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Horizontal biminimal general helices in the Lorentzian Heisenberg group $Heis^3$

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Abstract In this paper, we study timelike biminimal curves and we characterize non-geodesic timelike horizontal biminimal general helix in terms of its curvatures and torsions in the Lorentzian Heisenberg group Heis? We find out their explicit parametric equations.

Keywords Heisenberg group, biminimal curve, general helix.

§1. Introduction

Let $f:(M,g)\to(N,h)$ be a smooth function between two Lorentzian manifolds. f is harmonic over compact domain $\Omega\subset M$ if it is a critical point of the energy

$$E(f)=\int_{\Omega}h(df,df)\,dv_g,$$

where dv_g is the volume form of M. From the first variation formula it follows that is harmonic if and only if its first tension field $\tau(f)=\text{trace}_g\nabla df$ vanishes.

Harmonic maps between Riemannian manifolds were first introduced and established by Eells and Sampson [6] in 1964. Afterwards, there were two reports on harmonic maps by Eells and Lemaire [7,8] in 1978 and 1988.

The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_{\Omega}h(\tau(f),\tau(f))\,dv_g,$$

where $\tau(f)=\text{trace}_g\nabla df$ is the tension field of f. Using the first variational formula one sees that f is a biharmonic function if and only if its bitension field vanishes identically, i.e.

$$\widetilde{\tau}(f):=-\triangle^f(\tau(f))-\text{trace}_gR^N(df,\tau(f))df=0,$$

where

$$\triangle^f=-\text{trace}_g(\nabla^f)^2=-\text{trace}_g\left(\nabla^f\nabla^f-\nabla^f_{\nabla^M}\right) \tag{1}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z.$$

Biharmonic maps, which generalized harmonic maps, were first studied by Jiang [11,12] in 1986.

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations [2,13,16,21,22] , because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE. In differential geometry, harmonic maps, candidate minimisers of the Dirichlet energy, can be described as constraining a rubber sheet to fit on a marble manifold in a position of elastica equilibrium, i.e. without tension [7]. However, when this scheme falls through, and it can, as corroborated by the case of the two-torus and the two-sphere [9], a best map will minimise this failure, measured by the total tension, called bienergy. In the more geometrically meaningful context of immersions, the fact that the tension field is normal to the image submanifold, suggests that the most effective deformations must be sought in the normal direction.

An isometric immersion $f:(M,g)\longrightarrow(N,h)$ is called a $\lambda-$biminimal immersion if it is a critical point of the functional:

$$E_{2,\lambda}(f)=E_2(f)+\lambda E(f)\ \ ,\lambda\in\mathbb{R}.$$

The Euler-Lagrange equation for $\lambda-$biminimal immersions is

$$\widetilde{\tau}(f)^{\perp}=\lambda\tau(f). \tag{2}$$

Particularly, f is called a biminimal immersion if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{f_t\}$ through $f=f_0$ such that the variational vector field $V=df_t/dt|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is$\widetilde{\tau}(f)^{\perp}=0$. Here $\widetilde{\tau}(f)^{\perp}$ is the normal component of $\widetilde{\tau}(f)$.

In this paper, we study biminimal curves in Heisenberg group $Heis^3$ and we characterize non geodesic biminimal general helix in Heisenberg group $Heis^3$. Then we prove that the non-geodesic biminimal general helices are circular helices. Finally, we describe horizontal biminimal general helices in $Heis^3$ and we find out their explicit parametric equations.

§2. Lorentzian Heisenberg group $Heis^3$

The Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by:

$$g=-dx^2+dy^2+(xdy+dz)^2,$$

where

$$\omega^1=dz+xdy,\qquad \omega^2=dy,\qquad \omega^3=dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x} \tag{3}$$

for which we have the Lie products

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = 0, \quad [e_2, e_1] = 0,$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1. \tag{4}$$

Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{5}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1, 2, 3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Moreover we put

$$R_{abc} = R(e_a, e_b)e_c, \; R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a, b, c and d take the values $1, 2$ and 3.

Then the non-zero components of the Riemannian curvature tensor field and of the Riemannian curvature tensor are, respectively,

$$R_{121} = -e_2, \quad R_{131} = -e_3, \quad R_{232} = 3e_3,$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{6}$$

§3. Biminimal curves in Lorentzian Heisenberg group $Heis^3$

Let $\gamma : I \longrightarrow Heis^3$ be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to Lorentzian Heisenberg group $Heis^3$ along γ defined as follows: T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_T T &= kN, \\ \nabla_T N &= kT + \tau B, \\ \nabla_T B &= -\tau N, \end{aligned} \tag{7}$$

where k is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned}$$

Theorem 3.1. Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve parametrized by arc length. Then, γ is a timelike non-geodesic biminimal curve if and only if

$$k'' + k^3 - k\tau^2 = k(1 - 4B_1^2), \tag{8}$$

$$2\tau k' + k\tau' = 2kN_1 B_1. \tag{9}$$

Proof. Using (1) we have

$$\begin{aligned} \widetilde{\tau}(\gamma) &= \nabla_T^3 T - kR(T, N)T \\ &= (-3kk')T + (k'' + k^3 - k\tau^2)N + (2\tau k' + k\tau')B - kR(T, N)T. \end{aligned}$$

From the vanishing of the normal components of $\widetilde{\tau}(\gamma)$ we get

$$\begin{aligned} k'' + k^3 - k\tau^2 - kR(T, N, T, N) &= 0, \\ 2\tau k' + k\tau' - kR(T, N, T, B) &= 0. \end{aligned} \tag{10}$$

Since $k \neq 0$ by the assumption that is non-geodesic. A direct computation using (6), yields

$$R(T, N, T, N) = 1 - 4B_1^2,$$

and

$$R(T, N, T, B) = 2N_1 B_1,$$

these, together with (10), complete the proof of the theorem.

§4. Biminimal general helix in the Lorentzian Heisenberg group $Heis^3$

Definition 4.1. Let $\gamma : I \longrightarrow Heis^3$ be a curve and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. If k and τ are positive constant along γ, then γ is called circular helix with respect to Frenet frame [115].

Definition 4.2. Let $\gamma : I \longrightarrow Heis^3$ be a curve and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. A curve γ such that

$$\frac{k}{\tau} = \text{constant} \tag{11}$$

is called a general helix with respect to Frenet frame [115].

Theorem 4.3. Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biminimal general helix parametrized by arc length. If $N_1 B_1$ =constant, then γ is circular helix.

Proof. We can use (7) to compute the covariant derivatives of the vector fields T, N and B as:

$$\begin{aligned} \nabla_T T &= T_1' e_1 + (T_2' + 2T_1 T_3)e_2 + (T_3' + 2T_1 T_2)e_3, \\ \nabla_T N &= (N_1' + T_2 N_3 - T_3 N_2)e_1 + (N_2' + T_1 N_3 + T_3 N_1)e_2 \\ &\quad +(N_3' + T_2 N_1 + T_1 N_2)e_3, \\ \nabla_T B &= (B_1' + T_2 B_3 - T_3 B_2)e_1 + (B_2' + T_1 B_3 + T_3 B_1)e_2 \\ &\quad +(B_3' + T_2 B_1 + T_1 B_2)e_3. \end{aligned} \tag{12}$$

It follows that the first components of these vectors are given by

$$\begin{aligned} &< \nabla_T T, e_1 >= T_1', \\ &< \nabla_T N, e_1 >= N_1' + T_2 N_3 - T_3 N_2, \\ &< \nabla_T B, e_1 >= B_1' + T_2 B_3 - T_3 B_2. \end{aligned} \tag{13}$$

On the other hand, using Frenet formulas (7) we have

$$\begin{aligned} &< \nabla_T T, e_1 >= \kappa N_1, \\ &< \nabla_T N, e_1 >= \kappa T_1 + \tau B_1, \\ &< \nabla_T B, e_1 >= -\tau N_1. \end{aligned} \tag{14}$$

These, together with (13) and (14) give

$$\begin{aligned} T_1' &= \kappa N_1, \\ N_1' + T_2 N_3 - T_3 N_2 &= \kappa T_1 + \tau B_1, \\ B_1' + T_2 B_3 - T_3 B_2 &= -\tau N_1. \end{aligned} \tag{15}$$

From (15), we have

$$B_1' = (1 - \tau)N_1. \tag{16}$$

Suppose that γ is a be a non-geodesic biminimal general helix with respect to the Frenet frame $\{T, N, B\}$. Then,

$$\frac{k}{\tau} = c. \tag{17}$$

We have

$$k'\tau = \tau' k. \tag{18}$$

We substitue (18) in (9), we obtain

$$k' = \frac{2c}{3} N_1 B_1, \quad \tau' = \frac{2}{3} N_1 B_1. \tag{19}$$

From $N_1 B_1$ =constant, it follows that

$$k'' = 0. \tag{20}$$

We substitue (20) in (8), we obtain

$$k^2 - \tau^2 = 1 - 4B_1^2. \tag{21}$$

Next we replace $\tau = \frac{k}{c}$ in (21)

$$k^2 = \frac{c^2}{c^2 - 1}(1 - 4B_1^2). \tag{22}$$

Equation (22) derived and taking into account (16) and (19), becomes

$$k = \frac{6c}{c^2 - 1}(\tau - 1). \tag{23}$$

Substituting (17) in (23) we have

$$k = \text{constant}.$$

From (17) we obtain

$$\tau = \text{constant},$$

which implies γ circular helix.

Corollary 4.4. $\gamma : I \longrightarrow Heis^3$ is non-geodesic timelike biminimal general helix if and only if

$$\begin{aligned} k &= \text{constant} \neq 0, \\ \tau &= \text{constant}, \\ N_1 B_1 &= 0, \\ k^2 - \tau^2 &= -1 + 4B_1^2. \end{aligned} \tag{24}$$

Proof. Using Theorem 3.1 and Theorem 4.3, we have (24).

Corollary 4.5. **i)** If $N_1 \neq 0$, then γ is not biminimal general helix.

ii) If $N_1 = 0$, then

$$T(s) = \sinh\phi_0 e_1 + \cosh\phi_0 \sinh\psi(s) e_2 + \cosh\phi_0 \cosh\psi(s) e_3, \tag{25}$$

where $\phi_0 \in \mathbb{R}$.

Proof. i) We use the third equation of (15) we obtain

$$(1 - \tau)N_1 = 0. \tag{26}$$

Using $N_1 \neq 0$ we have

$$\tau = 1. \tag{27}$$

Assume now that γ is biminimal general helix. If we substitue (26) in (24) we obtain

$$k^2 = 4B_1^2. \tag{28}$$

By multipliying both side of (27) with N_1 we obtain

$$k^2 N_1 = 4B_1(B_1 N_1).$$

Using (24) and $N_1 \neq 0$ we have $k = 0$. These, together with Corollary 4.4 complete the proof of the corollary.

ii) Since γ is parametrized by arc length, we can write

$$T(s) = \sinh\phi(s) e_1 + \cosh\phi(s) \sinh\psi(s) e_2 + \cosh\phi(s) \cosh\psi(s) e_3. \tag{29}$$

From (15) we obtain

$$T_1' = kN_1.$$

Since $N_1 = 0$ we have

$$T_1' = 0.$$

Then T_1 is constant. Using (29) we get

$$T_1 = \sinh\phi_0 = \text{constant}.$$

We obtain (25) and corollary is proved.

§5. Horizontal biminimal general helix in the Lorentzian Heisenberg group $Heis^3$

Consider a nonintegrable 2-dimensional distribution $(x, y) \longrightarrow \mathcal{H}_{(x,y)}$ in $\mathbb{R}^3 = \mathbb{R}^2_{(x,y)} \times \mathbb{R}_z$ defined as $\mathcal{H} = \ker\omega$, where ω is a 1-form on $\mathbb{R}^3$. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $s \longrightarrow \gamma(s) = (x(s), y(s), z(s))$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s.

Lemma 5.1. Let $\gamma : I \longrightarrow Heis^3$ be a horizontal curve and ω is a 1-form on $Heis^3$. Then,

$$\omega(\gamma'(s)) = 0. \tag{30}$$

Proof. We use the equation of γ,

$$\gamma'(s) = x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z. \tag{31}$$

From (3) we have

$$\frac{\partial}{\partial x} = e_3, \quad \frac{\partial}{\partial y} = e_2 + xe_3, \quad \frac{\partial}{\partial z} = e_1. \tag{32}$$

Substituting (32) into (31) we obtain

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s))\partial_z.$$

Since γ is assumed to be a non-geodesic horizontal curve we have (30).

Lemma 5.2. $\gamma : I \longrightarrow Heis^3$ be a horizontal curve if and only if

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2, \quad \omega(\gamma'(s)) = z'(s) + x(s)y'(s). \tag{33}$$

If $\gamma(s)$ is horizontal curve, then we have

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} - x(s)y'(s)\frac{\partial}{\partial z}. \tag{34}$$

Using (3) and (34) we obtain

$$T = T_3\frac{\partial}{\partial x} + T_2\frac{\partial}{\partial y} + (T_1 - x(s)T_2)\frac{\partial}{\partial z}. \tag{35}$$

Theorem 5.3. Let $\gamma : I \longrightarrow Heis^3$ be a timelike horizontal biminimal general helix. Then explicit parametric equations of γ are:

$$\begin{aligned} x(s) &= \frac{1}{|k|}\sinh(|k|\,s + c) + a_1, \\ y(s) &= \frac{1}{|k|}\cosh(|k|\,s + c) + a_2, \\ z(s) &= \frac{1}{4\,|k|}\sinh 2(|k|\,s + c) - \frac{a_1}{|k|}\cosh(|k|\,s + c) - \frac{1}{|k|}s + a_3, \end{aligned} \tag{36}$$

where $\phi_0, a_1, a_2, a_3, c \in \mathbb{R}$.

Proof. The covariant derivative of the vector field T is:

$$\nabla_T T = T_1' e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3.$$

From (25) we have

$$\begin{aligned} \nabla_T T &= (\psi' \cosh\phi_0 \cosh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))e_2 + \\ &\quad (\psi' \cosh\phi \sinh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))e_3. \end{aligned}$$

Since $|\nabla_T T| = k$ we obtain

$$\psi(s) = (\frac{|k|}{\cosh\phi_0} - 2\sinh\phi_0)s + c, \tag{37}$$

where $c \in \mathbb{R}$.

To find equations for timelike horizontal biminimal general helix $\gamma(s) = (x(s), y(s), z(s))$ on the Lorentzian Heisenberg group $Heis^3$ we note that if

$$\frac{d\gamma}{ds} = T = T_1e_1 + T_2e_2 + T_3e_3, \tag{38}$$

and our left-invariant vector fields are

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x},$$

then

$$\frac{\partial}{\partial x} = e_3, \quad \frac{\partial}{\partial y} = e_2 + xe_3, \quad \frac{\partial}{\partial z} = e_1.$$

Therefore we easily have:

$$\begin{aligned} \frac{dx}{ds} &= \cosh\phi_0 \cosh[(\frac{|k|}{\cosh\phi_0} - 2\sinh\phi_0)s + c], \\ \frac{dy}{ds} &= \cosh\phi_0 \sinh[(\frac{|k|}{\cosh\phi_0} - 2\sinh\phi_0)s + c], \\ \frac{dz}{ds} &= \cosh\phi_0 \cosh[(\frac{|k|}{\cosh\phi_0} - 2\sinh\phi_0)s + c] \\ &\quad -x(s)\cosh\phi_0 \sinh[(\frac{|k|}{\cosh\phi_0} - 2\sinh\phi_0)s + c]. \end{aligned} \tag{39}$$

On the other hand, using (34) and (35) we have

$$T_1 = \sinh\phi_0 = 0. \tag{40}$$

Thus, we have

$$\cosh\phi_0 = 1. \tag{41}$$

Substituting (40) and (41) into (39) we get (36). Hence, the proof is completed.

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Robust stability of switched linear systems with time-varying delay

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Abstract We use a Lyapunov-Krasovskii functional approach to establish the exponential stability of linear systems with time-varying delay. Our delay-dependent condition allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. A simple procedure for constructing switching rule is also presented.

Keywords Switched system, time delay, exponential stability, Lyapunov equation.

§1. Introduction

As an important class of hybrid systems, switched system is a family of differential equations together with rules to switch between them. A switched system can be described by a differential equation of the form

$$\dot{x} = f_\alpha(t, x),$$

where $\{f_\alpha(.) : \alpha \in \Omega\}$, is a family of functions that is parameterized by some index set Ω, and $\alpha(\cdot) \in \Omega$ depending on the system state in each time is a switching rule/signal. The set Ω is typically a finite set. Switched systems arise in many practical models in manufacturing, communication networks, automotive engine control, chemical processes, and so on; see for example [3, 6, 9]. During the previous decade, the stability problem of switched linear systems has attracted a lot of attention (see e.g. [4, 7, 8] and the references therein). The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and LMI approach for constructing common Lyapunov function [7]. Although many important results have been obtained for switched linear systems, there are few results concerning the stability of the systems with time delay. Under commutative assumption on the system matrices, the authors of [5] showed that when all subsystems are asymptotically stable, the switched system is asymptotically stable under arbitrary switching rule. The paper [14] studied the asymptotic stability for switched linear systems with time delay, but the result was limited to symmetric systems. In [7, 12], delay-dependent asymptotic stability conditions are extended to switched linear discrete-time linear systems with time delay. The exponential stability problem was considered in [15] for switched linear systems with impulsive effects by using matrix measure concept and in [16] for nonholonomic chained systems with strongly nonlinear input/state driven disturbances and drifts. In recent paper [2], studying a switching system composed of a finite number of linear delay differential equations, it was shown that the asymptotic stability of this kind of switching systems may be achieved by using a common Lyapunov function method switching rule. There are some other results concerning asymptotic stability for switched linear systems with time delay, but we do not find any result on exponential stability even for the switched systems without delay except [15, 16]. On the other hand, it is worth noting that the existing stability conditions for time-delay systems must be solved upon a grid on the parameter space, which results in testing a nonlinear Riccati-like equation or a finite number of LMIs. In this case, the results using finite gridding points are unreliable and the numerical complexity of the tests grows rapidly. Therefore, finding simple stability conditions for switched linear systems with time-delay is of interest.

In this paper, we study the exponential stability of a class of switched linear systems with time-varying delay. The system studied in this paper is time-varying under a switching rule dependent on continuous system states. A delay-dependent condition for the exponential stability are formulated in terms of a generalized Lyapunov equation for linear systems with time-varying delay, which allows easily to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. A simple procedure for constructing switching rule is presented. The results obtained in this paper can be partly considered as extensions of existing results for linear time-delay systems and for switched linear systems without time delays.

The organization of this paper is as follows. Following the introduction and the problem motivation, Section 2 presents the notation, definitions and auxiliary propositions. The main result is given in Section 3 and followed by an numerical example and conclusion.

§2. Problem formulation

The following notations will be used throughout this paper. $\mathbb{R}^+$ denotes the set of all non-negative real numbers; $\mathbb{R}^n$ denotes the $n-$finite-dimensional Euclidean space, with the Euclidean norm $\|.\|$ and scalar product x^Ty of two vectors x, y. $\mathbb{R}^{n\times m}$ denotes the set of all $(n \times m)-$matrices; $\lambda_{\max}(A)(\lambda_{\min}(A)$, resp.) denotes the maximal number (the minimum number, resp.) of the real part of eigenvalues of A; A^T denotes the transpose of the matrix A; $Q \geq 0(Q > 0$, resp.) means Q is semi-positive definite (positive definite, resp.).

Consider a switched linear system with time-varying delay of the form

$$\begin{cases} \dot{x}(t) = A_\alpha x(t) + D_\alpha x(t-h(t)), \quad t \in \mathbb{R}^+, \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous trajectory of system, $A_\alpha, D_\alpha \in \mathbb{R}^{n\times n}$ are given constant matrices, $\phi(t) \in C([-h, 0], \mathbb{R}^n)$ is the initial function with the norm $\|\phi\| = \sup_{s\in[-h,0]} \|\phi(s)\|$. $\alpha(x) : \mathbb{R}^n \to \Omega := \{1, 2, \ldots, N\}$ is the switching rule, which is a piece-wise constant function depending on the state in each time. A switching rule is a rule which determines a switching sequence for a given switching system. Moreover, $\alpha(x) = i$ implies that the system realization is chosen as $[A_i, B_i]$, $i = 1, 2, \ldots, N$. It is seen that the system (1) can be viewed as an linear autonomous switched system in which the effective subsystem changes when the state $x(t)$ hits predefined boundaries; i.e., the switching rule is dependent on the system trajectory. The time-varying delay function $h(t)$ satisfies the following assumption

$$0 \leq h(t) \leq h, \quad h > 0, \quad \dot{h}(t) \leq \mu < 1.$$

This assumption means that the time delay may change from time to time, but the rate of changing is bounded. Also, due to the upper bound, the delay can not increase as fast as the time itself. In fact, the function $h(t)$ can be different for each subsystem; i.e., it should be denoted by $h_i(t)$, but we assume $h_i(t)$ be the same value in this paper for convenient formulation.

The exponential stability problem for switched linear system (1) is to construct a switching rule that makes the system exponentially stable.

Definition 2.1. Switched linear system (1) is exponentially stable if there exists switching rule $\alpha(.)$ such that every solution $x(t,\phi)$ of the system Σ_α satisfies the condition

$$\exists M > 0,\ \delta > 0 : \ \|x(t,\phi)\| \leq Me^{-\delta t}\|\phi\|, \quad \forall t \in \mathbb{R}^+.$$

The numbers $N > 0$ and $\delta > 0$ are called the stability factor and decay rate of the system.

Definition 2.2.[11] System of symmetric matrices $\{L_i\}$, $i = 1, 2, \ldots, N$, is said to be strictly complete if for every $0 \neq x \in \mathbb{R}^n$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T L_i x < 0$.

Let us define the sets

$$\Omega_i = \{x \in \mathbb{R}^n : x^T L_i x < 0\}, \quad i = 1, 2, \ldots, N.$$

It is easy to show that the system $\{L_i\}$, $i = 1, 2, \ldots, N$, is strictly complete if and only if $\cup_{i=1}^N \Omega_i = \mathbb{R}^n \setminus \{0\}$.

Remark 2.3. As shown in [10], a sufficient condition for the strict completeness of the system $\{L_i\}$ is that there exist numbers $\tau_i \geq 0$, $i = 1, 2, \ldots, N$, such that $\sum_{i=1}^N \tau_i > 0$ and

$$\sum_{i=1}^N \tau_i L_i < 0,$$

and in the case if $N = 2$, then the above condition is also necessary for the strict completeness.

Before presenting the main result, we recall the following well-known matrix inequality and Lyapunov stability theorem for time delay systems.

Proposition 2.4. For any $\epsilon > 0$, $x, y \in \mathbb{R}^n$, we have

$$-2x^T y \leq \epsilon^{-1} x^T x + \epsilon y^T y.$$

Proposition 2.5.[1] Let $x_t := \{x(t+s), s \in [-h, 0]\}$. Consider nonlinear delay system

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \\ x(t) &= \phi(t), \\ f(t, 0) &= 0, t \in \mathbb{R}^+. \end{aligned}$$

If there exists a Lyapunov function $V(t, x_t)$ satisfying the following conditions:

(i) There exists $\lambda_1 > 0$, $\lambda_2 > 0$ such that $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$, for all $t \in \mathbb{R}^+$,

(ii) $\dot{V}_f(t, x_t) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x_t) \leq 0$, for all solutions $x(t)$ of the system,

then the solution $x(t, \phi)$ is bounded: There exists $N > 0$ such that $\|x(t, \phi)\| \leq N\|\phi\|$, for all $t \in \mathbb{R}^+$.

§3. Main result

For given positive numbers μ, δ, h, ϵ we set

$$\beta = (1-\mu)^{-1},$$
$$L_i(P) = A_i^T P + PA_i + \epsilon^{-1} e^{2\delta h} D_i^T P D_i + (\epsilon\beta + 2\delta)P.$$
$$S_i^P = \{x \in \mathbb{R}^n : \quad x^T L_i(P) x < 0\},$$
$$\bar{S}_1^P = S_1^P, \quad \bar{S}_i^P = S_i^P \setminus \bigcup_{j=1}^{i-1} \bar{S}_j^P, \quad i = 2, 3, \ldots, N,$$
$$\lambda_{\max}(D) = \max_{i=1,2,\ldots,N} \lambda_{\max}(D_i^T D_i),$$
$$M = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\epsilon^{-1}(e^{2\delta h}-1)\lambda_{\max}(P)\lambda_{\max}(D)}{2\delta}}.$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. Switched linear system (1) is exponentially stable if there exist positive numbers ϵ, δ and symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that one of the following conditions holds

(i) The system of matrices $\{L_i(P)\}$, $i = 1, 2, \ldots, N$, is strictly complete.

(ii) There exists $\tau_i \geq 0$, $i = 1, 2, \ldots, N$, with $\sum_{i=1}^N \tau_i > 0$ and

$$\sum_{i=1}^N \tau_i L_i(P) < 0. \tag{2}$$

The switching rule is chosen in case (i) as $\alpha(x(t)) = i$ whenever $x(t) \in \bar{S}_i$, and in case (ii) as

$$\alpha(x(t)) = \arg\min\{x^T(t) L_i(P) x(t)\}.$$

Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq M e^{-\delta t}\|\phi\|, \quad t \in \mathbb{R}^+.$$

In the case $N = 2$, the conditions (i) and (ii) are equivalent.

Proof. For $\delta > 0$, we utilize the following state transformation $y(t) = e^{\delta t}x(t)$. The system (1) is transformed into

$$\begin{aligned} \dot{y}(t) &= \bar{A}_\alpha y(t) + e^{\delta h(t)} D_\alpha y(t-h(t)), t \in \mathbb{R}^+, \\ y(t) &= e^{\delta t}\phi(t), \quad t \in [-h, 0], \end{aligned} \tag{3}$$

where $\bar{A}_\alpha := A_\alpha + \delta I$. Assume that the condition (i) of the theorem holds. For every $i = 1, 2, \ldots, N$, we consider the Lyapunov-Krasovskii functional

$$V(t, y_t) = \langle Py(t), y(t)\rangle + \epsilon^{-1} e^{2\delta h} \int_{t-h(t)}^t \langle D_i^T P D_i y(s), y(s)\rangle ds. \tag{4}$$

It is easy to see that there are positive numbers λ_1, λ_2 such that

$$\lambda_1\|y(t)\|^2 \le V(t, y_t) \le \lambda_2\|y_t\|^2, \quad \forall t \in \mathbb{R}^+.$$

The derivative along the trajectory of the system (3) is

$$\begin{aligned}\dot V(t, y_t) &= 2\langle P\dot y(t), y(t)\rangle + \epsilon^{-1}e^{2\delta h}\langle D_i^T P D_i y(t), y(t)\rangle \\ &\quad - \epsilon^{-1}e^{\delta h}(1-\dot h(t))\langle D_i^T P D_i y(t-h(t)), y(t-h(t))\rangle \\ &\le \langle (A_i^T P + PA_i + 2\delta P + \epsilon^{-1}e^{\delta h}D_i^T P D_i)y(t), y(t)\rangle \\ &\quad + 2e^{2\delta h(t)}\langle PD_i y(t-h(t)), y(t)\rangle \\ &\quad - \epsilon^{-1}e^{\delta h}(1-\mu)\langle D_i^T P D_i y(t-h(t)), y(t-h(t))\rangle.\end{aligned} \tag{5}$$

Since P is a symmetric positive definite matrix, there is $\bar P = P^{1/2}$ such that $P = \bar P^T \bar P$. Using Proposition 2.4, for any $\xi > 0$, we have

$$\begin{aligned}&2e^{\delta h(t)}\langle PD_i y(t-h(t)), y(t)\rangle \\ &= 2y^T(t)\bar P^T \bar P e^{\delta h(t)} D_i y(t-h(t)) \\ &\le \xi y^T(t)\bar P^T \bar P y(t) + \xi^{-1}e^{2\delta h(t)}y^T(t-h(t))D_i^T \bar P^T \bar P D_i y(t-h(t)).\end{aligned}$$

Taking $\xi = \epsilon\beta > 0$, and since $h(t) \le h$, we obtain

$$\begin{aligned}&2e^{\delta h(t)}\langle PD_i y(t-h(t)), y(t)\rangle \\ &\le \epsilon\beta y^T(t)Py(t) + \epsilon^{-1}(1-\mu)e^{2\delta h}y^T(t-h(t))D_i^T P D_i y(t-h(t)).\end{aligned}$$

Then from (5) it follows that

$$\begin{aligned}\dot V(t, y_t)) &\le \langle (A_i^T P + PA_i + \epsilon^{-1}e^{2\delta h}D_i^T P D_i + (\epsilon\beta + 2\delta)P)y(t), y(t)\rangle \\ &= y^T(t)L_i(P)y(t).\end{aligned} \tag{6}$$

By the assumption, the system of matrices $\{L_i(P)\}$ is strictly complete. We have

$$\bigcup_{i=1}^{N} S_i^P = \mathbb{R}^n \setminus \{0\}. \tag{7}$$

Based on the sets S_i^P we constructing the sets $\bar S_i^P$ as above and we can verify that

$$\bar S_i^P \bigcap \bar S_j^P = \{0\}, i \ne j, \quad \bar S_i^P \bigcup \bar S_j^P = \mathbb{R}^n \setminus \{0\}. \tag{8}$$

We then construct the following switching rule: $\alpha(x(t)) = i$, whenever $x(t) \in \bar S_i^P$ (this switching rule is well-defined due to the condition (8)). From the state transformation $y(t) = e^{\delta t}x(t)$ and taking (6) into account we obtain

$$\dot V(t, y_t) = y^T(t)L_\alpha(P)y(t) = e^{2\delta t}x^T(t)L_\alpha(P)x(t) \le 0, \quad \forall t \in \mathbb{R}^+, \tag{9}$$

which implies that the solution $y(t)$ of system (3), by Proposition 2.5, is bounded. Returning to the state transformation of $y(t) = e^{\delta t}x(t)$ guarantees the exponentially stability with the decay rate δ of the system (1). To define the stability factor M, integrating both sides of (9) from 0 to t and using the expression of $V(t, y_t)$ from (4) we have

$$\langle Py(t), y(t)\rangle + \epsilon^{-1}e^{\delta h}\int_{t-h(t)}^{t}\langle D_i^T PD_i y(s), y(s)\rangle ds \le V(y(0)).$$

Since $D_i^T PD_i > 0$, and

$$\lambda_{\min}\|y(t)\|^2 \le \langle Py(t), y(t)\rangle,$$

we have

$$\lambda_{\min}\|y(t)\|^2 \le \lambda_{\max}(P)\|y(0)\|^2 + \epsilon^{-1}e^{2\delta h}\int_{h(0)}^{0}\langle D_i^T PD_i y(s), y(s)\rangle ds.$$

We have

$$\begin{aligned}\int_{-h(0)}^{0}\langle D_i^T PD_i y(s), y(s)\rangle ds &\le \lambda_{\max}(P)\lambda_{\max}(D)\|\phi\|^2\int_{-h(0)}^{0} e^{2\delta s}ds\\ &= \frac{\lambda_{\max}(P)\lambda_{\max}(D)}{2\delta}(1-e^{-2\delta h(0)})\|\phi\|^2\\ &\le \frac{\lambda_{\max}(P)\lambda_{\max}(D)}{2\delta}(1-e^{-2\delta h})\|\phi\|^2,\end{aligned}$$

we have

$$\|y(t)\|^2 \le [\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\epsilon^{-1}e^{2\delta h}\lambda_{\max}(P)\lambda_{\max}(D)}{2\delta}(1-e^{-2\delta h})]\|\phi\|^2.$$

Therefore,

$$\|y(t)\| \le M\|\phi\|, \quad \forall t \in \mathbb{R}^+,$$

where

$$M = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\epsilon^{-1}(e^{2\delta h}-1)\lambda_{\max}(P)\lambda_{\max}(D)}{2\delta}}.$$

We now assume the condition (ii), then we have

$$\sum_{i=1}^{N}\tau_i L_i(P) < 0.$$

where $\tau_i \ge 0,\ i = 1, 2, \ldots, N,\ \sum_{i=1}^{N}\tau_i > 0$. Since the numbers τ_i are non-negative and $\sum_{i=1}^{N}\tau_i > 0$, there is always a number $\epsilon > 0$ such that for any nonzero $y(t)$ we have

$$\sum_{i=1}^{N}\tau_i y^T(t)L_i(P)y(t) \le -\epsilon y^T(t)y(t).$$

Therefore,

$$\sum_{i=1}^{N}\tau_i \min_{i=1,\ldots,N}\{y^T(t)L_i(P)y(t)\} \le \sum_{i=1}^{N}\tau_i y^T(t)L_i(P)y(t) \le -\epsilon y^T(t)y(t). \tag{10}$$

The Lyapunov-Krasovskii functional $V(.)$ is defined by (4) and the switching rule is designed as follows

$$\alpha(x(t)) = \arg\min_{i=1,\ldots,N}\{x^T(t)L_i(P)x(t)\}.$$

Combining (6) and (10) gives

$$\dot{V}(t, y_t)) \leq -\eta\|y(t)\|^2 \leq 0, \quad \forall t \in \mathbb{R}^+,$$

where $\eta = \epsilon(\sum_{i=1}^{N} \tau_i)^{-1}$. This implies that all the solution $y(t)$ of system (3) are bounded. The proof is then completed by the same way as in the part (i).

Remark 3.2. Note that conditions (i) and (ii) involve linear Lyapunov-type matrix inequality, which is easy to solve. The following simple procedure can be applied to construct switching rule and define the stability factor and decay rate of the system.

Step 1. Define the matrices $L_i(P)$.

Step 2. Find the solution P of the generalized Lyapunov inequality (2).

Step 3. Construct the sets S_i^P, and then $\bar{S}_i^P$, and verify the condition (7), (8).

Step 4. The switching signal $\alpha(.)$ is chosen as $\alpha(x) = i$, whenever $x \in \bar{S}_i^P$ or as

$$\alpha(x(t)) = \arg\min\{x^T(t)L_i(P)x(t)\}.$$

Example 3.3. Consider the switched linear system defined by

$$\dot{x}(t) = A_i x(t) + D_i x(t - h(t)), \quad i = 1, 2$$

where

$$(A_1, D_1) = \left(\begin{bmatrix} 0.39 & 0.09 \\ 0.09 & -1.49 \end{bmatrix}, \begin{bmatrix} 0.39 & 0.03 \\ 0.03 & -0.29 \end{bmatrix} \right),$$

$$(A_2, D_2) = \left(\begin{bmatrix} -1.99 & -0.13 \\ -0.13 & 0.59 \end{bmatrix}, \begin{bmatrix} -0.09 & -0.01 \\ -0.01 & 0.19 \end{bmatrix} \right),$$

$$h(t) = \sin^2(0.1t).$$

We have $h = 1$, $\dot{h}(t) = 0.1\sin(0.2t)$ and then $\mu = 0.1$. By choosing the positive definite matrix $P > 0$ as

$$P = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$$

and $\epsilon = 0.1$, $\delta = 0.001$, one can verify condition (2) with $\tau_1 = \tau_2 = 0.5$. The switching regions are given as

$$\overline{S}_1 = \{(x_1, x_2) : 2.468x_1^2 - 0.002x_1x_2 - 2.015x_2^2 < 0\},$$

$$\overline{S}_2 = \{(x_1, x_2) : 2.468x_1^2 - 0.002x_1x_2 - 2.015x_2^2 > 0\}.$$

According to Theorem 3.1, the system with the switching rule $\alpha(x(t)) = i$ if $x(t) \in \overline{S}_i$ is exponentially stable.

In this case, it can be checked that

$$(L_1(P), L_2(P)) = \left(\begin{bmatrix} 2.468 & -0.001 \\ -0.001 & -2.015 \end{bmatrix}, \begin{bmatrix} -3.809 & -0.416 \\ -0.416 & 1.626 \end{bmatrix} \right).$$

Moreover, the sum

$$\tau_1 L_1(P) + \tau_2 L_2(P) = \begin{bmatrix} -0.6705 & -0.2085 \\ -0.2085 & -0.1945 \end{bmatrix}$$

is negative definite; i.e. the first entry in the first row and the first column $-0.6705 < 0$ is negative and the determinant of the matrix is positive. The sets S_1 and S_2 (without bar) are given as

$$S_1 = \{(x_1, x_2) : 2.468x_1^2 - 0.002x_1x_2 - 2.015x_2^2 < 0\},$$
$$S_2 = \{(x_1, x_2) : -3.809x_1^2 - 0.831x_1x_2 + 1.626x_2^2 < 0\}.$$

These sets are equivalent to

$$S_1 = \{(x_1, x_2) : (x_2 + 1.107x_1)(x_2 - 1.106x_1) > 0\},$$

and

$$S_2 = \{(x_1, x_2) : (x_2 + 1.296x_1)(x_2 - 1.807x_1) < 0\};$$

Obviously, the union of these sets is equal to $\mathbb{R}^2$.

Conclusions

We have presented a delay-dependent condition for the exponential stability of switched linear systems with time-varying delay. The condition is formulated in terms of a generalized Lyapunov equation, which allows easily to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. A simple procedure for constructing switching rule has been given.

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Majority neighborhood number of a graph

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Abstract A majority neighborhood set of a graph G is defined and the majority neighborhood number is determined for certain standard graphs. Moreover, majority vertex cover of G and its number are surveyed. Also certain relationships with these parameters of G are investigated.

Keywords Majority dominating set, majority neighborhood number, majority vertex cover.

§1. Preliminaries

By a graph we mean a finite undirected graph without loops or multiple edges. Let $G = (V, E)$ be a finite graph and v be a vertex in V. The open neighborhood of v is defined to be the set of vertices adjacent to v in G, and is denoted as $N(v)$. Further, the closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. The closed neighborhood of a set of vertices S is denoted as $N[S]$ and is $\cup_{s\in S} N[s]$.

In decision making process sometimes domination concept may not be necessary to have the consent of all but a majority opinion will do. In any democratic set up, the party which has majority of seats is given the opportunity to rule the state. To model such instances, the concept majority domination is introduced.

Definition 1.1. A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating Set if at least half of the vertices of V are either in S or adjacent to elements of S. (i. e.) $|N[S]| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$.

Observation 1.2.

(i) $V(G)$ is always a majority dominating set for any G.

(ii) Majority dominating sets have super hereditary property (i. e.), every super set of a majority dominating set is a majority dominating set.

Definition 1.3. A majority dominating set S is minimal if no proper subset of S is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the majority domination number and is denoted by $\gamma_M(G)$. The maximum cardinality of a minimal majority dominating set is denoted by $\Gamma_M(G)$.

Definition 1.4. In [3], A set S of vertices in a graph G is a neighborhood set of G if $G = \underset{v\in S}{\cup} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighborhood set of G is written as n-set of G.

Theorem 1.5.[4] Let S be a majority dominating set of G. S is minimal if and only if for every $v \in S$ one of the following conditions holds:

(i) $|N[S]| > \lceil \frac{p}{2} \rceil$ and $|Pn[v,S]| > |N[S]| - \lceil \frac{p}{2} \rceil$.

(ii) $|N[S]| = \lceil \frac{p}{2} \rceil$ and either v is an isolate of S or $Pn[v,S] \cap (V-S) \neq \phi$.

§2. Majority neighborhood set

Definition 2.1. Let $G=(V,E)$ be a graph. A set S of vertices of G is called a **majority neighborhood set** if $\underset{u\in S}{\cup}\langle N[u]\rangle$ has at least $\lceil \frac{p}{2} \rceil$ vertices and at least $\lceil \frac{q}{2} \rceil$ edges. A majority neighborhood set S is called a **minimal majority neighborhood set** if no proper subset of S is a majority neighborhood set. The minimum cardinality of a majority neighborhood set is called the **majority neighborhood number** of G and is denoted by $n_M(G)$.

Proposition 2.2. For $G = K_p,\ \ K_{1,p-1},\ p \geq 2,\ F_p,\ p\geq 3$ and $W_p,\ p\geq 5$, $n_M(G)=1$.

Proposition 2.3. For a graph $G=C_p$, a cycle on p vertices, $p\geq 3$, $n_M(G)=\lceil \frac{p}{4} \rceil$.

