!

1.5 Market Information Reconciliation Based on Neutrosophic Reasoning

Neutrosophy is a new branch of philosophy that is concerned with *neutralities* and their interaction with various ideational spectra. Let T, I, F be real subsets of the *non-standard interval* $]^{-}0, 1^{+}[$. If $\varepsilon > 0$ is an infinitesimal such that for all positive integers n and we have $|\varepsilon| < 1/n$, then the non-standard finite numbers $1^{+} = 1+\varepsilon$ and $0^{-} = 0-\varepsilon$ form the boundaries of the non-standard interval $]^{-}0, 1^{+}[$. Statically, T, I, F are *subsets* while dynamically they may be viewed as *set-valued vector functions*. If a logical proposition is said to be t% true in T, i% indeterminate in I and f% false in F then T, I, F are referred to as the *neutrosophic components*. Neutrosophic probability is useful to events that are shrouded in a *veil of indeterminacy* like the *actual* implied volatility of long-term options. As this approach uses a *subset-approximation* for truth-values, indeterminacy and falsity-values it provides a better approximation than classical probability to uncertain events.

The neutrosophic probability approach also makes a distinction between "relative sure event", event that is true only in certain world(s): NP (rse) = 1, and "absolute sure event", event that is true for all possible world(s): NP (ase) $=1^{+}$. Similar relations can be drawn for "relative impossible event" / "absolute impossible event" and "relative indeterminate event" / "absolute indeterminate event". In case where the truth- and falsity-components are complimentary i.e. they sum up to unity, and there is no indeterminacy and one is reduced to classical probability. Therefore, neutrosophic probability may be viewed as a generalization of classical and imprecise probabilities (Smarandache, 2000).

When a long-term option priced by the collective action of the market players is observed to be deviating from the theoretical price, three possibilities must be considered:

(1) The theoretical price is obtained by an inadequate pricing model, which means that the market price may well *be* the true price,

(2) An unstable rationalization loop has taken shape that has pushed the market price of the option 'out of sync' with its true price, or

(3) The nature of the deviation is indeterminate and could be due to either (a) or (b) or a super-position of both (a) and (b) and/or due to some random white noise.

However, it is to be noted that in none of these three possible cases are we referring to the efficiency or otherwise of the market as a whole. *The market can only be as efficient as the information it receives to process*. We term the systematic risk associated with the efficient market as ***resolvable risk***. Therefore, if the information about the true price of the option is misinterpreted (perhaps due to an inadequate pricing model), the market cannot be expected to process it into something useful just because the market is operating at a certain level of efficiency – after all, financial markets can't be expected to pull jack-rabbits out of empty hats!

The perceived risk resulting from the imprecision associated with how human psycho-cognitive factors subjectively interpret information and use the processed information in decision-making is what we term as ***irresolvable risk***.

With T, I, F as the neutrosophic components, let us now define the following events:

H = {p: p is the true option price determined by the theoretical pricing model}; and

M = {p: p is the true option price determined by the prevailing market price} ... (1.10)

Then there is a t% chance that the event ($H \cap M^c$) is true, or corollarily, the corresponding complimentary event ($H^c \cap M$) is untrue, there is a f% chance that the event ($M^c \cap H$) is untrue, or corollarily, the complimentary event ($M \cap H^c$) is true and there is a i% chance that neither ($H \cap M^c$) nor ($M \cap H^c$) is true/untrue; i.e. the determinant of the true market price is indeterminate. This would fit in neatly with possibility (c) enumerated above – that the nature of the deviation could be due to either (a) or (b) or a super-position of both (a) and (b) and/or due to some random white noise.

Illustratively, a set of AR(1) models used to extract the mean reversion parameter driving the volatility process over time have *coefficients of determination* in the range say between 50%-70%, then we can say that t varies in the set T (50% - 70%). If the subjective probability assessments of well-informed market players about the weight of the current excursions in implied volatility on short-term options lie in the range say between 40%-60%, then f varies in the set F (40% - 60%). Then unexplained variation in the temporal volatility driving process together with the subjective assessment by the market players will make the event indeterminate by either 30% or 40%. Then the neutrosophic probability of the true price of the option being determined by the theoretical pricing model is NP $(H \cap M^c)$ = [(50 – 70), (40 – 60), {30, 40}].

1.6 Implication of a Neutrosophic Interpretation of Financial Behavior

Finally, in terms of our behavioral conceptualization of the market anomaly primarily as manifestation of mass cognitive dissonance, the joint neutrosophic probability denoted by **NP $(H \cap M^c)$** will also be indicative of the extent to which an unstable rationalization loop has formed out of such mass cognitive dissonance that is causing the market price to deviate from the true price of the option. Obviously increasing strength of the non-linear feedback process fuelling the rationalization loop will tend to increase this deviation.

As human psychology; and consequently a lot of subjectivity; is involved in the process of determining what drives the market prices, neutrosophic reasoning will tend to reconcile market information much more realistically than classical probability theory. Neutrosophic reasoning approach will also be an improvement over rule-based reasoning possibly avoiding pitfalls like that brought out by Zadeh's argument. This has particularly significant implications for the vast majority of market players who rely on signals generated by some automated trading system following simple rule-based logic.

However, the fact that there is inherent subjectivity in processing the price information coming out of financial markets, given that the way a particular piece of information is subjectively interpreted by an individual investor may not be the globally correct interpretation, there is always the matter of irresolvable risk that will tend to pre-dispose the investor in favour of some *safe* investment alternative that offers some protection against both resolvable as well as irresolvable risk. This highlights the rapidly increasing importance and popularity of *safe* investment options that are based on some form of insurance i.e. an investment mechanism where the investor has some kind of in-built downside protection against adverse price movements resulting from erroneous interpretation of market information. Such portfolio insurance strategies offer protection against all possible downsides – whether resulting out of resolvable or irresolvable risk factors and therefore make the investors feel at ease and confident about their decision-making. We look at one such strategy called a *protective put* strategy in the next chapter.

2. Theorem of Consistent Preference and the Utility Structures Underlying a Simple Portfolio Insurance Strategy

It is well known that the possibility to lend or borrow money at a risk-free rate widens the range of investment options for an individual investor. The inclusion of the risk-free asset makes it possible for the investor to select a portfolio that *dominates* any other portfolio made up of only risky securities. This implies that an individual investor will be able to attain a higher *indifference curve* than would be possible in the absence of the risk-free asset. The risk-free asset makes it possible to *separate* the investor's decision-making process into two distinct phases – identifying the market portfolio and funds allocation. The market portfolio is the portfolio of risky assets that includes each and every available risky security. As all investors who hold any risky assets at all will choose to hold the market portfolio, this choice is independent of an individual investor's utility preferences (Tobin, 1958). Extending the argument to a case where the investor wants to have a floor to his or investment below which he or she would not want his or her portfolio value to fall then a natural choice for this floor would be the risk-free rate.

While theoretical finance literature is replete with models of portfolio choice under risk, there is surprisingly little work with respect to models of investor behavior where acquiring portfolio insurance through the usage of financial derivatives could reduce or remove investment risk.

In this section we take a look at a simple portfolio insurance strategy using a protective put and in Appendix (i), computationally derive the investor's governing utility structures underlying such a strategy under alternative market scenarios. Investor utility is deemed to increase with an increase in the excess equity generated by the portfolio insurance strategy over a simple investment strategy without any insurance.

Three alternative market scenarios (probability spaces) have been explored – "Down", "Neutral" and "Up", categorized according to whether the price of the underlying security is most likely to go down, stay unchanged or go up. The methodology used is computational, primarily based on simulation and numerical extrapolation. We have used the Arrow-Pratt measure of risk aversion in order to determine how individual investors react towards risk under the different market scenarios with varying risk assessments.

While countless research papers have been written and published on the mathematics of option pricing formulation, surprisingly little work has been done in the area of exploring the exact nature of investor utility structures that underlie investment in derivative financial assets. This is an area we deem to be of tremendous interest both from the point of view of mainstream financial economics as well as from the point of view of a more recent and more esoteric perspective of *behavioral economics*.

2.1 Brief Review of Financial Derivatives

Basically, a derivative financial asset is a legal contract between two parties – a buyer and a seller, whereby the former receives a rightful claim on an underlying asset while the latter has the corresponding liability of making good that claim, in exchange for a mutually agreed consideration. While many derivative securities are traded on the floors of exchanges just like ordinary securities, some derivatives are not exchange-traded at all. These are called OTC (Over-the-Counter) derivatives, which are contracts not traded on organized exchanges but rather negotiated privately between parties and are especially tailor-made to suit the nature of the underlying assets and the pay-offs desired therefrom.

Forward Contract

A contract to buy or sell a specified amount of a designated commodity, currency, security, or financial instrument at a known date in the future and at a price set at the time the contract is made. Forward contracts are negotiated between the contracting parties and are not traded on organized exchanges.

Futures Contract

Quite similar to a forwards contract – this is a contract to buy or sell a specified amount of a designated commodity, currency, security, or financial instrument at a known date in the future and at a price set at the time the contract is made. What primarily distinguishes forward contracts from futures contracts is that the latter are traded on organized exchanges and are thus standardized. These contracts are marked to market daily, with profits and losses settled in cash at the end of the trading day.

Swap Contract

This is a private contract between two parties to exchange cash flows in the future according to some prearranged formula. The most common type of swap is the "plain vanilla" interest rate swap, in which the first party agrees to pay the second party cash flows equal to interest at a predetermined f*ixed* rate on a notional principal. The second party agrees to pay the first party cash flows equal to interest at a *floating* rate on the same notional principal. Both payment streams are denominated in the same currency. Another common type of swap is the currency swap. This contract calls for the counter-parties to exchange specific amounts of two different currencies at the outset, which are repaid over time according to a prearranged formula that reflects amortization and interest payments.

Options Contract

A contract that gives its owner the right, but not the obligation, to buy or sell a specified asset at a stipulated price, called the strike price. Contracts that give owners the right to buy are referred to as *call* options and contracts that give the owner the right to sell are called *put* options. Options include both standardized products that trade on organized exchanges and customized contracts between private parties.

The simplest option contracts (also called *plain vanilla options*) are of two basic types – *call* and *put*. The call option is a right to buy (or call up) some underlying asset at or within a specific future date for a specific price called the strike price. The put option is a right to sell (or put through) some underlying asset at or within a specified date – again for a pre-determined strike price. The options come with no obligations attached – it is totally the discretion of the option holder to decide whether or not to exercise the same.

The pay-off function (from an option buyer's viewpoint) emanating from a call option is given as $P_{call} = Max\ [(S_T - X),\ 0]$. Here, S_T is the price of the underlying asset on maturity and X is the strike price of the option. Similarly, for a put option, the pay-off function is given as $P_{put} = Max\ [(X - S_T),\ 0]$. The implicit assumption in this case is that the options can only be exercised on the maturity date and not earlier. Such options are called *European options*. If the holder of an option contract is allowed to exercise the same any time on or before the day of maturity, it is termed an *American option*. A third, not-so-common category is one where the holder can exercise the option only on specified dates prior to its maturity. These are termed *Bermudan options*. The options we refer to in this paper will all be European type only but methodological extensions are possible to extend our analysis to also include American or even Bermudan options.

In our present analysis we will be restricted exclusively to portfolio insurance strategy using a long position in put options and explore the utility structures derivable therefrom.

2.2 Investor's Utility Structures Governing the Purchase of Plain Vanilla Option Contracts

Let us assume that an underlying asset priced at S at time t will go up or down by Δs or stay unchanged at time T either with probabilities p_U (u), p_U (d) and p_U (n) respectively contingent upon the occurrence of event U, or with probabilities p_D (u), p_D (d) and p_D (n) respectively contingent upon the occurrence of event D, or with probabilities p_N (u), p_N (d) and p_N (n) respectively contingent upon the occurrence of event N, in the time period (T – t). This, by the way, is comparable to the analytical framework that is exploited in option pricing using the numerical method of *trinomial trees*, which is a common numerical algorithm used in the pricing of the non-European options where it is difficult to find a closed-form pricing formula akin to the famous *Black-Scholes* pricing equation.

2.3 ***Theorem of Consistent Preference***

Let p_U, p_D and p_N be the three probability distributions contingent upon events U, D and N respectively where $p_U(u) = \text{Max}_i\, p_U(i)$, $p_N(n) = \text{Max}_i\, p_N(i)$ and $p_D(d) = \text{Max}_i\, p_D(i)$, for respective instances $i \in (u, n, d)$. Then we have *a consistent preference relation* for a call buyer such that *p_U is strictly preferred to p_N <u>and</u> p_N is strictly preferred to p_D* and *a corresponding consistent preference relation* for a put buyer such that *p_D is strictly preferred to p_N <u>and</u> p_N is strictly preferred to p_U* (Khoshnevisan, Bhattacharya and Smarandache, 2003). The *theorem of consistent preference* naturally implies that a~~n~~ protective put portfolio insurance strategy would be preferable when the market is expected to go down rather than when it is expected to either go up or retain status quo.

<u>*Proof:*</u>

<u>Case I:</u> Investor buys a call option for \$C maturing at time T having a strike price of \$X on the underlying asset. We modify the call pay-off function slightly such that we now have the pay-off function Y_{call} as: $Y_{call} = Max\ (S_T - X - C, - C)$.

Event U:

$$E_U\,(\text{Call}) = [(S + e^{-r\,(T-t)}\,\Delta s)\, p_U\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\, p_U\,(d) + S\, p_U\,(n)] - C - Xe^{-r\,(T-t)}$$

$$= [S + e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\}] - C - Xe^{-r\,(T-t)} \ldots \text{ since } p_U\,(u) > p_U\,(d)$$

Therefore, $E\,(Y_{call}) = \text{Max}\,[S + e^{-r\,(T-t)}\,\{\Delta s\,(p_U\,(u) - p_U\,(d)) - X\} - C, - C]$ … (I)

Event D:

$$E_D\,(\text{Call}) = [(S + e^{-r\,(T-t)}\,\Delta s)\, p_D\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\, p_D\,(d) + S\, p_D\,(n)] - C - Xe^{-r\,(T-t)}$$

$$= [S + e^{-r\,(T-t)}\,\Delta s\ \{p_D\,(u) - p_D\,(d)\}] - C - Xe^{-r\,(T-t)} \ldots \text{ since } p_D\,(u) < p_D\,(d)$$

Therefore, $E\,(Y_{call}) = \text{Max}\,[S - e^{-r\,(T-t)}\,\{\Delta s\,(p_D\,(d) - p_D\,(u)) + X\} - C, - C]$ … (II)

Event N:

$$E_N\,(\text{Call}) = [(S + e^{-r\,(T-t)}\,\Delta s)\, p_N\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\, p_N\,(d) + S\, p_N\,(n)] - C - Xe^{-r\,(T-t)}$$

$$= [S + e^{-r\,(T-t)}\,\Delta s\ \{p_N\,(u) - p_N\,(d)\}] - C - Xe^{-r\,(T-t)}$$

$$= S - C - Xe^{-r\,(T-t)} \ldots \text{ since } p_N\,(u) = p_N\,(d)$$

Therefore, $E\,(Y_{call}) = \text{Max}\,[S - Xe^{-r\,(T-t)} - C, - C]$ … (III)

<u>Case II:</u> Investor buys a put option for \$P maturing at time T having a strike price of \$X on the underlying asset. Again we modify the pay-off function such that we now have the pay-off function Y_{put} as: $Y_{put} = Max\,(X - S_T - P, - P)$.

Event U:

$$E_U\,(\text{Put}) = Xe^{-r\,(T-t)} - [\{(S + e^{-r\,(T-t)}\,\Delta s)\, p_U\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\, p_U\,(d) + S\, p_U\,(n)\} + P]$$

$$= Xe^{-r\,(T-t)} - [S + e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} + P]$$

$= Xe^{-r\,(T-t)} - [S + e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} + (C + Xe^{-r\,(T-t)} - S)]$ … *put-call parity*

$= - e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} - C$

Therefore, $E\,(Y_{put}) = \text{Max}\,[- e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} - C, - P]$

$= \text{Max}\,[- e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} - C, - (C + Xe^{-r\,(T-t)} - S)]$ … (IV)

Event D:

$E_D\,(\text{Put}) = Xe^{-r\,(T-t)} - [\{(S + e^{-r\,(T-t)}\,\Delta s)\,p_D\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\,p_D\,(d) + S\,p_D\,(n)\} + P]$

$= Xe^{-r\,(T-t)} - [S + e^{-r\,(T-t)}\,\Delta s\ \{p_D\,(u) - p_D\,(d)\} + P]$

$= Xe^{-r\,(T-t)} - [S + e^{-r\,(T-t)}\,\Delta s\ \{p_U\,(u) - p_U\,(d)\} + (C + Xe^{-r\,(T-t)} - S)]$ … *put-call parity*

$= e^{-r\,(T-t)}\,\Delta s\ \{p_D\,(d) - p_D\,(u)\} - C$

Therefore, $E\,(Y_{put}) = \text{Max}\,[e^{-r\,(T-t)}\,\Delta s\ \{p_D\,(d) - p_D\,(u)\} - C, - P]$

$= \text{Max}\,[e^{-r\,(T-t)}\,\Delta s\ \{p_D\,(d) - p_D\,(u)\} - C, - (C + Xe^{-r\,(T-t)} - S)]$ … (V)

Event N:

$E_N\,(\text{Put}) = Xe^{-r\,(T-t)} - [\{(S + e^{-r\,(T-t)}\,\Delta s)\,p_N\,(u) + (S - e^{-r\,(T-t)}\,\Delta s)\,p_N\,(d) + S\,p_N\,(n)\} + P]$

$= Xe^{-r\,(T-t)} - [S + e^{-r\,(T-t)}\,\Delta s\ \{p_N\,(u) - p_N\,(d)\} + P]$

$= Xe^{-r\,(T-t)} - (S + P)$

$= (Xe^{-r\,(T-t)} - S) - \{C + (Xe^{-r\,(T-t)} - S)\}$ … *put-call parity*

$= - C$

Therefore, $E\,(Y_{put}) = \text{Max}\,[- C, - P]$

$= \text{Max}\,[-C, - (C + Xe^{-r\,(T-t)} - S)]$ … (VI)

From equations (IV), (V) and (VI) we see that E_U (Put) < E_N (Put) < E_D (Put) and hence it is proved why we have the *consistent preference relation P_D is strictly preferred to P_N <u>and</u> P_N is strictly preferred to P_U* from a put buyer's point of view. The call buyer's consistent preference relation is also explainable likewise.

