Vol.9,No.4,2013

ISSN 1556-6706

SCIENTIA MAGNA

An international journal

Edited by Department of Mathematics
Northwest University, P.R.China

Vol. 9, No. 4, 2013 ISSN 1556-6706

SCIENTIA MAGNA

An international journal

Edited by

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

Scientia Magna is published annually in 400-500 pages per volume and 1,000 copies.

It is also available in **<u>microfilm</u> format** and can be ordered (online too) from:

Books on Demand
ProQuest Information & Learning
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, Michigan 48106-1346, USA
Tel.: 1-800-521-0600 (Customer Service)
URL: http://wwwlib.umi.com/bod/

Scientia Magna is a **referred journal**: reviewed, indexed, cited by the following journals: "Zentralblatt Für Mathematik" (Germany), "Referativnyi Zhurnal" and "Matematika" (Academia Nauk, Russia), "Mathematical Reviews" (USA), "Computing Review" (USA), Institute for Scientific Information (PA, USA), "Library of Congress Subject Headings" (USA).

Printed in the United States of America
Price: US$ 69.95

Information for Authors

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word XP (or lower), WordPerfect 7.0 (or lower), LaTeX and PDF 6.0 or lower.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 x 11 inches (21,6 x 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writing with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an abstract, followed by the keywords.

All manuscripts are subject to anonymous review by three independent reviewers.

Every letter will be answered.

Each author will receive a free copy of the journal.

Contributing to Scientia Magna

Authors of papers in science (mathematics, physics, philosophy, psychology, sociology, linguistics) should submit manuscripts, by email, to the

Editor-in-Chief:

Prof. Wenpeng Zhang
School of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China
E-mail: wpzhang@nwu.edu.cn

Or anyone of the members of

Editorial Board:

Dr. W. B. Vasantha Kandasamy, Department of Mathematics, Indian Institute of Technology, IIT Madras, Chennai - 600 036, Tamil Nadu, India.

Dr. Larissa Borissova and Dmitri Rabounski, Sirenevi boulevard 69-1-65, Moscow 105484, Russia.

Prof. Yuan Yi, Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China.
E-mail: yuanyi@mail.xjtu.edu.cn

Dr. Zhefeng Xu, School t of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China. E-mail: zfxu@nwu.edu.cn; zhefengxu@hotmail.com

Prof. József Sándor, Babes-Bolyai University of Cluj, Romania.
E-mail: jjsandor@hotmail.com; jsandor@member.ams.org

Dr. Danfang Wang, School of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China. E-mail: henanwdf@163.com

Contents

Scientia Magna
Vol. 9 (2013), No. 4, 1-6

A note on the length of maximal arithmetic progressions in random subsets

Y. Shang

Institute for Cyber Security, University of Texas at San Antonio
TX 78249, USA
E-mail: shylmath@hotmail.com

Abstract Let $U^{(n)}$ denote the maximal length arithmetic progression in a non-uniform random subset of $\{0,1\}^n$, where 1 appears with probability p_n. By using dependency graph and Stein-Chen method, we show that $U^{(n)} - c_n \ln n$ converges in law to an extreme type distribution with $\ln p_n = -2/c_n$. Similar result holds for $W^{(n)}$, the maximal length aperiodic arithmetic progression (mod n).

Keywords Arithmetic progression, random subset, Stein-Chen method.

§1. Introduction

An arithmetic progression is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. A celebrated result of Szemerédi [5] says that any subset of integers of positive upper density contains arbitrarily long arithmetic progressions. The recent work [6] reviews some extremal problems closely related with arithmetic progressions and prime sequences, under the name of the Erdös-Turán conjectures, which are known to be notoriously difficult to solve.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be a uniformly chosen random word in $\{0,1\}^n$ and Ξ_n be the random set consisting elements i such that $\xi_i = 1$. Benjamini et al. [3] studies the length of maximal arithmetic progressions in Ξ_n. Denote by $U^{(n)}$ the maximal length arithmetic progression in Ξ_n and $W^{(n)}$ the maximal length aperiodic arithmetic progression (mod n) in Ξ_n. They show, among others, that the expectation of $U^{(n)}$ and $W^{(n)}$ is roughly $2 \ln n / \ln 2$.

In view of the random graph theory [4], a natural extension of [3] is to consider non-uniform random subset of $\{0,1\}^n$, which is the main interest of this note. Let $\xi_i = 1$ with probability p_n and $\xi_i = 0$ with probability $1 - p_n$, where $p_n \in [0,1]$ is a function of n. Following [3], the key to our work is to construct proper dependency graph and apply the Stein-Chen method of Poisson approximation (see e.g. [1, 4]). Our result implies that, in the non-uniform scenarios, the expectation of $U^{(n)}$ and $W^{(n)}$ is roughly $c_n \ln n$, with $\ln p_n = -2/c_n$. Obviously, taking $p_n \equiv 1/2$ and $c_n \equiv 2/\ln 2$, we then recover the main result of Benjamini et al..

The rest of the note is organized as follows. We present the main results in Section 2. Section 3 is devoted to the proofs.

§2. Results

Let $\xi_1, \xi_2, \cdots$ be i. i. d. random variables with $P(\xi_i = 1) = p_n$ and $P(\xi_i = 0) = 1 - p_n$. For integers $1 \le s, t \le n$, define

$$W_{s,t}^{(n)} := \max \left\{ 1 \le k \le n : \xi_s = 0, \prod_{i=1}^{k} \xi_{s+it(\bmod n)} = 1 \right\}. \tag{1}$$

Therefore, $W_{s,t}^{(n)}$ is the length of the longest arithmetic progression (mod n) in $\{1, 2, \cdots, n\}$ starting at s with difference t. Moreover, set $W^{(n)} = \max_{1 \le s,t \le n} W_{s,t}^{(n)}$. Similarly, define

$$U_{s,t}^{(n)} := \max \left\{ 1 \le k \le \left\lfloor \frac{n-s}{t} \right\rfloor : \xi_s = 0, \prod_{i=1}^{k} \xi_{s+it} = 1 \right\}, \tag{2}$$

and $U^{(n)} = \max_{1 \le s,t \le n} U_{s,t}^{(n)}$, where $\lfloor a \rfloor$ is the integer part of a.

Theorem 2.1. Suppose that $\ln p_n = -2/c_n$ and $\alpha < c_n = o(\ln n)$ for some $\alpha > 0$. Let $\{x_n\}$ be a sequence such that $c_n \ln n + x_n \in \mathbb{Z}$ for all n, and $\inf_n x_n \ge \beta$ for some $\beta \in \mathbb{R}$. We have

$$\lim_{n \to \infty} e^{\lambda(x_n)} P(W^{(n)} \le c_n \ln n + x_n) = 1, \tag{3}$$

where $\lambda(x) = p_n^{x+2}$. In particular, $W^{(n)}/c_n \ln n$ converges to 1 in probability, as $n \to \infty$.

Theorem 2.2. Suppose that $\ln p_n = -2/c_n$ and $\alpha < c_n = o(\ln n)$ for some $\alpha > 0$. Let $\{y_n\}$ be a sequence such that $c_n \ln n - \ln(2c_n \ln n) + y_n \in \mathbb{Z}$ for all n, and $\inf_n y_n \ge \beta$ for some $\beta \in \mathbb{R}$. We have

$$\lim_{n \to \infty} e^{\lambda(y_n)} P(U^{(n)} \le c_n \ln n - \ln(2c_n \ln n) + y_n) = 1, \tag{4}$$

where $\lambda(x) = p_n^{x+2}$. In particular, $U^{(n)}/c_n \ln n$ converges to 1 in probability, as $n \to \infty$.

The relationship between p_n and c_n is depicted in Fig. 1. We observe that the probability p_n, by our assumptions, should within the regime $e^{-2/\alpha} < p_n = e^{-2/o(\ln n)}$ for $\alpha > 0$. For the case $p_n = o(1)$ (i.e., $c_n = o(1)$), by letting $\alpha \to 0$, we can infer that $W^{(n)} \ll \ln n$ and $U^{(n)} \ll \ln n$ whp.

§3. Proofs

In this section, we will only consider Theorem 2.1 since the proofs are very similar. Theorem 2.1 will be proved through a series of lemmas by similar reasoning to [3] with some modifications.

For a collection of random variables $\{X_i\}_{i=1}^n$, a graph G of order n is called a dependency graph [4] of $\{X_i\}_{i=1}^n$ if for any vertex i, X_i is independent of the set $\{X_j :$ vertices i and j are not adjacent$\}$. The following is a result of Arratia et al. [2], which is a instrumental version of the Stein-Chen method in numerous probabilistic combinatorial problems [1].

Lemma 3.1.[2] Let $\{X_i\}_{i=1}^n$ be n Bernoulli random variables with $EX_i = p_i > 0$. Let G be a dependency graph of $\{X_i\}_{i=1}^n$. Set $S_n = \sum_{i=1}^n X_i$ and $\lambda = ES_n = \sum_{i=1}^n p_i$. Define

$$B_1(G) = \sum_{i=1}^{n} \sum_{j:j \sim i} EX_i EX_j. \tag{5}$$

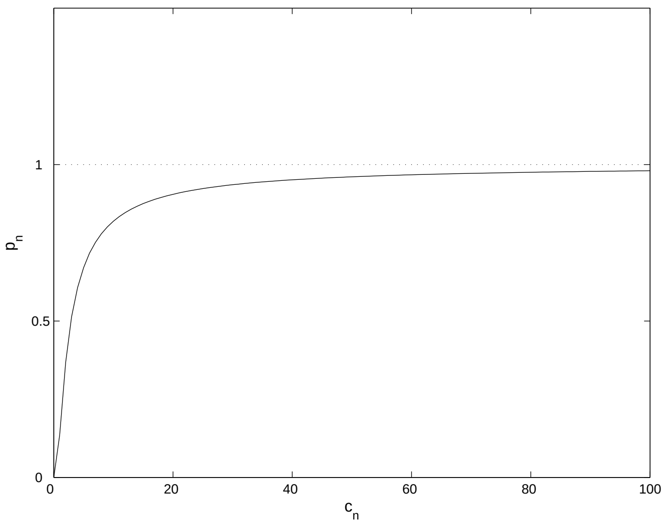


Figure 1: The probability p_n versus c_n.

and

$$B_2(G) = \sum_{i=1}^{n} \sum_{j \neq i : j \sim i} E(X_i X_j). \tag{6}$$

Let Z be a Poisson random variable with $EZ = \lambda$. For any $A \subset \mathbb{N}$, we have

$$|P(S_n \in A) - P(Z \in A)| \leq B_1(G) + B_2(G). \tag{7}$$

Fix $\varepsilon > 0$ and set $m = \lfloor (c_n + \varepsilon) \ln n \rfloor$. Define the truncated version

$$W'^{(n)}_{s,t} := \max \left\{ 1 \leq k \leq m : \xi_s = 0, \prod_{i=1}^{k} \xi_{s+it(\bmod n)} = 1 \right\} \tag{8}$$

and $W'^{(n)} = \max_{1 \leq s,t \leq n} W'^{(n)}_{s,t}$. For $x \in \mathbb{R}$ define the indicator variable

$$I_{s,t}(x) = 1_{\{W'^{(n)}_{s,t} > c_n \ln n + x\}} \quad \text{and} \quad S(x) = \sum_{1 \leq s,t \leq n} I_{s,t}(x). \tag{9}$$

By definition, it is clear that $W'^{(n)} > c_n \ln n + x$ if and only if $S(x) > 0$. Set $A(s,t) = \{s + it\}_{i=0}^{m}$. Fix $x \in \mathbb{R}$ such that $x < \varepsilon \ln n$. Hence, as in [3], we can construct a dependency graph G of random variables $\{I_{s,t}(x)\}_{s,t=1}^{n}$ by setting the vertex set $\{(s,t)\}_{s,t=1}^{n}$ and edges $(s_1,t_1) \sim (s_2,t_2)$ if and only if $A(s_1,t_1) \cap A(s_2,t_2) \neq \emptyset$.

The following combinatorial lemma is useful.

Lemma 3.2.[3] Let $D_{s,t}(k)$ be the number of pairs (s_1,t_1) such that $t \neq t_1$ and $|A(s,t) \cap A(s_1,t_1)| = k$. Then we have

$$D_{s,t}(k) \leq \begin{cases} (m+1)^2 n, & k = 1, \\ (m+1)^2 m^2, & 2 \leq k \leq \frac{m}{2} + 1, \\ 0, & k > \frac{m}{2} + 1. \end{cases} \tag{10}$$

Recall the definitions (5) and (6). Let

$$B_1(x,G)=\sum_{s_1,t_1}\sum_{\substack{s_2,t_2\\(s_1,t_1)\sim(s_2,t_2)}} EI_{s_1,t_1}(x)EI_{s_2,t_2}(x) \tag{11}$$

and

$$B_1(x,G)=\sum_{s_1,t_1}\sum_{\substack{(s_1,t_1)\neq(s_2,t_2)\\(s_1,t_1)\sim(s_2,t_2)}} E[I_{s_1,t_1}(x)I_{s_2,t_2}(x)]. \tag{12}$$

Lemma 3.3. For all $x<\varepsilon\ln n$ and $\delta>0$, we have

$$B_1(x,G)+B_2(x,G)=O(p_n^{2(x+1)}n^{\delta-1}). \tag{13}$$

Proof. From (9), we have $EI_{s,t}(x)=P(W'^{(n)}_{s,t}>c_n\ln n+x)\le p_n^{c_n\ln n+x+1})$. Since for fixed s and t, the number of pairs (s_1,t_1) such that $|A(s,t)\cap A(s_1,t_1)|=k$ is at most $D_{s,t}(k)+1$, we obtain by Lemma 3.2

$$\begin{aligned} B_1(x,G) &\le \sum_{s,t}\sum_{k=1}^{m+1}(D_{s,t}(k)+1)p_n^{2(c_n\ln n+x+1)}\\ &\le p_n^{2(x+1)}\cdot\frac{1}{n^4}\sum_{s,t}\left((m+1)^2n+1+\sum_{k=2}^{m/2+1}((m+1)^2m^2+1)\right)\\ &= p_n^{2(x+1)}\cdot O\left(\frac{m^2n+m^6}{n^2}\right)\\ &= O(p_n^{2(x+1)}n^{\delta-1}), \end{aligned} \tag{14}$$

for all $\delta>0$, where the last equality holds using the assumption $c_n=o(\ln n)$.

Next, we have $E(I_{s,t}(x)I_{s_1,t_1}(x))\le p_n^{2(c_n\ln n+x+1)-k}$ when $|A(s,t)\cap A(s_1,t_1)|=k$. Therefore, by Lemma 3.2

$$\begin{aligned} B_2(x,G) &\le \sum_{s,t}\sum_{k=1}^{m}D_{s,t}(k)p_n^{2(c_n\ln n+x+1)-k}\\ &\le p_n^{2(x+1)}\cdot\frac{1}{n^4}\sum_{s,t}\left(2(m+1)^2n+(m+1)^2m^2\sum_{k=2}^{m/2+1}p_n^{-k}\right). \end{aligned} \tag{15}$$

Since $c_n>\alpha>0$, we obtain

$$\sum_{k=2}^{m/2+1}p_n^{-k}=O\big(p_n^{-\frac{m}{2}}\big)=O\big(n^{\frac{c_n+\varepsilon}{c_n}}\big). \tag{16}$$

Combining (15), (16) and the assumption $c_n=o(\ln n)$, we derive

$$\begin{aligned} B_2(x,G) &= p_n^{2(x+1)}\cdot O\left(\frac{m^2n+m^4n^{\frac{c_n+\varepsilon}{c_n}}}{n^2}\right)\\ &= O(p_n^{2(x+1)}n^{\delta-1}) \end{aligned} \tag{17}$$

for all $\delta > 0$.

The following lemma is a simplified version of Theorem 2.1.

Lemma 3.4. $W^{(n)}/c_n \ln n$ converges to 1 in probability, as $n \to \infty$; i.e., for any $\delta > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{W^{(n)}}{c_n \ln n} - 1\right| > \delta\right) = 0. \tag{18}$$

Proof. Fix $\varepsilon > 0$, we have

$$P(W_{s,t}^{(n)} > (c_n + \varepsilon)\ln n) \le p_n^{(c_n+\varepsilon)\ln n+1}. \tag{19}$$

Since $c_n = o(\ln n)$, it follows that

$$P(W^{(n)} > (c_n + \varepsilon)\ln n) \le n^2 p_n^{(c_n+\varepsilon)\ln n+1} \le e^{-\frac{2\varepsilon \ln n}{c_n}} \to 0 \tag{20}$$

as $n \to \infty$.

Next, let $x = -\varepsilon \ln n$ and $Z(x)$ be a Poisson random variable with

$$EZ(x) = \lambda(x) = ES(x) = n^2 p_n^{\lfloor c_n \ln n+x+2\rfloor} \ge e^{\frac{2\varepsilon \ln n-4}{c_n}}. \tag{21}$$

Note that $\{W^{(n)} \le (c_n - \varepsilon)\ln n\}$ implies that $\{W'^{(n)} \le (c_n - \varepsilon)\ln n\}$. By Lemma 3.1 and Lemma 3.3,

$$\begin{aligned} P(W^{(n)} \le (c_n - \varepsilon)\ln n) &\le P(S(x) = 0) \\ &\le B_1(x, G) + B_2(x, G) + P(Z(x) = 0) \\ &= O(p_n^{2(x+1)} n^{\delta-1} + e^{-e^{\frac{2\varepsilon \ln n-4}{c_n}}}) \to 0, \end{aligned} \tag{22}$$

as $n \to \infty$, for $\delta > 0$ and $\varepsilon < \alpha/5$. Thus, by (20) and (22), it follows that

$$\lim_{n\to\infty} P\left(\left|\frac{W^{(n)}}{c_n \ln n} - 1\right| > \delta\right) = 0. \tag{23}$$

for any $0 < \delta < 1/5$.

To prove of Theorem 2.1, we need to further refine the proof of Lemma 3.4.

Proof of Theorem 2.1. As in the proof of Lemma 3.4, let $Z(x)$ be a Poisson random variable with

$$EZ(x) = \lambda(x) = ES(x) = n^2 p_n^{\lfloor c_n \ln n+x+2\rfloor}. \tag{24}$$

If $c_n \ln n + x \in \mathbb{Z}$, then $\lambda(x) = p_n^{x+2}$. Recall that $W'^{(n)} > c_n \ln n + x$ if and only if $S(x) > 0$. Thus, by Lemma 3.1 and Lemma 3.3

$$\begin{aligned} |P(W'^{(n)} > c_n \ln n + x) - P(Z(x) \ne 0)| &= |P(S(x) > 0) - P(Z(x) > 0)| \\ &\le B_1(x, G) + B_2(x, G) \\ &= O(p_n^{2(x+1)} n^{\delta-1}). \end{aligned} \tag{25}$$

Note that $x < \varepsilon \ln n$, and then we have

$$\{W^{(n)} > c_n \ln n + x\} = \{W^{(n)} > (c + \varepsilon)\ln n\} \cup \{W'^{(n)} > c_n \ln n + x\}. \tag{26}$$

Hence, by (20), (25) and (26), we obtain

$$\begin{aligned}|P(W^{(n)} \le c_n \ln n + x) - e^{-\lambda(x)}| &= |P(W^{(n)} > c_n \ln n + x) - P(Z(x) \ne 0)| \\ &\le P(W^{(n)} > (c_n + \varepsilon) \ln n) \\ &\quad + |P(W'^{(n)} > c_n \ln n + x) - P(Z(x) \ne 0)| \\ &\le e^{-\frac{2\varepsilon \ln n}{c_n}} + O(p_n^{2(x+1)} n^{\delta-1}), \end{aligned} \tag{27}$$

for $0 < \delta < 1$, where the first item on the right-hand side of (27) tends to 0 as $n \to \infty$.

Let $\{x_n\}$ be a sequence such that $c_n \ln n + x_n \in \mathbb{Z}$ for all n. If $\inf_n x_n \ge \beta \in \mathbb{R}$, then $p_n^{2(x_n+1)} n^{\delta-1} \to 0$ and $e^{\lambda(x_n)}$ is a bounded sequence. Thus, from (27) it follows that

$$\left|e^{\lambda(x_n)} P(W^{(n)} \le c_n \ln n + x_n) - 1\right| = O\left(e^{-\frac{2\varepsilon \ln n}{c_n}} + p_n^{2(x_n+1)} n^{\delta-1}\right) \to 0, \tag{28}$$

as $n \to \infty$.

Acknowledgements

The first version of the manuscript was finished when the author was a PhD candidate in the Department of Mathematics, Shanghai Jiao Tong University. The author really appreciates the reviewers careful and helpful concerns.

References

[1] N. Alon and J. H. Spencer, The Probabilistic Method, John Wiley & Sons, New York, 2008.

[2] R. Arratia, L. Goldstein and L. Gordon, Two moments suffice for Poisson approximations: the Chen-Stein method, Ann. Probab., **17**(1989), 9-25.

[3] I. Benjamini, A. Yadin and O. Zeitouni, Maximal arithmetic progressions in random subsets, Elect. Comm. in Probab., **12**(2007), 365-376.

[4] S. Janson, T. Łuczak and A. Rucinski, Random Graphs, John Wiley & Sons, New York, 2000.

[5] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta Arith., **27**(1975), 299-345.

[6] T. Tao, What is good mathematics? Bull. Amer. Math. Soc., **44**(2007), 623-634.

Scientia Magna
Vol. 9 (2013), No. 4, 7-12

Continuities on minimal space via ideals

Shyamapada Modak

Department of Mathematics, University of Gour Banga,
P.O. Mokdumpur, Malda-732103, India
E-mail: spmodak2000@yahoo.co.in

Abstract This paper will discuss about decompositions of continuity on minimal spaces with the help of ideals.

Keywords m-precontinuity, m-semicontinuity, αm-continuity, mM-I-continuity, m-I-continuity.
2000 Mathematics Subject Classification: 54A05, 54C10.

§1. Introduction and preliminaries

Generalization of topological space is not a new concept in literature. But A. Csaszar gives a new dimension generalizing this concept through generalized topology (GT) [1,2,3] on 1997. One of the important generalization of topology is minimal structure [5]. Min and Kim [6,7,8,9] Popa and Noiri [12,13] have defined different types of continuities on minimal spaces and discussed their properties. On 2009, Ozbakir and Yildirim [11] have introduced ideal [4] on minimal space and they defined ideal minimal spaces.

In this we considered m-continuity [13], m-precontinuity [9], m-semicontinuity [7] and αm-continuity and characterized them. We have also give some decompositions of the above continuities. For do this we shall define two continuities on ideal minimal spaces and discuss their relations.

Now we shall discuss some definitions and results for preparing of this paper.

Definition 1.1.[5] A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a minimal structure on X if $\phi \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call (X, m_X) a minimal space. Set $M(x) = \{U \in m_X : x \in U\}$.

Definition 1.2.[12,13] Let (X, m_X) be a minimal space. For a subset A of X, the closure of A and the interior of A are defined as the following:

(1) $mint(A) = \cup\{U : U \subseteq A,\ U \in m_X\}$.

(2) $mcl(A) = \cap\{F : A \subseteq F,\ X - F \in m_X\}$.

Definition 1.3.[13] Let $f : (X, m_X) \to (Y, \tau)$ be a function with a minimal space (X, m_X) and a topological space (Y, τ). Then f is said to be m-continuous if for each open set V containing $f(x)$, there exists an m-open set U containing x such that $f(U) \subseteq V$.

Definition 1.4.[9] Let (X, m_X) be a minimal space and $A \subseteq X$. Then a set A is called an

m-preopen set in X if $A \subseteq mint(mcl(A))$. A set A is called an m-preclosed if the complement of A is m-preopen.

The family of all m-preopen sets in X will be denoted by $MPO(X)$.

Definition 1.5.[9] Let $f : (X, m_X) \to (Y, \tau)$ be a function between a minimal space (X, m_X) and a topological space (Y, τ). Then f is said to be minimal precontinuous (briefly, m-precontinuous) if for each x and for each open set V containing $f(x)$, there exists a m-preopen set U containing x such that $f(U) \subseteq V$.

From Theorem 3.3 of [9], we get following results.

Theorem 1.1. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then f is m-precontinuous if and only if $f^{-1}(V)$ is m-preopen for each open set V in Y.

Definition 1.6.[6] Let (X, m_X) be a minimal space. A subset A of X is called an αm-open set if $A \subseteq mint(mcl(mint(A)))$. The complement of an αm-open set is called an αm-closed set. The family of all αm-open sets in X will be denoted by $\alpha M(X)$.

Here we mention a theorem from [6]:

Theorem 1.2. Let (X, m_X) be a minimal structure. Any union of αm-open sets is αm-open.

Definition 1.7.[7] Let (X, m_X) be a minimal space and $A \subset X$. Then A is called an m-semiopen set if $A \subseteq mcl(mint(A))$.

The complement of an m-semiopen set is called m-semiclosed set. The family of all m-semiopen sets in X will be denoted by $MSO(X)$.

Theorem 1.3.[7] Let (X, m_X) be a minimal structure. Any union of m-semiopen sets is m-semiopen.

From Theorem 3.3 of [7], we get following result:

Theorem 1.4. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then f is m-semicontinuous if and only if $f^{-1}(V)$ is m-semiopen for each open set V in Y.

Let I be an ideal [4] on X and m_X be a minimal structure on X, then (X, m_X, I) is called an ideal minimal space [12].

Definition 1.8.[11] Let (X, m_X, I) be an minimal structure and $(.)_*$ be a set operator from $P(X)$ to $P(X)$. For a subset $A \subseteq X, A_*(I, m_X) = \{x \in X : U \cap A \notin I$, for every $U \in M(x)\}$ is called minimal local function of A with respect to I and m_X. We will simply write A_* for $A_*(I, m_X)$.

Definition 1.9.[11] Let (X, m_X, I) be an minimal ideal space. Then the set operator m-cl^* is called a minimal $*$ -closure and is defined as m-$cl^*(A) = A \cup A_*$ for $A \subseteq X$. We will denote by $m^*_X(I, m_X)$ the minimal structure generated by m-cl^*, that is,

$m^*_X(I, m_X) = \{U \subseteq X : m\text{-}cl^*(X - U) = X - U\}$.

$m^*_X(I, m_X)$ is called $*$-minimal structure which is finer than m_X. The elements of $m^*_X(I, m_X)$ are called minimal $*$-open (briefly, m^*-open) and the complement of an m^*-open set is called minimal $*$-closed (briefly, m^*-closed).

Throughout the paper we write simply m^*_X for $m^*_X(I, m_X)$.

Definition 1.10.[11] A subset A of an ideal minimal minimal space (X, m_X, I) is m^*-dense in itself (resp. m^*-perfect) if $A \subseteq A_*$(resp. $A_* = A$).

Remark 1.1.[11] A subset A of an ideal minimal space (X, m_X, I) is m^*-closed if and only if $A_* \subseteq A$.

Definition 1.11.[10] Let (X, m_X, I) be an ideal topological space and $A \subset X$. Then A is called m-I-open is $A \subset mint((A)_*)$.

The family of all m-I-open sets in (X, m_X, I) will be denoted by $MIO(X)$.

Definition 1.12.[10] Let $f : (X, m_X, I) \to (Y, \tau)$ be a function between an ideal minimal space and a topological space (Y, τ). Then f is said to be m-I-continuous if for each x and each open set V containing $f(x)$, there exists an m-I-open set U containing x such that $f(U) \subseteq V$.

Theorem 1.5.[10] A function $f : (X, m_X, I) \to (Y, \tau)$ is m-I-continuous if and only if $f^{-1}(V)$ is m-I-open, for each open set V in (Y, τ).

Definition 1.13.[10] Let (X, m_X, I) be an ideal topological space and $A \subset X$. Then A is called M-I-open if $A \subset (mint(A))_*$.

The family of all M-I-open sets in (X, m_X, I) will be denoted by $MMIO(X)$.

Theorem 1.6.[10] Let (X, m_X, I) be an ideal topological space and $A_i \in MMIO(X)$ for all i. Then $\cup_i(A_i) \in MMIO(X)$.

§2. Continuities on minimal spaces

Definition 2.1. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a minimal space (X, m_X) and a topological space (Y, τ). Then f is said to be αm-continuous if for each x and for each open set V containing $f(x)$, there exists an αm-open set U containing x such that $f(U) \subseteq V$.

Theorem 2.1. A function $f : (X, m_X) \to (Y, \tau)$ is αm-continuous if and only if $f^{-1}(V)$ is αm-open, for each open set V in (Y, τ).

Proof. Let f be αm-continuous. Then for any open set V in Y and for each $x \in f^{-1}(V)$, there exists an αm-open set U containing x such that $f(U) \subseteq V$. This implies $x \in U \subseteq f^{-1}(V)$ for each $x \in f^{-1}(V)$. Since any union of αm-open sets is αm-open [6], $f^{-1}(V)$ is αm-open.

Converse part: Let $x \in X$ and for each open set V containing $f(x)$, $x \in f^{-1}(V) \subset mint(mcl(mint(f^{-1}(V))))$. So there exists an αm-open set U containing x such that $x \in U \subseteq f^{-1}(V)$, i.e., $f(U) \subseteq V$. Hence f is αm-continuous.

Now we mention a Result from [9]:

Result 2.1. Let $f : (X, m_X) \to (Y, \tau)$ be m-continuous map, then it is m-precontinuous.

Again Min and Kim [9] have shown by an example that the reverse part of the above result need hold in general.

Theorem 2.2. Let $f : (X, m_X) \to (Y, \tau)$ be m-continuous map. Then f is αm-continuous.

Proof. Proof is obvious from Lemma 3.3 of [6].

Again from Example 3.6 of [6], we have that the reverse part of the Theorem 2.2 need not hold in general.

Now we shall discuss two results on minimal space for further discussion.

Result 2.2. Let (X, m_X) be a minimal space. Then every αm-open set is a m-semiopen set.

Proof. Proof is obvious from definitions.

Result 2.3. Let (X, m_X) be a minimal space and $A \subseteq X$. Let A be m-preopen and m-semiopen then A is αm-open.

Proof. Proof is obvious from definitions.

Using these two results, we get following theorems:

Theorem 2.3. Let $f : (X, m_X) \to (Y, \tau)$ be αm-continuous map. Then f is m-semicontinuous.

Proof of this theorem is obvious from the fact that $\alpha M(X) \subseteq MSO(X)$. But reverse inclusion need not hold in general.

Example 2.1. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, \{a, b\}, \{d\}, X\}$. Then m-closed sets are: $\emptyset, \{a, b, c\}, \{c, d\}, X$. Consider the set $A = \{a, b, c\}$, then $A \subset mcl(mint(A))$, but A is not a subset of $mint(mcl(mint(A)))$. So, $A \in MSO(X)$ but $A \notin \alpha M(X)$. Hence we have that the reverse part of the Theorem 2.3 need not hold in general.

Theorem 2.4. Let $f : (X, m_X) \to (Y, \tau)$ be a map. Then f is m-precontinuous and m-semicontinuous if and only if f is αm-continuous.

Theorem 2.5. Let $f : (X, m_X, I) \to (Y, \tau)$ be a map. Then f is m-I-continuous and m-semicontinuous if and only if f is αm-continuous.

Proof. Proof is obvious from the fact that every m-I-continuous function is a m-precontinuous function.

For further discussion, we shall prove some results regarding M-I-open sets in ideal minimal space.

Theorem 2.6. Let (X, m_X, I) be an ideal minimal space. Then every M-I-open set is a m-semiopen set.

Hence we have $MMIO(X) \subseteq MSO(X)$, but converse need not hold in general.

Example 2.2. Let $X = \{a, b, c\}, m_X = \{\emptyset, \{a\}, X\}$ and $I = \{\emptyset, \{a\}\}$. Consider $A = \{a, c\}$, then $A \subset mcl(mint(A))$, but $(mint(A))_* = \emptyset$. So $A \in MSO(X)$ but $A \notin MMIO(X)$.

Theorem 2.7. Let (X, m_X, I) be an ideal minimal space and $A \subset X$. If A is a M-I-open then A is m^*-dense in it self.

Proof. Proof is obvious.

Following example shows that the reverse part of the above theorem need not hold in general.

Example 2.3. Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, \{a, b\}, \{d\}, X\}$ and $I = \{\emptyset, \{c\}\}$. Consider $A = \{a, c, d\}$. Then $A_* = \{a, b, c, d\}$, so $A \subseteq A_*$. Again $(mint(A))_* = \{d\}_* = \{c, d\}$ and A is not a subset of $(mint(A))_*$.

Definition 2.2. Let $f : (X, m_X, I) \to (Y, \tau)$ be a function between an ideal minimal space and a topological space (Y, τ). Then f is said to be mM-I-continuous if for each x and each open set V containing $f(x)$, there exists an M-I-open set U containing x such that $f(U) \subseteq V$.

Theorem 2.8. A function $f : (X, m_X, I) \to (Y, \tau)$ is mM-I-continuous if and only if $f^{-1}(V)$ is M-I-open, for each open set V in (Y, τ).

Proof. Let f be mM-I-continuous. Then for any open set V in Y and for each $x \in f^{-1}(V)$, there exists an M-I-open set U containing x such that $f(U) \subseteq V$. This implies $x \in U \subseteq f^{-1}(V)$ for each $x \in f^{-1}(V)$. Since any union of M-I-open sets is M-I-open [9], $f^{-1}(V)$ is M-I-open.

Converse part: Let $x \in X$ and for each open set V containing $f(x)$, $x \in f^{-1}(V) \subset (mint(f^{-1}(V)))_*$. So there exists an M-I-open set U containing x such that $x \in U \subseteq f^{-1}(V)$, i.e., $f(U) \subseteq V$. Hence f is mM-I-continuous.

Corollary 2.1. Let $f : (X, m_X, I) \to (Y, \tau)$ be mM-I-continuous function. Then $f^{-1}(V)$ is $*$-dense in it self for each open set V in Y.

Theorem 2.9. Let $f : (X, m_X, I) \to (Y, \tau)$ be a mM-I-continuous function, then f is a m-semicontinuous function.

From Example 2.2, we get that every m-semicontinuous function need be a mM-I-continuous function in general.

Theorem 2.10. Let (X, m_X, I) be an ideal minimal space and $A \subset X$. If A is M-I-open and m-preopen then A is αm-open.

Proof. Proof is obvious.

Since concept of M-I-open set and αm-open set are independent, then reverse part of this Theorem need not hold in general. Further we get following corollary:

Corollary 2.2. Let $f : (X, m_X, I) \to (Y, \tau)$ be a function. If f is m-precontinuous and mM-I-continuous then f is αm-continuous.

As the conclusion, we get following diagram:

m-continuity $\Longrightarrow$ αm-continuity $\Longrightarrow$ m-semicontinuity.
m-precontinuity + m-semicontinuity $\Longleftrightarrow$ αm-continuity.
m-I-continuity + m-semicontinuity $\Longleftrightarrow$ αm-continuity.
mM-I-continuity $\Longrightarrow$ m-semicontinuity $\Longleftarrow$ m-continuity.
mM-I-continuity + m-precontinuity $\Longrightarrow$ αm-continuity.

References

[1] A. Csaszar, Generalized open sets, Acta Math. Hungar., **75**(1997), No. 1-2, 65-87.

[2] A. Csaszar, Generalized topology, generalized continuity, Acta Math. Hungar., **96**(2002), No. 4, 351-357.

[3] A. Csaszar, Weak structures, Acta Math. Hungar., **131**(2011), No. 1-2, 193-195.

[4] K. Kuratowski, Topology, Vol. I, Academic Press (New York), 1966.

[5] H. Maki, J. Umehara and T. Noiri, Every topological space is pre$T1/2$, Men. Fac. Sci. Kochi. Univ. Ser. A Math., **17**(1996), 33-42.

[6] W. K. Min, αm-open sets and αM-continuous functions, Commun. Korean Math. Soc., **25**(2010), No. 2, 251-256.

[7] W. K. Min, On minimal semicontinuous functions, Commun. Korean Math. Soc., **27**(2012), No. 2, 341-345.

[8] W. K. Min and Y. K. Kim, m-preopen sets and M-precontinuity on spaces with minimal structures, Advances in Fuzzy Sets and System, **4**(2009), No. 3, 237-245.

[9] W. K. Min and Y. K. Kim, On minimal precontinuous functions, J. Chun Math. Soc., **22**(2009), No. 4, 667-673.

[10] S. Modak and S. Mistry, Continuities on ideal minimal spaces, Aryabhatta Journal of Mathematics and Informatics, **5**(2013), No. 1, 101-106.

[11] O. B. Ozbakir and E. D. Yildirim, On some closed sets in ideal minimal spaces, Acta Math. Hungar., **125**(2009), No. 3, 227-235.

[12] V. Popa and T. Noiri, On M-continuous functions, Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor. Fasc. II, **18**(2000), No. 23, 31-41.

[13] V. Popa and T. Noiri, On the definition of some generalized forms of continuity under minimal conditions, Men. Fac. Scr. Kochi. Univ. Ser. Math., **22**(2001), 9-19.

Scientia Magna
Vol. 9 (2013), No. 4, 13-15

On Sophie Germain's identity

A. K. S. Chandra Sekhar Rao

Department of Mathematics, DNR (Autonomous) College,
Bhimavaram, 534202, A. P., India
E-mail: chandhrasekhar_6@yahoo.co.in

Abstract A characterization of SS-elements of a ring by employing Sophie Germain's identity is given. Further, it is proved that there is a one-to-one correspondence between the set of SS-elements of a ring and the set of Sophie Germain's identities over that ring.

Keywords Ring, SS-element, SS-ring, factorization, one-to-one correspondence.

§1. Introduction

Marie-Sophie Germain, a French mathematician has contributed notably to the study of number theory and elasticity. Sophie Germain's identity states that for any integers x, y; $x^4+4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$. It is first appeared on everything [7]. Knowing how to express an object as a sum usually tells us nothing about how to express it as a product; finding clever factorizations is therefore helpful in number theory. One such enlightening identity is Sophie Germain's identity.

In the theory of rings, it is quite natural and interesting to bserve that there are elements of a ring satisfying the condition $a^2 = a + a$. These elements are called SS-elements of the ring. The concept of SS-element of a ring was first initiated by W. B. V. K. Samy in [8]. In a ring (with unit), we can always find two SS-elements 0 (additive identity) and $2 = (1 + 1)$, which are trivial SS-elements of the ring. An SS-element of a ring other than 0 and 2 is said to be a nontrivial SS-element. A ring is called an SS-ring if it contains at least one non-trivial SS-element. Examples of SS-rings can be easily found in the literature. It was proved that the SS-ring contains (always) nonzero divisors of zero [8]. It was also proved in [1] that if a is a non-trivial SS-element of a ring then it satisfies the condition $a^2 \neq a$. Consequently, the idempotent elements of a ring cannot be non-trivial SS-elements and the Boolean algebra can be regarded as a ring (Boolean ring) but it is not an SS-ring. In [6] we have obtained characterizations of SS-elements of a ring besides some applications of them.

In this paper, we present another characterization of SS-elements of a ring by employing Sophie Germain's identity. Further it is also proved that there is a one-to-one correspondence between the set of SS-elements of an SS-ring and the set of Sophie Germain's identities over that ring. For basic definitions and concepts, the reader can refer to Jacobson [5], A. K. S. C. S Rao [6] and W. B. V. K Samy [8].

§2. Proof of the theorem

Theorem 2.1. Let R be a commutative ring with unity. An element $a \in R$ is an SS-element of the ring R if and only if the Sophie Germain's identity, $x^4 + a^2y^4 = (x^2 + axy + ay^2)(x^2 - axy + ay^2)$ holds in R.

Proof. If a is an SS-element then $a + a = a^2$, now

$$\begin{aligned} & (x^2 + axy + ay^2)(x^2 - axy + ay^2) \\ = \; & x^4 - x^2axy + x^2ay^2 + axyx^2 - axyaxy + axyay^2 + ay^2x^2 - ay^2axy + ay^2ay^2 \\ = \; & x^4 + (a + a)y^2x^2 - a^2x^2y^2 + a^2y^4 \\ = \; & x^4 + a^2y^4. \end{aligned}$$

On the other hand, assume that $x^4 + a^2y^4 = (x^2 + axy + ay^2)(x^2 - axy + ay^2)$ holds in R. In contrary, if a is not an SS-element then Sophie Germain's identity $x^4 + a^2y^4 = (x^2 + axy + ay^2)(x^2 - axy + ay^2)$ cannot hold in R. Therefore, a is an SS-element of R. This completes the proof of the theorem.

Theorem 2.2. There is a one-to-one correspondence between the set of SS-elements of an SS-ring and the set of Sophie Germain's identities over that ring.

Proof. Let S be the set of SS-elemetns of an SS-ring R and I be the set of Sophie Germain's identities over the ring R.

The mapping $a \to (x^4 + a^2y^4) = (x^2 + axy + ay^2)(x^2 - axy + ay^2), \forall a \in S$. In view of the Theorem 2.1 it is evident that the mapping is a one-to-one and onto mapping. This completes the proof.