Proof. Let S be an independent set of vertices of C_p of cardinality $\lceil \frac{p}{4} \rceil$. Then $\langle N[S]\rangle$ covers at least $\lceil \frac{p}{2} \rceil$ vertices and at least $\lceil \frac{p}{2} \rceil$ edges since $p=q$. Therefore $n_M(G)\leq |S| = \lceil \frac{p}{4} \rceil$. Suppose T is a majority neighborhood set with $|T| < \lceil \frac{p}{4} \rceil$. Then the number of edges of C_p covered by $\langle N[T]\rangle$ is less than $\lceil \frac{p}{2} \rceil$. Therefore $|T| \geq \lceil \frac{p}{4} \rceil$. Hence $n_M(G)\geq \lceil \frac{p}{4} \rceil$.

Corollary 2.4. Let $G=P_p$ be a path on p vertices. Then $n_M(G)=\lceil \frac{p-1}{4} \rceil$.

Proposition 2.5. For every k-regular bipartite graph G, $n_M(G)=\lceil \frac{p}{4} \rceil$.

Proof. Let S be an independent set of vertices of cardinality $\lceil \frac{p}{4} \rceil$. Then S covers at least $k+\lceil \frac{p}{4} \rceil$ vertices and at least $k\lceil \frac{p}{4} \rceil$ edges.

Claim. Number of vertices covered by S is $\geq \lceil \frac{p}{2} \rceil$ and number of edges covered by $\langle N[S]\rangle$ is $\geq \lceil \frac{q}{2} \rceil$.

Case i. Let $k \geq \lceil \frac{p}{4} \rceil$. Then number of vertices covered by S is $\geq k+\lceil \frac{p}{4} \rceil \geq \lceil \frac{p}{2} \rceil$. Let $k<\lceil \frac{p}{4} \rceil$. Let $\lceil \frac{p}{4} \rceil = tk+r,\ \ 0\leq r<k$. Then number of vertices covered by S is

$$\geq \begin{cases} 2tk+k+\lceil \frac{p}{4} \rceil, \text{ if } r>0, \\ \\ 2tk+\lceil \frac{p}{4} \rceil, \text{ if } r=0. \end{cases}$$

$$= \begin{cases} 2\lceil \frac{p}{4} \rceil-2r+k+\lceil \frac{p}{4} \rceil, \text{ if } \ r>0, \\ \\ 2\lceil \frac{p}{4} \rceil+\lceil \frac{p}{4} \rceil, \text{ if } r=0. \end{cases}$$

$$\begin{aligned} 2\lceil \tfrac{p}{4} \rceil-2r+k+\lceil \tfrac{p}{4} \rceil &\geq 2\lceil \tfrac{p}{4} \rceil-k+\lceil \tfrac{p}{4} \rceil \\ &> 2\lceil \tfrac{p}{4} \rceil \text{ (since } \lceil \tfrac{p}{4} \rceil>k) \ \geq \lceil \tfrac{p}{2} \rceil. \end{aligned}$$

Therefore number of vertices covered by $S=k+\lceil \frac{p}{4} \rceil \geq \lceil \frac{p}{2} \rceil$.

Case ii. Let $p=2n$. Then $q=nk$. Then $k\lceil \frac{p}{4} \rceil = k\lceil \frac{n}{2} \rceil$; $\lceil \frac{q}{2} \rceil = \lceil \frac{nk}{2} \rceil$.

Let $n=2m$. Then $k\lceil \frac{n}{2} \rceil = mk$; $\lceil \frac{nk}{2} \rceil = \lceil \frac{2mk}{2} \rceil = mk$.

Let $n = 2m+1$. Then $k\left\lceil \frac{n}{2}\right\rceil = k(m+1)$; $\left\lceil \frac{nk}{2}\right\rceil = \left\lceil \frac{2mk+k}{2}\right\rceil = \left\lceil mk + \frac{k}{2}\right\rceil$. Therefore

$$\left\lceil \frac{nk}{2}\right\rceil \begin{cases} < mk+k, \text{ if } \ k>1, \\ \\ = mk+k, \text{ if } \ k=1. \end{cases}$$

implies

$$\left\lceil \frac{nk}{2}\right\rceil \begin{cases} < k\left\lceil \frac{n}{2}\right\rceil, \text{ if } \ k>1, \\ \\ = k\left\lceil \frac{n}{2}\right\rceil, \text{ if } \ k=1. \end{cases}$$

Hence $k\left\lceil \frac{p}{4}\right\rceil = k\left\lceil \frac{n}{2}\right\rceil \geq \left\lceil \frac{nk}{2}\right\rceil = \left\lceil \frac{q}{2}\right\rceil$. Then number of edges covered by $\langle N[S]\rangle$ is $\geq \left\lceil \frac{q}{2}\right\rceil$. This implies that S is a majority neighborhood set of G. Hence $n_M(G) \leq |S| = \left\lceil \frac{p}{4}\right\rceil$.

Let $D \subseteq V(G)$ with $|D| < \left\lceil \frac{p}{4}\right\rceil$. That is, $|D| \leq \left\lceil \frac{p}{4}\right\rceil - 1$. Let $|D| = r$ (say).

Case i. Let $n = 2m$. Number of edges covered by $\langle N[D]\rangle = kr < k(\left\lceil \frac{p}{4}\right\rceil - 1) = km - k$. Since $q = 2mk$, number of edges covered by D is $< \left\lceil \frac{q}{2}\right\rceil = mk$.

Case ii. Let $n = 2m+1$. Number of edges covered by $\langle N[D]\rangle = kr < k(\left\lceil \frac{p}{4}\right\rceil - 1) = mk$. Now,

$$\left\lceil \frac{q}{2}\right\rceil = \left\lceil \frac{nk}{2}\right\rceil \begin{cases} < mk+k, \text{ if } k>1, \\ \\ = mk+k, \text{ if } k=1. \end{cases}$$

Therefore the number of edges covered by $\langle N[D]\rangle$ is $< \left\lceil \frac{q}{2}\right\rceil$. Hence $n_M(G) \geq \left\lceil \frac{p}{4}\right\rceil$. Thus $n_M(G) = \left\lceil \frac{p}{4}\right\rceil$.

Corollary 2.6. For Q_n, the n-cube, which is a regular bipartite graph with 2^n vertices and $(n2^{n-1})$ edges, $n_M(Q_n) = 2^{n-2}$.

Proposition 2.7. Let $G = K_{m,n}$. Then $n_M(G) = \min\left\{\left\lceil \frac{m}{2}\right\rceil, \left\lceil \frac{n}{2}\right\rceil\right\}$.

Observation 2.8. Any superset of a majority neighbourhood set is a majority neighbourhood set. That is, The property of majority neighbourhood set is a super hereditary property.

Characterisation of Minimal Majority Neighbourhood Set

Theorem 2.9. Let S be a majority neighbourhood set of G. S is minimal if and only if for any $u \in S$, the following condition holds:

(i) $\left\lceil \frac{p}{2}\right\rceil - |N[u] - N[S-u]| \leq |V(\langle N[S-u]\rangle)| < \left\lceil \frac{p}{2}\right\rceil$,

or

(ii) $\left\lceil \frac{q}{2}\right\rceil - |E\langle N(u)\rangle| \leq |E(\langle N[S-u]\rangle)| < \left\lceil \frac{q}{2}\right\rceil$.

Proof. Suppose S is a minimal majority neighbourhood set. Suppose the conditions are not satisfied by $u \in S$. Then,

$$|V(\langle N[S-u]\rangle)| < \left\lceil \frac{p}{2}\right\rceil - |N[u] - N[S-u]|,$$

or

$$|V(\langle N[S-u]\rangle)| \geq \left\lceil \frac{p}{2}\right\rceil \text{ and } |E(\langle N[S-u]\rangle)| < \left\lceil \frac{q}{2}\right\rceil - |E\langle N(u)\rangle|,$$

or

$$|E(\langle N[S-u]\rangle)| \geq \left\lceil \frac{q}{2} \right\rceil.$$

In any case either S is not a majority neighbourhood set or $S-\{u\}$ is a majority neighbourhood set. It implies that S is not a minimal majority neighbourhood set of G, which is a contradiction to the assumption.

Conversely, assume that the condition (i) or (ii) is satisfied for any $u \in S$. Suppose S is not a minimal majority neighbourhood set. Then there exists $u \in S$ such that $S-\{u\}$ is a majority neighbourhood set.
$|V(N[S-u])| \geq \lceil \frac{p}{2} \rceil$ and $|E(\langle N[S-u]\rangle)| \geq \lceil \frac{q}{2} \rceil$, a contradiction. Therefore S is a minimal majority neighbourhood set of G.

Proposition 2.10. Let G be a graph. If there exists a n_o-set of G which can be partitioned into two subsets (S_1 and S_2, with the subset of lesser cardinality being a a majority neighbourhood set) then $n_M(G) \leq \left\lceil \frac{n_o(G)}{2} \right\rceil$.

Proposition 2.11. Let G be any graph. $n_M(G)=1$ if and only if G has a vertex u with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and $E(\langle N[u]\rangle) \geq \lceil \frac{q}{2} \rceil - \lceil \frac{p}{2} \rceil + 1$.

Proposition 2.12. Let G be a graph with $q \leq p-3$. Then $n_M(G)=1$ if and only if G has a vertex u with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$.

§3. Majority vertex cover

Definition 3.1. A set of vertices S which covers at least half of the edges is a **majority vertex cover** of G. The **majority vertex covering number** $\alpha_M(G)$ of G is the minimum number of vertices in a majority vertex cover.

Proposition 3.2.

(i) For $G = K_{1,p-1}$, $p \geq 2$. Then $\alpha_M(G) = 1$.

(ii) For $G = P_p$, a path on $p \geq 2$ vertices. Then $\alpha_M(G) = \lceil \frac{p-1}{4} \rceil$.

(iii) For $G = C_p$, a cycle on $p \geq 3$ vertices. Then $\alpha_M(G) = \lceil \frac{p}{4} \rceil$.

Proposition 3.3. Let $G = K_{m,n}$. Then $\alpha_M(G) = min\left\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\right\}$. Let G be a regular bipartite graph with p vertices. Then $\alpha_M(G) = \lceil \frac{p}{4} \rceil$.

Proposition 3.4. Let $G = K_p$ be a complete graph. Then

$$\alpha_M(K_p) = \begin{cases} \left\lceil \frac{(2p-1)-\sqrt{2p^2-2p+1}}{2} \right\rceil, & \text{if } p \equiv 0,3 (\text{mod } 4), \\ \left\lceil \frac{(2p-1)-\sqrt{2p^2-2p-1}}{2} \right\rceil, & \text{if } p \equiv 1 (\text{mod } 4), \\ \left\lceil \frac{(2p-1)-\sqrt{2p^2+2p-7}}{2} \right\rceil, & \text{if } p \equiv 2 (\text{mod } 4). \end{cases}$$

Proof. In K_p, any set of k vertices $\{v_1, v_2, ..., v_k\}$ covers $\{(p-1)+(p-2)+\cdots+(p-k)\}$ edges. $((p-1)+(p-2)+\cdots+(p-k)) \geq \left\lceil \frac{pC_2}{2} \right\rceil$, if and only if $pk - \frac{k(k+1)}{2} \geq \left\lceil \frac{p(p-1)}{4} \right\rceil$.

Case 1. $(\frac{2kp-k^2-k}{2}) \geq \frac{4l(4l-1)}{4}$ if $p \equiv 0, 3(\text{mod } 4)$.

Then

$$k^2 + k(1-8l) + (8l^2+2l) \leq 0 \ \ (i.e.),$$

$$(\frac{(8l-1) - \sqrt{32l^2-8l+1}}{2}) \leq k \leq (\frac{(8l-1) + \sqrt{32l^2-8l+1}}{2}).$$

It implies that $k = \left\lceil \frac{(2p-1) \pm \sqrt{2p^2-2p+1}}{2} \right\rceil$. Hence $\alpha_M(K_p) = \left\lceil \frac{(2p-1) - \sqrt{2p^2-2p+1}}{2} \right\rceil$, if p ≡ 0,3 (mod 4).

A similar argument can be given to get the result in the following cases.

Case 2. $(\frac{(8l+1) - \sqrt{32l^2+8l+1}}{2}) \leq k \leq (\frac{(8l+1) + \sqrt{32l^2+8l+1}}{2})$. Then

$$\alpha_M(K_p) = \left\lceil \frac{(2p-1) - \sqrt{2p^2-2p-1}}{2} \right\rceil \text{ if } p \equiv 1(\text{mod } 4).$$

Case 3. $(\frac{(8l+3) - \sqrt{32l^2+24l+5}}{2}) \leq k \leq (\frac{(8l+3) + \sqrt{32l^2+24l+5}}{2})$. Then

$$\alpha_M(K_p) = \left\lceil \frac{(2p-1) - \sqrt{2p^2+2p-7}}{2} \right\rceil \text{ if } p \equiv 2(\text{mod } 4).$$

Hence the result.

§4. Relationship between $\gamma_M(G), n_M(G)$ and $\alpha_M(G)$

Proposition 4.1. Every majority neighbourhood set is a majority dominating set. (i. e.), $\gamma_M(G) \leq n_M(G)$. But not conversely.

Theorem 4.2. For a connected graph G, every majority vertex cover is a majority neighbourhood set. (i. e.), $n_M(G) \leq \alpha_M(G)$.

Proof. Case 1. Let $p = q+1$. Let S be a majority vertex cover. Then $\lceil \frac{q}{2} \rceil$ edges have at least one of their ends in S. Therefore $\bigcup_{u \in S} \langle N[u] \rangle$ contains at least $\lceil \frac{q}{2} \rceil + 1$ vertices. (i. e.). It contains at least $\lceil \frac{p}{2} \rceil$ vertices.

Case 2. Let G be a connected graph but G is not a tree. Then $q \geq p$. If S is a majority vertex cover then clearly $\bigcup_{u \in S} \langle N[u] \rangle$ covers at least $\lceil \frac{q}{2} \rceil$ edges and hence contains at least $\lceil \frac{p}{2} \rceil$ vertices . Hence every majority vertex cover of G is a majority neighbourhood set.

Remark 4.3.

(i) In the above theorem, the connectedness of G can not be dropped. The above result is not true in the case of disconnected graph. For : $G = K_{1,6} \cup 4K_2$. $\alpha_M(G) = 1$; $n_M(G) = 2$.

(ii) Let G be a disconnected graph with $q(G) \geq p(G)$.
Then $n_M(G) \leq \alpha_M(G)$.

(iii) Let G be any graph. A majority vertex cover of G need not be a majority neighbourhood set of G.

Proposition 4.4. Let G be a connected graph without triangles. Then $n_M(G) = \alpha_M(G)$.

Proof. Consider any n_M-set $S = \{u_1, u_2, \cdots, u_r\}$ where $r = n_M(G)$. Then $\bigcup_{u \in S} \langle N[u] \rangle$ contains at least $\lceil \frac{q}{2} \rceil$ edges. Since the graph G has no triangle, number of edges in $\langle N[u] \rangle$ for any $u \in S$ is exactly the number of edges incident at u. Therefore S is a majority vertex cover of G and $\alpha_M(G) \leq n_M(G)$. But $n_M(G) \leq \alpha_M(G)$. Hence $n_M(G) = \alpha_M(G)$.

Corollary 4.5. If G is a disconnected graph without triangles in which $q(G) \geq p(G)$. Then $n_M(G) = \alpha_M(G)$.

Proposition 4.6. For a connected graph G, $\gamma_M(G) \leq n_M(G) \leq \alpha_M(G)$.

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Smarandache's Orthic Theorem

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Abstract In this paper We present the Smarandache's Orthic Theorem in the geometry of the triangle.

Keywords Smarandache's Orthic Theorem, triangle.

§1. The main result

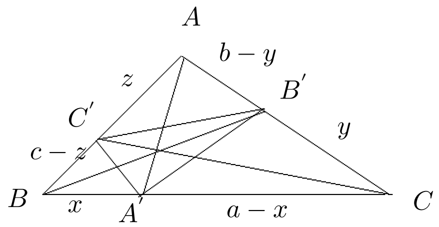
Smarandache's Orthic Theorem

Given a triangle ABC whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$\|A'B'\| \cdot \|B'C'\| + \|B'C'\| \cdot \|C'A'\| + \|C'A'\| \cdot \|A'B'\|$$

is maximum?



Proof. We have

$$\triangle ABC \sim \triangle A'B'C' \triangle AB'C \sim \triangle A'BC'. \tag{1}$$

We note

$$\|BA'\| = x, \|CB'\| = y, \|AC'\| = z.$$

It results that

$$\|A'C\| = a - x, \|B'A\| = b - y, \|C'B\| = c - z.$$

$$\widehat{BAC} = \widehat{B'A'C} = \widehat{BA'C'}; \widehat{ABC} = \widehat{AB'C'} = \widehat{A'B'C'}; \widehat{BCA} = \widehat{BC'A'} = \widehat{B'C'A}.$$

From these equalities it results the relation (1)

$$\triangle A'BC' \sim \triangle A'B'C \Rightarrow \frac{A'C'}{a-x} = \frac{x}{\|A'B'\|}, \tag{2}$$

$$\triangle A'B'C \sim \triangle AB'C' \Rightarrow \frac{A'C'}{z} = \frac{c-z}{\|B'c'\|}, \tag{3}$$

$$\triangle AB'C \sim \triangle A'B'C \Rightarrow \frac{B'C'}{y} = \frac{b-y}{\|A'B'\|}. \tag{4}$$

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

$$x(a-x) + y(b-y) + z(c-z) = \frac{1}{4}(a^2+b^2+c^2) - (x-\frac{a}{2})^2 - (y-\frac{b}{2})^2 - (z-\frac{c}{2})^2,$$

which will reach its maximum as long as $x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2}$, that is when the altitudes' legs are in the middle of the sides, therefore when the $\triangle ABC$ is equilateral. The maximum of the expression is $\frac{1}{4}(a^2+b^2+c^2)$.

§2. Conclusion (Smarandache's Orthic Theorem)

If we note the lengths of the sides of the triangle $\triangle ABC$ by $\|AB\| = c, \|BC\| = a, \|CA\| = b$, and the lengths of the sides of its orthic triangle $\triangle A^*B^*C^*$ by $\|A^*B^*\| = c^*, \|B^*C^*\| = a^*, \|C^*A^*\| = b^*$, then we proved that:

$$4(a^*b^* + b^*c^* + c^*a^*) \leq a^2 + b^2 + c^2.$$

§3. Open problems related to Smarandache's Orthic Theorem

1. Generalize this problem to polygons. Let $A_1A_2\cdots A_m$ be a polygon and P a point inside it. From P we draw perpendiculars on each side A_iA_{i+1} of the polygon and we note by $A_{i'}$ the intersection between the perpendicular and the side A_iA_{i+1}. A pedal polygon $A_{1'}A_{2'}\cdots A_{m'}$ is formed. What properties does this pedal polygon have?

2. Generalize this problem to polyhedrons. Let $A_1A_2\cdots A_n$ be a poliyhedron and P a point inside it. From P we draw perpendiculars on each polyhedron face F_i and we note by $A_{i'}$ the intersection between the perpendicular and the side F_i. A pedal polyhedron $A_{1'}A'_2\cdots A_{p'}$ is formed, where p is the number of polyhedron's faces. What properties does this pedal polyhedron have?

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On left C-wrpp semigroups [1]

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Abstract The aim of this paper is to study left C-wrpp semigroups. We prove that a left C-wrpp semigroup is a refined semilattice of left-$\mathcal{R}$ cancellative stripes if and only if it is a right strong semilattice of left-$\mathcal{R}$ cancellative stripes.

Keywords Left regular band, refined semilattice.

§1. Introduction and preliminaries

In 2001, Zhang L., Shum K. P. and Zhang R. H. introduced the concept of a refined semilattice of semigroups [4], which is a natural generalization of the notion of a strong semilattice of semigroups. They proved that any regular band is a refined semilattice of rectangular bands. Thus, a number of results in the literature concerning strong semilattice decomposition can be further developed. Zhang described a special class of left C-rpp semigroups [2] by using refined semilattices. Wang, Zhang and Xie established the structure of regular orthocryptou semigroups [3] in terms of refined semilattices. In this paper, we give some characterizations of left C-wrpp semigroups which have the refined semilattice decomposition. Throughout this paper, we shall use the notation and terminology of [1]. For other undefined terms, the reader is referred to [4, 5, 6].

Definition 1.[1] Let a, b be elements of a semigroup S. We define the $\mathcal{L}^{(+)}$-relation by $a\mathcal{L}^{(+)}b$ if for all $x, y \in S^1$, $(ax, ay) \in \mathcal{R}$ if and only if $(bx, by) \in \mathcal{R}$, where $\mathcal{R}$ is the usual Green's $\mathcal{R}-$relation on S .

Definition 2.[1] A semigroup S is called a wrpp semigroup if the following conditions are satisfied:

(1) each $\mathcal{L}^{(+)}-$class of S contains at least one idempotent of S;

(2) for all $e \in E(L_a^{(+)}), a = ae$ where $E(L_a^{(+)})$ is the set of idempotents in $L_a^{(+)}$.

Definition 3.[1] We call a wrpp semigroup S an adequate wrpp semigroup if for all $a \in S$, there exists a unique idempotent a^+ satisfying $a\mathcal{L}^{(+)}a^+$ and $a = a^+a$.

Definition 4.[1] An adequate wrpp semigroup S satisfying $aS \subseteq L^{(+)}(a)$ for all $a \in S$ is called a left C-wrpp semigroup.

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Definition 5.[1] A monoid M is called a left-$\mathcal{R}$ cancellative monoid if for $a, b, c \in M, (ab, ac) \in \mathcal{R}$ implies $(b, c) \in \mathcal{R}$. We call the direct product of a left-$\mathcal{R}$ cancellative monoid M and left zero band I a left-$\mathcal{R}$ cancellative stripe because the direct product looks like a two-dimensional stripe. We denote the left-$\mathcal{R}$ cancellative stripe by $M \times I$.

Definition 6.[4] Let Y be a semilattice and $\{S_\alpha : \alpha \in Y\}$ a family of disjoint semigroups of type T, indexed by Y. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $D(\alpha, \beta)$ be a set of index and

$$\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

a congruence partition of S_β (i.e., the relation σ on S_β defined by $(b_\beta, b'_\beta) \in \sigma$ if and only if $b_\beta, b'_\beta \in S_{d(\alpha,\beta)}$ for some $d(\alpha, \beta) \in D(\alpha, \beta)$ is a congruence on S_β), and for $\alpha \geq \beta \geq \gamma$, the partition

$$\{S_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}$$

is dense in the partition

$$\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\},$$

i.e., for any $d(\beta, \gamma) \in D(\beta, \gamma)$, there exists $D'(\alpha, \gamma) \subseteq D(\alpha, \gamma)$ such that

$$S_{d(\beta,\gamma)} = \bigcup_{d(\alpha,\gamma) \in D'(\alpha,\gamma)} S_{d(\alpha,\gamma)}.$$

Moreover, let

$$\{(\Phi_{d(\alpha,\beta)} : S_\alpha \to S_{d(\alpha,\beta)}) : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

be a family of homomorphisms. Suppose the following conditions are satisfied:

(a) $D(\alpha, \alpha)$ is singleton and $\Phi_{d(\alpha,\alpha)}$ is the identical automorphism of S_α for each $\alpha \in Y$, where $d(\alpha, \alpha)$ is the unique element of $D(\alpha, \alpha)$.

(b) (i) For every $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$,

$$\{\Phi_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)} : d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\} \subseteq \{(\Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}.$$

(ii) For any $d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$ and $d(\alpha\beta, \alpha\beta\gamma) \in D(\alpha\beta, \alpha\beta\gamma)$, $S_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)} \subseteq S_{d(\alpha,\alpha\beta\gamma)}$, where $d(\alpha\beta, \alpha\beta\gamma)$ satisfies $\Phi_{d(\alpha,\alpha\beta\gamma)} = \Phi_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)}$.

(c) For $\alpha, \beta, \gamma \in Y$ with $\gamma \leq \alpha\beta$ and for any fixed $a_\alpha \in S_\alpha, d(\alpha\beta, \gamma) \in D(\alpha\beta, \gamma)$, there exists $\bar{d}(\beta, \gamma) \in D(\beta, \gamma)$ such that $\{a_\alpha\Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \bigcap S_{d(\alpha\beta,\gamma)} \subseteq S_{\bar{d}(\beta,\gamma)}$.

(d) For $\alpha, \beta \in Y$ with $\alpha \geq \beta$, and $a_\alpha \in S_\alpha, b_\beta \in S_\beta, d(\alpha, \beta), d'(\alpha, \beta) \in D(\alpha, \beta)$,

$$b_\beta(a_\alpha\Phi_{d'(\alpha,\beta)}) \in S_{d(\alpha,\beta)} \Rightarrow b_\beta(a_\alpha\Phi_{d'(\alpha,\beta)}) = b_\beta(a_\alpha\Phi_{d(\alpha,\beta)}),$$

and

$$(a_\alpha\Phi_{d'(\alpha,\beta)})b_\beta \in S_{d(\alpha,\beta)} \Rightarrow (a_\alpha\Phi_{d'(\alpha,\beta)})b_\beta = (a_\alpha\Phi_{d(\alpha,\beta)})b_\beta.$$

We now form the set $S = \bigcup\{S_\alpha : \alpha \in Y\}$ and define a multiplication $\circ$ on S by the following statements. For any $a_\alpha \in S_\alpha, b_\beta \in S_\beta$, define

$$a_\alpha \circ b_\beta = (a_\alpha\Phi_{\bar{d}(\alpha,\alpha\beta)})(b_\beta\Phi_{\bar{d}(\beta,\alpha\beta)}),$$

where $\bar{d}(\alpha,\alpha\beta)\in D(\alpha,\alpha\beta), \bar{d}(\beta,\alpha\beta)\in D(\beta,\alpha\beta)$, that satisfy the following conditions:

$$\{a_\alpha\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta)\in D(\alpha,\alpha\beta)\}\subseteq S_{\bar{d}(\beta,\alpha\beta)},$$

and

$$\{b_\beta\Phi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta)\in D(\beta,\alpha\beta)\}\subseteq S_{\bar{d}(\alpha,\alpha\beta)}.$$

Then $(S,\circ)$ is a semigroup as it has been verified in [4]. Here, the semigroup $(S,\circ)$ is called the refined semilattice of semigroups and we denote it by

$$\{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha,\beta); S_\alpha\}.$$

§2. Characterizations

In this section we shall give some characterizations of left C-wrpp semigroups which have the refined semilattice decomposition. For our purpose, we need the following lemma.

Lemma 1. Let $S=\{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha,\beta); S_\alpha = M_\alpha\times I_\alpha\}$, where M_α is a left-$\mathcal{R}$ cancellative monoid M_α and I_α is a left zero band for any $\alpha\in Y$. For any $d(\alpha,\beta), d^{'}(\alpha,\beta)\in D(\alpha,\beta)(\alpha\geq\beta)$ if $(a_\alpha,i_\alpha)\Phi_{d(\alpha,\beta)}=(a_\beta,i_\beta)$ and $(a_\alpha,i_\alpha)\Phi_{d^{'}(\alpha,\beta)}=(a_\beta^{'},i_\beta^{'})$. Then

(1) $(e_\beta,i_\beta)\in S_{d(\alpha,\beta)}$;

(2) $a_\beta=a_\beta^{'}$;

(3) for any $\alpha\geq\beta$ and $d^1(\alpha,\beta)\in D(\alpha,\beta)$, if $(a_\beta^1,i_\beta^1)\in S_{d_1(\alpha,\beta)}$, Then $(e_\beta,i_\beta^1)\in S_{d_1(\alpha,\beta)}$;

(4) $(e_\alpha,i_\alpha)\Phi_{d(\alpha,\beta)}=(e_\beta,i_\beta)$.

Proof. The proof is the same as the proof of [2, Lemma 2.1].

We are now ready to give a characterization of left C-wrpp semigroups which have the refined semilattice decomposition.

Theorem 1. A semigroup S can be expressed as a refined semilattice of left-$\mathcal{R}$ cancellative stripes $M_\alpha\times I_\alpha$ if and only if it is a left C-wrpp semigroup such that the semi-spined product decomposition $S=M_S\times_\eta I_S$ of S is the spined product decomposition.

Proof. We need only to prove the sufficiency part since the proof of the necessity part is similar to the proof of the necessity part of [2, Theorem 1.5].

Let S be a refined semilattice of left-$\mathcal{R}$ cancellative stripes

$$M_\alpha\times I_\alpha,\ \text{i.e.,}\ S=\{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha,\beta); S_\alpha = M_\alpha\times I_\alpha\},$$

where M_α is a left-$\mathcal{R}$ cancellative monoid M_α and I_α is a left zero band. Then S is a semilattice of $M_\alpha\times I_\alpha$. By the proof of [2, Theorem 3.3], we know that M_α with identity e_α has a unique idempotent e_α and $E(S_\alpha)=\{e_\alpha\}\times I_\alpha$.We first prove that S is a wrpp semigroup. Let $(a_\alpha,i_\alpha),(x_\beta,j_\beta),(y_\gamma,\lambda_\gamma)\in S$ such that $(a_\alpha,i_\alpha)(x_\beta,j_\beta)\mathcal{R}(a_\alpha,i_\alpha)(y_\gamma,\lambda_\gamma)$. Hence $\alpha\beta=\alpha\gamma$ and

$$[(a_\alpha,i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}][(x_\beta,j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}]\mathcal{R}[(a_\alpha,i_\alpha)\Phi_{d^{'}(\alpha,\alpha\beta)}][(y_\gamma,\lambda_\gamma)\Phi_{d^{'}(\gamma,\alpha\beta)}].$$

where

$$\{(a_\alpha,i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta)\in D(\alpha,\alpha\beta)\}\subseteq S_{\bar{d}(\beta,\alpha\beta)},$$

$$\{(x_\beta,j_\beta)\Phi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta)\in D(\beta,\alpha\beta)\}\subseteq S_{\bar{d}(\alpha,\alpha\beta)},$$

$$\{(a_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{d'(\gamma,\alpha\beta)},$$

$$\{(y_\gamma, \lambda_\gamma)\Phi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)\} \subseteq S_{d'(\alpha,\alpha\beta)}.$$

Then

$$\begin{aligned} & (a_{\alpha\beta}, i_{\alpha\beta})(x_{\alpha\beta}, j_{\alpha\beta})\mathcal{R}(a_{\alpha\beta}, i'_{\alpha\beta})(y_{\alpha\beta}, \lambda_{\alpha\beta}) \\ \Rightarrow\ & a_{\alpha\beta}x_{\alpha\beta}\mathcal{R}(M_{\alpha\beta})a_{\alpha\beta}y_{\alpha\beta} \quad \text{and} \quad i_{\alpha\beta} = i'_{\alpha\beta}. \\ \Rightarrow\ & x_{\alpha\beta}\mathcal{R}(M_{\alpha\beta})y_{\alpha\beta} \quad \text{and} \quad \bar{d}(\alpha,\alpha\beta) = d'(\alpha,\alpha\beta). \\ \Rightarrow\ & e_{\alpha\beta}x_{\alpha\beta}\mathcal{R}(M_{\alpha\beta})e_{\alpha\beta}y_{\alpha\beta}. \\ \Rightarrow\ & (e_{\alpha\beta}, i_{\alpha\beta})(x_{\alpha\beta}, j_{\alpha\beta})\mathcal{R}(e_{\alpha\beta}, i_{\alpha\beta})(y_{\alpha\beta}, \lambda_{\alpha\beta}). \end{aligned}$$

By Lemma 1, we have

$$\{(e_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)},$$

and

$$\{(e_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{d'(\gamma,\alpha\beta)}.$$

Hence

$$\begin{aligned} (e_\alpha, i_\alpha)(x_\beta, j_\beta) &= [(e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}][(x_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}] \\ &= (e_{\alpha\beta}, i_{\alpha\beta})(x_{\alpha\beta}, j_{\alpha\beta}), \end{aligned}$$

and

$$\begin{aligned} (e_\alpha, i_\alpha)(y_\gamma, \lambda_\gamma) &= [(e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}][(y_\gamma, \lambda_\gamma)\Phi_{d'(\gamma,\alpha\beta)}] \\ &= (e_{\alpha\beta}, i_{\alpha\beta})(y_{\alpha\beta}, \lambda_{\alpha\beta}). \end{aligned}$$

Then $(e_\alpha, i_\alpha)(x_\beta, j_\beta)\mathcal{R}(e_{\alpha\beta}, i_{\alpha\beta})(y_{\alpha\beta}, \lambda_{\alpha\beta})$, that is $(a_\alpha, i_\alpha)\mathcal{L}^{(+)}(e_\alpha, i_\alpha)$. Let $(e_{\beta'}, j_{\beta'})\mathcal{L}^{(+)}(a_\alpha, i_\alpha)$. Then $(e_{\beta'}, j_{\beta'})\mathcal{L}^{(+)}(e_\alpha, i_\alpha)$ and so $(e_{\beta'}, j_{\beta'})(e_{\beta'}, j_{\beta'})\mathcal{R}(e_{\beta'}, j_{\beta'})\cdot 1 \Rightarrow (e_\alpha, i_\alpha)(e_{\beta'}, j_{\beta'})\mathcal{R}(e_\alpha, i_\alpha) \Rightarrow \beta \geq \alpha$. Similarly, $\alpha \geq \beta$. Hence $\alpha = \beta$. For all $(a_\alpha, i_\alpha) \in S$ and $(e_\alpha, j'_\alpha) \in E(L^{(+)}_{(a_\alpha,i_\alpha)})$, we have $(a_\alpha, i_\alpha)(e_\alpha, j'_\alpha) = (a_\alpha, i_\alpha)$. Then S is a wrpp semigroup. We have showed that for $(a_\alpha, i_\alpha) \in S, (e_\alpha, i_\alpha)\mathcal{L}^{(+)}(a_\alpha, i_\alpha)$ and we have $(e_\alpha, i_\alpha)(a_\alpha, i_\alpha) = (a_\alpha, i_\alpha)$. Let $(e_\alpha, i'_\alpha) \in E(L^{(+)}_{(a_\alpha,i_\alpha)})$ such that $(e_\alpha, i'_\alpha)(a_\alpha, i_\alpha) = (a_\alpha, i_\alpha)$. Then $i'_\alpha = i_\alpha$. Hence $(e_\alpha, i'_\alpha) = (e_\alpha, i_\alpha)$. Thus S is an adequate wrpp semigroup. By [2, Theorem 3.2], we know that S is a left C-wrpp semigroup.

The unique idempotent $e \in E(L^{(+)}_a)$ such that $ea = a$ in S is expressed by e_a. We now claim that S satisfies $e_x e_y = e_{xy}$ for any $x, y \in S$. In fact, if we let $x = (a_\alpha, i_\alpha)$ and $y = (a_\beta, j_\beta)$, then we have

$$\begin{aligned} xy &= (a_\alpha, i_\alpha)(a_\beta, j_\beta) \\ &= [(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}][(a_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}] \\ &= (a_{\alpha\beta}, i'_{\alpha\beta}), \end{aligned}$$

where $\bar{d}(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta), \bar{d}(\beta,\alpha\beta) \in D(\beta,\alpha\beta)$, satisfy that

$$\{(a_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)},$$

and

$$\{(a_\beta, j_\beta)\Phi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}.$$

Since $(a_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} = (e_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$, by Lemma 1, we have $(e_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$. This leads to $\{(e_\alpha, i_\alpha)\Phi_{d_1(\alpha,\alpha\beta)} : d_1(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$. Similarly, we have $\{(e_\beta, j_\beta)\Phi_{d_1(\beta,\alpha\beta)} : d_1(\beta, \alpha\beta) \in D(\beta, \alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}$, and consequently,

$$\begin{aligned} xy &= (a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \\ &= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \\ &= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(e_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, j_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \\ &= (e_\alpha, i_\alpha)(e_\beta, j_\beta)(a_\alpha, i_\alpha)(a_\beta, j_\beta) \\ &= e_x e_y xy = xy e_x e_y. \end{aligned}$$

Hence $e_x e_y = (e_{\alpha\beta}, i'_{\alpha\beta}) = e_{xy}$. Define a relation ρ on S by:

$$(a, i) \quad \rho \quad (b, j) \quad \text{if and only if} \quad \text{there exists } \alpha \in Y \quad \text{such that} \quad a, b \in M_\alpha \quad \text{and} \quad i = j \in I_\alpha. \tag{1}$$

Then $x\rho y$ if and only if $e_x = e_y$. By $e_{xy} = e_x e_y$, we deduce that ρ is a congruence. According to [1, Theorem 4.4], the semi-spined product decomposition $S = M_S \times_\eta I_S$ of S is the spined product decomposition. Our claim is established. Therefore a left C-wrpp semigroups which has the refined semilattice decomposition is a left regular orthocryptou semigroup.

By [3, Lemma 5.2] the relation ρ in(1) is exactly the same as $\widetilde{\mathcal{H}}$. Hence by Theorem 1 and [3, Corollary 5.8], we can get the following result.

Theorem 2. Let S be a semigroup. Then the following statements are equivalent.

(1) S is a left C-wrpp semigroup, in which $\widetilde{\mathcal{H}}$ is a congruence;

(2) S is a refined semilattice of some left-$\mathcal{R}$ cancellative stripes;

(3) S is a right strong semilattice of some left-$\mathcal{R}$ cancellative stripes.

Remarks 1. By [3, Corollary 5.8], a left C-wrpp semigroup, in which $\widetilde{\mathcal{H}}$ is a congruence is a left regular orthocryptou semigroup.

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Weyl type theorems for (α,β)-normal operators

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Abstract A bounded linear operator T is said to be an (α,β)-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$ if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$. This class of operators is defined by M. S. Moselehian [20]. In this paper some spectral properties of (α,β)-normal operators are discussed, generalised a-Weyl's theorem is proved for this kind of operators. An example of an (α,β)-normal operator which is neither normal nor hyponormal is given.

Keywords Generalised a-Weyl's theorem, a-Browder theorem, generalised Weyl's theorem, Riesz idempotent, (α,β)-normal operator, Single Valued Extension Property.

§1. Introduction and preliminaries

Let H be a complex Hilbert space and $B(H)$, the algebra of all bounded linear operators on H. An operator T is said to be an (α,β)- normal operator $(0\leq \alpha \leq 1 \leq \beta)$ if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$.

Following are the important facts about (α,β)- normal operators:

(i) If T is (α,β)-normal, then $ker(T) = ker(T^*)$, $ran(T) = ran(T^*)$ [11].

(ii) If T is (α,β)-normal, then so is $T-\lambda$ for each $\lambda \in$ C [11].

(iii) If T is (α,β)-normal and $M \subseteq H$ is invariant under T, then $T|_M$ is (α,β)-normal.

(iv) Every quasinilpotent (α,β)-normal operator is zero operator.

(v) Every (α,β)-normal operator is an isoloid.

(vi) If H is finite dimensional then an (α,β)-normal operator is normal.

S. S. Dragomir and M. S. Moselehain [11] has given an example of an (α,β)- normal operator which is neither normal nor hyponormal. Here we give another example of (α,β)- normal operator of the same kind. The operator matrix T = $\begin{pmatrix} U & K \\ 0 & U^* \end{pmatrix}$: $\ell_2 \oplus \ell_2 \longrightarrow \ell_2 \oplus \ell_2$ where U is the unilateral shift on ℓ_2 and K: $\ell_2 \longrightarrow \ell_2$ given by $K(x_1, x_2, x_3, \cdots) = (2x_1, 0, 0\cdots)$ is (α,β)-normal for $\alpha = 1/2$ and $\beta = 2$.

In the next section, we prove Riesz idempotent E_λ of T with respect to each non-zero isolated point spectrum λ defined by $E_\lambda = \int_{\partial D} (\lambda - T)^{-1} d\lambda$ is self adjoint and $E_\lambda H = ker(T-\lambda)$ $= ker(T-\lambda)^*$.

§2. Riesz idempotent

A complex number λ is said to be in the *joint point spectrum* of T if there exists a joint eigenvector x of T and T^* such that $Tx = \lambda x$ and $T^*x = \overline{\lambda}x$.

Let $\sigma_{jp}(T)$ and $\sigma_p(T)$ denote the *joint point spectrum* and the *point spectrum* of T respectively.

Theorem 2.1. If T is an (α,β) normal operator then the following assertions hold:

1. $Tx = \lambda x$, $\lambda \neq 0$, then $T^*x = \overline{\lambda}x$.
2. $\sigma_{jp}(T) \sim 0 = \sigma_p(T) \sim 0$.
3. If $Tx = \lambda x$ and $Ty = \mu y$, $\lambda \neq \mu$, then $(x, y) = 0$.

