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Almost $\nu g-$continuity

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Abstract The object of the present paper is to study the basic properties of almost $\nu g-$continuous functions.

Keywords $\nu g-$open sets, $\nu g-$continuity, $\nu g-$irresolute, $\nu g-$open map, $\nu g-$closed map, νg $-$homeomorphisms and almost $\nu g-$continuity.
AMS-classification Numbers: 54C10; 54C08; 54C05.

§1. Introduction

In 1963 M. K. Singhal and A. R. Singhal introduced Almost continuous mappings. In 1980, Joseph and Kwack introduced the notion of (θ, s)-continuous functions. In 1982, Jankovic introduced the notion of almost weakly continuous functions. Dontchev, Ganster and Reilly introduced a new class of functions called regular set-connected functions in 1999. Jafari introduced the notion of (p, s)-continuous functions in 1999. T. Noiri and V. Popa studied some properties of almost-precontinuity in 2005 and unified theory of almost-continuity in 2008. E. Ekici introduced almost-precontinuous functions in 2004 and recently have been investigated further by Noiri and Popa. Ekici E., introduced almost-precontinuous functions in 2006. Ahmad Al-Omari and Mohd. Salmi Md. Noorani studied Some Properties of almost-b-Continuous Functions in 2009. Recently S. Balasubramanian, C. Sandhya and P. A. S. Vyjayanthi introduced $\nu-$continuous functions in 2010. Inspired with these developments, I introduce a new class of functions called almost-$\nu g-$continuous fnctions and obtain its basic properties, preservation Theorems of such functions and relationship with other types of functions are verified. In the paper (X,τ) or simply X means a topological space.

§2. Preliminaries

Definition 2.1. $A \subset X$ is called

(i) closed if its complement is open.

(ii) regular open [pre-open; semi-open; α-open; β-open] if $A = (\overline{A})^0$ $[A \subseteq (\overline{A})^o$; $A \subseteq \overline{(A^o)}$; $A \subseteq (\overline{(A^o)})^o$; $A \subseteq \overline{((\overline{A})^o)}]$ and regular closed [pre-closed; semi-closed; α-closed; β-closed] if $A = \overline{A^0}[\overline{(A^o)} \subseteq A$; $(\overline{A})^o \subseteq A$; $\overline{((\overline{A})^o)} \subseteq A$; $(\overline{(A^o)})^o \subseteq A]$.

(iii) $\nu-$open [r$\alpha-$open] if $\exists O \in RO(X) \ni O \subset A \subset \overline{O}$ $[O \subset A \subset \alpha\overline{(O)}]$.

(iv) semi-$\theta-$open if it is the union of semi-regular sets and its complement is semi-$\theta-$closed.

(v) g-closed [resp: rg-closed] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(vi) sg-closed [resp: gs-closed] if $s(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is semi-open{open} in X.

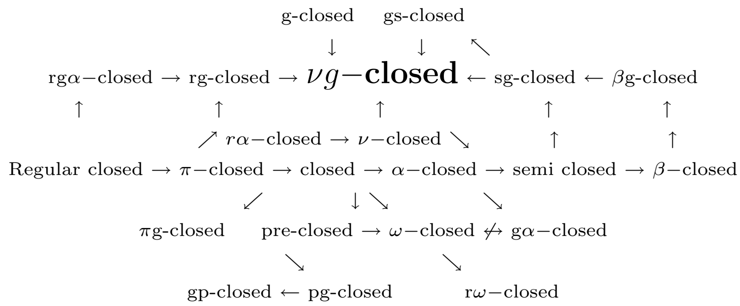
(vii) pg-closed [resp: gp-closed; gpr-closed] if $p(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is pre-open{open; regular-open} in X.

(viii) αg-closed [resp: g$\alpha-$closed; rgα-closed] if $\alpha(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is{$\alpha-$open; r$\alpha-$open}open in X.

(ix) βg-closed if $\beta(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

(x) νg-closed if $\nu(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is $\nu-$open in X.

Note 1. From the above definitions we have the following interrelations among the closed sets.



Definition 2.2. A function $f: X \to Y$ is called

(i) contra-[resp: contra-semi-; contra-pre-;contra-nearly-;contra-$r\alpha-$; contra-$\alpha-$; contra-$\beta-$; contra-$\omega-$; contra-pre-semi-; almost $\nu-$] continuuos if inverse image of every regular open set in Y is closed [resp: semi-closed; pre-closed; regular-closed; $r\alpha-$closed; $\alpha-$closed; $\beta-$closed; $\omega-$closed; pre-semi-closed; $\nu-$closed] in X.

(ii) almost pre-[resp: almost $\beta-$; almost $\lambda-$]continuous if $f^{-1}(V)$ is preclosed [resp: $\beta-$closed; $\lambda-$closed] in X for every regular open set V in Y.

Lemma 2.1. If V is an open [r-open] set, then $sCl_\theta(V) = sCl(V) = Int(Cl(V))$.

§3 Almost νg-continuous maps

Definition 3.1. A function $f: X \to Y$ is said to be almost $\nu g-$continuous if the inverse image of every regular open set is $\nu g-$open.

Note 2. Here after we call almost $\nu g-$ continuous function as al.νg.c function shortly.

Theorem 3.1. The following are true:

(i) f is al.νg.c. iff inverse image of each regular closed set in Y is $\nu g-$closed in X.

(ii) If f is νg.c., then f is al.νg.c. Converse is true if X is discrete space.

(iii) Let f be al.rg.c. and r-open, and $A \in \nu GO(X)$ then $f(A) \in \nu GO(Y)$.

(iv) If f is $\nu-$open [r-open] and al.νg.c.[al.rg.c.], then $f^{-1}(U) \in \nu GC(X)$ if $U \in \nu GC(Y)$.

Remark 1. 3.1(iii) is false if r-open is removed from the statement as shown by:

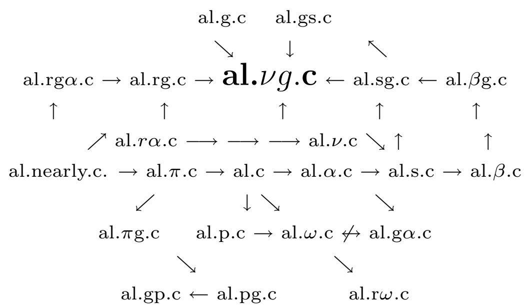
Example 1. Let $X = Y = \Re$ and f be defined by $f(x) = 1$ for all $x \in X$ then X is $\nu g-$open in X but $f(X)$ is not $\nu g-$open in Y.

Theorem 3.2. (i) f is al.νg.c. iff for each $x \in X$ and each $U_Y \in \nu GO(Y, f(x))$, $\exists A \in \nu GO(X, x)$ and $f(A) \subset U_Y$.

(ii) f is al.νg.c. iff for each $x \in X$ and each $V \in RGO(Y, f(x))$, $\exists\, U \in \nu GO(X, x) \ni f(U) \subset V$.

Proof. Let $U_Y \in RC(Y)$ and let $x \in f^{-1}(U_Y)$. Then $f(x) \in U_Y$ and thus $\exists\ A_x \in \nu GO(X, x)$ and $f(A_x) \subset U_Y$. Then $x \in A_x \subset f^{-1}(U_Y)$ and $f^{-1}(U_Y) = \cup A_x$. Hence $f^{-1}(U_Y) \in \nu$ GO(X).

Remark 2. We have the following implication diagram for a function $f\colon (X, \tau) \to (Y, \sigma)$



Theorem 3.3. (i) If R$\alpha C(X) = \nu gC(X)$ then f is al.rα.c. iff f is al.νg.c.

(ii) If $\nu gC(X) = RC(X)$ then f is al.r.c. iff f is al.νg.c.

(iii) If $\nu gC(X) = \alpha C(X)$ then f is al.α.c. iff f is al.νg.c.

(iv) If $\nu gC(X) = SC(X)$ then f is al.sg.c. iff f is al.νg.c.

(v) If $\nu gC(X) = \beta C(X)$ then f is c.βg.c. iff f is al.νg.c.

Example 2. $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let f be identity function, then f is al.νg.c; al.sg.c; al.gs.c; al.$r\alpha$.c; al.β.c; al.ν.c; al.s.c; but not al.gp.c; al.pg.c; al.rg.c; al.gr.c; al.g.c; al.gpr.c; al.gα.c; al.αg.c; al.rgα.c; al.α.c; al.p.c; al.c; and r-irresolute.

Example 3. $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let f be identity function, then f is al.gpr.c; al.rα.c; but not al.νg.c; al.sg.c; al.gs.c; al.β.c; al.ν.c; al.s.c; al.gp.c; al.pg.c; al.rg.c; al.gr.c; al.g.c; al.gα.c; al.αg.c; al.rgα.c; al.α.c; al.p.c; al.c; and rirresolute.

Example 4. $X = Y = \{a, b, c, d\} : \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, Y\}$. Let f be identity function, then f is al.gpr.c; al.rα.c; al.νg.c; al.sg.c; al.gs.c; al.β.c; al.ν.c; al.s.c; al.gp.c; al.pg.c; al.rg.c; al.gr.c; al.g.c; al.gα.c; al.αg.c; al.rgα.c; al.α.c; al.p.c; al.c; and r-irresolute.

Example 5. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Let f be defined as f(a) = b; f(b) = c; f(c) = a, then f is al.gpr.c; al.rα.c; but not al.νg.c; al.sg.c; al.gs.c; al.β.c; al.ν.c; al.s.c; al.gp.c; al.pg.c; al.rg.c; al.gr.c; al.g.c; al.gα.c; al.αg.c; al.rgα.c; al.α.c; al.p.c; al.c; and r-irresolute. Let f be identity map. then f is al.νg.c; al.sg.c; al.gs.c; al.rg.c; al.gr.c; al.gpr.c; al.gp.c; al.g.c; al.rgα.c; al.αg.c; but not al.rα.c; al.β.c; al.ν.c; al.s.c; al.gp.c; al.pg.c; al.g.c; al.gα.c; al.α.c; al.p.c; al.c; and rirresolute.

Example 6. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let f be defined as f(a) = c; f(b) = a; f(c) = b, then f is al.gpr.c; al.rα.c; al.νg.c; al.sg.c; al.gs.c; al.β.c; al.ν.c; al.s.c; al.gp.c; al.pg.c; al.rg.c; al.gr.c; al.g.c; al.gα.c; al.αg.c; al.rgα.c; al.α.c; al.p.c;

al.c; and r-irresolute. under usual topology on $\Re$ both al.c.g.c; al.rg.c; are same and both al.sg.c; al.νg.c; are same.

Theorem 3.4. Let $f_i : X_i \to Y_i$ be al.νg.c. for $i = 1, 2$. Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f : X_1 \times X_2 \to Y_1 \times Y_2$ is al.νg.c.

Proof. Let $U_1 \times U_2 \subset Y_1 \times Y_2$ where $U_i \in RO(Y_i)$ for $i = 1, 2$. Then $f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \times f_2^{-1}(U_2) \in \nu GO(X_1 \times X_2)$, since $f_i^{-1}(U_i) \in \nu GO(X_i)$ for $i = 1, 2$. Now if $U \in RO(Y_1 \times Y_2)$, then $f^{-1}(U) = f^{-1}(\cup U_i)$, where $U_i = U_1^i \times U_2^i$. Then $f^{-1}(U) = \cup f^{-1}(U_i) \in \nu GO(X_1 \times X_2)$, since $f^{-1}(U_i)$ is $\nu g-$open by the above argument.

Theorem 3.5. Let $h : X \to X_1 \times X_2$ be al.νg.c, where $h(x) = (h_1(x), h_2(x))$. Then $h_i : X \to X_i$ is al.νg.c for $i = 1, 2$.

Proof. Let $U_1 \in RO(X_1)$. Then $U_1 \times X_2 \in RO(X_1 \times X_2)$, and $h^{-1}(U_1 \times X_2) \in \nu GO(X)$. But $h_1^{-1}(U_1) = h^{-1}(U_1 \times X_2)$, therefore $h_1 : X \to X_1$ is al.νg.c. Similar argument gives $h_2 : X \to X_2$ is al.νg.c. and thus $h_i : X \to X_i$ is al.νg.c. for $i = 1, 2$.

In general we have the following extenstion of Theorem 3.4 and 3.5.

Theorem 3.6. (i) If $f : X \to \Pi Y_\lambda$ is al.νg.c, then $P_\lambda \circ f : X \to Y_\lambda$ is al.νg.c for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.

(ii) $f : \Pi X_\lambda \to \Pi Y_\lambda$ is al.νg.c, iff $f_\lambda : X_\lambda \to Y_\lambda$ is al.νg.c for each $\lambda \in \Lambda$.

Note 3. Converse of Theorem 3.6 is not true in general, as shown by the following example.

Example 7. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1 : X \to X_1$ be defined as follows: $f_1(x) = 1$ if $0 \leq x \leq \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x \leq 1$. Let $f_2 : X \to X_2$ be defined as follows: $f_2(x) = 1$ if $0 \leq x < \frac{1}{2}$ and $f_2(x) = 0$ if $\frac{1}{2} < x < 1$. Then $f_i : X \to X_i$ is clearly al.νg.c for $i = 1, 2$, but $h(x) = (f_1(x_1), f_2(x_2)) : X \to X_1 \times X_2$ is not al.νg.c., for $S_{\frac{1}{2}}(1, 0)$ is regular open in $X_1 \times X_2$, but $h^{-1}(S_{\frac{1}{2}}(1, 0)) = \{\frac{1}{2}\}$ which is not $\nu g-$open in X.

Remark 3. In general,

(i) The algebraic sum and product of two al.νg.c functions is not al.νg.c. However the scalar multiple of a al.νg.c function is al.νg.c.

(ii) The pointwise limit of a sequence of al.νg.c. functions is not al.νg.c. as shown by the following example.

Example 8. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1 : X \to X_1$ and $f_2 : X \to X_2$ are defined as follows: $f_1(x) = x$ if $0 < x < \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x < 1; f_2(x) = 0$ if $0 < x < \frac{1}{2}$ and $f_2(x) = 1$ if $\frac{1}{2} < x < 1$.

Example 9. Let $X = Y = [0, 1]$. Let f_n is defined as follows: $f_n(x) = x_n$ for $n \geq 1$ then f is the limit of the sequence where $f(x) = 0$ if $0 \leq x < 1$ and $f(x) = 1$ if $x = 1$. Therefore f is not al.νg.c. For $(\frac{1}{2}, 1]$ is open in Y, $f^{-1}((\frac{1}{2}, 1]) = (1)$ is not $\nu g-$open in X.

However we can prove the following theorem.

Theorem 3.7. Uniform Limit of a sequence of al.νg.c. functions is al.νg.c.

Problem. (i) Are sup$\{f, g\}$ and inf$\{f, g\}$ are al.νg.c. if f, g are al.νg.c.

(ii) Is $C_{al.\nu g.c}(X, R)$, the set of all al.νg.c. functions, (1) a Group. (2) a Ring. (3) a Vector space. (4) a Lattice.

(iii) Suppose $f_i : X \to X_i (i = 1, 2)$ are al.νg.c. If $f : X \to X_1 \times X_2$ defined by $f(x) = (f_1(x), f_2(x))$, then f is al.νg.c.

Solution. No.

Theorem 3.8. (i) If f is $\nu g-$irresolute and g is al.νg.c.[al.g.c], then $g \circ f$ is al.νg.c.

(ii) If f is al.νg.c.[ν.c; r.c.] and g is continuous [resp: nearly continuous] then $g \circ f$ is al.νg.c.

(iii) If f and g are r-irresolute then $g \circ f$ is al.νg.c.

(iv) If f is al.νg.c.[al.rg.c; ν.c; r.c.] g is al.rg.c and Y is r$T_{\frac{1}{2}}$ space, then $g \circ f$ is al.νg.c.

Theorem 3.9. If f is $\nu g-$irresolute, $\nu g-$open and $\nu GO(X) = \tau$ and g be any function, then $g \circ f$: is al.νg.c iff g is al.νg.c.

Proof. $A \in RO(Z)$. Since g is al.νg.c. $g^{-1}(A) \in \nu GO(Y) \Rightarrow f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A) \in \nu GO(X)$, since f is $\nu g-$irresolute. Hence $g \circ f$ is al.νg.c. Only if part: Let $A \in RO(Z)$. Then $(g \circ f)^{-1}(A)$ is a $\nu g-$open and hence open in X[by assumption]. Since f is $\nu g-$open $f\,(g \circ f)^{-1}(A) = g^{-1}(A)$ is $\nu g-$open in Y. Thus g is al.νg.c.

Theorem 3.10. If $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x)) \forall x \in X$ be the graph function of $f: X \rightarrow Y$. Then g is al.νg.c iff f is al.νg.c.

Proof. Let $V \in RC(Y)$, then $X \times V = X \times \overline{V^0} = \overline{X^0} \times \overline{V^0} = \overline{(X \times V)^0}$. Therefore, $X \times V \in RC(X \times Y)$. Since g is al.νg.c., then $f^{-1}(V) = g^{-1}(X \times V) \in \nu GC(X)$. Thus, f is al.νg.c.

Conversely, let $x \in X$ and $F \in RO(X \times Y, g(x))$. Then $F \cap (\{x\} \times Y) \in RO(\{x\} \times Y, g(x))$. Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in Y : (x, y) \in F\} \in RO(Y)$. Since f is al.νg.c. $\bigcup\{f^{-1}(y) : (x, y) \in F\} \in \nu GO(X)$. Further $x \in \bigcup\{f^{-1}(y) : (x, y) \in F\} \subseteq g^{-1}(F)$. Hence $g^{-1}(F)$ is $\nu g-$open. Thus g is al.νg.c.

Definition 3.2. f is said to be M-$\nu g-$open if the image of each $\nu g-$open set of X is $\nu g-$open in Y.

Example 10. f in Example 1 is not M-$\nu g-$open.

Corollary 3.3. (i) If f is $\nu g-$irresolute, M-$\nu g-$open and bijective, g is a function. Then g is al.νg.c. iff $g \circ f$ is al.νg.c.

(ii) If f be r-open, al.νg.c. and g be al.νg.c., then $g \circ f$ is al.νg.c.

Remark 4. In general, composition of two al.νg.c. functions is not al.νg.c. However we have the following example:

Example 11. Let $X = Y = Z = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$; $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$, and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$. Let f be identity map and g be be defined as g(a) = c; g(b) = c; g(c) = a; are al.νg.c. and $g \circ f$ is also al.νg.c.

Theorem 3.11. Let X, Y, Z be spaces and every $\nu g-$open set be r-open in Y, then the composition of two al.νg.c. maps is al.νg.c.

Note 4. Pasting Lemma is not true with respect to al.νg.c. functions. However we have the following weaker versions.

Theorem 3.12. Let X and Y be such that $X = A \cup B$. Let $f_{/A} : A \rightarrow Y$ and $g_{/B} : B \rightarrow Y$ are al.rg.c. functions such that $f\,(x) = g(x), \forall x \in A \cap B$. Suppose A and B are r-open sets in X and RO(X) is closed under finite unions, then the combination $\alpha : X \rightarrow Y$ is al.νg.c.

Theorem 3.13. Let X and Y be such that $X = A \cup B$. Let $f_{/A} : A \rightarrow Y$ and $g_{/B} : B \rightarrow Y$ are al.νg.c. maps such that f (x) = $g(x)$, $\forall x \in A \cap B$. Suppose A, B are r-open sets in X and νGO(X) is closed under finite unions, then the combination $\alpha : X \rightarrow Y$ is al.νg.c.

Proof. Let $F \in RO(Y)$, then $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) \in \nu GO(X)$ where $f^{-1}(F) \in \nu GO(A)$ and $g^{-1}(F) \in \nu GO(B) \Rightarrow f^{-1}(F)$ and $g^{-1}(F)$ are $\nu g-$open in X[since νgO(X) is closed under finite unions]. Hence α is al.νg.c.

Theorem 3.14. The following statements are equivalent for a function f:

(1) f is al.νg.c.;

(2) $f^{-1}(F) \in \nu GO(X)$ for every $F \in RO(Y)$;

(3) for each $x \in X$ and each $F \in RO(Y, f(x))$, $\exists$ a $U \in \nu GO(X, x) \ni f(U) \subset F$;

(4) for each $x \in X$ and each $V \in RC(Y)$ non-containing f (x), $\exists$ $K \in \nu GC(X)$ non-containing x $\ni f^{-1}(V) \subset K$;

(5) $f^{-1}(int(cl(G))) \in \nu GO(X)$ for every $G \in RO(Y)$;

(6) $f^{-1}(cl(int(F))) \in \nu GC(X)$ for every $F \in RC(Y)$.

Proof. (1)⇔(2): Let $F \in RO(Y)$. Then $Y - F \in RC(Y)$. By (1), $f^{-1}(Y - F) = X - f^{-1}(F) \in \nu GC(X)$. We have $f^{-1}(F) \in \nu GO(X)$ and so $f^{-1}(F) \in \nu GO(X)$ Reverse can be obtained similarly.

(2)⇒(3): Let $F \in RO(Y, f(x))$. By (2), $f^{-1}(F) \in \nu GO(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $f(U) \subset F$.

(3)⇒(2): Let $F \in RO(Y)$ and $x \in f^{-1}(F)$. From (3), $\exists$ $U_x \in \nu GO(X, x) \ni U \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x \in \nu GO(X)$.

(3)⇔(4): Let $V \in RC(Y)$ not containing f(x). Then, $Y - V \in RO(Y, f(x))$. By (3), $\exists$ $U \in \nu GO(X, x) \ni f(U) \subset Y - V$. Hence, $U \subset f^{-1}(Y - V) \subset X - f^{-1}(V)$ and then $f^{-1}(V) \subset X - U$. Take $H = X - U$, then H is a $\nu g-$closed set in X non-containing x. The converse can be shown easily.

(1)⇔(5): Let $G \in \sigma$. Since $(\overline{G})^o \in RO(Y)$, by (1), $f^{-1}((\overline{G})^o) \in \nu GO(X)$. The converse can be shown easily.

(2)⇔(6): It can be obtained similar as (1)⇔(5).

Example 12. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then the function f defined on X as f(a) = b; f(b) = c; f(c) = a; is al.νg.c. But it is not regular set-connected.

Example 13. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and f on X be the identity function. Then f is al.νg.c. function which is not c.νg.c.

Theorem 3.15. If f is al.νg.c., and $A \in RO(X)[A \in RC(X)]$, then $f_{|A} : A \to Y$ is al.νg.c.

Proof. Let $V \in RO(Y) \Rightarrow f^{-1}_{/A}(V) = f^{-1}(V) \cap A \in \nu GC(A)$. Hence $f_{/A}$ is al.νg.c.

Remark 5. Every restriction of an al.νg.c. function is not necessarily al.νg.c. in general.

Theorem 3.16. Let f be a function and $\Sigma = \{U_\alpha : \alpha \in I\}$ be a $\nu g-$cover of X. If for each $\alpha \in I$, $f_{|U_\alpha}$ is al.νg.c., then f is al.νg.c.

Proof. Let $F \in RO(Y)$. $f_{|U_\alpha}$ is al.νg.c. for each $\alpha \in I$, $f^{-1}_{|U_\alpha}(F) \in \nu GO_{|U_\alpha}$. Since $U_\alpha \in \nu GO(X)$, $f^{-1}_{|U_\alpha}(F) \in \nu GO(X)$ for each $\alpha \in I$. Then $f^{-1}(F) = \bigcup_{\alpha \in I} f^{-1}_{|U_\alpha}(F) \in \nu GO(X)$. This gives f is an al.νg.c.

Theorem 3.17. Let f and g be functions. Then, the following properties hold:

(1) If f is c.νg.c. and g is regular set-connected, then $g \circ f$ is al.νg.c.

(2) If f is νg.c. and g is regular set-connected, then $g \circ f$ is al.νg.c.

(3) If f is c.νg.c. and g is perfectly continuous, then $g \circ f$ is ν.c. and c.νg.c.

Proof. (1) Let$V \in RO(Z)$. Since g is regular set-connected, $g^{-1}(V)$ is clopen. Since f is c.νg.c., $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\nu g-$open and $\nu g-$open. Therefore, $g \circ f$ is al.νg.c.

(2) and (3) can be obtained similarly.

Theorem 3.18. If f is a surjective M-$\nu g-$open[resp: M-$\nu g-$closed] and g is a function such that $g \circ f$ is al.νg.c., then g is al.νg.c.

Proof. Let $V \in RC(Z)$. Since $g \circ f$ is al.νg.c., $(g \circ f)^{-1}(V) = f^{-1} \circ g^{-1}(V)$ is $\nu g-$open. Since f is surjective M-$\nu g-$open, $f(f^{-1} \circ g^{-1}(V)) = g^{-1}(V)$ is $\nu g-$open. Therefore, g is al.νg.c.

Theorem 3.19. If f is al.νg.c., then for each point $x \in X$ and each filter base Λ in X $\nu-$converging to x, the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.

Theorem 3.20. Let f be a function and $x \in X$. If there exists $U \in \nu GO(X, x)$ and $f_{|U}$ is al.νg.c. at x, then f is al.νg.c. at x.

Proof. Suppose that $F \in RO(Y, f(x))$. Since $f_{|U}$ is al.νg.c. at x, $\exists\ V \in \nu GO(U, x) \ni f(V) = (f_{|U})(V) \subset F$. Since $U \in \nu GO(X, x)$, it follows that $V \in \nu GO(X, x)$. Hence f is al.νg.c. at x.

Theorem 3.21. For f, the following properties are equivalent:

(1) f is (νg, s)-continuous;

(2) f is al.νg.c.;

(3) $f^{-1}(V)$ is $\nu g-$open in X for each θ-semi-open set V of Y;

(4) $f^{-1}(F)$ is $\nu g-$closed in X for each θ-semi-closed set F of Y.

Proof. (1)$\Rightarrow$(2): Let $F \in RO(Y)$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and F is semi-open. Since f is (νg, s)-continuous, $\exists U \in \nu GO(X, x) \ni f(U) \subset \overline{F} = F$. Hence $x \in U \subset f^{-1}(F)$ which implies that $x \in \nu(f^{-1}(F))^0$. Therefore, $f^{-1}(F) \subset \nu g(f^{-1}(F))^0$ and hence $f^{-1}(F) = \nu(gf^{-1}(F))^0$. This shows that $f^{-1}(F) \in \nu GO(X)$. It follows that f is al.νg.c.

(2)$\Rightarrow$(3): Follows from the fact that every θ-semi-open set is the union of regular closed sets.

(3)$\Leftrightarrow$(4): This is obvious.

(4)$\Rightarrow$ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. Since $\overline{V}$ is regular closed, it is θ-semi-open. Now, put $U = f^{-1}(\overline{V})$. Then $U \in \nu GO(X, x)$ and $f(U) \subset \overline{V}$. Thus f is (νg, s)-continuous.

§4. Covering and separation properties

Theorem 4.1. (i) If f is al.νg.c.[resp: al.rg.c] surjection and X is $\nu g-$compact, then Y is nearly compact.

(ii) If f is al.νg.c., surjection and X is $\nu g-$compact [$\nu g-$lindeloff] then Y is mildly compact [mildly lindeloff].

Theorem 4.2. If f is al.νg.c.[al.rg.c.], surjection and

(i) X is locally $\nu g-$compact, then Y is locally nearly compact [resp: locally mildly compact.]

(ii) X is $\nu g-$Lindeloff [locally $\nu g-$lindeloff], then Y is nearly Lindeloff [resp: locally nearly Lindeloff; locally mildly lindeloff].

(iii) If f is al.νg.c., surjection and X is s-closed then Y is mildly compact [mildly lindeloff].

(iv) X is $\nu g-$compact [resp: countably $\nu g-$compact] then Y is S-closed [resp: countably S-closed].

(v) X is $\nu g-$Lindelof, then Y is S-Lindelof and nearly Lindelof.

(vi) X is $\nu g-$closed [resp: countably $\nu g-$closed], then Y is nearly compact [resp: nearly countably compact].

Theorem 4.3. (i) If f is al.νg.c. surjection and X is $\nu g-$connected, then Y is connected.

(ii) If X is $\nu g-$ultra-connected and f is al.νg.c. and surjective, then Y is hyperconnected.

(ii) The inverse image of a disconnected [$\nu g-$disconnected] space under a al.νg.c, [almost $\nu g-$irreolute] surjection is $\nu g-$disconnected.

Proof. If Y is not hyperconnected. Then $\exists\ V \in \sigma \ni \overline{V} \neq A$. Then there exist disjoint non-empty regular open subsets B_1 and B_2 in Y. Since f is al.νg.c. and onto, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty $\nu g-$open subsets of X. By assumption, the $\nu-$ultra-connectedness of X implies that A_1 and A_2 must intersect, which is a contradiction. Therefore Y is hyperconnected.

Theorem 4.4. If f is al.νg.c., injection and

(i) Y is UT_i, then X is $\nu g - T_i$, $i = 0, 1, 2$.

(ii) Y is UR_i, then X is $\nu g - R_i$, $i = 0, 1$.

(iii) If f is closed; Y is UT_i, then X is $\nu g - T_i$, $i = 3, 4$.

(iv) Y is $UC_i[resp : UD_i]$, then X is $\nu g - T_i[resp : \nu g - D_i]$, $i = 0, 1, 2$.

Theorem 4.5. If f is al.νg.c.[resp: al.g.c.; al.sg.c.; al.rg.c] and Y is UT_2, then the graph G(f) of f is $\nu g-$closed in the product space $X \times Y$.

Proof. Let $(x_1, x_2) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint clopen sets V and $W \ni f(x) \in V$ and y$\in W$. Since f is al.νg.c., $\exists U \in \nu GO(X) \ni x \in U$ and $f(U) \subset W$. Therefore $(x, y) \in U \times V \subset X \times Y - G(f)$. Hence G($f$) is $\nu g-$closed in $X \times Y$.

Theorem 4.6. If f is al.νg.c.[al.rg.c] and Y is UT_2, then $A = \{(x_1, x_2)|f(x_1) = f(x_2)\}$ is $\nu g-$closed in the product space $X \times X$.

Proof. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_j \in CO(\sigma) \ni f(x_j) \in V_j$, and since f is al.νg.c., $f^{-1}(V_j) \in \nu GO(X, x_j)$ for each $j = 1, 2$. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in \nu GO(X \times X)$ and $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A$. Hence A is $\nu g-$closed.

Theorem 4.7. If f is al.rg.c.{al.g.c}; g is al.νg.c; and Y is UT_2, then $E = \{x \in X : f(x) = g(x)\}$ is $\nu g-$closed in X.

Theorem 4.8. If f is an al.νg.c. injection and Y is weakly Hausdorff, then X is νg_1.

Proof. Suppose Y is weakly Hausdorff. For any $x \neq y \in X, \exists V, W \in RC(Y) \ni f(x) \in V, f(y) \notin V, f(x) \notin W$ and $f(y) \in W$. Since f is al.νg.c., $f^{-1}(V)$ and $f^{-1}(W)$ are $\nu g-$open subsets of X such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. Hence X is νg_1.

Theorem 4.9. If for each $x_1 \neq x_2$ in a space X there exists a function f of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ and f is al.νg.c, at x_1 and x_2, then X is νg_2.

Proof. Let $x_1 \neq x_2$. Then by the hypothesis $\exists$ a function f which satisfies the condition of this theorem. Since Y is Urysohn and $f(x_1) \neq f(x_2)$, there exist open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively, $\ni \overline{V_1} \cap \overline{V_2} = \phi$. Since f is al.νg.c. at x_i, $\exists U_i \in \nu O(X, x_i) \ni f(U_i) \subset \overline{V_i}$ for $i = 1, 2$. Hence, we obtain $U_1 \cap U_2 = \phi$. Therefore, X is νg_2.

Corollary 4.1. If f is an al.rg.c, injection and Y is Urysohn, then X is νg_2.

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Separation axioms in intuitionistic bifuzzy topological space

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Abstract In this paper the concepts of completely induced intuitionistic bifuzzy topological spaces are introduced and some of their basic properties are studied. We construct some weaker axioms, by using the concepts of regular open sets in a bitopological space (X, S_1, S_2) and that of its corresponding completely induced intuitionistic bifuzzy topological space $(X, \tau(S_1), \tau(S_2))$ are shown.

Keywords Intuitionistic bifuzzy topological space, completely lower semi-continuous functi -ons, completely induced intuitionistic bifuzzy topology, strong and weak (α, β)-cut, pairwise intuitionistic fuzzy continuous, pairwise intuitionistic fuzzy T_0, pairwise intuitionistic fuzzy T_1 and pairwise intuitionistic fuzzy almost regular space.
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§1. Introduction

The concept of completely induced fuzzy topological (CIFT) space was introduced by R. N. Bhowmik and A. Mukherjee [5]. These spaces were defined with the notions of completely lower semi-continuous (CLSC) functions introduced in R. N. Bhowmik and A. Mukherjee [4]. The concept of completely induced bifuzzy topological space was introduced by Anjan mukherjee [3]. In this paper the concept of completely induced intuitionistic bifuzzy topological space is introduced and some of their basic properties are studied. We construct some weaker axioms, by using the concepts of regular open sets in a bitopological space (X, S_1, S_2) and that of its corresponding completely induced intuitionistic bifuzzy topological space $(X, \tau(S_1), \tau(S_2))$ are shown.

We use P-denotes pairwise (P-regular stands for pairwise regular). If A is a subset of an ordinary topological space then we denote its characteristic function by χ_A.

§2. Preliminaries

Definition 2.1. (i) [4,7] The function from a topological space (X, S) to the real number space (R, σ) is called a completely lower semicontinuous (in short CLSC) function at $x_0 \in X$ iff for each $\epsilon > 0$, there exists a regular open neighbourhood $N(x_0)$ such that $x \in N(x_0)$ implies $f(x) > f(x_0) - \epsilon$ or equivalently iff for each $a \in R$, $\{x \in X : f(x) > a\}$ is a union of regular open sets.

(ii) [4] The characteristic functions of a regular open set is CLSC.

Definition 2.2.[5] The family $\tau(S)$ of all CLSC functions from a fuzzy topological space (X, S) to the unit closed interval $I = [0, 1]$ forms a fuzzy topology called completely induced fuzzy topology (CIFT) and is denoted $\tau(S)$. The space $(X, \tau(S))$ is called completely induced fuzzy topological (CIFT) space.

Definition 2.3.[5] A fuzzy subset μ in a CIFT space $(X, \tau(S))$ is fuzzy open (resp. fuzzy closed) iff for each $r \in I$, the strong r-cut, $\sigma_r(\mu) = \{x \in X; \mu(x) > r\}$ (resp. weak r-cut $W_r(\mu) = \{x \in X; \mu(x) \geq r\}$) is a union of regular open set (resp. is a intersection of regular closed sets) in the topological space (X, S).

Proposition 2.1.[8] Let $A_i (i \in I)$ be fuzzy sets in X, let $B_i (i \in I)$ be subsets of X, let $r \in [0, 1]$. Then the following relations hold:

(1) $\sigma_r(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \sigma_r(A_i)$.

(2) $W_r(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} W_r(A_i)$.

(3) $\sigma_r(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} \sigma_r(A_i)$.

(4) $W_r(\bigcup_{i\in I} A_i) \supset \bigcup_{i\in I} W_r(A_i)$.

(5) $\chi_{\bigcup_{i\in I} B_i} = \bigcup_{i\in I} \chi_{B_i}$.

(6) $\chi_{\bigcap_{i\in I} B_i} = \bigcap_{i\in I} \chi_{B_i}$.

(7) $\sigma_s(r\chi_{B_i}) = B_i$ for $0 \leq s < r$.

(8) $W_s(r\chi_{B_i}) = B_i$ for $0 < s \leq r$.

Proposition 2.2.[8] Let $f : X \to Y$ be a function, let A and $A_i (i \in I)$ be fuzzy sets in Y and let B be a subsets of Y. Then the following relations hold:

(1) $f^{-1}(\sigma_r(A)) = \sigma_r(f^{-1}(A))$.

(2) $f^{-1}(W_r(A)) = W_r(f^{-1}(A))$.

(3) $f^{-1}(r\chi_B) = r\chi_{f^{-1}(B)}$.

(4) $f^{-1}(\bigcup_{i\in I}(A_i)) = \bigcup_{i\in I} f^{-1}(A_i)$.

Definition 2.4.[9] (i) Let X be a nonempty set. If $r \in I_0$, $s \in I_1$ are fixed real number, such that $r + s \leq 1$ then the intuitionistic fuzzy set $x_{r,s}$ is called an intuitionistic fuzzy point (in short IFP) in X given by

$$x_{r,s}(x_p) = \begin{cases} (r, s), & \text{if } x = x_p; \\ (0, 1), & \text{if } x \neq x_p. \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,s}$. where r denotes the degree of membership value and s is the degree of non-membership value of $x_{r,s}$.

(ii) An intuitionistic fuzzy point $x_{r,s}$ is said to be belong to an intuitionistic fuzzy set A if $r \leq \mu_A(x)$ and $s \geq \gamma_A(x)$.

(iii) An intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, S) is intuitionistic fuzzy open iff for all intuitionistic fuzzy point $x_{r,s} \in A$, there exists a basic intuitionistic fuzzy open set $U \in T$, such that $x_{r,s} \in U \subseteq A$.

(iv) An intuitionistic fuzzy singleton $x_{r,s} \in X$ is an intuitionistic fuzzy set in X taking value $r \in (0, 1]$ and $s \in [0, 1)$ at x and $0\sim$ elsewhere.

(v) Two intuitionistic fuzzy point or intuitionistic fuzzy singleton are said to be distinct if their support are distinct. (i.e $x_p \neq x_q$)

Definition 2.5.[3] Let (X, S_1, S_2) be a bitopological space and $(\tau(S_i))$ (i=1,2) be the set of all completely lower semi-continuous defined from X into the closed unit interval $I = [0, 1]$. Then $\tau(S_i)$ is a fuzzy topology in X. The triple $(X, \tau(S_1), \tau(S_2))$ is called a completely induced bifuzzy topological (CIBFT) space.

Definition 2.6.[6] A function $f : (X, S) \to (Y, K)$ from a topological space (X, S) to another topological space (Y, K) is called R-map iff the inverse image of each regular open subset of Y is regular open in X.

Definition 2.7.[3] A CIBFT space $(X, \tau(S_1), \tau(S_2))$ is said to be pairwise fuzzy T_0(in short P-F T_0) iff for any distinct fuzzy points p, q in X, there exists a fuzzy set $\mu \in \tau(S_1) \cup \tau(S_2)$ such that $p \in \mu$, $q \cap \mu = 0$ or $q \in \mu$, $p \cap \mu = 0$.

Definition 2.8.[3] A CIBFT space $(X, \tau(S_1), \tau(S_2))$ is said to be pairwise fuzzy T_1 (in short P-F T_1) iff for any two distinct fuzzy points p, q there exists a $\mu_1 \in \tau(S_1) \cup \tau(S_2)$ and $\mu_2 \in \tau(S_1) \cup \tau(S_2)$ such that $p \in \mu_1$, $q \cap \mu_1 = 0$ or $q \in \mu_2$, $p \cap \mu_2 = 0$.

Definition 2.9.[3] A CIBFT space $(X, \tau(S_1), \tau(S_2))$ is called a P-fuzzy regular space iff for each $\tau(S_i)$ fuzzy open subset δ of X is a union of $\tau(S_i)$ fuzzy open subsets δ_n's of X such that $cl(\delta_n) \subseteq \delta$ for each n.

§3. Completely induced intuitionistic bifuzzy topological space

Definition 3.1. The family $\tau(S)$ of all CLSC functions from a topological space (X, S) to the unit closed interval $I = [0, 1]$, forms an intuitionistic fuzzy topology called completely induced intuitionistic fuzzy topology (in short CIIFT) and it denoted by $\tau(S)$. The space $(X, \tau(S))$ is called a CIIFT space.

Definition 3.2. Let (X, S_1, S_2) be a bitopological space and $\tau(S_i), (i = 1, 2)$ be the set of all completely lower semicontinuous defined from X into the closed unit interval $I = [0, 1]$. Then $\tau(S_i), (i = 1, 2)$ is an intuitionistic fuzzy topology in X. The triple $(X, \tau(S_1), \tau(S_2))$ is called a completely induced intuitionistic bifuzzy topological space.

Definition 3.3. An intuitionistic fuzzy subset A in a CIIFT space $(X, \tau(S))$ is intuitionistic fuzzy open, (resp. intuitionistic fuzzy closed) iff for each $(\alpha, \beta) \in I$, the strong (α, β)-cut

$$\sigma_{(\alpha,\beta)}(A) = \{x \in X | \mu_A(x) > \alpha \text{ and } \gamma_A(x) < \beta\}$$

(resp. Weak (α, β)-cut) $\mathcal{W}_{(\alpha,\beta)}(A) = \{x \in X | \mu_A(x) \geq \alpha \text{ and } \gamma_A(x) \leq \beta\}$

is a union of regular open (resp. is a intersection of regular closed) in the topological space (X, S).

Definition 3.4. Let $(X,\tau(S_1),\tau(S_2))$ and $(Y,\tau(K_1),\tau(K_2))$ are the completely induced intuitionistic bifuzzy topological spaces. The function $f:(X,\tau(S_1),\tau(S_2))\rightarrow(Y,\tau(K_1),\tau(K_2))$ is said to be pairwise intuitionistic fuzzy continuous (in short P-IF continuous) iff the mapping $f:(X,\tau(S_i))\rightarrow(Y,\tau(K_i))$ $(i=1,2)$ are intuitionistic fuzzy continuous.

Definition 3.5.[3] A function $f:(X,S_1,S_2)\rightarrow(Y,K_1,K_2)$ from a bitopological space (X,S_1,S_2) to another bitopological space (Y,K_1,K_2) is said to be a P-R-map iff the function $f:(X,S_i)\rightarrow(Y,K_i)\,(i=1,2)$ are R-maps.

Proposition 3.1. Let (X,S_1,S_2) and (Y,K_1,K_2) be two bitopological spaces. Then $f:(X,S_1,S_2)\rightarrow(Y,K_1,K_2)$ is P-R-map iff $f:(X,\tau(S_1),\tau(S_2))\rightarrow(Y,\tau(K_1),\tau(K_2))$ is P-IF continuous.

Proof. Let $A=\langle x,\mu_A(x),\gamma_A(x)\rangle\in\tau(K_i)$ $(i=1,2)$ and $(\alpha,\beta)\in I$ by the Definition 3.3, the strong (α,β)-cut of A, $\sigma_{(\alpha,\beta)}(A)$ is a union of regular open sets in (Y,K_i) $(i=1,2)$. Now $f:(X,S_1,S_2)\rightarrow(Y,K_1,K_2)$ is a P-R-map. Hence $f^{-1}(\sigma_{(\alpha,\beta)}(A))=\sigma_{(\alpha,\beta)}(f^{-1}(A))$ (by propositon 2.2) is a union of a regular open sets in (X,S_i) $(i=1,2)$. Thus $f^{-1}(A)\in\tau(S_i)$, which implies $f:(X,\tau(S_1),\tau(S_2))\rightarrow(Y,\tau(K_1),\tau(K_2))$ is P-IF-continuous.

Conversely, let B be a regular open subset in $(Y,K_i)\,(i=1,2)$. Then by Definition 2.1(ii), $\chi_B\in\tau(K_i)(i=1,2)$. Now $f^{-1}(B)=\{x\in X:\chi_B(f(x))=1_\sim\}=\{x\in X:f^{-1}(\mu_{\chi_B}(x))>\alpha$ and $f^{-1}(\gamma_{\chi_B}(x))<\beta\}=\{x\in X:\mu_{\chi_B}(f(x))>\alpha$ and $\gamma_{\chi_B}(f(x))<\beta\}=\sigma_{(\alpha,\beta)}(f^{-1}(\chi_B))$. Since $\chi_B\in\tau(K_i)$, by P-IF-continuous of f, $f^{-1}(\chi_B)\in\tau(S_i)$. Thus $\sigma_{(\alpha,\beta)}(f^{-1}(\chi_B))$ is a union of regular open sets in (X,S_i). Since union of regular open sets is regular open set, $f^{-1}(B)$ is regular open in (X,S_i) $(i=1,2)$. Hence $f:(X,S_1,S_2)\rightarrow(Y,K_1,K_2)$ is a P-R-map.

