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Exploring the extension of natural operations on intervals, matrices and complex numbers

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DEDICATION

PROF. IQBAL UNNISA
(21.11.1936 – 12.06.2011)

We dedicate this book to Prof. Iqbal Unnisa on her first death anniversary which falls on 12 June 2012. She was the first Muslim woman in India to receive her doctorate degree in the Mathematical sciences. As a brilliant researcher, she was the first woman to become director of the Ramanujan Institute for Advanced Study in Mathematics, University of Madras. Today we are proud of her mathematical legacy and we are inspired by her courageous spirit that never compromised and always challenged injustice.

PREFACE

This book extends the natural operations defined on intervals, finite complex numbers and matrices. The intervals [a, b] are such that $a \leq b$. But the natural class of intervals [a, b] introduced by the authors are such that $a \geq b$ or a need not be comparable with b. This way of defining natural class of intervals enables the authors to extend all the natural operations defined on reals to these natural class of intervals without any difficulty. Thus with these natural class of intervals working with interval matrices like stiffness matrices finding eigen values takes the same time as that usual matrices.

Secondly the authors introduce the new notion of finite complex modulo numbers just defined as for usual reals by using the simple fact $i^2 = -1$ and -1 in case of Z_n is $n - 1$ so $i_F^2 = n - 1$ where i_F is the finite complex number and i_F's value depends on the integer n of Z_n. Using finite complex numbers several interesting results are derived.

Finally we introduced the notion of natural product $\times_n$ on matrices. This enables one to define product of two column matrices of same order. We can find also product of $m \times n$ matrices even if $m \neq n$. This natural product $\times_n$ is nothing but the usual product performed on the row matrices. So we have extended this type of product to all types of matrices.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this chapter we just give references and also indicate how the arithmetic operations '+', ×, '–' and ÷ can be in a natural way extended to intervals once we define a natural class of intervals [a, b]; to be such that a > b or a < b or a = b or a and b cannot be compared. This has been studied and introduced in [2 books]. By making this definition of natural class of intervals it has become very easy to work with interval matrices; for working for interval eigen values or determinants or products further, it takes the same time as that of usual matrices (we call all matrices with entries from C or R or Q or Z or Z_n as usual matrices). The operations the authors have defined on the natural class of intervals are mere extensions of operations existing on R. Thus this had made working with intervals easy and time saving.

Further the authors have made product on column matrices of same order, since column matrices can be added what prevents one to have multiplication so we have defined this sort of product on column matrices as natural product. Another reason is if the transpose of a row vector (matrix) is the column matrix so it is natural, one can take the transpose of a column matrix and find the product and then transpose it. Thus the introduction of the natural product on matrices have paved way for nice algebraic structures on matrices and this is also a natural extension of product on matrices.

We call the existing product on matrices as usual product. On the row matrices both the natural product and the usual product coincide. Further the natural product is like taking max (or min) of two matrices of same order. Finally this natural product permits the product of any two matrices of same order. This is another advantage of using natural product.

Thus if two rectangular array of numbers of same order are multiplied the resultant is again a rectangular array of numbers of the same order. Natural product on square matrices of same order is commutative where as the usual product on square matrices of same order is non commutative.

Next we have from the definition of complex number i, where $i^2 = -1$ developed to the case of finite modulo integers. For if $Z_n = \{0, 1, 2, \ldots, n - 1\}$ the role of -1 is played by $n - 1$ so we define finite complex number i_F to be such that $i_F^2 = n - 1$ so that $i_F^3 = (n - 1)i_F$ and $i_F^4 = 1$ and so on.

One of the valid observations in this case is that the square value of the finite complex number depends on n for the given Z_n. Thus $i_F^2 = 7$ for Z_8, $i_F^2 = 11$ for Z_{12} and $i_F^2 = 18$ for Z_{19}. When we have polynomial ring $C(Z_n)[x]$ where $C(Z_n) = \{a + bi_F \mid a, b \in Z_n \text{ and } i_F^2 = n - 1\}$, we see the number of roots for any polynomial in $C(Z_n)[x]$ can be only from the n^2 elements from $C(Z_n)$; otherwise the equation has no root in $C(Z_n)$. For instance in $C(Z_2)[x]$; $x^2 + x + 1 = p(x)$ has no root in $C(Z_2)$. Thus we cannot be speaking of algebraically closed field etc as in case of reals. Introduction of finite complex numbers happen to be very natural and interesting [4].

Finally authors have constructed matrices using the Boolean algebra P(X); the power set of a set X. Study in this direction is also carried out and these matrices of same order with entries from P(X) happen to be a lattice under min and max operations. Thus this book explores the possibilities of extending natural operation on matrices, construction of natural class of intervals and employing all the existing operations on reals on them and finally defining a finite complex modulo number [4].

Chapter Two

EXTENSION OF NATURAL OPERATIONS TO INTERVALS

In this chapter we just give a analysis of why we need the natural operations on intervals and if we have to define natural operations existing on reals to the intervals what changes should be made in the definition of intervals. Here we redefine the structure of intervals to adopt or extend to the operations on reals to these new class of intervals.

Infact authors of this book often felt that the operations on the intervals (addition, subtraction multiplication and division) happen to be defined in such a way that compatability of these operations alone is sought. But it is suprising to see why we cannot define an interval [a, b] to be such that a > b or a < b or a = b. If operations can be so artificially defined to cater to the compatability what is wrong in accepting intervals of the form [a, b] where a > b; so if we make a very small change by defining an interval [a, b] in which a > b can also occur, certainly all the four operations defined on the reals can be very naturally extended.

So we have over come the artificial way of defining product on intervals by redefining the intervals in a natural way, for these natural operations pave way for a simple working with interval matrices or any other form of interval structures like interval polynomials and so on.

Also we have already if f (x) is a function defined on [a, b] (a < b) then integral of f(x) in that interval is $\int_a^b f(x)\,dx$ and if we take integral from b to a we only make a small change by saying $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

So by all means the redefining has not resulted in any change or contradictions to the existing classical structures.

Infact these natural ways has only helped one with an easy computations of interval algebras or to be precise it takes the same time for computing with intervals or otherwise.

Thus $-\int_b^a f(x)\,dx$ can be redefined as the integral defined on the interval [b, a], b > a. So defining interval [x, y] with x > y will in no way contradict the working of integral or differential calculus.

Thus by defining [x, y]; x > y has become only natural in case of calculus. Only we need to make simple and appropriate modifications while transferring the increasing to decreasing intervals and vice versa.

So in our opinion it is essential to have intervals of the form [a, b]; a > b (a, b $\in$ R) also in the collection of intervals. This must be recognized as a decreasing interval and in school curriculum and in college curriculum this must be introduced so that a sudden change does not take place in researchers mind but the essential change is present in the their minds from the school days.

Further this study in no way is going to make the school children in any way affected they will only appreciate the natural extension or natural way how the negative reals and positive reals in a reversed way occur.

Thus making this sort of study at the school level is not only essential but is an absolute necessicity.

Thus we call the collection of intervals

$\{[a, b] \mid a > b$ or $a < b$ or $a = b$ where $a, b \in R\}$ to be the natural class of closed intervals and denote it by $N_c(R)$.

Thus $N_c(R) = \{[a, b] \mid a > b$ or $a < b$ or $a = b; a, b \in R\}$ we can define in a similar way the natural class of open intervals $N_o(R) = \{(a, b) \mid a > b$ or $a < b$ or $a = b; a, b \in R\}$.

Also $N_{oc}(R) = \{(a, b] \mid a > b$ or $a < b$ or $a = b; a, b \in R\}$ denotes the natural class of open closed intervals.

For example $(5, 2]$, $(\infty, 7]$, $(0, -7]$, $(9, 11]$, $(-\infty, 8]$ and so on.

$N_{co}(R) = \{[a, b) \mid a > b$ or $a < b$ or $a = b; a, b \in R\}$ denotes the natural class of closed open intervals.

$[7, 0)$, $[9, 12)$, $[-\infty, 0)$, $[0, -\infty)$ and $[11, -5)$ are some of the examples.

Now we can as in case of reals R define operations on $N_c(R)$, $N_o(R)$, $N_{oc}(R)$ and $N_{co}(R)$ in a natural way.

Further we can also replace R by Q or Z and still these can be defined. For instance $N_o(Q)$ denotes the natural class of open intervals with entries from the rationals Q.

$N_c(R) = \{[a, b] \mid a > b \text{ or } a < b \text{ or } a = b;\ a, b \in Q\}$ denotes the natural class of closed rational intervals.

Similarly $N_{oc}(Q)$ and $N_{co}(Q)$ denotes the natural class of open-closed rational intervals and closed-open rational intervals respectively.

Likewise we can define $N_c(Z)$, $N_o(Z)$, $N_{oc}(Z)$ and $N_{co}(Z)$ to be the natural class of integer closed intervals, integer open intervals, integer open-closed intervals and integer closed-open intervals respectively.

Now just for the sake of completeness we just recall the definition of the four arithmetic operations defined on the usual intervals. (Throughout this book by the term usual interval we mean an interval of the form [a, b] ([a, b), (a, b], (a, b); a < b or a = b). That is only increasing intervals will be termed as usual intervals.

We just recall the classical operation done on these usual intervals.

$$[a, b] + [c, d] = [a + c, b + d]$$
$$(a, b) + (c, d) = (a + c, b+d)$$
$$(a, b] + (c, d] = (a+c, b+d]$$
$$[a, b) + [c, d) = [a+c, b+d)$$

$$[a, b] - [c, d] = [a-c, b-d]$$
$$(a, b) - (c, d) = (a-c, b-d)$$
$$(a, b] - (c, d] = (a-c, b-d]$$
$$[a, b) - [c, d) = [a-c, b-d)$$

$$[a, b] \times [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$$
$$[a, b) \times [c, d) = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\})$$
$$(a, b] \times (c, d] = (\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$$
$$(a, b) \times (c, d) = (\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\})$$

$$[a, b] \div [c, d] = [\min\{a/c, a/d, b/c, b/d\}, \max\{a/c, a/d, b/c, b/d\}]$$
$$(a, b) \div (c, d) = (\min\{a/c, a/d, b/c, b/d\}, \max\{a/c, a/d, b/c, b/d\})$$

$[a, b) \div [c, d) = [\min \{a/c, a/d, b/c, b/d\}, \max \{a/c, a/d, b/c,b/d\})$
$(a, b] \div (c, d] = (\min \{a/c, a/d, b/c, b/d\}, \max \{a/c, a/d, b/c,b/d\}]$

($c \neq 0$ and $d \neq 0$).

Now we proceed on to define and extended those classical operations on R to these 4 natural class of intervals $N_c(R)$, $N_o(R)$, $N_{oc}(R)$ and $N_{co}(R)$.

However it is pertinent to mention here that we cannot have any compatible operations defined in between the four natural classes $N_c(Q)$, $N_o(Q)$, $N_{oc}(Q)$ and $N_{co}(Q)$ as in case of usual intervals. We just define and describe only for one class say $N_c(R)$ and they can be developed for all the four classes.

So we illustrate this by examples for other classes.

Let $[a, b]$ and $[c, d] \in N_c(R)$ we define
$[a, b] + [c, d] = [a+c, b+d]$; clearly $[a+c, b+d] \in N_c(R)$.

Suppose $[–5, 0]$ and $[7, 2] \in N_c(R)$; $[–5, 0]$ is an increasing interval as $–5 < 0$ and $[7, 2]$ is a decreasing interval as $7 > 2$; now we find the sum of $[5, 0] + [7, 2] = [2, 2] = 2$ is a degenerate interval.

This is the marked difference between the usual closed intervals and natural class of closed intervals; for in the usual class of intervals only two degenerate intervals can add up to a degenerate interval.

Consider $[3, 1]$ and $[–7, 5] \in N_c(R)$, clearly $[3, 1]$ is a decreasing interval and $[–7, 5]$ is an increasing closed interval; we see their sum $[3, 1] + [–7, 5] = [3+(–7), 1+5] = [–4, 6]$ is an increasing interval.

Consider $[8, –5]$ and $[–9, 0]$ in $N_c(R)$, $[8, –5]$ is a decreasing interval where as $[–9, 0]$ is an increasing interval, but their sum $[8, –5] + [–9, 0] = [–1, –5]$ is a decreasing interval.

However the sum of two increasing intervals is again an increasing interval. Likewise the sum of two decreasing intervals is a decreasing interval.

Same type of addition and analysis hold good in case of the natural class of closed intervals $N_o(R)$, $N_{oc}(R)$ and $N_{co}(R)$. (similar results hold good even if R is replaced by Q or Z).

Now we proceed onto define the operation of subtraction on $N_c(R)$. Let [a, b] and [c, d] $\in N_c(R)$.

$$[a,b] - [c, d] = [a-c, b-d] \in N_c(R).$$

We define the ordinary subtraction in $N_c(R)$.

Consider [8, 1] and [8, 3] $\in N_c(R)$. [8, 1] is a decreasing interval and [8, 3] is a decreasing interval.

We see [8, 1] – [8, 3] = [8–8, 1–3] = [0, –2] is a decreasing interval and [8, 3] – [8, 1] = [8–8, 3–1] = [0 2] is an increasing interval. Let [3, –2] and [–4, 0] $\in N_c(R)$. Clearly [3, –2] is a decreasing interval and [–4, 0] is an increasing interval.

Now [3, –2] – [–4, 0] = [3+4, –2–0] = [7, –2] is a decreasing interval [–4, 0] – [3, –2] = [–4, –3, 0 – (–2)] = [–7, 2] is an increasing interval.

Clearly the subtraction operation is non commutative.

We can define in the same way for the other three new classes of intervals $N_o(R)$, $N_{oc}(R)$ and $N_{co}(R)$.

Now we proceed onto define the product on the natural class of intervals $N_c(R)$.

Consider [a, b] and [c, d] $\in N_c(R)$. We do not define product as in case of usual intervals.

We will illustrate this by some examples.

Let [3, 0] and [–3, 7] be two intervals in $N_c(R)$ where [3, 0] is decreasing and [–3, 7] is an increasing interval.

[3, 0] × [–3, 7] = [–9, 0] is an increasing interval.

Take [1, –5] and [–2, 10] ∈ $N_c(R)$, [1, –5] is a decreasing interval and [–2, 10] is an increasing interval. Now the product [1, –5] × [–2, 10] = [–2, –50] ∈ $N_c(R)$; we see [–2, –50] is a decreasing interval in $N_c(R)$.

Take [3/7, 1/9] and [7/3, 9] ∈ $N_c(R)$; [3/7, 1/9] is a decreasing interval and [7/3, 9] is an increasing interval.

[3/7, 1/9] × [7/3, 9] = [1, 1] is a degenerate interval 1.

In the same way for all other natural class of intervals the operation of product can be defined.

Now we proceed onto define the division on the natural class of intervals $N_c(R)$ (or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$).

Let [a, b] and [c, d] ∈ $N_c(R)$ with $c \neq 0$ and $d \neq 0$.
[a, b] / [c, d] = [a/c, b/d] ∈ $N_c(R)$.

(Clearly as in the case of integers we cannot define division of natural class of intervals on $N_c(Z)$, $N_o(Z)$, $N_{oc}(Z)$ and $N_{oc}(Z)$). Let [5, 3] and [–7, 2] be in $N_c(R)$ (or $N_c(Q)$). Now [5, 3] / [–7, 2] = [5/–7, 3/2] is in $N_c(R)$ (or $N_c(Q)$).

Further [5, 3] is a decreasing interval and [–7, 2] is an increasing interval. [–5/7, 3/2] is an increasing interval. We find [–7, 2] / [5, 3] = [–7/5, 2/3] is again an increasing interval. Consider [2, 1] and [3,7] in $N_c(Q)$.

[2,1] / [3,7] = [2/3, 1/7] is a decreasing interval.

Now we give the highest algebraic structures enjoyed by these operations on $N_c(R)$ (or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$).

Result 1: $N_c(R)$ ($N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$) is a commutative ring with unit and zero divisors.

Clearly $R \subseteq N_c(R)$ ($Q \subseteq N_c(Q)$ and $Z \subseteq N_c(Z)$). Thus $N_c(R)$ is a Smarandache ring as R is a field contained in $N_c(R)$. (Clearly this is not true in case of $N_c(Z)$ or $N_o(Z)$ or $N_{oc}(Z)$ or $N_{co}(Z)$). We can define interval polynomials with natural closed interval coefficients or natural closed open interval coefficients).

Let

$$N_c(R)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_c(R) \right\},\ N_c(R)$$

is a commutative ring with unit which is a Smarandache ring. On similar lines we can define $N_c(Q)[x]$, $N_o(R)[x]$, $N_{oc}(Q)[x]$, $N_{oc}(R)[x]$, $N_{co}(R)[x]$ and so on. However $N_c(Z)[x]$, $N_o(Z)[x]$, $N_{oc}(Z)[x]$ and $N_{co}(Z)[x]$ are not Smarandache rings.

Now we solve polynomial in the variable x with coefficient from the natural class of intervals.

For if

$$P(x) = \sum_{i=0}^{\infty} a_i x^i = [p_1(x), q_1(x)] \text{ where } a_i \in N_c(R)[x]$$

where $p_1(x) = \sum_{i=0}^{\infty} a_i^1 x^i$ and $q_1(x) = \sum_{i=0}^{\infty} a_i^2 x^i$ where $a_i = [a_i^1, a_i^2] \in N_c(R)$.

Thus every interval coefficient polynomial is an interval polynomial and vice versa.

We will illustrate this by a simple example.

Let $p(x) = [0,8]x^7 + [-3, 2]x^3 + [5,1]x + [6,7]$

$$= [p_1(x), p_2(x)] = [-3x^3 + 5x + 6,\ 8x^7 + 2x^3 + x + 7];$$

p(x) is a polynomial with interval coefficient and $[p_1(x), p_2(x)]$ is an interval polynomial. So solving p(x) is equivalent to solving $p_1(x)$ and $p_2(x)$ and writing them in intervals. So the roots are intervals.

Suppose

$$p(x) = [-3, -4]x^2 + [2,4]x + [1, 8] \in N_c(Q)[x]$$

be a polynomial with interval coefficient.

$$p(x) \quad = ([3, -2]x + [1, 4])\,([-1, 2]x + [1, 2])$$

so that x = [1, –1] and [–1/3, 2].

$$\begin{aligned} \text{Now } p([1, -1]) &= [-3, -4]\,[1, 1] + [2, 4]\,[1, -1] + [1, 8] \\ &= [-3, -4] + [2, -4] + [1, 8] \\ &= 0. \end{aligned}$$

$$\begin{aligned} p([-1/3, 2]) &= [-3, -4]\,[1/9, 4] + [2, 4]\,[-1/3, 2] + [1, 8] \\ &= [-1/3 \;\; -16] + [-2/3+8] + [1, 8] \\ &= 0. \end{aligned}$$

Suppose we are interested in applying the solution of quadratic equations using the formula for $ax^2 + bx + c = 0$; this gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where}$$

a, b, c $\in$ R, we have to now define root of an interval.

$\sqrt{[7,4]} = [\sqrt{7}, 2]$; $\sqrt{[16,9]} = [4, 3]$ and

$\sqrt{[5,-3]} = [\sqrt{5}, 3i]$ and so on.

Let $p(x) = [7, 2]x^2 + [5, 3]x + [-3, 2]$;

$$x = \frac{-[5,3] \pm \sqrt{[25,9] - 4[7,2][-3,2]}}{2[7,2]}$$

$$= \frac{-[5,3] \pm \sqrt{[25,9] - [84,-16]}}{2[7,2]}$$

$$= \frac{-[5,3] \pm \sqrt{[109,-5]}}{2[7,2]}.$$

We just recall $[4, 5]^2 = [16, 25]$ and $[3^2, 4^2] = [9, 16]$ likewise for any power of an interval $[2, 5]^7 = [2^7, 5^7]$ and so on.

Now if we write the polynomial p(x)

$$= [7,2]x^2 + [5, 3]x + [-3, 2]$$
$$= [p_1 (x), p_2(x)]$$
$$= [7x^2 + 5x - 3, 2x^2 + 3x + 2].$$

Now roots of p(x) = [roots of $p_1(x)$, roots of $p_2(x)$]

$$= \left[\frac{-5 \pm \sqrt{25 + 12 \times 7}}{14}, \frac{-3 \pm \sqrt{9 - 4 \times 4}}{4}\right].$$

So working with interval polynomials or polynomials with interval coefficients can by no means make the solving time greater than that of usual polynomials (we say usual polynomials if the polynomials take its coefficients from R or Z or Q) [3, 5].

Now we can discuss about the matrices with interval entries. The matrices with entries from Z or Q or R are called usual matrices.

Now we see $P = (a_{ij})_{m \times n}$ matrix is an interval matrix if it takes its entries from the natural class of intervals that is; $a_{ij} \in$

$N_c(R)$; $1 \leq i \leq m$ and $1 \leq j \leq n$. Suppose $a_{ij} = [a_{ij}^1, a_{ij}^2]$ then $P = (a_{ij})_{m\times n} = ([a_{ij}^1, a_{ij}^2])$

$$= [P_1, P_2] = [(a_{ij}^1), (a_{ij}^2)].$$

We will illustrate this situation by an example.

Consider A = ([3, 2], [0, 5], [9, 4], [11,2]) row interval matrix.

$$A = (A_1, A_2) = [A_1, A_2]$$
$$= [(3, 0, 9, 11), (2, 5, 4, 2)].$$

As in case of usual matrices we can find the product of two interval matrices of same order.

$$A \times A = [A_1^2, A_2^2]$$
$$= [(9, 0, 81, 121), (4, 25, 16, 4)]$$
$$= ([9, 4], [0, 25]. [81, 16], [121, 4]).$$

By this way the product with interval row matrices is similar to those of usual matrices.

Likewise addition of two interval row matrices are carried out. More so is the addition of any $m \times n$ interval matrices or matrices with interval entries.

Now for more about these structures please refer [3, 5].

Working again for interval eigen values and interval eigen vectors can be easily carried out as in case of usual matrices after writing the interval matrix as two usual matrices. For instance if

$$A = \begin{bmatrix} [8,6] & [3,-2] \\ [1,4] & [0,7] \end{bmatrix} = \left(\begin{bmatrix} 8 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 6 & -2 \\ 4 & 7 \end{bmatrix}\right)$$

$= [(A_1, A_2)]$.

Eigen values for A_1 and A_2 can be found individually and finally the interval eigen values can be given [3, 5].

One of the important and valid results about polynomials with interval coefficients and matrices with entries from the natural class of intervals is that, if p(x) and q(x) are polynomials in R[x] then

$$P(x) = [p(x), q(x)] = ([p_0, q_0] + [p_1, q_1]x + \ldots + [p_n\ q_n]\ x^n)$$

is an interval polynomial with coefficients from $N_c(R)$, where some of the q_i's and p_j's are non zero.

Thus from a pair of usual polynomials from R[x] we can construct an interval polynomial with coefficients from $N_c(R)$ or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ and from interval polynomials with coefficients from $N_c(R)[x]$ (or $N_o(R)[x]$ or $N_{oc}(R)[x]$ or $N_{co}(R)[x]$) we can get back to usual polynomials in R[x].

Likewise if we have two usual matrices A and B of same order with entries from R, we can get the interval matrix [A, B] ((A,B), [A,B), (A,B]) and vice versa [3, 5].

We will just illustrate this situation by a simple example.

$$\text{Let } A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 7 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 5 & -2 \\ 1 & 0 \end{bmatrix}$$

be any two usual matrices with entries from R.

$$\text{Now } C = [A, B] = \left[\begin{pmatrix} 3 & 2 \\ 1 & 5 \\ 7 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & -2 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} [3,0] & [2,1] \\ [1,5] & [5,-2] \\ [7,1] & [3,0] \end{pmatrix}$$

is an interval matrix resulting from the usual matrices A and B.

$$\text{Also if P} = \begin{bmatrix} [3,9) & [5,2) & [8,0) & [0,11) \\ [1,2) & [0,1) & [10,1) & [9,7) \\ [3,3) & [5,-7) & [-3,0) & [2,1) \end{bmatrix}$$

$$\text{then} \left[\begin{pmatrix} 3 & 5 & 8 & 0 \\ 1 & 0 & 10 & 9 \\ 3 & 5 & -3 & 2 \end{pmatrix}, \begin{pmatrix} 9 & 2 & 0 & 11 \\ 2 & 1 & 1 & 7 \\ 3 & -7 & 0 & 1 \end{pmatrix}\right] = [P_1, P_2)$$

where P_1 and P_2 are usual matrices. For more please refer [3,5].

We can replace R by Q or Z, still the results hold good. Thus we by defining the natural class of intervals $N_c(R)$ (or $N_o(R)$, $N_{oc}(R)$ or $N_{co}(R)$) we have made it not only easy, but time saving to work with interval matrices and polynomials with interval coefficients.

We can also define integration and differentiation of interval coefficient polynomials [3,5].

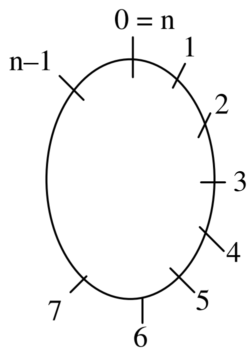
Now suppose we have intervals of the form [a, b] where a cannot be compared with b but still we have some elements lying between a and b then also we can work with these intervals. When we say intervals of the type we will not call them as increasing intervals or decreasing intervals.

First we will illustrate this situation by some examples.

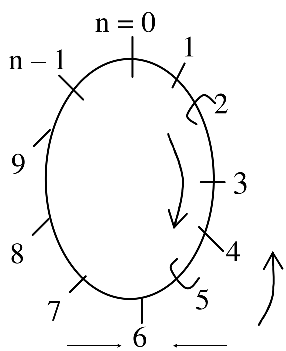
[I, 7] where I is a neutrosophic number and 7 the integer is an interval which has only the two element I and 7. Clearly we cannot compare I with 7 or 7 with I. Likewise [–7 + 5i, 8I + 3i] is an interval in which no comparison is possible.

Also [3, 12] and [7, 5] where 3, 12, 7, 5 ∈ Z_{25} (set of modulo integers) are intervals which is neither decreasing nor increasing. However [3, 12] has elements (3, 4, 5, 6, 7, 8, 9, 10, 12) in between 3 and 12 and including 3 and 12 and [7, 5] has (7, 6, 5) as its elements.

Thus when we use intervals in Z_n ($n < \infty$, n an integer) those intervals are not comparable as ordering cannot be made as they lie on the arc of a circle.



So interval is an arc



$\overrightarrow{[2,6]}$ or $\overleftarrow{[6,2]}$.

However if $N_c(Z_n) = \{[a, b] \mid a, b \in Z_n\}$ then $N_c(Z_n)$ is closed under the operations of interval addition and multiplication.

We will illustrate this situation by some simple examples.

Consider $N_c(Z_5) = \{[0,1]\ [0,2],\ldots,[4,3], [3,4], 0, 1, 2, 3, 4\}$.

Now $[4, 3] + [2, 1] = [1, 4]$.
$[4, 3] \times [2, 1] = [3, 3]$.

So product of two non degenerate intervals can lead to a degenerate interval.

$$[2, 2]\ [3, 2] = [1, 4].$$

Take $N_c(Z_3) = \{[0,0], [1,1], [2,2], [0,1], [1,0], [0,2], [2,0], [1,2], [2,1]\}$. $N_c(Z_3)$ has 9 elements.

We see $N_o(Z_3)$ is a commutative ring with (0, 0) as its additive identity and (1, 1) as its multiplicative identity. Infact $N_o(Z_3)$ is a ring with zero divisors. All elements are not invertible.

Only $(2, 2) \times (2, 2) = (1, 1)$,
$(2, 1) \times (2, 1) = (1, 1)$
and $(1, 2) \times (1, 2) = (1, 1)$ are invertible.

Infact $N_o(Z_3)$ is a Smarandache ring.

Let $N_o(Z_n) = \{(a, b) \mid a, b \in Z_n\}$ be a ring of open intervals. We see $o\ (N_o(Z_n)) = |N_o(Z_n)| = n^2$; $1 < n < \infty$. If n is a prime then certainly $N_o(Z_n)$ is a Smarandache ring. If Z_n is a Smarandache ring then $N_o(Z_n)$ (or $N_{co}(Z_n)$ or $N_{oc}(Z_n)$ or $N_c(Z_n)$) is also a Smarandache ring. Every $N_o(Z_n)$ has zero divisors and units [3, 5].

We can construct polynomials with interval coefficients from $N_o(Z_n)$.

$$N_o(Z_n)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_o(Z_n)\right\}$$

is also an interval polynomial ring of infinite order. These interval polynomial rings have zero divisors.

We can replace open intervals using Z_n by closed intervals, open-closed intervals and closed-open intervals.

We can also define interval matrices with entries from $N_o(Z_n)$ A = {3 × 3 interval matrices with entries from $N_o(Z_n)$}. A is a non commutative ring with unit and zero divisors.

B = {7 × 4 interval matrices with entries from $N_c(Z_{20})$}. B is only an abelian group with respect to addition modulo 20.

M = {5 × 9 interval matrices with entries from $N_{oc}(Z_{11})$}. M is additive abelian group.