Proof follows by using (i) of Section 1. Let $\sigma_{iso}(T)$ be the set of all isolated points of $\sigma(T)$. If $\lambda_0 \in \sigma_{iso}(T)$, the Riesz idempotent E_λ of T with respect to λ_0 is defined by

$$E_\lambda = \int_{\partial D} (\lambda - T)^{-1} d\lambda,$$

where $\lambda_0 \in D$ is a closed disk of center λ_0 which contains no other points of $\sigma(T)$.

Stampfli [22] proved that if T is hyponormal, E_λ is self adjoint and $E_\lambda H = ker(T - \lambda) = ker(T - \lambda)^*$. This result has been proved for various operators in [8],[12], [23], [24].

The next theorem proves the same result for (α, β)-normal operators. We need the following lemma to prove the same.

Lemma 2.2. If T is an (α, β)-normal operator with $\sigma(T) = \lambda$ then $T = \lambda$.

Proof. Suppose $\sigma(T) = \lambda$, since T is (α, β)-normal, then so is $T - \lambda$. Since every (α, β)-normal quasinilpotent operator is zero operator, $T - \lambda = 0$. Hence $T = \lambda$.

Theorem 2.3. Let T be an (α, β)-normal operator and λ_0 be an isolated point of $\sigma(T)$. If E is the Riesz idempotent for λ_0, then E is self adjoint and $E(H) = ker(T - \lambda_0)= ker(T^* - \overline{\lambda})$.

Proof. The second equality follows from the definition of (α, β)-normal operators. Any (α, β)-normal operator T can be represented as a block matrix $\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$ where C : $\overline{ran(T)} \longrightarrow \overline{ran(T)}$ has zero kernel.

$$\begin{aligned} E &= \frac{1}{2\Pi i} \int_{\partial D} (\lambda - T)^{-1} d\lambda \\ &= \frac{1}{2\Pi i} \int_{\partial D} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & (\lambda - C)^{-1} \end{pmatrix} d\lambda \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

on $E(H) \oplus E(H)^\perp$. Hence E is self adjoint.

Definition 2.4. For $T \in B(H)$, $\lambda \in \sigma(T)$ is said to be a *regular point* if there exists $S \in B(H)$ such that $T - \lambda = (T - \lambda)S(T - \lambda)$. If every isolated point of $\sigma(T)$ is a *regular point*, then T is called a *reguloid*.

Following lemma is used to prove the Corollary below.

Lemma 2.5.[13] $T - \lambda$ has a closed range if and only if $T - \lambda$= $(T - \lambda)S(T - \lambda)$.

Corollary 2.6. If T is a (α,β)- normal operator, then T is a *reguloid.*

Proof. Let λ_0 be an isolated point of $\sigma(T)$. Using Reisz idempotent E_λ, we can represent T as a direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{0\}$ and $\sigma(T_2 = \sigma(T)\backslash\{\lambda_0\}$.
Since T_1 is also (α,β) normal operator, it follows from Lemma 2.2 that $T_1 = \lambda_0$. Therefore by Theorem 2.3, $H = E(H) \oplus E(H)^\perp = ker(T-\lambda_0) \oplus ker(T-\lambda_0)^\perp$. Hence $T = \lambda_0 \oplus T_2$. Therefore $T - \lambda_0 = 0 \oplus (T_2 - \lambda_0)$ and hence

$$ran(T-\lambda_0) = (T-\lambda_0)(H) = 0 \oplus (T_2 - \lambda_0)\ (ker(T-\lambda_0)^\perp).$$

Since T_2 - λ_0 is invertible, $T - \lambda_0$ has a closed range.

§3. Weyl type theorems

For $T \in B(H)$, let $\alpha(T) = dim(ker(T))$, $\beta(T) = dim(ker(T^*))$, and $\sigma(T), \sigma_a(T)$, $\pi_0(T)$ denote the *spectrum*, *approximate point spectrum* and the *point spectrum* of T respectively. An operator $T \in B(H)$ is called *Fredholm* if it has closed range, finite dimensional null space and its range has finite co-dimension. The *index* of a Fredholm operator is given by $ind(T) = \alpha(T)$ - $\beta(T)$.

A bounded linear operator T is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent; equivalently , if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $\mid \lambda \mid \geq 0$, $\lambda \in C$ [14]. For $T \in B(H)$, for each non-negative integer n, define T_n to be the restriction of T to $ran(T^n)$ into $ran(T^n)$. If for some n, the space $ran(T^n)$ is closed and T_n is a Fredholm operator, then T is called a *B - Fredholm* operator [3]. $T \in B(H)$ is called a *B - Weyl* operator if it is a B - Fredholm operator of index zero.

$$\text{Let } \Phi_+(H) = \{T \in B(H) : \alpha(T) < \infty \text{ and } T(H) \text{ is closed } \}$$
$$\text{Let } \Phi_-(H) = \{T \in B(H) : \beta(T) < \infty\}$$

denote the class of all *upper semi - Fredholm* operators and *lower semi - Fredholm* operators. And let Φ_+^-(H) is the class of all *left - semi - Fredholm* operators, such that for every $T \in \Phi_+^-(H)$, $ind\ T \leq 0$.

Let SBF_+ be the class of all *upper semi - B - Fredholm* operators, SBF_+^- be the class of all *semi - B - Fredholm* operators such that for every $T \in SBF_+^-$, $ind(T) \leq 0$. The *essential spectrum* $\sigma_e(T)$, the *Weyl spectrum* $\sigma_w(T)$ and the *Browder spectrum* $\sigma_b(T)$ of T are defined by [13, 14]:

$$\sigma_e(T) = \{\lambda \in C : T - \lambda \text{ is not Fredholm } \};$$
$$\sigma_w(T) = \{\lambda \in C : T - \lambda \text{ is not Weyl } \};$$
$$\sigma_b(T) = \{\lambda \in C : T - \lambda \text{ is not Browder } \};$$

respectively. And the *B - Weyl spectrum, a - Browder spectrum, essential approximate spectrum and Weyl(essential) approximate point spectrum* are defined by

$$\sigma_{Bw}(T) = \{\lambda \in C : T - \lambda \text{ is not B-Weyl } \} ;$$
$$\sigma_{ea}(T) = \{\lambda \in C : T - \lambda \notin \Phi_+^-(H)\};$$
$$\sigma_{ab}(T) = \bigcap \sigma_a(T + K) : TK = KT \text{ , } K \text{ is a compact operator;}$$
$$\sigma_{wa}(T) = \{\lambda \in C : T - \lambda \notin \Phi_+(H) \text{ and } ind(T - \lambda) \leq \infty\};$$

Evidently, $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\ \sigma(T)$;

where we write $acc\ \sigma(T)$ for the accumulation points of $\sigma(T)$.

If we write $isoK = K\backslash\ accK$, then we write

$$\pi_{00}(T) = \{\lambda \in iso\ \sigma(T) : 0 < \alpha(T - \lambda) < \infty\},$$
$$p_{00}(T) = \sigma(T)\backslash\sigma_b(T),$$
$$\pi^a_{00}(T) = \{\lambda \in iso\ \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},$$
$$\pi_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T).$$

We say that *Weyl's theorem* holds for $T \in B(H)$ if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$.

We say that *a - Weyl's theorem* holds for $T \in B(H)$ if $\sigma_{ap}(T)\backslash\sigma_{wa}(T) = \pi_{00}(T)$.

We say that *Browder's theorem* holds for $T \in B(H)$ if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$.

We say that *a - Browder's theorem* holds for $T \in B(H)$ if $\sigma_{ea}(T) = \sigma_{ab}(T)$.

We say that *generalised a - Weyl's theorem* holds for $T \in B(H)$ if $\sigma_{SBF^-_+}(T) = \sigma_{ap}(T)\backslash E^a(T)$, where E^a is the set of all eigen values of T that are isolated in $\sigma_{ap}(T)$.

It is clear that [10]

generalized a - Weyl's theorem $\Rightarrow$ generalized Weyl's theorem
$\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem.
and
generalized a - Weyl's theorem $\Rightarrow$ a - Weyl's theorem
$\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem.
and
generalized a - Weyl's theorem $\Rightarrow$ a - Weyl's theorem
$\Rightarrow$ generalized Browder's theorem $\Rightarrow$ Browder's theorem.

S. Mecheri [18] has proved generalised a - Weyl's theorem for some classes of operators. Here in this paper we prove generalised a - Weyl's theorem for (α,β)-normal operators.

§3. Main results

We say that $T \in B(H)$ has the Single Valued Extension Property ($SVEP$) if, for every open set $U \subseteq C$, the only analytic function $f : U \to H$ that satisfies the equation $(T - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$.

T is said to have a finite ascent if $ker\ T^m = ker\ T^{m+1}$ for some positive integer m, and finite descent if $ran\ T^n = ran\ T^{n+1}$ for some positive integer n.

We prove the following lemmas to prove the generalised a-Weyl's theorem.

Lemma 3.1. If T is (α, β)-normal such that $\alpha\ \beta = 1$, then T^* is also (α, β)-normal.

Lemma 3.2. If T is (α, β)-normal, then T has $SVEP$.

Proof. Let T be (α,β)-normal. If $\pi_0(T) = \phi$, then T has $SVEP$.

Suppose $\pi_0(T) \neq \phi$, let $\triangle(T) = \{\lambda \in \pi_0 : ker(T - \lambda) \subseteq ker(T^*$ - $\overline{\lambda}\})$.

Since T is (α,β)-normal and $\pi_0(T) \neq \emptyset, \triangle(T) \neq \emptyset$. Let M be a closed linear span of the subspaces $ker(T - \lambda)$ with $\lambda \in \triangle(T)$. Then M reduces T, and so we can write T as a direct sum $T = T_1 \oplus T_2$ on $M \oplus M^\perp$. Here T_1 is normal and hence has $SVEP$. $\pi_0(T_2) = \emptyset$ implies T_2 has $SVEP$ and hence T has $SVEP$.

Corollary 3.3. If T is an (α,β)-normal operator such that $\alpha\ \beta = 1$, then T^* has $SVEP$.

Definition 3.4. Let $T \in B(H)$. We say that T is of *stable sign index* if for each λ, $\mu \in \rho_{BF}(T)$, *ind* $(T-\lambda I)$ and *ind* $(T-\mu I)$ have the same sign. Here $\rho_{BF}(T)$ is the B - Fredholm resolvent set of T, i.e. $\rho_{BF}(T) = C \sim \sigma_{BF}(T)$.

Theorem 3.5. Let H be a Hilbert space and let $T \in B(H)$ be (α,β)-normal operator. Then T is of *stable sign index.*

Proof. Let T be an (α,β)-normal operator. Then $ker(T) = ker(T^*) = ran(T)^{\perp}$. Since $ker(T^2)/ker(T) \simeq ker(T) \cap ran(T)$, then $ker(T^2) = ker(T)$. If T is also a B - Fredholm operator, then there exists an integer n such that $ran(T^n)$ is closed and such that $T_n : ran(T^n) \longrightarrow ran(T^n)$ is a Fredholm operator. We have $ind(T) = ind(T_n) \leq 0$.

And if $\lambda \in \rho_{BF}(T)$, then $T-\lambda I$ is a B - Fredholm operator, and $T-\lambda I$ is also an (α,β)-normal operator, which proves $ind(T-\lambda I) \leq 0$. Therefore T is of *stable sign index.*

Corollary 3.6. Let $T \in B(H)$ be an (α,β)-normal operator, and let f be a function analytic in the neighbourhood of the usual spectrum $\sigma(T)$ of T then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$.

Proof. If $T \in B(H)$ is of stable sign index and f is a function analytic in a neighbourhood of the spectrum $\sigma(T)$ of T then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ [6].

A bounded linear operator $T \in B(H)$ is said to be *Drazin invertible* if and only if it has a finite ascent and descent. This is also equivalent to the fact that $T = U \oplus V$ where U is an invertible operator and V is nilpotent.

Theorem 3.7. Let $T \in B(H)$ be an (α,β)-normal operator, then T satisfies generalised Weyl's theorem.

Proof. Let $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{BW}(T)$, $T-\lambda I$ is a B - Fredholm operator of index 0 , there exist two closed subspaces M, N of H such that $H = M \oplus N$, and $T-\lambda I = U \oplus V$ with $U = (T-\lambda I)|_M$ a Fredholm operator of index 0 and $V = (T-\lambda I)|_N$ is a nilpotent operator [4].

Let $S = T|_M$ and $I_M = I|_M$. Since T is an (α,β)-normal operator, then S is also a (α,β)-normal operator and $(S-\lambda I)_{|M} = U$ is a Fredholm operator of index 0.

If $\lambda \in \sigma(S)$, since S is a (α,β)-normal operator, we have $\sigma_W(S) = \sigma(S)\backslash E_0(S)$ [4]. As $\lambda \notin \sigma_W(S)$ we have $\lambda \in E_0(S)$. In particular, λ is isolated in $\sigma(S)$. Since $T-\lambda I = U \oplus V = (S-\lambda I)_{|M} \oplus V$, and V is a nilpotent operator, we have $\sigma(U)\backslash\{0\}$. Therefore 0 is isolated in $\sigma(T-\lambda I)$ or equivalently λ is isolated in $\sigma(T)$. As $\lambda \in E_0(S)$ then $\lambda \in E(T)$.

Conversely, if $\lambda \in E(T)$, then λ is isolated in $\lambda(T)$.We have $X = M \oplus N$, where M, N are closed subspaces of X, $U = (T-\lambda I)_{|M}$ is an invertible operator and $V = (T-\lambda I)_{|N}$ is a quasinilpotent operator From [15]. Since T is a (α,β)-normal operator, V is also (α,β)-normal operator. As V is quasinilpotent, we have $V = 0$ from [21, Chapter XI]. Hence $T-\lambda I$ is Dravin invertible. $T-\lambda I$ is a B - Fredholm operator of index 0 [4].

Theorem 3.8. Let $T \in B(H)$ be an (α,β)-normal operator, and let f be a function in the neighbourhood of the spectrum $\sigma(T)$ of T then $f(T)$ satisfies generalised Weyl's theorem.

Proof. If $T \in B(H)$ is an isoloid operator which satisfies generalised Weyl's theorem and if f is a function analytic in the neighbourhood of $\sigma(T)$ of T, then generalised Weyl's theorem holds for $f(T)$ if and only if $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ [6].

Theorem 3.9. Let T be an (α,β)-normal operator such that $\alpha\ \beta = 1$, then T satisfies generalised a - Weyl's theorem.

Proof. M. Lahrouz and M. Zohry [17] have proved that if T^* has $SVEP$, then T satisfies generalised a - Weyl's theorem if and only if it satisfies generalised Weyl's theorem.

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A construction of regular ortho−lc−monoids [1]

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Abstract In this paper, a construction of the regular ortho−lc−monoids is given, and it is shown that a semigroup S is a regular ortho−lc−monoid if and only if it is a refined semilattice of rectangular lc−monoids. Consequently, our main results generalize the corresponding ones in [1].

Keywords Regular ortho −lc− monoids, normal ortho −lc−monoids, refined semilattice, strong semilattice.

§1. Introduction

Refined semilattice of semigroups was firstly studied by Zhang, Shum and Zhang in [2]. It is a natural generalization of the notation of strong semilattice of semigroups. Hence, a lot of results in the literature concerning strong semilattice decomposition can be further developed (see [3]-[6]).

Wang, Zhang and Xie [5] investigated the refined semilattice structure of regular orthocryptou semigroups, they showed that a semigroup S is a regular orthocryptou semigroup if and only if it is a refined semilattice of rectangular u−semigroups.

In [7], Guo, Shum and Tsai investigated a class of semigroups called ortho−lc−monoids, and showed that a semigroup S is an ortho−lc−monoids if and only if it is a semilattice of rectangular lc− monoids, where a rectangular lc− monoid means a direct product of a rectangular band and a left cancellative monoid. Further, the authors introduced semi–spined product, and then proved that a semigroup S is a regular ortho−lc−monoid if and only if S is isomorphic to a semi–spined product of a lc−Clifford semigroup, a left regular band and right regular band.

In this paper, our aim is to give another construction of regular ortho−lc−monoids. We shall prove that a semigroup S is a regular ortho −lc− monoid if and only if it is a refined semilattice of rectangular lc− monoids. For notations and terminologies not mentioned in this paper,the reader is referred to [5], [7]-[9].

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§2. Preliminaries

First, we recall three families of generalized Green's relations used to study certain classes of semigroups. Let S be an arbitrary semigroup, the following Green'$*-$relations are due to McAlister [10] and Pastijn [11]. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are defined on S by the rule that the elements a, b of S are related by $\mathcal{L}^*$ ($\mathcal{R}^*$) in S if and only if they are related by Green's relation $\mathcal{L}$ ($\mathcal{R}$) in some oversemigroup of S.

Lemma 1.[8] Let a, b be elements of a semigroup S. Then the following are equivalent:

(1) $(a, b) \in \mathcal{L}^*$;

(2) for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

It means that

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall a, b \in S^1)ax = ay \Leftrightarrow bx = by\}.$$

Dual result for relation $\mathcal{R}^*$ is

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall a, b \in S^1)xa = ya \Leftrightarrow xb = yb\},$$

and $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$.

El-Qallali [12] generalized Green's $*-$relations to Green's $\sim -$relations:

$$\tilde{\mathcal{L}} = \{(a, b) \in S \times S : (\forall e \in E(S))ae = a \Leftrightarrow be = b\},$$

$$\tilde{\mathcal{R}} = \{(a, b) \in S \times S : (\forall e \in E(S))ea = a \Leftrightarrow eb = b\},$$

$$\tilde{\mathcal{H}} = \tilde{\mathcal{L}} \cap \tilde{\mathcal{R}},$$

$$\tilde{\mathcal{D}} = \tilde{\mathcal{L}} \vee \tilde{\mathcal{R}}.$$

In [7], Guo, Shum and Tsai defined Green's $(*, \sim)-$ relations as follows:

$$\mathcal{L}^{*,\sim} = \mathcal{L}^*, \mathcal{R}^{*,\sim} = \tilde{\mathcal{R}}$$

$$\mathcal{H}^{*,\sim} = \mathcal{L}^* \cap \tilde{\mathcal{R}}$$

$$\mathcal{D}^{*,\sim} = \mathcal{L}^* \vee \tilde{\mathcal{R}}$$

$$a\mathcal{J}^{*,\sim}b \Leftrightarrow J^{*,\sim}(a) = J^{*,\sim}(b)$$

where $J^{*,\sim}(a)$ is the smallest ideal of S containing a and saturated by $\mathcal{L}^*$ and $\tilde{\mathcal{R}}$.

It is clear that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}}$ ($\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}$) and there is at most one idempotent contained in each $\tilde{\mathcal{H}}-$class.

A semigroup S is abundant (semiabundant) if its each $\mathcal{L}^*-$class ($\tilde{\mathcal{L}}-$ class) and each $\mathcal{R}^*-$class ($\tilde{\mathcal{R}}-$ class) contains an idempotent. A semigroup S is called superabundant (semisuperabundant) if and only if its each $\mathcal{H}^*-$class ($\tilde{\mathcal{H}}-$ class) contains an idempotent.

A semisuperabundant semigroup S is called an orthocryptou semigroup if $\tilde{\mathcal{H}}$ on S is a congruence and $E(S)$ is a subsemigroup of S [5]. An orthoctyptou semigroup S is said to be a regular (respectively, left quasi–normal, right quasi–normal, normal) orthocryptou semigroup if its set of idempotents forms a regular (respectively, left quasi-normal, right quasi-normal,

normal) band [5], where a band is said to be regular (respectively, left quasi–normal, right quasi–normal, normal) if it satisfies the identity $axya = axaya$ ($axy = axay, yxa = yaxa, axya = ayxa$).

A semigroup S is called a rectangular $lc-$monoid if it is a direct product of left cancellative monoid and a rectangular band.

In the following, we call a semigroup S $\sigma-$ abundant [7], where $\sigma \in \varepsilon(S)$, the lattice of all equivlences on S, if every $\sigma-$ class of S contains idempotents of S.

Definition 1.[7] A semigroup S is called an ortho$-lc-$monoid (regular ortho$-lc-$monoid) if it is an $\mathcal{H}^{*,\sim}-$ abundant semigroup in which $E(S)$ forms a subsemigroup ($E(S)$ is a regular band), and the relation $\tilde{\mathcal{R}}$ is a left congruence on S.

By Definition 1, it is known that an ortho$-lc-$ monoid is actually an orthocryptou semigroup satisfying $\tilde{\mathcal{L}} = \mathcal{L}^*$.

Now, we introduce the concept of refined semilattice of semigroups .

Definition 2.[2,6] Let Y be a semilattce and $\{S_\alpha : \alpha \in Y\}$ a family of disjoint of semigroups of type T, indexed by Y. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $D(\alpha, \beta)$ be a set of index and

$$\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

a congruence partition of S_β (i.e., the relation σ on S_β defined by $(b_\beta, b'_\beta) \in \sigma$ if and only if $b_\beta, b'_\beta \in S_{d(\alpha,\beta)}$ for some $d(\alpha, \beta) \in D(\alpha, \beta)$ is a congruence on S_β), and for $\alpha \geq \beta \geq \gamma$, the partition

$$\{S_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}$$

is dense in the partition

$$\{S_{d(\beta,\gamma)} : d(\beta, \gamma) \in D(\beta, \gamma)\},$$

i.e., for any $d(\beta, \gamma) \in D(\beta, \gamma)$, there exists $D'(\alpha, \gamma) \subseteq D(\alpha, \gamma)$ such that

$$S_{d(\beta,\gamma)} = \cup_{d(\alpha,\gamma) \in D'(\alpha,\gamma)} S_{d(\alpha,\gamma)}.$$

Moreover, let

$$\{\Phi_{d(\alpha,\beta)} : S_\alpha \to S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

be a family of homomorphisms. Suppose the following conditions are satisfied :

(a) $D(\alpha, \alpha)$ is singleton and $\Phi_{d(\alpha,\alpha)}$ is the identical automorphism of S_α for each $\alpha \in Y$, where $d(\alpha, \alpha)$ is the unique element of $D(\alpha, \alpha)$.

(b) (i) For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$,

$$\{\Phi_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)} : d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\}$$

$$\subseteq \{\Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}.$$

(ii) For any $d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$ and $d(\alpha\beta, \alpha\beta\gamma) \in D(\alpha\beta, \alpha\beta\gamma)$,

$$S_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)} \subseteq S_{d(\alpha,\alpha\beta\gamma)}.$$

where $d(\alpha, \alpha\beta\gamma)$ satisfies $\Phi_{d(\alpha,\alpha\beta\gamma)} = \Phi_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)}$.

(c) For $\alpha, \beta, \gamma \in Y$ with $\gamma \leq \alpha\beta$ and for any fixed $a_\alpha \in S_\alpha$, $d(\alpha\beta, \gamma) \in D(\alpha\beta, \gamma)$, there exists $\bar{d}(\beta, \gamma) \in D(\beta, \gamma)$ such that

$$\{a_\alpha \Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap S_{d(\alpha\beta,\gamma)} \subseteq S_{\bar{d}(\beta,\gamma)}.$$

(d) For $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$, $d(\alpha, \beta) \in D(\alpha, \beta)$, $d^{'}(\alpha, \beta) \in D(\alpha, \beta)$,

$$b_\beta(a_\alpha \Phi_{d^{'}(\alpha,\beta)}) \in S_{d(\alpha,\beta)} \Rightarrow b_\beta(a_\alpha \Phi_{d^{'}(\alpha,\beta)}) = b_\beta(a_\alpha \Phi_{d(\alpha,\beta)}),$$

and

$$(a_\alpha \Phi_{d^{'}(\alpha,\beta)})b_\beta \in S_{d(\alpha,\beta)} \Rightarrow (a_\alpha \Phi_{d^{'}(\alpha,\beta)})b_\beta = (a_\alpha \Phi_{d(\alpha,\beta)})b_\beta.$$

We now form the set $S = \cup\{S_\alpha : \alpha \in Y\}$ and define a multiplication $\circ$ on S by the following statements.

For any $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$, define

$$a_\alpha \circ b_\beta = (a_\alpha \Phi_{\bar{d}(\alpha,\alpha\beta)})(b_\beta \Phi_{\bar{d}(\beta,\alpha\beta)}),$$

where $\bar{d}(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$, $\bar{d}(\beta, \alpha\beta) \in D(\beta, \alpha\beta)$, that satisfy the following conditions:

$$\{a_\alpha \Phi_{d(\alpha,\alpha\beta)} : d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$$

and

$$\{b_\beta \Phi_{d(\beta,\alpha\beta)} : d(\beta, \alpha\beta) \in D(\beta, \alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}.$$

Then $(S = \cup_{\alpha\in Y} S_\alpha, \circ)$ is a semigroup as it has been shown in [2]. Hence, the semigroup $(S, \circ)$ is called the refined semilattice of semigroups and is denoted by

$$\{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); S_\alpha\}.$$

Also, the concept of refined semilattice of semigroups is introduced in [5], where the definitions of left (right) strong semilattice of semigroups are also given. In the following, we will show our results by using the properties and results introduced in [2],[5] or [6].

§3. Regular ortho$-lc-$monoids

Firstly, the following lemmas and proposition are needed.

Lemma 1.[5] A semigroup S is a regular orthocryptou semigroup if and only if it is a refined semilattice of rectangular $u-$ semigroups.

Lemma 2.[7] A semigroup S is an ortho$-lc-$monoid if and only if it is rpp and is a semilattice of rectangular $lc-$ monoids.

Proposition 1. Let $S = \{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); S_\alpha = I_\alpha \times M_\alpha \times \Lambda_\alpha\}$, where M_α is a left cancellative monoid and $I_\alpha(\Lambda_\alpha)$ a left (right) zero band for any $\alpha \in Y$. For any $d(\alpha, \beta), d^{'}(\alpha, \beta) \in D(\alpha, \beta)$ $(\alpha \geq \beta)$, if $(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{d(\alpha,\beta)} = (i_\beta, a_\beta, \lambda_\beta)$ and $(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{d^{'}(\alpha,\beta)} = (i^{'}_\beta, b_\beta, \lambda^{'}_\beta)$, then we have $a_\beta = b_\beta$.

Proof. Firstly, $(i_\beta, e_\beta, \lambda_\beta)(i_\alpha, a_\alpha, \lambda_\alpha) = (i_\beta, e_\beta, i_\beta)[(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{d(\alpha,\beta)}] = (i_\beta, a_\beta, \lambda_\beta)$, and then,

$$\begin{aligned}
(i_\beta', b_\beta, \lambda_\beta') &= (i_\beta', e_\beta, \lambda_\beta')(i_\beta', b_\beta, \lambda_\beta')\\
&= (i_\beta', e_\beta, \lambda_\beta')[(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{d'(\alpha,\beta)}]\\
&= (i_\beta', e_\beta, \lambda_\beta')(i_\alpha, a_\alpha, \lambda_\alpha)\\
&= (i_\beta', e_\beta, \lambda_\beta')(i_\beta, e_\beta, \lambda_\beta)(i_\alpha, a_\alpha, \lambda_\alpha)\\
&= (i_\beta', e_\beta, \lambda_\beta')(i_\beta, a_\beta, \lambda_\beta)\\
&= (i_\beta', a_\beta, \lambda_\beta').
\end{aligned}$$

Hence, we have $a_\beta = b_\beta$.

Now, we begin to introduce our main theorem.

Theorem 1. A semigroup S is a regular ortho$-lc-$monoid if and only if it is a refined semilattice of rectangular $lc-$monoids.

Proof. $\Rightarrow$) Assume that S is a regular ortho$-lc-$monoid. By Lemma 2 and its proof in [7], it is a semilattice of rectangular $lc-$monoids, where the rectangular $lc-$monoids are all the $\mathcal{D}^{*,\sim} = \mathcal{J}^{*,\sim}-$classes of S. By its definition, it is known that a regular ortho $-lc-$ monoid is a regular orthocryptou semigroup. Note by Lemma 1 that, a regular orthocryptou semigroup is a refined semilattice of rectangular $u-$ semigroups, where the rectangular $u-$ semigroups are all its $\tilde{\mathcal{J}}-$ classes (see [5]). Now, using the similar arguments to prove the necessity of Lemma 1 in [5], it is a routine way for us to check that a regular ortho $-lc-$ monoid S is also a refined semilattice of its $\mathcal{J}^{*,\sim}-$classes, i.e., it is a refined semilattice of rectangular $lc-$ monoids.

$\Leftarrow$) Let $S = \{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); S_\alpha = I_\alpha \times M_\alpha \times \Lambda_\alpha\}$, where M_α is a left cancellative monoid and $I_\alpha(\Lambda_\alpha)$ is a left (right) zero band for any $\alpha \in Y$. Then S is a semilattice of $I_\alpha \times M_\alpha \times \Lambda_\alpha$. We will finish our proof by the following steps:

(1) S is an ortho $-lc-$ monoid.

To show this, by Lemma 2, we only need to prove that S is an rpp semigroup. Let $(i_\alpha, a_\alpha, \lambda_\alpha) \in I_\alpha \times M_\alpha \times \Lambda_\alpha$. Then clearly, $(i_\alpha, e_\alpha, \lambda_\alpha)(i_\alpha, a_\alpha, \lambda_\alpha) = (i_\alpha, a_\alpha, \lambda_\alpha)(i_\alpha, e_\alpha, \lambda_\alpha) = (i_\alpha, a_\alpha, \lambda_\alpha)$. Now, if $(i_\alpha, a_\alpha, \lambda_\alpha)x_\beta = (i_\alpha, a_\alpha, \lambda_\alpha)x_\gamma$, where $x_\beta \in I_\beta\times M_\beta\times\Lambda_\beta$, $x_\gamma \in I_\gamma\times M_\gamma\times\Lambda_\gamma$, then we have

$$(i_\alpha, a_\alpha, \lambda_\alpha)(i_\alpha, e_\alpha, \lambda_\alpha)x_\beta = (i_\alpha, a_\alpha, \lambda_\alpha)(i_\alpha, e_\alpha, \lambda_\alpha)x_\gamma,$$

and then,

$$(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(i_\alpha, e_\alpha, \lambda_\alpha)x_\beta]\Phi_{\bar{d}(\alpha\beta,\alpha\beta)} = (i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}[(i_\alpha, e_\alpha, \lambda_\alpha)x_\gamma]\Phi_{\bar{d}_1(\alpha\gamma,\alpha\gamma)} \quad (*)$$

where $\bar{d}(\alpha, \alpha\beta)$ and $\bar{d}_1(\alpha, \alpha\gamma)$ satisfy:

$$(i_\alpha, e_\alpha, \lambda_\alpha)x_\beta \in S_{\bar{d}(\alpha,\alpha\beta)}, (i_\alpha, e_\alpha, \lambda_\alpha)x_\gamma \in S_{\bar{d}_1(\alpha,\alpha\gamma)}.$$

Hence, we have $\alpha\beta = \alpha\gamma$. By Proposition 1 and $(*)$, $(i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)} = (i_\alpha, a_\alpha, \lambda_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}$.

Consequently, $\bar{d}(\alpha,\alpha\beta)=\bar{d}_1(\alpha,\alpha\beta)$. And then by $(*)$, we have

$$\begin{aligned}
& (i_\alpha,a_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta]\Phi_{\bar{d}(\alpha\beta,\alpha\beta)} \\
= & [(i_\alpha,e_\alpha,\lambda_\alpha)(i_\alpha,a_\alpha,\lambda_\alpha)]\Phi_{\bar{d}(\alpha,\alpha\beta)}[(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta] \\
= & (i_\alpha,e_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(i_\alpha,a_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta] \\
= & (i_\alpha,e_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(i_\alpha,a_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(i_\alpha,e_\alpha,\lambda_\alpha)x_\gamma].
\end{aligned}$$

Recall that $M_{\alpha\beta}$ is a left cancellative monoid, we can get

$$(i_\alpha,e_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta=(i_\alpha,e_\alpha,\lambda_\alpha)\Phi_{\bar{d}(\alpha,\alpha\gamma)}(i_\alpha,e_\alpha,\lambda_\alpha)x_\gamma,$$

i.e.,

$$(i_\alpha,e_\alpha,\lambda_\alpha)(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta=(i_\alpha,e_\alpha,\lambda_\alpha)(i_\alpha,e_\alpha,\lambda_\alpha)x_\gamma.$$

Thus, $(i_\alpha,e_\alpha,\lambda_\alpha)x_\beta=(i_\alpha,e_\alpha,\lambda_\alpha)x_\gamma$. So we have shown that S is an rpp semigroup.

(2) S is a regular ortho$-lc-$monoid.

To show this, we only need to show that $E(S)$ is a regular band. In fact, since rectangular $lc-$monoids are rectangular $u-$ semigroups, and notice that if a semigroup S is a refined semilattice of rectangular $lc-$ monoids, then it is a refined semillatice of rectangular $u-$ semigroups. By Lemma 1, we immediately have that S is a regular orthocryptou semigroup, and then $E(S)$ is a regular band.

Now, by Theorem 1 and its dual, we can obtain the following corollary at once.

Corollary 1. A semigroup S is a regular ortho$-c-$monoid if and only if it is a refined semilattice of rectangular $c-$monoids.

Restricting the previous discussions to the scope of completely regular semigroups, we can also have the following corollary.

Corollary 2. A semigroup S is a regular orthocryptogroup if and only if it is a refined semilattice of rectangular groups.

Next,we will show another structure theorem of the normal ortho $-lc-$ monoids.

Lemma 3.[5] Let S be a refined semilattice of S_α.Then S is both a left strong semilattice and a right strong semilattice of S_α if and only if S is a strong semilattice of S_α.

Lemma 4.[5] A semigroup S is a left (right) quasi-normal orthocryptou semigroup if and only if it is a right (left) strong semilattice of rectangular u-semigroups.

Now, using the similar arguments with Lemma 4 shown in [5], we can get the following proposition about left (right) quasi-normal ortho $-lc-$ monoids.

Proposition 2. A semigroup S is a left(right) quasi-normal ortho $-lc-$ monoid if and only if it is a right(left) strong semilattice of rectangular $lc-$ monoids .

Lemma 5.[13] $\mathcal{LQVB}\wedge\mathcal{RQVB}=\mathcal{NB}$, where $\mathcal{LQVB}(\mathcal{RQVB})$ represents the variety of left (right) quasi-normal bands,and $\mathcal{NB}$ represents the variety of normal bands.

It is known by Lemma 5 that a band is a normal band if and only if it is not only a left quasi-normal band but also a right quasi-normal band. Now, by Lemma 3, Lemma 4 and Proposition 2, we immediately have the following theorem.

Theorem 2. A semigroup S is a normal ortho$-lc-$monoid if and only if it is strong semilattice of rectangular $lc-$monoids.

By the above structure theorem, we can see that the normal ortho$-lc-$monoids are just the perfect rpp semigroups studied by Guo, Shum and Guo in [1], where a semigroup S is called a perfect rpp semigroup [1] if it satisfies the following conditions:
(i) S is a strongly rpp semigroup;
(ii) $E(S)$ forms a normal band under the multiplication of S;
(iii) $\mathcal{L}^*$ is a congruence.

And then, together with the main structure theorem in [1], we have the following theorem.

Theorem 3. The following conditions are equivalent:
(1) S is a normal ortho $-lc-$ monoid;
(2) S is a strong semilattice of rectangular $lc-$monoids;
(3) S is a perfect rpp semigroup;
(4) S is a spined product of a $C-$rpp (or $lc-$Clifford) semigroup and a normal band.

Remark. By Theorem 3, we can see that our main results generalize the corresponding ones in [1].

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Smarandache's Concurrent Lines Theorem

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Abstract In this paper we present the Smarandache's Concurrent Lines Theorem in the geometry of the triangle.

Keywords Smarandache's Concurrent Lines Theorem.

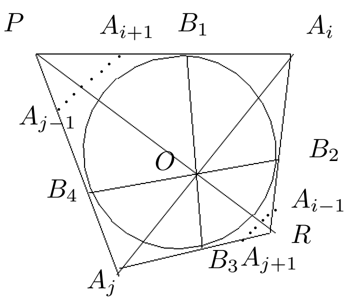
§1. The main result

Smarandache's Concurrent Lines Theorem

Let's consider a polygon (which has at least four sides) circumscribed to a circle, and the set of its diagonals and the lines joining the points of contact of two non-adjacent sides. Then contains at least three concurrent lines.

Proof. Let be the number of sides. If $n = 4$, then the two diagonals and the two lines joining the points of contact of two adjacent sides are concurrent (according to Newton's Theorem).

The case $n > 4$ is reduced to the previous case: we consider any polygon $A_1 \cdots A_n$ (see the figure)



circumscribed to the circle and we choose two vertices $A_i, A_j (i \neq j)$ such that

$$A_j A_{j-1} \bigcap A_i A_{i+1} = P,$$

and

$$A_j A_{j+1} \bigcap A_i A_{i_1} = R.$$

Let $B_h, h \in \{1, 2, 3, 4\}$ the contact points of the quadrilateral PA_jRA_i with the circle of center O. Because of the Newton's theorem, the lines A_iA_j, B_1B_3 and B_2B_4 are concurrent.

§2. Open problems related to the Smarandache Concurrent Lines Theorem

2.1. In what conditions there are more than three concurrent lines?

2.2. What is the maximum number of concurrent lines that can exist (and in what conditions)?

2.3. What about an alternative of this problem: to consider instead of a circle an ellipse, and then a polygon ellipsoscribed (let's invent this word, ellipso-scribed, meaning a polygon whose all sides are tangent to an ellipse which inside of it): how many concurrent lines we can find among its diagonals and the lines connecting the point of contact of two non-adjacent sides?

2.4. What about generalizing this problem in a 3D-space: a sphere and a polyhedron circumscribed to it?

2.5. Or instead of a sphere to consider an ellipsoid and a polyhedron ellipsoido-scribed to it?

Comments. Of course, we can go by construction reversely: take a point inside a circle (similarly for an ellipse, a sphere, or ellipsoid), then draw secants passing through this point that intersect the circle (ellipse, sphere, ellipsoid) into two points, and then draw tangents to the circle (or ellipse), or tangent planes to the sphere or ellipsoid) and try to construct a polygon (or polyhedron) from the intersections of the tangent lines (or of tangent planes) if possible.

For example, a regular polygon (or polyhedron) has a higher chance to have more concurrent such lines.

In the 3D space, we may consider, as alternative to this problem, the intersection of planes (instead of lines).

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$\approx$ - Aluthge transformation and adjoint of $*$ - Aluthge transformation

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Abstract In this paper, we have defined $\approx$ - Aluthge Transformation (i.e., $\widetilde{\widetilde{T}}^{(*)}=|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}^*|^{\frac{1}{2}}$) and have discussed some properties of it. Also by using the adjoint of $*$ - Aluthge transformation (i.e., $(\widetilde{T}^{(*)})^*=|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}$), we have discussed some properties of it.

Keywords Aluthge transformation, $*$-Aluthge transformation, polar decomposition.

§1. Introduction

In this paper, T is a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. An operator T is said to be p - hyponormal if $|T|^{2p} \geq |T^*|^{2p}$, where $|T|=(T^*T)^{\frac{1}{2}}$ for each $p > 0$, and it is also well known that "every p -hyponormal operator is q - hyponormal for $p \geq q > 0$. If p=1, T is called hyponormal, and if $p=\frac{1}{2}$, T is called semi - hyponormal."

Aluthge [1] has defined an transformation $\widetilde{T}$ of T as $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $\widetilde{T}$ is the Aluthge transformation, $T=U|T|$ is the polar decomposition of T, where U is a partial isometric operator, $|T|$ is a positive square root of T^*T and $N(T)=N(|T|)=N(U)$, where $N(S)$ denotes the kernel of an operator S. In [10], Yamazaki has defined $*$ - Aluthge transformation and have discussed some properties of $*$ - Aluthge transformation. Also parallelism between Aluthge transformation and powers of p - hyponormal is shown.

In Section 2, we have given the definition of Aluthge transformation and $*$ - Aluthge transformation.