§4. Intuitionistic bifuzzy separation axioms

Definition 4.1. A bitopological space (X,S_1,S_2) is said to be pairwise almost T_0 (in short P-A T_0) iff for any $x_p,\ x_q\in X$ such that $x_p\neq x_q$, there exists a regular open set A on S_1 satisfying $x_p\in A$, $x_q\notin A$ or there exists a regular open set A on S_2 satisfying $x_q\in A$, $x_p\notin A$.

Definition 4.2. Let $A=\langle x,\mu_A(x),\gamma_A(x)\rangle$ be an intuitionistic fuzzy set of X. A completely induced intuitionistic bifuzzy topological (CIIBFT) space $(X,\tau(S_1),\tau(S_2))$ is said to be pairwise intuitionistic fuzzy T_0 (in short P-IF T_0) iff for any distinct intuitionistic fuzzy point $x_{r,s},x_{u,v}\in X$, there exists a $A\in\tau(S_1)\cup\tau(S_2)$ such that $x_{r,s}\in A$, $x_{u,v}\cap A=0_\sim$ or $x_{u,v}\in A$, $x_{r,s}\cap A=0_\sim$.

Definition 4.3. A bitopological space (X,S_1,S_2) is said to be pairwise almost T_1 (in short P-A T_1) iff for any $x_p,x_q\in X$ such that $x_p\neq x_q$, there exists a regular open subset $A\in S_1$ and $B\in S_2$, such that $x_p\in A$, $x_q\notin A$ and $x_q\in B$, $x_p\notin B$.

Definition 4.4. Let $A=\langle x,\mu_A(x),\gamma_A(x)\rangle$ and $B=\langle x,\mu_B(x),\gamma_B(x)\rangle$ are the intuitionistic fuzzy sets of X. A completely induced intuitionistic bifuzzy topological (CIIBFT) space $(X,\tau(S_1),\tau(S_2))$ is said to be pairwise intuitionistic fuzzy T_1 (in short P-IF T_1) iff for any distinct intuitionistic fuzzy points $x_{r,s}$ and $x_{u,v}$, there exists a $A\in\tau(S_1)\cup\tau(S_2)$ and $B\in\tau(S_1)\cup\tau(S_2)$ such that $x_{r,s}\in A$,$x_{u,v}\cap A=0_\sim$ and $x_{u,v}\in B$,$x_{r,s}\cap B=0_\sim$.

Proposition 4.1. Let (X,S_1,S_2) be a bitopological space, then the following statements are equivalent:

(i) (X,S_1,S_2) is a P-A-T_0 space.

(ii) $(X, \tau(S_1), \tau(S_2))$ is a P-IF T_0 space.

Proposition 4.2. Let (X, S_1, S_2) be a bitopological space, then the following statements are equivalent:

(i) (X, S_1, S_2) is a P-A-T_1 space.

(ii) $(X, \tau(S_1), \tau(S_2))$ is a P-IF T_1 space.

Proof. $(i) \Rightarrow (ii)$

Let $x_{r,s}, x_{u,v}$ be two distinct intuitionistic fuzzy point in X. Since (X, S_1, S_2) is a P-A-T_1 space and $x_p \neq x_q$ there exists two p-regular open subsets A and B on X, such that $x_p \in A$, $x_q \notin A$ and $x_p \notin B$, $x_q \in B$. It is clear that $\chi_A : X \to [0,1]$ is a CLSC function thus $\chi_A \in \tau(S_i)$ $(i = 1, 2)$, also $x_{u,v} \notin \chi_A$ because $x_{u,v}(x_q) \supset \chi_A(x_q) = 0_\sim (x_q \notin A)$. To show that $x_{u,v} \cap \chi_A = 0_\sim$. Suppose $x_{u,v} \cap \chi_A \neq 0_\sim$, this implies $\chi_A(x_q) \supset 0_\sim$ which gives $x_q \in A$, a contradiction, therefore $x_{u,v} \cap \chi_A = 0_\sim$. Similarly we can show that the other case, when $x_q \in B, x_p \notin B$ and $x_{r,s}(x_p) \cap \chi_B(x_p) = 0_\sim$. Thus $(X, \tau(S_1), \tau(S_2))$ is a P-IF T_1 space.

$(ii) \Rightarrow (i)$

Let x_p, x_q be two distinct points in X. Since $(X, \tau(S_1), \tau(S_2))$ be a P-IF T_1 space, $x_{r,s}$ and $x_{u,v}$ are the two intuitionistic fuzzy points in X for which there exists an intuitionistic fuzzy set $A_1 = \langle x, \mu_{A_1}(x), \gamma_{A_1}(x) \rangle \in \tau(S_1) \cup \tau(S_2)$ and $B_1 = \langle x, \mu_{B_1}(x), \gamma_{B_1}(x) \rangle \in \tau(S_1) \cup \tau(S_2)$, such that $x_{r,s} \in A_1$, $x_{u,v} \cap A_1 = 0_\sim$ and $x_{u,v} \in B_1$, $x_{r,s} \cap B_1 = 0_\sim$, by $(i) \Rightarrow (ii)$ Since $A_1 \in \tau(S_1) \cup \tau(S_2)$, for $(\alpha, \beta) \in I$ $\sigma_{(\alpha,\beta)}(A_1)$ is a union of regular open sets in (X, S_i) $(i = 1, 2)$, hence regular open in (X, S_i) which contains x_p but not x_q, let $A = \sigma_{(\alpha,\beta)}(A_1)$ if $x_p \notin A$ then $x_{r,s}(x_p) \supset \chi_A(x_p) = 0_\sim$, which is contradiction, $x_{r,s} \in A_1$. To show that $x_{u,v} \cap A_1 = 0_\sim$. Suppose $x_{u,v} \cap A_1 \neq 0_\sim$, this implies $A_1(x_q) \supset 0_\sim$ which gives $x_q \in A$ a contradiction, therefore $x_{u,v} \cap A_1 = 0_\sim$. Similarly we can show that $x_{u,v} \cap B_1 = 0_\sim$ and $x_q \in B$, $x_p \notin B$. Thus (X, S_1, S_2) is a P-A-T_1 space.

Definition 4.5. A bitopological space (X, S_1, S_2) is called a P-almost regular space iff for each S_i regular open subsets A of X is a union of $S_i$regular open subsets A_j's of X, such that $cl(A_j) \subseteq A$ for each j.

Definition 4.6. Let $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$ be an intuitionistic fuzzy set of X. A completely induced intuitionistic bifuzzy topological (CIIBFT) space $(X, \tau(S_1), \tau(S_2))$ is a pairwise intuitionistic bifuzzy regular (in short P-IBF-regular) space iff for each $\tau(S_i)$ open subset A of X is a union of $\tau(S_i)$ open subset of A_j's of X, such that $IFcl(A_j) \subseteq A$ for each j.

Proposition 4.3. A completely induced intuitionistic bifuzzy topological space $(X, \tau(S_1), \tau(S_2))$ is P-IBF-regular space iff the bitopological space (X, S_1, S_2) is a P-almost regular space.

Proof. Let $(X, \tau(S_1), \tau(S_2))$ is a P-IF regular space and A be a p-regular open subset in (X, S_i) $(i = 1, 2)$. Then $\chi_A \in \tau(S_i)$ $(i = 1, 2)$, by Definition 2.1 (ii). Thus $\chi_A = \cup A_j$, where A_j's are $\tau(S_i)$ open subsets with $IFcl(A_j) \subseteq \chi_A$. Now for each $(\alpha, \beta) \in I$,

$$A = \sigma_{(\alpha,\beta)}(\bigcup_j (A_j)) = \bigcup_j \sigma_{(\alpha,\beta)}(A_j),$$

by Proposition 2.1 (i) where $\sigma_{(\alpha,\beta)}(A_j)$ is a union of P-regular open set in (X, S_i) for each j, hence union of regular open set is regular open in (X, S_i) for each j. Also

$$cl(\sigma_{(\alpha,\beta)}(A_j)) \subseteq \mathcal{W}_{(\alpha,\beta)}(cl(A_j))\mathcal{W}_{(\alpha,\beta)}(\chi_A) = A.$$

Thus (X, S_1, S_2) is a pairwise almost regular space.

Conversely, let (X, S_1, S_2) be a pairwise almost regular space and let $B = \langle x, \mu_B(x), \gamma_B(x)\rangle \in \tau(S_i)$ $(i = 1, 2)$. Then for each $(\alpha, \beta) \in I$, $\sigma_{(\alpha,\beta)}(B)$ is a union of regular open sets in (X, S_i) $(i = 1, 2)$. Now

$$\sigma_{(\alpha,\beta)}(B) = \bigcup U_j,$$

where $U_j's$ are regular open subsets in (X, S_i) $(i = 1, 2)$ with $cl(U_j) \subseteq \sigma_{(\alpha,\beta)}(B)$. By decomposition theorem [8].

$$\begin{aligned} A &= \bigcup_{(\alpha,\beta)\in I} (\alpha, \beta)\chi_{\sigma_{(\alpha,\beta)}(B)} \\ &= \bigcup_{(\alpha,\beta)\in I} (\alpha, \beta)\chi_{\bigcup_j U_j} = \bigcup_{(\alpha,\beta)\in I} (\alpha, \beta)\bigcup_j \chi_{U_j} \\ &= \bigcup_{(\alpha,\beta)\in I} \bigcup_j (\alpha, \beta)\chi_{U_j}, \end{aligned}$$

where $(\alpha, \beta)\chi_{U_j} \in \tau(S_i)$ with

$$(\alpha, \beta)IFcl(\chi_{U_j}) = (\alpha, \beta)\chi_{IFcl(U_j)} \subseteq A.$$

Hence $(X, \tau(S_1), \tau(S_2))$ is a P-IF-regular space.

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Topology of intuitionistic fuzzy rough sets

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Abstract In this paper topology of intuitionistic fuzzy rough sets is defined and different types of continuity are studied.

Keywords Fuzzy subsets, rough sets, intuitionistic fuzzy sets, intuitionistic fuzzy rough sets, topological space of IFRSs, IFR continuous mapping, IFR_1 continuous mapping, IFR_2 continu -ous mapping.
2000 Mathematics Subject Classification: 54A40.

§1. Introduction

Lotfi Zadeh [20] first introduced the idea of fuzzy subsets. After Zadeh different variations/generalisations of fuzzy subsets were made by several authors. Pawlak [18] introduced the idea of rough sets. Nanda [17] and Çoker [7] gave the definition of fuzzy rough sets. Atanassov [1] introduced the idea of intuitionistic fuzzy sets. Combining the ideas of fuzzy rough sets and intuitionistic fuzzy sets T. K. Mandal and S. K. Samanta [14] introduced the concept of intuitionistic fuzzy rough sets (briefly call IFRS, Definition 1.15). On the other hand fuzzy topology (we call it topology of fuzzy subsets) was first introduced by C. L. Chang [4]. Later many authors dealt with the idea of fuzzy topology of different kinds of fuzzy sets. M. K. Chakroborti and T. M. G. Ahsanullah [3] introduced the concept of fuzzy topology on fuzzy sets. T. K. Mondal and S. K. Samanta introduced the topology of interval valued fuzzy sets in [16] and the topology of interval valued intuitionistic fuzzy sets in [15].

In this paper, we introduce the concept of topology of intuitionistic fuzzy rough sets and study its various properties. In defining topology on an IFRS from the parent space, we have observed that two topologies are induced on the IFRS and accordingly two types of continuity are defined.

§2. Preliminaries

Unless otherwise stated, we shall consider $(V,\ \mathcal{B})$ to be a rough universe where V is a nonempty set and $\mathcal{B}$ is a Boolean subalgebra of the Boolean algebra of all subsets of V. Also consider a rough set $X = (X_L, X_U) \in \mathcal{B}^2$ with $X_L \subset X_U$. Moreover we assume that $\mathcal{C}_X$ be the collection of all IFRSs in X.

Definition 1.1.[17] A fuzzy rough set (briefly FRS) in X is an object of the form $A = (A_L, A_U)$, where A_L and A_U are characterized by a pair of maps $A_L : X_L \to \mathcal{L}$ and $A_U : X_U \to \mathcal{L}$ with $A_L(x) \leq A_U(x), \forall x \in X_L$, where ($\mathcal{L}, \leq$) is a fuzzy lattice (ie complete and completely distributive lattice whose least and greatest elements are denoted by 0 and 1 respectively with an involutive order reversing operation $' : \mathcal{L} \to \mathcal{L}$).

Definition 1.2.[17] For any two fuzzy rough sets $A = (A_L, A_U)$ and $B = (B_L, B_U)$ in X.

(i) $A \subset B$ iff $A_L(x) \leq B_L(x), \forall x \in X_L$ and $A_U(x) \leq B_U(x), \forall x \in X_U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

If $\{A_i : i \in J\}$ be any family of fuzzy rough sets in X, where $A_i = (A_{iL}, A_{iU})$, then

(iii) $E = \bigcup_i A_i$ where $E_L(x) = \vee A_{iL}(x), \forall x \in X_L$ and $E_U(x) = \vee A_{iU}(x), \forall x \in X_U$.

(iv) $F = \bigcap_i A_i$ where $F_L(x) = \wedge A_{iL}(x), \forall x \in X_L$ and $F_U(x) = \wedge A_{iU}(x), \forall x \in X_U$.

Definition 1.3.[14] If A and B are fuzzy sets in X_L and X_U respectively where $X_L \subset X_U$. Then the restriction of B on X_L and the extension of A on X_U (denoted by $B_{>L}$ and $A_{<U}$ respectively) are defined by $B_{>L}(x) = B(x),\ \forall x \in X_L$ and

$$A_{<U}(x) = \begin{cases} A(x), & \forall x \in X_L; \\ \vee_{\xi \in X_L}\{A(\xi)\}, & \forall x \in X_U - X_L. \end{cases}$$

Complement of an FRS $A = (A_L, A_U)$ in X are denoted by $\bar{A} = ((\bar{A})_L, (\bar{A})_U)$ and is defined by $(\bar{A})_L$(x)= $(A_{U>L})'(x),\ \forall x \in X_L$ and $(\bar{A})_U(x) = (A_{L<U})'(x),\ \forall x \in X_U$. For simplicity we write $(\bar{A}_L, \bar{A}_U)$ instead of $((\bar{A})_L, (\bar{A})_U)$.

Result 1.4. If $X_L = X_U$, then for any FRS $A\big(= (A_L, A_U)\big)$ in $X\big(= (X_L, X_U)\big)$, $\bar{\bar{A}} = A$.

Remark 1.5. Following example shows that, if $X_L \neq X_U$, then $\bar{\bar{A}}$ may not be equal to A.

Example 1.6. Let $X_U = X_L \cup \{a\},\ a \notin X_L$. Let $A = (A_L, A_U)$ are defined by $A_L(x) = 0, \forall x \in X_L$ and $A_U(x) = 0, \forall x \in X_L; A_U(a) = 0.3$. Then $\bar{A} = \tilde{1} = (\tilde{1}_L, \tilde{1}_U)$, the whole FRS in X and hence $\bar{\bar{A}} = \tilde{0} = (\tilde{0}_L, \tilde{0}_U)$, the null FRS in X. Thus $\bar{\bar{A}} \neq A$.

Theorem 1.7. If A is any FRS in $X\big(= (X_L, X_U)\big)$, then $\bar{\bar{\bar{A}}} = \bar{A}$.

The proof is straightforward .

Theorem 1.8.[14] If A, B, C, D and $B_i,\ i \in J$ are FRS in X, then

(i) $A \subset B$ and $C \subset D$ implies $A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.

(ii) $A \subset B$ and $B \subset C$ implies $A \subset C$.

(iii) $A \cap B \subset A, B \subset A \cup B$.

(iv) $A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)$ and $A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$.

(v) $A \subset B \Rightarrow \bar{A} \supset \bar{B}$.

(vi) $\overline{\bigcup_i B_i} = \bigcap_i \bar{B}_i$ and $\overline{\bigcap_i B_i} = \bigcup_i \bar{B}_i$.

Theorem 1.9.[14] If A be any FRS in X, $\tilde{0} = (\tilde{0}_L, \tilde{0}_U)$ be the null FRS and $\tilde{1} = (\tilde{1}_L, \tilde{1}_U)$ be the whole FRS in X, then (i) $\tilde{0} \subset A \subset \tilde{1}$ and (ii) $\bar{\tilde{0}} = \tilde{1}$, $\bar{\tilde{1}} = \tilde{0}$.

Notation 1.10. Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : V \to V_1$ be a mapping .

If $f(\lambda) \in \mathcal{B}_1, \forall \lambda \in \mathcal{B}$, then f maps $(V, \mathcal{B})$ to $(V_1, \mathcal{B}_1)$ and it is denoted by $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$.

If $f^{-1}(\mu) \in \mathcal{B}$, $\forall \mu \in \mathcal{B}_1$, then it is denoted by $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Definition 1.11.[14] Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$.

Let $A = (A_L, A_U)$ be a FRS in X . Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L),\ Y_U = f(X_U)$.

The image of A under f, denoted by $f(A) = (f(A_L), f(A_U))$ and is defined by

$$f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L$$

and

$$f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\},\ \forall y \in Y_U.$$

Next let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Let $B = (B_L, B_U)$ be a FRS in Y where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L), X_U = f^{-1}(Y_U)$. Then the inverse image of B, under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ and is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)), \forall x \in X_L$$

and

$$f^{-1}(B_U)(x) = B_U(f(x)), \forall x \in X_U.$$

Theorem 1.12.[14] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then for all FRSs A, A_1 and A_2 in X, we have

(i) $f(\bar{A}) \supset \overline{f(A)}$.

(ii) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

Theorem 1.13.[14] If $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRSs B, $B_i, i \in J$ in Y we have

(i) $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$.

(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any FRS C in Z where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set and gof is the composition of g and f.

(iv) $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$.

(v) $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.

Theorem 1.14.[14] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRS A in X and B in Y, we have

(i) $B = f(f^{-1}(B))$.

(ii) $A \subset f^{-1}(f(A))$.

Definition 1.15.[14] If A and B are two FRSs in X with $B \subset \bar{A}$ and $A \subset \bar{B}$, then the ordered pair (A, B) is called an intuitionistic fuzzy rough set(briefly IFRS) in X. The condition $A \subset \bar{B}$ and $B \subset \bar{A}$ are called intuitionistic condition (briefly IC).

Definition 1.16.[14] Let $P = (A, B)$ and $Q = (C, D)$ be two IFRSs in X. Then

(i) $P \subset Q$ iff $A \subset C$ and $B \supset D$.

(ii) $P = Q$ iff $P \subset Q$ and $Q \subset P$.

(iii) The complement of $P = (A, B)$ in X, denoted by P', is defined by $P' = (B, A)$.

(iv) For IFRSs $P_i = (A_i, B_i)$ in $X, i \in J$, define $\bigcup_{i\in J} P_i = (\bigcup_{i\in J} A_i, \bigcap_{i\in J} B_i)$ and $\bigcap_{i\in J} P_i = (\bigcap_{i\in J} A_i, \bigcup_{i\in J} B_i)$.

Theorem 1.17.[14] Let $P = (A, B),\ Q = (C, D),\ R = (E, F)$ and $P_i = (A_i, B_i), i \in J$ be IFRSs in X, then

(i) $P \cap P = P = P \cup P$.

(ii) $P \cap Q = Q \cap P$; $P \cup Q = Q \cup P$.

(iii) $(P \cap Q) \cap R = P \cap (Q \cap R)$; $(P \cup Q) \cup R = P \cup\ (Q \cup R)$.

(iv) $P \cap Q \subset P, Q \subset P \cup Q$.

(v) $P \subset Q$ and $Q \subset R \Rightarrow P \subset R$.

(vi) $P_i \subset Q, \forall i \in J \Rightarrow \bigcup_{i\in J} P_i \subset Q$.

(vii) $Q \subset P_i, \forall i \in J \Rightarrow Q \subset \bigcap_{i\in J} P_i$.

(viii) $Q \cup (\bigcap_{i\in J} P_i) = \bigcap_{i\in J}(Q \cup P_i)$.

(ix) $Q \cap (\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(Q \cap P_i)$.

(x) $(P')' = P$.

(xi) $P \subset Q \Leftrightarrow Q' \subset P'$.

(xii) $(\bigcup_{i\in J} P_i)' = \bigcap_{i\in J} P'$ and $(\bigcap_{i\in J} P_i)' = \bigcup_{i\in J} P'_i$.

Definition 1.18.[14] $0^* = (\tilde{0}, \tilde{1})$ and $1^* = (\tilde{1}, \tilde{0})$ are respectively called null IFRS and whole IFRS in X. Clearly $(0^*)' = 1^*$ and $(1^*)' = 0^*$.

Theorem 1.19.[14] If P be any IFRS in X, then $0^* \subset P \subset 1^*$. Slightly changing the definition of the image of an IFRS under f given by Samanta and Mondal [14] we give the following:

Definition 1.20. Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Let $P = (A, B)$ be an IFRS in $X(= (X_L, X_U))$ and $Y = f(X) \in \mathcal{B}_1^2$, where $Y_L = f(X_L)$ and $Y_U = f(X_U)$. Then we define image of P, under f by $f(P) = (\breve{f}\ (A), \hat{f}(B))$, where $\breve{f}\ (A) = (f(A_L), f(A_U))$, $A = (A_L, A_U)$ and $\hat{f}(B) = (C_L, C_U)$ (where $B = (B_L, B_U)$) is defined by

$$C_L(y) = \wedge\{B_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L.$$

$$C_U(y) = \begin{cases} \wedge\{B_U(x) : x \in X_L \cap f^{-1}(y)\}, & \forall y \in Y_L; \\ \wedge\{B_U(x) : x \in X_U \cap f^{-1}(y)\}, & \forall y \in Y_U - Y_L. \end{cases}$$

Definition 1.21.[14] Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Let $Q = (C, D)$ be an IFRS in Y, where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L)$ and $X_U = f^{-1}(Y_U)$. Then the inverse image $f^{-1}(Q)$ of Q, under f, is defined by $f^{-1}(Q) = (f^{-1}(C), f^{-1}(D))$, where $f^{-1}(C) = (f^{-1}(C_L), f^{-1}(C_U))$ and $f^{-1}(D) = (f^{-1}(D_L), f^{-1}(D_U))$.

The following three Theorems 1.22, 1.23, 1.24 of Samanta and Mondal [14] are also valid for this modified definition of functional image of an IFRS.

Theorem 1.22. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Then for all IFRSs P and Q, we have

(i) $f(P') \supset (f(P))'$.

(ii) $P \subset Q \Rightarrow f(P) \subset f(Q)$.

Theorem 1.23. Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all IFRSs R, S and $R_i, i \in J$ in Y,

(i) $f^{-1}(R') = (f^{-1}(R))'$.

(ii) $R \subset S \Rightarrow f^{-1}(R) \subset f^{-1}(S)$.

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ for any IFRS W in Z, where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set, gof is the composition of g and f.

(iv) $f^{-1}(\bigcup_{i\in J} R_i) = \bigcup_{i\in J} f^{-1}(R_i)$.

(v) $f^{-1}(\bigcap_{i\in J} R_i) = \bigcap_{i\in J} f^{-1}(R_i)$.

Theorem 1.24. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all IFRS P in X and R in Y, we have

(i) $R = f(f^{-1}(R))$.

(ii) $P \subset f^{-1}(f(P))$.

Theorem 1.25. If P and Q be two IFRSs in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cup Q) = f(P) \cup f(Q)$.

The proof is straightforward.

Corollary 1.26. If $P_1, P_2, \cdots, P_n$ be IFRSs in X, then $f(P_1 \cup P_2 \cup \cdots \cup P_n) = f(P_1) \cup f(P_2) \cup \cdots \cup f(P_n)$.

The proof follows by extending the Theorem 1.25.

Theorem 1.27. If P, Q be two IFRSs in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cap Q) \subset f(P) \cap f(Q)$.

The proof is straightforward.

Note 1.28. If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be one-one, then clearly $f(P \cap Q) = f(P) \cap f(Q)$. But in general $f(P \cap Q) \neq f(P) \cap f(Q)$, which can be shown by the following example.

Example 1.29. Let V $= \{x, y\}$, $X_L = X_U = V$, $X = (X_L, X_U)$, $V_1 = \{a\}$, $Y_L = Y_U = V_1$, $Y = (Y_L, Y_U)$.

Let $f : V \to V_1$ be defined by $f(x) = f(y) = a$.

Let $P = ((\{x/0.4, y/0.3\}, \{x/0.4, y/0.4\}), (\{x/0.2, y/0.4\}, \{x/0.3, y/0.4\}))$,

$Q = ((\{x/0.3, y/0.4\}, \{x/0.3, y/0.4\}), (\{x/0.3, y/0.2\}, \{x/0.3, y/0.3\}))$.

Clearly P, Q are IFRSs in X.

$P \cap Q = ((\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4\}))$.

Therefore $f(P \cap Q) = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.3\}, \{a/0.3\}))$,

$f(P) = ((\{a/0.4\}, \{a/0.4\}), (\{a/0.2\}, \{a/0.3\}))$,

$f(Q) = ((\{a/0.4\}, \{a/0.4\}), (\{a/0.2\}, \{a/0.3\}))$.

So $f(P) \cap f(Q) = ((\{a/0.4\}, \{a/0.4\}), (\{a/0.2\}, \{a/0.3\}))$.

Therefore $f(P \cap Q) \neq f(P) \cap f(Q)$.

§3. Topological space of IFRSs

Definition 2.1. Let $X = (X_L, X_U)$ be a rough set and τ be a family of IFRSs in X such that

(i) $0^*,\ 1^*\ \in \tau$.

(ii) $P \cap Q \in \tau,\ \forall P,\ Q \in \tau$.

(iii) $P_i \in \tau, i \in \Delta \Rightarrow \bigcup_{i\in\Delta} P_i \in \tau$.

Then τ is called a topology of IFRSs in X and the pair (X, τ) is called a topological space of IFRSs in X. Every member of τ is called open IFRS. An IFRS C is called closed IFRS if $C' \in \tau$. Let $\mathcal{F}$ denote the collection of all closed IFRSs in (X, τ). If $\tau_I = \{0^*, 1^*\}$, then τ_I is a topology of IFRSs in X. This topology is called the indiscrete topology. The discrete topology of IFRSs in X contains all the IFRSs in X.

Theorem 2.2. The collection $\mathcal{F}$ of all closed IFRSs satisfies the following properties:

(i) $0^*, 1^* \in \mathcal{F}$.

(ii) $P, Q \in \mathcal{F} \Rightarrow P \cup Q \in \mathcal{F}$.

(iii) $P_i \in \mathcal{F}, i \in \Delta \Rightarrow \bigcap_{i\in\Delta} P_i \in \mathcal{F}$.

The proof is straightforward.

Definition 2.3. Let P be an IFRS in X. The union of all open IFRSs in (X, τ) contained in P is called the interior of P in (X, τ) and is denoted by $Int_\tau P$. Clearly $Int_\tau P$ is the largest open IFRS contained in P and P is open iff $P = Int_\tau P$.

Definition 2.4. Let P be an IFRS in X. The closure of P in (X, τ), denoted by $cl_\tau P$, is defined by the intersection of all closed IFRSs in (X, τ) containing P. Clearly $cl_\tau P$ is the smallest closed IFRS containing P and P is closed iff $P = cl_\tau P$.

Theorem 2.5. (i) $cl_\tau 0^* = 0^*$,

(ii) $cl_\tau(cl_\tau P) = cl_\tau P$.

(iii) $P \subset Q \Rightarrow cl_\tau P \subset cl_\tau Q$.

(iv) $cl_\tau(P \cup Q) = cl_\tau P \cup cl_\tau Q$.

(v) $cl_\tau(P \cap Q) \subset cl_\tau P \cap cl_\tau Q$.

The proof is straightforward.

Definition 2.6. Let (X, τ) and (Y, u) be two topological spaces of IFRSs and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then $f : (X, \tau) \to (Y, u)$ is said to be IFR continuous if $f^{-1}(Q) \in \tau,\ \forall Q \in u$. Unless otherwise stated we consider (X, τ) and (Y, u) be topological spaces of IFRSs and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Theorem 2.7. The following statements are equivalent :

(i) $f : (X, \tau) \to (Y, u)$ is IFR continuous.

(ii) $f^{-1}(Q)$ is closed IFRS in (X, τ), for every closed IFRS Q in (Y, u).

(iii) $f(cl_\tau P) \subset cl_u(f(P))$, for every IFRS P in X.

The proof is straightforward.

§4. Topology on an IFRS

Definition 3.1. Let $P \in \mathcal{C}_X$. Then a subfamily T of $\mathcal{C}_X$ is said to be a topology on P if

(i) $Q \in T \Rightarrow Q \subset P$.

(ii) $0^*, P \in T$.

(iii) $P_1, P_2 \in T \Rightarrow P_1 \cap P_2 \in T$.

(iv) $P_i \in T, i \in \Delta \Rightarrow \bigcup_{i\in\Delta} P_i \in T$.

Then (P, T) is called a subspace topology of (X, τ).

Theorem 3.2. Let τ be a topology of IFRSs in X and let $P \in \mathcal{C}_X$. Then $\tau_1 = \{P \cap R : R \in \tau\}$ is a topology on P.

The proof is straightforward.

Every member of τ_1 is called open IFRS in (P, τ_1). If $Q \in \tau_1$, then Q'_P is called a closed IFRS in (P, τ_1), where $Q'_P = P \cap Q'$. We take 0^* as closed IFRS also in (P, τ_1). Let $C_1 = \{Q'_P = P \cap Q' : Q \in \tau_1\} \cup \{0^*\}$.

Theorem 3.3. C_1 is closed under arbitrary intersection and finite union.

The proof is straightforward.

Remark 3.4. Let $S \in \tau_1$. Then $S'_P = P \cap S' \in C_1$. Now

$$(S'_P)'_P = P \cap (S'_P)' = P \cap (P \cap S')' = P \cap (P' \cup S) = (P \cap P') \cup S,$$

since $S \subset P$.

$$(S'_P)'_P = (P \cap P') \cup (P \cap R), \text{ for some } R \in \tau$$
$$(S'_P)'_P = P \cap (P' \cup R).$$

Thus $(S'_P)'_P$ need not necessarily belongs to τ_1.

$$\begin{aligned} ((S'_P)'_P)'_P &= P \cap ((S'_P)'_P)' \\ &= P \cap ((P \cap P') \cup S)' \\ &= P \cap ((P \cap P')' \cap S') \\ &= P \cap ((P' \cup P) \cap S') \\ &= P \cap S' = S'_P \in C_1. \end{aligned}$$

Clearly the collection $\tau_2 = \{(S'_P)'_P = (P \cap P') \cup S : S \in \tau_1\} \cup \{0^*\}$ form a topology of IFRSs on P of which C_1 is a family of closed IFRSs. But C_1 is also a family of closed IFRSs in (P, τ_1). Thus $\exists$ two topologies of IFRSs τ_1 and τ_2 on P. τ_1 is called the first subspace topology of (X, τ) on P and τ_2 is called the second subspace topology of (X, τ) on P. We briefly write, τ_1 and τ_2 are first and second topologies respectively on P, where there is no confusion about the topological space (X, τ) of IFRSs.

Note 3.5. Let $P \in \mathcal{C}_X$ and τ_1, τ_2 be respectively first and second topologies on P. i.e.,

$$\tau_1 = \{P \cap R : R \in \tau\}$$

and

$$\tau_2 = \{(P \cap P') \cup S : S \in \tau_1\} \cup \{0^*\}$$

$$= \{P \cap (P' \cup R) : R \in \tau\} \cup \{0^*\}.$$

If τ be an indiscrete topology of IFRSs in X, then $\tau_1 = \{P, 0^*\}$ and $\tau_2 = \{P, 0^*, P \cap P'\}$. Thus $\tau_1 \subset \tau_2$ ($\tau_1 \neq \tau_2$).

If τ be discrete topology, then $\tau_1 = \{S \in \mathcal{C}_X : S \subset P\}$, $\tau_2 = \{S \in \mathcal{C}_X : P \cap P' \subset S \subset P\} \cup \{0^*\}$. Thus $\tau_2 \subset \tau_1$.

Note 3.6. Let $R \in \mathcal{C}_X$ such that $R \subset P$. The closure of R in (P, τ_1) and (P, τ_2) be denoted by $cl_{\tau_1}R$, $cl_{\tau_2}R$ respectively.

$$\begin{aligned} cl_{\tau_1}R = \bigcap\{G'_P : G'_P \supset R;\ G \in \tau_1\} &= \bigcap\{P \cap G' : P \cap G' \supset R;\ G \in \tau_1\} \\ &= P \cap (\bigcap\{G' : G' \supset R;\ G \in \tau_1\}) \\ &= P \cap (\bigcap\{(P \cap Q)' : (P \cap Q)' \supset R;\ Q \in \tau\}) \\ &= P \cap (\bigcap\{P' \cup Q' : P' \cup Q' \supset R;\ Q \in \tau\}) \\ &\subset P \cap (P' \cup (\bigcap\{Q' : Q' \supset R;\ Q \in \tau\})) \\ &= P \cap (P' \cup cl_\tau R) \\ &= (P \cap P') \cup (P \cap cl_\tau R). \end{aligned}$$

Since closed IFRS in (P, τ_1) and in (P, τ_2) are same,

$$cl_{\tau_1}R = cl_{\tau_2}R \subset (P \cap P') \cup (P \cap cl_\tau R).$$

Following example shows that, in general

$$cl_{\tau_1}R \neq (P \cap P') \cup (P \cap cl_\tau R).$$

Example 3.7. Let $X_L \neq \phi, X_U = X_L \cup \{a\} (a \notin X_L)$ be two crisp sets with $X_L \subset X_U$. Then $X = (X_L, X_U)$ is a rough set.

Let $C = (C_L, C_U), D = (D_L, D_U)$ are defined by

$$C_L(x) = 0.7 = C_U(x), \forall x \in X_L,$$

$$C_U(a) = 0.5,$$

$$D_L(x) = 0.3 = D_U(x), \forall x \in X_L,$$

$$D_U(a) = 0.3.$$

Then $Q = (C, D)$ is an IFRS in $X = (X_L, X_U)$.

Let $\tau = \{0^*, 1^*, Q\}$.

Then (X, τ) is a topological space of IFRSs.

Let $A = (A_L, A_U), B = (B_L, B_U)$ are defined by

$$A_L(x) = 0.5, \forall x \in X_L, \quad A_U(x) = 0.6, \forall x \in X_U,$$

$$B_L(x) = 0.4 = B_U(x), \forall x \in X_L, \quad B_U(a) = 0.4.$$

Then P = (A,B) is an IFRS in X.

Define $\tau_1 = \{P \cap S : S \in \tau\}$.
Then τ_1 is the first topology on P.

Let $M = (M_L, M_U)$ and $N = (N_L, N_U)$ are defined by

$$M_L(x) = 0.4 = M_U(x), \forall x \in X_L, \quad M_U(a) = 0.4,$$

$$N_L(x) = 0.6 = N_U(x), \forall x \in X_L, \quad N_U(a) = 0.5.$$

Then $R = (M, N)$ is an IFRS in X such that $R \subset P$.

Now $cl_{\tau_1} R =$ closure of R in $(P, \tau_1) = P \cap (\bigcap\{P' \cup Q' : P' \cup Q' \supset R;\ Q \in \tau\})$ (see Note 3.6).

Now

$$P' \cup (0^*)' = P' \cup 1^* = 1^* \supset R,$$

$$P' \cup (1^*)' = P' \cup 0^* = P' \not\supset R,$$

since $A \not\subset N$.

$$P' \cup Q' = (B \cup D, A \cap C) \supset R.$$

Thus $cl_{\tau_1} R = P \cap (1^* \cap (P' \cup Q')) = P \cap (P' \cup Q') = (F, G)$, say.

Therefore

$$F_L(x) = 0.4 = F_U(x), \forall x \in X_L, \quad F_U(a) = 0.4,\ G_L(x) = 0.5$$

and

$$G_U(x) = 0.6, \forall x \in X_L, \quad G_U(a) = 0.5.$$

$$cl_\tau R = \text{ Closure of } R \text{ in}(X, \tau) = \bigcap\{Q_i' : Q_i' \supset R;\ Q_i \in \tau\}.$$

Now $(0^*)' = 1^* \supset R$; $(1^*)' = 0^* \not\supset R$; $Q' \not\supset R$, since $C \not\subset N$.
Thus $cl_\tau R = 1^*$.

Therefore $(P \cap P') \cup (P \cap cl_\tau R) = (P \cap P') \cup (P \cap 1^*) = (P \cap P') \cup P = P = (A, B)$.
Clearly $(F, G) \subset (A, B)$ and $(F, G) \neq (A, B)$.

Thus $cl_{\tau_1} R$ is a proper subset of $(P \cap P') \cup (P \cap cl_\tau R)$.

§5. Continuity of a mapping over IFRSs

Definition 4.1. Let (X, τ) and (Y, u) be two topological spaces of IFRSs and $P \in \mathcal{C}_X$. Let τ_1 and u_1 are first topologies on P and $f(P)$ respectively. Then $f : (P, \tau_1) \to (f(P), u_1)$ is said to be IFR_1 continuous if $P \cap f^{-1}(Q) \in \tau_1, \forall Q \in u_1$.

Theorem 4.2. Let (X, τ) and (Y, u) be two topological spaces of IFRSs in X and Y respectively and $P \in \mathcal{C}_X$ and let τ_1 and u_1 be first topologies on P and $f(P)$ respectively. If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous.

The proof is straightforward.

Remark 4.3. The converse of the Theorem 4.2 is not necessarily true. This is shown by the following example.

Example 4.4. Let $V = \{x, y\}$, $X_L = X_U = V$, $V_1 = \{a\}$, $Y_L = Y_U = V_1$.

Let $f : V \to V_1$ be defined by $f(x) = f(y) = a$.

Let $Q = ((\{x/0.2, y/0.3\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.2\}, \{x/0.3, y/0.3\}))$,

$R = ((\{x/0.3, y/0.2\}, \{x/0.3, y/0.4\}), (\{x/0.3, y/0.1\}, \{x/0.3, y/0.4\}))$,

$T = Q \cup R = ((\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\}), (\{x/0.3, y/0.1\}, \{x/0.3, y/0.3\}))$,

$S = Q \cap R = ((\{x/0.2, y/0.2\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.2\}, \{x/0.3, y/0.4\}))$.

Clearly $\tau = \{0^*, 1^*, Q, R, T, S\}$ forms a topology of IFRSs in $X = (X_L, X_U)$.

Let $P = ((\{x/0.2, y/0.3\}, \{x/0.2, y/0.5\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\}))$

Therefore $\underline{Q} = P \cap Q = ((\{x/0.2, y/0.3\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\}))$,

$\underline{R} = P \cap R = ((\{x/0.2, y/0.2\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\}))$,

$\underline{T} = P \cap T = ((\{x/0.2, y/0.3\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\})) = \underline{Q}$,

$\underline{S} = P \cap S = ((\{x/0.2, y/0.2\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\})) = \underline{R}$.

Thus $\tau_1 = \{0^*, P, \underline{Q}, \underline{R}\}$ forms a first topology on P.

Let $\tilde{Q} = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.2\}, \{a/0.3\}))$,

$\tilde{R} = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.1\}, \{a/0.3\}))$,

$\tilde{S} = ((\{a/0.2\}, \{a/0.4\}), (\{a/0.2\}, \{a/0.3\}))$.

So $\tilde{S} \subset \tilde{Q} \subset \tilde{R}$.

Clearly $u = \{0^*, 1^*, \tilde{Q}, \tilde{R}, \tilde{S}\}$ forms a topology of IFRSs in $Y = (Y_L, Y_U)$.

Now $f(P) = ((\{a/0.3\}, \{a/0.5\}), (\{a/0.3\}, \{a/0.3\}))$.

Therefore $\underline{\tilde{Q}} = f(P) \cap \tilde{Q} = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.3\}, \{a/0.3\}))$,

$\underline{\tilde{R}} = f(P) \cap \tilde{R} = ((\{a/0.3\}, \{a/0.4\}), (\{a/0.3\}, \{a/0.3\})) = \underline{\tilde{Q}}$,

$\underline{\tilde{S}} = f(P) \cap \tilde{S} = ((\{a/0.2\}, \{a/0.4\}), (\{a/0.3\}, \{a/0.3\}))$.

Thus $u_1 = \{0^*, f(P), \underline{\tilde{Q}}, \underline{\tilde{S}}\}$ forms a first topology on $f(P)$.

Now $P \cap f^{-1}(0^*) = P \cap 0^* = 0^* \in \tau_1$, $\quad P \cap f^{-1}(f(P)) = P \in \tau_1$,

$P \cap f^{-1}(\underline{\tilde{Q}}) = ((\{x/0.2, y/0.3\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\})) = \underline{Q} \in \tau_1$,

$P \cap f^{-1}(\underline{\tilde{S}}) = ((\{x/0.2, y/0.2\}, \{x/0.2, y/0.4\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.4\})) = \underline{R} \in \tau_1$.

Thus $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous.

Since $f^{-1}(\tilde{Q}) = ((\{x/0.3, y/0.3\}, \{x/0.4, y/0.4\}), (\{x/0.2, y/0.2\}, \{x/0.3, y/0.3\})) \notin \tau$, but $\tilde{Q} \in u$, it follows that

$$f : (X, \tau) \to (Y, u)$$

is not IFR continuous.

Definition 4.5. Let (X, τ) and (Y, u) be two topological spaces of IFRSs in X and Y respectively and $P \in \mathcal{C}_X$ and let τ_2, u_2 be second topologies on P and $f(P)$ respectively. Then $f : (P, \tau_2) \to (f(P), u_2)$ is said to be IFR_2 continuous if $P \cap (P' \cup f^{-1}(Q)) \in \tau_2, \quad \forall Q \in u_2$.

Theorem 4.6. Let (X, τ) and (Y, u) be two topological spaces of IFRSs in X and Y respectively and $P \in \mathcal{C}_X$. Let τ_1 and u_1 be first topologies on P and $f(P)$ respectively and τ_2, u_2 be second topologies on P and $f(P)$ respectively.

If $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous, then

$f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

Proof. Let $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous. Let $Q \in u_2$. If $Q = 0^*$, then $P \cap (P' \cup f^{-1}(Q)) = P \cap P' \in \tau_2$. So let $Q \neq 0^*$. Then $Q = (f(P) \cap (f(P))') \cup R$, for some

$R \in u_1$. Therefore

$$\begin{aligned} f^{-1}(Q) &= f^{-!}(f(P) \cap (f(P))') \cup f^{-1}(R) \\ &= (f^{-1}(f(P)) \cap f^{-1}((f(P))')) \cup f^{-1}(R) \\ &= (f^{-1}(f(P)) \cap (f^{-1}(f(P)))') \cup f^{-1}(R). \end{aligned}$$

Therefore $P' \cup f^{-1}(Q) = P' \cup f^{-1}(R)$ and hence $P \cap (P' \cup f^{-1}(Q)) = P \cap (P' \cup f^{-1}(R)) = (P \cap P') \cup (P \cap f^{-1}(R))$. Since $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous and $R \in u_1$, we have $P \cap f^{-1}(R) \in \tau_1$. Thus $(P \cap P') \cup (P \cap f^{-1}(R)) \in \tau_2$ and hence $P \cap (P' \cup f^{-1}(Q)) \in \tau_2$. Thus $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

Remark 4.7. The converse of the Theorem 4.6 is not necessarily true. This can be shown by the following example.

Example 4.8. Let $V = \{x, y, z\}$, $X_L = \{x, y\}$, $X_U = V$, $V_1 = \{a, b\}$, $0.15 in Y_L = \{a\}$, $Y_U = V_1$.

Define $f : V \to V_1$ by $f(x) = f(y) = a, f(z) = b$.

