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On the Smarandache additive sequence

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Abstract In this paper, we use the elementary method to study the properties of the Smarandache additive sequence, and give several interesting identities for some infinite series involving the Smarandache additive sequence.

Keywords Smarandache additive sequence, infinite series, convergent properties.

§1. Introduction and results

For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \cdots < d_m$ are m positive integers, the Smarandache multiplicative sequence A_m is defined as: If $d_1, d_2, \cdots, d_m$ are the first m terms of the sequence A_m, then $d_k > d_{k-1}$, for $k \geq m+1$, is the smallest number equal to $d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdots d_m^{\alpha_m}$, where $\alpha_i \geq 1$ for all $i = 1, 2, \cdots, m$. For example, the Smarandache multiplicative sequence A_4 (generated by digits $2, 3, 5, 7$) is:

$$2, 3, 5, 7, 210, 420, 630, 840, 1050, 1260, 1470, 1680, 1890, 2100, \cdots.$$

In the book "Sequences of Numbers Involved Unsolved Problems", Professor F.Smarandache introduced many sequences, functions and unsolved problems, one of them is the Smarandache multiplicative sequence, and he also asked us to study the properties of this sequence. About this problem, Ling Li [3] had studied it, and proved the following conclusion:

For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \cdots < d_m$ are m positive integers, and A_m denotes the Smarandache multiplicative sequence generated by $d_1, d_2, \cdots, d_m$. Then for any real number $s > 0$, the infinite series

$$\sum_{n \in A_m} \frac{1}{n^s}$$

is convergent, and

$$\sum_{n \in A_m} \frac{1}{n^s} = \prod_{i=1}^{m} \frac{1}{d_i^s - 1} + \sum_{i=1}^{m} \frac{1}{d_i^s}.$$

In this paper, we define another sequence called the Smarandache additive sequence as follows: Let $1 \leq d_1 < d_2 < \cdots < d_m$ are m positive integers, we define the Smarandache additive sequence D_m as: If $d_1, d_2, \cdots, d_m$ are the first m terms of the sequence D_m, then $d_k > d_{k-1}$, for $k \geq m+1$, is the smallest number equal to $\alpha_1 \cdot d_1 + \alpha_2 \cdot d_2 + \cdots\cdots + \alpha_m \cdot d_m$, where $\alpha_i \geq 1$ for all $i = 1,\ 2,\ \cdots,\ m$. For example, the Smarandache additive sequence D_2 (generated by digits $3,\ 5$) is:

$$3,\ 5,\ 8,\ 11,\ 13,\ 14,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28,\ \cdots.$$

About the properties of this sequence, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because it has the close relationship with the positive integer solutions of the indefinite equation

$$d_1 \cdot x_1 + d_2 \cdot x_2 + d_3 \cdot x_3 + \cdots + d_{m-1} \cdot x_{m-1} + d_m \cdot x_m = n.$$

The main purpose of this paper is using the elementary method to study the convergent properties of some infinite series involving the Smarandache additive sequence, and get some interesting identities. For convenience, we use the symbol $a_m(n)$ denotes the n-th term of the Smarandache additive sequence D_m. In this paper, we shall prove the following:

Theorem. For any positive integer $m \geq 2$, let $1 \leq d_1 < d_2 < \cdots < d_m$ are m positive integers, and D_m denotes the Smarandache additive sequence generated by $d_1,\ d_2,\ \cdots,\ d_m$. Then for any real number $s \leq 1$, the infinite series

$$\sum_{n \in D_m} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} \tag{1}$$

is divergent; For any real number $s > 1$, the series (1) is convergent, and

$$\sum_{n \in D_m} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} = \zeta(s) + \sum_{a_m(n) < M} \frac{1}{a_m^s(n)} - \sum_{n < M} \frac{1}{n^s},$$

where $\zeta(s)$ is the Riemann zeta-function, and M is a computable positive integer.

From this Theorem we may immediately deduce the following three corollaries:

Corollary 1. Let D_2 be the Smarandache additive sequence generated by 2 and 3, then we have the identity

$$\sum_{n \in D_2} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{a_2^2(n)} = \frac{\pi^2}{6} - \frac{157}{144}.$$

Corollary 2. Let D_2 be the Smarandache additive sequence generated by 4 and 6, then we have the identity

$$\sum_{n \in D_2} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{a_2^2(n)} = \frac{\pi^2}{24} - \frac{157}{576}.$$

Corollary 3. Let D_3 be the Smarandache additive sequence generated by 3, 4 and 5, then we have the identity

$$\sum_{n \in D_3} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{a_3^2(n)} = \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} - \frac{1}{121} - \frac{1}{169} - \frac{1}{196}.$$

§2. Proof of the theorem

To prove our Theorem, we need a simple Lemma which stated as follows:

Lemma. Let a and b are two positive integers with $(a,\ b) = 1$, then for any positive integer $n > ab - a - b$, the indefinite equation

$$ax + by = n$$

must has nonnegative integer solution.