We can as in case of R write two usual matrices with entries from Z_n into an interval matrix with intervals from $N_o(Z_n)$ (or $N_c(Z_n)$ or $N_{oc}(Z_n)$ or $N_{co}(Z_n)$) and vice versa.

Now natural class of neutrosophic intervals can be constructed using the set

$\langle Z \cup I\rangle = \{a + bI \mid a, b \in Z\}$
or $\langle R \cup I\rangle = \{a + bI \mid a, b \in R\}$
or $\langle Q \cup I\rangle = \{a + bI \mid a, b \in Q\}$;

where I is the indeterminate such that $I^2 = I$.

Also we can construct using Z_n,
$\langle Z_n \cup I\rangle = \{a + bI \mid a, b \in Z_n\}$.

Thus $N_o(\langle Z \cup I\rangle) = \{(x, y) \mid x, y \in \langle Z \cup I\rangle\}$ and so on. Here some of the intervals in $N_o(\langle Z_n \cup I\rangle)$ may be comparable and some may not be comparable.

If we replace $\langle Z \cup I\rangle$ in $N_o(\langle Z \cup I\rangle)$ by $\langle Q \cup I\rangle$ or $\langle R \cup I\rangle$ still the results continue to be true in case of these interval rings, that is $N_o(\langle Q \cup I\rangle)$ or $N_o(\langle R \cup I\rangle)$. The open intervals can be replaced by closed and closed-open intervals.

We can build polynomial neutrosophic interval coefficients using $N_c(\langle Z \cup I\rangle)$ or $N_o(\langle Q \cup I\rangle)$ and so on.

$$N_c(\langle R \cup I\rangle)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_c(\langle R \cup I\rangle)\right\}$$

is a neutrosophic interval coefficient polynomial ring with zero divisors. Infact $N_c(\langle R \cup I\rangle)[x]$ is a commutative ring which is not an integral domain. Likewise we can construct neutrosophic interval polynomial coefficient rings using $\langle Z_n \cup I\rangle$.

Now we can also define neutrosophic interval matrices A = {all $n \times n$ neutrosophic matrices with entries from $N_o(\langle Z \cup I\rangle)$}; is a ring which is non commutative and has zero divisors.

B = {all $m \times n$ neutrosophic interval matrices with entries form $N_c(\langle Q \cup I\rangle)$} is only an additive abelian group if $m \neq n$.

Interested reader can refer [3, 5]. Also all the properties enjoyed by these algebraic structures built using neutrosophic intervals can be studied and developed as a matter of routine; with simple appropriate modifications.

Finally we can also develop intervals using $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$. We define

$N_c(Z^+ \cup \{0\}) = \{[a, b] \mid a, b \in Z^+ \cup \{0\}\}$. $N_c(Z^+ \cup \{0\})$ is only a semiring and not a semifield as it has zero divisors.

But if we consider $N_c(Z^+) \cup [0,0]$ where $N_c(Z^+) = \{[a, b] \mid a, b \in Z^+\}$ then $N_c(Z^+) \cup [0,0]$ is a semifield. $N_o(R^+ \cup \{0\})$ is only a semiring not a semifield. Likewise $N_{oc}(Q^+) \cup [0,0]$ is a semifield and $N_{oc}(Q^+ \cup \{0\})$ is a semiring and is not a semifield but however $N_{oc}(Q^+ \cup \{0\})$ is a Smarandache semiring.

We can also work with matrices whose entries are from $N_{oc}(Q^+ \cup \{0\})$. Suppose

P = {all m × n matrices with entries form $N_{oc}(Q^+ \cup \{0\})$}, then P is only a semiring of interval matrices. P has zero divisors. P is a S-semiring. Of course these results hold good if in $N_{oc}(Q^+ \cup \{0\})$, $Q^+ \cup \{0\}$ is replaced by $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

Further if open-closed interval is replaced by open intervals or closed intervals or closed-open intervals still all the results hold good.

Now we can develop the notion of semiring of interval coefficient polynomials.

$$\text{Let } N_o(Z^+ \cup \{0\})[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_c(Z^+ \cup \{0\}) \right\}$$

be the semiring of closed interval coefficient polynomials.

For example

$$p(x) = [7, 3] + [0,5]x + [12,1]x^3 + [17,120]x^7 \in N_c(Z^+ \cup \{0\})[x],$$

we can replace $Z^+ \cup \{0\}$ by $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ and still the results hold good.

Suppose

$$(N_c(Z^+) \cup \{0\})[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_c(Z^+) \text{ or } a_i = 0 \right\}$$

be the closed interval coefficient polynomial semiring then $(N_c(Z^+)\cup\{0\})[x]$ is a semifield. This result holds good if Z^+ is replaced by R^+ or Q^+.

Now having seen how the natural class of intervals gives various structures we can also define interval polynomials and interval matrices in the following in case of semirings.

Let $p(x) = [p_1(x), p_2(x)]$ where $p_1(x), p_2(x) \in (Z^+ \cup \{0\})[x]$ then p(x) is an interval polynomial also

$$p(x) = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in N_c(Z^+ \cup \{0\}) \right\} = [p_1(x), p_2(x)],$$

so even in case of semirings an interval polynomial is a polynomial interval of usual polynomials.

Likewise matrices with interval entries is the interval matrices of a semiring. If $A = ([a_{ij}^1, a_{ij}^2]_{n\times n}$ then $A = [A_1, A_2]$ where $A_1 = (a_{ij}^1)$ and $A_2 = (a_{ij}^2)$ are usual matrices with entries $a_{ij}^1, a_{ij}^2 \in Z^+ \cup \{0\}$ or $[a_{ij}^1, a_{ij}^2] \in N_c(Z^+ \cup \{0\})$. Thus one can go from interval matrices to matrix intervals and vice versa.

Now we can replace this study by complex numbers so that $N_o(C) = \{(a+ib, c+id) \mid a, b, c, d \in R; i^2 = -1\}$ is the collection of all open complex intervals. $N_o(C)$ is a ring with zero divisors which is commutative.

We can write (a+ib, c+id), also as (a, c) + i(c, d); this represents the interval complex number. It is easily verified that every complex interval is an interval complex number and vice versa. Of course open intervals can be replaced by closed complex intervals, open-closed complex intervals and closed-open complex intervals.

Thus we can redefine interval complex number as A+iB where $A, B \in N_o(R)$ (or $N_{oc}(R)$ or $N_{co}(R)$ or $N_c(R)$).

On similar lines we can define interval neutrosophic numbers as A + BI where A, B $\in$ $N_o(R)$ (or $N_{oc}(R)$ or $N_c(R)$ or $N_{co}(R)$).

Further if [a+bI, c+dI] is a neutrosophic interval we can rewrite this as [a, c] + [b,d]I where a, b, c, d $\in$ R. However we are not in a position to give any form of geometrical representation. Further we say we can represent interval complex numbers or interval neutrosophic numbers in this form just like we represent complex numbers as a+bi, a, b $\in$ R; we can write an interval neutrosophic number as A+BI where A and B are intervals from $N_o(R)$ or $N_c(R)$ or $N_{oc}(R)$ or $N_{co}(R)$.

Likewise our study can be made as in case of complex interval numbers.

Thus $I_o(C) = \{(a, b) + (c, d)I \mid (a, b) \text{ and } (c, d) \in N_o(R)\}$ denotes the collection of all interval complex numbers. We see $I_o(C)$ is a ring which is commutative and has zero divisors. Further $C \subseteq N_o(C) \subseteq I_o(C)$ and $R \subseteq N_o(R) \subseteq I_o(C)$.

Just we show how we define operations with them.

Suppose x = (3, 4) + (2, –1)i and
y = (–7, 2) + (0, 4)i $\in$ $I_o(C)$; then

$$\begin{aligned} x+y &= \{(3, 4) + (2, -1)i\} + \{(-7, 2) + (0, 4)i\} \\ &= [(3, 4) + (-7,2)] + [(2, -1) + (0,4)]i \\ &= (-4, 6) + (2, 3)i \in I_o(C). \end{aligned}$$

$$\begin{aligned} x \times y &= \{(3, 4) + (2, -1)i\} \times \{(-7, 2) + (0, 4)i\} \\ &= (3, 4)(-7,2) + (2, -1)(-7,2)i + (3, 4)(0, 4)i + (2, -1)(0, 4)(-1) \\ &= (-21, 8) + (-14, -2)i + (0, 16)i + (0, 4) \\ &= (-21, 12) + (-14, 14)i. \end{aligned}$$

Thus we have a ring structure on open interval complex number $I_o(C)$.

We can on similar lines define $I_c(C)$, $I_{oc}(C)$ and $I_{co}(C)$ and all these structures are also rings which are commutative having unit (1,1) = 1 and has zero divisors and units.

Now likewise $I_o(\langle R \cup I\rangle) = \{(a, b) + (c, d)I \mid (a, b)$ and $(c, d) \in N_o(R)$; I the indeterminate with $I^2 = I\}$ is a collection pf open interval neutrosophic numbers. $I_o(\langle R \cup I\rangle)$ is again a ring with unit, and is known as the open interval neutrosophic ring.

On similar lines we can define closed interval neutrosophic ring, open-closed interval neutrosophic ring and closed-open interval neutrosophic ring.

Just we show how the operations are carried out on elements of closed interval neutrosophic ring.

Let $x = [3, -2] + [5, 7]I$ and $y = [0, 4] + [2, -1]I$

be in $I_c(\langle R \cup I\rangle)$.

$$\begin{aligned} x+y &= [3, -2] + [5, 7]I + [0, 4] + [2, -1]I \\ &= ([3, -2] + [0, 4]) + ([5, 7] + [2, -1])\, I \\ &= [3, 2] + [7, 6]I \in I_c(\langle R \cup I\rangle). \end{aligned}$$

$$\begin{aligned} x \times y &= [3, -2] + [5, 7]I \times [0, 4] + [2, -1]I \\ &= [3, -2] \times [0, 4] + [5, 7]\,[0, 4]I + [3, -2]\,[2, -1]I + \\ &\quad [5, 7]I\,[2, -1]I \\ &= [0, -8] + ([0, 28] + [6, 2])\, I + [10, -7]I \\ &\qquad\qquad\qquad\qquad (\because\ I^2 = I) \\ &= [0, -8] + [16, 23]I \in I_c(\langle R \cup I\rangle). \end{aligned}$$

Thus we see $(I_c(\langle R \cup I\rangle), +, \times)$ is a commutative ring of closed interval neutrosophic numbers.

Now having seen these interval neutrosophic numbers we can construct polynomials with interval neutrosophic coefficients and matrices with entries from interval neutrosophic numbers.

Further we see using interval complex numbers also we can build polynomials and matrices.

We will only illustrate these structures and show how operations can be carried out on them.

In the first place we wish to state an interval complex number reduces to a complex number if both the intervals are degenerate intervals, that is if x = [a,a] + [b,b]i = a+bi $\in$ C $\subseteq$ $I_c(C)$. Infact this is true for all types of intervals open, open-closed, or closed-open.

Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in I_o(C) \text{ that is } a_i = (a_1^1, a_1^2) + (b_1^1, b_1^2)i,\right.$$

$$\left. a_1^i, b_1^j \in R;\ 1 \leq i, j \leq 2 \right\}$$

be the interval complex number coefficient polynomial in the variable x.

V is a ring defined as the interval complex number coefficient polynomial.

Let p(x) = (3, 0) + (2, 5)i + [(7, 5) + (2, 1)i]x + [(5, 1) + (4, 3)i] x^3 and

q(x) = (1,2) + (7,8)i + [(9, 0) + (1,2)i]x^2 be two interval complex coefficient polynomials in V.

Now we show how the operations '+' and '×' are defined on V.

p(x) + q(x) = {(3, 0) + (2, 5)i + [(7, 5) + (2,1)i]x + [(5, 1) + (4, 3)i]x^3 and q(x) = (1,2) + (7, 8)i + [(9,0) + (1,2)i]x^2 be two interval complex coefficient polynomials in V.

Now we show how the operations '+' and '×' are defined on V.

$p(x) + q(x) = \{(3,0) + (2,5)i + [(7,5) + (2,1)i]x + [(5,1) + (4,3)i]x^3\} + \{(1,2)+(7,8)i + [(9,0) + (1,2)i]x^2\} = [(3,0) + (2,5)i] + [(1,2) + (7,8)i] + [(7,5) + (2,1)i]x + [(9,0) + (1,2)i]x^2 + [(5,1) + (4,3)i]x^3 = (4,2) + (9,13)i + [(7,5) + (2,1)i]x + [(9,0) + (1,2)i]x^2 + [(5,1) + (4,3)i]x^3$ is again a interval complex number polynomial.

$p(x) \times (q(x)) = [(3,0) + (2,5)i + ((7,5) + (2,1)i)x + ((5,1) + (4,3)i]x^3] \times [(1,2) + (7,8)i + [(9,0) + (1,2)i]x^2]$

$= [(3, 0) + (2,5)i] [(1,2) + (7,8)i] + [(3,0) + (2,5)i] \times [(9,0) + (1,2)i]x^2 + [(7,5) + (2,1)i]x [(1,2) + (7,8)i] + [(7,5) + (2,1)i]x [(9,0) + (1,2)i]x^2 + [(5,1) + (4,3)i]x^3 \times (1,2) + (7,8)i + ((5,1) + (4,3)i]x^3 \times [(9,0) + (1,2)i]x^2$

$= (3,0) + (2,10)i + (21,0)i + (14,40) (–1) + \{(3,0) (9,0) + (2,5)i (9,0) + (3,0) (1,2)i + (2,5) (1,2)i^2] x^2 + \{(7,5) (1,2)x + (2,1) (1,2)i x + (7,5) (7,8)i x + (2,1) (7,8)i^2 x\} + \{(7,5) (9,0) + (2,1) (9,0)i + (7,5) (1,2)i + (2,1) (1,2) (–1)]x^2 + [(5,1) (1,2) + (4,3) (1,2)i + (5,1) (7,8)i + (4,3) (7,8)i^2] x^3 + \{(5,1) (9,0) + (4,3) (9,0)i + (5,1) (1,2)i + (4,3) (1,2)–1] x^2$

$= (–11, –40) + (23, 10)i+ \{(25, –10) + (21,0)i\} x^2 + \{(–7, 2) + (51, 42)i\} x + \{(63,0) + (18,0)i + (7,10)i + (–2–2)]x^2 + \{(–23–22) + (39,14)i\} x^3 + \{(41–6) + (41,2)i\} x^2$

$= (–11,–40) + (23,10)i + \{(–7,2) + (51,42)i\} x \{(127,–18) + (87,12)i\} x^2 + \{(–23, –2) + (39,14)i\} x^3.$

Now having seen how the operations are performed one can work with polynomial with complex interval coefficients.

Now we can replace $I_o(C)$ by $I_c(C)$ or $I_{oc}(C)$ or $I_{co}(C)$ we can for these rings study the properties. Clearly if $C = \{a+bi \mid a, b \in R \text{ or } Q\}$ then $I_o(C)$ is a Smarandache ring. If $C = \{a+bi \mid a, b \in Z\}$ then $I_o(C)$ is not a Smarandache ring.

Clearly I = {(a, 0) + (b,0)i | a, b ∈ R} ⊆ $I_o(C)$ is an ideal of $I_o(C)$. Also P = {(0,a) + (0,b)i | a, b ∈ R} ⊆ $I_o(C)$ is again an ideal of $I_o(C)$.

Thus these rings has ideals I and P and the ideals I.P = {0} and I ∩ P = {0}. Further I + P = $I_o(C)$. Such ideals happen to have very interesting substructures.

Similar study on polynomial rings with complex interval coefficients can be made and they have zero divisors and ideals.

If $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (a_{ij}^1, a_{ij}^2) + (b_{ij}^1, b_{ij}^2)i \in I_o(C) \right\}$

be the complex interval coefficient ring.

Consider

$$P = \left\{ \sum_{i=0}^{\infty} a_j x^j \,\middle|\, a_j = (a_1^j, 0) + (a_2^j, 0)i \in I_o(C) \right\} \subseteq V$$

is an ideal of V.

Likewise

$$S = \left\{ \sum_{i=0}^{\infty} a_k x^k \,\middle|\, a_k = (0, a_1^k) + (0, a_2^k)i \in I_o(C) \right\}$$

is an ideal of V. Further S.P = {0}. Every element p in P is such that for every s in S, p.s = (0). Thus these ideals happen to have interesting structure in these rings.

Clearly $I_o(C)$ can be replaced by $I_c(C)$ or $I_{oc}(C)$ or $I_{co}(C)$.

Now we proceed to define matrices with complex interval entries.

Consider A = {m × n matrices with entries from $I_o(C)$}. A is a group under addition. If m ≠ n or 1 ≠ m then A is not compatible under usual product.

We will illustrate this situation by some examples.

Let

M = ((0,3) + (2,5)i, (7,1) + (4,2)i, (6,1) + (1,2)i, (10,8) + (–3, 2)i) and

N = ((7,2) + (0,2)i, (8,4) + (3,3)i, (2,0) + (3,0)i, (2,7) + (–3,1)i) be the {set of all 1 × 4 matrices with entries from $I_o(C)$} = A.

MN = ((0,3) + (2,5)i × (7,2) + (0,2)i, (7,1) + (4,2)i × (8,4) + (3,3)i, (6,1) + (1,2)i × (2,0) + (3,0)i, (10,8) + (–3, 2)i ×(2,7) + (–3,1)i)

= ((0, 6) + (14,10)i + (0,6)i + (0,10) (–1), (56,4) + (32,8)I + (21,3)i + (12,6) (–1), (12,0) + (2, 0)i + (18,0)i + (3,0) (–1), (20,56) + (–6,14)i + (–30, 8)i + (9,2) (–1))

= ((0,–4) (14,16)i, (44,–2) + (53,11)i, (9,0) + (20,0)i, (11,54) + (–36,22)i).

Thus we see MN ∈ A

M+N = ((0,3) + (2,5)i, (7,1) + (4,2)i, (6,1) + (1,2)i, (10,8) + (–3,2)i) + ((7,2) + (0,2)i, (8,4) + (3,3)i, (2,0) + (3,0)i, (2,7) + (–3,1)i); is in A.

= ((7,5) + (2,7)i, (15,5) + (7,5)i, (8,1) + (4,2)i, (12,15) + (–6,3)i).

Thus A is a ring.

Likewise we can find product of m × m matrices with complex open interval entries.

We can also have $m \times n$ matrices with entries from complex closed intervals / complex open closed intervals or closed open intervals.

In all these cases they are abelian groups under addition and if these structures are compatible with respect to product they are rings which may be commutative or non commutative.

Now on similar lines we can define polynomials with interval neutrosophic coefficients and matrices with interval neutrosophic entries.

We define and illustrate this situation by some simple examples.

Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i = (a_{ij}^1, a_{ij}^2) + (b_{ij}^1, b_{ij}^2)I \in I_o(\langle R \cup I \rangle)\}$$

be a interval neutrosophic coefficient polynomial in the variable x. P is a commutative ring with zero divisors and ideals.

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i = (a_i^1, 0) + (a_i^2, 0)I \in I_o(\langle R \cup I \rangle);$$

$$a_{ij}^1, a_{ij}^2 \in R\} \subseteq P$$

is the ideal of P. Consider

$$N = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_{ki} = (0, a_i^1) + (0, a_i^2)I \in I_o(\langle R \cup I \rangle)$$

$$a_{ij}^1, a_{ij}^2 \in R\} \subseteq P$$

is again an ideal of P.

M and N are ideals of P such that $M \cap N = (0)$ and $MN = (0)$.

Thus every element x in M is such that x.y = (0) for every y in N.

So P has infinite number of zero divisors.

Let

$p(x) = (5,0) + (3,2)I + ((7,1) + (5,-1)I)x + ((3,11) + (2,0)I)x^3$

and

$q(x) = (3,1)\ (1,1)I + ((5,2) + (3,7)I)\ x + ((1,2) + (3,0)I)\ x^2$ be in P.

To find p(x) + q(x).

$p(x) + q(x) = (8,1) + (4,3)I + ((12,3) + (8,6)I)\ x + ((1,2) + (3,0)I)x^2 + ((3,11) + (2,0)I)\ x^3$ is in P.

Consider $p(x) \times q(x) = ((5,0) + (3,2)I) \times ((3,1)+(1,1)I) + ((7,1) + (5,-1)I) \times (3,1) + (1,1)I)x + ((3,11) + (2,0)I) \times ((3,1) + (1,1)I\ x^3 + ((5,0) + (3,2)I) \times ((5,2) + (3,7)Ix + ((5,0) + (3,2)I) \times ((1,2) + (3,0)I)\ x^2 + ((7,1) + (5,-1)I)x \times ((5,2) + (3,7)I)x + ((3,1) + (2,0)I)x^3 \times ((5,2) + (3,7)I)x + (7,1) + (5,-1)I)x \times ((1,2) + (3,0)I)\ x^2 + ((3,11) + (2,0)I)\ x^3 \times ((1,2) + (3,0)I)\ x^2$

$= [(15,0) + (9,2)I + (5,0)I + (3,2)I] + [(21,1) + (15,-1)I\ (7,1)I + (5,-1)I]x + [(9,11) + 6,0)I + (3,11)I + (2,0)I]x^3 + [(25,0) + (15,4)I + (15,0)I + (9,14)I]x + ((5,0) + (3,4)I + (15,0)I + (9,0)I]x^2\ ((35,2) + (25,-2)I + (21,7)I + (15,-7)I]x^2 + [(15,2) + (10,0)I + (9,7)I + (6,0)I]x^4 + [(7,2) + (5,-2)I + (21,0)I + (15,0)I\ x^3 + [(3,22) + (2,0)I + (9,0)I + (6,0)I]x^5$

$= (15,0) + (17,4)I + [(21,1) + (27,-1)Ix + [(9,11) + (11,11)I]x^3 + [(25,0) + (39,18)I]x + [5,0) + (27,4)I]x^2 + [(35,2) + (61,-2)I]x^2 + [(15,2) + (25,7)I]\ x^4 + ((7,2) + (41,-2)I]x^3 + [[(3,22) + (17,0)I]x^5$

= (15,0) + (17,4)I + [(46,1) + (66,17)I]x + [(40,2) + (88,2)I]x^2 + [(16,13) + (55,9)I]x^3 + [(15,2) + (25,7)I]x^4 + [(3,22) + (17,0)I]x^5 $\in$ P.

Thus P is a ring. $I_o(\langle R \cup I\rangle)$ can be replaced by $I_c(\langle R \cup I\rangle)$ or $I_{co}(\langle R \cup I\rangle)$ or $I_{oc}(\langle R \cup I\rangle)$.

Now we just show how matrices with open interval neutrosophic entires are defined and how operations on them are performed.

Let P = {all m $\times$ n matrices with entries from $I_c(\langle R \cup I\rangle)$}; P is an abelian group under addition. P is not compatible with respect to product. However if m = 1, then P is a commutative ring with zero divisors. If n = m then P is a non commutative ring.

Infact P has ideals and zero divisors. We will just illustrate how product and sum are performed on 2 $\times$ 2 matrices with interval neutrosophic numbers from $I_o(\langle R \cup I\rangle)$.

$$\text{Let } x = \begin{bmatrix} (6,1)+(2,3)I & (1,2)+(3,4)I \\ (7,5)+(11,6)I & (1,2)+(5,-1)I \end{bmatrix}$$

$$\text{and } y = \begin{bmatrix} (8,3)+(5,3)I & (7,3)+(2,4)I \\ (1,7)+(8,4)I & (4,-1)+(3,-2)I \end{bmatrix}$$

be any two 2 $\times$ 2 matrices.

$$x + y = \begin{bmatrix} (14,4)+(7,6)I & (8,5)+(5,8)I \\ (8,12)+(19,10)I & (5,1)+(8,-3)I \end{bmatrix}$$

is again a 2 $\times$ 2 matrix.

Now x × y

$$= \begin{bmatrix} (6,1)+(2,3)\mathrm{I}\times(8,3)+(5,3)\mathrm{I}+(1,2)+(3,4)\mathrm{I}\times(1,7)+(8,4)\mathrm{I} \\ (7,5)+(11,6)\mathrm{I}\times(8,3)+(5,3)\mathrm{I}+(1,2)+(5,-1)\mathrm{I}\times(1,7)+(8,4)\mathrm{I} \end{bmatrix}.$$

$$\begin{matrix} (6,1)+(2,3)\mathrm{I}\times(7,3)+(2,4)\mathrm{I}+(1,2)+(3,4)\mathrm{I}\times(4,-1)+(3,-7)\mathrm{I} \\ (7,4)+(11,6)\mathrm{I}\times(7,3)+(2,4)\mathrm{I}+(1,2)+(5,-1)\mathrm{I}\times(4,-1)+(3,-2)\mathrm{I} \end{matrix}\Bigg]$$

$$\Bigg[\begin{matrix} (48,3)+(16,9)\mathrm{I}+(30,3)\mathrm{I}+(10,9)\mathrm{I}+(1,14)+(3,28)\mathrm{I}+(8,8)\mathrm{I}+(24,16)\mathrm{I} \\ (56,15)+(88,18)\mathrm{I}+(35,15)\mathrm{I}+(55,18)\mathrm{I}+(1,14)+(5,-7)\mathrm{I}+(8,8)\mathrm{I}+(40,-4)\mathrm{I} \end{matrix}$$

$$\begin{matrix} (42,3)+(14,9)\mathrm{I}+(12,4)\mathrm{I}+(4,12)\mathrm{I}+(4,-2)+(12,-4)\mathrm{I}+(3,-4)\mathrm{I}+(9,-8)\mathrm{I} \\ (49,12)+(77,18)\mathrm{I}+(14,16)\mathrm{I}+(22,24)\mathrm{I}+(4,-2)+(20,1)\mathrm{I}+(3,-4)\mathrm{I}+(15,2)\mathrm{I} \end{matrix}\Bigg]$$

$$= \begin{bmatrix} (49,17)+(91,73)\mathrm{I} & (46,1)+(54,9)\mathrm{I} \\ (57,29)+(231,49)\mathrm{I} & (53,10)+(151,57)\mathrm{I} \end{bmatrix}$$

is in the set of all 2 × 2 matrices with interval neutrosophic entries form $I_o(\langle R \cup I\rangle)$.

Now we proceed onto first define the notion of complex neutrosophic intervals.

$N_c\langle C \cup I\rangle = \{[a,b] \mid a = a_1 + a_2 i + a_3 I + a_4 Ii$ and

$b = b_1 + b_2 i + b_3 I + b_4 iI$ where $a_i, b_j \in R$; $1 \leq i, j \leq 4\}$

denotes the collection of all closed intervals with complex neutrosophic integers.

Suppose x = [2+5i + 6I + 3iI, 7–10i + 12I + 14iI]

Then the interval can be rewritten as [2,7] + [5i, –10i] + [6I, 12I] + [3iI, 14iI] now each of these subintervals are comparable. Though we may not be in a position to compare elements in x as a totality.

Now we can define operations on $N_c(\langle C \cup I\rangle)$.

(It is pertinent to mention here that we can define on similar lines algebraic operations on $N_o(\langle C \cup I\rangle)$, $N_{oc}(\langle C\cup I\rangle)$ and $N_{co}(\langle C \cup I\rangle)$.

We can define both addition and product on $N_c(\langle C \cup I\rangle)$ and $N_c(\langle C \cup I\rangle)$ is a commutative ring with zero divisors and units. Just we show how addition is performed on $N_c(\langle C \cup I\rangle)$.

Now we proceed onto describe how operations on them are carried out.

Suppose X = [a + bi + cI + dIi, x + yi + zI + tiI] and

Y = [p + qi + rI + sIi, m + ni + tI + viI] are two neutrosophic complex number intervals then X + Y = [a + p + (b+q)i + (c+r)I + (d+s)Ii, x + m + (y+n)i + (z+t)I + (t+v)Ii].

`

Thus X + Y is again a complex neutrosophic interval.

Now we proceed to find the product of X with Y; X × Y = [(a + bi + cI + dIi) × (p + qi + rI + sIi), (x + yi + zI + tiI) × (m + ni + tI + viI)]

= [(ap + bip + cpI + dpIi + aqi – bq + cqiI – dqI + arI + rbiI + crI + drIi + saIi – sbI + scIi – dsI), (xm + ymi + zmI + tiIm + nxi – ny - nziI – ntI + tIx + tyIi + tzI + t^2iI + vziI – vyI + xvIi– tvI)]

= [(ap – bq) +(bp + aq)i + (cp – dq + ar + cr – sb – ds)I+ (dp + cq + rb + dr + sa + sc)iI, (xm – ny) + (ym + nx)i + (zm – nt + tx + tz – vy – tv)I + (tm + nz + ty + t^2 + vz + xv)Ii] is again an interval in $N_c(\langle C \cup I\rangle)$.

We can build like in case of other intervals in case of $N_c(\langle C \cup I\rangle)$ also polynomial rings with complex neutrosophic interval coefficients and matrices with complex neutrosophic entries. This is considered as a matter of routine and hence is left as an exercise to the reader.

Now we can also define interval complex neutrosophic numbers. Number of the form $[a, b] + [c, d]i + [e, f]I + [g, h]iI$ $a, b, c, d, e, f, g, h \in R$ (or Z or Q) is defined as the interval complex neutrosophic numbers.