Let $Q = ((\{x/0.3, y/0.4\}, \{x/0.4, y/0.5, z/0.2\}), (\{x/0.2, y/0.5\}, \{x/0.3, y/0.5, z/0.4\}))$,

Let $R = ((\{x/0.4, y/0.3\}, \{x/0.4, y/0.5, z/0.2\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.5, z/0.3\}))$,

$T = Q \cup R = ((\{x/0.4, y/0.4\}, \{x/0.4, y/0.5, z/0.2\}), (\{x/0.2, y/0.4\}, \{x/0.3, y/0.5, z/0.3\}))$,

$S = Q \cap R = ((\{x/0.3, y/0.3\}, \{x/0.4, y/0.5, z/0.2\}), (\{x/0.3, y/0.5\}, \{x/0.3, y/0.5, z/0.4\}))$.

Clearly $\tau = \{0^*, 1^*, Q, R, T, S\}$ forms a topology of IFRSs in $X = (X_L, X_U)$.

Let $P = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.3\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4, z/0.5\}))$,

$\underline{Q} = P \cap Q = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.2\}), (\{x/0.3, y/0.5\}, \{x/0.3, y/0.5, z/0.5\}))$,

$\underline{R} = P \cap R = ((\{x/0.2, y/0.3\}, \{x/0.4, y/0.4, z/0.2\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.5, z/0.5\}))$,

$\underline{T} = P \cap T = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.2\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.5, z/0.5\}))$,

$\underline{S} = P \cap S = ((\{x/0.2, y/0.3\}, \{x/0.4, y/0.4, z/0.2\}), (\{x/0.3, y/0.5\}, \{x/0.3, y/0.5, z/0.5\}))$.

Thus $\tau_1 = \{0^*, P, \underline{Q}, \underline{R}, \underline{T}, \underline{S}\}$ forms a first topology on P.

Clearly $P \cap P' = ((\{x/0.2, y/0.4\}, \{x/0.3, y/0.4, z/0.3\}), (\{x/0.3, y/0.4\}, \{x/0.4, y/0.4, z/0.5\}))$.

$\underline{\underline{Q}} = (P \cap P') \cup \underline{Q} = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/.3\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4, z/0.5\}))$,

$\underline{\underline{R}} = (P \cap P') \cup \underline{R} = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.3\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4, z/0.5\})) = \underline{\underline{Q}}$.

Similarly it can be proved that $\underline{\underline{T}} = (P \cap P') \cup \underline{T} = \underline{\underline{Q}}$, $\underline{\underline{S}} = (P \cap P') \cup \underline{S} = \underline{\underline{Q}}$.

Thus $\tau_2 = \{0^*, P \cap P', P, \underline{\underline{Q}}\}$ forms a second topology on P.

Let $\tilde{Q} = ((\{a/0.4\}, \{a/0.5, b/0.2\}), (\{a/0.2\}, \{a/0.3, b/0.4\}))$,

$\tilde{R} = ((\{a/0.4\}, \{a/0.5, b/0.2\}), (\{a/0.3\}, \{a/0.3, b/0.3\}))$,

$\tilde{T} = \tilde{Q} \cup \tilde{R} = ((\{a/0.4\}, \{a/0.5, b/0.2\}), (\{a/0.2\}, \{a/0.3, b/0.3\}))$,

$\tilde{S} = \tilde{Q} \cap \tilde{R} = ((\{a/0.4\}, \{a/0.5, b/0.2\}), (\{a/0.3\}, \{a/0.3, b/0.4\}))$.

Clearly $u = \{0^*, 1^*, \tilde{Q}, \tilde{R}, \tilde{T}, \tilde{S}\}$ forms a topology of IFRSs in $Y = (Y_L, Y_U)$.

Now $f(P) = ((\{a/0.4\}, \{a/0.4, b/0.3\}), (\{a/0.3\}, \{a/0.3, b/0.5\}))$.

Therefore $(f(P))' = ((\{a/0.3\}, \{a/0.3, b/0.5\}), (\{a/0.4\}, \{a/0.4, b/0.3\}))$.

$\underline{\tilde{Q}} = f(P) \cap \tilde{Q} = ((\{a/0.4\}, \{a/0.4, b/0.2\}), (\{a/0.3\}, \{a/0.3, b/0.5\}))$,

$\underline{\tilde{R}} = f(P) \cap \tilde{R} = ((\{a/0.4\}, \{a/0.4, b/0.2\}), (\{a/0.3\}, \{a/0.3, b/0.5\})) = \underline{\tilde{Q}}$.

Similarly it can be checked that $\underline{\tilde{T}} = f(P) \cap \tilde{T} = \underline{\tilde{Q}}$, $\underline{\tilde{S}} = f(P) \cap \tilde{S} = \underline{\tilde{Q}}$. Thus $u_1 = \{0^*, f(P), \underline{\tilde{Q}}\}$ forms a first topology on $f(P)$.

Now $f(P) \cap (f(P))' = ((\{a/0.3\}, \{a/0.3, b/0.3\}), (\{a/0.4\}, \{a/0.4, b/0.5\}))$.
Therefore $\underline{\underline{\tilde{Q}}} = (f(P) \cap (f(P))') \cup \underline{\tilde{Q}} = ((\{a/0.4\}, \{a/0.4, b/0.3\}), (\{a/0.3\}, \{a/0.3, b/0.5\}))$.
Thus $u_2 = \{0^*, f(P) \cap (f(P))', f(P), \underline{\underline{\tilde{Q}}}\}$ forms a second topology on $f(P)$.
It is clear that $P \cap (P' \cup f^{-1}(0^*)) = P \cap (P' \cup 0^*) = P \cap P' \in \tau_2$,
$P \cap (P' \cup f^{-1}(f(P) \cap (f(P))')) = P \cap (P' \cup (f^{-1}(f(P)) \cap (f^{-1}(f(P)))')) = P \cap P' \in \tau_2$,
$P \cap (P' \cup f^{-1}(f(P))) = P \in \tau_2$
and since $f^{-1}(\underline{\underline{\tilde{Q}}}) = ((\{x/0.4, y/0.4\}, \{x/0.4, y/0.4, z/0.3\}), (\{x/0.3, y/0.3\}, \{x/0.3, y/0.3, z/0.5\}))$
it follows that

$P \cap (P' \cup f^{-1}(\underline{\underline{\tilde{Q}}}))$
$= ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.3\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4, z/0.5\}))$
$= \underline{\underline{Q}} \in \tau_2$.

Thus $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.
Now $P \cap f^{-1}(\underline{\tilde{Q}}) = ((\{x/0.2, y/0.4\}, \{x/0.4, y/0.4, z/0.2\}), (\{x/0.3, y/0.4\}, \{x/0.3, y/0.4, z/0.5\}))$
$\notin \tau_1$, but $\underline{\tilde{Q}} \in u_1$.
Thus $f : (P, \tau_1) \to (f(P), u_1)$ is not IFR_1 continuous.

Corollary 4.9. If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous, where the symbols have usual meaning.

The proof follows from Theorems 4.2 and 4.6.

Remark 4.10. The converse of the corollary 4.9 is not true, which can be shown by the following:

(i) In Example 4.8, note that
$f^{-1}(\tilde{Q}) = ((\{x/0.4, y/0.4\}, \{x/0.5, y/0.5, z/0.2\}), (\{x/0.2, y/0.2\}, \{x/0.3, y/0.3, z/0.4\})) \notin \tau$, but $\tilde{Q} \in u$. Thus $f : (X, \tau) \to (Y, u)$ is not IFR continuous, but $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

(ii) In Example 4.4, note that
$f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous and hence $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous, but $f : (X, \tau) \to (Y, u)$ is not IFR continuous.

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On a limit for the product of powers of primes

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Abstract Denote the nth prime number by p_n. In this note we show a limit theorem involving the product of powers of prime numbers: If a sequence a_n has the limit a, under some conditions, we show that $p_n^{a_n}/\sqrt[n]{p_1^{a_1}\cdots p_n^{a_n}}$ has the limit e^a.

Keywords Prime, arithmetic function, asymptotics.

§1. Introduction

Denote the nth prime number by p_n. The prime number theorem (see e.g. [2]) indicates the following asymptotics:

$$p_n \sim n \ln n, \tag{1}$$

as $n \to \infty$; i.e. $\lim_{n\to} p_n/(n \ln n) = 1$.

The first Chebyshev's function is defined as

$$\theta(x) = \sum_{p \le x} \ln p, \tag{2}$$

where the sum extending over all primes p that are less than or equal to x. A classical estimate for Chebyshev's function is the following lemma.

Lemma 1.1.[3] There exists a positive constant $c > 0$ such that for all $x > 1$,

$$|\theta(x) - x| \le \frac{cx}{\ln^2 x}. \tag{3}$$

By virtue of Lemma 1.1, we establish a limit theorem involving the product of powers of primes in the present note.

Theorem 1.2. Let $\{a_n\}_{n\ge1}$ be a real-valued sequence. Let $a_{\max}(n) = \max\{a_1, \ldots, a_n\}$ and $a_{\min}(n) = \min\{a_1, \ldots, a_n\}$ such that

$$\lim_{n\to\infty} a_{\max}(n) = \lim_{n\to\infty} a_{\min}(n) = a \in \mathbb{R}, \tag{4}$$

and

$$|a_{\max}(n) - a_{\min}(n)| = o\left(\frac{1}{\ln n}\right), \tag{5}$$

as $n \to \infty$. Then

$$\frac{p_n^{a_n}}{\sqrt[n]{p_1^{a_1}\cdots p_n^{a_n}}} \to e^a, \tag{6}$$

as $n \to \infty$.

§2. Proof of Theorem 1.2

In this section, we present a proof of Theorem 1.2. We will need the following asymptotics:

Lemma 2.1.[1,4]

$$\ln p_n - \frac{p_n}{n} \to 1, \tag{7}$$

as $n \to \infty$.

First note that it follows from (4) that $\lim_{n\to\infty} a_n = a$. To prove (6), it suffices to show

$$\ln\left(\frac{p_n^{a_n}}{\sqrt[n]{p_1^{a_1}\cdots p_n^{a_n}}}\right) \to a, \tag{8}$$

as $n \to \infty$, which is tantamount to

$$a_n \ln p_n - \frac{a_1 \ln p_1 + \cdots + a_n \ln p_n}{n} \to a. \tag{9}$$

Since

$$\begin{aligned} \frac{a_{\min}(n)(\ln p_1 + \cdots + \ln p_n)}{n} &\le \frac{a_1 \ln p_1 + \cdots + a_n \ln p_n}{n} \\ &\le \frac{a_{\max}(n)(\ln p_1 + \cdots + \ln p_n)}{n}, \end{aligned} \tag{10}$$

we obtain

$$\frac{a_1 \ln p_1 + \cdots + a_n \ln p_n}{n} = \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a\theta(p_n)}{n}, \tag{11}$$

by using the assumptions (4), (5) and the definition (2).

Therefore, we obtain

$$\begin{aligned} a_n \ln p_n - \frac{a_1 \ln p_1 + \cdots + a_n \ln p_n}{n} &= a_n \ln p_n - \frac{a_n p_n}{n} + \frac{a_n p_n}{n} \\ &\quad - \frac{a_1 \ln p_1 + \cdots + a_n \ln p_n}{n} \\ &= a_n \ln p_n - \frac{a_n p_n}{n} + \frac{a_n p_n}{n} - \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a p_n}{n} \\ &\quad + \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a p_n}{n} - \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a\theta(p_n)}{n}. \end{aligned} \tag{12}$$

In view of (12), Lemma 2.1 and the fact that $a_n \to a$ as $n \to \infty$, it would be sufficient to show

(i)

$$\frac{a_n p_n}{n} - \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a p_n}{n} \to 0, \tag{13}$$

and

(ii)

$$\frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a p_n}{n} - \frac{\left(1 + o\left(\frac{1}{\ln n}\right)\right) a\theta(p_n)}{n} \to 0, \tag{14}$$

as $n \to \infty$.

For (i), we recall that $p_n = (1+o(1))n \ln n$ by (1). The limit (13) then follows immediately.

As for (ii), we have

$$\frac{|p_n - \theta(p_n)|}{n} \leq \frac{cp_n}{n \ln^2 p_n}, \tag{15}$$

by Lemma 1.1. It follows easily from (1) that

$$\frac{p_n}{n \ln^2 p_n} \sim \frac{1}{\ln p_n}. \tag{16}$$

Combining (15) and (16), we have

$$\frac{|p_n - \theta(p_n)|}{n} \to 0, \tag{17}$$

as $n \to \infty$, which is equivalent to (14). Thus, the proof of Theorem 1.2 is complete.

Finally, we remark that if we take the sequence $a_n \equiv 1$ above, we easily recover Theorem 2.1 in [4].

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Switching equivalence in symmetric n-sigraphs-IV

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Abstract An n-tuple $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function. In this paper, we define the *antipodal symmetric n-sigraph* $A(S_n) = (A(G), \sigma)$ of a given symmetric n-sigraph $S_n = (G, \sigma)$ and offer a structural characterization of antipodal symmetric n-sigraphs. Further, we characterize symmetric n-sigraphs S_n for which $S_n \sim A(S_n)$ and $\overline{S_n} \sim A(S_n)$ where $\sim$ denotes switching equivalence and $A(S_n)$ and $\overline{S_n}$ are denotes the antipodal symmetric n-sigraph and complementary symmetric n-sigraph of S_n respectively.

Keywords Symmetric n-sigraphs, symmetric n-marked graphs, balance, switching, antipod -al symmetric n-sigraphs, complementation.

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [3]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [8], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. S. K. Reddy [5]):

Definition. Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note. An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [8].

Proposition 1.1. (***E. Sampathkumar et al.***[8]) An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.

In [8], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [4,6,7,11-17])

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result.

Proposition 1.2. (***E. Sampathkumar et al.***[8]) Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Antipodal n-sigraphs

Singleton [10] has introduced the concept of antipodal graph of a graph G as the graph $A(G)$ having the same vertex set as that of G and two vertices are adjacent if they are at a distance of $diam(G)$ in G.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of antipodal graphs to n-sigraphs as follows: The *antipodal n-sigraph* $A(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $A(G)$ and the n-tuple of any edge uv in $A(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called antipodal n-sigraph, if $S_n \cong A(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $A(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be antipodal n-sigraphs.

Proposition 2.1. For any n-sigraph $S_n = (G, \sigma)$, its antipodal n-sigraph $A(S_n)$ is i-balanced.

Proof. Since the n-tuple of any edge uv in $A(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $A(S_n)$ is i-balanced.

For any positive integer k, the k^{th} iterated antipodal n-sigraph $A(S_n)$ of S_n is defined as follows:

$$A^0(S_n) = S_n,\ A^k(S_n) = A(A^{k-1}(S_n)).$$

Corollary 2.2. For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $A^k(S_n)$ is i-balanced.

In [1], the authors characterized those graphs that are isomorphic to their antipodal graphs.

Proposition 2.3. For a graph $G = (V, E)$, $G \cong A(G)$ if and only if G is complete.

We now characterize the n-sigraphs that are switching equivalent to their antipodal n-sigraphs.

Proposition 2.4. For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim A(S_n)$ if, and only if, $G = K_p$ and S_n is i-balanced signed graph.

Proof. Suppose $S_n \sim A(S_n)$. This implies, $G \cong A(G)$ and hence G is K_p. Now, if S_n is any n-sigraph with underlying graph as K_p, Proposition 2.1 implies that $A(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $A(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and G is K_p. Then, since $A(S_n)$ is i-balanced as per Proposition 3 and since $G \cong A(G)$, the result follows from Proposition 2 again.

Proposition 2.5. For any two n-sigraphs S_n and S'_n with the same underlying graph, their antipodal n-sigraphs are switching equivalent.

Proposition 2.6. (***Aravamudhan and Rajendran*** [1]) For a graph $G = (V, E)$, $\overline{G} \cong A(G)$ if, and only if, i). G is diameter 2 or ii). G is disconnected and the components of G are complete graphs.

In view of the above, we have the following result for n-sigraphs:

Proposition 2.7. For any n-sigraph $S_n = (G, \sigma)$, $\overline{S_n} \sim A(S_n)$ if, and only if, G satisfies conditions of Proposition 2.6.

Proof. Suppose that $A(S_n) \sim \overline{S_n}$. Then clearly we have $A(G) \cong \overline{G}$ and hence G satisfies conditions of Proposition 2.6.

Conversely, suppose that G satisfies conditions of Proposition 2.6. Then $\overline{G} \cong A(G)$ by Proposition 2.6. Now, if S_n is an n-sigraph with underlying graph satisfies conditions of Proposition 2.6, by definition of complementary n-sigraph and Proposition 2.1, $\overline{S_n}$ and $A(S_n)$ are i-balanced and hence, the result follows from Proposition 1.2.

The following result characterize n-sigraphs which are antipodal n-sigraphs.

Proposition 2.8. An n-sigraph $S_n = (G, \sigma)$ is an antipodal n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is an antipodal graph.

Proof. Suppose that S_n is i-balanced and G is a $A(G)$. Then there exists a graph H such that $A(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $A(S'_n) \cong S_n$. Hence S_n is an antipodal n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an antipodal n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $A(S'_n) \cong S_n$. Hence G is the $A(G)$ of H and by Proposition 2.1, S_n is i-balanced.

§3. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $A(S_n)$ is i-balanced (Theorem 3). We now examine, the condition under which m-complement of $A(S_n)$ is i-balanced, where for any $m \in H_n$.

Proposition 3.1. Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $A(G)$ is bipartite then $(A(S_n))^m$ is i-balanced.

Proof. Since, by Proposition 3.1, $A(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $A(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m \in H_n$. Hence $(A(S_n))^t$ is i-balanced. The following result due to B. D. Acharya [1] gives a characterization of graphs for which $L(G) \cong N(G)$.

Proposition 2.4 & 2.7 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

Corollary 3.2. For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim A((S_n)^m)$.

Corollary 3.3. For any n-sigraph $S_n = (G, \sigma)$, $\overline{S_n} \sim A((S_n)^m)$.

Problem 3.4. Characterize n-sigraphs for which

i) $(S_n)^m \sim A(S_n)$.

ii) $(\overline{S_n})^m \sim A(S_n)$.

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$(1,2)^* - \pi g\alpha^{**}$-closed sets and $(1,2)^* - \pi g\alpha^{**}$-continuous functions in bitopological spaces

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Abstract In this paper we introduce a new class of sets called $(1,2)^* - \pi g\alpha^{**}$-closed set in bitopological spaces and study some of their properties. Further we discuss and study $(1,2)^* - \pi g\alpha^{**}$-continuous functions in bitopological spaces. Also we investigate $(1,2)^* - \pi g\alpha^{**}$-irresolute functions in bitopological settings.

Keywords $(1,2)^* - \pi g\alpha$-closed sets, $(1,2)^* - \pi g\alpha^{**}$-closed sets, $(1,2)^* - \pi g\alpha^{**}$-continuous fu -nctions, $(1,2)^* - \pi g\alpha^{**}$-irresolute functions.

§1. Introduction and preliminaries

In 1970, Levine [6] introduced the concept of generalized closed sets as a generalization of closed sets in bitopological spaces. Using generalized closed sets, Dunham [3] introduced the concept of the closure operator cl* and a new topology τ^* and studied some of their properties. Further A. Pushpalatha et al [8,4,9] have developed the concept τ^*-generalized closed sets, τ^*-generalized continuous maps and strong forms of τ^*-generalized continuous maps in topological spaces. The author [1] introduced $(1,2)^* - \pi g\alpha$-closed sets in bitopological spaces. The purpose of the present paper is to study a new class of sets called $(1,2)^* - \pi g\alpha^{**}$-closed sets in bitopological spaces. Further we discuss and study $(1,2)^* - \pi g\alpha^{**}$-continuous functions in bitopological spaces. Also we study $(1,2)^* - \pi g\alpha^{**}$-irresolute functions in bitopological settings.

Throughout this paper by a space X we mean it is a bitopological space. We recall the following definitions which are useful in the sequel.

Definition 1.1.[5] A subset space (X, τ_1, τ_2) is said to be $\tau_{1,2}$-open if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$. A subset S of X is said to be

(1) $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open.

(2) $\tau_{1,2}$-clopen if S is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed.

Definition 1.2.[5] Let S be a subset of the bitopological space (X, τ_1, τ_2). Then

(1) The $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int (S) is defined by $\bigcup \{G : G \subseteq S$ and G is $\tau_{1,2}$-open$\}$.

(2) The $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2} - cl(S)$ is defined by $\bigcap \{F : S \subseteq F$ and F is $\tau_{1,2}$-closed$\}$.

Remark 1.3.[5] $\tau_{1,2}$-open sets need not form a topology.

Definition 1.4.[5] A subset A of a bitoplogical space (X, τ_1, τ_2) is called

(1) $(1,2)^*$-regular open if $A = \tau_{1,2}\text{-int}(\tau_{1,2} - cl(A))$.

(2) $(1,2)^* - \alpha$-open if $A \subseteq \tau_{1,2}\text{-int}(\tau_{1,2} - cl(\tau_{1,2} - int(A)))$.

The complement of the sets mentioned above are called their respective closed sets.

Definition 1.5.[1] A subset A of a bitoplogical space (X, τ_1, τ_2) is called $(1,2)^* - \pi g\alpha$-closed if $(1,2)^* - \alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2} - \pi$-open in X. The complement of the respective set is called the respective open set.

The class of all $(1,2)^* - \alpha$-open (resp.$(1,2)^* - \pi g\alpha$-open) subsets of X is denoted by $(1,2)^* - \alpha O(X)$ (resp. $(1,2)^* - \pi G\alpha O(X)$).

Definition 1.6.[2] A function $f : X \to Y$ is called

[1] $(1,2)^* - \pi g\alpha$-continuous if the inverse image of every $\sigma_{1,2}$-closed set in Y is $(1,2)^* - \pi g\alpha$-closed in X.

[2] $(1,2)^* - \pi g\alpha$-irresolute if the inverse image of every $(1,2)^* - \pi g\alpha$-closed in Y is $(1,2)^* - \pi g\alpha$-closed in X.

Definition 1.7.[2] Let S be a subset of the bitopological space (X, τ_1, τ_2). Then

(1) The $(1,2)^* - \alpha-$interior of S, denoted by $(1,2)^* - \alpha - int(S)$ is defined by $\bigcup \{G : G \subseteq S$ and G is $(1,2)^* - \alpha$-open$\}$.

(2) The $(1,2)^* - \alpha-$closure of S, denoted by $(1,2)^* - \alpha - cl(S)$ is defined by $\bigcap \{F : S \subseteq F$ and F is $(1,2)^* - \alpha$-closed$\}$.

Definition 1.8.[2] Let S be a subset of the bitopological space (X, τ_1, τ_2). Then

(1) The $(1,2)^* - \pi g\alpha$-interior of S, denoted by $(1,2)^* - \pi g\alpha$-int (S) is defined by $\bigcup \{G : G \subseteq S$ and G is $(1,2)^* - \pi g\alpha$-open$\}$.

(2) The $(1,2)^* - \pi g\alpha$-closure of S, denoted by $(1,2)^* - \pi g\alpha$-cl(S) is defined by $\bigcap\{F : S \subseteq F$ and F is $(1,2)^* - \pi g\alpha$-closed$\}$.

Definition 1.9.[7] A function $f : X \to Y$ is called strongly $-(1,2)^* - \pi g\alpha-$continuous if the inverse image of every $(1,2)^* - \pi g\alpha-$open set in Y is $\tau_{1,2}$-open set in X.

§2. $(1,2)^* - \pi g\alpha^{**}$-closed sets

Definition 2.1. For the subset A of a bitopological space X, the $(1,2)^* - \pi g\alpha$-closure operator $(1,2)^* - cl^*$ is defined by the intersection of all $(1,2)^* - \pi g\alpha$-closed sets containing A.

Definition 2.2. For the subset A of a bitopological space X, the bitopology $\tau^*_{1,2}$ is defined by $\tau^*_{1,2}$=$\{G : (1,2)^* - cl^*(G^C) = G^C\}$.

Definition 2.3. For the subset A of a bitopological space X, the $(1,2)^* - \alpha$-closure of A (briefly $(1,2)^* - \alpha cl(A)$) is defined by the intersection of all $(1,2)^* - \alpha$-closed sets containing A.

Next we introduce the concept of $(1,2)^* - \pi g\alpha^{**}$-closed sets in bitopological spaces.

Definition 2.4. A subset A of a bitopological space X is said to be $(1,2)^* - \pi g\alpha^{**}$-closed if $(1,2)^* - cl^* \subseteq U$ whenever $A \subseteq U$ and U is $\tau^*_{1,2}$-open in X. The complement of

$(1,2)^* - \pi g\alpha^{**}$-closed set is called the $(1,2)^* - \pi g\alpha^{**}$-open set.

Theorem 2.5. Every $\tau_{1,2}$-closed set in X is $(1,2)^* - \pi g\alpha^{**}$-closed set.

Proof. Let G be a $\tau^*_{1,2}$-open set. Let $A \subseteq G$. Since A is $\tau_{1,2}$-closed, $\tau_{1,2} - cl(A) = A \subseteq G$. But $(1,2)^* - cl^*(A) \subseteq \tau_{1,2} - cl(A)$. Thus, we have $(1,2)^* - cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is $\tau^*_{1,2}$-open. Therefore, A is $(1,2)^* - \pi g\alpha^{**}$-closed set.

Theorem 2.6. Every $\tau^*_{1,2}$-closed set in X is $(1,2)^* - \pi g\alpha^{**}$-closed set.

Proof. Let G be a $\tau^*_{1,2}$-open set and $A \subseteq G$, since A is $\tau^*_{1,2}$-closed, $(1,2)^* - cl^*(A) = A \subseteq G$. Thus we have $(1,2)^* - cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is $\tau^*_{1,2}$-open. Therefore, A is $(1,2)^* - \pi g\alpha^{**}$-closed set.

Remark 2.7. Every $(1,2)^* - \alpha$-closed set in X is $(1,2)^* - \pi g\alpha^{**}$-closed set. But the converse need not be true as shown by the following example.

Example 2.8. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$, $\tau_2 = \{\phi, X, \{a, c\}\}$. Then {a, b, d} is $(1,2)^* - \pi g\alpha^{**}$-closed set, but not $(1,2)^* - \alpha$-closed set in X.

Remark 2.9. From the above results and example, we have the following implications.

$(1,2)^* - \alpha$-closed set $\searrow$ | $\tau_{1,2}$-closed set $\downarrow$ | $\tau^*_{1,2}$-closed set $\swarrow$

$(1,2)^* - \pi g\alpha^{**}$-closed set

Theorem 2.10. For any two sets A and B, $(1,2)^* - cl^*(A \cup B) = (1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$.

Proof. Since $A \subseteq A \cup B$, we have $(1,2)^* - cl^*(A) \subseteq (1,2)^* - cl^*(A \cup B)$ and since $B \subseteq A \cup B$, we have $(1,2)^* - cl^*(B) \subseteq (1,2)^* - cl^*(A \cup B)$. Therefore $(1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B) \subseteq (1,2)^* - cl^*(A \cup B)$. Also, $(1,2)^* - cl^*(A)$ and $(1,2)^* - cl^*(B)$ are $\tau_{1,2}$-closed sets. Therefore, $(1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$ is also $\tau_{1,2}$-closed sets. Again, $A \subseteq (1,2)^* - cl^*(A)$ and $B \subseteq (1,2)^* - cl^*(B)$ implies $A \cup B \subseteq (1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$. Thus, $(1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$ is a $\tau_{1,2}$-closed set containing $A \cup B$. Since $(1,2)^* - cl^*(A \cup B)$ is the smallest $\tau_{1,2}$-closed set containing $A \cup B$ we have $(1,2)^* - cl^*(A \cup B) \subseteq (1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$. Thus, $(1,2)^* - cl^*(A \cup B) = (1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$.

Theorem 2.11. Union of two $(1,2)^* - \pi g\alpha^{**}$-closed sets in X is a $(1,2)^* - \pi g\alpha^{**}$-closed sets in X.

Proof. Let A and B be two $(1,2)^* - \pi g\alpha^{**}$-closed sets. Let $A \cup B \subseteq G$, where G is $\tau^*_{1,2}$-open. Since A and B are $(1,2)^* - \pi g\alpha^{**}$-closed sets, $(1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B) \subseteq G$. But by Theorem 2.10, $(1,2)^* - cl^*(A \cup B) = (1,2)^* - cl^*(A) \cup (1,2)^* - cl^*(B)$. Therefore, $(1,2)^* - cl^*(A \cup B) \subseteq G$. Hence $A \cup B$ is a $(1,2)^* - \pi g\alpha^{**}$-closed set.

Theorem 2.12. A subset A of X is $(1,2)^* - \pi g\alpha^{**}$-closed if and only if $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau^*_{1,2}$-closed in X.

Proof. Let A be a $(1,2)^* - \pi g\alpha^{**}$-closed set. Suppose that F is a non-empty $\tau^*_{1,2}$-closed subset of $(1,2)^* - cl^*(A) - A$. Now $F \subseteq (1,2)^* - cl^*(A) - A$. Then $F \subseteq (1,2)^* - cl^*(A) \cap A^c$, since $(1,2)^* - cl^*(A) - A \subseteq (1,2)^* - cl^*(A) - A \cap A^c$. Therefore $F \subseteq (1,2)^* - cl^*(A)$ and $F \subseteq A^c$. Since F^c is a $\tau^*_{1,2}$-open set and A is a $(1,2)^* - \pi g\alpha^{**}$-closed, $(1,2)^* - cl^*(A) \subseteq F^c$. That is, $F \subseteq ((1,2)^* - cl^*(A))^c$. Hence $F \subseteq (1,2)^* - cl^*(A) \cap ((1,2)^* - cl^*(A))^c = \phi$. That is, $F = \phi$, a contradiction. Thus, $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau^*_{1,2}$-closed in X.

Conversely, assume that $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau^*_{1,2}$-closed set. Let

$A \subseteq G$, G is $\tau^*_{1,2}$-open suppose that $(1,2)^* - cl^*(A)$ is not contained in G, then $(1,2)^* - cl^*(A) \cap G^c$ is a non-empty $\tau^*_{1,2}$-closed set of $(1,2)^* - cl^*(A) - A$ which is a contradiction. Therefore, $(1,2)^* - cl^*(A) \subseteq G$ and hence A is $(1,2)^* - \pi g\alpha^{**}$-closed.

Corollary 2.13. A subset A of X is $(1,2)^* - \pi g\alpha^{**}$-closed if and only if $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau_{1,2}$-closed in X.

Proof. The proof follows from the above theorem and the fact that every $\tau_{1,2}$-closed set is $\tau^*_{1,2}$-closed set in X.

Corollary 2.14. A subset A of $X\alpha^{**}$-closed if and only if $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau_{1,2}$-open in X.

Proof. The proof follows from the above theorem and the fact that every $\tau_{1,2}$-open set is $\tau^*_{1,2}$-open set in X.

Theorem 2.15. A subset A of X is $(1,2)^* - \pi g\alpha^{**}$-closed and $A \subseteq B \subseteq (1,2)^* - cl^*(A)$, then B is $(1,2)^* - \pi g\alpha^{**}$-closed set in X.

Proof. Let A be a $(1,2)^* - \pi g\alpha^{**}$-closed set such that $A \subseteq B \subseteq (1,2)^* - cl^*(A)$. Let U be a $\tau^*_{1,2}$-open set of X such that $B \subseteq U$. Since A is $(1,2)^* - \pi g\alpha^{**}$-closed, we have $(1,2)^* - cl^*(A) \subseteq U$. Now $(1,2)^* - cl^*(A) \subseteq (1,2)^* - cl^*(B) \subseteq (1,2)^* - cl^*((1,2)^* - cl^*(A)) = (1,2)^* - cl^*(A) \subseteq U$. That is, $(1,2)^* - cl^*(B) \subseteq U$, U is $\tau^*_{1,2}$-open. Therefore, B is $(1,2)^* - \pi g\alpha^{**}$-closed set in X.

The converse of the above theorem need not be true as seen from the following example.

Example 2.16. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$. Let $A = \{c\}$ and $B = \{a, c\}$. Then A and B are $(1,2)^* - \pi g\alpha^{**}$-closed sets in X. But $A \subseteq B$ is not a subset of $(1,2)^* - cl^*(A)$.

Example 2.17. Let A be a $(1,2)^* - \pi g\alpha^{**}$-closed in X. Then A is $(1,2)^* - \pi g\alpha$-closed if and only if is $(1,2)^* - cl^*(A) - A$ is $\tau^*_{1,2}$-open.

Proof. Let A be a $(1,2)^* - \pi g\alpha^{**}$-closed set in X. Then $(1,2)^* - cl^*(A) = A$ and so $(1,2)^* - cl^*(A) - A = \phi$. Which is $\tau^*_{1,2}$-open in X. Conversely, suppose $(1,2)^* - cl^*(A) - A$ is $\tau^*_{1,2}$-open in X. Since A is $(1,2)^* - \pi g\alpha^{**}$-closed, by the Theorem 2.12, $(1,2)^* - cl^*(A) - A$ contains no non-empty $\tau^*_{1,2}$-closed set in X. Then $(1,2)^* - cl^*(A) - A = \phi$. Hence A is $(1,2)^* - \pi g\alpha$-closed.

Theorem 2.18. For $x \in X$, the set $X - \{x\}$ is $(1,2)^* - \pi g\alpha^{**}$-closed or $\tau^*_{1,2}$-open.

Proof. Suppose $X - \{x\}$ is not $\tau^*_{1,2}$-open. Then X is the only $\tau^*_{1,2}$-open set containing $X - \{x\}$. This implies $(1,2)^* - cl^*(X - \{x\}) \subseteq X$. Hence $X - \{x\}$ is a $(1,2)^* - \pi g\alpha^{**}$-closed in X.

§3. $(1,2)^* - \pi g\alpha^{**}$-continuous functions

Definition 3.1. A function $f : X \to Y$ is said to be $(1,2)^* - \pi g\alpha^{**}$-continuous if the inverse image of every $\sigma_{1,2}$-closed V of Y is $(1,2)^* - \pi g\alpha^{**}$-closed in X.

Definition 3.2. A space X is called $(1,2)^* - \pi g\alpha^{**} - T_{1/2}$- space if every $(1,2)^* - \pi g\alpha^{**}$-closed set is $(1,2)^* - \alpha$-closed set.

Definition 3.3. Every $(1,2)^* - \pi g\alpha^{**}$-continuous function defined on a $(1,2)^* - \pi g\alpha^{**} - T_{1/2}$- space is every $(1,2)^* - \alpha$-continuous function.

Theorem 3.4. Let $f : X \longrightarrow Y$ be a function then the following statements are equivalent.

[1] f is $(1,2)^* - \pi g\alpha^{**}$-continuous function.

[2] The inverse image of every $\sigma_{1,2}$-open set in Y is also $(1,2)^* - \pi g\alpha^{**}$-open set in X.

Proof. Straight forward.

Remark 3.5. Composition of two $(1,2)^* - \pi g\alpha^{**}$-continuous functions need not be $(1,2)^* - \pi g\alpha^{**}$-continuous function.

Example 3.6. Let $X = \{a,b,c,d\}, Y = Z = \{a,b,c\}$ and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be the identity maps. τ_1=$\{\phi, X, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$, τ_2=$\{\phi, X, \{a,c\}\}$, σ_1=$\{\phi, X, \{a\}\}$, σ_2=$\{\phi, X, \{a,b\}\}$, η_1=$\{\phi, X, \{c\}\}$, η_2=$\{\phi, X, \{b\}\}$. Then clearly, f and g are $(1,2)^* - \pi g\alpha^{**}$-continuous, but gof is not $(1,2)^* - \pi g\alpha^{**}$-continuous. Since $(gof)^{-1}(\{a\}, \{a,c\}) = \{a\}, \{a,c\}$ are not $(1,2)^* - \pi g\alpha^{**}$-closed set in X.

Theorem 3.7. If a function $f : X \longrightarrow Y$ is $(1,2)^*$-continuous then it is $(1,2)^* - \pi g\alpha^{**}$-continuous but not conversely.

Proof. Let $f : X \longrightarrow Y$ be $(1,2)^*$-continuous. Let V be a $\sigma_{1,2}$-closed set in Y. Since f is $(1,2)^*$-continuous, $f^{-1}(V)$ is $\tau_{1,2}$-closed in X. Since every $\tau_{1,2}$-closed set is $(1,2)^* - \pi g\alpha^{**}$-closed set in X. Thus, f is $(1,2)^* - \pi g\alpha^{**}$-continuous function.

The converse of the theorem need not be true as seen from the following example.

Example 3.8. Let $X = Y = \{a,b,c,d\}$ and $f : X \longrightarrow Y$ be the indentity map. τ_1=$\{\phi, X, \{a,b,d\}\}$, τ_2=$\{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, σ_1=$\{\phi, Y, \{a\}, \{a,b,c\}\}$, σ_2=$\{\phi, X, \{a,b,d\}\}$. Then f is $(1,2)^* - \pi g\alpha^{**}$-continuous, but not $(1,2)^*$-continuous. Since for the $\sigma_{1,2}$-closed set {d} in Y, $f^{-1}(\{d\}) = \{d\}$ is not $\tau_{1,2}$-closed set in X.

Theorem 3.9. If a function $f : X \to Y$ is strongly $-(1,2)^* - \pi g\alpha-$continuous then it is $(1,2)^* - \pi g\alpha^{**}$-continuous but not conversely.

Proof. Let $f : X \to Y$ be strongly $(1,2)^*$-continuous. Let F be a $\sigma_{1,2}$-closed set in Y. Then F^C is $(1,2)^* - \pi g\alpha-$open in Y, since f is strongly$-(1,2)^* - \pi g\alpha-$continuous, $f^{-1}(F^C)$ is $\tau_{1,2}$-open in X. But $f^{-1}(F^C) = X - f^{-1}(F)$. Therefore $f^{-1}(F)$ is $\tau_{1,2}$-closed in X. By Theorem 2.5, $f^{-1}(F)$ is $(1,2)^* - \pi g\alpha^{**}$-closed in X. Thus, f is $(1,2)^* - \pi g\alpha^{**}$-continuous function.

The converse of the theorem need not be true as seen from the following example.

Example 3.10. In Example 3.8, f is $(1,2)^* - \pi g\alpha^{**}$-continuous, but not strongly$-(1,2)^* - \pi g\alpha-$continuous. Since for the $(1,2)^* - \pi g\alpha-$open set $\{a,b,c\}$ in Y, $f^{-1}(\{a,b,c\}) = \{a,b,c\}$ is not $\tau_{1,2}-$open set in X.

Next we introduce a new class of functions called $(1,2)^* - \pi g\alpha^{**}$-irresolute functions which is introduced in the class of $(1,2)^* - \pi g\alpha^{**}$-continuous functions. We investigate some basic properties also.

Definition 3.11. A function $f : X \to Y$ is said to be $(1,2)^* - \pi g\alpha^{**}$-irresolute if the inverse image of every $(1,2)^* - \pi g\alpha^{**}$-closed set V of Y is $(1,2)^* - \pi g\alpha^{**}$-closed set in X.

Theorem 3.12. A function $f : X \to Y$ is $(1,2)^* - \pi g\alpha^{**}$-irresolute if and only if the inverse image of every $(1,2)^* - \pi g\alpha^{**}$-open set in Y is $(1,2)^* - \pi g\alpha^{**}$-open in X.

Proof. Assume that f is $(1,2)^* - \pi g\alpha^{**}$-irresolute function. Let A be any $(1,2)^* - \pi g\alpha^{**}$-open set in Y. Then A^C is $(1,2)^* - \pi g\alpha^{**}$-closed in Y. Since f is $(1,2)^* - \pi g\alpha^{**}$-irresolute, $f^{-1}(A^c)$ is $(1,2)^* - \pi g\alpha^{**}$-closed in X. But $f^{-1}(A^c) = X - f^{-1}(A)$ and so $f^{-1}(A)$ is $(1,2)^* - \pi g\alpha^{**}$-open in X. Hence the inverse image of every $(1,2)^* - \pi g\alpha^{**}$-open set in Y is

$(1,2)^* - \pi g\alpha^{**}$-open in X.

Conversely, assume that the inverse image of every $(1,2)^* - \pi g\alpha^{**}$-open set in Y is $(1,2)^* - \pi g\alpha^{**}$-open in X. Let A be any $(1,2)^* - \pi g\alpha^{**}$-closed in Y. Then A^C is $(1,2)^* - \pi g\alpha^{**}$-open in Y. By assumption, $f^{-1}(A^c)$ is $(1,2)^* - \pi g\alpha^{**}$-open in X. But $f^{-1}(A^c) = X - f^{-1}(A)$ and so $f^{-1}(A)$ is $(1,2)^* - \pi g\alpha^{**}$-closed in X. Therefore, f is $(1,2)^* - \pi g\alpha^{**}$-irresolute function.

Theorem 3.13. If a function f is $(1,2)^* - \pi g\alpha^{**}$-irresolute then $(1,2)^* - \pi g\alpha^{**}$-continuous.

Proof. Assume that f is $(1,2)^* - \pi g\alpha^{**}$-irresolute. Let F be any $\sigma_{1,2}$-closed set in Y. By Theorem 2.5, F is $(1,2)^* - \pi g\alpha^{**}$-closed set in Y. Since f is $(1,2)^* - \pi g\alpha^{**}$-irresolute, $f^{-1}(F)$ is $(1,2)^* - \pi g\alpha^{**}$-closed in X. Therefore f is $(1,2)^* - \pi g\alpha^{**}$-continuous.

Converse of the theorem need not be true as seen from the following example.

Example 3.14. Let $X = Y = \{a, b, c, d\}$, let $f : X \longrightarrow Y$ be the indentity map. τ_1=$\{\phi, X, \{a\}, \{a, b, d\}\}$, τ_2=$\{\phi, X, \{b\}\}$, σ_1=$\{\phi, Y, \{a\}, \{a, b, c\}\}$, σ_2=$\{\phi, X, \{a, b, d\}\}$. Then f is $(1,2)^* - \pi g\alpha^{**}$-continuous, but not $(1,2)^* - \pi g\alpha^{**}$-irresolute. Since for the $\{a\}, \{b\}, \{a, b\}$ are $(1,2)^* - \pi g\alpha^{**}$-closed set in Y but not $(1,2)^* - \pi g\alpha^{**}$-closed set in X.

Theorem 3.15. Let X, Y and Z be any bitopological spaces. For any $(1,2)^* - \pi g\alpha^{**}$-irresolute function $f : X \to Y$ and any $(1,2)^* - \pi g\alpha^{**}$ -continuous function $g : Y \to Z$, the composition $gof : X \to Z$ is $(1,2)^* - \pi g\alpha^{**}$-continuous.

Proof. Let F be any $\eta_{1,2}$-closed set in Z. Since g is $(1,2)^* - \pi g\alpha^{**}$-continuous, $g^{-1}(F)$ is $(1,2)^* - \pi g\alpha^{**}$-closed in Y. Since f is $(1,2)^* - \pi g\alpha^{**}$- irresolute, $f^{-1}(g^{-1}(F))$ is $(1,2)^* - \pi g\alpha^{**}$-closed in X. But $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$. Therefore g o f is $(1,2)^* - \pi g\alpha^{**}$-continuous.

Theorem 3.16. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then

[1] g o f is $(1,2)^* - \pi g\alpha^{**}$-continuous if g is $(1,2)^*$-continuous and f is $(1,2)^* - \pi g\alpha^{**}$-continuous.

[2] g o f is $(1,2)^* - \pi g\alpha^{**}$-continuous if g is $(1,2)^* - \pi g\alpha^{**}$- continuous and f is $(1,2)^* - \pi g\alpha^{**}$-irresolute.

Proof. Follows from definitions.

Definition 3.17. A function $f : X \to Y$ is said to be pre-$(1,2)^* - \pi g\alpha^{**}$-continuous if the inverse image of every $(1,2)^* - \alpha$-closed V of Y is $(1,2)^* - \pi g\alpha^{**}$-closed in X.

Theorem 3.18. For a function $f : X \to Y$, the following implications hold. $(1,2)^* - \pi g\alpha^{**}$-irresolute $\to$ pre-$(1,2)^* - \pi g\alpha^{**}$-continuous $\to$ $(1,2)^* - \pi g\alpha^{**}$-continuous.