Proof. See Theorem 19 of Chapter one in reference [3].

Now we use this Lemma to prove our Theorem. First note that for any positive integer $n > m$, we have

$$a_m(n) = \alpha_1 \cdot d_1 + \alpha_2 \cdot d_2 + \cdots + \alpha_m \cdot d_m,$$

where $\alpha_i \geq 1$ for all $i = 1,\ 2,\ 3,\ \cdots,\ m$.

From Lemma and its generalization we know that there exist a positive integer M such that for any integer $n \geq M$, the indefinite equation

$$d_1 \cdot x_1 + d_2 \cdot x_2 + d_3 \cdot x_3 + \cdots + d_{m-1} \cdot x_{m-1} + d_m \cdot x_m = n$$

must have positive integer solution. So from the definition of the sequence D_m we know that for any integer $n \geq M$, there must exist one term $a_m(k)$ in sequence D_m such that $n = a_m(k)$. Therefore,

$$\begin{aligned}\sum_{n\in D_m} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} &= \sum_{a_m(n)\geq M} \frac{1}{a_m^s(n)} + \sum_{a_m(n)<M} \frac{1}{a_m^s(n)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n<M} \frac{1}{n^s} + \sum_{a_m(n)<M} \frac{1}{a_m^s(n)} \\ &= \zeta(s) - \sum_{n<M} \frac{1}{n^s} + \sum_{a_m(n)<M} \frac{1}{a_m^s(n)},\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, and the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$; It is divergent if $s \leq 1$. This completes the proof of Theorem.

Now we prove our Corollary 1. Taking $a = 2$, $b = 3$, note that $a\cdot b - a - b = 2\cdot 3 - 2 - 3 = 1$, so from Lemma we know that for all integer $n \geq 2$, the indefinite equation $2x + 3y = n$ has nonnegative integer solution. Therefore, for all integer $n \geq 7$, the indefinite equation $2x+3y = n$ has positive integer solution. So $M = 7$, from our Theorem we may immediately deduce that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{a_2^2(n)} &= \zeta(2) + \sum_{a_m(n)<7} \frac{1}{a_m^2(n)} - \sum_{n<7} \frac{1}{n^2} \\ &= \frac{\pi^2}{6} + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} - \sum_{n\leq 6} \frac{1}{n^2} \\ &= \frac{\pi^2}{6} - \frac{157}{144},\end{aligned}$$

where we have used the identity $\zeta(2) = \frac{\pi^2}{6}$. This proves Corollary 1.

To prove Corollary 2, note that the greatest common divisor of 4 and 6 is $(4,\ 6) = 2$, so from Corollary 1 we have the identity

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{a_2^2(n)} &= \frac{1}{4}\left\{\zeta(2) + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} - \sum_{n<7} \frac{1}{n^2}\right\} \\ &= \frac{\pi^2}{24} - \frac{157}{576}.\end{aligned}$$

Now we prove Corollary 3. Note that for all $n \geq 15$, the indefinite equation

$$3x + 4y + 5z = n$$

has positive integer solution. So we can take $M = 15$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{a_3^2(n)} &= \zeta(2) + \sum_{a_3(n)<15} \frac{1}{a_3^2(n)} - \sum_{n<15} \frac{1}{n^2} \\ &= \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} - \frac{1}{121} - \frac{1}{169} - \frac{1}{196}.\end{aligned}$$

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Contributions to classical differential geometry of the curves in E^3

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Abstract In this paper, a system of differential equation whose solution gives the components on the Frenet axis in E^3 is established by means of Frenet equations. In view of some special solutions of mentioned system, position vector of rectifying curves, osculating curves with constant first curvature, normal curves and special cases are presented. Moreover, a characterization for inclined spherical curves is expressed.

Keywords Euclidean space, Frenet frame.

§1. Introduction

It is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [2]).