§3. Examples

Example 3.1. $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is an SS-ring under the operations $+_8, X_8$. The SS-elements of the SS-ring Z_8 are $0, 2, 4, 6$. The one-to-one correspondance is as follows:

$$\begin{aligned} 0 \to x^4 + 0y^4 &= (x^2 + 0xy + 0y^2)(x^2 - 0xy + 0y^2), \\ 2 \to x^4 + 4y^4 &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2), \\ 4 \to x^4 + 16y^4 &= (x^2 + 4xy + 4y^2)(x^2 - 4xy + 4y^2), \\ 6 \to x^4 + 36y^4 &= (x^2 + 6xy + 6y^2)(x^2 - 6xy + 6y^2). \end{aligned}$$

Example 3.2. Let $Q = \{q_1 + q_2i + q_3j + q_4k\}$, where $q_i = 0$ or1, $i^2 = 1$, $j^2 = 1$, $k^2 = 1$, $ij = ji = ik = ki = j$ and $jk = kj = i$. Then Q forms a commutative ring (with unit) with respect to the operations addition and multiplication as follows:

$$(q_1 + q_2i + q_3j + q_4k) + (q_1^1 + q_2^1i + q_3^1j + q_4^1k) = (q_1 + q_1^1) + (q_2 + q_2^1)i + (q_3 + q_3^1)j + (q_4 + q_4^1)k,$$

$$\begin{aligned} (q_1 + q_2i + q_3j + a_4k)(q_1^1 + q_2^1i + q_3^1j + q_4^1k) \; = \; & q_1q_1^1 + q_1q_2^1i + q_1q_3^1j + q_1q_4^1k + q_2q_1^1i + q_2q_2^1i^2 \\ & + q_2q_3^1ij + q_2q_4^1ik + q_3q_1^1j + q_3q_2^1ji + q_3q_3^1j^2 \\ & + q_3q_4^1jk + q_4q_1^1k + q_4q_2^1ki + q_4q_3^1kj + q_4q_4^1k^2. \end{aligned}$$

The non-trivial SS-elements of the ring Q are $1+i, 1+j, 1+k, i+j+k, i+k, j+k, 1+i+j+k$. Thus Q is an SS-ring. The one-to-one correspondence is as follows:

$1+i \rightarrow x^4+(1+i)^2y^4 = (x^2+(1+i)xy+(1+i)y^2)(x^2-(1+i)xy+(1+i)y^2)$,

$1+j \rightarrow x^4+(1+j)^2y^4 = (x^2+(1+j)xy+(1+j)y^2)(x^2-(1+j)xy+(1+j)y^2)$,

$1+k \rightarrow x^4+(1+k)^2y^4 = (x^2+(i+k)xy+(1+k)y^2)(x^2-(1+ik)xy+(1+k)y^2)$,

$i+j \rightarrow x^4+(i+j)^2y^4 = (x^2+(i+j)xy+(i+j)y^2)(x^2-(i+j)xy+(i+j)y^2)$,

$i+k \rightarrow x^4+(i+k)^2y^4 = (x^2+(i+k)xy+(i+k)y^2)(x^2-(i+k)xy+(i+k)y^2)$,

$j+k \rightarrow x^4+(j+k)^2y^4 = (x^2+(j+k)xy+(j+k)y^2)(x^2-(j+k)xy+(j+k)y^2)$,

$1+i+j+k \rightarrow x^4+(1+i+j+k)^2y^4 = (x^2+(1+i+j+k)xy+(1+i+j+k)y^2)(x^2-(1+i+j+k)xy+(1+i+j+k)y^2)$.

References

[1] A. K. S. C. S Rao, A note on SS-rings, Math. India, 2000.

[2] Graeme Taylor, Identity of Sophie Germain, April 1, 2006.

[3] J Lambek, Lectures on Rings and Modules, Waltham, Massachusetts, Blasdell, 1966.

[4] L. Serge, Algebra, Addition-Wesley Publ. Company Reading, Massachusetts, USA.

[5] N. Jacobson, Structure of Rings, Amer. Math. Soc. Coll. Publ., **37**(1956).

[6] T. Srinivas and A. K. S. Chandra Sekhar Rao, SS-elements and their Applications, Southeast Asian Bulletin of Mathematics, SEAMS, Hong Kong, China, **33**(2009), 361-366.

[7] http://www.everything2.com/index.pl?node_id=1796348.

[8] W. B. Vasantha Kanda Samy, On SS-rings, Math. India, 1998.

Scientia Magna
Vol. 9 (2013), No. 4, 16-22

On the circulant solutions of the matrix equations $AX = C$ and $AXB = C$

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State University,
Science City of Muñoz 3120, Nueva Ecija, Philippines

Abstract In this paper, we present the sufficient and necessary conditions for the matrix equations $AX = C$ and $AXB = C$ for them to have circulant solutions. Moreover, we also present a possible extension of this problem.

Keywords Matrix equation, circulant matrix, left circulant matrix, right circulant matrix, skew-left circulant matrix, skew-right circulant matrix.

§1. Introduction

Given any sequence of complex numbers $c_0,\ c_1,\ \cdots, c_{n-1}$, a circulant matrix can be formed. Circulant matrices have four types namely right circulant, left circulant, skew-right circulant and skew-left circulant and they take the following froms, respectively.

$$C_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix} \quad (1)$$

$$C_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \end{pmatrix} \quad (2)$$

$$S_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ -c_{n-2} & -c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_3 & -c_4 & \cdots & c_0 & c_1 \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & c_0 \end{pmatrix} \tag{3}$$

$$S_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & -c_0 \\ c_2 & c_3 & c_4 & \cdots & -c_0 & -c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & -c_0 & \cdots & -c_{n-4} & -c_{n-3} \\ c_{n-1} & -c_0 & -c_1 & \cdots & -c_{n-3} & -c_{n-2} \end{pmatrix} \tag{4}$$

In each circulant matrix, the first row $\vec{c} = (c_0, c_1, \cdots, c_{n-1}) \in \mathbb{C}^n$ is called the circulant vector. It determines the circulant matrix.

§2. Preliminary results

In this section, we shall use $diag(c_0, c_1, \cdots, c_{n-1})$ to denote the diagonal matrix

$$\begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} \end{pmatrix}$$

and $adiag(c_0, c_1, \cdots, c_{n-1})$ to denote the anti diagonal matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & 0 & \cdots & c_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n-2} & 0 & \cdots & 0 & 0 \\ c_{n-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that these matrices are not necessarily circulant.

For the rest of the paper, we shall also use the following notations:

- $C_R(\mathbb{C})$:= the set of nxn complex right circulant matrices,
- $C_L(\mathbb{C})$:= the set of nxn complex left circulant matrices,
- $S_R(\mathbb{C})$:= the set of nxn complex skew-right circulant matrices,
- $S_L(\mathbb{C})$:= the set of nxn complex skew-left circulant matrices.

Now, we define the Fourier matrix to establish the relationship of the circulant matrices.

Definition 2.1. The unitary matrix F_n given by

$$F_n = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-2} & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-2)} & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{n-2} & \omega^{2(n-2)} & \cdots & \omega^{(n-2)(n-2)} & \omega^{(n-1)(n-2)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-2)(n-1)} & \omega^{(n-1)(n-1)} \end{pmatrix} \tag{5}$$

where $\omega = e^{2\pi i/n}$ is called the Fourier matrix.

The circulant matrices are related by the following equations:

$$C_R(\vec{c}) = F_n D F_n^{-1}, \tag{6}$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ and $D = diag(d_0, d_1, \cdots, d_{n-1})$.

$$C_L(\vec{c}) = \Pi C_R(\vec{c}) = \Pi F_n D F_n^{-1}, \tag{7}$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = diag(d_0, d_1, \cdots, d_{n-1})$, $\Pi = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & \tilde{I}_{n-1} \end{pmatrix}$ (an $n \times n$ matrix), $\tilde{I}_{n-1} = adiag\,(1, 1, \cdots, 1, 1)$ (an $(n-1) \times (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$ (an $(n-1) \times 1$ matrix).

$$S_R(\vec{c}) = \Delta F_n D (\Delta F_n)^* = \Delta F_n D F_n^{-1} \Delta^*, \tag{8}$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = diag(d_0, d_1, \cdots, d_{n-1})$, $\Delta = diag\left(1, \theta, \ldots, \theta^{n-1}\right)$, and $\theta = e^{i\pi/n}$.

$$S_L(\vec{c}) = \Sigma SRCIRC_n(\vec{c}) = \Sigma \Delta F_n D (\Delta F_n)^*, \tag{9}$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $\Delta = diag\left(1, \theta, \ldots, \theta^{n-1}\right)$, $\Sigma = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & -\tilde{I}_{n-1} \end{pmatrix}$(an $n \times n$ matrix), $\tilde{I}_{n-1} = adiag\,(1, 1, \cdots, 1, 1)$ (an $(n-1) \times (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$.

Remark 2.1. The matrices Π and Σ are orthonormal $\Pi = \Pi^T = \Pi^{-1}$ and $\Sigma = \Sigma^T = \Sigma^{-1}$.

Properties of circulant matrices regarding inverses.

- $C_R^{-1}(\vec{a}) \in C_R(\mathbb{C})$,

- $C_L^{-1}(\vec{a}) \in C_L(\mathbb{C})$,
- $S_R^{-1}(\vec{a}) \in S_R(\mathbb{C})$,
- $S_L^{-1}(\vec{a}) \in S_L(\mathbb{C})$.

The above properties show that circulant matrices are closed under matrix inverse.

Now we introduce the following notations for invertible circulant matrices which will also be used for the rest of the paper.

- $CL_R(\mathbb{C})$:= the set of invertible nxn complex right circulant matrices,
- $CL_L(\mathbb{C})$:= the set of invertible nxn complex left circulant matrices,
- $SL_R(\mathbb{C})$:= the set of invertible nxn complex skew-right circulant matrices,
- $SL_L(\mathbb{C})$:= the set of invertible nxn complex skew-left circulant matrices.

§3. Preliminary results

The following lemmas will be used to prove the main results.

Lemma 3.1. $C_R(\vec{a})\Pi = C_L(\vec{\gamma}) \in C_L(\mathbb{C})$ where $\vec{\gamma} = (c_0, c_{n-1}, c_{n-2}, \cdots, c_1)$.

Proof. Note that Π is a permutation matrix and right multiplication with a permutation matrix switches the colums. In Π the column C_1 is fixed while C_2 and C_{n-1}, C_3 and C_{n-2} etc are switched. This means that after right multiplication with Π, we have

$$S_L(\vec{\gamma}) = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ c_1 & c_0 & c_{n-1} & \cdots & c_3 & c_2 \end{pmatrix},$$

which is left circulant.

Lemma 3.2. $S_R(\vec{a})\Sigma = S_L(\vec{\rho}) \in S_L(\mathbb{C})$ where $\vec{\rho} = (c_0, -c_{n-1}, -c_{n-2}, \cdots, -c_1)$.

Proof. Σ has the same effect as Π but once the columns are switched, they will become negative of the original. Hence, after right multiplication we have

$$S_L(\vec{\rho}) = \begin{pmatrix} c_0 & -c_{n-1} & -c_{n-2} & \cdots & -c_2 & -c_1 \\ -c_{n-1} & -c_{n-2} & -c_{n-3} & \cdots & -c_1 & -c_0 \\ -c_{n-2} & -c_{n-3} & -c_{n-4} & \cdots & -c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_1 & -c_0 & \cdots & c_4 & c_3 \\ -c_1 & -c_0 & c_{n-1} & \cdots & c_3 & c_2 \end{pmatrix},$$

which is skew-left circulant.

Lemma 3.3. $C_R(\vec{a})C_R(\vec{b}) \in C_R(\mathbb{C})$.

Proof.

$$\begin{aligned} C_R(\vec{a})C_R(\vec{b}) &= (FD_1F^{-1})(FD_2F^{-1}) \\ &= FD_1D_2F^{-1} \in C_R(\mathbb{C}). \end{aligned}$$

Lemma 3.4. $C_R(\vec{a})C_L(\vec{b})$ and $C_L(\vec{b})C_R(\vec{a}) \in C_L(\mathbb{C})$.

Proof.

$$\begin{aligned} C_R(\vec{a})C_L(\vec{b}) &= C_R(\vec{a})\Pi C_R(\vec{b}) \\ &= C_L(\vec{\alpha})C_R(\vec{b}) \\ &= \Pi C_R(\vec{\alpha})C_R(\vec{b}) \in C_L(\mathbb{C}), \end{aligned}$$

$$C_L(\vec{b})C_R(\vec{a}) = \Pi C_R(\vec{b})C_R(\vec{a}) \in C_L(\mathbb{C}),$$

where $\vec{\alpha} = (a_, a_{n-1}, \cdots, a_1)$.

Lemma 3.5. $S_R(\vec{a})S_R(\vec{b}) \in S_R(\mathbb{C})$.

Proof.

$$\begin{aligned} S_R(\vec{a})S_R(\vec{b}) &= (\Delta FD_1(\Delta F)^*)(\Delta FD_2(\Delta F)^*) \\ &= \Delta FD_1D_2(\Delta F)^* \in S_R(\mathbb{C}). \end{aligned}$$

Lemma 3.6. $S_L(\vec{a})S_R(\vec{b})$ and $S_R(\vec{b})S_L(\vec{a}) \in S_L(\mathbb{C})$.

Proof.

$$S_L(\vec{a})S_R(\vec{b}) = \Sigma S_R(\vec{a})S_R(\vec{b}) \in S_L(\mathbb{C}),$$

$$\begin{aligned} S_R(\vec{b})S_L(\vec{a}) &= S_R(\vec{b})\Sigma S_R(\vec{a}) \\ &= S_L(\vec{\beta})S_R(\vec{a}) \\ &= \Sigma S_R(\vec{\beta})S_R(\vec{a}) \in S_L(\mathbb{C}), \end{aligned}$$

where $\vec{\beta} = (b_0, -b_{n-1}, \cdots, -b_1)$.

§4. Main results

The following are results on the circulant solutions of the matrix equations $AX = C$ and $AXB = C$. All of them are provable using combinations of some lemmas which were proven in the previous section.

Theorem 4.1. $AX = C$ has a right circulant solution if and only if one of the following is satisfied

- $A \in CL_R(\mathbb{C})$, $C \in C_R(\mathbb{C})$,

- $A \in CL_L(\mathbb{C})$, $C \in C_L(\mathbb{C})$.

Theorem 4.2. $AX = C$ has a left circulant solution if and only if one of the following is satisfied

- $A \in CL_R(\mathbb{C})$ and $C \in C_L(\mathbb{C})$,
- $A \in CL_L(\mathbb{C})$ and $C \in C_R(\mathbb{C})$.

Theorem 4.3. $AX = C$ has a skew-right circulant solution if and only if one of the following is satisfied

- $A \in SL_R(\mathbb{C})$, $C \in S_R(\mathbb{C})$,
- $A \in SL_L(\mathbb{C})$, $C \in S_L(\mathbb{C})$.

Theorem 4.4. $AX = C$ has a skew-left circulant solution if and only if one of the following is satisfied

- $A \in SL_R(\mathbb{C})$ and $C \in S_L(\mathbb{C})$,
- $A \in SL_L(\mathbb{C})$ and $C \in S_R(\mathbb{C})$.

Theorem 4.5. $AXB = C$ has right circulant solution if and only if one of the following is satisfied

- $A, B \in CL_R(\mathbb{C})$ and $C \in C_R(\mathbb{C})$,
- $A, B \in CL_L(\mathbb{C})$ and $C \in C_R(\mathbb{C})$,
- $A \in CL_R(\mathbb{C})$, $C \in C_L(\mathbb{C})$ and $C \in CL_L(\mathbb{C})$,
- $B \in CL_R(\mathbb{C})$, $C \in C_L(\mathbb{C})$ and $A \in CL_L(\mathbb{C})$.

Theorem 4.6. $AXB = C$ has left circulant solution if and only if one of the following is satisfied

- $A, B \in CL_R(\mathbb{C})$ and $C \in C_L(\mathbb{C})$,
- $A, B \in CL_L(\mathbb{C})$ and $C \in C_L(\mathbb{C})$,
- $A \in CL_R(\mathbb{C})$, $C \in C_R(\mathbb{C})$ and $B \in CL_L(\mathbb{C})$,
- $A \in CL_L(\mathbb{C})$, $C \in C_R(\mathbb{C})$ and $B \in CL_R(\mathbb{C})$.

Theorem 4.7. $AXB = C$ has skew-right circulant solution if and only if one of the following is satisfied

- $A, B \in SL_R(\mathbb{C})$ and $C \in S_R(\mathbb{C})$,
- $A, B \in SL_L(\mathbb{C})$ and $C \in S_R(\mathbb{C})$,
- $A \in SL_R(\mathbb{C})$, $C \in S_L(\mathbb{C})$ and $B \in SL_L(\mathbb{C})$,

- $B \in SL_R(\mathbb{C})$, $C \in S_L(\mathbb{C})$ and $A \in SL_L(\mathbb{C})$.

Theorem 4.8. $AXB = C$ has skew-left circulant solution if and only if one of the following is satisfied

- $A, B \in SL_R(\mathbb{C})$ and $C \in S_L(\mathbb{C})$,
- $A, B \in SL_L(\mathbb{C})$ and $C \in S_L(\mathbb{C})$,
- $A \in SL_R(\mathbb{C})$, $C \in S_R(\mathbb{C})$ and $B \in SL_L(\mathbb{C})$,
- $A \in SL_L(\mathbb{C})$, $C \in S_R(\mathbb{C})$ and $B \in SL_R(\mathbb{C})$.

§5. Conclusion

In summary, we have provided the sufficient and necessary conditions for the matrix equations $AX = C$ and $AXB = C$ for them to have right circulant, left circulant, skew-right circulant and skew-left circulant solutions.

§6. Recommendation

A possible extension on this paper is considering matrices of dimension $m \times n$. That is, to find the circulant solutions of the following:

- $AX = C$, where A and C are both $m \times n$ matrices,
- $AXB = C$, where A, B and C are $m \times n$, $n \times p$, and $m \times p$ matrices, respectively.

References

[1] R. Gray, Toeplitz and Circulant Matrices, A Review, Technical Report, Stanford University, USA.

[2] H. Karner, J. Schneid, C. Ueberhuber, Spectral decomposition of real circulant matrices, Linear Algebra and Its Applications, **367**(2003), 301-311.

Scientia Magna
Vol. 9 (2013), No. 4, 23-29

Coefficient bounds for certain subclasses of analytic functions

B. S. Mehrok† and Gagandeep Singhc‡

Department of Mathematics, Dhilwan(Kapurthala), Punjab, India
E-mail: kamboj.gagandeep@yahoo.in

Abstract In this paper, we introduce some subclasses of analytic functions and determine the sharp upper bounds of the functional $|a_2a_4 - a_3^2|$ for the functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to these classes in the unit disc $E = \{z :| z |< 1\}$.

Keywords Analytic functions, starlike functions, convex functions, alpha-convex functions, functions whose derivative has a positive real part, Bazilevic functions, Hankel determinant.

§1. Introduction and preliminaries

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the unit disc $E = \{z :| z |< 1\}$.

Let S be the class of functions $f(z) \in A$ and univalent in E.

Let $M_\alpha(\alpha \geq 0)$ be the class of functions in A which satisfy the conditions

$$\frac{f(z)f'(z)}{z} \neq 0$$

and

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{2}$$

The class M_α was introduced by Mocanu [15] and functions of this class are called α-convex functions. Obviously $M_0 \equiv S^*$, the class of starlike functions and $M_1 \equiv K$, the class of convex functions. Miller, Mocanu and Reade [14] have shown that α-convex functions are starlike in E, and for $\alpha \geq 1$, all α-convex functions are convex in E. Therefore α-convex functions are also called α-starlike functions. Concept of α-convex functions gives a continuous parametrization between starlike functions and convex functions.

$H_\alpha(\alpha \geq 0)$ be the class of functions in A which satisfy the condition

$$Re\left[(1-\alpha)f'(z) + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0. \tag{3}$$

This class was introduced by Al-Amiri and Reade [1]. In particular $H_0 \equiv R$, the class of functions whose derivative has a positive real part and studied by Macgregor [12]. Also $H_1 \equiv K$.

$B_\alpha(\alpha \geq 0)$ is the class of functions in A which satisfy the condition

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] > 0. \tag{4}$$

The class B_α was introduced by Singh [19] and studied further by Thomas [21] and El-Ashwah and Thomas [3]. Functions of this class are called Bazilevic functions. Particularly $B_0 \equiv S^*$ and $B_1 \equiv R$.

In 1976, Noonan and Thomas [16] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by Eq. (1) with bounded boundary. Ehrenborg [2] studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman [9]. Also Hankel determinant for different classes was studied by various authors including Hayman [5], Pommerenke [18], Janteng et al. ([6],[7],[8]) and recently by Mehrok and Singh [13].

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [4] then further generalised the estimate of $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

For μ complex, Szynal [21] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class M_α. Al-Amiri and Reade [1] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class H_α and Singh [19] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class B_α.

In this paper, we seek upper bound of the functional $|a_2a_4 - a_3^2|$ for the functions belonging to the classes M_α, H_α and B_α. Results due to various authors follows as special cases.

§2. Main result

Let P be the family of all functions p analytic in E for which $Re(p(z)) > 0 (z \in E)$ and

$$p(z) = 1 + p_1z + p_2z^2 + \cdots \tag{5}$$

Lemma 2.1. If $p \in P$, then $|p_k| \leq 2(k = 1, 2, 3, \cdots)$.

This result is due to Pommerenke [18].

Lemma 2.2. If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \leq 1,\ |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez ([10],[11]).

Theorem 2.1. If $f \in M_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}\left[\frac{3\alpha(1+\alpha)^3}{(1+3\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)} + 1\right]. \tag{6}$$

Proof. As $f \in M_\alpha$, so from (2)

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p(z). \tag{7}$$

On expanding and equating the coefficients of z, z^2 and z^3 in (7), we obtain

$$a_2 = \frac{p_1}{1+\alpha}, \tag{8}$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{(1+3\alpha)p_1^2}{2(1+2\alpha)(1+\alpha)^2} \tag{9}$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{(1+5\alpha)p_1p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4+6\alpha+17\alpha^2)p_1^3}{6(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \tag{10}$$

Using (8), (9) and (10), it yields

$$\begin{aligned} &|a_2a_4 - a_3^2| \\ = &\frac{1}{C(\alpha)}\left|\begin{array}{r} 4(1+2\alpha)^2(1+\alpha)^3p_1(4p_3) + 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2p_1^2(2p_2) \\ +8(1+2\alpha)(1+6\alpha+17\alpha^2)^2p_1^4 - 3(1+3\alpha)((1+\alpha)^2(2p_2) + 2(1+3\alpha)p_1^2)^2 \end{array}\right| \end{aligned} \tag{11}$$

where $C(\alpha) = 48(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4$.

Using Lemma 2.1 and Lemma 2.2 in (11) and replacing p_1 by p, it can be easily established that

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}\left[\begin{array}{r} [-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) \\ +3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) \\ -6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) \\ -(1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3p(4-p^2) \end{array}\right]$$

where $\delta = |x| \leq 1$.

Therefore

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}F(\delta),$$

where

$$\begin{aligned} F(\delta) &= [-4(1+2\alpha)(1+\alpha)^2 - 12(1+2\alpha)(1+5\alpha)(1+\alpha)^2 - 8(1+2\alpha)(1+6\alpha+17\alpha^2) \\ &\quad +3(1+3\alpha)(3+8\alpha+\alpha^2)]p^4 + [8(1+2\alpha)^2(1+\alpha)^3 + 12(1+2\alpha)(1+5\alpha) \\ &\quad -6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2]p^2(4-p^2)\delta + (1+\alpha)^3(2-p)[6(1+\alpha)(1+3\alpha) \\ &\quad -(1+4\alpha+7\alpha^2)](4-p^2)p^2\delta^2 + 8(1+2\alpha)^2(1+\alpha)^3 p(4-p^2). \end{aligned}$$

As $F'(\delta) > 0$, so $F(\delta)$ is an increasing function in $[0,1]$ and therefore Max$F(\delta) = F(1)$.

Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}G(p), \tag{12}$$

where $G(p) = F(1)$.

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4,$$

where

$A(\alpha) = 4\alpha(1+\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)$ and $B(\alpha) = 48\alpha(1+\alpha)^4$.

Now $G^{'}(p) = -4A(\alpha)p^3 + 2B(\alpha)p$ and $G^{''}(p) = -12A(\alpha)p^2 + 2B(\alpha)$.

$G^{'}(p) = 0$ gives $p[2A(\alpha)p^2 - B(\alpha)] = 0$.

$G^{''}(p)$ is negative at $p = \sqrt{\frac{6(1+\alpha)^3}{(2+15\alpha+24\alpha^2+7\alpha^3)}} = p^{'}$.

So Max$G(p) = G(p^{'})$.

Hence from (12), we obtain (6).

The result is sharp for $p_1 = p^{'}$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(2 - p_1^2)$.

For $\alpha = 0$ and $\alpha = 1$ respectively, Theorem 2.1 gives the following results due to Janteng et al. [8].

Corollary 2.1. If $f(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Corollary 2.2. If $f(z) \in K$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

Theorem 2.2. If $f \in H_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4}{9(1+\alpha)^2} & \text{if } 0 \leq \alpha \leq \frac{5}{17}, \\ \frac{(17\alpha-5)^2}{144(1+2\alpha)(1+20\alpha+7\alpha^2-4\alpha^3)} + \frac{4}{9(1+\alpha)^2} & \text{if } \frac{5}{17} \leq \alpha \leq 1. \end{cases} \tag{13}$$

Proof. Since $f \in H_\alpha$, so from (3)

$$(1-\alpha)f'(z) + \alpha\frac{(zf'(z))^{'}}{f'(z)} = p(z). \tag{14}$$

On expanding and equating the coefficients of z, z^2 and z^3 in (14), we obtain

$$a_2 = \frac{p_1}{2}, \tag{15}$$

$$a_3 = \frac{p_2 + \alpha p_1^2}{3(1+\alpha)}, \tag{16}$$

and

$$a_4 = \frac{p_3}{4(1+2\alpha)} + \frac{3\alpha p_1 p_2}{4(1+\alpha)(1+2\alpha)} + \frac{\alpha(2\alpha-1)p_1^3}{4(1+\alpha)(1+2\alpha)}. \tag{17}$$

Using (15), (16) and (17), it yields

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)}\left|\begin{array}{r} 9(1+\alpha)^3 p_1(4p_3) + 54(1+\alpha)p_1^2(2p_2) \\ +36\alpha(2\alpha-1)(1+\alpha)p_1^4 - 8(1+2\alpha)(2p_2+2\alpha p_1^2)^2 \end{array}\right| \tag{18}$$

where $C(\alpha) = 288(1+3\alpha)(1+2\alpha)(1+\alpha)^2$.

Using Lemma 2.1 and Lemma 2.2 in (18) and replacing p_1 by p, we obtain

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}\left[\begin{array}{r} (1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta \\ +(2-p)[16(1+2\alpha)-(1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2p(4-p^2) \end{array}\right]$$

where $\delta = |x| \le 1$.

Therefore

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}F(\delta),$$

where

$$F(\delta) = (1-12\alpha+3\alpha^2+8\alpha^3)p^4 + 2(1+13\alpha+4\alpha^2)p^2(4-p^2)\delta$$
$$+(2-p)[16(1+2\alpha)-(1+2\alpha+9\alpha^2)p^2](4-p^2)\delta^2 + 18(1+\alpha)^2p(4-p^2)$$

is an increasing function.

Therefore Max$F(\delta) = F(1)$.

Consequently

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)}G(p), \tag{19}$$

where $G(p) = F(1)$.

So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 128(1+2\alpha),$$

where $A(\alpha) = 2(1+20\alpha+17\alpha^2-4\alpha^3)$ and $B(\alpha) = 4(1+\alpha)(17\alpha-5)$.

Case I. For $0 \le \alpha \le \frac{5}{17}$, $B(\alpha) < 0$.

So $G(p)$ is maximum at $p = 0$ and it follows the first result of (13).

In this case, the result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

Case II. For $\frac{5}{17} \le \alpha \le 1$, as in Theorem 2.1, $G(p)$ is maximum for $p = \sqrt{\frac{(1+\alpha)(17\alpha-5)}{(1+20\alpha+7\alpha^2-4\alpha^3)}} = p'$.

So Max$G(p) = G(p')$.

In this case, the result is sharp for $p_1 = p'$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(p_1^2 - 3)$.

Hence the theorem.

For $\alpha = 0$ in Theorem 2.2, we obtain the following results due to Janteng et al. [6].

Corollary 2.3. If $f(z) \in R$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

Putting $\alpha = 1$ in Theorem 2.2, we get the following results due to Janteng et al. [8].

Corollary 2.4. If $f(z) \in K$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

On the same lines, we can easily prove the following theorem:

Theorem 2.3. If $f \in B_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{(2+\alpha)^2}.$$

The result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

For $\alpha = 0$, Theorem 2.3 gives the following result due to Janteng et al. [8].

Corollary 2.5. If $f(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

For $\alpha = 0$, Theorem 2.3 gives the following result due to Janteng et al. [6].

Corollary 2.6. If $f(z) \in R$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

References

[1] H. S. Al-Amiri and M. O. Reade, On a linear combination of some expressions in the theory of the univalent functions, Monatshafte fur Mathematik, **80**(1975), 257-264.

[2] R. Ehrenborg, The Hankel determinant of exponential polynomials, American Mathematical Monthly, **107**(2000), 557-560.

[3] R. M. El-Ashwah and D. K. Thomas, Growth results for a subclass of Bzilevic functions, Int. J. Math. and Math. Sci., **8**(1985), 785-793.

[4] M. Fekete and G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, J. London Math. Soc., **8**(1933), 85-89.

[5] W. K. Hayman, Multivalent functions, Cambridge Tracts in Math. and Math. Phys., Cambridge University Press, Cambridge, 1958, No. 48.

[6] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Ineq. in Pure and Appl. Math., **7**(2006), No. 2, 1-5.

[7] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Hankel determinant for functions starlike and convex with respect to symmetric points, J. Quality Measurement and Anal., **2**(2006), No. 1, 37-43.

[8] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal., **1**(2007), No. 13, 619-625.

[9] J. W. Layman, The Hankel transform and some of its properties, J. of Integer Sequences, **4**(2001), 1-11.

[10] R. J. Libera and E-J. Zlotkiewiez, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc., **85**(1982), 225-230.

[11] R. J. Libera and E-J. Zlotkiewiez, Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer. Math. Soc., **87**(1983), 251-257.

[12] T. H. MacGregor, Functions whose derivative has a positive real part, Tras. Amer. Math. Soc., **104**(1962), 532-537.

[13] B. S. Mehrok and Gagandeep Singh, Estimate of second Hankel determinant for certain classes of analytic functions, Scientia Magna, **8**(2012), No. 3, 85-94.

[14] S. S. Miller, P. T. Mocanu and M. O. Reade, All α-convex functions are univalent and starlike, Proc. Amer. Math. Soc., **37**(1973), 553-554.

[15] P. T. Mocanu, Une propriete'de convexite' généralisée dans la théorie de la représentation conforme, Mathematica(CLUJ), **11**(1969), No. 34, 127-133.

[16] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean p-valent functions, Trans. Amer. Math. Soc., **223**(1976), No. 2, 337-346.

[17] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., **28**(1983), No. 8, 731-739.

[18] Ch. Pommerenke, Univalent functions, Göttingen: Vandenhoeck and Ruprecht., 1975.

[19] R. Singh, On Bazilevic functions, Proc. Amer. Math. Soc., **38**(1973), 261-273.

[20] J. Szynal, Some remarks on coefficient inequality for α-convex functions, Bull. De L'Acad. Des Sci Ser des Sci Math. Astr et phys., **20**(1972), No. 11, 917-919.

[21] D. K. Thomas, On a subclass of Bazilevic functions, Int. J. Math. and Math. Sci., **8**(1985), 779-783.

Scientia Magna
Vol. 9 (2013), No. 4, 30-40

An intuitionistic $\mathcal{B}$-generalized closed symmetric member in intuitionistic uniform structure spaces

P. Saranya[†], M. K. Uma[‡] and E. Roja[♯]

Department of Mathematics, Sri Sarada College for Women,
Salem, Tamilnadu, India
E-mail: pssaranyalakshmi@gmail.com

Abstract The purpose of this paper is to introduce the concepts of intuitionistic t-open symmetric member, intuitionistic $\mathcal{B}$-open symmetric member, intuitionistic $\mathcal{AB}$-open symmetric member, intuitionistic $\mathcal{C}$-open symmetric member, intuitionistic $\mathcal{B}g$-closed symmetric member, intuitionistic $\mathcal{AB}$g-closed symmetric member and intuitionistic $\mathcal{C}$g-closed symmetric member. Besides providing some interesting properties, interrelations among the intuitionistic symmetric members are introduced and discussed. The concepts of intuitionistic uniformly $\mathcal{AB}$g-continuity, intuitionistic uniformly $\mathcal{B}$g-continuity, intuitionistic uniformly $\mathcal{C}$g-continuity, frontier of intuitionistic uniform $\mathcal{B}$g-symmetric member, exterior of intuitionistic uniform $\mathcal{B}$g-symmetric member, intuitionistic uniformly $\mathcal{B}g$-connected spaces and intuitionistic uniformly $\mathcal{B}g$-compact spaces are introduced and studied.

Keywords Intuitionistic symmetric member, intuitionistic uniform structure, intuitionistic t-open symmetric member, intuitionistic $\mathcal{B}$-open symmetric member, intuitionistic $\mathcal{C}$-open symmetric member, intuitionistic $\mathcal{AB}$-open symmetric member, intuitionistic $\mathcal{B}g$-closed symmetric member, intuitionistic $\mathcal{C}$g-closed symmetric member, intuitionistic $\mathcal{AB}g$-closed symmetric member, intuitionistic uniformly $\mathcal{AB}$g-continuity, intuitionistic uniformly $\mathcal{B}$g-continuity, intuitionistic uniformly $\mathcal{C}$g-continuity, frontier of intuitionistic uniform $\mathcal{B}$g-symmetric member, exterior of intuitionistic uniform $\mathcal{B}$g-symmetric member, intuitionistic uniformly $\mathcal{B}g$-connected spaces and intuitionistic uniformly $\mathcal{B}g$-compact spaces.
2000 Mathematics Subject Classification: 54A40, 54E55.

§1. Introduction and preliminaries

The concept of intuitionistic sets in topological spaces was introduced by Çoker in [3]. He studied topology on intuitionistic sets in [4]. In 1937, Andre Weil [10] formulated the concept of uniform space which is a generalization of a metric space. J. Tong [9] introduced the concept of $\mathcal{B}$-set in topological space. The purpose of this paper is to introduce the concepts of intuitionistic t-open symmetric member, intuitionistic $\mathcal{B}$-open symmetric member, intuitionistic $\mathcal{AB}$-open symmetric member, intuitionistic $\mathcal{C}$-open symmetric member, intuitionistic $\mathcal{B}g$-closed symmetric member, intuitionistic $\mathcal{AB}$g-closed symmetric member and intuitionistic $\mathcal{C}$g-closed symmetric member. Besides providing some interesting properties, interrelations among the intuitionistic symmetric members are introduced and discussed. The concepts of intuitionistic uniformly $\mathcal{AB}$g-continuity, intuitionistic uniformly $\mathcal{B}$g-continuity, intuitionistic uniformly $\mathcal{C}$g-continuity, frontier of intuitionistic uniform $\mathcal{B}$g-symmetric member, exterior of intuitionistic uniform $\mathcal{B}$g-symmetric member, intuitionistic uniformly $\mathcal{B}g$-connected spaces and intuitionistic uniformly $\mathcal{B}g$-compact spaces are introduced and studied.

Definition 1.1.[1] Let X be a non empty set. An intuitionistic set (IS for short) A is an object having the form $A = \langle x, A^1, A^2\rangle$, where A^1 and A^2 are subsets of X satisfying $A^1 \cap A^2 = \emptyset$. The set A^1 is called the set of members of A, while A^2 is called the set of nonmembers of A. Every crisp set A on a nonempty set X is obviously an intuitionistic set having the form $\langle x, A, A^c\rangle$.

Definition 1.2.[1] Let X be a non empty set and let the intuitionistic sets A and B be in the form $A = \langle x, A^1, A^2\rangle$, $B = \langle x, B^1, B^2\rangle$, respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_i = \langle x, {A_i}^1, {A_i}^2\rangle$. Then

(i) $A \subseteq B$ if and only if $A^1 \subseteq B^1$ and $A^2 \supseteq B^2$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $\overline{A} = \langle x, A^2, A^1\rangle$.

(iv) $\cup A_i = \langle x, \cup {A_i}^1, \cap {A_i}^2\rangle$.

(v) $\cap A_i = \langle x, \cap {A_i}^1, \cup {A_i}^2\rangle$.

(vi) $\emptyset_\sim = \langle x, \emptyset, X\rangle$; $X_\sim = \langle x, X, \emptyset\rangle$.

Definition 1.3.[2] An intuitionistic topology (IT for short) on a nonempty set X is a family T of intuitionistic sets in X satisfying the following axioms:

(i) $\emptyset_\sim$, $X_\sim \in T$.

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$.

(iii) $\cup G_i \in T$ for any arbitrary family $\{G_i : i \in J\} \subseteq T$.

In this case the pair (X, T) is called an intuitionistic topological space (ITS for short) and any intuitionistic set in T is called an intuitionistic open set (IOS for short) in X. The complement $\overline{A}$ of an intuitionistic open set A is called an intuitionistic closed set (ICS for short) in X.

Definition 1.4.[2] Let (X, T) be an intuitionistic topological space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic set in X. Then the closure of A are defined by

$$cl(A) = \cap\{K : K \text{ is an intuitionistic closed set in } X \text{ and } A \subseteq K\}.$$

It can be also shown that $cl(A)$ is an intuitionistic closed set in X, and A is an intuitionistic closed set in X if $cl(A) = A$.

Definition 1.5.[2] Let (X, T) be an intuitionistic topological space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic set in X. Then the interior of A are defined by

$$int(A) = \cup\{G : G \text{ is an intuitionistic open set in } X \text{ and } G \subseteq A\}.$$

It can be also shown that $int(A)$ is an intuitionistic open set in X, and A is an intuitionistic open set in X if $int(A) = A$.

Definition 1.6.[1] Let X be a nonempty set and $p \in X$ a fixed element in X. Then the intuitionistic set $\tilde{p}$ defined by $\tilde{p} = \langle x, \{p\}, \{p\}^c \rangle$ is called an intuitionistic point (IP for short) in X.

Definition 1.7.[1] (i) If $B = \langle y, B^1, B^2 \rangle$ is an intuitionistic set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the intuitionistic set in X defined by $f^{-1}(B) = \langle x, f^{-1}(B^1), f^{-1}(B^2) \rangle$.

(ii) If $A = \langle x, A^1, A^2 \rangle$ is an intuitionistic set in X, then the image of A under f, denoted by $f(A)$, is the intuitionistic set in Y defined by $f(A) = \langle y, f(A^1), \underline{f}(A^2) \rangle$ where $\underline{f}(A^2) = Y - (f(X - A^2))$.

Definition 1.8. Let (X, T) be a topological space. A subset S in X is said to be a

(i) semi-open set [8] if $S \subseteq cl(int(S))$ and semi-closed set [1] if $int(cl(A)) \subseteq A$.

(ii) semi regular [5] if it is both semi-open and semi-closed.

(iii) $\mathcal{AB}$-set [6] if $S = U \cap A$, where U is an open and A is semi regular.

(iv) t-set [9] if $intcl(S) = int(S)$.

(v) $\mathcal{B}$-set [9] if $S = U \cap A$, where U is an open and A is t-set.

(vi) α^*-open set [7] if $int(A) = int(cl(int(A)))$.

(vii) $\mathcal{C}$-set [7] if $S = U \cap A$, where U is an open and A is α^*-open set.

Definition 1.9.[2] A uniform space X with uniformity ξ is a set X with a nonempty collection ξ of subsets containing the diagonal Δ_x in $X \times X$ satisfying the following properties:

(i) If $E, F \in \xi$, then $E \cap F \in \xi$.

(ii) If $F \subset E$ and $E \in \xi$ then $F \in \xi$.

(iii) If $E \in \xi$ then $E^t = \{(x, y) : (y, x) \in E\} \in \xi$.

(iv) For any $E \in \xi$ there is some $F \in \xi$ such that $F^2 \subset E$.

§2. An intuitionistic $\mathcal{B}$-open symmetric member in intuitionistic uniform structure spaces

Definition 2.1. Let $X \times X$ be a non empty set. An intuitionistic symmetric member (ISM for short) A is an object having the form $A = \langle x, A^1, A^2 \rangle$, where A^1 and A^2 are subsets of $X \times X$ satisfying $A^1 \cap A^2 = \Delta$. The set A^1 is called the set of members of A, while A^2 is called the set of nonmembers of A.

Notation 2.1. Let $(X \times X, \xi)$ be an intuitionistic uniform structure space and it is simply denoted by $(\mathbb{X}, \xi)$.

Notation 2.2 Let $X \times X$ be a non empty set.

(i) $\Delta_\sim = \langle x, \Delta, \mathbb{X} \rangle$.

(ii) $\mathbb{X}_\sim = \langle x, \mathbb{X}, \Delta \rangle$.

Definition 2.2. An intuitionistic uniform structure (IUS for short) on a non-empty set $\mathbb{X}$ is a collection ξ of subsets in $\mathbb{X}$ which satisfies the following axioms

(i) $\Delta_\sim, \mathbb{X}_\sim \in \xi$.

(ii) $E_1 \cap E_2 \in \xi$ for any $E_1, E_2 \in \xi$.

(iii) $\cup E_i \in \xi$ for any arbitrary family $\{E_i : i \in J\} \subseteq \xi$.

(iv) If $E_1 \subset E_2$ and $E_1 \in \xi$ then $E_2 \in \xi$.

(v) If $E_1 \in \xi$ then $E_1^t = \{(x,y) : (y,x) \in E_1\} \in \xi$.

(vi) For any $E_1 \in \xi$ there is some $E_2 \in \xi$ such that $E_2^2 \subset E_1$.

In this case the pair $(\mathbb{X}, \xi)$ is called an intuitionistic uniform structure space ($IUSS$ for short) and any intuitionistic symmetric member in ξ is called an intuitionistic open symmetric member ($IOSM$ for short) in $\mathbb{X}$. The complement $\overline{E}$ of an intuitionistic open symmetric member E is called an intuitionistic closed symmetric member ($ICSM$ for short) in $\mathbb{X}$.

Definition 2.3. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform closure ($IUcl$ for short) of A are defined by

$IUcl(A) = \cap\{K : K$ is an intuitionistic closed symmetric member in $\mathbb{X}$ and $A \subseteq K\}$.

Definition 2.4. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform interior ($IUint$ for short) of A are defined by

$IUint(A) = \cup\{G : G$ is an intuitionistic open symmetric member in $\mathbb{X}$ and $G \subseteq A\}$.

Remark 2.1. (i) Finite intersection of intuitionistic open symmetric member is an intuitionistic open symmetric member.

(ii) Finite union of intuitionistic closed symmetric member is an intuitionistic closed symmetric member.

Proposition 2.1. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. For any two intutionistic symmetric member $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ the following statements are valid.

(i) $A \subseteq IUcl(A)$.

(ii) $IUcl(A \cap B) \subseteq IUcl(A) \cap IUcl(B)$.

(iii) $IUcl(A \cup B) = IUcl(A) \cup IUcl(B)$.

(iv) $IUint(A) \subseteq A$.

(v) $IUint(A \cap B) = IUint(A) \cap IUint(B)$.

(vi) $IUint(A \cup B) \supseteq IUint(A) \cup IUint(B)$.

Definition 2.5. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic t-open symmetric member if $IUint(IUcl(S)) = IUint(S)$. The complement of an intuitionistic t-open symmetric member S is called an intuitionistic t-closed symmetric member in $\mathbb{X}$.