However, the converse of the above are not always true as the following example shows.

Example 3.19. Let $X = Y = \{a, b, c, d\}$, let $f : X \longrightarrow Y$ be the indentity map. τ_1=$\{\phi, X, \{a, c\}, \{c, d\}, \{a, c, d\}\}$, τ_2=$\{\phi, X, \{a\}, \{d\}, \{a, d\}\}$, σ_1=$\{\phi, Y, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\}$, σ_2=$\{\phi, X, \{a, b, d\}\}$. Then f is $(1,2)^* - \pi g\alpha^{**}$-continuous, but not pre-$(1,2)^* - \pi g\alpha^{**}$-continuous. Since $f^{-1}(\{d\}) = \{d\}$ is not $(1,2)^* - \pi g\alpha^{**}$-closed in X.

Example 3.20. Let $X = Y = \{a, b, c, d\}$, τ_1=$\{\phi, X, \{a, c\}, \{c, d\}, \{a, c, d\}\}$, τ_2=$\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, σ_1=$\{\phi, Y, \{b\}, \{a, b\}\}$, σ_2=$\{\phi, X, \{a\}, \{a, b, d\}\}$. Define a map $f : X \longrightarrow Y$ by $f(a) = b, f(b) = c, f(c) = d, f(d) = a$. Then f is pre-$(1,2)^* - \pi g\alpha^{**}$-continuous, but not $(1,2)^* - \pi g\alpha^{**}$-irresolute. Since Since $f^{-1}(\{a, d\}, \{b, d\}, \{a, b, d\}) = \{c, d\}, \{a, c\}, \{a, c, d\}$ are not $(1,2)^* - \pi g\alpha^{**}$-closed in X.

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Slightly $gpr-$continuous functions

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Abstract In this paper we discuss a new type of continuous functions called slightly $gpr-$continuous functions; its properties and interrelation with other continuous functions are studied.

Keywords Slightly continuous functions, slightly semi-continuous functions, slightly $\beta-$con -tinuous functions, slightly $\gamma-$continuous functions and slightly $\nu-$continuous functions.
AMS-classification Numbers: 54C10; 54C08; 54C05.

§1. Introduction

T. M. Nour introduced slightly semi-continuous functions during the year 1995. After him T. Noiri and G. I. Ghae further studied slightly semi-continuous functions on 2000. During 2001 T. Noiri individually studied slightly $\beta-$continuous functions. C. W. Baker introduced slightly precontinuous functions. Erdal Ekici and M. Caldas studied slightly $\gamma-$continuous functions. Arse Nagli Uresin and others studied slightly $\delta-$continuous functions. Recently S. Balasubramanian and P. A. S. Vyjayanth studied slightly $\nu-$continuous functions. Inspired with these developements we introduce in this paper a new vairety of slightly continuous functions called slightly $gpr-$continuous function and study its basic properties; interrelation with other type of such functions available in the literature. Throughout the paper a space X means a topological space (X, τ).

§2. Preliminaries

Definition 2.1. $A \subset X$ is called

(i) r-closed if $A = \overline{A^0}$.

(ii) g-closed [resp: rg-closed] if $\overline{A} \subseteq U$, whenever $A \subseteq U$ and U is open in X.

(iii) gp-closed [resp: gpr-closed] if $p(\overline{A}) \subseteq U$, whenever $A \subseteq U$ and U is open [resp: regular-open] in X.

(iv) αg-closed if $\alpha(\overline{A}) \subseteq U$, whenever $A \subseteq U$ and U is open in X.

Note 1. We have the following interrelation among different closed sets.

closed $\rightarrow$ g-closed $\rightarrow$ αg-closed $\rightarrow$ gp-closed $\rightarrow$ gpr-closed

$\nwarrow$ $\nearrow$

regular-closed $\longrightarrow\longrightarrow$ rg-closed

none is reversible

Definition 2.2. A function $f\colon X \rightarrow Y$ is said to be

(i) nearly-continuous if inverse image of each open set is regular-open.

(ii) nearly-irresolute if inverse image of each regular-open set is regular-open.

(iii) weakly continuous [resp: weakly nearly-continuous; weakly pre-continuous] if for each $x \in X$ and each open set $(V, f(x))$, there exists an open[resp: regular-open; preopen] set $(U, x) \ni f(U) \subset \overline{V}$.

(iv) slightly continuous [resp: slightly semi-continuous; slightly pre-continuous; slightly $\beta-$continuous; slightly $\gamma-$continuous; slightly $\alpha-$continuous; slightly r-continuous; slightly $\nu-$continuous; slightly g-continuous; slightly sg-continuous; slightly pg-continuous; slightly $\beta g-$continuous; slightly $\gamma g-$continuous; slightly $\alpha g-$continuous; slightly rg-continuous; slightly $\nu g-$continuous] at $x \in X$ if for each clopen subset V in Y containing $f(x), \exists U \in \tau(X)[\exists U \in SO(X); \exists U \in PO(X); \exists U \in \beta O(X); \exists U \in \gamma O(X); \exists U \in \alpha O(X); \exists U \in RO(X); \exists U \in \nu O(X); \exists U \in GO(X); \exists U \in SGO(X); \exists U \in PGO(X); \exists U \in \beta GO(X); \exists U \in \gamma GO(X); \exists U \in \alpha GO(X); \exists U \in RGO(X); \exists U \in \nu GO(X);]$ containing x such that $f(U) \subseteq V$.

(v) slightly continuous [resp: slightly semi-continuous; slightly pre-continuous; slightly $\beta-$continuous; slightly $\gamma-$continuous; slightly $\alpha-$continuous; slightly r-continuous; slightly $\nu-$continuous; slightly g-continuous; slightly sg-continuous; slightly pg-continuous; slightly $\beta g-$continuous; slightly $\gamma g-$continuous; slightly $\alpha g-$continuous; slightly rg-continuous; slightly $\nu g-$continuous] if it is slightly-continuous [resp:slightly semi-continuous; slightly pre-continuous; slightly $\beta-$continuous; slightly $\gamma-$continuous; slightly $\alpha-$continuous; slightly r-continuous; slightly $\nu-$continuous; slightly g-continuous; slightly sg-continuous; slightly pg-continuous; slightly $\beta g-$continuous; slightly $\gamma g-$continuous; slightly $\alpha g-$continuous; slightly rg-continuous; slightly $\nu g-$continuous] at each $x \in X$.

Definition 2.3. X is said to be a

(i) compact [resp: nearly-compact; pre-compact; mildly-compact] space if every open [resp: regular-open; preopen; clopen] cover has a finite subcover.

(ii) countably-compact [resp: countably-nearly-compact; countably-pre-compact; mildly-countably compact] space if every countable open [resp: regular-open; preopen; clopen] cover has a finite subcover.

(iii) closed-compact [resp: closed-nearly-compact; closed-pre-compact] space if every closed [resp: regular-closed; preclosed] cover has a finite subcover.

(iv) Lindeloff [resp: nearly-Lindeloff; pre-Lindeloff; mildly-Lindeloff] space if every open [resp: regular-open; preopen; clopen] cover has a countable subcover.

(v) Extremally pre-disconnected [briefly e.p.d] if the preclosure of each preopen set is preopen.

Definition 2.4. X is said to be a

(i) Ultra T_0 space if for each $x \neq y \in X \exists U \in CO(X)$ containing either x or y.

(ii) Ultra T_1 space if for each $x \neq y \in X \exists U, V \in CO(X)$ such that $x \in U - V$ and $y \in V - U$.

(iii) Ultra T_2 space if for each $x \neq y \in X \exists U, V \in CO(X)$ such that $x \in U$; $y \in V$ and $U \cap V = \phi$.

Note 2. CO(U, x) represents U is a clopen set containing x. Similarly we can write GPRO(X, x).

§3. Slightly *gpr*−continuous functions

Definition 3.1. A function $f\colon X \to Y$ is said to be

(i) slightly *gpr*−continuous at $x \in X$ if for each $V \in CO(Y, f(x)), \exists U \in GPRO(X, x) \ni f(U) \subseteq V$.

(ii) slightly *gpr*−continuous function if it is slightly *gpr*−continuous at each $x \in X$.

Note 3. Here after we call slightly *gpr*−continuous function as *sl.gpr.c* function shortly.

Example 3.1. $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$.

(i) Let f defined by $f(a) = c; f(b) = a$; and $f(c) = b$; is *sl.gpr.c.*

(ii) Let f is identity function, then f is not *sl.gpr.c.*

Example 3.2. $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let f is identity function, then f is sl.gpr.c.

Theorem 3.1. The following are equivalent.

(i) f is sl.gpr.c.

(ii) $f^{-1}(V)$ is *gpr*−open for every clopen set V in Y.

(iii) $f^{-1}(V)$ is *gpr*−closed for every clopen set V in Y.

(iv) $f(gpr\overline{(A)}) \subseteq gpr\overline{(f(A))}$.

Corollary 3.1. The following are equivalent.

(i) f is sl.gpr.c.

(ii) For each $x \in X$ and each clopen subset $V \in (Y, f(x)) \exists U \in GPRO(X, x) \ni f(U) \subseteq V$.

Theorem 3.2. Let $\Sigma = \{U_i : i \in I\}$ be any cover of X by regular open sets in X. A function f is sl.gpr.c. iff $f_{/U_i}$: is sl.gpr.c., for each $i \in I$.

Proof. Let $i \in I$ be an arbitrarily fixed index and $U_i \in RO(X)$. Let $x \in U_i$ and $V \in CO(Y, f_{U_i}(x))$. Since f is sl.gpr.c., $\exists U \in GPRO(X, x) \ni f(U) \subset V$. Since $U_i \in RO(X)$, by Lemma 2.1 $x \in U \cap U_i \in GPRO(U_i)$ and $(f_{/U_i})U \cap U_i = f(U \cap U_i) \subset f(U) \subset V$. Hence $f_{/U_i}$ is sl.gpr.c.

Conversely Let $x \in X$ and $V \in CO(Y, f(x))$, $\exists i \in I \ni x \in U_i$. Since $f_{/U_i}$ is sl.gpr.c, $\exists U \in GPRO(U_i, x) \ni f_{/U_i}(U) \subset V$. By Lemma 2.1, $U \in GPRO(X)$ and $f(U) \subset V$. Hence f is sl.gpr.c.

Theorem 3.3. (i) If f is *gpr*−irresolute and g is sl.gpr.c.[sl.c.], then $g \circ f$ is sl.gpr.c.

(i) If f is *gpr*−irresolute and g is *gpr*.c., then $g \circ f$ is sl.gpr.c.

(iii) If f is *gpr*−continuous and g is sl.c., then $g \circ f$ is sl.gpr.c.

(iv) If f is rg-continuous and g is sl.gpr.c. [sl.c.], then $g \circ f$ is sl.gpr.c.

Theorem 3.4. If f is *gpr*−irresolute, *gpr*−open and $GPRO(X) = \tau$ and g be any function, then $g \circ f\colon X \to Z$ is sl.gpr.c. iff g is sl.gpr.c.

Proof. If part: Theorem 3.3 (i). Only if part: Let A be clopen subset of Z. Then $(g \circ f)^{-1}(A)$ is a $gpr-$open subset of Xand hence open in X [by assumption]. Since f is $gpr-$open $f(g \circ f)^{-1}(A)$ is $gpr-$open $Y \Rightarrow g^{-1}(A)$ is $gpr-$open in Y. Thus $g : Y \to Z$ is sl.gpr.c.

Corollary 3.2. If f is $gpr-$irresolute, $gpr-$open and bijective, g is a function. Then g is sl.gpr.c. iff $g \circ f$ is sl.gpr.c.

Theorem 3.5. If $g : X \to X \times Y$, defined by $g(x) = (x, f(x)) \forall x \in X$ be the graph function of $f : X \to Y$. Then g is *sl.gpr.c* iff f is *sl.gpr.c*.

Proof. Let $V \in CO(Y)$, then $X \times V$ is clopen in $X \times Y$. Since g is sl.gpr.c., $f^{-1}(V) = f^{-1}(X \times V) \in GPRO(X)$. Thus f is sl.gpr.c.

Conversely, let $x \in X$ and F be a clopen subset of $X \times Y$ containing g(x). Then $F \cap (\{x\} \times Y)$ is clopen in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in Y : (x, y) \in F\}$ is clopen subset of Y. Since f is sl.gpr.c. $\bigcup\{f^{-1}(y) : (x, y) \in F\}$ is $gpr-$open in X. Further $x \in \bigcup\{f^{-1}(y) : (x, y) \in F\} \subseteq g^{-1}(F)$. Hence $g^{-1}(F)$ is $gpr-$open. Thus g is sl.gpr.c.

Theorem 3.6. (i) If $f : X \to \Pi Y_\lambda$ is *sl.gpr.c*, then $P_\lambda \circ f : X \to Y_\lambda$ is *sl.gpr.c* for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.

(ii) $f : \Pi X_\lambda \to \Pi Y_\lambda$ is *sl.gpr.c*, iff $f_\lambda : X_\lambda \to Y_\lambda$ is *sl.gpr.c* for each $\lambda \in \Lambda$.

Remark 1. (i) Composition of two *sl.gpr.c* functions is not in general *sl.gpr.c*.

(ii) Algebraic sum and product of *sl.gpr.c* functions is not in general *sl.gpr.c*.

(iii) The pointwise limit of a sequence of *sl.gpr.c* functions is not in general *sl.gpr.c*.

Example 3.3. Let $X = Y = [0, 1]$. Let $f_n : X \to Y$ is defined as follows $f_n(x) = x_n$ for $n = 1, 2, 3, \ldots$, then f defined by $f(x) = 0$ if $0 \leq x < 1$ and $f(x) = 1$ if $x = 1$. Therefore each f_n is *sl.gpr.c* but f is not *sl.gpr.c*. For $(\frac{1}{2}, 1]$ is clopen in Y, but $f^{-1}((\frac{1}{2}, 1]) = \{1\}$ is not $gpr-$open in X.

However we can prove the following:

Theorem 3.7. The uniform limit of a sequence of *sl.gpr.c* functions is *sl.gpr.c*.

Note 4. Pasting lemma is not true for *sl.gpr.c* functions. However we have the following weaker versions.

Theorem 3.8. Let X and Y be topological spaces such that $X = A \cup B$ and let $f_{/A} : A \to Y$ and $g_{/B} : B \to Y$ are sl.r.c maps such that f(x) $= g$(x) $\forall x \in A \cap B$. Suppose A and B are r-open sets in X and RO(X) is closed under finite unions, then the combination $\alpha : X \to Y$ is *sl.gpr.c* continuous.

Theorem 3.9. Pasting lemma Let X and Y be spaces such that $X = A \cup B$ and let $f_{/A} : A \to Y$ and $g_{/B} : B \to Y$ are *sl.gpr.c* maps such that $f(x) = g(x)$ $\forall x \in A \cap B$. Suppose A, B are r-open sets in X and $GPRO(X)$ is closed under finite unions, then the combination $\alpha : X \to Y$ is *sl.gpr.c*.

Proof. Let $F \in CO(Y)$, then $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$, where $f^{-1}(F) \in GPRO(A)$ and $g^{-1}(F) \in GPRO(B) \Rightarrow f^{-1}(F); g^{-1}(F) \in GPRO(X) \Rightarrow f^{-1}(F) \cup g^{-1}(F) \in GPRO(X)$ [by assumption]. Therefore $\alpha^{-1}(F) \in GPRO(X)$. Hence $\alpha : X \to Y$ is *sl.gpr.c*.

Theorem 3.10. (i) If f is sl.rg.c, then f is sl.gpr.c.

(ii) If f is sl.gp.c, then f is sl.gpr.c.

(iii) If f is sl.αg.c, then f is sl.gpr.c.

(iv) If f is sl.g.c, then f is sl.gpr.c.

(v) If f is sl.c, then f is sl.gpr.c.

(vi) If f is sl.r.c, then f is sl.gpr.c.

Note 5. From Definition 3.1 and Note 1 we have the following implication diagram for closed maps.

closed map → g-closed map → αg-closed map → gp-closed map → gpr-closed map

↖ ↗

regular-closed map ⟶ ⟶ rg-closed map

none is reversible

Note 6. Converse implication is true if $GPRO(X) = RO(X)$.

Theorem 3.11. If f is sl.gpr.c., from a discrete space X into a e.p.d space Y, then f is w.p.c.

Example 3.4. In Example 3.1(ii) above f is *sl.ν.g.c*; sl.sg.c; sl.gs.c; sl.rα.c; sl.ν.c; sl.s.c. and sl.β.c; but not sl.g.c; sl.rg.c; sl.gr.c; sl.pg.c; sl.gp.c; sl.gpr.c; sl.gα.c; sl.αg.c; sl.rgα.c; sl.r.c; sl.p.c; sl.α.c; and sl.c;

Example 3.5. In Example 3.2 above f is sl.rα.c; and sl.gpr.c; but not *sl.ν.g.c*; sl.sg.c; sl.gs.c; sl.ν.c; sl.s.c; sl.β.c; sl.g.c; sl.rg.c; sl.gr.c; sl.pg.c; sl.gp.c; sl.gα.c; sl.αg.c; sl.rgα.c; sl.r.c; sl.p.c; sl.α.c; and sl.c.

§4. Covering and separation properties of sl.gpr.c. functions

Theorem 4.1. If f is sl.gpr.c.[resp: sl.rg.c] surjection and X is $gpr-$compact, then Y is compact.

Proof. Let $\{G_i : i \in I\}$ be any open cover for Y. Then each G_i is open in Y and hence each G_i is clopen in Y. Since f is sl.gpr.c., $f^{-1}(G_i)$ is $gpr-$open in X. Thus $\{f^{-1}(G_i)\}$ forms a $gpr-$open cover for X and hence have a finite subcover, since X is $gpr-$compact. Since f is surjection, $Y = f(X) = \bigcup_{i=1}^{n} G_i$. Therefore Y is compact.

Corollary 4.1. If f is sl.ν.c.[resp: sl.r.c] surjection and X is $gpr-$compact, then Y is compact.

Theorem 4.2. If f is sl.gpr.c., surjection and X is $gpr-$compact [$gpr-$lindeloff] then Y is mildly compact [mildly lindeloff].

Proof. Let $\{U_i : i \in I\}$ be clopen cover for Y. For each $x \in X, \exists \alpha_x \in I \ni f(x) \in U_{\alpha_x}$ and $\exists V_x \in GPRO(X, x) \ni f(V_x) \subset U_{\alpha_x}$. Since the family $\{V_i : i \in I\}$ is a cover of X by $gpr-$open sets of X, there exists a finite subset I_0 of $I \ni X \subset \{V_x : x \in I_0\}$. Therefore $Y \subset \bigcup\{f(V_x) : x \in I_0\} \subset \bigcup\{U_{\alpha_x} : x \in I_0\}$. Hence Y is mildly compact.

Corollary 4.2. (i) If f is sl.rg.c [resp: sl.g.c.; sl.r.c] surjection and X is $gpr-$compact [$gpr-$lindeloff] then Y is mildly compact [mildly lindeloff].

(ii) If f is sl.gpr.c.[resp: sl.rg.c; sl.g.c.; sl.r.c] surjection and X is locally $gpr-$compact{resp:g $pr-$Lindeloff; locally $gpr-$lindeloff}, then Y is locally compact{resp: Lindeloff; locally lindeloff}.

(iii) If f is sl.gpr.c., surjection and X is semi-compact [semi-lindeloff] then Y is mildly compact [mildly lindeloff].

(iv) If f is sl.gpr.c., surjection and X is $\beta-$compact [$\beta-$lindeloff] then Y is mildly compact [mildly lindeloff].

(v) If f is sl.gpr.c.[sl.r.c.], surjection and X is locally $gpr-$compact{resp: $gpr-$lindeloff; locally $gpr-$lindeloff} then Y is locally mildly compact{resp: locally mildly lindeloff}.

Theorem 4.3. If f is sl.gpr.c., surjection and X is s-closed then Y is mildly compact [mildly lindeloff].

Proof. Let $\{V_i : V_i \in CO(Y); i \in I\}$ be a cover of Y, then $\{f^{-1}(V_i) : i \in I\}$ is $gpr-$open cover of X [by Thm 3.1] and so there is finite subset I_0 of I, such that $\{f^{-1}(V_i) : i \in I_0\}$ covers X. Therefore $\{(V_i) : i \in I_0\}$ covers Y since f is surjection. Hence Y is mildly compact.

Corollary 4.3. If f is sl.rg.c [resp: sl.g.c.; sl.r.c.] surjection and X is s-closed then Y is mildly compact [mildly lindeloff].

Theorem 4.3. If f is sl.gpr.c.,[resp: sl.rg.c.; sl.g.c.; sl.r.c.] surjection and X is $gpr-$connecte -d, then Y is connected.

Proof. If Y is disconnected, then $Y = A \cup B$ where A and B are disjoint clopen sets in Y. Since f is sl.gpr.c. surjection, $X = f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ $f^{-1}(B)$ are disjoint $gpr-$open sets in X, which is a contradiction for X is $gpr-$connected. Hence Y is connected.

Corollary 4.4. The inverse image of a disconnected space under a sl.gpr.c.,[resp: sl.rg.c.; sl.g.c.; sl.r.c.] surjection is $gpr-$disconnected.

Theorem 4.4. If f is sl.gpr.c.[resp: sl.rg.c.; sl.g.c.], injection and Y is UT_i, then X is gpr_i, $i = 0, 1, 2$.

Proof. Let $x_1 \neq x_2 \in X$. Then $f(x_1) \neq f(x_2) \in Y$ since f is injective. For Y is $UT_2 \exists V_j \in CO(Y) \ni f(x_j) \in V_j$ and $\cap V_j = \phi$ for $j = 1, 2$. By Theorem 3.1, $x_j \in f^{-1}(V_j) \in GPRO(X)$ for $j = 1, 2$ and $\cap f^{-1}(V_j) = \phi$ for $j = 1, 2$. Thus X is $\nu - g_2$.

Theorem 4.5. If f is sl.gpr.c.[resp: sl.rg.c.; sl.g.c.], injection; closed and Y is UT_i, then X is $gprg_i$, $i = 3, 4$.

Proof. (i) Let $x \in X$ and F be disjoint closed subset of X not containing x, then $f(x)$ and $f(F)$ are disjoint closed subset of Y, since f is closed and injection. Since Y is ultraregular, $f(x)$ and $f(F)$ are separated by disjoint clopen sets U and V respectively. Hence $x \in f^{-1}(U)$; $F \subseteq f^{-1}(V)$, $f^{-1}(U); f^{-1}(V) \in GPRO(X)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Thus X is $gprg_3$.

(ii) Let F_j and $f(F_j)$ are disjoint closed subsets of X and Y respectively for $j = 1, 2$, since f is closed and injection. For Y is ultranormal, $f(F_j)$ are separated by disjoint clopen sets V_j respectively for $j = 1, 2$. Hence $F_j \subseteq f^{-1}(V_j)$ and $f^{-1}(V_j) \in GPRO(X)$ and $\cap f^{-1}(V_j) = \phi$ for $j = 1, 2$. Thus X is $gprg_4$.

Theorem 4.6. If f is sl.gpr.c.[resp: sl.rg.c.; sl.g.c.], injection and

(i) Y is UC_i[resp: UD_i] then X is $gprC_i$[resp: $gprD_i$], $i = 0, 1, 2$.

(ii) Y is UR_i, then X is $gpr - R_i$, $i = 0, 1$.

Theorem 4.7. If f is sl.gpr.c.[resp: sl.rg.c.; sl.g.c; sl.r.c] and Y is UT_2, then the graph $G(f)$ of f is $gpr-$closed in the product space $X \times Y$.

Proof. Let $(x_1, x_2) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint clopen sets V and $W \ni f(x) \in V$ and y$\in W$. Since f is sl.gpr.c., $\exists U \in GPRO(X) \ni x \in U$ and $f(U) \subset W$. Therefore $(x, y) \in$

$U \times V \subset X \times Y - G(f)$. Hence G$(f)$ is $gpr-$closed in $X \times Y$.

Theorem 4.8. If f is sl.gpr.c.[resp: sl.rg.c.; sl.g.c; sl.r.c] and Y is UT_2, then $A = \{(x_1, x_2)|f(x_1) = f(x_2)\}$ is $gpr-$closed in the product space $X \times X$.

Proof. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_j \in CO(Y) \ni f(x_j) \in V_j$, and since f is sl.gpr.c., $f^{-1}(V_j) \in GPRO(X, x_j)$ for each $j = 1, 2$. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in GPRO(X \times X)$ and $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A$. Hence A is $gpr-$closed.

Theorem 4.9. If f is sl.r.c.[resp: sl.c.]; g is sl.gpr.c.[resp: sl.rg.c; sl.g.c]; and Y is UT_2, then $E = \{x \in X : f(x) = g(x)\}$ is $gpr-$closed in X.

Conclusion

In this paper we defined slightly-$gpr-$continuous functions, studied its properties and their interrelations with other types of slightly-continuous functions.

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Results on majority dominating sets

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Abstract This paper deals with majority dominating sets in a graph G and majority domination number $\gamma_M(G)$. The characterisation of minimal majority dominating set and some results for $\gamma_M(G)$ are also established.

Keywords Majority dominating set, majority domination number $\gamma_M(G)$, minimal majority dominating set.
AMS Mathematics Subject Classification: 05C69.

§1. Introduction

In decision making process sometimes domination concept may not be necessary to have the consent of all but a majority opinion will do. In any democratic set up, the party which has majority of seats is given the opportunity to rule the state. All these lead to the concept of Majority Domination.

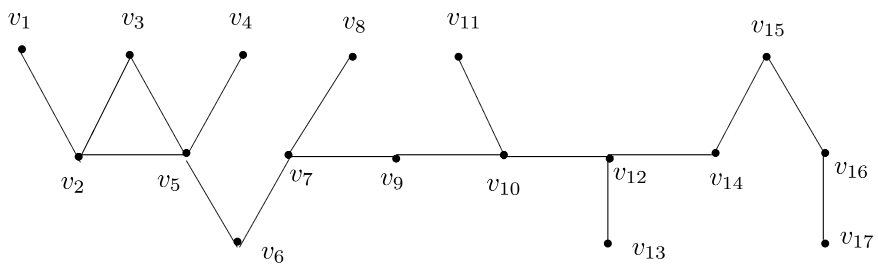
By a graph $G = (V, E)$ [2], we mean a finite undirected graph without loops or multiple edges. The open neighborhood of v is defined to be the set of vertices adjacent to v in G, and is denoted as $N(v)$. Further, the closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. The closed neighborhood of a set of vertices S is denoted as $N[s] = \cup_{s\in S} N[s]$.

Definition 1.1.[4] A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating Set if at least half of the vertices of V are either in S or adjacent to elements of S. (i. e.) $|N[S]| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$.

A majority dominating set S is minimal if no proper subset of S is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the majority domination number and is denoted by $\gamma_M(G)$. The maximum cardinality of a minimal majority dominating set is denoted by $\Gamma_M(G)$. Also, Majority dominating sets have super hereditary property.

Example 1.1.

$G:$



In this graph G, the following are some minimal majority dominating sets.

$D_1 = \{v_5, v_{12}\}$, $D_2 = \{v_2, v_7, v_9\}$, $D_3 = \{v_1, v_{10}, v_{15}\}$, $D_4 = \{v_1, v_8, v_{11}, v_{16}\}$ and $D_5 = \{v_1, v_8, v_{11}, v_{13}, v_{17}\}$ with cardinality $|D_1| = 2$, $|D_2| = 3 = |D_3|$, $|D_4| = 4$ and $|D_5| = 5$. $\gamma_M(G) = 2$ and $\Gamma_M(G) = 5$.

Majority Domination Number of Graphs

Proposition 1.1.[4] Let $G = K_p, K_{m,n}, m \leq n, K_{1,p-1}, D_{r,s}, F_p$ and W_p. Then $\gamma_M(G) = 1$.

Proposition 1.2.[4] Let $G = C_p$, $p \geq 3$. Then $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.

Corollary 1.3. Let $G = P_p$, $p \geq 2$. Then $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.

Theorem 1.4. Let C be a caterpillar with exactly one pendent edge at each internal vertex, $p > 5$. Then $\gamma_M(C) = \lceil \frac{p}{8} \rceil$.

Theorem 1.5. Let C be a caterpillar of order p with exactly k pendent edges at each internal vertex. Then $\gamma_M(C) = \left\lceil \frac{p}{2(k+3)} \right\rceil$.

Proof. Since $\gamma_M(C) \geq \left\lceil \frac{p}{2(k+3)} \right\rceil$ for any graph G, $\Delta(G) = k + 2$ where k denotes the number of pendents at each vertex of the spine. Therefore $\gamma_M(C) \geq \left\lceil \frac{p}{2(k+3)} \right\rceil$.

Let $D = \{u_1, u_2, u_3, ..., u_{\lceil \frac{p}{2(k+3)} \rceil}\}$, such that $N[u_i] \cap N[u_j] = \phi$ for $u_i, u_j \in D, i \neq j$. This is possible since we can choose the middle vertices of disjoint sets of three consecutive vertices starting from the left most pendent vertex on the spine. Therefore $|N[D]| = (k + 3) \left\lceil \frac{p}{2(k+3)} \right\rceil$. Then

$$|N[D]| = \begin{cases} \frac{p}{2} & \text{if p} \equiv 0(\text{mod}2(\text{k}+3)) \\ \frac{p}{2} + \frac{5}{2} + k & \text{if p} \equiv 1(\text{mod } 2(\text{k}+3)) \\ \frac{p}{2} + 2 + k & \text{if p} \equiv 2(\text{mod } 2(\text{k}+3)) \\ \cdots & \\ \cdots & \\ \frac{p}{2} + \frac{1}{2} & \text{if p} \equiv 2(\text{k}+3) - 1(\text{mod } 2(\text{k}+3)). \end{cases}$$

Hence D is a majority dominating set of C. Therefore $\gamma_M(C) \leq |D| = \left\lceil \frac{p}{2(k+3)} \right\rceil$.
Hence $\gamma_M(C) = \left\lceil \frac{p}{2(k+3)} \right\rceil$.

§2. Characterisation of minimal majority dominating sets

Theorem 2.1. Let S be a majority dominating set of G. Then S is minimal if and only if for every $v \in S$ either condition (i) or (ii) holds.

(i) $|N[S]| > \lceil \frac{p}{2} \rceil$ and $|pn[v,S]| > |N[S]| - \lceil \frac{p}{2} \rceil$.

(ii) $|N[S]| = \lceil \frac{p}{2} \rceil$ and either v is an isolate of S or $pn[v,S] \cap (V-S) \neq \phi$.

Proof. Suppose that $|N[S]| > \lceil \frac{p}{2} \rceil$. Let $v \in S$. Since S is minimal majority dominating set, $S - \{v\}$ is not a majority dominating set. Therefore $|N[S-v]| < \lceil \frac{p}{2} \rceil$. But $|N[S-v]| = |N[S]| - |pn[v,S]|$....... (1). It implies that $|N[S]| - |pn[v,S]| < \lceil \frac{p}{2} \rceil$. Thus condition (i) holds.

Let $|N[S]| = \lceil \frac{p}{2} \rceil$. Suppose that v is neither an isolate of S nor has a private neighbor in $(V-S)$. That is, $pn[v,S] = \phi$. Then $|N[S-v]| = |N[S]| - |pn[v,S]| = |N[S]| = \lceil \frac{p}{2} \rceil$, which implies $S - \{v\}$ is a majority dominating set, a contradiction to that S is minimal. Thus condition (ii) holds.

Conversely, let S be a majority dominating set and suppose that (i) or (ii) holds. If S is not a minimal majority dominating set, then $|N[S-v]| \geq \lceil \frac{p}{2} \rceil$ for some $v \in S$. Then by (1), $|N[S]| - |pn[v,S]| \geq \lceil \frac{p}{2} \rceil$. Therefore $|pn[v,S]| \leq |N[S]| - \lceil \frac{p}{2} \rceil$, a contradiction to condition (i) for some $v \in S$.

Next, Similarly if $|N[S-v]| \geq \lceil \frac{p}{2} \rceil$ for some $v \in S$ implies that $|N[S]| - |pn[v,S]| \geq \lceil \frac{p}{2} \rceil$. Then by (1), $|N[S]| \geq |pn[v,S]| + \lceil \frac{p}{2} \rceil$. If v is an isolate of S, then $v \in pn[v,S]$. Therefore $|pn[v,S]| \geq 1$. If $pn[v,S] \cap (V-S) \neq \phi$ then also $|pn[v,S]| \geq 1$. Hence $|N[S]| \geq \lceil \frac{p}{2} \rceil + 1$, a contradiction to condition (ii) for some $v \in S$. Thus S is a minimal majority dominating set.

Theorem 2.2.[4] Let G be a graph of order p. Let $G \neq \overline{K}_p$, when p is odd. If S is a minimal majority dominating set then $(V-S)$ is a majority dominating set.

Theorem 2.3. Let $G = (V,E)$ be any graph. Then $\gamma_M(G) \leq \left\lceil \frac{\gamma(G)}{2} \right\rceil$.

Proof. Let D be a γ set of G. Then $|N[D]| = V(G)$. Let $D = D_1 \cup D_2$, where $|D_1| = \left\lceil \frac{\gamma(G)}{2} \right\rceil$, $|D_2| = \left\lfloor \frac{\gamma(G)}{2} \right\rfloor$. Now, $N[D] = (N[D_1] - N[D_2]) \cup N[D_2]$. That is, $|N[D]| = |N[D_1] - N[D_2]| \cup |N[D_2]|$. Then either $|N[D_1] - N[D_2]| \geq \lceil \frac{p}{2} \rceil$ or $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$. If $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$, then D_2 is a majority dominating set. If $|N[D_1] - N[D_2]| \geq \lceil \frac{p}{2} \rceil$, then $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$. That implies, D_1 is a majority dominating set. Therefore $\gamma_M(G) \leq |D_1| = \left\lceil \frac{\gamma(G)}{2} \right\rceil$ or $\gamma_M(G) \leq |D_2| = \left\lfloor \frac{\gamma(G)}{2} \right\rfloor$. Hence $\gamma_M(G) \leq \left\lceil \frac{\gamma(G)}{2} \right\rceil$.

Construction 2.4. In fact, the difference between $\gamma(G)$ and $\gamma_M(G)$ can be made very large. For every integer $k \geq 0$, there exists a graph G such that $\left\lceil \frac{\gamma(G)}{2} \right\rceil - \gamma_M(G) = k$.

Proof. Let G be the graph obtained from $K_{1,2k+2}$ by dividing each edge exactly once. Then $\gamma(G) = 2k+2$ and $\gamma_M(G) = 1$.

Theorem 2.5. $\gamma_M(G) = \gamma(G)$ if and only if G has a full degree vertex.

Proof. Assume that $\gamma_M(G) = \gamma(G)$. Suppose $\Delta(G) \neq p-1$, then $\gamma(G) \neq 1$. Let D be a γ-set of G. Then $|D| \geq 2$. Let $D = D_1 \cup D_2$ where $D_1 \neq \phi$ and $D_2 \neq \phi$, $D_1 \cap D_2 = \phi$. Since $|N[D]| = p$, $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ or $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$. Hence D_1 or D_2 is a majority dominating set. $\Rightarrow$ $\gamma_M(G) \leq |D_1|$ or $\gamma_M(G) \leq |D_2|$. Since $D_1 \neq \phi$ and $D_2 \neq \phi$, $\gamma_M(G) < |D| = \gamma(G)$, a contradiction. Hence $\Delta(G) = p-1$. The converse is obvious.

§3. Some results for $\gamma_M(G)$

Proposition 3.1.[4] Let G be any graph with p vertices. Then $\gamma_M(G) = 1$ if and only if there exists a vertex of degree $\geq \lceil \frac{p}{2} \rceil - 1$.

Theorem 3.2.[4] Let G be a graph of order p and $G \neq \overline{K_p}$. Then $1 \leq \gamma_M(G) \leq \lfloor \frac{p}{2} \rfloor$.

Proposition 3.3. $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = \overline{K_p}$, $p \geq 3$.

Theorem 3.4. For any graph G,

(i) $\left\lceil \frac{p}{2(\Delta(G)+1)} \right\rceil \leq \gamma_M(G)$.

(ii) $\gamma_M(G) \leq \lceil \frac{p}{2} \rceil - \Delta(G)$ if $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$.

(iii) $\gamma_M(G) = 1 \leq \left\lceil \frac{p-\Delta(G)}{2} \right\rceil$ if $\Delta(G) \geq \lceil \frac{p}{2} \rceil - 1$. The bounds are sharp.

Proof. (i) Let $D = \{v_1, v_2, v_3, \ldots, v_{\gamma_M}\}$ be a γ_M-set. Then

$$|N[D]| \leq \sum_{v \in D} d(v) + \gamma_M(G).$$

$$\left\lceil \frac{p}{2} \right\rceil \leq |N[D]| \leq \sum_{v \in D} \Delta(G) + \gamma_M(G) = \gamma_M(G)(\Delta(G)+1).$$

Since $\gamma_M(G)$ is an integer, $\gamma_M(G) \geq \left\lceil \frac{p}{2(\Delta(G)+1)} \right\rceil$.

(ii) Let $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$. Let v be a vertex of maximum degree $\Delta(G)$. Let S be a subset of $V(G)$ containing v such that $S \cap N(v) = \phi$ and $|S| = \left\lceil \frac{|V(G)|}{2} \right\rceil - \Delta(G)$. Then S is a majority dominating set of cardinality $\lceil \frac{p}{2} \rceil - \Delta(G)$. Hence $\gamma_M(G) \leq \lceil \frac{p}{2} \rceil - \Delta(G)$.

(iii) Let $\Delta(G) \geq \lceil \frac{p}{2} \rceil - 1$. Then $\gamma_M(G) = 1$. Since $\Delta(G) \leq (p-1)$, $p - \Delta(G) \geq 1$. Therefore $\left\lceil \frac{p-\Delta(G)}{2} \right\rceil \geq 1 = \gamma_M(G)$.

For $G = C_7$, $G = K_{1,p-1}$ and $G = P_8$, the bounds are sharp.

Theorem 3.5.[4] Let G be a graph without isolates. Then $\gamma_M(G) \leq \lceil \frac{p}{4} \rceil$.

Theorem 3.6. Let G be a connected graph with $\delta(G) \geq 2$ and order $p \geq 3$. Then $\gamma_M(G) = \lceil \frac{p}{4} \rceil$ if and only if $G = C_3, C_4, C_7, C_8, K_4$ or $K_4 - e$.

Proof. Let $\gamma_M(G) = \lceil \frac{p}{4} \rceil$. Let $\delta(G) \geq 2$.

Case (i): If $\Delta(G) \geq \lceil \frac{p}{2} \rceil - 1$, then $\gamma_M(G) = 1$. Then $\gamma_M(G) = \lceil \frac{p}{4} \rceil = 1$, therefore $p = 3$ or 4. Since G is connected and $\delta(G) \geq 2$, $G = C_3$ or C_4 or K_4 or $K_4 - e$.

Case (ii): Let $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$. Then by Theorem 3.4, $\gamma_M(G) \leq \lceil \frac{p}{2} \rceil - \Delta(G)$. Let a vertex v such that $d(v) \geq 3$. Then the vertex v is with a vertex u of degree 2 in $G - N[v]$, v, u dominates 6 points. For dominating $\lceil \frac{p}{2} \rceil - 6$ points, it requires atmost $\lceil \frac{p}{4} \rceil - 3$ points. Therefore $|D| = \lceil \frac{p}{4} \rceil - 3 + 2$. Then $\gamma_M(G) \leq \lceil \frac{p}{4} \rceil - 1$, a contradiction. Hence $d(v) \leq 2$ for all $v \in V(G)$. Since G has no pendent vertices, $d(v) = 2$ for all $v \in V(G)$. Therefore G is a cycle. For C_p, $\gamma_M(G) = \lceil \frac{p}{6} \rceil$. Therefore, $\gamma_M(G) = \lceil \frac{p}{4} \rceil$ if and only if $p = 3, 4$ or $p = 7, 8$. Hence $G = C_4, C_7$ or C_8.

The converse is obvious.

Corollary 3.7. Let G be a connected graph with $\delta(G) = 2$, p is even, $p \geq 4$ and $\gamma_M(G) = \left\lceil \frac{O(G)}{4} \right\rceil$. Then $\gamma_M(G^+) = \left\lceil \frac{O(G)}{4} \right\rceil$.

Proposition 3.8. Let G be a connected graph with $\delta(G) = 1$ and order $p = 4n + 2$. Then $\gamma_M(G) = \lceil \frac{p}{4} \rceil$ if and only if $G = K_2$.

Proof. Let $\gamma_M(G) = \lceil \frac{p}{4} \rceil$. Since $\gamma_M(G) \leq \left\lceil \frac{\gamma(G)}{2} \right\rceil$, $\left\lceil \frac{\gamma(G)}{2} \right\rceil \geq \lceil \frac{p}{4} \rceil$. Since $\lceil \frac{p}{4} \rceil = n+1$, $\left\lceil \frac{\gamma(G)}{2} \right\rceil \geq n+1$. It implies that $\gamma(G) \geq 2n+1 = \frac{p}{2}$. But $\gamma(G) \leq \frac{p}{2}$. Therefore $\gamma(G) = \frac{p}{2}$. Hence $G = H^+$ for some connected graph H. Let $\Delta(G) \geq 3$. Since $O(G)$ is even, $O(G) \geq 4$. Since $O(G) = 4n+2, O(G) \geq 6$. Then $O(H) = 2n+1 \geq 3$. Take any set S of n vertices of H. Since H is connected, $|N_H[S]| \geq n+1$. It implies that $|N_{H^+}[S]| \geq 2n+1$. Hence $\gamma_M(G) \leq |S| = n$. But $\gamma_M(G) = n+1$, a contradiction. Therefore $O(G) < 6$. It gives that $O(G) = 2 \ \Rightarrow \ G = K_2$. The converse is obvious.

Theorem 3.9. For any connected graph G with p vertices $p \geq 2$, $\gamma_M(G) = p - \kappa(G)$, where $\kappa(G)$ is a vertex connectivity of G if and only if G is a complete graph of order p.

Proof. Let $\gamma_M(G) = p - \kappa(G)$. Let u be a vertex such that $d(u) = \Delta(G) = |N(u)|$. Since $N[V(G) - N(u)] = V(G)$, $(V(G) - N(u))$ is a majority dominating set of G. Therefore $(V(G) - N(u)) \in D(G)$, D(G) is the set of all majority dominating sets of G. Hence $\gamma_M(G) \leq |V(G) - N(u)| = p - \Delta(G)$. Then by hypothesis, $p - \kappa(G) \leq p - \Delta(G)$. This implies $\kappa(G) \geq \Delta(G)$. Since $\kappa(G) \leq \delta(G) \leq \Delta(G)$, $\kappa(G) = \delta(G) = \Delta(G)$. Therefore G is regular say κ-regular.

Let u be any vertex of κ-regular graph G. Then $(\kappa+1)$ vertices are dominated by u. If $(\kappa+1) < \lceil \frac{p}{2} \rceil$ then $(\{u\} \cup S)$ is a majority dominating set where S is a set of $\lceil \frac{p}{2} \rceil - (\kappa+1)$ vertices disjoint from $N[u]$. Therefore $\gamma_M(G) \leq 1 + \lceil \frac{p}{2} \rceil - (\kappa+1)$. By hypothesis, $(p - \kappa) \leq \lceil \frac{p}{2} \rceil - \kappa$ which is a absurd result. Hence $(\kappa+1) \geq \lceil \frac{p}{2} \rceil$ and $(\kappa+1)$ vertices are dominated by only one vertex u. Therefore $\gamma_M(G) = 1 = p - \kappa$ and $\kappa = p-1$. Thus G is a complete graph of order p. The converse part is obvious.

Corollary 3.10. For any connected graph G with p vertices $p \geq 2$, $\gamma_M(G) = p - \lambda(G)$, where $\lambda(G)$ is a edge connectivity of G if and only if G is a complete graph of order p.