In the Euclidean space E^3, it is well-known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields T, N and B are respectively, the tangent, the principal normal and the binormal vector fields[3]. At each of the curve, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane[5]. The curves $\varphi : I \subset R \to E^3$ for which the position vector φ always lie in their rectifying plane, are simplicity called rectifying curves [1]. Similarly, the curves for which the position vector φ always lie in their osculating plane, are for simplicity called osculating curves and the curves for which the position vector always lie in their normal plane, are for simplicity called normal curves [4].

In this paper, position vectors of some special curves in E^3 are investigated. First, a system of differential equation is established. In view of some special solutions, characterizations of some special curves are presented.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^3 are briefly presented. (A more complete elementary treatment can be found in [5].)
The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $a \in E^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E^3$ it is said to be orthogonal if and only if $\langle v, w\rangle = 0$. Let $\vartheta = \vartheta(s)$ be a curve in E^3. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \left\{ p = (p_1, p_2, p_3) \in E^3 : \langle p, p\rangle = r^2 \right\}.$$

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E^3, the following Frenet formulae are given in [3], [5]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1,$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0.$$

Here, first and second curvature are defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -\langle N, B'\rangle$.

It is well-known that for a unit speed curve with non vanishing curvatures following proposition holds [3].

Proposition 2.1. Let $\varphi = \varphi(s)$ be a regular curve with curvatures κ and τ, then;
i. It lies on a sphere (S^2) if and only if

$$\frac{\tau}{\kappa} + \left[\frac{1}{\tau} \left(\frac{1}{\kappa} \right)' \right]' = 0. \tag{2}$$

ii. It is an inclined curve if and only if

$$\frac{\kappa}{\tau} = constant. \tag{3}$$

§3. Position vectors of some special curves in E^3

Let $\varphi = \varphi(s)$ be a unit speed curve in E^3. We can write this curve respect to Frenet frame $\{T, N, B\}$ as

$$\varphi = \varphi(s) = m_1 T + m_2 N + m_3 B. \tag{4}$$

Differentiating (4) and considering Frenet equations, we have a system of differential equation as follow:

$$\left\{ \begin{array}{c} m_1' - m_2\kappa - 1 = 0 \\ m_2' + m_1\kappa - m_3\tau = 0 \\ m_3' + m_2\tau = 0 \end{array} \right\}. \tag{5}$$

This system's general solution have not been found. Due to this, we give some special values to the components.

Case 3.1. $m_1 = c_1$=constant. From $(5)_1$ we have second component

$$m_2 = -\frac{1}{\kappa}. \tag{6}$$

And $(5)_2$ gives us third component as

$$m_3 = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)' + c_1\frac{\kappa}{\tau}. \tag{7}$$

m_2 and m_3 satisfie $(5)_3$, if we rewrite it, we have following differential equation:

$$-\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' + c_1\left(\frac{\kappa}{\tau}\right)' - \frac{\tau}{\kappa} = 0. \tag{8}$$

Considering (8) and Proposition 2.1., suffice it to say that following theorem holds:

Theorem 3.1. Let $\varphi = \varphi(s)$ be a unit speed curve and first component of position vector on Frenet axis be constant in E^3:

i. If first component is zero, then φ is a spherical curve (lies on S^2).

ii. If φ is an inclined curve, then φ lies on S^2. Therefore, φ is an inclined spherical curve.

iii. All normal curves in the space E^3 are spherical curves.

iv. Position vector all of normal curves in E^3, can be written as follow:

$$\varphi = -\frac{1}{\kappa}N - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' B. \tag{9}$$

Here, it is not difficult to see that statement ii is a characterization for inclined spherical curves in E^3.

Case 3.2. $m_2 = c_2 = constant \neq 0$. From equation $(5)_1$, we obtain

$$c_2 = \frac{m_1' - 1}{\kappa}. \tag{10}$$

From $(5)_2$, we write

$$m_3 = m_1\frac{\kappa}{\tau}. \tag{11}$$

Now, let us suppose φ is an inclined curve. In this case, If (10) and (11) are used in $(5)_3$, we have a differential equation as follow:

$$m_1'(\frac{\kappa}{\tau}+\frac{\tau}{\kappa})-\frac{\tau}{\kappa}=0. \tag{12}$$

From solution of (12), we have the components

$$m_1=\frac{\tau^2 s}{\kappa^2+\tau^2}+l, \tag{13}$$

$$m_2=\frac{-\kappa^2}{\kappa^2+\tau^2} \tag{14}$$

and

$$m_3=\frac{\kappa}{\tau}(\frac{\tau^2 s}{\kappa^2+\tau^2}+l), \tag{15}$$

where $l \in R$.