<u>*Proved*</u>

We have computationally derived the associated utility structures using a Monte Carlo discrete-event simulation approach to estimate the change in investor utility (measured in terms of expected equity) following a particular investment strategy under each of the aforementioned event spaces *D*, *N* and *U* and the results are provided in Appendix (i).

The portfolio insurance strategy we have encountered here uses an exchange-traded vanilla put option. However, the same form of insurance can also be engineered by synthetically replicating the pay-off from a long put option and a long underlying asset by taking a long position in the underlying asset and long position in a risk-free asset such that a proportion equivalent to Δ_c of the total funds is invested in the underlying asset and the rest in the risk-free one. Here Δ_c is the delta of a call option on the same underlying asset i.e. if the price of the underlying asset moves marginally, the price of the call option moves by Δ_c units. This is known in investment parlance as *dynamic delta-hedging* and has been proven to be a very effective portfolio insurance strategy except in case of a crash situation. This dynamic-hedging model can be extended to a multi-asset scenario too, where a whole portfolio consisting of several underlying assets needs to be protected against adverse market movements, by extending the standard Black-Scholes argument to a multi-dimensional probability space. This dynamic hedging model will work efficiently every time because the Black-Scholes type of expected utility function has an *inherent biological superiority* over other utility forms and this is what we set out to computationally explore in our next chapter.

3. Exploring the Biological Basis of Utility Functions – Computational Implementation of a Genetic Algorithm Scheme to Illustrate the Efficiency of Black-Scholes Delta Hedging Using Multi-asset Options

3.1 Primer on Structured Products based on Multi-asset Options

For our purpose, a financial structured product is assumed to take the form of a complex, multi-asset option whereby an investor receives a terminal payoff that is tied to the price or performance of any one of the several assets enveloped within the structure. The premium to be paid by the investor is determined by means of a pricing formula akin to the *Black-Scholes* expected-payoff formulation but in a multi-dimensional form. However, the actual utility from the financial structured product at the end of the investment horizon depends on how well the assets within the structure have been 'managed' during the lock-up period to adapt to the changing asset market conditions.

It is apparent that the risk-adjusted returns to asset allocation increase as correlation between alternative assets decrease. However this is not strictly true in a mathematical sense. For example a financial structured product whose payoff is equivalent to that of a two-asset, best-of option (or an *exchange option*) payoff given by $Max[S_1 - S_2, 0]$, for the case of equal volatility, drift and correlation – i.e. exchangeability; computational results suggest that the *information ratio* (i.e. mean/standard deviation) of the returns is invariant to the degree of correlation. A formal proof for the two-asset case has been advanced (Martin, 2001) but there is no formal proof for this conjecture as yet for the n-assets case.

If the financial structured product is actually a multi-asset, *best-of* option whose payoff is linked to the best-performing asset enveloped within that structure, then an endogenous capital-guarantee may be built within this structure whereby the payoff from this structure is replicated using a portfolio consisting of investments in the various assets within the structure along with an investment in a risk-free asset like a commercial bank fixed-deposit or a government treasury bill. The proportion of invested funds in each of the risky assets as well as the risk-free asset is determined according to an *allocation formula*, which is re-calculated periodically so as to ensure a minimum-error replication.

Let a financial structured product be made up of an envelope of J different assets such that the investor has the right to claim the return on the best-performing asset out of that envelope after a stipulated lock-in period. Then, if one of the J assets in the envelope is the risk-free asset then the investor is assured of a minimum return equal to the risk-free rate i on his invested capital at the termination of the stipulated lock-in period. This effectively means that his or her investment becomes *endogenously capital-guaranteed* as the terminal wealth, even at its worst, cannot be lower in value to the initial wealth plus the return earned on the risk-free asset minus a finite cost of *portfolio insurance*, which is paid as the premium to the option writer.

Then the *expected present value* of the terminal option payoff is obtained as follows:

$$\mathbf{\hat{E}\ (r)_{t=T} = Max\ [w, Max_j\ \{e^{-it}\ E\ (r_j)_{t=T}\}],\ j = 1, 2 \ldots J - 1} \qquad \ldots (3.1)$$

In the above equation, i is the rate of return on the risk-free asset and T is the length of the investment horizon in continuous time and w is the initial wealth invested i.e. ignoring insurance cost, if the risk-free asset outperforms all other assets then we get:

$$\mathbf{\hat{E}\ (r)_{t=T} = we^{iT}/e^{iT} = w} \qquad \textbf{... (3.2)}$$

Now what is the probability of each of the (J – 1) risky assets performing worse than the risk-free asset? Even if we assume that there are some cross-correlations present among the (J – 1) risky assets, given the statistical nature of the risk-return trade-off, the joint probability of all these assets performing worse than the risk-free asset will be very low over even moderately long investment horizons. And this probability will keep diminishing with every additional risky asset added to the envelope. Thus this probability can become quite negligible if we consider sufficiently large values of n. However, in this paper, we have taken J = 3 mainly for computational simplicity, as closed-form pricing formulations become extremely difficult to obtain for J > 3. (Johnson, 1986)

For an option writer who is looking to hedge his or her position, the expected utility maximization criterion will require the *tracking error* to be at a minimum at each point of rebalancing, where the tracking error is the difference between the expected payoff on the best-of option and the replicating portfolio value at that point. Then, given a necessarily biological basis of the evolution of utility forms (Robson, 1968; Becker, 1976), a haploid genetic algorithm model, which, as a matter of fact, can be shown to be statistically equivalent to multiple *multi-armed bandit processes*, should show satisfactory convergence with the Black-Scholes type expected utility solution to the problem (Robson, 2001).

This is exactly what we have attempted to demonstrate in our present chapter. A Black-Scholes type expected utility maximization function is what is anticipated to gradually evolve out of the future generations as the largely predominant genotype.

3.2 A Genetic Algorithm Model for the Financial structured product Problem

A. The set-up

At each point of re-balancing, the tracking error has to be minimized if the difference between the expected option payoff and the replicating portfolio value is to be minimized. The more significant this difference, the more will be the *cost of re-balancing* associated with correcting the tracking error; and as these costs accumulate; the less will be the ultimate utility of the hedge to the option writer at the end of the lock-in period. Then the cumulative tracking error over the lock-in period is given as:

$$\Xi = \Sigma_t \left| E(r)_t - v_t \right| \quad \text{... (3.3)}$$

Here $E(r)_t$ is the expected best-of option payoff at time-point t and v_t is the replicating portfolio value at that point of time. Then the replicating portfolio value at time t is obtained as the following linear form:

$$\mathbf{v_t = (p_0)_t e^{it} + \Sigma_j \{(p_j)_t (S_j)_t\}, j = 1, 2 \ldots J - 1 \quad \ldots (3.4)}$$

Here $(S_j)_t$ is the realized return on asset j at time-point t and $p_1, p_2 \ldots p_{J-1}$ are the respective allocation proportions of investment funds among the J – 1 risky assets at time-point t and $(p_0)_t$ is the allocation for the risk-free asset at time-point t. Of course:

$$\mathbf{(p_0)_t = 1 - \Sigma_j (p_j)_t \quad \ldots (3.5)}$$

It is the *portfolio weights* i.e. the p_0 and p_j values that are of critical importance in determining the size of the tracking error. The correct selection of these portfolio weights will ensure that the replicating portfolio accurately tracks the option.

Then, over a few successive generations, the predominant genotype will evolve as the one that best meets the fitness criterion based on the magnitude of the tracking error. The option value on the best of two risky assets plus one risk-free asset is derived according to the standard Black-Scholes type formulation (Stulz; 1982).

The computational haploid genetic algorithm model we have devised for this purpose has been programmed in Borland C; Release 5.02 (Sedgewick, 1990) and performs the three basic genetic functions of *reproduction*, *crossover* and *mutation* with the premise that in each subsequent generation x number of *chromosomes* from the previous generation will be reproduced based on the principal of *natural selection*. Following the reproduction function, 2(x – 1) number of additional chromosomes will be produced through the crossover function, whereby every g^{th} chromosome included in the *mating pool* will be crossed with the $(g + 1)^{th}$ chromosome at a pre-assigned *crossover locus*. There is also a provision in our computer program to introduce a maximum number of mutations in each current chromosome population in order to enable rapid adaptation.

B The haploid genetic algorithm as a generalized bandit process

Let a simple two-armed bandit process be considered whereby it is known that one of the arms pays a reward m_1 with variance s_1^2 and the other arm pays a reward m_2 with variance s_2^2 such that $m_2 \leq m_1$. Then this gives rise to the classical bandit problem dilemma regarding which of the two arms to play so as to optimize the trade-off between *information usage* and *information acquisition* (Berry, 1985). A common solution method would be to allocate an equal number of trials between the two arms, say f. Suppose that we have a total number of F possible trials. Say we select f such that 2f < F. This will consist of the "training" phase. Thereafter, in the subsequent "testing" phase, the remaining F – 2f trials are allocated to the arm with the best-observed payoff in the "training" phase. Then the expected loss is calculated as follows:

$$\boldsymbol{\lambda\,(F, f) = |m_1 - m_2|\,[(F - f)\,q\,(f) + f\,\{1 - q\,(f)\}]} \qquad \textbf{... (3.6)}$$

In the above equation, q (f) is the probability of selecting the wrong arm as the best arm during the "training" phase (De Jong, 1976). The value of q (f) is fairly approximated by the tail of the normal distribution as follows:

$$\mathbf{q\ (f) \approx (1/\sqrt{2\pi})\ \{exp\ (-z^2/2)/z\},\ where\ z = (\sqrt{f})\ \{(m_1 - m_2) / \sqrt{(s_1^2 + s_2^2)}\}\ \ ...\ (3.7)}$$

The optimal "training" sample size f* for minimizing the loss function λ (F, f) may be obtained by setting $\lambda_f = 0$; as per the usual, *first-order condition* for minimization.

However, though procedure sounds simple enough, it is not necessarily the most optimal as was shown by Holland. According to his calculations, expected losses could be further minimized by allocating exponentially increasing number of trials to the observed better arm (Holland, 1975). Though this approach is untenable because it entails perfect future knowledge, it at least sets an *upper bound* to the best solution technique, whereby any good technique should asymptotically approach this bound.

The three-function genetic algorithm performs best in terms of asymptotically approaching the upper bound calculated by Holland as it allocates an exponentially increasing number of trials to the best-observed chromosomes in terms of the fitness criterion. However, with the genetic algorithm, one is actually solving several multi-armed bandit processes. To make this point clear, let us consider a set of four *schemata* that mutually compete at loci 3 and 5. Two schemata A and B with individual loci i, j are said to be *mutually competing* if at all loci i = j = 1, 2… L either $A_i = B_j = *$ or $A_i \neq *$, $B_j \neq *$ and $A_i \neq B_j$ for at least one locus i = j, where L is the length of each chromosome encoded as a bit-string.

* * 0 * 0 * *

* * 0 * 1 * *

* * 1 * 0 * *

* * 1 * 1 * *

Therefore there are $2^2 = 4$ competing schemata over the two loci 3 and 5 as each one of these two loci may be occupied by a 0 or a 1. As they are defined over the same loci, they compete for *survival* within each population. In order to make sure the best schemata survive in the population every time, exponentially increasing numbers of trials must be allocated to the observed best schemata. This is exactly the same procedure for allocating exponentially increasing number of trials to a multi-armed bandit process. In our illustration, this is akin to a four-armed bandit process. However, with a genetic algorithm, the problem is more complex than a multi-armed bandit process – it is actually akin to solving multiple multi-armed bandit problems! For example, in our above illustration, for 2 fixed loci in a bit-string of length 7, there are actually ${}^7C_2 = 21$ of the eight-armed bandit processes! In general, for encoded bit-strings of length L there are ${}^LC_k$ different 2^k-armed bandit problems being played out.

We have based our computational model on the premise that $2(x - 1)$ number of additional chromosomes will be produced through the crossover function, for x number of chromosomes included in the mating pool through the process of natural selection. Every g^{th} chromosome in the mating pool is crossed with the $(g + 1)^{th}$ chromosome at a predetermined *crossover locus*. Then, given that x number of chromosomes are initially included in the mating pool following the reproduction function, we will obtain a total number of $x + 2(x - 1)$ chromosomes in the first generation's population. Here, x number are chromosomes retained from the previous generation based on the observed best fitness criterion. One or more of these x chromosomes can be, optionally, allowed to mutate (i.e. swap of bit positions between 1 and 0 from a particular locus onwards in the encoded bit-string). The remaining chromosomes in the first generation, $2(x - 1)$ in number, are the ones that come out as a direct result of our pre-formulated crossover function. Therefore we get the following linear difference equation as the governing equation determining chromosome population size in the n^{th} generation (Derivation provided in Appendix (ii):

$$\mathbf{G_n = G_{n-1} + 2\,(G_{n-1} - 1) = 3\,G_{n-1} - 2} \qquad \textbf{... (3.8)}$$

C. Constructing a hypothetical financial structured product as a best of 3 assets option

The terminal payoff from such a financial structured product would be that on the asset ending up as the best performer among the three assets within the envelope i.e.

$$\mathbf{r}_{t=T} = \mathbf{Max}\ (\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2)_{t=T} \qquad \textbf{... (3.9)}$$

Two assets with realized returns S_1 and S_2 can be considered *risky* i.e. σ_{S1}^2, $\sigma_{S2}^2 > 0$, while S_0 may be considered *risk-free* e.g. the return on a government treasury bill i.e. $\sigma_{S0}^2 \approx 0$. Then, a *dynamic hedging scheme* for the issuer of this financial structured product; i.e. the option writer; would be to invest in a replicating portfolio consisting of the three assets, with funds allocated in particular proportions in accordance with the objective of maximizing expected utility i.e. minimizing the tracking error. Then the replicating portfolio at t for our 3-asset financial structured product is:

$$\mathbf{v}_t = (\mathbf{p}_0)_t \mathbf{e}^{it} + (\mathbf{p}_1)_t (\mathbf{S}_1)_t + (\mathbf{p}_2)_t (\mathbf{S}_2)_t \qquad \textbf{... (3.10)}$$

Then the tracking error at time-point t is given as the difference between the payoff on the option at time-point t and the value of the replicating portfolio at that time:

$$|\varepsilon_t| = |\mathbf{Max}\ (\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2)_t - \mathbf{v}_t| \textbf{ i.e.}$$

$$|\varepsilon_t| = |\mathbf{Max}\ (\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2)_t - \{(\mathbf{p}_0)_t \mathbf{e}^{it} + (\mathbf{p}_1)_t (\mathbf{S}_1)_t + (\mathbf{p}_2)_t (\mathbf{S}_2)_t\}| \qquad \textbf{... (3.11)}$$

If $|\varepsilon_t| \approx 0$ then the option writer is perfectly hedged at time-point t. If $|\varepsilon_t| \approx 0$ can be maintained for the entire lock-in period, then one can say that the dynamic hedging scheme has worked perfectly, resulting in utility maximization for the option writer. We have used the following hypothetical data set for conducting our computational study with the haploid genetic algorithm scheme to minimize the tracking error $|\varepsilon_t|$:

$S_{1\ (t=0)}$	1.00
$S_{2\ (t=0)}$	1.00
$S_{0\ (t=0)}$	1.00
Correlation (S_1, S_2)	0.50
μ_{S_1}	20%
μ_{S_2}	18%
σ_{S1}	30%
σ_{S2}	20%
I	5%
T	12 months
Re-balancing frequency	Monthly

Table 3.1

We assume that all three assets start off with a realized return of unity at t = 0. The correlation between the two risky assets is assumed constant at a moderate 50%. The first risky asset is taken to have a slightly higher mean return (20%) but a rather high volatility (30%), compared to the mean return (18%) and volatility (20%) of the second one. The risk-free rate is 5% p.a. and the lock-in period is taken as one year. The replicating portfolio is re-balanced at the end of each month over the period.