We see if $W = \{[a, b] + [c,d]i + [e,f]I + [g,h]iI \mid a, b, c, d, e, f, g, h \in R$ (or Z or Q)$\}$ then W is a commutative ring with identity and W has zero divisors and ideals.

For instance $P = \{[a, 0] + [b,0]i + [c,0]I + [d,0]iI \mid a, b, c, d \in R\} \subseteq W$ is an ideal of W.

Likewise

$M = \{[0, a] + [0, b]i + [0,c]I + [0,d]iI \mid a, b, c, d \in R\} \subseteq W$ is again an ideal. Clearly $M \cap P = \{0\}$ and

$M \times P = \{mp \mid m \in M \text{ and } p \in P\} = \{0\}$. Several interesting properties in this direction can be derived.

It is pertinent to mention here that in W instead of taking closed interval [a,b], one can take the intervals from $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ (R also replaced by Z or Q).

Further we wish to state that now we have got a relation or mapping between interval complex numbers and complex intervals.

For if $x = ([a, b] + [c, d]i]$ then if $y = ([a + ci, b + di]) \in N_c(C)$ then a map $x \mapsto y$ as $[a, b] + [c, d]i = ([a + ci, b + di])$ is a one to one map so the study from one to another is equivalent.

Likewise in case of neutrosophic intervals if

$x = ([a, b] + [c, d]I$ and $y = ([a + cI, b + dI]) \in N_c(\langle R \cup I\rangle)$ then the map $x \mapsto y$ as $[a, b] + [c, d]I = ([a + cI, b + dI])$ is a one to one map and both are equivalent.

Finally if $x = ([a, b] + [c, d]i + [e, f]I + [g, h]iI)$ is a complex neutrosophic interval number than $y = ([a + ci + eI + giI, b + di + fI + hiI]) \in I_c(\langle C \cup I\rangle)$ then a map $x \mapsto y$ mapping $= ([a,b] + [c,d]i + [e,f]I + [g,h]iI) \to ([a + ci + eI + giI, b + di + fI + hiI])$ is again a one to one map so both intervals can be treated as equivalent.

It is pertinent to mention here that we can replace closed intervals by open intervals or open-closed intervals or closed-open intervals and all the results hold good.

Chapter Three

FINITE COMPLEX NUMBERS

Before we start to describe and discuss about complex numbers we now proceed onto recall the history of complex numbers as given by O.Merino, www.math.uri.edu/~merino/ .../ShortHistoryComplexNumbers2006.pdf. The need for the imaginary or complex numbers did not arise in a single situation, when mathematicians tried to find the root of negative numbers they encountered with the problem of what is the root of –1.

Yet another situation is finding a solution to the equation $ax^2 + bx + c = 0$ a, b, c $\in$ R.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; when $b^2 < 4ac$, the term under the square root was negative this also gave the mathematicians a problem and how to encounter it.

At the end of 15^{th} century an Italian mathematicians del Ferro and Nicolo Fontana (known as Tarlaglia) made progress towards solving cubic equations in 1539, Tartaglia communicated his insight to Girolamo Cardano. Thus the first record of complex numbers dates back to 1545.

Rane Descartes in 1637 gives an informal version of the fundamental theorem of algebra and thinks zeros of the polynomials do not always correspond to any real quantity.

Isaac Newton in 1728 interprets complex roots of a polynomial merely as an explicity symptom of solutions which are not possible.

Leonard Euler in 1768 says square roots of negative numbers are impossible numbers.

In 1835 the formal interpretation of a complex number as an ordered pair of real numbers appears for the first time in the work of Sir William Rowan Hamilton. To Hamilton we also owe the introduction to i as the square root of –1.

So we have no real number which solves the equation $x^2 = -1$. But imagine there is a number i with the special property such that $i^2 = -1$. Then the equation above has the solutions $x = \pm i$. This is the one of the germs the complex numbers grew. $i^2 = -1$ where i is an imagined existence number.

$z = x + iy$ was the expression given to a complex number $x, y \in R$.

A complex number i is defined as $\sqrt{-1}$ or equivalently $i^2 = -1$ so that $i = \sqrt{-1}$. Clearly one is very well aware of the fact $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ and so on.

Now we know using the ring of integers Z the notion of modulo integers can be defined. Suppose 2 is the integer; consider the ideal generated by 2.

$\langle 2 \rangle$ = {all positive and negative integers greater than 2 with zero}

Consider $Z / \langle 2 \rangle = \{0 + \langle 2 \rangle, 1 + \langle 2 \rangle\}$.

Now $(0 + \langle 2 \rangle) + (0 + \langle 2 \rangle) = (0 + \langle 2 \rangle)$.
$(1 + \langle 2 \rangle) + (1 + \langle 2 \rangle) = (2 + \langle 2 \rangle) = 0 + \langle 2 \rangle$
as $2 \in \langle 2 \rangle$; $1 + \langle 2 \rangle + 0 + \langle 2 \rangle = 1 + \langle 2 \rangle$.
$(0 + \langle 2 \rangle)\, (0 + \langle 2 \rangle) = 0 + \langle 2 \rangle$,

(using the fact $\langle 2 \rangle$ $(1 + \langle 2 \rangle) \times (1 + \langle 2 \rangle) = 1 + \langle 2 \rangle$

(once again using the fact $\langle 2 \rangle$ is an ideal of Z.

Thus $Z / \langle 2 \rangle = \{0 + \langle 2 \rangle, 1 + \langle 2 \rangle\}$ is isomorphic with the modulo integer 2 that is $Z_2 = \{ \overline{0}, \overline{1} \}$.

Without any confusion we can also omit the bar on 0 and 1, for by the context of study it is clear. Further if $Z_n = \{0, 1, 2, \ldots, n–1\}$ then $–1 = n–1 \pmod n$, $–2 = n–2 \pmod n$ and so on.

We see depending on n the value of –1 also varies. For in Z_5, $–1 = 4 \pmod 5$; $–1 \equiv 7 \pmod 8$ in Z_8 and in Z_{20} $– 1 \equiv 19 \pmod{20}$. Thus we see –1 also varies with the n we choose.

However no mention was made of finite complex number or modulo complex integers. We authors in [4, 6] were just thinking and discussing that in the set of modulo integers $Z_n = \{0, 1, 2, \ldots, n–1\}$ and $1 + n–1 = 0 \pmod n$ similar to $1 + (–1) = 0$ in reals, so we thought we can define a finite complex number $i_F^2 = n – 1$. Thus we are developing the theory of finite complex numbers [4, 6]. However, as Z_n is not representable in a plane the same problem will also be encountered by complex modulo integers, so based on the fact $n–1 = –1 \pmod n$ the authors define $(n–1) = i_F^2$, where i_F depends on the n which we choose.

Now we define finite modulo complex integer as follows:

Let Z_n be the ring of integer modulo n. The finite complex modulo integers,

$C(Z_n) = \{a + bi_F \mid a, b \in Z_n;\ i_F^2 = n-1\}$. We can define addition and product on $C(Z_n)$ in the following way.

$$(a+bi_F) + (c+di_F) = (a+c) \ (\text{mod } n) + (b+d)i_F \ (\text{mod } n).$$

$$\begin{aligned}(a + bi_F) \times (c+di_F) &= ac + bci_F + adi_F + bd\, i_F^2 \\ &= ac + (bc + ad)i_F + bd\,(n-1) \\ &\qquad\qquad (\because i_F^2 = n-1) \\ &= [ac + bd(n-1)] \ (\text{mod } n) + (bc + ad)\, i_F \ (\text{mod } n).\end{aligned}$$

We will illustrate this situation with elements from Z_{15}.

Consider $C(Z_{15}) = \{a + bi_F \mid a, b \in Z_{15},\ i_F^2 = 14\}$. Let $5 + 9i_F$ and $12 + 4i_F$ be in $C(Z_5)$.

$$(5 + 9i_F) + (12 + 4i_F) = 17 \ (\text{mod } n) + (13i_F) = 2+13i_F.$$

Now $(5 + 9i_F) \times (12 + 4i_F)$

$$\begin{aligned}&= (60 + 108i_F + 20i_F + 36\, i_F^2)\ (\text{mod } 15) \\ &= 0 + 3i_F + 5i_F + 6 \times 14 \ (\text{mod } 15) \\ &= 8i_F + 9.\end{aligned}$$

Thus $C(Z_n)$ is a commutative ring of finite order. For more about complex modulo integers please refer (WBV finite comp].

Now $C = \{a + ib \mid a, b \in R\}$ is a field but here we may have $C(Z_n)$ to be a ring or a field. $C(Z_n)$ need not be a field even if n is a prime.

Inview of this we illustrate this situation by some simple examples.

$C(Z_2) = \{0, 1, i_F, 1+i_F\}$;

Clearly $C(Z_2)$ is only a ring as $(i_F+1)^2 = i_F^2 + 2i_F + 1$
$= 1 + 2i_F + 1 \ (\text{mod } 2) = 0$.

Consider $C(Z_3) = \{0, 1, 2, i_F, 2i_F, 1+i_F, 2+i_F, 2+2i_F, 2i_F +1\}$.

We write the table of $C(Z_3) \setminus \{0\}$ under product.

$\times$	1	2	i_F	$2i_F$	$1+ i_F$	$2+ i_F$	$2i_F +1$	$2+2i_F$
1	1	2	i_F	$2i_F$	$1+ i_F$	$2+ i_F$	$2i_F +1$	$2+2i_F$
2	2	1	$2i_F$	i_F	$2+2i_F$	$1+2i_F$	$i_F +2$	$1+i_F$
i_F	i_F	$2i_F$	2	1	$i_F +2$	$2i_F +2$	$1+i_F$	$2i_F +1$
$2i_F$	$2i_F$	i_F	1	2	$2i_F +1$	$i_F +1$	$2+2i_F$	$i_F +2$
$1+i_F$	$1+i_F$	$2+2i_F$	$i_F +2$	$2i_F +1$	$2i_F$	1	2	$i_F +2$
$2+i_F$	$2+i_F$	$1+2i_F$	$2+2i_F$	$i_F +1$	1	i_F	$2i_F$	2
$2i_F +1$	$2i_F +1$	$i_F +2$	$1+i_F$	$2+2i_F$	2	$2i_F$	i_F	1
$2+2i_F$	$2+2i_F$	$1+i_F$	$2i_F+1$	$i_F +2$	i_F	2	1	$2i_F$

Clearly $C(Z_3)$ is a field of characteristic three. However $C(Z_3)$ is not a prime field for $Z_3 \underset{\neq}{\subset} C(Z_3)$.

Consider $C(Z_4)$; $C(Z_4)$ is only a ring.

$C(Z_n)$; n not a prime is only a ring.

Now consider $C(Z_5) = \{a + bi_F \mid a, b \in Z_5,\ i_F^2 = 4\}$; $C(Z_5)$ is a ring. Is it a field? $C(Z_5)$ is not a field only a ring as

$$\begin{aligned}(1+2i_F)\,(2+i_F) &= 2 + 4i_F + i_F + 2.\,i_F^2 \\ &= 2+5i_F + 8 \qquad (i_F^2 = 4) \\ &= 0 \bmod 5.\end{aligned}$$

Thus we can say $C(Z_p)$ will be a field if and only if $Z_p \setminus \{0\}$ does not contain a, b such that $a^2 + b^2 \equiv 0 \pmod p$.

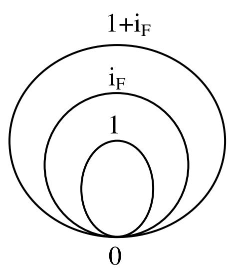
We see $C(Z_7)$ is a field.
$C(Z_{13})$ is not a field only a ring as $3^2 + 2^2 = 13 \pmod{13}$.

$C(Z_{17})$ is not a field as $17 \equiv 4^2 + 1$ and $C(Z_{189})$ is a field.

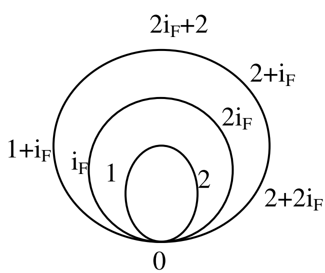
Consider $C(Z_{23})$ is a field. $C(Z_{29})$ is not a field as $29 = 5^2 + 2^2$ and so on.

We have given algebraic structures to them and worked with them, refer [,]. We also just recall the graphical representation of them. It has 3 layers, modulo integers which is the inner most layer and the middle layer consists of the only complex numbers and the outer layer consists of the complex modulo integers of the form $a+bi_F$. $a \neq 0$ and $b \neq 0$, $a, b \in Z_n$.

The diagram for $C(Z_2)$ is as follows:



The diagram for $C(Z_3)$ is as follows;



and so on.

Clearly $C(Z_n)$ contains Z_n. Also several interesting results about $C(Z_n)$ are derived in [4, 6].

We just give the statements of them.

THEOREM 3.1: *Let $C(Z_{2p})$ be the finite complex ring, p a prime, $p>2$. $C(Z_{2p})$ is only a ring and $(1+i_F)^2 = 2i_F$.*

THEOREM 3.2: *Let $C(Z_n)$ be the ring $Z_n \subseteq C(Z_n)$ is a subring of $C(Z_n)$ and not an ideal of $C(Z_n)$.*

THEOREM 3.3: *Let $C(Z_{2p})$ be a ring. $P = \{0, p+pi_F\} \subseteq C(Z_{2p})$ is an ideal of $C(Z_{2p})$, p a prime.*

THEOREM 3.4: *Let $C(Z_p)$, p a prime be a complex modulo integer ring.*

Then (i) $(a+ai_F)^2 = bi_F$ $b, a \in Z_p$ and

$$(ii) \left(\frac{p+1}{2}+\frac{p+1}{2}i_F\right)^2 = \left(\frac{p+1}{2}\right)i_F = \frac{(p+1)i_F}{2}.$$

THEOREM 3.5: *Let $S = C(Z_n)$ be a complex ring of modulo integers. For $x = a + (n-a)i_F \in C(Z_n)$ we have $x^2 = bi_F$ for some $b \in Z_n$ ($i_F^2 = n-1$).*

THEOREM 3.6: *Let $R = C(Z_n)$ be the ring of complex modulo integers (n not a prime). Every $x = a+bi_F$ in which $a.b = 0 \pmod n$ gives x^2 to be a real value.*

THEOREM 3.7: *$C(Z_n)$ has ideals and subrings if n is not a prime and if n is a prime then $C(Z_n)$ has ideals and subrings only if $n = a^2 + b^2$ ($a, b \in Z_n \setminus \{0\}$).*

Now we can construct algebraic structures using $C(Z_n)$.

First we define complex modulo integer matrices.

Let A = {all m × n matrices with entries from $C(Z_n)$}.

THEOREM 3.8: *A is an abelian group under addition.*

THEOREM 3.9: *If m = 1 in A; A is a semigroup under product ×.*

We will just illustrate this situation before we proceed onto define more algebraic structures on A.

$$\text{Let } x = \begin{bmatrix} 3+2i_F & 7+3i_F & 1+5i_F \\ 8+i_F & 1+4i_F & 2+i_F \end{bmatrix}$$

$$\text{and } y = \begin{bmatrix} 1+i_F & 8 & 7+3i_F \\ 3i_F & 4+2i_F & 8i_F \end{bmatrix}$$

where x and y take its entries from $C(Z_9)$.

$$\text{Now } x+y = \begin{bmatrix} 4+3i_F & 6+3i_F & 8+8i_F \\ 8+4i_F & 5+6i_F & 2 \end{bmatrix}.$$

This way addition is performed.

Infact product cannot be defined for x with y.

Let x = $(10+3i_F, 4+2i_F, 12+10i_F)$ and y = $(3+ i_F, 4i_F, 3)$

be two row vectors with entries from $C(Z_{13})$.

Now x + y = $(13+4i_F, 4+6i_F, 15+10i_F) = (4i_F, 4+6i_F, 2+10i_F)$

and x × y = $((10+3i_F) \times 3+ i_F, (4+2i_F) \times 4i_F, 12+10i_F \times 3)$

= $(30+9i_F +10i_F +3i_F^2, 16i_F + 8i_F^2, 36+30i_F)$

(mod 13 and $i_F^2 = 12$);

thus $x \times y = (1+6i_F, 3i_F +5, 10+4i_F)$.

Further $x \times y$ is defined hence we can have the following result the proof of which is left to the reader [4, 6].

THEOREM 3.10: *Let*

A = {collection of all $1 \times m$ row vectors with entries from $C(Z_n)$}. A is a commutative ring with unit. A has zero divisors. A is a Smarandache ring.

Next we proceed onto study the structure of column matrices. Let

P = {collection of all $m \times 1$ column matrices with entries from $C(Z_n)$}; P is only an additive abelian group.

We will illustrate this by an example.

Suppose $C(Z_{10}) = \{a+bi_F \mid a, b \in Z_{10},\ i_F^2 = 9\}$ be the complex modulo integers.

$$x = \begin{bmatrix} 3+2i_F \\ 9+i_F \\ 8 \\ 3i_F \\ 1+i_F \\ 7+9i_F \end{bmatrix} \text{ and } y = \begin{bmatrix} 7i_F \\ 8+6i_F \\ 9+i_F \\ 9i_F+1 \\ 8i_F \\ 6 \end{bmatrix}$$

be any two 6×1 column vectors.

$$\text{We define } x + y = \begin{bmatrix} 3+9i_F \pmod{10} \\ 17+7i_F \pmod{10} \\ 17+i_F \pmod{10} \\ 12i_F+1 \pmod{10} \\ 1+9i_F \pmod{10} \\ 13+9i_F \pmod{10} \end{bmatrix} = \begin{bmatrix} 3+9i_F \\ 7+7i_F \\ 7+i_F \\ 2i_F+1 \\ 1+9i_F \\ 3+9i_F \end{bmatrix}.$$

We see the set of all column vectors with entries from $C(Z_{10})$ is an additive abelian group. However usual product cannot be defined on these column vectors.

Consider now the collection of all n × n matrices with entries from $C(Z_m)$; suppose

P = {all n × n matrices with entries from $C(Z_m)$; $i_F^2 = m-1$}. P is a non commutative ring with zero divisors and units.

We will illustrate how sum and product are made.

Consider P = {all 3 × 3 matrices with entries from $C(Z_6)$; $i_F^2 = 5$}.

$$\text{Let } x = \begin{pmatrix} 3+i_F & 0 & 2i_F \\ 4 & 2+i_F & 1+4i_F \\ i_F & 2 & 0 \end{pmatrix} \text{ and}$$

$$y = \begin{pmatrix} i_F & 2i_F & 0 \\ 0 & 2 & 4 \\ 1+i_F & 2+i_F & 4i_F+1 \end{pmatrix}$$

be two elements from P. We find x + y ;

$$x+y = \begin{pmatrix} 3+2i_F & 2i_F & 2i_F \\ 4 & 4+i_F & 4i_F \\ 2i+1 & 4+i_F & 4i_F+1 \end{pmatrix} \in P.$$

Now we find product x × y =

$$\begin{pmatrix} 3+i_F & 0 & 2i_F \\ 4 & 2+i_F & 1+4i_F \\ i_F & 2 & 0 \end{pmatrix} \begin{pmatrix} i_F & 2i_F & 0 \\ 0 & 2 & 4 \\ 1+i_F & 2+i_F & 4i_F+1 \end{pmatrix}$$

$$= \begin{pmatrix} (3+i_F)i_F + 0 + 2i_F(1+i_F) & 2i_F(3+i_F) + 0 + 2i_F(2+i_F) & 0+0+2i_F(4i_F+1) \\ 4i_F + 0 + (1+i_F)(1+4i_F) & 4 \times 2i_F + 2(2+i_F) + (1+4i_F)(2+i_F) & 0+4(2+i_F)+(1+4i_F)(1+4i_F) \\ i_F^2 + 0 + 0 & 2i_F^2 + 4 + 0 & 0+8+0 \end{pmatrix}$$

$$= \begin{pmatrix} 5i_F+3 & 2+4i_F & 2i_F+4 \\ 3i_F+3 & 2+i_F & 4 \\ 5 & 2 & 3 \end{pmatrix},$$

using the fact $i_F^2 = 5$ and it is modulo 6 addition / multiplication. Now one can use these complex modulo integer matrices with entries from $C(Z_n)$ for any arbitrary integer n.

We see for any square complex modulo integer matrix only

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$ acts as the multiplicative identity.

Clearly the $n \times n$ zero matrix

$$(0) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$ acts as the additive identity.

The $n \times n$ matrix contains zero divisors.

These collection of matrices with complex modulo integers can be used to define vector spaces over $C(Z_p)$ where $p \neq a^2 + b^2$ for any $a, b \in Z_p$ (p a prime).

We have proved the following theorem [4, 6].

THEOREM 3.11: *Let*

$$V = \left\{ \left. \begin{bmatrix} a & a & \dots & a \\ a & a & \dots & a \\ \vdots & \vdots & & \vdots \\ a & a & \dots & a \end{bmatrix} \right| a \in C(Z_p);\ p \text{ a prime} \right\}$$

be the set of all $n \times n$ matrices of complex modulo integer linear algebra over Z_p.

(i) V has no complex modulo integer linear subalgebra with entries of the matrix $x = a+bi_F$, $a \neq 0$, $b \neq 0$; $a, b \in Z_p$.

$$\text{(ii) } M = \left\{ \left. \begin{bmatrix} x & x & \dots & x \\ x & x & \dots & x \\ \vdots & \vdots & & \vdots \\ x & x & \dots & x \end{bmatrix} \right| x \in Z_p \right\} \subseteq V$$

is a pseudo complex modulo integer linear subalgebra of V over Z_p.

Likewise we can define polynomials with coefficients from $C(Z_n)$.

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C(Z_n) \right.$$

$$\text{where } a_i = a_i^1 + b_i^1 i_F;\ i_F^2 = n-1,\ a_i^1,\ b_i^1 \in Z_n\}$$

denotes the collection of all polynomials with coefficients from $C(Z_n)$.

V is a commutative ring with unit. V has zero divisors if Z_n has zero divisors.

We now show if

$x^2 = (n-1)$ then the root of $x^2 = (n - 1)$ is $x = i_F$; since $i_F^2 = x^2 = (n-1)$ where $x^2 = n-1 \in C(Z_n)$.

Hence to solve equations in $C(Z_n)[x]$ and for finding the solution of $x^2 = (n-1)$ we have the root TO BE i_F.

Further it is pertinent to mention here that we cannot use the quadratic equations.

For the quadratic equation $ax^2 + bx + c = 0$ has

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where a, b, c} \in \text{R (reals only).}$$

So we have to use only the method of substitution of all values in Z_n in the equation $f(x) = ax^2 + bx + c$. For instance if

$f(x) = 2x^2 + 3x + 1 \in Z_5[x]$ then $x \neq 0$; $x \neq 1$, $x \neq 2$, $x \neq 3$ and $x \neq 4$ so f(x) has no solution in $Z_5[x]$.

Further in this book we are not using the concept of primitive roots of an irreducible polynomial $p(x) \in Z_p[x]$ $(C(Z_p)[x])$, p only a prime.

Suppose $f(x) \in C(Z_5)[x]$ then

$f(3+3i_F) = 2\,(3+3i_F)^2 + 3(3+3i_F) + 1$

$$= 2\,[9+9\times4+18i_F] + 9+9i_F +1$$
$$= 36i_F+9i_F+10 = 0 \pmod 5).$$

So $3+3i_F \in C(Z_5)$ is a root of f(x); several problems arise in this situation. Can one say every 2^{nd} degree equation (or polynomial of second degree) in $C(Z_n)[x]$ has two roots?

Or equivalently can one say a polynomial $p(x) \in C(Z_m)[x]$ of degree n can have n and only n roots?

This question remains a open problem for the authors are not in a position to solve this. Further can every polynomial $p(x) \in C(Z_n)[x]$ has atleast one root in $C(Z_m)$; this also remains open atleast till one constructs a polynomial which has no solution.

Consider $p(x) = x^2 + 2x + 2$, clearly $2+i_F$ is a root, for

$$\begin{aligned} p(2+i_F) &= (2|+i_F)^2 + 2(2+i_F) + 2 \\ &= 4 + 2 + i_F + 4 + 2i_F + 2 \\ &= 0. \end{aligned}$$

$2+2i_F$ is a root of p(x).

$$\begin{aligned} p(2+2i_F) &= (2+2i_F)^2 + 2(2+2i_F) + 2 \\ &= 4+ 4 \times 2 + 8i_F + 4 + 4i_F + 2 \\ &= 0 \text{ so } 2+2i_F \text{ is also a root of } p(x). \end{aligned}$$

However we wish to see whether the rule if α and β are the roots of a equation $x^2 + bx + c = p(x)$ then $b = -(\alpha + \beta)$ and $c = \alpha\beta$.

Let $\alpha = 2 + 2i_F$ and $\beta = 2 + i_F$. Now the equation

$$\begin{aligned} (x-\alpha)(x-\beta) &= (x + 1 + i_F)(x+1+2i_F) \\ &= x^2 + x + xi_F + 1\,x+1 + i_F + 2i_Fx + 2i_F + 4 \\ &= x^2 + 2x +2. \end{aligned}$$

$$\alpha + \beta = (2+2i_F) + (2+i_F) = 1$$

$$-(\alpha + \beta) = 2.$$

$$\alpha\beta = (2+2i_F)\ (2+i_F) = 2.$$

Since $-2 \equiv 2 \pmod 4$ we have the solution and equation behave in this case as that of reals. Let $2+3i_F$, $4+2i_F \in C(Z_6)$. Consider the quadratic equation $(x-(2+3i_F))\ (x-(4+2i_F)) = 0$

i.e., $(x+4+3i_F)\ (x+2+4i_F) = 0$.

$x^2 + 4x + 3i_Fx + 2x + 8 + 6i_F + 4i_Fx + 16i_F + 12 \times 5 = 0$; that is

$x^2 + i_Fx + 4i_F + 2 = 0$.

$(\alpha + \beta) = (2+3i_F)\ (4+2i_F) = 5i_F$.

$$\begin{aligned} -(\alpha + \beta) = +\, i_F \quad \text{and} \quad \alpha\beta &= (4+3i_F)\ (2+4i_F) \\ &= 8+6i_F + 16i_F + 12\, i_F^2 \\ &= 2+4i_F + 12 \times 5. \\ \alpha\beta &= 2+4i_F. \end{aligned}$$

Thus α and β are the roots of the equation $x^2 + i_Fx + 4i_F + 2 = 0$.

Let $\alpha = 3+7i_F$ and $\beta = 8+i_F \in C(Z_9)$ to find the quadratic equation satisfied by the roots α and β.

$$\begin{aligned} &(x-\alpha)\ (x-\beta) = (x + 6+2i_F)\ (x+1+8i_F) \\ &= x^2 + 6x + 2xi_F + x + 6 + 2i_F + 8i_Fx + 48i_F + 16 \times 8\ (i_F^2 = 8) \\ &= x^2 + 7x + 5i_F + i_Fx + 8 \\ &= x^2 + x\,(7+i_F) + 2 + 5i_F. \end{aligned}$$

$$\begin{aligned} -(\alpha + \beta) &= -(3+7i_F + 8 + i_F) \\ &= 7 + i_F. \end{aligned}$$

$$\begin{aligned} \alpha\beta &= (3+7i_F)\ (8+i_F) \\ &= 24 + 7 \times 8 + 56i_F + 3i_F \\ &= 8 + 5i_F. \end{aligned}$$

Thus the rule of sum of the roots is valid in this case also. Now we see if the equation in $C(Z_n)[x]$ of degree three how to tackle it. Let us consider $C(Z_7)$ and suppose $\alpha = 3+4i_F$, $\beta = 2+i_F$ and $\gamma = 5+2i_F \in C(Z_7)$.

Consider $(x-(3+4i_F)) \times (x-(2+i_F)) \times (x-5+2i_F)) = (x+4+3i_F) \times (x+5+6i_F) \times (x+2+5i_F)$

$= (x^2 + 4x + 3i_Fx + 5x + 20 + 15i_F + 6i_Fx + 24i_F + 18 \times 6) \times (x + 2+5i_F)$

$= (x^2 + 2x + 2i_Fx + 4i_F + 2) \times (x + 2 + 5i_F)$

$= x^3 + 2x^2 + 2x^2i_F + 4xi_F + 2x + 2x^2 + 4x + 4i_Fx + 8i_F + 4 + 5i_Fx^2 + 10xi_F + 10x \times 6 + 20 \times 6 + 10i_F$

$= x^3 + 4x^2 + x\,(3 + 4i_F) + 5 + 4i_F$.

$\alpha + \beta + \gamma = 3+4i_F + 2 + i_F + 5 + 2i_F = 3$.

$-(\alpha + \beta + \gamma) = -3 = 4 \pmod 7$.

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (3+4i_F)(2+i_F) + (2+i_F)(5+2i_F) + (3+4i_F)(5+2i_F)$

$= 6 + 8i_F + 3i_F + 4 \times 6 + 10 + 5i_F + 4i_F + 2 \times 6 + 15 + 20i_F + 6i_F + 8 \times 6 = 3 + 4i_F$.

$$\begin{aligned} \alpha\beta\gamma &= (3 + 4i_F)(2 + i_F)(5 + 2i_F) \\ &= (6 + 8i_F + 3i_F + 4 \times 6)(5 + 2i_F) \\ &= (2 + 4i_F)(5+2i_F) \\ &= 10 + 20i_F + 8 \times 6 + 4i_F \\ &= 3i_F + 2. \end{aligned}$$

$-\alpha\beta\gamma = -(3i_F + 2) = 4i_F + 5$.