Corollary 3.2. Let $\varphi=\varphi(s)$ be a unit speed curve and second component of position vector on Frenet axis be constant in E^3. Then position vector of φ can be formed by the equations (13), (14) and (15).

Case 3.2.1. $m_2=0$. From (5), we have, respectively,

$$m_1=s+l \tag{16}$$

and

$$m_3=\frac{\kappa}{\tau}(s+l)=constant, \tag{17}$$

where $l \in R$.

Considering (16) and (17), we give following theorem.

Theorem 3.3. Let $\varphi=\varphi(s)$ be a unit speed rectifying curve in E^3. Then;

i. There is a relation among curvatures as

$$\frac{\kappa}{\tau}(s+l)=constant.$$

ii. Position vector of φ can be written as follow:

$$\varphi=\varphi(s)=(s+l)T+\frac{\kappa}{\tau}(s+l)B, \tag{18}$$

where $l \in R$.

Remark 3.4. The case $m_3=constant \neq 0$ is similar to case 3.2.

Case 3.3. $m_3=0$. Thus $\tau=0$. And here, let us suppose $\kappa=constant$. Using these in the system of differential equation, we have second order homogenous differential equation with constant coefficient as follow:

$$m_2''+m_2\kappa^2+\kappa=0. \tag{19}$$

Solution of (19) is elementary. Then, we write the components

$$m_1=A_2\sin\kappa s-A_1\cos\kappa s \tag{20}$$

and

$$m_2 = A_1 \cos \kappa s + A_2 \sin \kappa s + \frac{1}{\kappa}, \tag{21}$$

where A_1 and A_2 are real numbers.

Corollary 3.5. Let $\varphi = \varphi(s)$ be a unit speed osculating curve with constant first curvature in E^3. The position vector of φ can be composed with the equations (20) and (21).

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Almost everywhere stability of the third-ordered nonlinear system

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Abstract In order to study the stability of the third-ordered nonlinear system, density function is constructed, and an inequality condition is obtained. When the considered system satisfies the inequality for almost all initial value, its equilibrium is asymptotically stable almost everywhere.
Keywords Amost everywhere stability, density function, nonlinear system.

§1. Introduction

Stability is a very important factor in the research of high-ordered nonlinear systems. The techniques currently available in the literature are mostly based on Lyapunov's direct method [1], contraction mapping method [2], nonlinear measure method [3,4], nonlinear spectral radius method [5] and so on. All these methods make the research of stable property everywhere. And if the equilibrium of the considered system is asymptotically stable, there exists a neighborhood of the equilibrium which is called attraction region. In fact, many real nonlinear systems do not need its equilibrium acting as an attracting actor attracting any points of its neighborhood, and it only needs attract "enough many" points in this neighborhood. That is, the equilibrium is asymptotically stable almost everywhere (the equilibrium attract all points of this neighborhood except a set with zero Lebesgue measure). In 2001, Anders Ranzter [6] proposed the concept of a "dual" to Lyapunov's stability theorem, and he called the stability derived by this way as stability almost everywhere. In control theory, most control systems are in high order, such as robotic multi-fingered grasping system [1], neural networks and so on, and up to their research background, it's unnecessary or impossible for the equilibrium to attract every points in its neighborhood even it's very small. So it's valuable to study the almost everywhere stability of high-ordered systems.

In this paper, the almost everywhere stability of the third-ordered system is studied. Density functions are constructed as a tool to analyze the stability of the given system.

For convenience, the following notations were used in this paper:

(1) $\nabla\rho = \text{grad}\rho = \left(\frac{\partial\rho}{\partial x_1}\ \frac{\partial\rho}{\partial x_2}\ \cdots\ \frac{\partial\rho}{\partial x_n}\right),\ \rho : R^n \to R;$

(2) $\nabla \bullet F = \text{div}F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n},\ F : R^n \to R^n;$

(3) "$\bullet$" represents inner product.

§2. Main result and its proof

Consider a third-ordered nonlinear system

$$x^{'''} + \psi(x, x^{'})x^{''} + f(x, x^{'}) = 0, \qquad f(0,0) = 0, \tag{1}$$

or its equivalent system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{'} = \begin{pmatrix} y \\ z \\ -\psi(x,y)z - f(x,y) \end{pmatrix} = F(x,y,z). \tag{2}$$

It is assumed that solutions of (1) exist and are unique.