§2. Some definition of Aluthge transformation and $*$ - Aluthge transformation

Definition 2.1.[9] **n - th Aluthge transformation**

Let $T=U|T|$ be the polar decomposition of an operator T. For each natural number n, $\widetilde{T_n}$ is defined as follows:

$$\widetilde{T_n} = (\widetilde{\widetilde{T_{n-1}}}) \text{ and } \widetilde{T_1} = \widetilde{T}.$$

Definition 2.2.[10] **∗ - Aluthge transformation**

Let $T=U|T|$ be the polar decomposition of an operator T. Then ∗ - Aluthge transformation of T is defined as follows:

(i) $\widetilde{T}^{(*)}=(\widetilde{T^*})^*=|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ (∗ - Aluthge transformation).

(ii) For each natural number n,

$$\widetilde{T_n}^{(*)}=(\widetilde{\widetilde{T_{n-1}}^{(*)}}^{(*)})=(\widetilde{T_n}^*)^* \text{ and } \widetilde{T_1}^{(*)}=\widetilde{T}^{(*)} \ (n \text{ th - } * \text{ - Aluthge transformation}) \quad .$$

In section 3, we shall discuss some properties on the ≈ -Aluthge transformation.

§3. ≈ - Aluthge transformation

In this section, we shall define ≈ - Aluthge transformation which is parallel to ∗ - Aluthge transformation. And we shall show some properties of ≈ - Aluthge transformation. Here we use the transformation $\widetilde{\widetilde{T}}$ of $\widetilde{T}$ as $\widetilde{\widetilde{T}}=|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{\widetilde{T}}$ is the Aluthge transformation, $\widetilde{T}=\widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.

If T is a bounded linear operator on a Hilbert space, then we know that

(i) $T=U|T| =|T^*|U$ is the polar decomposition of an operator T.

(ii) $T^*=|T|U^*=U^*|T^*|$ is the polar decomposition of T^*.

(iii) If $\widetilde{\widetilde{T}}=|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$ is the Aluthge transformation then the adjoint of Aluthge transformation $\widetilde{T}^*$ is given by $\widetilde{\widetilde{T}}^{(*)} = |\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}^*|^{\frac{1}{2}}$.

Theorem A.[10] Let $T \in B(H)$. Then the following assertions hold:

(i) $\sigma(\widetilde{T})=\sigma(\widetilde{T}^{(*)})=\sigma(T)$.

(ii) $w(\widetilde{T})=w(\widetilde{T}^{(*)})$, where $w(T)$ is the numerical radius of T.

(iii) $\|\widetilde{T}\|=\|\widetilde{T}^{(*)}\|$.

Theorem 3.1. Let $T \in B(H)$ and $\widetilde{T}=\widetilde{U}|\widetilde{T}|$ be the polar decomposition of an operator $\widetilde{T}$, and $\widetilde{\widetilde{T}}=|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$ is the Aluthge transformation then $\sigma(\widetilde{\widetilde{T}})=\sigma(\widetilde{\widetilde{T^{(*)}}})=\sigma(\widetilde{T})$.

Proof of the Theorem:

Theorem B.[5] Let $A, B \in B(H)$. If A has a decomposition $A=H+iK$ such that H commutes with K and $\sigma(H), \sigma(K) \in R$, then $\sigma(AB)=\sigma(BA)$. Hence, if A is normal, then $\sigma(AB)=\sigma(BA)$.

First we take and prove $\sigma(\widetilde{T})=\sigma(\widetilde{\widetilde{T}})$,

$$\begin{aligned}\sigma(\widetilde{T}) &= \sigma(\widetilde{U}|\widetilde{T}|)\\ &= \sigma(\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}|\widetilde{T}|^{\frac{1}{2}})\\ &= \sigma(|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}})\\ &= \sigma(\widetilde{\widetilde{T}}),\end{aligned}$$

Next we take $\sigma(\widetilde{T})=\sigma(\widetilde{\widetilde{T^{(*)}}})$,

$$\begin{aligned}\sigma(\widetilde{T}) &= \sigma(|\widetilde{T}^*|\widetilde{U}^*)\\ &= \sigma(|\widetilde{T}^*|^{\frac{1}{2}}|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}^*)\\ &= \sigma(|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}^*|\widetilde{T}^*|^{\frac{1}{2}})\\ &= \sigma(\widetilde{\widetilde{T}^*}).\end{aligned}$$

Theorem 3.2. Let $T \in B(H)$ and $\widetilde{T}=\widetilde{U}|\widetilde{T}|$ be the polar decomposition of an operator $\widetilde{T}$, and $\widetilde{\widetilde{T}}=|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$ is the Aluthge transformation then $w(\widetilde{\widetilde{T}})=w((\widetilde{\widetilde{T}}^{(*)}))$, where $w(T)$ is the numerical range of T.

Proof of the Theorem. To Prove: $w((\widetilde{\widetilde{T}}^{(*)})) \leq w(\widetilde{\widetilde{T}})$.

$$(\widetilde{\widetilde{T}}^{(*)})=|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}^*|^{\frac{1}{2}}=\widetilde{U}(|\widetilde{T}|)^{\frac{1}{2}}\widetilde{U}^*\widetilde{U}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}^*=\widetilde{U}[|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}]\widetilde{U}^*=\widetilde{U}\widetilde{\widetilde{T}}\widetilde{U}^* \tag{3.2.1}$$

Hence,

$$\begin{aligned}w((\widetilde{\widetilde{T}}^{(*)})) &= \sup_{\|x\|=1} |\langle \widetilde{T}^{(*)} x, x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{U}\widetilde{\widetilde{T}}\widetilde{U}^* x, x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}}\widetilde{U}^* x, \widetilde{U}^* x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}}\frac{\widetilde{U}^*x}{\|\widetilde{U}^*x\|}, \frac{\widetilde{U}^*x}{\|\widetilde{U}^*x\|}\rangle|\|U^*x\|^2\\ &\leq \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}}\frac{\widetilde{U}^*x}{\|\widetilde{U}^*x\|}, \frac{\widetilde{U}^*x}{\|\widetilde{U}^*x\|}\rangle|\\ &\leq w(\widetilde{\widetilde{T}}).\end{aligned}$$

To Prove: $w((\widetilde{\widetilde{T}}^{(*)})) \geq w(\widetilde{\widetilde{T}})$.

$$\widetilde{\widetilde{T}}=|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}=\widetilde{U}^*|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}\widetilde{U}\widetilde{U}^*|T^*|^{\frac{1}{2}}\widetilde{U}=\widetilde{U}^*|\widetilde{T}^*|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}^*|^{\frac{1}{2}}]\widetilde{U}=\widetilde{U}^*(\widetilde{\widetilde{T}}^{(*)})\widetilde{U}. \tag{3.2.2}$$

Hence,

$$\begin{aligned}w(\widetilde{\widetilde{T}}) &= \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}} x, x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{U}^*\widetilde{\widetilde{T}}^{(*)}\widetilde{U} x, x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}}^{(*)}\widetilde{U} x, \widetilde{U} x\rangle|\\ &= \sup_{\|x\|=1} |\langle \widetilde{\widetilde{T}}^{(*)}\frac{\widetilde{U}x}{\|\widetilde{U}x\|}, \frac{\widetilde{U}x}{\|\widetilde{U}x}\rangle|\|\widetilde{U}x\|^2\\ &\leq \sup_{\|x\|=1} |\langle (\widetilde{\widetilde{T}})^*\frac{\widetilde{U}x}{\|\widetilde{U}x\|}, \frac{\widetilde{U}x}{\|\widetilde{U}x\|}\rangle|\\ &\leq w(\widetilde{\widetilde{T}}^{(*)}).\end{aligned}$$

Therefore $w(\widetilde{\widetilde{T}})=w((\widetilde{\widetilde{T}}^{(*)}))$.

Corollary 3.3. Let $T \in B(H)$ and $T=U|T|$ be the polar decomposition of an operator T, and $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation then $\|\widetilde{\widetilde{T}}\|= \|\widetilde{\widetilde{T}}^{(*)}\|$.

Proof of the Corollary. Proof follows from above theorem using (3.2.1) and (3.2.2).

In Section 4, we shall discuss some properties on the adjoint of ∗ - Aluthge transformation.

§4. Some properties on the adjoint of ∗ - Aluthge transformation

If T is a bounded linear operator on a Hilbert space, then we know that

(i) $T=U|T| =|T^*|U$ is the polar decomposition of an operator T.

(ii) $T^*=|T|U^*=U^*|T^*|$ is the polar decomposition of T^*.

(iii) If $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation then the adjoint of Aluthge transformation $\widetilde{T}^*$ is given by $\widetilde{T}^{(*)}=|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}$.

(iv) If $\widetilde{T}^{(*)}=(\widetilde{T}^*)^*=|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ is the ∗ - Aluthge transformation then its adjoint is given by $(\widetilde{T}^{(*)})^*=|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}$.

Theorem A.[10] Let $T \in B(H)$. Then the following assertions hold:

(i) $\sigma(\widetilde{T})=\sigma(\widetilde{T}^{(*)})=\sigma(T)$.

(ii) $w(\widetilde{T})=w(\widetilde{T}^{(*)})$, where $w(T)$ is the numerical radius of T.

(iii) $\|\widetilde{T}\|=\|\widetilde{T}^{(*)}\|$.

Theorem 4.1. Let $T \in B(H)$ and $T=U|T|$ be the polar decomposition of an operator T, and $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation then $\sigma(\widetilde{T}^*)=\sigma((\widetilde{T}^{(*)})^*) =\sigma(T^*)$.

Proof of the Theorem.

Theorem B.[5] Let $A, B \in B(H)$. If A has a decomposition $A=H+iK$ such that H commutes with K and $\sigma(H), \sigma(K) \in R$, then $\sigma(AB)=\sigma(BA)$. Hence, if A is normal, then $\sigma(AB)=\sigma(BA)$.

$$\begin{aligned}\sigma(T^*) &= \sigma(|T|U^*)\\ &= \sigma(|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U^*)\\ &= \sigma(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\\ &= \sigma(\widetilde{T}^*),\end{aligned}$$

and

$$\begin{aligned}\sigma(T^*) &= \sigma(U^*|T^*|)\\ &= \sigma(U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}})\\ &= \sigma(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\\ &= \sigma((\widetilde{T}^{(*)})^*).\end{aligned}$$

Since $|T|^{\frac{1}{2}}$ and $|T^*|^{\frac{1}{2}}$ are normal and by using Theorem B.

Theorem 4.2. Let $T \in B(H)$ and $T{=}U|T|$ be the polar decomposition of an operator T, and $\widetilde{T}{=}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation then $w(\widetilde{T}^*){=}\ w((\widetilde{T}^{(*)})^*)$, where $w(T)$ is the numerical range of T.

Proof of the Theorem. To Prove: $w((\widetilde{T}^{(*)})^*) \leq w(\widetilde{T}^*)$.

$$(\widetilde{T}^{(*)})^*{=}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}{=}U|T|^{\frac{1}{2}}U^*U^*U|T|^{\frac{1}{2}}U^*{=}U[|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}]U^*{=}U\widetilde{T}^*U^* \quad (4.2.1)$$

Hence,

$$\begin{aligned} w((\widetilde{T}^{(*)})^*) &= \sup_{\|x\|=1} |\langle(\widetilde{T}^{(*)})^*x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle U\widetilde{T}^*U^*x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle \widetilde{T}^*U^*x, U^*x\rangle| \\ &= \sup_{\|x\|=1} |\langle \widetilde{T}^*\frac{U^*x}{\|U^*x\|}, \frac{U^*x}{\|U^*x\|}\rangle|\|U^*x\|^2 \\ &\leq \sup_{\|x\|=1} |\langle \widetilde{T}^*\frac{U^*x}{\|U^*x\|}, \frac{U^*x}{\|U^*x\|}\rangle| \\ &\leq w(\widetilde{T}^*). \end{aligned}$$

To Prove: $w((\widetilde{T}^{(*)})^*) \geq w(\widetilde{T}^*)$.

$$\widetilde{T}^*{=}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}{=}U^*|T^*|^{\frac{1}{2}}UU^*U^*|T^*|^{\frac{1}{2}}U{=}\ U^*[|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}]U{=}U^*(\widetilde{T}^{(*)})^*U. \quad (4.2.2)$$

Hence,

$$\begin{aligned} w(\widetilde{T}^*) &= \sup_{\|x\|=1} |\langle \widetilde{T}^*x, x\rangle| \\ &= \sup_{\|x\|=1} |\langle U^*(\widetilde{T}^{(*)})^*Ux, x\rangle| \\ &= \sup_{\|x\|=1} |\langle (\widetilde{T}^{(*)})^*Ux, Ux\rangle| \\ &= \sup_{\|x\|=1} |\langle (\widetilde{T}^{(*)})^*\frac{Ux}{\|Ux\|}, \frac{Ux}{\|Ux}\rangle|\|Ux\|^2 \\ &\leq \sup_{\|x\|=1} |\langle (\widetilde{T}^{(*)})^*\frac{Ux}{\|Ux\|}, \frac{Ux}{\|Ux}\rangle| \\ &\leq w((\widetilde{T}^{(*)})^*). \end{aligned}$$

Therefore $w(\widetilde{T}^*){=}w((\widetilde{T}^{(*)})^*)$

Corollary 4.3. Let $T \in B(H)$ and $T{=}U|T|$ be the polar decomposition of an operator T, and $\widetilde{T}{=}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation then $\|\widetilde{T}^*\|{=}\|(\widetilde{T}^{(*)})^*\|$.

Proof of the Corollary. Proof follows from above theorem using (4.2.1) and (4.2.2).

Theorem C.[10] Let $T \in B(H)$. Then for each $p > 0$, $\widetilde{T}$ is p - hyponormal $\Leftrightarrow$ $\widetilde{T}^{(*)}$ is p - hyponormal.

Theorem 4.4. Let $T \in B(H)$, then for each $p > 0$, then $\widetilde{T}^*$ is p - hyponormal $\Leftrightarrow$ $(\widetilde{T}^{(*)})^*$ is p - hyponormal.

Proof of the Theorem.

Proof of ($\Rightarrow$) part. If $\widetilde{T}^*$ is p - hyponormal then

$$((\widetilde{T}^*)^*\widetilde{T}^*)^p \geq (\widetilde{T}^*(\widetilde{T}^*)^*)^p,$$

$$\Rightarrow (\widetilde{T}\widetilde{T}^*)^p \geq (\widetilde{T}^*\widetilde{T})^p.$$

Then,

$$\begin{aligned}
(\{(\widetilde{T}^{(*)})^*\}^*(\widetilde{T}^{(*)})^*)^p &= (\widetilde{T}^{(*)}(\widetilde{T}^{(*)})^*)^p \\
&= (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})^p \\
&= (|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}})^p \\
&= (U|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}U^*)^p \\
&= U(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})^pU^* \\
&= U(\widetilde{T}\widetilde{T}^*)^pU^* \\
&\geq U(\widetilde{T}^*\widetilde{T})^pU^* \text{since } \widetilde{T}^* \text{is p - hyponormal.} \\
&\geq U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^pU^* \\
&\geq (U|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}U^*)^p \\
&\geq (|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}})^p \\
&\geq (|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^p \\
&\geq ((\widetilde{T}^{(*)})^*\widetilde{T}^{(*)})^p \\
&\geq ((\widetilde{T}^{(*)})^*\{(\widetilde{T}^{(*)})^*\}^*)^p.
\end{aligned}$$

Therefore $(\widetilde{T}^{(*)})^*$ is p - hyponormal.

Proof of $\Leftarrow$ part. If $(\widetilde{T}^{(*)})^*$ is p - hyponormal then

$$(\{(\widetilde{T}^{(*)})^*\}^*(\widetilde{T}^{(*)})^*)^p \geq ((\widetilde{T}^{(*)})^*\{(\widetilde{T}^{(*)})^*\}^*)^p$$
$$\Rightarrow (\widetilde{T}^{(*)}(\widetilde{T}^{(*)})^*)^p \geq ((\widetilde{T}^{(*)})^*\widetilde{T}^{(*)})^p.$$

Then,

$$\begin{aligned}
((\widetilde{T}^*)^*\widetilde{T}^*)^p &= (\widetilde{T}\widetilde{T}^*)^p \\
&= (|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})^p \\
&= (|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^p \\
&= (U^*|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}}U^*)^p \\
&= U^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})^pU \\
&= U^*(\widetilde{T}^{(*)}(\widetilde{T}^{(*)})^*)^pU \\
&\geq U^*((\widetilde{T}^{(*)})^*\widetilde{T}^{(*)})^pU, \text{since } (\widetilde{T}^{(*)})^* \text{is p - hyponormal.} \\
&\geq U^*(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^pU \\
&\geq (U^*|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}U)^p \\
&\geq (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^p \\
&\geq (|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^p \\
&\geq (\widetilde{T}^*\widetilde{T})^p \\
&\geq (\widetilde{T}^*(\widetilde{T}^*)^*)^p.
\end{aligned}$$

Therefore $\widetilde{T}^*$ is p - hyponormal.

Hence the proof.

Theorem 4.5. If $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is hyponormal for $\frac{1}{2} \leq p \leq 1$, U is unitary then $\widetilde{T}^{(*)}$ is also hyponormal.

Proof of the Theorem.

$\widetilde{T}$ is hyponormal for $\frac{1}{2} \leq p \leq 1$

$$\Rightarrow (\widetilde{T}^*\widetilde{T})^{\frac{1}{2}} \geq (\widetilde{T}\widetilde{T}^*)^{\frac{1}{2}}$$

(or) Equivalently

$$U^*|T^*|U \geq |\widetilde{T}| \geq U|T^*|U^*.$$

Then,

$$\begin{aligned}
(\widetilde{T}^{(*)})^*\widetilde{T}^{(*)} - \widetilde{T}^{(*)}(\widetilde{T}^{(*)})^* &= [|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}] - [|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}] \\
&= [|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}] - [|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}}] \\
&= |T^*|^{\frac{1}{2}}[U^*|T^*|U]|T^*|^{\frac{1}{2}} - |T^*|^{\frac{1}{2}}[U|T^*|U^*]|T^*|^{\frac{1}{2}} \\
&= |T^*|^{\frac{1}{2}}[U^*|T^*|U - U|T^*|U^*]|T^*|^{\frac{1}{2}} \\
&\geq 0 \\
&\Rightarrow \widetilde{T}^{(*)} \text{is also a hyponormal.}
\end{aligned}$$

Hence the proof.

Lemma 4.6. If $A \geq 0$ and x is a non - zero vector then $\|Ax\| \geq \|A^{\frac{1}{2}}x\|^2\|x\|^{-1}$.

Theorem 4.7. Let $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be the polar decomposition of the operator $\widetilde{T}$. If $|\widetilde{\widetilde{T}^{(*)}}|=|\widetilde{T}^{(*)}|$, then $\widetilde{T}^{(*)}$ is paranormal.

Proof of the Theorem. Let x be a unit vector. Since $\|\widetilde{\widetilde{T}^{(*)2}}x\| \geq \|\widetilde{T}^{(*)}x\|^2$. Clearly holds if Tx=0. We may assume $\widetilde{T}^{(*)}x \neq 0$. Thus

$$\begin{aligned}
\|\widetilde{T}^{(*)^2}x\| &= \||\widetilde{T}^{(*)}|\widetilde{T}^{(*)}x\| \\
&\geq \||\widetilde{T}^{(*)}|^{\frac{1}{2}}\widetilde{T}^{(*)}x\|^2\|\widetilde{T}^{(*)}x\|^{-1} \text{by Lemma 4.6.} \\
&= \|\widetilde{\widetilde{T}^{(*)}}|\widetilde{T}^{(*)}|^{\frac{1}{2}}x\|^2\|\widetilde{T}^{(*)}x\|^{-1} \\
&= \||\widetilde{\widetilde{T}^{(*)}}||\widetilde{T}^{(*)}|^{\frac{1}{2}}x\|^2\|\widetilde{T}^{(*)}x\|^{-1} \\
&\geq \||\widetilde{\widetilde{T}^{(*)}}|^{\frac{1}{2}}|\widetilde{T}^{(*)}|^{\frac{1}{2}}x\|^4\||\widetilde{T}^{(*)}|^{\frac{1}{2}}x\|^{-2}\|\widetilde{T}^{(*)}x\|^{-1} \text{by Lemma 4.6.} \\
&\geq \||\widetilde{T}^{(*)}|x\|^4\||\widetilde{T}^{(*)}|^{\frac{1}{2}}x\|^{-2}\|\widetilde{T}^{(*)}x\|^{-1} \text{Since } |\widetilde{\widetilde{T}^{(*)}}| = |\widetilde{T}^{(*)}|. \\
&\geq \||\widetilde{T}^{(*)}|x\|^4\||\widetilde{T}^{(*)}|x\|^{-1}\|\widetilde{T}^{(*)}x\|^{-1} \quad \text{by Lemma 4.6.} \\
&= \|\widetilde{T}^{(*)}x\|^2.
\end{aligned}$$

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Note on regularity and ν-lindeloffness

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Abstract The aim of the paper is to study some properties and interrelations of ν-lindeloff, lindeloff and weakly lindeloff spaces.

Keywords $\nu-$lindeloff, nearly lindeloff, weak lindeloff, semi lindeloff and almost lindeloff spaces.

§1. Introduction

Cammaroto, Faro and T. Noiri introduced and studied weakly compact spaces which are strictly weaker than the notions of almost-compact spaces due to Singhal and Singhal. Almost compact spaces were called quasi-H-closed spaces by Porter and Thomas. Almost compact spaces are well known as H-closed spaces. M. K. Singhal and A. Mathur studied nearly compact spaces and the authors of the present paper studied basic properties of ν-compact and ν-Lindeloff spaces using ν-open sets which are introduced by Cameron and V. K. Sharma. Also recently the authors studied about interrelation of ν-compactness and regularity. In the present paper we tried to extend the concept of compactness to lindeloffness and studied interrelation between ν-lindeloff, lindeloff and weakly lindeloff spaces. Throughout the paper X means a topological space unless otherwise mentioned without any separation axioms defined in it.

§2. Preliminaries

Definition 2.1. $A \subset X$ is said to be
1. Semi open(ν-open) if there exists an open (regular open) set U such that $U \subseteq A \subseteq \overline{U}$.
2. Regular open if $A = (\overline{A})^o$.

Note 1.

$\nearrow$ $\nu-$open $\searrow$

Regular open semiopen

$\searrow$ open $\nearrow$

Definition 2.2. $A \subset X$ is said to be

1. α-Hausdorff iff for any points $a, b \in X$, where $a \in A$ and $b \in X - A$, there are disjoint open sets U and V containing a and b respectively.
2. α-regular iff for any point $a \in A$ and any open set U containing a, there exists an open set V such that $a \in V \subset \overline{V} \subset U$.
3. α-almost regular [resp. Weak almost regular] iff for any point $a \in A$ and any regular-open set U containing a, there exist a regular-open set V such that $a \in V \subset \overline{V} \subset U$.
4. Almost ν-regular iff for any point $a \in A$ and any ν-open set U containing a, there exists a ν-open set V such that $a \in V \subset \overline{V} \subset U$.

Note 2. From the above definition we have the following interrelations among the spaces. α-almost regular $\rightarrow$ α-regular $\rightarrow$ α-Hausdorff where none of the relation is reversible.

Definition 2.3. $A \subset X$ is said to be

1. Compact (resp. semi compact, nearly compact, ν-compact) if every open (resp. semi open, regular open, ν-open) cover has a finite sub cover.
2. Lindeloff (resp. semi Lindeloff, nearly Lindeloff, ν-Lindeloff) if every open (resp. semi open, regular open, ν-open) cover has a countable sub cover.
3. Almost compact [lindeloff] if every open cover has a finite [countable] subclass whose closure covers X.
4. Weakly compact [lindeloff] if every regular open cover has a finite [countable] subclass whose closure covers X.

Example 1. Any closed and bounded subset of $\Re$ with usual topology is $\nu-$compact.

Example 2. $\Re$ with usual topology is not $\nu-$compact.

Note 3.

$\nearrow$ ν-lindeloff $\searrow$

semi lindeloff nearly lindeloff $\rightarrow$ almost lindeloff $\rightarrow$ weakly lindeloff

$\searrow$ lindeloff $\nearrow$

where none of the relation is reversible.

Definition 2.4. A mapping $f\colon X \rightarrow Y$ is said to be

(i) Almost continuous if inverse image of every regular open set is open.
(ii) $\nu-$continuous if inverse image of every $\nu-$open set is open.
(iii) $\nu-$irresolute if inverse image of every $\nu-$open set is $\nu-$open.

Lemma 2.1. Let $f\colon X \rightarrow Y$ is almost continuous then for each $A \subset Y\overline{(f^{-1}(A))} \subset f^{-1}(\overline{A})$.

Definition 2.5. A subset S of a space X is ν-lindeloff relative to X if for each ν-open cover $\{V_\alpha : \alpha \in \nabla\}$ of S by ν-open sets of X $\ni$ for each $\alpha \in \nabla, \exists F_\alpha \in RC(X) \ni F_\alpha \subset V_\alpha$ and $S \subset \cup\{(F_\alpha)^o : \alpha \in \nabla\}$ there exists a finite subset ∇_0 of ∇ such that $S \subset \cup\{F_\alpha : \alpha \in \nabla_0\}$.

§3. Properties of ν-lindeloff spaces

Definition 3.1. $A \subset X$ is said to be

(i) ν-lindeloff space if every $\nu-$open cover of A has a countable subcover.
(ii) Countably ν-lindeloff space if every countable $\nu-$open cover of A has a countable subcover.
(iii) σ $\nu-$lindeloff if A is the countable union of ν-lindeloff spaces.

Example 3. Every $\nu-$compact space is ν-lindeloff.

Example 4. $\Re$ with semi-open interval topology is ν-lindeloff as well lindeloff.

Definition 3.2. Let $S \subset X$. A point $x \in X$ is said to be

(i) $\nu-[\omega$-]accumulation point of S if every $\nu-$[regular-open]open nbd of x intersects S.

(ii) The union of S and the set of all $\nu-[\omega$-]accumulation points of S is called $\nu-[\omega$-]closed set.

Remark 1. Every ω-closed set is $\nu-$closed.

Theorem 3.1. Let A be r-open. $A \subseteq X$ is a ν-lindeloff subset of X iff the subspace $(A, \tau_{/A})$ is ν-lindeloff.

Proof. Assume A is ν-lindeloff. Let $\{G_i : i \in I\}$ be any $\nu-$open cover in $(A, \tau_{/A})$. Since each G_i is $\nu-$open in A and $G_i \subseteq A \subseteq X$, each G_i is $\nu-$open in X. By ν-lindeloffenss of X, this cover have a countable subcover, which in turn give a countable subcover for A in $(A, \tau_{/A})$. Hence $(A, \tau_{/A})$ is ν-lindeloff.

Conversely, Assume $(A, \tau_{/A})$ is ν-lindeloff. Let $\{H_i : i \in I\}$ be any $\nu-$open cover for A in X. Since each H_i is $\nu-$open in X and $A \subset X$ is r-open, each $G_i = H_i \cap A$ is $\nu-$open in A and $\{G_i : i \in I\}$ form a $\nu-$open cover for A. By ν-lindeloffenss of $(A, \tau_{/A})$, this cover have a countable subcover, which in turn gives a countable subcover for A in X. Hence A is a ν-lindeloff subset of X.

Theorem 3.2.

(i) $\nu-$closed subset of a (countably) ν-lindeloff space is (countably) ν-lindeloff.

(ii) $\nu-$irresolute image of a (countably) ν-lindeloff space is (countably) ν-lindeloff.

(iii) Countable union of (countably) ν-lindeloff spaces is (countably) ν-lindeloff.

Proof. (i) Let X be ν-lindeloff and $A \subset X$ be $\nu-$closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any $\nu-$open cover for A. Let $U_0 = X - A$ and $\Delta' = \Delta \cup \{0\}$. Then $\{U_\alpha : \alpha \in \Delta'\}$ is a $\nu-$open cover for X. Since X is ν-lindeloff, there exists a countable subset $\Delta'' \subset \Delta' \ni X \subset \bigcup_{\alpha\in\Delta''} U_\alpha$. Since $A \subset X$ & $A \cap U_0 = \phi$, also $\Delta'' - \{0\} \subset \Delta$. $A \subset \bigcup_{\alpha\in\Delta''-\{0\}} U_\alpha$. Hence $\{U_\alpha : \alpha \in \Delta'' - \{0\}\}$ is a countable subcover of A from $\{U_\alpha : \alpha \in \Delta\}$. Hence A is ν-lindeloff.

If X is countably ν-lindeloff, Δ will be countable set.

(ii) Let $f : X \to Y$ be $\nu-$irresolute and let X be ν-lindeloff. Let $\{V_i : i \in I\}$ be any $\nu-$open cover for f(X), then each V_i is $\nu-$open in f(X). Since f is $\nu-$ irresolute, each $f^{-1}(V_i)$ is $\nu-$open in X, By ν-lindeloffness of X, we have $X \subseteq \cup_{i=1}^{\infty} f^{-1}(V_i)$ implies $X \subseteq \cup_{n=1}^{\infty} f^{-1}(V_n)$. Thus $f(X) \subseteq \cup_{n=1}^{\infty} V_n$.

Remark 2. (countably) ν-lindeloffness is a weakly hereditary property.

Theorem 3.3.

(i) $\nu-$continuous image of a (countably) ν-lindeloff space is (countably) lindeloff.

(i) $\nu-$continuous image of a (countably) ν-compact space is (countably) lindeloff.

Proof. (i) Let X be ν-lindeloff and $f : X \to f(X)$ is $\nu-$continuous. Let $\{U_i : i \in I\}$ be any open cover for f(X) $\Rightarrow$ each U_i is open set in f(X) $\Rightarrow$ each $f^{-1}(U_i)$ is $\nu-$open set in X $\Rightarrow$ $\{f^{-1}(U_i) : i \in I\}$ form a $\nu-$open cover for X. By ν-lindeloffness of X we have $X \subset \bigcup_{i=1}^{\infty} f^{-1}(U_i)$ $\Rightarrow f(X) \subset \bigcup_{n=1}^{\infty} U_n \Rightarrow \{U_n : n = 1 \text{ to } \infty\}$ is a countable subcover for f(X). Hence f(X) is lindeloff.

(ii) Clear from (i) above and Example 3.

Definition 3.3. X is said to be locally ν-lindeloff space if every $x \in X$ has a $\nu-$nbd whose closure is $\nu-$lindeloff.

Note 4. Every ν-lindeloff space is locally ν-lindeloff.

Theorem 3.4. If $f\colon(X,\tau)\to(Y,\sigma)$ is $\nu-$irresolute, $\nu-$open and X is locally ν-lindeloff, then so is Y.

Proof. Let $y\in Y$. Then $\exists x\in X\ni f(x)=y$. Since X is locally ν-lindeloff x has a ν-lindeloff nbd V Then by $\nu-$irresolute, $\nu-$openness of $f, f(V)$ is a ν-lindeloff nbd of y. Hence Y is locally $\nu-$lindeloff.

Corollary 3.1. If $f\colon(X,\tau)\to(Y,\sigma)$ is $\nu-$irresolute, $\nu-$open and X is $\nu-$ compact, then Y is locally ν-lindeloff.

Proof. Follows from above two theorems.

Theorem 3.5. $A\subseteq X$ be r-open. Then A is locally ν-lindeloff subset of X iff the subspace $(A,\tau_{/A})$ is locally ν-lindeloff.

Proof. Follows from the Definition 3.3 and Theorem 3.1.

Theorem 3.6. (i) $\nu-$closed subset of a locally ν-lindeloff space is locally ν-lindeloff.
(ii) Countable product of locally ν-lindeloff spaces is locally ν-lindeloff.
(iii) Countable union of locally ν-lindeloff spaces is locally ν-lindeloff.

Proof. Follows from the Definition 3.3 and Theorem 3.2.

From the Definitions and Remark 1, we have the following:

Remark 3. For any space nearly-compact $\Rightarrow$ ν-lindeloff $\Rightarrow$ semi-lindeloff but the converse is not true in general.

The proof of the following theorems is routine, hence omitted.

Theorem 3.7. If A is ν-lindeloff subspace of X, then A is ν-lindeloff relative to X.

Theorem 3.8. The following are equivalent:
(i) X is ν-lindeloff.
(ii) For every family of ν-closed sets in X satisfying empty intersection, there is countable subfamily whose intersection is empty.
(iii) Every family of ν-closed sets with countable intersection condition has a non-empty intersection.

Proof. $(i)\Leftrightarrow(ii)$ X is ν-lindeloff and $\{V_i : i\in I\}$ is a family of $\nu-$closed sets with $\cap_{i=1}^{\infty}V_i=\phi$. If $\cap_{n=1}^{\infty}V_n\neq\phi$ then by De Morgan's Law we have $\cup_{n=1}^{\infty}V_n^c\neq X$ implies $\cup_{i=1}^{\infty}V_i^c\neq X$ which in turn gives that $\cap_{i=1}^{\infty}V_i\neq\phi$ which is a contradiction. Hence $\cap_{n=1}^{\infty}V_n=\phi$.

$(ii)\Leftrightarrow(iii)$ Assume (ii) and let $\{V_i : i\in I\}$ is a family of $\nu-$closed sets with $\cap_{n=1}^{\infty}V_n\neq\phi$. If $\cap_{i=1}^{\infty}V_i=\phi$ then by De Morgan's law we have $\cup_{i=1}^{\infty}V_i^c=X$ implies $\cup_{n=1}^{\infty}V_n^c=X$ which in turn gives that $\cap_{n=1}^{\infty}V_n=\phi$ which is a contradiction for (ii). Hence $\cap_{i=1}^{\infty}V_i\neq\phi$.

$(iii)\Leftrightarrow(i)$ Similar argument using De Morgan's Law gives the proof of this part and hence omitted.

Alexandroff's Subbase theorem for ν-lindeloff spaces

Theorem 3.9. ΠX_i is ν-lindeloff if and only if every X_i is ν-lindeloff.

Proof. Assume $\Pi_i X_i$ is ν-lindeloff and Fix $j\in I$. Let $P_j : \Pi_i X_i\to X_j$ be a projection and let $\{V_j^i : i\in I\}$ be a $\nu-$open cover of X_j. Then $\{\Pi_{i\neq j}X_i\times V_j^i : i\in I\}$ is a $\nu-$open cover of $\Pi_i X_i$. Since $\Pi_\alpha X_\alpha$ is ν-lindeloff, $\exists$ a countable subfamily such that $\Pi_i X_i=\bigcup_{i=1}^{n}(\Pi_{i\neq j}X_i\times V_j)$. By projection $P_j : X_j=\bigcup_{i=1}^{n}V_j^i$. Therefore X_j is ν-lindeloff.

Conversely, Let $\{U_j = \Pi_{\alpha\neq\alpha_{ij}} X_\alpha \times U_{\alpha 1j} \times \cdots \times U_{\alpha nj} : U_{\alpha ij}$ is $\nu-$open in $X_{\alpha ij}$ for each i $= 1$ to n, $j \in I\}$ be a $\nu-$open cover of $\Pi_\alpha X_\alpha$. Then $\{P_i(u_j) : j \in I\}$ is a $\nu-$open cover of X_i. By Assumption, $\exists$ a countable subfamily $\{P_i(u_j) : j = 1 \text{ to } n\} \ni X_i = \bigcup P_i(U_j)$.

Case 1. If $\Pi_i X_i = \bigcup_{j=1}^{n} P_i(U_j)$ then $\Pi_i X_i$ is ν-lindeloff.

Case 2. If not, there exists atmost countable $\ell_1, \ell_2, \cdots, \ell_n \ni X_{\ell_s} = \bigcup_{k=1^s}^{m} P_{\ell_s}(U_{k_s})$ for each $\ell_s \in \{\ell_1, \ell_2, \cdots, \ell_n\}$. Therefore $\Pi_\alpha X_\alpha = \bigcup_{k=1}^{n}(U_{jk}) \cup \bigcup_{k^1=1}^{n}(U_{jk}) \cup \cdots \cup \bigcup_{k^s=1}^{n}(U_{jk})$. Hence $\Pi_\alpha X_\alpha$ is ν-lindeloff.

But Product of any two $\nu-$lindeloff spaces is not $\nu-$lindeloff.

Example 5. Sorgenfrey Plane.

Theorem 3.10. If S is an arbitrary ν-lindeloff subset of X, then every infinite subset of S has a ν-accumulation point.

Proof. Let X be ν-lindeloff and let A be an infinite subset of X not having $\nu-$accumulation point then $D_\nu(A) = \phi$ and so A is $\nu-$closed in X. If B is any infinite subset of A, then for each $b \in B$ there exist a $\nu-$open set V_b containing b such that V_b contains no point of A other than b. Now the family $\{V_b : b \in B\}$ forms a $\nu-$open cover of B. By Theorem 3.2; B itself is ν-lindeloff, but any countable subfamily of $\{V_b : b \in B\}$ do not cover B, which is a contradiction. Therefore the infinite subset A of X has a $\nu-$accumulation point.

Theorem 3.11. If S is an arbitrary ν-lindeloff subset of X, then every infinite subset of S has a ω-accumulation point.

Proof. Let X be ν-lindeloff and let A be an infinite subset of X not having $\omega-$accumulation point in S, then for each $x \in S$ there exist a regular-open set V_x containing x such that $(V_x - x) \cap A = \phi$. Now the family $\{V_x : x \in A\}$ forms a $\nu-$open cover of A. Since S is ν-lindeloff, any countable subfamily of $\{V_x : x \in S\}$ do not cover S, which is a contradiction. Therefore the infinite subset A of S has a $\omega-$accumulation point.

Theorem 3.12. Let X be a ν-lindeloff space, and let $\{S_i\}$ be a descending chain of ω-closed subsets of X, then $\cap_{n\geq 1} S_n \neq \phi$.

Proof. Choose a point $x_n \in S_n$ for each $n = 1, 2, ...$ then x_n will have a ω-accumulation point x_0 in X by Theorem 1 [11], since X is ν-lindeloff, X is nearly compact and hence it is ω-closed. On the other hand, for each $n = 1, 2, \cdots$; x_0 becomes a ω-accumulation point of $\{x_k, x_{k+1}, \cdots\}$ also, hence of S_k. Since each S_k is ω-closed, we know that $x_0 \in S_k$ for each k, hence the intersection of all S_k is not empty.

Theorem 3.13. If $f\colon X \to Y$ is almost continuous, X is ν-lindeloff and Y = f(X) then Y is ν-lindeloff.

Proof. Let $\{V_\alpha\}$ be ν-open cover of Y, then for each α there exists a regular open set A_α such that $A_\alpha \subset V_\alpha \subset \overline{A_\alpha}$ implies that $f^{-1}(A_\alpha) \subset f^{-1}(V_\alpha) \subset f^{-1}(\overline{A_\alpha}) = \overline{(f^{-1}(A_\alpha))}$ implies that $\{f^{-1}(V_\alpha)\}$ is semi-open cover of $X \Rightarrow \{f^{-1}(V_\alpha) \cup (f^{-1}(V_\alpha)^{-0}\}$ is ν-open cover for X $\Rightarrow X = \cup_{n=1}^{\infty}\{f^{-1}(V_{\alpha n}) \cup (f^{-1}\overline{(V_{\alpha n})})^o\} = \cup_{n=1}^{\infty}\{(f^{-1}\overline{(V_{\alpha n})})^o \Rightarrow X \subset \cup_{n=1}^{\infty}(f^{-1}\overline{(A_{\alpha n})})^o \Rightarrow Y \subset \cup_{n=1}^{\infty}(\overline{(A_{\alpha n})})^o \Rightarrow Y \subset \cup_{n=1}^{\infty} A_{\alpha n} \Rightarrow Y \subset \cup_{n=1}^{\infty} V_{\alpha n}$. Hence Y is $\nu-$lindeloff.

Corollary 3.2. If $f\colon X \to Y$ is almost continuous, X is ν-compact and Y = f(X) then Y is ν-lindeloff.

Proof. By Theorem 3.13 [3] Y is $\nu-$compact and so by Example 3, Y is $\nu-$lindeloff.

Theorem 3.14. If f is an almost continuous open mapping of a topological space X into a ν-lindeloff space Y with $f^{-1}(f(A_\alpha)) \subset \overline{(A_\alpha)}$ for each regular open set A_α of X, then X is ν-lindeloff.