Proposition 2.2. (i) Finite intersection of intuitionistic t-open symmetric member is an intuitionistic t-open symmetric member.

(ii) Finite union of intuitionistic t-closed symmetric member is an intuitionistic t-closed symmetric member.

Definition 2.6. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic $\mathcal{B}$-open symmetric member if there is a $U \in \xi$ and an intuitionistic t-open symmetric member A in $\mathbb{X}$ such that $S = U \cap A$. The complement of an intuitionistic $\mathcal{B}$-open symmetric member S is called an intuitionistic $\mathcal{B}$-closed symmetric member in $\mathbb{X}$.

Proposition 2.3. (i) Finite intersection of intuitionistic $\mathcal{B}$-open symmetric member is an intuitionistic $\mathcal{B}$-open symmetric member.

(ii) Finite union of intuitionistic $\mathcal{B}$-closed symmetric member is an intuitionistic $\mathcal{B}$-closed symmetric member.

Definition 2.7. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform $\mathcal{B}$-closure ($IU\mathcal{B}cl$ for short) of A are defined by

$IU\mathcal{B}cl(A) = \cap\{K : K$ is an intuitionistic $\mathcal{B}$-closed symmetric member in $\mathbb{X}$ and $A \subseteq K\}$.

Proposition 2.4. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. For any two intutionistic symmetric member $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ the following statements are valid.

(i) $IUBcl(\Delta_\sim) = \Delta_\sim$.

(ii) $IUBcl(A) \supseteq A$.

(iii) $A \subseteq B \Rightarrow IUBcl(A) \subseteq IUBcl(B)$.

(iv) $IUBcl(IUBcl(A)) = IUBcl(A)$.

(v) $IUBcl(A \cap B) \subseteq IUBcl(A) \cap IUBcl(B)$.

(vi) $IUBcl(A \cup B) = IUBcl(A) \cup IUBcl(B)$.

Definition 2.8. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform $\mathcal{B}$-interior ($IU\mathcal{B}int$ for short) of A are defined by

$IU\mathcal{B}int(A) = \cup\{G : G$ is an intuitionistic $\mathcal{B}$-open symmetric member in $\mathbb{X}$ and $G \subseteq A\}$.

Proposition 2.5. For any intuitionistic symmetric member $A = \langle x, A^1, A^2\rangle$ of an intuitionistic uniform structure space $(\mathbb{X}, \xi)$.

(i) $IUBcl(\overline{A}) = \overline{IUBint(A)}$.

(ii) $IUBint(\overline{A}) = \overline{IUBcl(A)}$.

Proposition 2.6. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. For any two intutionistic symmetric member $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ the following statements are valid.

(i) $IUBint(A) \subseteq A$.

(ii) $A \subseteq B \Rightarrow IUBint(A) \subseteq IUBint(B)$.

(iii) $IUBint(IUBint(A)) = IUBint(A)$.

(iv) $IUBint(A \cap B) = IUBint(A) \cap IUBint(B)$.

(v) $IUBint(A \cup B) \supseteq IUBint(A) \cup IUBint(B)$.

(vi) $IUBint(\mathbb{X}_\sim) = \mathbb{X}_\sim$.

Remark 2.2. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space.

(i) If A is an intuitionistic $\mathcal{B}$-open symmetric member then$A = IUBint(A)$.

(ii) If A is an intuitionistic $\mathcal{B}$-closed symmetric member then $A = IUBcl(A)$.

(iii) $IUBint(A) \subseteq A \subseteq IUBcl(A)$.

Definition 2.9. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic semi-open symmetric member if $A \subseteq IUcl(IUint(A))$. The complement of an intuitionistic semi-open symmetric member is an intuitionistic semi-closed symmetric member.

Definition 2.10. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2\rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic semi-regular symmetric member if it is both intuitionistic semi-open symmetric member and intuitionistic semi-closed symmetric member.

Proposition 2.7. Every intuitionistic semi-regular symmetric member is an intuitionistic t-open symmetric member.

Definition 2.11. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic $\mathcal{AB}$-open symmetric member if there is a $U \in \xi$ and an intuitionistic semi-regular symmetric member A in $\mathbb{X}$ such that $S = U \cap A$. The complement of an intuitionistic $\mathcal{AB}$-open symmetric member S is called an intuitionistic $\mathcal{AB}$-closed symmetric member in $\mathbb{X}$.

Proposition 2.8. Every intuitionistic $\mathcal{AB}$-open symmetric member is an intuitionistic $\mathcal{B}$-open symmetric member.

Remark 2.3. The converse of Proposition (2.8) need not be true as shown in the Example (2.1).

Example 2.1. Let $X = \{0, 1, 2\}$ and $\xi = \{\Delta_\sim, P, Q, R, \mathbb{X}_\sim\}$ where

$\Delta_\sim = \langle \{(0,0),(1,1),(2,2)\}, \{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\} \rangle$,

$P = \langle \{\Delta, (1,0)\}, \{\Delta, (0,1),(0,2),(1,2),(2,0),(2,1)\} \rangle$,

$Q = \langle \{\Delta, (0,1)\}, \{\Delta, (1,0),(0,2),(1,2),(2,0),(2,1)\} \rangle$,

$R = \langle \{\Delta, (1,0),(0,1)\}, \{\Delta, (0,2),(1,2),(2,0),(2,1)\} \rangle$.

Clearly $(\mathbb{X}, \xi)$ is an intuitionistic uniform structure space. Let $A = \langle \{\Delta, (2,1)\}, \{\Delta, (0,1), (0,2),(1,0),(1,2),(2,0)\} \rangle$ be an intuitionistic symmetric member. Here, A is an intuitionistic $\mathcal{B}$-open symmetric member but it is not intuitionistic $\mathcal{AB}$-open symmetric member.

Definition 2.12. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic α^*-open symmetric member if $IUint(A) = IUint(IUcl(IUint(A)))$. The complement of an intuitionistic α^*-open symmetric member S is called an intuitionistic α^*-closed symmetric member in $\mathbb{X}$.

Proposition 2.9. Every intuitionistic t-open symmetric member is an intuitionistic α^*-open symmetric member.

Definition 2.13. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $S = \langle x, S^1, S^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$ is said to be an intuitionistic $\mathcal{C}$-open symmetric member if there is a $U \in \xi$ and an intuitionistic α^*-open symmetric member A in $\mathbb{X}$ such that $S = U \cap A$. The complement of an intuitionistic $\mathcal{C}$-open symmetric member S is called an intuitionistic $\mathcal{C}$-closed symmetric member in $\mathbb{X}$.

Proposition 2.10. Every intuitionistic $\mathcal{B}$-open symmetric member is an intuitionistic $\mathcal{C}$-open symmetric member.

Remark 2.4. The converse of Proposition (2.10) need not be true as shown in the Example (2.2).

Example 2.2. Let $X = \{0, 1, 2\}$ and $\xi = \{\Delta_\sim, P, Q, R, \mathbb{X}_\sim\}$ where

$\Delta_\sim = \langle \{(0,0),(1,1),(2,2)\}, \{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\} \rangle$,

$P = \langle \{\Delta, (1,2),(2,0)\}, \{\Delta, (0,1),(1,0)(0,2),(2,1)\} \rangle$,

$Q = \langle \{\Delta, (2,1),(0,2)\}, \{\Delta, (1,0),(0,1),(1,2),(2,0)\} \rangle$,

$R = \langle \{\Delta, (0,2),(1,2),(2,0),(2,1)\}, \{\Delta, (1,0),(0,1)\}, \rangle$.

Clearly $(\mathbb{X}, \xi)$ is an intuitionistic uniform structure space. Let $A = \langle \{\Delta, (0,2),(2,0)\}, \{\Delta,$

$(0,1)(1,0),(1,2),(2,1)\}\rangle$ be an intuitionistic symmetric member. Here, A is an intuitionistic $\mathcal{C}$-open symmetric member but it is not intuitionistic $\mathcal{B}$-open symmetric member.

Remark 2.5. Clearly the following implication diagram holds.

intuitionistic $\mathcal{AB}$-open $\Rightarrow$ intuitionistic $\mathcal{B}$-open $\Rightarrow$ intuitionistic $\mathcal{C}$-open.

§3. An intuitionistic $\mathcal{B}g$-closed symmetric member in intuitionistic uniform structure spaces

Definition 3.1. An intuitionistic symmetric member A of an intuitionistic uniform structure space $(\mathbb{X}, \xi)$ is called an intuitionistic generalized closed symmetric member (brief Ig-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an intuitionistic open symmetric member. A set $A \subseteq \mathbb{X}$ is called Ig-open if and only if its complement is Ig-closed.

Definition 3.2. An intuitionistic symmetric member A of an intuitionistic uniform structure space $(\mathbb{X}, \xi)$ is called an intuitionistic $\mathcal{AB}$-generalized closed symmetric member (brief $I\mathcal{AB}g$-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an intuitionistic $\mathcal{AB}$-open symmetric member. A set $A \subseteq \mathbb{X}$ is called $I\mathcal{AB}g$-open if and only if its complement is $I\mathcal{AB}g$-closed.

Definition 3.3. An intuitionistic symmetric member A of an intuitionistic uniform structure space $(\mathbb{X}, \xi)$ is called an intuitionistic $\mathcal{B}$-generalized closed symmetric member (brief $I\mathcal{B}g$-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an intuitionistic $\mathcal{B}$-open symmetric member. A set $A \subseteq \mathbb{X}$ is called $I\mathcal{B}g$-open if and only if its complement is $I\mathcal{B}g$-closed.

Proposition 3.1. Every $I\mathcal{AB}g$-closed symmetric member is an $I\mathcal{B}g$-closed symmetric member.

Remark 3.1. The converse of the above Proposition (3.1) need not be true as shown in Example (3.1).

Example 3.1. Let $X = \{0, 1, 2\}$ and $\xi = \{\Delta_\sim, P, Q, R, \mathbb{X}_\sim\}$ where

$\Delta_\sim = \langle\{(0,0),(1,1),(2,2)\},\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(1,2)\}\rangle$,

$P = \langle\{\Delta,(2,1)\},\{\Delta,(0,1),(2,0),(1,0),(0,2),(2,1)\}\rangle$,

$Q = \langle\{\Delta,(1,2)\},\{\Delta,(0,2),(0,1),(1,0),(2,1),(2,0)\}\rangle$,

$R = \langle\{\Delta,(1,2),(2,1)\},\{\Delta,(0,1),(1,0),(2,0),(0,2)\},\rangle$.

Clearly $(\mathbb{X}, \xi)$ is an intuitionistic uniform structure space. Let $A = \langle\{\Delta,(0,1),(2,0)\},\{\Delta,(0,2),(1,0),(1,2),(2,1)\}\rangle$ be an intuitionistic symmetric member.

Here, A is an intuitionistic $\mathcal{B}g$-closed symmetric member but it is not an intuitionistic $\mathcal{AB}g$-closed symmetric member.

Definition 3.4. An intuitionistic symmetric member A of an intuitionistic uniform structure space $(\mathbb{X}, \xi)$ is called an intuitionistic $\mathcal{C}$-generalized closed symmetric member(brief $I\mathcal{C}g$-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an intuitionistic $\mathcal{C}$-open symmetric member. A set $A \subseteq \mathbb{X}$ is called $I\mathcal{C}g$-open if and only if its complement is $I\mathcal{C}g$-closed.

Proposition 3.2. Every $I\mathcal{B}g$-closed symmetric member is an $I\mathcal{C}g$-closed symmetric member.

Remark 3.2. The converse of the above Proposition (3.2) need not be true as shown in Example (3.2).

Example 3.2. Let $X = \{0, 1, 2\}$ and $\xi = \{\Delta_\sim, P, Q, R, \mathbb{X}_\sim\}$ where

$\Delta_{\sim} = \langle\{(0,0),(1,1),(2,2)\},\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle$,
$P = \langle\{\Delta,(0,1),(1,2)\},\{\Delta,(2,0),(1,0),(0,2),(2,1)\}\rangle$,
$Q = \langle\{\Delta,(1,0),(2,1)\},\{\Delta,(0,2),(0,1),(1,2),(2,0)\}\rangle$,
$R = \langle\{\Delta,(0,1),(1,2),(1,0),(2,1)\},\{\Delta,(2,0),(0,2)\},\rangle$.

Clearly $(\mathbb{X},\xi)$ is an intuitionistic uniform structure space. Let $A = \langle\{\Delta,(0,1),(2,0)\},\{\Delta,(0,2),(1,0),(1,2),(2,1)\}\rangle$ be an intuitionistic symmetric member. Here, A is an intuitionistic $\mathcal{C}g$-closed symmetric member but it is not an intuitionistic $\mathcal{B}g$-closed symmetric member.

Remark 3.3. Clearly the following implication diagram holds.

intuitionistic $\mathcal{AB}g$-closed $\Rightarrow$ intuitionistic $\mathcal{B}g$-closed $\Rightarrow$ intuitionistic $\mathcal{C}g$-closed.

§4. An intuitionistic uniformly $\mathcal{B}g$-contiuous functions in intuitioistic uniform structure spaces

Definition 4.1. Let $(\mathbb{X},\xi_1)$ and $(\mathbb{Y},\xi_2)$ be an intuitionistic uniform structure spaces. A function $f:(\mathbb{X},\xi_1)\to(\mathbb{Y},\xi_2)$ is said to be an intuitionistic uniformly $\mathcal{AB}g$-continuous if for each intuitionistic open (resp.intuitionistic closed) symmetric member A of $(\mathbb{Y},\xi_2)$, $f^{-1}(A)$ is an $I\mathcal{AB}g$-open ($I\mathcal{AB}g$-closed) symmetric member in $(\mathbb{X},\xi_1)$.

Definition 4.2. Let $(\mathbb{X},\xi_1)$ and $(\mathbb{Y},\xi_2)$ be an intuitionistic uniform structure spaces. A function $f:(\mathbb{X},\xi_1)\to(\mathbb{Y},\xi_2)$ is said to be an intuitionistic uniformly $\mathcal{B}g$-continuous if for each intuitionistic open (resp.intuitionistic closed) symmetric member A of $(\mathbb{Y},\xi_2)$, $f^{-1}(A)$ is an $I\mathcal{B}g$-open ($I\mathcal{B}g$-closed) symmetric member in $(\mathbb{X},\xi_1)$.

Proposition 4.1. Every intuitionistic uniformly $\mathcal{AB}g$-continuous function is an intuitionistic uniformly $\mathcal{B}g$-continuous function.

Remark 4.1. The converse of the above Proposition (4.1) need not be true as shown in Example (4.1).

Example 4.1. Let $(\mathbb{X},\xi_1)$ be an intuitionistic uniform structure space as in Example (4.1). Let $Y=\{a,b,c\}$ and $\xi_2=\{\Delta_{\sim},A,\mathbb{Y}_{\sim}\}$, where
$\Delta_{\sim} = \langle\{(a,a),(b,b),(c,c)\},\{(a,a),(b,b),(c,c),(a,b),(a,c),(b,a),(b,c),(c,a),(c,b)\}\rangle$,
$A = \langle\{\Delta,(a,b),(b,a),(b,c),(c,b)\},\{\Delta,(c,a),(a,c)\}\rangle$.

Clearly $(\mathbb{Y},\xi_2)$ is an intuitionistic uniform structure space. Let $f:(\mathbb{X},\xi_1)\to(\mathbb{Y},\xi_2)$ be a function defined by $f(0)=a,\ f(1)=b,\ f(2)=c$. Now, f is an intuitionistic uniformly $\mathcal{B}g$-continuous function but not an intuitionistic uniformly $\mathcal{AB}g$-continuous function.

Definition 4.3. Let $(\mathbb{X},\xi_1)$ and $(\mathbb{Y},\xi_2)$ be an intuitionistic uniform structure spaces. A function $f:(\mathbb{X},\xi_1)\to(\mathbb{Y},\xi_2)$ is said to be an intuitionistic uniformly $\mathcal{C}g$-continuous if for each intuitionistic open (resp.intuitionistic closed) symmetric member A of $(\mathbb{Y},\xi_2)$, $f^{-1}(A)$ is an $I\mathcal{C}g$-open ($I\mathcal{C}g$-closed) symmetric member in $(\mathbb{X},\xi_1)$.

Proposition 4.2. Every intuitionistic uniformly $\mathcal{B}g$-continuous function is an intuitionistic uniformly $\mathcal{C}g$-continuous function.

Remark 4.2. The converse of the above Proposition (4.2) need not be true as shown in Example (4.2).

Example 4.2. Let $X=\{0,1,2\}$ and $\xi=\{\Delta_{\sim},P,Q,R,\mathbb{X}_{\sim}\}$ where
$\Delta_{\sim} = \langle\{(0,0),(1,1),(2,2)\},\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}\rangle$,

$A = \langle \{\Delta, (0,1), (1,0)\}, \{\Delta, (2,1), (2,0), (0,2), (1,2)\} \rangle$,
$B = \langle \{\Delta, (0,2), (2,0)\}, \{\Delta, (1,0), (0,1), (1,2), (2,1)\} \rangle$,
$C = \langle \{\Delta, (0,1), (1,0), (2,0), (0,2)\}, \{\Delta, (1,2), (2,1)\}, \rangle$.

Clearly $(\mathbb{X}, \xi)$ is an intuitionistic uniform structure space. Let $Y = \{a, b, c\}$ and $\xi_2 = \{\Delta_\sim, A, \mathbb{Y}_\sim\}$, where

$\Delta_\sim = \langle \{(a,a), (b,b), (c,c)\}, \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,a), (b,c), (c,a), (c,b)\} \rangle$,
$A = \langle \{\Delta, (a,b), (b,a), (b,c), (c,b)\}, \{\Delta, (c,a), (a,c)\} \rangle$.

Clearly $(\mathbb{Y}, \xi_2)$ is an intuitionistic uniform structure space. Let $f : (\mathbb{X}, \xi_1) \to (\mathbb{Y}, \xi_2)$ be a function defined by $f(0) = a, f(1) = b, f(2) = c$. Now, f is an intuitionistic uniformly $\mathcal{C}g$-continuous function but not an intuitionistic uniformly $\mathcal{B}g$-continuous function.

Remark 4.3. Clearly the following implication diagram holds.

intuitionistic uniformly $\mathcal{AB}g$-continuity
$\Downarrow$
intuitionistic uniformly $\mathcal{B}g$-continuity
$\Downarrow$
intuitionistic uniformly $\mathcal{C}g$-continuity

§5. An intuitionistic uniform $\mathcal{B}g$-connected spaces in intuitionistic uniform structure spaces

Definition 5.1. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform $\mathcal{B}$ generalized closure ($IU\mathcal{B}_g cl$ for short) of A are defined by

$IU\mathcal{B}_g cl(A) = \cap\{K : K$ is an intuitionistic $\mathcal{B}$ generalized closed in $\mathbb{X}$ and $A \subseteq K\}$.

Definition 5.2. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the intuitionistic uniform $\mathcal{B}$ generalized interior ($IU\mathcal{B}_g int$ for short) of A are defined by $IU\mathcal{B}_g int(A) = \cup\{G : G$ is an intuitionistic $\mathcal{B}$ generalized open in $\mathbb{X}$ and $G \subseteq A\}$.

Definition 5.3. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the frontier of intuitionistic uniform $\mathcal{B}$ generalized symmetric member ($Fr_{IU\mathcal{B}_g}$ for short) of A are defined by

$Fr_{IU\mathcal{B}_g}(A) = IU\mathcal{B}_g cl(A) \setminus IU\mathcal{B}_g int(A) = IU\mathcal{B}_g cl(A) \cap IU\mathcal{B}_g cl(\overline{A})$.

Proposition 5.1. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be a subset of $\mathbb{X}$. A is intuitionistic $\mathcal{B}$ generalized closed if and only if $Fr_{IU\mathcal{B}_g}(A) \subseteq A$.

Proof. Let A be a subset of $\mathbb{X}$. Assume that A is an intuitionistic $\mathcal{B}$ generalized closed symmetric member. Thus $\langle x, IU\mathcal{B}_g cl(A^1), IU\mathcal{B}_g cl(A^2) \rangle = \langle x, A^1, A^2 \rangle$.

$$\begin{aligned} Fr_{IU\mathcal{B}_g}(A) \cap \overline{A} &= \langle x, Fr_{IU\mathcal{B}_g}(A^1) \cap \overline{A^1}, Fr_{IU\mathcal{B}_g}(A^2) \cup \overline{A^2} \rangle \\ &= \langle x, IU\mathcal{B}_g cl(A^1) \cap IU\mathcal{B}_g cl(\overline{A^1}) \cap \overline{A^1}, IU\mathcal{B}_g int(A^2) \cup IU\mathcal{B}_g int(\overline{A^2}) \cup \overline{A^2} \rangle \\ &= \langle x, A^1 \cap IU\mathcal{B}_g cl(\overline{A^1}) \cap \overline{A^1}, A^2 \cup IU\mathcal{B}_g int(\overline{A^2}) \cup \overline{A^2} \rangle \\ &= \langle x, \Delta, \mathbb{X} \rangle. \end{aligned}$$

Therefore, $Fr_{IU\mathcal{B}_g}(A) \subseteq A$.

Assume that $Fr_{IU\mathcal{B}_g}(A) \subseteq A$. Thus

$$\begin{aligned}
Fr_{IU\mathcal{B}_g}(A) \cap \overline{A} &= \langle x, Fr_{IU\mathcal{B}_g}(A^1) \cap \overline{A^1}, Fr_{IU\mathcal{B}_g}(A^2) \cup \overline{A^2} \rangle \\
&= \langle x, \Delta, \mathbb{X} \rangle, \text{ and also} \\
IU\mathcal{B}_g cl(A) \cap \overline{A} &= \langle x, IU\mathcal{B}_g cl(A^1) \cap \overline{A^1}, IU\mathcal{B}_g int(A^2) \cup \overline{A^2} \rangle \\
&= \langle x, IU\mathcal{B}_g cl(A^1) \cap [IU\mathcal{B}_g cl(\overline{A^1}) \cap \overline{A^1}], IU\mathcal{B}_g int(A^2) \cup [IU\mathcal{B}_g int(\overline{A^2}) \cup \overline{A^2}] \rangle \\
&= \langle x, Fr_{IU\mathcal{B}_g}(A^1) \cap \overline{A^1}, Fr_{IU\mathcal{B}_g}(A^2) \cup \overline{A^2} \rangle \\
&= \langle x, \Delta, \mathbb{X} \rangle.
\end{aligned}$$

Therefore, $IU\mathcal{B}_g cl(A) \subseteq A$. But $A \subseteq IU\mathcal{B}_g cl(A)$.

Consequently, $\langle x, A^1, A^2 \rangle = \langle x, IU\mathcal{B}_g cl(A^1), IU\mathcal{B}_g int(A^2) \rangle$. Hence A is an intuitionistic $\mathcal{B}$ generalized closed symmetric member.

Proposition 5.2. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be a subset of $\mathbb{X}$. A is intuitionistic $\mathcal{B}$ generalized open if and only if $Fr_{IU\mathcal{B}_g}(A) \subseteq \overline{A}$.

Definition 5.4. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic symmetric member in $\mathbb{X}$. Then the exterior of intuitionistic uniform $\mathcal{B}$-generalized symmetric member ($Ext_{IU\mathcal{B}_g}$ for short) of A are defined by

$$Ext_{IU\mathcal{B}_g}(A) = \mathbb{X} \setminus IU\mathcal{B}_g cl(A).$$

Proposition 5.3. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be a subset of $\mathbb{X}$. A is intuitionistic $\mathcal{B}$ generalized closed if and only if $Ext_{IU\mathcal{B}_g}(A) \cup Ext_{IU\mathcal{B}_g}(B) = Ext_{IU\mathcal{B}_g}(A \cap B)$.

Proof. Let $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ are subsets of $\mathbb{X}$. Assume that A and B are intuitionistic $\mathcal{B}$ generalized closed symmetric member. Thus $\langle x, IU\mathcal{B}_g cl(A^1), IU\mathcal{B}_g int(A^2) \rangle = \langle A^1, A^2 \rangle$ and $\langle x, IU\mathcal{B}_g cl(B^1), IU\mathcal{B}_g int(B^2) \rangle = \langle x, B^1, B^2 \rangle$. Then $A \cap B = \langle x, A^1 \cap B^1, A^2 \cup B^2 \rangle$ is also an intuitionistic $\mathcal{B}$ generalized closed symmetric member.

$$\begin{aligned}
Ext_{IU\mathcal{B}_g}(A \cap B) &= \langle x, Ext_{IU\mathcal{B}_g}(A^1 \cap B^1), Ext_{IU\mathcal{B}_g}(A^2 \cup B^2) \rangle \\
&= \langle x, \mathbb{X} \setminus IU\mathcal{B}_g cl(A^1 \cap B^1), \mathbb{X} \setminus IU\mathcal{B}_g int(A^2 \cup B^2) \rangle \\
&= \langle x, \mathbb{X} \setminus (A^1 \cap B^1), \mathbb{X} \setminus (A^2 \cup B^2) \rangle \\
&= \langle x, (\mathbb{X} \setminus A^1), (\mathbb{X} \setminus A^2) \rangle \cup \langle x, (\mathbb{X} \setminus B^1), (\mathbb{X} \setminus B^2) \rangle \\
&= Ext_{IU\mathcal{B}_g}(A) \cup Ext_{IU\mathcal{B}_g}(B).
\end{aligned}$$

Proposition 5.4. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space and $A = \langle x, A^1, A^2 \rangle$ be a subset of $\mathbb{X}$.

(i) If A is intuitionistic $\mathcal{B}$g-closed then $Ext_{IU\mathcal{B}_g}(\mathbb{X} \setminus Ext_{IU\mathcal{B}_g}(A)) = Ext_{IU\mathcal{B}_g}(A)$.

(ii) If A is intuitionistic $\mathcal{B}$g-closed then $Ext_{IU\mathcal{B}_g}(\mathbb{X} \setminus Ext_{IU\mathcal{B}_g}(A)) \cap Fr_{IU\mathcal{B}_g}(A) = \langle x, \Delta, \mathbb{X} \rangle$.

Definition 5.5. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. Then $(\mathbb{X}, \xi)$ is said to be an intuitionistic uniform $\mathcal{B}$g-disconnected space if there exist non-empty intuitionistic symmetric member A and B containing Δ such that $A \cup B = \mathbb{X}_\sim$ and $A \cap B = \Delta_\sim$.

Definition 5.6. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. $\mathbb{X}$ is called intuitionistic uniform $\mathcal{B}$g-connected, if $\mathbb{X}$ is not intuitionistic uniform $\mathcal{B}$g-disconnected.

Proposition 5.5. Let $(\mathbb{X}, \xi)$ be an intuitionistic uniform structure space. Then $(\mathbb{X}, \xi)$ is an intuitionistic uniform $\mathcal{B}$g-connected space if it has no non-zero intuitioniistic $\mathcal{B}$g-open symmetric member $P = \langle x, P^1, P^2 \rangle$ and $Q = \langle x, Q^1, Q^2 \rangle$ such that $P \cup Q = \mathbb{X}_\sim$.

Definition 5.7. An intuitionistic uniform structure space $(\mathbb{X}, \xi)$ is called an intuitionistic uniform $\mathcal{B}$g-compact space if $\cup_{i \in I}(P_i) = \mathbb{X}_\sim$, for each intuitionistic $\mathcal{B}$g-open symmetric member $P_i, i \in I$, there is a finite subset J of I with $\cup_{j \in J}(P_i) = \mathbb{X}_\sim$.

Proposition 5.6. If $f : (\mathbb{X}, \xi_1) \to (\mathbb{Y}, \xi_2)$ is an intuitionistic uniformly $\mathcal{B}$g-continuous bijective function and $(\mathbb{X}, \xi_1)$ is an intuitionistic uniform $\mathcal{B}$g-compact space then $(\mathbb{Y}, \xi_2)$ is an intuitionistic uniform $\mathcal{B}$g-compact space.

References

[1] N. Biswas, On characterization of semi-continuous functions, Atti. Accad. Naz. Lincei Rend, Cl. Sci. Fis. Mat. Natur., **48**(1970), 399-402.

[2] N. Bourbaki, Elements of mathematics, General topology, Springer, Berlin, 1989.

[3] D. Coker, A note on intuitionistic sets and intuitionistic points, Turkish J. Math., **20**(1996), 343-351.

[4] D. Coker, An introduction to intuitionistic topological spaces, BUSEFAL, **81**(2000), 51-56.

[5] G. Di Maio and T. Noiri, On s-closed spaces, Indian J Pure App. Math. Sci., **18**(1987), 226-233.

[6] J. Dontchev, Between A- and B- sets. Math Balkanica, **12**(1998), 295-302.

[7] E. Hatir, T. Noiri and S. Yuskel, A decomposition of continuity, Acta. Math. Hungar., **70**(1996), 145-150.

[8] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, **70**(1963), 36-41.

[9] J. Tong, On decomposition of continuity in topological spaces, Acta. Math. Hungar., **54**(1989), 51-55.

[10] A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Act. Scient. et Ind., Vol. 551, Hermann, Paris, 1937.

Scientia Magna
Vol. 9 (2013), No. 4, 41-48

On intuitionistic dimension functions

M. Kousalyaparasakthi†, E. Roja‡ and M. K. Uma♯

Sri Sarada College for Women, Salem ,Tamilnadu, India
E-mail: koushi.1902@gmail.com

Abstract In this paper, the concepts of intuitionistic small inductive dimension, intuitionistic AB open set, intuitionistic partitions are introduced and studied. The concept of intuitionistic large inductive dimension, intuitionistic addition theorem and intuitionistic product theorem are introduced and studied.

Keywords Intuitionistic small inductive dimension, intuitionistic large inductive dimension, intuitionistic additition theorem and intuitionistic product theorem.
2000 Mathematics Subject Classification: 54A05-54E15

§1. Introduction

The concept of intuitionistic sets was introduced by çoker [1]. In 1998, J. Donchev [5] introduced the concept of AB-sets and decomposition of continuity. In this paper, the concepts of intuitionistic small inductive dimension, intuitionistic AB open set, intuitionistic partitions are introduced and studied.The concepts of intuitionistic large inductive dimension, intuitionistic addition theorem and intuitionistic product theorem are introduced and studied.

§2. Preliminaries

Definition 2.1.[1] Let X be a non empty set. An intuitionistic set (IS for short) A is an object having the form $A = \langle x, A^1, A^2\rangle$ for $x \in X$, where A^1 and A^2 are subsets of X satisfying $A^1 \cap A^2 = \emptyset$. The set A^1 is called the set of members of A, while A^2 is called the set of nonmembers of A. Every crisp set A on a nonempty set X is obviously an intuitionistic set having the form $\langle X, A, A^c\rangle$.

Definition 2.2.[2] Let X be a non empty set and let the intuitionistic sets A and B be in the form $A = \langle x, A^1, A^2\rangle$, $B = \langle x, B^1, B^2\rangle$, respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_i = \langle x, A_i^1, A_i^2\rangle$. Then

(i) $A \subseteq B$ if and only if $A^1 \subseteq B^2$ and $A^2 \supseteq B^2$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $\overline{A} = \langle x, A^2, A^1\rangle$.

(iv) $\bigcup A_i = \langle x, \cup A_i^1, \cap A_i^2\rangle$.

(v) $\bigcap A_i = \langle x, \cap A_i^1, \cup A_i^2\rangle$.

(vi) $\emptyset_\sim = \langle x, \emptyset, X\rangle$; $X_\sim = \langle x, X, \emptyset\rangle$.

Definition 2.3.[3] An intuitionistic topology (IT for short) on a nonempty set X is a family T of intuitionistic set in X satisfying the following axioms:

(i) $\emptyset_{\sim}$, $X_{\sim} \in T$.

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$.

(iii) $\cup G_i \in T$ for any arbitrary family $\{G_i : i \in J\} \subseteq T$.

In this case the pair (X, T) is called an intuitionistic topological space (ITS for short) and any intuitionistic set in T is called an intuitionistic open set (IOS for short) in X. The complement $\overline{A}$ of an intuitionistic open set A is called an intuitionistic closed set (ICS for short) in X.

Definition 2.4.[2] Let (X, T) be an intuitionistic topological space and $A = \langle x, A^1, A^2\rangle$ be an intuitionistic set in X. Then the closure and interior of A are defined by

$Icl(A) = \cap\{K : K$ is an intuitionistic closed set in X and $A \subseteq K\}$,

$Iint(A) = \cup\{G : G$ is an intuitionistic open set in X and $G \subseteq A\}$.

It can be also shown that $Icl(A)$ is an intuitionistic closed set and $Iint(A)$ is an intuitionistic open set in X, and A is an intuitionistic closed set in X if $Icl(A) = A$; and A is an intuitionistic open set in X if $Iint(A) = A$.

Definition 2.5.[2] Let X be a nonempty set and $p \in X$ a fixed element in X. Then the intuitionistic set $\tilde{p}$ defined by $\tilde{p} = \langle x, \{p\}, \{p\}^c\rangle$ is called an intuitionistic point (IP for short) in X.

Definition 2.6.[7] Let (X, T) be a topological space. A subset A in X is said to be ***semi-open*** if $A \subseteq cl(int(A))$.

Definition 2.7.[7] Let (X, T) be a topological space. A subset A in X is said to be ***semi-closed*** if $int(cl(A)) \subseteq A$.

Definition 2.8.[4] Let (X, T) be a topological space. A subset S in X is said to be ***semi-regular*** if both semi-open and semi-closed.

Definition 2.9.[5] Let (X, T) be a topological space. A subset A in X is said to be an ***AB-set*** if $A = U \cap V$, where $U \in T$ and V is semi-regular. The family of all AB-sets of a space X will be denoted by $AB(X)$.

Definition 2.10.[6] Let (X, T) be a topological spaces. An boundary of a subset V in X is denoted and defined as $\partial V = cl(V) \cap cl(\overline{V})$.

Definition 2.11.[6] The small inductive dimension of X is notated $ind(X)$, and is defined as follows:

(i) We say that the dimension of a space X ($ind(X)$) is -1 if $X = \phi$.

(ii) $ind(X) \leq n$ if for every point $x \in X$ and for every open set U exists an open V, $x \in V$ such that $\overline{V} \subseteq U$, and $ind(\partial V) \leq n - 1$. Where ∂V is the boundary of V.

(iii) $ind(X) = n$ if (ii) is true for n, but false for $n - 1$.

(iv) $ind(X) = \infty$ if for every n, $ind(X) \leq n$ is false.

Definition 2.12.[6] A partition L on between A and B exists, if there are open sets U, W, $A \subseteq U$, $B \subseteq W$ such that $W \cap U = \phi$ and $L = U^c \cap W^c$.

Definition 2.13.[6] The large inductive dimension of normal space X is notated $Ind(X)$, and is defined as follows:

(i) We say that the dimension of a space X ($Ind(X)$) is -1 if $X = \phi$.

(ii) $Ind(X) \leq n$ if for every closed set $C \subseteq X$ and for every open set U exists an open V, $C \subseteq V$ such that $\overline{V} \subseteq U$, and $ind(\partial V) \leq n-1$. Where ∂V is the boundary of V.

(iii) $Ind(X) = n$ if (ii) is true for n, but false for $n-1$.

(iv) $Ind(X) = \infty$ if for every n, $\mathcal{I}ind(X) \leq n$ is false.

§3. Intuitionistic small inductive dimension

Definition 3.1. Let (X, T) be an intuitionistic topological space. A subset $A = \langle x, A^1, A^2 \rangle$ in X is said to be intuitionistic semi-open if $A \subseteq Icl(Iint(A))$.

Definition 3.2. Let (X, T) be an intuitionistic topological space. A subset $A = \langle x, A^1, A^2 \rangle$ in X is said to be intuitionistic semi-closed if $Iint(Icl(A)) \subseteq A$.

Definition 3.3. Let (X, T) be an intuitionistic topological space. A subset $S = \langle x, S^1, S^2 \rangle$ in X is said to be intuitionistic semi-regular if both intuitionistic semi-open and intuitionistic semi-closed.

Definition 3.4. Let (X, T) be an intuitionistic topological space. A subset $A = \langle x, A^1, A^2 \rangle$ in X is said to be an intuitionistic $\mathcal{AB}$-set if $A = U \cap V$, where $U = \langle x, U^1, U^2 \rangle \in T$ and $V = \langle x, V^1, V^2 \rangle$ is an intuitionistic semi-regular. The family of all intuitionistic $\mathcal{AB}$-sets of a space X will be denoted by $I\mathcal{AB}(X)$.

Definition 3.5. Let (X, T) be an intuitionistic topological space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic set in X. Then the intuitionistic $\mathcal{AB}$ interior ($I\mathcal{AB}int$ for short) of A are defined by

$I\mathcal{AB}int(A) = \cup\{G = \langle x, G^1, G^2 \rangle : G$ is an intuitionistic $\mathcal{AB}$ open set in X and $G \subseteq A\}$.

Definition 3.6. Let (X, T) be an intuitionistic topological space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic set in X. Then the intuitionistic $\mathcal{AB}$ closure ($I\mathcal{AB}cl$ for short) of A are defined by

$I\mathcal{AB}cl(A) = \cap\{K = \langle x, K^1, K^2 \rangle : K$ is an intuitionistic $\mathcal{AB}$ closed set in X and $A \subseteq K\}$.

Notation 3.1. Let (X, T) be an intuitionistic topological space and $U = \langle x, U^1, U^2 \rangle$ be an intuitionistic set, $x \in U$ denotes $X \in U^1$ and $x \notin U^2$

Definition 3.7. Let (X, T) be an intuitionistic topological spaces. An intuitionistic $\mathcal{AB}$ boundary of a subset $V = \langle X, V^1, V^2 \rangle$ in X is denoted and defined as

$$\mathcal{IAB}\partial\mathcal{V} = I\mathcal{AB}cl(V) \cap I\mathcal{AB}cl(\overline{V}).$$

Definition 3.8. The intuitionistic small inductive dimension of X is notated $I - ind(X)$, and is defined as follows:

(i) We say that the intuitionistic dimension of a space X, $I - ind(X)$ is -1 iff $X = \phi_{\sim}$.

(ii) $I - ind(X) \leq n$ if for every point $x \in X$ and for every intuitionistic $\mathcal{AB}$-open set $U = \langle x, U^1, U^2 \rangle$ exists an intuitionistic $\mathcal{AB}$-open $V = \langle x, V^1, V^2 \rangle$, $x \in V$ such that $I\mathcal{AB}clV \subseteq U$ and $I - ind(\mathcal{AB}\partial\mathcal{V}) \leq n-1$. Where $I\mathcal{AB}\partial\mathcal{V}$ is the intuitionistic $\mathcal{AB}$ boundary of V.

(iii) $I - ind(X) = n$ if (ii) is true for n, but false for $n-1$.

(iv) $I - ind(X) = \infty$ if for every n, $I - ind(X) \leq n$ is false.

Definition 3.9. Let (X, T) be an intuitionistic topological space. A subset $A = \langle x, A^1, A^2 \rangle$ in X is said to be an intuitionistic $\mathcal{AB}$ neighborhood of x if there exists an intuitionistic open set $U = \langle x, U^1, U^2 \rangle$ such that $x \in U \subseteq A$.

Proposition 3.1. If $Y \subseteq X$ then $I - ind(Y) \leq I - ind(X)$.

Proof. By induction, it is true for $I - ind(X) = -1$. if $I - ind(X) = n$, for every point $y \in X$ there is an intuitionistic $\mathcal{AB}$ neighborhood $V = \langle x, V^1, V^2 \rangle$, with an intuitionistic $\mathcal{AB}$ open set $U = \langle x, U^1, U^2 \rangle \subseteq V$ such that $I - ind(\mathcal{AB}\partial\mathcal{V}) \leq n - 1$. Note that $V_Y = Y \cap V$ is an intuitionistic $\mathcal{AB}$ neighborhood in Y and U_Y is an intuitionistic AB open in Y. By induction, it is enough to show that $I\mathcal{AB}\partial U_Y \subseteq I\mathcal{AB}\partial\mathcal{U}$. Because now $I - ind(I\mathcal{AB}\partial U_Y) \leq n - 1$, and therefore $I - ind(Y) \leq n = I - ind(X)$ (By Definition 3.8). And indeed $I\mathcal{AB}cl(U_y) \subseteq I\mathcal{AB}cl(U)$, $I\mathcal{AB}\partial U_y = I\mathcal{AB}cl(U_y)/I\mathcal{AB}cl(U) \subseteq I\mathcal{AB}cl(U)\ /U = I\mathcal{AB}\partial\mathcal{U}$.

Definition 3.10. An intuitionistic partition L on between $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ exists, if there are intuitionistic $\mathcal{AB}$ open sets $U = \langle x, U^1, U^2 \rangle$, $W = \langle x, W^1, W^2 \rangle$, $A \subseteq U$, $B \subseteq W$ such that $W \cap U = \phi_\sim$ and $L = \overline{U} \cap \overline{W}$.

Proposition 3.2. $I - ind(X) \leq n \Longleftrightarrow$ for every point x and every intuitionistic $\mathcal{AB}$ closed set B there is an intuitionistic partition L, such that $I - ind(L) \leq n - 1$.

Proof. If $x \in X$ and $B = \langle x, B^1, B^2 \rangle$ is an intuitionistic $\mathcal{AB}$ closed, then there is an intuitionistic set $V = \langle x, V^1, V^2 \rangle$ such that $I\mathcal{AB}cl(V) \cap B = \phi_\sim$. By definition of $I - ind$, there is a $U = \langle x, U^1, U^2 \rangle \subseteq V, x \in U$, $I - ind(\mathcal{AB}\partial\mathcal{V}) \leq n - 1$. now, let $W = I\mathcal{AB}cl(U)$ and $L = I\mathcal{AB}\partial\mathcal{V}$.

Conversely, let $x \in X$ and V be an intuitionistic $\mathcal{AB}$ neigbhourhood of x. An intuitionistic set $B = I\mathcal{AB}cl(V)$ is an intuitionistic $\mathcal{AB}$ closed, and therefore there are $U = \langle x, B^1, B^2 \rangle$, $W = \langle x, W^1, W^2 \rangle$ such that $x \in U$, $B \subseteq W$. Note that by definition of B, $U \subseteq V$. Now $\overline{W}$ is an intuitionistic $\mathcal{AB}$ closed, therefore $I\mathcal{AB}cl(U) \subseteq \overline{W}$, and obviously $I\mathcal{AB}\partial\mathcal{U} \subseteq I\mathcal{AB}cl(U) \subseteq \overline{W}$ and by definition $I\mathcal{AB}\partial\mathcal{U} = I\mathcal{AB}cl(U)/U = IABcl(U)/U \subseteq \overline{U}$. Therefore $I\mathcal{AB}\partial\mathcal{U} \subseteq \overline{U} \cap \overline{W} =$ LI - ind(U)$\leq n - 1$.

