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A new refinement of the inequality $\sum \sin \frac{A}{2} \leq \sqrt{\frac{4R+r}{2r}}$

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Abstract The purpose of this paper is to give a new provement to inequality $\sum \sin \frac{A}{2} \leq \sqrt{\frac{4R+r}{2r}}$, who are given in [1], to prove that this is the better inequality of type: $\sum \sin \frac{A}{2} \leq \sqrt{\frac{\alpha R+\beta r}{R}}$, when

$$\sqrt{\frac{\alpha R+\beta r}{R}} \leq \frac{3}{2}, \tag{1}$$

and to refine this inequality with an equality of type: "better of the type" :

$$\sum \sin \frac{A}{2} \leq \frac{\alpha R+\beta r}{R},$$

when

$$\frac{\alpha R+\beta r}{R} \leq \frac{3}{2} \tag{2}$$

or in an equivalent from:

$$\sum \sin \frac{A}{2} \leq \frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{R}. \tag{3}$$

We denote

$$\frac{R}{r}=x,\ d=\sqrt{R^2-2Rr},\ d_x=\sqrt{x^2-2x}.$$

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§1. Main results

Lemma 1.1. In all triangle ABC holds:

$$\frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{R} \leq \sqrt{\frac{4R+r}{2R}}. \tag{4}$$

Proof. The inequality (4) will be written as:

$$\sqrt{2}+\frac{3-2\sqrt{2}}{x} \leq \sqrt{\frac{4x+1}{2x}}$$

and after squaring we shall obtain:

$$\begin{aligned} & 2+\frac{17-12\sqrt{2}}{x^2}+\frac{6\sqrt{2}-8}{x}\leq\frac{4x+1}{2x} \\ \Leftrightarrow\quad & 4x^2+34-24\sqrt{2}+\left(12\sqrt{2}-16\right)x\leq 4x^2+x \\ \Leftrightarrow\quad & \left(17-12\sqrt{2}\right)(x-2)\geq 0. \end{aligned}$$

Theorem 1.1. In all triangle ABC holds:

$$\sum\sin\frac{A}{2}\leq\frac{r}{R+d}+\sqrt{\frac{R+d}{R}}. \tag{5}$$

Proof. The inequality (5) will be given in [2].

We shall give a new prove of this inequality. In [3] it was proved the following inequality:

$$\sum\sqrt{\frac{p-a}{a}}\leq\sqrt{\frac{R-r+d}{2r}}+2\sqrt{\frac{R-d}{2R}}.$$

We have:

$$\begin{aligned} \left(\sum\sin\frac{A}{2}\right)^2 &= \sum\left(\sin\frac{A}{2}\right)^2+2\sum\sin\frac{B}{2}\sin\frac{C}{2} \\ &= \sum\frac{(p-b)(p-c)}{bc}+2\sqrt{\frac{(p-a)(p-b)(p-c)}{abc}}\sum\sqrt{\frac{p-a}{a}}, \end{aligned}$$

or in an equivalent form:

$$\begin{aligned} \sum\sin\frac{A}{2} &= \sqrt{\frac{2R-r}{2R}+\sqrt{\frac{r}{R}}\sum\sqrt{\frac{p-a}{a}}} \\ &\leq \sqrt{\frac{2R-r}{2R}+\sqrt{\frac{r}{R}}\left(\sqrt{\frac{R-r+d}{2r}}+2\sqrt{\frac{R-d}{2R}}\right)} \\ &= \sqrt{\frac{2R-r}{2R}+\sqrt{\frac{R-r+d}{2R}}+\frac{2}{R}\sqrt{\frac{r(R-d)}{2}}} \end{aligned}$$

and because

$$(R+d)^2=2R(R-r+d)(R-d)(R+d)=2Rr$$

and

$$(R-r-d)(R-r+d)=r^2,$$

it shall result

$$\begin{aligned} \sum\sin\frac{A}{2} &\leq \sqrt{\frac{R+d}{R}+\frac{2R-r}{2R}-\frac{R+d}{2R}+\frac{2r}{R+d}\sqrt{\frac{R+d}{R}}} \\ &= \frac{r}{R+d}+\sqrt{\frac{R+d}{R}}. \end{aligned}$$

In the following we shall prove (1).

From inequality (5):

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \le \sqrt{\frac{R+d}{R}} + \frac{r}{R+d} \le \frac{3}{2},$$

because the inequality (5) is the better of the type: $\sum \sin \frac{A}{2} \le f(R,r)$. It follows that:

$$\sqrt{\frac{R+d}{R}} + \frac{1}{R+d} \le \sqrt{\frac{\alpha R + \beta r}{R}} \le \frac{3}{2}$$

or

$$\frac{1}{1+d_x} + \sqrt{\frac{x+d_x}{x}} \le \sqrt{\frac{\alpha x + \beta}{x}} \le \frac{3}{2}, \ \forall x \ge 2. \tag{6}$$

In the case of equilateral triangle we have $x = 2$. We shall obtain

$$4\alpha + \frac{2}{3} = 9.$$

The inequality (6) may be written in the case of the isosceles triangle with sides: $b - c = 1$, $a = 0$ ($R = \frac{1}{2}$, $r = 0$) or putting $x \to \infty$ in an equivalent from as: $a \ge 2$.
Because: $(2\alpha - 4)x + 4x + 9 - 4\alpha \ge 2(2\alpha - 4) + 4x + 9 - 4\alpha = 4x + 1$ it shall result:

$$\sqrt{\frac{\alpha x + \beta}{x}} \ge \sqrt{\frac{4x+1}{2x}}.$$

In the following will be sufficient to prove that:

$$\sqrt{\frac{x+d_x}{x}} + \frac{1}{x+d_x} \le \sqrt{\frac{4x+1}{2x}}. \tag{7}$$

Theorem 1.2. In all triangle ABC holds

$$\sum \sin \frac{A}{2} \le \sqrt{\frac{4R+r}{2r}}.$$

Proof. The inequality (7) may be written in an equivalent form as:

$$\begin{aligned}
\sqrt{\frac{x+d_x}{x}} + \frac{1}{x+d_x} \le \sqrt{\frac{4x+1}{2\lambda}} \quad &\Leftrightarrow \quad \frac{1}{x+d_x} \le \sqrt{\frac{4x+1}{2x}} - \sqrt{\frac{x+d_x}{x}} \\
&\Leftrightarrow \quad \sqrt{\frac{4x+1}{2x}} + \sqrt{\frac{x+d_x}{x}} \\
&\le \quad (x+d_x)\left(\frac{4x+1-2x-2d_x}{2x}\right) \\
&= \quad \frac{d_x+5x}{2x} \\
&\Leftrightarrow \quad d_x + 5x \ge \sqrt{2x(4x+1)} + \sqrt{4x(x+d_x)}.
\end{aligned}$$

After squaring we shall obtain:

$$\begin{aligned}
x^2-2x+25x^2+10xd_x &\geq 8x^2+2x+4x^2+4xd_x+4x\sqrt{2\left(4x+1\right)\left(x+d_x\right)}\\
&\Leftrightarrow 14x^2-4x+6xdx\geq 4x\sqrt{2\left(4x+1\right)\left(x+d_x\right)}\\
&\Leftrightarrow 7x-2+3d_x\geq 2\sqrt{\left(8x+2\right)\left(x+d_x\right)}\\
&\Leftrightarrow 49x^2+4+9x^2-18x-28x-12d_x+42xd_x\\
&\geq 32x^2+32xd_x+8x+8d_x\\
&\Leftrightarrow 26x^2-54x+4+10xd_x-20d_x\geq 0\\
&\Leftrightarrow \left(13x-1+5d_x\right)\left(x-2\right)\geq 0.
\end{aligned}$$

In the following we shall determine $\alpha,\ \beta$ with the property (2).

According with the inequality (6) it follows that

$$\sqrt{\frac{R+d}{R}}+\frac{r}{R+d}\leq\frac{\alpha R+\beta r}{R}\leq\frac{3}{2}$$

or in an equivalent form

$$\sqrt{\frac{x+d_x}{x}}+\frac{1}{x+dx}\leq\alpha+\frac{\beta}{x}\leq\frac{3}{2}. \tag{8}$$

In the case of equilateral triangle we shall obtain $2\alpha+\beta=3$.

In the inequality (8) we shall consider $x\to\infty$. It follows that $a\geq\sqrt{2}$.

Because

$$\left(\alpha-\sqrt{2}\right)R+\left(3-2\alpha\right)r\geq\left(2\alpha-2\sqrt{2}+3-2\alpha\right)r=\left(3-2\sqrt{2}\right)r,$$

it follows that

$$\frac{\alpha R+\beta r}{R}\geq\frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{R}.$$

In the following will be sufficient to prove the inequality (3).

Theorem 1.3. In all triangle ABC holds

$$\sum\sin\frac{A}{2}\leq\frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{R}.$$

Proof. From the inequality (5) it follows that in order to prove the inequality (3) it will be sufficient to prove:

$$\sqrt{\frac{R+d}{R}}+\frac{r}{R+d}\leq\frac{\sqrt{2}R=\left(3-2\sqrt{2}\right)r}{R}$$

or in an equivalent form:

$$\begin{aligned}
&\sqrt{\frac{x+d_x}{x}}+\frac{1}{x+dx}\leq\frac{\sqrt{2}x+3-2\sqrt{2}}{x}\\
\Leftrightarrow\quad &\sqrt{\frac{x+d_x}{x}}\leq\frac{\sqrt{2}x^2+\left(3-2\sqrt{2}\right)x+\sqrt{2}xd_x+\left(3-2\sqrt{2}\right)d_x-x}{x\left(x+d_x\right)}
\end{aligned}$$

$$\Leftrightarrow \quad \sqrt{\frac{x+d_x}{x}} \leq \frac{\sqrt{2}x^2 + (2-2\sqrt{x})\,x + \sqrt{2}xd_x + \left(3-2\sqrt{2}\right)d_x}{x\,(x+d_x)}. \tag{9}$$

We denote

$$u_x = \frac{d_x}{x}.$$

The inequality (9) may be written in an equivalent form as:

$$\begin{aligned}
\frac{x+d_x}{x} \quad & \leq \quad \frac{x^2\left[\sqrt{2}x + 2 - 2\sqrt{2} + \sqrt{2}d_x + \left(3-2\sqrt{2}\right)u_x\right]^2}{2x^3\,(x+d_x-1)} \\
& \Leftrightarrow \quad (2x+2d_x)\,(x+d_x-1) \\
& \leq \quad \left[\sqrt{2}x + 2 - 2\sqrt{2} + \sqrt{2}d_x + \left(3-2\sqrt{2}\right)u_x\right] \\
& \Leftrightarrow \quad 2x^2 + 2xd_x - 2x + 2xd_x + 2x^2 - 4x - 2x \\
& \leq \quad 2x^2 + 12 - 8\sqrt{2} + 2x^2 - 4x + \left(17-12\sqrt{2}\right)u_x^2 \\
& + \quad \left(4\sqrt{2}-8\right)x + 4xd_x + \left(6\sqrt{2}-8\right)xu_x + \left(4\sqrt{2}-8\right)d_x \\
& + \quad \left(28-20\sqrt{2}\right)u_x + \left(4\sqrt{2}-8\right)d_x + \left(6\sqrt{2}-8\right)u_xd_x \\
& \Leftrightarrow \quad 4xd_x - 6x - 2d_x \\
& \leq \quad 12 - 8\sqrt{2} + \frac{(x-2)}{x}\left(17-12\sqrt{2}\right) - 4x + \left(4\sqrt{2}-8\right)x \\
& + \quad \left(6\sqrt{2}-8\right)d_x + 4xd_x + \left(28-20\sqrt{2}\right)\frac{d_x}{x} \\
& + \quad \left(4\sqrt{2}-8\right)d_x + (x-2)\left(6\sqrt{2}-8\right) \\
& \Leftrightarrow \quad 4x^2d_x - 6x^2 - 2xd_x \\
& \leq \quad \left(12-8\sqrt{2}\right)x + \left(17-12\sqrt{2}\right)x\sqrt{-34+}24\sqrt{2} - 4x^2 \\
& + \quad \left(4\sqrt{2}-8\right)x^2 + \left(6\sqrt{2}-8\right)xd_x + 4x^2d_x + \left(28-20\sqrt{2}\right)d_x \\
& + \quad \left(4\sqrt{2}-8\right)xd_x + \left(6\sqrt{2}-8\right)x^2 - \left(12\sqrt{2}-16\right)x \\
& \Leftrightarrow \quad \left[4x^2 - 2x - \left(6\sqrt{2}-8\right)x - 4x^2 - 28 + 20\sqrt{2} - \left(4\sqrt{2}-8\right)x\right]d_x \\
& \leq \quad \left(6 - 4 + 4\sqrt{2} - 8 + 6\sqrt{2} - 8\right)x^2 \\
& + \quad \left(12 - 8\sqrt{2} + 17 - 12\sqrt{2} - 12\sqrt{2} + 16\right)x - 34 + 24\sqrt{2} \\
& \Leftrightarrow \quad d_x\left[\left(-2 - 6\sqrt{2} + 8 - 4\sqrt{2} + 8\right)x - 28 + 20\sqrt{2}\right] \\
& \leq \quad \left(10\sqrt{2}-14\right)x^2 + \left(45-32\sqrt{2}\right)x - 34 + 24\sqrt{2} \\
& \Leftrightarrow \quad d_x\left[\left(14-10\sqrt{2}\right)x + 20\sqrt{2} - 28\right] \\
& \leq \quad \left(10\sqrt{2}-14\right)x^2 + \left(45-32\sqrt{2}\right)x - 34 + 24\sqrt{2} \\
& \Leftrightarrow \quad d_x\left[\left(14-10\sqrt{2}\right)x + 20\sqrt{2} - 28\right] \\
& \leq \quad \left(10\sqrt{2}-14\right)x^2 + \left(45-32\sqrt{2}\right)x - 34 + 24\sqrt{2} \\
& \Leftrightarrow \quad d_x\left(14-10\sqrt{2}\right)(x-2) \leq (x-2)\left[\left(10\sqrt{2}-14\right)x + 17 - 12\sqrt{2}\right]
\end{aligned}$$

$$\Leftrightarrow \quad (x-2)\left[\left(10\sqrt{2}-14\right)x+\left(10\sqrt{2}-14\right)d_x+17-12\sqrt{2}\right]\geq 0.$$

Corollary 1.1. In all triangle ABC holds

$$\sum\cos\frac{A}{2}\geq\frac{3S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}.$$

Proof. Using the Chebyshev's inequality to $\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}$ and $\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}$ we get

$$\left(\frac{1}{3}\sum\sin\frac{A}{2}\right)\left(\frac{1}{3}\sum\cos\frac{A}{2}\right)\geq\left(\frac{1}{3}\sum\sin\frac{A}{2}\cos\frac{A}{2}\right)$$

or

$$\sum\sin\frac{A}{2}\sum\cos\frac{A}{2}\geq\frac{3}{2}\sum\sin A$$

or

$$\sum\cos+\geq\frac{3\sum\sin A}{2\sum\sin\frac{A}{2}}\geq\frac{3\sum\sin AR}{2\left[\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right]}=\frac{3S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}.$$

Corollary 1.2. If $\lambda\in(0,1]$, then in all triangle ABC holds:

$$\sum\left(\sin\frac{A}{2}\right)^{\lambda}\leq 3\left(\frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{3R}\right)^{\lambda}.$$

Proof. Using the Jensen's inequality we get:

$$\sum\left(\sin\frac{A}{2}\right)^{\lambda}\leq 3\left(\frac{1}{3}\sum\sin\frac{A}{2}\right)^{\lambda}=3\left(\frac{\sqrt{2}R+\left(3-2\sqrt{2}\right)r}{3R}\right)^{\lambda}.$$

Corollary 1.3. If $\lambda\geq 1$ then in all triangle ABC holds

$$\sum\left(\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}\right)^{\lambda}.$$

Proof. From Jensen's inequality we get:

$$\sum\left(\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{1}{3}\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}\right)^{\lambda}.$$

Corollary 1.4. In all triangle ABC holds

$$\sum\cos\frac{A}{2}\geq\frac{2S+3\sqrt{3}R}{4R}.$$

Proof. From Popoviciu's inequality

$$\sum f(x)+3f\left(\frac{x+y+z}{3}\right)\geq 2\sum f\left(\frac{x+y}{2}\right).$$

applied to function $f:(0,\pi)\to R,\ f(x)=\sin x$ we get:

$$\sum\sin A+3\sin\frac{\pi}{3}\leq 2\sum\sin\frac{A+B}{2}$$

or

$$\frac{S}{R} + \frac{3\sqrt{3}}{2} \leq 2\sum \cos \frac{A}{2}$$

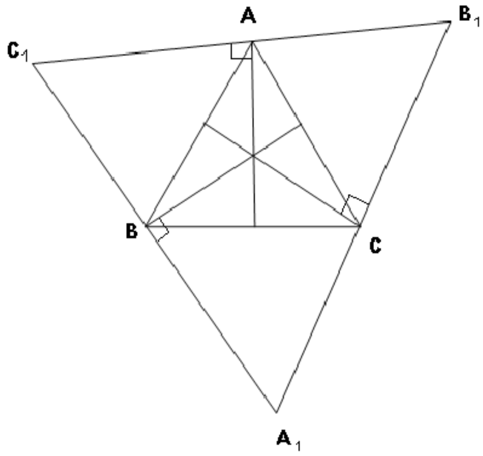
or

$$\sum \cos \frac{A}{2} \geq \frac{2S + 3\sqrt{3}R}{4R}.$$

Corollary 1.5. In all triangle ABC are true the following equality:

$$2R\sum \cos \frac{A}{2} \left(\sum \sin \frac{A}{2} - 1 \right) = S. \tag{10}$$

Proof. We shall consider the triangle ABC obtained with the exterior bisectors of the triangle ABC.



We denote with $S_{A_1B_1C_1}$ the semiperimeter of $A_1B_1C_1$ trianle.

We have: $2S_{A_1B_1C_1} = A_1B_1 + B_1C_1 + A_1C_1$.

We shall calculate:

$$\begin{aligned} B_1C_1 &= AB_1 + AC_1 \\ &= 4R\sin \frac{C}{2}\cos \frac{B}{2} + 4R\sin \frac{B}{2}\cos \frac{C}{2} \\ &= 4R\cos \frac{A}{2} \end{aligned}$$

and the others.

We shall obtain: $S_{A_1B_1C_1} = 2R\sum \cos \frac{A}{2}$.

We have:

$$S_{ABC_1} + S_{BCA_1} + S_{ACB_1} = S_{ABC} + S_{A_1B_1C_1}$$

and

$$\begin{aligned} 2S_{ABC_1} &= 4R\sin \frac{C}{2}\cos \frac{B}{2} + 4R\sin \frac{C}{2}\cos \frac{C}{2} \\ &= 4R\sin \frac{C}{2}\sum \cos \frac{A}{2}. \end{aligned}$$

It follows that

$$2R\left(\sum\cos\frac{A}{2}\right)\sin\frac{C}{2}+2R\left(\sum\cos\frac{A}{2}\right)\sin\frac{B}{2}+2R\left(\sum\cos\frac{A}{2}\right)\sin\frac{A}{2}=S+2R\sum\cos\frac{A}{2}$$

or

$$2R\sum\cos\frac{A}{2}\left(\sum\sin\frac{A}{2}-1\right)=S.$$

Corollary 1.6. In all triangle ABC holds:

$$\sum\cos\frac{A}{2}\geq\frac{S}{2\left[\left(\sqrt{2}-1\right)R+\left(3-2\sqrt{2}\right)r\right]}. \tag{11}$$

Proof. Using the equality (10) and inequality (3) we shall obtain:

$$\begin{aligned}\sum\cos\frac{A}{2} &= \frac{S}{2R}\cdot\frac{1}{\sum\sin\frac{A}{2}-1}\geq\frac{S}{2R}\cdot\frac{R}{\left(\sqrt{2}-1\right)R+\left(3-2\sqrt{2}\right)r}\\ &= \frac{S}{2\left[\left(\sqrt{2}-1\right)R+\left(3-2\sqrt{2}\right)r\right]}.\end{aligned}$$

Corollary 1.7. If $\lambda\geq 1$ then in all triangle ABC holds:

$$\sum\left(\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{S}{6\left(\left(\sqrt{2}-1\right)R+\left(3-2\sqrt{2}\right)r\right)}\right)^{\lambda}.$$

Proof. From Jensen's inequality and inequality (11) we get:

$$\sum\left(\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{1}{3}\sum\cos\frac{A}{2}\right)^{\lambda}\geq 3\left(\frac{S}{6\left(\left(\sqrt{2}-1\right)R+\left(3-2\sqrt{2}\right)r\right)}\right)^{\lambda}.$$

Corollary 1.8. In all triangle ABC holds:

$$\sum\cos\frac{A}{2}\geq\frac{2S+3\sqrt{3}R}{4R}\geq\frac{3S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}. \tag{12}$$

Proof. The first side of the inequality (12) i just Corollary 3.4.
The right side

$$\frac{2S+3\sqrt{3}R}{4R}\geq\frac{3S}{2\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)}$$

will may be written in an equivalent form as:

$$\left[\left(6-2\sqrt{2}\right)R-\left(6-4\sqrt{2}\right)r\right]S\leq 3\sqrt{6}R^2+\left(9\sqrt{3}-6\sqrt{6}\right)Rr. \tag{13}$$

From Blundon's inequality $S\leq 2R+\left(3\sqrt{3}-4\right)r$ it follows that to demonstrate the inequality (13) it will be sufficient to prove that:

$$\left[\left(6-2\sqrt{2}\right)R-\left(6-4\sqrt{2}\right)r\right]\left[2R+\left(3\sqrt{3}-4\right)r\right]\leq 3\sqrt{6}R^2+\left(9\sqrt{3}-6\sqrt{6}\right)Rr$$

or

$$\left(3\sqrt{6}+4\sqrt{2}-12\right)x^2+\left(36-9\sqrt{3}-16\sqrt{2}\right)x+18\sqrt{3}-12\sqrt{6}+16\sqrt{2}-24\geq -24x$$

or

$$(x-2)\left[\left(3\sqrt{6}+4\sqrt{2}-12\right)x+6\sqrt{6}-8\sqrt{2}-9\sqrt{3}+12\right]\geq 0,$$

But

$$\begin{aligned}
& (3\sqrt{6}+4\sqrt{2}-24+6\sqrt{6}-8\sqrt{2}-9\sqrt{3}+12) \\
\geq\ & \sqrt{6}+8\sqrt{2}-24+6\sqrt{6}-8\sqrt{2}-9\sqrt{3}+12 \\
=\ & 12\sqrt{6}-9\sqrt{3}-12>0,
\end{aligned}$$

who are true.

Corollary 1.9. In all triangle ABC holds:

$$S^2 \leq \frac{2\,(4R+r)\left(\sqrt{2}R+\left(3-2\sqrt{2}\right)r\right)^2}{3R}.$$

Proof. From the identity:

$$\sum \cos^2 \frac{A}{2} = \frac{4R+r}{2R}, \tag{14}$$

Corollary 1.3 and

$$\sum \cos^2 \frac{A}{2} \geq \frac{1}{3}\left(\sum \cos \frac{A}{2}\right)^2,$$

it shall result inequality of the statement.

Corollary 1.10. In all triangle ABC holds:

$$S \leq \sqrt{6R\,(4R+r)} - \frac{3\sqrt{3}}{2}R.$$

Proof. Result from the identity (14), Jensen's inequality

$$\sum \cos^2 \frac{A}{2} \geq 3\left(\frac{1}{3}\sum \cos \frac{A}{2}\right)^2$$

and Corollary 1.4 we get:

$$\frac{4R+r}{2R} \geq \frac{1}{3}\cdot\frac{\left(2S+3\sqrt{3}R\right)^2}{16R^2}.$$

After performing some calculation we shall obtain the inequality of the statement.

Corollary 1.11. In all triangle ABC holds:

$$S^2 \leq \frac{6\left(3-2\sqrt{2}\right)(4R+r)\left(R+\left(\sqrt{2}-1\right)r\right)^2}{R}.$$

Proof. Result from inequality

$$\sum \cos^2 \frac{A}{2} \geq \frac{1}{3}\left(\sum \cos \frac{A}{2}\right)^2,$$

equality (14) and (12).

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On certain subclasses of analytic functions

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Abstract This paper is an application of a lemma due to Miller and Mocanu [1], using which we study two certain subclasses of analytic functions and improve certain known results.

Keywords Analytic function, univalent function, differential subordination.

§1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions f which are analytic in $\mathbb{E}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$, has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{S}$ denote the class of all analytic functions $f \in \mathcal{A}$ which are univalent in $\mathbb{E}$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to : $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\phi(p(z),\ zp'(z); z) \prec h(z),\ \phi(p(0), 0; 0) = h(0), \tag{1}$$

then p is called a solution of the first order differential subordination (1). The univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

In 2005, Kyohei Ochiai [2] studied the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ defined below: Let

$$\mathcal{M}(\alpha) = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha},\ 0 < \alpha < 1,\ z \in \mathbb{E} \right\},$$

and they defined the class $\mathcal{N}(\alpha)$ as $f(z) \in \mathcal{N}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}(\alpha)$. They proved the following results.

Theorem 1.1. If $f \in \mathcal{A}$ satisfies

$$\left|\frac{zf'(z)}{f'(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < 1 - 2\alpha$$

for some $\alpha(\ 1/4 \leq \alpha < 1/2)$, then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < \frac{1}{2\alpha} - 1,\ z \in \mathbb{E},$$

therefore, $f \in \mathcal{M}(\alpha)$.

Theorem 1.2. If $f \in \mathcal{A}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)}\right| < 1 - 2\alpha$$

for some $\alpha(\ 1/4 \leq \alpha < 1/2)$, then

$$\left|\frac{f'(z)}{f'(z) + zf''(z)} - 1\right| < \frac{1}{2\alpha} - 1,\ z \in \mathbb{E},$$

therefore, $f \in \mathcal{N}(\alpha)$.

To prove our main result, we shall use the following lemma of Miller and Mocanu [1].

Lemma 1.1. Let $q,\ q(z) \neq 0$ be univalent in $\mathbb{E}$ such that $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$. If an analytic function $p,\ p(z) \neq 0$ in $\mathbb{E}$, satisfies the differential subordination

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)} = h(z),$$

then

$$p(z) \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t}\ dt\right]$$

and q is the best dominant.

§2. Main results and applications

Theorem 2.1. Let $q,\ q(z) \neq 0$ be univalent in $\mathbb{E}$ such that $\dfrac{zq'(z)}{q(z)}(= h(z))$ is starlike in $\mathbb{E}$.
If $f \in \mathcal{A},\ \dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec h(z),\ z \in \mathbb{E},$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t}\ dt\right].$$

Proof. By setting $p(z) = \dfrac{f(z)}{zf'(z)}$ in Lemma 1.1, proof follows.

Theorem 2.2. Let q, $q(z) \neq 0$ be univalent in $\mathbb{E}$ such that $\dfrac{zq'(z)}{q(z)}(= h(z))$ is starlike in $\mathbb{E}$.
If $f \in \mathcal{A}$,

$$\frac{f'(z)}{f'(z) + zf''(z)} \neq 0$$

for all z in $\mathbb{E}$, satisfies

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \prec h(z), \ z \in \mathbb{E},$$

then

$$\frac{f'(z)}{f'(z) + zf''(z)} \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t}\, dt\right].$$

Proof. By setting $p(z) = \dfrac{f'(z)}{f'(z) + zf'(z)}$ in Lemma 1.1, proof follows.

Remark 2.1. Consider the dominant

$$q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \ 0 \leq \alpha < 1, \ z \in \mathbb{E}$$

in above theorem, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(1 + \frac{1}{1 - z} - \frac{(1 - 2\alpha)z}{1 + (1 - 2\alpha)z}\right) > 0, \ z \in \mathbb{E}$$

for all $0 \leq \alpha < 1$. Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we immediately get the following result.

Theorem 2.3. If $f \in \mathcal{A}$, $\dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{2(1 - \alpha)z}{(1 - z)(1 + (1 - 2\alpha)z)},$$

then

$$\frac{f(z)}{zf'(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \ 0 \leq \alpha < 1, \ z \in \mathbb{E}.$$

Theorem 2.4. Let $f \in \mathcal{A}$, $\dfrac{f'(z)}{f'(z) + zf''(z)} \neq 0$ for all z in $\mathbb{E}$, satisfy

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \prec \frac{2(1 - \alpha)z}{(1 - z)(1 + (1 - 2\alpha)z)},$$

then

$$\frac{f'(z)}{f'(z) + zf''(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \ 0 \leq \alpha < 1, \ z \in \mathbb{E}.$$

Note that Theorem 2.3 is more general than the result of Kyohei Ochiai [2] stated in Theorem 1.1 and similarly Theorem 2.4 is the general form of Theorem 1.2.

Remark 2.2. Selecting the dominant $q(z) = \left(\dfrac{1 + z}{1 - z}\right)^{\delta}$, $0 < \delta \leq 1$, $z \in \mathbb{E}$, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1 + z^2}{1 - z^2}\right) > 0, \ z \in \mathbb{E}.$$

Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and from Theorem 2.1 and Theorem 2.2, we obtain the following results, respectively.

Theorem 2.5. Suppose that $f \in \mathcal{A}$, $\dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{2\delta z}{1-z^2},$$

then

$$\frac{f(z)}{zf'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \ 0 < \delta \leq 1, \ z \in \mathbb{E}.$$

Theorem 2.6. Let $f \in \mathcal{A}$, $\dfrac{f'(z)}{f'(z) + zf''(z)} \neq 0$ for all z in $\mathbb{E}$, satisfy

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \prec \frac{2\delta z}{1-z^2},$$

then

$$\frac{f'(z)}{f'(z) + zf''(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \ 0 < \delta \leq 1, \ z \in \mathbb{E}.$$

Remark 2.3. When we select the dominant $q(z) = \dfrac{\alpha(1-z)}{\alpha - z}$, $\alpha > 1$, $z \in \mathbb{E}$ in Theorem 2.1 and Theorem 2.2. A little calculation yields

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1-z} + \frac{z}{\alpha - z}\right) > 0, \ z \in \mathbb{E}.$$

Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we get the following results.

Theorem 2.7. Suppose that $\alpha > 1$ is a real number and if $f \in \mathcal{A}$, $\dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{(1-\alpha)z}{(1-z)(\alpha - z)},$$

then

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha - z}, \ z \in \mathbb{E}.$$

Theorem 2.8. Let $\alpha > 1$ be a real number and let $f \in \mathcal{A}$, $\dfrac{f'(z)}{f'(z) + zf''(z)} \neq 0$ for all z in $\mathbb{E}$, satisfy

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \prec \frac{(1-\alpha)z}{(1-z)(\alpha - z)},$$

then

$$\frac{f'(z)}{f'(z) + zf''(z)} \prec \frac{\alpha(1-z)}{\alpha - z}, \ z \in \mathbb{E}.$$

Remark 2.4. Consider the dominant $q(z) = 1 + \lambda z$, $0 < \lambda \leq 1$, $z \in \mathbb{E}$ in Theorem 2.1 and Theorem 2.2, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1+\lambda z}\right) > 0, \ z \in \mathbb{E}$$

for all $0 < \lambda \leq 1$. Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we have the following results.

Theorem 2.9. Suppose $f \in \mathcal{A}$, $\dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{\lambda z}{1 + \lambda z},$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < \lambda,\ 0 < \lambda \leq 1,\ z \in \mathbb{E}.$$

Theorem 2.10. Suppose $f \in \mathcal{A}$, $\dfrac{f'(z)}{f'(z) + zf''(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \prec \frac{\lambda z}{1 + \lambda z},$$

then

$$\left|\frac{f'(z)}{f'(z) + zf''(z)} - 1\right| < \lambda,\ 0 < \lambda \leq 1,\ z \in \mathbb{E}.$$

Remark 2.5. We, now claim that Theorem 2.9 extends Theorem 1.1 in the sense that the operator $\dfrac{zf'(z)}{f(z)} - \left(1 + \dfrac{zf''(z)}{f'(z)}\right)$, now, takes values in an extended and Theorem 2.10 gives the same extension for Theorem 1.2. We, now, compare the results by taking the following particular cases. Setting $\lambda = 1$ in Theorem 2.9, we obtain:
Suppose $f \in \mathcal{A}$, $\dfrac{f(z)}{zf'(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{z}{1 + z}, \tag{2}$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < 1,\ z \in \mathbb{E}.$$

For $\alpha = 1/4$, Theorem 1.1 reduces to the following result:
If $f \in \mathcal{A}$ satisfies

$$\left|\frac{zf'(z)}{f'(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{1}{2}, \tag{3}$$

then

$$\left|\frac{f(z)}{zf'(z)} - 1\right| < 1,\ z \in \mathbb{E},$$

According to the result stated in (3), the operator $\dfrac{zf'(z)}{f(z)} - \left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ takes values within the disk of radius $1/2$ and centered at origin (as shown by the dark shaded portion in Figure 2.1) to give the conclusion that $\left|\dfrac{f(z)}{zf'(z)} - 1\right| < 1$, whereas in view of the result stated above in (2), the same operator can take values in the entire shaded region (dark + light) in Figure 2.1 to get the same conclusion. Thus the result stated in (2) extends the result stated above in (3) in the sense that the region in which the operator $\dfrac{zf'(z)}{f(z)} - \left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ takes values is extended. The claimed extension is given by the light shaded portion of Figure 1. In the same fashion, the above explained extension also holds in comparison of results in Theorem 2.10 and Theorem 1.2.