Thus we see if α, β, γ are roots of a cubic equation

$x^3 + ax^2 + cx + d = 0$ in $C(Z_n)[x]$ then

$$a = -(\alpha + \beta + \gamma) \quad c = \alpha\beta + \beta\gamma + \gamma\alpha \quad \text{and } d = -\alpha\beta\gamma.$$

However we are not in a position to talk about "conjugates" of a root or any other properties enjoyed by polynomials with coefficients from $C = \{a + bi \mid a, b \in R, i^2 = -1\}$.

As in case of usual polynomials even in case of polynomials in $C(Z_n)[x]$ it is not an easy task to solve. However one of the advantages is that there can be only n^2 roots and all the n^2 values can be substituted in any polynomial to get the roots.

With the advent of superfast computers it is easy to solve these equations. However the authors wish to keep on record that interested researchers / computer scientists can give a programme for solving equations of polynomials p(x) of nth degree in $C(Z_m)[x]$. This will simplify the problem of finding roots of the complex modulo integer coefficient polynomials. We do not discuss about the case when the roots are not in $C(Z_n)$.

We can differentiate and integrate these polynomials also. Further we cannot define a continuous function or any notion of usual intervals but we use only the basic concept of derivative of every element in $C(Z_n)$ is zero. With this in mind we can differentiate and integrate in a special way all polynomials in $C(Z_n)[x]$.

We will just illustrate this by some simple examples.

Suppose $p(x) = 3x^4 + (3+5i_F)x^2 + (7+3i_F)x + 9 \in C(Z_{10})[x]$.

Now we find

$$\frac{dp(x)}{dx} = 4.3x^3 + 2(8+5i_F)\,x + (7+3i_F)$$

$$= 2x^3 + (6x) + 7 + 3i_F.$$

$$\frac{d^2p(x)}{dx^2} = 6x^2 + 6 \text{ and so on.}$$

However while integrating we face with several problems.

(1) If $C(Z_p)$ is a field we can integrate all polynomial of all degree except of degree p.

(2) Further we assume if $a \in C(Z_p)$; $\frac{1}{a} = a^{-1}$ only with this in mind we can integrate. However if we integrate in $C(Z_n)$ where $C(Z_n)$ is not a field we encounter several problems for take $C(Z_{25})$. We see $5 \in C(Z_{25})$, we do not know what 1/5 means to us. Also we are yet to study and analyse whether the integration is the reverse process of differentiation in $C(Z_n)$ [x]; n not a prime.

As this study is at a very dormant state we are yet to find proper methods of doing these operations. We will however describe these situations by some examples.

Let $C(Z_7)[x]$ be a polynomial ring with coefficients from $C(Z_7)$.

Let $(3 + 4i_F)\, x^3 + 3x^2 + 5i_Fx + 3 = p(x)$ be in $C(Z_7)$ [x].

$$\int p(x)\, dx = \frac{(3+4i_F)x^4}{3+1} + \frac{3x^3}{2+1} + \frac{5i_Fx^2}{1+1} + \frac{3x}{0+1} + C$$

$$= 2(3+ 4i_F) + 5.3\, x^3 + 4 \times 5i_Fx^2 + 3x + C$$

$$\text{Now } \frac{d}{dx}\int p(x)\, dx = \frac{4(3+4i_F)x^3}{4} + 3x^2 + 5i_Fx + 3$$

$$= (6+i_F)x^4 + x^3 + 6i_Fx^2 + 3x + C$$

$$\frac{d}{dx}\int p(x)\, dx = 4\,(6+i_F)x^3 + 3x^2 + 12i_F\, x + 3$$

$$= (3+4i_F)x^3 + 3x^2 + 5i_Fx + 3.$$

Simplify using inverse of 4. Thus we see as long as we use the polynomial ring with complex coefficients from $C(Z_p)$ where $C(Z_p)$ is a field all results will hold good for polynomials of degree different from p–1.

Consider $p(x) = (7+8i_F)\, x^5 + (4+2i_F)x^4 + (8+5i_F)x^3 + 3i_Fx^2 + 5x + 8 + 9i_F \in C(Z_{11})[x]$

We find $\dfrac{dp(x)}{dx} = 5\,(7+8i_F)x^4 + 4(4+2i_F)x^3 + 3(8+5i_F)x^2 + 6i_Fx + 5$

$$= (2+7i_F)x^4 + (5+8i_F)x^3 + (2+4i_F)x^2 + 6i_Fx + 5.$$

Now $\int p(x)\,dx = \dfrac{(7+8i_F)x^6}{6} + \dfrac{(4+2i_F)x^5}{5} + \dfrac{(8+5i_F)x^4}{4} +$

$$\frac{3i_Fx^3}{3} + \frac{5x^2}{2} + (8+9i_F)x + C$$

$$= \frac{2(7+8i_F)x^6}{2\times 6} + \frac{9(4+2i_F)x^5}{9\times 5} + \frac{3(8+5i_F)x^4}{4\times 3} +$$

$$\frac{4.3i_Fx^3}{3\times 4} + \frac{6\times 5x^2}{2\times 6} + (8+9i_F)x + C$$

$$= (3+5i_F)x^6 + (3+7i_F)x^5 + (2+4i_F)x^4 + i_Fx^3 + 8x^2 + (8+9i_F)x + C \in C(Z_{11})\,[x].$$

If in $C(Z_n)[x]$ where $C(Z_n)$ is only a ring we cannot integrate. Consider $p(x) = 3x^3 + 7x^2 + (4+3i_F) + 8 \in C(Z_{12})$. We now try to integrate p(x);

$$\int p(x)\,dx = \frac{3x^4}{4} + \frac{7x^3}{3} + \frac{(4+3i_F)x^2}{2} + 8x + C$$

Now clearly $\int$ p(x) dx has no meaning as $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ has no place in $C(Z_{12})$; now we cannot write $\frac{1}{4}$ as an element in Z_{12} as 4 is a zero divisor; similarly 3 and 2 are zero divisors, so $\int$ p(x)dx cannot be defined. This is the justification for we cannot find in general integral of a polynomial in $C(Z_n)[x]$ if $C(Z_n)$ is a ring.

However we can find the derivative of p(x),

$$\frac{dp(x)}{dx} = 9x^2 + 14x + (4+3i_F) = 9x^2 + 2x^2 + (4+3i_F).$$

We can always find the derivative of p(x), however we see the important result. If $p(x) \in C(Z_n)[x]$ and $C(Z_n)$ only a ring with p(x) is of degree n then $\frac{dp(x)}{dx}$ will be of degree less than n and we cannot always say that $\frac{dp(x)}{dx}$ will be of degree n–1. Consider

$$p(x) = 3x^4 + (6+6i_F)x^2 + 7x + (5+3i_F)x^3 + 8 \in C(Z_{12})[x].$$

We find $\frac{dp(x)}{dx}$ $= 3.4x^3 + 2.(6+6i_F)x + 7 + 3(5+3i_F)x^2$

$$= 0 + 0 + (3+9i_F)x^2 + 7 \in C(Z_{12})\,[x];$$

however degree of p(x) is 4 and that of $p'(x) = \frac{dp(x)}{dx}$ is only two. Hence the claim.

Inview of this we have the following theorem.

THEOREM 3.12: *Let p(x) be a polynomial of degree n in $C(Z_m)[x]$; where $C(Z_m)$ is only a ring and m a composite number, then the derivative of p(x) that is $\frac{dp(x)}{dx}$ is of degree less than n and in general it need not be n–1.*

Proof: We prove this only by a counter example.

Consider $p(x) \in C(Z_{24})[x]$ where

$p(x) = (8+6i_F)x^6 + (12+12i_F)x^4 + (8+8i_F)x^3 + (7+3i_F)x^2 + (5i_F+4)x + 9i_F$.

Clearly p(x) is a degree 6. We find the derivative of p(x). $\frac{dp(x)}{dx} = p'(x) = 6(8+6i_F)x^5 + 4(12+12i_F)x^3 + 3(8+8i_F)x^2 + 2+(7+3i_F)x + (5i_F+4)$

$$= 0 + 0 + 0 + (14 + 6i_F)x + 5i_F + 4$$

$$= (14+6i_F)x + 5i_F + 4 \in C(Z_{24})[x].$$

We see degree of $\frac{dp(x)}{dx}$, that is $p'(x)$, the derivative of p(x) is one, however degree of p(x) was six. Hence the theorem.

This vital theorem leads to yet another result, which is not true in case of polynomials with coefficients from the complex modulo integer ring $C(Z_n)$.

Suppose p(x) is a polynomial in $C(Z_n)[x]$ of degree n and p(x) has n roots of which m roots are repeating then

$p'(x) = \frac{dp(x)}{dx}$ need not in general have (m–1) repeating roots.

We first illustrate this by an example.

Suppose $p(x) = [x+(3+3i_F)]^6 \in C(Z_6)[x]$. We see p(x) is a polynomial of degree 6.

Clearly p(x) has six repeating roots given by $3+3i_F$. Consider $p'(x) = \frac{dp(x)}{dx} = 6[x+(3+3i_F)]^5 = 0$

Since 6 is zero mod 6 in $C(Z_6)[x]$. Thus $p'(x)$ does not even have single repeating root.

Now another natural question would be what is the expansion of $(x+3+3i_F)^6$. First of all it is pertinent to mention here binomial expansion of $(x+a)^2$ and its higher powers may not be valid if the polynomial is in $C(Z_n)[x]$, n a composite number of $\underline{|n}$ cannot be defined nC_r cannot be defined to this effect we find $(x+3+3i_F)^6$ just by finding the product $(x+3+3i_F) \times \ldots \times (x+3+3i_F)$ six times.

$$
\begin{aligned}
&\text{We know } (x+3+3i_F)^6 \in C(Z_6)[x] \\
&(x+3+3i_F) \times (x+3+3i_F) \\
&\quad = x^2 + 2(3+3i_F)\,x + (3+3i_F)\,(3+3i_F) \\
&\quad = x^2 + 0 + 9 + 9i_F + 9i_F + 9 \times 5 \\
&\quad = x^2.
\end{aligned}
$$

$$
\begin{aligned}
x^2 \times (x+3+3i_F) &= x^3 + (3 + 3i_F)x^2\ (x^3 + 3 + 3i_F)x^2\ (x + 3 + 3i_F) \\
&= x^4 + (3 + 3i_F)x^3 + (3 + 3i_F)x^3 + (3 + 3i_F)2x^2 \\
&= x^4 + 0 + (0) \\
&= x^4 \text{ so we get}
\end{aligned}
$$

$$(x + 3 + 3i_F)^6 = x^6.$$

How to solve this? Can be assume an equation with repeating root is such that $x^n - (\alpha + ... + \alpha)x^{n-1} + \ldots +$ so on. From this example it is very clear that such a theory cannot be stated so we can say at least for repeating roots we cannot write the polynomial of degree n as x^n –(sum of the roots takes one at a time)x^{n-1} + (sum of the product of roots taken two at a time) + … + $(-1)^n$ (product of the n repeating roots). However it is pertinent to note that the power of the linear polynomial in

$C(Z_n)$ [x] is the same as that of n the characteristic of the ring Z_n then the result is true.

We will study the case of repeating roots.

$$p(x) = (x + 2 + 2i_F)^4 \in C(Z_4)[x].$$

$$p'(x) = 0 \text{ so no root exists for } p'(x).$$

Now $p(x) = (x+2+2i_F) \times (x+2+2i_F) \times (x+2+2i_F) \times (x+2+2i_F)$

$$\begin{aligned}
&= [(x+2+2i_F) \times (x+2+2i_F)] \times [(x+2+2i_F) \times (x+2+2i_F)] \\
&= [x^2 + x(2+2i_F) + x(2+2i_F) + (2+2i_F)^2] \times [x^2 + x\,(2+2i_F) \\
&\quad + x\,(2+2i_F) + (2+2i_F)^2] \\
&= [x^2 + 0 + 4 + 4 + 4 \times 3] \times [x^2 + 0 + 4 + 4 + 4 \times 3] \\
&= x^4 \text{ so,}
\end{aligned}$$

$(x+2+2i_F)^4 = x^4 = p(x)$.

So we see these sort of polynomials are not well defined in $C(Z_n)[x]$.

Hence we see we can have $C(Z_n)[x]$ to be a polynomial ring where Z_n is a ring but finding roots or well definedness of them or usual expansion of them using binomial theorem cannot hold good. So we restrict ourselves to solving polynomials only if $C(Z_n)$ is a field or that Z_n is a field.

Consider $(x+2+2i_F)^3 \in C(Z_3)[x]$;

$$\begin{aligned}
&(x+2+2i_F)^3 \\
&= (x+2+2i_F)^2 \times (x+2+2i_F) \\
&= [x^2 + (2+2i_F)x + (2+2i_F)x + (2+2i_F)^2]\ [x+2+2i_F] \\
&= [x^2 + (1 + i_F)x + (4 + 4 \times 2 + 8i_F)] \times [x+2+2i_F] \\
&= (x^2 + (1 + i_F)x + 2i_F)\ (x + 2 + 2i_F) \\
&= x^3 + (1 + i_F)x^2 + 2i_Fx + (2+2i_F)x^2 + (2+2i_F)\ (1+i_F)x \\
&\quad + 2i_F\ (2+di_F) \\
&= x^3 + (1+i_F + (2+2i_F))\ x^2 + [2i_F + (2+2i_F)(1+i_F)]\ x \\
&= x^3 + 2 + i_F.
\end{aligned}$$

$2+2i_F$ is a root repeating 3 times of $(x+2+2i_F)^3 = p(x)$ then sum of the roots is 0 and sum of the product of the two roots is also zero. Now product of the roots taken all at a time is $i_F + 2$. Thus the rule if α is a 3 repeated root then we get values of $(x+2+2i_F)^3 = x^3+2+i_F$. It is clear $C(Z_3)[x]$ is such that $C(Z_3)$ is a field.

Consider $C(Z_5)$; $C(Z_5)$ is not a field. Take the polynomial ring $C(Z_5)[x]$ with coefficients from $C(Z_5)$. Take $(x+2+2i_F)^5 = p(x)$. To find the expansion of $(x+2+2i_F)^5$ and verify in the expansion coefficient of x^4 corresponds to sum of the roots taken 4 times, the coefficient of x^3 is the sum of the product of the roots taken two at a time and so on.

Consider

$$
\begin{aligned}
(x+2+2i_F)^5 &= (x+2+2i_F)^2 \times (x+2+2i_F)^3 \\
&= (x^2 + 2(2+2i_F)x + (2+2i_F)^2) \times (x+2+2i_F)^3 \\
&= (x^2 + (4+4i_F)x + 3i_F)^2 \times (x+2+2i_F) \\
&= (x^4 + (4+4i_F)^2x^2 + 9 \times 4 + 2(4+4i_F)x^3 + \\
&\quad 2(4+4i_F)3i_Fx + 2.3i_Fx^2) \times (x+2+2i_F) \\
&= (x^4 + (16 + 16 \times 4 + 32i_F)x^2 + 1 + (3 + 3i_F)x^3 \\
&\quad + (4i_F + 4 \times 4)x + i_Fx^2)\ (x+2+2i_F) \\
&= (x^4+2i_Fx^2 + 1 + (3+3i_F)x^3 + (1+4i_F)x + i_Fx^2) \times \\
&\quad (x+2+2i_F) \\
&= x^5+ 2i_Fx^3 + x + (3+3i_F)x^4 + (1+4i_F)x^2 + i_Fx^3 + \\
&\quad (2+2i_F)x^4 + (2+2i_F)2i_Fx^2 + 2+2i_F+ (3+3i_F) \\
&\quad (2+2i_F)x^3 + (1+4i_F)\ (2+2i_F)x + i_F\ (2+2i_F)x^2 \\
&= x^5 + 0x^4 + (2i_F + i_F + 2i_F)x^3 + (1 + 4i_F + \\
&\quad (4i_F+16) + 2i_F + 2 \times 4)x^2 + (1+2+2i_F + 8i_F + \\
&\quad 8 \times 4)\ x + 2 + 2i_F \\
&= x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 2 + 2i_F \\
&= x^5 + 2 + 2i_F.
\end{aligned}
$$

Thus we get $(x+2+2i_F)^5$ is such that if $\alpha = 2+2i_F$ then coefficient of x^4 is $\alpha + \alpha + \alpha + \alpha + \alpha = 0$.

Coefficient of x^3 is $\alpha^2 + \alpha^2 + \alpha^2 + \alpha^2 + \alpha^2=0$.

Coefficient of x^2 is $\alpha^3 + \alpha^3 + \alpha^3 + \alpha^3 + \alpha^3=0$.

Coefficient of x is $\alpha^4 + \alpha^4 + \alpha^4 + \alpha^4 + \alpha^4=0$ and the constant term is $- \alpha^5 = 2+2i_F$.

Inview of this we can have the following theorem.

THEOREM 3.13: *Let $C(Z_p)$ be the ring of complex modulo integers. $C(Z_p)$ has no nilpotent elements.*

Proof: We know if $a + bi_F \in C(Z_p)$ then $(a+bi_F)^p = a^p + b^p (i_F)^p$ and $a^p \neq 0$ for $a \neq 0$ in Z_p and $b^p \neq 0$ for any $b \neq 0$ in Z_p and $(i_F)^p \neq 0$. Hence the claim.

Using the above theorem we have the following result.

THEOREM 3.14: *Let Z_p be the finite field (p a prime). $C(Z_p)$ be the complex modulo integer ring. Then $(x+a+bi_F)^p = x^p + c + di_F$ for any $(x+a+bi_F)^p$ in $C(Z_p)[x]$.*

Proof is simple using number theoretic methods and the fact Z_p is a field.

Thus we face with simple means of solving these equations. Now how to solve equations in the $C(Z_p)[x]$ if $C(Z_p)$ is not a field.

Interested reader can work in this direction to get several properties which is not a matter of routine but can be done by appropriate changes.

Thus we have to show $C(Z_p)$, when Z_p is a field has no nilpotents but has zero divisors when $p = n^2 + m^2$, $m, n \in Z_p \setminus \{0\}$. Now solving equations in $C(Z_p)[x]$ is not a difficult task as we have only a finite number of elements in $C(Z_p)$; viz, p^2 elements in it.

However it is an open problem "can $p(x) \in C(Z_p)[x]$ such that p(x) has no root in $C(Z_p)$? This problem is also little difficult at the outset as the number of elements in $C(Z_p)[x]$ is infinite.

Whenever $C(Z_p)$ is a field we can also realize $C(Z_p)[x]$ as a linear algebra of infinite dimension over $C(Z_p)$.

Always $C(Z_p)[x]$ is a commutative module over $C(Z_p)$ even if $C(Z_p)$ is a ring.

One of the nice results is always $Z_p[x] \subseteq C(Z_p)[x]$ but $Z_pi_F = \{ai_F \mid a \in Z_p\}$ does not enjoy an algebraic structure except it is an abelian group under addition modulo p but is not closed under product. However Z_pi_F is a vector space over Z_p in this case $Z_p \subseteq Z_pi_F$. We see $Z_pi_F[x] \subseteq C(Z_p)[x]$ is only closed under addition. Further Z_pi_F is not even a vector space over $C(Z_p)$ even if $C(Z_p)$ is a field. $Z_pi_F[x]$ is also a vector space over Z_p and not over $C(Z_p)$, even if $C(Z_p)$ is a field.

Thus while integrating or differentiating these polynomials in $C(Z_p)[x]$ we have problems and the several results true in case of usual $C[x]$ are not true in case of $C(Z_p)[x]$. Only with these limitations we work with these structures as they are not only interesting but gives an alternative way of approaching problems as all the problems encountered in general are finite and finite dimensional. This alternate study will certainly lead to several applications and appropriate solutions.

Further these structures are not orderable though finite. Now having seen properties about polynomial with complex modulo integer coefficients now we proceed onto study matrices with these elements as its entries as vector space and a few properties enjoyed by them.

In the earlier part of this chapter we have introduced the notion of matrices with entries from $C(Z_n)$.

Now suppose $V = \{(x_1, \ldots, x_n) \mid x_i \in C(Z_p), 1 \leq i \leq n\}$ (Z_p a field) then V is a vector space over Z_p of finite dimension. Also V is a linear algebra over Z_p. If $C(Z_p)$ itself is a field V is a strong complex modulo integer vector space / linear algebra over $C(Z_p)$.

Interested reader can study the dimension, basis etc of these structures.

We can also find / define orthogonal subspaces if the pseudo inner product is zero [4, 6].

If $C(Z_p)$ is only a ring then we define V to be a Smarandache vector space / linear algebra over $C(Z_p)$ as $C(Z_p)$ is a Smarandache ring.

Suppose

$$M = \left\{ \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_m \end{array}\right] \middle| \; y_i \in C(Z_p) \; 1 \leq i \leq m \right\};$$

M is only a vector space over Z_p however M is a strong complex modulo integer vector space over $C(Z_p)$ if $C(Z_p)$ is a field. If $C(Z_p)$ is only a ring then M is a Smarandache vector space over $C(Z_p)$.

Clearly M is not a linear algebra over Z_p or $C(Z_p)$ however in the next chapter we show how this M is also made into a linear algebra over Z_p or $C(Z_p)$.

Next if S = {collection of all $m \times n$ matrices $m \neq n$; with entries from $C(Z_p)$ | Z_p is a field}, then S is a vector space over Z_p or a strong vector space over $C(Z_p)$ if $C(Z_p)$is a field and S is a S-vector space if $C(Z_p)$ is a ring as $C(Z_p)$ is always a S-ring. One can study the dimension, basis and other properties of these vector spaces.

However S is only a vector space and not a linear algebra over Z_p (or $C(Z_p)$). But by defining a natural product in the next chapter we can make S also a linear algebra or a S-linear algebra over Z_p or $C(Z_p)$. Let K = {collection of all $n \times n$ matrices with entries from $C(Z_p)$}; K is a vector space as well as a linear algebra over Z_p and over $C(Z_p)$ if $C(Z_p)$ is a field and

Smarandache vector space / linear algebra over the Smarandaceh ring $C(Z_p)$. We have the usual multiplication defined on K.

Now we can for any $n \times n$ matrix A with $C(Z_p)$ define eigen values, eigen vectors and the characteristic equation. Clearly the characteristic equation is of degree n. A is a matrix of order A. However the solution of the characteristic equation can take only p values if K is defined over Z_p if A is over $C(Z_p)$ as a vector space the characteristic equation can take p^2 values from $C(Z_p)$.

Now we will illustrate this situation by some simple examples.

Consider $A = \begin{bmatrix} 2+2i_F & 0 & 2+2i_F \\ 0 & i_F & 2 \\ 0 & 1 & 1+i_F \end{bmatrix}$ to be a 3×3 matrix with entries from $C(Z_3)$. To find the eigen values and eigen vectors associated with A.

$$|A - \lambda| = |\lambda - A| = \begin{vmatrix} \lambda+1+i_F & 0 & 1+i_F \\ 0 & \lambda+2i_F & 1 \\ 0 & 2 & \lambda+2+2i_F \end{vmatrix}$$

$= |\lambda + 1 + i_F|\ [(\lambda + 2i_F)(\lambda + 2+2i_F)+1]$
$= (\lambda + 1+i_F)(\lambda^2 + 2i_F\lambda + 2\lambda + 2i_F\lambda + (i_F+2)+1)$
$= (\lambda + 1+i_F)(\lambda^2 + \lambda i_F + 2\lambda + i_F)$
$= 0;$ $\qquad \lambda = 2+2i_F$ and

$\lambda^2 + (2+i_F) + i_F = 0$ gives
$\lambda \neq 0$ $\qquad \lambda \neq 1$ if $\lambda = 2$
$2^2 + 2(2+i_F) + i_F = 1+1+i_F+i_F \neq 0.$
So $\lambda \neq 2$ if $\lambda = i_F$
$2+i_F(2+i_F) + i_F = 2+2i_F + 2 + i_F \neq 0$ so $\lambda \neq i_F$.
If $\lambda = 2i_F$

$(2i_F)^2 + 2i_F\,(2+i_F) + i_F = 4 \times 2 + 4i_F + 2 \times 2 + i_F \neq 0.$

If
$\lambda = 1+i_F;\ (1+i_F)^2 + (1+i_F)\,(2+i_F) + i_F$
$= 1+2i_F + 2 + 2 + 2i_F + i_F + 2 + i_F \neq 0$ so $\lambda \neq 1 + i_F$.

If $\lambda = 2+2i_F$ then
$(2+2i_F)^2 + (2+2i_F)\,(2+i_F)+ i_F$
$= 4 + 4 \times 2 + 8i_F + 4 + 4i_F + 2i_F +2 \times 2 + i_F \neq 0.$
So $\lambda \neq 2+2i_F$

Suppose $\lambda = 1+2i_F$ then $(1+2i_F)^2 + (1+2i_F)\,(2+i_F) + i_F$
$= 1+4 \times 2 + 4i_F + 2+4i_F + i_F + 2 \times 2 + i_F \neq 0.$
So $\lambda \neq 1+2i_F$.
Take $\lambda = 2+i_F$;
$(2+i_F)^2 + (2+i_F)^2 + i_F = 4+2 + 4i_F + 4+2 + 4i_F + i_F = 0.$

So $2+i_F$ is a root of the equation and it is a repeated root.
In this way eigen values are found. Now we as in case of usual vector spaces define and find eigen values and eigen vectors.

We can define linear transformation provided they are defined over the same Z_p or $C(Z_p)$.

At times we have special linear transformation if one space is defined over Z_p and other over $C(Z_p)$ [4, 6]. Now having seen how vector spaces / linear algebra using complex modulo integers function we proceed onto study intervals using them.

Let $N_o(C(Z_p)) = \{(a + bi_F, c+di_F) \mid a, b, c, d \in Z_p\}$ be the collection of open intervals, we do not have any ordering in these intervals. Clearly $N_c(C(Z_p))$, $N_{oc}(C(Z_p))$ and $N_{co}(C(Z_p))$ can be defined. Clearly $N_o(C(Z_p))$ is a commutative ring with zero divisors and units. However we can replace Z_p by any Z_n; $1 < n < \infty$ still the results hold good. We can build polynomials with coefficients from $N_o(C(Z_p))$ or $N_c(C(Z_p))$ or $N_{oc}(C(Z_p))$ or $N_{co}(C(Z_p))$.

Now $N_o(C(Z_p))[x]$ is a commutative ring with zero divisors. This ring has two ideals $J = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (a_i^1, 0) \in N_o(C(Z_p))\right\}$ and $I = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (0, a_i^2) \in N_o(C(Z_p))\right\}$ in $N_o(C(Z_p))[x]$. I and J are such that $I.J = \{0\}$ and $I \cap J = \{0\}$.

Now we can in case of $N_o(C(Z_p))[x]$ solve equations in the following way.

If $p(x) = \sum_{i=0}^{n} a_i x^i$, $a_i = (a_i^1, a_i^2)$ in $N_o(C(Z_p))$ then we can rewrite p(x) as $[p_1(x), p_2(x)]$ where $p_1(x) = \sum_{i=0}^{n} a_i^1 x^i$ and $p_2(x) = \sum_{i=0}^{n} a_i^2 x^i$ we can solve $p_i(x)$ in $C(Z_p)$ and we have p^2 possibilities for each i=1,2.

So we get interval solutions for p(x) in $N_o(C(Z_p))$.

Similarly we can build matrices with entries from $N_o(C(Z_p))$ or $N_c(C(Z_p))$ or $N_{oc}(C(Z_p))$ or $N_{co}(C(Z_p))$.

Let $P = \{(a_1, \ldots, a_n) \mid a_i = (a_i^1, a_i^2) \in N_o(C(Z_p))$ with a_i^1 and $a_i^2 \in C(Z_p)$; $1 \leq i \leq n\}$, P is a commutative ring with unit and zero divisors.

If $x = (a_1, \ldots, a_n)$

$$= ((a_1^1, a_1^2), (a_2^1, a_2^2), \ldots, (a_n^1, a_n^2))$$
$$= ((a_1^1, a_2^1, \ldots, a_n^1), (a_1^2, a_2^2, \ldots, a_n^2)).$$

Then all working can be done as on intervals with entries from Z_p or as usual row matrices.

If

$$P = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \right| a_i = (a_i^1, a_i^2) \text{ in } N_o(C(Z_p)),\ 1 \leq i \leq m \right\}$$

then P is only an abelian group with respect to addition. However product is not defined on P; only natural product is defined on these structures in the following chapter so that P becomes a commutative ring.

M = {all $m \times n$ matrices with entries from $N_o(C(Z_p))$} $(m \neq n)$, M is only an additive abelian group with no product defined on it. Natural product on M is defined in chapter four. Suppose

P = {all $n \times n$ square matrices with entries from $N_o(C(Z_p))$}, P is a non commutative ring with zero divisors.

Now we can define using these structures vector spaces / linear algebras.

Let $M = \{(a_1, a_2, \ldots, a_n) \mid a_i = (a_i^1, a_i^2) \in N_o(C(Z_p))$, Z_p a field, $1 \leq i \leq n\}$. M is a linear algebra over the field Z_p. If $C(Z_p)$ is a field; M is a linear algebra over $C(Z_p)$ also. We see M is a Smarandache linear algebra over the S-ring $N_o(C(Z_p))$.