Proof. Let $\{V_\alpha\}$ be ν-open cover of X, then $\{f(V_\alpha)\}$ is semi-open cover of Y so $\{f(V_\alpha) \cup (f(V_\alpha)^{-0}\}$ is ν-open cover for $Y \Rightarrow Y = \cup_{n=1}^{\infty}\{f(V_{\alpha n}) \cup f(V_{\alpha n})^{-o}\} = \cup_{n=1}^{\infty}\{f(V_{\alpha n})^{-o}\} \subset \cup_{n=1}^{\infty}(f(A_{\alpha n})^{-o}\}$, it follows that, $X = \cup_{n=1}^{\infty}(f^{-1}(f(A_{\alpha n}))^{-})^{o}$. By Lemma 2.1 and hypothesis for f, $X = \cup_{n=1}^{\infty}(f(A_{\alpha n}))^{-o} \subset \cup_{n=1}^{\infty}(f^{-1}(f(A_{\alpha n})^{-o}) \subset \cup_{n=1}^{\infty}(A_{\alpha n})^{-o} = \cup_{n=1}^{\infty}(A_{\alpha n}) \subset \cup_{n=1}^{\infty}V_{\alpha n}$. Hence X is ν-lindeloff.

Corollary 3.3. If f is an almost continuous open mapping of a topological space X into a ν-compact space Y with $f^{-1}(f(A_\alpha)) \subset \overline{(A_\alpha)}$ for each regular open set A_α of X, then X is ν-lindeloff.

Proof. By Theorem 3.14 [3] X is $\nu-$compact and hence X is ν-lindeloff.

Combaining 3.13 and 3.14 we have the following corollary.

Corollary 3.2. If f is an almost continuous open bijection, then X is ν-lindeloff if and only if Y is ν-lindeloff.

§4. Relation between ν-lindeloff and compact spaces

Definition 4.0. An open base is said to be a regular open base if the elements of the base are regular open sets.

Lemma 4.1. If X is ν-lindeloff and semiregular then X is lindeloff.

Proof. Let $\{O_i : i \in I\}$ be an open cover of X. Since X is semiregular, there is a regular open basis B and we have ν-open cover $\{B_i^j : O_i = \cup_i B_i^j$ for each i, where $B_i^j \in B\}$. By ν-lindeloffness of X, $X \subset \cup_{k=1}^{\infty} B_{ik}^j \Rightarrow \cup_{k=1}^{\infty} O_k$. Therefore X is lindeloff.

Lemma 4.2. Every nearly lindeloff and semiregular space is lindeloff.

Proof. Let $\{O_i : i \in I\}$ be an open cover of X. Since X is semiregular, there is a regular open basis B and we have ν-open cover $\{B_i^j : O_i = \cup_i B_i^j$ for each i, where $B_i^j \in B\}$. By nearly lindeloffness of X, $X \subset \cup_{k=1}^{\infty} B_{ik}^j \Rightarrow \cup_{k=1}^{\infty} O_k$. Therefore X is lindeloff.

Corollary 4.1.

(i) If X is ν-compact and semiregular then X is lindeloff.

(ii) Every nearly compact and semiregular space is lindeloff.

Proof. Consequence of Lemma 4.1, Lemma 4.2 and Lemma 4.1 [3], Corollary 4.1 [3].

Theorem 4.1. If $A \subset X$ is Almost ν-regular and lindeloff, then $\overline{A}$ is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a ν-open set U_x containing x and by almost ν-regularity there exists an open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is lindeloff, $X = \cup_{n=1}^{\infty} V_{xn}$. Thus $\overline{A} \subseteq \overline{(\cup_{n=1}^{\infty} V_{xn})} = \cup_{n=1}^{\infty}\overline{(V_{xn})} \subseteq \cup_{n=1}^{\infty} U_{xn}$, which implies that $\overline{A}$ is ν-lindeloff.

Theorem 4.2. If $A \subset X$ is Almost ν-regular and compact, then $\overline{A}$ is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a ν-open set U_x containing x and by almost ν-regularity there exists an open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is compact, $X = \cup_{i=1}^{n} V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^{n} V_{xi})} = \cup_{i=1}^{n}\overline{(V_{xi})} \subseteq \cup_{i=1}^{n} U_{xi}$, which implies that $\overline{A}$ is $\nu-$compact and thus $\overline{A}$ is ν-lindeloff.

Theorem 4.3. If $A \subset X$ is Almost regular and lindeloff, then $\overline{A}$ is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a ν-open set U_x containing x and by almost-regularity there exists a regular open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a regular open cover and X is lindeloff, $\{V_x\}$ forms an open cover and X is lindeloff gives $X = \cup_{n=1}^{\infty} V_{xn}$. Thus $A^- \subseteq \overline{(\cup_{n=1}^{\infty} V_{xn})} = \cup_{n=1}^{\infty} \overline{(V_{xn})} \subseteq \cup_{n=1}^{\infty} U_{xn}$, which implies that $\overline{A}$ is ν-lindeloff.

Corollary 4.2. If $A \subset X$ is Almost regular and compact, then $\overline{A}$ is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a ν-open set U_x containing x and by almost-regularity there exists a regular open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a regular open cover and X is compact, $\{V_x\}$ forms an open cover and X is compact gives $X = \cup_{i=1}^{n} V_{xi}$. Thus $A^- \subseteq \overline{(\cup_{i=1}^{n} V_{xi})} = \cup_{i=1}^{n} \overline{(V_{xi})} \subseteq \cup_{i=1}^{n} U_{xi}$, which implies that $\overline{A}$ is ν-lindeloff.

Theorem 4.4. If $A \subset X$ is weak almost regular and nearly lindeloff, then $\overline{A}$ is lindeloff.

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x. By weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{(V_x)} \subset U$. For $\{V_x\}$ forms an open cover and X is nearly lindeloff, X $= \cup_{n=1}^{\infty} V_{xn}$. Thus $\overline{A} \subseteq \overline{(\cup_{n=1}^{\infty} V_{xn})} = \cup_{n=1}^{\infty}(\overline{(V_{xn})}) \subseteq \cup_{n=1}^{\infty} U_{xn}$, which implies that $\overline{A}$ is lindeloff.

Corollary 4.3. If $A \subset X$ is weak almost regular and nearly compact, then $\overline{A}$ is lindeloff.

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x. By weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{(V_x)} \subset U$. For $\{V_x\}$ forms an open cover and X is nearly compact $\exists N \ni X = \cup_{i=1}^{N} V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^{N} V_{xi})} = \cup_{i=1}^{N}(\overline{(V_{xi})}) \subseteq \cup_{i=1}^{N} U_{xi}$, which implies that $\overline{A}$ is compact. Thus $\overline{A}$ is lindeloff.

Corollary 4.4. If $A \subset X$ is weak almost regular and ν-lindeloff, then $\overline{A}$ is lindeloff.

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x and by weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is ν-lindeloff, $X = \cup_{n=1}^{\infty} V_{xn}$. Thus $\overline{A} \subseteq \overline{(\cup_{n=1}^{\infty} V_{xn})} = \cup_{n=1}^{\infty}(\overline{(V_{xn})}) \subseteq \cup_{n=1}^{\infty} U_{xn}$, which implies that $\overline{A}$ is lindeloff.

Corollary 4.5. If $A \subset X$ is weak almost regular and ν-compact, then $\overline{A}$ is lindeloff.

Proof. Let $\{U_i\}$ be any open cover of A and let $x \in A$ be any point, then for $x \in A$ there exists a regular-open set U_x containing x and by weak almost regularity there exists a regular-open set V_x such that $x \in V_x \subset \overline{V_x} \subset U$. For $\{V_x\}$ forms a open cover and X is ν-compact, $X = \cup_{i=1}^{n} V_{xi}$. Thus $\overline{A} \subseteq \overline{(\cup_{i=1}^{n} V_{xi})} = \cup_{i=1}^{n}(\overline{(V_{xi})}) \subseteq \cup_{i=1}^{n} U_{xi}$, which implies that $\overline{A}$ is compact and hence $\overline{A}$ is lindeloff.

Theorem 4.5. Every almost ν-regular and almost lindeloff subset A of X is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then there exists $i_x \in I$ such that $x \in U_{ix}$ then there exists a open set V_x such that $x \in V_x \subset \overline{V_x} \subset U_{ix}$. Now $\{V_x\}$ forms a open cover and X is almost lindeloff, $A \subseteq \cup_{j=1}^{\infty} \overline{(V_{xij})}$. Thus $\{U_{xij}\}_{j=1}^{\infty}$ is a countable subcovering of $\{U_i\}$. Hence A is ν-lindeloff.

Theorem 4.6. Every almost ν-regular and almost compact subset A of X is ν-lindeloff.

Proof. Let $\{U_i\}$ be any ν-open cover of A and let $x \in A$ be any point, then there exists

$i_x \in I$ such that $x \in U_{ix}$ then there exists a open set V_x such that $x \in V_x \subset \overline{V_x} \subset U_{ix}$. Now $\{V_x\}$ forms a open cover and X is almost compact, $A \subseteq \cup_{j=1}^{n} \overline{(V_{xij})}$. Thus $\{U_{xij}\}_{j=1}^{n}$ is a finite subcovering, which in turn a countable subcovering of $\{U_i\}$. Hence A is ν-lindeloff.

Theorem 4.7. Every weak almost regular and nearly lindeloff subset A of X is ν-lindeloff.

Corollary 4.6. Every weak almost regular and ν-lindeloff subset A of X is lindeloff.

Theorem 4.8. If in X, there exist a dense weak almost regular, regular lindeloff subset A of X, then X is lindeloff.

Theorem 4.9. Let A be any dense almost ν-regular subset of X such that every ν-open covering of A is a ν-open covering of X. Then X is almost lindeloff if and only if X is ν-lindeloff.

Theorem 4.10. Each ν-lindeloff metrizable space is finite.

§5. Relation between ν-lindeloff and weakly lindeloff spaces

Theorem 5.1. If X is weakly lindeloff and almost regular, then X is ν-lindeloff.

Proof. Let $\{V_i\}$ be any ν-open cover of X. For each $x \in X$, there exists $i_x \in I$ such that $x \in V_{ix}$. Since X is almost regular, there exists a regular open set G_{ix} such that $x \in G_{ix} \subset \overline{(G_{ix})} \subset V_{ix}$. This implies $x \in G_{ix} \subset \overline{(G_{ix})} \subset V_{ix}$ where G_{ix} are open. Since X is weakly lindeloff, $X = \cup_{n=1}^{\infty} \overline{(G_{nx})}$. Thus $X = \cup_{n=1}^{\infty} (V_{nx})$. Hence X is ν-lindeloff.

The following two corollaries are immediate consequences of above theorem and the proofs are thus omitted.

Corollary 5.1. An almost regular space X is weakly lindeloff if and only if X is ν-lindeloff.

Corollary 5.2. A Hausdorff space X is almost regular and weakly lindeloff if and only if X is ν-lindeloff.

Theorem 5.2. If X is weakly compact and almost regular, then X is ν-lindeloff.

Proof. Let $\{V_i\}$ be any ν-open cover of X. For each $x \in X$, there exists $i_x \in I$ such that $x \in V_{ix}$. Since X is almost regular, there exists a regular open set G_{ix} such that $x \in G_{ix} \subset \overline{(G_{ix})} \subset V_{ix}$. This implies $x \in G_{ix} \subset \overline{(G_{ix})} \subset V_{ix}$ where G_{ix} are open. Since X is weakly compact, $X = \cup_{i=1}^{n} \overline{(G_{ix})}$. Thus $X = \cup_{i=1}^{n} (V_{ix})$. Hence X is ν-lindeloff.

The following two corollaries are immediate consequences of above theorem and the proofs are thus omitted.

Corollary 5.3. An almost regular space X is weakly compact if and only if X is ν-lindeloff.

Corollary 5.4. A Hausdorff space X is almost regular and weakly compact if and only if X is ν-lindeloff.

Conclusion

In this paper we studied some properties and interrelations of ν-lindeloff, lindeloff and weakly lindeloff spaces.

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c^*-Normal and ss-quasinormal subgroups of finite groups

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Abstract Suppose G is a finite group and H is a subgroup of G. H is called c^*-normal in G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G; H is called ss-quasinormal in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. We investigate the influence of c^*-normal and ss-quasinormal subgroups on the structure of finite groups. Some recent results are generalized.

Keywords c^*-normal, ss-quasinormal, p-nilpotent, supersolvable.

§1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be s-quasinormal in G if H permutes with every Sylow subgroups of G. This concept was introduced by Kegel in [1]. Morerecently, Ballester-Bolinches and Pedraza-Aguilera [2] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. H is said to be s-quasinormally embedded in a group G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Yanming Wang [3] introduced the concepts of c-normal subgroup (a subgroup H of a group G is said to be a c-normal if there exists a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H). In 2002, Huaquan Wei [4] introduced the concepts of c^*-normal subgroup (a subgroup H of a group G is called c^*-normal in G if there is a normal subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G). In 2008, Shirong Li [5] introduced the concepts of ss-quasinormal subgroup (a subgroup H of a group G is said to be an ss-quasinormal subgroup of G if there is a subgroup B such that $G = HB$ and H permutes with every Sylow subgroup of B). There are examples to show that c^*-normal subgroups are not ss-quasinormal subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using c^*-normal and ss-quasinormal subgroups.

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§2. Preliminaries

Lemma 2.1. ([4], Lemma 2.3) Let H be a c^*-normal subgroup of a group G.

(1) If $H \leq L \leq G$, then H is c^*-normal in L.

(2) If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is c^*-normal in G/N.

(3) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is c^*-normal in G/N.

Lemma 2.2. ([5], Lemma 2.1) Let H be an ss-quasinormal subgroup of a group G.

(1) If $H \leq L \leq G$, then H is ss-quasinormal in L.

(2) If $N \trianglelefteq G$, then HN/N is ss-quasinormal in G/N.

Lemma 2.3. ([5], Lemma 2.2) Let H be a nilpotent subgroup of G. Then the following statements are equivalent:

(1) H is s-quasinormal in G.

(2) $H \leq F(G)$ and H is ss-quasinormal in G.

(3) $H \leq F(G)$ and H is s-quasinormally embedded in G.

Lemma 2.4. ([4], Lemma 2.7) Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If N is normal in G of order p, then $N \leq Z(G)$.

(2) If G has cyclic Sylow p-subgroup, then G is p-nilpotent.

(3) If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.

Lemma 2.5. ([4], Corollary 3.2) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every maximal subgroup of P is c^*-normal in G.

Lemma 2.6. ([8], Lemma 2.3) Let G be a group and $N \leq G$.

(1) If $N \trianglelefteq G$, then $F^*(N) \leq F^*(G)$.

(2) If $G \neq 1$, then $F^*(G) \neq 1$. In fact, $F^*(G)/F(G) = \text{Soc }(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is Solvable, then $F^*(G) = F(G)$.

Lemma 2.7. ([13], Lemma 2.3.) Suppose that H is s-quasinormal in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-quasinormal in G.

Lemma 2.8. ([13], Lemma 2.2.) If P is an s-quasinormal p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.

§3. p-nilpotentcy

Theorem 3.1. Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either c^*-normal or ss-quasinormal in G, then G is p-nilpotent.

Proof of Theorem 3.1. Let H be a maximal subgroup of P. We will prove H is c^*-normal in G.

If H is ss-quasinormal in G, then there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \text{Syl}(B)$. From $G = HB$, we obtain $|B : H \cap B|_p = |G : H|_p = p$, and hence $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \nsubseteq H$

for all $S \in \mathrm{Syl}_p(B)$ and $HS = SH$ is a Sylow p-subgroup of G. In view of $|P : H| = p$ and by comparison of orders, $S \cap H = B \cap H$, for all $S \in \mathrm{Syl}_p(B)$. So $B \cap H = \bigcap_{b \in B} (S^b \cap H) =\leq \bigcap_{b \in B} S^b = O_p(B)$.

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As $(|G|, p-1) = 1$, then $B/O_p(B)$ is p-nilpotent by Lemma 2.4, and hence B is p-solvable. So B has a Hall p'-subgroup. Thus the claim holds.

Now, let K be a p'-subgroup of B, $\pi(K) = \{p_2, \cdots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. By the condition, H permutes with every P_i and so H permutes with the subgroup $< P_2, \cdots, P_s >= K$. Thus $HK \leq G$. Obviously, K is a Hall p'-subgroup of G and HK is a subgroup of index p in G. Let $M = HK$ and so $M \trianglelefteq G$ by Lemma 2.4. It follows that H is s-quasinormally embedded, and so c^*-normal in G.

Since every maximal subgroup of P is c^*-normal in G, we have G is p-nilpotent by Lemma 2.5.

Corollary 3.2. Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either c^*-normal or ss-quasinormal in G, then G is p-nilpotent.

Proof of Corollary 3.2. By Lemma 2.1 and Lemma 2.2, every maximal subgroup of P is either c^*-normal or ss-quasinormal in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \lhd G$. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our Corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemma 2.1 and Lemma 2.2. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 3.1, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof.

Corollary 3.3. Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either c^*-normal or ss-quasinormal in G, then G is p-nilpotent.

Proof of Corollary 3.3. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and therefore Corollary 3.3 follows immediately from Theorem 3.1.

Corollary 3.4. Suppose that every maximal subgroup of any Sylow subgroup of a group G is either c^*-normal or ss-quasinormal in G, then G is a Sylow tower group of supersolvable type.

Proof of Corollary 3.4. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Corollary 3.3, G is p-nilpotent. Let U be the normal p-complement of G. By Lemma 2.1 and Lemma 2.2, U satisfies the hypothesis of the Corollary. It follows by induction that U, and hence G is a Sylow tower group of supersolvable type.

Corollary 3.5. ([6], Theorem 3.1) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either c-normal or s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 3.6. ([9], Theorem 3.1) Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either c-normal or ss-quasinormal in G, then G is p-nilpotent.

Theorem 3.7. Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroups of P is either c^*-normal or ss-quasinormal in G, then G is p-nilpotent.

Proof of Theorem 3.7. It is easy to see that the theorem holds when $p = 2$ by Corollary 3.3, so it suffices to prove the theorem for the case when p is odd. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. By Lemma 2.1 and 2.2, it is easy to see that every maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ is either c^*-normal or ss-quasinormal in $G/O_{p'}(G)$. Since

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent, $G/O_{p'}(G)$ satisfies all the hypotheses of our theorem. The minimality of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is clear to see $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is p-nilpotent. Applying Lemma 1, we immediately see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(3) $G = PQ$ is solvable, where Q is a Sylow q-subgroup of G with $p \neq q$.

Since G is not p-nilpotent, by a result of Thompson [11, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Now, T char $P \trianglelefteq N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leq N_G(T)$. By (2), we get $N_G(T) = G$ and $T = O_P(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that PQ is a subgroup of G [12, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Consequently, $PQ = G$, as desired.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

By (3), G is solvable. Let N be a minimal subgroup of G. Then $N \leq O_p(G)$ by (1). Consider G/N. It is easy to see that every maximal subgroups of P/N is either c^*-normal or ss-quasinormal in G/N. Since $N_{G/N}(P/N) = N_G(P)/N$ is p-nilpotent, we have G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(5) The final contradiction.

By step (4), there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since N is elementary abelian p-group, $N \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$. By the uniqueness of N, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence

$N = O_p(G) = C_G(N)$. If $|N| = p$, then $\mathrm{Aut}(N)$ is a cyclic group of order $p-1$. If $q > p$, then NQ is p-nilpotent and therefore $Q \le C_G(N) = N$, a contradiction. On the other hand, if $q < p$, then, since $N = C_G(N)$, we see that $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and therefore M, and in particular Q, is cyclic. Since Q is a cyclic group and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction. So we may assume N is not a cyclic subgroup of order p. Obviously $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we take a maximal subgroup P_1 of P such that $P \cap M \le P_1$. By our hypotheses, P_1 is either c^*-normal or ss-quasinormal in G. If P_1 is c^*-normal, then there is a normal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is s-quasinormally embedded in G. So there is an s-quasinormal subgroup K of G such that $P_1 \cap T$ is a Sylow p-subgroup of K. If $K_G \ne 1$, then $N \le K_G \le K$. It follows that $N \le P_1 \cap T \le P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.7, $P_1 \cap T$ is s-quasinormal in G and so $P_1 \cap T \le O_p(G) = N$. From Lemma 2.8 we have $O^p(G) \le N_G(P_1 \cap T)$. Since $P_1 \cap T$ is normalized by both P and $O^p(G)$, $P_1 \cap T \trianglelefteq G$. It follows that $P_1 \cap T = N$ or $P_1 \cap T = 1$. If $P_1 \cap T = N$, then $P = NP_1 \le P_1$, a contradiction. Thus $P_1 \cap T = 1$, then $P_1 \cap N = 1$. Since $|N/P_1 \cap N| = |NP_1/P_1| = |P/P_1| = p$, we have $|N| = p$, a contradiction. Now we assume P_1 is c^*-normal in G. By [5, Lemma 2.5], P_1Q is a subgroup of G. As $N \trianglelefteq G$, we have $P_1 \cap N = N \cap P_1Q \trianglelefteq P_1Q$, and it follows that $P_1 \cap N \trianglelefteq < P_1Q, N >= G$. Moreover, since N is a minimal normal subgroup of G, we have $P_1 \cap N = 1$ and N is a cyclic subgroup of order p, a contradiction.

Corollary 3.8. Let p be a prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $N_G(P)$ is p-nilpotent and there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either c^*-normal or ss-quasinormal in G, then G is p-nilpotent.

Proof of Corollary 3.8. By Theorem 3.7, H is p-nilpotent. If N is a normal Hall p'-subgroup of H, then N is normal in G. By the using the arguments as in the proof of Corollary 3.2, we may assume $N = 1$ and $H = P$. In the case, by our hypotheses, $N_G(P) = G$ is p-nilpotent.

Corollary 3.9. ([13], Theorem 3.2) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroups of P is s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 3.10. ([14], Theorem 3.1) Let P be a Sylow p-subgroup of a group G, where p is an odd prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroups of P is c-normal in G, then G is p-nilpotent.

§4. supersolvability

Theorem 4.1. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either c^*-normal or ss-quasinormal in G.

Proof of Theorem 4.1. The necessity is obvious. We only need to prove the sufficiency.

Suppose that the assertion is false and let G be a counterexample of minimal order.

By Lemma 2.1 and Lemma 2.2, every maximal subgroup of any Sylow subgroup of H is either c^*-normal or ss-quasinormal in H. By Corollary 3.4, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. We consider G/P. It is easy to see that $(G/P, H/P)$ satisfies the hypothesis of the Theorem. By the minimality of G, we have $G/P \in \mathcal{F}$. If the maximal P_1 of P is ss-quasinormal in G, then P_1 is s-quasinormal in G by Lemma 2.3. Thus every maximal subgroup of P is c^*-normal in G. By [4, Theorem 4.1], $G \in \mathcal{F}$, a contradiction.

Corollary 4.2. ([7], Theorem 3.2) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either s-quasinormally embedded or c-normal in G.

Corollary 4.3. ([9], Theorem 3.2) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either ss-quasinormally embedded or c-normal in G.

Corollary 4.4. Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is either c^*-normal or ss-quasinormal in G, then G is supersolvable.

Theorem 4.5. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroups of any Sylow subgroups of $F^*(H)$ is either c^*-normal or ss-quasinormal in G, then $G \in \mathcal{F}$.

Proof of Theorem 4.5. By Lemma 2.1 and 2.2, every maximal subgroups of any Sylow subgroups of $F^*(H)$ is either c^*-normal or ss-quasinormal in $F^*(H)$. Thus $F^*(H)$ is supersolvable by Corollary 4.4. In particular, $F^*(H)$ is solvable. By Lemma 2.6, $F^*(H) = F(H)$. It follows that every maximal subgroups of any Sylow subgroups of $F^*(H)$ is c^*-normal in G by Lemma 2.3. Thus the result is a corollary of Theorem 4.3 in [4].

Corollary 4.6. ([6], Theorem 3.9) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroups of any Sylow subgroups of $F^*(H)$ is either s-quasinormally embedded or c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.7. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroups of any Sylow subgroup of $F(H)$ are either c^*-normal or ss-quasinormal in G, then $G \in \mathcal{F}$.

Corollary 4.8. ([6], Theorem 3.7) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroups of any Sylow subgroup of $F(H)$ is either s-quasinormally embedded or c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.9. ([9], Theorem 3.3) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroups of any Sylow subgroup of $F(H)$ is either

ss-quasinormally embedded or c-normal in G, then $G \in \mathcal{F}$.

Theorem 4.10. If every cyclic subgroups of any Sylow subgroup of a group G of prime order or order 4 is either c^*-normal or ss-quasinormal in G, then G is supersolvable.

Proof of Theorem 4.10. Assume the theorem is false and let G be a counterexample of minimal order. It is obvious that the hypotheses of the Lemma are inherited for subgroups of G. Our minimal choice yields that G is not supersolvable but every proper subgroup of G is supersolvable. A well-known result of Doerk implies that there exists a normal Sylow p-subgroup of G such that $G = PM$, where M is supersolvable and if $p > 2$ then the exponent of P is p, if $p = 2$, the exponent of P is 2 or 4. Let x be an arbitrary element of P. If $< x >$ is ss-quasinormal in G, then $< x >$ is s-quasinormally embedded in G by Lemma 2.3. If $< x >$ is c^*-normal in G, then there is a normal subgroup T of G such that $G =< x > T$ and $< x > \cap T$ is s-quasinormally embedded in G. Hence $P = P \cap G = P\cap < x > T =< x > (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $< x >= P \trianglelefteq G$ and so $< x >$ is s-quasinormally embedded in G. If $P = P \cap T$, then $T = G$ and so $< x >$ is also s-quasinormally embedded in G. We have proved that every cyclic subgroups of any Sylow subgroup of G of prime order or order 4 is s-quasinormally embedded in G. Applying Theorem 3.3 in [10], we have G is supersolvable, a contradiction.

Theorem 4.11. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroups of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is either c^*-normal or ss-quasinormal in G, then $G \in \mathcal{F}$.

Proof of Theorem 4.11. By Lemma 2.1 and 2.2, every cyclic subgroups of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c^*-normal or ss-quasinormal in $F^*(H)$. Thus $F^*(H)$ is supersolvable by Theorem 4.10. In particular, $F^*(H)$ is solvable. By Lemma 2.6, $F^*(H) = F(H)$. Since $G/H \in \mathcal{F}$, we have that $G^{\mathcal{F}}$, the $\mathcal{F}$-residual subgroup of G, is contained in H. Hence, for any cyclic subgroup $< x >$ of $F^*(G^{\mathcal{F}}) \leq F^*(H)$ of prime order or order 4, $< x >$ is c^*-normal or ss-quasinormal in G. If $< x >$ is c^*-normal in G, then there exists a normal subgroup K of G such that $G =< x > K$ and $< x > \cap K$ is s-quasinormally embedded in G. Hence G/K is cyclic, then $G/K \in \mathcal{F}$ by the hypotheses. Therefore $G^{\mathcal{F}} \leq K$. This implies that $< x >\leq K$, so $< x >=< x > \cap K =< x >$ is s-quasinormally embedded in G. If $< x >$ is ss-quasinormal in G, then $< x >$ is also s-quasinormally embedded in G by lemma 2.3. Hence we have proved that every cyclic subgroup of prime order or order 4 of $F^*(G^{\mathcal{F}})$ is s-quasinormally embedded in G. Applying Theorem 1.2 in [10], we have $G \in \mathcal{F}$.

Corollary 4.12. ([6], Theorem 4.3) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroups of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is either c-normal or s-quasinormally embedded in G, then $G \in \mathcal{F}$.

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Pseudo commutative near-rings

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Abstract A near-ring N is called weak commutative (Definition 9.4, p. 289, Pilz [7]) if $xyz = xzy$ for every $x, y, z \in N$. It is quite natural to investigate the properties of N if $xyz = zyx$ for every $x, y, z \in N$. We call such a near-ring a pseudo commutative near-ring. In this paper, we furnish examples of such near-rings and probe certain properties of pseudo commutative near-rings. We also discuss the properties of pseudo commutative near-ring which are regular. We obtain some equivalent conditions for a near-ring to be pseudo commutative. Also we obtain a characterization of a field vis-a-vis subdirect irreducibility and use it to prove structure theorems for pseudo commutative near-rings.

Keywords Near-Rings, pseudo commutative, idempotents, nilpotents, regular, reduced, primary ideal.

§1. Introduction

Throughout this paper, N denotes a right near-ring $(N, +, .)$ with at least two elements. We denote by 0, the additive identity of the group $(N, +)$ and write xy for $x.y$ for $x, y \in N$. For any non-empty set A, we denote $A - \{0\}$ by A^*. We freely make use of the following terms, notations and results.

Known Results

- (K1) An element $a \in N$ is said to be

  (i) Nilpotent if there exists a positive integer k such that $a^k = 0$.

  (ii) Idempotent if $a^2 = a$.

  We denote by E, the set of all idempotents of N.

- (K2) (Theorem 1.62, p. 26, Pilz [7]) Each near-ring N is isomorphic to a subdirect product of subdirectly irreducible near-rings.

- (K3) If N is a zero symmetric near-ring, then

(i) Every left ideal A of N is an N-subgroup of N.

(ii) Every ideal I of N satisfies $NIN \subseteq I$.
i.e., every ideal is an N-subgroup.

(iii) $N^*I^*N^* \subseteq I^*$.

- (K4) Let N be a near-ring. Then the following are true:

  (i) If A is an ideal of N and B is any subset of N, then $(A : B) = \{n \in N/nB \subseteq A\}$ is always a left ideal.

  (ii) If A is an ideal of N and B is an N-subgroup, then $(A : B)$ is an ideal. In particular if A and B are ideals of a zero symmetric near-ring N, then $(A : B)$ is an ideal.

- (K5) Let N be a regular near-ring, $a \in N$ and $a = axa$. Then

  (i) $ax, xa \in E$ and $E \neq \{0\}$.

  (ii) $axN = aN$ and $Nxa = Na$.

  (iii) N is S and $S\,'$ near-rings.

- (K6) (Lemma 4 in Dheena [1]) Let N be a zero-symmetric reduced near-ring. For any $a, b \in N$ and $e \in E$, $abe = aeb$.

- (K7) N is said to be weak commutative if $xyz = xzy$ for every $x, y, z \in N$.

- (K8) (Gratzer [4] and Fain [3]) A near-ring N is subdirectly irreducible if and only if the intersection of all non-zero ideals of N is not zero.

- (K9) (Gratzer [4], p. 124) Each simple near-ring is subdirectly irreducible.

- (K10) (Theorem 9.3, p. 289, Pilz [7]) A zero symmetric near-ring N has IFP if and only if $(0 : S)$ is an ideal for any subset S of N.

- (K11) (Oswald [6]) An N-subgroup A of N is essential if $A \cap B = \{0\}$, where B is any N-subgroup of N, implies $B = \{0\}$.

Basic concepts and terms used but not defined in this paper can be found in Pilz [7].

§2. Definition and basic properties

In this section, we introduce the concept of pseudo commutative near-ring and distinguish it from commutative and weak commutative near-rings with illustrations.

Definition 2.1. A near-ring N is said to be pseudo commutative near-ring if $xyz = zyx$ for all $x, y, z \in N$.

Example 2.2.

1. Consider the near-ring $(N, +, .)$, where $(N, +)$ is the Klein's four group with $N = \{0, a, b, c\}$ and $(N, .)$ is given below (scheme 16, p. 408, Pilz [7]).

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	a

One can check that N is pseudo commutative, but not commutative.

2. When $N = \{0, 1, 2, 3\}$, the near-ring $(N, +, .)$ with a.b = a for all $a, b \in N$ (scheme 9, p. 407, Pilz [7]) is weak commutative but not pseudo commutative. (since $123 \neq 321$).

3. The near-ring $(N, +, .)$ where $(N, +)$ is Kleins four group and '.' is defined as per scheme 14, p. 408 of Pilz [7]

.	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	0	0
c	0	a	0	c

is not weak commutative (since $aca \neq aac$) and not pseudo commutative ($caa = a \neq aac$).

4. Every commutative near-ring is pseudo commutative.

Now we derive some properties of pseudo commutative near-rings. We investigate the behaviour of ideals and N-subgroups of pseudo commutative near-rings.

Proposition 2.3. Every pseudo commutative near-ring is zero symmetric.

Proof. For every $n \in N, n0 = n00 = 00n = 0$.

Proposition 2.4. Let N be a pseudo commutative near-ring. If $E \neq \{0\}$, then $E \subseteq C(N)$.

Proof. Let e be an idempotent in N. Then, for every $a \in N$, $ae = aee = eea = ea$ and so $e \in C(N)$.

One can see the following in a straight forward way.

Proposition 2.5. Homomorphic image of a pseudo commutative near-ring is also pseudo commutative.

As an immediate consequence, we have the following corollary.

Corollary 2.6. Let N be a pseudo commutative near-ring. If I is any ideal of N, then N/I is also pseudo commutative.

Now we have the following useful characterization of pseudo commutative near-rings.

Theorem 2.7. Every pseudo commutative near-ring N is isomorphic to a sub-direct product of sub-directly irreducible pseudo commutative near-rings.

Proof. By $(K2)$, N is isomorphic to a sub-direct product of sub-directly irreducible near-rings N_k and each N_k is a homomorphic image of N under the projection map π_k. Now the desired result follows from Proposition 2.5.

Proposition 2.8. Any weak commutative near-ring (K7) with left identity is a pseudo commutative near-ring.

Proof. Let $a, b, c \in N$ and e be a left identity.
Since N is weak commutative, $abc = (ea)bc = e(abc) = e(acb) = (eac)b = (eca)b = e(cab) = e(cba) = cba$. Hence N is pseudo commutative.

Proposition 2.9. Any pseudo commutative near-ring with right identity is weak commutative.

Proof. Let $a, b, c \in N$ and e be a right identity.
Then $abc = abce = a(bce) = a(ecb)$ (since N is pseudo commutative) $= (ae)cb = acb$. This completes the proof.

In the following theorem we prove sixteen of the properties of pseudo commutative near-rings.

Theorem 2.10.

Let N be a regular pseudo commutative near-ring. Then

(i) $A = \sqrt{A}$ for every N-subgroup A of N.

(ii) N is reduced.

(iii) N has $(*, IFP)$.

(iv) Every N subgroup is an ideal.

(v) $N = Na = Na^2 = aN = aNa$ for all $a \in N$.

(vi) Any ideal of N is completely semi prime.

(vii) N has property P_4.

(viii) For every ideal I of N, $(I : S)$ is an ideal of N where S is any subset of N.

(ix) For every ideal I of N, $x_1, x_2, \cdots, x_n \in N$, if $x_1x_2 \cdots x_n \in I$ then $\langle x_1\rangle\langle x_2\rangle \ \ldots \ \langle x_n\rangle \subseteq I$.

(x) N has strong IFP.

(xi) N is a semi prime near-ring.

(xii) $P \cap Q = PQ$ for any two N-subgroups P, Q of N.

(xiii) $P = P^2$ for every N-subgroup (ideal) P of N.

(xiv) If P is a proper N-subgroup of N then each element of P is a zero-divisor.

(xv) $NaNb = Na \cap Nb = Nab$ for all $a, b \in N$.

(xvi) Every N-subgroup of N is essential (K11) if N is integral.

Proof. Since N is regular for every $a \in N$ there exists $b \in N$ such that

$$a = aba. \tag{1}$$

$$\text{Since } E \subseteq C(N), \text{ by Proposition 2.4, } a = aba = (ab)a = a(ab) = a^2 b \tag{2}$$

(i) Let A be an N-subgroup of N and $a \in \sqrt{A}$.
Then there exists some positive integer k such that $a^k \in A$.
Similar to Equation (2), $a = aba = a(ba) = (ba)a = ba^2$.
Hence $a = b(ba^2)a = b^2a^3 = \cdots = b^{(k-1)}a^k \in NA \subseteq A$.
Therefore $\sqrt{A} \subseteq A$. And obviously $A \subseteq \sqrt{A}$.
This completes the proof of (i).

(ii) If $a^2 = 0$ then by Equation (2), $a = 0b = 0$. This completes the proof of (ii).

(iii) Let $ab = 0$. Then by Proposition 2.3, $(ba)^2 = ba(ba) = b(ab)a = b0a = b0 = 0$. Now by (ii) $ba = 0$. And for every $n \in N$, $(anb)^2 = anb(anb) = an(ba)nb = 0$. Again by (ii) $anb = 0$. This completes the proof of (iii).

(iv) Let $a \in N$. Since N is regular $a = aba$ for some $b \in N$.
And by (K5) (i), ba is an idempotent. Let $ba = e$.

$$\text{Therefore } Ne = Nba = Na \text{ (by } (K5)(ii)) \tag{3}$$

Let $S = \{n - ne/n \in N\}$. We claim that $(0 : S) = Ne$.
Since $(n - ne)e = 0$ for all $n \in N$, $(n - ne)Ne = 0$ (by (ii)).

$$\text{This implies } Ne \subset (0 : S). \tag{4}$$

Now, let $y \in (0 : S)$. Then $y = yxy$ for some $x \in N$ and $yx - (yx)e \in S$.
Therefore $(yx - yxe)y = 0$ and this implies, $yxy - yxey = 0$.
i.e. $y - y(xey) = 0$ and by (K6) we get, $y - y(xye) = 0$.
i.e. $y - (yxy)e = 0$. i.e. $y - ye = 0$. Hence $y = ye \in Ne$.

$$\text{It follows } \textit{that } (0 : S) \subset Ne. \tag{5}$$

Combining Equations (3), (4) and (5) we get, $(0 : S) = Ne = Na$.
Using (K10) we get, Na is an ideal of N.
Now if M is any N-subgroup of N then $M = \sum_{a \in M} Na$.
Thus M becomes an ideal of N.

(v) Since N is regular for every $a \in N$ there exists $b \in N$ such that
$a = aba = a(ba) = (ba)a$ (by Proposition 2.4) $= ba^2 \in Na^2$.
Therefore $N \subseteq Na^2$.
Now, $Na \subseteq N \subseteq Na^2 \subseteq Na \subseteq N$ implies

$$Na = Na^2 = N \tag{6}$$

Now we shall claim that $Na^2 = aN$
Let $x \in Na^2$. Then by the definition of pseudo commutative near-rings, for some $n \in N$, $x = na^2 = naa = aan \in aN$.

$$\text{Therefore } Na^2 \subseteq aN. \tag{7}$$

Let $an \in aN$. Since N is regular, $an = (aba)n = a(ba)n = (ba)an$
(by Proposition 2.4) $= b(aan) = b(naa)$ (by the definition of pseudo commutative near-rings) $= (bn)a^2 \in Na^2$
i.e., $aN \subseteq Na^2$. This together with (7) implies

$$Na^2 = aN. \tag{8}$$

Now we claim that $aN = aNa$. Since Na is an ideal, by (iv),

$$\text{for every } a \in N, \ (Na)N \subseteq Na. \tag{9}$$

Since N is regular for every $n \in N$,
$an = (aba)n = a(ban) \in a(NaN) \subseteq aNa$. (by Equation (9))
Thus $aN \subseteq aNa$. Obviously $aNa \subseteq aN$.

$$\text{Hence } aN = aNa \text{ for every } a \in N. \tag{10}$$

Equations (6),(8) and (10) completes (v) .

(vi) Let $a^2 \in I$. Then by Equation (2), $a = a^2b \in IN \subseteq I$. Therefore every ideal of N is completely semi prime.

(vii) Let $ab \in I$ then $(ba)^2 = ba(ba) = b(ab)a \in NIN \subseteq I$ (By (K3))
Now by (vi) $ba \in I$. This completes the proof of (vii).

(viii) Let I be an ideal of N and S be any subset of N.
By (K4), $(I : S) = \{n \in N/nS \subseteq I\}$ is a left ideal of N.
Let $s \in S$. If $a \in (I : S)$ then $as \in I$. And by (vii), $sa \in I$
Then for any $n \in N$, $(sa)n \in I$. i.e., $s(an) \in I$.
Again by (vii), $ans \in I$. i.e., $an \in (I : S)$ for any $n \in I$ and hence $(I : S)$ is a right ideal.
Consequently $(I : S)$ is an ideal. This completes the proof of (viii).