Theorem 3.11. For any tree T, $\gamma_M(T) + \Delta(T) = p$ if and only if T is a star.

Proof. Let $\gamma_M(T) + \Delta(T) = p$. Let $v \in V(T)$ such that $d(v) = \Delta(T)$. If $\Delta(T) < \lceil \frac{p}{2} \rceil - 1$, then by theorem (3.4), $\gamma_M(T) \leq \lceil \frac{p}{2} \rceil - \Delta(T)$ and $\gamma_M(T) + \Delta(T) \leq \lceil \frac{p}{2} \rceil$, a contradiction. Therefore $\Delta(T) \geq \lceil \frac{p}{2} \rceil - 1$. In this case $\gamma_M(T) = 1$. It implies that $\Delta(T) = p-1$. Hence T is a star. The converse is obvious.

Proposition 3.12. Given any two positive integers r, p with $r \leq \lceil \frac{p}{4} \rceil$, there exists a graph G with $\gamma_M(G) = r$ and $|V(G)| = p$.

Proof. Let $p = 4r + t$, $t \geq 0$. Let $G = K_{1, \lceil \frac{t}{2} \rceil + 1} \bigcup (2r-1)K_2 \bigcup \lfloor \frac{t}{2} \rfloor K_1$. Then $V(G) = \lceil \frac{t}{2} \rceil + 2 + (2r-1)2 + \lfloor \frac{t}{2} \rfloor = 4r + t$. Let $D = \{v_1, v_2, ..., v_r\}$, where v_1 is the center of the star $K_{1, \lceil \frac{t}{2} \rceil + 1}$, $v_2, ..., v_r$ are the centers of $(r-1)$ stars which are K_2's. $|N[D]| = 2r + \lceil \frac{t}{2} \rceil = \lceil \frac{p}{2} \rceil$. Then D is a majority dominating set and $\gamma_M(G) \leq r$. If D' is a set with $(r-1)$ vertices then $|N[D']| \leq 2r + \lceil \frac{t}{2} \rceil - 2 < \lceil \frac{p}{2} \rceil$. Therefore no subset of $V(G)$ with $(r-1)$ vertices is a majority dominating set. It implies that $\gamma_M(G) \geq r$.
Hence $\gamma_M(G) = r$.

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Some new generalized difference double sequence spaces via Orlicz functions

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Abstract In this article, the author defines the generalized difference double paranormed sequence spaces $w^2(M,\Delta^r,p,q), w_o^2(M,\Delta^r,p,q)$ and $w_\infty^2(M,\Delta^r,p,q)$ defined over a semi-normed sequence space (X,q), where Δ^r is generalized difference operator. The author also studies their properties and inclusion relations between them.

Keywords P-convergent, difference sequence, Orlicz function.

§1. Introduction

Let l_∞, c and c_o be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup_k|x_k|$. Kizmaz [14] introduced the notion of difference sequence spaces as follows:

$$X(\Delta)=\{x=(x_k):\ (\Delta x_k)\in X\}$$

for $X=l_\infty$, c and c_o. Later on, the notion was generalized by Et and Colak [15] as follows:

$$X(\Delta^m)=\{x=(x_k):\ (\Delta^m x_k)\in X\}$$

for $X=l_\infty$, c and c_o, where $\Delta^m x=(\Delta^m x_k)=\left(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}\right)$, $\Delta^0 x=x$ and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}\ \text{ for all } k\in\mathbb{N}.$$

Subsequently, difference sequence spaces were studied by Esi [4], Esi and Tripathy [5], Tripathy et.al [13] and many others.

An *Orlicz function* M is a function $M:[0,\infty)\to[0,\infty)$, which is continuous, convex, nondecreasing function define for $x>0$ such that $M(0)=0$, $M(x)>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called the *modulus function* and characterized by Ruckle [17]. An Orlicz function M

is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [12], Esi [1,2], Esi and Et [3], Parashar and Choudhary [16] and many others.

Let w^2 denote the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$ [7]. We shall describe such an $x = (x_{k,l})$ more briefly as "$P-$*convergent*". We shall denote the space of all $P-$*convergent* sequences by c^2. The double sequence $x = (x_{k,l})$ is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. We shall denote all bounded double sequences by l_∞^2.

§2. Definitions and results

In this presentation our goal is to extend a few results known in the literature from ordinary (single) difference sequences to difference double sequences. Some studies on double sequence spaces can be found in 8-10.

Definition 2.1. Let M be an Orlicz function and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers. Let X be a seminormed space over the complex field $\mathbb{C}$ with the seminorm q. We now define the following new generalized difference sequence spaces:

$$w^2(M, \Delta^r, p, q) = \left\{ x = (x_{k,l}) \in w^2 : \ P - \lim_{m,n} (mn)^{-1} \sum_{k=1}^{m} \sum_{l=1}^{n} \left[M\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } \rho > 0 \text{ and } L \right\},$$

$$w_o^2(M, \Delta^r, p, q) = \left\{ x = (x_{k,l}) \in w^2 : \ P - \lim_{m,n} (mn)^{-1} \sum_{k=1}^{m} \sum_{l=1}^{n} \left[M\left(\frac{q(\Delta^r x_{k,l})}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } \rho > 0 \right\},$$

and

$$w^2_\infty\left(M,\Delta^r,p,q\right)=\Bigg\{x=(x_{k,l})\in w^2:\ \sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^r x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}}<\infty,\ \text{for some }\rho>0\Bigg\},$$

where $\Delta^r x=(\Delta^r x_{k,l})=\left(\Delta^{r-1}x_{k,l}-\Delta^{r-1}x_{k,l+1}-\Delta^{r-1}x_{k+1,l}+\Delta^{r-1}x_{k+1,l+1}\right)$, $\left(\Delta^1 x_{k,l}\right)=(\Delta x_{k,l})=(x_{k,l}-x_{k,l+1}-x_{k+1,l}+x_{k+1,l+1})$, $\Delta^0 x=(x_{k,l})$ and also this generalized difference double notion has the following binomial representation [6]:

$$\Delta^r x_{k,l}=\sum_{i=0}^{r}\sum_{j=0}^{r}(-1)^{i+j}\binom{r}{i}\binom{r}{j}x_{k+i,l+j}.$$

Some double spaces are obtained by specializing M, p, q and r. Here are some examples:

(i) If $M(x)=x$, $r=0$, $p_{k,l}=1$ for all $k,l\in\mathbb{N}$, and $q(x)=|x|$, then we obtain ordinary double sequence spaces w^2, w^2_o and w^2_∞.

(ii) If $M(x)=x$, $r=0$ and $q(x)=|x|$, then we obtain new double sequence spaces as follows:

$$w^2\left(p\right)=\left\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left(|x_{k,l}-L|\right)^{p_{k,l}}=0,\ \text{for some }L\right\},$$

$$w^2_o\left(p\right)=\left\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left(|x_{k,l}|\right)^{p_{k,l}}=0\right\},$$

and

$$w^2_\infty\left(p\right)=\left\{x=(x_{k,l})\in w^2:\ \sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left(|x_{k,l}|\right)^{p_{k,l}}<\infty\right\}.$$

(iii) If $r=0$ and $q(x)=|x|$, then we obtain new double sequence spaces as follows:

$$w^2\left(M,p\right)=\Bigg\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}}=0,\ \text{for some }\rho>0\text{ and }L\Bigg\},$$

$$w^2_o\left(M,p\right)=\Bigg\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}=0,\ \text{for some }\rho>0\Bigg\},$$

and

$$w^2_\infty\left(M,p\right)=\left\{x=(x_{k,l})\in w^2:\sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}<\infty,\text{for some }\rho>0\right\}.$$

(iv) If $r=1$ and $q(x)=|x|$, then we obtain new double sequence spaces as follows:

$$w^2\left(M,\Delta,p\right)=\Bigg\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|\Delta x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}}=0,\ \text{for some }\rho>0\text{ and }L\Bigg\},$$

$$w_o^2(M,\Delta,p)=\left\{x=(x_{k,l})\in w^2:\ P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}=0,\right.$$
$$\left.\text{for some } \rho>0\right\},$$

and

$$w_\infty^2(M,\Delta,p)=\left\{x=(x_{k,l}):\ \sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}<\infty,\ \text{for some } \rho>0\right\},$$

where $(\Delta x_{k,l})=(x_{k,l}-x_{k,l+1}-x_{k+1,l}+x_{k+1,l+1})$.

§3. Main results

Theorem 3.1. Let $p=(p_{k,l})$ be bounded. The classes of $w^2(M,\Delta^r,p,q)$, $w_o^2(M,\Delta^r,p,q)$ and $w_\infty^2(M,\Delta^r,p,q)$ are linear spaces over the complex field $\mathbb{C}$.

Proof. We give the proof only $w_\infty^2(M,\Delta^r,p,q)$. The others can be treated similarly. Let $x=(x_{k,l}),y=(y_{k,l})\in w_\infty^2(M,\Delta^r,p,q)$. Then we have

$$\sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q(\Delta^r x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}}<\infty,\ \text{for some } \rho_1>0 \tag{1}$$

and

$$\sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}<\infty,\ \text{for some } \rho_2>0 \tag{2}$$

Let $\alpha,\beta\in\mathbb{C}$ be scalars and $\rho=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since M is non-decreasing convex function, we have

$$\begin{aligned}\left[M\left(\frac{q(\Delta^r(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}}&\le D\left\{\left[M\left(\frac{q(\Delta^r x_{k,l})}{2\rho_1}\right)\right]^{p_{k,l}}+\left[M\left(\frac{q(\Delta^r y_{k,l})}{2\rho_2}\right)\right]^{p_{k,l}}\right\}\\&\le D\left\{\left[M\left(\frac{q(\Delta^r x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}}+\left[M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}\right\},\end{aligned}$$

where $D=\max(1,2^H)$, $H=\sup_{k,l}p_{k,l}<\infty$. Now, from (1) and (2), we have

$$\sup_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q(\Delta^m(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}}<\infty.$$

Therefore $\alpha x+\beta y\in w_\infty^2(M,\Delta^r,p,q)$. Hence $w_\infty^2(M,\Delta^r,p,q)$ is a linear space.

Theorem 3.2. The double sequence spaces $w^2(M,\Delta^r,p,q)$, $w_o^2(M,\Delta^r,p,q)$ and $w_\infty^2(M,\Delta^r,p,q)$ are seminormed spaces, seminormed by

$$f((x_{k,l}))=\sum_{k=1}^{r}q(x_{k,1})+\sum_{l=1}^{r}q(x_{1,l})+\inf\left\{\rho>0:\ \sup_{k,l}M\left(q\left(\frac{\Delta^r x_{k,l}}{\rho}\right)\right)\le 1\right\}.$$

Proof. Since q is a seminorm, so we have $f((x_{k,l}))\ge 0$ for all $x=(x_{k,l})$; $f(\theta^2)=0$ and $f((\lambda x_{k,l}))=|\lambda|f((x_{k,l}))$ for all scalars λ.

Now, let $x = (x_{k,l})$, $y = (y_{k,l}) \in w_o^2(M, \Delta^r, p, q)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k,l} M\left(q\left(\frac{\Delta^r x_{k,l}}{\rho_1}\right)\right) \leq 1$$

and

$$\sup_{k,l} M\left(q\left(\frac{\Delta^r y_{k,l}}{\rho_2}\right)\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have,

$$\begin{aligned}\sup_{k,l} M\left(q\left(\frac{\Delta^r (x_{k,l} + y_{k,l})}{\rho}\right)\right) \leq &\left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sup_{k,l} M\left(q\left(\frac{\Delta^r x_{k,l}}{\rho_1}\right)\right) \\ &+ \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sup_{k,l} M\left(q\left(\frac{\Delta^r y_{k,l}}{\rho_2}\right)\right) \leq 1.\end{aligned}$$

Since ρ_1, $\rho_2 > 0$, so we have,

$$\begin{aligned}f\left((x_{k,l}) + (y_{k,l})\right) =& \sum_{k=1}^{r} q\left(x_{k,1} + y_{k,1}\right) + \sum_{l=1}^{r} q\left(x_{1,l} + y_{1,l}\right) \\ &+ \inf\left\{\rho = \rho_1 + \rho_2 > 0 : \sup_{k,l} M\left(q\left(\frac{\Delta^r (x_{k,l} + y_{k,l})}{\rho}\right)\right) \leq 1\right\} \\ \leq& \sum_{k=1}^{r} q\left(x_{k,1}\right) + \sum_{l=1}^{r} q\left(x_{1,l}\right) + \inf\left\{\rho_1 > 0 : \sup_{k,l} M\left(q\left(\frac{\Delta^r x_{k,l}}{\rho_1}\right)\right) \leq 1\right\} \\ &+ \sum_{k=1}^{r} q\left(y_{k,1}\right) + \sum_{l=1}^{r} q\left(y_{1,l}\right) + \inf\left\{\rho_2 > 0 : \sup_{k,l} M\left(q\left(\frac{\Delta^r y_{k,l}}{\rho_2}\right)\right) \leq 1\right\} \\ =& f\left((x_{k,l})\right) + f\left((y_{k,l})\right).\end{aligned}$$

Therefore f is a seminorm.

Theorem 3.3. Let (X, q) be a complete seminormed space. Then the spaces $w^2(M, \Delta^r, p, q)$, $w_o^2(M, \Delta^r, p, q)$ and $w_\infty^2(M, \Delta^r, p, q)$ are complete seminormed spaces seminormed by f.

Proof. We prove the theorem for the space $w_o^2(M, \Delta^r, p, q)$. The other cases can be establish following similar technique. Let $x^i = \left(x_{k,l}^i\right)$ be a Cauchy sequence in $w_o^2(M, \Delta^r, p, q)$. Let $\varepsilon > 0$ be given and for $r > 0$, choose x_o fixed such that $M\left(\frac{r x_o}{2}\right) \geq 1$ and there exists $m_o \in \mathbb{N}$ such that

$$f\left(\left(x_{k,l}^i - x_{k,l}^j\right)\right) < \frac{\varepsilon}{r x_o}, \text{ for all } i, j \geq m_o.$$

By definition of seminorm, we have

$$\sum_{k=1}^{r} q\left(x_{k,1}^i\right) + \sum_{l=1}^{r} q\left(x_{1,l}^j\right) + \inf\left\{\rho > 0 : \sup_{k,l} M\left(q\left(\frac{\Delta^r x_{k,l}^i - \Delta^r x_{k,l}^j}{\rho}\right)\right) \leq 1\right\} < \frac{\varepsilon}{r x_o} \quad (3)$$

This shows that $q\left(x_{k,1}^i\right)$ and $q\left(x_{1,l}^j\right)$ $(k, l \leq r)$ are Cauchy sequences in (X, q). Since (X, q) is complete, so there exists $x_{k,1}, x_{1,l} \in X$ such that

$$\lim_{i\to\infty} q\left(x_{k,1}^i\right) = x_{k,1} \text{ and } \lim_{j\to\infty} q\left(x_{1,l}^j\right) = x_{1,l} \quad (k, l \leq m).$$

Now from (3), we have

$$M\left(q\left(\frac{\Delta^r\left(x^i_{k,l}-x^j_{k,l}\right)}{f\left(\left(x^i_{k,l}-x^j_{k,l}\right)\right)}\right)\right)\leq 1\leq M\left(\frac{rx_o}{2}\right),\text{ for all } i,j\geq m_o. \quad (4)$$

This implies

$$q\left(\Delta^m\left(x^i_{k,l}-x^j_{k,l}\right)\right)\leq\frac{rx_o}{2}.\frac{\varepsilon}{rx_o}=\frac{\varepsilon}{2},\text{ for all } i,j\geq m_o.$$

So, $q\left(\Delta^r\left(x^i_{k,l}\right)\right)$ is a Cauchy sequence in (X,q). Since (X,q) is complete, there exists $x_{k,l}\in X$ such that $\lim_i\Delta^r\left(x^i_{k,l}\right)=x_{k,l}$ for all $k,l\in\mathbb{N}$. Since M is continuous, so for $i\geq m_o$, on taking limit as $j\to\infty$, we have from (4),

$$M\left(q\left(\frac{\Delta^r\left(x^i_{k,l}\right)-\lim_{j\to\infty}\Delta^r x^j_{k,l}}{\rho}\right)\right)\leq 1\Rightarrow M\left(q\left(\frac{\Delta^r\left(x^i_{k,l}\right)-x_{k,l}}{\rho}\right)\right)\leq 1.$$

On taking the infimum of such $\rho's$, we have

$$f\left(\left(x^i_{k,l}-x_{k,l}\right)\right)<\varepsilon,\text{ for all } i\geq m_o.$$

Thus $\left(x^i_{k,l}-x_{k,l}\right)\in w^2_o\left(M,\Delta^r,p,q\right)$. By linearity of the space $w^2_o\left(M,\Delta^r,p,q\right)$, we have for all $i\geq m_o$,

$$\left(x_{k,l}\right)=\left(x^i_{k,l}\right)-\left(x^i_{k,l}-x_{k,l}\right)\in w^2_o\left(M,\Delta^r,p,q\right).$$

Thus $w^2_o\left(M,\Delta^r,p,q\right)$ is a complete space.

Proposition 3.4. **(a)** $w^2\left(M,\Delta^r,p,q\right)\subset w^2_\infty\left(M,\Delta^r,p,q\right)$,

(b) $w^2_o\left(M,\Delta^r,p,q\right)\subset w^2_\infty\left(M,\Delta^r,p,q\right)$.

The inclusions are strict.

Proof. It is easy, so omitted.

To show that the inclusions are strict, consider the following example.

Example 3.5. Let $M(x)=x^p$, $p\geq 1$, $r=1,q\left(x\right)=|x|$, $p_{k,l}=2$ for all k, $l\in\mathbb{N}$ and consider the double sequence

$$x_{k,l}=\begin{cases}0, & \text{if } k+l \text{ is odd;}\\ k, & \text{otherwise.}\end{cases}$$

Then

$$\Delta^r x_{k,l}=\begin{cases}2k+1, & \text{if } k+l \text{ is even;}\\ -2k-1, & \text{otherwise.}\end{cases}$$

Here $x=\left(x_{k,l}\right)\in w^2_\infty\left(M,\Delta^r,p,q\right)$, but $x=\left(x_{k,l}\right)\notin w^2\left(M,\Delta^r,p,q\right)$.

Theorem 3.6. The double spaces $w^2\left(M,\Delta^r,p,q\right)$ and $w^2_o\left(M,\Delta^r,p,q\right)$ are nowhere dense subsets of $w^2_\infty\left(M,\Delta^r,p,q\right)$.

Proof. The proof is obvious in view of Theorem 3.3 and Proposition 3.4.

Theorem 3.7. Let $r\geq 1$, then for all $0<i\leq r$, $z^2\left(M,\Delta^i,p,q\right)\subset z^2\left(M,\Delta^r,p,q\right)$, where $z^2=w^2$, w^2_o and w^2_∞. The inclusions are strict.

Proof. We establish it for only $w_o^2\left(M,\Delta^{r-1},p,q\right) \subset w_o^2\left(M,\Delta^r,p,q\right)$. Let $x=(x_{k,l}) \in w_o^2\left(M,\Delta^{r-1},p,q\right)$. Then

$$P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}}=0, \text{ for some } \rho>0. \tag{5}$$

Thus from (5) we have

$$P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l+1}}=0,$$

$$P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k+1,l}}=0,$$

and

$$P-\lim_{m,n}(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k+1,l+1}}=0.$$

Now for

$$\Delta^r x=(\Delta^r x_{k,l})=\left(\Delta^{r-1}x_{k,l}-\Delta^{r-1}x_{k,l+1}-\Delta^{r-1}x_{k+1,l}+\Delta^{r-1}x_{k+1,l+1}\right),$$

we have

$$\begin{aligned}
&(mn)^{-1}\left[M\left(q\left(\frac{\Delta^r x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}\\
\le&(mn)^{-1}\left[M\left(q\left(\frac{\Delta^{r-1}x_{k,l}}{\rho}\right)+q\left(\frac{\Delta^{r-1}x_{k,l+1}}{\rho}\right)+q\left(\frac{\Delta^{r-1}x_{k+1,l}}{\rho}\right)+q\left(\frac{\Delta^{r-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}\\
\le&D^2(mn)^{-1}\left\{\left[M\left(q\left(\frac{\Delta^{r-1}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[M\left(q\left(\frac{\Delta^{r-1}x_{k+1,l}}{\rho}\right)\right)\right]^{p_{k,l}}\right.\\
&\left.+\left[M\left(q\left(\frac{\Delta^{r-1}x_{k,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[M\left(q\left(\frac{\Delta^{r-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k,l}}\right\}\\
\le&D^2\left\{\left[(mn)^{-1}M\left(q\left(\frac{\Delta^{r-1}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}+\left[(mn)^{-1}M\left(q\left(\frac{\Delta^{r-1}x_{k+1,l}}{\rho}\right)\right)\right]^{p_{k+1,l}}\right.\\
&\left.+\left[(mn)^{-1}M\left(q\left(\frac{\Delta^{r-1}x_{k,l+1}}{\rho}\right)\right)\right]^{p_{k,l+1}}+\left[(mn)^{-1}M\left(q\left(\frac{\Delta^{r-1}x_{k+1,l+1}}{\rho}\right)\right)\right]^{p_{k+1,l+1}}\right\}
\end{aligned}$$

from which it follows that $x=(x_{k,l}) \in w_o^2\left(M,\Delta^r,p,q\right)$ and hence $w_o^2\left(M,\Delta^{r-1},p,q\right) \subset w_o^2\left(M,\Delta^r,p,q\right)$. On applying the principle of induction, it follows that $w_o^2\left(M,\Delta^i,p,q\right) \subset w_o^2\left(M,\Delta^r,p,q\right)$ for $i=0,1,2,...,r-1$. The proof for the rest cases are similar. To show that the inclusions are strict, consider the following example.

Example 3.8. Let $M(x)=x^p$, $r=1$, $q(x)=|x|$, $p_{k,l}=1$ for all k odd and for all $l\in\mathbb{N}$ and $p_{k,l}=2$ otherwise. Consider the sequence $x=(x_{k,l})$ defined by $x_{k,l}=k+l$ for all $k,l\in\mathbb{N}$. We have $\Delta^r x_{k,l}=0$ for all $k,l\in\mathbb{N}$. Hence $x=(x_{k,l})\in w_o^2\left(M,\Delta,p,q\right)$ but $x=(x_{k,l})\notin w_o^2\left(M,p,q\right)$.

Theorem 3.9. **(a)** If $0<\inf_{k,l}p_{k,l}\le p_{k,l}<1$, then $z^2\left(M,\Delta^r,p,q\right)\subset z^2\left(M,\Delta^r,q\right)$,

(b) If $1 < p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$, then $z^2(M, \Delta^r, q) \subset z^2\left(M, \Delta^{r-1}, p, q\right)$, where $Z^2 = w^2$, w_o^2 and w_∞^2.

Proof. The first part of the result follows from the inequality

$$(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n} M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right) \leq (mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}}$$

and the second part of the result follows from the inequality

$$(mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}} \leq (mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n} M\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right).$$

Theorem 3.10. Let M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. If $\beta = \lim_{t\to\infty}\frac{M_2(t)}{t} \geq 1$, then $z^2(M_1, \Delta^r, p, q) = z^2(M_2 \circ M_1, \Delta^r, p, q)$, where $z^2 = w^2$, w_o^2 and w_∞^2.

Proof. We prove it for $z^2 = w_o^2$ and the other cases will follows on applying similar techniques. Let $x = (x_k) \in w^2(M_1, \Delta^r, p, q)$, then

$$P - (mn)^{-1}\sum_{k=1}^{m}\sum_{l=1}^{n}\left[M_1\left(\frac{q\left(\Delta^{r-1}x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}} = 0.$$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M_2(t) < \varepsilon$ for $0 \leq t < \delta$. Let

$$y_{k,l} = M_1\left(q\left(\frac{\Delta^r x_{k,l}}{\rho}\right)\right)$$

and consider

$$[M_2(y_{k,l})]^{p_{k,l}} = [M_2(y_{k,l})]^{p_{k,l}} + [M_2(y_{k,l})]^{p_{k,l}}, \tag{6}$$

where the first term is over $y_{k,l} \leq \delta$ and the second is over $y_{k,l} > \delta$. From the first term in (6), using the Remark

$$[M_2(y_{k,l})]^{p_{k,l}} < [M_2(2)]^H [(y_{k,l})]^{p_{k,l}} \tag{7}$$

On the other hand, we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.$$

Since M_2 is non-decreasing and convex, it follows that

$$M_2(y_{k,l}) < M_2\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M_2(2) + \frac{1}{2}M_2\left(\frac{2y_{k,l}}{\delta}\right).$$

Since M_2 satisfies Δ_2-condition, we have

$$M_2(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2) = K\frac{y_{k,l}}{\delta}M_2(2).$$

Hence, from the second term in (6)

$$[M_2(y_{k,l})]^{p_{k,l}} \leq \max\left(1, \left(KM_2(2)\delta^{-1}\right)^H\right)[(y_{k,l})]^{p_{k,l}} \tag{8}$$

By (7) and (8), taking limit in the Pringsheim sense, we have $x = (x_{k,l}) \in w_o^2(M_2 \circ M_1, \Delta^r, p, q)$. Observe that in this part of the proof we did not need $\beta \geq 1$. Now, let $\beta \geq 1$ and $x =$

$(x_{k,l}) \in w_o^2(M_1, \Delta^r, p, q)$. Since $\beta \geq 1$ we have $M_2(t) \geq \beta t$ for all $t \geq 0$. It follows that $x = (x_{k,l}) \in w_o^2(M_2 \circ M_1, \Delta^r, p, q)$ implies $x = (x_{k,l}) \in w_o^2(M_1, \Delta^r, p, q)$. This implies $w_o^2(M_2 \circ M_1, \Delta^r, p, q) = w_o^2(M_1, \Delta^r, p, q)$.

Theorem 3.11. Let M, M_1 and M_2 be Orlicz functions, q, q_1 and q_2 be seminorms. Then

(i) $z^2(M_1, \Delta^r, p, q) \cap z^2(M_2, \Delta^r, p, q) \subset z^2(M_1 + M_2, \Delta^r, p, q)$.

(ii) $z^2(M, \Delta^r, p, q_1) \cap z^2(M, \Delta^r, p, q) \subset z^2(M, \Delta^r, p, q_1 + q_2)$.

(iii) If q_1 is stronger than q_2, then $z^2(M, \Delta^r, p, q_1) \subset z^2(M, \Delta^r, p, q_2)$, where $z^2 = w^2$, w_o^2 and w_∞^2.

Proof. **(i)** We establish it for only $z^2 = w_o^2$. The rest cases are similar. Let $x = (x_{k,l}) \in w_o^2(M_1, \Delta^r, p, q) \cap w_o^2(M_2, \Delta^r, p, q)$. Then

$$P - \lim_{m,n} (mn)^{-1} \sum_{k=1}^{m} \sum_{l=1}^{n} \left[M_1 \left(\frac{q(\Delta^r x_{k,l})}{\rho_1} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_1 > 0,$$

$$P - \lim_{m,n} (mn)^{-1} \sum_{k=1}^{m} \sum_{l=1}^{n} \left[M_2 \left(\frac{q(\Delta^r x_{k,l})}{\rho_2} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the following inequality

$$\left[(M_1 + M_2) \left(q \left(\frac{\Delta^r x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} \leq D \left\{ \left[M_1 \left(q \left(\frac{\Delta^r x_{k,l}}{\rho_1} \right) \right) \right]^{p_{k,l}} + \left[M_2 \left(q \left(\frac{\Delta^r x_{k,l}}{\rho_2} \right) \right) \right]^{p_{k,l}} \right\}.$$

The proofs of (ii) and (iii) follow obviously.

The proof of the following result is also routine work.

Proposition 3.12. For any modulus function, if $q_1 \approxeq$ (equivalent to) q_2, then $z^2(M, \Delta^r, p, q_1) = z^2(M, \Delta^r, p, q_2)$, where $z^2 = w^2$, w_o^2 and w_∞^2.

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On a classification of a structure algebra: SU-Algebra

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Abstract In this paper, we introduce a new algebraic structure, called SU-algebra and a concept of ideal in SU-algebra. Moreover, the relations of SU-algebra and the other algebra structure are investigated.

Keywords SU-algebra, classification, structure algebra.

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§1. Introduction

In 1966, K. Iseki introduced the notion of a BCI-algebras which is a generalization of BCK-algebras. He defined a BCI-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:
(BCI 1) $((x * y) * (x * z)) * (z * y) = 0$, (BCI 2) $(x * (x * y)) * y = 0$, (BCI 3) $x * x = 0$, (BCI 4) $x * y = 0 = y * x$ imply $x = y$, (BCI 5) $x * 0 = 0$ imply $x = 0$, for all $x, y, z \in X$. If (BCI 5) is replaced by (BCI 6) $0 * x = 0$ for all $x \in X$, the algebra $(X, *, 0)$ is called BCK-algebra. In 1983, Hu and Li introduced the notion of a BCH-algebras which is a generalization of the notions of BCK-algebra and BCI-algebras. They have studied a few properties of these algebras and defined a BCH-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (BCH 1) $x * x = 0$, (BCH 2) $(x * y) * z = (x * z) * y$, (BCH 3) $x * y = 0 = y * x$ imply $x = y$, for all $x, y, z \in X$. In 1999 Ahn and Kim introduced the notion of a QS-algebras. They defined a QS-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (QS 1) $x * x = 0$, (QS 2) $x * 0 = x$, (QS 3) $(x * y) * z = (x * z) * y$, (QS 4) $(x * y) * (x * z)) = (z * y)$, for all $x, y, z \in X$. They presented some properties of QS-algebras and G-part of QS-algebras. In 2007, A. Walendziak introduced the notion of a BF-algebras, ideal and a normal ideal in BF-algebras. He defined a BF-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (BF 1) $x * x = 0$, (BF 2) $x * 0 = x$, (BF 3) $0 * (x * y) = y * x$, for all $x, y, z \in X$. He also studied the properties and characterizations of them. In 2010, Megalai and Tamilarasi introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and the several results are presented. They defined a TM-algebra by a TM-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (TM 1) $x * 0 = x$, (TM 2) $(x * y) * (x * z) = z * y$,

for any $x, y, z \in X$.

In this paper, we introduce a new algebraic structure, called SU-algebra and a concept of ideal in SU-algebra. We also describe connections between such ideals and congruences. Moreover, the relations of SU-algebra and the other algebra structure are investigated.

§2. The structure of SU-algebra

Definition 2.1. A SU-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(1) $((x * y) * (x * z)) * (y * z) = 0$,

(2) $x * 0 = x$,

(3) if $x * y = 0$ imply $x = y$,

for all $x, y, z \in X$.

From now on, X denotes a SU-algebra $(X, *, 0)$ and a binary operation will be denoted by juxtaposition.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined by the following:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a SU-algebra.

Theorem 2.3. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$.

(1) $xx = 0$,

(2) $xy = yx$,

(3) $0x = x$,

(4) $((xy)x)y = 0$,

(5) $((xz)(yz))(xy) = 0$,

(6) $xy = 0$ if and only if $(xz)(yz) = 0$,

(7) $xy = x$ if and only if $y = 0$.

Proof. Let X be a SU-algebra and $x, y, z \in X$.

(1) $xx = (xx)0 = ((x0)(x0))(00) = 0$.

(2) Since $(xy)(yx) = ((xy)0)(yx) = ((xy)(xx))(yx) = 0$
and $(yx)(xy) = ((yx)0)(xy) = ((yx)(yy))(xy) = 0, xy = yx$.

(3) $0x = x0 = x$.

(4) $((xy)x)y = ((xy)(x0))(y0) = 0$.

(5) $((xz)(yz))(xy) = ((zx)(zy))(xy) = 0$.

(6) Assume $xy = 0$. Then $(xz)(yz) = (zx)(zy) = ((zx)(zy))(0) = ((zx)(zy))(xy) = 0$.
Conversely, let $(xz)(yz) = 0$. Then $xy = (0)(xy) = ((xz)(yz))(xy) = 0$.

(7) Assume $xy = x$. Then $y = (0)y = (xx)y = ((xy)x)y = 0$.
Conversely, let $y = 0$. Since $0 = ((xy)x)y = ((xy)x)0 = (xy)x, xy = x$.

Theorem 2.4. Let X be a SU-algebra. A relation $\leq$ on X is defined by $x \leq y$ if $xy = 0$. Then $(X, \leq)$ is a partially ordered set.

Proof. Let X be a SU-algebra and let $x, y, z \in X$. 1) Since $xx = 0, x \leq x$. 2) Suppose that $x \leq y$ and $y \leq x$. Then $xy = 0$ and $yx = 0$. By Definition 2.1 (3) , $x = y$. 3) Suppose that $x \leq y$ and $y \leq z$. Then $xy = 0$ and $yz = 0$. By Definition 2.1.(1), $0 = ((xy)(xz))(yz) = (0(xz))0 = 0(xz) = xz$. Hence $x \leq z$. Thus $(X, \leq)$ is a partially ordered set on X.

Theorem 2.5. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$.

(1) $x \leq y$ if and only if $y \leq x$.

(2) $x \leq 0$ if and only if $x = 0$.

(3) if $x \leq y$, then $xz \leq yz$.

Proof. Let X be a SU-algebra and $x, y, z \in X$.

(1) Clearly, $x \leq y$ if and only if $y \leq x$.

(2) Suppose that $x \leq 0$. Then $0 = x0 = x$. Conversely, let $x = 0$. Then $x0 = 00 = 0$. Hence $x \leq 0$.

(3) Let $x \leq y$. Then $xy = 0$. By Theorem 2.3 (6), $(xz)(yz) = 0$. Hence $xz \leq yz$.

Theorem 2.6. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$

(1) $(xy)z = (xz)y$.

(2) $x(yz) = z(yx)$.

(3) $(xy)z = x(yz)$.

Proof. Let X be a SU-algebra and $x, y, z \in X$.

(1) Since $((xy)(xz))(yz) = 0, (xy)(xz) \leq yz$.

By Theorem 2.5 (3), $((xy)(xz))u \leq (yz)u$, for all $u \in X$.

By Theorem 2.5 (1), $(yz)u \leq ((xy)(xz))u$.

We substitute xu for x, xz for z and $((xu)y)zu$ for u, then we have

$$\begin{aligned} [y(xz)][((xu)y)zu] &\leq [((xu)y)((xu)(xz))][((xu)y)zu] \\ &\leq [(xu)(xz)](zu) && \text{(by Definition 2.1 (1) and Theorem 2.4.)} \\ &= 0 && \text{(because zu = uz and Definition 2.1 (1)).} \end{aligned}$$

By Theorem 2.5 (2), $[y(xz)][((xu)y)zu] = 0$. Thus $y(xz) \leq ((xu)y)zu$.

By Theorem 2.5 (1), $((xu)y)zu \leq y(xz)$.

We substitute z for u and xy for z, then we have

$$[(xz)y][(xy)z] \leq y(x(xy)) = (x(xy))y = ((xy)x)y = 0 \text{ (by Theorem 2.3 (4))}.$$

By Theorem 2.5 (2), $[(xz)y][(xy)z] = 0$. Hence $(xy)z = (xz)y$.

(2) $x(yz) = (yz)x = (yx)z = z(yx)$.

(3) $(xy)z = (yx)z = (yz)x = x(yz)$.

Theorem 2.7. Let X be a SU-algebra. If $xz = yz$, then $x = y$ for all $x, y, z \in X$.

Proof. Let X be a SU-algebra and let $x, y, z \in X$.

Suppose that $xz = yz$. Then $x = x0 = x(zz) = (xz)z = (yz)z = y(zz) = y0 = y$.

Theorem 2.8. Let X be a SU-algebra and $a \in X$. If $ax = x$ for all $x \in X$, then $a = x$.

Proof. Let X be a SU-algebra, $a \in X$ and $ax = x$ for all $x \in X$. Since $ax = x = 0x$, $a = x$ (by theorem 2.7).

§3. Ideal and congruences in SU-algebra

Definition 3.1. Let X be a SU-algebra. A nonempty subset I of X is called an ideal of X if it satisfies the following properties :

(1) $0 \in I$.

(2) if $(xy)z \in I$ and $y \in I$ for all $x, y, z \in X$, then $xz \in I$.

Clearly, X and $\{0\}$ are ideals of X.

Example 3.2. Let X be a SU-algebra as defined in example 2.2. If $A = \{0, 1\}$, then A is an ideal of X. If $B = \{0, 1, 2\}$, then B is not an ideal of X because $(1 * 1) * 2 = 0 * 2 = 2 \in B$ but $1 * 2 = 3 \notin B$.

Theorem 3.3. Let X be a SU-algebra and I be an ideal of X. Then

(1) if $xy \in I$ and $y \in I$ then $x \in I$ for all $x,\ y \in X$.

(2) if $xy \in I$ and $x \in I$, then $y \in I$ for all $x,\ y \in X$.

Proof. Let X be a SU-algebra and I be an ideal of X.

(1) Let $xy \in I$ and $y \in I$. Since $(xy)0 = xy \in I$ and $y \in I$, $x = x0 \in I$ (by definition 3.1).

(2) It is immediately followed by (1) and Theorem 2.3 (2).

Theorem 3.4. Let X be a SU-algebra and A_i be ideal of X for $i = 1, 2, \cdots, n$. Then $\bigcap_{i=1}^{n} A_i$ is an ideal of X.

Proof. Let X be a SU-algebra and A_i be ideal of X for $i = 1, 2, \cdots, n$. Clearly, $0 \in \bigcap_{i=1}^{n} A_i$. Let $x, y, z \in X$ be such that $(xy)z \in \bigcap_{i=1}^{n} A_i$ and $y \in \bigcap_{i=1}^{n} A_i$. Then $(xy)z \in A_i$ and $y \in A_i$ for all $i = 1, 2, \cdots, n$. Since A_i is an ideal, $xz \in A_i$ for all $i = 1, 2, \cdots, n$. Thus $xz \in \bigcap_{i=1}^{n} A_i$. Hence $\bigcap_{i=1}^{n} A_i$ is a ideal of X.

Definition 3.5. Let X be a SU-algebra. A nonempty subset S of X is called a SU-subalgebra of X if $xy \in S$ for all $x, y \in S$.

Theorem 3.6. Let X be a SU-algebra and I be an ideal of X. Then I is a SU-subalgebra of X.

Proof. Let X be a SU-algebra and I be an ideal of X. Let $x, y \in I$. By Theorem 2.3 (4), $((xy)x)y = 0 \in I$. Since I is an ideal and $x \in I$, $(xy)y \in I$. Since I is an ideal and $y \in I,\ xy \in I$.

Definition 3.7. Let X be a SU-algebra and I be an ideal of X. A relation $\sim$ on X is defined by $x \sim y$ if and only if $xy \in I$.

Theorem 3.8. Let X be a SU-algebra and I be an ideal of X. Then $\sim$ is a congruence on X.

Proof. A reflexive property and a symmetric property are obvious. Let $x, y, z \in X$. Suppose that $x \sim y$ and $y \sim z$. Then $xy \in I$ and $yz \in I$. Since $((xz)(xy))(zy) = 0 \in I$ and I is an ideal, $(xz)(yz) = (xz)(zy) \in I$. By Theorem 3.3 (1), $xz \in I$. Thus $x \sim z$. Hence $\sim$ is an equivalent relation.

Next, let $x, y, u, v \in X$ be such that $x \sim u$ and $y \sim v$. Then $xu \in I$ and $yv \in I$. Thus $ux \in I$ and $vy \in I$. Since $((xy)(xv))(yv) = 0 \in I$ and Theorem 3.3 (1) , $(xy)(xv) \in I$. Hence $xy \sim xv$. Since $((uv)(ux))(vx) = 0 \in I$ and Definition 3.1 (2), $(uv)(vx) \in I$. Since $vx = xv$, $(uv)(vx) = (uv)(xv)$. Hence $(uv)(xv) \in I$ and so $xv \sim uv$. Thus $xy \sim uv$. Hence $\sim$ is a congruence on X.

Let X be a SU-algebra, I be an ideal of X and $\sim$ be a congruence relation on X. For any $x \in X$, we define $[x]_I = \{y \in X | x \sim y\} = \{y \in X | xy \in I\}$. Then we say that $[x]_I$ is an equivalence class containing x.

Example 3.9. Let X be a SU-algebra as defined in example 2.2. It is easy to show that $I = \{0, 1\}$ is an ideal of X, then $[0]_I = \{0, 1\}$, $[1]_I = \{0, 1\}$, $[2]_I = \{2, 3\}$, $[3]_I = \{2, 3\}$.

Remarks 3.10. Let X be a SU-algebra. Then

1. $x \in [x]_I$ for all $x \in X$,
2. $[0]_I = \{x \in X | 0 \sim x\}$ is an ideal of X.

Theorem 3.11. Let X be a SU-algebra, I be an ideal of X and $\sim$ be a congruence relation on X. Then $[x]_I = [y]_I$ if and only if $x \sim y$ for all $x,\ y \in X$.

Proof. Let $[x]_I = [y]_I$. Since $y \in [y]_I$, $y \in [x]_I$. Hence $x \sim y$. Conversely Let $x \sim y$. Then $y \sim x$. Let $z \in [x]_I$. Then $x \sim z$. Since $y \sim x$ and $x \sim z, y \sim z$. Hence $z \in [y]_I$. Similarly, let $w \in [y]_I$. Then $w \in [x]_I$. Therefore $[x]_I = [y]_I$.

The family $\{[x]_I | x \in X\}$ gives a partition of X which is denoted by X/I. For $x, y \in X$ we define $[x]_I[y]_I = [xy]_I$. The following theorem show that X/I is a SU-algebra.

Theorem 3.12. Let X be a SU-algebra and I be an ideal of X. Then X/I is a SU-algebra.

Proof. Let $[x]_I, [y]_I, [z]_I \in X/I$.

(1) $(([x]_I[y]_I)([x]_I[z]_I))([y]_I[z]_I) = ([xy]_I[xz]_I)[yz]_I = [((xy)(xz))(yz)]_I = [0]_I$.

(2) $[x]_I[0]_I = [x0]_I = [x]_I$.

(3) Suppose that $[x]_I[y]_I = [0]_I$. Then $[xy]_I = [0]_I = [yx]_I$. Since $xy \in [xy]_I, 0 \sim xy$. Hence $xy \in [0]_I$. Since $[0]_I$ is an ideal, $x \sim y$. Hence $[x]_I = [y]_I$. Thus X/I is a SU-algebra.

§4. Classification of a SU- Algebra

In this section, we investigated the relations between SU-algebra and the other algebras such as BCI-algebra, TM-algebra, BCH-algebra, QS-algebra and BF-algebra.

Theorem 4.1. Let X be a SU-algebra. Then X is a BCI-algebra.

Proof. Let X be a SU-algebra and $x, y, z \in X$. Since $zy = yz, ((xy)(xz))(zy) = ((xy)(xz))(yz) = 0$. Since $x0 = x$ and $0y = y$, $(x(xy))y = ((x0)(xy))(0y) = 0$. By Theorem 2.3 (1) , $xx = 0$. Suppose that $xy = yx = 0$. Then $x = y$ (by Definition 2.1 (3)). Hence X is a BCI-algebra.

By theorem 4.1, every SU-algebra is a BCI-algebra but the following example show that the converse is not true.

Example 4.2. Let $X = \{0, 1, 2, 3\}$ be a BCI-algebra in which operation * is defined by the following:

*	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Then X is not SU-algebra because $((0 * 1) * (0 * 2)) * (1 * 2) = (0 * 0) * 3 = 0 * 3 = 2 \neq 0$.

Theorem 4.3. Let X be a BCI-algebra and $xy = yx$ for all $x, y \in X$. Then X is a SU-algebra .

Proof. Let X be a BCI-algebra and $yz = zy$ for all $x, y \in X$. Let $x, y, z \in X$. Then $((xy)(xz))(yz) = ((xy)(xz))(zy) = 0$ (by BCI 1). Since X is a BCI-algebra, $x0 = x$. Suppose that $xy = 0$. Then $yx = xy = 0$. Thus $x = y$. Hence X is a SU-algebra.