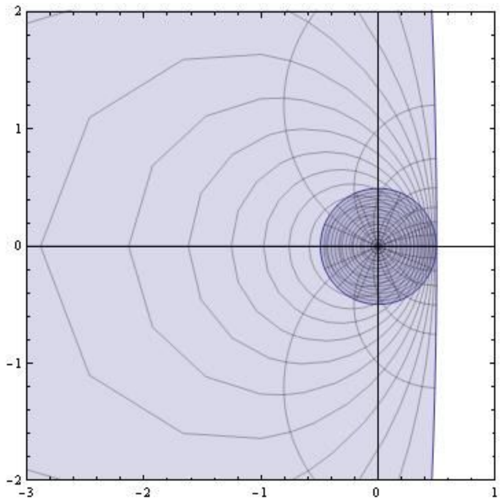

Figure 1

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Certain indefinite integral associate to elliptic integral and complex argument

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Abstract In this paper we have developed some indefinite integrals. The results represent here are assume to be new.
Keywords Floor function, elliptic integral.

§1. Introduction and preliminaries

Definition 1.1. Floor and ceiling functions. In mathematics and computer science, the floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively. More precisely, $floor(x) = \lfloor x \rfloor$ is the largest integer not greater than x and $ceiling(x) = \lceil x \rceil$ is the smallest integer not less than x. The floor function is also called the greatest integer or entier (French for "integer") function, and its value at x is called the integral part or integer part of x. In the following formulas, x and y are real numbers, k, m, and n are integers, and Z is the set of integers (positive, negative, and zero). Floor and ceiling may be defined by the set equations $\lfloor x \rfloor = \max\{m \in Z|\ m \leq x\}$, $\lceil x \rceil = \min\{n \in Z|\ n \geq x\}$.

Since there is exactly one integer in a half-open interval of length one, for any real x there are unique integers m and n satisfying $x - 1 < m \leq x \leq n < x + 1$.

Then $\lfloor x \rfloor = m$ and $\lceil x \rceil = n$ may also be taken as the definition of floor and ceiling. These formulas can be used to simplify expressions involving floors and ceilings.

$\lfloor x \rfloor = m$, if and only if, $m \leq x < m + 1$.

$\lceil x \rceil = n$, if and only if, $n - 1 < x \leq n$.

$\lfloor x \rfloor = m$, if and only if, $x - 1 < m \leq x$.

$\lceil x \rceil = n$, if and only if, $x \leq n < x + 1$.

In the language of order theory, the floor function is a residuated mapping, that is, part of a Galois connection: it is the upper adjoint of the function that embeds the integers into the reals.

$x < n$, if and only if, $\lfloor x \rfloor < n$.

$n < x$, if and only if, $n < \lceil x \rceil$.

$x \leq n$, if and only if, $\lceil x \rceil \leq n$.

$n \leq x$, if and only if, $n \leq \lfloor x \rfloor$.

These formulas show how adding integers to the arguments affect the functions:

$\lfloor x+n \rfloor = \lfloor x \rfloor + n,$

$\lceil x+n \rceil = \lceil x \rceil + n.$

The above are not necessarily true if n is not an integer, however:

$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1,$

$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil.$

Definition 1.2. Elliptic integral. In integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an "elliptic integral" as any function f which can be expressed in the form $f(x) = \int_c^x R\Big(t, \sqrt{P(t)}\Big)dt$, where R is a rational function of its two arguments, P a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when $R(x,y)$ contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).

Besides the Legendre form , the elliptic integrals may also be expressed in Carlson symmetric form. Additional insight into the theory of the elliptic integral may be gained through the study of the Schwarz-Christoffel mapping. Historically, elliptic functions were discovered as inverse functions of elliptic integrals. Incomplete elliptic integrals are functions of two arguments, complete elliptic integrals are functions of a single argument.

Definition 1.3. The incomplete elliptic integral of the first kind F is defined as

$$F(\psi, k) = F(\psi \mid k^2) = F(\sin\psi; k) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

This is the trigonometric form of the integral, substituting $t = \sin\theta,\ x = \sin\psi$, one obtains Jacobi's form

$$F(x; k) = \int_0^{x} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Equivalently, in terms of the amplitude and modular angle one has:

$$F(\psi \backslash \alpha) = F(\psi, \sin\alpha) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-(\sin\theta\sin\alpha)^2}}.$$

In this notation, the use of a vertical bar as delimiter indicates that the argument following it is the "parameter" (as defined above), while the backslash indicates that it is the modular angle. The use of a semicolon implies that the argument preceding it is the sine of the amplitude:

$$F(\psi, \sin\alpha) = F(\psi \mid \sin^2\alpha) = F(\psi \backslash \alpha) = F(\sin\psi; \sin\alpha).$$

Definition 1.4. Incomplete elliptic integral of the second kind E is defined as

$$E(\psi, k) = E(\psi \mid k^2) = E(\sin\psi; k) = \sqrt{1-k^2\sin^2\theta}\ d\theta.$$

Substituting $t = \sin\theta$ and $x = \sin\psi$, one obtains Jacobi's form:

$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt.$$

Equivalently, in terms of the amplitude and modular angle:

$$E(\psi\backslash\alpha) = E(\psi, \sin\alpha) = \int_0^\psi \sqrt{1-(\sin\theta\sin\alpha)^2}\, d\theta.$$

Definition 1.5. Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n;\psi\backslash\alpha) = \int_0^\psi \frac{1}{1-n\sin^2\theta}\,\frac{d\theta}{1-(\sin\theta\sin\alpha)^2}$$

or

$$\Pi(n;\psi \mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2}\,\frac{dt}{(1-mt^2)(1-t^2)}.$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Definition 1.6. Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n;\psi\backslash\alpha) = \int_0^\psi \frac{1}{1-n\sin^2\theta}\,\frac{d\theta}{1-(\sin\theta\sin\alpha)^2}$$

or

$$\Pi(n;\psi \mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2}\,\frac{dt}{(1-mt^2)(1-t^2)}.$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Definition 1.7. Complete elliptic integral of the first kind is defined as Elliptic Integrals are said to be complete when the amplitude

$$\psi = \frac{\pi}{2}$$

and therefore x=1. The complete elliptic integral of the first kind K may thus be defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

or more compactly in terms of the incomplete integral of the first kind as

$$K(k) = F\Big(\frac{\pi}{2}, k\Big) = F(1;k).$$

It can be expressed as a power series

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\Big[P_{2n}(0)\Big]^2 k^{2n},$$

where P_n is the Legendre polynomial, which is equivalent to

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \,.........\, + \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 k^{2n} + \,......\right],$$

where $n!!$ denotes the double factorial. In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; k^2\Big).$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean

$$K(k) = \frac{\frac{\pi}{2}}{agm(1-k, 1+k)}.$$

Definition 1.8. Complete elliptic integral of the second kind is defined as The complete elliptic integral of the second kind E is proportional to the circumference of the ellipse C

$$C = 4aE(e),$$

where a is the semi-major axis, and e is the eccentricity. E may be defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt$$

or more compactly in terms of the incomplete integral of the second kind as

$$E(k) = E\Big(\frac{\pi}{2}, k\Big) = E(1; k).$$

It can be expressed as a power series

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{1-2n},$$

which is equivalent to

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\ .\ 3}{2\ .\ 4}\right)^2 \frac{k^4}{3} - \dots\dots\dots - \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 \frac{k^{2n}}{2n-1} - \dots\dots\right].$$

In terms of the Gauss hypergeometric function, the complete elliptic integral of the second kind can be expressed as

$$E(k) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}; 1; k^2\Big).$$

Definition 1.9. Complete elliptic integral of the third kind is defined as The complete elliptic integral of the third kind Π can be defined as

$$\Pi(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.$$

Argument. In mathematics, arg is a function operating on complex numbers (visualised as a flat plane). It gives the angle between the line joining the point to the origin and the positive real axis, shown as ψ in figure 1 , known as an argument of the point (that is, the angle between the half-lines of the position vector representing the number and the positive real axis).

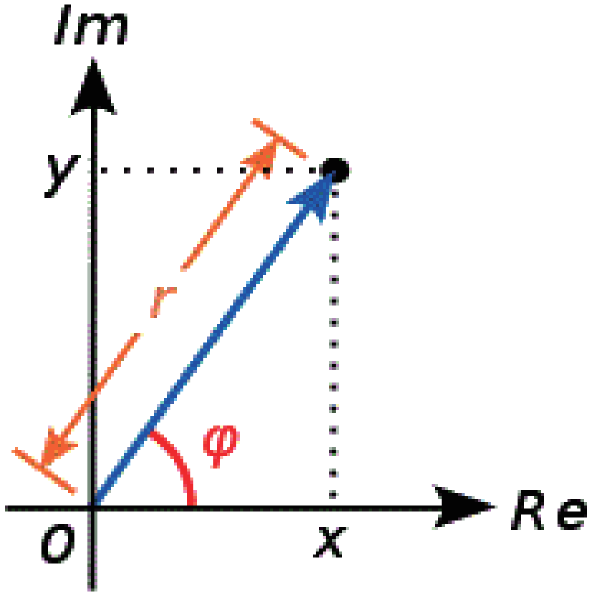

Figure 1.

In figure 1, The Argand diagram represents the complex numbers lying on a plane. For each point on the plane, arg is the function which returns the angle ψ. A complex number may be represented as $z = x + \iota y = |z|e^{\iota\theta}$ where $|z|$ is a positive real number called the complex modulus of z, and ψ is a real number called the argument. The argument is sometimes also known as the phase or, more rarely and more confusingly, the amplitude [7]. The complex argument can be computed as

$$\arg(x + \iota y) \equiv \tan^{-1}\left(\frac{y}{x}\right).$$

§2. Main integrals

$$\begin{aligned}
&\int \frac{\mathrm{d}y}{\sqrt{(1 - x\cosh 2y)}} \\
= &-\frac{\iota\sqrt{\frac{x\cosh 2y-1}{x-1}}F\left(\iota y\middle|\frac{2x}{x-1}\right)}{\sqrt{1 - x\cosh 2y}} + C \\
= &-\frac{\iota\sqrt{x\cosh 2y - 1}F\left(\iota y\middle|\frac{2x}{x-1}\right)exp\left(\iota\pi\left\lfloor\frac{\arg(x-1)}{2\pi} - \frac{\arg(x\cosh 2y-1)}{2\pi} + \frac{1}{2}\right\rfloor\right)}{\sqrt{x-1}\sqrt{1 - x\cosh 2y}} + C. \qquad (1)
\end{aligned}$$

$$
\begin{aligned}
&\int \frac{\mathrm{d}y}{\sqrt{(1-x\sinh 2y)}} \\
=\;& \frac{\iota\sqrt{\frac{x\sinh 2y-1}{-1-\iota x}}F\left(\frac{1}{4}(\pi-4\iota y)\middle|\frac{2x}{x-\iota}\right)}{\sqrt{1-x\sinh 2y}}+C \\
=\;& \frac{\iota\sqrt{x\sinh 2y-1}F\left(\frac{1}{4}(\pi-4\iota y)\middle|\frac{2x}{x-\iota}\right)exp\left(\iota\pi\left\lfloor\frac{\arg(-\iota x-1)}{2\pi}-\frac{\arg(x\sinh 2y-1)}{2\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{-\iota x-1}\sqrt{1-x\sinh 2y}}+C.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&\int \frac{\mathrm{d}y}{\sqrt{(1-x\tanh 2y)}} \\
=\;& \frac{\left[\frac{\tanh^{-1}\left(\frac{\sqrt{x\tanh 2y-1}}{\sqrt{x-1}}\right)}{\sqrt{x-1}}-\frac{\tanh^{-1}\left(\frac{\sqrt{x\tanh 2y-1}}{\sqrt{-x-1}}\right)}{\sqrt{-x-1}}\right]\sqrt{x\tanh 2y-1}}{2\sqrt{1-x\tanh 2y}}+C.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&\int \frac{\mathrm{d}y}{\sqrt{(1-x\coth 2y)}} \\
=\;& \frac{\left[\frac{\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{\iota(x-1)}}\right)}{\sqrt{\iota(x-1)}}-\frac{\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{-\iota(x+1)}}\right)}{\sqrt{-\iota(x+1)}}\right]\sqrt{\iota(x\coth 2y-1)}}{2\sqrt{1-x\coth 2y}}+C \\
=\;& -\frac{\iota\sqrt{\iota(x-1)}\sqrt{\iota(x\coth 2y-1)}\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{\iota(x-1)}}\right)}{2(x-1)\sqrt{1-x\coth 2y}} \\
&-\frac{\iota\sqrt{-\iota(x+1)}\sqrt{\iota(x\coth 2y-1)}\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{-\iota(x+1)}}\right)}{2(x+1)\sqrt{1-x\coth 2y}}+C.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&\int \sqrt{1-x\sinh 2y}\,\mathrm{d}y \\
=\;& -\frac{(x-\iota)\sqrt{\frac{x\sinh 2y-1}{-1-\iota x}}E\left(\frac{1}{4}(\pi-4\iota y)\middle|\ \frac{2x}{x-\iota}\right)}{\sqrt{1-x\sinh 2y}}+Constant \\
=\;& \frac{\iota\sqrt{x\sinh 2y-1}E\left(\frac{1}{4}(\pi-4\iota y)\middle|\frac{2x}{x-\iota}\right)exp\left(\iota\pi\left\lfloor\frac{\arg(-\iota x-1)}{2\pi}-\frac{\arg(x\sinh 2y-1)}{2\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{-\iota x-1}\sqrt{1-x\sinh 2y}} \\
&-\frac{x\sqrt{x\sinh 2y-1}E\left(\frac{1}{4}(\pi-4\iota y)\middle|\frac{2x}{x-\iota}\right)exp\left(\iota\pi\left\lfloor\frac{\arg(-\iota x-1)}{2\pi}-\frac{\arg(x\sinh 2y-1)}{2\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{-\iota x-1}\sqrt{1-x\sinh 2y}} \\
&+C.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&\int \sqrt{1-x\cosh 2y}\mathrm{d}y \\
= \quad & \frac{\iota(x-\iota)\sqrt{\frac{x\cosh 2y-1}{x-1}}E\Big(\iota y\Big|\frac{2x}{x-1}\Big)}{\sqrt{1-x\cosh 2y}} + Constant \\
= \quad & \frac{1}{\sqrt{1-x\cosh 2y}}\iota\ \sqrt{x-1}\sqrt{x\cosh 2y-1} \\
& \times E\Big(\iota y\Big|\frac{2x}{x-1}\Big) exp\Big(\iota\pi\Big\lfloor\frac{\arg(x-1)}{2\pi}-\frac{\arg(x\cosh 2y-1)}{2\pi}+\frac{1}{2}\Big\rfloor\Big)+C.
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\int \sqrt{1-x\tanh 2y}\mathrm{d}y \\
= \quad & \frac{1}{2\sqrt{x\tanh 2y-1}}\Bigg[\Bigg\{\sqrt{x-1}\ \tanh^{-1}\left(\frac{\sqrt{x\tanh 2y-1}}{\sqrt{x-1}}\right) \\
& -\sqrt{-x-1}\tanh^{-1}\left(\frac{\sqrt{x\tanh 2y-1}}{\sqrt{-x-1}}\right)\Bigg\}\sqrt{1-x\tanh 2y-1}\Bigg]+C.
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\int \sqrt{1-x\text{cosech } 2y}\mathrm{d}y \\
= \quad & \frac{1}{\sqrt{2-\iota\text{ cosech } y\text{sech } y}\sqrt{2-x\text{cosech } y\text{sech } y}} \\
& \times\Bigg[2\text{ sech } 2y\sqrt{\frac{\iota(x\text{cosech}2y-1)}{x-\iota}}\Bigg\{x\sqrt{1+\iota\text{ cosech } 2y}(1+2\iota\sinh y\cosh y) \\
& \times F\Bigg(\sin^{-1}\left(\frac{1}{2}\sqrt{2-\iota\text{cosech } y\text{sech } y}\right)\Bigg|\frac{2x}{x-\iota}\Bigg) \\
& +\sinh 2y\sqrt{\coth^2 2y}\sqrt{1-\iota\text{cosech } 2y} \\
& \times\Pi\Bigg(2;\sin^{-1}\left(\frac{1}{2}\sqrt{2-\iota\text{cosech } y\text{sech } y}\right)\Bigg|\frac{2x}{x-\iota}\Bigg)\Bigg\}\Bigg]+C.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\int \sqrt{1-x\coth 2y}\mathrm{d}y \\
= \quad & \frac{1}{2\sqrt{\iota(x\coth 2y-1)}}\Bigg[\Bigg\{\sqrt{\iota(x-1)}\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{\iota(x-1)}}\right) \\
& -\sqrt{-\iota(x+1)}\tanh^{-1}\left(\frac{\sqrt{\iota(x\coth 2y-1)}}{\sqrt{-\iota(x+1)}}\right)\Bigg\}\sqrt{1-x\coth 2y}\Bigg]+C.
\end{aligned}
\tag{9}
$$

where C denotes constant.

Derivations. By involving the method of [18] one can derived the integrals.

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On the types and sums of self-intergrating functions

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Abstract In this paper, two types of self-inetgrability of functions and the underlying conditions for each type was presented. Furthermore, the sum of self-integrable functions was investigated.

Keywords self-integrability, absolutely self-integrating, conditionally self-integrating.

§1. Introduction and preliminaries

Graham [1] shown that the polynomials of the form $p_k(x) = x^k + \frac{k}{k+1}$ has the property that $\int_0^1 p_k(x)\,dx = p_k(1) - p_k(0)$. The said polynomial is self-integrating in the closed interval [0,1].

Definition 1.1. A function f is said to be self-integrating on an interval [a,b] if and only if

$$\int_a^b f(x)\,dx = f(b) - f(a).$$

The following are the types of self-integrability:

Definition 1.2. Absolute self-integrability is the self-integrability everywhere in $\mathbb{R}$.

Definition 1.3. Conditional self-integrability is the self-integrability in some interval [a,b].

Definition 1.4. A function f is said to be absolutely self-integrating if and only if

$$\int_a^b f(x)\,dx = f(b) - f(a)$$

for every [a,b].

Definition 1.5. A function f is said to be conditionally self-integrating if and only if

$$\int_a^b f(x)\,dx = f(b) - f(a)$$

for some [a,b].

The natural exponential function and the zero function are absolutely self-integrating while any polynomial of the form $p_k(x) = x^k + \frac{k}{k+1}$ is conditionally self-integrating.

§2. Main results

Theorem 2.1. The sum of absolutely self-integrating functions is also absolutely (conditionally) self-integrating.

Proof. Suppose $f_0(x),\ f_1(x),\ ..., f_n(x)$ are absolutely self-integrating functions. Taking their sum and the definite integral for any closed interval [a,b] yields

$$\sum_{k=0}^{n-1} f_k(x)$$

and

$$\int_a^b \sum_{k=0}^{n-1} f_k(x)\ dx.$$

Since each function is absolutely self-integrating,

$$\int_a^b \sum_{k=0}^{n-1} f_k(x)\ dx = \sum_{k=0}^{n-1} f_k(b) - \sum_{k=0}^{n-1} f_k(a).$$

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Creation of a summation formula enmeshed with contiguous relation

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Abstract The main aim of the present paper is to create a summation formula enmeshed with contiguous relation and recurrence relation.

Keywords Gaussian hypergeometric function, contiguous function, recurrence relation, Bai -ley summation theorem and Legendre duplication formula.
2010 Mathematics Subject Classification: 33C60, 33C70, 33D15, 33D50, 33D60.

§1. Introduction

Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{c} a_1, a_2, \cdots, a_A; \\ b_1, b_2, \cdots, b_B; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{c} (a_A); \\ (b_B); \end{array} z\right] \equiv {}_AF_B\left[\begin{array}{c} (a_j)_{j=1}^A; \\ (b_j)_{j=1}^B; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}, \tag{1} $$

where the parameters $b_1,\ b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Definition 1.1. Contiguous relation [1] is defined as follows

$$ (a-b)\,{}_2F_1\left[\begin{array}{c} a,\ b; \\ c; \end{array} z\right] = a\,{}_2F_1\left[\begin{array}{c} a+1,\ b; \\ c; \end{array} z\right] - b\,{}_2F_1\left[\begin{array}{c} a,\ b+1; \\ c; \end{array} z\right]. \tag{2} $$

Definition 1.2. Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\Gamma(z). \tag{3} $$

Definition 1.3. Legendre duplication formula [3] is defined as follows

$$ \sqrt{\pi}\Gamma(2z) = 2^{(2z-1)}\Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right). \tag{4} $$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\Gamma(\frac{b}{2})\Gamma(\frac{b+1}{2})}{\Gamma(b)}. \tag{5}$$

$$\Gamma\left(\frac{1}{2}\right) = \frac{2^{(a-1)}\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}{\Gamma(a)}. \tag{6}$$

Definition 1.4. Bailey summation theorem [4] is defined as follows

$${}_2F_1\left[\begin{array}{cc} a, 1-a; & \frac{1}{2} \\ c; & \end{array}\right] = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\Gamma(c)}{2^{c-1}\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})}. \tag{7}$$

§2. Main results of summation formula

$$\begin{aligned}
{}_2F_1\left[\begin{array}{cc} a, & -a-47; \quad \frac{1}{2} \\ & c; \end{array}\right] &= \frac{\sqrt{\pi}\Gamma(c)}{2^{c+47}}\Bigg[\frac{42570157797673022485956840652800000 0}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-78537678825961371341492648312832000 0a}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{469309812545814233523297494114304000a^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-123774601029572534832795487522406400a^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{14625111924087816559179044743557120a^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-438347898583893962516353068868608a^5 - 4963045426867524863361690394214 4a^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{27663009042922306604202519436 80a^7 + 10438858393462191140292708864 0a^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-5303825310692857409567798208a^9 - 192895817207563664676708864a^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{3668538409366573790807760a^{11} + 211050196952959990276560a^{12}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{8379051431422106154 72a^{13} - 846414641853574849 44a^{14} - 150229677358731696 0a^{15}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{183035906280480a^{16} + 255280002782112a^{17} + 2687135353056a^{18} + 5174011920a^{19}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-82882800a^{20} - 570768a^{21} - 1104a^{22} + 108534383559859551664631958208512000 0c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{-149148292903295477473240769632665600 0ac}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})} \\
&+ \frac{70125408817258854426773581520732160 0a^2c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}
\end{aligned}$$

$$+ \quad \frac{-14554836965071641455189162650902528 0a^3c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{1258137789805840973574700014734131 2a^4c - 49858371156931628953152632509440a^5c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-4936679620937371702529617082265 6a^6c + 10121100672934334264714054617 60a^7c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{102659896561333470464292018176a^8c - 1263722647268647716236766144a^9c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-13505826019152250902620486 4a^{10}c - 4681120464349303598176 80a^{11}c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{840036525541234521583 20a^{12}c + 140815017135169030139 2a^{13}c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-10650452865122096752a^{14}c - 498647901368205600a^{15}c - 4011014871347616a^{16}c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{18224511511008a^{17}c + 472341402720a^{18}c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{2500418800a^{19}c + 1237808a^{20}c - 24816a^{21}c}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-48a^{22}c + 112328366596826313483540350631936 0000c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-1217237979186914577980813832683520000ac^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{459637802733991276684591641631457280a^2c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-75109343077380125960104624446357504a^3c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{4495948758102126856436240850358272a^4c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{110820877622010145392305020830720a^5c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-18505209015771704206625420019200a^6c^2 - 561213695213569239726952043 52a^7c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{33391871637919457044328022528a^8c^2 + 2637349548789408462397814 40a^9c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-30231055096849118604799200a^{10}c^2 - 5534535274656380052128 64a^{11}c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{8832091759377462745312a^{12}c^2 + 32768041712693821248 0a^{13}c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{169866366416906016 0a^{14}c^2 - 42511893386751360a^{15}c^2 - 651264806677056a^{16}c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{-2173016514240a^{17}c^2 + 16949664160a^{18}c^2 + 148399680a^{19}c^2 + 315744a^{20}c^2}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \quad \frac{6614556472981625199502950672629760 00c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-580963609797969401418991196472606720ac^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{1778383011792744446473477864070184 96a^2c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-226870732766968058385496042293657 60a^3c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{83467912069355825524288001520 8448a^4c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{5513488815293124614315634180 6080a^5c^3 - 353326780158432792821307284 2240a^6c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-87930473391433199561778719232a^7c^3 + 50775587593400749973962951 68a^8c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{1314566490731106159767376 00a^9c^3 - 2668104364057380084491616a^{10}c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-10696975900717375351596 8a^{11}c^3 - 2107903506627799932 48a^{12}c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{287456996444853664 00a^{13}c^3 + 364055841588845120a^{14}c^3 - 262125361829760a^{15}c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-33950475434304a^{16}c^3 - 222631780800a^{17}c^3 - 206217440a^{18}c^3 + 2150720a^{19}c^3}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{4576a^{20}c^3 + 2555768901964046193668025105776640 00c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-18457288589621695797353471799774 4128ac^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{45919445607132166131729284288544768a^2c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-44921192922093924345184435047628 80a^3c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{71420905733397579565811848294400a^4c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{12764859399187376072264398755840a^5c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-3565128936131831291737736970 24a^6c^4 - 2033131423769297416427466 2400a^7c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{350066261008398378442602240a^8c^4 + 20583161809106285436727680a^9c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-583365805934197699 2128a^{10}c^4 - 9291024467830008583680a^{11}c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{-10386073478979021 8240a^{12}c^4 + 897998110208559360a^{13}c^4 + 23816930790132480a^{14}c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{117051731596800a^{15}c^4 - 518691613440a^{16}c^4 - 6232786560a^{17}c^4 - 14734720a^{18}c^4}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$
$$+ \frac{70089163175122628558183721785622528c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$
\begin{aligned}
&+ \frac{-41922184541690479012551814150619136ac^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{8456755898874383206059513606635520a^2c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-615129901082531819911363834675200a^3c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-2647376356551940301758915661824a^4c^5 + 1794521532945591781937363011584a^5c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-11993561039452441138517864448a^6c^5 - 2458829738109436684070507520a^7c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-1995348997366689088121088a^8c^5 + 1687599926775970318919808a^9c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{18264351429909752911744a^{10}c^5 - 383300198083775132160a^{11}c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-8239334206574062592a^{12}c^5 - 18231457591995648a^{13}c^5 + 666965775700224a^{14}c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{5605172052480a^{15}c^5 + 7689986304a^{16}c^5 - 54198144a^{17}c^5 - 128128a^{18}c^5}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{14329159303517648781796272142024704c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-7118091568289703849052075256709120ac^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{1156582375156654482639067835596800a^2c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-59332304115662378340202427154432a^3c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-1556665917774611161587824971776a^4c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{166118513127876475491303014400a^5c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{1582847367566272466617620480a^6c^6 - 180153547682729260869605376a^7c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-2426884513333370974211328a^8c^6 + 76011904401847599144960a^9c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{1657160946405167723520a^{10}c^6 - 3148455651530821632a^{11}c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-307775278536101376a^{12}c^6 - 2141685186600960a^{13}c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{5207957314560a^{14}c^6 + 99724584960a^{15}c^6 + 265224960a^{16}c^6}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{2257232977580864083344904993898496c^7 - 930849406880543735987990112501760ac^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}
\end{aligned}
$$

$$+ \frac{1204938978889930032879022139637 76a^2c^7 - 394511409444295806940044037324 8a^3c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-2205689687863318556199942758 40a^4c^7 + 10261750967429421015080878080a^5c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{262413376738778861358077952a^6c^7 - 79816726149876049191137 28a^7c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-21759411425882723840332 8a^8c^7 + 13963190641556023296 00a^9c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{736143194132507166 72a^{10}c^7 + 38864098179043737 6a^{11}c^7 - 5447002767478272a^{12}c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-62233922150400a^{13}c^7 - 116380492800a^{14}c^7 + 619407360a^{15}c^7 + 1647360a^{16}c^7}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{2803950694866760398312880839065 60c^8 - 957024827228936782565666067578 88ac^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{971066733878007531907229889331 2a^2c^8 - 159367171451056817372217999360a^3c^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-190481130539842333467829862 40a^4c^8 + 3952739171094091635562905 60a^5c^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{197152347345141772745441 28a^6c^8 - 1692853629154331760230 40a^7c^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-104519972675848384512 00a^8c^8 - 3553087317319507968 0a^9c^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{180623415802935705 6a^{10}c^8 + 18134716325806080a^{11}c^8 - 16934790512640a^{12}c^8}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-797796679680a^{13}c^8 - 2424913920a^14c^8 + 279239568756987155656690714542 08c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-784628966597320372690857073049 6ac^9 + 609792647333324865919533449216a^2c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-101799764013484194117058560 0a^3c^9 - 1141917579520224502976348160a^4c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{59961480056805151316377 60a^5c^9 + 9306096984788959347998 72a^6c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{2396785715316114636800a^7c^9 - 30509028538926080000 0a^8c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-2902564333668679680a^9c^9 + 20997021486465024a^{10}c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{364885993472000a^{11}c^9 + 901447106560a^{12}c^9}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-3854090240a^{13}c^9 - 11714560a^{14}c^9 + 225575898602859630347485079142 4c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-517853350273617644920284119040ac^{10} + 298089285684258901843142246 40a^2c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{3657660721988044333748060l6a^3c^{10} - 4935299077939399568929587 2a^4c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-290716417841497401262080a^5c^{10} + 292356224826703958835 20a^6c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{299167153591153655808a^7c^{10} - 5189724758938976256a^8c^{10} - 81538011850014720a^9c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-35333151293440a^{10}c^{10} + 3616678281216a^{11}c^{10} + 12825100288a^{12}c^{10}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{149006928341289435076960976896c^{11} - 276677720709756930073244467 20ac^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{1120772023535946324592558080a^2c^{11} + 299024139610518897843240 96a^3c^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-15384700427426444336168 96a^4c^{11} - 235558464766592827392 00a^5c^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{59593186095531809177 6a^6c^{11} + 1043576208684200755 2a^7c^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-35268905398272000a^8c^{11} - 1229742091468800a^9c^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{4001177829376a^10c^11 + 14295171072a^{11}c^{11} + 50692096a^{12}c^{11}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{80885775482915268387353395 20c^{12} - 11992107900659556971477729 28ac^{12}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{313694292047442630703841 28a^2c^{12} + 139348500768641273561088 0a^3c^{12}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-3332399668789114699776 0a^4c^{12} - 852515260346324090880a^5c^{12}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{6701161737746251776a^6c^{12} + 206328208054026240a^7c^{12} + 402101871575040a^8c^{12}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-9863668039680a^9c^{12} - 41973055488a^{10}c^{12} + 3616168112688299360995246 08c^{13}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-42115063183514318757953536ac^{13} + 5975739666056969086566 40a^2c^{13}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{4502584416722819481600 0a^3c^{13} - 439220239580485386240a^4c^{13}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{-19449223515157954560a^5c^{13} + 1661681962450944a^6c^{13} + 2458439535820800a^7c^{13}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{10751091671040a^8c^{13} - 32988856320a^9c^{13} - 140378112a^{10}c^{13}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$+ \frac{1330869134419024622806630 4c^{14} - 1192579250017188194549760ac^{14}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}$$