Let

$$P = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \right| a_i = (a_i^1, a_i^2) \text{ in } N_o(C(Z_p)),\ 1 \leq i \leq m \right\}.$$

P is only a vector space over the field Z_p or a S-vector space over $N_o(C(Z_p))$.

Consider

P = {all m × n matrices with entries from $N_{oc}(C(Z_p))$} ($m \neq n$); P is only a vector space (not a linear algebra) over Z_p or over $C(Z_p)$ if $C(Z_p)$ is a field or a S-vector space over $N_{oc}(C(Z_p))$.

Finally

S = {all m × m matrices with entries from $N_{oc}(C(Z_p))$}; S is a linear algebra over Z_p or over $C(Z_p)$ if $C(Z_p)$ is a field. However S is a S-linear algebra over the S-ring $N_{oc}(C(Z_p))$.

We can find subspaces, dimension and other properties of these structures, but they are direct and hence left as an exercise to the reader.

Now we just show how eigen values and eigen vectors of a square matrix which entries from

$$N_c(C(Z_n)) = \{[a+bi_F, c+di_F] \mid a, b, c, d \in Z_3,\ i_F^2 = 2\}.$$

Consider the matrix A with its entries from $N_c(C(Z_3))$.

$$A = \begin{bmatrix} [2+i_F, 2i_F] & [2, i_F] \\ [0, 2+i_F] & [1+i_F, 0] \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2+i_F & 2 \\ 0 & 1+i_F \end{bmatrix}, \begin{bmatrix} 2i_F & i_F \\ i_F & 0 \end{bmatrix} \right).$$

$$[|\lambda - A|] = \left[\begin{vmatrix} \lambda+1+2i_F & 1 \\ 0 & \lambda+2+2i_F \end{vmatrix}, \begin{vmatrix} \lambda+i_F & 2i_F \\ 2i_F & \lambda \end{vmatrix} \right]$$

$$= [(\lambda + 1 + 2i_F)(\lambda + 2 + 2i_F), \lambda(\lambda + i_F) - 4\, i_F^2]$$

$= [(\lambda^2 + \lambda + 2i_F \lambda + 2\lambda + 2 + 4i_F + 2i_F \lambda + 2i_F + 4 \times 2,$

$\lambda + \lambda i_F + 1] = [0,0].$

So $\lambda^2 + i_F\lambda + 1 = 0$ and $\lambda^2 + \lambda i_F + 1 = 0$

Now $\lambda \neq 0, \lambda \neq 1, \lambda \neq i_F, \lambda \neq 2$
$\lambda \neq 2i_F, \lambda = 1+i_F, \lambda \neq 2i_F + 1,$
$\lambda = i_F + 2$; so $\lambda = 1 + i_F$ and $\lambda = i_F + 2$

are the two roots of $\lambda^2 + i_F \lambda + 1 = 0$.

Sum of the roots $= (1 + i_F) + (i_F + 2) = 2i_F$ so sum of the roots is i_F and product of the roots

$(1 + i_F)(2 + i_F) = 2 + 2i_F + i_F + 2 = 1.$

Thus the characteristic values associated with the matrix A is $[1+i_F, 2+i_F]$ and $[2+i_F, 1+i_F]$.

One can work with interval matrices for its eigen values. Now we have seen how to solve the polynomial equations with its coefficients from $N_c(C(Z_p))$.

Now we have complex neutrosophic numbers $\langle C \cup I\rangle$, we can also define complex modulo integer neutrosophic numbers.

Thus $\langle C(Z_p) \cup I\rangle = C(\langle Z_p \cup I\rangle)$
$= \{a + bi_F + cI + dIi_F \mid a, b, c, d \in Z_p.\ i_F^2 = p - 1$ and $I^2 = I\}$.

We have this structure $C(\langle Z_p \cup I\rangle)$ to be a commutative ring with unit.

$C(\langle Z_3 \cup I\rangle) = \{0, I, i_F, 2i_F\ 2I, 1, 2, 1 + I, 2 + I, 1 + 2I, 2 + 2I, i_F + 1, 2 + i_F, 2 + 2i_F, 2i_F + 1, I + i_F, 2I + i_F, 2i_F + 2I, I + 2i_F, 1 + I + i_F, 1 + I + 2i_F, \ldots, 2 + 2I + 2i_F + 2i_F I\}$.

Likewise we have $C(\langle Z_4 \cup I\rangle)$, $C(\langle Z_5 \cup I\rangle)$, $C(\langle Z_6 \cup I\rangle)$ and so on.

Study of these structures are interesting and innovative we can use these structures to build polynomials with coefficients from $C(\langle Z_n \cup I\rangle)$ and matrices with entries from $C(\langle Z_n \cup I\rangle)$.

We see $C(\langle Z_n \cup I\rangle)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = a_i^1 + a_i^2 I + a_i^3 i_F + a_i^4 i_F I \text{ where } a_i^j \in Z_n; 1 \leq j \leq 4 \right\}$. Clearly $C(\langle Z_n \cup I\rangle)[x]$ is a ring which is commutative and contains the multiplicative identity 1.

The interesting problems associated with this ring are

(1) if n = p, a prime can $C(\langle Z_p \cup I\rangle)[x]$ be an integral domain?

Will $C(\langle Z_p \cup I\rangle)$ be a field if p is a prime; justify?

Study $C(\langle Z_n \cup I\rangle)$ for varying n. What is the structure enjoyed by $C(\langle Z_p \cup I\rangle)[x]$?

Study in these directions are interesting [4, 6].

Now finding roots of these polynomials with coefficients from $C(\langle Z_p \cup I\rangle)$ can be programmed for the number of elements in $C(\langle Z_p \cup I\rangle)$ is finite.

Is $o(C(\langle Z_p \cup I\rangle)) = p^4$?

Now we can define matrices with entries from $C(\langle Z_n \cup I\rangle)$.

$M = \{(a_1, a_2, \ldots, a_n) \mid a_i = a_i^1 + a_i^2 I + a_i^3 i_F + a_i^4 i_F I$ where $a_i^t \in Z_n; 1 \leq t \leq 4\}$ is a ring with zero divisors and units.

For if n = 2 and x = $(2+3I + 4i_F + i_FI, 4 + 2I + 2i_F + 3i_FI)$ and y = $(3 + 2I + i_F + 4i_FI, 1+3I + 2i_F + 4i_FI)$ where entries are from $C(\langle Z_5 \cup I\rangle)$.

We find $x + y = (0 + 0 + 0 + 0, 0 + 0 + 4i_F, 2i_FI)$

$= (0, 4i_F + 2i_FI)$ and

$$x \times y = ((2+3I + 4i_F + i_FI) (3 + 2I + i_F + 4i_FI), (4 + 2I + 2i_F + 3i_FI) (1 + 3I + 2i_F + 4i_FI))$$

$$= ((6 + 4I + 2i_F + 8i_FI + 9I + 6I + 3Ii_F + 12i_FI + 12i_F + 8i_FI + 4 \times 4 + 16I \times 4 + 3i_FI + 2Ii_F + 4I + 16I), (4 + 12I + 8i_F + 16i_FI + 2I + 6I + 4i_FI + 8i_FI + 2i_F + 6i_FI + 4 \times 4 + 4 \times 8I + 3i_FI + 9i_FI + 6 \times 4I + 12 \times 4I)$$

$$= (2 + 3I + 4i_F + i_FI, 0 + 4I + 0 + 2i_FI).$$

Thus we see M is a commutative ring. Clearly M has zero divisors and units.

We can define $V = \{(a_1, a_2, \ldots, a_n) \mid a_i \in C(\langle Z_m \cup I\rangle)$ where $a_i = a_i^1 + a_i^2 i_F + a_i^3 I + a_i^4 i_FI$; $a_i^t \in Z_m$; $1 \le t \le 4\}$ to be a vector space / linear algebra over Z_m (or $C(Z_m)$) if Z_m (or $C(Z_m)$) is a field. However if Z_m is not a field V will be a S-vector space or S-linear algebra over the S-ring $C(\langle Z_m \cup I\rangle)$. Study in the direction can be carried out by any interested reader.

Now let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} \middle| \; a_i = a_1^i + a_2^i I + a_3^i i_F + a_4^i \; i_FI; \right.$$

$$\left. a_j^i \in Z_p;\ 1 \le j \le 4 \right\}$$

be a collection of column vectors. We say P is a vector space over Z_p or $C(Z_p)$ if Z_p or $C(Z_p)$ is a field otherwise only a Smarandache vector space over the S-ring $C(\langle Z_p \cup I\rangle)$.

We perform the operation of addition as follows:

$$\text{Suppose } x = \begin{bmatrix} 3+2i_F+7i_FI+2I \\ 4+i_F+I+5Ii_F \\ 0+7i_F+2I+3i_FI \\ 9+3i_F+2I+4i_FI \end{bmatrix}$$

$$\text{and } y = \begin{bmatrix} 10+5i_F+2i_FI+7I \\ 3+8i_F+10I+10i_FI \\ 4+0+2i_FI+4I \\ 3+i_F+0+2I \end{bmatrix}$$

be two column vectors with entries from $C(\langle Z_{11} \cup I\rangle)$.

$$\text{We find } x + y = \begin{bmatrix} 2+7i_F+9i_FI+9I \\ 7+9i_F+10I+4i_FI \\ 4+7i_F+6I+5i_FI \\ 1+4i_F+4I+4i_FI \end{bmatrix},$$

x + y is again a column vectors with entries from $C(\langle Z_{11} \cup I\rangle)$. However we are not in a position to define usual product on these column vectors.

Thus the collection of all column vectors of order $t \times 1$ is a vector space over $C(Z_p)$ or Z_p ($C(Z_p)$ if $C(Z_p)$ is a field) and the entries are from $C(\langle Z_p \cup I\rangle)$. Thus this cannot be a linear algebra over the usual matrix product, however under natural product these column vectors are a linear algebra.

Now we proceed onto define $m \times n$ matrices $m \neq n$ with entries from $C(\langle Z_p \cup I\rangle)$. Z_p is a field or a ring when we assume it as a ring it should be a Smarandache ring so that we can define Smarandache vector space over the S-ring. We just show

how the sum is done; the sum is got as that of the usual matrices adding componentwise.

Finally we define the set of square matrices with entries from $C(\langle Z_n \cup I\rangle)$.

Let P = {all n × n matrices with entries from $C(\langle Z_p \cup I\rangle)$ | p is a prime}. P is a non commutative ring with unit P contains zero divisors. Now P has ideals and subrings which are not ideals. Suppose Z_p is a field and $C(Z_p)$ is a field, then P is a linear algebra over Z_p as well as $C(Z_p)$. However P is only a Smarandache linear algebra over $C(\langle Z_p \cup I\rangle)$.

We now as in case of usual vector spaces / linear algebras find basis, subspaces, linear transformation and other properties.

Now we just show how we find the usual product of two matrices in M.

Consider P = $\begin{bmatrix} 2+3i_F+4I+2i_FI & 0+3i_F+8I+3i_FI \\ 7+4i_F+2I+7i_FI & 1+0+5I+10i_FI \end{bmatrix}$

and Q = $\begin{bmatrix} 1+2i_F+3I+4Ii_F & 5+6i_F+7I+8i_FI \\ 3+I+i_F+4i_FI & 6+2i_F+I+3i_FI \end{bmatrix}$ any

2 × 2 matrices with entries from $C(\langle Z_{13} \cup I\rangle)$. PQ =

$$\begin{bmatrix} \begin{array}{l}(2+3i_F+4I+2i_FI)(1+2i_F+3I+4i_FI) \\ +(3i_F+8I+3i_FI)(3+I+i_F+4i_FI)\end{array} & \begin{array}{l}(2+3i_F+4I+2i_FI)(5+6i_F+7I+8i_FI) \\ +(3i_F+8I+3i_FI)\times(6+2i_F+I+3i_FI)\end{array} \\ \\ \begin{array}{l}(7+4i_F+2I+7i_FI)(1+2i_F+3I+4Ii_F) \\ (1+5I+10i_FI)(3+I+i_F+4i_FI)\end{array} & \begin{array}{l}(7+4i_F+2I+7i_FI)\times(5+6i_F+7I+8i_FI) \\ +(1+5I+10i_F)\times(6+2i_F+I+3i_F)\end{array} \end{bmatrix}$$

It can be simplified and the resultant is again a 2 × 2 matrix.

Also given any $n \times n$ matrix with entries from $C(\langle Z_n \cup I\rangle)$ we can find the eigen values, eigen vectors and the characteristic polynomials, we just indicate it for a 2×2 matrix.

$$\text{Let } A = \begin{bmatrix} 2i_F + 3I + 2 & 3I + 4i_F + 3 + 4i_F I \\ 0 & 2i_F I + 3 + i_F \end{bmatrix}$$

be a 2×2 matrix with entries from $C(\langle Z_5 \cup I\rangle)$.

$$|\lambda - A| = \left|\begin{bmatrix} \lambda + 3 + 2I + 3i_F & 2I + i_F + 2 + i_F I \\ 0 & \lambda + 3i_F I + 2 + 4i_F \end{bmatrix}\right|$$

$= (\lambda + 3 + 2I + 3i_F)\ (\lambda + 3i_F I + 2 + 4i_F)$
$= \lambda^2 + \lambda\ (3+2I+3i_F + 3i_F I + 2+4i_F) + (3+2I + 3i_F) \times (3i_F I + 2+4i_F)$
$= \lambda^2 + \lambda\ (2I+2i_F+3i_F I) + (9i_F I + 6i_F I + 9 \times 4 \times I + 6 + 4I + 6i_F + 12i_F + 8i_F I + 12 \times 4)$
$= \lambda^2 + \lambda\ (2I + 2i_F + 3i_F I) + (3i_F I + 4 + 3i_F)$.

The roots are $\lambda = 2+3I + 2i_F$ and $\lambda = 2i_F I + 3 + i_F$. Thus we can find eigen values in case of square matrices with entries from $C(\langle Z_5 \cup I\rangle)$. On similar lines one can work with these structures.

Now we proceed onto define natural class of intervals using $C(\langle Z_n \cup I\rangle)$ and build algebraic structures using them.

Let $N_c C(\langle Z_N \cup I\rangle) = \{[a, b] \mid a, b \in C(\langle Z_N \cup I\rangle)$ where $a = a_1 + a_2 I + a_3 i_F + a_4 i_F I$ and $b = b_1 + b_2 I + b_3 i_F + b_4 i_F I$ with $a_i, b_i \in Z_N$, $1 \leq i \leq 4$ and $i_F^2 = N-1$ where $I^2 = I\}$.

Clearly $N_c(C(\langle Z_N \cup I\rangle))$ is a commutative ring with zero divisors and units.

We just show how sum and product are formed.

If $x = (a + bi_F + cI + di_F I,\ x+yi_F + zI + ti_F I)$

and $y = (m+ni_F + pI + qi_FI, e+fi_F + gI + ri_FI)$ are in

$N_o(\langle Z_N \cup I\rangle)$ with a, b, c, d, …, r $\in Z_N$, then

$x + y = (a + m + (b + n)\, i_F + (c + p)I + (d + q)\, i_FI, x + e + (y + f)\, i_F + (z + g)I + (t + r)\, i_FI) \in N_o(\langle Z_N \cup I\rangle)$.

Now we find product

$$\begin{aligned}
x \times y &= ((a + bi_F + cI + di_FI) \times (m + ni_F + pI + qi_FI),\\
&\quad (x + yi_F + zI + ti_FI) \times (e + fi_F + gI + ri_FI))\\
&= ((ma + mbi_F + mcI + mdi_FI + nai_F + nb \times (N-1) + ncIi_F\\
&\quad + ndI\,(N-1) + paI + pbIi_F + pcI + pdi_FI + aqi_FI\\
&\quad + bq(N-1)I + cqi_FI + qd(N-1)I)\ (\text{mod } (N), (xe + yei_F\\
&\quad + ezI + eti_FI + fxi_F + fy\,(N-1) + zfi_FI + tfI\,(N-1) + gxI\\
&\quad + gyi_FI + gzI + gtIi_F + xri_FI + ry\,(N-1)I + rzi_FI\\
&\quad + tr(N-1)I)\ (\text{mod } N)\\
&= (ma + nb\,(N-1) + (mc + nd\,(N-1) + pa + pc + bq\,(N-1)\\
&\quad + qd\,(N-1)I + (mb+na)\, i_F + md + nc + pb + pd + aq\\
&\quad + cq)\, i_F\ [\text{mod } N], (xe + fy\,(N-1) + (eZ + tf\,(N-1)\\
&\quad + gx + gz + ry\,(N-1) + tr\,(N-1)]I + (ye + fx)i_F + (et\\
&\quad + zf + gy + gt + xr + rz)\, i_FI]\ (\text{mod } N)]\\
&= ((ma + nb\,(N-1) + mc + pa + pc + (nd + bq + qd)\\
&\quad (N-1)]\, I + (mb + na)\, i_F + md + nc + pb + pd + aq + cq)\\
&\quad (\text{mod } N), ((xe + fy\,(N-1) + (ez + gx + gz + (tf + ry + tr)\\
&\quad (N-1))I + (ye + fx)\, i_F + (et + zf + gy + gt + xr + rz)\, i_FI\\
&\quad (\text{mod } N)).
\end{aligned}$$

Suppose we take $J = \{(a, 0) \mid a = a_1 + a_2i_F + a_3I + a_4Ii_F$ where $a_i \in Z_N;\ i_F^2 = N-1$ and $I^2 = I\} \subseteq N_o(C(\langle Z_n \cup I\rangle))$.

Consider $K = \{(0, a) \mid a = a_1 + a_2i_F + a_3I + a_4i_FI$ with $a_i \in Z_N;\ 1 \le i \le 4;\ I^2 = I$ and $i_F^2 = N-1\} \subseteq N_o((\langle Z_n \cup I\rangle))$.

We see $K \cap J = (0)$ also $KJ = (0)$; so the two ideals are orthogonal.

On similar lines we can work with $N_o(C\langle Z_N \cup I\rangle)$ or $N_{oc}(C\langle Z_n \cup I\rangle)$ or $N_{co}(C\langle Z_n \cup I\rangle)$.

Now we can also view these $N_o(C\langle Z_N \cup I\rangle)$ etc as vector space / linear algebra over Z_n or $C(Z_n)$ is a field otherwise they will be Smarandache vector spaces / linear algebra if Z_n or $C(Z_n)$ or $C(\langle Z_n \cup I\rangle))$ is a S-ring.

Now we can build polynomial with coefficients from $N_o(C(\langle Z_n \cup I\rangle))$. Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (a_i^1, a_i^2) \in N_o(C(\langle Z_n \cup I\rangle))\right.$ that is $a_i^1 = a_1^1 + a_2^1 i_F + a_3^1 I + a_4^1 Ii_F$ and $a_i^2 = a_1^2 + a_2^2 i_F + a_3^2 I + a_4^2 Ii_F$; $a_j^t \in Z_n$; $1 \leq t \leq 2$ and $1 \leq j \leq 4\}$;

V is a commutative ring with ideals and subrings. Any polynomial of finite degree if it has a root will lie in $N_o(C(\langle Z_n \cup I\rangle))$. Now likewise

$P = \{$all $m \times m$ matrices with entries from $N_o(C(\langle Z_n \cup I\rangle))\}$, P is a non commutative ring with zero divisors. P has both ideals and subrings.

Finding eigen values related to a $m \times m$ matrix with entries from $N_o(C(\langle Z_n \cup I\rangle))$ is a matter of routine and is left as an exercise to the reader. Study in this direction will lead to several applications and once the problem of polynomial with coefficients from $C(\langle Z_n \cup I\rangle)$ has its roots in $C(\langle Z_n \cup I\rangle)$ is established we have several nice applications as finding roots for any finite degree polynomial can be programmed as the number of elements in $C(\langle Z_n \cup I\rangle)$ is finite. However we have not discussed with $p(x) \in V$ such that $p(x)$ is an irreducible polynomial.

Chapter Four

NATURAL PRODUCT ON MATRICES

In this chapter we just define natural product on matrices [7]. In a compilation of the history of matrix theory, [1] provides the following brief and interesting history, “The history of matrices goes back to ancient times. The term matrix was not applied to the concept until 1850. The origin of mathematical matrices lie with the study of simultaneous linear equations. An important Chinese text from between 300 BC and 200 AD. Nine chapters of the mathematical art (Chiu Chang Suan Shu) gives the first known example of the use of matrix methods to solve simultaneous equations. In the treatise’s of seventh chapter “Too much and not enough” the concept of a determinant first appears, nearly too millennia before its supposed invention by the Japanese mathematician Seki Kowa in 1683 or his German contemporary Gottfried Leibnitz. More use of matrix-like arguments of numbers appear in chapter eight, method of rectangular arrays in which a method is given for solving simultaneous equations using a counting board that is mathematically identified to the modern matrix method of solution outlined by Carl Friedrich Gauss (1777 - 1855), known as Gaussian elimination.”

"Since their first appearance in ancient China, matrices have remained important mathematical tools. Today they are used not simply solving systems of simultaneous linear equations, but also for describing quantum mechanics of atomic structure, designing computer game graphics, analyzing relationships and even plotting complicated dance steps. (1906 - 1995) Olga Taussky Todd a female mathematician was the torch bearer for matrix theory who began to use matrices to analyse vibrations on airplanes during world war II."

Thus matrix theory has been very vital from the day of world war. We have made three types of exploration in matrix theory.

1. Define the notion of natural product on matrices.
2. Construct matrix using the subsets of a set X; that is the power set P(X) of X using '$\cup$' and '$\cap$' operations on matrices '$\cup$' akin to '+' and '$\cap$' just like natural product $\times_n$.
3. Using rectangular /square array of linguistic terms as matrices. This structure can be used in linguistic modeling leading to fuzzy linguistic graphs, fuzzy linguistic matrices and fuzzy linguistic models.

Now in the first place the authors give justification for defining the natural product on matrices.

Suppose we have two row matrices of same order say;

$X = (a_1, a_2, \ldots, a_n)$ and $Y = (b_1, b_2, \ldots, b_n)$ then we can find $X \times Y = (a_1, a_2, \ldots, a_n) \times (b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n)$. If we transpose X we get X^t and if we transpose Y we get Y^t however $X^t \times Y^t$ is not defined. Also if we can multiply two row matrices of same order then why cannot we multiply two column matrices of same order, keeping this in mind and $(XY)^t = Y^t X^t = X^t Y^t$ we multiply two column matrices and call / define that product as natural product and denote it by $\times_n$.

We just illustrate this by a simple example.

Let $X = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \\ -1 \end{bmatrix}$ be two 5×1 column matrices.

$$X \times_n Y = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 5\times1 \\ 3\times2 \\ -2\times0 \\ 1\times5 \\ 0\times-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix}.$$

It is pertinent to mention that we always find the sum of two column matrices of same order, when that is allowed what prevents us to define product or replace '+' sum of two column vectors by '$\times_n$' in the column vectors?

Also if X and Y are two row vectors matrices and $X^t Y^t$ is a column vector which is the natural product of two column vectors which are transposes of X and Y respectively.

Thus this natural product $\times_n$ on column vectors enables us to define the following algebraic structures.

(1) If S = {Collection of all $n \times 1$ column vectors with entries from R or C or Q or Z}, then S is a group under addition. Infact (S, +) is an abelian group.

(2) By defining the natural product $\times_n$ on S; $(S, \times_n)$ is a monoid

$$I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times1}$$ acts as the multiplicative identity.

$(S, \times_n)$ has zero divisors and units only if S takes its entries from Q or C or R. Infact $(S, \times_n)$ is a commutative monoid.

(3) Consider $(S, +, \times_n)$; S is a commutative ring with unit and zero divisors.

$$\text{Consider } x = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 5 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 7 \end{bmatrix} \text{ in S.}$$

$$\text{Clearly } x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly x^{-1} and y^{-1} does not exist.

$$\text{If } x = \begin{bmatrix} 7 \\ 3 \\ 5/2 \\ -1/5 \\ 2 \end{bmatrix} \text{ in S and take its entries from Q then}$$

$$x^{-1} = \begin{bmatrix} 1/7 \\ 1/3 \\ 2/5 \\ -5 \\ 1/2 \end{bmatrix};$$

$$\text{clearly } xx^{-1} = x^{-1}x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(4) {S, min} is an idempotent semigroup.

$$\text{If } x = \begin{bmatrix} 8 \\ 0 \\ 5 \\ -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } y = \begin{bmatrix} 7 \\ 1 \\ 2 \\ 0 \\ 1 \\ 4 \end{bmatrix} \text{ are in S, then}$$

$$\min\{x, y\} = \begin{bmatrix} \min\{8,7\} \\ \min\{0,1\} \\ \min\{5,2\} \\ \min\{-1,0\} \\ \min\{2,1\} \\ \min\{3,4\} \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

(5) {S, max} operation is an idempotent semigroup;

$$\max\{x, y\} = \begin{bmatrix} 8 \\ 1 \\ 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}.$$

(6) $\{S, +, \times_n\}$, the commutative ring, has both ideals and subrings which are not ideals.

(7) $\{S, +\}$ is a linear algebra over Q, if S takes its entries from Q or R.

(8) If P = {all $n \times 1$ column matrices with entries from $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$}, then P is a semigroup under '+'.

(9) P is a semigroup under $\times_n$.

(10) $\{P, +, \times_n\}$ is a semiring, P is infact a strict semiring but P is not a semifield.

(11) If we take K = {all $n \times 1$ column matrices with entries from Z^+ or Q^+ or R^+} $\cup \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$, then K is a semifield.

(12) P mentioned in (8) is a semivector space over $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ depending on the fact; from which P takes its entries. Infact P is a semilinear algebra over the appropriate semifield.

(13) P is also a semilinear algebra over the semifield K defined in (11)

Now thus we see giving a natural extension of product '$\times_n$' to column matrices one can define several nice algebraic structures on them.

We will be calling these product which exists / defined on matrices as usual product and the product $\times_n$ defined on matrices will be known as the natural product. It is pertinent to record that usual products cannot be defined on column matrices.

Now we proceed onto work with the natural product $\times_n$ on rectangular matrices.

Let

P = {all m × n matrices with entries from C or Z or Q or R} $m \neq n$. We know we can define only addition on P. The usual product on matrices cannot be defined for we do not have the compatibility.

However we can define the natural product.

When we have a row vector we see the natural product and the usual product coincide so we extend the natural product on P with m = 4 and n = 6.

We will illustrate it by an example.

$$\text{Let } x = \begin{bmatrix} 3 & 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 5 & 2 \\ 0 & 7 & 2 & 5 & 0 & 1 \\ 8 & 0 & 1 & 2 & 1 & 3 \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 7 & 2 & 0 & 1 & 0 & 5 \\ 4 & -1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ -7 & -8 & 0 & 1 & 0 & -2 \end{bmatrix}$$

be any two 4 × 6 matrices. Certainly we cannot define the usual product ×. Now we define the natural product $\times_n$.

$$x \times_n y = \begin{bmatrix} 3 & 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 5 & 2 \\ 0 & 7 & 2 & 5 & 0 & 1 \\ 8 & 0 & 1 & 2 & 1 & 3 \end{bmatrix} \times_n \begin{bmatrix} 7 & 2 & 0 & 1 & 0 & 5 \\ 4 & -1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ -7 & -8 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 0 & 2 & 0 & 5 \\ 4 & -1 & 0 & 0 & 10 & 6 \\ 0 & 14 & 6 & 20 & 0 & 6 \\ -56 & 0 & 0 & 2 & 0 & -6 \end{bmatrix} \in P.$$

This is the way the natural product is defined. Consider

$$x + y = \begin{bmatrix} 3 & 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 5 & 2 \\ 0 & 7 & 2 & 5 & 0 & 1 \\ 8 & 0 & 1 & 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 7 & 2 & 0 & 1 & 0 & 5 \\ 4 & -1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ -7 & -8 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 2 & 1 & 3 & 0 & 6 \\ 5 & 0 & 1 & 1 & 7 & 5 \\ 1 & 9 & 5 & 9 & 5 & 7 \\ 1 & -8 & 1 & 3 & 0 & 1 \end{bmatrix} \in P.$$

We see if + is replaced by $\times_n$ the product we get is $x \times_n y$.

It is important to note that when + and $\times$ are related on the reals on row vectors we can give a natural extension of product on a rectangular array of numbers of same order.

Further we see the unit in case of a $m \times n$ matrix is a $m \times n$ rectangular array of ones.

That is $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ acts as the unit under natural

product $\times_n$ on 4×6 matrices with entries from Q or Z or R or C.

$$\text{Thus if } x = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 3 & 2 & 1 & 6 & 1 \\ 7 & 0 & 8 & 0 & 9 & 0 \end{bmatrix} \text{ and}$$

$$I = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ then } x \times_n I$$

$$= I \times_n x = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 3 & 2 & 1 & 6 & 1 \\ 7 & 0 & 8 & 0 & 9 & 0 \end{bmatrix} = x.$$

Rectangular array of all ones acts as the multiplicative identity.

Now we give the algebraic structure enjoyed by
P = {all $m \times n$ matrices with entries from Q or Z or R or C}; $m \neq n$.

(1) P is an abelian group under addition.