Theorem 4.4. Let X be a SU-algebra. Then X is a TM-algebra.

Proof. Let X be a SU-algebra and $x, y, z \in X$. Then $(zy)((xy)(xz)) = ((xy)(xz))(zy)$ $= ((xy)(xz))(yz) = 0$. Hence $(xy)(xz) = zy$. By definition 2.1.(2), $x0 = x$. Hence X is a TM-algebra.

By theorem 4.4, every SU-algebra is a TM-algebra but the following example show that the converse is not true.

Example 4.5. Let $X = \{0, 1, 2, 3\}$ be a TM-algebra in which operation * is defined by the following:

*	0	1	2	3
0	0	0	3	2
1	1	0	2	3
2	2	2	0	3
3	3	3	2	0

Then X is not SU-algebra because $((1 * 0) * (1 * 2)) * (0 * 2) = (1 * 2) * 3 = 2 * 3 = 3 \neq 0$.

Theorem 4.6. Let X be a TM-algebra and $xy = yx$, for all $x, y \in X$. Then X is a SU-algebra.

Proof. Let X be a TM-algebra and $xy = yx$ for all $x, y \in X$. Let $x, y, z \in X$. Then $((xy)(xz))(yz) = ((xy)(xz))(zy)$ (because $yz = zy$). By TM 2, $(xy)(xz) = zy$ and so $((xy)(xz))(zy) = 0$. Hence $((xy)(xz))(yz) = 0$. By TM 1, $x0 = x$. Suppose that $xy = 0$. Then $yx = xy = 0$. Thus $x = y$ (because X is a TM-algebra, $xy = 0 = yx$ imply $x = y$). Hence X is a SU-algebra.

Theorem 4.7. [Megalai and Tamilarasi.(2010)]

(1) Let X be a TM-algebra. Then X is a BCH-algebra.

(2) Let X be a BCH-algebra and $(xy)(xz) = zy$ for all $x, y, z \in X$. Then X is a TM-algebra.

Corollary 4.8. Let X be a SU-algebra. Then X is a BCH-algebra.

Theorem 4.9. Let X be a BCH-algebra X and $(xy)(yz) = (xz)$ for all $x, y, z \in X$. Then X is a SU-algebra.

Proof. Let X be a BCH-algebra and $x, y, z \in X$ be such that $(xy)(yz) = (xz)$. Then $((xy)(yz))(xz) = 0$. By BCH 2,$((xy)(xz))(yz) = ((xy)(yz))(xz)$. Hence $((xy)(xz))(yz) = 0$. Since X is a BCH-algebra, $x0 = x$ Suppose that $xy = 0$. Then $yx = xy = 0$. Thus $x = y$ (by BCH 3). Hence X is a SU-algebra.

Theorem 4.10. Let X be a SU-algebra. Then X is a QS-algebra.

Proof. Let X be a SU-algebra and $x, y, z \in X$. By Theorem 2.3 (1), $xx = 0$. By Definition 2.1 (2), $x0 = x$. By Theorem 2.6 (3), $(xy)z = (xz)y$. Since $((xy)(xz))(zy) = 0$, $(xy)(xz) = (zy)$. Hence X is a QS-algebra.

By Theorem 4.10, every SU-algebra is a QS-algebra but the following example show that the converse is not true.

Example 4.11. Let $X = \{0, 1, 2\}$ be a QS-algebra in which operation * is defined by the following:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is not SU-algebra because $1 * 2 = 0$ but $1 \neq 2$.

Theorem 4.12. Let X be a QS-algebra and $xy = yx$ for all $x, y \in X$.

Then X is a SU-algebra.

Proof. Let X be a QS-algebra and $xy = yx$ for all $x, y \in Xb$. Let $x, y, z \in X$. By QS 1 and QS 4, $((xy)(xz))(zy) = 0$. Hence $((xy)(xz))(yz) = 0$ (because $zy = yz$). By QS 2 , $x0 = x$. Since X is a QS-algebra, $xy = 0$ imply $x = y$. Hence X is a SU-algebra.

Theorem 4.13. Let X be a SU-algebra. Then X is a BF-algebra.

Proof. Let X be a SU-algebra and $x, y, z \in X$. By Theorem 2.3 (1), $xx = 0$. By Definition 2.1 (2), $x0 = x$. By Theorem 2.3 (3), $0(xy) = xy = yx$. Hence X is a BF-algebra.

By theorem 4.13, every SU-algebra is a BF-algebra but the following example show that the converse is not true.

Example 4.14. Let $X = (R, *, 0)$ be a BF-algebra where R be the set of real numbers and operation * is defined by the following:

$$x * y = \begin{cases} x & if \;\; y = 0 \\ y & if \;\; x = 0 \\ 0 & otherwise. \end{cases}$$

Then X is not SU-algebra because $1 * 2 = 0 = 2 * 1$ but $1 \neq 2$.

Theorem 4.15. Let X be a BF-algebra satisfies

(1) $(xy)(xz) = yz$,

(2) if $xy = 0$ imply $x = y$ for all $x, y, z \in X$.

Then X is a SU-algebra.

Proof. Let X be a BF-algebra and $x, y, z \in X$ satisfying $(xy)(xz) = yz$ and $xy = 0$ imply $x = y$. Since $(xy)(xz) = yz, ((xy)(xz))(yz) = 0$. By BF 2, $x0 = x$. Hence X is a SU-algebra.

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A view on intuitionistic fuzzy regular G_δ set

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Abstract The purpose of this paper is to study the concepts of an intuitionistic fuzzy regular G_δ set. Here we introduce intuitionistic fuzzy regular G_δ set, intuitionistic fuzzy regular G_δ spaces, intuitionistic fuzzy regular G_δ border, intuitionistic fuzzy regular G_δ frontier and intuitionistic fuzzy regular G_δ exterior. Besides many properties and basic results, properties related to sets and spaces are established. Some interrelations are discussed with suitable examples.

Keywords Intuitionistic fuzzy regular G_δ set, intuitionistic fuzzy regular semi open set, intui -tionistic fuzzy regular G_δ spaces, intuitionistic fuzzy regular G_δ border, intuitionistic fuzzy re -gular G_δ frontier and intuitionistic fuzzy regular G_δ exterior.

2000 Mathematics subject classification: 54A40-03E72.

§1. Introduction

The concept of fuzzy sets was introduced by Zadeh [10]. Atanassov [1] introduced and studied intuitionistic fuzzy sets. On the other hand, Cocker [5] introduced the notions of intuitionistic fuzzy topological space, intuitionistic fuzzy continuity and some other related concepts. The concept of an intuitionistic fuzzy semi θ continuity in an intuitionistic fuzzy topological space was introduced and studied by I. M. Hanafy, A. M. ABD El. Aziz and T. M. Salman [7]. G. Balasubramanian [3] introducd and studied the concept of fuzzy G_δ set. M. Caldas, S. Jafari and T. Noiri [4] were introduced the toplogical properties of g-border, g-frontier and g-exterior of a set using the concept of g-open sets. In this paper we introduced intuitionistic fuzzy regular G_δ set, intuitionistic fuzzy regular G_δ spaces, intuitionistic fuzzy regular G_δ border, intuitionistic fuzzy regular G_δ frontier and intuitionistic fuzzy regular G_δ exterior. Besides many properties and basic results, properties related to sets and spaces are established. Suitable examples are given wherever necessary.

§2. Preliminaries

Definition 2.1.[1] Let X be a non empty fixed set and I the closed interval $[0,1]$. An intuitionistic fuzzy set (IFS), A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, where the mapping $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) for each element $x \in X$ to the set

A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each x $\in X$. Obviously, every fuzzy set A on a nonempty set X is an IFS of the following form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x) : x \in X\rangle\}$.

Definition 2.2.[1] Let A and B be IFS's of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x) :\rangle x \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$.

(ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$.

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$.

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$.

Definition 2.3.[1] The IFSs $0_\sim$ and $1_\sim$ are defined by $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.4.[5] An intuitionistic fuzzy topology (IFT) in Coker's sense on a non empty set X is a family τ of IFSs in X satisfying the following axioms.

(T_1) $0_\sim$,$1_\sim \in \tau$.

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

(T_3) $\cup G_i \in \tau$ for arbitrary family $\{G_i/i \in I\} \subseteq \tau$.

In this paper by (X, τ) or simply by X we will denote the Cocker's intuitionistic fuzzy topological space (IFTS). Each IFSs in τ is called an intuitionistic fuzzy open set (IFOS) in X. The complement $\overline{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 2.5.[5] Let A be an IFS in IFTS X. Then

int $A = \cup\{G/G$ is an IFOS in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A.

cl $A = \cap\{G/G$ is an IFCS in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Definition 2.6.[3] Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called fuzzy G_δ set if $\lambda = \bigwedge_{i=1}^{i=\infty} \lambda_i$ where each $\lambda_i \in T$.

The complement of a fuzzy G_δ is a fuzzy F_σ set.

Proposition 2.1.[5] For any IFS A in (X, τ) we have

(i) $cl(\overline{A}) = \overline{int(A)}$.

(ii) $int(\overline{A}) = \overline{cl(A)}$.

Definition 2.7.[6,8] Let A be an IFS of an IFTS X. Then A is called an

(i) intuitionistic fuzzy regular open set (IFROS) if $A = int(cl(A))$.

(ii) intuitionistic fuzzy semi open set (IFSOS) if $A \subseteq cl(int(A))$.

(iii) intuitionistic fuzzy β open set (IFβOS) if $A \subseteq cl(int(cl(A)))$.

(iv) intuitionistic fuzzy preopen set (IFPOS) if $A \subseteq int(cl(A))$.

(v) intuitionistic fuzzy α open set (IFαOS) if $A \subseteq int(cl(int(A)))$.

Definition 2.8.[6,8] Let A be an IFS in IFTS. Then

(i) $\beta int(A) = \cup\{G/G$ is an IFβOS in X and $G \subseteq A\}$ is called an intuitionistic fuzzy β-interior of A.

(ii) $\beta cl(A) = \cap\{G/G$ is an IFβCS in X and $G \supseteq A\}$ is called an intuitionistic fuzzy β -closure of A.

(iii) $int_s(A) = \cup\{G/G$ is an IFSOS in X and $G \subseteq A\}$ is called an intuitionistic fuzzy semi interior of A.

(iv) $cl_s(A) = \cap\{G/G$ is an IFCS in X and $G \supseteq A\}$ is called an intuitionistic fuzzy semi closure of A.

Definition 2.9.[9] A fuzzy topological space (X,T) is said to be fuzzy $\beta - T_{1/2}$ space if every gfβ-closed set in (X,T) is fuzzy closed in (X,T).

§3. Intuitionistic fuzzy regular G_δ set and it's properties

Definition 3.1. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionstic fuzzy set of an intuitionistic fuzzy topological space X. Then A is said to be an intuitionistic fuzzy G_δ set (inshort, IFG_δS) if $A = \bigcap_{i\in I} A_i$ where each $A_i \in T$ and $A_i = \{\langle x, \mu_{A_i}(x), \gamma_{A_i}(x) : x \in X\rangle\}$.

The complement of an intuitionistic fuzzy G_δ set is an intuitionistic fuzzy F_σ set (inshort, IFF_σS).

Definition 3.2. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy σ closure of an intuitionistic fuzzy set A is denoted and defined by

$IFcl_\sigma(A) = \cap$ { $K/K = \{\langle x, \mu_K(x), \gamma_K(x)\rangle : x \in X\}$ is an intuitionistic fuzzy F_σ set in X and $A \subseteq K\}$.

Remark 3.1. Every intuitionistic fuzzy closed set is an intuitionistic fuzzy F_σ set.

Definition 3.3. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy σ interior of an intuitionistic fuzzy set A is denoted and defined by

$IFint_\sigma(A) = \cup$ { $G/G = \{\langle x, \mu_G(x), \gamma_G(x)\rangle : x \in X\}$ is an intuitionistic fuzzy G_δ set in X and $G \subseteq A\}$.

Remark 3.2. Every intuitionistic fuzzy open set is an intuitionistic fuzzy G_δ set.

Definition 3.4. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy regular closure of an intuitionistic fuzzy set A is denoted and defined by:

IF $rcl(A) = \cap$ { $K/K = \{\langle x, \mu_K(x), \gamma_K(x)\rangle : x \in X\}$ is an intuitionistic fuzzy regular closed set in X and $A \subseteq K\}$.

Definition 3.5. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy regular interior of an intuitionistic fuzzy set A is denoted and defined by

IF $rint(A) = \cup$ { $G/G = \{\langle x, \mu_G(x), \gamma_G(x)\rangle : x \in X\}$ is an intuitionistic fuzzy regular open set in X and $G \subseteq A\}$.

Definition 3.6. Let (X,T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then A is said to be an intuitionistic fuzzy regular G_δ set (inshort, IFrG_δS) if there is an intuitionistic fuzzy regular open set $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$ such that $U \subseteq A \subseteq IFcl_\sigma(U)$.

The complement of an intuitionistic fuzzy regular G_δ set is said to be an intuitionistic fuzzy regular F_σ set (inshort,$IFrF_\sigma S$).

Definition 3.7. Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy regular G_δ closure of an intuitionistic fuzzy set A is denoted and defined by $IFrG_\delta cl(A) = \cap \{\ K/K = \{\langle x, \mu_K(x), \gamma_K(x)\rangle : x \in X\}$ is an intuitionistic fuzzy regular F_σ set in X and $A \subseteq K\}$.

Proposition 3.1. Let (X, T) be an intuiionistic fuzzy topological space. For any two intuitionistic fuzzy sets $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$ of an intuitionistic fuzzy topological space (X, T), the following are valid.

(i) $IFrG_\delta cl(0_\sim) = 0_\sim$.

(ii) If $A \subseteq B$ then $IFrG_\delta cl(A) \subseteq IFrG_\delta cl(B)$.

(iii) $IFrG_\delta cl(IFrG_\delta cl(A)) = IFrG_\delta cl(A)$.

Definition 3.8. Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then intuitionistic fuzzy regular G_δ interior of an intuitionistic fuzzy set A is denoted and defined by $IFG_\delta int(A) = \cup \{\ G/G = \{\langle x, \mu_G(x), \gamma_G(x)\rangle : x \in X\}$ is an intuitionistic fuzzy regular G_δ set in X and $G \subseteq A\}$.

Proposition 3.2. Let (X, T) be an intuiionistic fuzzy topological space. For any two intuitionistic fuzzy sets $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$ of an intuitionistic fuzzy topological space (X, T), the following are valid.

(i) If A is an intuitionistic fuzzy regular G_δ then $A = IFrG_\delta int(A)$.

(ii) If A is an intuitionistic fuzy regular G_δ, then $IFrG_\delta int(IFrG_\delta int(A)) = IFrG_\delta int(A)$.

(ii) $\overline{IFrG_\delta int(A)} = IFrG_\delta cl(\overline{A})$.

(iv) $\overline{IFrG_\delta cl(A)} = IFrG_\delta int(\overline{A})$.

(v) If $A \subseteq B$ then $IFrG_\delta int(A) \subseteq IFrG_\delta int(B)$.

Remark 3.3. (i) Let A be an intuitionistic fuzzy set. Then

$$IFrint(A) \subseteq IFrG_\delta int(A) \subseteq A \subseteq IFrG_\delta cl(A) \subseteq IFrcl(A).$$

(ii) $IFrG_\delta cl(A) = A$ if and only if A is an intuitionistic fuzzy regular F_σ set.

(iii) $IFrG_\delta cl(1_\sim) = 1_\sim$.

(iv) $IFrG_\delta int(0_\sim) = 0_\sim$.

(v) $IFrG_\delta int(1_\sim) = 1_\sim$.

Proposition 3.3. Let (X, T) be an intuiionistic fuzzy topological space. For any intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ of an intuitionistic fuzzy topological space (X, T), the following statements are equivalent:

(i) A be an intuitionistic fuzzy regular G_δ set.

(ii) $B \subseteq A \subseteq IFcl_\sigma(B)$ and $A = A \cap IFcl_\sigma(B)$, for some intuitionistic fuzzy regular open set B in (X, T).

(iii) $A \subseteq IFcl_\sigma(A)$, for any intuitionistic fuzzy set A in (X, T).

Proposition 3.4. Every intuitionistic fuzzy regular open set is an intuitionistic fuzzy regular G_δ set.

Remark 3.4. The converse of the Proposition 3.4 need not be true. See Example 3.1.

Example 3.1. Let $X = \{a, b\}$ be a non empty set, $A = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$ and $B = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$. Then the family $T = \{\ 0_\sim, 1_\sim, A, B\}$ of intuitionistic fuzzy sets of X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space.Here A and B are intuitionistic fuzzy regular open sets in (X, T). Then,

$$\langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.6})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.6})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.5})\rangle.$$

Let $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.6})\rangle$. Clearly C is an intuitionistic fuzzy regular G_δ set, but it is not an intuitionistic fuzzy regular open set. Hence **intuitionistic fuzzy regular G_δ set need not be an intuitionistic fuzzy regular open set.**

Definition 3.9. Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then A is said to be an intuitionistic fuzzy regular semi open set (inshort,IFrSOS) if there is an intuitionistic fuzzy regular open set $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$ such that $U \subseteq A \subseteq IFcl_s(U)$.

The complement of an intuitionistic fuzzy regular semi open set is said to be an intuitionistic fuzzy regular semi closed set (inshort, IFrSCS).

Definition 3.10. Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of an intuiionistic fuzzy topological space X. Then A is said to be an intuitionistic fuzzy regular β open set (inshort, IFrβOS) if there is an intuitionistic fuzzy regular open set $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$ such that $U \subseteq A \subseteq IF\beta cl(U)$.

The complement of an intuitionistic fuzzy regular β open set is said to be an intuitionistic fuzzy regular β closed set (inshort, IFrβCS).

Remark 3.5. Intuitionistic fuzzy regular G_δ set and intuitionistic fuzzy regular semi open set are independent to each other, as can be seen from the following Examples 3.2 and 3.3.

Example 3.2. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.6})\rangle \text{ and } B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.5}), (\tfrac{a}{0.3}, \tfrac{b}{0.4})\rangle.$$

Then the family $T = \{\ 0_\sim, 1_\sim, A, B\}$ of intuionistic fuzzy sets is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A and B are intuitionistic fuzzy regular open sets in (X, T). Now,

$$\langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.6})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.6})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.5})\rangle.$$

Let $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.6})\rangle$. Here C is an intuitionistic fuzzy regular G_δ set in (X, T). Now $cl_s(\langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle) = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$. Then,

$$\langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.6})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.6})\rangle \nsubseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.6})\rangle.$$

Clearly C is not an intuitionistic fuzzy regular semi open set in (X, T). **Hence intuitionistic fuzzy regular G_δ set need not be an intuitionistic fuzzy regular semi open set.**

Example 3.3. Let $X = \{a\}$ be a non empty set.Consider the intuitionistic fuzzy sets $(A_n),\ n = 0, 1, 2, 3...$ as follows.We define the intuitionistic fuzzy sets $A_n = \{\langle x, \mu_{A_n}(x), \gamma_{A_n}(x)\rangle : x \in X, n = 0, 1, 2, ...\}$ by

$$\mu_{A_n}(x) = \tfrac{n}{10n+1} \text{ and } \gamma_{A_n}(x) = 1 - \tfrac{n}{10n+1}.$$

Then the family $T = \{\ 0_\sim,\ 1_\sim,\ A_n : n = 0, 1, 2, ...\}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Let $A = \langle x, (\frac{a}{0.900008}), (\frac{a}{0.099992})\rangle$ be an intuitionistic fuzzy regular semi open set in (X, T) but it is not an intuitionistic fuzzy regular G_δ set in (X, T).

Remark 3.6. Intuitionistic fuzzy regular G_δ set and intuitionistic fuzzy regular β open set are independent to each other, as can be seen from the Examples 3.4 and 3.5.

Example 3.4. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \text{ and } B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.3}), (\tfrac{a}{0.5}, \tfrac{b}{0.4})\rangle.$$

Then the family $T = \{\ 0_\sim,\ 1_\sim,$ A, B $\}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A is an intuitionistic fuzzy regular open set in (X, T). Now,

$$\langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.5}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.5}, \tfrac{b}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.3})\rangle.$$

Let $C = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$. Here C is an intuitionistic fuzzy regular G_δ set in (X, T).
Now $IF\beta cl(\langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle) = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$. Then,

$$\langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.5}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \nsubseteq \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle.$$

Clearly C is not an intuitionistic fuzzy regular β open set in (X, T).
Hence intuitionistic fuzzy regular G_δ set need not be an intuitionistic fuzzy regular β open set.

Example 3.5. Let $X = \{a\}$ be a non empty set.Consider the intuitionistic fuzzy sets (A_n), $n = 0, 1, 2, 3...$ as follows. We define the intuitionistic fuzzy sets $A_n = \{\langle x, \mu_{A_n}(x), \gamma_{A_n}(x)\rangle : x \in X, n = 0, 1, 2, ...\}$ by

$$\mu_{A_n}(x) = \tfrac{n}{10n+1} \text{ and } \gamma_{A_n}(x) = 1 - \tfrac{n}{10n+1}.$$

Then the family $T = \{\ 0_\sim,\ 1_\sim, A_n : n = 0, 1, 2, ...\}$ of intuitionistic fuzzy sets is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Let $A = \langle x, (\frac{a}{0.900008}), (\frac{a}{0.099992})\rangle$ be an intuitionistic fuzzy regular β open set in (X, T) but it is not an intuitionistic fuzzy regular G_δ set in (X, T).

Definition 3.11. (X, T) be an intuitionistic fuzzy topological space and Y be any intuitionistic fuzzy subset of X. Then $T_Y = (A/Y : A \in T)$ is an intuitionistic fuzzy topological space on Y and is called an induced or relative intuitionistic fuzzy topology. The pair (Y, T_Y) is called an intuitionistic fuzzy subspace of $(X, T) : (Y, T_Y)$ is called an intuitionistic fuzzy open intuitionistic fuzzy closed intuitionistic fuzzy regular G_δ intuitionistic fuzzy subspace if the charecteristic function of (Y, T_Y) viz χ_Y is an intuitionistic fuzzy open / intuitionistic fuzzy closed intuitionistic fuzzy regular G_δ set respectively.

Proposition 3.5. Let (X, T) be an intuitionistic fuzzy topological space. Suppose $Z \subseteq Y \subseteq X$ and (Y, T_Y) is an intuitionistic fuzzy regular G_δ intuitionistic fuzzy subspace of an intuitonistic fuzzy topolgical space (X, T). If (Z, T_Z) is an intuitionistic fuzzy regular G_δ intuitionistic fuzzy subspace in an intuitionistic fuzzy topological space $(X, T) \Leftrightarrow (Z, T_Z)$ is an intuitionistic fuzzy regular G_δ intuitionistic fuzzy subspace of an intuitionistic fuzzy topological space (Y, T_Y).

Definition 3.12. An intuitionistic fuzzy topological space (X, T) is said to be intuitionistic fuzzy regular G_δ $T_{1/2}$ space if every intuitionistic fuzzy regular G_δ set is an intuitionistic fuzzy open set.

Definition 3.13. An intuitionistic fuzzy topological space (X, T) is said to be an intuitionistic fuzzy regular G_δ S (resp. β, α and pre) space, if every intuitionistic fuzzy regular G_δ set is an intuitionistic fuzzy semi (resp. β, α and pre) open set.

Proposition 3.6. Every intuitionistic fuzzy regular G_δ $T_{1/2}$ space is an intuitionistic fuzzy regular G_δ S space.

Remark 3.7. The converse of the Proposition 3.6 need not be true. See Example 3.6.

Example 3.6. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.4}, \tfrac{b}{0.3})\rangle \text{ and } B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.2}), (\tfrac{a}{0.3}, \tfrac{b}{0.3})\rangle.$$

Then the family $T = \{\ 0_\sim,\ 1_\sim,\ A, B\ \}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A and B are intuitionistic fuzzy regular open sets in (X, T). Since every intuitionistic fuzzy regular G_δ set in (X, T) is an intuitionistic fuzzy semi open set, (X, T) is an intuitionistic fuzzy regular G_δ S space. Now, Let $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.2}), (\frac{a}{0.4}, \frac{b}{0.2})\rangle$ be any intuitionistic fuzzy set.

$$\langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.4}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.4}, \tfrac{b}{0.2})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.3}), (\tfrac{a}{0.4}, \tfrac{b}{0.2})\rangle.$$

Now C is an intuitionistic fuzzy regular G_δ set in (X, T) but it is not an intuitionistic fuzzy open set in (X, T). Hence (X, T) is not an intuitionistic fuzzy regular G_δ $T_{1/2}$ space. **Thus intuitionistic fuzzy regular G_δ S space need not be an intuitionistic fuzzy regular G_δ $T_{1/2}$ space.**

Proposition 3.7. Every intuitionistic fuzzy regular G_δ $T_{1/2}$ space is an intuitionistic fuzzy regular G_δ β space.

Remark 3.8. The converse of the Proposition 3.7 need not be true. See Example 3.7.

Example 3.7. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \text{ and } B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.3}), (\tfrac{a}{0.4}, \tfrac{b}{0.2})\rangle.$$

Then the family $T = \{\ 0_\sim,\ 1_\sim, A, B\}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A is an intuitionistic fuzzy regular open set in (X, T). Since every intuitionistic fuzzy regular G_δ set in (X, T) is an intuitionistic fuzzy β open set, (X, T) is an intuitionistic fuzzy regular G_δ β space. Now, Let $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.2}), (\frac{a}{0.4}, \frac{b}{0.3})\rangle$ be any intuitionistic fuzzy set.

$$\langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.5}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.2}), (\tfrac{a}{0.4}, \tfrac{b}{0.3})\rangle \subseteq \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.2}), (\tfrac{a}{0.4}, \tfrac{b}{0.3})\rangle.$$

Cleraly C is an intuitionistic fuzzy regular G_δ set in (X, T) but it is not an intuitionistic fuzzy open set in (X, T). Hence (X, T) is not an intuitionistic fuzzy regular G_δ $T_{1/2}$ space. **Thus intuitionistic fuzzy regular G_δ β space need not be an intuitionistic fuzzy regular G_δ $T_{1/2}$ space.**

Proposition 3.8. Every intuitionistic fuzzy regular G_δ α space is an intuitionistic fuzzy regular G_δ S space.

Remark 3.9. The converse of the Proposition 3.8 need not be true. See Example 3.8.

Example 3.8. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5})\rangle \text{ and } B = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.5})\rangle.$$

Then the family $T = \{ 0_\sim, 1_\sim, A, B\}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A and B are intuitionistic fuzzy regular open sets in (X, T). Since every intuitionistic fuzzy regular G_δ set in (X, T) is an intuitionistic fuzzy semi open set, (X, T) is an intuitionistic fuzzy regular G_δ S space. Now, Let $C = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$ be any intuitionistic fuzzy set.

$$\langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5})\rangle \subseteq \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle \subseteq \langle x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle.$$

Now C is an intuitionistic fuzzy regular G_δ set in (X, T) but it is not an intuitionistic fuzzy α open set in (X, T). Hence (X, T) is not an intuitionistic fuzzy regular G_δ α space. **Thus intuitionistic fuzzy regular G_δ S space need not be an intuitionistic fuzzy regular G_δ α space.**

Proposition 3.9. Every intuitionistic fuzzy regular G_δ pre space is an intuitionistic fuzzy regular G_δ β space.

Remark 3.10. The converse of the Proposition 3.9 need not be true. See Example 3.9.

Example 3.9. Let $X = \{a, b\}$ be a non empty set.

$$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle \text{ and } B = \langle x, (\frac{a}{0.4}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.5})\rangle.$$

Then the family $T = \{ 0_\sim, 1_\sim, A, B\}$ of intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X. Clearly (X, T) is an intuitionistic fuzzy topological space. Here A and B are intuitionistic fuzzy regular open sets in (X, T). Since every intuitionistic fuzzy regular G_δ set in (X, T) is an intuitionistic fuzzy β open set, (X, T) is an intuitionistic fuzzy regular G_δ β space. Now, Let $C = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.3})\rangle$ be any intuitionistic fuzzy set

$$\langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle \subseteq \langle x, (\frac{a}{0.5}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.3})\rangle \subseteq \langle x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.3})\rangle.$$

Clearly C is an intuitionistic fuzzy regular G_δ set in (X, T) but it is not an intuitionistic fuzzy pre open set in (X, T). Hence (X, T) is not an intuitionistic fuzzy regular G_δ pre space. **Thus intuitionistic fuzzy regular G_δ β space need not be an intuitionistic fuzzy regular G_δ pre space.**

§4. Intuitionistic fuzzy regular G_δ border, intuitionistic fuzzy regular G_δ frontier and Intuitionistic fuzzy regular G_δ exterior

Definition 4.1. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, T). The intuitionistic fuzzy regular border of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as
IF rb$(A) = A \cap$ IF $rcl(\overline{A})$.

Definition 4.2. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological

space (X, T). Then intuitionistic fuzzy regular G_δ border of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as
IF rG_δb$(A) = A\cap$ IF $rG_\delta cl(\overline{A})$.

Definition 4.3. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, T). Then intuitionistic fuzzy regular frontier of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as

$$IFrFR(A) = IFrcl(A) \cap IFrcl(\overline{A}).$$

Definition 4.4. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, T). Then intuitionistic fuzzy regular G_δ frontier of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as
IF rG_δFR(A)= IF r$G_\delta$cl(A) $\cap$ IF $rG_\delta cl(\overline{A})$.

Proposition 4.1. Let (X, T) be an intuiionistic fuzzy topological space. For any intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ of an intuitionistic fuzzy topological space (X, T), the following statements hold.

(i) IF rG_δb$(A) \subseteq$ IF rb(A).

(ii) IF rG_δFR$(A) \subseteq$ IF rFR(A).

(iii) IF rG_δb$(A) \subseteq$ IF rG_δFR(A).

(iv) IF rG_δFR$(A) =$ IF rG_δFR$(\overline{A})$.

(v) IF rG_δFR(IF r$G_\delta$int$(A)) \subseteq$ IF rG_δFR(A).

(vi) IF rG_δFR(IF r$G_\delta$cl$(A)) \subseteq$ IF rG_δFR(A).

Definition 4.5. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, T). Then intuitionistic fuzzy regular exterior of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as IF rExt(A)=IF rint$(\overline{A})$.

Definition 4.6. Let (X, T) be an intuitionistic fuzzy topological space and let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space (X, T). Then intuitionistic fuzzy regular G_δ exterior of an intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is denoted and defined as IF rG_δExt(A)=IF r$G_\delta$int$(\overline{A})$.

Proposition 4.2. Let (X, T) be an intuiionistic fuzzy topological space. For any intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ of an intuitionistic fuzzy topological space (X, T), the following statements hold.

(i) IF rExt$(A) \subseteq$ IF rG_δExt(A).

(ii) IF rG_δExt$(A) =$ IF r$G_\delta$int$(\overline{A}) = \overline{IFrG_\delta cl(A)}$.

(iii) IF rG_δExt(IF rG_δExt$(A)) =$ IF r$G_\delta$int(IF r$G_\delta$cl$(A))$.

(iv) If $A \subseteq B$ then IF rG_δExt$(A) \supseteq$ IF rG_δExt(B).

(v) IF rG_δExt$(0_\sim)$= $1_\sim$.

(vi) IF rG_δExt$(1_\sim)$= $0_\sim$.

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b-connectedness in smooth fuzzy topological spaces

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Abstract The purpose of this paper is to introduce the concept of b-connectedness in smooth fuzzy topological space by using r-fuzzy b-open sets and study some of their fundamental properties. Further, several characterizations of these spaces are established.

Keywords Fuzzy b-connectedness, fuzzy super b-connectedness, fuzzy strongly b-connectedness, fuzzy weakly b-connectedness, fuzzy upper* b-continuous, fuzzy lower* b-continuous.

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§1. Introduction and preliminaries

The concept of the fuzzy set was introduced by Zadeh [8] in his classical paper. Fuzzy sets have applications in many fields such as information [4] and control [6]. In 1996, Andrijevic [1] introduced a new class of generalized open sets called b-open sets into the field of Topology. Later on Caldas. M and Jafari. S [2] investigated some applications of b-open sets in topological spaces. The concept of semi-connectedness in fuzzy topological spaces was introduced by Uma. M. K, Roja. E and Balasubramanian. G [7]. In this paper, making use of r-fuzzy b-open sets, we introduce the concept of b-connectedness in the sense of Sostak. A. P [5]. Further, several fundamental properties and characterizations of the spaces are introduced.

Throughout this paper, let X be a non-empty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$,$\overline{\alpha}(x) = \alpha$, for all $x \in X$.

Definition 1.1.[5] A function $T : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:

(a) $T(\overline{0}) = T(\overline{1}) = 1$.

(b) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.

(c) $T(\vee_{i\in\Gamma}\mu_i) \geq \wedge_{i\in\Gamma}T(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma} \in I^X$.

The pair (X, T) is called a smooth fuzzy topological space.

Remark 1.1. Let (X, T) be a smooth fuzzy topological space. Then, for each $r \in I$, $T_r = \{\mu \in I^X : T(\mu) \geq r\}$ is Chang's fuzzy topology on X.

Proposition1.1.[3] Let (X, T) be a smooth fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$ an operator $C_T : I^X \times I_0 \to I^X$ is defined as follows:

$$C_T(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, T(\overline{1} - \mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $C_T(\overline{0}, r) = \overline{0}$.

(2) $\lambda \leq C_T(\lambda, r)$.

(3) $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.

(4) $C_T(\lambda, r) \leq C_T(\lambda, s)$, if $r \leq s$.

(5) $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.2.[3] Let (X, T) be a smooth fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$ an operator $I_T : I^X \times I_0 \to I^X$ is defined as follows:

$$I_T(\lambda, r) = \vee\{\mu : \mu \leq \lambda, T(\mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $I_T(\overline{1} - \lambda, r) = \overline{1} - C_T(\lambda, r)$.

(2) $I_T(\overline{1}, r) = \overline{1}$.

(3) $\lambda \geq I_T(\lambda, r)$.

(4) $I_T(\lambda, r) \wedge I_T(\mu, r) = I_T(\lambda \wedge \mu, r)$.

(5) $I_T(\lambda, r) \geq I_T(\lambda, s)$, if $r \leq s$.

(6) $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

§2. Fuzzy b-connected spaces and fuzzy super b-connected spaces

Definition 2.1. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$:

(1) λ is called r-fuzzy b-open if $\lambda \leq C_T(I_T(\lambda, r), r) \bigvee I_T(C_T(\lambda, r), r)$.

(2) The complement of an r-fuzzy b-open set is r-fuzzy b-closed.

(3) The r-fuzzy b-interior of λ, denoted by $b - I_T(\lambda, r)$ is defined by
$b - I_T(\lambda, r) = \bigvee\{\mu : \mu \leq \lambda, \mu$ is r-fuzzy b-open$\}$.

(4) The r-fuzzy b-closure of λ, denoted by $b - C_T(\lambda, r)$ is defined by
$b - C_T(\lambda, r) = \bigwedge\{\mu : \mu \geq \lambda, \mu$ is r-fuzzy b-closed$\}$.

Remark 2.1. Let (X, T) be a smooth fuzzy topological space. Every $\lambda \in I^X$ with $T(\overline{1} - \lambda, r) \geq r$ is r-fuzzy b-closed and hence $b - C_T(\lambda, r) \leq C_T(\lambda, r)$.

Definition 2.2. A smooth fuzzy topological space (X, T) is fuzzy b-connected iff it has no r-fuzzy b-open sets $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$ such that $\lambda_1 + \lambda_2 = \overline{1}$.

Proposition 2.1. A smooth fuzzy topological space (X, T) is fuzzy b-connected iff it has no $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$ such that $\lambda_1 + \lambda_2 = \overline{1}, b - C_T(\lambda_1, r) + \lambda_2 = \lambda_1 + b - C_T(\lambda_2, r) = \overline{1}$.

Definition 2.3. Let (X, T) be a smooth fuzzy topological space and let $A \subset X$. Then A is said to be a fuzzy b-connected subset of X if (A, T_A) is a fuzzy b-connected space as a fuzzy subspace of (X, T).

Proposition 2.2. Let (X, T) be a smooth fuzzy topological space and let A be a fuzzy b-connected subset of X. If $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$ are r-fuzzy b-open sets satisfying $\lambda_1 + \lambda_2 = \overline{1}$, $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ then either λ_1/A or $\lambda_2/A = \overline{1}$.

Definition 2.4. Let (X, T) be a smooth fuzzy topological space and let $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. λ_1 and λ_2 are said to be fuzzy b-separated if $b{-}C_T(\lambda_1, r) + \lambda_2 \leq \overline{1}$ and $\lambda_1 + b{-}C_T(\lambda_2, r) \leq \overline{1}$.

Proposition 2.3. Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be a family of fuzzy b-connected subsets of (X, T) such that for each α and β in Γ with $\alpha \neq \beta$, 1_{A_α} and 1_{A_β} are not fuzzy b-separated from each other. Then $\bigcup_{\alpha \in \Gamma} A_\alpha$ is a fuzzy b-connected subset of X.

Proof. Suppose that $Y = \bigcup_{\alpha \in \Gamma} A_\alpha$ is not a fuzzy b-connected subset of X. Then there exist r-fuzzy b-open sets $\lambda_1, \lambda_2 \in I^Y$ with $\lambda_1 \neq \overline{0}, \lambda_2 \neq \overline{0}$ and $\lambda_1 + \lambda_2 = \overline{1}$. Fix $\alpha_0 \in \Gamma$, then A_{α_0} is a fuzzy b-connected subset of Y as it is in X. By Proposition 2.2, either $\lambda_1/A_{\alpha_0} = 1_{A_\alpha}/A_\alpha$ or $\lambda_2/A_{\alpha_0} = 1_{A_\alpha}/A_\alpha$. Without loss of generality, we shall assume that $\lambda_1/A_{\alpha_0} = 1_{A_{\alpha_0}}/A_{\alpha_0} = \overline{1}$.

Define μ_1 and μ_2 on X as follows:

$$\mu_1(x) = \begin{cases} \lambda_1(x), & x \in Y; \\ 0, & x \in X/Y. \end{cases}$$

$$\mu_2(x) = \begin{cases} \lambda_2(x), & x \in Y; \\ 0, & x \in X/Y. \end{cases}$$

Now $\mu_1/Y = \lambda_1$. This implies $b - C_T(\mu_1/Y, r) = b - C_T(\lambda_1, r)$. Similarly $b - C_T(\mu_2/Y, r) = b - C_T(\lambda_2, r)$. Further, $1_{A_{\alpha_0}} \leq \mu_1$. Therefore, $b - C_T(1_{A_{\alpha_0}}, r) \leq b - C_T(\mu_1, r)$. Let $\alpha \in \Gamma - \alpha_0$. Now A_α is a fuzzy b-connected subset of Y. Then either $\lambda_1/A_\alpha = \overline{1}$ or $\lambda_2/A_\alpha = \overline{1}$. We shall show that $\lambda_2/A_\alpha \neq 1_{A_\alpha}/A_\alpha$. Suppose that $\lambda_2/A_\alpha = 1_{A_\alpha}/A_\alpha$. Now $1_{A_\alpha} \leq \mu_2$. Therefore $b - C_T(1_{A_\alpha}, r) \leq b - C_T(\mu_2, r)$. Since $b - C_T(\lambda_1, r) + \lambda_2 = \lambda_1 + b - C_T(\lambda_2, r) = \overline{1}$ it follows that $b - C_T(\mu_1, r) + \mu_2 \leq \overline{1}$ and $\mu_1 + b - C_T(\mu_2, r) \leq \overline{1}$. Now, $b - C_T(1_{A_{\alpha_0}}, r) + 1_{A_\alpha} \leq b - C_T(\mu_1, r) + \mu_2 \leq \overline{1}$ and $b - C_T(1_{A_\alpha}, r) + 1_{A_{\alpha_0}} \leq b - C_T(\mu_2, r) + \mu_1 \leq \overline{1}$. This implies $1_{A_{\alpha_0}}$ and 1_{A_α} are fuzzy b-separated from each other, contradicting the hypothesis. Hence $\lambda_2/A_\alpha \neq 1_{A_\alpha}/A_\alpha$. Therefore $\lambda_1/A_\alpha = 1_{A_\alpha}/A_\alpha$ for each $\alpha \in \Gamma$. This implies $\lambda_1 = 1_{A_\alpha}$. But $\lambda_1 + \lambda_2 = \overline{1}$, hence $\lambda_2(x) = 0$ for every $x \in Y$. But $\lambda_2 \neq \overline{0}$. This implies that Y is not a fuzzy b-connected subset of X is false.

Proposition 2.4. Let A and B be any two subsets of a smooth fuzzy topological space (X, T) such that $1_A \leq 1_B \leq b - C_T(1_A, r)$ and if A is a fuzzy b-connected subset of X, then so is B.

Proof. Let us suppose that B is not a fuzzy b-connected subset of X. Then there exist r-fuzzy b-open sets $\lambda_1, \lambda_2 \in I^X$ such that $\lambda_1/B \neq \overline{0}, \lambda_2/B \neq \overline{0}$ and

$$\lambda_1/B + \lambda_2/B = \overline{1}. \tag{1}$$

To show that $\lambda_1/A \neq \overline{0}$. If $\lambda_1/A = \overline{0}$, then $\lambda_1 + 1_A \leq \overline{1}$. This implies $\lambda_1 + b - C_T(1_A, r) \leq \overline{1}$. Now $\lambda_1 + 1_B \leq \lambda_1 + b - C_T(1_A, r) \leq \overline{1}$. So $\lambda_1 + 1_B \leq \overline{1}$. This implies $\lambda_1/B = \overline{0}$, which is a contradiction. Therefore $\lambda_1/A \neq \overline{0}$. Similarly $\lambda_2/A \neq \overline{0}$. Now by (1) and $1_A \leq 1_B$, we find that $\lambda_1/A + \lambda_2/A = \overline{1}$. This implies that A is not a fuzzy b-connected subset of (X, T), which is a contradiction.

Proposition 2.5. If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family of fuzzy b-connected subsets of a smooth fuzzy topological space (X, T) and $\bigcap_{\alpha\in\Gamma} A_\alpha \neq \phi$, then $\bigcup_{\alpha\in\Gamma} A_\alpha$ is a fuzzy b-connected subset of (X, T).

Definition 2.5. A smooth fuzzy topological space (X, T) is called fuzzy super b-connected if it has no r-fuzzy b-open sets $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$ such that $\lambda_1 + \lambda_2 \leq \overline{1}$.

Proposition 2.6. If (X, T) is a smooth fuzzy topological space, then the following statements are equivalent

(1) (X, T) is fuzzy super b-connected.

(2) $b - C_T(\lambda, r) = \overline{1}$, for every r-fuzzy b-open set $\overline{0} \neq \lambda \in I^X$.

(3) $b - I_T(\lambda, r) = \overline{0}$, for every r-fuzzy b-closed set $\overline{1} \neq \lambda \in I^X$.

(4) (X, T) has no r-fuzzy b-open sets $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$, such that $b - C_T(\lambda_1, r) + \lambda_2 = \lambda_1 + b - C_T(\lambda_2, r) = \overline{1}$.

(5) (X, T) has no r-fuzzy b-closed sets $\lambda_1 \neq \overline{1}$ and $\lambda_2 \neq \overline{1}$, such that $b - I_T(\lambda_1, r) + \lambda_2 = \lambda_1 + b - I_T(\lambda_2, r) = \overline{1}$.

Definition 2.6. Let (X, T) be a smooth fuzzy topological space and let $A \subset X$. A is said to be fuzzy super b-connected subset of X if (A, T_A) is a fuzzy super b-connected space as fuzzy subspace of (X, T).

Proposition 2.7. Let (X, T) be a smooth fuzzy topological space. Let A be a fuzzy super b-connected subset of X. If there exist r-fuzzy b-closed sets $\lambda_1, \lambda_2 \in I^X$ such that $\overline{1} - \lambda_1 + \lambda_2 = \lambda_1 + \overline{1} - \lambda_2 = \overline{1}$, then $\lambda_1/A = \overline{1}$ or $\lambda_2/A = \overline{1}$.