$$
\begin{aligned}
&+ \frac{5164576925934330839040a^2c^{14} + 1056168947710033920000a^3c^{14}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-128046542799372288 0a^4c^{14} - 294089056321536000a^5c^{14} - 1182327161487360a^6c^{14}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{16475577384960a^7c^{14} + 87636049920a^8c^{14} + 4018915219723253 03975936c^{15}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-2698048373764369940 4800ac^{15} - 983420154043086929 92a^2c^{15}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{180495303143451525 12a^3c^{15} + 765811379145277 44a^4c^{15} - 2873436206530560a^5c^{15}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-17760398082048a^6c^{15} + 47755296768a^7c^{15} + 254017536a^8c^{15}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{989331090674368380 9280c^{16} - 4804621189268562247 68ac^{16}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-477219169942005350 4a^2c^{16} + 21980067789275136 0a^3c^{16}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{1693011636387840a^4c^{16} - 16475577384960a^5c^{16}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-116848066560a^6c^{16} + 196451621640022786048c^{17} - 6580790922886250496ac^{17}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-96742888427225088a^2c^{17} + 1810417346150400a^3c^{17} + 17609392128000a^4c^{17}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-42137026560a^5c^{17} - 298844160a^6c^{17} + 309672624364349030 4c^{18}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-66854662522798080ac^{18} - 1209872374824960a^2c^{18} + 9045415034880a^3c^{18}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{96227819520a^4c^{18} + 37830514710675456c^{19} - 474177077248000ac^{19}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-9602450063360a^2c^{19} + 20698890240a^3c^{19} + 220200960a^4c^{19} + 345110101360640c^{20}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-2094727692288ac^{20} - 44568674304a^2c^{20} + 2211455172608c^{21} - 4336910336ac^{21}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-92274688a^2c^{21} + 8875147264c^{22} + 16777216c^{23}}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c+a+47}{2})}\\
&+ \frac{-18191739523752586165066617647923200 00a}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{20756797525313415613899125902172160 00a^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-84167829435283464643877669462231040 0a^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{14961207606774549614863085489038848 0a^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}
\end{aligned}
$$

$$+ \quad \frac{-977008091098561434935328839124556 8a^5 - 293840026022695023557463681921600a^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{5250908312577810021633242163782 4a^7 + 214381848184003097097200360176a^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-1265565168583077960940262981 28a^9 - 117839275654440732282505406 0a^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{1601567318498980808871689 64a^{11} + 34975566419620914299138 41a^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-701798211242953977700 08a^{13} - 3187258922190359349170a^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-203646117143977692 36a^{15} + 704510224912815871a^{16} + 14467181376528672a^{17}}{\Gamma(\frac{c-a+1}{2}\ \Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{677126326109 20a^{18} - 835985767716a^{19} - 12326738369a^{20} - 53614968a^{21} - 2090a^{22}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{564a^{23} + a^{24} + 181917395237528446852340064976896 0000c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-547234267630390527810632543738265 6000ac}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{403078670183197861464544622440611 8400a^2c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-120897919314215463240111590063013 8880a^3c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{157943190504630697055314002831310 848a^4c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-563487733226497765893021615624192 0a^5c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-523567150386864001386596900898816a^6c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{333094147864927590361148662732 80a^7c + 11210978251548806323399327211 52a^8c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-645869278725944615866201098 24a^9c - 2193887426465282761967105664a^{10}c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{4618430289701151522674400 0a^{11}c + 249669782656991658931257 6a^{12}c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{87492066959831053153 92a^{13}c - 1022358942570156430272a^{14}c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-178568599863536467 20a^{15}c + 4798580730768000a^{16}c + 3072017195837568a^{17}c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{32177743065600a^{18}c + 61494544320a^{19}c - 995856576a^{20}c - 6849216a^{21}c - 13248a^{22}c}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{339666292377266025568306534023168 0000c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-599576313503382597734428832484556 8000ac^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{325169180139318211812702003891535 8720a^2c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-74553561440691994500684844717139558 4a^3c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{704562904124948352859800932516659 20a^4c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-605469622715762658411047240466432a^5c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-275184927500939293112075947222016a^6c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{65663155341601084330604869370 88a^7c^2 + 5854794434548954573710115399 68a^8c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-8347537271628761720847987456a^9c^2 - 7940897885123027627117842 56a^{10}c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-2172243856327755605557344a^{11}c^2 + 5051716448758795201946 56a^{12}c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{8276175446578024937952a^{13}c^2 - 660777075091711913 28a^{14}c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-29902362220327766 40a^{15}c^2 - 23862278817807936a^{16}c^2 + 110682100976832a^{17}c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{2835283927168a^{18}c^2 + 14989608480a^{19}c^2 + 7399392a^{20}c^2 - 148896a^{21}c^2 - 288a^{22}c^2}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{2806654727554841548987894861922304000c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-3576617725095301676076645072418897920ac^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{1502019245190770592871716138596696064a^2c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-2661348444403769407125796607115264 00a^3c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{17371915895554619686161169643470848a^4c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{345750175970640380357257123614720a^5c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-71127628355693579436763980527616a^6c^3 - 6685082328605698322922260275 2a^7c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{1307007238434591083583190394 88a^8c^3 + 892408608741546117803162880a^9c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-120831449300126314744395648a^{10}c^3 - 21395770636394363145530 88a^{11}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{36150638027459979484032a^{12}c^3 + 1303477848343243921920a^{13}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{6604265434079089920a^{14}c^3 - 170981195111400960a^{15}c^3 - 2600902267501824a^{16}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{-8642203764480a^{17}c^3 + 67916534400a^{18}c^3 + 593598720a^{19}c^3 + 1262976a^{20}c^3}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\ \frac{1384293040776686070100396020754022400c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$
\begin{aligned}
&+ \frac{-1366474980994844804281508917137113088ac^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{4546293600042006361188787344369254400a^2c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-6210273035936050455592978733673676 8a^3c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{25201216210723327925667877915525 12a^4c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{14795994692508031698326009568768 0a^5c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-1046735322829115965027786489497 6a^6c^4 - 2395950727825934057211281280 00a^7c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{152420810014555559115030627 84a^8c^4 + 3775634421879902383836028 80a^9c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-818335733247159242739894 4a^{10}c^4 - 317057367310254287416128a^{11}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-5502449481840788551 36a^{12}c^4 + 8641636548091225536 0a^{13}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{1085492168723202240a^{14}c^4 - 842397455053440a^{15}c^4 - 101928245445312a^{16}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-667353360960a^{17}c^4 - 617371040a^{18}c^4 + 6452160a^{19}c^4 + 13728a^{20}c^4}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{4618182406174540790214364287247319 04c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-3646874615669714332365219587854172 16ac^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{9706713397594662115946985262940160 0a^2c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-1008185134734249971678430140104704 0a^3c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{18863317639489005275426215565721 6a^4c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{286734376483191089437202320343 04a^5c^5 - 87310112236002206308894710988 8a^6c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-466468040788181040484386201 60a^7c^5 + 868466069177799368437604352a^8c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{4848472043364423316479436 8a^9c^5 - 337450760756104748712 96a^{10}c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-2225925818402234876928 0a^{11}c^5 - 24557379738251411251 2a^{12}c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{218083313948006707 2a^{13}c^5 + 5709794829529190 4a^{14}c^5 + 279727460812800a^{15}c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\\
&+ \frac{-1248042571776a^{16}c^5 - 14958687744a^{17}c^5 - 35363328a^{18}c^5}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}
\end{aligned}
$$

$$+ \frac{1116653729489913547592647744967 14752c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-71698367570652488373650548542406656ac^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{15296793177131728249823697952833536a^2c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-11751877353707612825973819996 24192a^3c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-233898779170829684056646839500 8a^4c^6 + 344611540700205213121191890534 4a^5c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-27552282459296713726038683648a^6c^6 - 48056483784333543751906652 16a^7c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-980210671702040354774016a^8c^6 + 33553117217374027883850 24a^9c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{35500577092308075075328a^{10}c^6 - 771998478818397422592a^{11}c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-16402304433367716864a^{12}c^6 - 35592093514897920a^{13}c^6 + 1335559500667392a^{14}c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{11201672401920a^{15}c^6 + 15356909568a^{16}c^6 - 108396288a^{17}c^6 - 256256a^{18}c^6}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{20438575748507189419885677275774976c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-107554497020821624257557148519 62880ac^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{18322767451542738383637368340 48000a^2c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-9910441046259696864749802474 7008a^3c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-237201377492544171826600452096 0a^4c^7 + 278403351525037406847381995520a^5c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{24029047382985221243286896 64a^6c^7 - 30607630901813301477145804 8a^7c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-3994014305316102353814528a^8c^7 + 130860940480593127833600a^9c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{2811941752341682262016a^{10}c^7 - 56869004047430615 04a^{11}c^7 - 527341651939694592a^{12}c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-3658695573012480a^{13}c^7 + 8966725447680a^{14}c^7 + 170956431360a^{15}c^7}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{454671360a^{16}c^7 + 2916297206121527964486662711410688c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{-12612743166146322414108361194 20928ac^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \frac{1700211990893903913129692871720 96a^2c^8 - 589260934085889900604541430988 8a^3c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-3106526481977888576772426117 12a^4c^8+1527932805326324482215776 2560a^5c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{377006743030727474991645696a^6c^8-12013249741780084010763264a^7c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-320907734198698317390720a^8c^8+214644080916453906432 0a^9c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{1100426089944378362 88a^{10}c^8+576396826987677696a^{11}c^8-8186718489261312a^{12}c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-93281509601280a^{13}c^8-174359877120a^{14}c^8+929111040a^{15}c^8+2471040a^{16}c^8}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{3311589550255909633345783584522 24c^9-1176086207803854596706969122242 56ac^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{123610084505559661368671427624 96a^2c^9-21938190076540411574514352128 0a^3c^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-24420680233427054821567365120a^4c^9+53258537900453404233302016 0a^5c^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{25704338125545841564188672a^6c^9-231633708585966064435200a^7c^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-138320785322120390246 40a^8c^9-45746775337016033280a^9c^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{240900594603058790 4a^{10}c^9+24107288202117120a^{11}c^9-22836222689280a^{12}c^9}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-1063728906240a^{13}c^9-3233218560a^{14}c^9+303718722259536694020968409989 12c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-882348189725499712725770148249 6ac^{10}+707334308986630959539026919424a^2c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-186158150745955808522954342 4a^3c^{10}-1336631292888781215756451840a^4c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{770584422568565289792307 2a^5c^{10}+110362195996099388499558 4a^6c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{264754002826049490124 8a^7c^{10}-36532812406336536576 0a^8c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-345603608312817254 4a^9c^{10}+252843954886082 56a^{10}c^{10}+437548698402816a^{11}c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{1080621301760a^{12}c^{10}-4624908288a^{13}c^{10}-14057472a^{14}c^{10}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{227335789477980180171231697305 6c^{11}-536751437943572449742920089600ac^{11}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{317686577869349343590854164448a^2c^{11}+366109285925362035789398016a^3c^{11}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+\frac{-5304052470365233334294937 6a^4c^{11}-2968344721655590669516 80a^5c^{11}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{3173197745597660332032 0a^6c^{11} + 321420686514924552192a^7c^{11}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-5671123704429281280a^8c^{11} - 88713830348881920a^9c^{11} - 37537902624768a^{10}c^{11}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{3945467215872a^{11}c^{11} + 13991018496a^{12}c^{11} + 13985733125605848573713108172 8c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-265898470730027959083262279 68ac^{12} + 1105174217826109051762638848a^2c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{287432065016420164648304 64a^3c^{12} - 15270018558996421713592 32a^4c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-2305199627431335198720 0a^5c^{12} + 595877349310399676416a^6c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{10371873079034707968a^7c^{12} - 355482719725486 08a^8c^{12} - 1228884381204480a^9c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-3997527998464a^{10}c^{12} + 14295171072a^{11}c^{12} + 50692096a^{12}c^{12}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{710141427802463744138045030 4c^{13} - 1073946580422210336929611776ac^{13}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{288080771978297532324249 60a^2c^{13} + 125689321763176461631488 0a^3c^{13}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-3072343440835630596096 0a^4c^{13} - 778715586672273653760a^5c^{13}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{62187013095570800 64a^6c^{13} + 18999549126770688 0a^7c^{13} + 368717148979200a^8c^{13}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-9104924344320a^9c^{13} - 38744358912a^{10}c^{13} + 2980862476186129326952939 52c^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-3529545971446871007795609 6ac^{14} + 5150647478415309259407 36a^2c^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{3805381761205138882560 0a^3c^{14} - 378755168624679321600a^4c^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-1658460004124590080 0a^5c^{14} + 195682626777907 2a^6c^{14} + 2105801228943360a^7c^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{9207600906240a^8c^{14} - 28276162560a^9c^{14} - 120324096a^{10}c^{14}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{1033388904864872018844057 6c^{15} - 93876489109109030780928 0ac^{15}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{4282991557001670033408a^2c^{15} + 83791186995759493939 2a^3c^{15}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-1078438784098369536a^4c^{15} - 23474402658091008 0a^5c^{15} - 942122591059968a^6c^{15}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{13180461907968a^7c^{15} + 70108839936a^8c^{15} + 29479047441217160531148 8c^{16}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$+ \quad \frac{-2001225891992473750732 8ac^{16} - 70480274486783115264a^2c^{16}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}$$

$$
\begin{aligned}
&+ \frac{1347562162610425036 8a^3c^{16} + 568374328281661 44a^4c^{16} - 2153644495994880a^5c^{16}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-13310137860096a^6c^{16} + 35816472576a^7c^{16} + 190513152a^8c^{16}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{6872581542156492079104c^{17} - 3367463063417687900 16ac^{17}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-332512163158346956 8a^2c^{17} + 154827784747745280a^3c^{17}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{1191602835947520a^4c^{17} - 11629819330560a^5c^{17}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-82480988160a^6c^{17} + 1295343042638139883 52c^{18} - 43691832949824552 96ac^{18}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-64130537664544768a^2c^{18} + 1206158339604480a^3c^{18} + 11731227115520a^4c^{18}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-28091351040a^5c^{18} - 199229440a^6c^{18} + 19420418320115957 76c^{19}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-42140208275128320ac^{19} - 762347173969920a^2c^{19} + 5712893706240a^3c^{19}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{60775464960a^4c^{19} + 22605469306585088c^{20} - 284324096114688ac^{20}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-5757594501120a^2c^{20} + 12419334144a^3c^{20} + 132120576a^4c^{20} + 196815265726464c^{21}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \frac{-1196987252736ac^{21} - 25467813888a^2c^{21} + 1205476524032c^{22} - 2365587456ac^{22}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})} \\
&+ \left.\frac{-50331648a^2c^{22} + 4630511616c^{23} + 8388608c^{24}}{\Gamma(\frac{c-a+1}{2})\Gamma(\frac{c+a+48}{2})}\right]. \qquad (8)
\end{aligned}
$$

Derivation of result (8):

Putting

$$b = -a - 47, \; z = \frac{1}{2}$$

in established result (2), we get

$$
\begin{aligned}
(2a+47)\,{}_2F_1\left[\begin{matrix} a, & -a-47; \\ & c; \end{matrix} \; \frac{1}{2}\right] &= a\,{}_2F_1\left[\begin{matrix} a+1, & -a-47; \\ & c; \end{matrix} \; \frac{1}{2}\right] \\
&\quad + (a+47)\,{}_2F_1\left[\begin{matrix} a, & -a-46; \\ & c; \end{matrix} \; \frac{1}{2}\right].
\end{aligned}
$$

Now proceeding same parallel method which is applied in [6], we can prove the main formula.

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Distar decompositions of a symmetric product digraph

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Abstract In this paper, we decompose the symmetric crown, $C^*_{n,n-1}$ into distars, some sufficient conditions are also given. Furthermore, we consider the problem of decomposing the $(P_m \circ K_n)^*$ into distar $S_{k,l}$-decomposition.

§1. Introduction

A distar $S_{k,l}$ is the digraph obtained from the star S_{k+l} by directing k of the edges out of the center and l of the edges into the center. A multiple star is a star with multiple edges allowed. A directed multiple star is a digraph whose underlying graph is a multiple star recall that for a digraph H and $x \in V(H)$, $d(x) = d^+(x) + d^-(x)$.

The problem of isomorphic $S_{k,l}$-decomposition of symmetric complete digraph K^*_n was posed by Caetano and Heinrich [1], and was solved by Colbourn in [3]. The problem of decomposing the symmetric complete bipartite digraph $K^*_{m,n}$ into $S_{k,l}$ was solved by Hung-Chih Lee in [5]. For positive integers $k \le n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_1,\ a_2, \ldots, a_n,\ b_1,\ b_2, \ldots, b_n\}$ and the edge set $\{a_i\ b_j :\ 1 \le i \le n,\ j \equiv i+1,\ i+2, \ldots, i+k\ (\text{mod } n)\}$. For a graph G, G^* denotes the symmetric digraph of G, that is, the digraph obtained from G by replacing each of its edge by a symmetric pair of arcs. Definitions which are not seen here can be found in [2]. Throughout this paper, we assume that $k,\ l \ge 1$ and $n \ge 1$ are integers, and let (X, Y) be the bipartition of $C^*_{n,n-1}$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The weak product of the graphs (resp. digraphs) G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge (resp. arc) whenever (g_1, g_2) is an edge (resp. arc) in G and (h_1, h_2) is an edge (resp. arc) in H. We use the following results in the proof of the main theorem.

Lemma 1.1. Let $k,\ l,\ t$ be positive integers. Suppose H is a directed multiple star with center w of in degree lt and out degree kt. Then H has an $S_{k,l}$-decomposition if and only if $e(w, x) \le t$ for every end vertex x of H where $e(w, x)$ denotes the number of arcs joining w and x.

Lemma 1.2. $\{((p_1,q_1),(p_2,q_2),\ldots,(p_m,q_m)),((s_1,t_1),(s_2,t_2),\ldots,(s_n,t_n))\}$ is bipartite digraphical if and only if $\{(p_1,p_2,\ldots,p_m),(t_1,t_2,\ldots,t_n)\}$ and $\{(q_1,q_2,\ldots,q_m),(s_1,s_2,\ldots,s_n)\}$ are both bigraphical.

Theorem 1.1. The pair $\{(a_1,a_2,\ldots,a_m),(b_1,b_2,\ldots,b_n)\}$ of the non-negative integral sequences with $a_1 \geq a_2 \geq \ldots \geq a_m$ is bigraphical if and only if $\sum_{i=1}^{r} a_i \leq \sum_{j=1}^{n} \min\{b_j, r\}$ for $1 \leq r \leq m$ and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Definition 1.1. Let $(a_1,a_2,\ldots,a_m),(b_1,b_2,\ldots,b_n)$ be two sequences of non-negative integers. The pair $\{(a_1,a_2,\ldots,a_m),(b_1,b_2,\ldots,b_n)\}$ is called bigraphical if there exists a bipartite graph G with bipartition (M,N), where $M=\{x_1,x_2,\ldots,x_m\}$ and $N=\{y_1,y_2,\ldots,y_n\}$ such that for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. $d_G(x_i)=a_i$ and $d_G(y_j)=b_j$. A simple digraph is a digraph in which each ordered pair of vertices occurs as an arc at most once. Let $((p_1,q_1),(p_2,q_2),\ldots,(p_m,q_m)),((s_1,t_1),(s_2,t_2),\ldots,(s_n,t_n))$ be two sequences of non-negative integral ordered pair. The pair $\{((p_1,q_1),(p_2,q_2),\ldots,(p_m,q_m)),((s_1,t_1),(s_2,t_2),\ldots,(s_n,t_n))\}$ is bipartite digraphical if there exists a simple bipartite digraph H with bipartition (M,N), where $M=\{x_1,x_2,\ldots,x_m\}$ and $N=\{y_1,y_2,\ldots,y_n\}$ such that for $i=1,\ 2,\ldots,m$ and $j=1,\ 2,\ldots,n$. $d_H^-(x_i)=p_i$, $d_H^+(x_i)=q_i$, $d_H^-(y_j)=s_j$, $d_H^+(y_j)=t_j$.

§2. Distar decompositions

Theorem 2.1. Suppose $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition. Then

(i) If $k+l \leq n-1$, then $k=l$ and $k|n-1$.

(ii) If $k+l \leq n-1$, $k=l$, then $k|n-1$.

(iii) If $k+l \leq n-1$, $k \neq l$, then $(k+l)|n(n-1)$ and $\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}$.

(iv) If $k+l = n-1$, $k \neq l$, then $(k+l)|n(n-1)$.

Proof. Let $\mathfrak{D}$ be an $S_{k,l}$-decomposition of $C^*_{n,n-1}$. For $i=1,\ 2,\ldots,n$, let p_i and q_j be the number of $S_{k,l}$ in $\mathfrak{D}$ with centers at x_i and y_j, respectively.

First prove (i). Suppose $k+l \leq n-1$, then any distar $S_{k,l}$ in $\mathfrak{D}$ may have its center in M or in N. Hence $n-1=d^-(x_i)=lp_i$, $n-1=d^+(x_i)=kp_i$.
Thus $k=l$ and $k|n-1$. This completes (i).

Consider (ii). Suppose $k+l \leq n-1$. Let G be a sub digraph of $C^*_{n,n-1}$ such that the arcs of G are those of $S_{k,l}$ in $\mathfrak{D}$ which are oriented to the centers. Then $d_G^-(x_i)=lp_i$, $d_G^+(x_i)=n-1-kp_i$, $d_G^-(y_j)=lq_j$ $d_G^+(y_j)=n-1-kq_j$. Thus

$$\sum_{i=1}^{n} lp_i = \sum_{i=1}^{n} d_G^-(x_i) = \sum_{j=1}^{n} d_G^+(y_j) = \sum_{j=1}^{n}(n-1-kq_j),$$

$$\sum_{j=1}^{n} lq_j = \sum_{j=1}^{n} d_G^-(y_j) = \sum_{i=1}^{n} d_G^+(x_i) = \sum_{i=1}^{n}(n-1-kp_i).$$

Hence

$$l\sum_{i=1}^{n} p_i = n(n-1) - k\sum_{j=1}^{n} q_j, \tag{1}$$

$$k\sum_{i=1}^{n} p_i = n(n-1) - l\sum_{j=1}^{n} q_j. \tag{2}$$

Since $k = l$ (1) implies $k(\sum_{i=1}^{n} p_i + \sum_{j=1}^{n} q_j) = n(n-1)$ Thus $k|n(n-1)$. This completes (ii).

Consider (iii). By assumption, $k \neq l$. Then (1) and (2) imply $(l-k)\sum_{i=1}^{n} p_i = (l-k)\sum_{j=1}^{n} q_j$ $\Rightarrow \sum_{i=1}^{n} p_i = \sum_{j=1}^{n} q_j$, Thus by (1),

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{n} q_j = n(n-1)|k+l, \tag{3}$$

Thus $(k+l)|n(n-1)$.

Note that the number of distars in $\mathfrak{D}$ with centers in M is $n(n-1)|(k+l)$ and so is the number of distars in $\mathfrak{D}$ with centers in N. Thus there is a vertex x_i in M which is the center of atleast $\lceil \frac{n-1}{k+l} \rceil$ distars.

Hence

$$\lceil \frac{n-1}{k+l} \rceil l \leq d^{-}_{C^*_{n,n-1}}(x_i) = n-1, \lceil \frac{n-1}{k+l} \rceil k \leq d^{+}_{C^*_{n,n-1}}(x_i) = n-1.$$

This implies

$$\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}. \tag{4}$$

Similarly, (4) follows if there is a vertex y_j in N which is also the center of atleast $\lceil \frac{n-1}{k+l} \rceil$ distars. This completes (iii). The proof of (iv) is obvious. Since $gcd(n, n-1) = 1$ and $k + l = n - 1$.

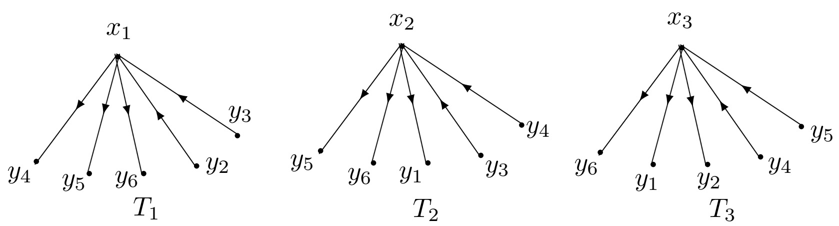
Theorem 2.2. If $2k \leq n-1$ and $k|(n-1)$, then $C^*_{n,n-1}$ has an $S_{k,k}$-decomposition.

Proof. By the assumption, $n - 1 = tk$ for some integer $t \geq 2$. For $j = 1, 2, \ldots, n$, let H_j be the sub digraph of $C^*_{n,n-1}$ induced by $M \cup \{y_j\}$. Then H_j is a directed multiple star with center y_j and $d^+_{H_j}(y_j) = d^-_{H_j}(y_j) = n - 1 = kt$ and $e_{H_j}(y_j, x_i) = 2 \leq t$ $e_{H_j}(y_j, x_j) = 0 \leq t$ (for every $x_i \in M$) and $i \neq j$. Thus, by Lemma 1.1, H_j has an $S_{k,k}$-decomposition. Since $C^*_{n,n-1}$ can be decomposed into $H_1, H_2, \ldots, H_n$. $C^*_{n,n-1}$ has an $S_{k,k}$-decomposition.

Corollary 2.1. Suppose $k + l \leq n - 1$. Then $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition if and only if $k = l$ and $k|(n-1)$.

Theorem 2.3. If $k + l = n - 1$ and $k \neq l$, then $C^*_{n,n-1}$ has $S_{k,l}$-decomposition.

Proof. We consider $C^*_{n,n-1}$ as the symmetric digraph with vertex set $\{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$. Let the distar $S_{k,l}$ decomposition of $C^*_{n,n-1}$ be as follows: For $1 \leq i \leq n$, $T_i = \{(y_{j+1}, x_i) : i \leq j \leq i+l-1\} \cup \{(x_i, y_{j+l}) : i+1 \leq j \leq i+k\}$, $T_{n+i} = \{(y_i, x_{j+1}) : i \leq j \leq i+k-1\} \cup \{(x_{j+k}, y_i) : i+1 \leq j \leq i+l\}$ where the subscripts of x and y are taken modulo n. As an example, the distar $S_{3,2}$-decomposition of $C^*_{6,5}$ are given in Fig. 1.



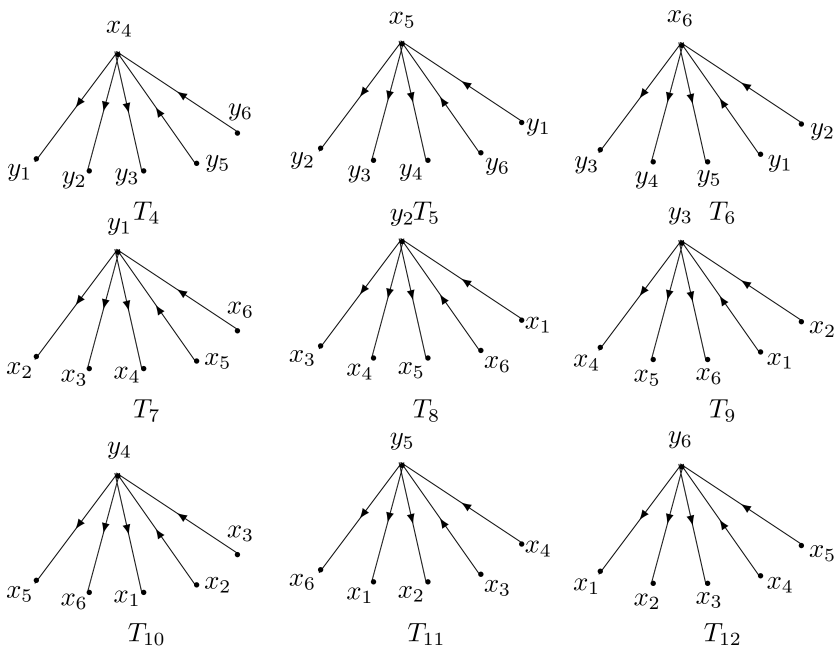


Fig. 1

Corollary 2.2. Suppose $k + l = n - 1$ and $k \neq l$, then $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition if and only if $(k + l)|n(n - 1)$.

Lemma 2.1. $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition if and only if there exists a non-negative integral function f defined on $V(C^*_{n,n-1})$, and $C^*_{n,n-1}$ contains a spanning sub digraph G such that

(i) for every $u \in V(C^*_{n,n-1})$, $d^-_G(u) = lf(u)$ and $d^+_G(u) = d^+_{C^*_{n,n-1}}(u) - kf(u)$.

(ii) if$f(u) = 1$ and $(v, u) \in E(G)$ then $(u, v) \in E(G)$.

Proof. (Necessity) Suppose $\mathfrak{D}$ is an $S_{k,l}$-decomposition of $C^*_{n,n-1}$. For a vertex u, let $f(u)$ be the number of distars in $\mathfrak{D}$ with center at u. Then f is a non-negative integral function defined on $V(C^*_{n,n-1})$. Let G be the spanning sub digraph of $C^*_{n,n-1}$ of which the arcs are those of distars in $\mathfrak{D}$ which are oriented to their centers. Then $d^-_G(u) = lf(u)$ and $d^+_G(u) = d^+_{C^*_{n,n-1}}(u) - kf(u)$. Suppose $f(u) = 1$ and $(v, u) \in E(G)$. Then (v, u) is an arc of a distar in $\mathfrak{D}$ with center at v. Thus $(u, v) \in E(G)$.

(Sufficiency) For each $u \in V(C^*_{n,n-1})$, let $H(u)$ be the directed multiple star with center at u and arc set $\{(v, u) : (v, u) \in E(G)\} \cup \{(u, w) : (u, w) \notin E(G)\}$.

The center of $H(u)$ has in degree $lf(u)$ and out degree $kf(u)$. If $f(u) \geq 2$, then $e_{H(u)}(v, u) \leq e_{C^*_{n,n-1}}(v, u) = 2 \leq f(u)$. If $f(u) = 1$, then (u, v) and (v, u) cannot both in $H(u)$. Hence $e_{H(u)}(v, u) \leq 1 = f(u)$. Thus, by Lemma 1.1, $H(u)$ has an $S_{k,l}$-decomposition. Since $\{H(u) : u \in V(C^*_{n,n-1})\}$ partitions $E(C^*_{n,n-1})$, $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition.

Theorem 2.4. Suppose k, l, n are positive integers such that $k + l \leq n - 1$, $k \neq l$, $(k + l)|n(n - 1)$, $2 \leq \lfloor \frac{n-1}{k+l} \rfloor$ and $\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}$. Then $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition.

Proof. Let $p = \lfloor \frac{n-1}{k+l} \rfloor$ and $n' = \frac{n(n-1)}{k+l} - np$. Note that $n' \geq 0$. Define the numbers a_i, b_i, c_i, d_i $(1 \leq i \leq n)$ as follows: If $n' = 0$, let $a_i = c_i = pl$ if $1 \leq i \leq n$, $b_i = d_i = n - 1 - pk$

if $1 \le i \le n$. If $n' \ge 1$, let

$$a_i = c_i = \begin{cases} (p+1)l, & \text{if} \quad 1 \le i \le n', \\ pl, & \text{if} \quad n'+1 \le i \le n. \end{cases}$$

$$b_i = d_i = \begin{cases} n-1-(p+1)k, & \text{if} \quad 1 \le i \le n', \\ n-1-pk, & \text{if} \quad n'+1 \le i \le n. \end{cases}$$

Since $\lfloor \frac{n-1}{k+l} \rfloor \ge 2$, $p \ge 2$. Now apply the sufficiency of Lemma 2.1. By suitably choosing the non-negative integer function f, we can see that $C^*_{n,n-1}$ has an $S_{k,l}$-decomposition if $\{((a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)),((c_1,d_1),(c_2,d_2),\ldots,(c_n,d_n))\}$ is bipartite digraphical. By Lemma 1.2, it suffices to show that $\{(a_1,a_2,\ldots,a_n),(d_1,d_2,\ldots,d_n)\}$ and $\{(b_1,b_2,\ldots,b_n),(c_1,c_2,\ldots,c_n)\}$ are both bigraphical.

We first prove that $\{(a_1,a_2,\ldots,a_n),(d_1,d_2,\ldots,d_n)\}$ is bigraphical.

$$\begin{aligned} \sum_{i=1}^{n} a_i &= (p+1)ln' + pl(n-n') \\ &= pln' + ln' + pln - pln' \\ &= pln + ln' \\ &= l(n'+np) = \frac{ln(n-1)}{k+l} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{n} d_i &= (n-1-(p+1)k)n' + (n-1-pk)(n-n') \\ &= (n-1)n' - (p+1)kn' + (n-1)n + pkn' - npk - (n-1)n' \\ &= -kn' + n(n-1) - npk \\ &= n(n-1) - (np+n')k \\ &= n(n-1) - \frac{n(n-1)}{k+l}k \\ &= \frac{n(n-1)}{k+l}l. \end{aligned}$$

Thus $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} d_i$.

Next we show that $\sum_{i=1}^{r} a_i \le \sum_{i=1}^{n} \min\{d_i, r\}$ for $1 \le r \le n$. We distinguish three cases depending on the relative position of r and the magnitudes of d_i $(i = 1,2,\ldots,n)$. We first evaluate $\sum_{i=1}^{r} a_i$.