(2) P under $\times_n$; the natural product is a semigroup with unit or a monoid.

$(P, \times_n)$ is a commutative semigroup with zero divisors. Further this semigroup in general has subsemigroups and ideals.

(3) $(P, +, \times_n)$ is a commutative ring with unit and has zero divisors. Elements in P has inverse only if the entries of P are from Q or R or C.

P has ideals and subrings which are not ideals. P is a S-ring if P takes its entries from Q or R or C.

(4) We know P is a vector space over Q (or R or C) if it takes its entries from Q (or R or C).

However P is not a linear algebra under usual product. P can be given the linear algebra status by defining natural product $\times_n$ on P.

(5) If S = {all $m \times n$ matrices $m \neq n$ with entries from $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$}, then S is a semigroup under usual addition.

(6) $(S, \times_n)$ is a semigroup under the natural product $\times_n$.

(7) $(S, +, \times_n)$ is a semiring.

(8) $(S, +, \times_n)$ is not a semifield however $(S, +, \times_n)$ is a strict semiring with unit under $\times_n$.

(9) Suppose we define T = {all $m \times n$ $(m \neq n)$ matrices with entries from Z^+ or Q^+ or R^+} $\cup$ $\{(0)_{m\times n}\}$ then $(T, +, \times_n)$ is a semifield.

(10) S in (5) is a semivector space or semilinear algebra over the semifield T.

Several other interesting properties can be derived and (S, $\times_n$) is a commutative semigroup having ideals and subsemigroups.

All properties associated with semivector spaces / semilinear algebras can be studied in the case of these structures also.

Now without the notion of natural product $\times_n$ we will not be in any position to study these algebraic structure.

Further we can use on these collection S or P 'min' or 'max' operations and they under 'min' (or max) operation are idempotent semigroups of infinite order. Also usual operations can be done on these matrices:

This sort of using min or max operation of rectangular matrices (under usual product) can be used in mathematical models when the entries in specific are taken from the unit interval [0, 1].

Thus solving or working with natural product will become popular in due course of time, that is once the working becomes common among researchers.

Next we proceed onto define natural product on square matrices.

Let
V = {all n × n matrices with entries from Z or Q or C or R}.
V is a non commutative ring under usual product of matrices. But (V, +, $\times_n$) is a commutative ring under natural product.

$$I = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$ acts as the identity for natural product $\times_n$.

However $I_{n \times n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$ is a zero divisor in V under the natural product $\times_n$.

The following are the algebraic structures enjoyed by V.

(i) (V, +) is an abelian group under addition.

(ii) (V, +) is a vector space / linear algebra over Q or R or C if V has entries in Q or R or C respectively.

(iii) (V, $\times$) is a non commutative monoid with zero divisors and ideals.

(iv) (V, $\times_n$) is a commutative monoid with zero divisors ideals and subsemigroups.

(v) (V, +, $\times$) is a non commutative ring with unit and zero divisors.

(vi) (V, +, $\times_n$) is a commutative ring with unit, zero divisors ideals and subrings.

(vii) (V, $\times_n$) is also a linear algebra over the field Q or R or C if the entries of V are from Q or R or C respectively.

(viii) If S = {all $n \times n$ matrices with entries from $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$} then V is a semiring under $\times_n$ and not a semifield.

(ix) S is a non commutative semiring under the natural product and is not a semifield.

(x) If P = {all $n \times n$ matrices with entries from Z^+ or R^+ or Q^+} $\cup$ {(0)} then P is a semifield under natural product.

(xi) S is a semilinear algebra / semivector space over the semifield $Z^+ \cup \{0\}$ (or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$).

(xii) S is infact a semilinear algebra / semivector space over the semifield P mentioned in (x).

Several properties enjoyed by these structures can be derived by any interested reader. Thus using natural product on matrices enables the natural extension of product defined on row matrices. However the usual product can be realized as a special product that can be used to determine operations on them and finding product of two order of matrices of different orders.

Now having seen the notion of natural product we just make a mention in case of fuzzy entries the natural product $\times_n$ on them is compatible. Further we can say that if A is a matrix with entries from [0, 1] and no entry in A is 1 then we can say

$A \times_n A \times_n A \ldots \times_n A$ say A^m (m very large) will tend to the zero matrix of same order as that of A.

So this property of fuzzy matrices may be helpful to us if we are interested in finding approximately a matrix tending to a zero matrix. However if we replace $\times_n$ by min (or max) A will continue only to be A that is min {A, A} = A and max {A, A} = A.

We will illustrate the natural product $\times_n$ on a fuzzy matrix.

Let $A = \begin{bmatrix} 0.003 & 0.02 & 0.0001 \\ 0.001 & 0.001 & 0.004 \\ 0.012 & 0.0015 & 0.01 \end{bmatrix}$ be a 3×3 fuzzy matrix.

$$A \times_n A = \begin{bmatrix} 0.000009 & 0.0004 & 0.00000001 \\ 0.000001 & 0.000001 & 0.000016 \\ 0.000144 & 0.00000225 & 0.0001 \end{bmatrix}$$

$A \times_n A \times_n A \times_n A =$

$$\begin{bmatrix} 8.1\times10^{-11} & 1.6\times10^{-7} & 1\times10^{-16} \\ 1\times10^{-12} & 1\times10^{-12} & 2.56\times10^{-10} \\ 2.0736\times10^{-8} & 5.0625\times10^{-12} & 1\times10^{-8} \end{bmatrix};$$

so we see A under natural product that is $A^m = \underbrace{A \times_n A \times_n \ldots \times_n A}_{n-\text{times}}$ tends to (0) for m just large.

Now we proceed onto make a mention that this natural product can be used for matrices with polynomial entries.

$$\text{That is if } A = \begin{bmatrix} x^3+1 & 5x-1 & 7x^2+x+1 \\ 3x^2-4 & 6 & 4x \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} x^2 & 3x+1 & 0 \\ 4 & 7x^2+2x+1 & 13x^3+1 \end{bmatrix}$$

are two matrices with polynomial entries we can find

$$A \times_n B = \begin{bmatrix} x^5+x^2 & 15x^2+2x-1 & 0 \\ 12x^2-16 & 4x^2+12x+6 & 52x^4+4x \end{bmatrix}.$$

Also we know differentiation and integration can be done on A.

$$\frac{dA}{dx} = \begin{bmatrix} 3x^2 & 5 & 14x+1 \\ 6x & 0 & 4 \end{bmatrix}.$$

$$\int A\,dx = \begin{bmatrix} \dfrac{x^4}{4}+x+c_1 & \dfrac{5x^2}{2}-x+c_3 & \dfrac{7x^3}{3}+\dfrac{x^2}{2}+x+c_3 \\ \dfrac{3x^3}{3}-4x+c_2 & 6x+c_5 & \dfrac{4x^2}{2}+c_6 \end{bmatrix}.$$

Now if A is matrix of any order the product $\times_n$, differentiation and integration can be performed, where A takes its entries from R[x].

Now we can also work with polynomials with matrix coefficients.

For instance

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} a_i^1 \\ a_i^2 \\ a_i^3 \\ a_i^4 \end{bmatrix},\ a_i^t \in R, 1 \le t \le 4\right\},$$

P is a commutative ring with unit $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ under natural product $\times_n$.

$$\text{Take } p(x) = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix} x^3$$

and

$$q(x) = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 7 \\ -2 \end{bmatrix} x^8 \text{ in P.}$$

$$p(x) + q(x) = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 1 \\ 7 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix} x^3 + \begin{bmatrix} 1 \\ 0 \\ 7 \\ -2 \end{bmatrix} x^8.$$

$$p(q) \times_n q(x) = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} \times_n \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} x^4$$

$$+ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 0 \\ 7 \\ 2 \end{bmatrix} x^8 + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 0 \\ 7 \\ 2 \end{bmatrix} x^9 + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 0 \\ 7 \\ 2 \end{bmatrix} x^{11}$$

$$= \begin{bmatrix} 12 \\ 0 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 20 \\ 2 \\ 2 \\ 0 \end{bmatrix} x^4 + \begin{bmatrix} 3 \\ 0 \\ 7 \\ 10 \end{bmatrix} x^8 + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 14 \end{bmatrix} x^9 + \begin{bmatrix} 5 \\ 0 \\ 14 \\ 0 \end{bmatrix} x^{11}.$$

Thus P under $\times_n$ is a semigroup thus P is a commutative polynomial ring with 4×1 column matrix coefficients.

We can also replace the column matrix coefficients with a row matrix coefficients or a rectangular matrix coefficients or a $n \times n$ square matrix coefficients still P will continue to be a commutative ring with zero divisors.

We see usual polynomial rings over R or Z or Q do not have zero divisors however these polynomial rings have zero divisors.

Suppose

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} a_j \in R; 1 \le j \le 15 \right\}$$

be a polynomial ring with coefficients as 5 × 3 rectangular matrices. (S, +, $\times_n$) is a ring which is communicative and has zero divisors. We can integrate or differentiate these polynomials provided the entries of these matrices are from R or Q.

We just give a simple illustration.

$$p(x) = \begin{bmatrix} 7 & 0 & 1 \\ 5 & 3 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 7 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & -1 \\ 6 & 7 & 8 \\ 3 & 0 & 5 \end{bmatrix} x^2 + \begin{bmatrix} 11 & 0 & 1 \\ -1 & 2 & 3 \\ 5 & 6 & -1 \\ 7 & 8 & 3 \\ 0 & 1 & 4 \end{bmatrix} x^4$$

be in S.

Now differentiating with respect to x we get

$$\frac{dp(x)}{dx} = 2\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & -1 \\ 6 & 7 & 8 \\ 3 & 0 & 5 \end{bmatrix} x + 4\begin{bmatrix} 11 & 0 & 1 \\ -1 & 2 & 3 \\ 5 & 6 & -1 \\ 7 & 8 & 3 \\ 0 & 1 & 4 \end{bmatrix} x^3$$

$$= \begin{bmatrix} 2 & 2 & 4 \\ 4 & 2 & 6 \\ 8 & 10 & -2 \\ 12 & 14 & 16 \\ 6 & 0 & 10 \end{bmatrix} x + \begin{bmatrix} 44 & 0 & 4 \\ -4 & 8 & 12 \\ 20 & 24 & -4 \\ 28 & 32 & 12 \\ 0 & 4 & 16 \end{bmatrix} x^3.$$

$$\text{Clearly } \frac{dp(x)}{dx} \text{ is in S.}$$

Consider $\int p(x)\, dx =$

$$\begin{bmatrix} 7 & 0 & 1 \\ 5 & 3 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 7 & 5 \end{bmatrix} x + \frac{1}{3} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & -1 \\ 6 & 7 & 8 \\ 3 & 0 & 5 \end{bmatrix} x^3 + \frac{1}{5} \begin{bmatrix} 11 & 0 & 1 \\ -1 & 2 & 3 \\ 5 & 6 & -1 \\ 7 & 8 & 3 \\ 0 & 1 & 4 \end{bmatrix} x^5 + c$$

$$= \begin{bmatrix} 7 & 0 & 1 \\ 5 & 3 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 7 & 5 \end{bmatrix} + \begin{bmatrix} 1/3 & 1/3 & 2/3 \\ 2/3 & 1/3 & 1 \\ 4/3 & 5/3 & -1/3 \\ 2 & 7/3 & 8/3 \\ 1 & 0 & 5/3 \end{bmatrix} x^3$$

$$+ \begin{bmatrix} 11/5 & 0 & 1/5 \\ -1/5 & 2/5 & 3/5 \\ 1 & 6/5 & -1/5 \\ 7/5 & 8/5 & 3/5 \\ 0 & 1/5 & 4/5 \end{bmatrix} x^5 + c \text{ is in S.}$$

Thus integration and differentiation on these polynomial rings with matrix coefficients can be done provided these matrices take their entries from Q or R.

We can also define polynomial rings with square matrix coefficients. We can using usual matrix product get a non commutative ring and using natural product we get a commutative ring.

We just see they are not isomorphic further if

$$p(x), q(x) \in P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } a, b, c, d \in R \right\}$$

is the polynomial ring with matrix coefficients and if

$$p(x) = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} x^3 \text{ and}$$

$$q(x) = \begin{bmatrix} 9 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 2 & 0 \end{bmatrix} x^2 \text{ are in P;}$$

to find $p(x) \times q(x)$ the usual product.

$$p(x) \times q(x) = \begin{bmatrix} 18 & 4 \\ 11 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 14 \\ 4 & 7 \end{bmatrix} x^2 + \begin{bmatrix} 11 & 2 \\ 0 & 0 \end{bmatrix} x$$

$$+ \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} x^3 + \begin{bmatrix} 9 & 14 \\ 1 & 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 & 7 \\ 2 & 0 \end{bmatrix} x^5 \in P.$$

$$\text{Now } p(x) \times_n q(x) = \begin{bmatrix} 18 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 9 & 4 \\ 0 & 0 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0 & 14 \\ 0 & 0 \end{bmatrix} x^3 + \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x^5 \in P.$$

Clearly $p(x) \times q(x)$ is of degree 5 where as $p(x) \times_n q(x)$ is of degree three.

We see in general the rules of usual polynomials need not in general be true for polynomials with matrix coefficients.

However if the matrix coefficients are column matrices or rectangular matrices those are not polynomial rings under usual product they are polynomial commutative rings only under the natural product, $\times_n$.

We see we can define integration and differentiation, however we cannot say all polynomials can be made monic even under natural product $\times_n$.

Under natural product $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ or (1 1 1 1 1 1) or $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ or

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

are the units and under usual product $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to be more precise $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ are the units; but they are

zero divisors under natural product $\times_n$.

Next we can say

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i\text{'s are row matrices with entries from R or Q} \right\}$$

is a vector space as well as linear algebra over R or Q both under usual product as well as under natural product.

Consider

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ where } b_j \in Q \text{ or } R;\ 1 \leq j \leq n \right\};$$

M is only a vector space under as usual product is not defined where as M is a linear algebra under the natural product. We see likewise if the coefficients of the polynomial are rectangular matrices then they are only vector spaces but under natural product we see they are linear algebras.

Finally if

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i\text{'s are square matrices with entries from R or}\right.$$

Q} then S is a linear algebra over R both under the usual product and under the natural product.

We see we can also define semirings and semifields.

Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (x_1, x_2, \ldots, x_n) \text{ where } x_i \in R^+ \cup \{0\} \right.$$

$$\text{or } Q^+ \cup \{0\} \text{ or } Z^+ \cup \{0\},\ 1 \leq i \leq n\},$$

be a semiring under + and $\times$ (or under $\times_n$). P is only a semiring; P is a strict semiring but is not a semifield as $p(x), q(x) \in P$ can be such that $p(x) \times q(x) = (0)$.

Now some modification be made so that P can be a semifield; this is done in a special way.

Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (x_1, \ldots, x_n) \text{ where } x_i \in R^+ \text{ or } Q^+ \text{ or } Z^+ \right.$$
$$\left. \text{together with } a_0 = (0, 0, \ldots, 0),\ 1 \leq i \leq n \right\}.$$

S is a semifield under $\times_n$ or $\times$ as both the natural product and the usual product on row matrices are identical.

S is a semilinear algebra over $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ depending on where the entries of a_i's are taken.

Also if $F = \{(x_1, x_2, \ldots, x_n)$ where $x_i \in Q^+$ or R^+ or Z^+; $1 \leq i \leq n\} \cup \{(0, 0, \ldots, 0)\}$ then F is a semifield.

We see S is also a semilinear algebra over the semifield F; depending on the choice of the semifield we see the dimension of S varies.

Consider

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \text{ where } y_i \in Q^+ \cup \{0\} \text{ or } Z^+ \cup \{0\} \right.$$
$$\left. \text{or } R^+ \cup \{0\};\ 1 \leq i \leq m \right\};$$

T is a semilinear algebra if $\times_n$ the natural product is defined on T; however T is only a semivector space for the usual product cannot be defined on T over the semifield $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ according as the set over which T is defined.

$$\text{Let } B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \,\middle|\, a_i \in Z^+ \text{ or } Q^+ \text{ or } R^+;\ 1 \leq i \leq m \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\};$$

then B is a semifield under '+' and $\times_n$.

Now T is a semilinear algebra over the semifield B. Likewise we can define semivector space / semilinear algebra over $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ or

A = {all $m \times n$ ($m \neq n$) matrices with entries from Z^+ or Q^+ or R^+} $\cup$ {(0); $m \times n$ zero matrix}. A is a semifield under $\times_n$, the natural product.

Consider $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i\text{'s are } m \times n\ (m \neq n) \text{ rectangular}\right.$ matrices with entries from $Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$)}; P is a semivector space over $Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$). But P is a semilinear algebra over the semifield; under the natural product $\times_n$. This varying definition directly affect the dimension of P over which ever semifield they are defined.

Finally when we take a square matrix as coefficients of a polynomial say $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_s = n \times n \text{ matrices with entries}\right.$ from $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$}. P is a semivector space / semilinear algebra over the semifield $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$.

We can find all properties associated with it. If we take F = {all $n \times n$ matrices with entries from Z^+ or R^+ or Q^+} $\cup$ {(0)} then F is a semifield and P is a semivector space or a semilinear algebra over the semifield F.

Interested reader can find dimension of P over F.

Now having seen polynomials with matrix coefficients and polynomials as semivector spaces / semilinear algebras we now proceed onto define and describe interval matrices and matrices with interval entries.

Let V = $\{([a_1^1, a_2^1], [a_1^2, a_2^2], \ldots, [a_1^n, a_2^n]) \mid a_i^t \in R; 1 \leq i \leq 2$ and $1 \leq t \leq n\}$ be a row interval matrix with entries from $N_c(R)$. V is a ring of row interval matrices.

Consider W = $\{[(a_1^1, a_2^1, \ldots, a_1^n), (a_2^1, a_2^2, \ldots, a_2^n)] \mid a_i^t \in R; 1 \leq i \leq 2$ and $1 \leq t \leq n\}$; W is a interval row matrix. W is also a ring.

We see there is a one to one correspondence between V and W.

Define $\eta : V \to W$ as $\eta(([a_1^1, a_2^1], [a_1^2, a_2^2], \ldots, [a_2^n, a_2^n]))$
$= ([a_1^1, a_2^1, \ldots, a_1^n], [a_2^1, a_2^2, \ldots, a_2^n])$;

η is one to one and on to.

Suppose we define

$\phi : W \to V$ by $\phi([a_1^1, a_2^1, \ldots, a_1^n], [a_2^1, a_2^2, \ldots, a_2^n])$
$= ([a_1^1, a_2^1], [a_1^2, a_2^2], \ldots, [a_1^n, a_2^n])$

then ϕ is also one to one and onto and $\eta \circ \phi$ = identity map V and $\phi \circ \eta$ is the identity map on W.

Now having got a one to one on to correspondence we can work with V or W; they are infact, isomorphic as rings.

Suppose

$$M = \left\{ \left. \begin{bmatrix} [a_1^1, a_2^1] \\ [a_1^2, a_2^2] \\ \vdots \\ [a_1^m, a_2^m] \end{bmatrix} \right| [a_1^t, a_2^t] \in N_c(R); 1 \leq t \leq m \right\};$$

we define M as a column matrix with interval entries.

Consider

$$N = \left\{\left(\begin{bmatrix} a_1^1 \\ a_1^2 \\ \vdots \\ a_1^m \end{bmatrix}, \begin{bmatrix} a_2^1 \\ a_2^2 \\ \vdots \\ a_2^m \end{bmatrix}\right) \middle| \; a_i^t \in R;\ 1 \leq i \leq 2 \text{ and } 1 \leq t \leq m\right\};$$

N is an interval column matrix. Clearly both M and N are rings under natural product of matrices.

$$\text{If } x = \begin{bmatrix} [a_1^1, a_2^1] \\ [a_1^2, a_2^2] \\ \vdots \\ [a_1^m, a_2^m] \end{bmatrix} \text{ and } y = \begin{bmatrix} [b_1^1, b_2^1] \\ [b_1^2, b_2^2] \\ \vdots \\ [b_1^m, b_2^m] \end{bmatrix} \text{ are in M then}$$

$$x \times_n y = \begin{bmatrix} [a_1^1 b_1^1, a_2^1 b_2^1] \\ [a_1^2 b_1^2, a_2^2 b_2^2] \\ \vdots \\ [a_1^m b_1^m, a_2^m b_2^m] \end{bmatrix}.$$

M is a commutative ring.

$$\text{Consider } a = \left(\begin{bmatrix} a_1^1 \\ a_1^2 \\ \vdots \\ a_1^m \end{bmatrix}, \begin{bmatrix} a_2^1 \\ a_2^2 \\ \vdots \\ a_2^m \end{bmatrix}\right) \text{ and } b = \left(\begin{bmatrix} c_1^1 \\ c_1^2 \\ \vdots \\ c_1^m \end{bmatrix}, \begin{bmatrix} c_2^1 \\ c_2^2 \\ \vdots \\ c_2^m \end{bmatrix}\right) \text{ in N.}$$

$$a \times_n b = \left(\begin{bmatrix} a_1^1 c_1^1 \\ a_1^2 c_1^2 \\ \vdots \\ a_1^m c_1^m \end{bmatrix}, \begin{bmatrix} a_2^1 c_2^1 \\ a_2^2 c_2^2 \\ \vdots \\ a_2^m c_2^m \end{bmatrix}\right) \in N.$$

N is also a commutative ring with $\left(\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix},\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right)$ as unit we can define

$\eta : M \to N$ by

$$\eta\begin{pmatrix}[a_1^1,a_2^1]\\[a_1^2,a_2^2]\\\vdots\\[a_1^m,a_2^m]\end{pmatrix}=\left(\begin{bmatrix}a_1^1\\a_1^2\\\vdots\\a_1^m\end{bmatrix},\begin{bmatrix}a_2^1\\a_2^2\\\vdots\\a_2^m\end{bmatrix}\right).$$

η is a one to one onto homomorphism from the ring M to N.

Consider $\phi : N \to M$ given by

$$\phi\left(\begin{bmatrix}a_1^1\\a_1^2\\\vdots\\a_1^m\end{bmatrix},\begin{bmatrix}a_2^1\\a_2^2\\\vdots\\a_2^m\end{bmatrix}\right)=\begin{pmatrix}[a_1^1,a_2^1]\\[a_1^2,a_2^2]\\\vdots\\[a_1^m,a_2^m]\end{pmatrix}.$$

ϕ is also a one to one onto homomorphism from N to M. Clearly $\phi \circ \eta$ is an identity on N and $(\eta \circ \phi)$ is an identity on M.

Thus we see we can go from an interval column matrix ring to a column interval matrix ring and vice versa.

Likewise we can define a rectangular matrix with intervals entries and interval rectangular matrix as follows:

$$\text{Let } X = \left\{\begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \ldots & [a_{1n}^1, a_{1n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{m1}^1, a_{m1}^2] & [a_{m2}^1, a_{m2}^2] & \ldots & [a_{mn}^1, a_{mn}^2] \end{pmatrix}\right.$$

$$\text{where } [a_{ij}^1, a_{ij}^2] \in N_c(R);\ m \neq n,\ 1 \leq i \leq m \text{ and } 1 \leq j \leq n\Big\}$$

be the collection of all $m \times n$ matrices with interval entries from $N_c(R)$. It is easily verified X under natural product $\times_n$, and componentwise addition is a commutative ring with unit;

$$I = \begin{pmatrix} [1,1] & [1,1] & \ldots & [1,1] \\ [1,1] & [1,1] & \ldots & [1,1] \\ \vdots & \vdots & & \vdots \\ [1,1] & [1,1] & \ldots & [1,1] \end{pmatrix}.$$

Let

$$Y = \left\{\left(\begin{bmatrix} a_{11}^1 & a_{12}^1 & \ldots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \ldots & a_{2n}^1 \\ \vdots & \vdots & & \vdots \\ a_{m1}^1 & a_{m2}^1 & \ldots & a_{mn}^1 \end{bmatrix}, \begin{bmatrix} a_{11}^2 & a_{12}^2 & \ldots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \ldots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{m1}^2 & a_{m2}^2 & \ldots & a_{mn}^2 \end{bmatrix}\right)\right|$$

$$a_{ij}^t \in R;\ 1 \leq t \leq 2,\ 1 \leq i \leq m \text{ and } 1 \leq j \leq n,\ (m \neq n)\Big\}$$

be the collection of all rectangular or $m \times n$ interval matrices Y under matrix addition and natural product $\times_n$ is a commutative ring with

$$I = \left(\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}\right)$$

as the multiplicative identity. It can be easily proved that X is isomorphic to Y as rings. Thus one can always identity a interval rectangular matrix with interval entries and vice versa.

This is infact the main property which enables one to find easily the eigen value or eigen vectors finding determinant values etc., for square matrices with interval entries.

We will just describe this situation in a line or two.

Let

$$P = \left\{ \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \dots & [a_{1n}^1, a_{1n}^2] \\ [a_{21}^1, a_{21}^2] & [a_{22}^1, a_{22}^2] & \dots & [a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{nn}^1, a_{nn}^2] & [a_{nn}^1, a_{nn}^2] & \dots & [a_{nn}^1, a_{nn}^2] \end{pmatrix} \right.$$

$$\text{where } [a_{ij}^1, a_{ij}^2] \in N_c(R);\ 1 \leq i, j \leq n\}$$

be the collection of all $n \times n$ matrices with interval entries from $N_c(R)$. P is a non commutative ring under usual product.

However

$$L = \left\{ \left(\begin{bmatrix} a_{11}^1 & a_{12}^1 & \dots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \dots & a_{2n}^1 \\ \vdots & \vdots & & \vdots \\ a_{n1}^1 & a_{n2}^1 & \dots & a_{nn}^1 \end{bmatrix}, \begin{bmatrix} a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \end{bmatrix} \right) \right|$$

$$a_{ij}^t \in R;\ 1 \leq t \leq 2,\ 1 \leq i, j \leq n\}$$

is the collection of all interval $n \times n$ square matrices. L under usual product is a non commutative ring. Infact P is isomorphic with L when both P and L enjoy the usual matrix product. However if we define the natural product $\times_n$ on both P and L we see both P and L are commutative rings and P is isomorphic with L and vice versa.

We will just illustrate this situation by 2×2 matrices.

Let

$$X = \left(\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}\right) \text{ and } Y = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix}\right)$$

be two elements in L. We find $X \times Y$ using the usual matrix product.

$$X \times Y = \left(\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 6 & 10 \\ 13 & 16 \end{bmatrix}, \begin{bmatrix} -23 & 3 \\ 25 & 9 \end{bmatrix}\right) \text{ and}$$

$$Y \times X = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}, \begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} -1 & 11 \\ 1 & 23 \end{bmatrix}, \begin{bmatrix} -34 & -5 \\ 40 & 2 \end{bmatrix}\right).$$

Clearly $X \times Y \neq Y \times X$.

Now we consider the natural product of X with Y; that is

$$X \times_n Y = \left(\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}\right) \times_n \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} \times_n \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix} \times_n \begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 3 & 2 \\ -6 & 20 \end{bmatrix}, \begin{bmatrix} -35 & 2 \\ 6 & -10 \end{bmatrix} \right) \in \text{L}.$$

Now we can write the interval matrix

$$\text{X} = \left(\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)$$

as the matrix with interval entries as

$$\text{X}' = \begin{pmatrix} [3,1] & [1,2] \\ [-2,3] & [5,4] \end{pmatrix} \text{ and } \text{Y} = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 1 \\ 6 & -2 \end{bmatrix} \right) \text{ as}$$

$$\text{Y}' = \begin{pmatrix} [1,-5] & [2,1] \\ [3,6] & [4,-2] \end{pmatrix}.$$

$$\text{Now } \text{X}' \times_n \text{Y}' = \begin{pmatrix} [3,1] & [1,2] \\ [-2,3] & [5,4] \end{pmatrix} \times_n \begin{pmatrix} [1,-5] & [2,1] \\ [3,6] & [4,-2] \end{pmatrix}$$

$$= \begin{pmatrix} [3,1][1,-5] & [1,2][2,1] \\ [2,3][3,6] & [5,4][4,-2] \end{pmatrix}$$

$$= \begin{pmatrix} [3,-5] & [2,2] \\ [-6,18] & [20,-8] \end{pmatrix}.$$

$$\text{X}' \times \text{Y}' = \begin{pmatrix} [3,1] & [1,2] \\ [-2,3] & [5,4] \end{pmatrix} \times \begin{pmatrix} [1,-5] & [2,1] \\ [3,6] & [4,-2] \end{pmatrix}$$

$$= \begin{pmatrix} [3,1][1,-5]+[1,2][3,6] & [3,1][2,1]+[1,2][4,-2] \\ [-2,3][1,-5]+[5,4][3,6] & [-2,3][2,1]+[5,4][4,-2] \end{pmatrix}$$

$$= \begin{pmatrix} [3,-5]+[3,12] & [6,1]+[4,-4] \\ [-2,-5]+[15,24] & [-4,3]+[20,-8] \end{pmatrix}$$

$$= \begin{pmatrix} [2,2] & [10,10] \\ [12,12] & [6,6] \end{pmatrix}.$$

$$A \times B = \begin{pmatrix} [2,1]\times[1,2]+[5,2]\times[4,3] & [2,1]\times[2,5]+[5,2]\times[1,6] \\ [3,4]\times[1,2]+[6,1]\times[4,3] & [3,4]\times[2,5]+[6,1]\times[1,6] \end{pmatrix}$$

$$= \begin{pmatrix} [22,8] & [9,17] \\ [27,11] & [12,26] \end{pmatrix}.$$

Suppose $A' = \left(\begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \right)$ and $B' = \left(\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} \right)$

$$A' \times_n B' = \left(\begin{bmatrix} 2 & 10 \\ 12 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 10 \\ 12 & 6 \end{bmatrix} \right)$$

$$\text{and } A' \times B' = \left(\begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 22 & 9 \\ 27 & 12 \end{bmatrix}, \begin{bmatrix} 8 & 17 \\ 11 & 26 \end{bmatrix} \right)$$

$$= \begin{pmatrix} [22,8] & [9,17] \\ [27,11] & [12,26] \end{pmatrix} \text{ and so on.}$$

Thus the value of $A \times B$ and $A' \times B'$ are the same likewise $A \times_n B$ and $A' \times_n B'$ are also the same.