(ix) Let $x_1x_2 \ldots x_n \in I$.
$\Rightarrow x_1 \in (I : x_2 \ldots x_n)$.
$\Rightarrow \langle x_1 \rangle \subseteq (I : x_2 \ldots x_n)$.
$\Rightarrow \langle x_1 \rangle x_2 \ldots x_n \subseteq I$.
$\Rightarrow x_2 \ldots x_n \langle x_1 \rangle \subseteq I$, (by (viii)).
$\Rightarrow x_2 \in (I : x_3 \ldots x_n \langle x_1 \rangle)$.
$\Rightarrow \langle x_2 \rangle \subseteq (I : x_3 \ldots x_n \langle x_1 \rangle)$.
$\Rightarrow \langle x_2 \rangle x_3 \ldots x_n \langle x_1 \rangle \subseteq I$.
$\Rightarrow x_3 \ldots x_n \langle x_1 \rangle \langle x_2 \rangle \subseteq I$.
Continuing like this we get $\langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_n \rangle \subseteq I$.

(x) Let I be an ideal of N. Since N is zero symmetric, by Proposition , $NI \subseteq I$. By (v), $aN = Na^2$. Hence $an = ma^2$ for some $m, n \in N$. Hence if $ab \in I$ then for every $n \in N, anb = ma^2b = (ma)ab \in NI \subseteq I$. Therefore N has strong IFP.

(xi) Let M be an N-subgroup of N. Then M is an ideal (by (iv)).
Let I be any ideal of N such that $I^2 \subseteq M$.
By Proposition and (K3), $NI \subseteq I$.
If $a \in I$, then $a = aba \in I(NI) \subseteq I^2 \subseteq M$.
Therefore any N-subgroup M of N is a semi prime ideal.
In particular $\{0\}$ is a semi prime ideal and hence N is a semi prime near-ring.

(xii) Let P, Q be two N-subgroups of N. Then by (iv), they are ideals too.
Hence $PQ \subseteq P$ and $PQ \subseteq Q$. Therefore $PQ \subseteq P \cap Q$
Let $a \in P \cap Q$. By (1), $a = (ab)a \in (PN)Q \subseteq PQ$
Hence $P \cap Q \subseteq PQ$ and consequently $P \cap Q = PQ$.
This completes (xii).

(xiii) Taking $Q = P$ in (xii) we get $P = P^2$.

(xiv) Let P be a proper N-subgroup of N and $a \in P^*$.
By (xiii), $Na = (Na)^2 = NaNa$.
Therefore for every $n \in N$, there exist $x, y \in N$ such that $na = xaya$.
This implies $(n - xay)a = 0$. If a is not a zero divisor, then $n - xay = 0$.
i.e., $n = xay \in NPN \subseteq P$. Hence $N = P$ this is a contradiction to P is proper. Therefore a is a zero-divisor of N and the (xiv) follows.

(xv) Since Na, Nb are N-subgroups, by (xii), $Na \cap Nb = (Na)(Nb)$.
Since $Na \subseteq N$, $Na \cap N = Na = Na \cap Na = NaNa \subseteq (Na)N = NaN$.
and Na is an ideal implies $NaN = (Na)N \subseteq Na = Na \cap N$.
Therefore $Na = Na \cap N = NaN$. This implies that $Nab = (Na)b = (NaN)b = NaNb = Na \cap Nb$. Hence (xv) follows.

(xvi) Let P be a non-zero N-subgroup of N.
Suppose there exists an N-subgroup Q of N such that $P \cap Q = \{0\}$.
Then, by (xii), $PQ = \{0\}$ and since N is an integral near-ring, $Q = \{0\}$.
This completes (xvi).

Remark 2.11. In a regular pseudo commutative near-ring,
(i) The concepts of N-subgroups, left ideals, right ideals and ideals are all equivalent.
(ii) Since N has (*, IFP), by Theorem 2.10 (iii), the concept of left zero divisors, right zero divisors and zero divisors are equivalent in N.

Theorem 2.12. Let N be a regular pseudo commutative near-ring and P be a proper N-subgroup of N. Then the following are equivalent

(i) P is a prime ideal.

(ii) P is a completely prime ideal.

(iii) P is a primary ideal.

(iv) P is a maximal ideal.

Proof. (i) $\Rightarrow$ (ii) Let $ab \in P$. By Theorem 2.10 (xv), $NaNb = Nab \subseteq NP \subseteq P$.

By Theorem 2.10 (iv), Na and Nb are ideals in N.

Since P is prime, $NaNb \subseteq P$ implies $Na \subseteq P$ or $Nb \subseteq P$.

Since N is regular for some $x, y \in N,\ a = axa \in Na \subseteq P$ and $b = byb \in Nb \subseteq P$. Therefore either $a \in P$ or $b \in P$ and hence (ii) follows.

(ii) $\Rightarrow$ (i) is obvious.

(ii) $\Rightarrow$ (iii) By Theorem 2.10 (xv), for any $a, b \in N,\ Nab = Na \cap Nb$.

Since $Na \cap Nb = Nb \cap Na,\ Nab = Nba$ for all $a, b \in N$.

Hence for all $a, b, c \in N,\ Nabc = Nbca = Ncab = Nacb = Nbac = Ncba$.

Suppose $abc \in P$ and $ab \notin P$ then $c \in P$, by (ii).

Again suppose $abc \in P$ and $ac \notin P$.

Since N is regular, $acb \in Nacb = Nabc \subseteq NP \subseteq P$.

Thus $acb = (ac)b \in P$ implies $b \in P$ (since $ac \notin P$).

Continuing in the same vein, we easily obtain that if $abc \in P$ and if the product of any two of a, b, c does not fall in P then the third falls in P.
This proves (iii).

(iii) $\Rightarrow$ (ii) Let $ab \in P$ and $a \notin P$.

Since N is regular $a = axa$ for some $x \in N$.

First we claim that $xa \notin P$.

Suppose $xa \in P$. This implies $a = a(xa) \in NP \subseteq P$ which is a contradiction.

Therefore $xa \notin P$. Also $x(ab) \in NP \subseteq P$. Thus $xab \in P$ and $xa \notin P$.

As P is a primary ideal of N, $b^k \in P$ for some integer k.

Now $b^k \in P$ implies $b \in \sqrt{P}$ and $\sqrt{P} = P$ (by Theorem 2.10 (i)).

Therefore $b \in P$ and (ii) follows.

(i) $\Rightarrow$ (iv) Let J be an ideal of N such that $J \neq P$ and that $P \subseteq J \subseteq N$.

Let $a \in J - P$. Since N is regular there exists $x \in N$ such that $a = a(xa) = (xa)a$ (by Proposition 2.4).

Thus for all $n \in N$, $na = nxa^2$ and this implies $(n - nxa)a = 0$.

Since N has (*, IFP), we get $(n - nxa)ya = 0$ for all $y \in N$.

Consequently $N(n - nxa)Na = N0 = \{0\}$, If $b = (n - nxa)$ then $NbNa = \{0\} \subseteq P$.

Since P is a prime ideal and $Na,\ Nb$ are ideals in N, $Na \subseteq P$ or $Nb \subseteq P$.

If $Na \subseteq P$ then $a = axa \in P$ which is a contradiction.

Hence $Nb \subseteq P$ then $Nb \subseteq J$ and this demands that for some $y \in N,\ b = byb \in J$. i.e., $n - nxa \in J$. Now since $a \in J,\ nxa \in NJ \subseteq J$ (by (K3)) and therefore $n \in J$. Hence $J = N$ and (vi) follows.

(iv)$\Rightarrow$ (i) is obvious.

This completes the proof.

§3. Structure theorems for a pseudo commutative near-ring

In this section we discuss conditions under which a pseudo commutative near-ring N becomes a field.

Proposition 3.1. Any pseudo commutative near-ring with identity is commutative.

Proof. Let $a, b \in N$ and u be the identity of N.
Then $ab = abu = uba$ (By the definition of pseudo commutative near-ring) $= ba$. Hence N is commutative.

With a view to establishing a structure theorem we prove the following two theorems.

Theorem 3.2. Let N be a subdirectly irreducible pseudo commutative near-ring. Then either N is simple with each non-zero idempotent is an identity or the intersection of the non-zero ideals of N has no non-zero idempotents.

Proof. Let N be a subdirectly irreducible pseudo commutative near-ring. Suppose that N is simple.

If e is a non-zero idempotent in N. Then by Theorem 2.10 (iii) and (K10), $(0 : e)$ is an ideal Since $e \notin (0 : e)$ and N is simple, $(0 : e) = \{0\}$.

Hence $(ne-n)e = 0$ for all $n \in N$ implies $ne-n = 0$. And by Proposition 2.4, $ne = en = n$. This means that e is an identity of N.

Suppose N is not simple.

Let I be the intersection of non-zero ideals of N. Since N is subdirectly irreducible, we have $I \neq \{0\}$.

Suppose that I contains a non-zero idempotent e. We claim that e is a right identity.

If not, then there exists $n \in N$ such that $ne \neq n$. Hence $ne - n \neq 0$.

Since $(ne - n)e = 0$ we have $ne - n \in (0 : e)$ and hence $(0 : e)$ is a non-zero ideal of N.

Therefore $I \subseteq (0 : e)$. Hence $e \in (0 : e)$. This contradiction leads us to conclude that e is a right identity of N.

Hence for all $n \in N,\ n = ne \in NI \subseteq I$. This implies that $I = N$, again a contradiction. Hence the intersection of the non zero ideals of N has no nonzero idempotents. This completes the desired result.

In the following theorem we give some equivalent conditions for a simple pseudo commutative near-ring.

Theorem 3.3. Let N be a regular pseudo commutative near-ring. Then the following are equivalent

(i) N is subdirectly irreducible

(ii) non-zero idempotents of N are not zero divisors.

(iii) N is simple.

Proof. (i) $\Rightarrow$ (ii) Let J be the set of all nonzero idempotents in N which are zero divisors too. We claim that J is empty.

If J is non-empty, let us assume that $I = \bigcap\{(0 : e)/e \in J\}$.

Since N is sub directly irreducible, $I \neq 0$ (by (K8)).

Let $a \in I^*$. Since N is regular there exists some $b \in N$ such that

$$a = aba. \tag{11}$$

Also $ab, ba \in E$. Again $a \in I^*$ implies

$$ae = 0 \text{ for all e in J} \tag{12}$$

$\Rightarrow$ bae = 0 (Since N is zero symmetric, by Proposition 2.3)
$\Rightarrow$ (ba)e = 0
Therefore ba is a zero divisor too. Hence $ba \in J$.

Now by Equation (12), $a(ba) = 0$ and by Equation (11), $a = 0$.

It is a contradiction to $a \in I^*$. Hence J is empty.

(The above proof is very similar to that of Claim 1 in Theorem 4.7 in [9]).

(ii)$\Rightarrow$ (iii) Let I be a non-zero ideal and $0 \neq x \in I$.

Since N is regular $x = xyx$ for some $y \in N$ and $yx \in E$.

Therefore for every $n \in N$, $nx = nxyx$. i.e., $(n - nxy)x = 0$.

Now by Proposition (v), $(n - nxy)yx = 0$. And by (ii), $n - nxy = 0$.

Hence for every $n \in N$, $n = nxy \in NIN \subseteq I$.

Thus $N \subseteq I$, this leads to the result that N has no nontrivial ideal of N and hence N is simple.

(iii) $\Rightarrow$ (i) By (K9), each simple near-ring is subdirectly irreducible.

As an immediate consequence we have

Corollary 3.4. Let N be a regular pseudo commutative near-ring. Then N is subdirectly irreducible if and only if N is a field.

Proof. By Theorem (3.2) and (3.3), every non zero idempotent is an identity.

Since N is regular $a = a(ba) = (ba)a$ (by Proposition 2.4).

That is inverse exists for every $a \in N$. Hence N is a near field.

Now Proposition guarantees that N is a field. The converse is obvious.

Now we are in a position to prove the structure theorems.

Theorem 3.5. Let N be a regular pseudo commutative near-ring. Then N is isomorphic to a subdirect product of fields.

Proof. By Theorem 2.7, N is isomorphic to a subdirect product of subdirectly irreducible pseudo commutative near-rings, N_k's, each N_k is regular and pseudo commutative. The rest of the proof is taken care of by Corollary 3.4.

Corollary 3.6. Let N be a regular pseudo commutative near-ring. Then N has no non-zero zero divisors if and only if N is a field.

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Divisibility tests for Smarandache semigroups

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Abstract Two divisibility tests for Smarandache semigroups are given. Further, the notion of divisibility of elements in a semigroup is applied to characterize the Smarandache semigroups. Examples are provided for justification.

Keywords Pseudo semigroup, divisibility, right divisor, left divisor, group, Smarandache semigroup.

§1. Introduction and results

Padilla Raul introduced the notion of Smarandache semigroups in the year 1998 in the paper entitled Smarandache algebraic structures [2]. Since groups are the prefect structures under a single closed associative binary operation, it has become infeasible to define Smarandache groups. Smarandache semigroups are the analog in the Smarandache ideologies of the groups. The Smarandache notions in groups and the concept of Smarandache semigroups have been studied in [9].

In [7], the notion of divisibility of elements in a semigroup is introduced and the properties of the elements which are both left and right divisiors of every element of a semigroup are studied. Further, the properties of the elements which are divisible both on the right and on the left by all elements of the semigroup are studied as well. Such elements were first considered in a paper by Clifford and Miller, where they were termed zeroid elements. The concept of divisibility of elements in a semigroup is very much useful to study the Smarandache semigroups. In this paper we give two tests called divisibility tests for Smarandache semigroups, and a characterization of Smarandache semigroups by applying the notion of divisibility of elements in a semigroup. Examples are provided for justification.

In section 2 we give the basic definitions and properties of divisibility of elements in a semigroup. The definition and example of Smarandache semigroups are given as well. In section 3 we present our theorems and in section 4 we provide examples for justification. For, more basic definitions and concepts please refer [7] and [9]. In this paper, we denote the operation multiplication (product) of elements in a semigroup by jux-ta-position.

§2. Preliminaries

Definition 2.1.[7] A semigroup is a nonempty set S, in which for every ordered pair of elements x, y in S there is defined a new element called their product $xy \in S$, where for all $x, y, z \in S$, we have $(xy)z = x(yz)$.

Definition 2.2.[7] An element b of the semigroup S is called a right divisor of the element a of the semigroup if there exists in S an element x such that $xb = a$. b is called a left divisor of a if there exists in S an element y such that $by = a$. If b is a right divisor of a, we say that a is divisible on the right by b. If b is a left divisor of a, we say that a is divisible on the left by b.

The following observations in the divisibility of elements in a semigroup are well known in [7].

2.2.1. An element a of a semigroup S will be a right (left) divisor of every element of S, if and only if in the column (row) of the multiplication table corresponding to the element a, all elements of S occur.

2.2.2. The element a is divisible on the left (right) by every element of S, if and only if a occurs in every row (column) of the multiplication table.

Definition 2.3.[7] A nonempty subset H of the semigroup S is called a subsemigroup of S, if $H \subset S$.

Definition 2.4.[6] A nonempty set G, together with an associative binary operation $*$ on G such that equations $a * x = b$ and $y * a = b$ have solutions in G for all $a, b \in G$ is a group.

Several examples of Semigroups, subsemigroups and groups can be found in the literature.

Definition 2.5.[9] The Smarandache semigroup defined to be a semigroup S such that a proper subset B of S is a group with respect to the same operation on S.

Example 2.6. Let us consider the semigroup $S = \{0, 1, 2, 3, 4, 5\}$ under the operation multiplication modulo 6. The multiplication table is as follows:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Table 1.

S is a Smarandache semigroup as the proper subsets $\{2, 4\}$, $\{1, 5\}$ of S are groups.

§3. Proofs of the theorems

In this section we present divisibility test 1, divisibility test 2 for Smarandache semigroups. Further, a characterization of Smarandache semigroups, using the notion of divisibility of elements in a semigroup, is given as well.

Theorem 3.1. (Divisibility test-1) A semigroup S (not a group) containing the elements that are both left and right divisors of every element of S is always a Smarandache semigroup.

Proof. Let G be the set of all elements that are both left and right divisors of every element of S. Cleary $G \subset S$. (If $S = G$, then G is a group as every element of S is a left and right divisors of every element of S). To show that S is a Smarandache semigroup, it is sufficient to show that G is a group under the operation on S.

Suppose that $b_1, b_2 \in G$. For every $a \in S$ there exists $x_2 \in S$ such that $x_2 b_2 = a$. Corresponding to $x_2 \in S$ there exists in S an element x_1 such that $x_1 b_1 = x_2$. Therefore, $x_1 b_1 b_2 = x_2 b_2 = a$. Accordingly, $b_1 b_2$ is a right divisor of a. Similarly, we can show that $b_1 b_2$ is a left divisor of a. Therefore, $b_1 b_2 \in G$. The multiplication in G is associative as $G \subset S$.

Suppose that $b_1, b_2 \in S$. There exist in S elements x and y such that $x b_2 = b_1$, $b_2 y = b_1$. If we show that $x, y \in G$ the proof that G is a group will be completed. Let α be an arbitrary element of S. Since, for some $t \in S$, we have $b_1 t = \alpha$; $x b_2 t = b_1 t = \alpha$. i.e., x is a left divisor of α. There exist also in S elements e, c_1, c_2 such that $b_2 e = b_2$, $c_1 b_1 = e$, $b_2 c_2 = e$. Since for arbitrary z in S there exists an element $z^1 \in S$ such that $z = z^1 b_2$, we have $ze = z^1 b_2 e = z^1 b_2 = z$. i.e., e is such that $ze = z$ for arbitrary $z \in S$. Making use of this property of e, we obtain, $\alpha = \alpha e = \alpha b_2 c_2 = b_2 e c_2 = b_2 c_1 b_1 c_2 = b_2 c_1 x b_2 c_2 = b_2 c_1 x e = \alpha b_2 c_1 x$. i.e., x turns to be also a right divisor of α. Therefore, $x \in G$. Analogously, we can show that $y \in G$. Hence, S is a Smarandache semigroup.

Theorem 3.2. (Divisibility test-2) A semigroup S (not a group) containing the elements that are divisible both on the right and on the left by every element of S is always a Smarandache semigroup.

Proof. Let G be the set of all elements of S that are divisible both on the right and on the left by every element of S. Clearly, $G \subset S$. Let $c_1, c_2 \in G$, for every $a \in S$, there is some x in S such that $c_1 = ax$. Therefore, $c_1 c_2 = a(xc_2)$, i.e., $c_1 c_2$ is divisible on the left by an arbitrary element a. Analogously, we can show that $c_1 c_2$ is divisible on the right by a. Therefore, $c_1 c_2 \in G$. The associativity of the operation in G follows from the fact that $G \subset S$. Since c_1 and c_2 belong to G, there exist elements u, v, w in S such that $uc_2^2 = c_2$, $c_2^2 v = c_2$, $c_1 = c_2 w$, We have then $uc_2 = uc_2 c_2 v = c_2 v$. This yields, $c_2(c_2 v^2 c_1) = (c_2^2 v)(vc_1) = c_2 v c_1 = uc_2 c_1 = uc_2 c_2 w = c_2 w = c_1$. We have shown that the element $y = c_2 v^2 c_1$ is a solution of the equation $c_2 y = c_1$. Let us show that $y \in G$. In fact, for arbitrary z in S there exist u^1 and v^1 in S such that $c_1 = u^1 z$, $c_2 = zv^1$. Therefore, $y = c_2 v^2 c_1 = zv^1 v^2 u^1 z$, i.e., y is divisible by z both on the right and on the left. Analogously, we may find a solution x of the equation $xc_2 = c_1$, such that $x \in G$. Therefore, G is a group. Hence, the semigroup S is a Smarandache semigroup.

Theorem 3.3. Let S be a semigroup. S is a Smarandache semigroup if and only if there exists a proper subsemigroup T of S such that T possesses an element which is both a right and left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T.

Proof. Let S be a semigroup. If S is a Smarandache semigroup then there exists a proper subset T of S such that T is a group with respect to the same operation on S. T possesses an element which is both a right and a left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T as T is a group with respect to the same operation on S.

On the other hand suppose that there exists a subsemigroup T of S such that T possesses an element which is both a right and a left divisor of every element of T and is at the same time itself divisible both on the left and on the right by every element of T. Now we show that S is a Smarandache semigroup. For, it is sufficient to prove that the proper subsemigroup T is a group. Let a and b be arbitrary elements of T and let d be an element having the properties postulated in the hypothesis of the theorem. There exist elements x_1, y_1, x_2, y_2 in T such that $a = x_1 d, a = dy_1, d = x_2 b, d = by_2$. It follows that $x = x_1 x_2$ and $y = y_1 y_2$ are solutions of the equations $a = xb$, $a = by$ in T. Therefore, T is a group and hence S is a Smarandache semigroup.

§4. Examples

In this section we provide examples to justify our divisibility test 1, divisibility test 2 for Smarandache semigroups, and the characterization of Smarandache semigroups as well. Further, we give examples to show that the conditions stated in Theorem 3.1 and Theorem 3.2 are sufficient conditions but not necessary conditions.

Example 4.1. Let $z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a semigroup under multiplication mod 8. The composition table is as follows:

X_8	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Table 2.

In view of (2.2.1.) the elements 1 and 7 in z_8 are both left and right divisors of every element of the semigroup z_8. So, in view of the Theorem 3.1, the proper subset $G = \{1, 7\}$ is a group under the multiplication mod 8. Hence, the semigroup (z_8, x_8) is a Smarandache semigroup.

Example 4.2. Let $z_7 = \{0, 1, 2, 3, 4, 5, 6, \}$ be a semigroup under multiplication mod 7. The composition table is as follows:

X_7	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Table 3.

In view of (2.2.1.) the elements $1, 2, 3, 4, 5$ and 6 in z_7 are both left and right divisors of every element of the semigroup z_7. So, in view of the Theorem 3.1, the proper subset $G = \{1, 2, 3, 4, 5, 6\}$ is a group under the multiplication mod 7. Hence, the semigroup (z_7, x_7) is a Smarandache semigroup.

Example 4.3. Let $z_4 = \{0, 1, 2, 3\}$ be a semigroup under the multiplication mod 4. The composition table is as follows:

X_4	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Table 4.

In view of (2.2.1) the elements 1 and 3 in z_4 are both left and right divisors of every element of the semigroup z_4. So, in view of the Theorem 3.1, the proper subset $G = \{1, 3\}$ is a group under the multiplication mod 4. Hence, the semigroup (z_4, x_4) is a Smarandache semigroup.

Example 4.4. Let $S = \{e, a, b, c\}$ be a semigroup under the operation defined by the following composition table.

	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	c	b
c	c	c	b	c

Table 5.

In view of (2.2.2.) the elements b and c are divisible both on the right and on the left by every element of the semigroup S. In view of the Theorm 3.2, the proper subset $G = \{b, c\}$ of S is a group under the operation on S. Hence, S is a Smarandache semigroup.

Example 4.5. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be a semigroup under multiplication mod 9. Consider the proper subset $T = \{1, 2, 4, 5, 7, 8\}$ of S. The composition table on T of the multiplication mod 9 is as follows:

X_9	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

Table 6.

From table 6 it is evident that T is a subsemigroup possessing the properties postulated in the hypothesis of the Theorem 3.3. Therefore, T is a group and hence S is a Smarandache semigroup. Finally, we show by providing examples that the conditions stated in Theorem 3.1 and Theorem 3.2 are sufficient but not necessary.

Example 4.6. Let $S = \{1, 2, 3, 4, 5, 6\}$ be a semigroup under the operation defined by xy = the great common divisor of x, y for all $x, y \in S$. The composition table is as follows:

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

Table 7.

This semigroup is not satisfying the conditions stated in Theorem 3.1 but this semigroup S is Smarandache semigroup. (see [1])

Example 4.7. Let us consider the Example 2.6. From the Table 1 it is evident that the semigroup S in the Example 2.6 is not satisfying the conditions stated in Theorem 3.2, but this semigroup S is Smarandahce semigroup.

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On isomorphisms of binary algebras

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Abstract In this paper, we study homomorphisms of binary algebras and investigate its properties. Moreover, some consequences of the relations between quotient binary algebras and isomorphisms are shown.

Keywords Homomorphism, isomorphism, ideal, congruence, binary algebras.

§1. Introduction

In [1], S. Asawasamrit and U. Leerawat introduced a new algebraic structure which is called binary algebras. And we described the relation between ideals and congruences. Furthermore, we define quotient binary algebra and study its properties.

In this paper, we gave the concept of homomorphisms of binary algebras and investigated some related properties. The purpose of this paper is to derive some straightforward consequences of the relations between quotient binary algebras and isomorphisms and also investigate some of its properties.

§2. Preliminaries

In this section we introduced an algebraic structure called a *binary algebra* which is an algebra $(X, *, 0)$ with a binary operation $*$ and a nullary operation 0 such that for all $x, y, z \in X$, satisfies the following properties:

(B-1) $((x * y) * (x * z)) * (z * y) = 0$;

(B-2) $x * x = 0$;

(B-3) $x * 0 = x$;

(B-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

(B-5) $(x * y) * z = (x * z) * y$;

It is easy to show that the following properties are true for a binary algebra. For all x, y, z in X :

(1) $(x * (x * y)) * y = 0$;

(2) $((x * z) * (y * z)) * (x * y) = 0$;

(3) $x * y = 0$ imply $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$.

A subset A of a binary algebra X is called *closed* of X if $x * y \in A$ whenever $x, y \in A$. A non-empty subset A of a binary algebra X is called an *ideal* of X if it satisfies the following conditions:

(I-1) $0 \in A$.

(I-2) for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply $x \in A$.

On binary algebra $(X, *, 0)$. We define a binary relation $\leq$ on G by putting $x \leq y$ if and only if $x * y = 0$. Let I be an ideal of binary algebra X. Define the relation $\sim$ on X by $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then the relation $\sim$ is a congruence relation on X. And $[0]_I = \{x \in G \mid x \sim 0\}$ is an ideal of G.

Let $\sim$ be a congruence relation on a binary algebra X and let I be an ideal of X. Define $[x]_I$ by $[x]_I = \{y \in X \mid x \sim y\} = \{y \in X \mid x * y \in I, y * x \in I\}$. Then the family $\{[x]_I : x \in X\}$ gives a partition of X which is denoted by X/I. For any $x, y \in X$, we define $[x]_I \circ [y]_I = [x * y]_I$. Since $\sim$ has the substitution property, the operation $*$ is well-defined. It is easily checked that $(X/I, \circ, [0]_I)$ is a binary algebra. Moreover, the set X/I is called the *quotient binary algebra.*

If I is a closed ideal of binary algebra X, then it is clear that $[a]_I = I$ for all a in I.

Let $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ be binary algebras. A *(binary algebras) homomorphism* is a map $f : X \to Y$ satisfying $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X$. An injective homomorphism is called *monomorphism*, a surjective homomorphism is called *epimorphism* and a bijective homomorphism is called *isomorphism.* The *kernel* of the homomorphism f, denoted by Ker f, is the set of elements of X that map to 0_Y in Y.

§3. Results

In this section, we describe properties of binary algebra homomorphism .

Definition 3.1. Let f be a mapping of a binary algebra X into a binary algebra Y, and let $I \subseteq X$ and $A \subseteq Y$. The image of I in X under f is

$$f(I) = \{f(x) \mid x \in I\}$$

and the inverse image of A in Y is

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Next, the basic properties of homomorphism are considered as the following theorem.

Theorem 3.2. Let f be a homomorphism of a binary algebras X into a binary algebra Y. Then

(i) $f(0_X) = 0_Y$.

(ii) If 0_X is the identity in X, then $f(0_X)$ is the identity in Y.

(iii) If I is an ideal of X, then $f(I)$ is an ideal of Y.
(iv) If I is a closed of X, then $f(I)$ is a closed of Y.
(v) If A is an ideal in Y, then $f^{-1}(A)$ is an ideal in X.
(vi) If A is a closed of Y, then $f^{-1}(A)$ is a closed of X.
(vii) Ker f is a closed ideal of X.
(viii) f is injective if and only if Ker $f = \{0\}$.

Proof. Assume that $f : X \to Y$ is a homomorphism.

(i) Since $0_X * 0_X = 0_X$, then $f(0_X) = f(0_X * 0_X) = f(0_X) \circ f(0_X) = 0_Y$.

(ii) Assume that 0_X is the identity in X and 0_Y the identity in Y. From (B-3), $f(0_X) \circ 0_Y = 0_Y$ and $0_Y \circ f(0_X) = [f(0_X) \circ f(0_X)] \circ f(0_X) = f(0_X * 0_X) \circ f(0_X) = f(0_X) \circ f(0_X) = 0_Y$. By (B-4), we get that $f(0_X) = 0_Y$.

(iii) Let I be an ideal of X. We see that $0_X \in I$, and by (i), $0_Y = f(0_X) \in f(I)$, so $0_Y \in f(I)$. Now, assume that $f(x) \circ f(y) \in f(I)$ and $f(y) \in f(I)$, then $f(x * y) \in f(I)$ and $f(y) \in f(I)$, so $x * y, y \in I$. Since I is an ideal of X, $x \in I$, it follows that $f(x) \in f(I)$. Hence $f(I)$ is an ideal of Y.

(iv) Let I be a closed of X and $x, y \in f(I)$. That means $x = f(a)$ and $y = f(b)$ for some a, b in I. Then $x \circ y = f(a) \circ f(b) = f(a * b) \in f(I)$. Thus $f(I)$ is a closed of Y.

(v) Let A be an ideal in Y. Then $0_Y \in A$, we get that $0_X = f^{-1}(0_Y) \in f^{-1}(A)$. Now, let $x * y \in f^{-1}(A)$ and $y \in f^{-1}(A)$ for all $x, y \in X$. Then $f(x) \circ f(y) = f(x * y) \in A$ and $f(y) \in A$. Since A is an ideal of Y, we obtain that $f(x) \in A$. Consequently $x \in f^{-1}(A)$, proving that $f^{-1}(A)$ is an ideal of X.

(vi) Let A be a closed of Y and $x, y \in f^{-1}(A)$. Then $f(x) = a$ and $f(y) = b$ for some $a, b \in A$. Thus $f(x * y) = f(x) \circ f(y) = a * b \in A$, because A is a closed. Hence $x * y \in f^{-1}(A)$.

(vii) It is clear that Ker $f \subseteq X$. Since $f(0_X) = 0_Y$, so $0_X \in$ Ker f. It follows that Ker $f \neq \emptyset$. Let $x * y \in$ Ker f and $y \in$ Ker f. We get that $f(x) = f(x) \circ 0_Y = f(x) \circ f(y) = f(x * y) = 0_Y$. Thus $x \in$ Ker f. Now, we will show Ker f is closed of X. Let $x, y \in$ Ker f. Then $f(x * y) = f(x) \circ f(y) = 0_Y \circ 0_Y = 0_Y$, these imply that $x * y \in$ Ker f. Therefore Ker f is a closed ideal of X.

(viii) Suppose that f is injective and $x \in$ Ker f. Then $f(x) = 0_Y$. Since $f(0_X) = 0_Y$, so $f(x) = f(0_X)$. By assumption, 0_X. Thus Ker $f = 0_X$.

Conversely, suppose that Ker $f = 0_X$. Let $x, y \in X$ be such that $f(x) = f(y)$. Then we get that $f(x * y) = f(x) \circ f(y) = 0_Y$ and $f(y * x) = f(y) \circ f(x) = 0_Y$. Thus $x * y, y * x \in$ Ker f, this means that $x * y = 0_X = y * x$. From B-4, $x = y$, and shows that f is injective.

Theorem 3.3. Let I be a closed ideal of binary algebra X. Defined the map $f : X \to X/I$ by $f(x) = [x]_I$, for all $x \in X$. Then f is epimorphism, we call f is the *natural homomorphism* of X onto X/I. Furthermore, Ker $f = I$

Proof. Let I be a closed ideal of binary algebra X and $x, y \in X$. We have that $f(x * y) = [x * y]_I = [x]_I \circ [y]_I = f(x) \circ f(y)$, proving that f is a homomorphism. Next we will show f is surjective, let $[x]_I \in X/I$ and $x \in X$. Then $f(x) = [x]_I$, so f is surjective. Finally, to show that Ker $f = I$, let $x \in$ Ker f. We get that $[x]_I = f(x) = [0]_I$, then $x \sim 0$. It follows that $x * 0 \in I$ and $0 * x \in I$. By hypothesis, $0 \in I$. Hence, $x \in I$, this mean Ker $f \subseteq I$. To show that $I \subseteq$ Ker f, choose $x \in I$. Since I is a closed ideal of X, we have $0 \in I$. Thus $x * 0 \in I$

and $0 * I \in I$. It follows that $x \sim 0$, so $[x]_I = [0]_I$. Since $f(x) = [x]_I = [0]_I$, then $x \in \text{Ker } f$. Accordingly, $\text{Ker } f = I$.

Theorem 3.4. Let f be a homomorphism of a binary algebra $(X, *, 0_X)$ onto a binary algebra $(Y, \cdot, 0_Y)$ and I be an ideal of X contain in Ker f. Let g be the natural homomorphism of X onto X/I then there exists a unique homomorphism h of X/I onto Y such that $f = hog$. Furthermore, h is an injective if and only if $I = \text{Ker } f$.

Proof. Define the map $h : X/I \to Y$ by $h([a]_I) = f(a)$ for all $[a]_I \in X/I$.

We first show that, h is well-defined, let $[a]_I, [b]_I \in X/I$ be such that $[a]_I = [b]_I$. We get that $a \sim b$, so $a * b \in I$ and $b * a \in I$. Since $I \subseteq \text{Ker } f$, $a * b \in \text{Ker } f$ and $b * a \in \text{Ker } f$. Thus $f(a) \cdot f(b) = f(a * b) = 0_Y$ and $f(b) \cdot f(a) = f(b * a) = 0_Y$. From B-4, $f(a) = f(b)$. Hence h is well-defined.

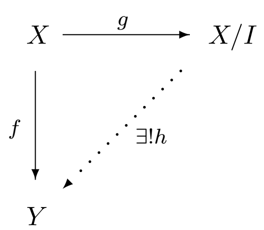
We will show that h is homomorphism. Let $[a]_I, [b]_I \in X/I$. Then $h([a]_I \circ [b]_I) = h([a*b]_I) = f(a * b) = f(a) \cdot f(b) = h([a]_I) \cdot h([b]_I)$.

Next, to show that $f = hog$, let $a \in X$. Then $(hog)(a) = h(g(a)) = h([a]_I) = f(a)$. Hence $hog = f$.

Finally, if $h' : X/I \to Y$ is another function such that $f = h'og$. Let $[a]_I \in X/I$. The equation $h([a]_I) = f(a) = (h'og)(a) = h'(g(a)) = h'([a]_I)$. Thus $h([a]_I) = h'([a]_I)$, for all $[a]_I \in X/I$. This completes the proof.

Now, we will show that h is injective if and only if $I = \text{Ker } f$. Suppose firstly that h is injective and $a \in \text{Ker } f$. Then $h([0_X]_I) = 0_Y = f(a) = h([a]_I)$ and since h is an injective, $[0_X]_I = [a]_I$. It follows that $0_X \sim a$, then $0_X * a \in I$ and $a * 0_X \in I$. By hypothesis, $0_X \in I$. Hence, $a \in I$, this mean $\text{Ker } f \subseteq I$. This show that $\text{Ker } f = I$.

On the other hand, suppose that $\text{Ker } f = I$ and $[a]_I, [b]_I \in X/I$ such that $h([a]_I) = h([b]_I)$. Then $f(a) = f(b)$, it follows that $f(a*b) = f(a) \cdot f(b) = 0_Y$. Thus $a*b \in \text{Ker } f$. Since $Kerf = I$, so $a * b \in I$. Similarly, $b * a \in I$. Hence $a \sim b$, proving that $[a]_I = [b]_I$. This show that h is injective. This completes the proof. And we can describe Theorem 3.4. by the following figure:



Next, we state the first isomorphism of binary algebras as the following theorem:

Theorem 3.5. (First Isomorphism Theorem)

If f be a homomorphism of a binary algebra $(X, *, 0_X)$ into a binary algebra $(Y, \cdot, 0_Y)$, then the quotient binary algebra $X/\text{Ker } (\phi)$ is isomorphic to $\phi(X)$.

Proof. Let $\phi : X \to Y$ be a homomorphism and let $K = \text{Ker}(\phi) = \{a \in X \mid \phi(a) = 0_Y\}$. We get that $X/K = \{[a]_K | a \in X\}$, where $[a]_K = \{b \in X \mid a \sim b\}$. From theorem 3.2(vii), we have $\text{Ker}(\phi)$ is an ideal of X. And since [1, Theorem 3.7], $(X/K, \circ, [0]_K)$ is a binary algebra and $\phi(X) = \{\phi(a) \mid a \in X\}$.

Assume that $f : X/K \to \phi(X)$ defined by $f([a]_K) = \phi(a)$, where $[a]_K \in X/K$.

Let $[a]_K, [b]_K \in X/K$ be such that $[a]_K = [b]_K$. Then $a \sim b$, it follows that $a * b \in K$ and $b * a \in K$. Thus $\phi(a) \cdot \phi(b) = 0_Y = \phi(b) \cdot \phi(a)$. By (B-4), we get that $\phi(a) = \phi(b)$. Hence f is well-defined.

Let $[a]_K, [b]_K \in X/K$. We get that $f([a]_K \circ [b]_K) = f([a * b]_K) = \phi(a * b) = \phi(a) \cdot \phi(b) = f([a]_K) \cdot f([b]_K)$. This show that f is a homomorphism.

Let $[a]_K, [b]_K \in X/K$ be such that $f([a]_K) = f([b]_K)$. Then $\phi(a) = \phi(b)$, it follows that $\phi(a * b) = \phi(a) \cdot \phi(b) = 0_Y$. Thus $a * b \in \mathrm{Ker}(\phi) = K$. Similarly, $b * a \in K$. We see that $a \sim b$, this mean $[a]_K = [b]_K$. Hence f is an injective.

Let $a \in \phi(X)$. Then there exists $b \in X$ such that $a = \phi(b)$ and $[b]_K \in X/K$. Thus $f([b]_K) = \phi(b) = a$. Therefore f is a surjective, proving our theorem.

Theorem 3.6. (Second Isomorphism Theorem)

Let X be a binary algebra. Let A, B be ideals of X. If $A \cup B$ is a binary algebra, then the quotient binary algebras $(A \cup B)/B$ and $A/(A \cap B)$ are isomorphic.

Proof. Let $\phi : A \to (A \cup B)/B$ defined by $\phi(x) = [x]_B$ for all $x \in A$. It is obvious that ϕ is well defined. Let $[x]_B \in (A \cup B)/B$. If $x \in A$, then $[x]_B = \phi(x)$. If $x \in B$, then $[x]_B = [0]_B = \phi(0)$. Thus ϕ is onto $(A \cup B)/B$. Consider the equation

$$\begin{aligned} \phi(x * y) &= [x * y]_B \\ &= [x]_B \circ [y]_B \\ &= \phi(x) \circ \phi(y) \end{aligned}$$

shows that ϕ is a homomorphism.

Now let $x \in \mathrm{Ker}(\phi)$. Then we get $\phi(x) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Since $\mathrm{Ker}(\phi) \subseteq A$, so $x \in A \cap B$. Hence $\mathrm{Ker}(\phi) \subseteq A \cap B$. On the other hand, let $x \in A \cap B$. Then $x \in B$. Thus $\phi(x) = [x]_B = [0]_B$, so $x \in \mathrm{Ker}\ (\phi)$. Hence $A \cap B \subseteq \mathrm{Ker}\ (\phi)$. Therefore, $\mathrm{Ker}(\phi) = A \cap B$. Theorem 3.5. immediately gives us that $(A \cup B)/B \cong A/(A \cap B)$.

Next, we state the third isomorphism theorem of binary algebras.

Theorem 3.7. (Third Isomorphism Theorem)

Let X be a binary algebra. Let A and B be ideals of X, with $A \subseteq B \subseteq X$. Then

(i) the quotient B/A is an ideal of the quotient X/A, and

(ii) the quotient KU-algebra $(X/A)/(B/A)$ is isomorphic to X/B.

Proof.

(i) It is clear that $B/A \subseteq X/A$ and $[0]_A \in B/A$. Let $[x]_A \circ [y]_A \in B/A$ and $[y]_A \in B/A$. Then $x * y \in B$ and $y \in B$. Since B is an ideal of X, $x \in B$, so $[x]_A \in B/A$. Therefore, B/A is an ideal of X/A.