Proposition2.8. Let (X, T) be a smooth fuzzy topological space and let $A \subset X$ be a fuzzy super b-connected subset of X such that 1_A is a r-fuzzy b-open set in (X, T). If $\lambda \in I^X$ is a r-fuzzy b-open set, then either $1_A \leq \lambda$ or $1_A \leq \overline{1} - \lambda$.

Proposition 2.9. Let $\{O_\alpha\}_{\alpha\in A}$ be a family of subsets of a smooth fuzzy topological space (X, T). If $\bigcap_{\alpha\in A} O_\alpha \neq \phi$ and each O_α is r-fuzzy super b-connected subset of X, then $\bigcup_{\alpha\in A} O_\alpha$ is also a fuzzy super b-connected subset of X.

Proposition 2.10. Let (X, T) be a smooth fuzzy topological space. If A and B are fuzzy super b-connected subsets of X with $b - I_T(1_B/A, r) \neq \overline{0}$ or $b - I_T(1_A/B, r) \neq \overline{0}$, then $A \bigcup B$ is a fuzzy super b-connected subset of X.

Proof. Suppose that $Y = A \bigcup B$ is not a fuzzy super b-connected subset of X. Then there exist r-fuzzy b-open sets $\lambda_1, \lambda_2 \in I^X$ such that $\lambda_1/Y \neq \overline{0}, \lambda_2/Y \neq \overline{0}$ and $\lambda_1/Y + \lambda_2/Y \leq \overline{1}$. Since A is a fuzzy super b-connected subset of X, either $\lambda_1/A = \overline{0}$ or $\lambda_2/A = \overline{0}$. Without loss of generality, assume that $\lambda_2/A = \overline{0}$. Since B is a fuzzy super b-connected subset we have

(1) $\lambda_1/A \neq \overline{0}$ (2) $\lambda_2/B \neq \overline{0}$ (3) $\lambda_2/A = \overline{0}$ (4) $\lambda_1/B = \overline{0}$.

Therefore, we get (5) $\lambda_1/A + b - I_T(1_B/A, r) \leq \overline{1}$. If $b - I_T(1_B/A, r) \neq \overline{0}$, then (1) and (5) implies that A is not a fuzzy super b-connected subset of X. Similarly if $b - I_T(1_A/B, r) \neq \overline{0}$, then (2) and $\lambda_2/B + b - I_T(1_A/B, r) \leq \overline{1}$ implies that B is not a fuzzy super b-connected subset of X, which is a contradiction.

Proposition 2.11. Let (X,T) be a smooth fuzzy topological space. If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family of fuzzy super b-connected subsets of X and $\bigwedge_{\alpha\in\Gamma} 1_{A_\alpha} \neq \overline{0}$, then $\bigcup_{\alpha\in\Gamma} A_\alpha$ is a fuzzy super b-connected subset of X.

Proposition 2.12. Let (X,T) be a smooth fuzzy topological space. Suppose that X is fuzzy super b-connected and C is a fuzzy super b-connected subset of X. Further, suppose that $X\backslash C$ contains a set V such that $1_V/X\backslash C$ is a r-fuzzy b-open set in the fuzzy subspace $X\backslash C$ of X. Then $C\bigcup V$ is a fuzzy super b-connected subset of X.

Proof. Suppose that $Y = C\bigcup V$ is not a fuzzy super b-connected subset of X. Then there exist r-fuzzy b-open sets λ_1 and λ_2 such that $\lambda_1/Y \neq \overline{0}$, $\lambda_2/Y \neq \overline{0}$ and $\lambda_1/Y + \lambda_2/Y \leq \overline{1}$. Now C is a fuzzy super b-connected subset of X. Then either $\lambda_1/C = \overline{0}$ or $\lambda_2/C = \overline{0}$. Without loss of generality, we assume that $\lambda_1/C = \overline{0}$. Then $\lambda_1/V \neq \overline{0}$. Let $\lambda_V \in I^X$ be defined as follows

$$\lambda_V(x) = \begin{cases} \lambda_1(x), & x \in V; \\ 0, & x \in X/V. \end{cases}$$

Now, $\lambda_V = \lambda_1 \bigwedge 1_V$ and hence λ_V is r-fuzzy b-open. We show that $b - C_T(\lambda_V, r) \neq \overline{1}$. Now, $\lambda_1/Y + \lambda_2/Y \leq \overline{1}$ implies $\lambda_V + \lambda_2 \leq \overline{1}$. Further, $b - C_T(\lambda_V, r) \leq b - C_T(\overline{1} - \lambda_2, r) = \overline{1} - \lambda_2$. Hence $b - C_T(\lambda_V, r) \neq \overline{1}$ as $\lambda_2 \neq \overline{0}$. Thus (X,T) is not a fuzzy super b-connected space which is a contradiction.

Proposition 2.13. Let (X,T) be a smooth fuzzy topological space. If A and B are subsets of X such that, $1_A \leq 1_B \leq b - C_T(1_A, r)$ and A is a fuzzy super b-connected subset of X, then B is also a fuzzy super b-connected subset of X.

§3. Fuzzy strongly b-connected space

Definition 3.1. A smooth fuzzy topological space (X,T) is said to be fuzzy strongly b-connected, if it has no r-fuzzy b-closed sets $\lambda_1 \neq \overline{0}, \lambda_2 \neq \overline{0}$ such that $\lambda_1 + \lambda_2 \leq \overline{1}$.

If (X,T) is not fuzzy strongly b-connected, then it is called as fuzzy weakly b-connected.

Proposition 3.1. A smooth fuzzy topological space (X,T) is said to be fuzzy strongly b-connected, if it has no r-fuzzy b-open sets $\lambda \neq \overline{1}, \mu \neq \overline{1}$, such that $\lambda + \mu \geq \overline{1}$.

Proposition 3.2. Let (X,T) be a smooth fuzzy topological space and let $A \subset X$. Then A is a fuzzy strongly b-connected subset of X iff for any r-fuzzy b-open sets $\lambda_1, \lambda_2 \in I^X$ with $\lambda_1 + \lambda_2 \geq 1_A$, either $1_A \leq \lambda_1$ or $1_A \leq \lambda_2$.

Proposition 3.3. If F is a subset of a smooth fuzzy topological space (X,T) such that 1_F is r-fuzzy b-closed in X, then X is fuzzy strongly b-connected implies that F is fuzzy strongly b-connected.

Proof. Suppose that F is not fuzzy strongly b-connected. Then there exist r-fuzzy b-closed sets $\lambda_1, \lambda_2 \in I^X$ such that (i) $\lambda_1/F \neq \overline{0}$, (ii) $\lambda_2/F \neq \overline{0}$ and (iii) $\lambda_1/F + \lambda_2/F \leq \overline{1}$. Now (iii) implies that $(\lambda_1 \bigwedge 1_F) + (\lambda_2 \bigwedge 1_F) \leq \overline{1}$. (i) and (ii) implies that $\lambda_1 \bigwedge 1_F \neq \overline{0}$ and $\lambda_2 \bigwedge 1_F \neq \overline{0}$. Hence (X,T) is not fuzzy strongly b-connected, which is a contradiction.

Definition 3.2. Let (X,T) and (Y,S) be any two smooth fuzzy topological spaces and let $f : (X,T) \to (Y,S)$ be a onto mapping. Then f is said to be a M-fuzzy b-continuous map if

$f^{-1}(\lambda)$ is an r-fuzzy b-open set for each r-fuzzy b-open set $\lambda \in I^Y$.

Proposition 3.4. Let (X,T) and (Y,S) be any two smooth fuzzy topological spaces. If $f:(X,T) \to (Y,S)$ is M-fuzzy b-continuous then (X,T) is fuzzy/ fuzzy super/ fuzzy strongly b-connected implies (Y,S) is fuzzy/ fuzzy super/ fuzzy strongly b-connected.

Definition 3.3. Let (X,T) and (Y,S) be any two smooth fuzzy topological spaces and let $F:(X,T) \to (Y,S)$ be a mapping. For each $\alpha \in I^Y$, the fuzzy dual F^* of F is defined as $F^*(\alpha) = sup\{\beta \in I^X / \alpha > F(\beta)\}$.

Definition 3.4. Let (X,T) and (Y,S) be any two smooth fuzzy topological spaces and let $F:(X,T) \to (Y,S)$ be a mapping. F is called fuzzy upper* (lower*) b-continuous iff for each $\lambda \in I^Y$ with $S(\lambda) \geq r (S(\overline{1}-\lambda) \geq r)$, $\mu \in I^X$ is such that $F(\mu) \leq \lambda$ is r-fuzzy b-open (b-closed).

Proposition 3.5. Let F be a mapping from a smooth fuzzy topological space (X,T) to another smooth fuzzy topological space (Y,S). Then

(1) F is fuzzy upper* b-continuous iff $\bigwedge\{F^*(\alpha) : \alpha \leq \lambda\}$ is r-fuzzy b-open for $S(\overline{1}-\lambda) \geq r$, $\lambda \in I^Y$ and $r \in I_0$.

(2) F is fuzzy lower* b-continuous iff $\bigwedge\{F^*(\alpha) : \alpha \leq \mu\}$ is r-fuzzy b-closed for $S(\mu) \geq r$, $\mu \in I^Y$ and $r \in I_0$.

Proof. (1)Suppose that $S(\overline{1}-\lambda) \geq r$, $\lambda \in I^Y$ and $r \in I_0$. F is fuzzy upper* b-continuous

$\Leftrightarrow \mu \in I^X$ such that $F(\mu) \leq \overline{1} - \lambda$ is r-fuzzy b-open.

$\Leftrightarrow \{\mu \in I^X : F(\mu) \leq \gamma\}$ is r-fuzzy b-open for any $\gamma \leq \lambda$.

$\Leftrightarrow \bigwedge\{\mu \in I^X : F(\mu) \leq \gamma \,, \gamma \leq \lambda\}$ is r-fuzzy b-open.

$\Leftrightarrow \bigwedge\{F^*(\gamma) : \gamma \leq \lambda\}$ is r-fuzzy b-open for $S(\overline{1}-\lambda) \geq r$, $\lambda \in I^Y$.

(2) The proof of (2) is similar to (1).

Proposition 3.6. Let (X,T) and (Y,S) be any two smooth fuzzy topological spaces and let $F:(X,T) \to (Y,S)$ be a mapping. If $\bigwedge\{F(\mu) : \mu \in I^X\} \neq \overline{0}$, then

(1) $S(\overline{1} - F(\mu)) \geq r$, for each $\mu \in I^X$.

(2) F is fuzzy upper* b-continuous.

(3) The subset A is fuzzy b-connected for each $\overline{0} \neq \lambda \in I^X$ is such that $1_A = \bigwedge\{F(\mu) : \mu \leq \lambda\}$.

(4) The subset B is fuzzy b-connected for each $\gamma \in I^Y$ with $1_B = \bigwedge\{F^*(\eta) : \eta \leq \gamma\}$ is fuzzy b-connected.

Proof. Let $\delta = \bigwedge\{F(\mu) : \mu \in I^X\}$ and $r = \frac{1}{4}$.

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0}, \overline{1}; \\ 0, & otherwise. \end{cases}$$

$$S(\theta) = \begin{cases} 1, & \theta = \overline{0}, \overline{1}; \\ \frac{1}{2}, & \theta = \zeta < \delta; \\ 0, & otherwise. \end{cases}$$

(1) For each $\mu \in I^X, S(\overline{1} - F(\mu)) \geq r$ because $F(\mu) \geq \delta$.

(2) Let $S(\zeta) \geq r$, then $\delta > \zeta$. Then there exist $\mu \in I^X$ such that $F(\mu) \leq \zeta$ is r-fuzzy b-open and hence F is fuzzy upper* b-continuous.

(3) For each $\overline{0} \neq \lambda \in I^X$ and the subset A with $1_A = \bigwedge\{F(\mu) : \mu \leq \lambda\}$, $1_A \neq \overline{0}$ or $1_A \neq \overline{1}$. This implies A is fuzzy b-connected.

(4) Obvious.

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Strong Lucas graceful labeling for some graphs

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Abstract Let G be a (p,q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, ..., l_a\}(a \epsilon N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling. Also an injective function $f : V(G) \to \{l_0, l_1, l_2, ..., l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$.Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling. In this paper, we show that some graphs namely, C_n, $C_n^{(t)}$, $F_n^{(t)}$, $C_n@2P_1$, $K_{1,n} \odot C_3$, $C_3@2P_n$, $C_n@K_{1,t}$, $B(m,n)$, $S_{m,n}@C_t$ and $S_{m,n}^{(t)}$ admit strong Lucas graceful labeling.

Keywords Graceful labeling, Lucas graceful labeling, strong Lucas graceful labeling.

§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_n. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching a pendant vertex to each vertex of G. The concept of graceful labeling was introduced by Rosa [3] in 1967.

A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{1, 2, 3, \ldots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K. M. Kathiresan and S. Amutha [4]. We call a function, a Fibonacci graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, ...$, and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concepts, we define the following [5].

Let G be a (p,q) -graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, ..., l_a\}$ $(a \epsilon N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$ with the assumption of $l_0 = 0,\ l_1 = 1,\ l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, ...$[1].

Then G is called Lucas graceful graph if it admits Lucas graceful labeling. Also an injective function $f : V(G) \to \{l_0, l_1, l_2, ..., l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$. Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling.

In this paper, we show that some graphs namely, $C_n, C_n^{(t)}, F_n^{(t)}, C_n@2P_1, K_{1,n}\odot C_3, C_3@2P_n, C_n@K_{1,t}, B(m,n), S_{m,n}@C_t$ and $S_{m,n}^{(t)}$ admit strong Lucas graceful labeling.

§2. Main Results

In this section, we show that some special graphs are strong Lucas graceful graphs.

Definition 2.1.[5] Let G be a (p,q)-graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, ..., l_q\}$, is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, ...$ [1]. Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling.

Theorem 2.2. $C_n (n \geq 3)$ is a strong Lucas graceful graph when $n \equiv 0 (\text{mod } 3)$.

Proof. Let $G = C_n$.

Let $u_1, u_2, ..., u_n$ be the vertices of C_n. So, $|V(G)| = n$ and $|E(G)| = n$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_n\}$ by $f(u_1) = l_0$.

For $s = 1, 2, ..., \frac{n}{3} - 1$, $f(u_i) = l_{2i-3s}$, $3s - 1 \leq i \leq 3s + 1$,

For $s = \frac{n}{3}$, $f(u_i) = l_{2i-3s}$, $3s - 1 \leq i \leq 3s$.

Next, we claim that the edge labels are distinct

$$\begin{aligned} \text{Let } E_1 &= \{f_1(u_1u_2)\} \\ &= \{|f(u_1) - f(u_2)|\} \\ &= \{|l_0 - l_1|\} \\ &= \{l_1\}. \end{aligned}$$

For $s = 1$

$$\begin{aligned} \text{Let } E_2 &= \{f_1(u_iu_{i+1}) : 2 \leq i \leq 3\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|\} \\ &= \{l_2, l_4\}. \end{aligned}$$

$$\begin{aligned} \text{Let } E_3 &= \{f_1(u_{3s+1}u_{3s+2})\} \\ &= \{f_1(u_4u_5)\} \\ &= \{|f(u_4) - f(u_5)|\} \\ &= \{|l_5 - l_4|\} \\ &= \{l_3\}. \end{aligned}$$

For $s = 2$

$$\begin{aligned}
\text{Let } \mathrm{E}_4 &= \{f_1(u_iu_i+1) : 5 \le i \le 6\} \\
&= \{|f(u_5) - f(u_6)|\,, |f(u_6) - f(u_7)|\} \\
&= \{|l_4 - l_6|\,, |l_6 - l_8|\} \\
&= \{l_5, l_7\}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } \mathrm{E}_5 &= \{f_1(u_{3s+1}u_{3s+2}\} \\
&= \{f_1(u_7u_8)\} \\
&= \{|f(u_7) - f(u_8)|\} \\
&= \{|l_8 - l_7|\} \\
&= \{l_6\}.
\end{aligned}$$

In the same way, we find $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, etc.

For $s = \frac{n}{3} - 1$

$$\begin{aligned}
\text{Let } \mathrm{E}_{2(\frac{n}{3}-1)} &= \{f_1(u_iu_{i+1}) : 3s - 1 \le i \le 3s\} \\
&= \{f_1(u_iu_{i+1}) : n - 4 \le i \le n - 3\} \\
&= \{|f(u_i) - f(u_{i+1})| : n - 4 \le i \le n - 3\} \\
&= \{|f(u_{n-4}) - f(u_{n-3})|\,, |f(u_{n-3}) - f(u_{n-2})|\} \\
&= \{|l_{n-5} - l_{n-3}|, |l_{n-3} - l_{n-1}|\} \\
&= \{l_{n-4}, l_{n-2}\}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } \mathrm{E}_{2(\frac{n}{3}-1)+1} &= \{f_1(u_iu_{i+1}) : i = n - 2\} \\
&= \{|f(u_i) - f(u_{i+1})| : i = n - 2\} \\
&= \{|f(u_{n-2}) - f(u_{n-1})|\} \\
&= \{|l_{n-1} - l_{n-2}|\} \\
&= \{l_{n-3}\}.
\end{aligned}$$

For $s = \dfrac{n}{3}$

$$\begin{aligned}
\text{Let } \mathrm{E}_{\frac{2n}{3}} &= \{f_1(u_iu_{i+1}) : n - 1 \le i \le n\} \\
&= \{|f(u_i) - f(u_{i+1})| : n - 1 \le i \le n\ \} \\
&= \{|f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_{n+1})|\} \\
&= \{|f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_1)|\} \\
&= \{|l_{n-2} - l_n|, |l_n - l_0|\} \\
&= \{l_{n-1}, l_n\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{\frac{2n}{3}} E_i = \{l_1,\ l_2,\ \ldots,\ l_n\}$.

So, the edges of $C_n, n \equiv 0(\text{mod } 3)$, receive the distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, $C_n,\ n \equiv 0(\text{mod } 3)$, is a strong Lucas graceful graph.

Example 2.3. C_{18} admits strong Lucas graceful Labeling.

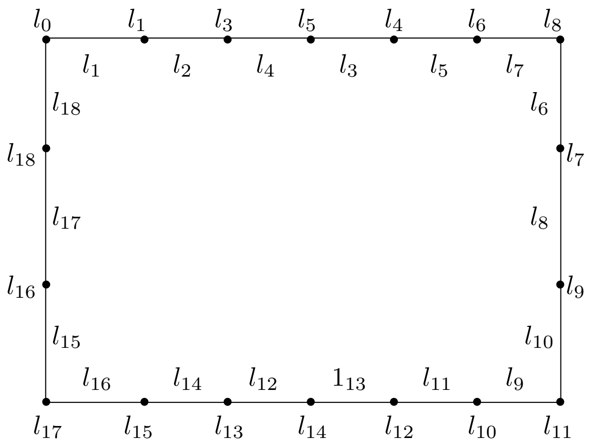

Fig. 1 C_{18}

Definition 2.4.[2] The graph $C_n^{(t)}$ denotes the one point union of t copies of a cycle C_n.

Theorem 2.5. $C_n^{(t)}$ is a strong Lucas graceful graph, when $n \equiv 0(\text{mod } 3)$.

Proof. Let $G = C_n^{(t)}$. Let $V(G) = \{u_0\} \bigcup \left\{ \bigcup_{s=1}^{\frac{n}{3}} \{u_j^i\} : 1 \le i \le t \text{ and } 3\text{s} - 1 \le \text{j} \le 3\text{s} + 1 \right\}$ be the vertex set of G.

Then $|V(G)| = (n-1)t + 1$ and $|\text{E(G)}| = \text{nt}$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{nt}\}$ by $f(u_0^i) = l_0$ for $i = 1, 2, 3, ..., t$.

For $s = 1,\ 2,\ ...,\dfrac{n}{3} - 1$, $f(u_j^i) = l_{n(i-1)+2j-3s}$, $3s - 1 \le j \le 3s + 1$ and for $i = 1, 2, 3, ..., t$.

For $s = \frac{n}{3}$, $f(u_j^i) = l_{n(i-1)+2j-3s}$, $3s - 1 \le j \le 3s$ and for $i = 1, 2, 3, ..., t$.

Next, we claim that the edge labels are distinct.

$$
\begin{aligned}
\text{Let } \mathrm{E}_0 &= \{f_1({u_1}^i {u_2}^i) : 1 \le i \le t\} \cup \{f_1({u_1}^i {u_n}^i) : 1 \le i \le t\} \\
&= \{|f({u_1}^i) - f({u_2}^i)| : 1 \le i \le t\} \cup \{|f({u_1}^i) - f({u_n}^i)| : 1 \le i \le t\} \\
&= \bigcup_{i=1}^{t} \{\{|f({u_1}^i) - f({u_2}^i)|\} \cup |f({u_1}^i) - f({u_n}^i)|\} \\
&= \bigcup_{i=1}^{t} \{\{|l_0 - l_{n(i-1)+1}|\} \cup \{|l_0 - l_{ni}|\}\} \\
&= \bigcup_{i=1}^{t} \{\{l_{n(i-1)+1}\} \cup \{l_{ni}\}\}.
\end{aligned}
$$

For $s = 1$

$$
\begin{aligned}
E_1^1 &= \bigcup_{i=1}^{t} \{f_1(u^i_{(n-1)(i-1)+j} u^i_{(n-1)(i-1)+j+1}) : 2 \le j \le 3\} \\
&= \bigcup_{i=1}^{t} \{|\ f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1})\ | : 2 \le j \le 3\}
\end{aligned}
$$

$$\begin{aligned}
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid: 2 \le j \le 3\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3} - l_{n(i-1)+2j-1} \mid: 2 \le j \le 3\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+2}, l_{n(i-1)+4}\}\\
&= \{l_2, l_{n+2}, l_{2n+2}, \dots, l_{n(t-1)+2}, l_4, l_{n+4}, l_{2n+4}, \dots, l_{n(t-1)+4}\}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } \mathrm{E}_1^2 &= \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : j = 4\}\\
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid: j = 4\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid: j = 4\}\\
&= \{\mid l_{n(i-1)+5} - l_{n(i-1)+4} \mid\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+3}\}\\
&= \{l_3, l_{n+3}, l_{2n+3}, \dots, l_{n(t-1)+3}\}.
\end{aligned}$$

For $s = 2$

$$\begin{aligned}
E_2^1 &= \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : 5 \le j \le 6\}\\
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid: 5 \le j \le 6\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid: 5 \le j \le 6\}\\
&= \{\mid l_{n(i-1)+2j-6} - l_{n(i-1)+2j-4} \mid 5 \le j \le 6\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+2j-5} : 5 \le j \le 6\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+5}, l_{n(i-1)+7}\}\\
&= \{l_5, l_{n+5}, l_{2n+5}, \dots, l_{n(t-1)+5}, l_7, l_{n+7}, l_{2n+7}, \dots, l_{n(t-1)+7}\}.
\end{aligned}$$

$$\text{Let } \mathrm{E}_2^2 = \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : j = 7\}$$

$$
\begin{aligned}
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid : j = 7\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid : j = 7\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-6} - l_{n(i-1)+2j+2-9} \mid : j = 7\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-6} - l_{n(i-1)+2j+2-7} \mid : j = 7\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+8} - l_{n(i-1)+7} \mid\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+6}\}\\
&= \{l_6, l_{n+6}, l_{2n+6}, \ldots, l_{n(t-1)+6}\}.
\end{aligned}
$$

For $s = \frac{n}{3} - 1$

$$
\begin{aligned}
E_1^{\frac{n}{3}-1} &= \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : n-4 \leq j \leq n-3\}\\
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid : n-4 \leq j \leq n-3\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid : n-4 \leq j \leq n-3\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+2n-8-n+3+1} - l_{n(i-1)+2n-6-n+3+1}\}\\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+n-4}, l_{n(i-1)+n-2}\}\\
&= \bigcup_{i=1}^{t}\{l_{ni-4}, l_{ni-2}\}\\
&= \{l_{n-4}, l_{2n-4}, \ldots, l_{nt-4}, l_{n-2}, l_{2n-2}, \ldots, l_{nt-2}\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } \mathrm{E}_2^{\frac{n}{3}-1} &= \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : j = n-2\}\\
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid : j = n-2\}\\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2j-3s} - l_{n(i-1)+2j+2-3s} \mid : j = n-2\}
\end{aligned}
$$

$$
\begin{aligned}
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2(n-2)-n+3} - l_{n(i-1)+2(n-2)+2-n} \mid\} \\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+n-1} - l_{n(i-1)+n-2}\} \\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+n-3}\} \\
&= \bigcup_{i=1}^{t}\{l_{ni-3}\} \\
&= \{l_{n-3}, l_{2n-3}, \ldots, l_{nt-3}\}.
\end{aligned}
$$

For $s = \frac{n}{3}$

$$
\begin{aligned}
\text{Let } \mathrm{E}_1^{\frac{n}{3}} &= \bigcup_{i=1}^{t}\{f_1(u^i_{(n-1)(i-1)+j}u^i_{(n-1)(i-1)+j+1}) : j = n\} \\
&= \bigcup_{i=1}^{t}\{\mid f(u^i_{(n-1)(i-1)+j}) - f(u^i_{(n-1)(i-1)+j+1}) \mid : j = n\} \\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+2n-2-n} - l_{n(i-1)+2n-2+2-n} \mid\} \\
&= \bigcup_{i=1}^{t}\{\mid l_{n(i-1)+n-2} - l_{n(i-1)+n} \mid\} \\
&= \bigcup_{i=1}^{t}\{l_{n(i-1)+n-1}\} \\
&= \bigcup_{i=1}^{t}\{l_{ni-1}\} \\
&= \{l_{n-1}, l_{2n-1}, \ldots, l_{nt-1}\}.
\end{aligned}
$$

Now,

$$
E = \left(\bigcup_{i=1}^{\frac{2n}{3}}\left(E_1^i \bigcup E_2^i\right)\right)\bigcup E_1^{\frac{n}{3}}.
$$

So, the edges of $C_n^{(t)}, n \equiv 0 \pmod 3$, receive the distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, $C_n^{(t)}, n \equiv 0 \pmod 3$, is a strong Lucas graceful graph.

Example 2.6. $C_{12}^{(4)}$ admits strong Lucas graceful labeling.

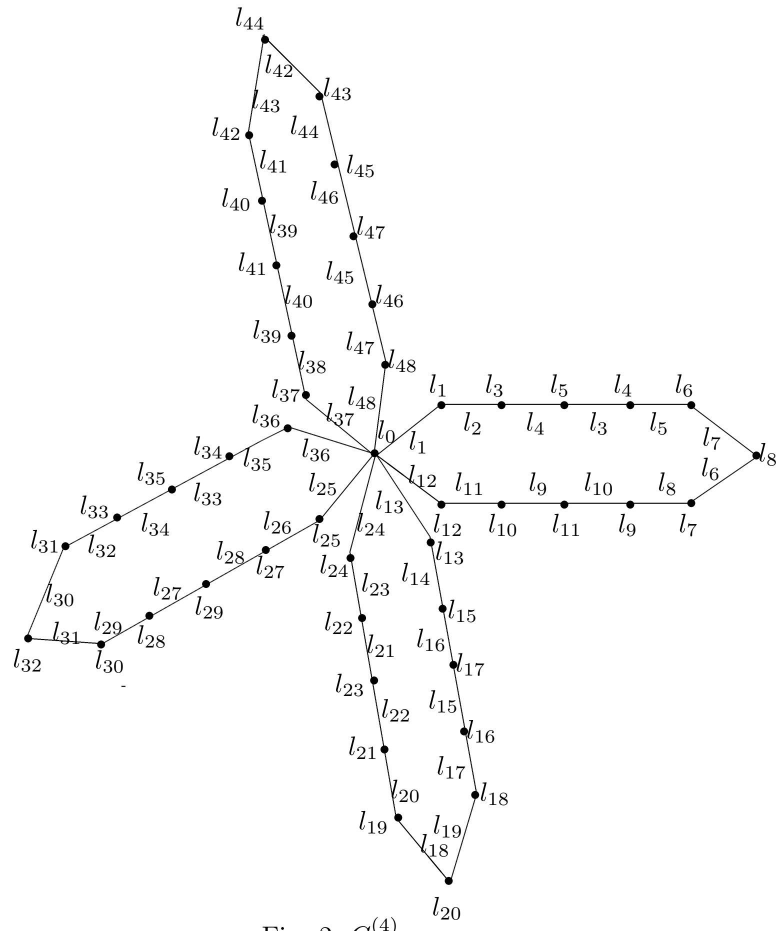

Fig. 2 $C_{12}^{(4)}$

Definition 2.7.[2] The graph $F_n^{(t)}$ denotes the one point union of t copies of a fan F_n.

Theorem 2.8. $F_n^{(t)}$ is a strong Lucas graceful graph.

Proof. Let $G = F_n^{(t)}$. Let $V(G) = \left\{u_j^i : 1 \leq i \leq t \text{ and } 0 \leq \mathrm{j} \leq \mathrm{n}\right\}$ be the vertex set of $F_n^{(t)}$. Now, $|V(G)| = nt + 1$ and $|\mathrm{E(G)}| = (2\mathrm{n}-1)\mathrm{t}$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{(2n-1)t}\}$ by: $f(u_0^i) = f(u_0) = l_0, 1 \leq i \leq t$.

For $i = 1, 2, ..., t,$ and $\mathrm{i} \equiv 1 \pmod 2$, $f\left(u_j^i\right) = l_{(i-1)(2n-1)+2j-1},\ 1 \leq j \leq n$.

For $i = 1, 2, ..., t,$ and $\mathrm{i} \equiv 0 \pmod 2$, $f\left(u_j^i\right) = l_{(i-2)(2n-1)+2(n-1+j)},\ 1 \leq j \leq n$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } \mathrm{E}_1 &= \bigcup_{\substack{i=1 \\ i\equiv 0 (\mathrm{mod}\ 2)}}^{t} \{f_1(u_j^i\ u_{j+1}^i) : 1 \leq j \leq n-1\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (\mathrm{mod}\ 2)}}^{t} \{|\ f(u_j^i) - f(u_{j+1}^i)\ | : 1 \leq j \leq n-1\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (\mathrm{mod}\ 2)}}^{t} \{|\ l_{(2n-1)(i-1)+2j-1} - l_{(2n-1)(i-1)+2j+1}\ | : 1 \leq j \leq n-1\}
\end{aligned}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0 (\mathrm{mod}\ 2)}}^{t} \{l_{(2n-1)(i-1)+2j} : 1 \leq j \leq n-1\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0 (\mathrm{mod}\ 2)}}^{t} \{l_{(2n-1)(i-1)+2}, l_{(2n-1)(i-1)+4}, ..., l_{(2n-1)(i-1)+2n}\}$$

$$= \{l_2, l_{2(2n-1)+2}, ..., l_{(2n-1)(t-1)+2}\} \cup \{l_4, l_{2(2n-1)+4}, ..., l_{(2n-1)(t-1)+4}\} \cup$$

$$\ldots \cup \{l_{2n-2}, l_{2(2n-1)+2n-2}, ..., l_{(2n-1)(t-1)+2n-2}\}.$$

$$\text{Let } E_2 = \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{f_1(u_j^i\ u_{j+1}^i) : 1 \leq j \leq n-1\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{\mid f(u_j^i) - f(u_{j+1}^i) \mid : 1 \leq j \leq n-1\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{\mid l_{(2n-1)(i-2)+2(n-1+j)} - l_{(2n-1)(i-2)+2(n+j)} \mid : 1 \leq j \leq n-1\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{l_{(2n-1)(i-2)+2(n+j)-1} : 1 \leq j \leq n-1\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{l_{(2n-1)(i-2)+2n+1}, l_{(2n-1)(i-2)+2n+3}, ..., l_{(2n-1)(i-2)+4n-3}\}$$

$$= \{l_{2n+1}, l_{2(2n-1)+2n+1}, ..., l_{(2n-1)(t-2)+2n+1}\} \cup \{l_{2n+3}, l_{2(2n-1)+2n+3}, ...,$$

$$l_{(2n-1)(t-2)+2n+3}\} \cup \ldots \cup \{l_{4n-3}, l_{2(2n-1)+4n-3}, ..., l_{(2n-1)(t-2)+4n-3}\}.$$

$$\text{Let } E_3 = \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{f_1(u_0\ u_j^i) : 1 \leq j \leq n\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{\mid f(u_0) - f(u_j^i) \mid : 1 \leq j \leq n\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{\mid l_0 - l_{(2n-1)(i-1)+2j-1)} \mid : 1 \leq j \leq n\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1 (\mathrm{mod}\ 2)}}^{t} \{l_{(2n-1)(i-1)+2j-1} : 1 \leq j \leq n\}$$

$$= \{l_1, l_{2(2n-1)+1}, ..., l_{(2n-1)(t-1)+1}\} \cup \{l_3, l_{2(2n-1)+3}, ..., l_{(2n-1)(t-1)+3}\} \cup$$

$$\ldots \cup \{l_{2n-1}, l_{2(2n-1)+2n-1}, ..., l_{(2n-1)(t-1)+2n-1}\}.$$

$$\begin{aligned}
\text{Let } E_4 &= \bigcup_{\substack{i=1 \\ i\equiv 0 (\text{mod } 2)}}^{t} \{f_1(u_0\ u_j^i) : 1 \leq j \leq n\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (\text{mod } 2)}}^{t} \{| f(u_0) - f(u_j^i) | : 1 \leq j \leq n\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (\text{mod } 2)}}^{t} \{| l_0 - l_{(2n-1)(i-2)+2(n-1+j)} | : 1 \leq j \leq n\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (\text{mod } 2)}}^{t} \{l_{(2n-1)(i-2)+2n}, l_{(2n-1)(i-2)+2n+2}, \ldots, l_{(2n-1)(i-2)+4n-2}\} \\
&= \{l_{2n}, l_{2n+2}, ..., l_{4n-2}\} \cup \{l_{2(2n-1)+2n}, l_{2(2n-1)+2n+2}, ..., l_{2(2n-1)+4n-2}\} \\
&\quad \cup \ldots \cup \{l_{(2n-1)(t-2)}, l_{(2n-1)(t-2)+2n+2}, ..., l_{(2n-1)(t-2)+4n-2}\}.
\end{aligned}$$

Now, $E = E_1 \bigcup E_2 \bigcup E_3 \bigcup E_4$.

So, the edges of $F_n^{(t)}$ receive the distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, $F_n^{(t)}$ is a strong Lucas graceful graph.

Example 2.9. $F_4^{(5)}$ admits strong Lucas graceful labeling.

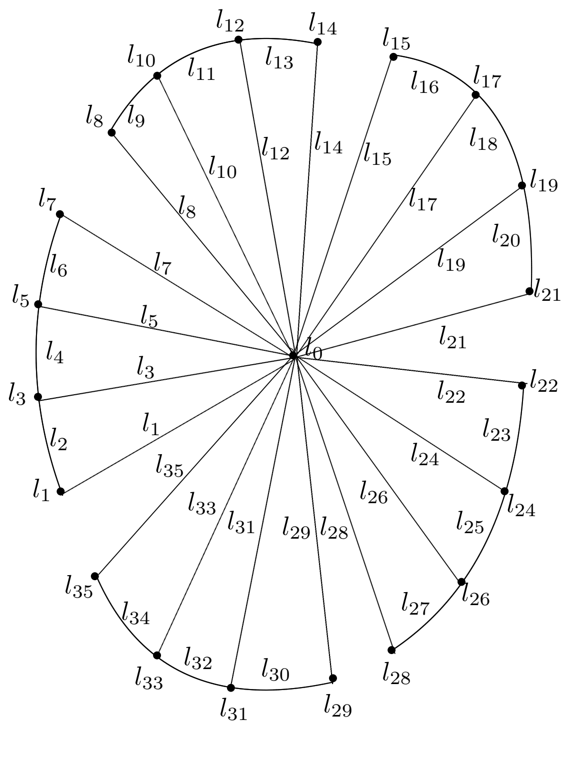

Fig. 3 $F_4^{(5)}$

Definition 2.10.[2] An $(n, 2t)$ graph consists of a cycle of length n with two copies of t edge path attached to two adjacent vertices and it is denoted by $C_n@2P_t$.

Theorem 2.11. $C_n@2P_1$, $n \geq 3$, is a strong Lucas graceful graph.

Proof. Let $G = C_n@2P_1$. Let $V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq 2\}$ be the vertex set of G. Let $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be the edge set of G. So, $|V(G)| = n+2$ and $|E(G)| = n+2$.

Case (i) Suppose $n \equiv 0 (\text{mod } 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{n+2}\}$, by $f(u_1) = l_0$.

For $s = 1, 2, ..., \frac{n-3}{3}$, $f(u_i) = l_{2i+2-3s}$, $3s-1 \leq i \leq 3s+1$,

For $s = \frac{n}{3}$, $f(u_i) = l_{2i+2-3s}$, $3s-1 \leq i \leq 3s$. $f(v_1) = l_1$ and $f(v_2) = l_4$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } E_1 &= \{f_1(u_1v_1), f_1(u_2v_2), f_1(u_1u_2)\} \\
&= \{|f(u_1) - f(v_1)|, |f(u_2) - f(v_2)|, |f(u_1) - f(u_2)|\} \\
&= \{|l_0 - l_1|, |l_3 - l_4|, |l_0 - l_3|\} \\
&= \{l_1, l_2, l_3\}. \\
\text{Let } E_2 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_iu_{i+1}) : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|l_{2i+2-3s} - l_{2i+4-3s}| : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{l_{2i-3s+3} : 3s-1 \leq i \leq 3s\} \\
&= \{l_4, l_6, l_7, l_9, ..., l_{n-2}, l_n\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ... \frac{n-3}{3}$.

$$\begin{aligned}
\text{Let } E_3 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{3s+1}u_{3s+2})\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{3s+1}) - f(u_{3s+2})|\} \\
&= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \ldots, |f(u_{n-2}) - f(u_{n-1})|\} \\
&= \{|l_7 - l_6|, |l_{10} - l_9|, \ldots, |l_{n+1} - l_n|\} \\
&= \{l_5, l_8, ..., l_{n-1}\}.
\end{aligned}$$

For $s = \frac{n}{3}$

$$\text{Let } E_4 = \{f_1(u_iu_{i+1}) : i = 3s-1\}$$

$$\begin{aligned}
&= \{|f(u_i) - f(u_{i+1})| : i = 3s-1\} \\
&= \{|f_{u_{n-1}} - f_{u_n}|\} \\
&= \{l_{2n-2+2-n} l_{2n+2-n}\} \\
&= \{|l_n - l_{n+2}|\} \\
&= \{l_{n+1}\}. \\
\text{Let } E_5 &= \{f_1(u_i u_{i+1}) : i = 3s\} \\
&= \{|f(u_i) - f(u_{i+1})| : i = 3s\} \\
&= \{|f_{u_n} - f_{u_{n+1}}|\} \\
&= \{l_{2n+2-n} l_0\} \\
&= \{|l_{n+2} - l_0|\} \\
&= \{l_{n+2}\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{5} E_i = \{l_1, l_2, \ldots, l_{n+2}\}$. So, the edge labels of G are distinct.

Case (ii) Suppose $n \equiv 1(\text{mod } 3)$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_q\}$ by $f(u_1) = l_0$, $f(v_1) = l_{n+2}$, $f(v_2) = l_n$.

For $s = 1, 2, ... \frac{n-1}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s+1$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } E_1 &= \{f_1(u_1v_1), f_1(u_1u_2), f_1(u_nv_2), f_1(u_nv_1)\} \\
&= \{|f(u_1) - f(v_1)|, |f(u_1) - f(u_2)|, |f(u_n) - f(v_n)|, |f(u_n) - f(v_1)|\} \\
&= \left\{|l_0 - l_{n+2}|, |l_0 - l_1|, |l_{2n-(n-1)} - l_n|, |l_{2n-(n+1)} - l_n|\right\} \\
&= \{l_{n+2}, l_1, l_{n-1}, l_{n+1}\}. \\
\text{Let } E_2 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_i u_{i+1}) : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|l_{2i-3s} - l_{2i+2-3s}| : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{l_{2i-3s+1} : 3s-1 \leq i \leq 3s\} \\
&= \{l_2, l_4\} \cup \{l_5, l_7\} \cup \{l_8, l_{10}\} \cup ... \cup \{l_{n-2}, l_n\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ... \frac{n-1}{3}$.

$$\begin{aligned}
\text{Let } E_3 &= \{f_1(u_i u_{i+1}) : i = 3s+1\} \\
&= \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\
&= \{|f(u_4) - f(u_5)|, |f(u_7), -f(u_8)|, ..., |f(u_{n-3}) - f(u_{n-2})|\}
\end{aligned}$$

$$= \{|l_5 - l_4|, |l_8 - l_7|, ..., |l_{n-3} - l_{n-2}|\}$$
$$= \{l_3, l_6, ..., l_{n-4}\}.$$

Now, $E = E_1 \cup E_2 \cup E_3 = \{l_1, l_2, ..., l_{n+2}\}$. So, the edge labels of G are distinct.

Case (iii) Suppose $n \equiv 2(\text{mod } 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_q\}$ by $f(u_1) = l_0, f(v_1) = l_{n+1}, f(v_2) = l_{n-1}, f(u_n) = l_{n+2}$.

For $s = 1, 2, ..., \frac{n-2}{3}$, $f(u_i) = l_{2i-3s}$, $3s - 1 \leq i \leq 3s + 1$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } E_1 &= \{f_1(u_1u_2), f_1(u_nu_1), f_1(u_{n-1}v_2), f_1(u_nv_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_n) - f(u_1)|, |f(u_{n-1}) - f(v_2)|, |f(u_n) - f(v_1)|\} \\
&= \{|l_0 - l_1|, |l_{n+2} - l_0|, |l_n - l_{n-1}|, |l_{n+2} - l_{n+1}|\} \\
&= \{l_1, l_{n+2}, l_{n-2}, l_n\}. \\
\text{Let } E_2 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_iu_{i+1}) : 3s - 1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_i) - f(u_{i+1})| : 3s - 1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{2i-3s} - l_{2i+2-3s}| : 3s - 1 \leq i \leq 3s\} \\
&= \{l_{2i-3s+1} : 3s - 1 \leq i \leq 3s\} \\
&= \{l_2, l_4\} \cup \{l_5, l_7\} \cup ... \cup \{l_{n-3}, l_{n-1}\} \\
&= \{l_2, l_4, l_5, l_7, ..., l_{n-3}, l_{n-1}\}.
\end{aligned}$$

We find the edge labeling the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ..., \frac{n-2}{3}$.

$$\begin{aligned}
\text{Let } E_3 &= \{f_1(u_iu_{i+1}) : i = 3s + 1\} \\
&= \{|f(u_i) - f(u_{i+1})| : i = 3s + 1\} \\
&= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, ..., |f(u_{n-4}) - f(u_{n-3})|\} \\
&= \{|l_5 - l_4|, |l_8 - l_7|, ..., |l_{n-3} - l_{n-4}|\} \\
&= \{l_3, l_6, ..., l_{n-5}\}. \\
\text{Let } E_4 &= \{f_1(u_{n-1}u_n)\} \\
&= \{|f(u_{n-1}) - f(u_n)|\} \\
&= \{|l_n - l_{n+2}|\} \\
&= \{l_{n+1}\}.
\end{aligned}$$

Now, $E = E_1 \cup E_2 \cup E_3 \cup E_4 = \{l_1, l_2, ..., l_{n+2}\}$.

So, the edge labels of G are distinct. By observing three cases given above, f is a strong Lucas graceful labeling. Hence, $G = C_n@2P_1$, $n \geq 3$, is a strong Lucas graceful graph.

Example 2.12. $C_{12}@2P_1$ admits strong Lucas graceful graph.