Thus we can go from interval matrices to matrix with interval entries and vice versa.

Now we just find the determinant of both the interval square matrix and the square matrix with interval entries.

$$\text{Let } A = \left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix}\right) \text{ and } A' = \begin{pmatrix} [3,7] & [4,6] \\ [5,5] & [2,4] \end{pmatrix}$$

be the interval square matrix and matrix with interval entries respectively.

$$\text{Now det } A = \left(\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} 7 & 6 \\ 5 & 4 \end{vmatrix}\right)$$

$$= [-14, -2].$$

$$\text{det } A' = |A'| = \begin{vmatrix} [3,7] & [4,6] \\ [5,5] & [2,4] \end{vmatrix}$$

$$= [6, 28] - [20, 30]$$

$$= [-14, -2].$$

$$\text{We see } |A'| = |A|.$$

Now this method of writing a matrix with interval entries into an interval matrix helps us to find all properties in a feasible time for all matrices with interval entries can be realized as interval matrices and vice versa.

At this juncture the authors wish to state that instead of defining product of intervals in a round about manner one can extend the product in a natural way using the natural class of intervals. This definition happens to be in keeping with the operations on the reals and so only we define these new class of intervals as natural class of intervals and product on these matrices with interval entries or product of any m × n matrix with itself as the natural product. This has helped us in many ways by solving the compabity of the product.

Next we proceed onto define matrices with entries from subsets of a set.

Let X be a finite set.

P(X) = {collection of all subsets of X together with ϕ and X}; P(X) is a Boolean algebra.

Now we can define matrices with entries from P(X) and define operations of union '$\cup$' and intersection '$\cap$' and complementation. We will just see the algebraic structure enjoyed by these matrices.

Let M = $\{(a, b) \mid a, b \in P(X) = P(\{1, 2, 3\})\}$.

$|M| = o(M) = 8 \times 8 = 64.$

$A = (\{1\}, \{3\}),\ A^C = (\{2, 3\}, \{1, 2\}).$

$A \cup A^C = \{(X, X)\}$ and $A \cap A^C = \{(\phi, \phi)\}$.

$$\begin{aligned} A \cap (B \cup C) &= (\{1,2\}, \{1\}) \cap ((\{2\},\{3\}) \cup (\{1,2\}, \{1,3\})) \\ &= (\{1, 2\}\ \{1\}) \cap (\{1, 2\}, \{1, 3\}) \\ &= (\{1, 2\}, \{1\}). \end{aligned}$$

$$\begin{aligned} A \cup (B \cap C) &= (\{1,2\}, \{1\}) \cup \{(\{2\},\{3\}) \cap (\{1,2\}, \{1,3\}) \\ &\quad (\{1, 2\}\ \{1\}) \cup (\{2\}, \{3\}) \\ &= (\{1, 2\}, \{1, 3\}). \end{aligned}$$

Consider $(A \cap B) \cup (A \cap C)$

$$\begin{aligned} &= [(\{1,2\},\{1\}) \cap (\{2\},\{3\})] \cup [(\{1, 2\},\{1\}) \cap \\ &\quad (\{1,2\}, \{1, 3\})] \\ &= (\{2\}, \phi) \cup (\{1, 2\}, \{1\}) \\ &= (\{1, 2\}, \{1\}). \end{aligned}$$

Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We see M is again a Boolean algebra. We can extend the row matrix with entries from the power set P (X) of a set X.

Let P = $\{(a_1, a_2, \ldots, a_n) \mid a_i \in P(X); 1 \leq i \leq n\}$ be the collection of all row matrices with entries from the sets in P(X) or subsets of X.

P under the operation $\cup$ and $\cap$ is a Boolean algebra will be known as the row matrix Boolean algebra.

Let

$$S = \left\{\left[\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{matrix}\right] \middle| a_i \in P(X); P(X) \text{ the power set of a set } X; 1 \leq i \leq m\right\}$$

be the set of all column matrices with subsets of X or from the elements of the power set P(X). We can define '$\cup$' and '$\cap$' on S as follows:

$$\text{If } x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ are in S then}$$

$$x \cap y = \begin{bmatrix} a_1 \cap b_1 \\ a_2 \cap b_2 \\ \vdots \\ a_m \cap b_m \end{bmatrix} \text{ and } x \cup y = \begin{bmatrix} a_1 \cup b_1 \\ a_2 \cup b_2 \\ \vdots \\ a_m \cup b_m \end{bmatrix} \text{ are in S.}$$

We will illustrate this situation by an example.

Let X = {1, 2, 3, 4} P (X) = {ϕ, X, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, …, {2, 3, 4}}.

Clearly |P (X)| = 2^4 = 16.

Consider S = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_i \in P(X), 1 \le i \le 3 \right\}$.

We just show how '$\cup$' and '$\cap$' are defined on S.

Let x = $\begin{bmatrix} \{3,4\} \\ \{1,2,3\} \\ \{4,2\} \end{bmatrix}$ and y = $\begin{bmatrix} \{2,3\} \\ \{3,4,2\} \\ \{1,2,3\} \end{bmatrix}$ be in S.

$$x \cup y = \begin{bmatrix} \{2,3,4\} \\ \{1,2,3,4\} = X \\ \{1,2,3,4\} = X \end{bmatrix}$$

$$= \begin{bmatrix} \{2,3,4\} \\ X \\ X \end{bmatrix}.$$

$x \cap y = \begin{bmatrix} \{3\} \\ \{3,2\} \\ \{2\} \end{bmatrix}$ both $x \cap y$ and $x \cup y$ are in S.

Now $x^c = \begin{bmatrix} \{1,2\} \\ \{4\} \\ \{1,3\} \end{bmatrix}$ and $x \cup x^c = \begin{bmatrix} X \\ X \\ X \end{bmatrix}$.

Consider

$$x \cap x^c = \begin{bmatrix} \phi \\ \phi \\ \phi \end{bmatrix}.$$

Likewise

$$y^c = \begin{bmatrix} \{1,4\} \\ \{1\} \\ \{4\} \end{bmatrix}, \; y \cup y^c = \begin{bmatrix} X \\ X \\ X \end{bmatrix} \text{ and } y \cap y^c = \begin{bmatrix} \phi \\ \phi \\ \phi \end{bmatrix}.$$

Clearly S is a Boolean algebra of column matrices.

Let us define the collection of all $m \times n$ matrices with entries from P (X).

P = {all $m \times n$ ($m \neq n$) matrices with entries from P (X); the power set of a set X}.

$(P, \cup, \cap, ')$ is a Boolean algebra.

We will just illustrate this situation for X = {1, 2, 3, 4, 5}. P (X) = {ϕ, X, {1}, …, {5}, {1, 2}, …, {4, 5}, …, {1, 2, 3, 4}}.

Clearly |P (X)| = 32.

Take

$$x = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \text{ where } a_{ij} \in P(X);$$

$$1 \leq i \leq 3 \text{ and } 1 \leq j \leq 4.$$

Suppose

P = {collection of all 3×4 matrices with entries from P(X)}; P is a Boolean algebra of 3×4 matrices.

We see if $x = \begin{pmatrix} \phi & \{1\} & X & \{2,3\} \\ \{2\} & \{3,5\} & \{1,2,3\} & X \\ \{1,3\} & X & \phi & \{4,5,1\} \end{pmatrix}$ is in P, then

$$x^c = \begin{pmatrix} X & \{2,3,4,5\} & \phi & \{1,4,5\} \\ \{1,3,4,5\} & \{2,1,4\} & \{4,5\} & \phi \\ \{2,4,5\} & \phi & X & \{2,3\} \end{pmatrix} \in P.$$

Now $x \cup x^c = \begin{pmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{pmatrix}$ and

$$x \cap x^c = \begin{pmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{pmatrix} \in P.$$

Clearly $\begin{pmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{pmatrix}^c = \begin{pmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{pmatrix}$

and $\begin{pmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{pmatrix}^c = \begin{pmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{pmatrix}$.

Suppose $y = \begin{pmatrix} \{1,2,3\} & \{4,5\} & \phi & X \\ \{2,3\} & \{1,5,2\} & \{3\} & \{4,2\} \\ \{1,4,3\} & \{1,5\} & \{1,2\} & \{3,4\} \end{pmatrix} \in P$.

We find $x \cup y = \begin{pmatrix} \{1,2,3\} & \{1,4,5\} & X & X \\ \{2,3\} & \{1,3,5,2\} & \{1,2,3\} & X \\ \{1,4,3\} & X & \{1,2\} & \{1,4,5,3\} \end{pmatrix}$

$$\text{and } x \cap y = \begin{pmatrix} \phi & \{1\} & \phi & \{2,3\} \\ \{2\} & \{5\} & \{3\} & \{4,2\} \\ \{1,3\} & \{1,5\} & \phi & \{4\} \end{pmatrix} \text{are in P.}$$

One can verify (P, $\cup$, $\cap$, $'$) is a Boolean algebra of 3×4 matrices.

Finally let us consider S = {all $n \times n$ matrices with entries from P(X), the power set of a set X}; S is a Boolean algebra.

We can also define usual product on these square matrices with entries from P(X).

Let M = {all 3×3 matrices with entries from P (X) where X = {1, 2, 3}}.

Consider

$$x = \begin{pmatrix} \{1,2\} & X & \phi \\ \{2\} & \{1,3\} & \{3,2\} \\ X & \{3\} & \{1\} \end{pmatrix} \text{ in M. } x^c = \begin{pmatrix} \{3\} & \phi & X \\ \{1,3\} & \{2\} & \{1\} \\ \phi & \{1,2\} & \{2,3\} \end{pmatrix}$$

is in M.

$$x^c \cup x = \begin{pmatrix} X & X & X \\ X & X & X \\ X & X & X \end{pmatrix} \text{ and } x^c \cap x = \begin{pmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix} \text{ is in M.}$$

$$\text{Let } y = \begin{pmatrix} \{3,1\} & \{1,2\} & \{2,3\} \\ X & \{1\} & \{2\} \\ \{3\} & \{1,3\} & \{1\} \end{pmatrix} \text{ be in M.}$$

We find $x \cap y = \begin{pmatrix} \{3\} & \phi & \{2,3\} \\ \{1,3\} & \phi & \phi \\ \phi & \{1\} & \phi \end{pmatrix} \in M.$

$$x^c \cup y = \begin{pmatrix} \{1,3\} & \{1,2\} & X \\ X & \{1,2\} & \{1,2\} \\ \{3\} & \{1,2,3\} = X & X \end{pmatrix} \in M.$$

$$(x^c \cup y)^c = \begin{pmatrix} \{2\} & \{3\} & \phi \\ \phi & \{3\} & \{3\} \\ \{1,2\} & \phi & \phi \end{pmatrix}$$

$$x^c \cup y^c = \begin{pmatrix} \{3\} & \phi & X \\ \{1,3\} & \{2\} & \{1\} \\ \phi & \{1,2\} & \{2,3\} \end{pmatrix} \cup \begin{pmatrix} \{2\} & \{3\} & \{1\} \\ \phi & \{2,3\} & \{1,3\} \\ \{1,2\} & \{2\} & \{2,3\} \end{pmatrix}$$

$$= \begin{pmatrix} \{2,3\} & \{3\} & X \\ \{1,3\} & \{2,3\} & \{1,3\} \\ \{1,2\} & \{1,2\} & \{2,3\} \end{pmatrix} \in M.$$

$$(x \cup y)^c = \begin{pmatrix} X & X & \{2,3\} \\ X & \{1,3\} & \{2,3\} \\ X & \{1,3\} & \{1\} \end{pmatrix}^c$$

$$= \begin{pmatrix} \phi & \phi & \{1\} \\ \phi & \{2\} & \{1\} \\ \phi & \{2\} & \{2,3\} \end{pmatrix} \in M.$$

$$x^c \cap y^c = \begin{pmatrix} \phi & \phi & \{1\} \\ \phi & \{2\} & \{1\} \\ \phi & \{2\} & \{2,3\} \end{pmatrix} \in M.$$

$$(x \cup y)^c = x^c \cap y^c.$$

Several related results can be obtained.

Now we show how usual multiplication can be carried out on square matrices with entries from a power set P(X).

$$\text{Let } x = \begin{pmatrix} \{2\} & \phi & \{1,3\} \\ X & \{3\} & \{2,1\} \\ \{1\} & \{2\} & \phi \end{pmatrix} \text{ and } y = \begin{pmatrix} \{1,3\} & \{2\} & X \\ \phi & \{2,3\} & \{1\} \\ \{1,2\} & \phi & \{3\} \end{pmatrix}$$

be two elements in M. To find $x \times y$ the usual matrix product of x with y.

$$x \times y = \begin{pmatrix} \{2\} & \phi & \{1,3\} \\ X & \{3\} & \{2,1\} \\ \{1\} & \{2\} & \phi \end{pmatrix} \times \begin{pmatrix} \{1,3\} & \{2\} & X \\ \phi & \{2,3\} & \{1\} \\ \{1,2\} & \phi & \{3\} \end{pmatrix}$$

$$= \begin{pmatrix} (\{2\} \cap \{1,3\}) \cup (\phi \cap \phi) \cup \{1,3\} \cap \{1,2\} & \phi \cup \phi \cup \{1,3\} & \{2\} \cup \phi \cup \{3\} \\ \{1,3\} \cup \phi \cup \{1,2\} & \{2\} \cup \{3\} \cup \phi & X \cup \{\phi\} \cup \phi \\ \{1\} \cup \phi \cup \phi & \phi \cup \{2\} \cup \phi & \{1\} \cup \phi \cup \phi \end{pmatrix}$$

$$= \begin{pmatrix} \{1\} & \{1,3\} & \{2,3\} \\ X & \{2,3\} & X \\ \{1\} & \{2\} & \{1\} \end{pmatrix}.$$

$$\text{Consider } y \times x = \begin{pmatrix} \{1,3\} & \{2\} & X \\ \phi & \{2,3\} & \{1\} \\ \{1,2\} & \phi & \{3\} \end{pmatrix} \times \begin{pmatrix} \{2\} & \phi & \{1,3\} \\ X & \{3\} & \{2,1\} \\ \{1\} & \{2\} & \phi \end{pmatrix}$$

$$= \begin{pmatrix} \phi \cup \{2\} \cup \{1\} & \phi \cup \phi \cup \{2\} & \{1,3\} \cup \{2\} \cup \phi \\ \phi \cup \{2,3\} \cup \{1\} & \phi \cup \{3\} \cup \phi & \phi \cup \{2\} \cup \phi \\ \{2\} \cup \phi \cup \{\phi\} & \phi \cup \phi \cup \phi & \{1\} \cup \phi \cup \{0\} \end{pmatrix}$$

$$= \begin{pmatrix} \{1,2\} & \{2\} & X \\ X & \{3\} & \{2\} \\ \{2\} & \phi & \{1\} \end{pmatrix} \in M.$$

We see $x \times y \neq y \times x$ in general for $x, y \in M$. We do not call this structure with $\{M, \cup, \times\}$ as a Boolean algebra for (i) '$\times$' is not commutative on M. (ii) $\times$ is not an idempotent operation on M.

Further, will $x \times (y \cup z) = x \times y \cup x \times z$? We first verify this.

Consider S = {all 2×2 matrices with entries from P(X) where X = {1, 2, 3, 4}}.

Take

$$x = \begin{pmatrix} \{1\} & X \\ \{2,3\} & \{4\} \end{pmatrix}, \; y = \begin{pmatrix} \phi & \{1,2,3\} \\ \{3,4\} & \{1,4\} \end{pmatrix} \text{ and}$$

$$z = \begin{pmatrix} \{1,2\} & \{3,4\} \\ \{4,1\} & \{1,3,4\} \end{pmatrix} \text{ in S.}$$

Consider $x \times (y \cup z)$

$$= \begin{pmatrix} \{1\} & X \\ \{2,3\} & \{4\} \end{pmatrix} \times \begin{pmatrix} \{1,2\} & X \\ \{1,3,4\} & \{1,3,4\} \end{pmatrix}$$

$$= \begin{pmatrix} \{1,3,4\} & \{1,3,4\} \\ \{2,4\} & \{2,3,4\} \end{pmatrix} \qquad \text{I}$$

Consider $x \times y \cup x \times z$

$$= \begin{pmatrix} \{1\} & X \\ \{2,3\} & \{4\} \end{pmatrix} \times \begin{pmatrix} \phi & \{1,2,3\} \\ \{3,4\} & \{1,4\} \end{pmatrix} \cup$$

$$\begin{pmatrix} \{1\} & X \\ \{2,3\} & \{4\} \end{pmatrix} \times \begin{pmatrix} \{1,2\} & \{3,4\} \\ \{4,1\} & \{1,3,4\} \end{pmatrix}$$

$$= \begin{pmatrix} \{3,4\} & \{1,4\} \\ \{4\} & \{2,3,4\} \end{pmatrix} \cup \begin{pmatrix} \{1,4\} & \{1,3,4\} \\ \{2,4\} & \{3,4\} \end{pmatrix}$$

$$= \begin{pmatrix} \{1,3,4\} & \{1,3,4\} \\ \{2,4\} & \{2,3,4\} \end{pmatrix} \qquad \text{II}$$

I and II are equal. Thus S can only be a non commutative semiring.

Thus using matrices with entries from the power set we can build semirings which are also semifields under rational product. Using usual product we may get a non commutative semiring provided the matrices are square matrices. Finally just see we can build polynomials with matrix coefficients and the entries of these matrices are from P(X), X a power set we define polynomial semiring with coefficients from P(X).

Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P(X);\ X \text{ a set and } P(X) \text{ the power set of } X \right\}.$$

We define operation on S. For any $a_i\, x^n$ and $b_i x^m$ $(m > n)$;

$$a_i x^n \cap b_i x^m = (a_i \cap b_i) x^{(\text{min n and m})}$$

$$a_i x^n \cup b_i x^m = (a_i \cup b_i) x^{(\text{min n and m})}.$$

We see $(S, \cup, \cap)$ is a semiring which is commutative. This will be known as the Boolean polynomial semiring.

We can define several properties associated with it.

Further we see several properties associated with these Boolean polynomial semirings can be derived in case of these semirings also, this task is left as an exercise to the reader.

We now define matrix coefficient polynomial semiring where the entries of the matrices are from the power set. Thus we can have polynomial matrix coefficient Boolean semiring. We can as in case of usual semirings derive several of the properties associated with it. Since the power set of a set X is only a Boolean algebra it cannot be a semifield so keeping this in mind we can only define Smarandache polynomial semivector spaces / semilinear algebras.

Thus we can realize polynomial Boolean semirings as Smarandache Boolean polynomial semivector spaces / semilinear algebras. Such study will certainly contribute to several nice results.

We will just show how matrix polynomial coefficient Boolean semiring functions.

Let us take X = {1, 2, 3, 4} to be a set, P (X) the power set of X. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in$ P(X) where X = {1, 2, 3, 4} be a polynomial semiring}.

Take

$$p(x) = \{1, 4\} \cup \{3, 2, 1\}x \cup \{1,2\}x^3 \text{ and}$$
$$q(x) = \{4,2\} \cup \{1,3\}x^2 \text{ in S.}$$

We find p(x) ∩ q(x)

$$= (\{1,4\} \cup \{3,2,1\}x \cup \{1,2\}x^3) \cap (\{4,2\} \cup \{1,3\}x^2)$$

$$= \{1,4\} \cap \{4,2\} \cup \{3,2,1\} \cap \{4,2\}\ x^o \cup \{1,2\} \cap [4,2\}x^o \cup \{1,4\} \cap \{1,3\}\ x^o \cup (\{1,2,3\} \cap \{1,3\})\ x \cup \{1,2\} \cap \{1,3\}x^2$$

$$= \{4\} \cup \{2\} \cup \{2\} \cup \{1\} \cup \{1,3\}x \cup \{1\}x^2$$

$$= \{1, 2, 4\} \cup \{1,3\}x \cup \{1\}x^2.$$

Thus p(x) ∪ q(x) = $\{1,2,4\} \cup \{1,2,3\}x \cup \{1,2\}x^3 \cup \{1,3\}x^2$.

Hence (S, ∪, ∩) is a semiring we have several interesting properties associated with it. We have to work on these polynomial semirings with coefficients from P(X).

Now we can define also intervals with entries from P(X). For we see any two subsets in P(X) are not always comparable so our interval can be of the following types.

If A, B ∈ P(X) we may have [A, B] in which A ⊆ B or B ⊆ A or A and B are comparable. For example if we take X = {1, 2, 3} .

P(X) = {ϕ, {1}, {2}, {3}, X, {1, 2}, {2, 3}, {1,3}} we see [{1}, {2}] is an interval in which the terms are not comparable where as [{2}, {1,2}] is such that $\{2\} \subseteq \{1,2\}$ is a comparable set. [{3,1} {1}] is an interval in which $\{3,1\} \supseteq \{1\}$. We can define increasing or decreasing or not comparable interval.

Let $N_c(P(x))$ denote the collection of all closed intervals with entries from P(X). $N_c(P(x))$ is an idempotent semigroup under '$\cup$' or '$\cap$'. $N_c(P(X))$ is a semiring will be known as the interval semiring. Likewise using the interval semiring we can construct polynomial semiring with interval coefficients from $N_c(P(X))$ or $N_o(P(X))$ or $N_{oc}(P(X))$ or $N_{co}(P(X))$.

Also we can construct matrices whose entries are from $N_c(P(X))$ or $N_o(P(X))$ or $N_{oc}(P(X))$ or $N_{co}(P(X))$. Another advantage of using P(X) is we can vary the set X and by varying X we can get different interval semirings of different finite orders.

$P = \left\{\begin{bmatrix} [a_1, b_1] & [a_2, b_2] \\ [a_3, b_3] & [a_4, b_4] \end{bmatrix} \middle| [a_i, b_i] \in N_c(P(X))\right.$ where X = {1, 2, 3, 4, 5} and P(X) is the power set of a set X}. We can define $\cup$ and $\cap$ on P so that P is a semiring.

We just show how this is carried out.

$$\text{Let } T = \begin{bmatrix} [\{3\},\phi] & [X,\{1,2,4\}] \\ [\{2,3\},\{5\}] & [\{1,2,3,4\},\{5\}] \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} [\{2,3\},X] & [\{1,2\},\{3,5\}] \\ [\{5,2\},\{2,1\}] & [\{1,2,3\},\{1,2\}] \end{bmatrix}$$

$$T \cap B = \begin{bmatrix} [\{3\},\phi] & [\{1,2\},\{\phi\}] \\ [\{2\},\phi] & [\{1,2,3\},\phi] \end{bmatrix} \in P.$$

$$\text{Consider } T \cup B = \begin{bmatrix} [\{2,3\},X] & [X,X] \\ [\{2,3,5\},\{1,2,5\}] & [\{1,2,3,4\},\{1,2,5\}] \end{bmatrix},$$

Clearly $T \cup B$ also $\in$ P.

$$\text{The matrix } \begin{bmatrix} [X,X] & [X,X] \\ [X,X] & [X,X] \end{bmatrix} = I$$

is such that $I \cap A = A$ and $I \cup A = I$.

The matrix $\bar{\phi} = \begin{bmatrix} [\phi,\phi] & [\phi,\phi] \\ [\phi,\phi] & [\phi,\phi] \end{bmatrix}$ in P is such that $I \cap A = I$ and

$$\phi \cup A = A.$$

I acts as the zero under $\cap$ and I acts as identity under '$\cup$'.

Also we see $\{P, \cup, \cap\}$ is a semiring. P can be any $m \times n$ matrix with $m \neq n$.

Thus we can get several Boolean semirings of varying order. Further these matrices can also be found; product in different ways.

Suppose $x = (a_1, a_2, a_3, \ldots, a_n)$ where $a_i \in P(X)$ then

$$x^t = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ we can also find } x \times x^t \text{ or } x^t \times x.$$

We will first illustrate this by some examples.

Let X = {1, 2, 3, 4}; P(X) the power set of X. x = ({1, 2}, ϕ, {2, 4}, X, {3, 2}) and y = ({43}, {3}, {1,2}, {3,4}, X) be in P= {collection of all 1×5 row vectors with entries from P(X)}.

Now $x \cap y = (\phi, \phi, \{2\}, \{3, 4\}, \{3,2\})$.

$x \cup y = (X, \{3\}, \{1,2,4\}, X, X)$.

Now $x \cap y^t = (\{1, 2\}, \phi, \{2, 4\}, X, \{3, 2\}) \cap \begin{bmatrix} \{4,3\} \\ \{3\} \\ \{1,2\} \\ \{3,4\} \\ X \end{bmatrix}$

$= (\phi \cup \phi \cup \{2\} \cup \{3, 4\} \cup \{3, 2\})$
$= \{2, 3, 4\}.$

$x \cup y^t = (\{4, 1, 2, 3\} \cap \{3\} \cap \{1, 2, 4\} \cap X \cap X$

$= \phi.$

We find $x^t \cap x = \begin{bmatrix} \{1,2\} \\ \phi \\ \{2,4\} \\ X \\ \{3,2\} \end{bmatrix} \cap (\{1,2\}, \phi, \{2, 4\}, X, \{3, 2\})$

$$= \begin{pmatrix} \{1,2\} & \phi & \{2\} & \{1,2\} & \{2\} \\ \phi & \phi & \phi & \phi & \phi \\ \{2\} & \phi & \{2,4\} & \{2,4\} & \{2\} \\ \{1,2\} & \phi & \{2,4\} & X & \{3,2\} \\ \{2\} & \phi & \{2\} & \{3,2\} & \{3,2\} \end{pmatrix}.$$

Suppose

$$x^t \cup x = \begin{pmatrix} \{1,2\} & \{1,2\} & \{1,2,4\} & X & \{1,2,3\} \\ \{1,2\} & \phi & \{2,4\} & X & \{3,2\} \\ \{1,2,4\} & \{2,4\} & \{2,4\} & X & \{3,2,4\} \\ X & X & X & X & X \\ \{1,2,3\} & \{3,2\} & \{2,3,4\} & X & \{3,2\} \end{pmatrix}.$$

Clearly $x^t \cup x \neq x^t \cap x$.

We can also use the same type of product with interval matrices. This will be just illustrated.

Let x = ([{2}, {3,4}] [ϕ, {2}] [X, {3, 1}])

$$\text{and } x^t = \begin{bmatrix} [\{2\},\{3,4\}] \\ [\phi,\{2\}] \\ [X,\{3,1\}] \end{bmatrix}$$

be the row interval matrix and its transpose.

$$\begin{aligned} x \cap x^t &= [\{2\}, \{3, 4\}] \cap [\phi, \{2\}] \cap [X, \{3, 1\}] \\ &= [\{2\} \cap \phi \cap X, \{3, 4\} \cap \{2\} \cap \{3, 1\}] \\ &=[\phi, \phi]. \end{aligned}$$

$$\begin{aligned} \text{Consider } x \cup x^t &= \{[\{2\}, \{3,4\}] \cup [\phi, \{2\}] \cup [X, \{3, 1\}] \\ &= [\{2\} \cup \phi \cup X, \{3, 4\} \cup \{2\} \cup \{3, 1\}] \\ &= [X, X]. \end{aligned}$$

We find

$$x^t \cap x = \begin{bmatrix} [\{2\},\{3,4\}] \\ [\phi,\{2\}] \\ [X,\{3,1\}] \end{bmatrix} \cap ([\{2\}, \{3,4\}], [\phi, \{2\}], [X, \{3,1\}])$$

$$= \begin{bmatrix} [\{2\},\{3,4\}] & [\phi,\phi] & [\{2\},\{3\}] \\ [\phi,\phi] & [\phi,\{2\}] & [\phi,\phi] \\ [\{2\},\{3\} & [\phi,\phi] & [X,\{3,1\}] \end{bmatrix}.$$

We can also find product of a row interval matrix with a suitable square / rectangular interval matrix with entries from P(X).

Let us consider a row matrix with entries from P(X), where X = {1,2,3,4,5} and a 5 × 3 matrix with entries from P(X).

Let x = ({3,2}, {1,5},{4,2,3},{4,1}, X) and

$$A = \begin{bmatrix} \{3\} & \{4,3\} & \{5,1\} \\ \{4,1\} & \phi & \{2,3,1\} \\ \{5\} & X & \{1,2\} \\ X & \{1,2,3\} & \phi \\ \{2,3\} & \{4,3\} & X \end{bmatrix}$$

be two matrices with entries from the power set P(X).