For the case $(k+l)|(n-1)$, we have $p = \frac{n-1}{k+l}$, which implies $n' = \frac{n(n-1)}{k+l} - np = 0$ and $a_i = pl$ for all $1 \le i \le n$. Hence

$$\sum_{i=1}^{r} a_i = \begin{cases} plr, & \text{if} \quad (k+l)|(n-1), \\ (p+1)lr, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } r \le n', \\ (p+1)lr - (r-n')l, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } n'+1 \le r \le n. \end{cases}$$

Case (1): $r \geq d_i$ for all d_i. Then for $1 \leq r \leq n$, we have

$$\begin{aligned}\sum_{i=1}^{n} \min\{d_i, r\} &= \sum_{i=1}^{n} d_i \\ &= \sum_{i=1}^{n} a_i \\ &\geq \sum_{i=1}^{r} a_i.\end{aligned}$$

This completes Case(1).

In the following, we define, for $r = 1, 2, \ldots, n$, $\Delta(G) = \sum_{i=1}^{n} \min\{d_i, r\} - \sum_{i=1}^{r} a_i$. It suffices to show each $\Delta(r) \geq 0$.

Case (2): $r \leq d_i$ for all d_i. Then $\sum_{i=1}^{n} \min\{d_i, r\} = \sum_{i=1}^{n} r = nr$.

$$\Delta(r) = \begin{cases} (n - pl)r, & \text{if} \quad (k+l)|(n-1), \\ (n - (p+1)l)r, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } r \leq n', \\ (n - (p+1)l)r + (r - n')l, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } n' \leq r \leq n. \end{cases}$$

Note that

$$\lceil \frac{n-1}{k+l} \rceil = \begin{cases} p, & \text{if} \quad (k+l)|(n-1), \\ p+1, & \text{if} \quad (k+l) \nmid (n-1). \end{cases}$$

By the assumption that $\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}$, we have

$$\frac{n-1}{l} \geq \begin{cases} p, & \text{if} \quad (k+l)|(n-1), \\ p+1, & \text{if} \quad (k+l) \nmid (n-1). \end{cases}$$

Thus $n - 1 - pl \geq 0$ if $(k+l)|(n-1)$, and $n - 1 - (p+1)l \geq 0$ if $(k+l) \nmid (n-1)$. This implies $\Delta(r) > 0$, which completes Case (2).

Case (3): Neither $r \geq d_i$ for all d_i nor $r \leq d_i$ for all d_i. Then $n-1-(p+1)k < r < n-1-pk$, and $\sum_{i=1}^{n} \min\{d_i, r\} = (n - 1 - (p+1)k)n' + (n - n')r$. Let $\Delta(r)$ be defined as in Case(2). Thus

$$\Delta(r) = \begin{cases} (n - 1 - (p+1)k)n' + (n - n' - pl)r, & \text{if} \quad (k+l)|(n-1), \\ (n - 1 - (p+1)k)n' + \\ (n - n' - (p+1)l)r, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } r \leq n', \\ (n - 1 - (p+1)k)n' + \\ (n - n' - (p+1)l)r + (r - n')l, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } n' \leq r \leq n. \end{cases}$$

Since $r > n - 1 - (p+1)k$, we have

$$\Delta(r) > \begin{cases} (n - 1 - (p+1)k)(n - pl), & \text{if} \quad (k+l)|(n-1), \\ (n - 1 - (p+1)k)(n - (p+1)l), & \text{if} \quad (k+l) \nmid (n-1) \text{ and } r \leq n', \\ (n - 1 - (p+1)k)(n - (p+1)l) + (r - n')l, & \text{if} \quad (k+l) \nmid (n-1) \text{ and } n' \leq r \leq n. \end{cases}$$

Note that in this case $n' \neq 0$. Thus $(k+l) \nmid (n-1)$ and $p+1 = \lceil \frac{n-1}{k+l} \rceil$. By the assumption that $\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}$, we have $n-1|k \geq \lfloor \frac{n-1}{k} \rfloor \geq p+1$, which implies $(n-1)-(p+1)k \geq 0$. Note that (3) also holds in Case (3). Thus $\Delta(r) > 0$. This completes Case (3). By Theorem 1.1, $\{(a_1, a_2, \ldots, a_n), (d_1, d_2, \ldots, d_n)\}$ is bigraphical. Similarly, $\{(b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n)\}$ is bigraphical. This completes the proof.

Theorem 2.5. If $k = l = n-1$, then $(P_3 \circ K_n)^*$ has $S_{k,l}$-decomposition.

Proof. Let $V(P_3) = \{x_1, x_2, x_3\}$ and $V(K_n) = \{y_1, y_2, \ldots y_n\}$. Then $V((P_3 \circ K_n)^*) = \cup_{i=1}^{3}\{x_i \times V(K_n)\} = V_1 \cup V_2 \cup V_3$ and $V_i = \cup_{j=1}^{n}\{x_i^j\}$, $1 \leq i \leq 3$, where x_i^j stands for (x_i, y_j). The distar $S_{k,l}$-decomposition of $(P_3 \circ K_n)^*$ is given as follows: For $1 \leq k \leq n$, $T_k = \{(x_1^j, x_2^k, x_3^j) : 1 \leq j \leq n \text{ and } j \neq k\}$ and $T'_k = \{(x_3^j, x_2^k, x_1^j) : 1 \leq j \leq n \text{ and } j \neq k\}$.

Theorem 2.6. Suppose $(P_3 \circ K_n)^*$ has an $S_{k,l}$-decomposition and if $n \leq k+l \leq 2(n-1)$, then $k = l$ and $k|2(n-1)$.

Proof. Let $\mathfrak{D}$ be an $S_{k,l}$- decomposition of $(P_3 \circ K_n)^*$. Let $V((P_3 \circ K_n)^*) = X \cup Z \cup Y$. For $i = 1,\ 2, \ldots, n$, let r_i be the number of distars $S_{k,l}$ in $\mathfrak{D}$ with centers z_i. Since $n \leq k+l \leq 2(n-1)$, any distar $S_{k,l}$ in $\mathfrak{D}$ must have its center in Z. Hence $2(n-1) = d^-(z_i) = lr_i$, $2(n-1) = d^+(z_i) = kr_i$ Thus $k = l$ and $k|2(n-1)$.

Corollary 2.3. Suppose $n \leq k+l \leq 2(n-1)$. Then $(P_3 \circ K_n)^*$ has $S_{k,l}$-decomposition if and only if $k = l = n-1$ and $k|2(n-1)$.

Theorem 2.7. $(P_m \circ K_n)^*$ has $S_{k,l}$-decomposition if any one of the following conditions hold

(i) If $k+l \leq n-1$, $k = l$ and $k|n-1$.

(ii) If $k+l = n-1$ and $k \neq l$.

(iii) If $k+l \leq n-1$, $k \neq l$, $(k+l)|n(n-1)$, $2 \leq \lfloor \frac{n-1}{k+l} \rfloor$ and $\lceil \frac{n-1}{k+l} \rceil \leq \min\{\lfloor \frac{n-1}{k} \rfloor, \lfloor \frac{n-1}{l} \rfloor\}$.

(iv) If $k = l = n-1$ and m is odd.

Proof. For (i), (ii) and (iii), we have $C^*_{n,n-1}$ has $S_{k,l}$-decomposition. Also we know that

$$(P_m \circ K_n)^* = \underbrace{C^*_{n,n-1} \oplus C^*_{n,n-1} \oplus \ldots \oplus C^*_{n,n-1}}_{(m-1)\text{times}}.$$

Thus $(P_m \circ K_n)^*$ has $S_{k,l}$-decomposition. Similarly, if condition (iv) holds then $(P_3 \circ K_n)^*$ has $S_{k,l}$-decomposition. Also

$$(P_m \circ K_n)^* = \underbrace{(P_3 \circ K_n)^* \oplus \ldots \oplus (P_3 \circ K_n)^*}_{\frac{m-1}{2}\text{times}}.$$

Hence $(P_m \circ K_n)^*$ has $S_{k,l}$-decomposition for this case also. Thus the result follows.

References

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The generalized difference gai sequences of fuzzy numbers defined by orlicz functions

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Abstract In this paper we introduce the classes of gai sequences of fuzzy numbers using generalized difference operator Δ^m(m fixed positive integer) and the Orlicz functions. We study its different properties and also we obtain some inclusion results these classes.

Keywords Fuzzy numbers, difference sequence, Orlicz space, entire sequence, analytic sequ -ence, gai sequence, complete.
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§1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh [18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

In this paper we introduce and examine the concepts of Orlicz space of entire sequence of fuzzy numbers generated by infinite matrices.

Let $C(R^n) = \{A \subset R^n : \text{A compact and convex}\}$. The space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A,\ B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}.$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space.

The fuzzy number is a function X from R^n to [0,1] which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α-level set $[X]^\alpha = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of R^n, with support $X^c = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X,\ \lambda \in R$, in terms of α-level sets, by $|X+Y|^\alpha = |X|^\alpha + |Y|^\alpha,\ |\lambda X|^\alpha = \lambda |X|^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty (X^\alpha, Y^\alpha)^q \, d\alpha\right)^{1/q} \text{ and } d_\infty = \sup_{0\le\alpha\le1} \delta_\infty (X^\alpha, Y^\alpha),$$

where δ_∞ is the Hausdorff metric. Clearly $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \le d_r$, if $q \le r$ [11]. Throughout the paper, d will denote d_q with $1 \le q \le \infty$.

The additive identity in $L(R^n)$ is denoted by $\bar{0}$. For simplicity in notation, we shall write throughout d instead of d_q with $1 \le q \le \infty$.

A metric on $L(R^n)$ is said to be translation invariant if $d(X+Z, Y+Z) = d(X,Y)$ for all $X,\ Y,\ Z \in L(R^n)$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(R^n)$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$. We denotes by $W(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

A complex sequence, whose k^{th} terms is x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. Let $\ell_\infty,\ c,\ c_0$ be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of $\ell_\infty,\ c,\ c_0$ we have $\|x\| = \sup_k |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ. A sequence x is called gai sequence if $\lim_{k\to\infty} (k! |x_k|)^{1/k} = 0$. The vector space of all gai sequences will be denoted by χ. Orlicz [26] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary [28], Mursaleen [29], Bektas and Altin [30], Tripathy [31], Rao and subramanian [32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [33].

Recall [26,33] an Orlicz function is a function $M : [0,\infty) \to [o,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$ then this function is called modulus function, introduced by Nakano [34] and further discussed by Ruckle [35] and Maddox [36] and many others.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \le KM(u)$ $(u \ge 0)$. The Δ_2-condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$. Lindenstrauss and Tzafriri [27] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}. \tag{1}$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}, \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \le p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p. Given a sequence $x = \{x_k\}$ its n^{th}

section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$, $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ...)$, 1 in the n^{th} place and zero's else where.

§2. Remark

An Orlicz function M satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. Let $m \in \mathbb{N}$ be fixed, then the generalized difference operation

$$\Delta^m : W(F) \to W(F)$$

is defined by

$$\Delta X_k = X_k - X_{k+1} \text{ and } \Delta^m X_k = \Delta\left(\Delta^{m-1} X_k\right) \ (m \geq 2) \text{ for all } k \in \mathbb{N}.$$

§3. Definitions and preliminaries

Let P_s denotes the class of subsets of $\mathbb{N}$, the natural numbers, those do not contain more than s elements. Throughout (ϕ_n) represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The sequence $\chi(\phi)$ for real numbers is defined as follows

$$\chi(\phi) = \left\{(X_k) : \tfrac{1}{\phi_s}\left(k!\,|X_k|\right)^{1/k} \to 0 \text{ as } k,\ s \to \infty \text{ for } k \in \sigma \in P_s\right\}.$$

The generalized sequence space $\chi(\Delta_n, \phi)$ of the sequence space $\chi(\phi)$ for real numbers is defined as follows

$$\chi(\Delta_n, \phi) = \left\{(X_k) : \tfrac{1}{\phi_s}\left(k!\,|\Delta X_k|\right)^{1/k} \to 0 \text{ as } k,\ s \to \infty \text{ for } k \in \sigma \in P_s\right\},$$

where $\Delta_n X_k = X_k - X_{k+n}$ for $k \in \mathbb{N}$ and fixed $n \in \mathbb{N}$.

In this article we introduce the following classes of sequences of fuzzy numbers.

Let M be an Orlicz function, then

$$\Lambda_M^F(\Delta^m) = \left\{(X_k) \in W(F) : \sup_k M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k}\right), \bar{0}\right)}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

$$\chi_M^F(\Delta^m) = \left\{(X_k) \in W(F) : M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0\right\}.$$

$$\Gamma_M^F(\Delta^m) = \left\{(X_k) \in W(F) : M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k}\right), \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k \to \infty, \text{ for some } \rho > 0\right\}.$$

$$\chi_M^F\left(\Delta^m,\phi\right)$$
$$= \left\{ (X_k) \in W(F) : \frac{1}{\phi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho} \right) \to 0 \text{ as } k,\ s\to\infty,\ \text{for } k\in\sigma\in P_s \right\}.$$
$$\Gamma_M^F\left(\Delta^m,\phi\right)$$
$$= \left\{ (X_k) \in W(F) : \frac{1}{\phi_s} M\left(\frac{d\left(\left(\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho} \right) \to 0 \text{ as } k,\ s\to\infty,\ \text{for } k\in\sigma\in P_s \right\}.$$

§4. Main results

In this section we prove some results involving the classes of sequences of fuzzy numbers $\chi_M^F\left(\Delta^m,\phi\right),\ \chi_M^F\left(\Delta^m\right)$ and $\Lambda_M^F\left(\Delta^m\right)$.

Theorem 4.1. If d is a translation invariant metric, then $\chi_M^F\left(\Delta^m,\phi\right)$ are closed under the operations of addition and scalar multiplication.

Proof. Since d is a translation invariant metric implies that

$$d\left(\left(k!\left(\Delta^m X_k+\Delta^m Y_k\right)\right)^{1/k},\bar{0}\right) \le d\left(\left(k!\left(\Delta^m X_k\right)\right)^{1/k},\bar{0}\right) + d\left(\left(k!\left(\Delta^m Y_k\right)\right)^{1/k},\bar{0}\right) \tag{3}$$

and

$$d\left(\left(k!\left(\Delta^m \lambda X_k\right)\right)^{1/k},\bar{0}\right) \le |\lambda|^{1/k} d\left(\left(k!\left(\Delta^m X_k\right)\right)^{1/k},\bar{0}\right), \tag{4}$$

where λ is a scalar and $|\lambda|^{1/k} > 1$. Let $X=(X_k)$ and $Y=(Y_k)\in\chi_M^F\left(\Delta^m,\phi\right)$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\frac{1}{\phi_s} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_1}\right) \to 0 \text{ as } k,s\to\infty,\ \text{for } k\in\sigma\in P_s.$$
$$\frac{1}{\phi_s} M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho_2}\right) \to 0 \text{ as } k,s\to\infty,\ \text{for } k\in\sigma\in P_s.$$

Let $\rho_3=\max\left(2\rho_1,2\rho_2\right)$. By the equation (3) and since M is non-decreasing convex function, we have

$$\begin{aligned}
& M\left(\frac{d\left(\left(k!\left|\Delta^m X_k+\Delta^m Y_k\right|\right)^{1/k},\bar{0}\right)}{\rho}\right) \\
\le\ & M\left(\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k},\bar{0}\right)}{\rho_3}\right)+\left(\frac{d\left(\left(k!\left|\Delta^m Y_k\right|\right)^{1/k},\bar{0}\right)}{\rho_3}\right)\right) \\
\le\ & \frac{1}{2}M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k},\bar{0}\right)}{\rho_1}\right)+\frac{1}{2}M\left(\frac{d\left(\left(k!\left|\Delta^m Y_k\right|\right)^{1/k},\bar{0}\right)}{\rho_2}\right) \\
\Rightarrow\ & \frac{1}{\phi_s}M\left(\frac{d\left(\left(k!\left|\Delta^m X_k+\Delta^m Y_k\right|\right)^{1/k},\bar{0}\right)}{\rho_3}\right) \\
\le\ & \frac{1}{\phi_s}M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k},\bar{0}\right)}{\rho_1}\right)+\frac{1}{\phi_s}M\left(\frac{d\left(\left(k!\left|\Delta^m Y_k\right|\right)^{1/k},\bar{0}\right)}{\rho_2}\right).
\end{aligned}$$

for $k \in \sigma \in P_s$, Hence $X + Y \in \chi_M^F(\Delta^m, \phi)$. Now, let $X = (X_k) \in \chi_M^F(\Delta^m, \phi)$ and $\lambda \in$ R with $0 < |\lambda|^{1/k} < 1$. By the condition (4) and Remark, we have

$$\begin{aligned} & M\left(\frac{d\left((k!\,|\Delta^m \lambda X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \\ \leq \; & M\left(\frac{|\lambda|^{1/k}\, d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \\ \leq \; & |\lambda|^{1/k} M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right). \end{aligned}$$

Therefore $\lambda X \in \chi_M^F(\Delta^m, \phi)$. This completes the proof.

Theorem 4.2. The space $\chi_M^F(\Delta^m, \phi)$ is a complete metric space with the metric by

$$\begin{aligned} g(X, Y) &= d\left((k!\,|X_k - Y_k|)^{1/k}\right) \\ &+ \inf\left\{\rho > 0 : \sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\,(|\Delta^m X_k - \Delta^m Y_k|)^{1/k}\right)}{\rho}\right)\right) \leq 1\right\}. \end{aligned}$$

Proof. Let (X^i) be a cauchy sequence in $\chi_M^F(\Delta^m, \phi)$. Then for each $\epsilon > 0$, there exists a positive integer n_0 such that $g(X^i, Y^j) < \epsilon$ for $i,\ j \geq n_0$, then $\Rightarrow$
$d\left(\left(k!\left|X_k^i - Y_k^j\right|\right)^{1/k}\right) + \inf\left\{\rho > 0 : \sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m Y_k^j\right|\right)^{1/k}\right)}{\rho}\right)\right) \leq 1\right\} < \epsilon$,
for all $i,\ j \geq n_0$

$$d\left(\left(k!\left|X_k^i - Y_k^j\right|\right)^{1/k}\right) < \epsilon$$

for all $i,\ j \geq n_0$ and $\inf\left\{\rho > 0 : \sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!(|\Delta^m X_k - \Delta^m Y_k|)^{1/k}\right)}{\rho}\right)\right) \leq 1\right\} < \epsilon$ for all $i,\ j \geq n_0$.

By (5), $d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}\right) < \epsilon$ for all $i,\ j \geq n_0$ and $k = 1,\ 2,\ 3, ..., m$. It follows that (X_k^i) is a cauchy sequence in $L(R)$ for $k = 1,\ 2,\ 3, ..., m$. Since $L(R)$ is complete, then (X_k^i) is convergent in $L(R)$. Let $\lim_{i\to\infty} X_k^i = X_k$ for $k = 1,\ 2, ..., m$. Now (6) for a given $\epsilon > 0$, there exists some $\rho_\epsilon\ (0 < \rho_\epsilon < \epsilon)$ such that

$$\sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right)}{\rho_\epsilon}\right)\right) \leq 1.$$

Thus

$$\begin{aligned} & \left(\sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right)}{\rho}\right)\right) \leq 1\right) \\ \leq \; & \left(\sup_{k\in\sigma\in P_s} \frac{1}{\phi_s}\left(M\left(\frac{d\left(k!\left(\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right)}{\rho_\epsilon}\right)\right) \leq 1\right), \end{aligned}$$

we have $d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right) < \epsilon$ and the fact that

$$\begin{aligned} d\left(\left(k!\left|X_{k+m}^i - X_{k+m}^j\right|\right)^{1/k}\right) &\leq d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right) \\ &= \binom{m}{0} d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}\right) \\ &\quad + \binom{m}{1} d\left(\left(k!\left|X_{k+1}^i - X_{k+1}^j\right|\right)^{1/k}\right) \\ &\quad + \cdots + \binom{m}{m-1} d\left(\left(k!\left|X_{k+m-1}^i - X_{k+m-1}^j\right|\right)^{1/k}\right). \end{aligned}$$

So, we have $d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}\right) < \epsilon$ for each $k \in \mathbb{N}$. Therefore $\left(X^i\right)$ is a cauchy sequence in $L(R)$. Since $L(R)$. is complete, then it is convergent in $L(R)$. Let $\lim_{i\to\infty} X_k^i = X_k$ say, for each $k \in \mathbb{N}$. Since $\left(X^i\right)$ is a cauchy sequence, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that $g\left(X^i, X^j\right) < \epsilon$ for all $i,\ j \geq n_0$. So we have

$$\lim_{j\to\infty} d\left(\left(k!\left|X_k^i - X_k^j\right|\right)^{1/k}\right) = d\left(\left(k!\left|X_k^i - X_k\right|\right)^{1/k}\right) < \epsilon$$

and

$$\lim_{j\to\infty} d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k^j\right|\right)^{1/k}\right) = d\left(\left(k!\left|\Delta^m X_k^i - \Delta^m X_k\right|\right)^{1/k}\right) < \epsilon$$

for all $i,\ j \geq n_0$. This implies that $g\left(X^i, X\right) < \epsilon$ for all $i \geq n_0$. That is $X^i \to X$ as $i \to \infty$, where $X = (X_k)$. Since

$$d\left(\left(k!\left|\Delta^m X_k - X_0\right|\right)^{1/k}\right) \leq d\left(\left(k!\left|\Delta^m X_k^{n_0} - X_0\right|\right)^{1/k}\right) + d\left(\left(k!\left|\Delta^m X_k^{n_0} - \Delta^m X_k\right|\right)^{1/k}\right),$$

we obtain $X = (X_k) \in \chi_M^F$. Therefore $\chi_M^F(\Delta^m, \phi)$ is complete metric space. This completes the proof.

Proposition 4.1. The space $\Lambda_M^F(\Delta^m)$ is a complete metric space with the metric by

$$h(X, Y) = \inf\left\{\rho > 0 : \sup_k\left(M\left(\tfrac{d\left(\left(\left|\Delta^m X_k - \Delta^m Y_k\right|\right)^{1/k}\right)}{\rho}\right)\right) \leq 1\right\}.$$

Theorem 4.3. If $\left(\frac{\phi_s}{\psi_s}\right) \to$ as $s \to \infty$ then $\chi_M^F(\Delta^m, \phi) \subset \chi_M^F(\Delta^m, \psi)$

Proof. Let $\left(\frac{\phi_s}{\psi_s}\right) \to 0$ as $s \to \infty$ and $X = (X_k) \in \chi_M^F(\Delta^m, \phi)$. Then, for some $\rho > 0$,

$$\frac{1}{\phi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k,\ s \to \infty,\ \text{for } k \in \sigma \in P_s$$

$$\Rightarrow\ \frac{1}{\psi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right) \leq \left(\frac{\phi_s}{\psi_s}\right)\left(\frac{1}{\phi_s} M\left(\frac{d\left(\left(k!\left|\Delta^m X_k\right|\right)^{1/k}, \bar{0}\right)}{\rho}\right)\right).$$

Therefore $X = (X_k) \in \chi_M^F(\Delta^m, \psi)$. Hence $\chi_M^F(\Delta^m, \phi) \subset \chi_M^F(\Delta^m, \psi)$. This completes the proof.

Proposition 4.2. If

$$\left(\frac{\phi_s}{\psi_s}\right) \text{ and } \left(\frac{\psi_s}{\phi_s}\right) \to 0 \text{ as } s \to \infty,$$

then

$$\chi_M^F(\Delta^m, \phi) = \chi_M^F(\Delta^m, \psi).$$

Theorem 4.4. $\chi_M^F(\Delta^m) \subset \Gamma_M^F(\Delta^m, \phi)$.

Proof. Let $X = (X_k) \in \chi_M^F(\Delta^m)$. Then we have

$$M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k \to \infty \text{ for some } \rho > 0.$$

Since (ϕ_n) is monotonic increasing, so we have

$$\begin{aligned} & \frac{1}{\phi_s} M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \\ \leq\ & \frac{1}{\phi_1} M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \\ \leq\ & \frac{1}{\phi_s} M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right). \end{aligned}$$

Therefore

$$\frac{1}{\phi_s} M\left(\frac{d\left((k!\,|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k,\ s \to \infty$$

for $k \in \sigma \in P_s$. Hence

$$\frac{1}{\phi_s} M\left(\frac{d\left((|\Delta^m X_k|)^{1/k}, \bar{0}\right)}{\rho}\right) \to 0 \text{ as } k,\ s \to \infty$$

for $k \in \sigma \in P_s$ and $(k!)^{1/k} \to 1$. Thus

$$X = (X_k) \in \Gamma_M^F(\Delta^m, \phi).$$

Therefore

$$\chi_M^F(\Delta^m) \subset \Gamma_M^F(\Delta^m, \phi).$$

This completes the proof.

Theorem 4.5. Let M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. Then

$$\chi_{M_2}^F(\Delta^m, \phi) \subset \chi_{M_1 \circ M_2}^F(\Delta^m, \phi).$$

Proof. Let $X = (X_k) \in \Gamma_{M_2}^F(\Delta^m, \phi)$. Then there exists $\rho > 0$ such that

$$\frac{1}{\phi_s} M_2 \left(\frac{d\big((k!|\Delta^m X_k|)^{1/k}, \bar{0}\big)}{\rho} \right) \to 0 \text{ as } k,\ s \to \infty \text{ for } k \in \sigma \in P_s.$$

Let $0 < \epsilon < 1$ and δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \le 1 < \delta$. Let

$$y_k = M_2 \left(\frac{d\big((k!|\Delta^m X_k|)^{1/k}, \bar{0}\big)}{\rho} \right) \text{ for all } k \in \mathbb{N}.$$

Now, let

$$M_1(y_k) = M_1(y_k) + M_1(y_k), \tag{5}$$

where the equation (7) RHS of the first term is over $y_k \le \delta$ and the equation of (7) RHS of the second term is over $y_k > \delta$. By the Remark, we have

$$M_1(y_k) \le M_1(1)\, y_k + M_1(2)\, y_k. \tag{6}$$

For $y_k > \delta$,

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

Since M_1 is non-decreasing and convex, so

$$M_1(y_k) < M_1\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1\left(\frac{2y_k}{\delta}\right).$$

Since M_1 satisfies Δ_2-condition, so

$$M_1(y_k) < \frac{1}{2} K M_1(2) \frac{y_k}{\delta} + \frac{1}{2} K M_1(2) \frac{y_k}{\delta}.$$

Hence the equation (7) in RHS of second terms is

$$M_1(y_k) \le max\left(1, K\delta^{-1} M_1(2)\right) y_k. \tag{7}$$

By equation (8) and (9), we have

$$X = (X_k) \in \chi^F_{M_1 \circ M_2}(\Delta^m, \phi).$$

Thus,

$$\chi^F_{M_2}(\Delta^m, \phi) \subset \chi^F_{M_1 \circ M_2}(\Delta^m, \phi).$$

This completes the proof.

Propositon 4.3. Let M be an Orlicz function which satisfies Δ_2-condition. Then $\chi^F(\Delta^m, \phi) \subset \chi^F_M(\Delta^m, \phi)$

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Hankel determinant for a new subclass of analytic functions

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Abstract We introduce a new subclass of analytic-univalent functions and determine the sharp upper bounds of the second Hankel determinant for the functions belonging this class.

Keywords Analytic functions, starlike functions, convex functions, Hankel determinant, coefficient bounds.

§1. Introduction and preliminaries

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the unit disc $E = \{z :| z |< 1\}$.

Let S be the class of functions $f(z) \in A$ and univalent in E.

Let $M(\alpha)$ $(0 \leq \alpha \leq 1)$ be the class of functions $f(z) \in A$ which satisfy the condition

$$Re \left\{ \frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \right\} > 0,\ z \in E. \tag{2}$$

This class was introduced by Singh [13] and Fekete-Szegö inequality for functions of this class was establihed by him. Obviously $M(\alpha)$ is the subclass of the class of α-convex functions introduced by Mocanu [9]. In particular $M(0) \equiv S^*$, the class of starlike functions and $M(1) \equiv K$, the class of convex functions.

In 1976, Noonan and Thomas [10] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by Eq. (1) with bounded boundary. Ehrenborg [1] studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman [5]. Also Hankel determinant for various classes was studied by several authors including Hayman [3], Pommerenke [12] and recently by Mehrok and Singh [8].

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [2] then further generalised the estimate of $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In this paper, we seek upper bound of the functional $|a_2a_4 - a_3^2|$ for functions belonging to the above defined class.

§2. Main result

Let P be the family of all functions p analytic in E for which $Re(p(z)) > 0$ and

$$p(z) = 1 + p_1(z) + p_2z^2 + ... \tag{3}$$

for $z \in E$.

Lemma 2.1. If $p \in P$, then $|p_k| \leq 2(k = 1, 2, 3, ...)$.

This result is due to Pommerenke [12].

Lemma 2.2. If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x and z satisfying $|x| \leq 1,\ |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [6,7].

Theorem 2.1. If $f \in M(\alpha)$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+\alpha)(1+3\alpha)}. \tag{4}$$

Proof. As $f \in M(\alpha)$, so from (2)

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} = p(z). \tag{5}$$

On expanding and equating the coefficients of z, z^2 and z^3 in (5), we obtain

$$a_2 = \frac{p_1}{1+\alpha}, \tag{6}$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{p_1^2}{2(1+2\alpha)} \tag{7}$$

and

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{p_1 p_2}{2(1+3\alpha)} + \frac{p_1^3}{6(1+3\alpha)}. \tag{8}$$

Using (6), (7) and (8), it yields

$$a_2a_4 - a_3^2 = \frac{1}{C(\alpha)}\begin{bmatrix} 4(1+2\alpha)^2 p_1p_3 + 6((1+2\alpha)^2 - (1+\alpha)(1+3\alpha))p_1^2p_2 \\ +(2(1+2\alpha)^2 - 3(1+\alpha)(1+3\alpha))p_1^4 - 3(1+\alpha)(1+3\alpha)p_2^2 \end{bmatrix}, \tag{9}$$

where $C(\alpha) = 12(1+\alpha)(1+3\alpha)(1+2\alpha)^2$.
Using Lemma 2.1 and Lemma 2.2 in (9), we obtain

$$|a_2a_4 - a_3^2| = \frac{1}{4C(\alpha)}\begin{vmatrix} -[-4(1+2\alpha)^2 - 12\alpha^2 + 4(1+4\alpha+\alpha^2) + 3(1+\alpha)(1+3\alpha)]p_1^4 \\ +[8(1+2\alpha)^2 + 12\alpha^2 - 6(1+\alpha)(1+3\alpha)]p_1^2(4-p_1^2)x \\ -[(4(1+2\alpha)^2 - 3(1+\alpha)(1+3\alpha))p_1^2 + 12(1+\alpha)(1+3\alpha)](4-p_1^2)x^2 \\ +8(1+2\alpha)^2 p_1(4-p_1^2)(1-|x|^2)z \end{vmatrix}.$$

Assume that $p_1 = p$ and $p \in [0,2]$, using triangular inequality and $|z| \leq 1$, we have

$$|a_2a_4 - a_3^2| \leq \frac{1}{4C(\alpha)}\begin{bmatrix} [-4(1+2\alpha)^2 - 12\alpha^2 + 4(1+4\alpha+\alpha^2) + 3(1+\alpha)(1+3\alpha)]p^4 \\ +[8(1+2\alpha)^2 + 12\alpha^2 - 6(1+\alpha)(1+3\alpha)]p^2(4-p^2)|x| \\ +[(4(1+2\alpha)^2 - 3(1+\alpha)(1+3\alpha))p^2 + 12(1+\alpha)(1+3\alpha)](4-p^2)|x|^2 \\ +8(1+2\alpha)^2 p(4-p^2)(1-|x|^2) \end{bmatrix}$$

or

$$|a_2a_4 - a_3^2| \leq \frac{1}{4C(\alpha)}\begin{bmatrix} [-4(1+2\alpha)^2 - 12\alpha^2 + 4(1+4\alpha+\alpha^2) + 3(1+\alpha)(1+3\alpha)]p^4 + \\ 8(1+2\alpha)^2 p(4-p^2) + [8(1+2\alpha)^2 + 12\alpha^2 - 6(1+\alpha)(1+3\alpha)] \\ p^2(4-p^2)\delta + [(4(1+2\alpha)^2 - 3(1+\alpha)(1+3\alpha))p^2 \\ -8(1+2\alpha)^2 p + 12(1+\alpha)(1+3\alpha)](4-p^2)\delta^2 \end{bmatrix}.$$

Therefore

$$|a_2a_4 - a_3^2| = \frac{1}{4C(\alpha)}F(\delta),$$

where $\delta = |x| \leq 1$ and

$$\begin{aligned} F(\delta) \;=\; & [-4(1+2\alpha)^2 - 12\alpha^2 + 4(1+4\alpha+\alpha^2) + 3(1+\alpha)(1+3\alpha)]p^4 \\ & +8(1+2\alpha)^2 p(4-p^2) + [8(1+2\alpha)^2 + 12\alpha^2 - 6(1+\alpha)(1+3\alpha)]p^2(4-p^2)\delta \\ & +[(4(1+2\alpha)^2 - 3(1+\alpha)(1+3\alpha))p^2 - 8(1+2\alpha)^2 p + 12(1+\alpha)(1+3\alpha)](4-p^2)\delta^2 \end{aligned}$$

is an increasing function.
Therefore Max$F(\delta) = F(1)$.
Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{4C(\alpha)}G(p), \tag{10}$$

where $G(p) = F(1)$. So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 48(1+\alpha)(1+3\alpha),$$

where

$$A(\alpha) = 48\alpha^2$$

and

$$B(\alpha) = 96\alpha^2.$$

Now

$$G^{'}(p) = -4A(\alpha)p^3 + 2B(\alpha)p$$

and

$$G^{''}(p) = -12A(\alpha)p^2 + 2B(\alpha).$$

$G^{'}(p) = 0$ gives

$$p[2A(\alpha)p^2 - B(\alpha)] = 0.$$

$G^{''}(p)$ is negative at

$$p = \sqrt{\frac{B(\alpha)}{2A(\alpha)}} = 1.$$

So Max$G(p) = G(1)$.
Hence from (10), we obtain (4).
The result is sharp for $p_1 = 1$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(p_1^2 - 3)$.
For $\alpha = 0$ and $\alpha = 1$ respectively, we obtain the following results due to Janteng [4].