Proposition 3.3. (an intuitionistic separation theorem) For every intuitionistic $\mathcal{AB}$ closed sets A and B of an intuitionistic separable metric space X, $I - ind(X) \leq n$, there is an intuitionistic partition L, such that $I - ind(L) \leq n - 1$.

Proof. We can decompose X into two intuitionistic spaces Y, $I - ind(Y) \leq n - 1$; Z, $I - ind(Z) = 0$. There is an intuitionistic partition L between an intuitionistic sets $A = \langle x, A^1, A^2 \rangle$, $B = \langle x, B^1, B^2 \rangle$ and $L \cap Z = \phi_\sim \Rightarrow L \subseteq U$ and $I - ind(L) \leq n - 1$ as an intuitionistic subspace of Y.

Proposition 3.4. Let X be an intuitionistic separable metric spaces, $I - ind(X) \leq n \Longleftrightarrow$ Then for every sequence of $(A_1, B_1) \cdots (A_{n+1}, B_{n+1})$ of an intuitionistic $\mathcal{AB}$ closed disjoint sets. An intuitionistic partitions $L_1 \cdots L_{n+1}$ exists, such that they have an empty intersection.

Proof.

Let us take $A_1 = \langle x, A_1^1, A_1^2 \rangle$,$B_1 = \langle x, B_1^1, B_1^2 \rangle$ by the separation theorem, we can find an intuitionistic partition L_1, $I - ind(L_1) \leq n - 1$. Let us take $A_2 = \langle x, A_2^1, A_2^2 \rangle$,$B_2 = \langle x, B_2^1, B_2^2 \rangle$ and $M = L_1$. we can find an intuitionistic partition L_2 such that $I - ind(M \cap L_2) \leq n - 2$.

Let us take $A_i = \langle x, A_i^1, A_i^2 \rangle$, $B_i = \langle x, B_i^1, B_i^2 \rangle$ and let $M = L_1 \cap L_2 \cap \cdots L_{i-1}$. we can find an intuitionistic partition L_i such that $I - ind(M \cap L_i) = n - i$. When we go to do $A_{n+1} = \langle x, A_{n+1}^1, A_{n+1}^2 \rangle$, $B_{n+1} = \langle x, B_{n+1}^1, B_{n+1}^2 \rangle$, we have $I - ind(L_1 \cap L_2 \cap \cdots L_{n+1}) \leq -1 \Longleftrightarrow L_1 \cap L_2 \cap \cdots \cap L_{n+1} = \phi_\sim$.

§4. Intuitionistic large inductive dimension

Definition 4.1. The intuitionistic large inductive dimension of intuitionistic $\mathcal{AB}$ normal space X is notated $I - \mathcal{I}nd(X)$, and is defined as follows:

(i) We say that the intuitionistic dimension of a space X, $I - \mathcal{I}nd(X)$ is -1 if $X = \phi_{\sim}$.

(ii) $I - \mathcal{I}nd(X) \leq n$ if for every intuitionistic $\mathcal{AB}$ closed set $C = \langle x, C^1, C^2\rangle \subseteq X$ and for every intuitionistic $\mathcal{AB}$ open set $U = \langle x, U^1, U^2\rangle$ exists an intuitionistic $\mathcal{AB}$ open $V = \langle x, V^1, V^2\rangle$, $C \subseteq V$ such that $I\mathcal{AB}cl(V) \subseteq U$ and $I - \mathcal{I}nd(\mathcal{AB}\partial\mathcal{V}) \leq n - 1$. Where $I\mathcal{AB}\partial\mathcal{V}$ is the intuitionistic boundary of V.

(iii) $I - \mathcal{I}nd(X) = n$ if (ii) is true for n, but false for n-1.

(iv) $I - \mathcal{I}nd(X) = \infty$ if for every n, $I - \mathcal{I}nd(X) \leq n$ is false.

Definition 4.2. Let (X, T) be an intuitionistic topological space. A subset $A = \langle x, A^1, A^2\rangle$ in X is said to be an intuitionistic $\mathcal{AB}$ neighborhood of x if there exists an intuitionistic $\mathcal{AB}$ open set$U = \langle x, U^1, U^2\rangle$ such that $x \in U \subseteq A$.

Definition 4.3. Let (X, T) is an intuitionistic $\mathcal{AB}$ normal space if, given any intuitionistic disjoint AB closed sets $E = \langle x, E^1, E^2\rangle$ and $F = \langle x, F^1, F^2\rangle$, there are an intuitionistic $\mathcal{AB}$ neighborhood $U = \langle x, U^1, U^2\rangle$ of E and an intuitionistic $\mathcal{AB}$ neighborhood $V = \langle x, V^1, V^2\rangle$ of F that are also intuitionistic disjoint.

Proposition 4.1. An intuitionistic $\mathcal{AB}$ normal space X satisfies the inequality $I - \mathcal{I}nd(X) \leq n$ iff for every pair E, F of disjoint intuitionistic $\mathcal{AB}$ closed subset of X exists a partition L between E and F such that $\mathcal{I} - Ind(X) \leq n - 1$.

Proof. Proof is obvious.

Proposition 4.2. For every intuitionistic separable space X we have $I - ind(X) = I - \mathcal{I}nd(X)$.

Proof. For every intuitionistic $\mathcal{AB}$ normal space X we have $I - ind(X) \leq I - \mathcal{I}nd(X)$ by definition.

To show that $I - \mathcal{I}nd(X) \leq I - ind(X)$ we will use induction with respect to $I - ind(X)$, clearly one can suppose that $I - ind(X) < \infty$.

If $I - ind(X) = -1 \Rightarrow I - ind(X) \leq I - \mathcal{I}nd(X)$. Assume that the inequality is proven for all intuitionistic separable metric space X of $I - ind(X) < n$ and consider an intuitionistic separable metric space X such that $I - ind(X) = n$.

Let $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ be a pair of intuitionistic disjoint $\mathcal{AB}$ closed subset of X, according to the first intuitionistic separation theorem there exists an intuitionistic partition L between A and B such that $I - ind(L) \leq n - 1$, by the inductive assumption for every $k < n, I - \mathcal{I}nd(L) \leq n - 1$ and according to the previous Proposition 4.1 $I - \mathcal{I}nd(X) \leq n$ and finally we get $I - \mathcal{I}nd(X) \leq I - ind(X) \Rightarrow \mathcal{I} - Ind(X) = I - ind(X)$.

§5. Intuitionistic addition theorem

Proposition 5.1. Let X be an intuitionistic $\mathcal{AB}$ normal space and Y be an dense intuitionistic subset of X. Let also A, B be disjoint intuitionistic subset of X; O_1, O_2 are intuitionistic $\mathcal{AB}$ open subsets of X and $A \subset O_1$, $B \subset O_2$, $I\mathcal{AB}cl_X O_1 \cap I\mathcal{AB}cl_X O_2 = \phi$. Assume that C_Y be an intuitionistic partition between the intuitionistic sets $I\mathcal{AB}cl_X O_1 \cap Y$ and $I\mathcal{AB}cl_X O_2 \cap Y$ in Y. Then there exists an intuitionistic partition C in the intuitionistic space X between A and B such that $C \cap Y = C_Y$.

Proof. Let $U_1 = \langle x, U_1^1, U_1^2 \rangle$ and $V_1 = \langle x, V_1^1, V_1^2 \rangle$ be any two disjoint intuitionistic $\mathcal{AB}$-open subsets of Y such that $Y \setminus C_Y = U_1 \cup V_1, I\mathcal{AB}cl_X O_1 \cap Y \subset U_1$ and $I\mathcal{AB}cl_X O_2 \cap Y$ $\subset V_1$. Put$Y_1 = U_1 \cup C_Y$, $Y_2 = V_1 \cup C_Y$ and $X_i = I\mathcal{AB}cl_X Y_i$ for each i = 1, 2. It is easy to see that

(i) $Y_1 = \langle x, Y_1^1, Y_1^2 \rangle$ and $Y_2 = \langle x, Y_1^1, Y_1^2 \rangle$ are intuitionistic $\mathcal{AB}$-closed subset of Y and $Y = Y_1 \cup Y_2, C_Y = Y_1 \cap Y_2$.

(ii) $X_1 = \langle x, X_1^1, X_1^2 \rangle$ and $X_2 = \langle x, X_1^1, X_1^2 \rangle$ are intuitionistic $\mathcal{AB}$-closed subset of X and $X = I\mathcal{AB}cl_X Y = I\mathcal{AB}cl_X(Y_1 \cup Y_2)= I\mathcal{AB}cl_X(Y_1) \cup I\mathcal{AB}cl_X(Y_2) = X_1 \cup X_2$.

(iii) $X_i \cap Y = Y_i$ for each $i = 1, 2$.

Moreover, we have $X_1 \cap B = \phi$ (because $Y_1 \cap (I\mathcal{AB}cl_X O_2 \cap Y) = \phi$, so $Y_1 \cap O_2 = \phi$, and consequently, $I\mathcal{AB}cl_X Y_1 \cap O_2 = \phi$). Analogously, we get $X_2 \cap A = \phi$.

Put $U = \langle x, U^1, U^2 \rangle = X \setminus X_2, V = \langle x, V^1, V^2 \rangle = X \setminus X_1$ and $C = X_1 \cap X_2$. Observe that U and V are disjoint intuitionistic $\mathcal{AB}$ open subsets of the intuitionistic space X and $A = \langle x, A^1, A^2 \rangle \subset U$, $B = \langle x, B^1, B^2 \rangle \subset V$ and $C = X \setminus (U \cup V)$. In addition we have $C \cap Y = (X_1 \cap Y) \cap (X_2 \cap Y) = Y_1 \cap Y_2 = C_Y$.

Remark 5.1. Let X be an intuitionistic space, x be a point of X, $B = \langle x, B^1, B^2 \rangle$ an intuitionistic $\mathcal{AB}$ closed subset of X and $x \notin B$. Put $U = \langle x, U^1, U^2 \rangle = X \setminus B$. In the intuitionistic space X let us consider intuitionistic $\mathcal{AB}$ open subsets $O_1 = \langle x, O^1, O^2 \rangle$, $V = \langle x, V^1, V^2 \rangle$, $W = \langle x, W^1, W^2 \rangle$ such that $x \in O_1 \subset I\mathcal{AB}cl(O_1) \subset V \subset I\mathcal{AB}cl(V) \subset W \subset I\mathcal{AB}cl(W) \subset U$. put $O_2 = \langle x, O^1, O^2 \rangle = X \setminus I\mathcal{AB}cl(W)$. It is evident that $B \subset O_2$ and $I\mathcal{AB}cl(O_1) \cap I\mathcal{AB}cl(O_2)$ $= \phi$. Let Y be an intuitionistic subset of X. In Y let us consider an intuitionistic $\mathcal{AB}$ open subset $P = \langle x, P^1, P^2 \rangle$ such that $I\mathcal{AB}cl(O_1) \cap Y \subset P \subset V \cap Y$. It is evident that $I\mathcal{AB}clO_Y P \subset I\mathcal{AB}clO_Y(V \cap Y) \subset I\mathcal{AB}clO_Y V \cap Y \subset W \cap Y \subset Y \setminus I\mathcal{AB}cl(O_2)$. So the $I\mathcal{AB}\partial_Y P$ is an intuitionistic partition between the sets $I\mathcal{AB}cl(O_1) \cap Y$, $I\mathcal{AB}cl(O_2) \cap Y$ in Y. By $trI - ind$, $trI - \mathcal{I}nd$ we will denote the natural intuitionistic transfinite extension of I-ind $I - \mathcal{I}nd$.

Proposition 5.2. Let X be an intuitionistic space and Y be a dense intuitionistic subset of X with $trI - \mathcal{I}ndY = \alpha$, α is an intuitionistic ordinal number ≥ 0. Then for each point $x \in X$and every intuitionistic $\mathcal{AB}$ closed set $B \subset X$ such that $x \notin X$there exist an intuitionistic partition C in X between the intuitionistic point x and the intuitionistic set B such that $trI - \mathcal{I}nd(C \cap Y) < \alpha$. In particular, if $trI - \mathcal{I}nd(Y)= 0$ then $C \subset X \setminus Y$.

Proof. Let us choose intuitionistic $\mathcal{AB}$ open subsets $O_1 = \langle x, O^1, O^1 \rangle$, $O_2 = \langle x, O^2, O^2 \rangle$ and $V = \langle x, V^1, V^2 \rangle$ of the intuitionistic space X as in Remark 5.1. In the intuitionistic space Y by our assumption there exists an intuitionistic $\mathcal{AB}$ open subset $P = \langle x, P^1, P^2 \rangle$ such that $I\mathcal{AB}cl(O_1) \cap Y \subset P \subset V \cap Y$ and $trI - \mathcal{I}ndI\mathcal{AB}\partial_{\mathcal{Y}} P < \alpha$.

Moreover, the intuitionistic set $I\mathcal{AB}\partial_{\mathcal{Y}} P$ is an intuitionistic partition between the sets $I\mathcal{AB}cl(O_1) \cap Y$, $I\mathcal{AB}cl(O_2) \cap Y$ in Y. By Proposition 5.1 there is an intuitionistic partition$C$ in X between the intuitionistic point x and the intuitionistic set B such that $C \cap Y = I\mathcal{AB}\partial_{\mathcal{Y}} P$.

§6. Intuitionistic product theorem

Definition 6.1. Let d be an intuitionistic dimension function which is monotone with respect to intuitionistic $\mathcal{AB}$-closed subsets. Then the intuitionistic finite sum dimension function for d in an intuitionistic $\mathcal{AB}$ normal space X (in intuitionistic dimension $k \geq 0$ ($I\mathcal{F}SDF(d)$ in short) (respectively ($I\mathcal{F}SDF(d,k)$, if $d(A \cup B) = \max\{dA, dB\}$ for every intuitionistic AB-closed in X intuitionistic sets $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ (such that $dA, dB \leq k$). For any intuitionistic space X, let us define

$$I\mathcal{F}SDF(d,X) = \begin{cases} \infty, & \text{if } I\mathcal{F}SDF(d) \text{ hold in } X; \\ \min\{k \geq 0 : I\mathcal{F}SDF(d,k) \text{does not hold in } X\}, & \text{otherwise.} \end{cases}$$

It is evident that either $0 \leq (I\mathcal{F}SDF(d,k) \leq dX - 1$ or $(I\mathcal{F}SDF(d,k) = \infty$. Moreover, for every intuitionistic $\mathcal{AB}$ closed set A in X we have $(I\mathcal{F}SDF(d,A) \geq (I\mathcal{F}SDF(d,X)$, and if $dA \leq (I\mathcal{F}SDF(d,X)$ then $(I\mathcal{F}SDF(d)$ holds in A.

Definition 6.2. Let(X,T) and (Y,T) be any two intuitionistic $\mathcal{AB}$ normal space, $I - ind(X \times Y) \leq I - ind(X) + I - ind(Y)$ if $I\mathcal{F}SDF(ind)$ holds in the intuitionistic space X.

Definition 6.3. Let (X,T) and (Y,T) be any two intuitionistic $\mathcal{AB}$ normal spaces with $I - indX = m \geq 0$ and $I - indY = n \geq 0$ then

$$I - ind(X \times Y) = \begin{cases} m+n, & \text{if } m = 0 \text{ or } n = 0; \\ 2(m+n) - 1, & \text{otherwise.} \end{cases}$$

Definition 6.4. Let (X,T) be an intuitionistic $\mathcal{AB}$ normal space. Let $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ be any two intuitionistic $\mathcal{AB}$ closed subsets such that $A \cup B = X$ then $I - indX \leq \max\{I - indA, I - indB\}+1$.

Proposition 6.1. Let X and Y be an intuitionistic space and A, B are intuitionistic $\mathcal{AB}$ closed subsets of X, Y, respectively. Assume that $I - indA \leq I\mathcal{F}SDF(I - ind, X)$ and $I - indB \leq I\mathcal{F}SDF(I - ind, Y)$. Then $I - ind(A \times B) \leq I - indA + I - indB$.

Proof. Observe that $I - indA \leq I\mathcal{F}SDF(ind)$ holds in the subspaces $A = \langle x, A^1, A^2\rangle$, $B = \langle x, B^1, B^2\rangle$. Hence by Definition 6.1, $I - ind(A \times B) \leq I - indA + I - indB$. Hence proved.

Proposition 6.2. Let X be an intuitionistic space and $trI - indX = 0$. Then $trI - ind(X \times Y) = trI - indY$ for any intuitionistic space Y.

Proof. Proof is obvious.

Proposition 6.3. Let X and Y be any two intuitionistic spaces with $I - indX \leq m \geq 0$ and $I - indY \leq n \geq 0$. Assume that $I\mathcal{F}SDF(I - ind, X)$, $I\mathcal{F}SDF(I - ind, Y) \geq k$ for some $k = 0, 1, \cdots or\ \infty$.Then

$$I - ind(X \times Y) \leq \begin{cases} m+n, & \text{if } n = 0,\ m = 0, \text{ or } m, n \leq k\ ; \\ 2(m+n) - k - 1, & \text{otherwise.} \end{cases}$$

Proof. Observe that the proposition is valid for $k = 0$(by Definition 6.2) and $k = \infty$ (by Definition 6.2). Assume now that k is an integer ≥ 1. Note that the proposition holds if either

$m, n \leq k$ (by proposition 6.2) or $m = 0$, or $n = 0$ (by proposition 4.2). Let us prove the statement for the remained part of the set $\{m, n \geq 0\}$. Put $s = m + n$. It would be enough if we prove that $I - ind(X \times Y) \leq 2s - k - 1$, for $s \geq k + 1$. Apply induction. Observe that for $s = k + 1$ the inequality evidently holds. Suppose that the inequality is valid for all $s : k + 1 \leq s < r$. Let now $s = r$ and $m, n \geq 1$ for each point $p \in X \times Y$ and every intuitionistic $\mathcal{AB}$ open neighborhood W of p let us choose a rectangular intuitionistic $\mathcal{AB}$ open neighborhood $U \times V \subset W$ of this point such that $I - ind\mathcal{AB}\partial\mathcal{U} \leq m - 1$ and $I - ind\mathcal{AB}\partial\mathcal{V} \leq n - 1$. Observe that $I\mathcal{AB}\partial(U \times V) = (I\mathcal{AB}\partial\mathcal{U} \times I\mathcal{AB}cl(V)) \cup (I\mathcal{AB}cl(U) \times I\mathcal{AB}\partial\mathcal{V})$, and $I - ind\mathcal{AB}\partial(U \times V) \leq max\{I - ind(\mathcal{AB}\partial\mathcal{U} \times I\mathcal{AB}cl(V)),\ I - ind(I\mathcal{AB}cl(U) \times I\mathcal{AB}\partial\mathcal{V})\} + 1$ (by Definition 6.3). Moreover, we have $I\mathcal{F}SDF(I - ind, I\mathcal{AB}\partial\mathcal{U})$, $I\mathcal{F}SDF(I - ind, \mathcal{AB}cl(U))$, $I\mathcal{F}SDF(I - ind, I\mathcal{AB}\partial\mathcal{V})$, $I\mathcal{F}SDF(I - ind, \mathcal{AB}cl(V)) \geq k$. By the proposition condition $I\mathcal{F}SDF(I - ind, X) \geq k$and $I\mathcal{F}SDF(I - ind, Y) \geq k)$. This allow us to apply induction. By induction assumption, $\max\{I - ind(\mathcal{AB}\partial\mathcal{U} \times I\mathcal{AB}cl(V)), I - ind(I\mathcal{AB}cl(U) \times I\mathcal{AB}\partial\mathcal{V})\} \leq 2(r-1) - 1 - k = 2r - 3 - k$. So by (1) we get $I - ind(\partial U \times V) \leq 2r - 3 - k + 1 = 2r - 2 - k$. Hence the inequality $I - ind(X \times Y) \leq 2r - 1 - k$ holds.

References

[1] D. Coker, A note on intuitionistic sets and intuitionistic points, Turkish J. Math., **20**(1996), 343–351.

[2] D. Coker, An introduction to intuitionistic topological spaces, BUSEFAL, **81**(2000), 51–56.

[3] D. Coker and Sadik. Bayhan, On separation axioms in intuitionistic topological spaces, IJMMS, **27**(2001), 621–630.

[4] G. Di Maio, T. Noiri. On s-closed spaces, Indian J. Pure Appl. Maths., **18**(1987),226-233.

[5] J. Donchev, Between A- and B-sets, Math Balkanica, New Ser., **12**(1988), 295-302.

[6] R. Engelking, Dimension theory, 1978.

[7] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math., **70**(1963), 36-41.

Scientia Magna
Vol. 9 (2013), No. 4, 49-56

On μ-b-connectedness and μ-b-compactness

V. Dhanya†, S. Krishnaprakash‡ and R. Ramesh‡

† Department of Mathematics,
NGM College, Pollachi- 642001, Tamil Nadu, India
E-mail: velmurugan.dhanya@gmail.com
‡ Department of Science and Humanities,
Karpagam College of Engineering, Coimbatore-641032, Tamil Nadu, India

Abstract In this paper, we introduce and study the notion of μ-b-separateness. Using this notion, the concept of μ-b-connectedness and μ-b-compactness are studied.

Keywords μ-b-separateness and μ-b-connectedness, μ-b-compactness.
2000 Mathematics Subject Classification: 54A05.

§1. Introduction and preliminaries

By a space (X, μ), we mean a generalized topological space (briefly GTS).

Definition 1.1.[5] If $X \in \mu$ then (X, μ) is called a strong generalized topological space.

Definition 1.2.[9] A subset A of a GTS (X, μ) is μ-b-open if $A \subseteq c_\mu(i_\mu(A)) \cup i_\mu(c_\mu(A))$. The complement of μ-b-open set is μ-b-closed. The class of μ-b-open sets of a GTS (X, μ) is denoted by $b(\mu)$.

Definition 1.3.[9] The μ-b-interior of a subset A of a GTS (X, μ) denoted by $i_b(A)$ is the union of all μ-b-open sets contained in A.

Definition 1.4.[9] The μ-b-closure of a subset A of a GTS (X, μ) denoted by $c_b(A)$ is the intersection of all μ-b-closed sets containing A.

Definition 1.5.[4] A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is (b, μ_Y)-continuous if $f^{-1}(V)$ is μ_X-b-open in X for each μ_Y-open set V in Y.

Definition 1.6.[4] A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is (μ_X, μ_Y)-b-closed in X if for each μ_X-closed set U in X, $f(U)$ is μ_Y-b-closed in Y.

Definition 1.7. A GTS (X, μ) is μ-connected [8] (γ-connected [3]) if there are no non-empty disjoint sets $U, V \in \mu$ such that $U \cup V = X$.

Definition 1.8.[1] Two subsets A and B of a GTS (X, μ) are μ-separated if and only if $A \cap c_\mu(B) = \emptyset$ and $B \cap c_\mu(A) = \emptyset$.

Definition 1.9.[6] Let $\mathcal{B} \subseteq exp(X)$ and $\emptyset \in \mathcal{B}$. Then $\mathcal{B}$ is a base for μ if $\{\cup A : A \subseteq \mathcal{B}\} = \mu$.

Definition 1.10.[7] If (X, μ) is a GTS and Y is a subset of X then the collection $\mu_{|Y} = \{U \cap Y : U \in \mu\}$ is a GT on Y called the subspace generalized topology and $(Y, \mu_{|Y})$ is the subspace of X.

§2. μ-b-Separateness

Definition 2.1. Two subsets A and B of a GTS (X, μ) are μ-b-separated if and only if $A \cap c_b(B) = \emptyset$ and $B \cap c_b(A) = \emptyset$.

Obviously, every μ-separated set is μ-b-separated but converse need not be true as shown in the following example.

Example 2.1. Let $X = \mathbb{R}$ and $\mu = \{\emptyset, \mathbb{Q}\}$ where $\mathbb{R}$ and $\mathbb{Q}$ denotes the set of all real numbers and rational numbers respectively. Consider $\mathbb{Q}, \mathbb{R} - \mathbb{Q} \subseteq X$. Then $\mathbb{Q} \cap c_b(\mathbb{R} - \mathbb{Q}) = c_b(\mathbb{Q}) \cap (\mathbb{R} - \mathbb{Q}) = \emptyset$ but $c_\mu(\mathbb{Q}) \cap (\mathbb{R} - \mathbb{Q}) \neq \emptyset$. Hence $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are μ-b-separated but not μ-separated.

Theorem 2.1. Every two μ-b-separated sets are always disjoint but not conversely.

Proof. Let A and B be μ-b-separated sets. Then $A \cap B \subseteq A \cap c_b(B) = \emptyset$. Hence A and B are disjoint.

Example 2.2. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{a, b\}\}$. Consider the disjoint sets $\{a\}, \{b\} \subseteq X$. Then $\{a\} \cap c_b(\{b\}) = \emptyset$ but $c_b(\{a\}) \cap \{b\} \neq \emptyset$. Thus $\{a\}$ and $\{b\}$ are not μ-b-separated. Hence disjoint sets are not μ-b-separated.

Theorem 2.2. Let A and B be non-empty sets in a GTS (X, μ). Then the following hold:

1. If A and B are μ-b-separated, $A_1 \subseteq A$ and $B_1 \subseteq B$ then A_1 and B_1 are so.

2. If $A \cap B = \emptyset$ such that each of A and B are both μ-b-closed then A and B are μ-b-separated.

3. If each of A and B are both μ-b-open and if $H = A \cap (X - B)$ and $G = B \cap (X - A)$ then H and G are μ-b-separated.

Proof. 1. Since $A_1 \subseteq A$, $c_b(A_1) \subseteq c_b(A)$. Since A and B are μ-b-separated, $c_b(A) \cap B = \emptyset$ implies $c_b(A_1) \cap B = \emptyset$ which implies $c_b(A_1) \cap B_1 = \emptyset$ as $B_1 \subseteq B$. Similarly $c_b(B_1) \cap A_1 = \emptyset$. Hence A_1 and B_1 are μ-b-separated.

2. Since A and B are μ-b-closed, $A = c_b(A)$ and $B = c_b(B)$. Then $A \cap B = \emptyset$ implies $c_b(A) \cap B = \emptyset$ and $c_b(B) \cap A = \emptyset$. Hence A and B are μ-b-separated.

3. Let $A, B \in b(\mu)$. Then $X - A$ and $X - B$ are μ-b-closed. Since $H = A \cap (X - B)$, $H \subseteq X - B$ which implies $c_b(H) \subseteq c_b(X - B) = X - B$ and so $c_b(H) \cap B = \emptyset$. Since $G \subseteq B$, $c_b(H) \cap G = \emptyset$. Similarly, $c_b(G) \cap H = \emptyset$. Hence H and G are μ-b-separated.

Corollary 2.1. Let A and B be non-empty sets in a GTS (X, μ). Then the following hold:

1. If $A \cap B = \emptyset$ such that each of A and B are both μ-b-open then A and B are μ-b-separated.

2. If each of A and B are both μ-b-closed and if $H = A \cap (X - B)$ and $G = B \cap (X - A)$ then H and G are μ-b-separated.

Theorem 2.3. Two disjoint sets A and B are μ-b-separated in a GTS (X, μ) if and only if they are both μ-b-open and μ-b-closed in $A \cup B$.

Proof. Assume that A and B are disjoint and μ-b-separated and let $E = A \cup B$. Then $c_b(A) = c_b(A) \cap E = c_b(A) \cap (A \cup B) = (c_b(A) \cap A) \cup (c_b(A) \cap B) = A \cup \emptyset = A$. Thus $c_b(A) = A$. Similarly, $c_b(B) = B$. Hence A and B are μ-b-closed in E. Since $E = A \cup B$ and A, B are disjoint and μ-b-separated, A and B are complements of each other in E. Consequently, A and B is μ-b-open in E.

Conversely, assume that A and B are disjoint sets which are both μ-b-open and μ-b-closed in $E = A \cup B$. Since A is μ-b-closed in E, $A = c_b(A) = c_b(A) \cap E = c_b(A) \cap (A \cup B) = (c_b(A) \cap A) \cup (c_b(A) \cap B) = A \cup (c_b(A) \cap B)$. Thus $A = A \cup (c_b(A) \cap B)$. Since $A \cap B = \emptyset$, $(A \cap A) \cap B = \emptyset$ which implies $(A \cap c_b(A)) \cap B = \emptyset$. Thus $A \cap (c_b(A) \cap B) = \emptyset$. Since $A = A \cup (c_b(A) \cap B)$ and $A \cap (c_b(A) \cap B) = \emptyset$, $c_b(A) \cap B = \emptyset$. Similarly, $c_b(B) \cap A = \emptyset$. Hence A and B are μ-b-separated.

Theorem 2.4. The subsets A and B of a GTS (X, μ) are μ-b-separated if and only if there exist $U, V \in b(\mu)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$, $B \cap U = \emptyset$.

Proof. Let A and B are μ-b-separated sets. Set $V = X - c_b(A)$ and $U = X - c_b(B)$. Then $U, V \in b(\mu)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$, $B \cap U = \emptyset$.

Conversely, assume that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset$, $B \cap U = \emptyset$. Since $U, V \in b(\mu)$, $X - U$ and $X - V$ are μ-b-closed. Since $A \cap V = \emptyset$, $A \subseteq X - V$ which implies $c_b(A) \subseteq c_b(X - V) = X - V \subseteq X - B$. Thus $B \cap c_b(A) = \emptyset$. Similarly, $A \cap c_b(B) = \emptyset$. Hence A and B are μ-b-separated sets.

§3. μ-b-connectedness

Definition 3.1. A subset S of a GTS (X, μ) is μ-b-connected if there exists no μ-b-separated subsets A and B and $S = A \cup B$. Otherwise A is μ-b-disconnected.

Example 3.1. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{a, b\}\}$. Then $\{a, c\}$ is μ-connected but not μ-b-connected.

Theorem 3.1. A GTS (X, μ) is μ-b-disconnected if and only if there exists a non-empty proper subset of X which is both μ-b-open and μ-b-closed in X.

Proof. Since (X, μ) is μ-b-disconnected, there exist μ-b-separated sets A and B and $A \cup B = X$. Since $A \subseteq c_b(A)$ and $c_b(A) \cap B = \emptyset$, $A \cap B = \emptyset$. Hence $A = X - B$. Since B is non-empty and $B \cup (X - B) = X$, $B = X - A$ is a proper subset of X. Since $A \cup B = X$ and $B \subseteq c_b(B)$, $X \subseteq A \cup c_b(B)$. But $A \cup c_b(B) \subseteq X$. Thus $A \cup c_b(B) = X$. Since $A \cap c_b(B) = \emptyset$ and $B \cap c_b(A) = \emptyset$, $A = X - c_b(B)$ and $B = X - c_b(A)$. Since $c_b(A)$ and $c_b(B)$ are μ-b-closed, $X - c_b(A)$ and $X - c_b(B)$ are μ-b-open. Thus A and B are μ-b-open. Since $A = X - B$ and $B = X - A$, A and B are μ-b-closed.

Conversely, assume that there exist non-empty proper subset of X which is both μ-b-open and μ-b-closed in X. Let $B = X - A$. Then $A \cap B = \emptyset$ and $A \cup B = X$. Since A is a proper subset of X, B is non-empty. Since A is both μ-b-open and μ-b-closed in X, B is μ-b-open and μ-b-closed in X. Hence $A = c_b(A)$ and $B = c_b(B)$. Since $A \cap B = \emptyset$, $c_b(A) \cap B = \emptyset$ and $c_b(B) \cap A = \emptyset$. Thus A and B are μ-b-separated. Hence (X, μ) is μ-b-disconnected.

Theorem 3.2. A GTS (X, μ) is μ-b-disconnected if and only if any one of the following holds :

1. X is the union of two non-empty disjoint μ-b-open sets,
2. X is the union of two non-empty disjoint μ-b-closed sets.

Proof. Since (X, μ) is μ-b-disconnected, by Theorem 3.1, there exist a non-empty proper subset of X which is both μ-b-open and μ-b-closed in X. Also, $A \cup (X - A) = X$. Hence A and $X - A$ satisfy the requirements of 1 and 2.

Conversely, assume $A \cup B = X$ and $A \cap B = \emptyset$ where A and B are non-empty μ-b-open sets. Then $A = X - B$ is μ-b-closed. Since B is non-empty, A is a proper subset of X. Thus A is a non-empty proper subset of X which is both μ-b-open and μ-b-closed. Hence by Theorem 3.1, X is μ-b-disconnected. Again, let $X = C \cup D$ and $C \cap D = \emptyset$ where C, D are non-empty μ-b-closed sets. Then $C = X - D$ so that C is μ-b-open. Since D is non-empty, C is a proper subset of X which is both μ-b-open and μ-b-closed. Hence by Theorem 3.1, X is μ-b-disconnected.

Theorem 3.3. The real line $\mathbb{R}$ is μ-b-connected.

Proof. Let $\mathbb{R}$ be an interval. By method of contradiction, assume that $\mathbb{R}$ is μ-b-disconnected. Then by 2 of Theorem 3.2, $\mathbb{R}$ is the union of two non-empty disjoint μ-b-closed sets A and B. Let $a \in A$ and $b \in B$. Since $A \cap B = \emptyset$, $a \neq b$. So either $a < b$ or $b < a$. Without loss of generality, assume that $a < b$. Since $a, b \in \mathbb{R}$ and $\mathbb{R}$ is an interval, $[a, b] \subseteq \mathbb{R} = A \cup B$. Let $k = \sup\{[a, b] \cap A\}$. Then $a \leq k \leq b$ which implies $k \in \mathbb{R}$. Since A is μ-b-closed, $k \in A$. Therefore $k \neq b$ which implies $k < b$. Therefore $k + \epsilon \in B$ for every $\epsilon > 0$ and $k + \epsilon \leq b$. Since B is μ-b-closed, $k \in B$. Thus $k \in A \cap B$ which implies $A \cap B \neq \emptyset$ which is a contradiction. Hence $\mathbb{R}$ is μ-b-connected.

Theorem 3.4. If E is a μ-b-connected subset of a GTS (X, μ) such that $E \subseteq A \cup B$ where A and B are μ-b-separated sets then $E \subseteq A$ or $E \subseteq B$, that is E cannot intersect both A and B.

Proof. Since A and B are μ-b-separated sets, $A \cap c_b(B) = \emptyset$ and $B \cap c_b(A) = \emptyset$. Since $E \subseteq A \cup B$, $E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$. Now, suppose that $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$. Then $(E \cap A) \cap c_b(E \cap B) \subseteq (E \cap A) \cap (c_b(E) \cap c_b(B)) = (E \cap c_b(E)) \cap (A \cap c_b(B)) = \emptyset$. Similarly, $(E \cap B) \cap c_b(E \cap A) = \emptyset$. Hence $E \cap A$ and $E \cap B$ are μ-b-separated. Thus E is μ-b-disconnected, which is a contradiction. Hence atleast one of the sets $E \cap A$ and $E \cap B$ is empty. If $E \cap A = \emptyset$ then $E = E \cap B$ which implies that $E \subseteq B$. Similarly if $E \cap B = \emptyset$ then $E \subseteq A$. Hence either $E \subseteq A$ or $E \subseteq B$.

Corollary 3.1. If E is a μ-b-connected subset of a GTS (X, μ) such that $E \subseteq A \cup B$ where A and B are disjoint μ-b-open (μ-b-closed) subsets of X then A and B are μ-b-separated.

PROOF. Let A and B be μ-b-open with $A \cap B = \emptyset$. Then $A \subseteq X - B$ implies $c_b(A) \subseteq c_b(X - B) = X - B$ since $X - B$ is μ-b-closed. Thus $B \cap c_b(A) = \emptyset$. Similarly $A \cap c_b(B) = \emptyset$. Hence A and B are μ-b-separated.

Theorem 3.5. If E is a μ-b-connected subset of a GTS (X, μ) and C is any subset such that $E \subseteq C \subseteq c_b(E)$ then C is also μ-b-connected.

Proof. Suppose that C is not μ-b-connected. Then there exist μ-b-separated sets A and B such that $C = A \cup B$. Since $E \subseteq C$, $E \subseteq A \cup B$. Then by Theorem 3.4, $E \subseteq A$ or $E \subseteq B$. Consider $E \subseteq A$. Then $c_b(E) \subseteq c_b(A)$ which implies $c_b(E) \cap B \subseteq c_b(A) \cap B = \emptyset$ since A and B are μ-b-separated. Since $C = A \cup B$, $B \subseteq C$. Then since $C \subseteq c_b(E)$, $B \subseteq C \subseteq c_b(E)$ which implies $c_b(E) \cap B = B$. Thus $c_b(E) \cap B = \emptyset$ and $c_b(E) \cap B = B$ implies $B = \emptyset$. Similarly considering $E \subseteq B$, we obtain $A = \emptyset$ which contradicts A and B are non-empty as A and B are μ-b-separated. Hence C is μ-b-connected.

Theorem 3.6. If E is μ-b-connected then $c_b(E)$ is μ-b-connected.

Proof. By contradiction, assume that $c_b(E)$ is μ-b-disconnected. Then there exist μ-b-

separated sets G and H in X such that $c_b(E) = G \cup H$. Since $E = (G \cap E) \cup (H \cap E)$, $G \cap E \subseteq G$ and $H \cap E \subseteq H$, $c_b(G \cap E) \cap (H \cap E) \subseteq c_b(G) \cap H = \emptyset$. Thus $c_b(G \cap E) \cap (H \cap E) = \emptyset$. Similarly, $c_b(H \cap E) \cap (G \cap E) = \emptyset$. Therefore, $G \cap E$ and $H \cap E$ are μ-b-separated. Thus E is μ-b-disconnected, which is a contradiction. Hence $c_b(E)$ is μ-b-connected.

Theorem 3.7. If every two points of a subset E of a GTS (X, μ) are contained in some μ-b-connected subset of E then E is μ-b-connected subset of X.

Proof. Suppose E is not μ-b-connected. Then there exist non-empty subsets A and B of X such that $A \cap c_b(B) = \emptyset$, $B \cap c_b(A) = \emptyset$ and $E = A \cup B$. Since A, B are non-empty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis, a and b must be contained in some μ-b-connected subset F of E. Since $F \subseteq A \cup B$, either $F \subseteq A$ or $F \subseteq B$ by Theorem 3.4. It follows that either a, b are both in A or both in B. Let $a, b \in A$. Then $A \cap B \neq \emptyset$ which is a contradiction. Hence E is μ-b-connected.

Theorem 3.8. The union of any family of μ-b-connected sets having a non-empty intersection is a μ-b-connected set.

Proof. Let $\{E_\alpha\}$ be any family of μ-b-connected sets in such a way that $\bigcap \{E_\alpha\} \neq \emptyset$. Let $E = \bigcup \{E_\alpha\}$. Suppose E is not μ-b-connected. Then there exist μ-b-separated sets A and B such that $E = A \cup B$. Since $\bigcap \{E_\alpha\} \neq \emptyset$, $x \in \bigcap \{E_\alpha\}$. Then x belongs to each E_α and so $x \in E$. Consequently, $x \in A$ or $x \in B$. Without loss of generality, assume that $x \in A$. Then since x belongs to each of $\{E_\alpha\}$, $x \in E_\alpha \cap A$ and so $E_\alpha \cap A \neq \emptyset$ for each α. Since $E = \bigcup \{E_\alpha\}$ and $E_\alpha \cap A \neq \emptyset$, $E_\alpha \subseteq A$ for each α which implies $\bigcup E_\alpha \subseteq A$ and so $E \subseteq A$. Thus $A \cup B \subseteq A$. But $A \subseteq A \cup B$. Hence $A = E$ which implies $B = \emptyset$ which is a contradiction, since A and B being μ-b-separated are non-empty. Thus E is μ-b-connected.

Theorem 3.9. The union of any family of μ-b-connected subsets of a GTS (X, μ) with the property that one of the members of the family intersects every other member is a μ-b-connected set.

Proof. Let $\{E_\alpha\}$ be any family of μ-b-connected sets of a GTS (X, μ) with the property that one of the member say E_{α_0} intersects every other member. That is, $E_{\alpha_0} \cap E_\alpha \neq \emptyset$ for every α. Then by Theorem 3.8, $E_{\alpha_0} \cup E_\alpha$ is μ-b-connected. Now let E_{α_p} and E_{α_q} be any two members of the family so that $E_{\alpha_0} \cap E_{\alpha_p} \neq \emptyset$, $E_{\alpha_0} \cap E_{\alpha_q} \neq \emptyset$ and $(E_{\alpha_0} \cup E_{\alpha_p}) \cap (E_{\alpha_0} \cap E_{\alpha_q}) = E_{\alpha_0} \cup (E_{\alpha_p} \cap E_{\alpha_q}) \neq \emptyset$. That is, $E_{\alpha_0} \cup E_\alpha \neq \emptyset$ for each α. Then by Theorem 3.8, $\bigcup (E_{\alpha_0} \cap E_\alpha)$ for each α is μ-b-connected. Hence $\bigcup E_\alpha$ for each α is μ-b-connected.

Theorem 3.10. If $A \subseteq B \cup C$ such that A is a non-empty μ-b-connected set in a GTS (X, μ) and B, C are μ-b-separated then only one of the following holds:

1. $A \subseteq B$ and $A \cap C = \emptyset$,
2. $A \subseteq C$ and $A \cap B = \emptyset$.

Proof. If $A \cap C = \emptyset$ then $A \subseteq B$. Also if $A \cap B = \emptyset$ then $A \subseteq C$. Since $A \subseteq B \cup C$, both $A \cap C = \emptyset$ and $A \cap B = \emptyset$ cannot hold simultaneously. Similarly, suppose that $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$. Then $A \cap B \subseteq B$ and $A \cap C \subseteq C$. Then by 1 of Theorem 2.2, $A \cap B$ and $A \cap C$ are μ-b-separated such that $A = (A \cap B) \cup (A \cap C)$. Thus A is μ-b-disconnected, which is a contradiction. Hence $A \cap C = \emptyset$ or $A \cap B = \emptyset$.

Theorem 3.11. Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a (b, μ_Y)-continuous function. If K is μ_X-b-connected in X then $f(K)$ is μ_Y-connected in Y.