Thus, if the best performer is the first risky asset at time-point t, $|\varepsilon_t|$ is minimized when maximum allocation is made to the first risky asset at t while if the best performer is the second risky asset at time-point t, $|\varepsilon_t|$ is minimized if the maximum allocation is made to that asset at t. If neither of the two risky assets can outperform the guaranteed return on the risk-free asset at t, then $|\varepsilon_t|$ is minimized if maximum allocation is made to the risk-free asset. Short selling is not permitted in our model.

To minimize programming involvement our haploid genetic algorithm model is designed to handle only univariate optimization models. However, the problem we are studying is one of multivariate optimization with three parameters corresponding to the portfolio weights of the three assets constituting our financial structured product. Therefore, we have taken the allocation for the risk-free asset as given. This then essentially reduces our problem to one of univariate optimization, whereby we have to minimize the cumulative tracking error given by the following objective function:

$$\Sigma_t |\varepsilon_t| = \Sigma_t |\mathbf{Max}\ (S_0, S_1, S_2)_t - \{c\ e^{it} + p_t\ (S_1)_t + (1 - c - p_t)\ (S_2)_t\}| \qquad \ldots (3.12)$$

Here, p_t is the allocation to be made to the first risky asset at every point of re-balancing and c is the given allocation always made to the risk-free asset thereby allowing us to substitute $(p_2)_t$ with $(1 - c - p_t)$; as portfolio weights sum up to unity. Then the utility maximizing behavior of the option writer will compel him or her to find out the optimal functional values of p_t at every t so as to minimize the total error.

It is quite logical to assume that the p_t values will have to be related in some way to the *sensitivity* of the change in potential option payoff to the change in performance of the observed best asset within the envelope. With *continuous re-balancing*, one can theoretically achieve $|\varepsilon_t| \approx 0$ if portfolio weights are selected in accordance with the *partial derivatives* of the option value with respect to the underlying asset returns, as per usual dynamic hedging technique in a Black-Scholes environment. Thus the utility maximization goal would be to suppress $|\varepsilon_t|$ to a value as close as possible to zero at every t so that overall $\Sigma_t |\varepsilon_t|$ is consequently minimized. Of course, *re-balancing costs* make continuous re-balancing a mathematical artifact rather than a practicable financial strategy. However, monthly re-balancing is mostly practicable even for some asset types which are considered relatively illiquid like many of the offshore funds.

Then, if one accepts the biological origin of utility forms, any sufficiently robust evolutionary algorithm should show convergence with a Black Scholes type expected utility formulation. In other words, embedding such a utility function in any simple heuristic model involving no prior beliefs about probabilities should be evolutionarily optimal. This is exactly what our present study is purported to reveal computationally.

3.3 Computational Observations

A *Monte Carlo simulation* algorithm is used to generate the *potential* payoffs for the option on best of three assets at the end of each month for t = 0, 2 … 11. The word potential is crucial in the sense that our option is essentially *European* and *path-independent* i.e. basically to say only the terminal payoff counts. However the replicating portfolio has to track the option all through its life in order to ensure an optimal hedge and therefore we have evaluated potential payoffs at each t. The potential payoffs are computed as Max $[(S_1)_t - (S_0)_t, (S_2)_t - (S_0)_t, 0]$ in Table 3.2.

The risky returns S_1 and S_2 are assumed to evolve over time following the stochastic diffusion process of a *geometric Brownian motion*. The risk-free return S_0 is continuously compounded approximately at a rate of 0.41% per month giving a 5% annual yield. We have run our Monte Carlo simulation model with the hypothetical data in Table 3.1 over the one-year lock-in period and calculated the potential option payoffs. All formal definitions pertaining to the model are given in Appendix (ii).

t	S_1	S_2	S_0	Best Asset	Potential Payoff
0	1	1	1	-	-
1	1.0169	1.0219	1.0003	Asset 2	0.0216
2	1.0311	1.0336	1.0010	Asset 2	0.0326
3	1.0801	1.0820	1.0021	Asset 2	0.0799
4	1.1076	1.0960	1.0035	Asset 1	0.1041
5	1.1273	1.1280	1.0052	Asset 2	0.1228

6	1.1694	1.1694	1.0073	Asset 2	0.1621
7	1.2008	1.1923	1.0098	Asset 1	0.1910
8	1.2309	1.2160	1.0126	Asset 1	0.2183
9	1.2836	1.2489	1.0157	Asset 1	0.2679
10	1.3362	1.3030	1.0193	Asset 1	0.3169
11	1.3617	1.3196	1.0232	Asset 1	0.3385

Table 3.2

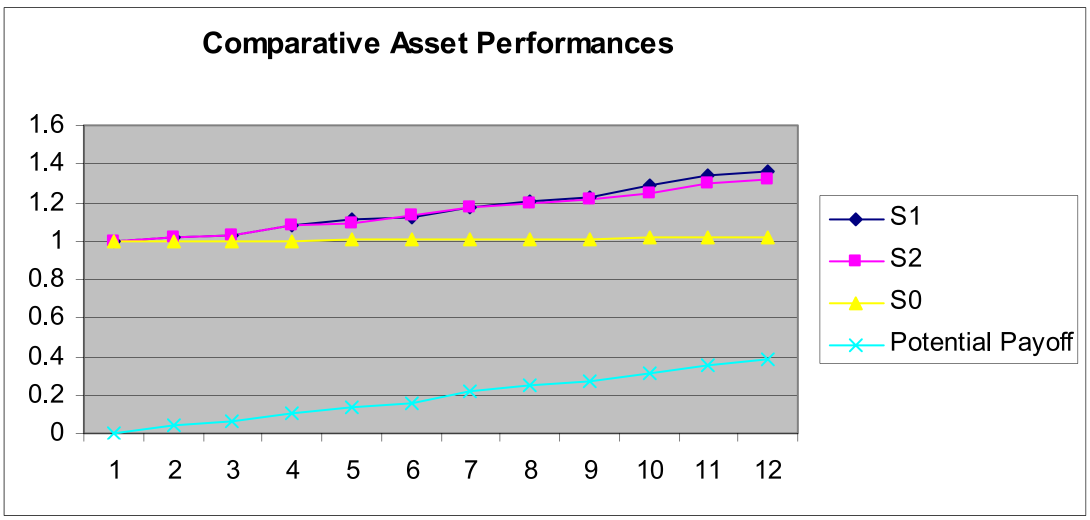

Figure 3.1

The results shown in Table 3.2 and Figure 3.1 show that the second risky asset is the best performer towards the beginning and in the middle of lock-in period but the first one catches up and in fact outperforms the second one towards the end of the period.

t	p_1S_1	p_2S_2	p_0S_0	v_t

0	0.4500	0.4500	0.1000	1.0000
1	0.4576	0.4598	0.1000	1.0175
2	0.0000	0.9303	0.1001	1.0304
3	0.0000	0.9738	0.1002	1.0740
4	0.0000	0.9864	0.1003	1.0867
5	1.0145	0.0000	0.1005	1.1151
6	0.0000	1.0524	0.1007	1.1532
7	0.0000	1.0731	0.1008	1.1741
8	1.1078	0.0000	0.1013	1.2091
9	1.1552	0.0000	0.1016	1.2568
10	1.2025	0.0000	0.1019	1.3045
11	1.2255	0.0000	0.1023	1.3279

Table 3.3

For an initial input of \$1, apportioned at t = 0 as 45% between S_1 and S_2 and 10% for S_0, we have constructed five replicating portfolios according to a simple *rule-based logic*: k% of funds are allocated to the observed best performing risky asset and the balance proportion i.e. (90 – k) % to the other risky asset (keeping the portfolio self-financing after the initial investment) at every monthly re-balancing point. We have reduced k by 10% for each portfolio starting from 90% and going down to 50%. As is shown in Appendix (ii), this simple hedging scheme performs quite well over the lock-in period when k = 90%. But the performance falls away steadily as k is reduced every time.

3.4 Inferential Remarks based on the Computation Results

Choice of k	Cumulative Tracking Error
90%	0.1064
80%	0.1172
70%	0.1281
60%	0.1390
50%	0.1498

Table 3.4

The fact that a dominance pattern can be noticed for $80\% < k^* \leq 90^*$ is quite evident from the output data in Table 3.4 and the graphical plot of that data and accompanying tabulated figures in Appendix (ii). It may be noted to be of significance that $k^* \approx 90\%$ indeed comes closest to the percentage allocation for the best performing asset made according to the sensitivity of change in the potential payoff on the option to a change in performance of the observed best performer. This indeed satisfies the dynamic hedging principle in a Black Scholes environment to the maximal extent possible given monthly re-balancing, a fixed allocation to S_0 and no shorting, as were our imposed conditions. This does seem to lend credence to the belief that embedding a Black Scholes type of expected payoff (utility) maximization function is indeed *evolutionarily optimal*.

Having computationally verified the biological basis of a Black-Scholes type expected utility function in the dynamic delta hedging situation, we now proceed to computationally derive an Information Theoretic model of utility applicable to dynamic risk management engineered by a multi-asset options-based portfolio insurance strategy.

4. A Proposed Information Theoretic Model of Utility Applicable to Active Risk Management Engineered by Multi-asset Options-based Portfolio Insurance Strategy for Distinct Investor Profiles

In early nineteenth century most economists conceptualized utility as a psychic reality – cardinally measurable in terms of *utils* like distance in kilometers or temperature in degrees centigrade. In the later part of nineteenth century Vilfredo Pareto discovered that all the important aspects of Demand theory could be analyzed ordinally using geometric devices, which later came to be known as "indifference curves". The indifference curve approach effectively did away with the notion of a cardinally measurable utility and went on to form the methodological cornerstone of modern microeconomic theory.

An indifference curve for a two-commodity model is mathematically defined as the locus of all such points in $\boldsymbol{E}^2$ where different combinations of the two commodities give the same level of satisfaction to the consumer so as the consumer is indifferent to any particular combination. Such indifference curves are always *convex to the origin* because of the operation of the *law of substitution*. This law states that the scarcer a commodity becomes, the greater becomes its relative substitution value so that its marginal utility rises relative to the marginal utility of the other commodity that has become comparatively plentiful.

In terms of the indifference curves approach, the problem of utility maximization for an individual consumer may be expressed as a constrained non-linear programming problem that may be written in its general form for an n-commodity model as follows:

$$\textbf{Maximize } U = U(C_1, C_2 \ldots C_n)$$

$$\textbf{Subject to } \Sigma\, C_j P_j \leq B$$

$$\textbf{and } C_j \geq 0, \textbf{ for } j = 1, 2 \ldots n \qquad \ldots (4.1)$$

If the above problem is formulated with a strict equality constraint i.e. if the consumer is allowed to use up the entire budget on the n commodities, then the utility maximizing condition of consumer's equilibrium is derived as the following first-order condition:

$$\partial U/\partial C_j = (\partial U/\partial C_j) - \lambda P_j = 0 \textbf{ i.e.}$$

$$(\partial U/\partial C_j)/P_j = \lambda^* = \textbf{constant, for } j = 1, 2 \ldots n \qquad \ldots (4.2)$$

This pertains to the classical economic theory that in order to maximize utility, individual consumers necessarily must allocate their budget so as to *equalize the ratio of marginal utility to price* for every commodity under consideration, with this ratio being found equal to the optimal value of the *Lagrangian multiplier* λ^*.

However a mathematically necessary pre-condition for the above indifference curve approach to work is **$(U_{C1}, U_{C2} \ldots U_{Cn}) > 0$** i.e. the marginal utilities derived by the consumer from each of the n commodities must be positive. Otherwise of course the problem degenerates. To prevent this from happening one needs to strictly adhere to the law of substitution under all circumstances. This however, at times, could become an untenable proposition if measure of utility is strictly restricted to an intrinsic one. This is because, for the required condition to hold, each of the n commodities necessarily *must always have a positive intrinsic utility* for the consumer. However, this invariably leads to anomalous reasoning like the intrinsic utility of a woolen jacket being independent of the temperature or the intrinsic utility of an umbrella being independent of rainfall!

Choice among alternative courses of action consist of trade-offs that confound subjective probabilities and marginal utilities and are almost always too coarse to allow for a meaningful separation of the two. From the viewpoint of a classical statistical decision theory like that of *Bayesian inference* for example, failure to obtain a correct representation of the underlying behavioral basis would be considered a major pitfall in the aforementioned analytical framework.

Choices among alternative courses of action are largely determined by the relative degrees of belief an individual attaches to the prevailing uncertainties. Following Vroom (Vroom; 1964), the *motivational strength* S_n of choice c_n among N alternative available choices from the choice set $C = \{c_1, c_2 \ldots c_N\}$ may be ranked with respect to the multiplicative product of the relative reward $r(c_n)$ that the individual attaches to the consequences resulting from the choice c_n, the likelihood that the choice set under consideration will yield a positive intrinsic utility and the respective probabilities given by $p\{r(c_n)\}$ associated with $r(c_n)$ such that:

$$\mathbf{S_{max} = Max_n\ [r\ (c_n)\ x\ p\ (U_{r(C)} > 0)\ x\ p\{r\ (c_n)\}],\ n = 1, 2 \ldots N} \qquad \ldots\ \mathbf{(4.3)}$$

Assuming for the time-being that the individual is calibrated with perfect certainty with respect to the intrinsic utility resulting from a choice set such that we have the condition $\mathbf{p\ (U_{r(C)} > 0) = \{0, 1\}}$, the above model can be reduced as follows:

$$\mathbf{S_{max} = Max_k\ [r\ (c_k)\ x\ p\{r\ (c_k)\}],\ k = 1, 2 \ldots K} \text{ such that } K < N \qquad \ldots\ \mathbf{(4.4)}$$

Therefore, choice A, which entails a large reward with a low probability of the reward being actualized, could theoretically yield the same motivational strength as choice B, which entails a smaller reward with a higher probability of the reward being actualized.

However, we recognize the fact that the *information conveyed* to the decision-maker by the outcomes would be quite different for A and B though their values may have the same mathematical expectation. Therefore, whereas intrinsic utility could explain the ranking with respect to *expected value* of the outcomes, there really has to be another dimension to utility whereby the ***expected information*** is considered – that of *extrinsic utility*. So, though there is a very low probability of having an unusually cold day in summer, the information conveyed to the likely buyer of a woolen jacket by occurrence of such an aberration in the weather pattern would be quite substantial, thereby validating an extended substitution law based on an *expected information measure* of utility. The specific objective of this paper is to formulate a mathematically sound theoretical edifice for the formal induction of extrinsic utility into the folds of statistical decision theory.

4.1 A few essential working definitions

Object: Something with respect to which an individual may perform a specific goal-oriented behavior

Object set: The set O of a number of different objects available to an individual at any particular point in space and time with respect to achieving a goal where $n\{O\} = K$

Choice: A path towards the sought goal emanating from a particular course of action - for a single available object within the individual's object set, there are two available choices - either the individual takes that object or he or she does not take that object. Therefore, generalizing for an object set with K alternative objects, there can be 2^K alternative courses of action for the individual

Choice set: The set C of all available choices where $C = \boldsymbol{P}\,O$, $n\{C\} = 2^K$

Outcome: The relative reward resulting from making a particular choice

Decision-making is nothing but goal-oriented behavior. According to the celebrated *theory of reasoned action* (Fishbain, 1979), the immediate determinant of human behavior is the intention to perform (or not to perform) the behavior. For example, the simplest way to determine whether an individual will invest in Acme Inc. equity shares is to ask whether he or she *intends* to do so. This does not necessarily mean that there will always be a perfect relationship between intention and behavior. However, there is no denying the fact that people usually tend to act in accordance with their intentions.

However, though intention may be shaped by a positive intrinsic utility expected to be derived from the outcome of a decision, the *ability* of the individual to actually act according to his or her intention also needs to be considered. For example, if an investor truly intends to buy a call option on the equity stock of Acme Inc. even then his or her intention cannot get translated into behavior if there is no exchange-traded call option available on that equity stock.

Thus we may view the *additional element of choice* as a measure of extrinsic utility. *Utility is not only to be measured by the intrinsic want-satisfying capacity of a commodity for an intending individual but also* by *the availability of the particular commodity at that point in space and time to enable that individual to act according to his or her intension*. Going back to our woolen jacket example, though the intrinsic utility of such a garment in summer is practically zero, the extrinsic utility afforded by its mere availability can nevertheless suffice to uphold the law of substitution.

4.2 Utility and Thermodynamics

In our present paper we have attempted to extend the classical utility theory applying the entropy measure of information (Shannon, 1948; Jaynes, 1957), which by itself bears a direct constructional analogy to the well-known *Boltzmann equation* in thermodynamics.

There is some uniformity in views among economists as well as physicists that a functional correspondence exists between the formalisms of economic theory and classical thermodynamics. The laws of thermodynamics can be intuitively interpreted in an economic context and the correspondences do show that thermodynamic entropy and economic utility are related concepts sharing the same formal framework. Utility is said to arise from that component of thermodynamic entropy whose change is due to irreversible transformations. This is the standard *Carnot entropy* given by $dS = \delta Q/T$ where S is the entropy measure, Q is the thermal energy of state transformation (irreversible) and T is the absolute temperature (Smith, 1998). In this chapter however we will keep to the information theoretic definition of entropy rather than the purely thermodynamic one.