Corollary 2.1. If $f(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Corollary 2.2. If $f(z) \in K$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

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On non-unit speed curves in Minkowski 3-space

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Abstract In this work, by means of the method used to obtain Frenet apparatus and Darboux frame for non-unit speed curve in $\mathbb{E}^3$ in [3], we present Frenet apparatus for non-unit speed curves in $\mathbb{E}_1^3$. Also, Darboux frame for non-unit speed curves and surface pair is given in $\mathbb{E}_1^3$.

Keywords Non-unit speed curve, spacelike non-unit speed curve, timelike non-unit speed c -urve, Darboux frame on non-unit speed curve-surface pair.
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§1. Introduction

Suffice it to say that the many important results in the theory of the curves in $\mathbb{E}^3$ were initiated by G. Monge and the moving frames idea was due to G. Darboux. In the classical differential geometry of curves in $\mathbb{E}^3$, the unit speed curve which is obtained by using the norm of the curve, is a well known concept. In the light of this idea, there are many studies on the unit speed curve in Minkowski space. Frenet apparatus for non-unit speed curve $\mathbb{E}^3$ has been studied by Sabuncuoglu in [3]. Also he has studied Darboux frame and its derivative formulas for non-unit speed curves and surface pair in $\mathbb{E}^3$ [3].

In this work, by means of the method used to obtain Frenet apparatus and Darboux frame for non-unit speed curves in $\mathbb{E}^3$ in [3], we present Frenet apparatus for non-unit speed curves and Darboux frame and its derivative formulas for non-unit speed curves and surface pairs in Minkowski 3-space $\mathbb{E}_1^3$.

§2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{E}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{E}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} > > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} > < 0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} > = 0$

and $\mathbf{x}=\mathbf{0}$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x}\in\mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\|=\sqrt{|<\mathbf{x},\mathbf{x}>|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $<\mathbf{x},\mathbf{x}>=1$ and a timelike unit vector if $<\mathbf{x},\mathbf{x}>=-1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [2]. For any two vectors $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $<\mathbf{x},\mathbf{y}>=x_1y_1+x_2y_2-x_3y_3$ and the vector product is defined by $\mathbf{x}\times\mathbf{y}=((x_2y_3-x_3y_2),(x_3y_1-x_1y_3),-(x_1y_2-x_2y_1))$.

The unit Lorentzian and hyperbolic spheres in $\mathbb{E}_1^3$ are defined by

$$\begin{aligned} H_+^2 &= \{\mathbf{x}\in E_1^3 \mid x_1^2+x_2^2-x_3^2=-1,\ x_3\geq 1\}, \\ S_1^2 &= \{\mathbf{x}\in E_1^3 \mid x_1^2+x_2^2-x_3^2=1\}. \end{aligned} \tag{1}$$

Let $\{T,n,b\}$ be the moving Frenet frame along the curve α with arc-length parameter s. For a spacelike curve α, the Frenet Serret equations are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where $\langle T,T\rangle=1$, $\langle N,N\rangle=\varepsilon=\pm1$, $\langle B,B\rangle=-\varepsilon$, $\langle T,N\rangle=\langle T,B\rangle=\langle N,B\rangle=0$. Furthermore, for a timelike non-unit speed curve α in $\mathbb{E}_1^3$, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3}$$

where $\langle T,T\rangle=-1$, $\langle N,N\rangle=\langle B,B\rangle=1$, $\langle T,N\rangle=\langle T,B\rangle=\langle N,B\rangle=0$ [1].

Let M be an oriented surface in three-dimensional Minkowski space $\mathbb{E}_1^3$ and let consider a non-null curve $\alpha(t)$ lying on M fully. Since the curve $\alpha(t)$ is also in space, there exists Frenet frame $\{T,N,B\}$ at each points of the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. Since the curve $\alpha(t)$ lies on the surface M there exists another frame of the curve $\alpha(t)$ which is called Darboux frame and denoted by $\{T,b,n\}$. In this frame T is the unit tangent of the curve, n is the unit normal of the surface M and b is a unit vector given by $b=n\times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, b and n lie on the same plane. Then, if the surface M is an oriented timelike surface, the relations between these frames can be given as follows.

(i) If the surface M is a timelike surface, then the curve $\alpha(t)$ lying on M can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(t)$ is given by

$$\begin{bmatrix} T' \\ b' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & -\varepsilon\kappa_n \\ \kappa_g & 0 & \varepsilon\tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix}, \tag{4}$$

where $\langle T,T\rangle = \varepsilon = \pm 1$, $\langle b,b\rangle = -\varepsilon$, $\langle n,n\rangle = 1$.

(ii) If the surface M is a spacelike surface, then the curve $\alpha(t)$ lying on M is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(t)$ is given by

$$\begin{bmatrix} T' \\ b' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix}, \tag{5}$$

where $\langle T,T\rangle = 1$, $\langle b,b\rangle = 1$, $\langle n,n\rangle = -1$ [1].

In these formulae at (4) and (5) , the functions κ_n, κ_g and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve. Here, we have the following properties of α characterized by the conditions of κ_g, κ_n and τ_g :

α is a geodesic curve if and only if $\kappa_g = 0$.

α is a asymptotic curve if and only if $\kappa_n = 0$.

α is line of curvature if and only if $\tau_g = 0$.

(see [2]).

§3. Some properties of non-unit speed curves in $\mathbb{E}_1^3$

Let $\alpha(t)$ be an arbitrary curve with speed

$$v = \left|\overrightarrow{\alpha}'\right| = \frac{ds}{dt}.$$

Theoretically we may reparametrize α to get a unit speed curve $\overline{\alpha}(s(t))$, so, we define curvature and torsion of α in terms of its arclength reparametrization $\overline{\alpha}(s(t))$. Moreover, the tangent vector $\alpha'(t)$ is in the direction of the unit tangent $T(s)$ of the reparametrization, so $T(s(t)) = \alpha'(t)/\left|\alpha'(t)\right|$. This says that we should define a non-unit speed curve's invariants in terms of its unit speed reparametrization's invariants.

Definition 3.1. Let α be any non unit speed curve in Minkowski 3-space $\mathbb{E}_1^3$, so

(1) The unit tangent of $\alpha(t)$ is defined to be $T(t) \overset{def}{=} \overline{T}(s(t))$.

(2) The curvature of $\alpha(t)$ is defined to be $\kappa(t) \overset{def}{=} \overline{\kappa}(s(t))$.

(3) If $\kappa > 0$, then the principal normal of $\alpha(t)$ is defined to be $N(t) \overset{def}{=} \overline{N}(s(t))$.

(4) If $\kappa > 0$, then the binormal of $\alpha(t)$ is defined to be $B(t) \overset{def}{=} \overline{B}(s(t))$.

(5) If $\kappa > 0$, then the torsion of $\alpha(t)$ is defined to be $\tau(t) \overset{def}{=} \overline{\tau}(s(t))$.

Theorem 3.1. If the curve $\alpha : I \to \mathbb{E}_1^3$ is a regular non-unit speed curve in $\mathbb{E}_1^3$, then there is a parameter map $h : I \to J$ so that $\|(\alpha \circ h)'\| = 1$ for all $s \in J$.

Proof. Let the arc-length function of the curve α be f. It is known that $f' = \|\alpha'\|$. The function f' has values differed from zero since α is a regular curve. Hence f is a one to one function because it is an increasing function. Let's say $f(I) = J$. Consequently, f becomes regular and bijective function. So has get the inverse, let h be the inverse of f and $\alpha \circ h = \beta$. So if we show that $\beta : J \to \mathbb{E}_1^3$ is a unit speed curve, the proof is completed as follows:

$$\|\beta'\| = \|(\alpha \circ h)'\| = \|h'(\alpha' \circ h)\| = |h'| \, \|\alpha' \circ h\| \,,$$

and

$$h'(s) = (f^{-1})'(s) = \frac{1}{f'(h(s))} = \frac{1}{\|\alpha'(h(s))\|} = \frac{1}{\|(\alpha' \circ h)(s)\|} = \frac{1}{\|(\alpha' \circ h)\|}(s).$$

Theorem 3.2. (The Frenet formulas for non-unit speed curves in $\mathbb{E}_1^3$)

For a regular curve α with speed

$$v = \frac{ds}{dt},$$

and curvature $\kappa > 0$,

(i) if α is a spacelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$\begin{aligned} T' &= v\kappa N, \\ N' &= v(-\varepsilon\kappa T + \tau B), \\ B' &= v\tau N. \end{aligned}$$

(ii) if α is a timelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$\begin{aligned} T' &= v\kappa N, \\ N' &= v(\kappa T + \tau B), \\ B' &= v\tau N. \end{aligned}$$

Proof. (i) For non-unit speed spacelike curve, the unit tangent $T(t)$ is $\overline{T}(s)$ by the definition 3.1. Now $T'(t)$ denotes differentiation with respect of t, so we must use the chain rule on the righthand side to determine κ and τ. From the definition 3.1,

$$T(t) = \overline{T}(s(t)). \tag{6}$$

If we differentiate (6), the expression is obtained as

$$\frac{dT(t)}{dt} = \frac{d\overline{T}(s(t))}{ds}\frac{ds}{dt} = \overline{\kappa}(s)\overline{N}(s)v = \kappa(t)N(t)v. \tag{7}$$

From (7), we get

$$T' = \kappa(t)vN(t).$$

So the first formula of (i) is proved. For the second and third,

$$N'(t) = \frac{d\overline{N}(s)}{ds}\frac{ds}{dt} = (-\overline{\kappa}(s)\overline{T}(s) + \overline{\tau}(s)\overline{B}(s))v,$$

by the unit speed Frenet formulas, we have

$$N'(t) = -\kappa(t)vT(t) + \tau(t)vB(t).$$

And we get

$$B'(t) = \frac{d\overline{B}(s)}{ds}\frac{ds}{dt} = -\overline{\tau}(s)\overline{N}(s)v = -\tau(t)vN(t).$$

(ii) For non-unit speed timelike curve, if the computations are made as follows:
by the definition 3.1,

$$T(t) = \overline{T}(s(t)). \tag{8}$$

If we differentiate (8), the expression is found as

$$\frac{dT(t)}{dt} = \frac{d\overline{T}(s(t))}{ds}\frac{ds}{dt} = \frac{d\overline{T}(s(t))}{ds}v = v\kappa(t)N(t). \tag{9}$$

From (9), we obtain

$$T' = \kappa(t)vN(t).$$

So the first formula of (ii) for non-unit speed timelike curve is proved. For the second and third, we get

$$N'(t) = \frac{d\overline{N}(s)}{ds}\frac{ds}{dt} = (\overline{\kappa}(s)\overline{T}(s) + \overline{\tau}(s)\overline{B}(s))v = (\kappa(t)T(t) + \tau(t)B(t))v$$

and finally

$$B'(t) = \frac{d\overline{B}(s)}{ds}\frac{ds}{dt} = \overline{\tau}(s)\overline{N}(s)v = \tau(t)vN(t).$$

Lemma 3.1. For a non-unit speed curve α in $\mathbb{E}_1^3$,

$$\alpha' = vT \quad \text{and} \quad \alpha'' = \tfrac{dv}{dt}T + \kappa v^2 N.$$

Proof. Since $\alpha(t) = \overline{\alpha}(s(t))$, the first calculation is

$$\alpha'(t) = \overline{\alpha}'(s)\frac{ds}{dt} = v\overline{\alpha}'(s) = v\overline{T}(s) = vT,$$

while the second is

$$\alpha''(t) = \frac{dv}{dt}T(t) + v(t)T'(t) = \frac{dv}{dt}T(t) + \kappa(t)v^2(t)N(t).$$

§4. Non-unit speed curve and surface pair in $\mathbb{E}_1^3$

In this part, we will define the curvatures of curve-surface pair (α, M) while α is a non-unit speed curve in $\mathbb{E}_1^3$. Let's consider the non-unit speed curve $\alpha : I \to M$. Previously, we proved that there is a parameter map $h : J \to I$ so that $\alpha \circ h : J \to M$ is a unit speed curve. The function h is the inverse of $f(t) = \int_{t_0}^{t} \|\alpha'(r)\|\, dr$. Let's say $\alpha \circ h = \beta$. β is a unit speed curve. Let's the unit tangent vector field of β be denoted by $\overline{T}$. The set $\{\overline{T}, \overline{b}, n \circ \beta\}$ is the frame of curve-surface pair (β, M), where $\overline{b} = (n \circ \beta) \times \overline{T}$. Let $s \in J$, $h(s) = t$. $s = f(t)$ because of $h(s) = f^{-1}(s)$. So $\beta(s) = \alpha(h(s)) = \alpha(t)$.

Definition 4.1. Let β be the unit speed curve obtained from the non-unit speed curve $\alpha : I \to M$ in $\mathbb{E}_1^3$. While the frame of curve-surface pair (β, M) is $\{\overline{T}, \overline{b}, n\circ\beta\}$, the set $\{T, b, n\circ\alpha\}$ defined by

$$T(t) = \overline{T}(f(t)), \qquad b(t) = \overline{b}(f(t)), \qquad n(\alpha(t)) = n(\beta(f(t)))$$

is called Darboux frame of curve-surface pair (α, M) in $\mathbb{E}_1^3$. Also, while the curvatures of curve-surface pair (β, M) are $\overline{\kappa}_n,\ \overline{\kappa}_g,\ \overline{\tau}_g$, the curvatures of curve-surface pair (α, M) are as follows:

$$\kappa_n(t) = \overline{\kappa}_n(f(t)), \qquad \kappa_g(t) = \overline{\kappa}_g(f(t)), \qquad \tau_g(t) = \overline{\tau}_g(f(t)).$$

Theorem 4.1. Given the non-unit speed curve $\alpha:\ I \to M$, the curvatures $\kappa_g,\ \kappa_n$ and τ_g of curve-surface pair (α, M) in $\mathbb{E}_1^3$ are as follows:

$$\kappa_n = \tfrac{1}{v^2}\left\langle \alpha'', n\circ\alpha\right\rangle, \quad \kappa_g = \tfrac{1}{v^2}\left\langle \alpha'', b\right\rangle, \quad \tau_g = \tfrac{1}{v^2}\left\langle (n\circ\alpha)', b\right\rangle.$$

Proof. Let $f(t) = s$, by definition 4.1, we get

$$\kappa_n(t) = \overline{\kappa}_n(s) = \left\langle \beta''(s), (n\circ\beta)(s)\right\rangle. \tag{10}$$

Let's calculate the components of (10),

$$\beta'(s) = \overline{T}(s) = T(t) = \frac{1}{v(t)}\alpha'(t), \tag{11}$$

we write (11) as follows:

$$\alpha'(t) = v(t)\beta'(s) = v(t)\beta'(f(t)) = v(t)(\beta'\circ f)(t). \tag{12}$$

If we differentiate (12), we get

$$\alpha''(t) = v'(t)(\beta'\circ f)(t) + v(t)f'(t)\beta''(f(t)) = v'(t)\beta'(s) + (v(t))^2\beta''(s). \tag{13}$$

By differentiating (11) and using (13), we get

$$\beta''(s) = \frac{1}{v^2(t)}(\alpha''(t) - v'(t)\beta'(s)). \tag{14}$$

By using (14) in (10),

$$\begin{aligned}
\kappa_n(t) &= \left\langle \beta''(s), (n\circ\beta)(s)\right\rangle \\
&= \left\langle \frac{1}{v^2(t)}(\alpha''(t) - v'(t)\beta'(s)), (n\circ\beta)(s)\right\rangle \\
&= \frac{1}{v^2(t)}\{\left\langle \alpha''(t), n\circ\beta(s)\right\rangle - \left\langle v'(t)\beta'(s), n\circ\beta(s)\right\rangle\} \\
&= \frac{1}{v^2(t)}\left\langle \alpha''(t), n\circ\beta\circ f(t)\right\rangle \\
&= \frac{1}{v^2(t)}\left\langle \alpha''(t), n\circ\alpha(t)\right\rangle \\
&= (\frac{1}{v^2}\left\langle \alpha'', n\circ\alpha\right\rangle)(t).
\end{aligned}$$

Let's see the equation of $\kappa_g(t)$. So

$$\kappa_g(t) = \overline{\kappa}_g(s) = \left\langle \beta''(s), \overline{b}(s)\right\rangle. \tag{15}$$

By using (14) in (15), we get

$$\begin{aligned}
\kappa_g(t) &= \left\langle \frac{1}{v^2(t)}(\alpha''(t) - v'(t)\beta'(s)), \overline{b}(s)\right\rangle \\
&= \frac{1}{v^2(t)}\left\langle \alpha''(t), \overline{b}(f(t))\right\rangle \\
&= \frac{1}{v^2(t)}\left\langle \alpha''(t), b(t)\right\rangle.
\end{aligned}$$

Finally, let's see the equation of $\tau_g(t)$. So

$$\tau_g(t) = \overline{\tau}_g(s) = -\left\langle (n \circ \beta)'(s), \overline{b}(s) \right\rangle = -\left\langle (n \circ \beta)'(s), b(t) \right\rangle. \tag{16}$$

Also we get

$$(n \circ \alpha)'(t) = (n \circ \beta \circ f)'(t) = f'(t)(n \circ \beta)'(f(t)) = v(t)(n \circ \beta)'(s)$$

or

$$(n \circ \beta)'(s) = \frac{1}{v(t)}(n \circ \alpha)'(t). \tag{17}$$

By using (17) in (16), we get

$$\tau_g(t) = -\left\langle \frac{1}{v(t)}(n \circ \alpha)'(t), b(t) \right\rangle = -\frac{1}{v} \left\langle (n \circ \alpha)', b \right\rangle (t).$$

Theorem 4.2. Given non-unit speed curve $\alpha : I \to M$,

(i) If (α, M) is timelike or spacelike curves on timelike surface pair, the derivative formula of Darboux frame $\{T, b, n \circ \alpha\}$ is as follows:

$$\begin{aligned} T'(t) &= v(t)[\kappa_g(t)b(t) - \varepsilon\kappa_n(t)(n \circ \alpha)(t)], \\ b'(t) &= v(t)[\kappa_g(t)T(t) + \varepsilon\tau_g(t)(n \circ \alpha)(t)], \\ (n \circ \alpha)'(t) &= v(t)[\kappa_n(t)T(t) + \tau_g(t)b(t)], \end{aligned}$$

for curve-surface pair (α, M) in $\mathbb{E}_1^3$.

(ii) If (α, M) is spacelike curve on spacelike surface pair, the derivative formula of Darboux frame $\{T, b, n \circ \alpha\}$ is as follows:

$$\begin{aligned} T'(t) &= v(t)[\kappa_g(t)b(t) + \kappa_n(t)(n \circ \alpha)(t)], \\ b'(t) &= v(t)[-\kappa_g(t)T(t) + \tau_g(t)(n \circ \alpha)(t)], \\ (n \circ \alpha)'(t) &= v(t)[\kappa_n(t)T(t) + \tau_g(t)b(t)], \end{aligned}$$

for curve-surface pair (α, M) in $\mathbb{E}_1^3$.

Proof. (i) For timelike surface:

$$T(t) = \overline{T}(f(t)) = (\overline{T} \circ f)(t). \tag{18}$$

By differentiating (18), we get

$$\begin{aligned} T'(t) &= (\overline{T} \circ f)'(t) \\ &= f'(t)\overline{T}'(f(t)) \\ &= v(t)[\overline{\kappa}_g(s)\overline{b}(s) - \varepsilon\overline{\kappa}_n(s)(n \circ \beta)(s)] \\ &= v(t)[\kappa_g(t)b(t) - \varepsilon\kappa_n(t)(n \circ \alpha)(t)] \end{aligned}$$

and

$$b(t) = \overline{b}(f(t)) = (\overline{b} \circ f)(t). \tag{19}$$

By differentiating (19), we get

$$\begin{aligned} b'(t) &= (\bar{b}\circ f)'(t) \\ &= f'(t)\overline{b}'(f(t)) \\ &= v(t)[\overline{\kappa}_g(s)\overline{T}(s)+\varepsilon\overline{\tau}_g(s)(n\circ\beta)(s)] \\ &= v(t)[\kappa_g(t)T(t)+\varepsilon\tau_g(t)(n\circ\alpha)(t)]. \end{aligned}$$

For the third formula of (i),

$$(n\circ\alpha)(t)=(n\circ\beta\circ f)(t). \tag{10}$$

By differentiating (10), we get

$$\begin{aligned} (n\circ\alpha)'(t) &= (n\circ\beta\circ f)'(t) \\ &= f'(t)(n\circ\beta)'(f(t)) \\ &= v(t)[\overline{\kappa}_n(s)\overline{T}(s)+\overline{\tau}_g(s)\bar{b}(s)] \\ &= v(t)[\kappa_n(t)T(t)+\tau_g(t)b(t)]. \end{aligned}$$

(ii) For spacelike surface:

$$T(t)=\overline{T}(f(t))=(\overline{T}\circ f)(t). \tag{11}$$

By differentiating (11), we get

$$\begin{aligned} T'(t) &= (\overline{T}\circ f)'(t) \\ &= f'(t)\overline{T}'(f(t)) \\ &= v(t)[\overline{\kappa}_g(s)\bar{b}(s)+\overline{\kappa}_n(s)(n\circ\beta)(s)] \\ &= v(t)[\kappa_g(t)b(t)+\kappa_n(t)(n\circ\alpha)(t)] \end{aligned}$$

and

$$b(t)=\bar{b}(f(t))=(\bar{b}\circ f)(t). \tag{12}$$

By differentiating (12), we get

$$\begin{aligned} b'(t) &= (\bar{b}\circ f)'(t) \\ &= f'(t)\overline{b}'(f(t)) \\ &= v(t)[-\overline{\kappa}_g(s)\overline{T}(s)+\overline{\tau}_g(s)(n\circ\beta)(s)] \\ &= v(t)[-\kappa_g(t)T(t)+\tau_g(t)(n\circ\alpha)(t)]. \end{aligned}$$

For the third formula of (ii),

$$(n\circ\alpha)(t)=(n\circ\beta\circ f)(t). \tag{13}$$

By differentiating (13), we get

$$\begin{aligned} (n\circ\alpha)'(t) &= (n\circ\beta\circ f)'(t) \\ &= f'(t)(n\circ\beta)'(f(t)) \\ &= v(t)[\overline{\kappa}_n(s)\overline{T}(s)+\overline{\tau}_g(s)\bar{b}(s)] \\ &= v(t)[\kappa_n(t)T(t)+\tau_g(t)b(t)]. \end{aligned}$$

Corollary 4.1. Given non-unit speed curve $\alpha : I \to M$, the curvatures of curve-surface pair (α, M) in $\mathbb{E}_1^3$ as follows:

(i) If (α, M) is timelike or spacelike curves on timelike surface pair, then

$$\kappa_n = -\frac{\varepsilon}{v}\langle T', n \circ \alpha\rangle, \quad \kappa_g = -\frac{\varepsilon}{v}\langle T', b\rangle, \quad \tau_g = \frac{\varepsilon}{v}\langle b', n \circ \alpha\rangle.$$

(ii) If (α, M) is spacelike curve on spacelike surface pair, then

$$\kappa_n = -\frac{1}{v}\langle T', n \circ \alpha\rangle, \quad \kappa_g = \frac{1}{v}\langle T', b\rangle, \quad \tau_g = -\frac{1}{v}\langle b', n \circ \alpha\rangle.$$

Proof. The direct result of Theorem 4.3.

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Self-integrating Polynomials in two variables

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Abstract In this paper, a condition for self-integrability of a polynomial in two variables on a rectangular region R was obtained.

Keywords Double integral, polynomial in two variables, rectangular region, self-integrability, self-integrable.

§1. Introduction and preliminaries

Graham [1] shown that the polynomials of the form $p_k(x) = x^k + \frac{k}{k+1}$ has the property that $\int_0^1 p_k(x)\,dx = p_k(1) - p_k(0)$. The said polynomial is self-integrating in the closed interval [0,1].

Definition 1.1. A polynomial P in x and y with degree m, n is given by

$$P(x,y) = \sum_{i=0,j=0}^{m,n} a_{ij}x^i y^j,$$

where $a_{ij} \in \mathbb{R}$.

Definition 1.2. A function f is said to be self-integrable in the rectangular region R bounded the lines $x = l_1,\ y = l_2,\ x = u_1$ and $y = u_2$ if and only if the double integral given by

$$\int_{l_2}^{u_2} \int_{l_1}^{u_1} f(x,y)dxdy \ = f(x,y)||_{x=l_1}^{x=u_1}||_{y=l_2}^{y=u_2}.$$

Definition 1.3. We introduce the following terms for simplifying the double integrals and equations.

$$\text{Lower limit integration vector} : \vec{L} = (l_1, l_2),$$
$$\text{Upper limit integration vector} : \vec{U} = (l_1, l_2),$$
$$\text{Variable vector} : \vec{V} = (x, y),$$
$$\text{Integration vector} : d\vec{V} = dxdy.$$

Definition 1.4.

$$\int_{l_2}^{u_2} \int_{l_1}^{u_1} f(x,y)dxdy \ = \int_{\vec{L}}^{\vec{U}} f(\vec{V})d\vec{V}.$$

Definition 1.5.

$$f(x,y)||_{x=l_1}^{x=u_1}||_{y=l_2}^{y=u_2} = f(\vec{V})||_{\vec{L}}^{\vec{U}}.$$

§2. Preliminary results

Lemma 2.1.

$$\int_{\vec{L}}^{\vec{U}} f(\vec{V})d\vec{V} = \sum_{i=0,j=0}^{m+n} \frac{a_{ij}(u_1^{i+1} - l_1^{i+1})(u_2^{i+1} - l_2^{i+1})}{(i+1)(j+1)},$$

where $f(\vec{V}) = P(x,y) = \sum_{i=0,j=0}^{m+n} a_{ij}x^i y^j$.

Lemma 2.2.

$$f(\vec{V})|_{\vec{L}}^{\vec{U}} = \sum_{i=1,j=1}^{m+n} a_{ij}(u_1^i - l_1^i)(u_2^j - l_2^j),$$

where $f(\vec{V}) = P(x,y) = \sum_{i=0,j=0}^{m+n} a_{ij}x^i y^j$.

§3. Main results

Theorem 3.1. The function

$$f(\vec{V}) = P(x,y) = \sum_{i=0,j=0}^{m+n} a_{ij}x^i y^j$$

is self-integrable in the rectangular region R bounded the lines $x = l_1,\ y = l_2,\ x = u_1$ and $y = u_2$ if and only if the coefficients satisfy the homogeneous equation given by

$$a_{00}(u_1 - l_1)(u_2 - l_1) + \sum_{i=0,j=0}^{m+n} [a_{ij}(u_1^{i+1} - l_1^{i+1})(u_2^{j+1} - l_2^{j+1}) - (i+1)(j+1)(u_1^i - l_1^i)(u_2^j - l_2^j)] = 0.$$

Corollary 3.1. The function

$$f(\vec{V}) = P(x,y) = \sum_{i=0,j=0}^{m+n} a_{ij}x^i y^j$$

is self-integrable in the rectangular region R bounded the lines $x = 0,\ y = 0,\ x = 1$ and $y = 1$ if and only if the coefficients satisfy the homogeneous equation given by

$$a_{00} + \sum_{i=0,j=0}^{m+n} a_{ij}[1 - (i+1)(j+1)] = 0.$$

Theorem 3.2. The sum of self-integrable polynomials in the rectangular region R is also self-integrable in the region R where R is the region bounded by the lines $x = l_1,\ y = l_2,\ x = u_1$ and $y = u_2$.

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Signed product cordial graphs in the context of arbitrary supersubdivision

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Abstract In this paper we discuss signed product cordial labeling in the context of arbitrary supersubdivision of some special graphs. We prove that the graph obtained by arbitrary supersubdivision of $(P_n \times P_m) \odot P_s$ is signed product cordial. We also prove that the tadpole $T_{n,l}$ is signed product cordial except m_i $(1 \leq i \leq n)$ are odd, m_i $(n+1 \leq i \leq n+l)$ are even and n is odd and $C_n \odot P_m$ is signed product cordial except m_i $(1 \leq i \leq n)$ are odd, m_i $(n+1 \leq i \leq nm)$ are even and n is odd.

Keywords Signed product cordial labeling, signed product cordial graph, tadpole, arbitrary supersubdivision.

§1. Introduction and preliminaries

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For all other terminology and notations we follow Harary [3]. Given below are some definitions useful for the present investigations.

Sethuraman [6] introduced a new method of construction called supersubdivision of graph.

Definition 1.1. Let G be a graph with q edges. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then supersubdivision is called arbitrary supersubdivision of G.

In the same paper Sethuraman proved that arbitrary supersubdivision of any path and cycle C_n are graceful. Kathiresan [4] proved that arbitrary supersubdivision of any star are graceful.

Definition 1.2. The assignment of values subject to certain conditions to the vertices of a graph is known as graph labeling.

Definition 1.3. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \longrightarrow \{0, 1\}$ is called binary vertex labeling of G and $f\ (v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \longrightarrow \{0, 1\}$ is given by $f^*\ (e) = |f(u) - f(v)|$.

Let $v_f(0),\ v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0),\ e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.4. A binary vertex labeling of a graph G is called a cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [2]. Vaidya [8] proved that arbitrary supersubdivision of any path, star and cycle C_n except when n and all m_i are simultaneously odd numbers are cordial. Vaidya [7] proved that arbitrary supersubdivision of any tree, grid graph, complete bipartite graph star and $C_n \odot P_m$ except when n and all m_i are simultaneously odd numbers are cordial.

The concept of signed product cordial labeling was introduced by Baskar Babujee [1].

Definition 1.5. A vertex labeling of graph G, $f : V(G) \longrightarrow \{-1, 1\}$ with induced edge labeling $f^* : E(G) \longrightarrow \{-1, 1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a signed product cordial labeling if $\mid v_f(-1) - v_f(1) \mid \leq 1$ and $\mid e_f(-1) - e_f(1) \mid \leq 1$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(1)$ is the number of vertices labeled with 1, $e_f(-1)$ is the number of edges labeled with -1 and $e_f(1)$ is the number of edges labeled with 1. A graph G is signed product cordial if it admits signed product cordial labeling.

In the same paper Baskar Babujee proved that the path graph, cycle graph, star, and bistar are signed product cordial and some general results on signed product cordial labeling are also studied.

Lawrence Rozario [5] proved that arbitrary supersubdivision of tree, complete bipartite graph, grid graph and cycle C_n except when n and all m_i are simultaneously odd numbers are signed product cordial.

Vaidya [9] proved that arbitrary supersubdivision path, cycle, star and tadpole graph are strongly multiplicative.

Definition 1.6. Tadpole $T_{n,l}$ is a graph in which path P_l is attached by an edge to any one vertex of cycle C_n. $T_{n,l}$ has $n + l$ vertices and edges.

Here we prove that the graphs obtained by arbitrary supersubdivision of $(P_n \times P_m) \odot P_s$ is signed product cordial. We also prove that the tadpole $T_{n,l}$ is signed product cordial except m_i $(1 \leq i \leq n)$ are odd, m_i $(n + 1 \leq i \leq n + l)$ are even and n is odd and $C_n \odot P_m$ is signed product cordial except m_i $(1 \leq i \leq n)$ are odd, m_i $(n + 1 \leq i \leq nm)$ are even and n is odd.