Proof. Suppose that $f(K)$ is μ_Y-disconnected in Y. Then there exist μ_Y-separated sets G and H of Y such that $f(K) = G \cup H$. Set $A = K \cap f^{-1}(G)$ and $B = K \cap f^{-1}(H)$. Since $f(K) = G \cup H$, $K \cap f^{-1}(G) \neq \emptyset$ which implies $A \neq \emptyset$. Similarly, $B \neq \emptyset$. Since G and H are μ-separated, $G \cap H = \emptyset$. Now, $A \cap B = (K \cap f^{-1}(G)) \cap (K \cap f^{-1}(H)) = K \cap (f^{-1}(G) \cap f^{-1}(H)) = K \cap (f^{-1}(G \cap H)) = \emptyset$ and so $A \cap B = \emptyset$. Now, $A \cap c_b(B) \subseteq f^{-1}(G) \cap c_b(f^{-1}(H))$. Since f is (b, μ_Y)-continuous, $A \cap c_b(B) \subseteq f^{-1}(G) \cap f^{-1}(c_{\mu_Y}(H)) \subseteq f^{-1}(G \cap c_{\mu_Y}(H)) = \emptyset$. Thus $A \cap c_b(B) = \emptyset$. Similarly, $B \cap c_b(A) = \emptyset$. Thus A and B are μ_X-b-separated in X, which is a contradiction. Hence $f(K)$ is μ_Y-connected in Y.

Corollary 3.2. For a (b, μ_Y)-continuous function $f : (X, \mu_X) \to (Y, \mu_Y)$, if K is μ_X-disconnected in X then $f(K)$ is μ_Y-b-disconnected in Y.

Corollary 3.3. For a bijective (μ_X, μ_Y)-b-closed function $f : (X, \mu_X) \to (Y, \mu_Y)$, if K is μ_Y-b-connected in Y then $f^{-1}(K)$ is μ_X-connected in X.

§4. μ-b-Compactness

Definition 4.1. A collection $\mathcal{F}$ of subsets of a strong GTS (X, μ) is a μ-b-cover of X if the union of the elements of $\mathcal{F}$ is equal to X.

Definition 4.2. A μ-b-subcover of a μ-b-cover $\mathcal{F}$ is a subcollection $\mathcal{G}$ of $\mathcal{F}$ which itself is a μ-b-cover.

Definition 4.3. A μ-b-cover $\mathcal{F}$ of a strong GTS (X, μ) is a μ-b-open cover if the elements of $\mathcal{F}$ are μ-b-open subsets of X.

Definition 4.4. A strong GTS (X, μ) is μ-b-compact if and only if each μ-b-open cover of X has a finite μ-b-open subcover.

Theorem 4.1. If (X, μ) is a finite strong GTS then X is μ-b-compact.

Proof. Let $X = \{x_1, x_2, \cdots, x_n\}$. Let $\mathcal{F}$ be a μ-b-open covering of X. Then each element in X belongs to one of the members of $\mathcal{F}$, say $x_1 \in G_1$, $x_2 \in G_2, \cdots$ $x_n \in G_n$ where $G_i \in \mathcal{F}$, $i = 1, 2, \cdots, n$. Thus the collection $\{G_1, G_2, \cdots, G_n\}$ is a finite μ-b-open subcover of X. Hence X is μ-b-compact.

Theorem 4.2. If $\mu = \{\ U \subseteq X : X - U$ is either finite or is all of $X\}$ in a strong GTS (X, μ) then X is μ-b-compact.

Proof. Let $\mathcal{F}$ be a μ-b-open covering of X and let G be an arbitrary member of $\mathcal{F}$. Since $G \in \mu$, $X - G$ is finite. Let $X - G = \{x_1, x_2, \cdots, x_m\}$. Since $\mathcal{F}$ is a μ-b-open covering of X, each x_i belongs to one of the members of $\mathcal{F}$, say, $x_1 \in G_1$, $x_2 \in G_2, \cdots$ $x_m \in G_m$ where $G_i \in \mathcal{F}$, $i = 1, 2, \cdots, m$. Thus the collection $\{G_1, G_2, \cdots, G_m\}$ is a finite μ-b-open subcover of X. Hence X is μ-b-compact.

Example 4.1. Let X be any infinite set and $\mu = \{\emptyset, X\}$. Then (X, μ) is μ-compact since the only μ-open covering of X is $\{X\}$ which consists of only one set and therefore finite. But (X, μ) is not μ-b-compact since every singletons are μ-b-open and singletons do not have finite subcover. Hence (X, μ) is μ-compact but not μ-b-compact.

Theorem 4.3. Every finite union of μ-b-compact sets is μ-b-compact.

Proof. Let U and V be any two μ-b-compact subsets of X and $\mathcal{F}$ be a μ-b-open covering of $U \cup V$. Then $\mathcal{F}$ is a μ-b-open cover of both U and V. So by hypothesis, there exists a finite subcollections of $\mathcal{F}$ of μ-b-open sets, say, $\{U_1, U_2, ..., U_n\}$ and $\{V_1, V_2, \cdots, V_m\}$ covering U and V respectively. Thus the collection $\{U_1, U_2, \cdots, U_n, V_1, V_2, \cdots, V_m\}$ is a finite collection of μ-b-open sets covering $U \cup V$. Hence by induction, every finite union of μ-b-compact sets is μ-b-compact.

Theorem 4.4. Let (X, μ) be a strong GTS. Then every μ-b-closed subset of a μ-b-compact space (X, μ) is μ-b-compact.

Proof. Let F be μ-b-closed in the μ-b-compact space (X, μ) and $\mathcal{F}$ be a μ-b-open covering of F. Then the collection $\mathcal{G} = (X - F) \cup \mathcal{F}$ is a μ-b-open cover of X. Since X is μ-b-compact, the collection $\mathcal{G}$ has a finite μ-b-open subcover. If this finite μ-b-open subcover contains the set $X - F$ then discard it. Otherwise leaving the finite μ-b-open subcover alone, the resulting collection is a finite μ-b-subcover of $\mathcal{F}$.

Theorem 4.5. In a strong GTS (X, μ), the real line $\mathbb{R}$ is not μ-b-compact.

Proof. Let $C = \{(-n, n) : n \in \mathbb{N}\}$. Then each member of C is clearly μ-b-open interval and therefore a μ-b-open set. Also if x is any real number, we may choose a positive integer $n_\infty > |x|$. Then clearly, $x \in (-n_\infty, n_\infty) \in C$. Thus each point of $\mathbb{R}$ is contained in some member of C and therefore C is a μ-b-open covering of $\mathbb{R}$. Now consider a family C^* of finite number of sets in C, say, $C^* = \{(-n_1, n_1), (-n_2, n_2), \cdots, (-n_k, n_k)\}$ and if $n^* = \max\{n_1, n_2, \cdots, n_k\}$ then $n^* \notin \bigcup_{i=1}^{k}(-n_i, n_i)$. Thus it follows that no finite subfamily of C covers $\mathbb{R}$. Hence $(\mathbb{R}, \mu)$ is not μ-b-compact.

Theorem 4.6. If $f : (X, \mu_X) \to (Y, \mu_Y)$ is a surjective (b, μ_Y)-continuous function and X is μ_X-b-compact then f(X) is μ_Y-b-compact.

Proof. Assume that $\mathcal{F}$ is any μ_Y-b-open cover of Y. Since f is (b, μ_Y)-continuous function, $f^{-1}(\mathcal{F})$ is μ_X-b-open in X. Then the collection $\{f^{-1}(G) : G \in \mathcal{F}\}$ forms a μ_X-b-open cover of X. Since X is μ_X-b-compact, there exists a finite μ_X-b-open cover of X, say $\{f^{-1}(G_1), f^{-1}(G_2), \cdots, f^{-1}(G_n)\}$ where $G_1, G_2, \cdots, G_n \in \mathcal{F}$. Since f is surjective, the collection $\{G_1, G_2, \cdots, G_n\}$ is a μ_Y-b-open subcover of Y. Hence f(X) is μ_Y-b-compact.

Theorem 4.7. A strong GTS (X, μ) is μ-b-compact if and only if there exists a base $\mathcal{B}$ for it such that every μ-b-open cover of X by members of $\mathcal{B}$ has a finite μ-b-open subcover.

Proof. Since X is μ-b-compact, every μ-b-open cover of X has a finite μ-b-open subcover and hence there exists a base $\mathcal{B}$ for it such that every μ-b-open cover of X by members of $\mathcal{B}$ has a finite μ-b-open subcover.

Conversely, assume that $\mathcal{F} = \{G_\lambda : \lambda \in \wedge\}$ be any μ-b-open cover of X. If $\mathcal{B} = \{B_\alpha : \alpha \in \wedge\}$ be any μ-b-open base for X. Then each G_i is the union of some members of $\mathcal{B}$. Thus the collection of members of $\mathcal{B}$ has a finite subcover, say $\{B_{\alpha_i} : i = 1, 2, \cdots, n\}$. For each B_{α_i}, we can select a G_{λ_i} from $\mathcal{B}$ such that $B_{\alpha_i} \subseteq G_{\lambda_i}$. Thus $\{G_{\lambda_i} : i = 1, 2, \cdots, n\}$ is a μ-b-subcover of $\mathcal{B}$. Hence X is μ-b-compact.

Theorem 4.8. If (X, μ) is a strong GTS and $(Y, \mu_{|Y})$ is a subspace of the strong GTS (X, μ) then $(Y, \mu_{|Y})$ is $\mu_{|Y}$-b-compact if and only if every μ-b-open covering of Y contains a finite subcollection covering Y.

Proof. Assume that $(Y, \mu_{|Y})$ is $\mu_{|Y}$-b-compact. Let $\mathcal{F} = \{G_\alpha : \alpha \in \wedge\}$ be a μ-b-open covering of Y. Then the collection $\mathcal{G} = \{G_\alpha \cap Y : \alpha \in \wedge\}$ is a $\mu_{|Y}$-b-covering of Y. Since $(Y, \mu_{|Y})$ is $\mu_{|Y}$-b-compact, a finite subcollection $\{G_{\alpha_1} \cap Y, G_{\alpha_2} \cap Y, \cdots, G_{\alpha_n} \cap Y\}$ covers Y.

Then $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ is a finite collection of μ-b-open sets that covers Y.

Conversely, assume that every μ-b-open covering of Y contains a finite subcollection covering Y. Let $\mathcal{G} = \{G_\alpha : \alpha \in \wedge\}$ be a $\mu_{|Y}$-b-open covering of Y. Then $G_\alpha = F_\alpha \cap Y$ for each α and $F_\alpha \in \mu$. Then the collection $\{F_\alpha : \alpha \in \wedge\}$ is μ-b-open covering of Y. So by hypothesis, a finite subcollection $\{F_{\alpha_1}, F_{\alpha_2}, \cdots, F_{\alpha_n}\}$ covers Y. Then the collection $\{F_{\alpha_1} \cap Y, F_{\alpha_2} \cap Y, \cdots, F_{\alpha_n} \cap Y\}$ is a finite subcollection that covers Y. That is, $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ is a finite subcollection of $\mu_{|Y}$-b-open sets that covers Y. Hence Y is $\mu_{|Y}$-b-compact.

References

[1] R. Baskaran, M. Murugalingam and D. Sivaraj, Separated sets in generalized topological spaces, J. Adv. Res. Pure Math., **2**(2010), No. 1, 74-83.

[2] A. Csaszar, Generalized topology, generalized continuity, Acta Math. Hungar., **96**(2002), No. 4, 351-357.

[3] A. Csaszar, γ-connected spaces, Acta Math. Hungar., **101**(2003), No. 4, 273-279.

[4] V. Dhanya, S. Krishnaprakash and Ennis Rosas, On (b, μ_Y)-continuous functions, (Communicated).

[5] R. Jamunarani and P. Jeyanthi, Regular sets in generalized topological spaces, Acta Math. Hungar., **135**(2012), No. 4, 342-349.

[6] Jyothis Thomas and Sunil Jacob John, μ-compactness in generalized topological spaces, J. Adv. Stu. Top., **3**(2012), No. 3, 18-22.

[7] J. Li, Generalized topologies generated by subbases, Acta Math. Hungar., **114**(2007), No. 1-2, 01-12.

[8] R. X. Shen, A note on generalized connectedness, Acta Math. Hungar., **122**(2009), No. 3, 231-235.

[9] P. Sivagami, Remarks on γ-interior, Acta Math. Hungar., **119**(2008), No. 1-2, 81-94.

Scientia Magna
Vol. 9 (2013), No. 4, 57-63

An improved particle swarm optimization algorithm based on domination theory for multi-objective optimization

Li Liu

Department of basic courses,
Anhui Technical College of Mechanical and Electrical Engineering,
Wuhu, Anhui, P. R. Chinaa
E-mail: jsslliuli@163.com

Abstract As the need for engineering design optimization techniques has become widely used, the need for multi-objective optimization also becomes more pertinent, because it allows a Pareto optimal solution to be determined. The objective of this paper is to identify a need for faster and more precise algorithms within the multi-objective optimization community for engineering applications. This paper proposes a new algorithm by combining past research and using domination theory. The newly proposed algorithm has demonstrated to be very successful at combining multi-objective optimization theory with particle swarm theory to develop a single algorithm that is capable of determining a good representation of a Pareto Front.

Keywords Particle swarm optimization, multi-objective optimization, Pareto Front.
2000 Mathematics Subject Classification: 68W99 90C29
Chinese Library Classification: TP301.6

§1. Introduction

Derivative-optimization methods typically use high computational cost derivatives, for example in the simulation. In addition, they are also widely used when the mathematical model of the design space is not reliable. A non-reliable design space lead to a discontinuous and/or noise, so derivatives are as reliable as possible. Many different methods have been developed derivative-free, such as direct method [1] of the genetic algorithm, simulated annealing [2] and particle swarm and other. The particle swarm optimization algorithm is fast becoming an important derivative-free optimization algorithm in engineering design optimization modeling, because it can provide reliable solutions or box constraints and easy calculation model of constraints. Engineering design optimization has been used in the automotive, aerospace and medical industry. Use of the optimization in the auto industry, reduce resistance and the minimum weight [3].

§2. Particle swarm theory

Particle Swarm Optimization (PSO) is an evolutionary algorithm developed by R. Eberhart and J. Kennedy in 1995 [2]. Many variations of the Particle Swarm have been proposed since then, but they all tend to have the same formulation. The varied algorithms consist of the difference between a particle's personal best location and current location, in addition to the difference between the global or neighborhood best and a particle's current location.

The PSO algorithm is widely sought for a variety of engineering problems, because of its reliability and simplicity to implement [4]. The PSO algorithm is reliable, because it performs a thorough search of the design space and the communication between the particles allows the particles to converge upon an optimal solution, which tends to be a robust. However, there is no proof demonstrating that it will always locate the global optimal solution. The simplicity lies in the lack of parameters to initialize and manipulate at each iteration of algorithm. There are two main parameters, position x_{id}, and velocity v_{id}. The x_{id} parameter gives the current position of the particle, and then the particles are "flown" through the problem space at velocity v_{id}, for one time increment per iteration.

The following general PSO algorithm, consisting of two equations, effectively demonstrates the simplicity of the optimization technique [5]:

$$v_{id} = wv_{id} + c_1\text{rand}()(p_{id} - x_{id}) + c_2\text{Rand}()(p_{gd} - x_{id}) \tag{1}$$

$$x_{id} = x_{id} + v_{id} \tag{2}$$

Equations (1) and (2) describe the updating behavior of the PSO algorithm. If there are n variables in x_{id}, then x_{id} is a $n \times 1$ vector, and as a result there are n elements in v_{id}, which is also a $n \times 1$ vector. The PSO algorithm requires that the particles remember their personal best P_{id}, as well as the local best P_{id}, or "global" best P_{gd}. The local best may also be referred to as the neighborhood best. A particle's personal best is the best position determined thus far by each individual particle. The swarm best is the best position determined thus far by the overall swarm.

The PSO algorithm uses a randomized population. In Equation (1), rand() denotes a randomized number for multiplication by a particle's personal performance, whereas Rand() denotes a randomized number for multiplication by the performance of the best in the local or global swarm. The current position and personal best values change with each variable, for each particle and each iteration. For example, if the design space was of 5 variables, then the vector for the position of each individual particle would consist of 5 elements. Each element in the vector would denote a control variable having a boundary.

§3. Multi-objective optimization

Multi-Objective Optimization (MOO) is the objective function of two or more of the optimization . We will mathematically define an MOO model as follows:

$$\begin{gathered} \text{minimize } \overrightarrow{f_i}(x) \ \text{ for } \ i = 1, 2, \cdots, n, \\ \text{with } \ \overrightarrow{g(x)} = 0 \ \text{ and } \ \overrightarrow{h(x)} \leq 0, \\ \text{where } \ \overrightarrow{x} \in \mathbb{R}^n, \overrightarrow{f}(\overrightarrow{x}) \in \mathbb{R}^m \ \text{ and } \ \overrightarrow{h}(\overrightarrow{x}) \in \mathbb{R}^p. \end{gathered} \tag{3}$$

The above definition for a MOO model has the objective functions defined as $\overrightarrow{f_i}(x)$ where i is the number of the objective functions. The quality constraints are denoted as $\overrightarrow{g(x)}$, and the inequality constraints are denoted as $\overrightarrow{h(x)}$.

A MOO problem may have multiple solutions as we all know. In single objective optimization there is one true solution known as a global optimal, however there may be many local solutions. A MOO model has Pareto optimal solution, because there are two or more objective functions for the design space, thus causing a tradeoff between the objective functions, where solution $\overrightarrow{x}$ is not dominated by $\overrightarrow{x}'$ per the definition of the Global Optimality in the Pareto Sense. A vector $\overrightarrow{x}$ is globally optimal in the Pareto Sense or is an optimal in the Pareto Sense, if there does not exist a vector $\overrightarrow{x}'$, such that $\overrightarrow{x}'$ dominates vector $\overrightarrow{x}$.

Solutions in an objective space of two or more objectives are considered to be interesting if there exists a domination relationship multiple solutions. The domination relationship is defined as follows: a vector $\overrightarrow{x}_1$, dominates a vector $\overrightarrow{x}_2$ if $\overrightarrow{x}_1$ is at least as good as $\overrightarrow{x}_2$ for all the objectives, and $\overrightarrow{x}_1$ is strictly better than $\overrightarrow{x}_2$ for at least one objective. If a solution is not dominated, then it is said to have a rank of 1. The trade-off surface or Pareto Front between two or more objectives can be determined by determining a number of solutions with rank 1 from the definition of solution domination. Therefore, all particles with a rank of 1, r_1, form a subset of all known solutions S, to be defined as $r_1 \subseteq S$. When the subset r_1 is plotted in the design space, the plots form a trade-off curve also known as the Pareto Front.

The combination of combining a meta-heuristic algorithm, such as Particle Swarm, with MOO is in its infancy stages, primarily because Swarm Theory is still very young in the development. Combining MOO with the Genetic Algorithm, originally proposed by D. E. Goldberg in 1989 [6], is more developed because it was developed before the PSO algorithm. The Neighborhood Cultivation Genetic Algorithm ($NCGA$) is widely used for a variety of MOO problems. Goldberg denoted that the most important advantage of the genetic algorithms as an optimization method is the robustness against ill-natured property of the target problems. Particle Swarm Theory has been combined with multi-objective optimization in a variety of ways, and one standout algorithm is the Elitist Mutated Multi-Objective Particle Swarm Optimization algorithm ($EMMOPSO$) [7].

§4. Multi-objective optimization

The proposed algorithm, Particle Swarm Interpretation of Pareto Front Algorithm (PSI of Pareto Front or Ψ of Pareto Front), is used to determine the Pareto Front with a single algorithm. The presented algorithm also develops upon the notion of Sub- and Super- swarms. A Sub-swarm, denoted by s_i, is the swarm for the analysis of each objective function. The Super-swarm, denoted by S, is developed by plotting the solution for each particle in each Sub-swarm against each other, therefore developing a new graph, where the axis are the objectives. Being referred to as the Particle Swarm Interpretation (PSI) graph, ψ graph. The positions of the particles in the Super-swarm are modified within the newly develop algorithm to determine the Pareto Front. The advantage of using Sub-swarms as opposed to the traditional all inclusive swarm is that the global best of the individual objectives can be exploited as opposed to trying to exploit those by using a scaling or penalty function as done traditionally. The bi-objective optimization model is to be defined as follows for the Sub-swarms.

$$P_{id} = \overrightarrow{x}_j \in s_i \text{ where } j = 1, 2, \cdots, n \text{ and } i = 1, 2, s_1 \cup s_2 = S. \tag{4}$$

The ψ of Pareto Front algorithm defines the particle velocity equations as follows.

$$\Delta_i(x_j) = |\frac{F_i(\overrightarrow{fg}_i) - F_i(\overrightarrow{x}_j)}{F_i(\overrightarrow{fg}_i)}|, \tag{5}$$

$$v_{id} = wv_{id} + c_2(r(fg1_{gd} - x_{id}) + R(M_{gd} - x_{id})), \tag{6}$$

$$v_{id} = wv_{id} + c_2(r(fg2_{gd} - x_{id}) + R(M_{gd} - x_{id})). \tag{7}$$

There exist two velocity functions, because the particles use the closest determined global solution in each Sub-swarm. The closest solution is determined as a percentage to remove the scaling of the solutions as seen in equation 5, where $\overrightarrow{x}_j$ denotes the position of the j^{th} particle and $\Delta_i(\overrightarrow{x}_j)$ denotes the distance in the Super-swarm from particle $\overrightarrow{x}_j$ to global solution per the i^{th} objective, which is denoted by $\overrightarrow{F_i}$. Again, we will let fgi denote the determined global solution of the i^{th} objective function. The initialization of the particle's positions is consistent for all objective functions. The Super-swarm separates into two subclasses by determining the closest rank 1 particle. The rank 1 particle with respect to objective function F_1 is denoted by $fg1_{gd}$, whereas the rank 1 particle with respect to objective function F_2 is denoted by $fg2_{gd}$. The Median Particle is denoted by M_{gd}, and is used in both velocity function calculations (6) and (7). The Median Particle is determined by the following algorithm.

Algorithm 4.1. Median Particle is chosen from the Super-swarm.

```
RI = 1 Feasibility flag
for i = 1 : 2 Number of objectives do
  for j = 1 : pop Population of particles do
    F_i = f_i(x_j) Calculate objective, denoted by F
  end for
end for
for j = 1 : pop do
  r = 0
  if x_j = Feasible then
    RI = 0
    for m = 1 : pop do
      if j! = m then
        if F(x_j) < F(x_m) Dominates then r = r + 1 end if
      end if
```

if m = pop then $D_j = r$ end if
end for
if j = 1 then
$\overrightarrow{MidF} = \overrightarrow{F}$ DomR = D_j
else if $MidF_1 < F_1$ and $MidF_2 < F_2$ then
$\overrightarrow{MidF} = \overrightarrow{F}$ DomR =D_j
end if
end if
end for
if RI = 1 then
Exit, Re-initialize particles
end if.

The following algorithm is newly developed velocity function, as seen in equations (6) and (7). The position function remains consistent as in equation (2).

Algorithm 4.2. Algorithm for Trade-off Curve development by Super-swarm.
Require: fg_1 denotes global best particle of objective f_1,
Require: fg_2 denotes global best particle of objective f_2,
Require: MidPart by previous algorithm,
MidPart = M_{gd}
for $j = 1 : n$ do
calculate $diff_i(x_j)$, equation (5)
if $diff_1(x_j) > diff_2(x_j)$ then
$v_{id} = wv_{id} + c_2(r(fg1_{gd} - x_{id}) + R(M_{gd} - x_{id}))$
else
$v_{id} = wv_{id} + c_2(r(fg2_{gd} - x_{id}) + R(M_{gd} - x_{id}))$
end if
$x_{id} = x_{id} + v_{id}$
end for.

The weight parameter w, as well as the social parameter c_2, are the same as they appear in the traditional Particle Swarm Optimization algorithm.

§5. Test problem

To demonstrate the capability of this new algorithm we will examine a structural optimization problem, a plate girder or I-beam, where the cross-section and cost will be minimized. The problem will be subject to 2 constraints and 5 variables.

$$\begin{aligned} &\text{minimize } F_1 : ht_w + 2bt_f(\text{Cross} - \text{Section}),\\ &\text{minimize } F_2 : |(\text{ Initial Mass} - \text{Current Mass })|^2,\\ &\text{Subject to } D - D_\alpha < 0(\text{Deflection}),\\ &\text{where } 1.0 \le h \le 2.5,\\ &\qquad 1.0 \le h \le 2.5,\\ &\qquad .03 \le t_f \le .1, \end{aligned} \tag{8}$$

$.03 \leq t_w \leq .1,$

$22.0 \leq L \leq 25.0.$

Other relevant equations:

$E = 210, \quad P_m = 104,$

$I = \frac{1}{12}t_w h_3 + \frac{2}{3}bt_f + \frac{1}{2}bt_f h(h + 2t_f),$

$D_\alpha = \frac{L}{800},$

$D = \frac{L_3}{384e6EI}(8P_m + WL),$

$W = 19 + 77F_1.$

The above figures demonstrate the ψ of Pareto Front algorithm against the Neighborhood Cultivation Genetic Algorithm ($NCGA$). The $NCGA$ is very popular for MOO problems, because of its reliability and capability in producing accurate Pareto Fronts. Comparisons were made at intervals of 200 and 300 function evaluations in figures 1 and 2 as well as Figures 3 and 4 respectively. Those figures which were developed with $NCGA$ in insight by Simulia, being figures 1, 3 were unable to be scaled, because of the lack of capability in the software.

A Pareto Front that consists of more points is viewed as more reliable, because there is less speculation to the behavior of the design space among the points. A representation of a tradeoff surface or Pareto Front is satisfactory if the spread of points on the tradeoff surface is uniform [8]. The above results of the ψ of Pareto Front algorithm was determined with a population of 30, and c2 = .5. The weight inertia, w was initialized to .9 and then remained static for all iterations. The valued remained static simplify the algorithm, but it was just as easy as discussed in [9] which could have been linearly or exponentially decrease.

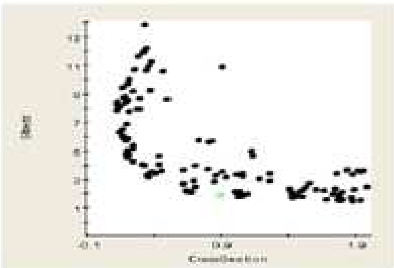


Fig. 1: Simulation Results with NCGA at 200 iterations

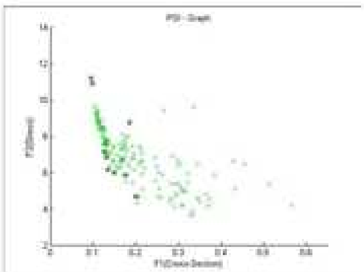


Fig.2: Simulation Results with of Pareto Front at 200 iterations

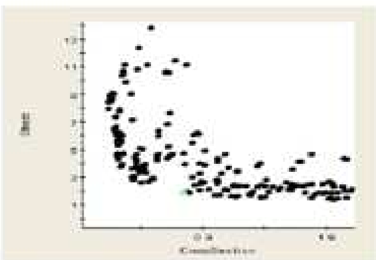


Fig. 3: Simulation Results with NCGA at 300 iterations

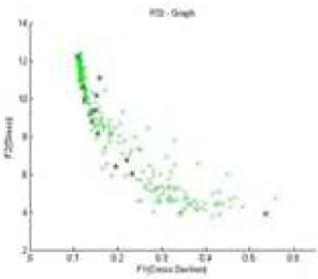


Fig.4: Simulation Results with of Pareto Front at 300 iterations

§6. Conclusion

The notion of combining different subsets of swarms is newly proposed and shows much promise for a variety of mathematical purposes, particularly multi-objective optimization. The PSI of Pareto Front or ψ of Pareto Front algorithm is in the beginning stages, therefore there is much room for improvement and expansion of the algorithm. The efficiency of an algorithm is of concern for any algorithm, but the ψ of Pareto Front algorithm has made a large leap forward in determining a Pareto Front with fewer simulations. However, future research could be done to improve upon the convergence speed in later iterations. The proposed algorithm has been verified for bi-objective optimization, but needs to be further enhanced for multi-objective optimization.

References

[1] M. Ferris and G. Deng, Extension of the direct method for noisey functions, In Proceedings of 39th Winter Simulation Conference 2010, 497-504.

[2] R. Eberhart and J. Kennedy, A new optimizer using particle swarm theory, In Proceedings of the Sixth Internation Symposium on Micro Machine and Human Science 1995, 39-43.

[3] L. Dumas, V. Herbert and F. Muy, Comparison of global optimization methods for drag reduction in the automotive industry, Lecture Notes in Computer Science 2010, 948-957.

[4] R. Perez and K. Behdinan, Swarm Intelligence, Focus on Ant and Particle Swarm Optimization, I-Tech Education and Publishing, Vienna, 2011, 373-394.

[5] X. Hu and R. Eberhart, Solving constrained nonlinear optimization problems with particle swarm optimization, In Proceedings of the Sixth World Multiconference on Systemics, Cybernetics and Informatics, 2002, 203-206.

[6] D. Goldberg, Genetic Algorithm in Search Optimization and Machine Learning, Addison-Welsey, Inc., 1989.

[7] J. Reddy and N. Kumar, Multi-objective particle swarm optimization for generating optimal trade-offs in reservoir operation, Hydrological Processes, **21**(2010), No. 6, 2897-2909.

[8] Y. Collette and P. Siarry, Multiobjective Optimization - Principles and Case Studies, Springer, 2004.

[9] W. Lodwick, A. Neumaier, and F. Newman, Optimization under uncertainty: methods and applications in radiation therapy, In Proceedings of the Tenth IEEE International Conference on Fuzzy Systems, 2011, 1219-1222.

Scientia Magna
Vol. 9 (2013), No. 4, 64-70

The minimal equitable dominating signed graphs

P. Siva Kota Reddy†, U. K. Misra‡ and P. N. Samanta♯

† Department of Mathematics, Siddaganga Institute of Technology,
Tumkur, 572103, India
‡ ♯ Department of Mathematics, Berhampur University,
Berhampur, 760007, Orissa, India
E-mail: reddy_math@yahoo.com umakanta_misra@yahoo.com
dr.pns.math@gmail.com

Abstract In this paper, we define the minimal equitable dominating signed graph of a given signed graph and offer a structural characterization of minimal equitable dominating signed graphs. In the sequel, we also obtained switching equivalence characterization: $\overline{\Sigma} \sim MED(\Sigma)$, where $\overline{\Sigma}$ are $MED(\Sigma)$ are complementary signed graph and minimal equitable dominating signed graph of Σ respectively.

Keywords Signed graphs, balance, switching, complement, minimal equitable dominating signed graph, negation.
2000 Mathematics Subject Classification: 05C22.

§1. Introduction and preliminaries

For standard terminology and notation in graph theory we refer Harary [6] and Zaslavsky [30] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Signed graphs, in which the edges of a graph are labelled positive or negative, have developed many applications and a flourishing literature (see [30]) since their first introduction by Harary in 1953 [7]. Their natural extension to multisigned graphs, in which each edge gets an n-tuple of signs—that is, the sign group is replaced by a direct product of sign groups—has received slight attention, but the further extension to gain graphs (also known as voltage graphs), which have edge labels from an arbitrary group such that reversing the edge orientation inverts the label, have been well studied [30]. Note that in a multisigned group every element is its own inverse, so the question of edge reversal does not arise with multisigned graphs.

A signed graph $\Sigma = (\Gamma, \sigma)$ is a graph $\Gamma = (V, E)$ together with a function $\sigma : E \to \{+, -\}$, which associates each edge with the sign + or −. In such a signed graph, a subset A of $E(\Gamma)$ is said to be positive if it contains an even number of negative edges, otherwise is said to be negative. A signed graph $\Sigma = (\Gamma, \sigma)$ is balanced [7] if in every cycle the product of the edge signs is positive. Σ is antibalanced [8] if in every even (odd) cycle the product of the edge signs is positive (resp., negative); equivalently, the negated signed graph $-\Sigma = (\Gamma, -\sigma)$ is balanced. A marking of Σ is a function $\mu : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking μ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\mu(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking μ of Σ is called canonical marking of Σ. In a signed graph $\Sigma = (\Gamma, \sigma)$, for any $A \subseteq E(\Gamma)$ the sign $\sigma(A)$ is the product of the signs on the edges of A.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Proposition 1.1. A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:

(i) Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [7]).

(ii) There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$. (Sampathkumar [13]).

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. Complement of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Proposition 1.1. For more new notions on signed graphs refer the papers ([10, 11, 14, 15], [17-25]).

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [30].

If $\mu : V(\Gamma) \to \{+, -\}$ is switching function, then switching of the signed graph $\Sigma = (\Gamma, \sigma)$ by μ means changing σ to σ^μ defined by:

$$\sigma^\mu = \mu(u)\sigma(uv)\mu(v).$$

The signed graph obtained in this way is denoted by Σ^μ and is called μ-witched signed graph or just switched signed graph. Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be isomorphic, written as $\Sigma_1 \cong \Sigma_2$ if there exists a graph isomorphism $f : \Gamma_1 \to \Gamma_2$ (that is a bijection $f : V(\Gamma_1) \to V(\Gamma_2)$ such that if uv is an edge in Γ_1 then $f(u)f(v)$ is an edge in Γ_2) such that for any edge $e \in E(\Gamma_1)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $\Sigma_1 = (\Gamma_1, \sigma_1)$ switches to a signed graph $\Sigma_2 = (\Gamma_2, \sigma_2)$ (or that Σ_1 and Σ_2 are switching equivalent) written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $\Sigma_1^\mu \cong \Sigma_2$. Note that $\Sigma_1 \sim \Sigma_2$ implies that $\Gamma_1 \cong \Gamma_2$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be weakly isomorphic (see [26]) or cycle isomorphic (see [29]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [29]):

Proposition 1.2.(**T. Zaslavsky** [29]) Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.

In [16], the authors introduced the switching and cycle isomorphism for signed digraphs.

§2. Minimal equitable dominating signed graphs

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [4] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [12] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950 and 1960, beginning with Berge [2] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability", which is now known as the domination number of a graph. Oystein Ore [9] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [28]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [3] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

A subset D of $V(\Gamma)$ is called an equitable dominating set of a graph Γ, if for every $v \in V-D$ there exists a vertex $v \in D$ such that $uv \in E(\Gamma)$ and $|d(u)-d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of Γ. An equitable dominating set D is minimal, if for any vertex $u \in D$, $D-\{u\}$ is not a equitable dominating set of Γ. A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $v \in S-u$. If a vertex $u \in V$ be such that $|d(u)-d(v)| \geq 2$, for all $v \in N(u)$ then u is in equitable dominating set. Such vertices are called equitable isolates. An equitable dominating set D is minimal, if for any vertex $u \in D$, $D-\{u\}$ is not a equitable dominating set of Γ.

Let $\mathcal{S}$ be a finite set and $F=\{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n\}$ be a partition of S. Then the intersection graph $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices $\mathcal{S}_i$ and $\mathcal{S}_j$ are adjacent if and only if $\mathcal{S}_i \cap \mathcal{S}_j \neq \phi$, $i \neq j$.

In [27], the authors introduced a new class of intersection graphs in the field of domination theory. The minimal equitable dominating graph $MED(\Gamma)$ of a graph Γ is the intersection graph defined on the family of all minimal equitable dominating sets of vertices in Γ.

Motivated by the existing definition of complement of a signed graph, we extend the notion of minimal equitable dominating graphs to signed graphs as follows: The minimal equitable dominating signed graph $MED(\Sigma)$ of a signed graph $\Sigma=(\Gamma,\sigma)$ is a signed graph whose underlying graph is $MED(\Gamma)$ and sign of any edge PQ in $MED(\Sigma)$ is $\mu(P)\mu(Q)$, where μ is the canonical marking of Σ, P and Q are any two minimal equitable dominating sets of vertices in Γ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called minimal equitable dominating signed graph, if $\Sigma \cong MED(\Sigma')$ for some signed graph Σ'. In the following section, we shall present a characterization of minimal equitable dominating signed graphs. The purpose of this paper is to initiate a study of this notion.

We now give a straightforward, yet interesting, property of minimal equitable dominating signed graphs.

Proposition 2.1. For any signed graph $\Sigma = (\Gamma, \sigma)$, its minimal equitable dominating signed graph $MED(\Sigma)$ is balanced.

Proof. Since sign of any edge PQ in $MED(\Sigma)$ is $\mu(P)\mu(Q)$, where μ is the canonical marking of Σ, by Proposition 1.1, $MED(\Sigma)$ is balanced.

For any positive integer k, the k^{th} iterated minimal equitable dominating signed graph $MED(\Sigma)$ of Σ is defined as follows:

$$MED^0(\Sigma) = \Sigma,\ MED^k(\Sigma) = MED(MED^{k-1}(\Sigma)).$$

Corollary 2.1. For any signed graph $\Sigma = (\Gamma, \sigma)$ and any positive integer k, $MED^k(\Sigma)$ is balanced.

In [27], the authors characterized graphs for which $MED(\Gamma) \cong \overline{\Gamma}$.

Proposition 2.2. (**Sumathi & Soner** [27]) For any graph $\Gamma = (V, E)$, $MED(\Gamma) \cong \overline{\Gamma}$ if and only if Γ is complete with p vertices.

We now characterize signed graphs whose common minimal equitable dominating signed graphs and complementary signed graphs are switching equivalent.

Proposition 2.3. For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\Sigma} \sim MED(\Sigma)$ if and only if Γ is K_p.

Proof. Suppose $\overline{\Sigma} \sim MED(\Sigma)$. This implies, $\overline{\Gamma} \cong MED(\Gamma)$ and hence by Proposition 2.2, Γ is iK_p.

Conversely, suppose that Γ is K_p. Then $\overline{\Gamma} \cong MED(\Gamma)$ by Proposition 5. Now, if Σ is a signed graph with underlying graph Γ is K_p, by the definition of complementary signed graph and Proposition 3, $\overline{\Sigma}$ and $MED(\Sigma)$ are balanced and hence, the result follows from Proposition 1.2.

Proposition 2.4. For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their minimal equitable dominating signed graphs are switching equivalent.

Proof. Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Proposition 2.1, $MED(\Sigma_1)$ and $MED(\Sigma_2)$ are balanced and hence, the result follows from Proposition 1.2.

The notion of negation $\eta(\Sigma)$ of a given signed graph S defined in [8] as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

Proposition 2.3 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 2.2. For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\eta(\Sigma)} \sim MED(\Sigma)$ if and only if Γ is K_p.

Corollary 2.3. For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\Sigma} \sim MED(\eta(\Sigma))$ if and only if Γ is K_p.

Corollary 2.4. For any signed graph $\Sigma = (\Gamma, \sigma)$, $\overline{\eta(\Sigma)} \sim MED(\eta(\Sigma))$ if and only if Γ is K_p.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $MED(\Sigma)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation of $MED(\Sigma)$ is balanced.

Proposition 2.5. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $MED(\Gamma)$ is bipartite then $\eta(MED(\Sigma))$ is balanced.

Proof. Since, by Proposition 2.1, $MED(\Sigma)$ is balanced, each cycle C in $MED(\Sigma)$ contains even number of negative edges. Also, since $MED(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $MED(\Sigma)$ is also even. Hence $\eta(MED(\Sigma))$ is balanced.

§3. Characterization of minimal equitable dominating signed graphs

The following result characterize signed graphs which are minimal equitable dominating signed graphs.

Proposition 3.1. A signed graph $\Sigma = (\Gamma, \sigma)$ is a minimal equitable dominating signed graph if and only if Σ is balanced signed graph and its underlying graph Γ is a $MED(\Gamma)$.

Proof. Suppose that Σ is balanced and its underlying graph Γ is a minimal equitable dominating graph. Then there exists a graph Γ' such that $MED(\Gamma') \cong \Gamma$. Since Σ is balanced, by Proposition 1.1, there exists a marking μ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $MED(\Sigma') \cong \Sigma$. Hence Σ is a minimal equitable dominating signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a minimal equitable dominating signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $MED(\Sigma') \cong \Sigma$. Hence by Proposition 2.1, Σ is balanced.

Problem 3.1. Characterize signed graphs for which $\overline{\Sigma} \cong MED(\Sigma)$.

Acknowledgments

We thank the anonymous referees for many detailed and valuable suggestions that helped us to improve this paper. Also, the first author is grateful to Honorable Shri Dr. M. N. Channabasappa, Director, Siddaganga Institute of Technology, Tumkur, for his constant support and encouragement for research and development.

References

[1] R. P. Abelson and M. J. Rosenberg, Symoblic psychologic: A model of attitudinal cognition, Behav. Sci., **3**(1958), 1-13.

[2] C. Berge, Theory of Graphs and its Applications, Methuen, London, 1962.

[3] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, Networks, **7**(1977), 247-261.

[4] C. F. De Jaenisch, Applications de lnalyse mathematique an Jen des Echecs, 1862.

[5] David Easley and Jon Kleinberg, Networks, Crowds, and Markets: Reasoning About a Highly Connected World, Cambridge University Press, 2010.

[6] F. Harary, Graph Theory, Addison-Wesley Publishing Co., 1969.

[7] F. Harary, On the notion of balance of a signed graph, Michigan Math. J., **2**(1953), 143-146.

[8] F. Harary, Structural duality, Behav. Sci., **2**(1957), No. 4, 255-265.

[9] O. Ore, Theory of Graphs. Amer. Math. Soc. Colloq. Publ., **38**, 1962.

[10] R. Rangarajan and P. Siva Kota Reddy, The edge C_4 signed graph of a signed graph, Southeast Asian Bulletin of Mathematics, **34**(2010), No. 6, 1077-1082.

[11] R. Rangarajan, M. S. Subramanya and P. Siva Kota Reddy, Neighborhood signed graphs, Southeast Asian Bulletin of Mathematics, **36**(2012), No. 3, 389-397.

[12] W. W. Rouse Ball, Mathematical Recreation and Problems of Past and Present Times, 1892.

[13] E. Sampathkumar, Point signed and line signed graphs, Nat. Acad. Sci. Letters, **7** (1984), No. 3, 91-93.

[14] E. Sampathkumar, P. Siva Kota Reddy and M. S. Subramanya, Directionally n-signed graphs, Ramanujan Math. Soc., Lecture Notes Series (Proc. Int. Conf. ICDM 2008), **13** (2010), 155-162.

[15] E. Sampathkumar, P. Siva Kota Reddy and M. S. Subramanya, Directionally n-signed graphs-II, International J. Math. Combin., **4**(2009), 89-98.