(ii) Let $\phi : X/A \to X/B$ defined by $\phi([x]_A) = [x]_B$. Assume that $[x]_A = [y]_A$. Then $x \sim y$ determined by A, that is $x * y, y * x \in A$. Since $A \subseteq B$, $x * y, y * x \in B$. Thus $x \sim y$ determined by B, and hence $[x]_B = [y]_B$. Then $\phi([x]_A) = \phi([y]_A)$. Therefore, ϕ is well defined. Next, to show that ϕ is onto X/B, let $[x]_B \in X/B$. If $x \in X$ and $x \notin B$, then $[x]_B = \phi([x]_A)$. If $x \in B$, then $[x]_B = [0]_B = \phi([0]_B)$. Hence ϕ is onto. Consider the equation

$$\begin{aligned} \phi([x]_A \circ [y]_A) &= \phi([x * y]_A) \\ &= [x * y]_B \\ &= [x]_B \circ [y]_B \\ &= \phi([x]_A) \circ \phi([y]_A) \end{aligned}$$

shows that ϕ is a homomorphism.

Finally, to show that $\text{Ker}(\phi) = B/A$, let $[x]_A \in \text{Ker}(\phi)$. Then $\phi([x]_A) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Now we have $[x]_A \in B/A$. Hence $\text{Ker}(\phi) \subseteq B/A$. Going the other hand, let $[x]_A \in B/A$. We get that $\phi([x]_A) = [x]_B = [0]_B$, since $x \in B$. Thus $[x]_A \in \text{Ker}\phi)$, and hence $A/B \subseteq \text{Ker}(\phi)$. Consequently, $\text{Ker}(\phi) = B/A$. By Theorem 3.5, $(X/A)/(B/A)$ is isomorphic to X/B.

It turns out that an analogous result of the third isomorphism theorem for groups is also true for binary algebras.

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Two accelerated approximations to the Euler-Mascheroni constant [1]

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Abstract Let $\gamma = 0.577215\ldots$ be the Euler-Mascheroni constant, and let $P_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2+n+\frac{1}{3}\right)$ and $Q_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{4}\ln\left[\left(n^2+n+\frac{1}{3}\right)^2 - \frac{1}{45}\right]$. We prove that for all integers $n \geq 1$,

$$\frac{1}{180(n+1)^4} < \gamma - P_n < \frac{1}{180n^4} \quad \text{and} \quad \frac{8}{2835(n+1)^6} < Q_n - \gamma < \frac{8}{2835n^6}.$$

This provides the higher order estimate for the Euler-Mascheroni constant.

Keywords Euler-Mascheroni constant, harmonic numbers, inequality, psi function, asymptotic formula.

§1. Introduction and preliminaries

The Euler-Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = H_n - \ln n, \tag{1}$$

where H_n denotes the nth harmonic number, defined for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Several bounds for $D_n - \gamma$ have been given in the literature. We recall some of them:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)} \quad \text{for} \quad n \geq 2 \quad ([15]);$$

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad \text{for} \quad n \geq 1 \quad ([13,\ 20]);$$

$$\frac{1-\gamma}{n} \leq D_n - \gamma < \frac{1}{2n} \quad \text{for} \quad n \geq 1 \quad ([4]);$$

$$\frac{1}{2n+\frac{2}{5}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for} \quad n \geq 1 \quad ([16,\ 17]);$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} \leq D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for} \quad n \geq 1 \quad ([2,\ 7,\ 16,\ 17]).$$

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The convergence of the sequence D_n to γ is very slow. The next step in the study of the convergence speed is to find other sequences which converge faster to γ. One method is to change the logarithmic term in (1). In 1993, DeTemple [11] studied the sequence

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right),$$

and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{2}$$

Recently, Chen [8] obtained a sharp form of the inequality (2) as follows: For all integers $n \geq 1$, then

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2} \tag{3}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

The second inequality with $b = \frac{1}{2}$ in (3) comes out from what DeTemple [11] wrote on page 470 of the article. Sîntămărian [14] gave results for a generalization of the Euler-Mascheroni constant and taken $a = 1$ in [14, Theorem 3.1, part (iii)], we obtain the second inequality with $b = \frac{1}{2}$ in (3). Chen [9] proved the second inequality in (3). Moreover, the author showed that $b = \frac{1}{2}$ is the best possible.

In 1997, Negoi [12] proved that the sequence

$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right)$$

is strictly increasing and convergent to γ. Moreover,

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \tag{4}$$

Later, Vernescu [18] have found a fast convergent sequence to γ, by having the idea to replace the last term of the harmonic sum. He proved that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n$$

is strictly increasing and convergent to γ. Moreover,

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}. \tag{5}$$

Recently, Chen [10] obtained a sharp form of the inequality (5) as follows: For all integers $n \geq 1$, then

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n < \frac{1}{12(n+b)^2} \tag{6}$$

with the best possible constants

$$a = \frac{1}{\sqrt{12\gamma - 6}} - 1 = 0.038859\ldots \quad \text{and} \quad b = 0.$$

Cesàro [6] proved that for every positive integer $n \geq 1$, there exists a number $c_n \in (0,1)$ such that the following approximation is valid:

$$\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2+n\right) - \gamma = \frac{c_n}{6n(n+1)}.$$

Entry 9 of Chapter 38 of Berndt's edition of Ramanujan's Notebooks [5, p. 521] reads,

"Let $m := \frac{n(n+1)}{2}$, where n is a positive integer. Then, as n approaches infinity,

$$\sum_{k=1}^{\infty} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5}$$
$$- \frac{191}{360360m^6} + \frac{1}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - [\cdots]."$$

For the history and the development of Ramanujan's formula, see [19].

For $n \in \mathbb{N}$, let

$$P_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2+n+\frac{1}{3}\right) \tag{7}$$

and

$$Q_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{4}\ln\left[\left(n^2+n+\frac{1}{3}\right)^2 - \frac{1}{45}\right]. \tag{8}$$

In this paper, we establish the bounds for $\gamma - P_n$ and $Q_n - \gamma$.

Before we state and prove the main theorem, we need the following preliminary results.

The constant γ is deeply related to the gamma function $\Gamma(z)$ thanks to the Weierstrass formula:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-1} e^{z/k}\right\} \quad (z \in \mathbb{C} \setminus Z_0^-; Z_0^- := \{-1,-2,-3,\ldots\}).$$

The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)dt$$

is known as the psi (or digamma) function. The following recurrence and asymptotic formulas are well known for the psi function:

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{9}$$

(see [1, p. 258], and

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \ldots \quad (z \to \infty \,\text{in}\, |argz| < \pi) \tag{10}$$

(see [1, p. 259]. From (9) and (10), we get

$$\psi(n+1) \sim \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \ldots \quad (n \to \infty). \tag{11}$$

It is also well-known [1, p. 258] that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{12}$$

Alzer [3] proved that let $k \geq 1$ and $n \geq 0$ be integers, then for all real numbers $x > 0$:

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{13}$$

where

$$S_k(p; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} \left[B_{2i} \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i+k}},$$

B_i $(i = 0, 1, 2, \dots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}.$$

In particular, it follows from (13) that

$$\begin{aligned} \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} &< \psi'(x) \\ &< \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} &< \psi'(x+1) \\ &< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \end{aligned} \tag{14}$$

§2. Bounds for $P_n - \gamma$

By (12), we obtain

$$P_n - \gamma = \psi(n+1) - \ln n - \frac{1}{2} \ln \left(1 + \frac{1}{n} + \frac{1}{3n^2} \right). \tag{15}$$

It is easy to see that for $n \geq 2$,

$$\begin{aligned} \ln \left(1 + \frac{1}{n} + \frac{1}{3n^2} \right) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n} + \frac{1}{3n^2} \right)^k \\ &= \frac{1}{n} - \frac{1}{6n^2} + \frac{1}{36n^4} - \frac{1}{45n^5} + \frac{1}{81n^6} - \frac{1}{189n^7} + O\left(\frac{1}{n^8} \right). \end{aligned} \tag{16}$$

Upon substituting from (11) and (16) into (15), we obtain the asymptotic formula:

$$P_n - \gamma \sim -\frac{1}{180n^4} + \frac{1}{90n^5} - \frac{23}{2268n^6} + O\left(\frac{1}{n^7} \right) \quad (n \to \infty). \tag{17}$$

Motivated by (17), we establish the higher order estimate below.

Theorem 1. Let P_n be defined by (7). Then

$$\frac{1}{180(n+1)^4} < \gamma - P_n < \frac{1}{180n^4} \quad (n \in \mathbb{N}). \tag{18}$$

Proof. The upper and lower bounds are both obtained by considering the function f defined for $x \geq 0$ by

$$f(x) = \frac{1}{x+1} + \frac{1}{2}\log\left(x^2 + x + \frac{1}{3}\right) - \frac{1}{2}\log\left((x+1)^2 + x + \frac{4}{3}\right).$$

Direct calculations show that

$$P_{n+1} - P_n = f(n)$$

and

$$-f'(x) = \frac{1}{(x+1)^2(3x^2+3x+1)(3x^2+9x+7)}.$$

To derive the upper bound, we first observe that

$$\begin{aligned}
&\frac{1}{(x+1)^2(3x^2+3x+1)(3x^2+9x+7)} - \frac{1}{9(x+\frac{1}{2})^6} \\
&= -\frac{1728x^5 + 6288x^4 + 9312x^3 + 7012x^2 + 2708x + 439}{9(x+1)^2(3x^2+3x+1)(3x^2+9x+7)(2x+1)^6} < 0,
\end{aligned}$$

from which it follows that

$$-f'(x) < \frac{1}{9(x+\frac{1}{2})^6}.$$

Therefore, since $f(\infty) = 0$,

$$f(k) = -\int_k^\infty f'(x)dx < \frac{1}{9}\int_k^\infty \frac{1}{(x+\frac{1}{2})^6}dx = \frac{1}{45(k+\frac{1}{2})^5}.$$

Since

$$\begin{aligned}
&\frac{1}{4}\left(\frac{1}{k^4} - \frac{1}{(k+1)^4}\right) - \frac{1}{(k+\frac{1}{2})^5} \\
&= \frac{160k^6 + 480k^5 + 552k^4 + 304k^3 + 86k^2 + 14k + 1}{4k^4(k+1)^4(2k+1)^5} > 0,
\end{aligned}$$

it follows that

$$\frac{1}{(k+\frac{1}{2})^5} < \frac{1}{4}\left(\frac{1}{k^4} - \frac{1}{(k+1)^4}\right) = \int_k^{k+1} \frac{1}{x^5}dx.$$

Altogether then,

$$\begin{aligned}
\gamma - P_n &= \sum_{k=n}^\infty (P_{k+1} - P_k) = \sum_{k=n}^\infty f(k) \\
&< \frac{1}{45}\sum_{k=n}^\infty \frac{1}{(k+\frac{1}{2})^5} < \frac{1}{45}\int_n^\infty \frac{1}{x^5}dx = \frac{1}{180n^4}.
\end{aligned}$$

To derive the lower bound, we require the inequality

$$\begin{aligned}&\frac{1}{(x+1)^2(3x^2+3x+1)(3x^2+9x+7)}-\frac{1}{9(x+1)^6}\\ =\ &\frac{3x^2+6x+2}{9(x+1)^6(3x^2+3x+1)(3x^2+9x+7)}>0,\end{aligned}$$

from which it follows that

$$-f'(x)>\frac{1}{9(x+1)^6}.$$

Proceeding as before, we find that

$$f(k)>\frac{1}{9}\int_k^\infty\frac{1}{(x+1)^6}dx=\frac{1}{45(k+1)^5},$$

and then obtain

$$\gamma-P_n>\frac{1}{45}\sum_{k=n}^\infty\frac{1}{(k+1)^5}>\frac{1}{45}\int_{n+1}^\infty\frac{1}{x^5}dx=\frac{1}{180(n+1)^4}.$$

This completes the proof of Theorem 1.

§3. Bounds for $Q_n-\gamma$

By (12), we obtain

$$Q_n-\gamma=\psi(n+1)-\frac{1}{4}\ln\left[\left(n^2+n+\frac{1}{3}\right)^2-\frac{1}{45}\right]. \tag{19}$$

It is easy to see that for $n\geq 3$,

$$\begin{aligned}&\frac{1}{4}\ln\left[\left(n^2+n+\frac{1}{3}\right)^2-\frac{1}{45}\right]\\ &=\frac{1}{4}\ln\left(n^4+2n^3+\frac{5n^2}{3}+\frac{2n}{3}+\frac{4}{45}\right)\\ &=\ln n+\frac{1}{4}\ln\left(1+\frac{2}{n}+\frac{5}{3n^2}+\frac{2}{3n^3}+\frac{4}{45n^4}\right)\\ &=\ln n+\frac{1}{4}\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k}\left(\frac{2}{n}+\frac{5}{3n^2}+\frac{2}{3n^3}+\frac{4}{45n^4}\right)^k\\ &=\ln n+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{11}{1620n^6}+\frac{8}{945n^7}-\frac{217}{32400n^8}+O\left(\frac{1}{n^9}\right).\end{aligned} \tag{20}$$

Upon substituting from (11) and (20) into (19), we obtain the asymptotic formula:

$$Q_n-\gamma\sim\frac{8}{2835n^6}+O\left(\frac{1}{n^7}\right)\quad(n\to\infty). \tag{21}$$

Motivated by (21), we establish the higher order estimate below.

Theorem 2. Let Q_n be defined by (8). Then

$$\frac{8}{2835(n+1)^6} < \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{4} \ln \left[\left(n^2 + n + \frac{1}{3} \right)^2 - \frac{1}{45} \right] - \gamma < \frac{8}{2835n^6} \quad (n \in \mathbb{N}). \tag{22}$$

Proof. The lower bound is obtained by considering the function F defined for $x \geq 0$ by

$$F(x) = \psi(x+1) - \frac{1}{4} \ln \left[\left(x^2 + x + \frac{1}{3} \right)^2 - \frac{1}{45} \right] - \frac{8}{2835(x+1)^6}.$$

Differentiation and applying the right-hand inequality of (14) yields

$$\begin{aligned}
F'(x) &= \psi'(x+1) - \frac{1}{2} \cdot \frac{(x^2+x+\frac{1}{3})(2x+1)}{(x^2+x+\frac{1}{3})^2 - \frac{1}{45}} + \frac{16}{945(x+1)^7} \\
&< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{2} \cdot \frac{(x^2+x+\frac{1}{3})(2x+1)}{(x^2+x+\frac{1}{3})^2 - \frac{1}{45}} + \frac{16}{945(x+1)^7} \\
&= -\left[\frac{3906458548 + 12062825974(x-4) + 15321695883(x-4)^2}{1890x^7(45x^4+90x^3+75x^2+30x+4)(x+1)^7} \right. \\
&\qquad + \frac{10943775905(x-4)^3 + 4957139255(x-4)^4 + 1503157257(x-4)^5}{1890x^7(45x^4+90x^3+75x^2+30x+4)(x+1)^7} \\
&\qquad\quad + \frac{310973761(x-4)^6 + 43522339(x-4)^7 + 3955161(x-4)^8}{1890x^7(45x^4+90x^3+75x^2+30x+4)(x+1)^7} \\
&\qquad\qquad \left. + \frac{211197(x-4)^9 + 5040(x-4)^{10}}{1890x^7(45x^4+90x^3+75x^2+30x+4)(x+1)^7} \right] \\
&< 0 \text{ for } x \geq 4.
\end{aligned}$$

For $n = 1, 2, 3, 4$, we compute $F(n)$ directly:

$$F(1) = 0.000113809\ldots, \quad F(2) = 0.000005661\ldots,$$
$$F(3) = 0.000000693\ldots, \quad F(4) = 0.000000138\ldots.$$

Hence, the sequence

$$F(n) = \psi(n+1) - \frac{1}{4} \ln \left[\left(n^2 + n + \frac{1}{3} \right)^2 - \frac{1}{45} \right] - \frac{8}{2835(n+1)^6}$$

is strictly decreasing for all integers $n \geq 1$. This leads to

$$F(n) > \lim_{n \to \infty} F(n) = 0,$$

by using the asymptotic formula (21). Hence, the left-hand side of inequality (22) is valid for $n \in \mathbb{N}$.

The upper bound is obtained by considering the function G defined for $x \geq 0$ by

$$G(x) = \psi(x+1) - \frac{1}{4} \ln \left[\left(x^2 + x + \frac{1}{3} \right)^2 - \frac{1}{45} \right] - \frac{8}{2835x^6}.$$

Differentiation and applying the left-hand inequality of (14) yields

$$\begin{aligned}G'(x) &= \psi'(x+1) - \frac{1}{2}\cdot\frac{(x^2+x+\frac{1}{3})(2x+1)}{(x^2+x+\frac{1}{3})^2-\frac{1}{45}} + \frac{16}{945x^7}\\ &> \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} - \frac{1}{2}\cdot\frac{(x^2+x+\frac{1}{3})(2x+1)}{(x^2+x+\frac{1}{3})^2-\frac{1}{45}} + \frac{16}{945x^7}\\ &= \frac{720x^5+384x^4-480x^3-631x^2-270x-36}{270x^9(45x^4+90x^3+75x^2+30x+4)}\\ &= \frac{22244+61334(x-2)+63305(x-2)^2+31392(x-2)^3+7584(x-2)^4}{270x^9(45x^4+90x^3+75x^2+30x+4)}\\ &\quad + \frac{720(x-2)^5}{270x^9(45x^4+90x^3+75x^2+30x+4)}\\ &> 0 \text{ for } x \geq 2.\end{aligned}$$

For $n = 1, 2$, we compute $G(n)$ directly:

$$G(1) = -0.002663968\ldots, \quad G(2) = -0.000034559\ldots.$$

Hence, the sequence

$$G(n) = \psi(n+1) - \frac{1}{4}\ln\left[\left(n^2+n+\frac{1}{3}\right)^2 - \frac{1}{45}\right] - \frac{8}{2835n^6}$$

is strictly increasing for all integers $n \geq 1$. This leads to

$$G(n) < \lim_{n\to\infty} G(n) = 0,$$

by using the asymptotic formula (21). Hence, the right-hand side of inequality (22) is valid for $n \in \mathbb{N}$.

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Character graph in Brauer graph's model

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Abstract In this paper I demonstrate a new and original way to apply the representation theory of finite groups to the development of a new graph called the inducted character graph. I conclude a summary of the relevant method for finding a graph and its properties were studied till signed graph. A new finite simple graph $\Gamma(G, H)$ is called the Relative Character Graph of a finite group G relative to a subgroup H (RC - graph, in short). (See [3], [4], [6], [7], [8]). The vertices of $\Gamma(G, H)$ are the complex irreducible characters of G and two vertices Φ and ψ, are adjacent if and only if their restrictions Φ_H and ψ_H contain at least one irreducible character θ of H in common. They also obtained many interesting results giving a criterion for connectivity, tree problem, triangulation.

Keywords Character graph, Brauer graph.

§1. Introduction

The representation theory of finite groups has found relevance in many applications especially in wireless communications and in multiple antenna system. The theory has relevance crystals in chemistry to diffraction patterns in physics.

In the early 40's, R. Brauer while studying the ordinary irreducible representation and the -modular irreducible representation of a finite group G, constructed a simple finite graph. Based on that, as a variation, a new character graph is constructed using the vertex set namely the set irr G of complex irreducible characters of G, which is the same for Brauer graph.

Definition 1. The vertices of $I(G, H)$ are the complex irreducible characters of H and two vertices θ_1 and θ_2 are adjacent if and only if the induced characters θ_1^G and θ_2^G contain at least one (complex) irreducible character Φ of G in common.

By the Interwining Number Theorem (see eg. [1]), we easily see that the above definition of adjacency is equivalent to $(\theta_1^G, \theta_2^G) = \sum_{s\in D}(\theta_1^x, \theta_2^x)^x_{H\cap H}$ (D = a set of double coset representatives of H) and $H^x = xHx^{-1}$ positive. Obviously the vertices of $\Gamma(G, H)$ and $I(G, H)$ are different. However due to same nice symmetry and duality properties which exist between induction and restriction such as Frobenius Reciprocity, Mackey's subgroup theorem, Interwining number theorem etc, it is quite interesting to get certain results which are similar to same properties of $\Gamma(G, H)$, but are also different in some other respects. For details of character Theory of Finite Groups, we refer to [5].

§2. Connectivity properties of $I(G, H)$

1. When H is a normal subgroup of G, Gnanaseelan proved that $\Gamma(G, H)$ is disconnected.

2. If H is arbitrary, then $\Gamma(G, H)$ is connected if and only if Core $_G H = (1)$, where Core $_G H$ is the largest normal subgroup of G contain in H. We prove below that the 'topological' property of disconnectedness, when H is normal carries through in our $I(G, H)$ graph also.

Main results

Theorem 1. Let H be a normal subgroup of G. Then $I(G, H)$ is disconnected. Indeed, it has as many components as there are orbits of $IrrH$ under the G-action.

Proof of Theorem 1. Let $g \in G$ and $\theta \in IrrH$. Then one can construct another character $\theta^g \in IrrH$ as follows: $\theta^g(s) = \theta(gsg^{-1})$, $s \in H$. θ^g is called the conjugate θ under g. Collection of distinct characters of the form θ^g as g runs through G is called the orbit of θ under the G - action on, $IrrH$. (We use the celebrated Clifford's theory here, concerning restriction and induction of characters with respect to a normal subgroup H.)

(It is known that the two induced characters) θ^g and $(\theta^g)^G$ are equal. Also by the same theory, if x is a G-irreducible character belonging to θ^G, then the restriction x_H is a sum of all the distinct conjugates of θ under the G-action an $IrrH$ and the multiplicities of all the H - irreducible constituents are equal.

It is now clear that if X and Φ lie in θ^G, then their restrictions to H are of the form $x_H = e_1(\theta_1 + \theta_2 + \cdots + \theta_t)$. $\Phi_H = e_2(\theta_1 + \theta_2 + \cdots + \theta_t)$ where $\theta_1 = \theta$, $\theta_2 = \theta_t$ are the distinct conjugates of θ (members of the orbit of θ) and e_1 and e_2 are positive integers, which may be different. By the same argument if $x \in \theta^G$ and $\psi \in \xi^G$ where ξ is not in the orbit of θ, then x_H and ψ_H do not have any members of the two orbits of θ and ξ are adjacent if and only if they belong to the same G-orbit of $IrrH$.

Hence it immediately follows that the number of connected components of $I(G, H)$ is the same as the number of distinct G-orbits of $IrrH$. In particular, $I(G, H)$ is disconnected.

We would like to obtain a characterization of connectivity of $I(G, H)$ as in the work of Gnanaseelan. The immediate subgroup that appears to be the best candidate is $N_G(H)$, the normalizer of H in G. We propose the following conjecture.

Conjecture 1. $I(G, H)$ is connected if and only if $N_G(H)$ is a proper subgroup of G. (Note that if $N_G(H)$ is the whole of G, then N is normal and by the above theorem, $I(G, H)$ is disconnected.)

The conjecture is well-formulated due to the basic transitivity of induction namely $(\theta_H^K)^G = \theta_H^G$ for $H \subset K \subset G$ and has been verified in several example.

§3. $I(G, H)$ for Frobenius groups

A group G is a Frobenius group if it has a subgroup H with the property that $H \cap H^x = (1)$ for all $x \notin H$, where $H^x = xHx^{-1}$. Just as these groups give very beautiful results in the TC-graph case, the corresponding results in our $I(G, H)$ case is not for behind.

Theorem 2. Let $G = NH$ be a Frobenius group. Then $I(G, H)$ is a complete graph.

Proof of Theorem 2. We recall the Interwining Number Theorem.

If $\theta_1, \theta_2 \in IrrH$, then $(\theta_1^G, \theta_2^G) = \sum_{x \in D} (\theta_1, \theta_2^x)^x_{H \cap H}$, where D is a set of double cosets for H. Since $G = NH$ is a semi direct product with N normal, the elements of N can be taken as D (and here the usual coset theory will do). By definition, $H \cap H^x = (1)$ for any $x \in D$ and hence $(\theta_1, \theta_2^x)_{(1)} = deg\theta_1 deg\theta_2 \neq 0$. Hence any two elements of $IrrH$ are adjacent. This shows that $I(G, H)$ is a complete graph.

Remark 1.

We can use the above proof for all cases of subgroups H where $H \cap H^x = (1)$ for some $x \neq H$. For example, the Dihedral group $D_{2n} = C_n$. H never with H a subgroup of order 2 is not Frobenius, but satisfies the above condition. Also for prime cyclic groups H the same result is true since $H \cap H^x = (1)$ for every $x \neq H$. In all the above cases H should not be normal, which is of course true in the Frobenius and D_{2n} cases.

Just like the path Lemma in the $\Gamma(G, H)$ case, which says that $\Phi, \Psi \in IrrH$ belong to the same connected component if and only if $\Phi \subset \Psi x^s$ for some $s \geq 1$, $x = 1_H$, we can obtain a similar result in the $I(G, H)$ case also which can be proved exactly in the same way as in the RC-graph version.

Theorem 3. θ_1, θ_2 in $IrrH$ belong to the same connected component of $I(G, H)$ if and only if $\theta_1^G \subset \theta_2^G x^s$ for some $s \geq 1$.

Even though $I(G, H)$ and $\Gamma(G, H)$ are completely different graphs, there is at least one instance where both these graphs are one and the same.

Theorem 4.[7] Consider the sequence of subgroups $S_{n-1} \subset S_n \subset S_{n+1}$ where S_n denotes the symmetric group on n letters. Then $\Gamma(S_n, S_{n-1})$ is isomorphic to $I(S_{n+1}, S_n)$.

$I(G, H)$ as signed graphs. Let $\theta_1, \theta_2 \in IrrH$ be adjacent in $I(G, H)$. Then (θ_1^G, θ_2^G) is a positive integer r we introduce a sign for each edge of $I(G, H)$ as follows: The edge θ_1, θ_2 is assigned a + sign if r is an even integer and a - sign if r is an odd integer.

Acharya's Result

Theorem 5.[1] Let K_n denote the complete graph an n vertices.

(i) No signed graph K_p, $p \geq 6$ is graceful

(ii) Every signed graph K_p, $p \leq 3$ is graceful.

(iii) A signed graph K_4 is graceful if and only if the number of negative edges is not 3.

(iv) A signed graph K_5 is graceful if and only if the number of negative edges n is odd, except when $n = 3$ and the 3 negative edges are incident at the same vertex or when $n = 7$ and the 3 +ve edges are incident on the same vertex or $n = 5$.

Definition 5. If an injection f assigns distinct labels to the vertices of a (p, m, n) - signed graph (p = number of vertices, m = number of +ve edges, n = number of -ve edges) S from $\{0, 1, ..., q = m + n\}$ such that when each edge $uv \in E(s)$ is assigned $g_f(uv) = s(uv)|f(u) - f(v)|$, the positive edges receive the labels $\{1, 2, ..., m\}$ and the negative edges receive the labels

$\{-1, -2, ..., -n\}$ such a function f, if it exists, is called a graceful labeling of S. The signed graph admitting such a labeling is called a graceful signed graph.

We have proved that every $I(G, H)$ is a naturally signed graph and also, by our Theorem 2, when $G = NH$ is Frobenius, then $I(G, H)$ is complete.

In other words, $I(G, H)$ is a complete signed graph.

We can now use the above general theorem of Acharya et al for the complete graph $I(G, H)$ and conclude the appropriate gracefulness or otherwise of the corresponding graphs.

We give the following example: Let $G = D_{2m}$ (m odd) $= \{a, b | a^m = 1, b^2 = 1, bab = a^{-1}\}$. Let $H = < b >$. Then it is known that G is Frobenius and hence the graph $I(G, H)$ is complete with just 2 vertices. By Theorem 5, $I(G, H)$ is graceful. Likewise, the alternating group A_4 is Frobenius, with $H =$ cyclic subgroup of order 3. $I(A_4, H)$ is complete signed graph K_3, which must be graceful. Given any prime p, there exists a Frobenius group G of order $p(p-1)$ with the Frobenius complement H, an abelian subgroup of order $p-1$. Hence $|IrrH| = p-1$ and $I(G, H)$ is a complete signed graph with $p-1$ vertices. For $I(G, H)$ is not graceful, according to Theorem 5.

Problem 1. Study gracefulness of arbitrary $I(G, H)$ (which is always a signed graph).

Problem 2. Introduce a suitable sign for the vertices of $I(G, H)$ to make the graph into a (vertex) signed graph.

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On L-fuzzy ω-extremally disconnected spaces

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Abstract In this paper a new class of L-fuzzy topological spaces called L-fuzzy ω-extremally disconnected spaces are introduced. Several characterizations and some interesting properties of these spaces are also given.

Keywords L-fuzzy ω-closed set, L-fuzzy ω-open set, L-fuzzy ω-extremally disconnected space, L-fuzzy ω-continuous map, L-fuzzy ω-irresolute, strong F_σ L-fuzzy ω-continuous map, lower (upper) L-fuzzy ω-continuous map.

§1. Introduction

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh [13]. Fuzzy sets have applications in many fields such as information [9] and control [10]. The theory of fuzzy topological spaces was introduced and developed by Chang [2] and since then various important notions in classical topology have been extended to fuzzy topological spaces. Rodabaugh [6] discussed normality and the L-fuzzy unit interval. He [7] also studied fuzzy addition in the L-fuzzy real line. Hoeche [5] studied the characterizations of L-topologies by L-valued neighbourhoods. An L-fuzzy normal spaces and Tietze extension theorem were discussed by Tomash Kubiak [11]. The concept of ω-open set was studied in [8]. The purpose of this paper is to introduce L-fuzzy ω-closed, L-fuzzy ω-open sets and a new class of L-fuzzy topological spaces called L-fuzzy ω-extremally disconnected space. In this connection several characterizations and some interesting properties are also given.

§2. Preliminaries

Definition [2]. Throughout this paper (L, $\leq$, $'$) stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. Let X be a non-empty set. An L-fuzzy set in X is an element of the set L^X of all functions from X to L.

Definition. The L-fuzzy real line R(L) [3] is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\vee$ { λ(t)/t$\in$R } = 1 and $\wedge$ { λ(t)/t$\in$R } = 0, after the identification of λ, $\mu \in L^R$ iff $\lambda(t -) = \mu(t -)$ and $\lambda(t +) = \mu(t +)$ for all t$\in$R where $\lambda(t -) = \wedge \{ \lambda(s)/s < t \}$ and $\lambda(t +) = \vee \{ \lambda(s)/s > t \}$. The natural L-fuzzy topology on R(L) is generated from the subbases $\{L_t, R_t/t \in R\}$, where $L_t(\lambda) = \lambda(\ t\ -\ )'$ and $R_t(\lambda) = \lambda(t+)$. The L-fuzzy unit interval I(L) [4] is a subset of R(L) such that [λ] $\in$ I [L] if λ(t)=1 for t < 0 and λ(t) = 0 for t > 1. It is equipped with the subspace L-fuzzy topology.

Definition [11]. If A$\in$ L^X is crisp, then (A, T_A) is an L-fuzzy topological space called a crisp subspace of (X, T), where T_A={ U/A|U$\in$T } is called the subspace L-fuzzy topology.

Definition [8]. A subset of a topological space (X, T) is called ω-closed in (X, T) if cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, T). A subset A is called ω-open in (X, T) if its complement, A^C is ω-closed.

Definition [12]. Let (X, T) be any L-fuzzy topological space. (X, T) is called L-fuzzy extremally disconnected if the closure of every L-fuzzy open set is L-fuzzy open.

§3. Characterizations and properties of L-fuzzy ω-extremally disconnected spaces

Definition 3.1. Let λ be any L-fuzzy set in the L-fuzzy topological space (X, T). Then we define

L-int (λ) = $\vee$ { μ / $\mu \leq \lambda$ and μ is L-fuzzy open },

L-cl (λ) = $\wedge$ { μ / $\mu \geq \lambda$ and μ is L-fuzzy closed }.

Definition 3.2. Let λ be any L-fuzzy set in the L-fuzzy topological space (X, T). λ is called L-fuzzy semi-open if $\lambda \leq$ L-cl (L- int (λ)).

Definition 3.3. An L-fuzzy set λ of an L-fuzzy topological space (X, T) is called L-fuzzy ω-closed in (X, T) if L-cl (λ) $\leq \mu$ whenever $\lambda \leq \mu$ and μ is L-fuzzy semi-open in (X, T). The complement of L-fuzzy ω-closed set is L-fuzzy ω-open.

Definition 3.4. Let (X, T) be an L-fuzzy topological space. For any L-fuzzy set λ in (X, T) L-fuzzy ω-closure of λ (briefly, Lω-cl (λ)) is defined as Lω-cl (λ) = $\wedge$ { μ : $\mu \geq \lambda$ and μ is L-fuzzy ω-closed }.

Definition 3.5. Let (X, T) be an L-fuzzy topological space. For any L-fuzzy set λ in (X, T) L-fuzzy ω-interior of λ (briefly, Lω-int (λ)) is defined as Lω-int (λ) = $\vee$ { μ : $\mu \leq \lambda$ and μ is L-fuzzy ω-open }.

Remark 3.1. Let (X, T) be an L-fuzzy topological space. For any L-fuzzy set λ in (X, T)

1. 1 $-$ Lω-int (λ) = Lω-cl (1 $-$ λ).
2. 1 $-$ Lω-cl (λ) = Lω-int (1 $-$ λ).

Definition 3.6. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces. A mapping f : (X, T) $\rightarrow$ (Y, S) is called L-fuzzy ω-continuous if f^{-1}(λ) is L-fuzzy ω-closed in (X, T) for every L-fuzzy closed set λ in (Y, S).

Definition 3.7. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces. A mapping f : (X, T) $\rightarrow$ (Y, S) is called L-fuzzy ω-irresolute if the inverse image of every L-fuzzy ω-open set in (Y, S) is L-fuzzy ω-open in (X, T).

Definition 3.8. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces. A mapping f : (X, T) → (Y, S) is said to be L-fuzzy ω-open if the image of every L-fuzzy ω-open set in (X, T) is L-fuzzy ω-open in (Y, S).

Definition 3.9. Let (X, T) be an L-fuzzy topological space. Let λ be any L-fuzzy ω-open set in (X, T). If Lω-cl (λ) is L-fuzzy ω-open then (X, T) is said to be L-fuzzy ω-extremally disconnected space.

Proposition 3.1. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces. Then f : (X, T) → (Y, S) is L-fuzzy ω-irresolute iff f (Lω-cl (λ)) $\leq$ Lω-cl (f (λ)), for every L-fuzzy set λ in (Y, S).

Proposition 3.2. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces and let f : (X, T) → (Y, S) be an L-fuzzy ω-open surjective function. Then f^{-1}(Lω-cl (λ)) $\leq$ Lω-cl (f^{-1} (λ)), for each L-fuzzy set λ in (Y, S).

Proposition 3.3. For an L-fuzzy topological space (X, T) the following statements are equivalent:

(a) (X, T) is an L-fuzzy ω-extremally disconnected space.

(b) For each L-fuzzy ω-closed set λ, Lω-int (λ) is L-fuzzy ω-closed.

(c) For each L-fuzzy ω-open set λ, Lω-cl (λ) + Lω-cl (1 − Lω-cl (λ)) =1.

(d) For every pair of L-fuzzy ω-open sets λ and μ with Lω-cl (λ) + μ = 1, we have

$$L\omega - cl(\lambda) + L\omega - cl(\mu) = 1.$$

Proof. (a) ⇒ (b) Let λ be any L-fuzzy ω-closed set. Then 1 − λ is L-fuzzy ω-open. Now Lω-cl (1 − λ)= 1 − Lω-int (λ). By (a), Lω-cl (1 − λ) is L-fuzzy ω-open, which implies that Lω-int (λ) is L-fuzzy ω-closed.

(b) ⇒ (c) Let λ be any L-fuzzy ω-open set. Then 1− λ is L-fuzzy ω-closed. By (b), Lω-int (1 − λ) is L-fuzzy ω-closed. Now

$$L\omega - cl(\lambda) + L\omega - cl(1 - L\omega - cl(\lambda)) = L\omega - cl(\lambda) + L\omega - cl(L\omega - \mathrm{int}(1 - \lambda)) \tag{1}$$

Therefore by (1),

$$\begin{aligned} L\omega - cl(\lambda) + L\omega - cl(1 - L\omega - cl(\lambda)) &= L\omega - cl(\lambda) + L\omega - \mathrm{int}(1 - \lambda) \\ &= L\omega - cl(\lambda) + 1 - L\omega - cl(\lambda) \\ &= 1. \end{aligned}$$

Therefore, Lω-cl (λ) + Lω-cl (1 −Lω-cl (λ)) = 1.

(c) ⇒ (d) Let λ and μ be L-fuzzy ω-open sets with

$$L\omega - cl(\lambda) + \mu = 1. \tag{2}$$

Then by (c), 1= Lω-cl (λ) + Lω-cl (1 − Lω-cl (λ))
By (2), 1 − Lω-cl (λ) = μ.
Therefore, Lω-cl (λ) + Lω-cl (μ) =1.

(d) ⇒ (a) Let λ be any L-fuzzy ω-open set. Put μ = 1 − Lω-cl (λ). Then clearly μ is L-fuzzy ω-open and Lω-cl (λ) + μ = 1. Therefore by (d), Lω-cl (λ) + Lω-cl (μ) =1. This implies that Lω-cl (λ) is L-fuzzy ω-open and so (X, T) is L-fuzzy ω-extremally disconnected.

Proposition 3.4. The image (Y, S) of an L-fuzzy ω-extremally disconnected space (X, T) under L-fuzzy ω-irresolute, L-fuzzy ω-open and surjective mapping is also L-fuzzy ω-extremally disconnected.

Proof. The proof follows from the concepts of L-fuzzy ω-irresolute, L-fuzzy ω-open maps and by the Propositions 3.1 and 3.2.

Definition 3.10. Let { (X_α, T_α) / $\alpha\in\Delta$ } be a family of disjoint L-fuzzy topological spaces. Let X = $\underset{\alpha\in\Delta}{U} X_\alpha$. Define T = { $\lambda \in L^X$/ λ /X_α is L-fuzzy ω-open in (X_α, T_α) }. Then (X, T) is an L-fuzzy topological space called the L-fuzzy topological sum of { (X_α, T_α) / $\alpha\in\Delta$ }.

Proposition 3.5. Let { (X_α, T_α)/ $\alpha\in\Delta$ } be a family of disjoint L-fuzzy ω-extremally disconnected spaces and let (X, T) be their L-fuzzy topological sum. Then (X, T) is L-fuzzy ω-extremally disconnected.

Proof. Let λ be an L-fuzzy ω-open F_σ set in (X, T). Then λ/ X_α is L-fuzzy ω-open F_σ in (X_α, T_α). Since (X_α, T_α) is L-fuzzy ω-extremally disconnected Lω-cl_{X_α} (λ/X_α) is L-fuzzy ω-open in (X_α, T_α). Now Lω-cl_X(λ)/X_α = Lω-cl_{X_α}(λ/X_α), which implies that Lω-cl_X(λ) is L-fuzzy ω-open in (X, T).Therefore (X, T) is L-fuzzy ω-extremally disconnected.

Proposition 3.6. Let (X, T) be an L-fuzzy topological space. Then (X, T) is L-fuzzy ω-extremally disconnected iff for all L-fuzzy ω-open set λ and an L-fuzzy ω-closed set μ such that $\lambda \leq \mu$, Lω-cl (λ) $\leq$ Lω-int(μ).

Proof. Let (X, T) be L-fuzzy extremally disconnected. Let λ be L-fuzzy ω-open and μ be L-fuzzy ω-closed with $\lambda \leq \mu$. Then by (b) of Proposition 3.3, Lω-int (μ) is L-fuzzy ω-closed. Also since λ is L-fuzzy ω-open and $\lambda \leq \mu$, it follows that $\lambda \leq$ Lω-int (μ). Again, since Lω-int (μ) is L-fuzzy ω-closed it follows that Lω- cl (λ) $\leq$ Lω-int (μ). Conversely let μ be any L-fuzzy ω-closed set. Then Lω-int (μ) is L-fuzzy ω-open in (X, T) and Lω-int (μ) $\leq \mu$. Therefore by assumption, Lω-cl (Lω-int(μ)) $\leq$ Lω-int (μ). This implies that Lω-int (μ) is L-fuzzy ω-closed. Hence by (b) of Proposition 3.3, it follows that (X, T) is L-fuzzy ω-extremally disconnected.