§2. Signed product cordial labeling of arbitrary supersubdivision of graphs

Theorem 2.1. Arbitrary supersubdivision of grid graph $(P_n \times P_m) \odot P_s$ is signed product cordial.

Proof. Let v_{ij} be the vertices of $(P_n \times P_m)$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Let v_{ij}^k $(1 \leq i \leq n,\ 1 \leq j \leq m$ and $2 \leq k \leq s)$ be the vertices of paths. Arbitrary supersubdivision of $(P_n \times P_m) \odot P_s$ is obtained by replacing every edge of $(P_n \times P_m) \odot P_s$ with K_{2,m_i} and we denote the resultant graph by G.

Let

$$\alpha = \sum_{1}^{mn(s+1)-m-n} m_i.$$

Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. Here $\mid V(G) \mid = \alpha + smn$, $\mid E(G) \mid = 2\alpha$. We define vertex labeling $f : V(G) \longrightarrow \{-1, 1\}$ as follows.

$$\left.\begin{aligned} f(v_{i,j}) &= -1, \quad \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ &= 1, \quad \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even.} \end{aligned}\right\} \begin{array}{l} 1 \leq i \leq n \text{ and} \\ 1 \leq j \leq m. \end{array}$$

$$\left.\begin{aligned} f(v^k_{i,j}) &= 1, \quad \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ &= -1, \quad \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even.} \end{aligned}\right\} \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq m \\ \text{and } k \text{ is even.} \end{array}$$

$$\left.\begin{aligned} f(v^k_{i,j}) &= -1, \quad \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ &= 1, \quad \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even.} \end{aligned}\right\} \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq m \\ \text{and } k \text{ is odd.} \end{array}$$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} 1 \leq i \leq \alpha.$$

In view of the above defined function f, the graph G satisfies the following conditions. When

(i) α and mn are even and s is odd/even,

(ii) α and s are even and mn is odd,

(iii) α and mn are odd and s is odd, then

$$v_f(-1) = v_f(1) = \frac{\alpha + smn}{2}, \ e_f(-1) = e_f(1) = \alpha.$$

When

(i) α and mn are odd and s is even,

(ii) α is odd, mn is even and s is odd/even, then

$$v_f(-1) + 1 = v_f(1) = \frac{\alpha + smn + 1}{2}, \ e_f(-1) = e_f(1) = \alpha.$$

(iii) When α is even and mn and s are odd, then

$$v_f(-1) = v_f(1) + 1 = \frac{\alpha + smn + 1}{2}, \ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G. Hence the result.

Illustration 2.1 Consider the arbitrary supersubdivision of graph $(P_2 \times P_3) \odot P_3$. Here $n = 2,\ m = 3,\ m_1 = 3,\ m_2 = 2,\ m_3 = 2,\ m_4 = 2,\ m_5 = 2,\ m_6 = 2,\ m_7 = 2,\ m_8 = 2,\ m_9 = 3,\ m_{10} = 1,\ m_{11} = 1,\ m_{12} = 2,\ m_{13} = 2,\ m_{14} = 2,\ m_{15} = 2,\ m_{16} = 1,\ m_{17} = 1,\ m_{18} = 2,\ m_{19} = 2$ and $\alpha = 36$. The signed product cordial labeling of the arbitrary supersubdivision of graph $(P_2 \times P_3) \odot P_3$ as shown in Figure 1.

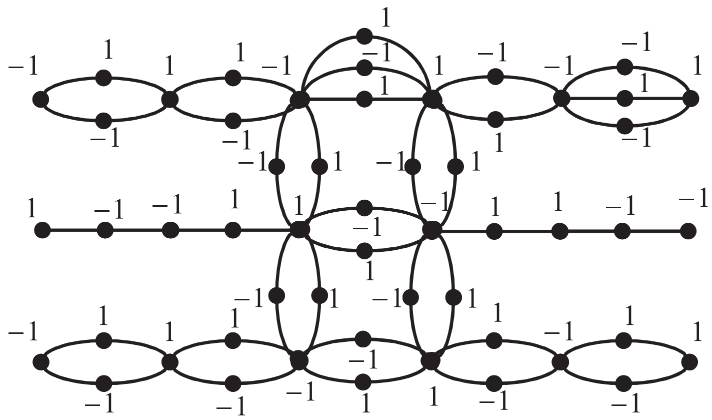

Figure 1: Signed product cordial labeling of the arbitrary supersubdivision of $(P_2 \times P_3) \odot P_3$

Theorem 2.2. Arbitrary supersubdivision of any tadpole $T_{n,l}$ is a signed cordial product graph except m_i $(1 \leq i \leq n)$ are odd, m_i $(n+1 \leq i \leq n+l)$ are even and n is odd.

Proof. Let $v_1,\ v_2,\ v_3, ..., v_n$ be the vertices of C_n and $v_{n+1},\ v_{n+2},\ v_{n+3}, ..., v_{n+l}$ be the vertices of path P_l. Arbitrary supersubdivision of $T_{n,l}$ is obtained by replacing every edge of $T_{n,l}$ with K_{2,m_i} and we denote this graph by G. Let $\alpha = \sum_1^{n+l} m_i$ and u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. Here $\mid V(G) \mid = \alpha + n + l$, $\mid E(G) \mid = 2\alpha$. To define binary vertex labeling $f : V(G) \longrightarrow \{-1, 1\}$.
We consider following cases.

Case 1: For n even.

$$\left.\begin{aligned} f(v_i) &= -1, \quad \text{if } i \text{ is odd,} \\ &= 1, \quad \text{if } i \text{ is even.} \end{aligned}\right\} 1 \leq i \leq n+l.$$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} 1 \leq j \leq \alpha.$$

In view of the case, the graph G satisfies the following conditions. When

(i) $n,\ l$ and α are even,

(ii) n is even, l and α are odd, then

$$v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n and l are even, α is odd, then

$$v_f(-1) + 1 = v_f(1) = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n and α are even, l is odd, then

$$v_f(-1) = v_f(1) + 1 = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 2: For n odd.

Subcase 2.1 : m_i $(1 \leq i \leq n)$ are even.

$$\left.\begin{array}{rcrl} f(v_i) & = & -1, & \text{if } i \text{ is odd,} \\ & = & 1, & \text{if } i \text{ is even.} \end{array}\right\} 1 \leq i \leq n+l.$$

$$\left.\begin{array}{rcrl} f(u_j) & = & -1, & \text{if } j \text{ is even,} \\ & = & 1, & \text{if } j \text{ is odd.} \end{array}\right\} 1 \leq j \leq \alpha.$$

In view of the above case, the graph G satisfies the following conditions. When

(i) n and l are odd, α is even,

(ii) n and α are odd, l is odd, then

$$v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n is odd, l and α are even, then

$$v_f(-1) = v_f(1) + 1 = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n, l and α are odd, then

$$v_f(-1) + 1 = v_f(1) = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Subcase 2.2: At least one m_i $(1 \leq i \leq n)$ is even. m_i is even for some i, where $1 \leq i \leq n$. In this case at least one m_i must be even so label all the vertices $v_{i+1},\ v_{i+2}, ..., v_n,\ v_1, ..., v_i$ alternately by using -1 and 1 starting with -1.

Subcase 2.2 (a): m_i is even for some i, where $1 \leq i \leq n$ and i is even or $i = n$.

$$\left.\begin{array}{rcrl} f(v_i) & = & -1, & \text{if } i \text{ is odd,} \\ & = & 1, & \text{if } i \text{ is even.} \end{array}\right\} n+1 \leq i \leq n+l.$$

$$\left.\begin{array}{rcrl} f(u_j) & = & -1, & \text{if } j \text{ is even,} \\ & = & 1, & \text{if } j \text{ is odd.} \end{array}\right\} 1 \leq j \leq \alpha.$$

In view of the above case, the graph G satisfies the following conditions. When

(i) n and l are odd, α is even,

(ii) n and α are odd, l is even, then $v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha$. When

(i) n is odd, l and α are even,

(ii) n, l and α are odd, then $v_f(-1) = v_f(1) + 1 = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha$.

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Subcase 2.2 (b): m_i is even for some i, where $1 \leq i < n$, i is odd and l is even.

$$\left.\begin{array}{rcrl} f(v_i) & = & -1, & \text{if } i \text{ is even,} \\ & = & 1, & \text{if } i \text{ is odd.} \end{array}\right\} n+1 \leq i \leq n+l.$$

$$\left.\begin{array}{rcrl} f(u_j) & = & -1, & \text{if } j \text{ is even,} \\ & = & 1, & \text{if } j \text{ is odd.} \end{array}\right\} 1 \leq j \leq \alpha.$$

In view of the above case, the graph G satisfies the following conditions. When n is odd, l and α is even, then

$$v_f(-1) = v_f(1) + 1 = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n and α are odd, l is even, then

$$v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Subcase 2.2 (c): m_i is even for some i, where $1 \leq i \leq n$, i and l are odd.

$$\left.\begin{aligned} f(v_i) &= -1, \quad \text{if } i \text{ is even,} \\ &= 1, \quad \text{if } i \text{ is odd.} \end{aligned}\right\} n+1 \leq i \leq n+l.$$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} 1 \leq j \leq \alpha-2.$$

$f(u_j) = 1$, for $j = \alpha - 1,\ \alpha$.

In view of the above two cases graph G satisfies the following conditions. When n and l are odd, α is even, then

$$v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When n, l and α are odd, then

$$v_f(-1) + 1 = v_f(1) = \frac{n+l+\alpha+1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 3: If n is odd and m_i $(1 \leq i \leq n+l)$ are odd.

Subcase 3.1: For n odd and $n+l$ is even.

$$\left.\begin{aligned} f(v_i) &= -1, \quad \text{if } i \text{ is odd,} \\ &= 1, \quad \text{if } i \text{ is even.} \end{aligned}\right\} 1 \leq i \leq n+l-1.$$

$f(v_{n+l}) = -1.$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} 1 \leq j \leq \alpha - m_{n+l} - 2.$$

$f(u_j) = 1, \text{for} j = \alpha - m_{n+l} - 1,\ \alpha - m_{n+l}.$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} \alpha - m_{n+l} + 1 \leq j \leq \alpha.$$

Subcase 3.2: For n odd and $n+l$ is odd.

$$\left.\begin{aligned} f(v_i) &= -1, \quad \text{if } i \text{ is odd,} \\ &= 1, \quad \text{if } i \text{ is even.} \end{aligned}\right\} 1 \leq i \leq n+l-1.$$

$f(v_{n+l}) = 1.$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is even,} \\ &= 1, \quad \text{if } j \text{ is odd.} \end{aligned}\right\} 1 \leq j \leq \alpha.$$

In view of above two cases, the graph G satisfies the following condition.

$$v_f(-1) = v_f(1) = \frac{n+l+\alpha}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 4: If n is odd and m_i $(1 \leq i \leq n)$ are odd, m_i $(n+1 \leq i \leq n+l)$ are even. Consider v_i and v_j are adjacent. If $f(v_i)$ and $f(v_j)$ have different values and assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have different values, i.e., if m_i is either even or odd, then $e_f(-1) = e_f(1) = \alpha$. If $f(v_i)$ and $f(v_j)$ have same value and assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have same value. i.e., if m_i is even, then $e_f(-1) = e_f(1) = \alpha$ and if m_i is odd, then $e_f(-1) = e_f(1)+2$ or $e_f(-1)+2 = e_f(1)$. Here n is odd and all m_i $(1 \leq i \leq n)$ are odd in cycle C_n. Then there exist atleast one pair of adjacent vertices v_i and v_j of cycle C_n have $f(v_i)$ and $f(v_j)$ are same. Therefore, assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have same value and $e_f(-1) = e_f(1)+2$ or $e_f(-1)+2 = e_f(1)$. But m_i $(n+1 \leq i \leq n+l)$ are even. Therefore, assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have different values. Here n is odd, m_i $(1 \leq i \leq n)$ are odd and m_i $(n+1 \leq i \leq n+l)$ are even, then either $e_f(-1) = e_f(1)+2$ or $e_f(-1)+2 = e_f(1)$. Therefore, G is not signed product cordial.

Illustration 2.2 Consider the arbitrary supersubdivision of $T_{4,3}$. Here $n = 4,\ l = 3,$ $m_1 = 2,\ m_2 = 2,\ m_3 = 2,\ m_4 = 2,\ m_5 = 1,\ m_6 = 2,\ m_7 = 3$ and $\alpha = 14$. The signed product cordial labeling of the arbitrary supersubdivision of $T_{4,3}$ as shown in Figure 2.

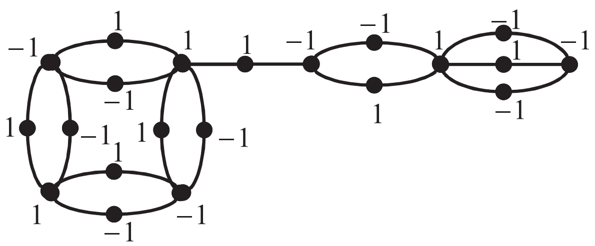

Figure 2: Signed product cordial labeling of the arbitrary supersubdivision of $T_{4,3}$

Theorem 2.3: Arbitrary supersubdivision of $C_n \odot P_m$ is signed product cordial except m_i $(1 \leq i \leq n)$ are odd, m_i $(n+1 \leq i \leq nm)$ are even and n is odd.

Proof. Let $v_1,\ v_2,\ v_3, ..., v_n$ be the vertices of C_n and v_{ij} $(1 \leq i \leq n,\ 2 \leq j \leq m)$ be the vertices of paths. Arbitrary supersubdivision of $C_n \odot P_m$ is obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote this graph by G.

Let $\alpha = \sum_1^{mn} m_i$ and u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq \alpha$. Here $\mid V(G) \mid = \alpha + mn$, $\mid E(G) \mid = 2\alpha$. To define binary vertex labeling $f : V(G) \longrightarrow \{-1, 1\}$. We consider following cases.

Case 1: For n even.

$$\left.\begin{array}{rcll} f(v_i) & = & -1, & \text{if } i \text{ is odd,} \\ & = & 1, & \text{if } i \text{ is even ,} \\ f(v_{ij}) & = & -1, & \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ & = & 1, & \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even.} \end{array}\right\} \begin{array}{l} 1 \le i \le n \text{ and} \\ 2 \le j \le m. \end{array}$$

$$\left.\begin{array}{rcll} f(u_j) & = & -1, & \text{if } j \text{ is even,} \\ & = & 1, & \text{if } j \text{ is odd.} \end{array}\right\} 1 \le j \le \alpha.$$

In view of the above case, the graph G satisfies the following conditions. When $\alpha + mn$ is even, then

$$v_f(-1) = v_f(1) = \frac{\alpha + mn}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When α is odd and mn is even, then

$$v_f(-1) + 1 = v_f(1) = \frac{\alpha + mn + 1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 2: For n odd and at least one m_i $(1 \le i \le n)$ is even. Without loss of generality we assume that m_1 is even.

$f(v_1) = -1.$

$$\left.\begin{array}{rcll} f(v_i) & = & -1, & \text{if } i \text{ is even,} \\ & = & 1, & \text{if } i \text{ is odd,} \\ f(v_{1j}) & = & -1, & \text{if } j \text{ is odd,} \\ & = & 1, & \text{if } j \text{ is even,} \\ f(v_{ij}) & = & -1, & \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even,} \\ & = & 1, & \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even.} \end{array}\right\} \begin{array}{l} 2 \le i \le n \text{ and} \\ 2 \le j \le m. \end{array}$$

$$\left.\begin{array}{rcll} f(u_j) & = & -1, & \text{if } 1 \le j \le \frac{m_1}{2}, \\ & = & 1, & \text{if } \frac{m_1}{2} + 1 \le j \le m_1, \\ f(u_j) & = & -1, & \text{if } m_1 + 1 \le j \le \lfloor \frac{\alpha + m_1}{2} \rfloor, \\ & = & 1, & \text{if } \lceil \frac{\alpha + m_1}{2} \rceil \le j \le \alpha. \end{array}\right\} 1 \le j \le \alpha.$$

In view of the above case, the graph G satisfies the following conditions. When $\alpha + mn$ is even, then

$$v_f(-1) = v_f(1) = \frac{\alpha + mn}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When α is odd and mn is even, then

$$v_f(-1) + 1 = v_f(1) = \frac{\alpha + mn + 1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

When α is even and mn is odd, then

$$v_f(-1) = v_f(1) + 1 = \frac{\alpha + mn + 1}{2},\ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 3: If n is odd and m_i $(1 \leq i \leq nm)$ are odd.

Subcase 3.1: For n odd and nm is even.

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } j \text{ is odd,} \\ &= 1, \quad \text{if } j \text{ is even.} \end{aligned}\right\} 1 \leq j \leq m_1.$$

$$\left.\begin{aligned} f(u_j) &= -1, \quad \text{if } m_1 + 1 \leq j \leq m_1 + \tfrac{\alpha - m_{nm} - m_1}{2}, \\ &= 1, \quad \text{if } m_1 + \tfrac{\alpha - m_{nm} - m_1}{2} + 1 \leq j \leq \alpha - m_{nm}, \\ f(u_j) &= -1, \quad \text{if } \alpha - m_{nm} + 1 \leq j \leq \alpha - m_{nm} + \lceil \tfrac{m_{nm}}{2} \rceil, \\ &= 1, \quad \text{if } \alpha - m_{nm} + \lceil \tfrac{m_{nm}}{2} \rceil + 1 \leq j \leq \alpha. \end{aligned}\right\} 1 \leq j \leq \alpha.$$

$$\left.\begin{aligned} f(v_1) &= -1, \\ f(v_i) &= -1, \quad \text{if } i \text{ is even,} \\ &= 1, \quad \text{if } i \text{ is odd,} \\ f(v_{1j}) &= -1, \quad \text{if } j \text{ is odd,} \\ &= 1, \quad \text{if } j \text{ is even,} \\ f(v_{ij}) &= -1, \quad \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd} \\ &\qquad\quad \text{and } j \text{ is even and } i \neq n \text{ and } j \neq m, \\ &= 1, \quad \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ f(v_{nm}) &= 1. \end{aligned}\right\} \begin{array}{l} 2 \leq i \leq n \text{ and} \\ \quad 2 \leq j \leq m. \end{array}$$

In view of above, the graph G satisfies the following condition. Here $\alpha + mn$ is even, then

$$v_f(-1) = v_f(1) = \frac{\alpha + mn}{2}, \ e_f(-1) = e_f(1) = \alpha.$$

That is, f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Subcase 3.2: For n odd and nm is odd. Here $\beta = m_1 + m_2 + ... + m_n + m_{n+1} + ... + m_{n+(m-2)}$,

$$\left.\begin{aligned} f(v_1) &= -1, \\ f(v_i) &= -1, \quad \text{if } i \text{ is even,} \\ &= 1, \quad \text{if } i \text{ is odd,} \\ f(v_{1j}) &= -1, \quad \text{if } j \text{ is odd and } j \neq m, \\ &= 1, \quad \text{if } j \text{ is even,} \\ f(v_{ij}) &= -1, \quad \text{if } i \text{ is even and } j \text{ is odd or } i \text{ is odd and } j \text{ is even,} \\ &= 1, \quad \text{if } i \text{ and } j \text{ are odd or } i \text{ and } j \text{ are even,} \\ f(v_{1m}) &= 1. \end{aligned}\right\} \begin{array}{l} 2 \leq i \leq n \text{ and} \\ \quad 2 \leq j \leq m. \end{array}$$

$$\left.\begin{aligned}
f(u_j) &= -1, \quad \text{if } 1 \le j \le \lceil \tfrac{m_1}{2} \rceil, \\
&= 1, \quad \text{if } \lceil \tfrac{m_1}{2} \rceil + 1 \le j \le m_1, \\
f(u_j) &= -1, \quad \text{if } m_1 + 1 \le j \le m_1 + \lfloor \tfrac{\beta - m_1}{2} \rfloor, \\
&= 1, \quad \text{if } m_1 + \lfloor \tfrac{\beta - m_1}{2} \rfloor + 1 \le j \le \beta, \\
f(u_j) &= -1, \quad \text{if } \beta + 1 \le j \le \beta + \lceil \tfrac{m_{n+(m-1)}}{2} \rceil, \\
&= 1, \quad \text{if } \beta + \lceil \tfrac{m_{n+(m-1)}}{2} \rceil + 1 \le j \le \beta + m_{n+(m-1)}, \\
f(u_j) &= -1, \quad \text{if } \beta + m_{n+(m-1)} + 1 \le j \le \beta + m_{n+(m-1)} \\
&\qquad + \tfrac{\alpha - \beta - m_{n+(m-1)}}{2}, \\
&= 1, \quad \text{if } \beta + m_{n+(m-1)} + \tfrac{\alpha - \beta - m_{n+(m-1)}}{2} + 1 \le j \le \alpha.
\end{aligned}\right\} 1 \le j \le \alpha.$$

In view of above, the graph G satisfies the following condition. Here $\alpha + mn$ is even, then

$$v_f(-1) = v_f(1) = \frac{\alpha + mn}{2}, \; e_f(-1) = e_f(1) = \alpha.$$

i.e., f is a signed product cordial labeling for G and consequently G is a signed product cordial graph.

Case 4: If n is odd and m_i $(1 \le i \le n)$ are odd, m_i $(n+1 \le i \le nm)$ are even. Here n is odd and all m_i $(1 \le i \le n)$ are odd in cycle C_n. Then there exist at least one pair of adjacent vertices v_i and v_j of cycle C_n have $f(v_i)$ and $f(v_j)$ are same. Therefore, assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have same value and $e_f(-1) = e_f(1) + 2$ or $e_f(-1) + 2 = e_f(1)$. But m_i $(n+1 \le i \le nm)$ are even. Therefore, assign any value to $f(u_j)$, then induce edge labeling of the adjacent edges have different values. Here n is odd, m_i $(1 \le i \le n)$ are odd and m_i $(n+1 \le i \le nm)$ are even, then either $e_f(-1) = e_f(1) + 2$ or $e_f(-1) + 2 = e_f(1)$. Therefore, G is not signed product cordial.

Illustration 2.3 : Consider the arbitrary supersubdivision of $C_4 \odot P_3$. Here $n = 4$, $m = 3$, $m_1 = 2$, $m_2 = 2$, $m_3 = 2$, $m_4 = 2$, $m_5 = 1$, $m_6 = 1$, $m_7 = 2$, $m_8 = 2$, $m_9 = 2$, $m_{10} = 2$, $m_{11} = 1$, $m_{12} = 1$ and $\alpha = 20$. The signed product cordial labeling of the arbitrary supersubdivision of graph $C_4 \odot P_3$ as shown in Figure 3.

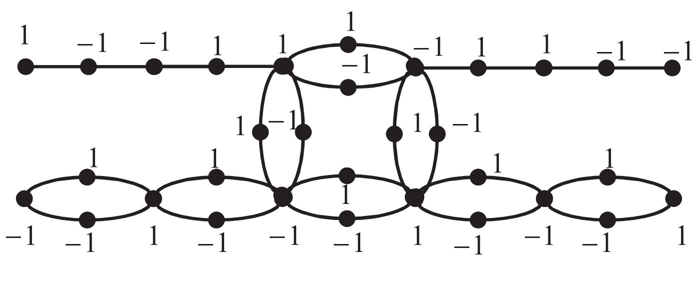

Figure 3: Signed product cordial labeling of the arbitrary supersubdivision of $C_4 \odot P_3$

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Smarandache cyclic geometric determinant sequences

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Abstract In this paper, the concept of Smarandache cyclic geometric determinant sequence was introduced and a formula for its n^{th} term was obtained using the concept of right and left circulant matrices.

Keywords Smarandache cyclic geometric determinant sequence, determinant, right circulan -t matrix, left circulant matrix.

§1. Introduction and preliminaries

Majumdar [1] gave the formula for n^{th} term of the following sequences: Smarandache cyclic natural determinant sequence, Smarandache cyclic arithmetic determinant sequence, Smarandache bisymmetric natural determinant sequence and Smarandache bisymmetric arithmetic determinant sequence.

Definition 1.1. A Smarandache cyclic geometric determinant sequence $\{SCGDS(n)\}$ is a sequence of the form

$$\{SCGDS(n)\} = \left\{ |a|, \begin{vmatrix} a & ar \\ ar & a \end{vmatrix}, \begin{vmatrix} a & ar & ar^2 \\ ar & ar^2 & a \\ ar^2 & a & ar \end{vmatrix}, \ldots \right\}.$$

Definition 1.2. A matrix $RCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix},$$

where $\vec{c} = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ is the circulant vector.

Definition 1.3. A matrix $LCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a leftt circulant matrix if it is of the form

$$LCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & ... & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & ... & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & ... & c_0 & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & c_0 & ... & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & ... & c_{n-4} & c_{n-2} \end{pmatrix},$$

where $\vec{c} = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ is the circulant vector.

Definition 1.4. A right circulant matrix $RCIRC_n(\vec{g})$ with geometric sequence is a matrix of the form

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & ... & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a. \end{pmatrix}.$$

Definition 1.5. A left circulant matrix $LCIRC_n(\vec{g})$ with geometric sequence is a matrix of the form

$$LCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ ar^{n-1} & a & ar & ... & ar^{n-4} & ar^{n-2}. \end{pmatrix}.$$

The right and left circulant matrices has the following relationship:

$$LCIRC_n(\vec{c}) = \Pi RCIRC_n(\vec{c}).$$

where $\Pi = \begin{pmatrix} 1 & O_1 \\ O_2 & \tilde{I}_{n-1} \end{pmatrix}$ with $\tilde{I}_{n-1} = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{pmatrix}$, $O_1 = (0\ 0\ 0\ ...\ 0)$ and $O_2 = O_1^T$.

Clearly, the terms of $\{SCGDS(n)\}$ are just the determinants of $LCIRC_n(\vec{g})$. Now, for the rest of this paper, let $|A|$ be the notation for the determinant of a matrix A. Hence

$$\{SCGDS(n)\} = \{|LCIRC_1(\vec{g})|, |LCIRC_2(\vec{g})|, |LCIRC_3(\vec{g})|, \ldots\}.$$

§2. Preliminary results

Lemma 2.1.

$$|RCIRC_n(\vec{g})| = a^n(1-r^n)^{n-1}.$$

Proof.

$$\begin{aligned} RCIRC_n(\vec{g}) &= \begin{pmatrix} a & ar & ar^2 & \ldots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \ldots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \ldots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \ldots & a & ar \\ ar & ar^2 & ar^3 & \ldots & ar^{n-1} & a \end{pmatrix} \\ &= a \begin{pmatrix} 1 & r & r^2 & \ldots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \ldots & r^{n-3} & r^{n-2} \\ r^{n-2} & r^{n-1} & 1 & \ldots & r^{n-4} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \ldots & 1 & r \\ r & r^2 & r^3 & \ldots & r^{n-1} & 1. \end{pmatrix}. \end{aligned}$$

By applying the row operations $-r^{n-k}R_1 + R_{k+1} \to R_{k+1}$ where $k = 1,\ 2,\ 3, \ldots, n-1$,

$$RCIRC_n(\vec{g}) \sim a \begin{pmatrix} 1 & r & r^2 & \ldots & r^{n-2} & r^{n-1} \\ 0 & -(r^n-1) & -r(r^n-1) & \ldots & -r^{n-3}(r^n-1) & -r^{n-2}(r^n-1) \\ 0 & 0 & -(r^n-1) & \ldots & -r^{n-4}(r^n-1) & -r^{n-3}(r^n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -(r^n-1) & -r(r^n-1) \\ 0 & 0 & 0 & \ldots & 0 & -(r^n-1) \end{pmatrix}.$$

Since $|cA| = c^n|A|$ and its row equivalent matrix is a lower traingular matrix it follows that $|RCIRC_n(\vec{g})| = a^n(1-r^n)^{n-1}$.

Lemma 2.2.

$$|\Pi| = (-1)^{\lfloor \frac{n-1}{2} \rfloor},$$

where $\lfloor x \rfloor$ is the floor function.

Proof. Case 1: $n = 1, 2,$

$$|\Pi| = |I_n| = 1.$$

Case 2: n is even and $n > 2$ If n is even then there will be $n-2$ rows to be inverted because there are two 1's in the main diagonal. Hence there will be $\frac{n-2}{2}$ inversions to bring back Π to I_n so it follows that

$$|\Pi| = (-1)^{\frac{n-2}{2}}.$$

Case 3: n is odd and and $n > 2$ If n is odd then there will be $n-1$ rows to be inverted because of the 1 in the main diagonal of the frist row. Hence there will be $\frac{n-1}{2}$ inversions to bring back Π to I_n so it follows that

$$|\Pi| = (-1)^{\frac{n-1}{2}}.$$

But $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor$, so the lemma follows.

§3. Main results

Theorem 3.1. The n^{th} term of $\{SCGDS(n)\}$ is given by

$$SCGDS(n) = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} a^n (1 - r^n)^{n-1}$$

via the previous lemmas.

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convergence results for asymptotically generalized Φ-hemicontractive mappings

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Abstract In this paper, some strong convergence results are obtained for asymp totically generalized Φ-hemicontractive maps in real Banach spaces using a modified Mann iteration formula with errors. Our results extend and improve the convergence results of Kim, Sahu and Nam [7].

Keywords Modified mann iteration, noor iteration, uniformly lipschitzian, asymptotically p -seudocontractive, three-step iterative scheme, Banach spaces.
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§1. Introduction and preliminaries

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real normed linear space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see, [7]).

Let C be a nonempty subset of real normed linear space X.

Definition 1.1. A mapping $T : C \to X$ is called strongly pseudocontractive if for all $x,\ y \in C$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq (1-k)\|x-y\|^2.$$

A mapping T is called strongly ϕ-pseudocontractive if for all $x,\ y \in C$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$$

and is called generalized strongly Φ-pseudocontractive if for all $x,\ y \in C$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$

Every strongly ϕ-pseudocontractive operator is a generalized strongly Φ-pseudo contractive operator with $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(s) = \phi(s)s$, and every strongly pseudocontractive operator is strongly ϕ-pseudocontractive operator where ϕ is defined by $\phi(s) = ks$ for $k \in (0,1)$ while the converses need not be true. An example by Hirano and Huang [6] showed that a strongly ϕ-pseudocontractive operator T is not always a strongly pseudocontractive operator.

A mapping T is called generalized Φ-hemicontractive if $F(T) = \{x \in C : x = Tx\} \neq \emptyset$ and for all $x \in C$ and $x^* \in F(T)$, there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tx^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|).$$

Definition 1.2. A mapping $T : C \to C$ is called asymptotically generalized Φ-pseudocontractive with sequence $\{k_n\}$ if for each $n \in N$ and $x,\ y \in C$, there exist constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n\|x - y\|^2 - \Phi(\|x - y\|),$$

and is called asymptotically generalized Φ-hemicontractive with sequence $\{k_n\}$ if $F(T) \neq \emptyset$ and for each $n \in N$ and $x \in C$, $x^* \in F(T)$, there exist constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \Phi(\|x - x^*\|).$$

Clearly, the class of asymptotically generalized Φ-hemicontractive mappings is the most general among those defined above. (see, [2]).

Definition 1.3. A mapping $T : C \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x,\ y \in C$ and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1)$$

for all $x,\ y \in C$.

A mapping $T : C \to C$ is called uniformly L-Lipschitzian if for each $n \in N$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x,\ y \in C$.

It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Moreover, every mapping with a bounded range is a generalized Lipschitzian mapping.

Sahu [6] introduced the following new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings.

Fix a sequence $\{r_n\}$ in $[0,\infty]$ with $r_n \to 0$.

Definition 1.4. A mapping $T : C \to C$ is called nearly Lipschitzian with respect to $\{r_n\}$ if for each $n \in N$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + r_n)$$

for all $x,\ y \in C$.

A nearly Lipschitzian mapping T with sequence $\{r_n\}$ is said to be nearly uniformly L-Lipschitzian if $k_n = L$ for all $n \in N$.

Observe that the class of nearly uniformly L-Lipschitzian mapping is more general than the class of uniformly L-Lipschitzian mappings.

In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz type pseudocontractive type nonlinear mappings (see, [1-4, 7, 9-14]). Ofoedu [9] used the modified Mann iteration process introduced by Schu [12]

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 0 \tag{1}$$

to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudocontractive mapping in real Banach space setting. This result itself is a generalization of many of the previous results (see [9] and the references therein).

Recently, Chang [3] proved a strong convergence theorem for a pair of L-Lipschitzian mappings instead of a single map used in [9]. In fact, they proved the following theorem.