Underlying premises of our extrinsic utility model:

1. Utility derived from making a choice can be distinctly categorized into two forms:

(a) Intrinsic utility ($U_{r(C)}$) – the intrinsic, non-quantifiable capacity of the potential outcome from a particular choice set to satisfy a particular human want under given circumstances; in terms of expected utility theory $U_{r\,(C)} = \Sigma\, r\,(c_j)\, p\{r\,(c_j)\}$, where j = 1, 2 … K and

(b) Extrinsic utility (U_X) – the additional possible choices afforded by the mere availability of a specific object within the object set of the individual

2. An choice set with n (C) = 1 (i.e. when K = 0) with respect to a particular individual corresponds to lowest (zero) extrinsic utility; so U_X cannot be negative

3. The law of diminishing marginal utility tends to hold in case of U_x when an individual repeatedly keeps making the same choice to the exclusion of other available choices within his or her choice set

Expressing the frequency of alternative choices in terms of the probability of getting an outcome r_j by making a choice c_j, the generalized extrinsic utility function can be framed as a modified version of *Shannon's entropy function* as follows:

$$\mathbf{U_X = - K\, \Sigma_j\, p\, \{r\, (c_j)\}\, \log_2 p\, \{r\, (c_j)\},\; j = 1, 2 \ldots 2^K} \qquad \textbf{... (4.5)}$$

The multiplier $-K = -n(O)$ is a scale factor somewhat analogous to the ***Boltzmann constant*** in classical thermodynamics with a reversed sign. Therefore general extrinsic utility maximization reduces to the following non-linear programming problem:

$$\begin{aligned}
&\textbf{Maximize } U_X = -K \textstyle\sum_j p\{r(c_j)\} \log_2 p\{r(c_j)\} \\
&\textbf{Subject to } \textstyle\sum p\{r(c_j)\} = 1, \\
&p\{r(c_j)\} \geq 0; \textbf{ and} \\
&j = 1, 2 \ldots 2^K \qquad \ldots (4.6)
\end{aligned}$$

Putting the objective function into the usual Lagrangian multiplier form, we get

$$Z = -K \sum p\{r(c_j)\} \log_2 p\{r(c_j)\} + \lambda \left(\sum p\{r(c_j)\} - 1\right) \qquad \ldots (4.7)$$

Now, as per the first-order condition for maximization, we have

$$\partial Z / \partial p\{r(c_j)\} = -K(\log_2 p\{r(c_j)\} + 1) + \lambda = 0 \text{ i.e.}$$

$$\log_2 p\{r(c_j)\} = \lambda/K - 1 \qquad \ldots (4.8)$$

Therefore; for a pre-defined K; $p\{r(c_j)\}$ is independent of j, i.e. all the probabilities are necessarily equalized to the constant value $p\{r(c_j)\}^* = 2^{-K}$ at the point of maximum U_X.

The analytical reasoning we have employed here is related to the earlier work of Abbas (2002) where he utilized Shannon's definition of entropy as a measure for the spread of coordinates of a utility increment vector. He also applied the differential form of entropy expression in continous form to a utility density function to get a value of U_X.

It is also intuitively obvious that when $p\ \{r\ (c_j)\} = 2^{-K}$ for $j = 1, 2 \ldots 2^K$, the individual has the *maximum element of choice in terms of the available objects within his or her object set*. For a choice set with a single available choice, the extrinsic utility function will be simply given as follows:

$$\mathbf{U_X = -\ p\{r\ (c)\}\ log_2\ p\{r\ (c)\} - (1 - p\{r\ (c)\})\ log_2\ (1 - p\{r\ (c)\})} \qquad \textbf{... (4.9)}$$

Then the slope of the marginal extrinsic utility curve will as usual be given by the second-order condition $\mathbf{d^2U_X/dp\{r\ (c)\}^{2} < 0}$, and this can additionally serve as an alternative basis for intuitively deriving the generalized, downward-sloping demand curve and is thus a neat theoretical spin-off!

Therefore, though the mathematical expectation of a reward resulting from two mutually exclusive choices may be the same thereby giving them equal rank in terms of the intrinsic utility of the expected reward, the expected information content of the outcome from the two choices will be quite different given different probabilities of obtaining the relative rewards. The following vector will then give a *composite measure of total expected utility* from the object set:

$$\mathbf{U = [U_r, U_X] = [\Sigma r\ (c_j)\ p\{r\ (c_j)\}, - K\ \Sigma_j\ p\ \{r\ (c_j)\}\ \log_2 p\ \{r\ (c_j)\}], j = 1, 2 \ldots 2^K \ldots (4.10)}$$

Now, having established the essential premise of formulating an extrinsic utility measure, we can proceed to let go of the assumption that an individual is calibrated with perfect certainty about the intrinsic utility resulting from the given choice set so that we now look at the full Vroom model rather than the reduced version. If we remove the restraining condition that $p\ (U_{r\ (C)} > 0) = \{0, 1\}$ and instead we have the more general case of $0 \leq p\ (U_{r(C)} > 0) \leq 1$, then we introduce another probabilistic dimension to our choice set whereby the individual is no longer certain about the nature of the impact the outcomes emanating from a specific choice will have on his intrinsic utility. This can be intuitively interpreted in terms of the *likely opportunity cost* of making a choice from within a given choice set to the exclusion of all other possible choice sets. For the particular choice set C, if the likely opportunity cost is less than the potential reward obtainable, then $U_{r\ (c)} > 0$, if opportunity cost is equal to the potential reward obtainable, then $U_{r(C)} = 0$, else if the opportunity cost is greater than the potential reward obtainable then $U_{r\ (C)} < 0$. Writing $U_{r(C)} = \Sigma_j\ r\ (c_j)\ p\{r\ (c_j)\}$, $j = 1, 2 \ldots N$, the *total expected utility vector* now becomes:

$$\mathbf{[U_{r(C)}, U_X] = [\Sigma_j\ r\ (c_j)\ p\{r\ (c_j)\}, - K\ \Sigma\ p\ \{r\ (c_j) |\ U_{r(C)} > 0\}\ \log_2 p\ \{r\ (c_j) |\ U_{r(C)} > 0\}] \quad \ldots (4.11)}$$

Here $j = 1 \ldots N$ and $p\ \{r\ (c_j) |\ U_{r(C)} > 0\}$ may be estimated by the standard *Bayes criterion* as under:

$$\mathbf{p\ \{r\ (c_j)|\ U_{r(c)} > 0\} = [p\ \{(U_{r(C)} \geq 0|r\ (c_j)\}\ p\ \{(r\ (c_j)\}][\Sigma_j\ p\ \{(U_{r(C)} > 0|r\ (c_j)\}\ p\ \{(r\ (c_j)\}]^{-1}} \quad \ldots (4.12)$$

4.3 Evaluating an investor's extrinsic utility from capital-guaranteed, financial structured products

Let a financial structured product be made up of a basket of n different assets such that the investor has the right to claim the return on the best-performing asset out of that basket after a stipulated holding period. Then, if one of the n assets in the basket is the risk-free asset then the investor is assured of a minimum return equal to the risk-free rate i on his invested capital at the termination of the stipulated holding period (minus the stipulated costs). This effectively means that his or her investment becomes endogenously capital-guaranteed as the terminal wealth, even at its worst, cannot be lower in value to the initial wealth plus the return earned on the risk-free asset minus a finite cost of *portfolio insurance*.

Therefore, with respect to each risky asset, we can have a *binary response from the investor in terms of his or her funds-allocation decision* whereby the investor either takes funds out of an asset or puts funds into an asset. Since the overall portfolio has to be *self-financing* in order to pertain to a *Black-Scholes* kind of pricing model, funds added to one asset will also mean same amount of funds removed from one or more of the other assets in that basket. If the basket consists of a single risky asset s (and of course cash as the risk-free asset) then, if η_s is the amount of re-allocation effected each time with respect to the risky asset s, the two alternative, mutually exclusive choices open to the investor with respect to the risky asset s are as follows:

(1) C ($\eta_s \geq 0$) (funds left in asset s), with associated outcome r ($\eta_s \geq 0$); and

(2) C ($\eta_s < 0$) (funds removed from asset s), with associated outcome r ($\eta_s < 0$)

Therefore what the different assets are giving to the investor apart from their intrinsic utility in the form of higher expected terminal reward is some *extrinsic utility in the form of available re-allocation options*. Then the expected present value of the final return is given as follows:

$$\mathbf{E\ (r) = Max\ [w, Max_j\ \{e^{-it}\ E\ (r_j)_t\}],\ j = 1, 2 \ldots 2^{n-1}} \qquad \ldots (4.13)$$

In the above equation i is the rate of return on the risk-free asset and t is the length of the investment horizon in continuous time and w is the initial wealth invested i.e. ignoring insurance cost, if the risk-free asset outperforms all others E (r) = $we^{it}/e^{it} = w$.

Now what is the probability of each of the (n – 1) risky assets performing worse than the risk-free asset? Even if we assume that there are some cross-correlations present among the (n – 1) risky assets, given the statistical nature of the risk-return trade-off the joint probability of these assets performing worse than the risk-free asset will be very low over moderately long investment horizons. And this probability will keep going down with every additional risky asset added to the basket. Thus each additional asset will *empower the investor* with additional choices with regards to re-allocating his or her funds among the different assets according to their observed performances.

Intuitively we can make out that the extrinsic utility to the investor is indeed maximized when there is *an equal positive probability of actualizing each outcome r_j resulting from η_j given that the intrinsic utility $U_{r(C)}$ is greater than zero.* By a purely economic rationale, *each additional asset introduced into the basket will be so introduced if and only if it significantly raises the expected monetary value of the potential terminal reward*. As already demonstrated, the extrinsic utility maximizing criterion will be given as under:

$$\mathbf{p\ (r_j \mid U_{r(C)} > 0)^* = 2^{-(n-1)}\ for\ j = 1, 2 \ldots 2^{n-1}} \qquad \ldots (4.14)$$

The *composite utility vector from the multi-asset structured product* will be as follows:

$$\mathbf{[U_{r(C)}, U_X] = [E\ (\ r\ ), -\ (n - 1)\Sigma\ p\ \{r_j \mid U_{r(C)} > 0\}\ \log_2 p\ \{r_j \mid U_{r(C)} > 0\}],\ j = 1, 2 \ldots 2^{n-1}} \quad \ldots (4.15)$$

4.4 Choice set with a structured product having two risky assets (and cash):

0	0
1	0
0	1
1	1

Table 4.1

The first row of Table 4.1 shows that the investor can remove all funds from the two risky assets and convert it to cash (the risk-free asset), or funds can be taken out of asset 2 and put it in asset 1, or the investor can take funds out of asset 1 and put it in asset 2, or the investor can convert some cash into funds and put it in both the risky assets. Thus there are 4 alternative choices for the investor when it comes to re-balancing his portfolio. Therefore we may classify such a structured, financial product as *4-bit* product.

4.5 Choice set with a structured product having three risky assets (and cash):

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Table 4.2

That is, the investor can remove all funds from the three risky assets and convert it into cash (the risk-free asset), or the investor can take funds out of asset 1 and asset 2 and put it in asset 3, or the investor can take funds out from asset 1 and asset 3 and put it in asset 2, or the investor can take funds out from asset 2 and asset 3 and put it in asset 1, or the investor can take funds out from asset 1 and put it in asset 2 and asset 3, or the investor can take funds out of asset 2 and put it in asset 1 and asset 3, or the investor can take funds out of asset 3 and put it in asset 1 and asset 2, or the investor can convert some cash into funds and put it in all three of the assets. Thus there are 8 alternative choices for the investor when it comes to re-balancing the portfolio. Therefore we may classify such a structured, financial product as *8-bit* product. *An 8-bit product will impart greater utilty of empowerment to an individual investor due to the wider range of decision options*.

Of course, according to the Black-Scholes hedging principle, the re-balancing needs to be done each time by setting the optimal proportion of funds to be invested in each asset equal to the partial derivatives of the option valuation formula with respect to each of these assets. However, the total number of alternative choices available to the investor increases with every new risky asset that is added to the basket thereby contributing to the extrinsic utility in terms of the expected information content of the total portfolio.

4.6 Coding of financial structured product information

Extending the entropy measure of extrinsic utility, we may conceptualize the interaction between the buyer and the vendor as a two-way communication flow whereby the vendor *informs* the buyer about the expected utility derivable from the product on offer and the buyer *informs* the seller about his or her individual expected utility criteria. An economic transaction goes through if the two sets of information are compatible. Of course, the greater expected information content of the vendor's communication, the higher is the extrinsic utility of the buyer. Intuitively, the expected information content of the vendor's communication will increase with increase in the variety of the product on offer, as that will increase the likelihood of matching the buyer's expected utility criteria.

The product information from vendor to potential buyer may be transferred through some medium e.g. the vendor's website on the Internet, a targeted e-mail or a telephonic promotion scheme. But such transmission of information is subject to noise and distractions brought about by environmental as well as psycho-cognitive factors. While a *distraction* is prima facie predictable, (e.g. the pop-up windows that keep on opening when some commercial websites are accessed), *noise* involves unpredictable perturbations (e.g. conflicting product information received from any competing sources).

Transmission of information calls for some kind of *coding*. Coding may be defined as a mapping of words from a source alphabet A to a code alphabet B. A *discrete, finite memory-less channel* with finite inputs and output alphabets is defined by a set of transition probabilities $p_i (j)$, $i = 1, 2 \ldots a$ and $j = 1, 2 \ldots b$ with $\Sigma_j p_i (j) = 1$ and $pi (j) \geq 0$. Here $p_i (j)$ is the probability that for an input letter i output letter j will be received.

A *code word* of length n is defined as a sequence of n input letters, which are actually n integers chosen from 1, 2 … a. A *block code* of length n having M words is a mapping of the message integers from 1 to M into a set of code words each having a fixed length n. Thus for a structured product with N component assets, a block code of length n having N words would be used to map message integers from 1 to N, corresponding to each of the N assets, into a set of a fixed-length code words. Then there would be a total number of $C = 2^N$ possible combinations such that $\log_2 C = N$ binary-state devises (flip-flops) would be needed.

A *decoding system* for a block code is the inverse mapping of all output words of length n into the original message integers from 1 to M. Assuming all message integers are used with same probability 1/M, the probability of error P_e for a code and decoding system ensemble is defined as the probability of an integer being transmitted and received as a word which is mapped into another integer i.e. P_e is the probability of wrongly decoding a message.

Therefore, in terms of our structured product set up, P_e might be construed as the probability of misclassifying the best performing asset. Say within a structured product consisting of three risky assets - a blue-chip equity portfolio, a market-neutral hedge fund and a commodity future (and cash as the risk-free asset), while the original transmitted information indicates the hedge fund to be the best performer, due to erroneous decoding of the encoded message, the equity portfolio is interpreted as the best performer. Such erroneous decoding could result in investment funds being allocated to the wrong asset at the wrong time.

4.7 Relevance of Shannon-Fano coding to structured product information transmission

By the well-known Kraft's inequality we have $K = \Sigma_n 2^{-li} \leq 1$, where l_i stands for some definite code word lengths with a radix of 2 for binary encoding. For block codes, $l_i = l$ for $i = 1, 2 \ldots n$. As per *Shannon's coding theorem*, *it is possible to encode all sequences of n message integers into sequences of binary digits in such a way that the average number of binary digits per message symbol is approximately equally to the entropy of the source, the approximation increasing in accuracy with increase in n*. For efficient binary codes, $K = 1$ i.e. $\log_2 K = 0$ as it corresponds to the maximal entropy condition. Therefore the inequality occurs if and only if $p_i \neq 2^{-li}$. Though the Shannon-Fano coding scheme is not strictly the most efficient, it has the advantage of directly deriving the code word length l_i from the corresponding probability p_i. With source symbols $s_1, s_2 \ldots s_n$ and their corresponding probabilities $p_1, p_2 \ldots p_n$, where for each p_i there is an integer l_i, and then given that we have bounds that span a unit length, we have the following relationship:

$$\mathbf{\log_2 (p_i^{-1}) \leq l_i < \log_2 (p_i^{-1}) + 1} \qquad \textbf{… (4.16)}$$

Removing the logs, taking reciprocals and summing each term we therefore end up with, $\Sigma_n p_i \geq \Sigma_n 2^{li} \geq (\Sigma_n p_i)/2$, that is,

$$\mathbf{1 \geq \Sigma_n 2^{li} \geq 1/2} \qquad \textbf{… (4.17)}$$

Inequality (4.17) gets us back to the Kraft's inequality. This shows that there is an instantaneously decodable code having the Shannon-Fano lengths l_i. By multiplying inequality (4.16) by p_i and summing we get:

$$\mathbf{\Sigma_n (p_i \log_2 p_i^{-1}) \leq \Sigma_n p_i l_i < \Sigma_n (p_i \log_2 p_i^{-1}) + 1, \text{ i.e.}}$$

$$\mathbf{H_2 (S) \leq L \leq H_2 (S) + 1} \qquad \textbf{… (4.18)}$$

That is, in terms of the average Shannon-Fano code length L, we have conditional entropy as an effective lower bound while it is also the non-integral component of the upper bound of L. This underlines the relevance of a Shannon-Fano form of coding to our structured product formulation as this implies that the average code word length used in this form of product information coding would be *bounded by a measure of extrinsic utility* to the potential investor of the financial structured product itself, which is definitely an intuitively appealing prospect.