Remark 3.2. Let (X, T) be an L-fuzzy ω-extremally disconnected space. Let { λ_i, $1-\mu_i$ / i $\in$ N } be a collection such that $\lambda_i's$, are L-fuzzy ω-open and $\mu_i's$ are L-fuzzy ω-closed and let λ, μ are L-fuzzy ω-clopen sets respectively. If $\lambda_i \leq \lambda \leq \mu_j$ and $\lambda_i \leq \mu \leq \mu_j$ for all i, j $\in$ N, then there exists an L-fuzzy ω-clopen set γ such that Lω-cl (λ_i) $\leq \gamma \leq$ Lω-int (μ_j), for all i, j $\in$ N.

Proposition 3.7. Let (X, T) be an L-fuzzy ω-extremally disconnected space. Let $\{\lambda_r\}_{r\in Q}$ and $\{\mu_r\}_{r\in Q}$ be monotone increasing collections of L-fuzzy ω-open sets and L-fuzzy ω-closed sets of (X, T) and suppose that $\lambda_{q_1} \leq \mu_{q_2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{\gamma_r\}_{r\in Q}$ of L-fuzzy ω-clopen sets of (X, T) such that Lω-cl (λ_{q_1}) $\leq$ (γ_{q_2}) and $\gamma_{q_1} \leq$ Lω-int (μ_{q_2}) whenever $q_1 < q_2$.

Definition 3.10. Let (X, T) be an L-fuzzy topological space. A mapping f : X $\longrightarrow$ R(L) is called lower (resp. upper) L-fuzzy ω-continuous if f^{-1} (R_t) (resp. f^{-1} (L_t)) is L-fuzzy ω-open (resp. L-fuzzy ω-clopen) for each t $\in$ R.

Proposition 3.8. Let (X, T) be any L-fuzzy topological space; let $\lambda \in L^X$ and let f : X

$\rightarrow$ R(L) be such that

$$f(x)(t) = \begin{cases} 1, & \text{if } t < 0, \\ \lambda(x), & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } t > 1. \end{cases}$$

for all x $\in$ X. Then f is lower (resp. upper) L-fuzzy ω-continuous iff λ is L-fuzzy ω-open (resp. L-fuzzy ω-clopen).

Remark 3.2, Proposition 3.7 and Proposition 3.8 can be established by the concepts of L-fuzzy ω-clopen set, L-fuzzy ω-interior, L-fuzzy ω-closure and the lemmas given in [13] with some slight suitable modifications.

Definition 3.11. The characteristic function of $\lambda \in L^X$ is the map χ_λ : X $\rightarrow$ I(L) defined by χ_λ (x) = (λ(x)), x $\in$ X.

Proposition 3.9. Let (X, T) be an L-fuzzy topological space and let $\lambda \in L^X$. Then χ_λ is lower (resp.upper) L-fuzzy ω-continuous iff λ is L-fuzzy ω-open (resp. L-fuzzy ω-clopen).

Proof. The proof follows from Proposition 3.8.

Definition 3.12. Let (X, T) and (Y, S) be any two L-fuzzy topological spaces. A mapping f: (X, T)$\rightarrow$ (Y, S) is called strong L-fuzzy ω-continuous if f^{-1}(λ) is L-fuzzy ω-clopen set of (X, T) for every L-fuzzy ω-open set λ of (Y, S).

Proposition 3.10. Let (X, T) be an L-fuzzy topological space. Then the following statements are equivalent :

(a) (X, T) is an L-fuzzy ω-extremally disconnected space.

(b) If g, h : X $\rightarrow$ R(L) where g is lower L-fuzzy ω-continuous, h is upper L-fuzzy ω-continuous, then there exists f $\in$ CLω (X) such that g $\leq$ f $\leq$ h. [CLω (X) = collection of all strong L-fuzzy ω-continuous functions on X with values in R (L)].

(c) If 1$-\lambda$, μ are L-fuzzy ω-clopen sets such that $\mu \leq \lambda$ then there exists a strong L-fuzzy ω-continuous function f : X$\rightarrow$ I(L) such that $\mu \leq (1 - L_1)$f $\leq$ R$_o$f $\leq \lambda$.

Proof. (a) $\Rightarrow$ (b) can be established by the concept of L-fuzzy ω-clopen set and the Theorem 3.7 of Kubiak [11] with some slight suitable modifications.

(b) $\Rightarrow$ (c) Suppose λ is L-fuzzy ω-clopen and μ is L-fuzzy ω-open such that $\mu \leq \lambda$. Then $\chi_\mu \leq \chi_\lambda$ where χ_μ,χ_λ are lower and upper L-fuzzy ω-continuous respectively. Hence by (b), there exists a strong L-fuzzy ω-continuous function f : X $\rightarrow$ R(L) such that, $\chi_\mu \leq$ f $\leq \chi_\lambda$. Clearly f(x) $\in$ I(L), for all x $\in$ X and μ = (1$-$ L_1) χ_μ $\leq$ (1$-L_1$) f $\leq$ R$_o$f $\leq$ $R_o\chi_\lambda = \lambda$. Therefore $\mu \leq$ (1$-$ L_1)f $\leq$ R$_o$f $\leq \lambda$.

(c) $\Rightarrow$ (a) (1 $-L_1$)f and R$_o$f are L-fuzzy ω-clopen sets. By Proposition 3.6, (X, T) is an L-fuzzy ω-extremally disconnected space.

Proposition 3.11. Let (X, T) be an L-fuzzy ω-extremally disconnected space and let A $\subset$ X be such that χ_A is L-fuzzy ω-open. Let f : (A, T/A) $\longrightarrow$ I(L) be strong L-fuzzy ω-continuous. Then f has a strong L-fuzzy ω-continuous extension over (X, T).

Proof. Let g, h: X $\rightarrow$ I(L) be such that g = f = h on A and g (x) = $\langle 0 \rangle$, h (x)= $\langle 1 \rangle$ if x $\notin$ A. We now have

$$R_t g = \begin{cases} \mu_t \wedge \chi_A, & \text{if } t \geq 0, \\ 1, & \text{if } t < 0. \end{cases}$$

where μ_t is L-fuzzy ω-open and is such that $\mu_t/A = R_t f$ and

$$L_t h = \begin{cases} \lambda_t \wedge \chi_A, & \text{if } t \leq 1, \\ 1, & \text{if } t > 1. \end{cases}$$

where λ_t is L-fuzzy ω-clopen and is such that $\lambda_t/A = L_t f$. Thus g is lower L-fuzzy ω-continuous h is upper L-fuzzy ω-continuous and g $\leq$ h. By Proposition 3.10, there is a strong L-fuzzy ω-continuous function F: X $\to$ I(L) such that g $\leq$ F $\leq$ h. Hence F = f on A.

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Hopf bifurcation in a predator-prey model with distributed delays [1]

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Abstract In this paper, A mathematical model of two species with stage structure and distributed delays is investigated, the necessary and sufficient of the stable equilibrium point are studied. Further, by analyzing the associated characteristic equation, it is founded that Hopf bifurcation occurs when τ crosses some critical value.

Keywords Hopf bifurcation, time delay, predator-prey system, equilibrium point.

§1. Introduction

In recent years, the dynamical behaviors of population models with time delay has drawn much attention from researchers.

In this paper, we consider a predator-prey model with discrete and distributed delays

$$\begin{cases} \dot{x}(t) = x(t)[a - bx(t-\tau) - c\int_{-\infty}^{t} f(t-s)y(s)ds], \\ \dot{y}(t) = y(t)[-\alpha + kx(t-\tau) - \beta y(t)], \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ is called the population densities of prey and predator at time t, a denotes the intrinsic growth rate of the prey, α denotes the death rate of the predator. $f(t) = re^{-rt}$. In this paper, we will take the delay τ as the bifurcation parameter. And in Section 2, results on positivity and boundedness of solutions are presented. In Section 3, by analyzing the corresponding characteristic equations, we discuss the local stability of equilibria, and study the existence of Hopf bifurcation.

§2. Positive and bounded

We define a new variable as

$$z(t) = \int_{-\infty}^{t} re^{-r(t-s)}y(s)ds.$$

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Then using the linear chain trick technique, system (1) can be transformed into the following equivalent system

$$\begin{cases} \dot{x}(t) = x(t)[a - bx(t-\tau) - cz(t)], \\ \dot{y}(t) = y(t)[-\alpha + kx(t-\tau) - \beta y(t)], \\ \dot{z}(t) = r(y(t) - z(t))]. \end{cases} \tag{2}$$

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \tag{3}$$

Theorem 1.[7] The solution of system (2) with initial conditions (3) are positive for all $t \geq 0$.

§3. Existence of Hopf bifurcation

In this section, we mainly study the stability of positive equilibrium and the existence of a Hopf bifurcation at the positive equilibrium.

(H_1) $ka - \alpha b > 0$.

(H_2) $b\beta > ck$.

If the hypothesis (H_1) is satisfied, then the system (2) has a unique positive equilibrium $E^*(x^*, y^*, z^*)$, where $y^* = z^*$, and

$$x^* = \frac{a\beta + c\alpha}{ck + \beta b}, \quad y^* = \frac{ka - \alpha b}{ck + b\beta}.$$

Let $\overline{E} = (\overline{x}, \overline{y}, \overline{z})$ be any arbitrary equilibrium. Then the characteristic equation about $\overline{E}$ is given by

$$\begin{vmatrix} \lambda + b\overline{x}e^{-\lambda\tau} & 0 & c\overline{x} \\ -k\overline{y}e^{-\lambda\tau} & \lambda + \beta\overline{y} & 0 \\ 0 & -r & \lambda + r \end{vmatrix} = 0.$$

Immediately, we can get

$$\lambda^3 + A\lambda^2 + B\lambda + E\lambda^2 e^{-\lambda\tau} + D\lambda e^{-\lambda\tau} + [C + F]e^{-\lambda\tau} = 0, \tag{4}$$

where $A = r + \beta\overline{y}, B = \beta r\overline{y}, C = b\beta r\overline{xy}, D = br\overline{x} + b\beta\overline{xy}, E = b\overline{x}, F = ckr\overline{xy}$.

Theorem 2. If (H_1) and (H_2) are satisfied, then the equilibrium E^* of system (2) is asymptotically stable when $\tau = 0$.

Proof of Theorem 2. Following from the Routh-Hurwize we can get

(1) $M_1 = A + E > 0$,

(2) $M_2 = (A + E)(B + D) - (C + F) > 0$,

(3) $M_3 = (C + F)[(A + E)(B + D) - (C + F)] > 0$.

So, the theorem is proved.

We suppose $i\varpi(\varpi > 0)$ is a root of equation (4), so ϖ satisfies

$$-i\varpi^3 - A\varpi^2 + B\varpi i - E\varpi^2 e^{-i\varpi\tau} + Di\varpi e^{-i\varpi\tau} + (C + F)e^{-i\varpi\tau} = 0. \tag{5}$$

Separating the real and imaginary parts, we get

$$\begin{cases} D\varpi\cos\varpi\tau + [E\varpi^2 - (C+F)]\sin\varpi\tau = \varpi^3 - B\varpi, \\ (C+F-E\varpi^2)\cos\varpi\tau + D\varpi\sin\varpi\tau = A\varpi^2. \end{cases}$$

As is known to all, $sin^2\varpi\tau + cos^2\varpi\tau = 1$, we can obtain the following equations

$$\varpi^{10} + A_1\varpi^8 + A_2\varpi^6 + A_3\varpi^4 + A_4\varpi^2 + A_5 = 0, \tag{6}$$

where

$A_1 = \dfrac{2D^2 - 4CE - 2EF4 - E^4}{E^2}$

$A_2 = \dfrac{4C^2 + F^2 + 2A^2D^2 + 6CF + 2BC + 4CE^3 - 2BD^2 - 2A^2CF - 2D^2E^2 - 4E^3F}{E^2}$

$A_3 = \dfrac{B^2D^2 + (A^2 - 6E^2)(C+F)^2 + 4C^2D^2E + 4D^2EF - 4BC^2 - 4BCF - 2BF^2 -}{E^2}$

$\cdot \dfrac{2CEF - D^4}{E^2}$

$A_4 = \dfrac{B^2(C+F)^2 + 12CEF(C+F) + 4C^2E + 4EF^3 - 2C^2D^2 - 4CD^2F}{E^2}$

$A_5 = -\dfrac{(C+F)^2[(C-F)^2+1]}{E^2}$

Let $\theta = \varpi^2$, (6) becomes

$$\theta^5 + A_1\theta^4 + A_2\theta^3 + A_3\theta^2 + A_4\theta + A_5 = 0. \tag{7}$$

Let

$$G(\theta) = \theta^5 + A_1\theta^4 + A_2\theta^3 + A_3\theta^2 + A_4\theta + A_5. \tag{8}$$

Because $G(0) = A_5 < 0$, $\lim\limits_{\theta\to\infty} G(\theta) = +\infty$, equation (7) has at least one positive real root.

Generally, we assume equation (7) has five positive roots, defined by $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, respective, (6) has five positive roots $\varpi_1 = \sqrt{\theta_1},\ \varpi_2 = \sqrt{\theta_2},\ \varpi_3 = \sqrt{\theta_3},\ \varpi_4 = \sqrt{\theta_4},\ \varpi_5 = \sqrt{\theta_5}$.

So, we can denote

$$\tau_i^j = \frac{1}{\varpi_i}\{\arccos\frac{D\varpi_i^2(\varpi_i^2 - B) + A\varpi_i^2(C+F-E\varpi_i^2)}{(D\varpi_i)^2 + (C+F-E\varpi_i^2)^2} + 2\pi j\},$$

where $i = 1, 2, \cdots, 5, j = 0, 1, 2, \cdots$, then $\pm\varpi_i$ is a pair of purely imaginary roots of (6) with τ_i^j. Define

$\tau_0 = \tau_{i_0}^0 = \min\limits_{k\in 1,\cdots,5}\{\tau_i^0\},\ \varpi_0 = \varpi_{i_0}$

$(H_3)\ Re(\dfrac{d\lambda}{d\tau})|_{\tau=\tau_0}^{\lambda=i\varpi} > 0.$

Taking the derivative of λ with respect to τ in (5), we can obtain:

$$\frac{d\lambda}{d\tau} = \frac{[E\lambda^3 + D\lambda^2 + (C+F)\lambda]e^{-\lambda\tau}}{3\lambda^2 + 2A\lambda + B + (2E\lambda + D)e^{-\lambda\tau} - [E\lambda^2 + D\lambda + (C+F)]\tau e^{-\lambda\tau}}.$$

Obviously

$$\frac{d\lambda}{d\tau}^{-1} = \frac{3\lambda^2 + 2A\lambda + B + (2E\lambda + D)e^{-\lambda\tau}}{[E\lambda^3 + D\lambda^2 + (C+F)\lambda]e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Noting that $sign\{Re(\frac{d\lambda}{d\tau})|^{\lambda=i\varpi}_{\tau=\tau_0}\} = sign\{Re(\frac{d\lambda}{d\tau})^{-1}|^{\lambda=i\varpi}_{\tau=\tau_0}\}$, so

$$\begin{aligned}
& Re(\frac{d\lambda}{d\tau}^{-1})^{\lambda=i\varpi}_{\tau=\tau_0} \\
= & \frac{3E\varpi^5\sin\varpi\tau + D\varpi^4\cos\varpi\tau - 2E^2\varpi^4 + 2AD\varpi^3\sin\varpi\tau - (4C+3F)\varpi^3\sin\varpi\tau}{[(C+F-E\varpi^2)\varpi\sin\varpi\tau - D\varpi^2\cos\varpi\tau]^2 + [(C+F-E\varpi^3)\cos\varpi\tau + D\varpi^2\sin\varpi\tau]^2} \\
& + \frac{2E(C+F)\varpi^2 - D^2\varpi^2 + (AC+2AF)\varpi^2\cos\varpi\tau}{[(C+F-E\varpi^2)\varpi\sin\varpi\tau - D\varpi^2\cos\varpi\tau]^2 + [(C+F-E\varpi^3)\cos\varpi\tau + D\varpi^2\sin\varpi\tau]^2}.
\end{aligned}$$

Lemma 1.[2] Consider the exponential polynomial

$$\begin{aligned}
& P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) \\
= & \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} \\
& + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m},
\end{aligned}$$

where $\tau_i \geq 0 (i = 1, 2, \cdots, m)$ and $p_j^{(i)} (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ are constants. As $(\tau_1, \tau_2, \cdots, \tau_m)$ vary, the sum of order of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

So, we can obtain the following theorem:

Theorem 3. Assume equation (7) has at least one positive real root and (H_3) is satisfied, then the following results hold true:

(1) The positive equilibrium E^* of the system (2) is asymptotically stable for $\tau \in [0, \tau_0)$;

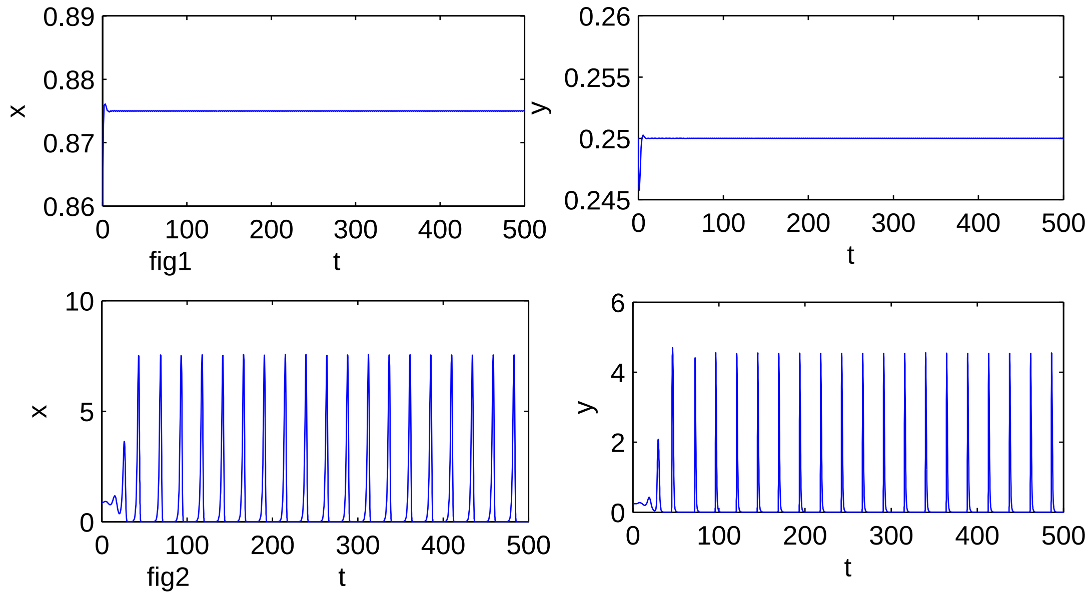
(2) The positive equilibrium E^* of system (2) undergoes a Hopf bifurcation when $\tau = \tau_0$, that is, system (2) has a branch of periodic solutions bifurcating from the positive equilibrium E^* near $\tau = \tau_0$.

§4. Numerical example

Example 1.

$$\begin{cases} \dot{x}(t) = x(t)[1 - x(t-\tau) - \frac{1}{2}z(t)], \\ \dot{y}(t) = y(t)[-1 + 2x(t-\tau) - 3y(t)], \\ \dot{z}(t) = 2(y(t) - z(t)). \end{cases} \tag{9}$$

Which has a positive equilibrium $E^*(0.8750, 0.2500, 0.2500)$, and we obtain $\tau^j = 2.2897 + 9.8670j (j = 1, 2, \cdots)$, and the $\omega_0 = 0.6368$, $\tau_0 = 2.2897$, $\lambda'(\tau_0) = 0.1554 - 0.1557i$, so the hypothesis of (H_3) holds. E^* satisfies the conditions of Theorem 3.



where fig1. $\tau = 0.4 < \tau_0$, the positive equilibrium is stable, fig2. $\tau = 3 > \tau_0$, oscillation solution.

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Some identities involving function $U_t(n)$

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Abstract In this paper, we use the elementary method to study the properties of pseudo Smarandache function $U_t(n)$, and obtain some interesting identities involving function $U_t(n)$, and for any fixed integer n, offer a method of calculating of the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

Keywords Pseudo Smarandache function, some identities, elementary method.

§1. Introduction and result

For any positive integer n and $U_t(n)$ fixed $t \geq 1$, we define function

$$U_t(n) = \min\left\{1^t + 2^t + 3^t + \cdots + n^t + k,\ n \mid m,\ k \in N^+,\ t \in N^+\right\},$$

where $n \in N^+,\ m \in N^+$. Wang Yu studied the properties of pseudo Smarandache function $U_t(n)$, and obtained calculation of the infinite series

$$\sum_{i=1}^{\infty} U_t(1),\ \sum_{i=1}^{\infty} U_t(2),\ \sum_{i=1}^{\infty} U_t(3).$$

In this paper we use the elementary method to study the calculating of the infinite series

$$\sum_{i=1}^{\infty} U_t(n),$$

where $n \geq 4$, obtain calculation of the infinite series $\sum_{i=1}^{\infty} U_t(4),\ \sum_{i=1}^{\infty} U_t(5)$, and for any fixed integer n, we offer a method of calculating the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

Theorem 1. For any real positive integer s, we have

$$\sum_{i=1}^{\infty} \frac{1}{U_4^s}(n) = 1 + (2 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{5^s})(1 - \frac{1}{5^s}) + \frac{1}{7^s}(2 - \frac{1}{2^s} - \frac{1}{3^s}) + (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(2 - \frac{1}{5^s}).$$

Theorem 2. For any real positive integer s, we have

$$\sum_{i=1}^{\infty} \frac{1}{U_5^s}(n) = (2 - \frac{2}{2^s} + \frac{1}{4^s} - \frac{1}{6^s})\zeta(s).$$

§2. Some lemmas

Lemma 1. If $S_r(n) = \sum_{i=1}^{n} k^r$, then

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n,\ S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,\ S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$

if $r \in Z^-$, let $S_r(n) = 0$, when $r \geq 4$, and $r \in Z$, we have

$$\begin{aligned} S_r(n) &= \frac{1}{r+1}n^{r+1} + \frac{1}{2}n^r + \frac{3r}{28}n^{r-1} + \frac{r(r-1)}{84}n^{r-2} \\ &+\frac{r(r-1)(r-2)}{8\cdot 7\cdot 6\cdot 5}n^{r-3} - \frac{r(r-1)}{42}S_{r-2}(n) \\ &-\frac{1}{8\cdot 7\cdot 6\cdot 5}\sum_{i=1}^{r-n}(-1)^j \left(\begin{array}{c} r \\ 8+j \end{array}\right) \frac{(j+1)(j+2)(j+3)(j+4)}{j+9} S_{r-8-j}(n). \end{aligned}$$

Proof. See reference [1].

Lemma 2. For any positive integer n, we have

$$\sum_{i=1}^{\infty} \frac{1}{U_4^s}(n) = \begin{cases} \frac{n}{30}, & \text{if } n \equiv 0(\text{mod } 30), \\ n, & \text{if } n \equiv 1(\text{mod } 30),\ \text{or } n \equiv 29(\text{mod } 30),\ \text{or } n \equiv 7(\text{mod } 30), \\ & \text{or } n \equiv 23(\text{mod } 30),\ \text{or } n \equiv 11(\text{mod } 30),\ \text{or } n \equiv 19(\text{mod } 30), \\ & \text{or } n \equiv 13(\text{mod } 30),\ \text{or } n \equiv 17(\text{mod } 30), \\ \frac{n}{2}, & \text{if } n \equiv 2(\text{mod } 30),\ n \equiv 28(\text{mod } 30),\ \text{or } n \equiv 4(\text{mod } 30), \\ & \text{or } n \equiv 26(\text{mod } 30),\ \text{or } n \equiv 8(\text{mod } 30) \text{ or } n \equiv 22(\text{mod } 30), \\ & \text{or } n \equiv 14(\text{mod } 30),\ \text{or } n \equiv 16(\text{mod } 30), \\ \frac{n}{3}, & \text{if } n \equiv 3(\text{mod } 30),\ \text{or } n \equiv 27(\text{mod } 30),\ \text{or } n \equiv 9(\text{mod } 30), \\ & \text{or } n \equiv 21(\text{mod } 30), \\ \frac{n}{5}, & \text{if } n \equiv 5(\text{mod } 30),\ \text{or } n \equiv 25(\text{mod } 30), \\ \frac{5n}{6}, & \text{if } n \equiv 6(\text{mod } 30),\ \text{or } n \equiv 24(\text{mod } 30),\ \text{or } n \equiv 12(\text{mod } 30), \\ & \text{or } n \equiv 18(\text{mod } 30), \\ \frac{7n}{10}, & \text{if } n \equiv 10(\text{mod } 30),\ \text{or } n \equiv 20(\text{mod } 30), \\ \frac{7n}{15}, & \text{if } n \equiv 15(\text{mod } 30), \end{cases}$$

Proof. (1) If $n \equiv 0(\text{mod } 30)$, then we have $n = 30h_1$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (2700h_1^3+90h_1^2)(30h_1+1)(60h_1+1)-(1800h_1^3+60h_1^2+31h_1),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{30}$.

(2) If $n \equiv 1(\text{mod } 30)$, then we have $n = 30h_1 + 1$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+1)(15h_1+1)(20h_1+1)(540h_1^2+54h_1+1),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = n$.

If $n \equiv 29(\text{mod } 30)$, then we have $n = 30h_1 + 29$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+29)(h_1+1)(60h_1+59)(2700h_1^2+5310h_1+2609),$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = n$.

(3) If $n \equiv 2(\text{mod } 30)$, then we have $n = 30h_1 + 2$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} &= (90h_1+9)(15h_1+1)(10h_1+1)(12h_1+1)(30h_1+2)\\ &\quad -5h_1(30h_1+2)(12h_1+1)+6h_1(30h_1+2)+(15h_1+1),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 28(\text{mod } 30)$, then we have $n = 30h_1 + 28$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ = &(15h_1+14)(30h_1+29)(20h_1+19)(30h_1+28)(18h_1+17)\\ &+(15h_1+14)(30h_1+28)(20h_1+19)(12h_1+11)+(30h_1+28)(12h_1+11)(10h_1+9)\\ &+(6h_1+5)(30h_1+28)+(15h_1+14),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

(4) If $n \equiv 3(\text{mod } 30)$, then we have $n = 30h_1 + 3$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\ = &(18h_1+2)(15h_1+2)(10h_1+1)(60h_1+7)(30h_1+3)\\ &+2(15h_1+2)(10h_1+1)(12h_1+1)(30h_1+3)+5h_1(12h_1+1)(30h_1+3)\\ &+8h_1(30h_1+3)+(20h_1+2),\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

If $n \equiv 27 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 27$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (10h_1+9)(15h_1+14)(12h_1+11)(30h_1+27)(90h_1+84) \\ & -(12h_1+11)(30h_1+27)(5h_1+4)-(30h_1+27)(8h_1-7)-(10h_1+9), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

(5) If $n \equiv 4 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 4$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (15h_1+2)(6h_1+1)(20h_1+3)(90h_1+15)(30h_1+4) \\ & -(6h_1+1)(30h_1+4)(10h_1+1)-3h_1(30h_1+4)-(15h_1+2), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 26 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 26$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (15h_1+13)(10h_1+9)(60h_1+53)(18h_1+16)(30h_1+26) \\ & +2(6h_1+9)(15h_1+13)(10h_1+9)(30h_1+26)+(5h_1+4)(30h_1+26)(6h_1+5) \\ & +(3h_1+2)(30h_1+3)+(15h_1+13), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

(6) If $n \equiv 5 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 5$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & 2(6h_1+1)(5h_1+1)(2700h_1^2+990h_1+89)(30h_1+5) \\ & +(6h_1+1)(5h_1+1)(90h_1+18)(30h_1+5)-h_1(30h_1+5)-(6h_1+1), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{n}{5}$.

If $n \equiv 25 (\mathrm{mod}\ 30)$, then we have $n = 30h_1 + 25$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (6h_1+5)(15h_1+13)(20h_1+17)(90h_1+78)(30h_1+25) \\ & -(6h_1+5)(10h_1+8)(30h_1+25)-(5h_1+4)(30h_1+25)-(6h_1+5), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{n}{5}$.

(7) If $n \equiv 6 (\bmod\ 30)$, then we have $n = 30h_1 + 6$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (5h_1+1)(30h_1+6)(60h_1+13)(540h_1^2+234h_1+25) \\ & +2(5h_1+1)(30h_1+6)(540h_1^2+234h_1+25) + (5h_1+1)(18h_1+4)(30h_1+6) \\ & +(30h_1+6)h_1 + (5h_1+1), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

If $n \equiv 24 (\bmod\ 30)$, then we have $n = 30h_1 + 24$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & 2(5h_1+4)(6h_1+5)(30h_1+24)(2700h_1^2+4410h_1+1799) \\ & +(5h_1+4)(6h_1+5)(90h_1+75)(30h_1+24) - h_1(30h_1+24) - (25h_1+20), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_5(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

(8) If $n \equiv 7 (\bmod\ 30)$, then we have $n = 30h_1 + 7$, $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+7)(15h_1+4)(4h_1+1)(2700h_1^2+1350h_1+167),$$

if and only if $U_4(n) = n$.

If $n \equiv 23 (\bmod\ 30)$, then we have $n = 30h_1 + 23$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+23)(5h_1+4)(60h_1+47)(540h_1^2+846h_1+333),$$

if and only if $U_4(n) = n$.

(9) If $n \equiv 8 (\bmod\ 30)$, then we have $n = 30h_1 + 8$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (15h_1+4)(10h_1+3)(60h_1+17)(30h_1+8)(18h_1+5) \\ & +2(15h_1+4)(10h_1+3)(30h_1+18)(12h_1+29) + (15h_1+4)(4h_1+9)(30h_1+8) \\ & +(12h_1+7)(30h_1+8) + (15h_1+4), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 22 (\bmod\ 30)$, then we have $n = 30h_1 + 22$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ &(15h_1+11)(4h_1+3)(30h_1+22)(2700h_1^2+4050h_1+1517)\\
&+(15h_1+11)(4h_1+3)(90h_1+69)(30h_1+22)-(2h_1+1)(30h_1+22)-(15h_1+11),
\end{aligned}$$

so if and only if $U_4(n) = \frac{n}{2}$.

(10) If $n \equiv 9 (\bmod\ 30)$, then we have $n = 30h_1 + 9$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ &2(10h_1+3)(3h_1+1)(30h_1+9)(2700h_1^2+1710h_1+269)\\
&+(3h_1+1)(10h_1+3)(90h_1+30)(30h_1+9)-h_1(30h_1+9)-(10h_1+3),
\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{3}$.

If $n \equiv 21 (\bmod\ 30)$, then we have $n = 30h_1 + 21$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ &2(10h_1+7)(15h_1+11)(30h_1+21)(540h_1^2+774h_1+277)\\
&+(10h_1+7)(15h_1+11)(18h_1+13)(30h_1+21)+(5h_1+3)(6h_1+4)(30h_1+21)\\
&+(4h_1+2)(30h_1+21)+(20h_1+14),
\end{aligned}$$

so if and only if $U_4(n) = \frac{n}{3}$.

(11) If $n \equiv 10 (\bmod\ 30)$, then we have $n = 30h_1 + 10$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ &(3h_1+1)(30h_1+11)(20h_1+7)(90h_1+33)(30h_1+10)\\
&-(3h_1+1)(20h_1+7)(30h_1+10)-2h_1(30h_1+10)-(21h_1+7),
\end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{7n}{10}$.

If $n \equiv 20 (\bmod\ 30)$, then we have $n = 30h_1 + 20$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ &(10h_1+7)(60h_1+41)(3h_1+2)(90h_1+63)(30h_1+20)\\
&-2(10h_1+7)(3h_1+2)(30h_1+20)-h_1(30h_1+20)-(20h_1+14),
\end{aligned}$$

so if and only if $U_4(n) = \frac{7n}{10}$.

(12) If $n \equiv 11(\bmod 30)$, then we have $n = 30h_1 + 11$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+11)(2h_1+2)(60h_1+23)(540h_1^2+378h_1+79),$$

if and only if $U_4(n) = n$.

If $n \equiv 19(\bmod 30)$, then we have $n = 30h_1 + 19$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+19)(3h_1+2)(20h_1+13)(2700h_1^2+3510h_1+1139),$$

if and only if $U_4(n) = n$.

(13) If $n \equiv 12(\bmod 30)$, then we have $n = 30h_1 + 12$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (5h_1+2)(30h_1+12)(12h_1+5)(2700h_1^2+2250h_1+467) \\ & +(5h_1+2)(12h_1+5)(90h_1+39)(30h_1+12)-2h_1(30h_1+12)-(25h_1+10), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{5n}{6}$.

If $n \equiv 18(\bmod 30)$, then we have $n = 30h_1 + 18$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & (5h_1+3)(30h_1+18)(60h_1+39)(540h_1^2+666h_1+205) \\ & +(5h_1+3)(18h_1+11)(30h_1+18)+(2h_1+1)(30h_1+18)+(5h_1+3), \end{aligned}$$

so if and only if $U_4(n) = \frac{5n}{6}$.

(14) If $n \equiv 13(\bmod 30)$, then we have $n = 30h_1 + 13$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+13)(15h_1+7)(20h_1+19)(540h_1^2+486h_1+109),$$

if and only if $U_4(n) = n$.

If $n \equiv 17(\bmod 30)$, then we have $n = 30h_1 + 17$ $(h_1 = 1, 2, 3, \cdots)$,

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = (30h_1+17)(5h_1+3)(12h_1+7)(2700h_1^2+3510h_1+917),$$

if and only if $U_4(n) = n$.

(15) If $n \equiv 14(\bmod 30)$, then we have $n = 30h_1 + 14$ $(h_1 = 1, 2, 3, \cdots)$,

$$\begin{aligned} & \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ = \ & 2(15h_1+7)(2h_1+1)(30h_1+14)(2700h_1^2+2610h_1+629) \\ & +(15h_1+7)(2h_1+1)(90h_1+45)(30h_1+14)-h_1(30h_1+14)-(15h_1+7), \end{aligned}$$

so

$$n \mid \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + U_4(n),$$

if and only if $U_4(n) = \frac{n}{2}$.

If $n \equiv 16(\text{mod } 30)$, then we have $n = 30h_1 + 16$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ & (15h_1+8)(30h_1+16)(20h_1+11)(540h_1^2+594h_1+163)\\
&+(15h_1+8)(20h_1+11)(18h_1+10)(30h_1+16)+(10h_1+5)(30h_1+16)(6h_1+3)\\
&+(3h_1+1)(30h_1+16)+(15h_1+8),
\end{aligned}$$

so if and only if $U_4(n) = \frac{n}{2}$.

(16) If $n \equiv 15(\text{mod } 30)$, then we have $n = 30h_1 + 15$ $(h_1 = 1,\ 2,\ 3, \cdots)$,

$$\begin{aligned}
&\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\\
= \ & 2(15h_1+8)(2h_1+1)(30h_1+15)(2700h_1^2+2790h_1+719)\\
&+(2h_1+1)(15h_1+8)(90h_1+48)(30h_1+15)-h_1(30h_1+15)-(16h_1+8),
\end{aligned}$$

so if and only if $U_4(n) = \frac{7n}{15}$.

Lemma 3. For any positive integer n, we have

$$\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n) = \begin{cases} n, & \text{if } n \equiv 0(\text{mod } 12),\ \text{or } n \equiv 1(\text{mod } 12), or\ n \equiv 11(\text{mod } 12),\\ & \text{or } n \equiv 3(\text{mod } 12),\ \text{or } n \equiv 9(\text{mod } 12),\ \text{or } n \equiv 4(\text{mod } 12),\\ & \text{or } n \equiv 8(\text{mod } 12),\ \text{or } n \equiv 5(\text{mod } 12),\ \text{or } n \equiv 7(\text{mod } 12),\\ \frac{n}{2}, & \text{if } n \equiv 2(\text{mod } 12),\ n \equiv 10(\text{mod } 12),\ \text{or } n \equiv 6(\text{mod } 12),\end{cases}$$

Proof. Using the same method of Lemma 1, we can complete the proof of Lemma 2.

§3. Proof of the Theorem 1

In this section, we shall use the Lemma 1 to complete the proof of the theorems. First we prove Theorem 1. For any real number $s > 1$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n) = \ & \sum_{\substack{i=1\\ n=30h_1}}^{\infty}\frac{1}{(\frac{n}{30})^s} + \sum_{\substack{i=1\\ n=30h_1+1}}^{\infty}\frac{1}{n^s} + \sum_{\substack{i=1\\ n=30h_1+2}}^{\infty}\frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1\\ n=30h_1+3}}^{\infty}\frac{1}{(\frac{n}{3})^s}\\
&+ \sum_{\substack{i=1\\ n=30h_1+4}}^{\infty}\frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1\\ n=30h_1+5}}^{\infty}\frac{1}{(\frac{n}{5})^s} + \sum_{\substack{i=1\\ n=30h_1+6}}^{\infty}\frac{1}{(\frac{5n}{6})^s} + \sum_{\substack{i=1\\ n=30h_1+7}}^{\infty}\frac{1}{n^s}\\
&+ \sum_{\substack{i=1\\ n=30h_1+8}}^{\infty}\frac{1}{(\frac{n}{2})^s} + \sum_{\substack{i=1\\ n=30h_1+9}}^{\infty}\frac{1}{(\frac{n}{3})^s} + \sum_{\substack{i=1\\ n=30h_1+10}}^{\infty}\frac{1}{(\frac{7n}{10})^s} + \sum_{\substack{i=1\\ n=30h_1+11}}^{\infty}\frac{1}{n^s}
\end{aligned}$$

$$+\sum_{\substack{i=1\\n=30h_1+12}}^{\infty}\frac{1}{(\frac{5n}{6})^s}+\sum_{\substack{i=1\\n=30h_1+13}}^{\infty}\frac{1}{n^s}+\sum_{\substack{i=1\\n=30h_1+14}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\n=30h_1+15}}^{\infty}\frac{1}{(\frac{7n}{15})^s}$$

$$+\sum_{\substack{i=1\\n=30h_1+16}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\n=30h_1+17}}^{\infty}\frac{1}{n^s}+\sum_{\substack{i=1\\n=30h_1+18}}^{\infty}\frac{1}{(\frac{5n}{6})^s}+\sum_{\substack{i=1\\n=30h_1+19}}^{\infty}\frac{1}{n^s}$$

$$+\sum_{\substack{i=1\\n=30h_1+20}}^{\infty}\frac{1}{(\frac{7n}{10})^s}+\sum_{\substack{i=1\\n=30h_1+21}}^{\infty}\frac{1}{(\frac{n}{3})^s}+\sum_{\substack{i=1\\n=30h_1+22}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\n=30h_1+23}}^{\infty}\frac{1}{n^s}$$

$$+\sum_{\substack{i=1\\n=30h_1+24}}^{\infty}\frac{1}{(\frac{5n}{6})^s}+\sum_{\substack{i=1\\n=30h_1+25}}^{\infty}\frac{1}{(\frac{n}{5})^s}+\sum_{\substack{i=1\\n=30h_1+26}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\n=30h_1+27}}^{\infty}\frac{1}{(\frac{n}{3})^s}$$

$$+\sum_{\substack{i=1\\n=30h_1+28}}^{\infty}\frac{1}{(\frac{n}{2})^s}+\sum_{\substack{i=1\\n=30h_1+29}}^{\infty}\frac{1}{n^s},$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{U_4^s}(n) &= 1+(1-\frac{1}{3^s})(1-\frac{1}{5^s})+(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})+\frac{1}{5^s}(1-\frac{1}{5^s})\\ &\quad +(1-\frac{1}{2^s})(1-\frac{1}{5^s})+\frac{1}{7^s}(1-\frac{1}{3^s})+\frac{1}{7^s}(1-\frac{1}{2^s})+(1-\frac{1}{2^s})(1-\frac{1}{3^s})\\ &= 1+(2-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{5^s})(1-\frac{1}{5^s})+\frac{1}{7^s}(2-\frac{1}{2^s}-\frac{1}{3^s})+(1-\frac{1}{2^s})(1-\frac{1}{3^s})(2-\frac{1}{5^s}).\end{aligned}$$

This completes the proof of Theorem 1.

Using the same method, we can completes the proof of Theorem 2. In addition, by Theorem 1 and Lemma 1, and for any fixed integer n, we can obtain the calculating of the infinite series $\sum_{i=1}^{\infty} U_t(n)$.

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