$x \cap A = (\{3\} \cup \{1\} \cup \{4,1\} \cup \{2,3\}, \{3\} \cup \phi \cup \{4,2,3\} \cup \{1\} \cup \{4, 3\}, \phi \cup \{1\} \cup \{2\} \cup \phi \cup \{X\})$

$= (\{1, 2, 3, 4\}, \{1, 4, 2, 3\}, X).$

Now consider $x \cup A$

$= (\{3, 2\} \cup \{1, 4, 5\} \cup \{4, 2, 3, 5\} \cup X \cup X, \{4, 2, 3\} \cup \{1, 5\} \cup X \cup \{1, 2, 3, 4\} \cup X, \{1, 5, 2, 3\} \cup \{1, 5, 2, 3\} \cup \{1, 2, 3, 4\} \cup \{1, 4\} \cup X)$

$= (X, X, X).$

We can also find the product of two matrices for which the product can be defined with entries from the power set P (X).

$$\text{Let } A = \begin{pmatrix} \{3\} & \{1,2\} & \{5,1\} \\ \{2,4\} & \phi & \{1,3,4\} \\ X & \{1,4\} & \phi \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \{1\} & X & \{4\} & \{1,5,2\} & X \\ \phi & \{2,1\} & X & \{4\} & \phi \\ \{3,4\} & \phi & \{1\} & \{1,2,3\} & \{5\} \end{pmatrix}$$

be two matrices with entries from P(X) where X = {1,2,3,4,5}.

To find

$$A \cap B = \begin{pmatrix} \phi & \{3,1,2\} & \{1,2\} & \{1\} & \{3,5\} \\ \{3,4\} & \{2,4\} & \{1,4\} & \{1,2,3\} & \{2,4\} \\ \{1\} & X & \{1,4\} & \{1,5,2,4\} & X \end{pmatrix}.$$

We now find $A \cup B = \begin{pmatrix} X & X & X & X & X \\ \{1,2,3,4\} & X & X & X & X \\ X & X & X & X & X \end{pmatrix}.$

We see $A \cap B \neq A \cup B$. We call this sort of product as usual product. However natural product cannot be defined for A with B.

Now we can define as in case of usual matrices with entries from P(X) for matrices with intervals entries from $N_c(P(x))$ (or $N_o(P(x))$ or $N_{oc}(P(x))$ or $N_{co}(P(x))$.

We will only illustrate these situations.

Consider X = {1, 2, 3, 4} and P (X) the power set of X. Take

$x = ([\{3\}, \phi], [\{1,2\}, X], [\{3, 4, 2\}, \{1\}])$
and $y = ([\{4\}, \{2, 3\}], [\{4\}, \{2, 4\}], [\{1\}, \phi])$ in

M = {all 1 × 3 row matrices with interval entries from $N_c(P(x))$}.

Now $x \cap y = ([\phi, \phi], [\phi, \{2, 4\}], [\phi,\phi])$ and
$x \cup y = ([\{3, 4\}, \{2, 3\}, [\{1, 2, 4\}, X], [X, \{1\}])$.

Clearly $x \cap y$ and $x \cup y \in M$ but $x \cap y \neq x \cup y$.

Consider $x^c = ([\{1, 2, 4\}, X], [\{3, 4\}, \phi], [\{1\}, \{2, 3, 4\}])$.

$x^c \cup x = ([X, X], [X, X], [X, X])$ and

$x^c \cap x = ([\phi,\phi],[\phi,\phi],[\phi,\phi])$.

Clearly $(M, \cup, \cap)$ is a semiring which is commutative and every element in M is an idempotent with respect to $\cup$ and $\cap$. Also M is a Boolean algebra.

Next we consider the set P = {all $m \times 1$ column matrices with entries from $N_c(P(X))$ where P(X) is the power set of X}.

We see $(P, \cup, \cap)$ is again a semiring and also a Boolean algebra.

Suppose we take X = {1, 2, 3, 4, 5, 6} and P to be the collection of all 5×1 column matrices with entries from $N_c(P(X))$.

$$\text{Let } x = \begin{pmatrix} [\{3,1\},\{1,2\}] \\ [\phi,\{1,2,3\}] \\ [X,\{4,6\}] \\ [\{3,6,2\},\{4,2\}] \\ [\{6,1\},\{1,3,5\}] \end{pmatrix} \text{ and } y = \begin{pmatrix} [\phi,X] \\ [\{1\},\{3,4\}] \\ [\{5\},\{6\}] \\ [\{5,1\},\phi] \\ [\{3,2\},\{4,3\}] \end{pmatrix} \text{ be in P.}$$

$$x^c = \begin{pmatrix} [\{2,4,5,6\},\{3,4,5,6\}] \\ [X,\{4,5,6\}] \\ [\phi,\{1,2,3,5\}] \\ [\{1,45\},\{1,3,5,6\}] \\ [\{2,3,4,5\},\{2,4,6\}] \end{pmatrix} \in \text{P}.$$

Clearly $x^c \cap x = \begin{bmatrix} [\phi,\phi] \\ [\phi,\phi] \\ [\phi,\phi] \\ [\phi,\phi] \\ [\phi,\phi] \end{bmatrix}$ and $x^c \cup x = \begin{bmatrix} [X,X] \\ [X,X] \\ [X,X] \\ [X,X] \\ [X,X] \end{bmatrix}$.

Consider $x \cap y = \begin{bmatrix} [\phi,\{1,2\}] \\ [\phi,\{3\}] \\ [\{5\},\{6\}] \\ [\phi,\phi] \\ [\phi,\{3\}] \end{bmatrix}$ and

$$x \cup y = \begin{bmatrix} [\{1,3\},X] \\ [\{1\},\{1,2,3,4\}] \\ [X,\{4,6\}] \\ [\{1,2,3,5,6\},\{4,2\}] \\ [\{1,6,2,3\},\{1,3,4,5\}] \end{bmatrix}.$$

Both $x \cup y$ and $x \cap y \in$ P.

We can in a similar way define for $N_c(P(X))$ take

x = {$1 \times m$ row matrix with entries from $N_c(P(X))$} and

y = {$m \times 1$ column matrix with entries from $N_c(P(X))$} we can find $x \cup y$, $x \cap y$, $y \cap x$ and $y \cup x$.

We will first illustrate this situation by an example.

Consider X = {1, 2, 3} and P(X) the power set of X and $N_c(P(X))$ the collection of all closed intervals.

Consider x = ([{3}, ϕ], [{1, 2}, x], [{1}, {2}], [{1, 3}, {3, 2}], [{2}, {3}])

$$y = \begin{bmatrix} [\{1\},\{2\}] \\ [\phi,\{3\}] \\ [\{1,3\},\phi] \\ [\{1,2\},X] \\ [\{3,2\},\{2\}] \end{bmatrix}$$

be two matrices with column entries from $N_c(P(X))$.

$x \cap y = ([\phi,\phi] \cup [\phi,\{3\}] \cup [\{1\},\phi] \cup [\{1\},\{3,2\}] \cup [\{2\},\phi])$

$= [\{1,2\}, \{3,2\}]$.

Consider

$$y \cap x = \begin{bmatrix} [\phi,\phi] & [\{1\},\{2\}] & [\{1\},\{2\}] & [\{1\},\{2\}] & [\phi,\phi] \\ [\phi,\phi] & [\phi,\{3\}] & [\phi,\phi] & [\phi,\{3\}] & [\phi,\{3\}] \\ [\{3\},\phi] & [\{1\},\phi] & [\{1\},\phi] & [\{1,3\},\phi] & [\phi,\phi] \\ [\phi,\phi] & [\{1,2\},X] & [\{1\},\{2\}] & [\{1\},\{3,2\}] & [\{2\},\{3\}] \\ [\{3\},\phi] & [\{2\},\{2\}] & [\phi,\{2\}] & [\{3\},\{2\}] & [\{2\},\phi] \end{bmatrix}.$$

We can find usual product using matrices.

Finally let

$x = ([\{3\}, X], [\{1,2\},\phi], [\{1,3\},\{3,2\}], \{\phi, \{1,2\}]$ and

$$y = \begin{bmatrix} [\{3,2\},X] & [\{2\},\phi] & [\{1,2\},\{3\}] & [\{1\},\phi] \\ [\{2,1\},\phi] & [\{3,1\},\{2\}] & [X,X] & [\{2\},\{3\}] \\ [\{1,3\},\{2\}] & [X,\{1,3\} & [\{1,2\},\phi] & [\{3,1\},\{2\}] \\ [\{1\},\{3\}] & [\{2\},\{3\}] & [\{1\},\{3\}] & [X,\phi] \end{bmatrix}$$

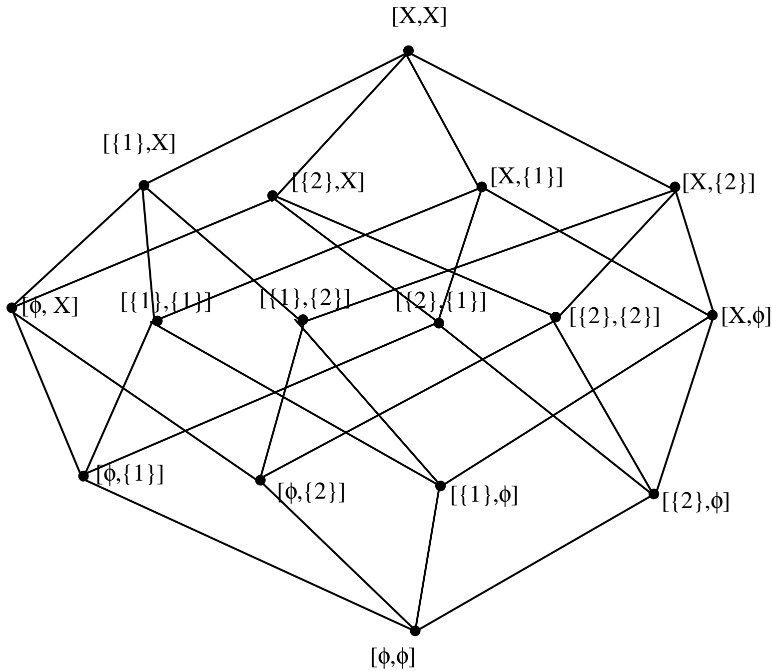
Now we find

$x \cap y$; $x \cap y = ([X, X], [\{1, 3\}, \{3\}], [X, \{3\}], [X, \{2\}])$.

Thus we can find the usual product by taking '$\cap$' as usual product. Now interested reader can work with this type of interval matrices with entries as the sets of a power set P(X).

Suppose X = {1, 2}, P(X) = {ϕ, X, {1}, {2}} be the power set of X. The interval set N_c(P(X)) = {[ϕ, ϕ], [X, X], [ϕ, {1}], [ϕ, {2}], [ϕ, X], [X. {1}], [X, {2}], [X, ϕ], [{1}, X], [{1}, ϕ], [{1}, {1}], [{1}, {2}], [{2}, X], [{2},ϕ], [{2}, {2}], [{2}, {1}]} where the number of elements in N_c(P(X)) is 16.

We now draw the lattice diagram associated with N_c(P(X)).



This is a Boolean algebra of order 16 = 2^4.

Likewise if X = {1, 2, 3}, then P(X) is a Boolean algebra of order 64. Thus every N_c(P(X)) is a Boolean algebra.

Similarly one can easily verify that a row matrix of order 1×3; we see we get a Boolean algebra of order 64 if X = {1, 2} and so on. Thus we can using X = {1, 2} get all finite Boolean algebras.

Now we proceed onto recall and discuss about the linguistic matrices using linguistic terms. We first briefly recall the linguistic set and its related concepts. Suppose we have a set S of linguistic terms say S = {good, bad, very bad, very good, fair, very fair, worst, best, 0} qualify the performance of a student in a class. We say S is a comparable fuzzy linguistic set / space. Mostly authors felt while using fuzzy models, we have a membership function from a set X to the unit interval [0, 1]. An element in X can be a member and may not be a member. If an element is not a member we say the membership is 0 otherwise it takes values form [0, 1] \ {0}. Likewise here our membership will be from the set of linguistic set / space in which every element is comparable. So we replace [0, 1] by the set S and '0' if the element has no membership.

So it is always assumed $0 \in S$ (for that matter any fuzzy linguistic set). The fuzzy linguistic set can be constructed mainly based on the problem under study. For instance our problem is to find the speed of the vehicles at peak hours in the important city roads, then the fuzzy linguistic set / space associated with this problem would be {0, very fast, often, very fast, fast, just slow, often slow, slow, medium, speed and so on}. At the signals the speed would be zero. While reaching the signal the speed would 'often medium' in the empty road the speed can be very fast or fast depending on the drivers capacity to control the vehicle in which he / she is traveling and the nature of the vehicle (car or bus or bicycle or scooter or so on).

Similarly we can measure the social stigma suffered by PWDs or by HIV/AIDS patients and so on.

Now we take L to be a fuzzy linguistic set together with 0 and any pair of elements in L are comparable.

We call $M = \{(a_1, a_2, \ldots, a_n) \mid a_i \in L, 1 \leq i \leq n\}$ the collection of all row matrices as the fuzzy linguistic matrices. We perform either the min or max operation.

Clearly [M, min, max} operation is a lattice.

We will just illustrate by a simple example. Consider L = {0, good, bad, fair, v.fair, v.bad, v.good, best, worst}. Take x = (0, good, bad, fair) and y = (bad, worst, best, bad) be any two row matrices with entries from L.

min {x, y} = {(0, worst, bad, bad)}.
max {x, y} = (bad, good, best, fair).

Now if
P = {collection of all $n \times 1$ column vectors with entries from L};
{P, min} is a semilattice {P, max} is also semilattice.

{P, min, max} is a lattice. We just show how the operations on them are carried out.

$$\text{Let } a = \begin{bmatrix} \text{good} \\ \text{good} \\ \text{worst} \\ \text{best} \\ \text{fair} \end{bmatrix} \text{ and } b = \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{worst} \\ \text{v.fair} \end{bmatrix}$$

be any two 5×1 column matrices.

$$\min\{a, b\} = \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{worst} \\ \text{worst} \\ \text{fair} \end{bmatrix} \text{ and } \max\{a, b\} = \begin{bmatrix} \text{good} \\ \text{good} \\ \text{best} \\ \text{best} \\ \text{v.fair} \end{bmatrix}.$$

We can also find the effect of a 1 × n row matrix with a n × 1 column matrix.

This will only be illustrated by an example.

Let x = (bad, good, best, best, v.fair)

$$\text{and } y = \begin{bmatrix} \text{good} \\ \text{bad} \\ \text{best} \\ \text{fair} \\ \text{good} \end{bmatrix} \text{ to find}$$

min {max {x, y}} and max {min {x, y}}.

min {max {x, y}} = good.

max {min {x, y}} = best.

We can also calculate min {y, x} and max {y, x}.

$$\min \{y, x\} = \begin{bmatrix} \text{bad} & \text{good} & \text{good} & \text{good} & \text{v.fair} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{good} & \text{best} & \text{best} & \text{v.fair} \\ \text{bad} & \text{fair} & \text{fair} & \text{fair} & \text{fair} \\ \text{bad} & \text{good} & \text{good} & \text{good} & \text{v.fair} \end{bmatrix} \text{ and}$$

$$\max \{y, x\} = \begin{bmatrix} \text{good} & \text{good} & \text{best} & \text{best} & \text{good} \\ \text{bad} & \text{good} & \text{best} & \text{best} & \text{v.fair} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{fair} & \text{good} & \text{best} & \text{best} & \text{v.fair} \\ \text{good} & \text{good} & \text{best} & \text{best} & \text{good} \end{bmatrix}.$$

Now having seen examples we can also define rectangular and square linguistic matrices with entries from L.

Let S = {all m × n (m ≠ n) matrices with entries from L}. We can define two types of operations on matrices from S. Let A, B ∈ M we have min {A, B} and max {A, B}.

$$\text{Consider } A = \begin{bmatrix} \text{good} & 0 & \text{bad} & \text{worst} & 0 \\ \text{best} & \text{bad} & 0 & \text{best} & \text{fair} \\ \text{fair} & \text{best} & \text{good} & 0 & \text{bad} \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 0 & \text{bad} & \text{good} & 0 & \text{best} \\ \text{best} & 0 & \text{good} & \text{best} & \text{bad} \\ \text{fair} & \text{good} & 0 & \text{fair} & 0 \end{bmatrix}$$

two linguistic matrices with entries from L.

$$\min \{A, B\} = \begin{bmatrix} 0 & 0 & \text{bad} & 0 & 0 \\ \text{best} & 0 & 0 & \text{best} & \text{bad} \\ \text{fair} & \text{good} & 0 & 0 & 0 \end{bmatrix} \text{and}$$

$$\max \{A, B\} = \begin{bmatrix} \text{good} & \text{bad} & \text{good} & \text{worst} & \text{best} \\ \text{best} & \text{bad} & \text{good} & \text{best} & \text{fair} \\ \text{fair} & \text{best} & \text{good} & \text{fair} & \text{bad} \end{bmatrix}.$$

Now for square matrices A, B we can have 6 types of operations on them namely, min {A, B}, max {A, B}, min {max {A, B}}, and max min {A, B}, min {min {A, B}} and max max {A, B}.

We will illustrate all the six operations by some examples.

$$\text{Let } A = \begin{bmatrix} \text{good} & \text{bad} & \text{best} \\ 0 & \text{good} & \text{fair} \\ \text{best} & 0 & \text{bad} \end{bmatrix} \text{ and } B = \begin{bmatrix} \text{bad} & \text{good} & \text{good} \\ \text{best} & 0 & \text{fair} \\ \text{best} & \text{best} & 0 \end{bmatrix}$$

be any two 3×3 matrices with entries from L.

$$\min \{A, B\} = \begin{bmatrix} \text{bad} & \text{bad} & \text{good} \\ 0 & 0 & \text{fair} \\ \text{best} & 0 & 0 \end{bmatrix},$$

$$\max \{A, B\} = \begin{bmatrix} \text{good} & \text{good} & \text{best} \\ \text{best} & \text{good} & \text{fair} \\ \text{best} & \text{best} & \text{bad} \end{bmatrix},$$

$$\max \{\min \{A, B\}\} = \begin{bmatrix} \text{best} & \text{best} & \text{good} \\ \text{good} & \text{fair} & \text{fair} \\ \text{bad} & \text{good} & \text{good} \end{bmatrix},$$

$$\min \{\min \{A, B\}\} = \begin{bmatrix} \text{bad} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\max \{\max \{A, B\}\} = \begin{bmatrix} \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{good} \\ \text{best} & \text{best} & \text{best} \end{bmatrix}.$$

We can also find products like $1 \times n$ row fuzzy linguistic matrix with $n \times m$ rectangular fuzzy linguistic matrix.

We will illustrate this first by an example with possible operations on them.

Let A = (good bad best fair)

$$\text{and B} = \begin{bmatrix} \text{bad} & 0 & \text{best} & \text{bad} & \text{best} \\ \text{good} & \text{bad} & 0 & \text{good} & \text{fair} \\ \text{fair} & \text{good} & \text{bad} & 0 & \text{good} \\ \text{bad} & \text{bad} & \text{good} & 0 & \text{good} \end{bmatrix}$$

be two fuzzy linguistic matrices.

max {min {A, B}} = (fair, good, good, bad, good).

max {max {A, B}} = (best, best, best, best, best).

min {max {A, B}} = {fair, bad, bad, fair, fair}.

min {min {A, B}} = (bad, 0, 0, 0, bad).

Now on similar lines we can find the product of a n × m fuzzy linguistic matrix with a m × 1 fuzzy linguistic column matrix.

This will also be illustrated by the following example.

Consider the fuzzy linguistic 3 × 5 matrix

$$A = \begin{bmatrix} \text{bad} & 0 & \text{fair} & \text{bad} & \text{good} \\ \text{best} & \text{good} & \text{bad} & 0 & \text{best} \\ 0 & \text{fair} & \text{v.good} & \text{fair} & \text{bad} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \text{bad} \\ \text{fair} \\ \text{good} \\ \text{best} \\ \text{bad} \end{bmatrix}$$

a fuzzy linguistic column matrix;

$$\min \{\max \{A, B\}\} = \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{bad} \end{bmatrix}.$$

$$\max \{\min \{A, B\}\} = \begin{bmatrix} 0 \\ \text{bad} \\ 0 \end{bmatrix}.$$

In the same way we can calculate max {max {A, B}} and min {min {A, B}}. This simple task is left as an exercise to the reader.

Now having defined fuzzy linguistic matrices and operations on them, we proceed onto define fuzzy linguistic intervals.

Let L be a fuzzy linguistic set in which every pair is comparable $N_c(L) = \{[a, b] \mid a, b \in L\}$ is the natural class of fuzzy linguistic intervals built using L.

On similar lines we can define $N_o(L)$, $N_{oc}(L)$ and $N_{co}(L)$.

We can define for any two intervals [a, b] and [c, d] in $N_c(L)$ min {[a, b], [c, d]}

= [min {a, c}, min {b, d}] and max {[a, b], [c, d]}
= [max {a, c}, max {b, d}}.

For instance [good, fair] and [bad good] be two fuzzy linguistic intervals then max {[good, fair], [bad, good]}

= [max {good, bad}, max {fair, good}]

= [good, good].

min {[good, fair], [bad, good]} = [bad, fair] {$N_c(L)$, min} is an idempotent semigroup. {$N_c(L)$, max} is again an idempotent semigroup. {$N_c(L)$, min, max} is a lattice.

Now using these intervals we can construct matrices called the fuzzy linguistic interval matrices.

Let L be a fuzzy linguistic space.

R = {$(a_1, a_2, \ldots, a_n)$ | $a_i \in N_c(L)$; $1 \leq i \leq n$} is a row fuzzy linguistic interval matrix.

We see {R, min, max} is a lattice.

Similarly

$$P = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \middle| \; b_i \in N_o(L); 1 \leq i \leq m \right\}$$

is a fuzzy linguistic interval column matrix {P, min, max} is a lattice.

Now

$$S = \{(a_{ij})_{m\times n}\ (m \neq n) \mid a_{ij} \in N_{oc}(L); 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$$

is a set of all fuzzy linguistic interval rectangular $m \times n$ matrices {S, min, max} is a lattice. Finally

T = {$(a_{ij})_{n\times n}$ | $a_{ij} \in N_o(L)$; $1 \leq i, j \leq n$} is the collection of all $n \times n$ fuzzy linguistic interval square matrix {T, min, max} is also a lattice.

We just describe how the operations are defined on these sets.

Let L be as before.

Let

A = ([good, bad], [fair, fair], [fair, good], [best, worst]) and

B = ([bad, bad], [good, fair], [best, good], [best, good]) be two 1×4 interval matrices.

min {A, B} = ([bad, bad], [fair, fair], [fair, good], [best, worst]).

max {A, B} = ([good, bad], [good, fair], [best, good], [best, good]).

We can take entries from $N_o(L)$ or $N_{oc}(L)$ or $N_{co}(L)$.

Let us consider two fuzzy linguistic interval column matrices of same order

$$A = \begin{bmatrix} [\text{good}, \text{best}] \\ [\text{best}, \text{bad}] \\ [\text{bad}, \text{good}] \\ [0, \text{fair}] \\ [\text{fair}, \text{worst}] \\ [\text{good}, 0] \end{bmatrix} \text{ and } B = \begin{bmatrix} [\text{bad}, \text{bad}] \\ [\text{good}, \text{good}] \\ [0, \text{fair}] \\ [\text{best}, 0] \\ [\text{best}, \text{bad}] \\ [\text{good}, \text{bad}] \end{bmatrix}$$

be two 6×1 column fuzzy linguistic interval matrices. We find both min {A, B} and max {A, B}.

$$\min\{A, B\} = \begin{bmatrix} [\text{bad}, \text{bad}] \\ [\text{good}, \text{bad}] \\ [0, \text{fair}] \\ [0, 0] \\ [\text{fair}, \text{worst}] \\ [\text{good}, 0] \end{bmatrix}$$

$$\max\{A, B\} = \begin{bmatrix} [\text{good}, \text{best}] \\ [\text{best}, \text{good}] \\ [\text{bad}, \text{good}] \\ [\text{best}, \text{fair}] \\ [\text{best}, \text{bad}] \\ [\text{good}, \text{bad}] \end{bmatrix}.$$

On similar lines we can find max {A, B} when A and B rectangular interval matrices of same order or square fuzzy linguistic interval matrices of same order.

Now we can find the product (i.e., max or min) of a n × 1 interval fuzzy linguistic matrix with a 1 × m interval fuzzy linguistic matrix. This is described by the following example.

$$\text{Let } A = \begin{bmatrix} [\text{good}, \text{bad}] \\ [0, \text{best}] \\ [\text{fair}, 0] \\ [\text{fair}, \text{good}] \\ [\text{bad}, \text{bad}] \\ [\text{good}, \text{best}] \end{bmatrix} \text{ and}$$

B = ([bad, good], [best, 0], [good, good], [0, good], [fair, good], [best, good] [0, bad]) be two fuzzy linguistic interval matrices.

To find max {A, B}

$$= \begin{bmatrix} [\text{good},\text{good}] & [\text{best},\text{bad}] & [\text{good},\text{good}] & [\text{good},\text{good}] \\ [\text{bad},\text{best}] & [\text{best},\text{best}] & [\text{good},\text{best}] & [0,\text{best}] \\ [\text{fair},\text{good}] & [\text{best},0] & [\text{good},\text{good}] & [\text{fair},\text{good}] \\ [\text{fair},\text{good}] & [\text{best},\text{good}] & [\text{good},\text{good}] & [\text{fair},\text{good}] \\ [\text{bad},\text{good}] & [\text{best},\text{bad}] & [\text{good},\text{good}] & [\text{bad},\text{good}] \\ [\text{good},\text{best}] & [\text{best},\text{best}] & [\text{good},\text{best}] & [\text{good},\text{best}] \end{bmatrix}$$

$$\begin{bmatrix} [\text{good},\text{good}] & [\text{best},\text{good}] & [\text{good},\text{bad}] \\ [\text{fair},\text{best}] & [\text{best},\text{best}] & [0,\text{best}] \\ [\text{fair},\text{good}] & [\text{best},\text{good}] & [\text{fair},\text{bad}] \\ [\text{fair},\text{good}] & [\text{best},\text{good}] & [\text{fair},\text{good}] \\ [\text{fair},\text{good}] & [\text{best},\text{good}] & [\text{bad},\text{bad}] \\ [\text{good},\text{best}] & [\text{best},\text{best}] & [\text{good},\text{best}] \end{bmatrix}.$$

On similar lines we can find min {A, B}.

Now we can find max {min {A, B}} where

A = ([good, bad], [0, best], [bad, 0], [bad, best], [best, good])

and

$$B = \begin{bmatrix} [0,\text{bad}] \\ [\text{good},\text{good}] \\ [\text{fair},\text{fair}] \\ [\text{best},0] \\ [\text{bad},\text{worst}] \end{bmatrix};$$

max {min {A, B}} = max {[0, bad], [0, good], [bad, 0], [best, 0], [bad, worst]}

= [max {0, 0, bad, best, bad}, max {bad, good, 0, 0, worst}]

= [best, good].

Likewise we can find min {min {A, B}}, max {max {A, B}} and min {max {A, B}}.

For square linguistic matrices with fuzzy linguistic entries of same order we can find max {max {A, B}}, max {min {A, B}}, min {max {A, B}}, and min{min {A, B}}.

$$\text{Let A} = \begin{bmatrix} [\text{good}, \text{best}] & [0, \text{good}] & [\text{bad}, \text{best}] \\ [\text{best}, \text{fair}] & [\text{good}, \text{good}] & [\text{worst}, \text{fair}] \\ [\text{fair}, 0] & [\text{fair}, \text{bad}] & [\text{best}, 0] \end{bmatrix}$$

$$\text{and B} = \begin{bmatrix} [\text{best}, \text{best}] & [\text{good}, \text{good}] & [\text{bad}, \text{bad}] \\ [\text{good}, 0] & [\text{good}, \text{bad}] & [0, \text{good}] \\ [0, \text{fair}] & [\text{best}, \text{fair}] & [\text{fair}, 0] \end{bmatrix}$$

be two square fuzzy linguistic interval matrices.

To find

$$\max \{\min \{A,B\}\} = \begin{bmatrix} [\text{good}, \text{best}] & [\text{good}, \text{good}] & [\text{bad}, \text{good}] \\ [\text{best}, \text{fair}] & [\text{good}, \text{fair}] & [\text{bad}, \text{good}] \\ [\text{fair}, 0] & [\text{best}, \text{bad}] & [\text{fair}, \text{bad}] \end{bmatrix}.$$

Likewise the reader is left with the task of finding min {min {A, B}} max {max {A, B}} and min {max {A, B}}. We can construct using these operations and matrices fuzzy linguistic models [8]. Thus we have introduced several types of matrices and different types of operations on them.

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Exploring the extension of natural operations on intervals, matrices and complex numbers

In this book we explore the possibility of extending the natural operations on reals to intervals and matrices. The extension to intervals make us define a natural class of intervals in which we accept [a, b], a > b. Further we introduce a complex modulo integer in Z_n (n, a +ve integer) and denote it by i_F with $i_F^2 = n - 1$.

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