[16] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Characterization of line sidigraphs, Southeast Asian Bulletin of Mathematics, **35**(2011), No. 2, 297-304.

[17] P. Siva Kota Reddy and M. S. Subramanya, Note on path signed graphs, Notes on Number Theory and Discrete Mathematics, **15**(2009), No. 4, 1-6.

[18] P. Siva Kota Reddy, S. Vijay and V. Lokesha, n^{th} Power signed graphs, Proceedings of the Jangjeon Math. Soc., **12**(2009), No. 3, 307-313.

[19] P. Siva Kota Reddy, t-Path Sigraphs, Tamsui Oxford J. of Math. Sciences, **26**(2010), No. 4, 433-441.

[20] P. Siva Kota Reddy, E. Sampathkumar and M. S. Subramanya, Common-edge signed graph of a signed graph, J. Indones. Math. Soc., **16**(2010), No. 2, 105-112.

[21] P. Siva Kota Reddy, B. Prashanth, and T. R. Vasanth Kumar, Antipodal signed directed Graphs, Advn. Stud. Contemp. Math., **21**(2011), No. 4, 355-360.

[22] P. Siva Kota Reddy and B. Prashanth, The Common Minimal Dominating Signed Graph, Trans. Comb., **1**(2012), No. 3, 39-46.

[23] P. Siva Kota Reddy and B. Prashanth, $\mathcal{S}$-Antipodal signed graphs, Tamsui Oxford J. of Inf. Math. Sciences, **28**(2012), No. 2, 165-174.

[24] P. Siva Kota Reddy and S. Vijay, The super line signed graph $\mathcal{L}_r(S)$ of a signed Graph, Southeast Asian Bulletin of Mathematics, **36**(2012), No. 2, 875-882.

[25] P. Siva Kota Reddy and B. Prashanth, Note on Minimal Dominating Signed Graphs, Preprint.

[26] T. Sozánsky, Enueration of weak isomorphism classes of signed graphs, J. Graph Theory, **4**(1980), No. 2, 127-144.

[27] M. P. Sumathi and N. D. Soner, Minimal equitable dominating graph, International J. of Math. Sci. & Engg. Appls., **6**(2012), No. 4, 273-278.

[28] A. M. Yaglom and I. M. Yaglom. Challenging mathematical problems with elementary solutions. Volume 1: Combinatorial Analysis and Probability Theory, 1964.

[29] T. Zaslavsky, Signed graphs, Discrete Appl. Math., **4**(1982), No. 1, 47-74.

[30] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, VIII Edition, Electron. J. Combin., Dynamic Surveys in Combinatorics (1998), No. DS8. Eighth ed. (2012).

Scientia Magna
Vol. 9 (2013), No. 4, 71-77

Generalized reciprocal power sums

Erping Song

Department of Mathematics, Northwest University
Xi'an, Shaanxi, China
E-mail: songerping@fox.com

Abstract Recently, Anthony Sofo had given integral forms of sums associated with harmonic numbers. Moreover, he proved an integral representation for a reciprocal power sum, and gave an identity for binomial coefficients. In this paper we use integral to establish a relationship with generalized reciprocal power sums, and obtain some identities for the derivative of the binomial coefficients.

Keywords Generalized reciprocal power, integral, derivative, binomial coefficient .
2010 Mathematics Subject Classification: 33B15.

§1. Introduction and preliminaries

coefficients play an important role in many subjects, such as number theory and graph theory. In fact, it is difficult to compute the value of combinatorial sums involving inverses of binomial coefficients. However, the idea of derivative operators of binomial coefficients has become a method to handle harmonic number identities which were presented by George E. Andrews [1]. Afterwards, Tiberiu Trif [2] gave some results about combinatorial sums and series involving inverses of binomial coefficients. In 2009, Anthony Sofo [3] given an integral integral representation for a reciprocal power sum and some beautiful results associated with harmonic numbers. In the meantime, he pointed out some identities for sums of derivatives about binomial coefficients. In addition, Anthony Sofo [4] proved an integral representation for a reciprocal power sum. We will use the idea of the consecutive derivative to improve these issues.

Some definitions are defined as follows.

Definition 1.1. The binomial coefficients are denoted by

$$\left(\begin{array}{c} n \\ m \end{array} \right) = \begin{cases} \dfrac{n!}{m!(n-m)!}, & n \geq m, \\ 0, & n < m. \end{cases}$$

Definition 1.2.

$$\Gamma(z) = \int_0^\infty \omega^{z-1} \mathrm{e}^{-\omega} d\omega, \text{ for } \mathrm{R}e(z) > 0,$$

and for $\mathrm{R}e(s) > 0$

$$B(s, z) = B(z, s) = \int_0^1 \omega^{s-1}(1-\omega)^{z-1} d\omega = \frac{\Gamma(s)\Gamma(z)}{\Gamma(s+z)}.$$

Besides, the binomial coefficients need to satisfy

$$\begin{pmatrix} z \\ \omega \end{pmatrix} = \frac{\Gamma(z+1)}{\Gamma(\omega+1)\Gamma(z-\omega-1)}.$$

Anthony Sofo [4] presented an identity for a reciprocal power sum given by Bhatnager, namely $\sum\limits_{n=1}^{p} \frac{1}{\begin{pmatrix} n+z \\ z \end{pmatrix}}$, and studied the derivatives about the identity.

Proposition 1.1. Let $p,\ q \in \mathbb{N}$ be a positive integer, and $Q(n,z) = \begin{pmatrix} n+z \\ z \end{pmatrix}$ be an analytic function of $z \in \mathbb{C} \setminus \{-1,-2,\cdots\}$. Then

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} = -Q^{(q)}(p,z),$$

where

$$Q^{(q)}(n,z) = \begin{cases} \dfrac{d^q}{dz^q}\left(\begin{pmatrix} n+z \\ z \end{pmatrix}^{-1}\right), & q > 1, \\ Q^{(0)}(n,z) = \begin{pmatrix} n+z \\ z \end{pmatrix}^{-1}, & q = 0, \\ Q^{(0)}(n,0) = 1, & z = 0 \text{ and } q = 0. \end{cases}$$

Our goal is to promote the reciprocal power sum,

$$\sum_{n=1}^{p} \frac{1}{(n+t)\begin{pmatrix} n+z+t \\ z \end{pmatrix}} \quad \text{where} \quad t \geq 1,$$

by introducing derivative and integral, there are some simple Theorems.

Theorem 1.1. If $z \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $p,\ q \in \mathbb{N}$, $F(n+t,z) = \sum\limits_{n=1}^{p} \frac{Q(n+t,z)}{n+t}$ and $f(z) = (z+1)(z+2)\cdots(z+t)$, then

$$\big(zF(n+t,z) + Q(p+t,z)\big)f(z) = t!.$$

Especially, when $t = 0$, differentiating finite powers with respect to z we have the result

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} = -Q^{(q)}(p,z), \quad \text{where} \quad Q^{(q)}(n,z) = \frac{d^q}{dz^q}\left(\begin{pmatrix} n+z \\ z \end{pmatrix}^{-1}\right).$$

Proof . It is not hard to show that

$$\sum_{n=1}^{p}\frac{1}{(n+t)\begin{pmatrix} n+z+t \\ z \end{pmatrix}} = \sum_{n=1}^{p}\frac{\Gamma(n+t)\Gamma(z+1)}{\Gamma(n+z+t+1)} = \sum_{n=1}^{p}B(n+t,z+1)$$
$$= \int_0^1 (1-x)^z\sum_{n=1}^{p}x^{n+t-1}dx = \int_0^1(1-x)^{z-1}(x^t-x^{p+t})dx$$
$$= \frac{t!}{z(z+1)...(z+t)} - \frac{(p+t)!}{z(z+1)...(z+p+t)}$$
$$= \frac{t!}{z(z+1)...(z+t)} - \frac{1}{z\begin{pmatrix} z+p+t \\ p+t \end{pmatrix}}, \quad (1.1)$$

Let $f(z)=(z+1)...(z+t)$ and multiple $f(z)$ of both side for (2.1), we may get

$$t!-f(z)Q(p+t,z)=zf(z)F(n+t,z).$$

This completes the proof.

Corollary 1.1. Taking $t=1$ in Theorems 1.1, the following result can be proved

$$(z+1)Q(p+1,z)+z(z+1)F(n+1,z)=1.$$

Differentiating q times with respect to z results in

$$-qQ^{(q-1)}(p+1,z)-(z+1)Q^{(q)}(p+1,z)$$
$$=(q-1)qF^{(q-2)}(n+1,z)+q(2z+1)F^{(q-1)}(n+1,z)+z(z+1)F^{(q)}(n+1,z).$$

Without loss of generality, we can choose $z=0$ such as

$$\sum_{n=1}^{p}\frac{(q-1)qQ^{(q-2)}(n+1,0)+qQ^{(q-1)}(n+1,0)}{n+1} = -qQ^{(q-1)}(p+1,0)-Q^{(q)}(p+1,0).$$

Corollary 1.2. If taking $z=0$ and $q=3$ in Corollary 1.1, then

$$\sum_{n=1}^{p}\frac{3(H_{n+1})^2+3H_{n+1}^{(2)}-6H_{n+1}}{n+1} = -3H_{p+1}^{(2)}-3(H_{p+1})^2+2H_{p+1}^{(3)}+3H_{p+1}H_{p+1}^{(2)}+(H_{p+1})^3.$$

Remark 1.1. For $1<t$, it is different to establish an identity about the derivative of finite powers in Theorem 1.1. However, a more interesting recurrence is the derivative of coefficient of Theorem 1.1, we shall consider this recurrence further in Section 3.

§2. Definition and properties

Definition 2.1. If $\alpha>1$, the series is called the generalized Harmonic numbers

$$H_n^{(\alpha)}=\sum_{r=1}^{n}\frac{1}{r^{\alpha}},$$

if $\alpha = 1$, we define $H_n^{(1)}$, the n-th Harmonic number as follows

$$H_n^{(1)} = \sum_{r=1}^{n} \frac{1}{r}.$$

Theorem 2.1. Let $z \in \mathbb{R} \setminus \mathbb{Z}_0^- \cup \{0\}$, $F_1(n,z) = \sum_{n=1}^{p} Q(n,z)$ and $p, q \in \mathbb{N}$, then

$$\begin{aligned}&(q-1)qF_1^{(q-2)}(n,z) + q(2z-1)F_1^{(q-1)}(n,z) + (z-1)zF_1^{(q)}(n,z)\\ &= -qQ^{(q-1)}(p,z-1) - zQ^{(q)}(p,z-1) - pqQ^{(q-1)}(p,z) - p(z-1)Q^{(q)}(p,z).\end{aligned}$$

The above equation is equivalent to

$$\begin{aligned}&\sum_{n=1}^{p}(q-1)qQ^{(q-2)}(n,z) + q(2z-1)Q^{(q-1)}(n,z) + (z-1)zQ^{(q)}(n,z)\\ &= -qQ^{(q-1)}(p,z-1) - zQ^{(q)}(p,z-1) - pqQ^{(q-1)}(p,z) - p(z-1)Q^{(q)}(p,z).\end{aligned}$$

We consider some special cases for Theorem 1.2 and obtain some corollaries.

Proof. To complete the proof of theorems, we first proof the equation. For positive integer $t,\ n$ and $Q(n+t,z) = \begin{pmatrix} n+z+t \\ z \end{pmatrix}^{-1}$ be an analytic function of $z \in \mathbb{C} \setminus \{-1,-2,\cdots\}$, it is easy to show that

$$Q(n+t,z) = \begin{pmatrix} n+z+t \\ t \end{pmatrix}^{-1} = \frac{\Gamma(z+1)\Gamma(n+t+1)}{\Gamma(n+t+z+1)} = \frac{\Gamma(n+t+1)}{\prod_{i=1}^{n+t}(z+i)},$$

taking logarithm and differentiating with respects to z we have the following results, namely

$$Q^{(1)}(n+t,z) = \begin{cases} -Q(n+t,z)P(n+t,z), & \text{where } \ P(n+t,z) = \sum_{r=1}^{n+t} \frac{1}{r+z}, \\ -H_{n+t}, & \text{for } \ z = 0, \end{cases} \tag{2.1}$$

if $m \leq 2$, we get

$$Q^{(s)}(n+p,z) = -\sum_{m=0}^{s-1} \begin{pmatrix} s-1 \\ m \end{pmatrix} Q^{(m)}(n+t,z)P^{(s-m-1)}(n+t,z), \tag{2.2}$$

where $P^{(0)}(n+t,z) = \sum_{r=1}^{n+t} \frac{1}{r+z}$, and for $i \in \mathbb{N}$,

$$P^{(i)}(n+t,z) = \frac{d^i}{dz^i}\left(\sum_{r=1}^{n+t} \frac{1}{r+z}\right) = (-1)^i i! \sum_{r=1}^{n+t} \frac{1}{(r+z)^{i+1}}.$$

On the other hand, we have

$$\begin{aligned}\sum_{n=1}^{p} Q(n,z) &= \sum_{n=1}^{p}\binom{n+z}{z}^{-1} = \sum_{n=1}^{p}\frac{\Gamma(z+1)\Gamma(n+1)}{\Gamma(n+z+1)}\\ &= \sum_{n=1}^{p} nB(n,z+1) = \int_0^1 (1-x)^z \sum_{n=1}^{p} nx^{n-1}dx\\ &= \int_0^1 (1-x^p)(1-x)^{z-2}dx - p\int_0^1 x^p(1-x)^{z-1}dx\\ &= \frac{1}{z-1} - \frac{1}{z-1}Q(p,z-1) - \frac{p}{z}Q(p,z). \end{aligned} \tag{2.3}$$

Let $F_1(n,z) = \sum_{n=1}^{p} Q(n,z)$ and multiple $(z-1)z$ of both side for (2.3), we can obtain

$$z - zQ(p,z-1) - (z-1)pQ(p,z) = (z-1)zF_1(n,z).$$

Differentiating with respects to z and combining (2.2) we get Theorem 1.2. Furthermore, combining (2.1) we can obtain the corollaries.

Corollary 2.1. Especially, for $z = 1$, we may get

$$\sum_{n=1}^{p}(q-1)qQ^{(q-2)}(n,1) + qQ^{(q-1)}(n,1) = -qQ^{(q-1)}(p,0) - Q^{(q)}(p,0) - pqQ^{(q-1)}(p,1).$$

Corollary 2.2. If we put $z = 1$ and $q = 2$ in Theorem 1.2, we have

$$2\sum_{n=1}^{p}\frac{H_n}{n} = (H_p)^2 + H_p^{(2)}.$$

Corollary 2.3. When $z = 1$ and $q = 3$, we also have

$$\begin{aligned}&\sum_{n=1}^{p}\frac{3H_{n+1}^{(2)} - 12H_{n+1} + 3(H_{n+1})^2}{n}\\ &= -6H_p - 3(H_p)^2 - 3H_p^{(2)} + 2H_p^{(3)} + 3H_pH_p^{(2)} + (H_p)^3 - 3H_{p+1}^{(2)} + 6H_{p+1} - 3(H_{p+1})^2.\end{aligned}$$

It is equivalent to

$$\begin{aligned}&\sum_{n=1}^{p}\left(\frac{3H_n^{(2)} - 6H_n + 3(H_n)^2}{n} - \frac{6H_n}{n+1}\right)\\ &= -3(H_p)^2 - 3H_p^{(2)} + 2H_p^{(3)} + 3H_pH_p^{(2)} + (H_p)^3 - 3(H_{p+1})^2\\ &= 2H_p^{(3)} - 6(H_p)^2 + 3H_pH_p^{(2)} + (H_p)^3 - \frac{6H_p}{p+1}.\end{aligned}$$

We have also extended the range of identities of finite sums with Harmonic numbers.

§3. Properties

As an extension of Theorem 1.1, we shall be concerned the relationship between derivative times and coefficients in this section, it is interesting that the coefficient satisfies the Pascal's triangle, which is able to explained by some examples. From Theorem 1.1 we get

$$\big(zF(n+t,z)+Q(p+t,z)\big)f(z)=t!.$$

Differentiating q times with respect to z for the above identity.

(a)

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & & & & \\
 & & & 1 & 2 & 1 & & & \\
 & & 1 & 3 & 3 & 1 & & & \\
 & & 1 & 4 & 6 & 4 & 1 & & \\
 & 1 & 5 & 10 & 10 & 5 & 1 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
1 & q & \cdots & \cdots & \cdots & \cdots & \cdots & q & 1
\end{array}$$

Suppose that (a) express the derivative of the coefficient of $f(z)Q(p+t,z)$, the derivative of $f(z)Q(p+t,z)$ should be written left to right in ascending.

(b)

$$\begin{array}{cccccccc}
 & & & 1 & & & & \\
 & & & 2 & 2 & & & \\
 & & 3 & 2\cdot 3 & 3 & & & \\
 & & 4 & 3\cdot 4 & 3\cdot 4 & 4 & & \\
 & 5 & 4\cdot 5 & 30 & 4\cdot 5 & 5 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q & (q-1)q & \cdots & \cdots & \cdots & \cdots & \cdots & (q-1)q
\end{array}$$

Similarly, (c) satisfies the Pascal's triangle,

(c)

$$\begin{array}{cccccccc}
 & & & 1 & & & & \\
 & & & 2 & 2 & & & \\
 & & 3 & 2\cdot 3 & 3 & & & \\
 & & 4 & 3\cdot 4 & 3\cdot 4 & 4 & & \\
 & 5 & 4\cdot 5 & 30 & 4\cdot 5 & 5 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q & (q-1)q & \cdots & \cdots & \cdots & \cdots & \cdots & (q-1)q
\end{array}$$

where (b) and (c) are the derivatives of the coefficient of $f(z)F(n+t,z)$ and $zf(z)F(n+t,z)$ respectively. For convenience, let $a(m,n)$ denote the m-th line and the n-th column of the above (a), and the method is also suitable for $b(m,n)$ and $c(m,n)$. According to (a) and (b) we can observe that

(i) $\quad a(u,v)=a(u-1,v-1)+a(u-1,v), \qquad 1<u,v<n;$

(ii) $\quad b(u,v)=b(u-1,v-1)+b(u-1,v), \qquad 1<u,v<n;$

(iii) $\quad c(u,1)=c(u,u)=u, c(u,2)=c(u,u-1)=(u-1)\cdot u, \qquad 1<u<n;$

(iv) $\quad c(u,v)=(u-1,v-1)+c(u-1,v)+b(u,v), \qquad 3\le u<n;$

Example 3.1. For $q=4$, we can deduce the following equation from the above conclusion

$$\begin{aligned}&-f^{(4)}(z)Q(p+t,z)-4f^{(3)}(z)Q^{'}(p+t,z)-6f^{''}(z)Q^{''}(p+t,z)-4f^{'}(z)Q^{(3)}(p+t,z)\\&\quad-f(z)Q^{(4)}(p+t,z)=4f^{(3)}(z)F(n+t,z)+12f^{(2)}(z)F^{'}(n+t,z)+12f^{'}(z)F^{''}(n+t,z)\\&\quad+4f(z)F^{(3)}(n+t,z)+zf(z)F^{(4)}(n+t,z)+4zf^{'}(z)F^{(3)}(n+t,z)+6zf^{''}(z)F^{''}(n+t,z)\\&\quad+4zf^{(3)}(z)F^{'}(n+t,z)+f^{(4)}(z)F(n+t,z).\end{aligned}$$

Example 3.2. When $q=5$, the above triangular recurrence imply that

$$\begin{aligned}&-f^{(5)}(z)Q(p+t,z)-5f^{(4)}(z)Q^{'}(p+t,z)-10f^{(3)}(z)Q^{''}(p+t,z)-10f^{''}(z)Q^{(3)}(p+t,z)\\&\quad-5f^{'}(z)Q^{(4)}(p+t,z)-f(z)Q^{(5)}(p+t,z)=5f^{(4)}(z)F(n+t,z)+20f^{(3)}(z)F^{'}(n+t,z)\\&\quad+30f^{''}(z)F^{''}(n+t,z)+20f^{'}(z)F^{(3)}(n+t,z)+5f(z)F^{(4)}(n+t,z)+zf(z)F^{(5)}(n+t,z)\\&\quad+5zf^{'}(z)F^{(4)}(n+t,z)+10zf^{''}(z)F^{(3)}(n+t,z)+10zf^{(3)}(z)F^{''}(n+t,z)\\&\quad+5zf^{(4)}(z)F^{'}(n+t,z)+f^{(5)}(z)F(n+t,z).\end{aligned}$$

It is obvious that Anthony Sofo [**?**] and Corollary 1.1 satisfy the above recurrence, but it is difficult to establish an identity for Theorem 1.1 to differentiate q times with $t>2$.

References

[1] George E. Andrews, Kanzō. Uchimura, Identities in combinatorics. IV:differentition and harmonic numbers, Utilitas Math, **26**(1995), 73-88.

[2] Tiberiu Trif, Combinatorial sums and series involving inverses of binomial coefficients, Fibnacci Quart, **38**(2000), 79-84.

[3] Anthony Sofo, Integral forms of sums associated with harmonic numbers, Mathematica computer, **207**(2009), 365-372.

[4] Anthony Sofo, Reciprocal power sums, Integers, **12**(2012), 1-9.

Scientia Magna
Vol. 9 (2013), No. 4, 78-91

Compactness in topology of intuitionistic fuzzy rough sets

H. Hazra[†] and S. K. Samanta[‡]

† Department of Mathematics, Bolpur College, Bolpur, Birbhum, India
‡ Department of Mathematics, Visva Bharati, Santiniketan, 731235, India
E-mail: h.hazra2010@gmail.com syamal_123@yahoo.co.in

Abstract In this paper compactness in topology of intuitionistic fuzzy rough sets are introduced. Also compactnesses in (P, τ_1) and (P, τ_2) and quasicompactness are studied and relation between them are established.

Keywords Fuzzy subsets, rough sets, instuitionistic fuzzy sets, intuitionistic fuzzy rough sets, fuzzy topology, open $IFRS$, closed $IFRS$, closure, IFR continuous, first topology, IFR_1 continuous, second topology, IFR_2 continuous, open cover, compact in (X, τ), compact in (P, τ_1), compact in (P, τ_2), quasicompact.
2000 Mathematics Subject Classification: 54A40.

§1. Introduction

Lotfi Zadeh [23] first introduced the idea of fuzzy subsets. Then different variations/generalisations of fuzzy subsets were made by several authors. Pawlak [21] introduced the idea of rough sets. Nanda [20] and Çoker [7] gave the definition of fuzzy rough sets. Atanassov [2] introduced the idea of intuitionistic fuzzy sets. Combining the ideas of fuzzy rough sets and intuitionistic fuzzy sets T. K. Mandal and S. K. Samanta [17] introduced the concept of intuitionistic fuzzy rough sets (briefly call $IFRS$). On the other hand fuzzy topology (we call it topology of fuzzy subsets) was first introduced by C. L. Chang [4]. Later many authors dealt with the idea of fuzzy topology of different kinds of fuzzy sets. M. K. Chakroborti and T. M. G. Ahsanullah [3] introduced the concept of fuzzy topology on fuzzy sets. T. K. Mondal and S. K. Samanta introduced the topology of interval valued fuzzy sets in [19] and the topology of interval valued intuitionistic fuzzy sets in [18]. In [11], we introduced the concept of topology of intuitionistic fuzzy rough sets and study its various properties. In defining topology on an $IFRS$ from the parent space, we observed that two topologies are induced on the $IFRS$ and accordingly two types of continuity are defined. In [13], we studied the connectedness in topology of intuitionistic fuzzy sets. In [12], we have defined Type-1 connectedness and Type-2 connectedness in topology of intuitionistic fuzzy rough sets and studied properties of two types of connectedness. In this paper we have defined compactness in topology of intuitionistic fuzzy rough sets, compactnesses in (P, τ_1) and (P, τ_2) and quasicompactness and studied properties of such compactnesses.

§2. Preliminaries

Unless otherwise stated, we shall consider $(V, \mathcal{B})$ to be a rough universe where V is a nonempty set and $\mathcal{B}$ is a Boolean subalgebra of the Boolean algebra of all subsets of V. Also consider a rough set $X = (X_L, X_U) \in \mathcal{B}^2$ with $X_L \subset X_U$. Moreover we assume that $\mathcal{C}_X$ be the collection of all $IFRSs$ in X.

Definition 2.1.[20] A fuzzy rough set (briefly FRS) in X is an object of the form $A = (A_L, A_U)$, where A_L and A_U are characterized by a pair of maps $A_L : X_L \to \mathcal{L}$ and $A_U : X_U \to \mathcal{L}$ with $A_L(x) \leq A_U(x), \forall x \in X_L$, where ($\mathcal{L}, \leq$) is a fuzzy lattice (i.e., complete and completely distributive lattice whose least and greatest elements are denoted by 0 and 1 respectively with an involutive order reversing operation $' : \mathcal{L} \to \mathcal{L}$).

Definition 2.2.[20] For any two fuzzy rough sets $A = (A_L, A_U)$ and $B = (B_L, B_U)$ in X,

(i) $A \subset B$ if $A_L(x) \leq B_L(x), \forall x \in X_L$ and $A_U(x) \leq B_U(x), \forall x \in X_U$.

(ii) $A = B$ if $A \subset B$ and $B \subset A$. If $\{A_i : i \in J\}$ be any family of fuzzy rough sets in X, where $A_i = (A_{iL}, A_{iU})$, then

(iii) $E = \bigcup_i A_i$, where $E_L(x) = \vee A_{iL}(x),\ \forall x \in X_L$ and $E_U(x) = \vee A_{iU}(x),\ \forall x \in X_U$.

(iv) $F = \bigcap_i A_i$, where $F_L(x) = \wedge A_{iL}(x),\ \forall x \in X_L$ and $F_U(x) = \wedge A_{iU}(x),\ \forall x \in X_U$.

Definition 2.3.[17] If A and B are fuzzy sets in X_L and X_U respectively where $X_L \subset X_U$. Then the restriction of B on X_L and the extension of A on X_U (denoted by $B_{>L}$ and $A_{<U}$ respectively) are defined by $B_{>L}(x) = B(x),\ \forall x \in X_L$ and

$$A_{<U}(x) = \begin{cases} A(x), & \forall x \in X_L, \\ \vee_{\xi \in X_L}\{A(\xi)\}, & \forall x \in X_U - X_L. \end{cases}$$

Complement of an FRS $A = (A_L, A_U)$ in X are denoted by $\bar{A} = ((\bar{A})_L, (\bar{A})_U)$ and is defined by $(\bar{A})_L$(x)= $(A_{U>L})'(x),\ \forall x \in X_L$ and $(\bar{A})_U(x) = (A_{L<U})'(x),\ \forall x \in X_U$. For simplicity we write $(\bar{A}_L, \bar{A}_U)$ instead of $((\bar{A})_L, (\bar{A})_U)$.

Theorem 2.1.[17] If $A,\ B,\ C,\ D$ and $B_i,\ i \in J$ are $FRSs$ in X, then

(i) $A \subset B$ and $C \subset D$ implies $A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$,

(ii) $A \subset B$ and $B \subset C$ implies $A \subset C$,

(iii) $A \cap B \subset A,\ B \subset A \cup B$,

(iv) $A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)$ and $A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$,

(v) $A \subset B \Rightarrow \bar{A} \supset \bar{B}$,

(vi) $\overline{\bigcup_i B_i} = \bigcap_i \bar{B}_i$ and $\overline{\bigcap_i B_i} = \bigcup_i \bar{B}_i$.

Theorem 2.2.[17] If A be any FRS in X, $\tilde{0} = (\tilde{0}_L, \tilde{0}_U)$ be the null FRS and $\tilde{1} = (\tilde{1}_L, \tilde{1}_U)$ be the whole FRS in X, then (i) $\tilde{0} \subset A \subset \tilde{1}$ and (ii) $\bar{\tilde{0}} = \tilde{1},\ \bar{\tilde{1}} = \tilde{0}$.

Notation 2.1. Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : V \to V_1$ be a mapping. If $f(\lambda) \in \mathcal{B}_1,\ \forall \lambda \in \mathcal{B}$, then f maps $(V, \mathcal{B})$ to $(V_1, \mathcal{B}_1)$ and it is denoted by $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$. If $f^{-1}(\mu) \in \mathcal{B},\ \forall \mu \in \mathcal{B}_1$, then it is denoted by $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Definition 2.4.[17] Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$. Let $A = (A_L, A_U)$ be a FRS in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L)$, $Y_U = f(X_U)$. The image of A under f, denoted by $f(A) = (f(A_L),\ f(A_U))$ and is defined by $f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L$ and $f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\},\ \forall y \in Y_U$.

Next let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Let $B = (B_L, B_U)$ be a FRS in Y, where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L),\ X_U = f^{-1}(Y_U)$. Then the inverse image of B, under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ and is defined by $f^{-1}(B_L)(x) = B_L(f(x)),\ \forall x \in X_L$ and $f^{-1}(B_U)(x) = B_U(f(x)),\ \forall x \in X_U$.

Theorem 2.3.[17] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then for all $FRSs$ A, A_1 and A_2 in X, we have

(i) $f(\bar{A}) \supset \overline{f(A)}$,

(ii) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

Theorem 2.4.[17] If $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $FRSs$ B, $B_i,\ i \in J$ in Y we have

(i) $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$,

(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any FRS C in Z where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set. gof is the composition of g and f,

(iv) $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$,

(v) $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.

Theorem 2.5.[17] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRS A in X and B in Y, we have

(i) $B = f(f^{-1}(B))$,

(ii) $A \subset f^{-1}(f(A))$.

Definition 2.5.[17] If A and B are two $FRSs$ in X with $B \subset \bar{A}$ and $A \subset \bar{B}$, then the ordered pair (A, B) is called an intuitionistic fuzzy rough set (briefly $IFRS$) in X.

The condition $A \subset \bar{B}$ and $B \subset \bar{A}$ are called intuitionistic condition (briefly IC).

Definition 2.6.[17] Let $P = (A, B)$ and $Q = (C, D)$ be two $IFRSs$ in X. Then

(i) $P \subset Q$ if $A \subset C$ and $B \supset D$,

(ii) $P = Q$ if $P \subset Q$ and $Q \subset P$,

(iii) The complement of P in X, denoted by P', is defined by $P' = (B, A)$,

(iv) For $IFRSs$ $P_i = (A_i, B_i)$ in X, $i \in J$, define

$$\bigcup_{i \in J} P_i = (\bigcup_{i \in J} A_i, \bigcap_{i \in J} B_i) \text{ and } \bigcap_{i \in J} P_i = (\bigcap_{i \in J} A_i, \bigcup_{i \in J} B_i).$$

Theorem 2.6.[17] Let $P = (A, B),\ Q = (C, D),\ R = (E, F)$ and $P_i = (A_i, B_i),\ i \in J$ be $IFRSs$ in X, then

(i) $P \cap P = P = P \cup P$,

(ii) $P \cap Q = Q \cap P$; $P \cup Q = Q \cup P$,

(iii) $(P \cap Q) \cap R = P \cap (Q \cap R)$; $(P \cup Q) \cup R = P \cup\ (Q \cup R)$,

(iv) $P \cap Q \subset P,\ Q \subset P \cup Q$,

(v) $P \subset Q$ and $Q \subset R \Rightarrow P \subset R$,

(vi) $P_i \subset Q,\ \forall i \in J \Rightarrow \bigcup_{i \in J} P_i \subset Q$,

(vii) $Q \subset P_i,\ \forall i \in J \Rightarrow Q \subset \bigcap_{i\in J} P_i$,

(viii) $Q \cup (\bigcap_{i\in J} P_i) = \bigcap_{i\in J}(Q \cup P_i)$,

(ix) $Q \cap (\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(Q \cap P_i)$,

(x) $(P')' = P$,

(xi) $P \subset Q \Leftrightarrow Q' \subset P'$,

(xii) $(\bigcup_{i\in J} P_i)' = \bigcap_{i\in J} P'$ and $(\bigcap_{i\in J} P_i)' = \bigcup_{i\in J} P_i'$.

Definition 2.7.[17] $0^* = (\tilde{0}, \tilde{1})$ and $1^* = (\tilde{1}, \tilde{0})$ are respectively called null $IFRS$ and whole $IFRS$ in X. Clearly $(0^*)' = 1^*$ and $(1^*)' = 0^*$.

Theorem 2.7.[17] If P be any $IFRS$ in X, then $0^* \subset P \subset 1^*$.

Slightly changing the definition of the image of an $IFRS$ under f given by Samanta and Mondal [17] we have given the following :

Definition 2.8.[11] Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Let $P = (A, B)$ be an $IFRS$ in $X(= (X_L, X_U))$ and $Y = f(X) \in \mathcal{B}_1^2$, where $Y_L = f(X_L)$ and $Y_U = f(X_U)$.

Then we define image of P, under f by $f(P) = (\breve{f}(A), \hat{f}(B))$, where $\breve{f}(A) = (f(A_L), f(A_U))$, $A = (A_L, A_U)$ and $\hat{f}(B) = (C_L, C_U)$ (where $B = (B_L, B_U)$) is defined by

$$C_L(y) = \wedge\{B_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L,$$

$$C_U(y) = \begin{cases} \wedge\{B_U(x) : x \in X_L \cap f^{-1}(y)\}, & \forall y \in Y_L, \\ \wedge\{B_U(x) : x \in X_U \cap f^{-1}(y)\}, & \forall y \in Y_U - Y_L. \end{cases}$$

Definition 2.9.[17] Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Let $Q = (C, D)$ be an $IFRS$ in Y, where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L)$ and $X_U = f^{-1}(Y_U)$.

Then the inverse image $f^{-1}(Q)$ of Q, under f, is defined by $f^{-1}(Q) = (f^{-1}(C), f^{-1}(D))$, where $f^{-1}(C) = (f^{-1}(C_L), f^{-1}(C_U))$ and $f^{-1}(D) = (f^{-1}(D_L), f^{-1}(D_U))$.

The following three Theorems (2.8, 2.9, 2.10) of Samanta and Mondal [17] are also valid for this modified definition of functional image of an $IFRS$.

Theorem 2.8. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Then for all $IFRSs$ P and Q, we have

(i) $f(P') \supset (f(P))'$,

(ii) $P \subset Q \Rightarrow f(P) \subset f(Q)$.

Theorem 2.9. Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $IFRSs$ R, S and $R_i,\ i \in J$ in Y,

(i) $f^{-1}(R') = (f^{-1}(R))'$,

(ii) $R \subset S \Rightarrow f^{-1}(R) \subset f^{-1}(S)$,

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ for any $IFRS$ W in Z, where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set, gof is the composition of g and f,

(iv) $f^{-1}(\bigcup_{i\in J} R_i) = \bigcup_{i\in J} f^{-1}(R_i)$,

(v) $f^{-1}(\bigcap_{i\in J} R_i) = \bigcap_{i\in J} f^{-1}(R_i)$.

Theorem 2.10. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $IFRS$ P in X and R in Y, we have

(i) $R = f(f^{-1}(R))$,

(ii) $P \subset f^{-1}(f(P))$.

Theorem 2.11.[11] If P and Q be two $IFRSs$ in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cup Q) = f(P) \cup f(Q)$.

Theorem 2.12.[11] If P, Q be two $IFRSs$ in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cap Q) \subset f(P) \cap f(Q)$.

Note 2.1. If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be one-one, then clearly $f(P \cap Q) = f(P) \cap f(Q)$. But in general $f(P \cap Q) \neq f(P) \cap f(Q)$.

Definition 2.10.[11] Let $X = (X_L, X_U)$ be a rough set and τ be a family of $IFRSs$ in X such that

(i) $0^*, 1^* \in \tau$,

(ii) $P \cap Q \in \tau,\ \forall P, Q \in \tau$,

(iii) $P_i \in \tau,\ i \in \Delta \Rightarrow \bigcup_{i \in \Delta} P_i \in \tau$. Then τ is called a topology of $IFRSs$ in X and (X, τ) is called a topological space of $IFRSs$ in X.

Every member of τ is called open $IFRS$. An $IFRS$ C is called closed $IFRS$ if $C' \in \tau$. Let $\mathcal{F}$ denote the collection of all closed $IFRSs$ in (X, τ). If $\tau_I = \{0^*, 1^*\}$, then τ_I is a topology of $IFRSs$ in X. This topology is called the indiscrete topology. The discrete topology of $IFRSs$ in X contains all the $IFRSs$ in X.

Theorem 2.13.[11] The collection $\mathcal{F}$ of all closed $IFRSs$ satisfies the following properties:

(i) $0^*, 1^* \in \mathcal{F}$,

(ii) $P, Q \in \mathcal{F} \Rightarrow P \cup Q \in \mathcal{F}$,

(iii) $P_i \in \mathcal{F},\ i \in \Delta \Rightarrow \bigcap_{i \in \Delta} P_i \in \mathcal{F}$.

Definition 2.11.[11] Let P be an $IFRS$ in X. The closure of P in (X, τ), denoted by $cl_\tau P$, is defined by the intersection of all closed $IFRSs$ in (X, τ) containing P. Clearly $cl_\tau P$ is the smallest closed $IFRS$ containing P and P is closed if $P = cl_\tau P$.

Definition 2.12.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then $f : (X, \tau) \to (Y, u)$ is said to be IFR continuous if $f^{-1}(Q) \in \tau,\ \forall Q \in u$.

Unless otherwise stated we consider (X, τ) and (Y, u) be topological spaces of $IFRSs$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Theorem 2.14.[11] The following statements are equivalent:

(i) $f : (X, \tau) \to (Y, u)$ is IFR continuous,

(ii) $f^{-1}(Q)$ is closed $IFRS$ in (X, τ), for every closed $IFRS$ Q in (Y, u),

(iii) $f(cl_\tau P) \subset cl_u(f(P))$, for every $IFRS$ P in X.

Definition 2.13.[11] Let $P \in \mathcal{C}_X$. Then a subfamily T of $\mathcal{C}_X$ is said to be a topology on P if

(i) $Q \in T \Rightarrow Q \subset P$,

(ii) $0^*, P \in T$,

(iii) $P_1, P_2 \in T \Rightarrow P_1 \cap P_2 \in T$,

(iv) $P_i \in T,\ i \in \Delta \Rightarrow \bigcup_{i \in \Delta} P_i \in T$.

Then (P, T) is called a subspace topology of (X, τ).

Theorem 2.15.[11] Let τ be a topology of $IFRSs$ in X and let $P \in \mathcal{C}_X$. Then $\tau_1 = \{P \cap R : R \in \tau\}$ is a topology on P. Every member of τ_1 is called open $IFRS$ in (P, τ_1). If $Q \in \tau_1$, then Q'_P is called a closed $IFRS$ in (P, τ_1), where $Q'_P = P \cap Q'$. We take 0^* as closed $IFRS$ also in (P, τ_1). Let $C_1 = \{Q'_P = P \cap Q' : Q \in \tau_1\} \cup \{0^*\}$.

Theorem 2.16.[11] C_1 is closed under arbitrary intersection and finite union.

Remark 2.1. Clearly the collection $\tau_2 = \{(S'_P)'_P = (P \cap P') \cup S : S \in \tau_1\} \cup \{0^*\}$ form a topology of $IFRSs$ on P of which C_1 is a family of closed $IFRSs$ [11]. But C_1 is also a family of closed $IFRSs$ in (P, τ_1). Thus $\exists$ two topologies of $IFRSs$ τ_1 and τ_2 on P. τ_1 is called the first subspace topology of (X, τ) on P and τ_2 is called the second subspace topology of (X, τ) on P. We briefly write, τ_1 and τ_2 are first and second topologies respectively on P, where there is no confusion about the topological space (X, τ) of $IFRSs$.

Definition 2.14.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ and $P \in \mathcal{C}_X$. Let τ_1 and u_1 be first topologies on P and $f(P)$ respectively.

Then $f : (P, \tau_1) \to (f(P), u_1)$ is said to be IFR_1 continuous if $P \cap f^{-1}(Q) \in \tau_1,\ \forall Q \in u_1$.

Theorem 2.17.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$ and let τ_1 and u_1 be first topologies on P and $f(P)$ respectively. If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous.

Definition 2.15.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$ and let $\tau_2,\ u_2$ be second topologies on P and $f(P)$ respectively. Then $f : (P, \tau_2) \to (f(P), u_2)$ is said to be IFR_2 continuous if $P \cap (P' \cup f^{-1}(Q)) \in \tau_2,\ \forall Q \in u_2$.

Theorem 2.18.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$. Let τ_1 and u_1 be first topologies on P and $f(P)$ respectively and $\tau_2,\ u_2$ be second topologies on P and $f(P)$ respectively. If $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous, then $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

Corollary 2.1.[11] If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous, where the symbols have usual meaning.

§3. Compactness of an $IFRS$

Compactness in a fuzzy topological space has been studied by several authors including Lowen [14–15]. In this section we study some aspects of compactness of an $IFRS$.

Definition 3.1. Let (X, τ) be a topological space of $IFRSs$ in X and $P \in \mathcal{C}_X$. A family of open $IFRSs$ $\{G_\alpha : \alpha \in \Delta\}$ is said to be an open cover of P if $\bigcup\limits_{\alpha \in \Delta} G_\alpha \supset P$.

Definition 3.2. Let (X, τ) be a topological space of $IFRSs$ in X and $P \in \mathcal{C}_X$. Then P is said to be compact in (X, τ) if every open cover of P in (X, τ) has a finite subcover in (X, τ).

Theorem 3.1. If $f : (X, \tau) \to (Y, u)$ is IFR continuous and P is compact in (X, τ), then $f(P)$ is compact in (Y, u).

The proof is straightforward.

Definition 3.3. Let τ_1 be the first topology on P. Then P is said to be compact in (P, τ_1) if every open cover of P in (P, τ_1) has a finite subcover in (P, τ_1).

Theorem 3.2. P is compact in (X, τ) iff P is compact in (P, τ_1).