4.8 Conceptualizing financial structured product information transmission as a generalized Markov process

The Black-Scholes option-pricing model is based on the underlying assumption that asset prices evolve according to the geometric diffusion process of a Brownian motion. The Brownian motion model has the following fundamental assumptions:

(1). $W_0=0$

(2). W_t-W_s is a random variable that is normally distributed with mean 0 and variance t-s

(3). W_t-W_s is independent of W_v-W_u if (s, t) and (u, v) are non-overlapping time intervals

Property (3) implies that the Brownian motion is a *Markovian process* with no long-term memory. The switching behavior of asset prices from “high” (Bull state) to “low” (Bear state) and vice versa according to Markovian transition rule constitutes a well-researched topic in stochastic finance. It has in fact been proved that steady-state equilibrium exists when the state probabilities are equalized for a stationary transition-probability matrix (Bhattacharya and Samanta, 2003). This steady-state equilibrium corresponds to the condition of strong efficiency in the financial markets whereby no historical market information can result in arbitrage opportunities over any significant length of time.

By logical extension, considering a structured portfolio with n assets, the best performer may be hypothesized to be traceable by a first-order Markov process, whereby the best performing asset at time t+1 is dependent on the best performing asset at time t. For example, with n = 3, we have the following state-transition matrix:

	Asset 1	Asset 2	Asset 3
Asset 1	P (1 | 1)	P (2 | 1)	P (3 | 1)
Asset 2	P (2 | 1)	P (2 | 2)	P (3 | 2)
Asset 3	P (3 | 1)	P (3 | 2)	P (3 | 3)

Table 4.3

In information theory also, a similar Markov structure is used to improve the encoding of a source alphabet. For each state in the Markov system, an appropriate code can be obtained from the corresponding transition probabilities of leaving that state. The efficiency gain will depend on how variable the probabilities are for each state (Wolfowitz, 1957). However, as the order of the Markov process is increased, the gain will tend to be less and less while the number of attainable states approaches infinity.

The strength of the Markov formulation lies in its capacity of handling correlation between successive states. If S_1, S_2 … S_m are the first m states of a stochastic variable, what is the probability that the next state will be S_i? This is written as the conditional probability $p(S_i \mid S_1, S_2 \ldots S_m)$. Then, the Shannon measure of information from a state S_i is given as usual as follows:

$$\mathbf{I\ (S_i \mid S_1, S_2 \ldots S_m) = \log_2 \{p\ (S_i \mid S_1, S_2 \ldots S_m)\}^{-1}} \qquad \ldots (4.19)$$

The entropy of a Markov process is then derived as follows:

$$\mathbf{H\ (S_i) = \sum_{S^{m+1}} p\ (S_1, S_2 \ldots S_m, S_i)\ I\ (S_i \mid S_1, S_2 \ldots S_m)} \qquad \ldots (4.20)$$

Then the extrinsic utility to an investor from the ith asset included within a financial structured product expressed in terms of the entropy of a Markov process governing the state-transition of the best performing asset over N component risky assets (and cash as the one risk-free asset) would be obtainable as follows:

$$\mathbf{U_x = H\ (S_i) = \sum_{S^{N+1}} p\ (S_1, S_2 \ldots S_m, S_i)\ I\ (S_i \mid S_1, S_2 \ldots S_m)} \qquad \ldots (4.21)$$

However, to find the entropy of a Markov source alphabet one needs to explicitly derive the stationary probabilities of being in each state of the Markov process. But these state probabilities may be hard to derive explicitly especially if there are a large number of allowable states (e.g. corresponding to a large number of elementary risky assets within a financial structured product). Using *Gibbs inequality*, it can be show that the following limit can be imposed for bounding the entropy of the Markov process:

$\mathbf{\Sigma_j\ p\ (S_j)\ H\ (S_i \mid S_j) \leq H\ (S^*)}$, where $\mathbf{H\ (S^*)}$ is the ***adjoint system*** **... (4.22)**

The entropy of the original message symbols given by the zero memory source adjoint system with $p\ (S_i) = p_i$ bound the entropy of the Markov process. The equality holds if and only if $p\ (S_j, S_i) = p_j p_i$ that is, in terms of the structured portfolio set up, the equality holds if and only if the joint probability of the best performer being the pair of assets i and j is equal to the product of their individual probabilities (Hamming, 1986). Thus a clear analogical parallel may be drawn between Markov structure of the coding process and performances of financial assets contained within a structured investment portfolio.

4.9 Conclusion

In this chapter we have basically outlined a novel methodological approach whereby expected information measure is used as a measure of utility derivable from a basket of commodities. We have illustrated the concepts with an applied finance perspective whereby we have used this methodological approach to derive a measure of investor utility from a structured financial portfolio consisting of many elementary risky assets combined with cash as the risk-free asset thereby giving the product a quasi - capital guarantee status. We have also borrowed concepts from mathematical information theory and coding to draw analogical parallels with the utility structures evolving out of multi-asset, financial structured products. In particular, principles of Shannon-Fano coding have been applied to the coding of structured product information for transmission from vendor (fund manager) to the potential buyer (investor). Finally we have dwelled upon the very similar Markovian structure of coding process and that of asset performances.

Appendix (iii) presents a computational exposition of our proposed information theoretic utility measure scheme for multi-asset option-based financial structured products with actual market price data for a product consisting of three risky assets – a long position in an equity portfolio closely replicated by the S&P 500 index, a long position in gold futures and a long position in the Lehman Brothers Growth Fund. The 3-month T-Bill rates act as a proxy for the return on the risk-free asset. A downside protection mechanism akin to a *dynamic threshold management* scheme is formulated to manage the overall portfolio and extrinsic utilities are measured in terms of *entropy*.

This chapter in many ways is a curtain raiser on the different ways in which tools and concepts from mathematical information theory can be applied in utility analysis in general and to analyzing investor utility preferences in particular. It seeks to extend the normal peripheries of utility theory to a new domain – *information theoretic utility*.

5. Concluding Perspectives

Our line of thinking started with the cognitive psychological basis of *perceived* risk in the first chapter where we formulated a neutrosophic notion of such risk which highlighted the intrinsic desirability of downside-protected investment strategies. Subsequently we mathematically investigated the intrinsic utility of a simple portfolio insurance strategy involving exchange-traded put options and proved a general theorem of consistent preference relating to option purchase decisions of an individual investor. We then computationally demonstrated the evolutionary optimality of a Black-Scholes multi-asset, dynamic hedging scheme that uses synthetic puts instead of exchange traded ones. Finally we took another close look at neoclassical utility theory and examined some of its shortcomings before letting our thought processes culminate in a cardinalized extrinsic utility concept in the form of the Shannon-Boltzmann entropy measure.

The entropic measure of utility we have postulated here is however not meant to be a theoretical alternative to the formal devices employed by neoclassical economists to model utility in general. In fact there is no clash of paradigms between our proposed entropic utility concept and that of neoclassical economics in the sense that our entropic model is basically offered as an extension of the notion of neoclassical intrinsic utility to a higher dimension; incorporating the utility of choice. Such a measure then becomes perfectly appropriate to measure the extrinsic utility of choice afforded by a complex, multi-asset financial structured product over and above the intrinsic utility offered by such a product which is measurable in terms of the standard framework of risk and return.

Our entropic measure of extrinsic utility, being a novel methodological approach, offers substantial scope of future academic research especially in exploring the analogical Markovian properties of asset performances and message transmission and devising an efficient coding scheme to represent the two-way transfer of utility information from vendor to buyer and vice versa in transactions involving financial structured products. The mathematical kinship between neoclassical utility theory and classical thermodynamics is also worth exploring, and is likely to be aimed at establishing some higher-dimensional, formal connectivity between isotherms and indifference curves. Our postulated *rationalization loop conjecture* to explain long-run financial market pricing anomalies also offers immediate scope of rigorous empirical research to test its veracity.

Apart from generating purely academic interest, we feel that our work shall also stimulate germination of new ideas on the applied side. Indeed it is not too far-fetched to envisage direct financial engineering applications of our extrinsic utility model whereby custom-made financial structured products could be developed which match the utility profile of an individual investor both intrinsically – in terms of risk-reward preferences, as well as extrinsically – in terms of the nature and extent of the available choice set.

Also, our evolutionary derivation of a Black-Scholes multi-asset, dynamic hedging scheme does provide significant food for thought as to the practical possibilities of using self-adaptive algorithms to generate optimal hedge parameters for insured portfolios in situations where analytical extraction of such parameters may prove too daunting a task.

The present work is mainly intended as a theoretical contribution to the conglomeration of disparate bodies of knowledge that are loosely encapsulated within the umbrella term of Computational Finance. While few scientific theories advanced to date have perennially withstood the test of time, many have left lasting impressions in the annals of human endeavour inspite of their revealed shortcomings. It may so happen that some of the key theoretical constructs we have proposed here like the neutrosophic notion of perceived risk and the entropic utility measure come to be replaced over time by better constructs through a simple process of natural selection. However we are optimistic that our work will merit its own humble position both within the hallowed portals of academia as well as the more prosaic realms of commerce and industry.

Appendix (i)

Monte Carlo output of utility forms evolving out of a simple options-based portfolio insurance strategy involving an exchange-traded put option

A. Monte Carlo computation of investor utility in terms of expected excess equity:

If the market is likely to move adversely, holding a long put alongside ensures that the investor is better off than just holding a long position in the underlying asset. The long put offers the investor some kind of *price insurance* in case the market goes down. This strategy is known in derivatives parlance as a *protective put*. The strategy effectively puts a floor on the downside deviations without cutting off the upside by too much. From the expected changes in investor's equity we can computationally derive his or her utility curves under the strategies A_1 and A_2 in each of the three probability spaces D, N and U as defined in Chapter 2.

The following hypothetical data have been assumed to calculate the simulated put price:

S = \$50.00 (purchase price of the underlying security)

X = \$55.00 (put strike price)

(T – t) = 1 (single period investment horizon)

Risk-free rate = 5%

The put options are valued by Monte Carlo simulation of a trinomial tree using a customized MS-Excel spreadsheet for a hundred independent replications in each space. *Note that no ex ante assumption is made about the volatility parameter.*

Event space: D

Strategy: A_1 (Long underlying asset)

Instance (i): (–) ΔS = \$5.00, (+) ΔS = \$15.00

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.1	\$1.50
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	-\$3.00
		Σ = –\$1.50

Table (i).1

To see how the expected change in investor's equity goes up with an increased upside potential we will double the possible up movement at each of the next two stages while keeping the down movement unaltered. This should enable us to account for any possible loss of investor utility by way of the cost of using a portfolio insurance strategy.

Instance (ii): (+) ΔS = \$30.00

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.1	\$3.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	–\$3.00
		Σ = \$0.00

Table (i).2

Instance (iii): (+) ΔS = \$60.00

Price movement	Probability	Expected Δ Equity
Up (+ \$60.00)	0.1	\$6.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	–\$3.00
		Σ = \$3.00

Table (i).3

Event space: D

Strategy: A_2 (Long underlying asset + long put)

Instance (i): (–) ΔS = \$5.00, (+) ΔS = \$15.00

Simulated put option price	\$6.99
Variance	($\$^2$)11.63
Simulated asset value	\$48.95
Variance	($\$^2$)43.58

Table (i).4

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$8.01)	0.1	\$0.801		
Neutral (–\$1.99)	0.3	–\$0.597		
Down (– \$1.99)	0.6	–\$1.194		
		Σ = –\$0.99	\$0.51	≈ 0.333

Table (i).5

Note that since the put option price has to be accounted for, the actual (+) ΔS in this case is \$15 - \$6.99 = \$8.01 and the actual (–) ΔS is (\$55 – \$50) – \$6.99 = – \$1.99.

Instance (ii): (+) ΔS = \$30.00

Simulated put price	\$6.75
Variance	($\$^2$)13.33
Simulated asset value	\$52.15
Variance	($\$^2$)164.78

Table (i).6

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$23.25)	0.1	\$2.325		
Neutral (– \$1.75)	0.3	–\$0.525		
Down (– \$1.75)	0.6	–\$1.05		
		Σ = \$0.75	\$0.75	≈ 0.666

Table (i).7

Instance (iii): (+) ΔS = \$60.00

Simulated put price	\$6.71
Variance	($\$^2$)12.38
Simulated asset value	\$56.20
Variance	($\$^2$)520.77

Table (i).8

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$53.29)	0.1	\$5.329		
Neutral (– \$1.71)	0.3	–\$0.513		
Down (– \$1.71)	0.6	–\$1.026		
		Σ = \$3.79	\$0.79	≈ 0.999

Table (i).9

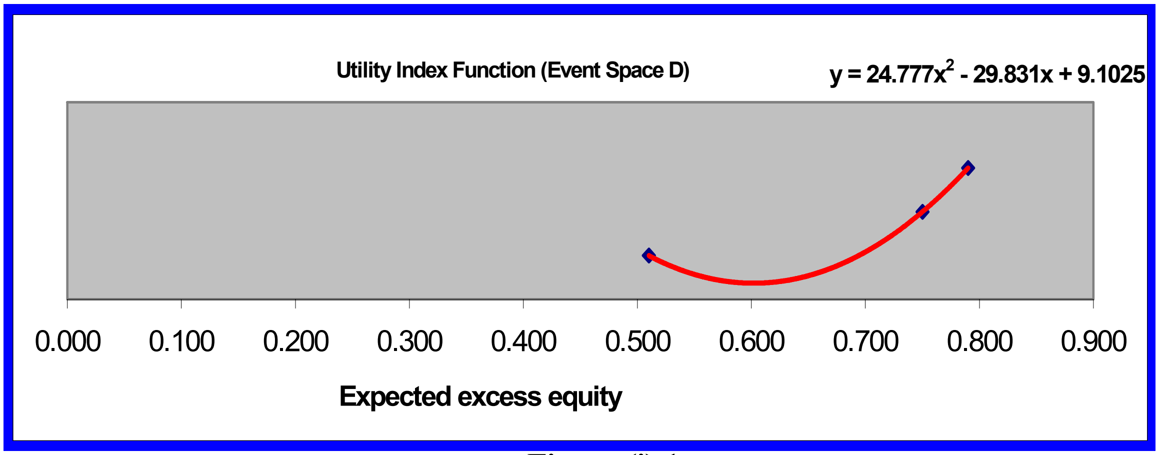

Figure (i).1

The utility function as obtained above is *convex* in probability space D, which indicates that the protective strategy can make the investor *risk-loving* even when the market is expected to move in an adverse direction, as the expected payoff from the put option largely neutralizes the likely erosion of security value at an affordable insurance cost! This is in line with the *intuitive behavioral reasoning* that as investors with a viable downside protection will become more aggressive in their approach than they would be without it implying markedly lowered risk aversion for the investors with insurance.

Event space: N

Strategy: A_1 (Long underlying asset)

Instance (i): (–) ΔS = \$5.00, (+) ΔS = \$15.00

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.2	\$3.00
Neutral (\$0.00)	0.6	\$0.00
Down (– \$5.00)	0.2	–\$1.00
		Σ = \$2.00

Table (i).10

Instance (ii): (+) ΔS = \$30.00

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.2	\$6.00
Neutral (\$0.00)	0.6	\$0.00
Down (– \$5.00)	0.2	–\$1.00
		Σ = \$5.00

Table (i).11

Instance (iii): (+) ΔS = $60.00

Price movement	Probability	Expected Δ Equity
Up (+ $60.00)	0.2	$12.00
Neutral ($0.00)	0.6	$0.00
Down (– $5)	0.2	–$1.00
		Σ = $11.00

Table (i).12

Event space: N

Strategy: A_2 (Long underlying asset + long put)

Instance (i): (–) ΔS = $5.00, (+)ΔS = $15.00

Simulated put price	$4.85
Variance	(^{2})9.59
Simulated asset value	$51.90
Variance	(^{2})47.36

Table (i).13

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $11.15)	0.2	$2.23		
Neutral (+ $0.15)	0.6	$0.09		
Down (+ $0.15)	0.2	$0.03		
		Σ = $2.35	$0.35	≈ 0.999

Table (i).14

Instance (ii): (+) ΔS = $30.00

Simulated put price	$4.80
Variance	($^{2})9.82
Simulated asset value	$55.20
Variance	($^{2})169.15

Table (i).14

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $25.20)	0.2	$5.04		
Neutral (+ $0.20)	0.6	$0.12		
Down (+ $0.20)	0.2	$0.04		
		Σ = $5.20	$0.20	≈ 0.333

Table (i).15

Instance (iii): (+) ΔS = $60.00

Simulated put price	$4.76
Variance	($^{2})8.68
Simulated asset value	$60.45
Variance	($^{2})585.40

Table (i).16

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $55.24)	0.2	$11.048		
Neutral (+ $0.24)	0.6	$0.144		
Down (+ $0.24)	0.2	$0.048		
		Σ = $11.24	$0.24	≈ 0.666

Table (i).17

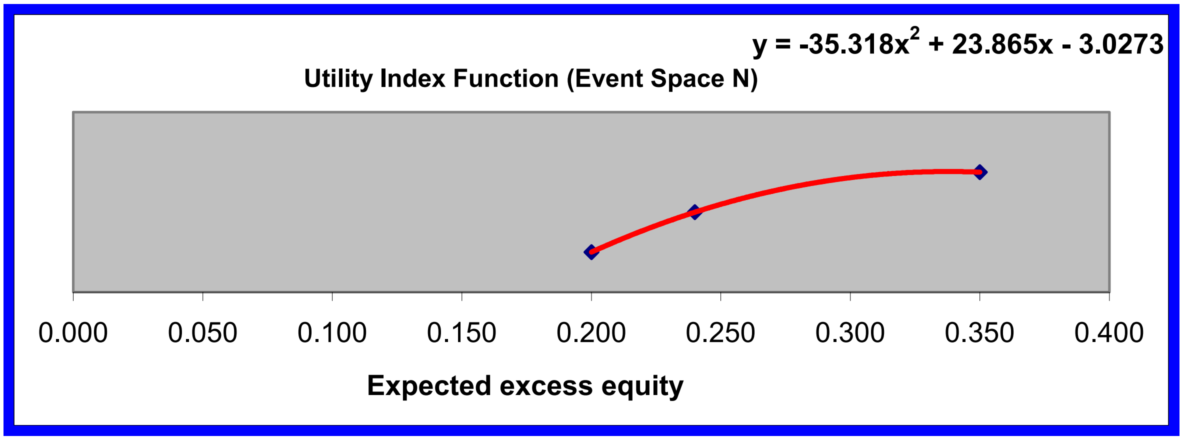

Figure (i).2

The utility function as obtained above is *concave* in probability space N, which indicates that the insurance provided by the protective strategy can no longer make the investor risk-loving as the expected value of the insurance is offset by the cost of buying the put! This is again in line with intuitive behavioral reasoning because if the market is equally likely to move up or down and more likely to stay unmoved the investor would deem himself or herself better off not buying the insurance because in order to have the insurance i.e. the put option it is necessary to pay an out-of-pocket cost, which may not be offset by the expected payoff from the put option under the prevalent market scenario.