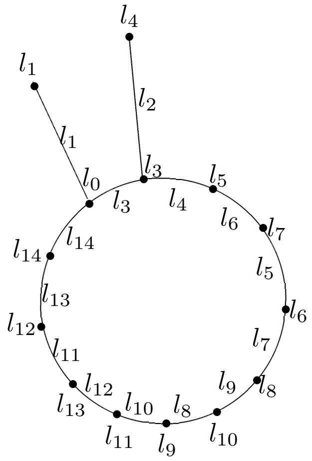

Fig. 4 $C_{12}@2P_1$

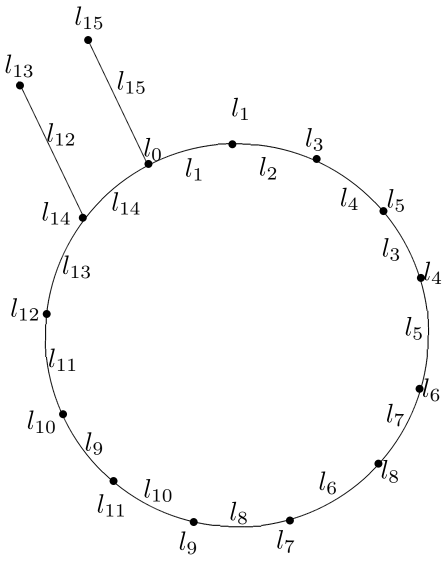

Fig. 5 $C_{13}@2P_1$

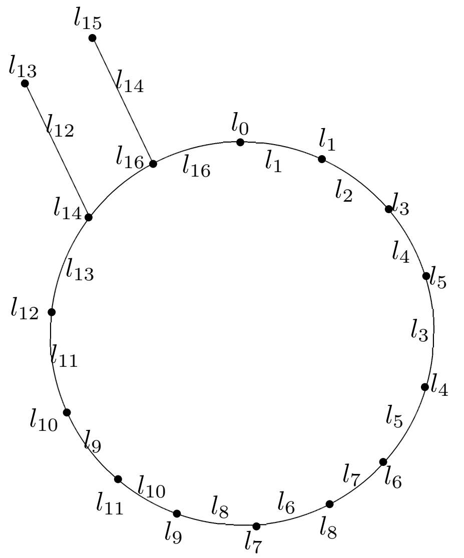

Fig. 6 $C_{14}@2P_1$

Definition 2.13.[2] $K_{1,n} \odot C_3$ means that one copy of a cycle C_3 is attached with each pendent vertex of $K_{1,n}$.

Theorem 2.14. $K_{1,n} \odot C_3$ is a strong Lucas graceful graph.

Proof. Let $G = K_{1,n} \odot C_3$. Let $V(G) = V_1 \cup V_2$ be a bipartition of G, such that $V_1 = \{w, u_i : 1 \le i \le n\}$ and $V_2 = \left\{u_i^{(1)}, u_i^{(2)} : 1 \le i \le n\right\}$.

Let $E(G) = \left\{wu_i, u_iu_i^{(1)}, u_iu_i^{(2)}, u_i^{(1)}u_i^{(2)} : 1 \le i \le n\right\}$ be the edge set of G. So, $|V(G)| = 3n + 1$ and $|E(G)| = 4n$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{4n}\}$ by $f(w) = l_0$, $f(u_i) = l_{4i}$, $f(u_i^{(1)}) = l_{4i-2}$ and $f(u_i^{(2)}) = l_{4i-1}$, $1 \le i \le n$.

Next, we claim that the edge labels are distinct.

$$
\begin{aligned}
\text{Let } \mathrm{E}_1 &= \bigcup_{i=1}^{n} \{f_1(wu_i)\} \\
&= \bigcup_{i=1}^{n} \{|f(w) - f(w_i)|\} \\
&= \bigcup_{i=1}^{n} \{|l_0 - l_{4i}|\} \\
&= \bigcup_{i=1}^{n} \{l_{4i}\} \\
&= \{l_4, l_8, ..., l_{4n}\}. \\
\text{Let } \mathrm{E}_2 &= \bigcup_{i=1}^{n} \left\{f_1\left(u_iu_i^{(1)}\right)\right\} \\
&= \bigcup_{i=1}^{n} \left\{\left|f\left(u_i\right) - f\left(u_i^{(1)}\right)\right|\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \bigcup_{i=1}^{n} \{|l_{4i} - l_{4i-2}|\} \\
&= \bigcup_{i=1}^{n} \{l_{4i-1}\} \\
&= \{l_3, l_7, ..., l_{4n-1}\}. \\
\text{Let } \mathrm{E}_3 &= \bigcup_{i=1}^{n} \left\{f_1\left(u_i u_i^{(2)}\right)\right\} \\
&= \bigcup_{i=1}^{n} \left\{\left|f(u_i) - f\left(u_i^{(2)}\right)\right|\right\} \\
&= \bigcup_{i=1}^{n} \{|l_{4i} - l_{4i-1}|\} \\
&= \{l_2, l_6, ..., l_{4n-2}\}. \\
\text{Let } \mathrm{E}_4 &= \bigcup_{i=1}^{n} \left\{f_1\left(u_i^{(1)} u_i^{(2)}\right)\right\} \\
&= \bigcup_{i=1}^{n} \left\{\left|f\left(u_i^{(1)}\right) - f\left(u_i^{(2)}\right)\right|\right\} \\
&= \bigcup_{i=1}^{n} \{|l_{4i-2} - l_{4i-1}|\} \\
&= \bigcup_{i=1}^{n} \{l_{4i-3}\} \\
&= \{l_1, l_5, ..., l_{4n-3}\}.
\end{aligned}
$$

Now, $E = E_1 \cup E_2 \cup E_3 \cup E_4 = \{l_1, l_2, l_3, ..., l_{4n-3}, l_{4n-2}, l_{4n-1}, l_{4n}\}$.

So, the edge labels of G are distinct. Therefore, f is a strong Lucas graceful labeling. Hence, $G = K_{1,n} \odot C_3$ is a strong Lucas graceful graph.

Example 2.15. $K_{1,6} \odot C_3$ admits strong Lucas graceful labeling.

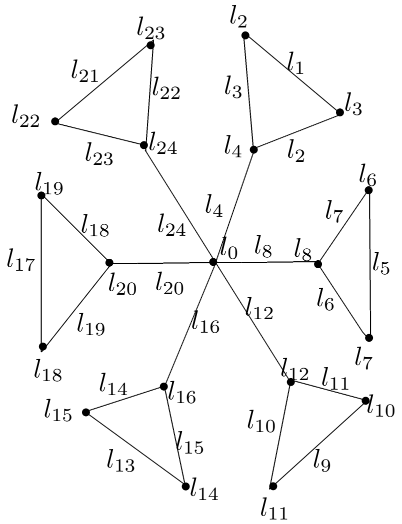

Fig. 7 $K_{1,6} \odot C_3$

Theorem 2.16. C_3@ $2P_n$ is a strong Lucas graceful graph, when $n \equiv 0, 2(\text{mod } 3)$.

Proof. Let $G = C_3$@ $2P_n$. Let $V(G) = \{w_i : 1 \le i \le 3\} \cup \{u_j, v_j : 1 \le j \le n+1\}$ be the vertex set of G. The vertices of w_2 and w_3 of C_3 are identified with v_1 and u_1 of two paths of length n respectively. Let $V(G) = \{w_i w_{i+1} : 1 \le i \le 2\} \cup \{u_i u_{i+1}, v_{i,i+1} : 1 \le i \le n\}$ be the edge set of G. So, $|V(G)| = 2n+3$ and $|E(G)| = 2n+3$.

Case (i) Suppose $n \equiv 0(\text{mod } 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{2n+3}\}$ by $f(w_1) = l_{n+4}$, $f(u_i) = l_{n+3-i}$, $1 \le i \le n+1$.

For $n \equiv 0(\text{mod } 3)$ and for $s = 1, 2, ..., \frac{n-3}{3}$, $f(v_j) = l_{n+4+2j-3s}$, $3s-2 \le j \le 3s$.

For $n \equiv 0(\text{mod } 3)$ and for $s = \frac{n}{3}$, $f(v_j) = l_{n+4+2j-3s}$, $3s-2 \le j \le 3s-1$ and $f(v_n) = l_0$, $f(v_{n+1}) = l_{2n+3}$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let E}_1 &= \bigcup_{i=1}^{n} \{f_1(u_i u_{i+1})\} \\
&= \{|f(u_i) - f(u_{i+1})|\} \\
&= \bigcup_{i=1}^{n} \{|l_{n+3-i} - l_{n+3-i-1}|\} \\
&= \bigcup_{i=1}^{n} \{l_{n+1-i}\} \\
&= \{l_n, l_{n-1}, ..., l_1\}. \\
\text{Let E}_2 &= \{f_1(u_{n+1} w_1), f_1(w_1 v_1), f_1(v_1 u_{n+1})\} \\
&= \{|f(u_{n+1}) - f(w_1)|, |f(w_1) - f(v_1)|, |f(v_1) - f(u_{n+1})|\} \\
&= \{|l_{n+2} - l_{n+4}|, |l_{n+4} - l_{n+3}|, |l_{n+3} - l_{n+2}|\} \\
&= \{l_{n+3}, l_{n+2}, l_{n+1}\}.
\end{aligned}$$

For $s = 1, 2, ..., \frac{n-3}{3}$

$$\begin{aligned}
\text{Let } \mathrm{E}_3 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(v_j v_{j+1}) : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(v_j) - f(v_{j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \cup \{|f(v_4) - f(v_5), \\
&\quad |f(v_5) - f(v_6)|\} \cup \{|f(v_{n-4}) - f(v_{n-3})|, |f(v_{n-3} - f(v_{n-2})|\} \\
&= \{|l_{n+3} - l_{n+5}|, |l_{n+5} - l_{n+7}|\} \cup \{|l_{n+6} - l_{n+8}|, |l_{n+8} - l_{n+10}|\} \cup \\
&\quad ... \cup \{|l_{n+4+2n-8-n+2} - l_{n+4+2n-6-n+2}|, |l_{n+4+2n-6-n+2} - l_{n+4+2n-4-n+2}|\ \} \\
&= \{l_{n+4}, l_{n+6}\} \cup \{l_{n+7}, l_{n+9}\} \cup ... \cup \{l_{2n-1}, l_{2n+1}\} \\
&= \{l_{n+4}, l_{n+6}, l_{n+7}, l_{n+9}, ..., l_{2n-1}, l_{2n+1}\}.
\end{aligned}$$

We find the edge labeling the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ..., \frac{n}{3}$.

$$\begin{aligned}
\text{Let } \mathrm{E}_4 &= \{f_1(v_j v_{j+1}) : j = 3s\} \\
&= \{|f(v_j) - f(v_{j+1})| : j = 3s\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, ..., |f(v_n) - f(v_{n+1})|\} \\
&= \{|l_{n+7} - l_{n+6}|, |l_{n+10} - l_{n+9}|, ..., |l_0 - l_{2n+3}|\} \\
&= \{l_{n+5}, l_{n+8}, ..., l_{2n}, l_{2n+3}\}.
\end{aligned}$$

Now, $E = E_1 \cup E_2 \cup E_3 \cup E_4 = \{l_1, l_2, ..., l_{2n+3}\}$.

So, the edge labels of G are distinct.

Case (ii) Suppose $n \equiv 2 (\text{mod } 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_a\}$ $a\epsilon N$, by $f(w_1) = l_{n+4}$, $f(u_i) = l_{n+3-i}$, $1 \le i \le n+1$.

For $s = 1, 2, ..., \frac{n-2}{3}$, $f(v_j) = l_{n+4+2j-3s}$, $3s-2 \le j \le 3s$ and $f(v_{n+1}) = l_0$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } \mathrm{E}_1 &= \bigcup_{i=1}^{n} \{f_1(u_i u_{i+1}\} \\
&= \bigcup_{i=1}^{n} \{|f(u_i) - f(u_{i+1})|\} \\
&= \bigcup_{i=1}^{n} \{|l_{n+3-i} - l_{n+3-i-1}|\} \\
&= \bigcup_{i=1}^{n} \{l_{n+1-i}\} \\
&= \{l_n, l_{n-1}, ..., l_1\}.
\end{aligned}$$

$$\text{Let } \mathrm{E}_2 = \{f_1(u_1 w_1), f_1(w_1 v_1), f_1(v_1 u_1)\}$$

$$= \{|f(u_1) - f(w_1)|, |f(w_1) - f(v_1)|, |f(v_1) - f(u_1)|\}$$
$$= \{|l_{n+2} - l_{n+4}|, |l_{n+4} - l_{n+3}|, |l_{n+3} - l_{n+2}|\}$$
$$= \{l_{n+3}, l_{n+2}, l_{n+1}\}.$$

For $s = 1, 2, ..., \frac{n-2}{3}$

$$\text{Let } E_3 = \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(v_j v_{j+1}) : 3s-2 \leq j \leq 3s-1\}$$
$$= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(v_j) - f(v_{j+1})| : 3s-2 \leq j \leq 3s-1\}$$
$$= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \cup \{|f(v_4) - f(v_5)|,$$
$$|f(v_5) - f(v_6)|\} \cup \{|f(v_{n-4}) - f(v_{n-3})|, |f(v_{n-3} - f(v_{n-2})|\}$$
$$= \{|l_{n+3} - l_{n+5}|, |l_{n+5} - l_{n+7}|\} \cup \{|l_{n+6} - l_{n+8}|, |l_{n+8} - l_{n+10}|\} \cup$$
$$... \cup \{|l_{n+4+2n-8-n+2} - l_{n+4+2n-6-n+2}|, |l_{n+4+2n-6-n+2} - l_{n+4+2n-4-n+2}|\ \}$$
$$= \{l_{n+4}, l_{n+6}\} \cup \{l_{n+7}, l_{n+9}\} \cup ... \cup \{l_{2n-1}, l_{2n+1}\}$$
$$= \{l_{n+4}, l_{n+6}, l_{n+7}, l_{n+9}, ..., l_{2n-1}, l_{2n+1}\}.$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ..., \frac{n-2}{3}$.

$$\text{Let } E_4 = \{f_1(v_j v_{j+1}) : j = 3s\} \cup \{f_1(v_n v_{n+1})\}$$
$$= \{|f(v_j) - f(v_{j+1})| : j = 3s\} \cup \{|f(v_n) - f(v_{n+1})|\}$$
$$= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, ..., |f(v_{n-2}) - f(v_{n-1})|\} \cup \{|f(v_n) - f(v_{n+1})|\}$$
$$= \{l_{n+5}, l_{n+8}, ..., l_{2n}, l_{2n+3}\}.$$

Now, $E = E_1 \cup E_2 \cup E_3 \cup E_4 = \{l_1, l_2, ..., l_{2n+3}\}$.

So, the edge labels of G are distinct. In both the cases, f is a strong Lucas graceful graph. Hence $G = C_3@2P_n$ is a strong Lucas graceful graph, when $n \equiv 0, 2 (\text{mod } 3)$.

Example 2.17. $C_3@2P_n$ admits strong Lucas graceful labeling, when $n \equiv 0, 2 (\text{mod } 3)$.

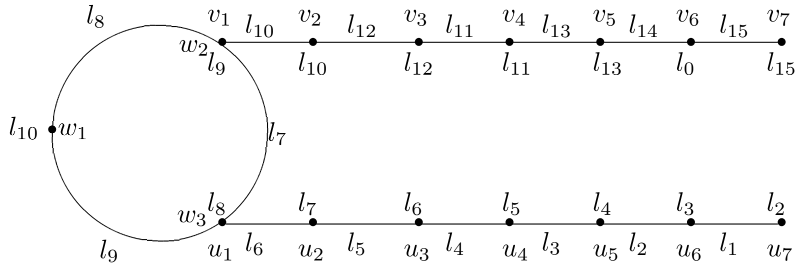

Fig. 8 C_3@ $2P_6$

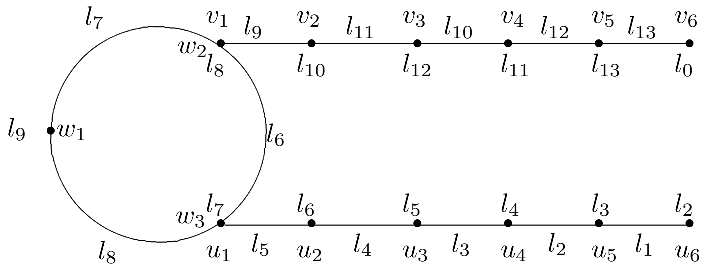

Fig. 9 C_3@ $2P_5$

Theorem 2.18. $C_n@K_{1,t}$ is a strong Lucas graceful graph, when $n \equiv 0 (\text{mod } 3)$.

Proof. Let $G = C_n@K_{1,t}$. Let $V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq t\}$ be the vertex set of G. Let $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_1v_j : 1 \leq j \leq t\}$ be the edge set of G. So, $|V(G)| = n + t$ and $|E(G)| = n + t$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{n+t}\}$ by $f(u_1) = l_0$.

For $s = 1, 2, ..., \frac{n-3}{3}$, $f(u_i) = l_{2i-3s},\ 3s - 1 \leq i \leq 3s + 1$.

For $s = \frac{n}{3}$, $f(u_i) = l_{2i-3s},\ 3s - 1 \leq i \leq 3s$ and $f(v_j) = l_{n+j}, 1 \leq j \leq t$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let } E_1 &= \{f_1(u_1u_2), f_1(u_nu_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_n) - f(u_1)|\} \\
&= \{|l_0 - l_1|, |l_n - l_0|\} \\
&= \{l_1, l_n\}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } E_2 &= \bigcup_{j=1}^{n} \{f_1(u_1v_j)\} \\
&= \bigcup_{i=1}^{n} \{|f(u_1) - f(v_j)|\} \\
&= \bigcup_{i=1}^{n} \{|l_0 - l_{n+j}|\}
\end{aligned}$$

$$= \bigcup_{i=1}^{n} \{l_{n+j}\}$$
$$= \{l_{n+1}, l_{n+2}, ..., l_{n+t}\}.$$

$$\text{Let } E_3 = \bigcup_{i=1}^{n} \{f_1(u_i u_{i+1}) : 3s-1 \leq i \leq 3s\}$$
$$= \bigcup_{i=1}^{n} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\}$$
$$= \{|f(u_2) - f(u_3)|, |f(u_3 - f(u_4)|\} \cup \{|f(u_{n-3}) - f(u_6)|, |f(u_6) - f(u_7)|\}$$
$$\cup ... \cup \{|f(u_{n-4}) - f(u_{n-3})|, |f(u_{n-3} - f(u_{n-2})|\}$$
$$= \{|l_1 - l_3|, |l_3 - l_5|\} \cup \{|l_4 - l_6|, |l_6 - l_8|\} \cup ... \cup \{|l_{n-5} - l_{n-3}|, |l_{n-3} - l_{n-1}|\}$$
$$= \{l_2, l_4\} \cup \{l_5, l_7\} \cup ... \cup \{l_{n-4}, l_{n-2}\}$$
$$= \{l_2, l_4, l_5, l_7, ..., l_{n-4}, l_{n-2}\}.$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and for $s = 1, 2, ..., \frac{n-3}{3}$.

$$\text{Let } E_4 = \{f_1(u_i u_{i+1}) : i = 3s+1\}$$
$$= \{|f(u_i) - f(u_{i+1})| : i = 3s+1\}$$
$$= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, ..., |f(u_{n-2} - f(u_{n-1})|\}$$
$$= \{|l_5 - l_4| \, |l_8 - l_7|, ..., |l_{n-1} - l_{n-2}|\}$$
$$= \{l_3, l_6, ..., l_{n-3}\}.$$

For $s = \frac{n}{3}$

$$\text{Let } E_5 = \{f_1(u_i u_{i+1}) : i = 3s-1\}$$
$$= \{f_1(u_i u_{i+1}) : i = n-1\}$$
$$= \{|f(u_{n-1}) - f(u_n)|\}$$
$$= \{|l_{2n-2-n} - l_{2n-n}|\}$$
$$= \{|l_{n-2} - l_n|\}$$
$$= \{l_{n-1}\}.$$

Now, $E = \bigcup_{i=1}^{5} E_i = \{l_1, l_2, ..., l_n, ..., l_{n+t}\}$.

So, the edge labels of G are distinct. Therefore, f is a strong Lucas graceful labeling. Hence, $G = C_n@K_{1,t}$ when n $\equiv 0 (\text{mod } 3)$, is a strong Lucas graceful graph.

Example 2.19. $C_{12}@K_{1,5}$ admits strong Lucas graceful graph.

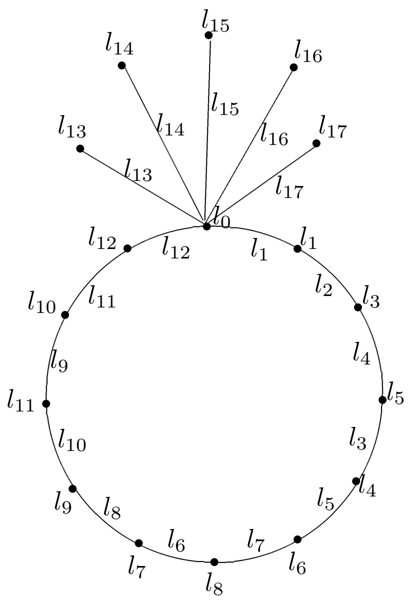

Fig. 10 $C_{12}@K_{1,5}$

Definition 2.20.[2] K_2 with m pendant edges at one end vertex and n pendant edges at another end vertex is called bistar and is denoted by $B(m,n)$.

Theorem 2.21. The bistar graph $B(m,n)$ is strong Lucas graceful, when $m \leq 3$.

Proof. Let $G = B(m,n)$ be a bistar graph with $m = 3$. Let $V(G) = \{u_i : 0 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq 3\}$ be the vertex set of G. Let $E(G) = \{u_0u_i\} \cup \{u_0v_0\} \cup \{v_0v_j : 1 \leq j \leq 3\}$ be the edge set of G.

So, $|V(G)| = n+5$ and $|E(G)| = n+4$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{n+4}\}$ by $f(u_0) = l_0,\ f(u_i) = l_i,\ 1 \leq i \leq n,\ f(v_0) = l_{n+2}$, $f(v_j) = l_{n+2+j},\ 1 \leq j \leq 3$.

Next, we claim that the edge labels are distinct.

$$\begin{aligned}
\text{Let }\ \mathrm{E}_1 &= \bigcup_{i=1}^{n} \{f_1(u_0u_i)\} \\
&= \bigcup_{i=1}^{n} \{|f(u_0) - f(u_i)|\} \\
&= \bigcup_{i=1}^{n} \{|l_0 - l_i|\} \\
&= \bigcup_{i=1}^{n} \{l_i\} \\
&= \{l_1, l_2, ..., l_n\}. \\
\text{Let }\ \mathrm{E}_2 &= \{f_1(u_0v_1), f_1(v_0v_1)\} \\
&= \{|f(u_0) - f(v_1)|, |f(v_0) - f(v_1)|\}
\end{aligned}$$

$$\begin{aligned} &= \{|l_0 - l_{n+3}|, |l_{n+2} - l_{n+3}|\} \\ &= \{l_{n+3}, l_{n+1}\}. \\ \text{Let } E_3 &= \{f_1(v_1v_2), f_1(v_1v_3)\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_1) - f(v_3)|\} \\ &= \{|l_{n+2+1} - l_{n+2+2}|, |l_{n+2+1} - l_{n+2+3}|\} \\ &= \{|l_{n+3} - l_{n+4}|, |l_{n+3} - l_{n+5}|\} \\ &= \{l_{n+2}, l_{n+4}\}. \end{aligned}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1 l_2, ..., l_{n+4}\}$.

So, the edge label of G are distinct. Therefore, f is a strong Lucas graceful labeling.

Hence, $G = B(m, n)$ is a strong Lucas graceful graph, when $m \leq 3$.

Example 2.22. $B(3, 7)$ admits strong Lucas graceful labeling.

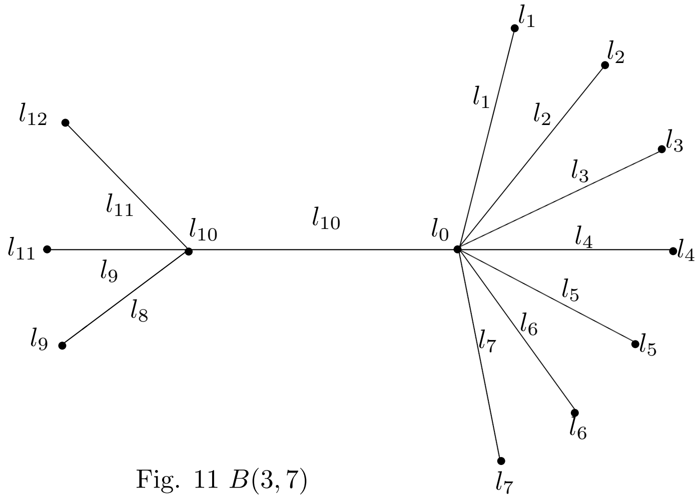

Fig. 11 $B(3, 7)$

Definition 2.23.[2] The graph $G = S_{m,n}@C_t$ consists of a graph $S_{m,n}$ and a cycle C_t is attached with the maximum degree vertex of $S_{m,n}$.

Theorem 2.24. $S_{m,n}@C_t$ is a strong Lucas graceful graph, when $m \equiv 0 (\text{mod } 2)$ and $t \equiv 0 (\text{mod } 3)$.

Proof. Let $G = S_{m,n}@C_t$. Let $V(G) = \{u_{i,j} : 1 \leq i \leq m \text{ and } 0 \leq \text{j} \leq \text{n}\} \cup \{v_k : 1 \leq k \leq t\}$ be the vertex set of G and u_0 is identified with v_1.

Let $E(G) = \{u_0 u_{i,1} : 1 \leq i \leq m\} \cup \{u_{i,j}\ u_{i,j+1} : 1 \leq j \leq n-1\} \cup \{v_1 v_2, v_t v_1\}$
$\cup \{v_k v_{k+1} : 2 \leq k \leq t-1\}$ be the edge set of G. So, $|V(G)| = mn + t$ and $|E(G)| = mn + t$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{mn+t}\}$ by $f(u_0) = f(v_1) = l_0$.

For $i = 1, 2, ...m, i \equiv 1 (\text{mod } 2)$ and $j = 1, 2, ...n$, $f(u_{i,j}) = l_{n(i-1)+2j-1}$.

For $i = 1, 2, ...m, i \equiv 0 (\text{mod } 2)$.

For $j = 1, 2, ..., n$, $f(u_{i,j}) = l_{ni-2j+2}$.

For $s = 1, 2, ..., \frac{t-3}{3}$, $f(v_k) = l_{mn+2k-3s}$, $3s - 1 \leq k \leq 3s + 1$.

For $s = \frac{t}{3}$, $f(v_k) = l_{mn+2k-3s}$, $3s - 1 \leq k \leq 3s$.

Next, we claim that the edge labels are distinct.

$$
\begin{aligned}
\text{Let } E_1 &= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \{f_1(u_0\ u_{i,1})\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{l_{n(i-1)+1}\right\}\\
&= \left\{l_1, l_{2n+1}, l_{4n+1}, ..., l_{(m-2)n+1}\right\}.\\
\text{Let } E_2 &= \bigcup_{\substack{i=1\\ i\equiv 0 (\text{mod } 2)}}^{m} \{f_1(u_0\ u_{i,1})\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0 (\text{mod } 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0 (\text{mod } 2)}}^{m} \{|l_0 - l_{ni}|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0 (\text{mod } 2)}}^{m} \{l_{ni}\}\\
&= \{l_{2n}, l_{4n}, ..., l_{mn}\}.\\
\text{Let } E_3 &= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{\bigcup_{j=1}^{n-1} \{f_1(u_{i,j}, u_{i,j+1})\}\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{\bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\}\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{\bigcup_{j=1}^{n-1} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right|\right\}\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{\bigcup_{j=1}^{n-1} \left\{l_{n(i-1)+2j}\right\}\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1 (\text{mod } 2)}}^{m} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, ..., l_{n(i-1)+2n-2}\right\}\\
&= \{l_2, l_4, ..., l_{2n-2}\} \cup \{l_{2n+2}, l_{2n+4}, ..., l_{4n-2}\}\\
&\quad \cup ... \cup \left\{l_{(m-2)n+2}, l_{(m-2)n+4}, ..., l_{mn-2}\right\}.
\end{aligned}
$$

$$\begin{aligned}
\text{Let } E_4 &= \bigcup_{\substack{i=1\\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}\ u_{i,j+1})\} \right\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \right\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} \right\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \right\} \\
&= \bigcup_{\substack{i=1\\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ l_{ni-1}, l_{ni-3}, ..., l_{ni-(2n-3)} \right\} \\
&= \left\{ l_{2n-1}, l_{2n-3}, ..., l_3 \right\} \cup \left\{ l_{4n-1}, l_{4n-3}, ..., l_{2n+3} \right\} \\
&\quad \cup ... \cup \left\{ l_{mn-1}, l_{mn-3}, ..., l_{(m-2)n+3} \right\}.
\end{aligned}$$

For $s = 1, 2, ... \frac{t-3}{3}$. $f(v_k) = l_{mn+2k-3s},\ 3s-1 \le k \le 3s+1$.

For $s = \frac{t}{3}$, $f(v_k) = l_{mn+2k-3s},\ 3s-1 \le k \le 3s$.

$$\begin{aligned}
\text{Let } E_1^{'} &= \{f_1(v_1 v_2), f_1(v_n v_1)\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_n) - f(v_1)|\} \\
&= \{|l_0 - l_{mn+1}|, |l_{mn+t} - l_0|\} \\
&= \{l_{mn+1}, l_{mn+t}\}.
\end{aligned}$$

$$\begin{aligned}
\text{Let } E_2^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(v_k\ v_{k+1} : 3s-1 \le k \le 3s\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|f(v_k) - f(v_{k+1})| : 3s-1 \le k \le 3s\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|l_{mn+2k-3s} - l_{mn+2k+2-3s}| : 3s-1 \le k \le 3s\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{l_{mn+2k-3s+1} : 3s-1 \le k \le 3s\} \\
&= \{l_{mn+2}, l_{mn+4}\} \cup \{l_{mn+5}, l_{mn+7}\} \cup ... \cup \{l_{mn+t-4}, l_{mn+t-2}\} \\
&= \{l_{mn+2}, l_{mn+4}, l_{mn+5}, l_{mn+7}, ..., l_{mn+t-4}, l_{mn+t-2}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of

$(s+1)^{th}$ loop and $s = 1, 2, ... \frac{t-3}{3}$.

$$\begin{aligned}
\text{Let } E_3^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(v_{3s+1}\ v_{3s+2}\} \\
&= \{f_1(v_4v_5), f_1(v_7v_8), ..., f_1(v_{t-2}v_{t-1})\} \\
&= \{|f(v_4) - f(v_5)|, |f(v_7) - f(v_8)|, ..., |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+5} - l_{mn+4}|, |l_{mn+8} - l_{mn+7}|, ..., |l_{mn+t-1} - l_{mn+t-2}|\} \\
&= \{l_{mn+3}, l_{mn+6}, ..., l_{mn+t-3}\}.
\end{aligned}$$

For $s = \frac{t}{3}$

$$\begin{aligned}
\text{Let } E_4^{'} &= \{f_1(v_kv_{k+1}) : k = 3s - 1\} \\
&= \{|f(v_k) - f(v_{k+1})| : k = 3s - 1\} \\
&= \{|f(v_{t-1}) - f(v_t)|\} \\
&= \{|l_{mn+2t-2-t} - l_{mn+2t-t}|\} \\
&= \{|l_{mn+t-2} - l_{mn+t}|\} \\
&= \{l_{mn+t-1}\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{4} \left(E_i \cup E_i^{'}\right)$. So, the edge labels of G are distinct. Therefore, f is a strong Lucas graceful labeling. Hence, $G = S_{m,n}@C_t$ is a strong Lucas graceful graph.

Example 2.25. $S_{6,5}@C_9$ admits strong Lucas graceful labeling.

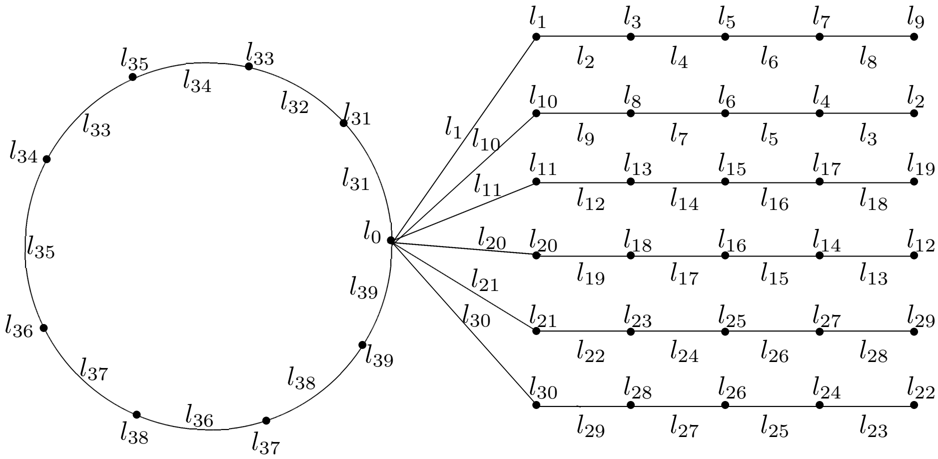

Fig. 12 $S_{6,5}@C_9$

Definition 2.26.[2] The graph $S_{m,n}^{(t)}$ denotes the one point union of t copies of $S_{m,n}$.

Theorem 2.27. $S_{m,n}^{(t)}$is a strong Lucas graceful graph, when $m \equiv 0(\text{mod } 2)$.

Proof. Let $G = S_{m,n}^{(t)}$. Let $V(G) = \{u_0\} \cup \{u_{i,j}^t : 1 \le j \le m, 1 \le j \le n \text{ and } 1 \le \text{k} \le \text{t}\}$ be the vertex set of G. Let $E(G) = \left\{u_0u_{i,1}^{(t)} : 1 \le i \le m\right\} \cup \left\{u_{i,j}^{(t)}u_{i,j+1}^{(t)} : 1 \le i \le m \text{ and } 1 \le \text{j} \le \text{n} - 1\right\}$ be the edge set of G.

So, $|V(G)| = mnt + 1$ and $|E(G)| = mnt$.

Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{mnt}\}$ by $f(u_0) = l_0$.

For $k = 1, 2, ..., t$ and for $i = 1, 2, ..., m, i \equiv 1(\text{mod } 2)$, $f\left(u_{i,j}^k\right) = l_{mn(k-1)+n(i-1)+2j-1}$, $1 \leq j \leq n$.

For $k = 1, 2, ..., t$ and $i = 1, 2, ..., m, i \equiv 0(\text{mod } 2)$, $f\left(u_{i,j}^k\right) = l_{mn(k-1)+ni-2j+2}$, $1 \leq j \leq n$.

Next, we claim that the edge labels are distinct.

$$
\begin{aligned}
\text{Let } \; E_1 &= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{f_1\left(u_0\ u_{i,1}^k\right)\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{\left|f(u_0) - f(u_{i,1}^k)\right|\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{\left|l_0 - l_{mn(k-1)+n(i-1)+1}\right|\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{l_{mn(k-1)+n(i-1)+1}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{l_{mn(k-1)+1}, l_{mn(k-1)+2n+1}, ..., l_{mn(k-1)+n(m-2)+1}\right\}\\
&= \left\{l_1, l_{2n+1}, ..., l_{n(m-2)+1}\right\} \cup \left\{l_{mn+1}, l_{mn+2n+1}, ..., l_{mn+n(m-2)+1}\right\}\\
&\quad \cup \left\{l_{2mn+1}, l_{2mn+2n+1}, ..., l_{2mn+n(m-2)+1}\right\} \cup \cdots\\
&\quad \cup \left\{l_{mn(t-1)+1}, l_{mn(t-1)+2n+1}, ..., l_{mn(t-1)+n(m-2)+1}\right\}.\\
\text{Let } \; E_2 &= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{f_1\left(u_0\ u_{i,1}^k\right)\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m}\left\{\left|f(u_0) - f(u_{i,1}^k)\right|\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{\left|l_0 - l_{mn(k-1)+ni}\right|\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\text{mod } 2)}}^{m}\left\{l_{mn(k-1)+ni}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{l_{mn(k-1)+2n}, l_{mn(k-1)+4n}, ..., l_{mn(k-1)mn}\right\}\\
&= \{l_{2n}, l_{4n}, ..., l_{mn}\} \cup \{l_{mn+2n}, l_{mn+4n}, ..., l_{mn+mn}\} \cup \ldots
\end{aligned}
$$

$$
\begin{aligned}
&\cup\left\{l_{mn(t-1)+2n}, l_{mn(t-1)+4n}, ..., l_{mn(t-1)+mn}\right\}\\
=\ &\{l_{2n}, l_{4n}, ..., l_{mn}\}\cup\{l_{mn+2n}, l_{mn+4n}, ..., l_{mn+mn}\}\cup\ldots\\
&\cup\left\{l_{mn(t-1)+2n}, l_{mn(t-1)+4n}, ..., l_{mnt}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } E_3 &= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{f_1(u_{i,j}^k\ u_{i,j+1}^k)\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{|f(u_{i,j}^k)-f(u_{i,j+1}^k)|\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{|l_{mn(k-1)+n(i-1)+2j-1}-l_{mn(k-1)+n(i-1)+2j+1}|\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 1(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{l_{mn(k-1)+n(i-1)+2j}\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\{\bigcup_{\substack{i=1\\ i\equiv 1(\mathrm{mod}\ 2)}}^{m}\{\bigcup_{j=1}^{n-1}\{l_{mn(k-1)+n(i-1)+2}, l_{mn(k-1)+n(i-1)+4}, ...,\\
&\quad l_{mn(k-1)+n(i-1)+2n-2}\}\}\}\\
&= \bigcup_{k=1}^{t}\{\{l_{mn(k-1)+2}, l_{mn(k-1)+4}, ..., l_{mn(k-1)+2n-2}\}\\
&\quad\cup\{l_{mn(k-1)+2n+2}, l_{mn(k-1)+2n+4}, ..., l_{mn(k-1)+4n-2}\}\\
&\quad\cup...\{l_{mn(k-1)+(m-2)n+2}, l_{mn(k-1)+(m-2)n+4}, ..., l_{mn(k-1)+mn-2}\}\}\\
&= \{l_2, l_4, ..., l_{2n-2}, l_{2n+2}, l_{2n+4}, ..., l_{4n-2}, ..., l_{(m-2)n+2}, l_{(m-2)n+4}, ..., l_{mn-2}\}\\
&\quad\cup\{l_{mn+2}, l_{mn+4}, ..., l_{mn+2n-2}, l_{mn+2n+2}, l_{mn+2n+4}, ..., l_{mn+4n-2}, ..., l_{mn+(m-2)n+2},\\
&\quad l_{mn+(m-2)n+4}, ..., l_{2mn-2}\}\cup ...\{l_{mn(t-1)+2}, l_{mn(t-1)+4}, ..., l_{mn(t-1)+2n-2},\\
&\quad l_{mn(t-1)+2n+2}, l_{mn(t-1)+(m-2)n+4}, ..., l_{mnt-2}\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } E_4 &= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 0(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{f_1(u_{i,j}^k\ u_{i,j+1}^k)\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 0(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{|f(u_{i,j}^k)-f(u_{i,j+1}^k)|\}\right\}\right\}\\
&= \bigcup_{k=1}^{t}\left\{\bigcup_{\substack{i=1\\ i\equiv 0(\mathrm{mod}\ 2)}}^{m}\left\{\bigcup_{j=1}^{n-1}\{|l_{mn(k-1)+ni-2j+2}-l_{mn(k-1)+ni-2j}|\}\right\}\right\}
\end{aligned}
$$

$$= \bigcup_{k=1}^{t} \left\{ \bigcup_{\substack{i=1 \\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{l_{mn(k-1)+ni-2j+1}\} \right\} \right\}$$

$$= \bigcup_{k=1}^{t} \left\{ \bigcup_{\substack{i=1 \\ i\equiv 0(\text{mod } 2)}}^{m} \left\{ \bigcup_{j=1}^{n-1} \{l_{mn(k-1)+ni-1}, l_{mn(k-1)+ni-3}, ..., l_{mn(k-1)+ni-(2n-3)}\} \right\} \right\}$$

$$= \bigcup_{k=1}^{t} \{l_{mn(k-1)+2n-1}, l_{mn(k-1)+2n-3}, ..., l_{mn(k-1)+3}, l_{mn(k-1)+4n-1}, l_{mn(k-1)+4n-3},$$
$$..., l_{mn(k-1)+2n+3}, ..., l_{mn(k-1)+mn-1}, l_{mn(k-1)+mn-3}, ..., l_{mn(k-1)+mn-2n+3}\}$$

$$= \{l_{2n-1}, l_{2n-3}, ..., l_3, l_{4n-1}, l_{4n-3}, ..., l_{2n+3}, ..., l_{mn-1}, l_{mn-3}, ..., l_{mn-(2n-3)}\} \cup$$
$$\{l_{mn+2n-1}, l_{mn+2n-3}, ..., l_{mn+3}, l_{mn+4n-1}, l_{mn+4n-3}, ..., l_{mn+2n+3}, ..., l_{2mn-1},$$
$$l_{2mn-3}, ..., l_{2mn-(2n-3)}\} \cup ... \cup \{l_{mn(t-1)+4n-1}, l_{mn(t-1)+4n-3}, ..., l_{mn(t-1)+2n+3},$$
$$l_{mnt-1}, l_{mnt-3}, ..., l_{mnt-(2n-3)}\}$$

Now, $E = E_1 \cup E_2 \cup E_3 \cup E_4 = \{l_1, l_2, ..., l_{mnt}\}$. So, the edges of G are distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, $G = S^{(t)}_{m,n}$ is a strong Lucas graceful graph.

Example 2.28. $S^{(4)}_{4,4}$ admits strong Lucas graceful labeling.

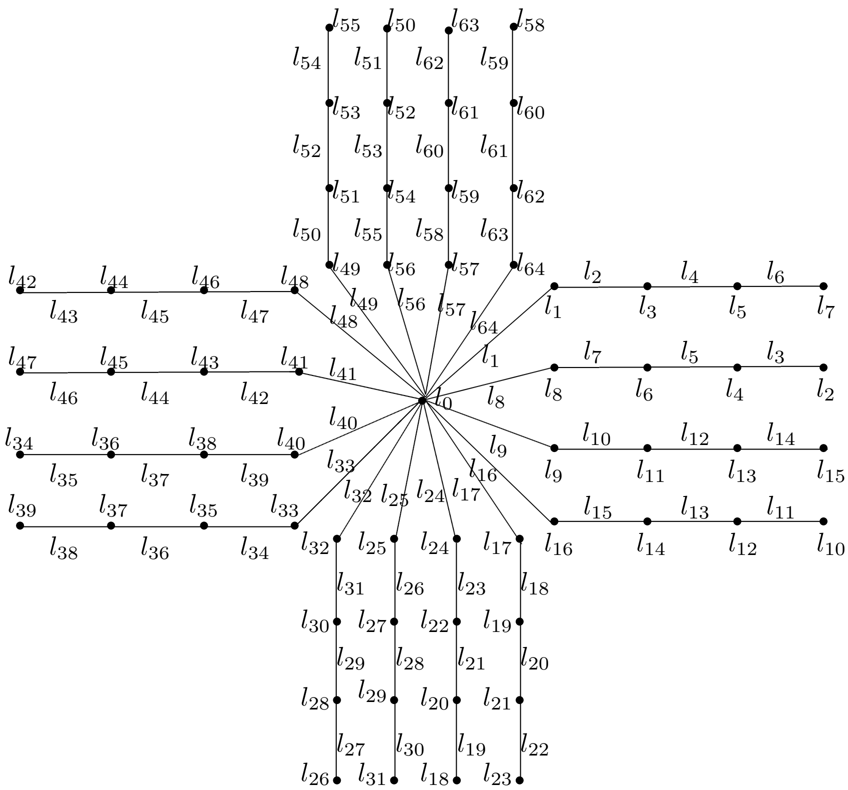

Fig. 13 $S^{(4)}_{4,4}$

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