Theorem 1.1.[3] Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i :\ K \to K$, $(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$,

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$,

(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n y_n.$$

$$y_n = (1 - \beta_n)x_n + \beta_n T_2^n x_n. \tag{2}$$

If there exists a strictly increasing function $\Phi :\ [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n\|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i = 1, 2), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.

The result above extends and improves the corresponding results of [9] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian asymptotically pseudocontractive mappings. In fact, if the iteration parameter $\{\beta_n\}_{n=0}^{\infty}$ in Theorem 1.1 above is equal to zero for all n and $T_1 = T_2 = T$ then, we have the main result of Ofoedu [9].

Very recently, Kim, Sahu and Nam [7] used the notion of nearly uniformly L-Lipschitzian to established a strong convergence result for asymptotically generalized Φ-hemicontractive mappings in a general Banach space. This result itself is a generalization of many of the previous results. (see, [7]). Indeed, they proved the following:

Theorem 1.2.[7] Let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a nearly uniformly L-Lipschitzian mappings with sequence $\{r_n\}$ and asymptotically generalized $\Phi-$hemicontractive mapping with sequence $\{k_n\}$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the conditions:

(i) $\{\frac{r_n}{\alpha_n}\}$ is bounded,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ in C defined by (1) converges to a unique fixed point of T.

Remark 1.1. Although Theorem 1.2 is a generalization of many of the previous results but, we observed that, the proof process of Theorem 1.2 in [7] can be extend and improve upon.

In this paper, we employed a simple analytical approach to prove the convergence of the modified Mann iteration with errors for fixed points of uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mapping and improve the results of Kim, Sahu and Nam [7]. For this, we need the following Lemmas.

Lemma 1.1.[3] Let X be a real Banach space and let $J : X \to 2^{X^*}$ be a normalized duality mapping. For all $x,\ y \in X$ and for all $j(x + y) \in J(x + y)$. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) > .$$

Lemma 1.2.[8] Let $\Phi : [0, \infty) \to [0, \infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geq N_0,$$

where

$$\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0,$$

then $\lim_{n\to\infty} a_n = 0$.

§2. Main results

Theorem 2.1. Let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{r_n\}$. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying

(i) $a_n + b_n + c_n = 1$,

(ii) $\frac{1}{b_n}$ is bounded,

(iii) $\sum_{n\geq 0} b_n = \infty$,

(iv) $c_n = o(b_n)$,

(v) $\lim_{n\to\infty} b_n = 0$.

Let T be asymptotically generalized Φ-hemicontractive mapping with sequence $k_n \subset [1,\infty)$, $k_n \to 1$ such that $\rho \in F(T) = \{x \in C : Tx = x\}$ and $\{x_n\}_{n=1}^{\infty}$ be the sequence generated for $x_1 \in C$ by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1. \tag{3}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ in C defined by (3) converges to a unique fixed point of T.

Proof. Since T is a nearly uniformly L-Lipschitzian, asymptotically generalized Φ-hemicontractive mapping, then there exists a strictly increasing continuous function $\Phi :\ [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\|T^n x_n - T^n \rho\| \leq L(\|x_n - \rho\| + r_n)$$

and

$$\langle T^n x_n - T^n \rho, j(x_n - \rho)\rangle \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|) \tag{4}$$

for $x \in C,\ \rho \in F(T)$. Equation (3) reduces to

$$x_{n+1} = (1 - b_n)x_n + b_n T^n x_n + c_n(u_n - x_n),\ n \geq 1. \tag{5}$$

Step 1. We first show that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. For this, if $x_{n_0} = Tx_{n_0}$, $n \geq 1$ then it clearly holds. So, let if possible, there exists a positive integer $x_{n_0} \in C$ such that $x_{n_0} \neq Tx_{n_0}$, thus set $x_{n_0} = x_0$ and $a_0 = \|x_0 - T^n x_0\|\|x_0 - \rho\| + (k_n - 1)\|x_0 - \rho\|^2$. Thus by (4),

$$\langle T^n x_0 - T^n \rho, j(x - \rho)\rangle \leq k_n \|x_0 - \rho\|^2 - \Phi(\|x_0 - \rho\|), \tag{6}$$

so that, on simplifying

$$\|x_0 - \rho\| \leq \Phi^{-1}(a_0). \tag{7}$$

Now, we claim that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_0)$, $n \geq 0$. Clearly, inview of (7), the claim holds for $n = 0$. We next assume that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_0)$, for some n and we shall prove that $\|x_{n+1} - \rho\| \leq 2\Phi^{-1}(a_0)$. Suppose this is not true, i.e. $\|x_{n+1} - \rho\| > 2\Phi^{-1}(a_0)$.

Since $\{r_n\} \in [0,\infty]$ with $r_n \to 0$, $\frac{1}{b_n}$ and $\{u_n\}$ are bounded sequences, set $M_1 = \sup\{r_n : n \in N\}$, $M_3 = \sup\{\frac{1}{b_n} : n \in N\}$ and $M_2 = \sup\{u_n : n \in N\}$. Denote

$$\begin{aligned}\tau_0 \quad = \quad & \min \frac{1}{3}\{1, \frac{\Phi(2(\Phi^{-1}(a_0)))}{18(\Phi^{-1}(a_0))^2}, \\ & \frac{\Phi(2(\Phi^{-1}(a_0)))}{6(1+L)[2(2+L)\Phi^{-1}(a_0) + 2M_1 L + M_2 + \frac{M_1 M_3 L}{(1+L)}](\Phi^{-1}(a_0))}, \\ & \frac{\Phi(2(\Phi^{-1}(a_0)))}{6\Phi^{-1}(a_0)(M_2 + 2\Phi^{-1}(a_0))}, \frac{3\Phi^{-1}(a_0)}{2(2+L)\Phi^{-1}(a_0) + M_1 L + M_2}\}.\end{aligned} \tag{8}$$

Since $\lim_{n\to\infty} b_n,\ c_n = 0$, and $c_n = o(b_n)$, without loss of generality, let $0 \leq b_n,\ c_n,\ k_n - 1 \leq \tau_0,\ c_n < b_n \tau_0$ for any $n \geq 1$. Then we have the following estimates from (3).

$$\begin{aligned}
\|x_n - T^n x_n\| &\le \|x_n - \rho\| + \|T^n x_n - \rho\| \\
&\le \|x_n - \rho\| + L(\|x_n - \rho\| + r_n) \\
&\le (1+L)\|x_n - \rho\| + r_n L \\
&\le 2(1+L)\Phi^{-1}(a_0) + M_1 L. \\
\|u_n - x_n\| &= \|(u_n - \rho + \rho - x_n\| \\
&\le \|u_n - \rho\| + \|x_n - \rho\|) \\
&\le M_2 + 2\Phi^{-1}(a_0)). \\
\|x_{n+1} - \rho\| &\le \|(1-b_n)x_n + b_n T^n x_n + c_n(u_n - x_n) - \rho\|, \\
&\le \|x_n - \rho\| + b_n\|T^n x_n - x_n\| + c_n\|u_n - x_n\|, \\
&\le 2\Phi^{-1}(a_0) + b_n[2(1+L)\Phi^{-1}(a_0) + M_1 L] \\
&\quad + c_n[M_2 + 2\Phi^{-1}(a_0))] \\
&\le 2\Phi^{-1}(a_0) + \tau_0[2(2+L)\Phi^{-1}(a_0) + M_1 L + M_2] \\
&\le 3\Phi^{-1}(a_0). \\
\|x_n - x_{n+1}\| &\le \tau_0[2(2+L)\Phi^{-1}(a_0) + 2M_1 L + M_2]. \\
&= \|b_n(T^n x_n - x_n) + c_n(u_n - x_n)\| \\
&\le b_n\|T^n x_n - x_n\| + c_n\|u_n - x_n\| \\
&\le b_n[2(1+L)\Phi^{-1}(a_0) + M_1 L] + c_n[M_2 + 2\Phi^{-1}(a_0))] \\
&\le \tau_0[2(2+L)\Phi^{-1}(a_0) + M_1 L + M_2]. \qquad (9)
\end{aligned}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{aligned}
\|x_{n+1} - \rho\|^2 &\le \|x_n - \rho\|^2 - 2b_n < x_n - T^n x_n, j(x_{n+1} - \rho) > \\
&\quad + 2c_n < u_n - x_n, j(x_{n+1} - \rho) > \\
&= \|x_n - \rho\|^2 + 2b_n < T^n x_{n+1} - \rho, j(x_{n+1} - \rho) > \\
&\quad - 2b_n < x_{n+1} - \rho, j(x_{n+1} - \rho) > + 2b_n < T^n x_n - T^n x_{n+1}, j(x_{n+1} - \rho) > \\
&\quad + 2b_n < x_{n+1} - x_n, j(x_{n+1} - \rho) > + 2c_n < u_n - x_n, j(x_{n+1} - \rho) > \\
&\le \|x_n - \rho\|^2 + 2b_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|)) \\
&\quad - 2b_n\|x_{n+1} - \rho\|^2 + 2b_n L(\|x_{n+1} - x_n\| + r_n)\|x_{n+1} - \rho\| \\
&\quad + 2b_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| + 2c_n\|u_n - x_n\|\|x_{n+1} - \rho\| \\
&\le \|x_n - \rho\|^2 + 2b_n(k_n - 1)\|x_{n+1} - \rho\|^2 - 2b_n\Phi(\|x_{n+1} - \rho\|)) \\
&\quad + 2b_n\{(1+L)\|x_{n+1} - x_n\| + M_1 L\}\|x_{n+1} - \rho\| \\
&\quad + 2c_n(M_2 + \|x_n - \rho\|)\|x_{n+1} - \rho\| \\
&\le \|x_n - \rho\|^2 - 2b_n\Phi(2(\Phi^{-1}(a_0)) + 18b_n(k_n - 1)(\Phi^{-1}(a_0))^2 \\
&\quad + 6b_n\Phi^{-1}(a_0)\{(1+L)\tau_0(2(2+L)\Phi^{-1}(a_0) \\
&\quad + M_1 L + M_2) + M_1 L\} + 6b_n\Phi^{-1}(a_0)(M_2 + 2\Phi^{-1}(a_0))\tau_0 \\
&= \|x_n - \rho\|^2 - 2b_n\Phi(2(\Phi^{-1}(a_0)) + 18b_n\tau_0(\Phi^{-1}(a_0))^2
\end{aligned}$$

$$
\begin{aligned}
& +6b_n\Phi^{-1}(a_0)(1+L)\tau_0\{2(2+L)\Phi^{-1}(a_0) \\
& +M_1L+M_2+\frac{M_1L}{(1+L)\tau_0}\}+6b_n\Phi^{-1}(a_0)(M_2+2\Phi^{-1}(a_0))\tau_0, \\
\leq\ & \|x_n-\rho\|^2-2b_n\Phi(2(\Phi^{-1}(a_0))+18b_n\tau_0(\Phi^{-1}(a_0))^2 \\
& +6b_n\Phi^{-1}(a_0)(1+L)\tau_0\{2(2+L)\Phi^{-1}(a_0) \\
& +M_1L+M_2+\frac{M_1L}{(1+L)b_n}\}+6b_n\Phi^{-1}(a_0)(M_2+2\Phi^{-1}(a_0))\tau_0, \\
=\ & \|x_n-\rho\|^2-2b_n\Phi(2(\Phi^{-1}(a_0))+18b_n\tau_0(\Phi^{-1}(a_0))^2 \\
& +6b_n\Phi^{-1}(a_0)(1+L)\tau_0\{2(2+L)\Phi^{-1}(a_0) \\
& +M_1ra|x_n-\rho\|^2-b_n\Phi(2(\Phi^{-1}(a_0)) \\
\leq\ & \|x_n-\rho\|^2 \\
\leq\ & (2(\Phi^{-1}(a_0)))^2, \qquad (10)
\end{aligned}
$$

which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

Step 2. We want to prove $\|x_n-\rho\|\to\infty$. Since $b_n,\ c_n\to 0$ as $n\to\infty$ and $\{x_n\}_{n=1}^{\infty}$ is bounded. From (9), we observed that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\ \lim_{n\to\infty}\|T^nx_n-T^nx_{n+1}\|=0,\ \lim_{n\to\infty}(k_n-1)=0$. So from (10), we have

$$
\begin{aligned}
\|x_{n+1}-\rho\|^2 \leq\ & \|x_n-\rho\|^2-2b_n<x_n-T^nx_n,j(x_{n+1}-\rho)> \\
& +2c_n<u_n-x_n,j(x_{n+1}-\rho)> \\
=\ & \|x_n-\rho\|^2+2b_n<T^nx_{n+1}-\rho,j(x_{n+1}-\rho)> \\
& -2b_n<x_{n+1}-\rho,j(x_{n+1}-\rho)> \\
& +2b_n<T^nx_n-T^nx_{n+1},j(x_{n+1}-\rho)> \\
& +2b_n<x_{n+1}-x_n,j(x_{n+1}-\rho)> \\
& +2c_n<u_n-x_n,j(x_{n+1}-\rho)> \\
\leq\ & \|x_n-\rho\|^2+2b_n(k_n-1)\|x_{n+1}-\rho\|^2-2b_n\Phi(\|x_{n+1}-\rho\|) \\
& 2b_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
& +2c_n\|u_n-x_n\|\|x_{n+1}-\rho\| \\
=\ & \|x_n-\rho\|^2-2b_n\Phi(\|x_{n+1}-\rho\|)+o(b_n),
\end{aligned}
$$

where

$$
\begin{aligned}
& 2b_n(k_n-1)\|x_{n+1}-\rho\|^2+2b_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
& +2b_n\|T^nx_{n+1}-T^nx_n\|\|x_{n+1}-\rho\|+2c_n\|u_n-x_n\|\|x_{n+1}-\rho\| \\
=\ & o(b_n). \qquad (11)
\end{aligned}
$$

By Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|x_n-\rho\|=0.$$

This completes the proof.

Corollary 2.1. Let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a nearly uniformly L-Lipschitzian mapping with $\{r_n\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0, 1]$ satisfying:

(i) $\frac{1}{\alpha_n}$ is bounded,

(ii) $\sum_{n\geq 0} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$.

Let T be asymptotically generalized $\Phi-$hemicontractive mapping with sequence $k_n \subset [1, \infty)$, $k_n \to 1$ such that $\rho \in F(T) = \{x \in C : Tx = x\}$ and $\{x_n\}_{n=1}^{\infty}$ be the sequence generated for $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1, \tag{12}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ in C defined by (12) converges to a unique fixed point of T.

Remark 2.1. Our Corollary removes the conditions in Theorem 2.1 of [7], (i.e. $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$) by replacing them with a weaker condition $\lim_{n\to\infty} \alpha_n = 0$. Therefore our result extends and improves the very recent results of Kim [7] which in turn is a correction, improvement and generalization of several results.

Application 2.1. Let $X = R$, $C = [0, 1]$ and $T :\ C \to C$ be a map defined by

$$Tx = \frac{x}{4}.$$

Clearly, T is nearly uniformly Lipschitzian ($r_n = \frac{1}{4^n}$) with $F(T) = 0$.

Define $\Phi :\ [0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = \frac{t^2}{4},$$

then Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in C,\ \rho \in F(T)$, we get

$$\begin{aligned} < T^n x - T^n\rho, j(x-\rho) > &= < \frac{x^n}{4^n} - 0, j(x-0) > \\ &= < \frac{x^n}{4^n} - 0, x > \\ &= \frac{x^{n+1}}{4^n} \\ &\leq x^2 - \frac{x^2}{4} \\ &\leq x^2 - \Phi(x). \end{aligned}$$

Obviously, T is asymptotically generalized Φ-hemicontractive mapping with sequence $\{k_n\} = 1$. If we take

$$b_n = \frac{1}{n+1},\ c_n = \frac{1}{(n+1)^2}$$

for all $n \geq 1$. For arbitrary $x_1 \in C$, the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by (3) converges strongly to the unique fixed point $\rho \in T$.

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The $\upsilon-$ invariant χ^2 sequence spaces

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Abstract In this paper we characterize υ-invariance of the sequence spaces $\Delta^2\left(\Lambda^2\right)$, and $\Delta^2\left(\chi^2\right)$.

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§1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m,\ n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn}-l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,\ n \in \mathbb{N}$ and $p-\lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,\ n \in \mathbb{N}$, $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and

$\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to multiply sequence spaces.

We need the following inequality in the sequel of the paper. For $a,\ b \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p. \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,\ n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m,\ n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i,\ j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable, locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

If X is a sequence space, we give the following definitions:

(i) $X^{'}$= the continuous dual of X,

(ii) $X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\},$

(iii) $X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X\right\},$

(iv) $X^{\gamma} = \left\{a = (a_{mn}) : \sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\},$

(v) Let$X$be an FK-space $\supset \phi$, then $X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\},$

(vi) $X^{\delta} = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\}.$

X^{α}, X^{β}, X^{γ} are called α- (or Köthe-Toeplitz) dual of X, β- (or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X respectively. X^{α} is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c,\ c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here $c,\ c_0$ and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2,\ \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,\ n \in \mathbb{N}$.

§2. Definitions and preliminaries

Definition 2.1. A sequence X is υ-invariant if $X_\upsilon = X$ where $X_\upsilon = \{x = (x_{mn}) : (\upsilon_{mn}x_{mn}) \in X\},$ where $X = \Lambda^2$ and χ^2.

Definition 2.2. We say that a sequence space $\Delta^2(X)$ is υ-invariant if $\Delta^2_\upsilon(X) = \Delta^2(X)$.

Definition 2.3. Let $A = \left(a^{mn}_{k,\ell}\right)$ denotes a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ \ell$-th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn},$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

§3. Main results

Lemma 3.1. $\sup_{mn} |x_{mn} - x_{mn+1} - x_{m+1n+1} + x_{m+1n+1}|^{1/m+n} < \infty$ if and only if

(i)

$$\sup_{mn} (mn)^{-1} |x_{mn}|^{1/m+n} < \infty,$$

(ii)

$$\sup_{mn} |x_{mn} - (mn)(mn+1)^{-1} x_{mn+1} - (mn)(m+1n)^{-1} x_{m+1n} + (mn)(m+1n+1)^{-1} x_{m+1n+1}|^{1/m+n} < \infty.$$

If we consider Lemma (3.1), then we have the following result.

Corollary 3.1.

$$\sup_{mn} |\,(mn)\frac{v_{mn}}{w_{mn}} - (mn+1)\frac{v_{mn+1}}{w_{mn+1}} - (m+1n)\frac{v_{m+1n}}{w_{m+1n}} + (m+1n+1)\frac{v_{m+1n+1}}{w_{m+1n+1}}|^{1/m+n} < \infty,$$

if and only if

(i)

$$\sup_{mn} \left|\frac{v_{mn}}{w_{mn}}\right|^{1/m+n} < \infty,$$

(ii)

$$\sup_{mn} (mn) \left|\frac{v_{mn}}{w_{mn}} - \frac{v_{mn+1}}{w_{mn+1}} - \frac{v_{m+1n}}{w_{m+1n}} + \frac{v_{m+1n+1}}{w_{m+1n+1}}\right|^{1/m+n} < \infty.$$

Theorem 3.1. $\Delta_w^2(\Lambda^2) \subset \Delta_v^2(\Lambda^2)$ if and only if the matrix $A = (a_{k\ell}^{mn})$ maps Λ^2 into Λ^2 where

$$a_{k\ell}^{mn} = \begin{cases} v_{i+1j+1} + v_{i+1j} + v_{ij+1} - v_{ij}, & \\ \frac{v_{i+1j+1}}{w_{i+1j+1}}, & \text{if } m,\ n\text{=i+1, j+1,} \\ \frac{v_{i+1j}}{w_{i+1j}}, & \text{if } m,\ n\text{=i+1, j,} \\ \frac{v_{ij+1}}{w_{ij+1}}, & \text{if } m,\ n \text{ i, j+1,} \\ 0, & \text{if } m,\ n \geq \text{i+2, j+2.} \end{cases} \tag{2}$$

Proof. Let $y \in \Lambda^2$. Define

$$x_{ij} = -\frac{\sum_{m=1}^{i}\sum_{n=1}^{j} |y_{mn}|^{1/m+n}}{w_{pq}}.$$

Then $w_{ij}x_{ij} - w_{ij+1}x_{ij+1} - w_{i+1j}x_{i+1j} + w_{i+1j+1}x_{i+1j+1} \in \Lambda^2$. Hence $\Delta_w^2 x \in \Lambda^2$, so by assumption $\Delta_v x \in \Lambda^2$. It follows that $Ay \in \Lambda^2$. This shows that $A = (a_{k\ell}^{mn})$ maps Λ^2 into Λ^2. Let $x \in \Delta_w\left(\Lambda^2\right)$, hence $(w_{ij}x_{ij} - w_{ij+1}x_{ij+1} - w_{i+1j}x_{i+1j} + w_{i+1j+1}x_{i+1j+1}) \in \Lambda^2$. Then also $y = (w_{i-1j-1}x_{i-1j-1} - w_{i-1j}x_{i-1j} - w_{ij-1}x_{ij-1} + w_{ij}x_{ij}) \in \Lambda^2$, where $w_{-1} = x_{-1} = 0$. By assumption we have $Ay \in \Lambda^2$. Hence $x \in \Delta_v^2\left(\Lambda^2\right)$. This completes the proof.

Theorem 3.2. Let X and Y be sequence spaces and assume that X is such that a sequence

$$\begin{pmatrix} x_{11} & x_{12} & x_{13}\cdots & x_{1n} & 0 \\ x_{21} & x_{22} & x_{23}\cdots & x_{2n} & 0 \\ \vdots & & & & \\ x_{m1} & x_{m2} & x_{m3}\cdots & x_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

belongs to X if and only if the sequence

$$\begin{pmatrix} x_{21} & x_{22} & x_{23}\cdots & x_{2n} & 0 \\ x_{31} & x_{32} & x_{33}\cdots & x_{3n} & 0 \\ \vdots & & & & \\ x_{m1} & x_{m2} & x_{m3}\cdots & x_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}.$$

Then we have $\Delta_w^2(X) \subset \Delta_v^2(Y)$ if and only if the matrix $A = (a_{k\ell}^{mn})$ maps X into Y, where $A = (a_{k\ell}^{mn})$ is defined by equation (2).

The proof is very similar to that of Theorem (3.3).

Corollary 3.2. We have $\Delta_w^2\left(\Lambda^2\right) \subset \Delta_v^2\left(\Lambda^2\right)$ if and only if

$$\sup_{mn}\left|mn\frac{v_{mn}}{w_{mn}} - mn+1\frac{v_{mn+1}}{w_{mn+1}} - m+1n\frac{v_{m+1n}}{w_{m+1n}} + m+1n+1\frac{v_{m+1n+1}}{w_{m+1n+1}}\right|^{1/m+n} < \infty.$$

Proof. This follows from Theorem 3.3, the well known characterization of matrices mapping Λ^2 into Λ^2 and corollary 3.2.

Corollary 3.3. We have $\Delta_w^2\left(\chi^2\right) \subset \Delta_v^2\left(\chi^2\right)$ if and only if

$$((m+n)!\left|mn\frac{v_{mn}}{w_{mn}} - mn+1\frac{v_{mn+1}}{w_{mn+1}} - m+1n\frac{v_{m+1n}}{w_{m+1n}} + m+1n+1\frac{v_{m+1n+1}}{w_{m+1n+1}}\right|)^{1/m+n}$$
$$\to 0 \text{ as } m,\ n \to \infty.$$

Proof. This follows from Theorem 3.5, the well-known characterization of matrices mapping χ^2 into χ^2 and corollary 3.2.

If we consider corollary 3.5 and corollary 3.6, then we have necessary and sufficient conditions for the v-invariant of $\Delta^2\left(\Lambda^2\right)$ and $\Delta^2\left(\chi^2\right)$.

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Weakly generalized compactness in intuitionistic fuzzy topological spaces

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Abstract The aim of this paper is to introduce and study the concepts of weakly generalized compact space, almost weakly generalized compact space using intuitionistic fuzzy weakly generalized open sets and nearly weakly generalized compact space using intuitionistic fuzzy weakly generalized closed sets in intuitionistic fuzzy topological space. Some of their properties are explored.

Keywords Intuitionistic fuzzy topology, intuitionistic fuzzy weakly generalized compact spa -ce, intuitionistic fuzzy almost weakly generalized compact space and intuitionistic fuzzy nea -rly weakly generalized compact space.
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§1. Introduction

Fuzzy set (FS) as proposed by Zadeh [13] in 1965, is a framework to encounter uncertainty, vagueness and partial truth and it represents a degree of membership for each member of the universe of discourse to a subset of it. After the introduction of fuzzy topology by Chang [3] in 1968, there have been several generalizations of notions of fuzzy sets and fuzzy topology. By adding the degree of non-membership to FS, Atanassov [1] proposed intuitionistic fuzzy set (IFS) in 1986 which looks more accurate to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. In 1997, Coker [4] introduced the concept of intuitionistic fuzzy topological space. In this paper, we introduce a new class of intuitionistic fuzzy topological space called weakly generalized compact space, almost weakly generalized compact space using intuitionistic fuzzy weakly generalized open sets and nearly weakly generalized compact space using intuitionistic fuzzy weakly generalized closed sets and study some of their properties.

§2. Preliminaries

Definition 2.1.[1] Let X be a non empty fixed set. An intuitionistic fuzzy set (IFS in short) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element x $\in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

Definition 2.2.[1] Let A and B be $IFSs$ of the forms $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and B $= \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$.

(ii) A = B if and only if A $\subseteq B$ and B $\subseteq$ A.

(iii) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

(iv) A $\cap$ B $= \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$.

(v) A $\cup$ B $= \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$.

For the sake of simplicity, the notation $A = \langle x, \mu_A, \nu_A\rangle$ shall be used instead of $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Also for the sake of simplicity, we shall use the notation $A = \langle x, (\mu_A, \mu_B), (\nu_A, \nu_B)\rangle$ instead of $A = \langle x, (A/\mu_A, B/\mu_B), (A/\nu_A, B/\nu_B)\rangle$.

The intuitionistic fuzzy sets $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$ are the empty set and the whole set of X, respectively.

Definition 2.3.[4] An intuitionistic fuzzy topology (IFT in short) on a non empty set X is a family τ of $IFSs$ in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$.

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space ($IFTS$ in short) and any IFS in τ is known as an intuitionistic fuzzy open set ($IFOS$ in short) in X.

The complement A^c of an $IFOSA$ in an $IFTS$ (X, τ) is called an intuitionistic fuzzy closed set ($IFCS$ in short) in X.

Definition 2.4.[4] Let (X, τ) be an $IFTS$ and $A = \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the intuitionistic fuzzy interior and an intuitionistic fuzzy closure are defined by

$$int(A) = \cup\{G/G \text{ is an } IFOS \text{ in } X \text{ and } G \subseteq A\},$$

$$cl(A) = \cap\{K/K \text{ is an } IFCS \text{ in } X \text{ and } A \subseteq K\}.$$

Note that for any IFS A in (X, τ), we have $cl(A^c) = (int(A))^c$ and $int(A^c) = (cl(A))^c$.

Definition 2.5.[5] An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in an $IFTS$ (X, τ) is said to be an intuitionistic fuzzy weakly generalized closed set ($IFWGCS$ in short) if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is an $IFOS$ in X.

The family of all $IFWGCSs$ of an $IFTS$ (X, τ) is denoted by $IFWGC(X)$.

Definition 2.6.[5] An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is said to be an intuitionistic fuzzy weakly generalized open set ($IFWGOS$ in short) in (X, τ) if the complement A^c is an $IFWGCS$ in (X, τ).

The family of all $IFWGOSs$ of an $IFTS$ (X, τ) is denoted by $IFWGO(X)$.

Result 2.1.[5] Every $IFCS$, $IF\alpha CS$, $IFGCS$, $IFRCS$, $IFPCS$, $IF\alpha GCS$ is an $IFWGCS$ but the converses need not be true in general.

Definition 2.7.[6] Let (X, τ) be an $IFTS$ and $A = \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the intuitionistic fuzzy weakly generalized interior and an intuitionistic fuzzy weakly generalized closure are defined by

$$wgint(A) = \cup\{G/G \text{ is an } IFWGOS \text{ in } X \text{ and } G \subseteq A\},$$

$$wgcl(A) = \cap\{K/K \text{ is an } IFWGCS \text{ in } X \text{ and } A \subseteq K\}.$$

Definition 2.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping from an $IFTS$ (X, τ) into an $IFTS$ (Y, σ). Then f is said to be

(i) Intuitionistic fuzzy weakly generalized continuous [7] ($IFWG$ continuous in short) if $f^{-1}(B)$ is an $IFWGOS$ in X for every $IFOS$ B in Y.

(ii) Intuitionistic fuzzy quasi weakly generalized continuous [8] (IF quasi WG continuous in short) if $f^{-1}(B)$ is an $IFOS$ in X for every $IFWGOS$ B in Y.

(iii) Intuitionistic fuzzy weakly generalized iresolute [6] ($IFWG$ irresolute in short) if $f^{-1}(B)$ is an $IFWGOS$ in X for every $IFWGOS$ B in Y.

(iv) Intuitionistic fuzzy perfectly weakly genralized continuous continuous [10] (IF perfectly WG continuous in short) if $f^{-1}(B)$ is an intuitionistic fuzzy clopen set in X for every $IFWGCS$ B in Y.

(v) Intuitionistic fuzzy weakly generalized * open mapping [9] ($IFWG^*OM$ in short) if $f(B)$ is an $IFWGOS$ in Y for every $IFWGOS$ B in X.

(vi) Intuitionistic fuzzy contra weakly generalized continuous [11] (IF contra WG continuous in short) if $f^{-1}(B)$ is an $IFWGCS$ in X for every $IFOS$ B in Y.

(vii) Intuitionistic fuzzy contra weakly generalized irresolute [12] (IF contra WG irresolute in short) if $f^{-1}(B)$ is an $IFWGCS$ in X for every $IFWGOS$ B in Y.

Definition 2.9.[2] Let (X, τ) be an $IFTS$. A family $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ of $IFOSs$ in X satisfying the condition $1_\sim = \cup\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ is called an intuitionistic fuzzy open cover of X.

Definition 2.10.[2] A finite sub family of an intuitionistic fuzzy open cover $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ of X which is also an intuitionistic fuzzy open cover of X is called a finite sub cover of $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$.

Definition 2.11.[2] An $IFTS$ (X, τ) is called intuitionistic fuzzy compact if every intuitionistic fuzzy open cover of X has a finite sub cover.

Definition 2.12.[2] An $IFTS$ (X, τ) is called intuitionistic fuzzy Lindelof if each intuitionistic fuzzy open cover of X has a countable sub cover for X.

Definition 2.13.[2] An $IFTS$ (X, τ) is called intuitionistic fuzzy countable compact if each countable intuitionistic fuzzy open cover of X has a finite sub cover for X.

Definition 2.14. An $IFTS$ (X, τ) is said to be

(i) Intuitionistic fuzzy S-closed [2] if each intuitionistic fuzzy regular closed cover of X has a finite sub cover for X.

(ii) Intuitionistic fuzzy S-Lindelof [2] if each intuitionistic fuzzy regular closed cover of X has a countable sub cover for X.

(iii) Intuitionistic fuzzy countable S-closed [2] if each countable intuitionistic fuzzy regular closed cover of X has a finite sub cover for X.

Definition 2.15. An $IFTS$ (X, τ) is said to be

(i) Intuitionistic fuzzy strongly S-closed [2] if each intuitionistic fuzzy closed cover of X has a finite sub cover for X.

(ii) Intuitionistic fuzzy strongly S-Lindelof [2] if each intuitionistic fuzzy closed cover of X has a countable sub cover for X.

(iii) Intuitionistic fuzzy countable strongly S-closed [2] if each countable intuitionistic fuzzy closed cover of X has a finite sub cover for X.

Definition 2.16. An $IFTS$ (X, τ) is sad to be

(i) Intuitionistic fuzzy almost compact [2] if each intuitionistic fuzzy open cover of X has a finite sub cover the closure of whose members cover X.

(ii) Intuitionistic fuzzy almost Lindelof [2] if each intuitionistic fuzzy open cover of X has a countable sub cover the closure of whose members cover X.

(iii) Intuitionistic fuzzy countable almost compact [2] if each countable intuitionistic fuzzy open cover of X has a finite sub cover the closure of whose members cover X.

§3. Intuitionistic fuzzy weakly generalized compact spaces

In this section, we introduce a new class of intuitionistic fuzzy topological spaces called weakly generalized compact spaces, almost weakly generalized compact spaces using intuitionistic fuzzy weakly generalized open sets and nearly weakly generalized compact spaces using intuitionistic fuzzy weakly generalized closed sets and study some of their properties.