Event space: U

Strategy: A_1 (Long underlying asset)

Instance (i): (–) ΔS = \$5.00, (+) ΔS = \$15.00

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.6	\$9.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.1	–\$0.50
		Σ = \$8.50

Table (i).18

Instance (ii): (+) ΔS = \$30.00

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.6	\$18.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.1	–\$0.50
		Σ = \$17.50

Table (i).19

Instance (iii): (+) ΔS = \$60.00

Price movement	Probability	Expected Δ Equity
Up (+ \$60.00)	0.6	\$36.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5)	0.1	–\$0.50
		Σ = \$35.50

Table (i).20

Event space: U

Strategy: A_2 (Long underlying asset + long put)

Instance (i): (−) ΔS = \$5.00, (+) ΔS = \$15.00

Simulated put price	\$2.28
Variance	(\$^{2})9.36
Simulated asset value	\$58.60
Variance	(\$^{2})63.68

Table (i).21

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$12.72)	0.6	\$7.632		
Neutral (+ \$2.72)	0.3	\$0.816		
Down (+ \$2.72)	0.1	\$0.272		
		Σ = \$8.72	\$0.22	≈ 0.333

Table (i).22

Instance (ii): (+) ΔS = \$30.00

Simulated put price	\$2.14
Variance	(\$^{2})10.23
Simulated asset value	\$69.00
Variance	(\$^{2})228.79

Table (i).23

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $27.86)	0.6	$16.716		
Neutral (+ $2.86)	0.3	$0.858		
Down (+ $2.86)	0.1	$0.286		
		Σ = $17.86	$0.36	≈ 0.666

Table (i).24

<u>Instance (iii):</u> (+) ΔS = $60.00

Simulated put price	$2.09
Variance	($^{2})9.74
Simulated asset value	$88.55
Variance	($^{2})864.80

Table (i).25

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $57.91)	0.6	$34.746		
Neutral (+ $2.91)	0.3	$0.873		
Down (+ $2.91)	0.1	$0.291		
		Σ = $35.91	$0.41	≈ 0.999

Table (i).26

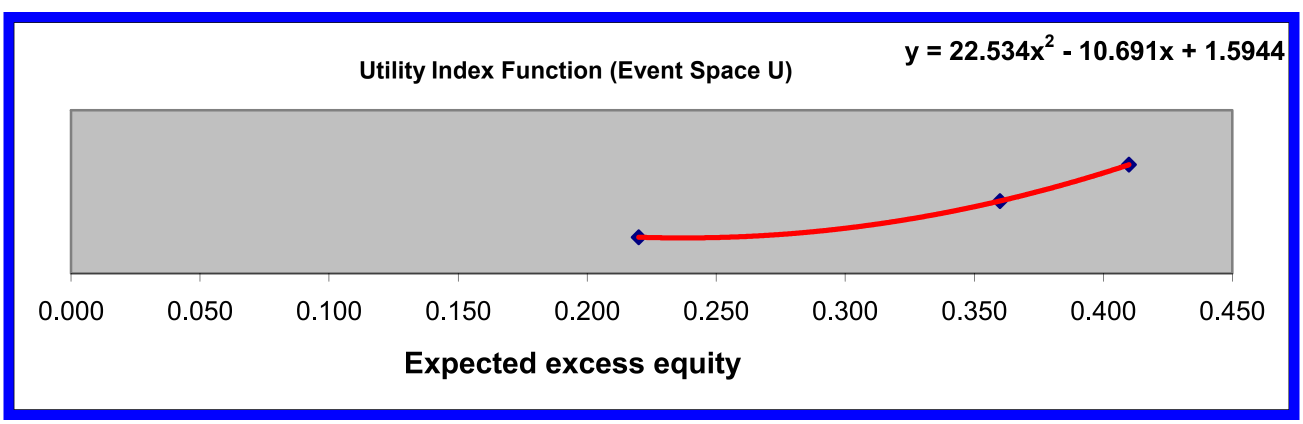

Figure (i).3

In accordance with intuitive, behavioral reasoning the utility function is again seen to be *convex* in the probability space U, which is probably attributable to the fact that while the market is expected to move in a favourable direction the put option nevertheless keeps the downside protected while costing less than the expected payoff on exercise thereby fostering a risk-loving attitude in the investors as they get the best of both worlds.

Note that particular values assigned to the utility indices won't affect the essential mathematical structure of the utility curve – but only cause a scale shift in the parameters. For example, the indices could easily have been taken as (0.111, 0.555 and 0.999) - these assigned values should not have any computational significance as long as all they all lie within the conventional interval (0, 1]. Repeated simulations have shown that the investor would be considered extremely unlucky to obtain an excess return less than the minimum excess return obtained or extremely lucky to get an excess return more than the maximum excess return obtained under each of the event spaces. Hence, the maximum and minimum expected excess equity within a particular event space should correspond to the lowest and highest utility indices and the utility derived from the median excess equity should then naturally occupy the middle position. As long as this is the case, there will be no alteration in the fundamental mathematical form of the investor's utility functions no matter what index values are assigned to his or her utility from expected excess equity.

B. Extrapolating the ranges of investor's risk aversion within each probability space:

For a continuous, twice-differentiable utility function u (x), the *Arrow-Pratt measure of absolute risk aversion* (ARA) is given as follows: (Arrow, 1971; Pratt and Zeckhauser, 1987)

$$\boldsymbol{\lambda (x) = - [d^2u (x)/dx^2] [du (x)/dx]^{-1}} \qquad \textbf{… (i.1)}$$

λ (x) > 0 if u is monotonically increasing and *strictly concave* as in case of a risk-averse investor having u'' (x) < 0. Obviously, λ (x) = 0 for the risk-neutral investor with a *linear* utility function having u'' (x) = 0 while λ (x) < 0 for the risk-loving investor with a *strictly convex* utility function having u'' (x) > 0 (Chiang, 1984).

<u>Case I: Probability Space D:</u>

u (x) = 24.777x^2 – 29.831x + 9.1025, u' (x) = 49.554x – 29.831 and u''(x) = 49.554. Thus λ (x) = – 49.554 / (49.554x – 29.831). Therefore, given the *convex utility function*, the defining range is λ (x) < 0 i.e. (49.554x – 29.831) < 0 or x < 0.60199.

<u>Case II: Probability Space N:</u>

u (x) = -35.318 x^2 + 23.865x – 3.0273, u' (x) = -70.636x + 23.865 and u''(x) = –70.636. Thus, λ (x) = – [–70.636 / (–70.636x + 23.865)] = 70.636 / (–70.636x + 23.865).

Therefore, given the *concave utility function*, the defining range is λ (x) > 0, i.e. we have the denominator (-70.636x + 23.865) > 0 or x > 0.33786.

Case III: Probability Space U:

$u(x) = 22.534x^2 - 10.691x + 1.5944$, $u'(x) = 45.068x - 10.691$ and $u''(x) = 45.068$. Thus $\lambda(x) = -45.068 / (45.068x - 10.691)$. Therefore, given the *convex utility function*, the defining range is $\lambda(x) < 0$ i.e. $(45.068x - 10.691) < 0$ or $x < 0.23722$.

Note that these ranges as evaluated above will however depend on the *parameters of the utility function* and will therefore be different for different investors according to the values assigned to his or her utility indices corresponding to the expected excess equity (Goldberger, 1987).

In general**,** if we have a parabolic utility function $u(x) = a + bx - cx^2$, where $c > 0$ ensures concavity, then we have $u'(x) = b - 2cx$ and $u''(x) = -2c$. The Arrow-Pratt measure is given by $\lambda(x) = 2c / (b-2cx)$. Therefore, for $\lambda(x) \geq 0$, we need $b \geq 2cx$, thus it can only apply for a limited range of x. Notice that$\lambda'(x) \geq 0$ up to where $x = b/2c$. Beyond that, marginal utility is negative - i.e. beyond this level of equity, utility *declines*. One more implication is that there is an increasing apparent unwillingness to take risk as their equity increases, i.e. with larger excess equity investors are less willing to take risks as concave, parabolic utility functions exhibit *increasing absolute risk aversion* (IARA).

Appendix (ii)

Genetic Algorithm demonstration of the biological basis of Black-Scholes type expected utility functions

A. Formal definitions:

A *geometric Brownian motion* is assumed to be the ubiquitous stochastic diffusion process driving asset prices and performances in organized financial markets. The discrete time version of this model, as is relevant in most computational applications, is given as follows:

$$\Delta S/S = \mu\Delta t + \varepsilon\sqrt{(\sigma^2\Delta t)} \qquad \text{... (ii.1)}$$

The variable ΔS is the change in the asset price S and ε is a random sample from the standard normal distribution *N* (0, 1). The expected return per unit of time on the asset is denoted as μ and the variance of the asset price is denoted as σ^2. $\Delta S/S$ is actually the proportional return provided by the asset in a very short interval of time Δt. The term $\mu\Delta t$ then stands for the expected value of this return, which is deterministic and $\varepsilon\sqrt{(\sigma^2\Delta t)}$ is then the stochastic component of this return.

Therefore the variance of the return on the asset is given by $\sigma^2\Delta t$. That is to say $\Delta S/S$ follows a normal distribution with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$. Thus a path for an asset price or performance can be simulated by repeated sampling from *N* ($\mu\Delta t$, $\sigma^2\Delta t$). This is exactly how the Monte Carlo simulation model we have used in this study was designed. Since we have used monthly rebalancing, we have $\Delta t = 1/12$ i.e. $\Delta t \approx 0.0833$. The asset performances S_1 and S_2 have been generated by repeatedly sampling from *N* (0.0167, 0.0075) and *N* (0.0150, 0.0033) respectively and averaging. As the performances of the two risky assets are correlated, the random samples were drawn according to the following formula using Excel's inbuilt RAND () function:

$$\mathbf{r_1 = RAND\ ()};\ \text{and}$$

$$\mathbf{r_2 = r_1 + RAND\ () * (1 - 0.50)^2 = r_1 + RAND\ () * 0.50^2} \qquad \mathbf{...\ (ii.2)}$$

B. Mathematical derivation of the size of G_n:

According to our haploid genetic algorithm reproduction and crossover functions, the size of the n^{th} generation i.e. the number of chromosomes in the population at the end of the n^{th} generation is given by the following first-order, linear difference equation:

$$\mathbf{G_n = G_{n-1} + 2\ (G_{n-1} - 1) = 3\ G_{n-1} - 2} \qquad \mathbf{...\ (ii.3)}$$

If x initial number of chromosomes are introduced at n = 0, we have $G_0 = x$. Then, obviously, $G_1 = x + 2(x - 1) = 3x - 2 = 3^1 (x - 1) + 1$. Extending the recursive logic to G_2

and G_3 we get $G_2 = 9x - 8 = 3^2 (x - 1) + 1$ and $G_3 = 27x - 26 = 3^3 (x - 1) + 1$. Therefore, extending to G_t we can write the following relation:

$$\mathbf{G_t = 3^t (x - 1) + 1} \qquad \textbf{... (ii.4)}$$

Therefore, $G_{t+1} = 3^{t+1} (x - 1) + 1$. But $G_{t+1} = 3G_t - 2$. Substituting for G_t we thereby get, $G_{t+1} = 3\{3^t (x - 1) + 1\} - 2 = 3^{t+1} (x - 1) + 3 - 2 = 3^{t+1} (x - 1) + 1$. Therefore the case is proved for G_{t+1}. But we have already proved it for G_1, G_2 and G_3. Therefore, by the *principle of mathematical induction*, the general formula is derived as follows:

$$\mathbf{G_n = 3^n (x - 1) + 1} \qquad \textbf{... (ii.5)}$$

We can verify that our genetic algorithm model has indeed reproduced this G_n number of chromosomes in each generation for an initial input of $G_0 = x = 5$. Accordingly, our algorithm reproduced 13, 37 and 109 chromosomes for n = 1, 2, 3.

C. Computational output of the GA performance:

For an initial input of \$1, apportioned at t = 0 as 45% between S_1 and S_2 and 10% for S_0, we have constructed five replicating portfolios according to a simple *rule-based logic*: k% of funds are allocated to the observed best performing risky asset and the balance (90 – k) % to the other risky asset (keeping the portfolio self-financing after the initial investment) at every monthly re-balancing point. We have reduced k by 10% for each portfolio starting from 90% and going down to 50%. As is shown in the following Figures III.1 – III.5, this simple hedging scheme performs quite well over the lock-in period when k = 90%. But the performance falls away steadily as k is reduced every time. Evaluation of the fitness criterion corresponding to choices of k is shown in Table II.1.