Proof. First suppose that P is compact in (X,τ). Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of P in (P,τ_1). Therefore $\bigcup_{\alpha\in\Delta} G_\alpha \supset P$. Since $G_\alpha \in \tau_1$, $\forall\alpha \in \Delta$, we have $G_\alpha = P \cap H_\alpha$ for some $H_\alpha \in \tau$, $\forall\alpha \in \Delta$. Therefore $\bigcup_{\alpha\in\Delta}(P \cap H_\alpha) \supset P$ and hence $\bigcup_{\alpha\in\Delta} H_\alpha \supset P$. Thus $\{H_\alpha : \alpha \in \Delta\}$ form an open cover of P in (X,τ). Since P is compact in (X,τ), $\exists$ a finite subfamily $\{H_{\alpha_1}, H_{\alpha_2}, \cdots, H_{\alpha_n}\}$ of $\{H_\alpha : \alpha \in \Delta\}$ which covers P. i.e., $\bigcup_{i=1}^{n} H_{\alpha_i} \supset P$. Therefore $P \cap (\bigcup_{i=1}^{n} H_{\alpha_i}) \supset P$ and hence $\bigcup_{i=1}^{n}(P \cap H_{\alpha_i}) \supset P$. i.e., $\bigcup_{i=1}^{n} G_{\alpha_i} \supset P$. Thus the open cover $\{G_\alpha : \alpha \in \Delta\}$ of P in (P,τ_1) has a finite subcover $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ in (P,τ_1). Hence P is compact in (P,τ_1). Next suppose that, P is compact in (P,τ_1). Let $\{H_\alpha : \alpha \in \Delta\}$ be an open cover of P in (X,τ). Therefore $\bigcup_{\alpha\in\Delta} H_\alpha \supset P$ and so $P \cap (\bigcup_{\alpha\in\Delta} H_\alpha) \supset P$ and hence $\bigcup_{\alpha\in\Delta}(P \cap H_\alpha) \supset P$. Therefore $\bigcup_{\alpha\in\Delta} G_\alpha \supset P$, where $G_\alpha = P \cap H_\alpha \in \tau_1$, since $H_\alpha \in \tau$. Therefore $\{G_\alpha : \alpha \in \Delta\}$ form an open cover of P in (P,τ_1). Since P is compact in (P,τ_1), $\exists$ a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ of $\{G_\alpha : \alpha \in \Delta\}$ which covers P. i.e., $\bigcup_{i=1}^{n} G_{\alpha_i} \supset P$. Therefore $\bigcup_{i=1}^{n}(P \cap H_{\alpha_i}) \supset P$ and hence $\bigcup_{i=1}^{n} H_{\alpha_i} \supset P$. Thus $\{H_{\alpha_1}, H_{\alpha_2}, \cdots, H_{\alpha_n}\}$ is a finite subcover of P in (X,τ) and hence P is compact in (X,τ).

Definition 3.4. Let τ_2 be the second topology on P. Then P is said to be compact in (P,τ_2) if every open cover of P in (P,τ_2) has a finite subcover in (P,τ_2).

Note 3.1. If P is compact in (X,τ), then P may not be compact in (P,τ_2). This is shown by the following Example.

Example 3.1. Let $X_L = \{x_1,\ x_2, \cdots\}$, $X_U = X_L \cup \{a\}$, where $a \notin X_L$ and $X = (X_L, X_U)$. Let $P_i = (A_i, B_i)$ $(i = 1, 2, \cdots)$, where $A_i,\ B_i$ are FRS in X defined by

$$A_{iL}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$A_{iU}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$A_{iU}(a) = 0.4.$$

$$B_{iL}(x_j) = \begin{cases} 0.3, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$B_{iU}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$B_{iU}(a) = 0.6.$$

Clearly each P_i is an $IFRS$ in X.

Let τ be the topology of $IFRSs$ in X generated by $P_i,\ i = 1, 2, 3, \cdots$. Now $\cup_i P_i = ((\{x_i/0.4\},\ \{x_i/0.4, a/0.4\}), (\{x_i/0.3\},\ \{x_i/0.4, a/0.6\}))$. Let $P = ((\{x_i/0.4\},\ \{x_i/0.4, a/0.5\}), (\{x_i/0.35\},\ \{x_i/0.4, a/0.5\}))$. Therefore every open cover of P in (X, τ) must contain 1^* and hence every open cover of P in (X, τ) has a finite subcover, say $\{1^*\}$ and hence P is compact in (X, τ). Let $P \cap (P' \cup P_i) = Q_i = (C_i, D_i)$. So $C_i = A \cap (B \cup A_i),\ D_i = B \cup (A \cap B_i)$ [Taking $P = (A, B)$]. Therefore $\{P \cap (P' \cup P_i) : i = 1, 2, \cdots\}$ forms an open cover of P in (P, τ_2), since $\bigcup_{i=1}^{\infty}(P \cap (P' \cup P_i)) \supset P$. But no finite subfamily of $\{P \cap (P' \cup P_i) : i = 1, 2, \cdots\}$ can cover P. Hence P can not be compact in (P, τ_2).

Note 3.2. Following example shows that, if P is compact in (P, τ_2), then P may not be compact in (X, τ).

Example 3.2. Let $X_L = X_U = \{x_1, x_2, \cdots\}$ and $X = (X_L, X_U)$. Let $A_i,\ B_i,\ (i = 1, 2, \cdots)$ are $FRSs$ in X defined by

$$A_{iL}(x_j) = A_{iU}(x_j) = \begin{cases} \frac{1}{2}, & i = j, \\ \frac{1}{3}, & i \neq j. \end{cases}$$

$$B_{iL}(x_j) = B_{iU}(x_j) = \begin{cases} \frac{1}{2}, & i = j, \\ \frac{2}{3}, & i \neq j. \end{cases}$$

Clearly each $P_i = (A_i, B_i)$ is an $IFRS$ in X. Let τ be the topology of $IFRSs$ in X generated by $P_i,\ i = 1, 2, \cdots$. Let $P = ((\{x_i/0.5\}, \{x_i/0.5\}), (\{x_i/0.5\}, \{x_i/0.5\}))$. Clearly P is an $IFRS$ in X. Note that $\bigcup_{i=1}^{\infty} P_i \supset P$. Therefore $\{P_i : i = 1, 2, \cdots\}$ is an open cover of P in (X, τ), but no finite subfamily of $\{P_i : i = 1, 2, \cdots\}$ can cover P. Thus P is not compact in (X, τ). Now $\tau_2 = \{(P \cap P') \cup R : R \in \tau_1\} \cup \{0^*\} = \{P, 0^*\}$, since $R \subset P = P \cap P',\ \forall R \in \tau_1$. Clearly P is compact in (P, τ_2). Thus the above example shows that, if P is compact in (P, τ_2), then P may not be compact in (X, τ).

Theorem 3.3. If $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous and P is compact in (P, τ_1), then $f(P)$ is compact in $(f(P), u_1)$. The proof is straightforward.

Note 3.3. If $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous and P is compact in (P, τ_2), then $f(P)$ may not be compact in $(f(P), u_2)$. This can be shown by the following example.

Example 3.3. Let $V = \{x_i, z_i : i = 1, 2, \cdots\},\ X_L = X_U = V,\ X = (X_L, X_U),\ V_1 = \{y_i : i = 1, 2, \cdots\},\ Y_L = Y_U = V_1,\ Y = (Y_L, Y_U)$. Let $f : V \to V_1$ be defined by $f(x_i) = f(z_i) = y_i,\ \forall i = 1, 2, \cdots$. Clearly $f(X_L) = Y_L,\ f(X_U) = Y_U,\ f^{-1}(Y_L) = X_L,\ f^{-1}(Y_U) = X_U$. Let $P_i = (A_i, B_i)$ be $IFRS$ in X $(i = 1, 2, \cdots)$, where A_i, B_i $(i = 1, 2, \cdots)$ are $FRSs$ in X defined by

$$A_{iL}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$A_{iL}(z_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$A_{iU}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$A_{iU}(z_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$B_{iL}(x_j) = \begin{cases} 0.3, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$B_{iL}(z_j) = \begin{cases} 0.3, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$B_{iU}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$B_{iU}(z_j) = \begin{cases} 0.4, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

Let τ be the topology of $IFRSs$ in $X = (X_L, X_U)$ generated by P_i, $i = 1, 2, \cdots$. Let $P = ((\{x_i/0.35, z_i/0.4\}, \{x_i/0.35, z_i/0.4\}), (\{x_i/0.35, z_i/0.4\}, \{x_i/0.4, z_i/0.4\}))$. Let τ_2 be the second topology on P. It is easy to check that $P \cap (P' \cup P_i) = P$, $\forall i$. Thus $\tau_2 = \{0^*, P \cap P', P\}$. Clearly P is compact in (P, τ_2). Let $Q_i = (C_i, D_i)$ $(i = 1, 2, \cdots)$ be $IFRSs$ in $Y = (Y_L, Y_U)$ defined by

$$C_{iL}(y_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$C_{iU}(y_j) = \begin{cases} 0.4, & i = j, \\ 0.3, & i \neq j. \end{cases}$$

$$D_{iL}(y_j) = \begin{cases} 0.3, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

$$D_{iU}(y_j) = \begin{cases} 0.4, & i = j, \\ 0.6, & i \neq j. \end{cases}$$

Let u be the topology of $IFRSs$ in Y generated by Q_i , $i = 1, 2, \cdots$. Now $f(P) = ((\{y_i/0.4\}, \{y_i/0.4\}), (\{y_i/0.35\}, \{y_i/0.4\}))$. Let u_2 be the second topology on $f(P)$. Let $f(P) \cap ((f(P))' \cup Q_i) = (R_i, S_i)$. Therefore

$$R_{iL}(y_j) = \begin{cases} 0.4, & i = j, \\ 0.35, & i \neq j. \end{cases}$$

$$R_{iU}(y_j) = 0.4, \ \forall i, j.$$

$$S_{iL}(y_j) = \begin{cases} 0.35, & i = j, \\ 0.4, & i \neq j. \end{cases}$$

$$S_{iU}(y_j) = 0.4,\ \forall i, j.$$

Clearly $\{(R_i, S_i) : i = 1, 2, \cdots\}$ forms an open cover of $f(P)$ in $(f(P), u_2)$, but no finite subfamily of $\{(R_i, S_i) : i = 1, 2, \cdots\}$ form an open cover of $f(P)$ in $(f(P), u_2)$. Thus $f(P)$ is not compact in $(f(P), u_2)$. Also $f^{-1}(Q_i) = P_i$. Since $f^{-1}(\bigcup_i Q_i) = \bigcup_i f^{-1}(Q_i) = \bigcup_i P_i$ and $f^{-1}(\bigcap_i Q_i) = \bigcap_i f^{-1}(Q_i) = \bigcap_i P_i$, it follows that $f : (X, \tau) \to (Y, u)$ is IFR continuous and hence $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

Definition 3.5. Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of P in (P, τ_2). Then $\bigcup_{\alpha\in\Delta} H_\alpha$, where $P \cap (P' \cup H_\alpha) = G_\alpha, H_\alpha \in \tau, \alpha \in \Delta$, is called a kernel of the cover.

Remark 3.1. An open cover may admit two different kernels, which can be shown by the following example.

Example 3.4. Let $V = \{x_i, y_i : i = 1, 2, 3, \cdots\}$, $X_L = X_U = V, X = (X_L, X_U)$. Let $P_i = (A_i, B_i), i = 1, 2, 3, \cdots$, where $A_i, B_i (i = 1, 2, 3, \cdots)$ are $FRSs$ in X, defined by

$$A_{iL}(x_j) = \begin{cases} 0.4, & i = j, \\ 0.25, & i \neq j. \end{cases}$$

$$A_{iU}(x_j) = 0.4 = A_{iU}(y_j),\ \forall i, j = 1, 2, \cdots$$

$$A_{iL}(y_j) = 0.4,\ \forall i, j = 1, 2, \cdots$$

$$B_{iL}(x_j) = \begin{cases} 0.25, & i = j, \\ 0.4, & i \neq j. \end{cases}$$

$$B_{iU}(x_j) = 0.4 = B_{iU}(y_j),\ \forall i, j = 1, 2, \cdots$$

$$B_{iL}(y_j) = 0.4,\ \forall i, j = 1, 2, \cdots$$

Let $Q_i = (C_i, D_i)(i = 1, 2, 3, \cdots)$, where $C_i, D_i(i = 1, 2, 3, \cdots)$ are $FRSs$ in X, defined by

$$C_{iL}(x_j) = \begin{cases} 0.35, & i = j, \\ 0.25, & i \neq j. \end{cases}$$

$$C_{iU}(x_j) = 0.4 = C_{iU}(y_j),\ \forall i, j = 1, 2, \cdots$$

$$C_{iL}(y_j) = 0.4,\ \forall i, j = 1, 2, \cdots$$

$$D_{iL}(x_j) = \begin{cases} 0.25, & i = j, \\ 0.4, & i \neq j. \end{cases}$$

$$D_{iU}(x_j) = 0.4 = D_{iU}(y_j),\ \forall i, j = 1, 2, \cdots$$

$$D_{iL}(y_j) = 0.4,\ \forall i, j = 1, 2, \cdots$$

Clearly $P_i, Q_i \in \mathcal{C}_X$. Let τ be the topology of $IFRSs$ in X generated by $P_i, Q_i (i = 1, 2, \cdots)$. Let $P = ((\{x_i/0.35, y_i/0.4\}, \{x_i/0.35, y_i/0.4\}), (\{x_i/0.25, y_i/0.4\}, \{x_i/0.4, y_i/0.4\}))$. Then $P \cap (P' \cup P_i) = (E_i, F_i)$ is defined by

$$E_{iL}(x_j) = \begin{cases} 0.35, & i = j, \\ 0.25, & i \neq j. \end{cases}$$

$$E_{iU}(x_j) = 0.35, \ \forall i, j = 1, 2, \cdots$$

$$E_{iL}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$E_{iU}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$F_{iL}(x_j) = \begin{cases} 0.25, & i = j, \\ 0.35, & i \neq j. \end{cases}$$

$$F_{iU}(x_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$F_{iL}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$F_{iU}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

Also $P \cap (P' \cup Q_i) = (G_i, H_i)$ is defined by

$$G_{iL}(x_j) = \begin{cases} 0.35, & i = j, \\ 0.25, & i \neq j. \end{cases}$$

$$G_{iU}(x_j) = 0.35, \ \forall i, j = 1, 2, \cdots$$

$$G_{iL}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$G_{iU}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$H_{iL}(x_j) = \begin{cases} 0.25, & i = j, \\ 0.35, & i \neq j. \end{cases}$$

$$H_{iU}(x_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$H_{iL}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

$$H_{iU}(y_j) = 0.4, \ \forall i, j = 1, 2, \cdots$$

Thus $P \cap (P' \cup P_i) = P \cap (P' \cup Q_i), \ \forall i = 1, 2, 3, \cdots$. Clearly $\{P \cap (P' \cup P_i) : i = 1, 2, 3, \cdots\}$ form an open cover of P in (P, τ_2) which has two different kernels $\bigcup\limits_i P_i, \ \bigcup\limits_i Q_i$.

Definition 3.6. An open cover $\{G_\alpha : \alpha \in \Delta\}$ of P in (P, τ_2) is called a (*) cover of P in (P, τ_2) if $\exists$ a kernel of the cover containing $P \cap P'$.

Definition 3.7. P is said to be quasicompact in (P, τ_2), if every (*) cover of P in (P, τ_2) has a finite subcover.

Theorem 3.4. If P is compact in (P, τ_2), then P is quasicompact in (P, τ_2).

The proof is straightforward.

Theorem 3.5. If P is compact in (X, τ), then P is quasicompact in (P, τ_2).

Proof . Let $\{G_\alpha : \alpha \in \Delta\}$ be a (*) cover of P in (P, τ_2) i.e., $\exists H_\alpha \in \tau,\ \alpha \in \Delta$ such that $G_\alpha = P \cap (P' \cup H_\alpha),\ \forall \alpha \in \Delta$ and $\bigcup_{\alpha \in \Delta} G_\alpha \supset P$, $\bigcup_{\alpha \in \Delta} H_\alpha \supset P \cap P'$. Thus $\bigcup_{\alpha \in \Delta} (P \cap (P' \cup H_\alpha)) \supset P$ i.e., $P \cap (\bigcup_{\alpha \in \Delta} (P' \cup H_\alpha)) \supset P$. Therefore $P \cap (P' \cup (\bigcup_{\alpha \in \Delta} H_\alpha)) \supset P$ and hence $(P \cap P') \cup (P \cap (\bigcup_{\alpha \in \Delta} H_\alpha)) \supset P$. Thus $P \cap (\bigcup_{\alpha \in \Delta} H_\alpha) \supset P$, since $P \cap (\bigcup_{\alpha \in \Delta} H_\alpha) \supset P \cap P'$. Therefore $\bigcup_{\alpha \in \Delta} H_\alpha \supset P$. Since P is compact in (X, τ), $\exists H_{\alpha_1}, H_{\alpha_2}, \cdots, H_{\alpha_n};\ \alpha_i \in \Delta (i = 1, 2, \cdots, n)$ such that $\bigcup_{i=1}^{n} H_{\alpha_i} \supset P$. Therefore $\bigcup_{i=1}^{n} (P' \cup H_{\alpha_i}) \supset P$ and hence $P \cap (\bigcup_{i=1}^{n} (P' \cup H_{\alpha_i})) \supset P$. Thus $\bigcup_{i=1}^{n} (P \cap (P' \cup H_{\alpha_i})) \supset P$. i.e., $\bigcup_{i=1}^{n} G_{\alpha_i} \supset P$. Thus $\{G_\alpha : \alpha \in \Delta\}$ has a finite subcover of P and hence P is quasicompact in (P, τ_2).

Remark 3.2. The converse of the above theorem is not true. This is supported by Example 3.2, where P is compact (and hence quasicompact) in (P, τ_2), but P is not compact in (X, τ).

Remark 3.3. If P is quasicompact in (P, τ_2) and $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f(P)$ may not be quasicompact in $(f(P), u_2)$. In fact in Example 3.3 , P is compact (and hence quasicompact) in (P, τ_2). Also $\bigcup_i Q_i \supset f(P) \cap (f(P))'$. So $\{(R_i, S_i) : i = 1, 2, \cdots\}$ is a (*) cover of $f(P)$ in $(f(P), u_2)$, but no finite subfamily of $\{(R_i, S_i) : i = 1, 2, \cdots\}$ form an open cover of $f(P)$ in $(f(P), u_2)$. Thus $f(P)$ is not quasicompact in $(f(P), u_2)$.

Definition 3.8. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping and $P \in \mathcal{C}_X$. Then f is said to satisfy complement condition with respect to P, if $P' \cup Q \supset P \Rightarrow (f(P))' \cup f(Q) \supset f(P),\ \forall Q \in \mathcal{C}_X$.

Theorem 3.6. If $P \in \mathcal{C}_X$ such that $f(P') = (f(P))'$, then f satisfies complement condition with respect to P.

The proof is straightforward.

Remark 3.4. The converse of the Theorem 3.6 is not true, which can be shown by the following example.

Example 3.5. Let $V = \{x, y\},\ X_L = X_U = V,\ X = (X_L, X_U);\ V_1 = \{a\},\ Y_L = Y_U = V_1,\ Y = (Y_L, Y_U)$. Let $f : V \to V_1$ be defined by $f(x) = f(y) = a$. Let $P = ((\{x/0.4, y/0.3\}, \{x/0.5, y/0.3\}), (\{x/0.3, y/0.4\}, \{x/0.4, y/0.4\}))$. Clearly $P \in \mathcal{C}_X$. Let $Q = (C, D)$ be an $IFRS$ in X such that $P' \cup Q \supset P$. Then $0.3 \vee C_L(x) \geq 0.4,\ 0.4 \vee C_L(y) \geq 0.3,\ 0.4 \vee C_U(x) \geq 0.5,\ 0.4 \vee C_U(y) \geq 0.3,\ 0.4 \wedge D_L(x) \leq 0.3,\ 0.3 \wedge D_L(y) \leq 0.4,\ 0.5 \wedge D_U(x) \leq 0.4,\ 0.3 \wedge D_U(y) \leq 0.4$. Thus $C_L(x) \geq 0.4, C_U(x) \geq 0.5,\ D_L(x) \leq 0.3, D_U(x) \leq 0.4$. ....(i).

Now $f(P) = ((\{a/0.4\}, \{a/0.5\}), (\{a/0.3\}, \{a/0.4\}))$. Therefore $(f(P))' = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.4\}, \{a/0.5\}))$. Clearly $(f(P))' \cup f(Q) \supset f(P)$ [by (i)]. Thus f satisfies complement condition with respect to P. Now $P' = ((\{x/0.3, y/0.4\}, \{x/0.4, y/0.4\}), (\{x/0.4, y/0.3\}, \{x/0.5, y/0.3\}))$. So $f(P') = ((\{a/0.4\}, \{a/0.4\}), (\{a/0.3\}, \{a/0.3\}))$. Thus $f(P') \neq (f(P))'$. Thus f satisfies complement condition with respect to P, but $f(P') \neq (f(P))'$.

Theorem 3.7. Let P be compact in (P, τ_2) and $f : (P, \tau_2) \to (f(P), u_2)$ be IFR_2 continuous satisfying complement condition with respect to P, then $f(P)$ is compact in $(f(P), u_2)$.

Proof . Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of $f(P)$ in $(f(P), u_2)$. Thus $\exists H_\alpha \in u,\ \alpha \in \Delta$ such that $G_\alpha = f(P) \cap ((f(P))' \cup H_\alpha),\ \forall \alpha \in \Delta$ and $\bigcup_{\alpha\in\Delta} G_\alpha \supset f(P)$. Now $f^{-1}(G_\alpha) = f^{-1}(f(P)) \cap (f^{-1}((f(P))') \cup f^{-1}(H_\alpha))$. Therefore $P \cap f^{-1}(G_\alpha) = P \cap ((f^{-1}(f(P)))' \cup f^{-1}(H_\alpha))$ $= (P \cap (f^{-1}(f(P)))') \cup (P \cap f^{-1}(H_\alpha))$. Therefore $(P \cap P') \cup (P \cap f^{-1}(G_\alpha)) = (P \cap P') \cup (P \cap f^{-1}(H_\alpha))$. Thus $P \cap (P' \cup f^{-1}(G_\alpha)) = P \cap (P' \cup f^{-1}(H_\alpha))$. .... (i).

Since $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous, $P \cap (P' \cup f^{-1}(G_\alpha)) \in \tau_2,\ \forall \alpha \in \Delta$ and hence $P \cap (P' \cup f^{-1}(H_\alpha)) \in \tau_2,\ \forall \alpha \in \Delta$. Since $\bigcup_{\alpha\in\Delta} G_\alpha \supset f(P),\ f^{-1}(\bigcup_{\alpha\in\Delta} G_\alpha) \supset f^{-1}(f(P)) \supset P$. Therefore $\bigcup_{\alpha\in\Delta} f^{-1}(G_\alpha) \supset P$ and hence $\bigcup_{\alpha\in\Delta}(P' \cup f^{-1}(G_\alpha)) \supset P$. Thus $P \cap (\bigcup_{\alpha\in\Delta}(P' \cup f^{-1}(G_\alpha))) \supset P$. i.e., $\bigcup_{\alpha\in\Delta}(P \cap (P' \cup f^{-1}(G_\alpha))) \supset P$. Therefore $\bigcup_{\alpha\in\Delta}(P \cap (P' \cup f^{-1}(H_\alpha))) \supset P$ [By (i)]. Thus $\{P \cap (P' \cup f^{-1}(H_\alpha)) : \alpha \in \Delta\}$ be an open cover of P in (P, τ_2). Since P is compact in (P, τ_2), $\exists \alpha_1, \alpha_2, \cdots, \alpha_n \in \Delta$ such that $\bigcup_{i=1}^{n}(P \cap (P' \cup f^{-1}(H_{\alpha_i}))) \supset P$. Therefore $\bigcup_{i=1}^{n}(P' \cup f^{-1}(H_{\alpha_i})) \supset P$. So $P' \cup (\bigcup_{i=1}^{n} f^{-1}(H_{\alpha_i})) \supset P$. Therefore $(f(P))' \cup f(\bigcup_{i=1}^{n} f^{-1}(H_{\alpha_i})) \supset f(P)$, since f satisfies complement condition with respect to P. Therefore $(f(P))' \cup (\bigcup_{i=1}^{n} f(f^{-1}(H_{\alpha_i}))) \supset f(P)$. So $(f(P))' \cup (\bigcup_{i=1}^{n} H_{\alpha_i}) \supset f(P)$. Therefore $\bigcup_{i=1}^{n}((f(P))' \cup H_{\alpha_i}) \supset f(P)$. Therefore $f(P) \cap (\bigcup_{i=1}^{n}((f(P))' \cup H_{\alpha_i})) \supset f(P)$ and hence $\bigcup_{i=1}^{n}(f(P) \cap ((f(P))' \cup H_{\alpha_i})) \supset f(P)$. i.e., $\bigcup_{i=1}^{n} G_{\alpha_i} \supset f(P)$.

Thus $\{G_\alpha : \alpha \in \Delta\}$ has a finite subcover $\{G_{\alpha_i} : i = 1, 2, \cdots n\}$ for $f(P)$ in $(f(P), u_2)$. Hence $f(P)$ is compact in $(f(P), u_2)$.

References

[1] N. Ajmal and J. K. Kohli, Connectedness in fuzzy topological spaces, Fuzzy Sets and Systems, **31**(1989), 369-388.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, **20**(1986), 87-96.

[3] M. K. Chakroborty and T. M. G. Ahsanullah, Fuzzy topology on fuzzy sets and tolerance topology, Fuzzy Sets and Systems, **45**(1992), 103-108.

[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., **24**(1968), 182-190.

[5] K. C. Chattopadhyay and S. K. Samanta, Fuzzy topology : Fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets and Systems, **54**(1993), 207-212.

[6] A. K. Chaudhuri and P. Das, Fuzzy connected sets in fuzzy topological spaces, Fuzzy Sets and Systems, **49**(1992), 223-229.

[7] Doğan Çoker, Fuzzy rough sets and Intuitionistic L-fuzzy sets, Fuzzy Sets and Systems, **96**(1998), 381-383.

[8] D. Çoker and S. ÖzcaĞ, On connectedness in intuitionistic fuzzy topological spaces, to appear in J. Math. and Math. Sci.

[9] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General systems, **17**(1990), 191-209.

[10] T. E. Gantner, R. C. Steinlage and W. H. Warren, Compactness in fuzzy topological spaces, J. Math. Anal. Appl., **62**(1978), 547-562.

[11] H. Hazra, S. K. Samanta and K. C. Chattopadhyay, Topology of intuitionistic fuzzy rough sets, Scientia Magna, **7**(2011), No. 3, 18-30.

[12] H. Hazra and S. K. Samanta, Connectedness in topology of intuitionistic fuzzy rough sets, Scientia Magna, **9**(2013), No. 2, 1-12.

[13] H. Hazra, T. K. Mondal and S. K. Samanta, Connectedness in topology of intuitionistic fuzzy sets, Advanced studies in contemporary Mathematics, **7**(2003), No. 2, 119-134.

[14] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl., **50**(1976), 621-623.

[15] R. Lowen, A comparision of different compactness notions in fuzzy topological spaces, J. Math. Anal. Appl., **64**(1978), 446-454.

[16] P. P. Ming and L. Y. Ming, Fuzzy topology I, Neighbourhood structure of a fuzzy point and Moore- Smith Convergence, J. Math. Anal. Appl., **76**(1980), 571-599.

[17] T. K. Mondal and S. K. Samanta, Intuitionistic fuzzy rough sets and Rough intuitionistic fuzzy sets, The Journal of fuzzy Mathematics, **9**(2001), No. 3, 561-582.

[18] T. K. Mondal and S. K. Samanta , Topology of interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, **119**(2001), 483-494.

[19] T. K. Mondal and S. K. Samanta , Topology of interval-valued fuzzy sets, Indian J. Pure and Appl. Math., **30**(1999), No. 1, 23-38.

[20] S. Nanda and S. Majumdar, Fuzzy rough sets , Fuzzy Sets and Systems, **45**(1992), 157-160.

[21] Z. Pawlak, Rough sets , Internat. J. Inform. Comput. Sci., **11**(1982), No. 5, 341-356.

[22] S. Saha, Local connectedness in fuzzy setting, Simon Stevin, **61**(1987), 3-13.

[23] L. A. Zadeh, Fuzzy sets, Information and control, **8**(1965), 338-353.

Scientia Magna
Vol. 9 (2013), No. 4, 92-97

On right circulant matrices with trigonometric sequences

A. C. F. Bueno

Department of Mathematics and Physics, Central Luzon State University, Science City of Muñoz 3120, Nueva Ecija, Philippines

Abstract In this paper, the eigenvalues and the upper bounds for spectral norm and Euclidean norm of right circulant matrices with sine and cosine sequences were derived.

Keywords right circulant matrix, sine and cosine sequences.

1 Introduction and preliminaries

In [1] and [2] the formulae for the determinant, eigenvalues, Euclidean norm, spectral norm and inverse of the right circulant matrices with arithmetic and geometric sequences were derived. The said right circulant matrices are as follows:

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & ... & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a. \end{pmatrix}$$

$$RCIRC_n(\vec{d}) = \begin{pmatrix} a & a+d & a+2d & ... & a+(n-2)d & a+(n-1)d \\ a+(n-1)d & ad & a+d & ... & a+(n-3)d & a+(n-2)d \\ a+(n-2)d & a+(n-1)d & a & ... & a+(n-4)d & a+(n-3)d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a+2d & a+3d & a+4d & ... & a & a+d \\ a+d & a+2d & a+3d & ... & a+(n-1)d & a \end{pmatrix}$$

with circulant vectors

$$\begin{aligned} \vec{g} &= (a, ar, ar^2, ..., ar^{n-2}, ar^{n-1}), \\ \vec{d} &= (a, a+d, a+2d, ..., a+(n-2)d, a+(n-1)d). \end{aligned}$$

The following are basic identities on sine and cosine:

1. $\sin^2 x + \cos^2 x = 1$,

2. $\sin^2 2x = \frac{1-\cos^2 2x}{2}$,

3. $\cos^2 2x = \frac{1+\cos^2 2x}{2}$,
   The following are identities regarding $\sin kx$ and $\cos kx$ that will be used later:

4. $\sin kx = 2\cos x \sin(k-1)x - \sin(k-2)x$,

5. $\cos kx = 2\cos x \cos(k-1)x - \cos(k-2)x$,

6. $\sum_{k=0}^{n} \sin kx = \frac{\sin\left(\frac{1}{2}nx\right)\sin\left[\frac{1}{2}(n+1)x\right]}{\sin\left(\frac{1}{2}x\right)}$,

7. $\sum_{k=0}^{n} \cos kx = \frac{\cos\left(\frac{1}{2}nx\right)\sin\left[\frac{1}{2}(n+1)x\right]}{\sin\left(\frac{1}{2}x\right)}$.

Definition 1.1. Let the following be the circulant vectors of the right matrices $RCIRC_n(\vec{s})$, $RCIRC_n(\vec{c})$, $RCIRC_n(\vec{t})$ and $RCIRC_n(\vec{h})$:

$$\begin{aligned}
\vec{s} &= (0, \sin\theta, \sin 2\theta, ..., \sin(n-1)\theta), &(1)\\
\vec{c} &= (1, \cos\theta, \cos 2\theta, ..., \cos(n-1)\theta), &(2)\\
\vec{t} &= (0, \sin^2\theta, \sin^2 2\theta, ..., \sin^2(n-1)\theta), &(3)\\
\vec{h} &= (1, \cos^2\theta, \cos^2 2\theta, ..., \cos^2(n-1)\theta). &(4)
\end{aligned}$$

Definition 1.2. Let $\sigma_m,\ \gamma_m,\ \tau_m$ *and* δ_m be the eigenvalues of $RCIRC_n(\vec{s})$, $RCIRC_n(\vec{c})$, $RCIRC_n(\vec{t})$ and $RCIRC_n(\vec{h})$, respectively.

Definition 1.3. Let A be an nxn matrix then the Euclidean norm and spectral norm of A is denoted by $||A||_E$ and $||A||_2$, respectivley. These two are given by

$$\begin{aligned}
\|A\|_E &= \sqrt{\sum_{i,j=0}^{m,n} a_{ij}^2},\\
\|A\|_2 &= max\{|\lambda_m|\},
\end{aligned}$$

where λ_m is an eigenvalue of A.

Lemma 1.1.

$$\sum_{k=0}^{n-1} \omega^{-mk} = 0,$$

where $\omega = e^{2\pi i/n}$.

Proof.

$$\begin{aligned}
\sum_{k=0}^{n-1} \omega^{-mk} &= \frac{1-\omega^{-mkn}}{1-\omega^{-mk}}\\
&= \frac{1-e^{2\pi ik}}{1-\omega^{-mk}}\\
&= 0.
\end{aligned}$$

§2. Main results

Theorem 2.1. The eigenvalues of $RCIRC_n(\vec{s})$ are the following:

$$\begin{aligned}\sigma_0 &= \frac{\sin\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)},\\ \sigma_m &= \sum_{k=0}^{n-1}\left[2\cos\theta\sin(k-1)\theta-\sin(k-2)\theta\right]\omega^{-mk},\end{aligned}$$

where m=1, 2, ..., n-1 and $\omega = e^{2\pi i/n}$.

Proof. For $m = 0$

$$\begin{aligned}\sigma_0 &= \sum_{k=0}^{n-1}\sin k\theta,\\ &= \frac{\sin\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.\end{aligned}$$

For $m \neq 0$

$$\begin{aligned}\sigma_m &= \sum_{k=0}^{n-1}\left[\sin k\theta\right]\omega^{-mk}\\ &= \sum_{k=0}^{n-1}\left[2\cos\theta\sin(k-1)\theta-\sin(k-2)\theta\right]\omega^{-mk}.\end{aligned}$$

Theorem 2.2. The eigenvalues of $RCIRC_n(\vec{s})$ are the following:

$$\begin{aligned}\gamma_0 &= \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}\\ \gamma_m &= \sum_{k=0}^{n-1}\left[2\cos\theta\cos(k-1)\theta-\cos(k-2)\theta\right]\omega^{-mk}.\end{aligned}$$

Proof. For $m = 0$

$$\begin{aligned}\gamma_0 &= \sum_{k=0}^{n-1}\cos k\theta\\ &= \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.\end{aligned}$$

For $m \neq 0$

$$\begin{aligned}\gamma_m &= \sum_{k=0}^{n-1}\left[\cos k\theta\right]\omega^{-mk}\\ &= \sum_{k=0}^{n-1}\left[2\cos\theta\cos(k-1)\theta-\cos(k-2)\theta\right]\omega^{-mk}.\end{aligned}$$

Theorem 2.3. The eigenvalues of $RCIRC_n(\vec{t})$ are the following:

$$\begin{aligned}\tau_0 &= \frac{n\sin\theta - 2\cos[(n-1)\theta]\sin n\theta}{2\sin\theta},\\ \tau_m &= \sum_{k=0}^{n-1}\{\cos[2(n-2)\theta] - \cos 2\theta\cos[2(k-1)\theta]\}\,\omega^{-mk}.\end{aligned}$$

Proof. For $m = 0$

$$\begin{aligned}\tau_0 &= \sum_{k=0}^{n-1}\sin^2 k\theta\\ &= \sum_{k=0}^{n-1}\frac{1-\cos 2k\theta}{2}\\ &= \frac{n}{2} - \sum_{k=0}^{n-1}\cos 2k\theta\\ &= \frac{n\sin\theta - 2\cos[(n-1)\theta]\sin n\theta}{2\sin\theta}.\end{aligned}$$

For $m \neq 0$

$$\begin{aligned}\tau_m &= \sum_{k=0}^{n-1}\left[\sin^2 k\theta\right]\omega^{-mk}\\ &= \sum_{k=0}^{n-1}\frac{1-\cos 2k\theta}{2}\omega^{-mk}\\ &= \frac{1}{2}\sum_{k=0}^{n-1}\omega^{-mk} - \sum_{k=0}^{n-1}[\cos 2k\theta]\,\omega^{-mk}\\ &= \sum_{k=0}^{n-1}\{\cos[2(n-2)\theta] - \cos 2\theta\cos[2(k-1)\theta]\}\,\omega^{-mk}.\end{aligned}$$

via Lemma 1.1 and identity 5.

Theorem 2.4. The eigenvalues of $RCIRC_n(\vec{h})$ are the following:

$$\begin{aligned}\delta_0 &= \frac{n\sin\theta + 2\cos[(n-1)\theta]\sin n\theta}{2\sin\theta},\\ \delta_m &= \sum_{k=0}^{n-1}\{\cos 2\theta\cos[2(k-1)\theta] - \cos[2(n-2)\theta]\}\,\omega^{-mk}.\end{aligned}$$

For $m = 0$

$$\begin{aligned}\tau_0 &= \sum_{k=0}^{n-1}\cos^2 k\theta\\ &= \sum_{k=0}^{n-1}\frac{1+\cos 2k\theta}{2}\\ &= \frac{n}{2} + \sum_{k=0}^{n-1}\cos 2k\theta\\ &= \frac{n\sin\theta + 2\cos[(n-1)\theta]\sin n\theta}{2\sin\theta}.\end{aligned}$$

For $m \neq 0$

$$\begin{aligned}
\tau_m &= \sum_{k=0}^{n-1} \left[\cos^2 k\theta\right] \omega^{-mk} \\
&= \sum_{k=0}^{n-1} \frac{1+\cos 2k\theta}{2} \omega^{-mk} \\
&= \frac{1}{2} \sum_{k=0}^{n-1} \omega^{-mk} + \sum_{k=0}^{n-1} \left[\cos 2k\theta\right] \omega^{-mk} \\
&= \sum_{k=0}^{n-1} \left\{\cos 2\theta \cos\left[2(k-1)\theta\right] - \cos\left[2(n-2)\theta\right]\right\} \omega^{-mk}.
\end{aligned}$$

via Lemma 1.1 and identity 5.

Theorem 2.5. $\|RCIRC_n(\vec{s})\|_2 \leq n-1$ and $\left\|RCIRC_n(\vec{t})\right\|_2 \leq n-1$

Proof. Let $RCIRC_n(\vec{p}) = RCIRC_n(\vec{s}) or RCIRC_n(\vec{t})$ and $\mu_m = \sigma_m or \tau_m$.

$$\begin{aligned}
\|RCIRC_n(\vec{p})\|_2 &= max\left\{|\mu_m|\right\} \\
&= \left|\sum_{k=0}^{n-1} p_k \omega^{-mk}\right| \\
&\leq \sum_{k=0}^{n-1} |p_k|.
\end{aligned}$$

Note that $|\sin k\theta|, |\sin^2 k\theta| \in [0,1]$, so the theorem follows.

Theorem 2.6. $\|RCIRC_n(\vec{c})\|_2 \leq n$ and $\left\|RCIRC_n(\vec{h})\right\|_2 \leq n$.

Proof. Let $RCIRC_n(\vec{q}) = RCIRC_n(\vec{c})$ or $RCIRC_n(\vec{h})$ and $\phi_m = \gamma_m$ or δ_m.

$$\begin{aligned}
\|RCIRC_n(\vec{q})\|_2 &= max\left\{|\phi_m|\right\} \\
&= \left|\sum_{k=0}^{n-1} q_k \omega^{-mk}\right| \\
&\leq \sum_{k=0}^{n-1} |q_k|.
\end{aligned}$$

Note that $|\cos k\theta|, |\cos^2 k\theta| \in [0,1]$, so the theorem follows.

Theorem 2.7. $\|RCIRC_n(\vec{s})\|_E \leq \sqrt{n(n-1)}$ and $\left\|RCIRC_n(\vec{t})\right\|_E \leq \sqrt{n(n-1)}$.

Proof. Let $RCIRC_n(\vec{p}) = RCIRC_n(\vec{s})$ *or* $RCIRC_n(\vec{t})$.

$$\begin{aligned}
\|RCIRC_n(\vec{p})\|_E &= \sqrt{n \sum_{k=0}^{n-1} p_k^2} \\
&\leq \sqrt{n \sum_{k=1}^{n-1} p_k^2} \\
&\leq \sqrt{n(n-1)}.
\end{aligned}$$

Theorem 2.8. $\|RCIRC_n(\vec{c})\|_E \leq n$ and $\left\|RCIRC_n(\vec{h})\right\|_E \leq n$.

Proof. Let $RCIRC_n(\vec{q}) = RCIRC_n(\vec{c})$ *or* $RCIRC_n(\vec{h})$.

$$\begin{aligned}\|RCIRC_n(\vec{q})\|_E &= \sqrt{n\sum_{k=0}^{n-1} q_k^2}\\ &\leq \sqrt{n\sum_{k=0}^{n-1} 1}\\ &= \sqrt{n^2}\\ &= n.\end{aligned}$$

References

[1] M. Bahsi and S. Solak, On the Circulant Matrices with Arithmetic Sequence, Int. J. Contemp. Math. Sciences, **5**(2010), No. 25, 121 - 122.

[2] A.C.F. Bueno, Right Circulant Matrices with Geometric Progression, International Journal of Applied Mathematical Research, Vol. 1, No. 4, 59 - 60.

[3] www.wolfram.mathworld.com/Cosine.html

[4] www.wolfram.mathworld.com/Sine.html

Scientia Magna
Vol. 9 (2013), No. 4, 98-106

Difference cordial labeling of subdivided graphs

R. Ponraj†, S. Sathish Narayanan† and R. Kala‡

† Department of Mathematics,
Sri Paramakalyani College, Alwarkurichi- 627 412, India
‡ Department of Mathematics,
Manonmaniam Sundaranar University, Tirunelveli- 627 012, India
E-mails: ponrajmaths@gmail.com sathishrvss@gmail.com
karthipyi91@yahoo.co.in

Abstract Let G be a (p, q) graph. Let f be a map from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $|f(x) - f(y)|$. f is called a difference cordial labeling if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph. In this paper we investigate the difference cordial labeling behavior of Subdivision of some graphs.

Keywords Subdivision, Web, Wheel, Book, corona.
2010 AMS: 05C78

§1. Introduction

Let $G = (V, E)$ be (p, q) graph. In this paper we have considered only simple and undirected graphs. The number of vertices of G is called the order of G and the number of edges of G is called the size G. The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The corona of G with H, $G \odot H$ is the graph obtained by taking one copy of G and p copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H. Graph labeling plays an important role of numerous fields of sciences and some of them are astronomy, coding theory, x-ray crystallography, radar, circuit design, communication network addressing, database management, secret sharing schemes etc.[3] The graph labeling problem was introduced by Rosa. In 1967 Rosa introduced Graceful labeling of graphs.[13] In 1980, Cahit [1] introduced the concept of Cordial labeling of graphs. Riskin [12], Seoud and Abdel Maqusoud [14], Diab [2], Lee and Liu [5], Vaidya, Ghodasara, Srivastav, and Kaneria [15] were worked in cordial labeling. Ponraj et al. introduced k-Product cordial labeling, [10] k-Total Product cordial labeling [11] recently. Motivated by the above work, R. Ponraj, S. Sathish Narayanan and R. Kala introduced difference cordial labeling of grpahs .[6] In [6,7,8,9] difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web, $B_m \odot K_1$, $T_n \odot K_2$ and some more standard graphs have been investigated. In this paper, we investigate the difference cordial behaviour of $S(K_{1,n})$, $S(K_{2,n})$, $S(W_n)$, $S(P_n \odot K_1)$, $S(W(t, n))$, $S(B_m)$, $S(B_{m,n})$. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms not defined here are used in the sense of Harary [4].