Definition 3.1. Let (X, τ) be an $IFTS$. A family $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ of $IFWGOSs$ in X satisfying the condition $1_\sim = \cup\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ is called an intuitionistic fuzzy weakly generalized open cover of X.

Definition 3.2. A finite sub family of an intuitionistic fuzzy weakly generalized open cover $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ of X which is also an intuitionistic fuzzy weakly generalized open cover of X is called a finite sub cover of $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$.

Definition 3.3. An intuitionistic fuzzy set A of an $IFTS(X, \tau)$ is said to be intuitionistic fuzzy weakly generalized compact relative to X if every collection $\{A_i : i \in I\}$ of intuitionistic fuzzy weakly generalized open subset of X such that $A \subseteq \cup\{A_i : i \in I\}$, there exists a finite subset I_0 of I such that $A \subseteq \cup\{A_i : i \in I_0\}$.

Definition 3.4. Let (X, τ) be an $IFTS$. A family $\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ of $IFWGCSs$ in X satisfying the condition $1_\sim = \cup\{\langle x, \mu_{Gi}, \nu_{Gi}\rangle : i \in I\}$ is called an intuitionistic fuzzy weakly generalized closed cover of X.

Definition 3.5. An $IFTS$ (X, τ) is said to be intuitionistic fuzzy weakly generalized compact if every intuitionistic fuzzy weakly generalized open cover of X has a finite sub cover.

Definition 3.6. An $IFTS$ (X, τ) is said to be intuitionistic fuzzy weakly generalized Lindelof if each intuitionistic fuzzy weakly generalized open cover of X has a countable sub cover for X.

Definition 3.7. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy countable weakly generalized compact if each countable intuitionistic fuzzy weakly generalized open cover of X has a finite sub cover for X.

Definition 3.8. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy nearly weakly generalized compact if each intuitionistic fuzzy weakly generalized closed cover of X has a finite sub cover.

Definition 3.9. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy nearly weakly generalized Lindelof if each intuitionistic fuzzy weakly generalized closed cover of X has a countable sub cover for X.

Definition 3.10. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy nearly countable weakly generalized compact if each countable intuitionistic fuzzy weakly generalized closed cover of X has a finite sub cover for X.

Definition 3.11. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy almost weakly generalized compact if each intuitionistic fuzzy weakly generalized open cover of X has a finite sub cover the closure of whose members cover X.

Definition 3.12. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy almost weakly generalized Lindelof if each intuitionistic fuzzy weakly generalized open cover of X has a countable sub cover the closure of whose members cover X.

Definition 3.13. An $IFTS$ (X,τ) is said to be intuitionistic fuzzy almost countable weakly generalized compact if each countable intuitionistic fuzzy weakly generalized open cover of X has a finite sub cover the closure of whose members cover X.

Definition 3.14. A crisp subset B of an $IFTS$ (X,τ) is said to be intuitionistic fuzzy weakly generalized compact if B is intuitionistic fuzzy weakly generalized compact as intuitionistic fuzzy subspace of X.

Theorem 3.1. An intuitionistic fuzzy weakly generalized closed crisp subset of an intuitionistic fuzzy weakly generalized compact space is intuitionistic fuzzy weakly generalized compact relative to X.

Proof. Let A be an intuitionistic fuzzy weakly generalized closed crisp subset of an intuitionistic fuzzy weakly generalized compact space (X,τ). Then A^c is an $IFWGOS$ in X. Let S be a cover of A by $IFWGOS$ in X. Then, the family $\{S, A^c\}$ is an intuitionistic fuzzy weakly generalized open cover of X. Since X is an intuitionistic fuzzy weakly generalized compact space, it has finite sub cover say $\{G_1, G_2 \ldots G_n\}$. If this sub cover contains A^c, we discard it. Otherwise leave the sub cover as it is. Thus we obtained a finite intuitionistic fuzzy weakly generalized open sub cover of A. Therefore A is intuitionistic fuzzy weakly generalized compact relative to X.

Theorem 3.2. Let $f : (X,\tau) \to (Y,\sigma)$ be an $IFWG$ continuous mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (X,τ) is intuitionistic fuzzy weakly generalized compact, then (Y,σ) is intuitionistic fuzzy compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy open cover of (Y,σ). Then $1_\sim = \cup_{i\in I} A_i$. From the relation, $1_\sim = f^{-1}(\cup_{i\in I} A_i)$ follows that $1_\sim = \cup_{i\in I} f^{-1}(A_i)$, so $\left\{f^{-1}(A_i) : i \in I\right\}$ is an intuitionistic fuzzy weakly generalized open cover of (X,τ). Since (X,τ) is intuitionistic fuzzy weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}$.

Therefore $1_\sim = \bigcup_{i=1}^{n} f^{-1}(A_i)$ Hence

$$1_\sim = f\left(\bigcup_{i=1}^{n} f^{-1}(A_i)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(A_i)\right) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \dots A_n\}$ is a finite sub cover of (Y, σ). Hence (Y, σ) is intuitionistic fuzzy compact.

Corollary 3.1. Let $f : (X, \tau) \to (Y, \sigma)$ be an $IFWG$ continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy weakly generalized Lindelof, then (Y, σ) is intuitionistic fuzzy Lindelof.

Proof. Obvious.

Theorem 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF quasi WG continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy compact, then (Y, σ) is intuitionistic fuzzy weakly generalized compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy weakly generalized open cover of (Y, σ). Then $1_\sim = \cup_{i\in I} A_i$. From the relation, $1_\sim = f^{-1}(\cup_{i\in I} A_i)$ follows that $1_\sim = \cup_{i\in I} f^{-1}(A_i)$, so $\left\{f^{-1}(A_i) : i \in I\right\}$ is an intuitionistic fuzzy open cover of (X, τ). Since (X, τ) is intuitionistic fuzzy compact, there exists a finite sub cover say, $\left\{f^{-1}(A_1), f^{-1}(A_2), \dots f^{-1}(A_n)\right\}$. Therefore $1_\sim = \bigcup_{i=1}^{n} f^{-1}(A_i)$. Hence

$$1_\sim = f\left(\bigcup_{i=1}^{n} f^{-1}(A_i)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(A_i)\right) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \dots A_n\}$ is a finite sub cover of (Y, σ). Hence (Y, σ) is intuitionistic fuzzy weakly generalized compact.

Corollary 3.2. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF perfectly WG continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy compact, then (Y, σ) is intuitionistic fuzzy weakly generalized compact.

Proof. Obvious.

Theorem 3.4. If $f : (X, \tau) \to (Y, \sigma)$ is an $IFWG$ irrresolute mapping and an intuitionistic fuzzy subset B of an $IFTS$ (X, τ) is intuitionistic fuzzy weakly generalized compact relative to an $IFTS$ (X, τ), then the image $f(B)$ is intuitionistic fuzzy weakly generalized compact relative to (Y, σ).

Proof. Let $\{A_i : i \in I\}$ be any collection $IFWGOSs$ of (Y, σ) such that $f(B) \subseteq \cup \{A_i : i \in I\}$. Since f is $IFWG$ irresolute, $B \subseteq \cup\left\{f^{-1}(A_i) : i \in I\right\}$ where $f^{-1}(A_i)$ is intuitionistic fuzzy weakly generalized open cover in (X, τ) for each i. Since B is intuitionistic fuzzy weakly generalized compact relative to (X, τ), there exists a finite subset I_0 of I such that $B \subseteq \cup\left\{f^{-1}(A_i) : i \in I_0\right\}$. Therefore $f(B) \subseteq \cup\{A_i : I \in I_0\}$. Hence $f(B)$ is intuitionistic fuzzy weakly generalized compact relative to (Y, σ).

Theorem 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be an $IFWG$ irresolute mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy nearly weakly generalized compact, then (Y, σ) is intuitionistic fuzzy nearly weakly generalized compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy weakly generalized closed cover of (Y, σ). Then $1_\sim = \cup_{i\in I} A_i$. From the relation, $1_\sim = f^{-1}(\cup_{i\in I} A_i)$ follows that $1_\sim = \cup_{i\in I} f^{-1}(A_i)$, so $\left\{f^{-1}\left(A_i\right) : i \in I\right\}$ is an intuitionistic fuzzy weakly generalized closed cover of (X, τ). Since (X, τ) is intuitionistic fuzzy nearly weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}$. Therefore $1_{\sim} = \bigcup_{i=1}^{n} f^{-1}\left(A_i\right)$. Hence

$$1_{\sim} = f\left(\bigcup_{i=1}^{n} f^{-1}\left(A_i\right)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}\left(A_i\right)\right) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \ldots A_n\}$ is a finite sub cover of (Y, σ). Hence (Y, σ) is intuitionistic fuzzy nearly weakly generalized compact.

Corollary 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be an $IFWG$ irresolute mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). Then the following statements hold.

(i) If (X, τ) is intuitionistic fuzzy weakly generalized compact, then (Y, σ) is intuitionistic fuzzy weakly generalized compact.

(ii) If (X, τ) is intuitionistic fuzzy countable weakly generalized compact, then (Y, σ) is intuitionistic fuzzy countable weakly generalized compact.

(iii) If (X, τ) is intuitionistic fuzzy weakly generalized Lindelof, then (Y, σ) is intuitionistic fuzzy weakly generalized Lindelof.

(iv) If (X, τ) is intuitionistic fuzzy nearly countable weakly generalized compact, then (Y, σ) is intuitionistic fuzzy nearly countable weakly generalized compact.

(v) If (X, τ) is intuitionistic fuzzy nearly weakly generalized Lindelof, then (Y, σ) is intuitionistic fuzzy nearly weakly generalized Lindelof.

Proof. Obvious.

Theorem 3.6. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy nearly weakly generalized compact, then (Y, σ) is intuitionistic fuzzy compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy open cover of (Y, σ). Then $1_{\sim} = \cup_{i \in I} A_i$. From the relation, $1_{\sim} = f^{-1}\left(\cup_{i \in I} A_i\right)$ follows that $1_{\sim} = \cup_{i \in I} f^{-1}\left(A_i\right)$, so $\left\{f^{-1}\left(A_i\right) : i \in I\right\}$ is an intuitionistic fuzzy weakly generalized closed cover of (X, τ). Since (X, τ) is intuitionistic fuzzy nearly weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}\left(A_1\right), f^{-1}\left(A_2\right),$ .. $f^{-1}\left(A_n\right)\}$. Therefore $1_{\sim} = \bigcup_{i=1}^{n} f^{-1}\left(A_i\right)$. Hence

$$1_{\sim} = f\left(\bigcup_{i=1}^{n} f^{-1}\left(A_i\right)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}\left(A_i\right)\right) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \ldots A_n\}$ is a finite sub cover of (Y, σ). Hence (Y, σ) is intuitionistic fuzzy compact.

Corollary 3.4. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy nearly weakly generalized Lindelof, then (Y, σ) is intuitionistic fuzzy Lindelof.

Proof. Obvious.

Theorem 3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). If (X, τ) is intuitionistic fuzzy weakly generalized compact, then (Y, σ) is intuitionistic fuzzy strongly S-closed.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy closed cover of (Y, σ). Then $1_{\sim} = \cup_{i \in I} A_i$. From the relation, $1_{\sim} = f^{-1}\left(\cup_{i \in I} A_i\right)$ follows that $1_{\sim} = \cup_{i \in I} f^{-1}\left(A_i\right)$, so $\left\{f^{-1}\left(A_i\right) : i \in I\right\}$ is an intuitionistic fuzzy weakly generalized open cover of (X,τ). Since(X,τ) is intuitionistic fuzzy weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}$. Therefore $1_\sim = \bigcup_{i=1}^n f^{-1}(A_i)$. Hence

$$1_\sim = f(\bigcup_{i=1}^n f^{-1}(A_i)) = \bigcup_{i=1}^n f(f^{-1}(A_i)) = \bigcup_{i=1}^n A_i.$$

That is, $\{A_1, A_2, \ldots A_n\}$ is a finite sub cover of (Y,σ). Hence (Y,σ) is intuitionistic fuzzy strongly S-closed.

Corollary 3.5. Let $f:(X,\tau)\to(Y,\sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). Then the following statements hold.

(i) If (X,τ) is intuitionistic fuzzy weakly generalized Lindelof, then (Y,σ) is intuitionistic fuzzy strongly S-Lindelof.

(ii) If (X,τ) is intuitionistic fuzzy countable weakly generalized compact, then (Y,σ) is intuitionistic fuzzy countable strongly S-Closed.

Proof. Obvious.

Theorem 3.8. Let $f:(X,\tau)\to(Y,\sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (X,τ) is intuitionistic fuzzy weakly generalized compact (respectively, intuitionistic fuzzy weakly generalized Lindelof, intuitionistic fuzzy countable weakly generalized compact), then (Y,σ) is intuitionistic fuzzy S-closed (respectively, intuitionistic fuzzy S-Lindelof, intuitionistic fuzzy countable S-closed).

Proof. It follows from the statement that each $IFRCS$ is an $IFCS$.

Theorem 3.9. Let $f:(X,\tau)\to(Y,\sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, then (Y,σ) is intuitionistic fuzzy almost compact.

Proof. Let $\{A_i : i\in I\}$ be an intuitionistic fuzzy open cover of (Y,σ). Then $1_\sim = \cup_{i\in I} A_i$. It follows that $1_\sim = \cup_{i\in I} cl(A_i)$. From the relation, $1_\sim = f^{-1}(\cup_{i\in I} cl(A_i))$ follows that $1_\sim = \cup_{i\in I} f^{-1}cl(A_i)$, so $\{f^{-1}cl(A_i) : i\in I\}$ is an intuitionistic fuzzy weakly generalized closed cover of (X,τ). Since (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}cl(A_1), f^{-1}cl(A_2), \ldots f^{-1}cl(A_n)\}$. Therefore $1_\sim = \bigcup_{i=1}^n f^{-1}cl(A_i)$. Hence

$$1_\sim = f(\bigcup_{i=1}^n f^{-1}cl(A_i)) = \bigcup_{i=1}^n f(f^{-1}cl(A_i)) = \bigcup_{i=1}^n cl(A_i).$$

Hence (Y,σ) is intuitionistic fuzzy almost compact.

Corollary 3.6. Let $f:(X,\tau)\to(Y,\sigma)$ be an IF contra WG continuous mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). Then the following statements hold.

(i) If (X,τ) is intuitionistic fuzzy nearly weakly generalized Lindelof, then (Y,σ) is intuitionistic fuzzy almost Lindelof.

(ii) If (X,τ) is intuitionistic fuzzy nearly countable weakly generalized compact, then (Y,σ) is intuitionistic fuzzy countable almost compact.

Proof. Obvious.

Theorem 3.10. Let $f:(X,\tau)\to(Y,\sigma)$ be an $IFWG^*$ open bijective mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (Y,σ) is intuitionistic fuzzy weakly generalized compact, then (X,τ) is intuitionistic fuzzy weakly generalized compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy weakly generalized open cover of (X,τ). Then $1_\sim = \cup_{i\in I} A_i$. From the relation, $1_\sim = f\left(\cup_{i\in I} A_i\right)$ follows that $1_\sim = \cup_{i\in I} f\left(A_i\right)$, so $\{f\left(A_i\right) : i \in I\}$ is an intuitionistic fuzzy weakly generalized open cover of (Y,σ). Since (Y,σ) is intuitionistic fuzzy weakly generalized compact, there exists a finite sub cover say, $\{f(A_1), f(A_2), \ldots f(A_n)\}$. Therefore $1_\sim = \bigcup_{i=1}^{n} f\left(A_i\right)$. Hence

$$1_\sim = f^{-1}\left(\bigcup_{i=1}^{n} f\left(A_i\right)\right) = \bigcup_{i=1}^{n} f^{-1}\left(f\left(A_i\right)\right) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \ldots A_n\}$ is a finite sub cover of (X,τ). Hence (X,τ) is intuitionistic fuzzy compact.

Theorem 3.11. Let $f : (X,\tau) \to (Y,\sigma)$ be an IF contra WG irresolute mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, then (Y,σ) is intuitionistic fuzzy weakly generalized compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy weakly generalized open cover of (Y,σ). Then $1_\sim = \cup_{i\in I} A_i$. From the relation, $1_\sim = f^{-1}(\cup_{i\in I} A_i)$ follows that $1_\sim = \cup_{i\in I} f^{-1}(A_i)$, so $\{f^{-1}(A_i) : i \in I\}$ is an intuitionistic fuzzy weakly generalized closed cover of (X,τ). Since (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, there exists a finite sub cover say, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots f^{-1}(A_n)\}$. Therefore $1_\sim = \bigcup_{i=1}^{n} f^{-1}(A_i)$. Hence

$$1_\sim = f(\bigcup_{i=1}^{n} f^{-1}(A_i)) = \bigcup_{i=1}^{n} f(f^{-1}(A_i)) = \bigcup_{i=1}^{n} A_i.$$

That is, $\{A_1, A_2, \ldots A_n\}$ is a finite sub cover of (Y,σ). Hence (Y,σ) is intuitionistic fuzzy weakly generalized compact.

Corollary 3.7. Let $f : (X,\tau) \to (Y,\sigma)$ be an IF contra WG irresolute mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). Then the following statements hold.

(i) If (X,τ) is intuitionistic fuzzy nearly weakly generalized Lindelof, then (Y,σ) is intuitionistic fuzzy weakly generalized Lindelof.

(ii) If (X,τ) is intuitionistic fuzzy nearly countable weakly generalized compact, then (Y,σ) is intuitionistic fuzzy countable weakly generalized compact.

(iii) If (X,τ) is intuitionistic fuzzy weakly generalized compact, then (Y,σ) is intuitionistic fuzzy nearly weakly generalized compact.

(iv) If (X,τ) is intuitionistic fuzzy weakly generalized Lindelof, then (Y,σ) is intuitionistic fuzzy nearly weakly generalized Lindelof.

(v) If (X,τ) is intuitionistic fuzzy countable weakly generalized compact, then (Y,σ) is intuitionistic fuzzy nearly countable weakly generalized compact.

Proof. Obvious.

Theorem 3.12. Let $f : (X,\tau) \to (Y,\sigma)$ be an IF contra WG irresolute mapping from an $IFTS$ (X,τ) onto an $IFTS$ (Y,σ). If (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, then (Y,σ) is intuitionistic fuzzy almost weakly generalized compact.

Proof. Let $\{A_i : i \in I\}$ be an intuitionistic fuzzy weakly generalized open cover of (Y,σ). Then $1_\sim = \cup_{i\in I} A_i$. It follows that $1_\sim = \cup_{i\in I} cl(A_i)$. From the relation, $1_\sim = f^{-1}\left(\cup_{i\in I} cl(A_i)\right)$ follows that $1_\sim = \cup_{i\in I} f^{-1} cl\left(A_i\right)$, so $\left\{f^{-1} cl\left(A_i\right) : i \in I\right\}$ is an intuitionistic fuzzy weakly generalized closed cover of (X,τ). Since (X,τ) is intuitionistic fuzzy nearly weakly generalized compact, there exists a finite sub cover say, $\left\{f^{-1}cl\left(A_1\right), f^{-1}cl\left(A_2\right), \ldots f^{-1}cl\left(A_n\right)\right\}$. Therefore $1_\sim = \bigcup_{i=1}^{n} f^{-1}cl\left(A_i\right)$. Hence

$$1_\sim = f\left(\bigcup_{i=1}^{n} f^{-1}cl\left(A_i\right)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}cl\left(A_i\right)\right) = \bigcup_{i=1}^{n} cl(A_i).$$

Hence (Y, σ) is intuitionistic fuzzy almost weakly generalized compact.

Corollary 3.8. Let $f : (X, \tau) \to (Y, \sigma)$ be an IF contra WG irresolute mapping from an $IFTS$ (X, τ) onto an $IFTS$ (Y, σ). Then the following statements hold.

(i) If (X, τ) is intuitionistic fuzzy nearly weakly generalized Lindelof, then (Y, σ) is intuitionistic fuzzy almost weakly generalized Lindelof.

(ii) If (X, τ) is intuitionistic fuzzy nearly countable weakly generalized compact, then (Y, σ) is intuitionistic fuzzy almost countable weakly generalized compact.

Proof. Obvious.

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On normal and fantastic filters of BL-algebras

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Abstract We further study the fantastic filter and normal filter of BL-algebras. By studying the equivalent condition of fantastic filter, we reveal the relation between fantastic filter and normal filter of BL-algebras and we solve two open problems that "Under what suitable condition a normal filter becomes a fantastic filter?" and "(Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?".

Keywords Non-classical logics, BL-algebras, filter, fantastic filter, normal filter.

§1. Introduction

The origin of BL-algebras is in Mathematical Logic. Hájek introduced BL-algebras as algebraic structures for his Basic Logic in order to investigate many-valued logic by algebraic means [3,9]. They play the role of Lindenbaum algebras from classical propositional calculus. The main example of a BL-algebra is the interval [0,1] endowed with the structure induced by a continuous t-norm. MV-algebras, Gödel algebras and Product algebras are the most known classes of BL-algebras [4].

In studying of algebras, filters theory plays an important role. From logical point of view, various filters correspond to various sets of provable formulae[11]. In [4], [5] and [8], the notions of prime filter, Boolean filter, implicative filters, normal filters, fantastic filters and positive implicative filters in BL-algebras were proposed, and the properties of the filters were investigated. In [2], several different filters of residuated lattices and triangle algebras were defined and their mutual dependencies and connections were examined. In [1], [6] and [12], filters of pseudo MV-algebras, pseudo BL-algebras, pseudo effect algebras and pseudo hoops were further studied. We found that filters are a useful tool to obtain results of BL-algebras.

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In Section 2, we present some basic definitions and results on filters and BL-algebras, then we introduce the some kinds of filters in BL-algebras.

In [8], Saeid and Motamed proposed two open problems that "Under what suitable condition a normal filter becomes a fantastic filter? " and "(Extension property for a normal filter) Under what suitable condition extension property for normal filter holds? " in BL-algebras. In Section 3, by studying the equivalent condition of fantastic filter, we reveal the relation between fantastic filter and normal filter of BL-algebras and we solve the open problems.

§2. Main results of summation formula

Here we recall some definitions and results which will be needed.

Definition 2.1.[3] A BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and the following conditions hold for all $x,\ y,\ z \in A$,

(A1) $x \odot y \leq z$ iff $x \leq y \rightarrow z$,

(A2) $x \wedge y = x \odot (x \rightarrow y)$,

(A3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Proposition 2.1.[8] In a BL-algebras A, the following properties hold for all $x,\ y,\ z \in A$,

(1) $y \rightarrow (x \rightarrow\rightarrow z) = x \rightarrow (y \rightarrow z)$,

(2) $1 \rightarrow x = x$,

(3) $x \leq y$ iff $x \rightarrow y = 1$,

(4) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,

(5) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$,

(6) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$,

(7) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,

(8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(9) $x \leq (x \rightarrow y) \rightarrow y$.

Definition 2.2.[3] A filter of a BL-algebra A is a nonempty subset F of A such that for all $x,\ y \in A$,

(F1) if $x,\ y \in F$, then $x \odot y \in F$,

(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

It is easy to prove the following equivalent conditions of filter in a BL-algebra.

Proposition 2.2.[3] Let F be a nonempty subset of a BL-algebra A. Then F is a filter of A if and only if the following conditions hold

(1) $1 \in F$,

(2) $x, x \rightarrow y \in F$ implies $y \in F$.

A filter F of a BL-algebra A is proper if $F \neq A$.

Definition 2.3.[8] Let A be a BL-algebra. A filter F of A is called normal if for any $x,\ y,\ z \in A$, $z \rightarrow ((y \rightarrow x) \rightarrow x) \in F$ and $z \in F$ imply $(x \rightarrow y) \rightarrow y \in F$.

Definition 2.4.[8] Let F be a nonempty subset of a BL-algebra A. Then F is called a fantastic filter of A if for all $x,\ y,\ z \in A$, the following conditions hold

(1) $1 \in F$,

(2) $z \to (y \to x) \in F$, $z \in F$ implies $((x \to y) \to y) \to x \in F$.

Definition 2.5.[8] Let F be a nonempty subset of a BL-algebra A. Then F is called an implicative filter of A if for all $x,\ y,\ z \in A$, the following conditions hold

(1) $1 \in F$,

(2) $x \to (y \to z) \in F$, $x \to y \in F$ imply $x \to z \in F$.

Definition 2.6.[8] Let F be a nonempty subset of a BL-algebra A. Then F is called a positive implicative filter of A if for all $x,\ y,\ z \in A$, the following conditions hold

(1) $1 \in F$,

(2) $x \to ((y \to z) \to y) \in F$, $x \in F$ imply $y \in F$.

Definition 2.7.[10] Let F be a filter of A. F is called an ultra filter of A if it satisfies $x \in F$ or $x^- \in F$ for all $x \in A$.

Definition 2.8.[10] Let F be a filter of A. F is called an obstinate filter of A if it satisfies $x \notin F$ and $y \notin F$ implies $x \to y \in F$ for all $x,\ y \in A$.

Definition 2.9.[5] A filter F of A is called Boolean if $x \vee x^- \in F$ for any $x \in A$.

Definition 2.10.[8] A proper filter F of A is prime if for all $x,\ y \in A,\ x \vee y \in F$ implies $x \in F$ or $y \in F$.

Theorem 2.1.[10] Let F be a filter of A. Then the following conditions are equivalent

(1) F is an obstinate filter of A,

(2) F is an ultra filter of A,

(3) F is a Boolean and prime filter of A.

§3. The relation among the filters of BL-algebras

On the relation between fantastic filter and normal filter of a BL-algebra, by far we have

Theorem 3.1.[8] Let F be a fantastic filter of A. Then F is a normal filter of A.

It is easy to find that the converse of the theorem is not true. In [8], there are two open problems that "Under what suitable condition a normal filter becomes a fantastic filter?" and "(Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?" In this section, we investigate the relation between fantastic filter and normal filter in a BL-algebras. After giving the equivalent conditions of fantastic filter, we present the relation between the two filters in a BL-algebra and solve the open problems.

Theorem 3.2.[8] Let F be an implicative filter of a BL-algebra A. Then F is a normal filter if and only if $(x \to y) \to x \in F$ implies $x \in F$ for any $x,\ y \in A$.

Theorem 3.3.[8] Let F be a filter of a BL-algebra A. Then F is a normal filter if only if $(y \to x) \to x \in F$ implies $(x \to y) \to y \in F$ for all $x,\ y \in A$.

Theorem 3.4.[8] Let F be a filter of a BL-algebra A, if $(x \to y) \to x \in F$ implies $x \in F$ for any $x,\ y \in A$, then F is a normal filter.

Theorem 3.5.[8] Let F be a filter of a BL-algebra A, if $x^- \to x \in F$ implies $x \in F$ for any $x \in A$, then F is a normal filter.

Theorem 3.6.[8] Let F be a filter of a BL-algebra A, if $(x^- \to x) \to x \in F$ for any $x \in A$, then F is a normal filter.

Theorem 3.7.[8] Let F be a filter of a BL-algebra A. Then F is a normal filter if only if $x^{--} \in F$ implies $x \in F$ for all $x \in A$.

Theorem 3.8.[8] Every normal and implicative filter of a BL-algebra is a positive implicative filter.

Corollary 3.1. Every implicative filter satisfying $(y \to x) \to x \in F$ implies $(x \to y) \to y \in F$ for all $x,\ y \in A$ of a BL-algebra A is a positive implicative filter.

Theorem 3.9. Let f be a fuzzy filter of a BL-algebra A. Then the followings are equivalent

(1) F is a fantastic filter,

(2) $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$ for any $x,\ y \in A$,

(3) $x^{--} \to x \in F$ for all $x \in A$,

(4) $x \to z \in F,\ y \to z \in F$ implies $((x \to y) \to y) \to z \in F$ for all $x,\ y,\ z \in A$.

Proof. (1) $\Rightarrow$ (2). Let F be a fantastic filter of A, and $y \to x \in F$. Then $1 \to (y \to x) = y \to x \in F$ and $1 \in F$, hence $((x \to y) \to y) \to x \in F$.

(2) $\Rightarrow$ (1). Let a filter F satisfy the condition and let $z \to (y \to x) \in F$ and $z \in F$. Then $y \to x \in F$, therefore also $((x \to y) \to y) \to x \in F$.

(2) $\Rightarrow$ (3). Let $y = 0$ in (2).

(3) $\Rightarrow$ (4). Similar to the proof of Theorem 4.4 [7].

(4) $\Rightarrow$ (2). Obvious when $z = x$.

Corollary 3.2.[8] Each fantastic filter of A is a normal filter.

By Theorem 3.9, we can give an answer to the first open problem presented in [8] in the following theorem.

Theorem 3.10. Every normal and implicative filter of a BL-algebra is a fantastic filter.

Proof. Suppose $y \to x \in F$. We have $x \leq ((x \to y) \to y) \to x$, thus $(((x \to y) \to y) \to x) \to y \leq x \to y$. Further, $((((xy \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. Since $y \to x \in F$, hence also $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in F$. By Theorem 3.2, we get $((x \to y) \to y) \to x \in F$, and hence F is a fantastic filter by Theorem 3.10.

Theorem 3.11.[7] Let F be a filter of a BL-algebra A. Then the followings are equivalent

(1) F is an implicative filter,

(2) $y \to (y \to x)$ implies $y \to x \in F$ for any $x,\ y \in A$,

(3) $z \to (y \to x)$ implies $(z \to y) \to (z \to x) \in F$ for any $x,\ y,\ z \in A$,

(4) $z \to (y \to (y \to x)) \in F$ and $z \in F$ implies $y \to x \in F$ for any $x,\ y,\ z \in A$,

(5) $x \to x \odot x \in F$ for all $x \in A$.

Theorem 3.12.[7] Let F be a filter of a BL-algebra A. Then the followings are equivalent

(1) F is a positive implicative filter,

(2) $(x \to y) \to x \in F$ implies $x \in F$ for any $x,\ y \in A$,

(3) $(x^{-} \to x) \to x \in F$ for any $x \in A$.

Theorem 3.13. Every positive implicative filter of A is a fantastic filter.

Proof. Suppose F is a positive implicative filter and for any $x,\ y \in A$, $y \to x \in F$. We have $x \leq ((x \to y) \to y) \to x$, thus $(((x \to y) \to y) \to x) \to y \leq x \to y$. Further, $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. Since $y \to x \in F$, hence also $((((x \to y) \to y) \to$

$x) \to y) \to (((x \to y) \to y) \to x) \in F$. Thus we get $((x \to y) \to y) \to x \in F$, and hence F is a fantastic filter.

Corollary 3.3. Let F be an implicative filter of a BL-algebra A. Then F is a normal filter if only if F is a positive implicative filter of A.

Corollary 3.4. Let F be a positive implicative filter of A. Then F is a normal filter.

Theorem 3.14. Every normal and obstinate filter of a BL-algebra is a fantastic filter.

Proof. Suppose F is a normal and obstinate filter of a BL-algebra A, then for any $x \in A$, if $x^{--} \in F$, then $x \in F$ since F is a normal filter, by $x^{--} \to x \geq x$, thus $x^{--} \to x \in F$. Further, if $x \notin F$, then $x^{--} \notin F$, since F is an obstinate filter, then $x^{--} \to x \in F$, and hence F is a fantastic filter.

By Theorem 2.1, we have:

Theorem 3.15. Every normal and ultra filter of a BL-algebra is a fantastic filter.

Theorem 3.16. Every normal, Boolean and prime filter of a BL-algebra is a fantastic filter.

Combining all the above results, we can give answers to the first open problem presented in [8] respectively.

Theorem 3.17. (1) Every normal filter of a MV-algebra is equivalent to a fantastic filter.

(2) Every normal and implicative filter of a BL-algebra is a fantastic filter.

(3) Every normal and obstinate filter of a BL-algebra is a fantastic filter.

(4) Every normal and ultra filter of a BL-algebra is a fantastic filter,

(5) Every normal, Boolean and prime filter of a BL-algebra is a fantastic filter.

For the second open problem presented in [8], since the extension property for a fantastic filter holds [7], and further by the above results, we have the following theorem.

Theorem 3.18. (Extension property for a normal filter) Under one of the following conditions extension property for normal filter holds

(1) If A is a MV-algebra.

(2) F is a normal and implicative filter of A.

(3) F is a normal and obstinate filter of A.

(4) F is a normal and ultra filter of A,

(5) F is a normal, Boolean and prime filter of A.

§4. Conclusion

By studying the equivalent condition of fantastic filter, we reveal the relation between fantastic filter and normal filter of BL-algebras and we solve an open problem that "Under what suitable condition a normal filter becomes a fantastic filter? " and "(Extension property for a normal filter) Under what suitable condition extension property for normal filter holds? ". In the future, we will extend the corresponding filter theory and study the congruence relations induced by the filters.

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