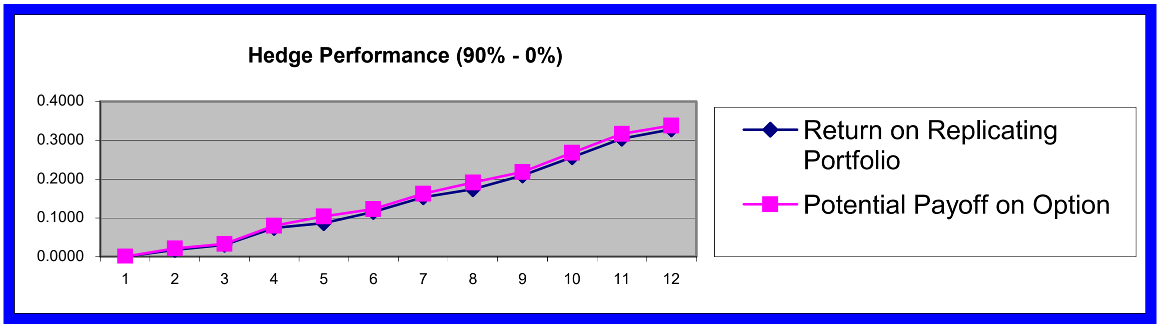

Figure (ii).1

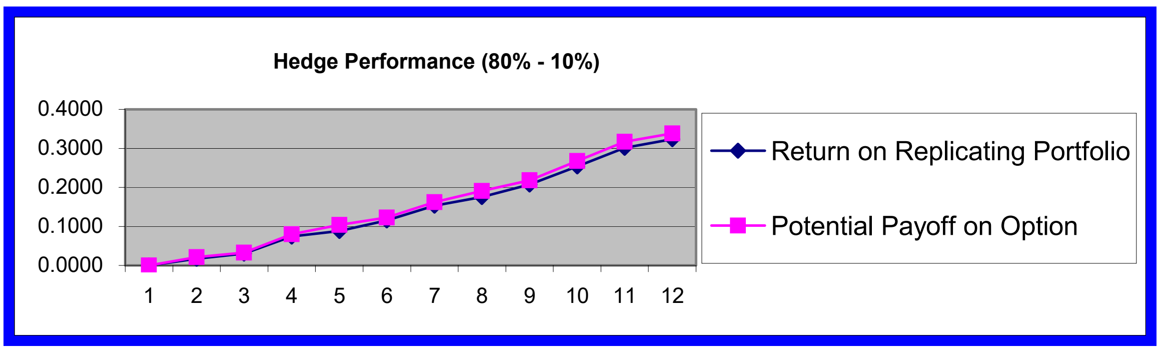

Figure (ii).2

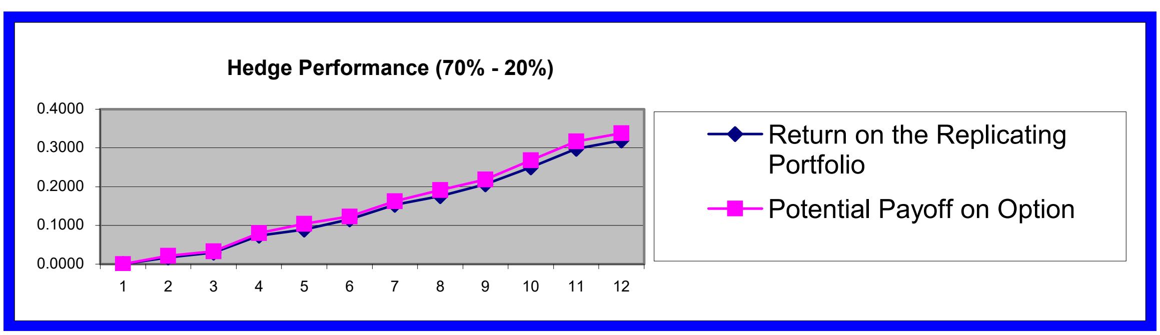

Figure (ii).3

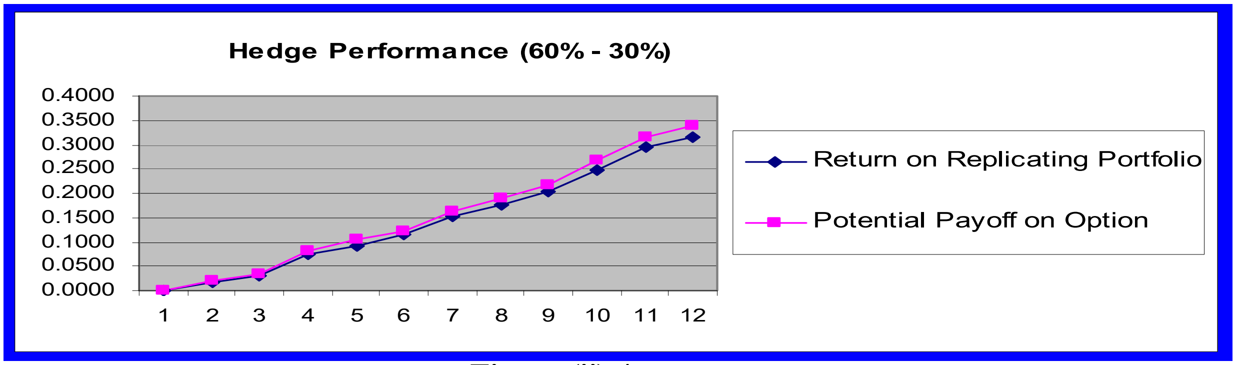

Figure (ii).4

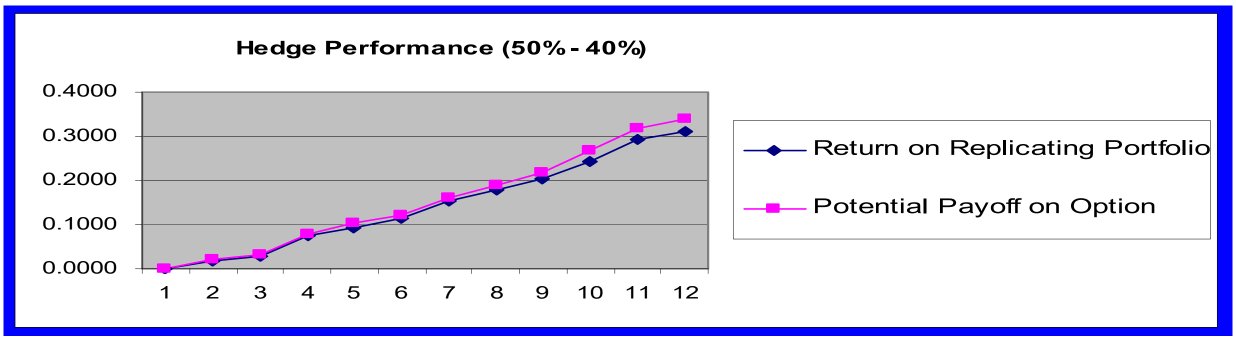

Figure (ii).5

Generation	Number of chromosomes	80% < k* <= 90%	% in range (fitness)
0	5	1	20%
1	13	4	31%
2	37	19	51%
3	109	61	56%

Table (ii).1

D. Haploid Genetic Algorithm program code in Borland C Release 5.02:

```
#include<stdio.h> /* Standard Header*/
#include<stdlib.h>/* Standard Header*/
#include<conio.h> /* Standard Header*/
#include<string.h>/* Binary Array - string */
#include<math.h>

#define DELAY 1500000000
#define MAX 32        /* 32 Bit MAX only 32 elements total */
#define SLICER 8
#define CAPACITY 1000

void crossover(int locus, int seed, int children, char daughter[][MAX]);
void reconvert(char new_str1[][MAX], char new_str2[][MAX], char daughter[][MAX],
int children);
int reproduction (float fitness[CAPACITY], int init_val, int r);
int randomizer1 (int seed, float tot_fitness);
int randomizer2 (int mute_seed, int children);

void main()
{
 char bit_str[CAPACITY][MAX], daughter[CAPACITY][MAX];
 int time_count, reply, b, init_val, j, input, count, gap, seed, children;
 int locus, choice[CAPACITY], simcount, r, index, rowcount;
 int binum [CAPACITY][MAX], chromosome[CAPACITY];
 int chrom_code[CAPACITY], chrom_count[CAPACITY];
 float fitness[CAPACITY], tot_fitness;

 printf("*** UNIVARIATE GENETIC OPTIMIZATION PROGRAM IN BORLAND C
***");
 printf("\n\n\n");
 printf(“Copiright:  Sukanto Bhattacharya  Date: 18 July 2003");
 time_count = 0;
 do
 {
  time_count++;
 } while (time_count <= DELAY);
 reply = 1;
 while(reply == 1||reply == 1)
 {
  clrscr();
  printf("Enter initial number of chromosomes : ");
  scanf("%d", &init_val);
```

```
printf("\n\n");
printf("Enter number of offsprings to be reproduced : ");
scanf("%d", &children);
printf("\n\n");
printf ("Enter the crossover locus : ");
scanf("%d", &locus);
clrscr();
tot_fitness=0;
rowcount = 0;
for(b=1; b<=init_val; b++)
{
 printf ("\n\n");
 printf("Enter chromosome value %d in decimal for encoding : ", b);
 scanf ("%d", &chromosome[b]);
 printf ("\n");
 printf("Enter the fitness value associated with chromosome %d : ", b);
 scanf("%f", &fitness[b]);
 tot_fitness = tot_fitness + fitness[b];
 rowcount++;
 if(rowcount > 5)
 {
  clrscr();
  rowcount = 0;
 }
}
clrscr();
for(b=1; b <= init_val; b++)
{
 input = chromosome[b];
 count=0;
 do
 {
  j = input%2;
  binum[b][count] = j;
  if(binum[b][count]==1)
   bit_str[b][count]='1';
  else
   bit_str[b][count]='0';
  input = input/2;
  count++;
 }while (input > 0);
 for(gap=0; gap<MAX; gap++)
 {
  if(bit_str[b][gap]!='0'&& bit_str[b][gap]!='1')
   bit_str[b][gap]='0';
 }
```

```
}
printf("Enter seed value for random number generation : ");
scanf("%d", &seed);
clrscr();
simcount = 1;
for(b=1; b <= init_val; b++)
{
 chrom_code[b]= b;
 chrom_count[b]=0;
}
do
{
 r = randomizer1(seed, tot_fitness);
 if(r == 0)
  r = 1;
 seed = r;
 choice[simcount] = reproduction(fitness, init_val, r);
 for (b=1; b <= init_val; b++)
 {
  if(choice[simcount] == chrom_code[b])
   chrom_count[b]++;
 }
 for (gap=0; gap < MAX; gap ++)
 {
  index = choice[simcount];
  daughter[simcount][gap]= bit_str[index][gap];
 }
 simcount++;
}while (simcount <= children);
rowcount = 0;
for(b=1; b <= init_val; b++)
{
 printf("\n");
 printf("%d of chromosome %d selected for new generation", chrom_count[b], b);
 rowcount++;
 if (rowcount > 5)
 {
  printf("\n\n");
  printf("Press any key to continue :");
  getch();
  gotoxy(1, 1);
  clrscr();
  rowcount = 0;
 }
}
crossover(locus, seed, children, daughter);
```

```
  printf("\n\n");
  printf("Do you want another run? (Yes -> 1/No -> 0) :");
  scanf ("%d", &reply);
 }
}

/* Random Number Generation */

int randomizer1 (int seed, float tot_fitness)
{
 int a1, c1, result1, m;

 a1 = 16807;
 c1 = 0;
 m = floor(tot_fitness);
 result1 = (a1*seed + c1)%m;
 return result1;
}

int randomizer2 (int mute_seed, int children)
{
 int a2, c2, result2;

 a2 = 16807;
 c2 = 0;
 result2 = (a2*mute_seed + c2)%children;
 return result2;
}
/* Reproduction function */

int reproduction (float fitness[CAPACITY], int init_val, int r)
{
 int b1, t, z, c[CAPACITY], select, increment;

 for(b1=1; b1 <= init_val; b1++)
  c[b1] = b1;
 t = 0;
 for (z=1; t<r; z++)
 {
  increment = floor(fitness[z]);
  t = t + increment;
 }
 select = c[z-1];
 return select;
}
```

```
/* Crossover function */

void crossover(int locus, int seed, int children, char daughter[][MAX])
{
 int gap2, b2, position, o, k, mute_seed, mutant[CAPACITY], rowcount;
 int number_mutant, j, pop_count;
 char new_str1[CAPACITY][MAX], new_str2[CAPACITY][MAX];

 for (o=1; o <= children; o++)
 {
  gap2=0;
  do
  {
   new_str1[o][gap2]='0';
   new_str2[o][gap2]='0';
   gap2++;
  } while (gap2 < MAX);
 }
 rowcount = 1;
 k=1;
 printf("\n\n\n");
 printf ("Child chromosomes after crossover between selected parents :");
 for(b2=1; b2 <= children-1; b2++)
 {
  for(position = MAX; position > locus; position--)
   new_str1[b2][position]= daughter[k][position];

  while (position>=0)
  {
   new_str1[b2][position]= daughter[k+1][position];
   position--;
  }
  for(position = MAX; position > locus; position--)
   new_str2[b2][position]= daughter[k+1][position];
  while (position>=0)
  {
   new_str2[b2][position]= daughter[k][position];
   position--;
  }
  printf("\n\n");
  gap2=MAX;
  do
  {
   printf ("%c", new_str1[b2][gap2-1]);
   gap2--;
  } while (gap2 > 0);
```

```
 printf("\n\n");
 gap2=MAX;
 do
 {
  printf ("%c", new_str2[b2][gap2-1]);
  gap2--;
 } while (gap2 > 0);
 k++;
 rowcount++;
 if(rowcount > 3)
 {
  printf("\n\n");
  printf("Press any key to continue :");
  getch();
  gotoxy(1, 1);
  clrscr();
  rowcount = 0;
 }
}
printf("\n\n\n");
printf("Enter the maximum number of mutants for next generation : ");
scanf("%d", &number_mutant);
gotoxy(1, 1);
clrscr();
mute_seed = seed;
for(j=1; j <= children; j++)
 mutant[j]=0;
j=1;
while (j <= number_mutant)
{
 mutant[j] =  randomizer2(mute_seed, children);
 mute_seed = mutant[j];
 if(mute_seed == 0)
  mute_seed = 1;
 j++;
}
pop_count=children + 2*(children - 1);
printf("%d chromosomes belonging to current population", pop_count);
printf("\n\n\n");
if(number_mutant > 0)
{
 for(j=1; j<= number_mutant; j++)
 {
  if(mutant[j]!=0)
  {
   printf("\n");
```

```
    printf("Chromosome selected for mutation: %d", mutant[j]);
   }
  }
 }
 printf ("\n\n\n");
 printf ("Press any key to generate output :");
 getch();
 gotoxy(1, 1);
 clrscr();
 for (b2=1; b2 <= children; b2++)
 {
  for(j=1; j <= number_mutant; j++)
  {
   if(b2==mutant[j])
   {
    gap2=MAX/SLICER;
    do
    {
     if(daughter[b2][gap2-1]!='0')
      daughter[b2][gap2-1] = '0';
     else
      daughter[b2][gap2-1] = '1';
     gap2--;
    }while (gap2 > 0);
   }
  }
 }
 gotoxy(1, 1);
 clrscr();
 reconvert(new_str1, new_str2, daughter, children);
 printf ("\n\n\n");
 printf ("Press any key to exit :");
 getch();
 clrscr();
}
void reconvert(char new_str1[][MAX], char new_str2[][MAX], char daughter[][MAX],
int children)
{
 char binary1[CAPACITY][MAX], binary2[CAPACITY][MAX],
binary3[CAPACITY][MAX];
 int i, z, dec1[CAPACITY], dec2[CAPACITY], dec3[CAPACITY], power[MAX];

 for(z=1; z<=children; z++)
 {
  dec1[z]=0;
  for(i=0; i<MAX; i++)
```

```
  binary1[z][i] = daughter[z][i];
 }
 for (z=1; z<=children; z++)
 {
  dec2[z]=0;
  dec3[z]=0;
  for(i=0; i<MAX; i++)
  {
   binary2[z][i] = new_str1[z][i];
   binary3[z][i]= new_str2[z][i];
  }
 }
 for (z=1; z<=children; z++)
 {
  for (i=MAX; i > 0; i--)
  {
   power[i]= (binary1[z][MAX-i]=='1'? 1:0)* pow(2, MAX-i);
   dec1[z]= dec1[z]+ power[i];
  }
 }
 for (z=1; z<=children; z++)
 {
  for (i=MAX; i > 0; i--)
  {
   power[i]= (binary2[z][MAX-i]=='1'? 1:0)* pow(2, MAX-i);
   dec2[z]= dec2[z]+ power[i];
  }
 }
 for (z=1; z<=children; z++)
 {
  for (i=MAX; i > 0; i--)
  {
   power[i]= (binary3[z][MAX-i]=='1'? 1:0)* pow(2, MAX-i);
   dec3[z]= dec3[z]+ power[i];
  }
 }
 printf ("Decimal values of chromosomes in current population are as follows :");
 printf ("\n\n");
 for (z=1; z<=children; z++)
  printf ("\n %d", dec1[z]);
 for (z=1; z<=children-1; z++)
  printf ("\n %d", dec2[z]);
 for (z=1; z<=children-1; z++)
  printf ("\n %d", dec3[z]);
}
```

Appendix (iii)

Computational exposition of the proposed *information theoretic utility measure* scheme for a multi-asset, capital-guaranteed financial structured product

Here we have considered actual historical market data to construct a three-asset, capital-guaranteed financial structured product. The three risky assets are gold futures, the Lehman Brothers Growth Fund and a well-diversifed portfolio of equities proxied by the S&P500 index. The return on the risk-free asset is proxied by the 3-month T bill rate.

The periodicity of the data is monthly and the span is two years – from 02/02/01 to 03/02/03. A notional amount of US$100,000 is invested on 02/02/01 in a dynamically managed structured product with a floor of $90,000. The risk-free rate is an annualized 2.40% determined as the average observed rate on 3-month T bills for the holding period.