§2. Difference cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, p\}$ be a bijection. For each edge uv, assign the label $|f(u) - f(v)|$. f is called a difference cordial labeling if f is $1-1$ and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

Theorem 2.1. $S(K_{1,n})$ is difference cordial.

Proof. Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$. Define a function $f : V(S(K_{1,n})) \to \{1, 2, \ldots, 2n+1\}$ by

$$\begin{aligned} f(u_i) &= 2i - 1, \quad 1 \leq i \leq n, \\ f(v_i) &= 2i, \quad\quad\ \ 1 \leq i \leq n. \end{aligned}$$

and $f(u) = 2n + 1$. Since $e_f(0) = e_f(1) = n$, f is a difference cordial labeling of $S(K_{1,n})$.

Theorem 2.2. $S(K_{2,n})$ is difference cordial.

Proof. Let $V(S(K_{2,n})) = \{u, v, u_i, v_i, w_i : 1 \leq i \leq n\}$, $E(S(K_{2,n})) = \{uv_i, v_iu_i, vw_i, w_iu_i : 1 \leq i \leq n\}$. Define $f : V(S(K_{2,n})) \to \{1, 2, \ldots, 3n+2\}$ by

$$\begin{aligned} f(u_i) &= 3i - 1, \quad 1 \leq i \leq n, \\ f(v_i) &= 3i - 2, \quad 1 \leq i \leq n, \\ f(w_i) &= 3i, \quad\quad\ \ 1 \leq i \leq n. \end{aligned}$$

$f(u) = 3n + 1$ and $f(v) = 3n + 2$. Since $e_f(0) = e_f(1) = 2n$, f is a difference cordial labeling of $S(K_{2,n})$.

Theorem 2.3. $S(W_n)$ is difference cordial.

Proof. Let $V(S(W_n)) = \{u, u_i, w_i, v_i : 1 \leq i \leq n\}$ and $E(S(W_n)) = \{uv_i, v_iu_i : 1 \leq i \leq n\} \cup \{u_iw_i, w_iu_{i+1}, u_nw_n, w_nu_1 : 1 \leq i \leq n-1\}$. Define $f : V(S(W_n)) \to \{1, 2, \ldots, 3n+1\}$ by

$$\begin{aligned} f(u_i) &= 2i - 1, \quad\quad\ 1 \leq i \leq n, \\ f(w_i) &= 2i, \quad\quad\quad\quad 1 \leq i \leq n, \\ f(v_i) &= 2n + 1 + i, \quad 1 \leq i \leq n. \end{aligned}$$

and $f(u) = 2n + 1$. Since $e_f(0) = e_f(1) = 2n$, $S(W_n)$ is difference cordial.

Theorem 2.4. Let G be a (p, q) graph and if $S(G)$ is difference cordial. Then $S(G \odot mK_1)$ is difference cordial.

Proof. Let f be a difference cordial labeling of $S(G)$. Let $V(S(G)) = \{u_i : 1 \leq i \leq n\}$, $V(S(G \odot mK_1)) = \{v_i^j, w_i^j : 1 \leq i \leq m, 1 \leq j \leq p\} \cup V(S(G))$ and $E(S(G \odot mK_1)) =$

$\{u_iv_i^j, v_i^jw_i^j : 1 \leq i \leq m, 1 \leq j \leq p\} \cup E(S(G))$. Define a 1-1 map $g : V(S(G \odot mK_1)) \rightarrow \{1, 2, \ldots, (m+2)p\}$ by

$$\begin{aligned} g(u_i) &= f(u_i), & 1 \leq i \leq 2p, \\ f(v_i^j) &= 2p + (2j-2)m + 2i, & 1 \leq i \leq m, 1 \leq j \leq p, \\ f(w_i^j) &= 2p + (2j-2)m + 2i - 1, & 1 \leq i \leq m, 1 \leq j \leq p. \end{aligned}$$

Obviously the above labeling g is a difference cordial labeling of $S(G \odot mK_1)$.

Theorem 2.5. $S(P_n \odot K_1)$ is difference cordial.

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and let $V(S(P_n \odot K_1)) = V(P_n \odot K_1) \cup \{u_i' : 1 \leq i \leq n-1\} \cup \{v_i' : 1 \leq i \leq n\}$ and $E(S(P_n \odot K_1)) = \{u_iu_i', u_i'u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i', v_i'v_i : 1 \leq i \leq n\}$. Define $f : V(S(P_n \odot K_1)) \rightarrow \{1, 2, \ldots, 4n-1\}$ by

$$\begin{aligned} f(u_i) &= 2i - 1, & 1 \leq i \leq n, \\ f(u_i') &= 2i, & 1 \leq i \leq n-1, \\ f(v_{n-i+1}') &= 2n - 1 + i, & 1 \leq i \leq n, \\ f(w_{n-i+1}') &= 3n - 1 + i, & 1 \leq i \leq n. \end{aligned}$$

Since $e_f(0) = e_f(1) = 2n - 1$, $S(P_n \odot K_1)$ is difference cordial.

Theorem 2.6. $S(P_n \odot 2K_1)$ is difference cordial.

Proof. Let $V(S(P_n \odot 2K_1)) = \{u_i, v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n-1\}$ and $E(S(P_n \odot 2K_1)) = \{u_iv_i, v_iw_i, u_ix_i, x_iy_i : 1 \leq i \leq n\} \cup \{u_iz_i, z_iu_{i+1} : 1 \leq i \leq n-1\}$. Define $f : V(S(P_n \odot 2K_1)) \rightarrow \{1, 2, \ldots, 6n-1\}$ by

$$\begin{aligned} f(u_i) &= 2i - 1, & 1 \leq i \leq n, \\ f(z_i) &= 2i, & 1 \leq i \leq n-1, \\ f(v_{n-i+1}) &= 2n + 2i - 2, & 1 \leq i \leq n, \\ f(w_{n-i+1}) &= 2n + 2i - 1, & 1 \leq i \leq n, \\ f(x_i) &= 4n + i - 1, & 1 \leq i \leq n, \\ f(y_i) &= 5n + i - 1, & 1 \leq i \leq n. \end{aligned}$$

Since $e_f(0) = e_f(1) = 3n - 1$, f is a difference cordial labeling of $S(P_n \odot 2K_1)$.

The Lotus inside a circle LC_n is a graph obtained from the cycle $C_n : u_1u_2 \ldots u_nu_1$ and a star $K_{1,n}$ with central vertex v_0 and the end vertices $v_1v_2 \ldots v_n$ by joining each v_i to u_i and $u_{i+1 \ (\mathrm{mod}\ n)}$.

Theorem 2.7. $S(LC_n)$ is difference cordial.

Proof. Let the edges $u_iu_{i+1(\mathrm{mod}\ n)}$, v_0v_i, v_iu_i and $v_iu_{i+1(\mathrm{mod}\ n)}$ be subdivided by w_i,

x_i, y_i and z_i respectively. Define a map $f : V(S(LC_n)) \to \{1, 2, \ldots, 6n+1\}$ as follows:

$$\begin{aligned} f(u_i) &= 4i-3, & 1 \le i \le n, \\ f(v_i) &= 4i-1, & 1 \le i \le n, \\ f(w_i) &= 5n+1+i, & 1 \le i \le n, \\ f(x_i) &= 4n+1+i, & 1 \le i \le n, \\ f(y_i) &= 4i-2, & 1 \le i \le n, \\ f(z_i) &= 4i, & 1 \le i \le n. \end{aligned}$$

and $f(v_0) = 4n+1$. Obviously, the above vertex labeling is a difference cordial labeling of $S(LC_n)$.

The graph P_n^2 is obtained from the path P_n by adding edges that joins all vertices u and v with $d(u, v) = 2$.

Theorem 2.8. $S\left(P_n^2\right)$ is difference cordial.

Proof. Let $V(S(P_n^2)) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n-1\} \cup \{w_i : 1 \le i \le n-2\}$ and $E(S(P_n^2)) = \{u_iv_i, v_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_iw_i, w_iu_{i+2} : 1 \le i \le n-2\}$. Define $f : V(S(P_n^2)) \to \{1, 2, \ldots, 3n-3\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, & 1 \le i \le n-1, \\ f(v_i) &= 2i, & 1 \le i \le n-1, \\ f(w_i) &= 2n-1+i, & 2 \le i \le n-2. \end{aligned}$$

$f(u_n) = 2n$ and $f(w_1) = 2n-1$. Since $e_f(0) = e_f(1) = 2n-3$, f is a difference cordial labeling of $S(P_n^2)$.

Theorem 2.9. $S\left(K_2 + mK_1\right)$ is difference cordial.

Proof. Let $V(S(K_2 + mK_1)) = \{u, v, w, u_i, v_i, w_i : 1 \le i \le m\}$ and $E(S(K_2 + mK_1)) = \{uu_i, u_iw_i, w_iv_i, v_iv : 1 \le i \le m\} \cup \{uw, vw\}$. Define an injective map $f : V(S(K_2 + mK_1)) \to \{1, 2, \ldots, 3n+3\}$ as follows:

$$\begin{aligned} f(u_i) &= 3i+1, & 1 \le i \le m, \\ f(v_i) &= 3i+3, & 1 \le i \le m, \\ f(w_i) &= 3i+2, & 2 \le i \le m. \end{aligned}$$

$f(u) = 1$, $f(v) = 2$ and $f(w) = 3$. Since $e_f(0) = e_f(1) = 2m+1$, f is a difference cordial labeling of $S(K_2 + mK_1)$.

The sunflower graph SF_n is obtained by taking a wheel with central vertex v_0 and the cycle $C_n : v_1v_2 \ldots v_nv_1$ and new vertices $w_1w_2 \ldots w_n$ where w_i is joined by vertices v_i, $v_{i+1 \ (\text{mod } n)}$.

Theorem 2.10. Subdivision of a sunflower graph $S(SF_n)$ is difference cordial.

Proof. Let $V(S(SF_n)) = \{u_i, u_i', v_i, w_i, w_i', x_i, u : 1 \le i \le n\}$ and $E(S(SF_n)) = \{ux_i, x_iu_i : 1 \le i \le n\} \cup \{u_iu_i', u_iw_i, w_iv_i, v_iw_i', u_i'u_{i+1 (mod\ n)}, w_i'u_{i+1 (\text{mod } n)} : 1 \le i \le n\}$. Define

$f : V(S(SF_n)) \to \{1, 2, \ldots, 6n+1\}$ by

$$\begin{aligned} f(u_i) &= 4i-3, & 1 \le i \le n, \\ f(u_i^{'}) &= 5n+1+i, & 1 \le i \le n, \\ f(w_i) &= 4i-2, & 1 \le i \le n, \\ f(v_i) &= 4i-1, & 1 \le i \le n, \\ f(w_i^{'}) &= 4i, & 1 \le i \le n, \\ f(x_i) &= 4n+1+i, & 1 \le i \le n. \end{aligned}$$

and $f(u) = 4n+1$. Since $e_f(0) = e_f(1) = 4n$, $S(SF_n)$ is difference cordial.

The helm H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. A flower Fl_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

Theorem 2.11. Subdivision of graph of a flower graph $S(Fl_n)$ is difference cordial.

Proof. Let $V(S(Fl_n)) = \{u, u_i, v_i, w_i, x_i, y_i, z_i : 1 \le i \le n\}$ and $E(S(Fl_n)) = \{u_iy_i, y_iw_i : 1 \le i \le n\} \cup \{w_iz_i, z_iu, ux_i, x_iu_i, u_iv_i, v_iu_{i+1(\text{mod } n)} : 1 \le i \le n\}$. Define $f : V(S(Fl_n)) \to \{1, 2, \ldots, 6n+1\}$ by

$$\begin{aligned} f(u_i) &= 5i-1, & 1 \le i \le n, \\ f(v_i) &= 5n+i, & 1 \le i \le n, \\ f(w_i) &= 5i-3, & 1 \le i \le n, \\ f(x_i) &= 5i, & 1 \le i \le n, \\ f(y_i) &= 5i-2, & 1 \le i \le n, \\ f(z_i) &= 5i-4, & 1 \le i \le n. \end{aligned}$$

and $f(u) = 6n+1$. Since $e_f(0) = e_f(1) = 4n$, $S(Fl_n)$ is difference cordial.

The book B_m is the graph $S_m \times P_2$ where S_m is the star with $m+1$ vertices.

Theorem 2.12. $S(B_m)$ is difference cordial.

Proof. Let $V(S(B_m)) = \{u, v, w\} \cup \{u_i, v_i, w_i, x_i, y_i : 1 \le i \le m\}$ and $E(S(B_m)) = \{uw, vw\} \cup \{ux_i, x_iu_i, u_iw_i, w_iv_i, v_iy_i, y_iv : 1 \le i \le m\}$. Define $f : V(S(B_m)) \to \{1, 2, \ldots, 5m+3\}$ by

$$\begin{aligned} f(u_i) &= 4i, & 1 \le i \le m, \\ f(w_i) &= 4i+1, & 1 \le i \le m, \\ f(v_i) &= 4i+2, & 1 \le i \le m, \\ f(y_i) &= 4i+3, & 1 \le i \le m-1, \\ f(x_i) &= 4m+2+i, & 1 \le i \le m. \end{aligned}$$

$f(u) = 1$, $f(w) = 2$, $f(v) = 3$ and $f(y_m) = 5m+3$. Since $e_f(0) = e_f(1) = 3m+1$, $S(B_m)$ is difference cordial.

Prisms are graphs of the form $C_n \times P_n$.

Theorem 2.13. $S(C_n \times P_2)$ is difference cordial.

Proof. Let $V(S(C_n \times P_2)) = \{u_i, v_i, w_i, x_i, y_i : 1 \le i \le n\}$ and $E(S(C_n \times P_2)) = \{u_ix_i, x_iu_{i+1(\text{mod} n)}, u_iw_i, w_iv_i, v_iy_i, y_iv_{i+1(\text{mod } n)} : 1 \le i \le n\}$. Define a function $f : V(S(C_n \times$

$P_2)) \to \{1, 2, \ldots, 5n\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, & 1 \le i \le n, \\ f(x_i) &= 2i, & 1 \le i \le n, \\ f(w_i) &= 2n+2i-1, & 1 \le i \le n, \\ f(v_i) &= 2n+2i, & 1 \le i \le n, \\ f(y_i) &= 4n+1+i, & 1 \le i \le n-1. \end{aligned}$$

and $f(y_n) = 4n+1$. Since $e_f(0) = e_f(1) = 3n$, f is a difference cordial labeling of $S(C_n \times P_2)$.

The graph obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge is called Bistar $B_{m,n}$.

Theorem 2.14. $S(B_{m,n})$ is difference cordial.

Proof. Let $V(S(B_{m,n})) = \{u, v, w\} \cup \{u_i, x_i : 1 \le i \le m\} \cup \{v_i, y_i : 1 \le i \le n\}$ and $E(S(B_{m,n})) = \{uw, wv\} \cup \{ux_i, x_iu_i : 1 \le i \le m\} \cup \{vy_i, y_iv_i : 1 \le i \le n\}$. Define an injective map $f : V(S(B_{m,n})) \to \{1, 2, \ldots, 2m+2n+3\}$ by

$$\begin{aligned} f(u_i) &= 2i, & 1 \le i \le m, \\ f(v_i) &= 2m+2i, & 1 \le i \le n, \\ f(x_i) &= 2i-1, & 1 \le i \le m, \\ f(y_i) &= 2m+2i-1, & 1 \le i \le n. \end{aligned}$$

$f(u) = 2m+2n+3$, $f(v) = 2m+2n+2$ and $f(w) = 2m+2n+1$. Clearly the above vertex labeling is a difference cordial labeling of $S(B_{m,n})$.

The graph $K_2 \times K_2 \times \cdots \times K_2$ (n copies) is called a n-cube.

Theorem 2.15. Subdivision of a n-cube is difference cordial.

Proof. Let $G = K_2 \times K_2 \times \cdots \times K_2$ (n copies). Let the edges u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1}, w_iw_{i+1}, u_iv_i, v_ix_i, x_iw_i and w_iu_i be subdivided by $u_i^{'}$, $v_i^{'}$, $x_i^{'}$, $w_i^{'}$, y_i, $y_i^{'}$, z_i and $z_i^{'}$ respectively. Define $f : V(S(G)) \to \{1, 2, \ldots, 12n-16\}$ by

$$\begin{aligned} f(u_i) &= 5n+5i-14, & 3 \le i \le n-1, \\ f(u_i^{'}) &= 5n+5i-10, & 2 \le i \le n-2, \\ f(v_i) &= 5n+5i-12, & 3 \le i \le n-1, \\ f(v_i^{'}) &= 5n+5i-6, & 2 \le i \le n-2, \\ f(x_i) &= 5i-4, & 3 \le i \le n-1, \\ f(x_i^{'}) &= 5i+2, & 2 \le i \le n-2, \\ f(w_i) &= 5i-6, & 3 \le i \le n-1, \\ f(w_i^{'}) &= 5i-2, & 2 \le i \le n-2, \\ f(y_i) &= 5n+5i-13, & 3 \le i \le n-1, \\ f(y_i^{'}) &= 10n-16+i, & 1 \le i \le n-1, \\ f(z_i) &= 5i-5, & 3 \le i \le n-1, \\ f(z_i^{'}) &= 11n-17+i, & 1 \le i \le n-1. \end{aligned}$$

$f(u_1) = 5n-7$, $f(u_2) = 5n-5$, $f(u_1^{'}) = 5n-6$, $f(v_1) = 5n-1$, $f(v_2) = 5n-3$, $f(v_1^{'}) = 5n-2$, $f(x_1) = 7$, $f(x_2) = 5$, $f(x_1^{'}) = 6$, $f(w_1) = 1$, $f(w_2) = 3$, $f(w_1^{'}) = 2$, $f(y_1) = 12n-17$, $f(y_2) = 5n-4$, $f(z_1) = 12n-16$ and $f(z_2) = 4$. Since $e_f(0) = e_f(1) = 8n-12$, f is a difference cordial labeling of $S(G)$.

Jelly fish graphs $J(m,n)$ are obtained from a cycle $C_4 : v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending m pendant edges to v_2 and n pendant edges to v_4.

Theorem 2.16. $S\left(J\left(m,n\right)\right)$ is difference cordial.

Proof. Let the edges v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3 be subdivided by u, v, w, x, y respectively and the edges v_2u_i $(1 \le i \le m)$ and v_4w_j $(1 \le j \le n)$ be subdivided by $u_i^{'}$ and $w_j^{'}$ respectively. Define a one-to-one map $f : V(S(J(m,n))) \to \{1, 2, \ldots, 2m+2n+9\}$ by $f(v_1) = 2$, $f(v_2) = 7$, $f(v_3) = 5$, $f(v_4) = 8$, $f(u) = 3$, $f(v) = 6$, $f(w) = 4$, $f(x) = 1$, $f(y) = 9$,

$$\begin{array}{rcll} f(u_i) & = & 2i+8, & 1 \le i \le m, \\ f(u_i^{'}) & = & 2i+9, & 1 \le i \le m, \\ f(w_i) & = & 2m+2i+8, & 1 \le i \le n, \\ f(w_i^{'}) & = & 2m+2i+9, & 1 \le i \le n. \end{array}$$

Clearly $e_f(0) = e_f(1) = m+n+5$. It follows that, f is a difference cordial labeling of $S(J(m,n))$.

The helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle Cn. Koh et al. [3] define a web graph. A web graph is obtained by joining the pendant points of the helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. Yang [3] has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendent point to the new cycle. $W(t,n)$ is the generalized web with t cycles C_n.

Theorem 2.17. $S(W(t,n))$ is difference cordial.

Proof. Let $C_n^{(i)}$ be the cycle $u_1^i, u_2^i \ldots u_n^i u_1^i$. Let $V(W(t,n)) = \bigcup_{i=1}^{t} V(C_n^{(i)}) \cup \{z_i : 1 \le i \le n\} \cup \{u\}$ and $V(S(W(t,n))) = \{v_i^j : 1 \le i \le n, 1 \le j \le t\} \cup \{w_i^j : 1 \le i \le n, 1 \le j \le t+1\} \cup V(W(t,n))$ and $E(S(W(t,n))) = \{u_i^j v_i^j, v_i^j u_{i+1}^j : 1 \le i \le n-1, 1 \le j \le t\} \cup \{u_n^j v_n^j, v_n^j u_1^j : 1 \le j \le t\} \cup \{uw_i^1 : 1 \le i \le n\} \cup \{w_i^j u_i^j, u_i^j w_{i+1}^j : 1 \le j \le t, 1 \le i \le n\} \cup \{w_i^{t+1} z_i : 1 \le i \le n\}$. Define a map $f : V(St(W(t,n))) \to \{1, 2, \ldots, 3nt+2n+1\}$ as follows:

$$\begin{array}{rclll} f(w_1^j) & = & 2j-1, & 1 \le j \le t+1, & \\ f(w_i^j) & = & f(w_{i-1}^t)+2j, & 1 \le j \le t, & 2 \le i \le n, \\ f(u_i^j) & = & f(w_i^j)+1, & 1 \le j \le t, & 2 \le i \le n, \\ f(z_i^j) & = & f(w_i^{t+1})+1, & 1 \le i \le n, & \\ f(v_1^j) & = & n(2t+2)+j, & 1 \le j \le t+1, & \\ f(v_i^j) & = & f(v_{i-1}^t)+j, & 1 \le j \le t+1 & 2 \le i \le n. \end{array}$$

and $f(u) = 3nt+2n+1$. Since $e_f(0) = e_f(1) = n(2t+1)$, f is a difference cordial labeling of $S(W(t,n))$.

A Young tableau $Y_{m,n}$ is a sub graph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting the vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a non increasing sequence. Let $V(P_m \times P_n) = \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(P_m \times P_n) = \{u_{i,j}ui,j+1 : 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{u_{i,j}u_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n\}$.

Theorem 2.18. $S(Y_{n,n})$ is difference cordial.

Proof. Let $Y_{n,n}$ be a young tableau graph obtained from the grid $P_n \times P_n$. Let the edges $u_{i,j}u_{i+1,j}$ be subdivided by $w_{i,j}$. The order and size of $S(Y_{n,n})$ are $\frac{3n^2+5n-6}{2}$ and $2(n-1)(n+2)$ respectively. Define a map $f : V(S(Y_{n,n})) \rightarrow \left\{1, 2, \ldots, \frac{3n^2+5n-6}{2}\right\}$ as follows:

$$
\begin{array}{rcllll}
f(u_{1,j}) & = & 2j-1, & 1 \leq j \leq n, & \\
f(v_{1,j}) & = & 2j, & 1 \leq j \leq n-1, & \\
f(w_{1,j}) & = & n^2-2n-2+j, & 1 \leq j \leq n, & \\
f(u_{2,j}) & = & 2n+2j-2, & 1 \leq j \leq n, & \\
f(v_{2,j}) & = & 2n+2j-1, & 1 \leq j \leq n-1, & \\
f(u_{i,j}) & = & f(u_{i-1,n-i+3})+2j-1, & 3 \leq i \leq n, & 1 \leq j \leq n-i+2, \\
f(v_{i,j}) & = & f(u_{i,j})+1, & 3 \leq i \leq n, & 1 \leq j \leq n-i+2, \\
f(w_{i,j}) & = & f(w_{i-1,n-i+2})+j, & 2 \leq i \leq n-1 & 1 \leq j \leq n-i+1.
\end{array}
$$

Since $e_f(0) = e_f(1) = (n-1)(n+2)$, f is a difference cordial labeling of $S(Y_{n,n})$.

References

[1] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, Ars combin., **23**(1987), 201-207.

[2] A. Diab, Study of some problems of cordial graphs, Ars Combin., **92**(2009), 255-261.

[3] J. A. Gallian, A Dynamic survey of graph labeling, The Electronic Journal of Combinatorics, **18**(2012) #Ds6.

[4] F. Harary, Graph theory, Addision wesley, New Delhi, (1969).

[5] S. M. Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, Ars Combin., **32**(1991), 209-214.

[6] R. Ponraj, S. Sathish Narayanan and R. Kala, Difference cordial labeling of graphs (Communicated).

[7] R. Ponraj and S. Sathish Narayanan, Further Results on Difference cordial labeling of corona graphs, The Journal of The Indian Academy of Mathematics, **35**(2013).

[8] R. Ponraj, S. Sathish Narayanan and R. Kala, Difference cordial labeling of graphs obtained from double snakes, International Journal of Mathematics Research, **5**(2013), 317-322.

[9] R. Ponraj and S. Sathish Narayanan, Difference cordiality of some graphs obtained from double alternate snake graphs, Global Journal of Mathematical Research: Theory and Practical, **5**(2013), 167-175.

[10] R. Ponraj, M. Sivakuamr and M. Sundaram, k-Product cordial labeling of graphs, Int. J. Contemp. Math. Sciences, **7**(2012), 733 -742.

[11] R.Ronraj, M.Sivakumar, M.Sundaram, k- Total product cordial labelings of graphs, Applications and Applied Mathematics, **7**(2012), 708- 716.

[12] A. Riskin, Cordial deficiency, Bull. Malays. Math. Sci. Soc., **30**(2007), 201-204.

[13] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349-355.

[14] M. A. Seoud and A. E. I. Abdel Maqsoud, On cordial and balanced labelings of graphs, J. Egyptian Math. Soc., **7**(1999), 127-135.

[15] S. K. Vaidya, G. Ghodasara, S. Srivastav, and V. Kaneria, Cordial labeling for two cycle related graphs, Math. Student, **76**(2007), 237-246.

Scientia Magna
Vol. 9 (2013), No. 4, 107-112

Slant ruled surface in the Euclidean 3-space E^3

Rashad A. Abdel-Baky†‡

† Present address: Department of Mathematics, Sciences Faculty for Girls,
King Abdulaziz University, P.O. Box 126300, Jeddah 21352, Saudi Arabia
‡ Department of Mathematics, Faculty of Science, University of Assiut
E-mails: rbaky@live.com

Abstract This paper presents a generalization of the slant curve for ruled surface.
Keywords Ruled surface, Blaschke frame, Riccati equation.
2000 Mathematics Subject Classification: 53A04, 53A05, 53A17

§1. Introduction and preliminaries

Several classical, well-known geometric objects are defined in terms of making a constant angle with a given, distinguished direction. Firstly, classical helices are curves making a constant angle with a fixed direction. A second example is the logarithmic spiral, the spiral marbles studied by Jacob Bernoulli, which makes a constant angle with the radial direction. In a third famous example which had applications to navigation, the loxodromes or rhumb lines are those curves in the sphere making a constant angle with the sphere meridians. Important contributions have been studied in [1-4].

In [5, 6], Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines makes a constant angle with a fixed direction. In the last few years, the study of surfaces making a constant angle with a fixed direction has some important applications to physics, namely in special equilibrium configurations of nematic and smectic liquid crystals [7−9].

As it is known, the other analytical tool in the study of three-dimensional kinematics and the differential geometry of ruled surfaces is based upon dual vector calculus as shown in [10-15]. Although the slant space curve in E^3 is known, however, the dual curve is not. So, this led us to give a generalization of the slant curve for ruled surface.in the Euclidean 3-space E^3. The condition for ruled surface to be slant ruled surface is developed and clarified using computer-aided example

There is a tight connection between spatial kinematics and the geometry of line in three-dimensional Euclidean space E^3. Therefore we start with recalling the use of appropriate line coordinates: An oriented line L in the Euclidean 3-space E^3 can be represented by a dual unit vector

$$\mathbf{E} = \mathbf{e} + \varepsilon \mathbf{e}^*,$$

where $\mathbf{e}$ is a unit vector along L, $\mathbf{e}^*$ is the moment vector of $\mathbf{e}$ about the origin of coordinates $\mathbf{o}$ and ε is a dual unit subject to the rules $\varepsilon \neq 0,\ \varepsilon^2 = 0,\ \varepsilon.1 = 1.\varepsilon = \varepsilon$. By the definitions,

$$< \mathbf{e}, \mathbf{e} >=1, \ \ < \mathbf{e}^*, \mathbf{e} >=0. \tag{1}$$

For an introduction to dual numbers and their applications in kinematics, the reader may please refer to [10-14]. The relation between line geometry and kinematics in the Euclidean 3-space E^3 can be used for understanding the ruled surface from a practical point of view. Then, a ruled surface is represented by a dual curve on the dual unit sphere [11−13]:

$$\mathbf{E}(u) = \mathbf{e}(u) + \varepsilon \mathbf{e}^*(u), \ \ u \in \mathbb{R}.$$

The differential geometry of the dual curve $\mathbf{E}(u)$ is identical to that of curve on the usual unit sphere. Thus, we can define an orthonormal moving frame along $\mathbf{E} = \mathbf{E}(u)$ as follows:

$$\mathbf{E} = \mathbf{E}(u), \ \mathbf{T}(u){=}\ \mathbf{t}{+}\varepsilon\mathbf{t}^* = \frac{\mathbf{E}'}{\|\mathbf{E}'\|}, \ \mathbf{G}(u){=}\ \mathbf{g}{+}\varepsilon\mathbf{g}^* = \mathbf{E}{\times}\frac{\mathbf{E}'}{\|\mathbf{E}'\|}, \tag{2}$$

(a prime denotes the derivative with respect to u), this frame is called the Blaschke frame. The dual unit vectors $\mathbf{E}$, $\mathbf{T}$, $\mathbf{G}$ corresponds to three concurrent mutually orthogonal lines in the Euclidean 3-space E^3. Their point of intersection is the striction point on the ruling $\mathbf{E}$. $\mathbf{G}$ is the limit position of the common perpendicular to $\mathbf{E}(u)$ and $\mathbf{E}(u+du)$ and is called the central tangent of the ruled surface at the central point. The line $\mathbf{T}$ is called the central normal of $\mathbf{E}(u)$ at the central point. By construction, the Blaschke formula is:

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{3}$$

where

$$P = p + \varepsilon p^* = \left\|\mathbf{E}'\right\|, \ Q = q + \varepsilon q^* = \frac{\det(\mathbf{E}, \mathbf{E}', \mathbf{E}'')}{\|\mathbf{E}'\|^3},$$

are called the curvature functions of the ruled surface $\mathbf{E}(u)$ and their derivatives define the shape of the ruled surface $\mathbf{E}(u)$ in vicinity of a line $\mathbf{E} = \mathbf{e}{+}\varepsilon\mathbf{e}^*$.

§2. Slant ruled surface

In the Euclidean 3-space E^3, it is well known that the determination of a (non-isotropiic) curve with curvature $k(s)$ and torsion $\tau(s)$, where s represents the arc-length, depends upon the integration of a Riccati equation; and that this can bee performed only if a particular integral of the equation is known.[1] Using the concept of dual numbers and their use in kinematics, this statement can be extended for ruled surface.

2.1. Dual Riccati equation

Since $\mathbf{E}(E_1, E_2, E_3)$, $\mathbf{T}(T_1, T_2, T_3)$ and $\mathbf{G}(G_1, G_2, G_3)$ are dual unit vectors, then

$$E_j^2 + T_j^2 + G_j^2 = 1, \ \ (j = 1, \ \ 2, \ \ 3), \tag{4}$$

can be decomposed as follows:

$$(E_j + iG_j)(E_j - iG_j) = (1 + T_j)(1 - T_j),\ i = \sqrt{-1}. \tag{5}$$

Let us now introduce the conjugate imaginary dual functions $M = m + \varepsilon m^*$ and $-N^{-1} = -(m + \varepsilon m^*)^{-1}$:

$$M = \frac{E_j + iG_j}{1 - T_j} = \frac{1 + T_j}{E_j - iG_j};\ -\frac{1}{N} = \frac{E_j - iG_j}{1 - T_j} = \frac{1 + T_j}{E_j + iG_j}. \tag{6}$$

On differentiating with respect to u and take into account the Blaschke formulae, we find for $M^{'}$:

$$M^{'} = \frac{E_j^{'} + iG_j^{'}}{1 - T_j} + \frac{E_j + iG_j}{(1 - T_j)^2},\ \ T_j^{'} = \frac{(P - iQ)T_j}{1 - T_j} + \frac{M(-PE_j + QG_j)M}{1 - T_j}. \tag{7}$$

Because of (6) we find for E_j and G_j:

$$E_j = \frac{1 + T_j + M^2 - M^2 T_j}{2M},\ \ G_j = \frac{i[1 + T_j - M^2 + M^2 T_j]}{2M}. \tag{8}$$

Substituting (8) into (7), we obtain:

$$M^{'} = -\frac{M^2}{2}(P + iQ) - \frac{1}{2}(P - iQ). \tag{9}$$

Performing the same type of elimination for $N^{'}$, we find that N salsifies the same equation as M. This is a dual version of the Riccati equation (see [1]).

It is assumed that the functions

$$P = \Omega\cos(\Psi u),\ \ Q = \Omega\sin(\Psi u),\ \Omega = \omega + \varepsilon\omega^* > 0,\ \ \Psi = \psi + \varepsilon\psi^*, \tag{10}$$

are known. Equations (10) are intrinsic equations of the ruled surface, which do not contain coordinates. Assigning the above functions defines the ruled surface accurate to its position in space.

If M is found as a result of integration of the equation, we shall be able to find E_j, G_j and T_j from (6) by separating the latter into parts containing and not containing i and applying (4). Substituting equations (10) into equation (9), we have

$$M^{'} = -\frac{\Omega}{2}[M^2 e^{i\Psi u} + e^{-i\Psi u}]. \tag{11}$$

It is immediately obvious that there exist two particular integrals of (11) of the form $Ae^{-i\Psi u}$. If we substitute $M = Ae^{-i\Psi u}$ into (11) and cancel $e^{-i\Psi u}$, we get the quadratic equation

$$\Omega A^2 - 2i\Psi A + \Omega = 0,$$

whose roots, in terms of $\Sigma := \sigma + \varepsilon\sigma^* = \sqrt{\Omega^2 + \Psi^2}$, are:

$$A_1 = i\sqrt{\frac{\Sigma + \Psi}{\Sigma - \Psi}},\ \ A_2 = -i\sqrt{\frac{\Sigma - \Psi}{\Sigma + \Psi}}.$$

If we choose $M = A_1 e^{-i\Psi u}$, $N = A_2 e^{-i\Psi u}$ and substitute into

$$T_j = \frac{M+N}{M-N},$$

we get

$$T_j = \frac{A_1 + A_2}{A_1 - A_2} = \frac{\Psi}{\Sigma}, \tag{12}$$

one of the components of the central normal $\mathbf{T}(u)$ be chosen to be constant. Hence the following theorem can be given:

Theorem 2.1. A ruled surface in the Euclidean 3-space E^3 defined by $P = \Omega\cos(\Psi u)$, $Q = \Omega\sin(\Psi u)$, where $\Omega > 0$ and Ψ are dual constants, has the property that its central normal makes a constant dual angle with a (suitably chosen) fixed direction.

According to Theorem 2.1, surfaces of this kind belong to a wider family of ruled surfaces distinguished by

$$P^2 + Q^2 = \Omega^2 = const. \tag{13}$$

We shall call them slant ruled surfaces and denoted them by (M).

2.2. Parametric equation of (M)

Since equation (11) admits two known particular integrals, its general integral can be obtained by one additional quadrature. Equations (6) will then enable us determine the parametric equations of (M). For this purpose, since one of the components of $\mathbf{T}$, i.e. $(\frac{\Psi}{\Sigma})$ can be chosen to be is constant, consider the linear differential equation satisfied by $\mathbf{T}$. If we start from the Blaschke formulae (3) and differentiae the second raw twice and write for $\mathbf{E}'$and $\mathbf{G}'$ their expressions given by the first and third, we obtain as the linear differential equation for $\mathbf{T}$.

$$\mathbf{T}^{(3)} + \Sigma^2\mathbf{T}' = 0. \tag{14}$$

The general integral of the differential equation (14) subject to the conditions $\|\mathbf{T}\| = 1$, $\left\|\mathbf{T}'\right\| = \Omega$, can be shown to be

$$\mathbf{T} = \frac{\Omega}{\Sigma}\cos(\Sigma u)\mathbf{F}_1 + \frac{\Omega}{\Sigma}\sin(\Sigma u)\mathbf{F}_2 + \frac{\Psi}{\Sigma}\mathbf{F}_3, \tag{15}$$

where $\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3$ are orthonormal dual unit vectors which are rigidly linked to fixed space. From (15), with the help of the Blaschke formulae, we find by integration that (taking into account that $\|\mathbf{E}\| = 1$, $<\mathbf{E},\mathbf{T}>= 0$):

$$\begin{aligned}\mathbf{E}(u) \;=\; & \frac{\Omega^2}{2\Sigma}\left(\frac{\sin(\Sigma+\Psi)u}{\Sigma+\Psi} + \frac{\sin(\Sigma-\Psi)u}{\Sigma-\Psi}\right)\mathbf{F}_1 \\ & -\frac{\Omega^2}{2\Sigma}\left(\frac{\cos(\Sigma+\Psi)u}{\Sigma+\Psi} + \frac{\cos(\Sigma-\Psi)u}{\Sigma-\Psi}\right)\mathbf{F}_2 + \frac{\Omega}{\Sigma}\sin(\Psi u)\mathbf{F}_3,\end{aligned} \tag{16}$$

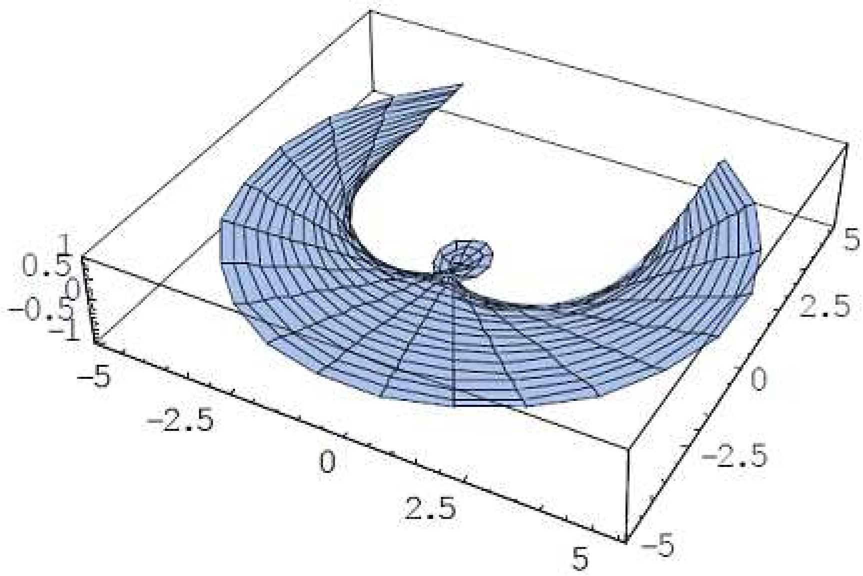
which is a canonical form for the family of slant ruled surfaces (M). Let $\mathbf{L}$ denote a point on (M). We can write:

$$\mathbf{L}(u,\mu) = \mathbf{e}(u)\times\mathbf{e}^*(u) + \mu\mathbf{e}(u);\quad \mu\in\mathbb{R}. \tag{17}$$

As a special case, if we take $\sigma = 1$, $\sigma^* = 0$, $\psi = 0$ and $\psi^* = 1$, then we obtain a member of this family as:

$$\mathbf{L}(u,\mu) = (-u\cos u, -u\sin u, 0) + \mu(\sin u, -\cos u, 0). \tag{18}$$

The graph of the surface is shown in Figure (1);



References

[1] D. J. Struik, Lectures on classical differential geometry, Addison-Wesley, Reading, Mass, (1961).

[2] M. I. and A. I. Nistor, A new approach on constant angle surfaces in E3, Turk. J. Math. **33**(2)(2009), 169–178.

[3] P. Cermelli and A. J. Di Scala, Constant-angle surfaces in liquid crystals, Philosophical Magazine, **87**(12)(2007), 1871- 1888.

[4] W. Kuhnel, Differential Geometry Curves-Surfaces-Manifolds, second ed. Friedr. Vieweg & Sohn Verlag, Wiesbaden, (2003).

[5] S. Izumiya and N. Takeuchi, Special Curves and Ruled surfaces, Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry, (2003), Vo. 44, No. 1, 203-212.

[6] S. Izumiya and N. Takeuchi, Generic properties of helices and Bertrand curves, Journal of Geometry, (2002), Vo. 74, 97–109.

[7] Lopez and Munteanu, Constant-angle surfaces in Minkowski space, Bulletin of the Belgian Mathematical Society, (2011), Vo. 18, No. 2, 271-286.

[8] P. Cermelli and A. J. Di Scala, Constant-angle surfaces in liquid crystals, Philosophical Magazine, **87**(12)(2007), 1871-1888.

[9] M. I. Munteanu and A. I. Nistor, A new approach on constant angle surface in E3, Turkish J. Math. 33.2, (2009), 1-10.

[10] O. Bottema and B. Roth, Theoretical Kinematics, North-Holland Press, New York,(1979).

[11] R. A. Abdel Baky and F. R. Al-Solamy, A new geometrical approach to one-parameter

spatial motion, J. of Eng. Maths, (2008), Vo. 60, 149-172.

[12] R. A. Abdel-Baky and R. A. Al-Ghefari, On the one-parameter dual spherical motions, Computer Aided Geometric Design 2011, **28**(2011), 23-37..

[13] R. A. Abdel-Baky and R. A. Al-Ghefari, On the kinematic geometry of relative screw motions, J. of Mechanical Science and Technology **26**(8)(2012), 2497-2503.

[14] J. M. McCarthy, On the scalar and dual formulations of curvature theory of line trajectories, ASME Journal of Mechanisms, Transmissions and Automation in Design 109,(1987), 101-106.

[15] G. Figlioini, H. Stachel and J. Angeles, On Martin Disteli's main achievements in spatial gearing: Disteli's diagram, proceeding of EuCoMES, the first European Conference on Mechanisms Science, Obeergurgal (Austria), February, (2006), 21-26.

ISBN 9781599731810
9 781599 731810

SCIENTIA MAGNA

An international journal

ISBN 9781599731810

9 781599 731810