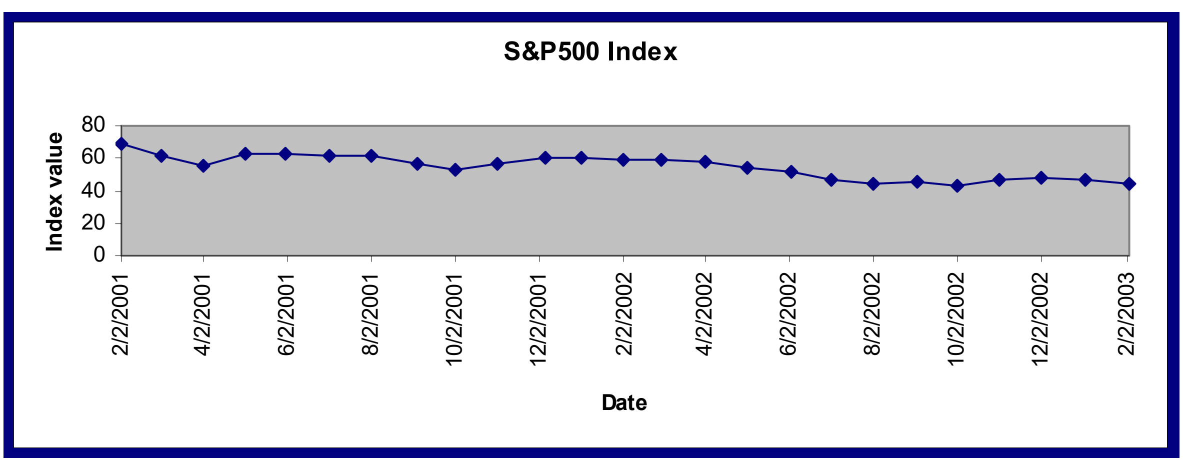

Figure (iii).1

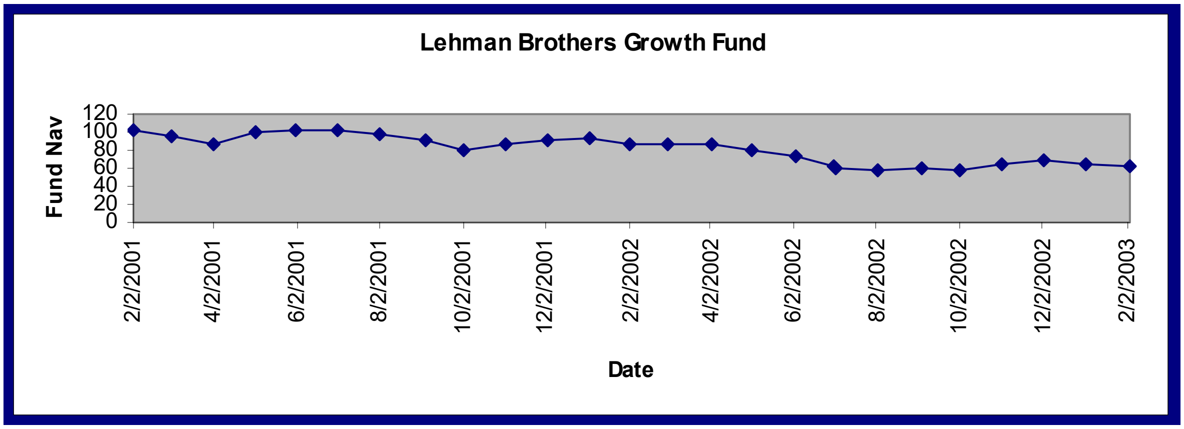

Figure (iii).2

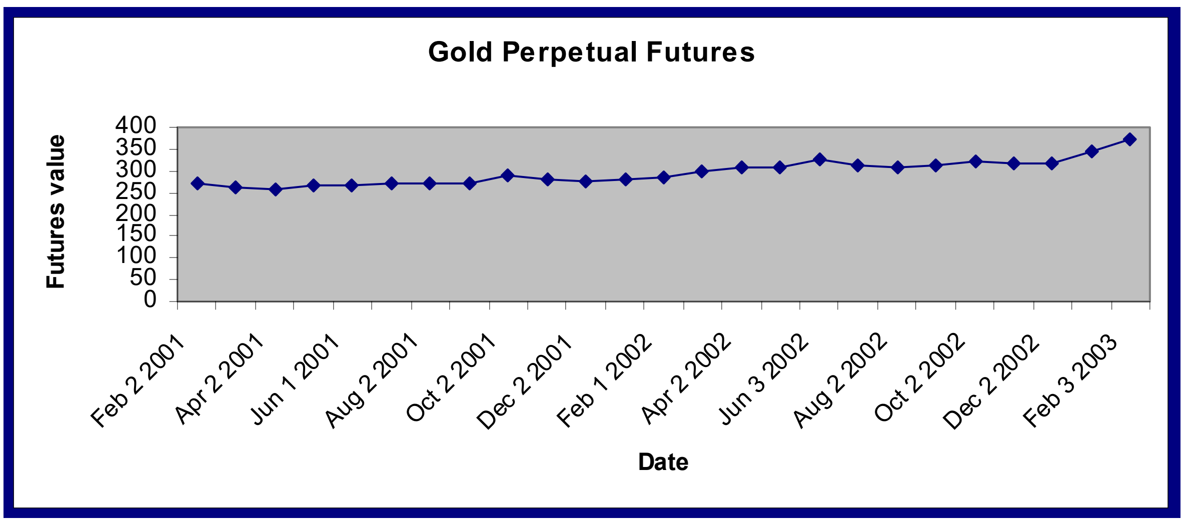

Figure (iii).3

Correlation Matrix	S&P500 Index	Lehman Brothers Growth Fund	Gold Futures
S&P500 Index	1	0.862887415	-0.19457
Lehman Brothers Growth Fund	0.862887415	1	-0.34702
Gold Futures	-0.194566788	-0.347019467	1

Table (iii).1

The overall market during the period under consideration was strongly bearish with the average monthly return on S&P 500 index at a miserable -1.61% and that on the Lehman Brothers Growth Fund was not much better at -1.49%! Gold futures performed reasonably well generating a monthly average return of 1.20%.

Relating to Chapter 4, the product we have constructed is an *8-bit* financial structured product consisting of three risky assets and one risk-free asset. We employ a mechanism of *dynamic threshold management*, which has become quite a popular investment vehicle with some of the larger commercial banks in Europe and also starting to make an entry of late into the Australian financial market.

The commonest threshold management mechanism goes by the name of *constant proportion portfolio insurance* (or CPPI), which expose a constant multiple (hence the name CPPI) of a cushion over an investor's floor or "insured" value to the performance of the risky asset. For example, an investor with a $100 million portfolio, a floor of $90 million and a constant multiple of 5 will allocate $50 million [= 5 x ($100 m - $90 m)] to the risky asset and the balance $50 million to the riskfree asset. The investor will rebalance the exposures as the portfolio value changes.

Unlike other portfolio insurance strategies, CPPI does not require the investor to specify a finite investment horizon. It has further been mathematically demonstrated (Black and Jones, 1987) that there is no "dominance" between CPPI and OBPI for the standard criteria of portfolio choices. It has also been shown that the CPPI is in fact algebraically generalizable to the OBPI by making the multiple constant some function of the Brownian price path of the underlying risky component (Bertrand and Prigent, 2003).

CPPI Portfolio Insurance Scheme								
Date	S&P500	Lehman Bros.	GC00	Exposure	Cash	Portfolio	Floor	Cushion
2-Feb-01	6667	6667	6667	20000	80000	100000	90000	10000
	5823	6257	6511	18592	80160	98752	90000	8752
2-Mar-01	5483	5891	6130	17504	81248	98752	90000	8752
	4969	5183	5977	16129	81411	97540	90000	7540
2-Apr-01	4645	4846	5588	15079	82460	97540	90000	7540
	5226	5572	5785	16583	82625	99208	90000	9208
2-May-01	5803	6188	6424	18416	80792	99208	90000	9208
	5724	6382	6477	18584	80954	99537	90000	9537
1-Jun-01	5875	6551	6648	19074	80463	99537	90000	9537
	5813	6528	6700	19042	80624	99665	90000	9665
2-Jul-01	5902	6627	6802	19331	80335	99665	90000	9665
	5847	6318	6774	18939	80495	99435	90000	9435
2-Aug-01	5825	6295	6749	18869	80565	99435	90000	9435
	5383	5831	6831	18045	80727	98771	90000	8771
4-Sep-01	5233	5669	6641	17543	81229	98771	90000	8771
	4912	4953	7080	16945	81391	98336	90000	8336
2-Oct-01	4833	4873	6966	16672	81664	98336	90000	8336
	5093	5290	6720	17102	81827	98930	90000	8930
2-Nov-01	5318	5524	7017	17859	81070	98930	90000	8930
	5724	5784	6945	18452	81232	99685	90000	9685
6-Dec-01	6008	6071	7290	19369	80315	99685	90000	9685
	5935	6193	7324	19452	80476	99928	90000	9928
2-Jan-02	6058	6322	7476	19856	80072	99928	90000	9928
	5954	5873	7677	19503	80232	99736	90000	9736
1-Feb-02	5944	5863	7664	19471	80264	99736	90000	9736
	5962	5796	7981	19740	80425	100165	90000	10165
1-Mar-02	6140	5970	8219	20329	79835	100165	90000	10165
	6026	6059	8453	20538	79995	100533	90000	10533
2-Apr-02	6181	6215	8670	21067	79467	100533	90000	10533
	5803	5659	8715	20177	79626	99803	90000	9803
2-May-02	5639	5498	8469	19606	80197	99803	90000	9803
	5382	4974	8977	19333	80358	99691	90000	9691
3-Jun-02	5395	4986	8999	19381	80309	99691	90000	9691
	4845	4308	8592	17745	80470	98215	90000	8215
2-Jul-02	4486	3989	7956	16430	81785	98215	90000	8215
	4232	3797	7822	15851	81948	97799	90000	7799
2-Aug-02	4164	3736	7698	15598	82201	97799	90000	7799
	4234	3804	7844	15882	82365	98247	90000	8247
3-Sep-02	4397	3951	8146	16495	81753	98247	90000	8247
	4173	3889	8374	16436	81916	98352	90000	8352
2-Oct-02	4241	3953	8511	16705	81648	98352	90000	8352

	4586	4386	8416	17387	81811	99198	90000	9198
1-Nov-02	4852	4640	8904	18396	80802	99198	90000	9198
	5005	4930	8884	18819	80964	99782	90000	9782
2-Dec-02	5203	5125	9236	19564	80218	99782	90000	9782
	5007	4630	10015	19651	80378	100030	90000	10030
2-Jan-03	5111	4726	10222	20059	79970	100030	90000	10030
	4844	4550	10937	20331	80130	100461	90000	10461
3-Feb-03	4984	4682	11255	20922	79539	100461	90000	10461

Table (iii).2

In the CPPI scheme we have depicted above, the constant multiple is 2 and the floor is \$90,000. The terminal portfolio value on 3rd February 2003 is seen to be \$100,061 approximately i.e. the capital guarantee mechanism endogenous to our 8-bit structured product has just about worked, ensuring that the value of the portfolio at the end of the horizon did not go below the initial capital invested.

Though the holding period return is lower in this case (only about 0.461%) as compared to return on the best-performing asset i.e. gold futures, the main benefit of the CPPI approach is that with a constant multiple of k > 0, the investor obtains a higher participation in the market component and therefore, is afforded an opportunity to reap a proportionately higher reward if and when one of the risky assets within the portfolio starts to perform really well over a period of time. This flexibility is not offered by most of the simple capital-guarantee mechanisms like the zero-coupon bond scheme.

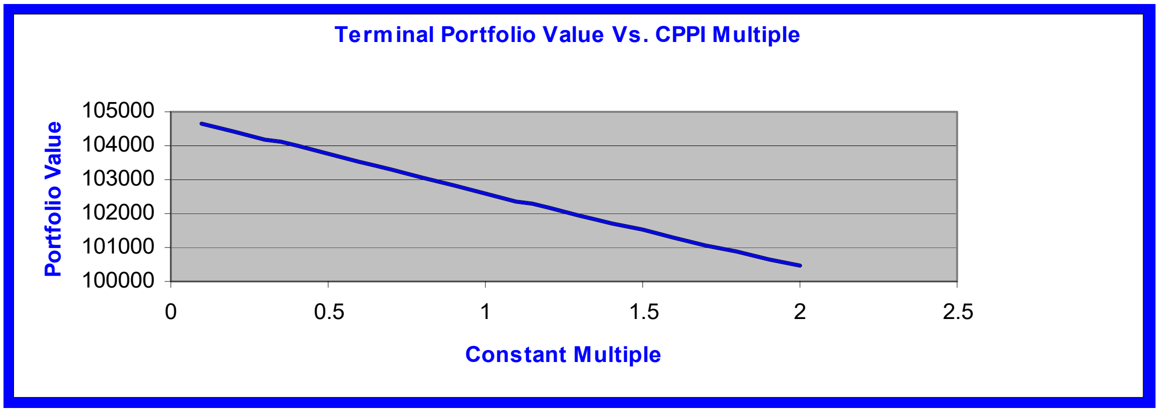

Figure (iii).4

The constant multiple and the terminal portfolio value were found to have a negative linear relationship during the period under consideration which is intuitively quite apparent given the rather poor performance of the three risky assets over that period. In practical terms, the holding period of two years we have considered here is ~~a tad~~ too short and most capital-guaranteed structured products in the markets are offered for a holding period of five years or more. The longer the holding period the higher is the probability of the best performing asset being any one of the risky assets thereby implying a higher probability of strong positive returns over and above the risk-free rate of return.

In the actual market data we have used here, the ideal investment would have been to invest everything in gold futures which would have generated a holding period return of about 33%! However this return cannot be obtained realistically as it calls for ~~a~~ perfect foresight. But the nature of the financial structured product guarantees the investor an assured return over the investment horizon anywhere roughly between 0.33% and 33% with a zero downside risk! We may verify from III.2 that the CPPI algorithm is striving to allocate more funds over time to the best performing asset within the structured product i.e. gold futures but obviously falls short of its target performance as the holding period horizon is too short. The rational objective is obviously to get as close possible to the upper limit of the return range. The investor's choice set (related to the number of assets within the envelope) then becomes an ideal vehicle for cardinalizing utility. It may be argued that a five-year time horizon instead of a two-year one would have provided a more comprehensive picture but then again we have treated the utility emanating from an effective capital-guarantee mechanism as our primary objective and it was primarily with respect to that objective that an overwhelmingly bearish period was chosen as we ideally wanted to measure the extrinsic utilty of the portfolio insurance scheme when the capital-guarantee mechanism is triggered.

In our numerical example here, if we construct a 4-bit product out of the 8-bit one by randomly leaving out one of the three risky assets, we would still have a capital guaranteed structure. But there will then be a higher probability that now the maximum return the investor could get is no longer 33% but only the 5.06% approximate return obtainable by investing all of the $100,000 in the risk-free asset for the period. This is because there is a probability of 0.33 that the best performer (i.e. gold futures) could be left out! This demonstrates that, in general, higher bit structured products will offer greater utility to the investor by expanding their choice set thereby broadening the return range. Of course, a trade-off between the increased overheads of having additional assets and the benefits of an expanded opportunity set is something open to future exploration.

In a structured product with three underlying risky assets plus one risk-free asset, the best performer may be hypothesized to be traceable by a first-order Markov process, whereby the best performing asset at time t+1 is dependent on the best performing asset at time t. In our numerical illustration, we have the following state-transition matrix:

Transition Matrix	S&P500	Lehman Bros.	GC00	Risk-free rate
S&P500	0.0000	1.0000	0.0000	0.0000
Lehman Bros.	0.1429	0.2857	0.4286	0.1429
GC00	0.0000	0.1818	0.6364	0.1818
Risk-free rate	0.0000	0.2000	0.4000	0.4000

Table (iii).3

Then for a one-step Markov process, the extrinsic utilities offered to the investor by each of the assets included in this structured product is computable as follows using the information theoretic measure of extrinsic utility developed in chapter 6:

H (S&P500) = Σ [p (S&P 500) I (S&P 500 | Lehman Bros., GC00, Risk-free rate)] = **0.0416**,

H (Lehman Bros.) = Σ [p (Lehman Bros.) I (Lehman Bros.| S&P500, GC00, Risk-free rate)] = **0.250,**

H (GC00) = Σ [p (GC00) I (GC00|S&P500, Lehman Bros., Risk-free rate)] = **0.500;** and

H (Risk-free rate) = Σ [p (Risk-free rate) I (Risk-free rate|S&P500, Lehman Bros., GC00)] = **0.208**

It is interesting to note that Lehman Brothers Growth Fund actually has a higher extrinsic utility than the risk-free rate though on an average although the risk-free rate outperformed the Lehman Brothers Growth Fund over the two-year investment period! This is due to the fact that though intrinsic utility of the risk-free rate was higher than that of the Lehman Brothers Growth Fund, the latter provided greater element of choice to the investor primarily due to a substantially wider range of volatility such that it ended up as the best performer more often compared to the risk-free asset though, on an average, it was outperformed by the risk-free rate.

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This work has been wholly adapted from the dissertation submitted by the author in 2004 to the Faculty of Information Technology, Bond University, Australia in fulfillment of the requirements for his doctoral qualification in Computational Finance.

This work covers a substantial mosaic of related concepts in utility theory as applied to financial decision-making.

The main body of the work is divided into four relevant chapters. The first chapter takes up the notion of *resolvable risk* i.e. systematic investment risk which may be attributed to actual market movements as against *irresolvable risk* which is primarily born out of the inherent imprecision associated with the information gleaned out of market data such as price, volume, open interest etc. A *neutrosophic* model of risk classification is proposed – neutrosophic logic being a new branch of mathematical logic which allows for a three-way generalization of binary fuzzy logic by considering a third, *neutral* state in between the *high* and *low* states associated with binary logic circuits.

A plausible application of the postulated model is proposed in reconciliation of price discrepancies in the long-term options market where the only source of resolvable risk is the long-term implied volatility. The chapter postulates that inherent imprecision in the way market information is subjectively processed by psycho-cognitive factors governing human decision-making actually contributes to the creation of heightened risk appraisals. Such heightened notions of *perceived risk* make investors predisposed in favor of *safe* investments even when pure economic reasoning may not entirely warrant such a choice.

To deal with this information fusion problem a new combination rule has been proposed -the *Dezert-Smarandache combination rule of paradoxist sources of evidence*, which looks for the basic probability assignment or *bpa* denoted as m (.) = m_1 (.) $\oplus$ m_2 (.) that *maximizes the joint entropy* of the two information sources.

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