Theoretically, this is a very useful nonlinear system since (1) is a rather general third-ordered nonlinear differential equation. For example, many third-ordered differential equations which have been discussed in [8] are special cases of (1). However, most available research all focused on the study of everywhere stability and it is not necessary or impossible for many real nonlinear equations.

In this section, the almost everywhere stability of third-ordered system is studied. Density functions are constructed as a tool to analyze the stability of the given system, and the condition was obtained that the given nonlinear system is asymptotically stable almost everywhere. That is, we will prove the following main result:

Theorem. If there exists a positive real number α, such that the following inequalities

(1) $xf(x,y) > 0, \quad yf(x,y) > 0$;

(2) $\psi(x,y) > 0$;

(3) $\int_0^y (f(x,0) + y\psi_x(x,v)v + yf_x(x,v) + z\psi(x,y))\,\mathrm{d}v +$
$+\alpha^{-1}\psi(x,y)\left(\int_0^x f(u,0)\mathrm{d}u + \int_0^y (v\psi(x,v) + f(x,v))\mathrm{d}v\right)$
$< \psi(x,y)z^2(1 - \frac{1}{2}\alpha^{-1})$

hold for almost all x, then the equilibrium of the nonlinear system (2) is asymptotically stable almost everywhere.

First, we need a basic Lemma that is necessary in proof of Theorem.

Lemma. Given the equation $\dot{x}(t) = f(x(t))$, where $f \in C^1(R^n, R^n)$ and $f(0) = 0$, suppose there exists a non-negative scalar function $\rho \in C^1(R^n \setminus \{0\}, R)$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in R^n : |x| \geq 1\}$ and $[\nabla \bullet (f\rho)](x) > 0$ for almost all x. Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in (0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Proof. (See reference [6]).

By this Lemma, now we shall complete the proof of Theorem.

The proof of Theorem.

Let $\tilde{x} = (x, y, z)^{\top}$ and

$$V(\tilde{x}) = \int_0^x f(u,0)\mathrm{d}u + \int_0^y v\psi(x,v)\mathrm{d}v + \int_0^y f(x,v)\mathrm{d}v + \frac{1}{2}z^2,$$

then $V(\tilde{x}) \geq 0$ and when $\tilde{x} \neq 0$ we have $V(\tilde{x}) \neq 0$. That is, $V(\tilde{x})$ is positive definite. Let $\rho = V^{-\alpha}$, then $\rho \in C^1(R^3\backslash\{0\}, R)$ and $\rho(\tilde{x})F(\tilde{x})/|\tilde{x}|$ is integrable on $\{\tilde{x} \in R^3 : |\tilde{x}| \geq 1\}$.

For $\mathrm{div}F = -\psi(x,y)$ and

$$[\nabla \bullet (F\rho)](\tilde{x}) = V^{-\alpha-1}(\tilde{x})(V\nabla \bullet F - \alpha\nabla V \bullet F)(\tilde{x}),$$

thus the condition $[\nabla \bullet (F\rho)](\tilde{x}) > 0$ in Lemma will be satisfied if and only if $(V\nabla \bullet F - \alpha\nabla V \bullet F)(\tilde{x}) > 0$. From

$$\begin{aligned} & \nabla V \bullet F \\ = \;& V_x x' + V_y y' + V_z z' \\ = \;& f(x,0)y + y\int_0^y \psi_x(x,v)v\mathrm{d}v + y\int_0^y f_x(x,v)\mathrm{d}v - \psi(x,y)z^2 + \psi(x,y)yz, \end{aligned}$$

and

$$\begin{aligned} & \alpha^{-1}V\nabla \bullet F \\ = \;& -\alpha^{-1}\psi(x,y)\int_0^x f(u,0)\mathrm{d}u - \alpha^{-1}\psi(x,y)\int_0^y v\psi(x,v)\mathrm{d}v \\ & -\alpha^{-1}\psi(x,y)\int_0^y f(x,v)\mathrm{d}v - \frac{1}{2}\alpha^{-1}z^2\psi(x,y), \end{aligned}$$

then by Lemma, we know that if the inequality

$$\begin{aligned} & \int_0^y \left(f(x,0) + y\psi_x(x,v)v + yf_x(x,v) + z\psi(x,y)\right)\mathrm{d}v \\ & \quad +\alpha^{-1}\psi(x,y)\left(\int_0^x f(u,0)\mathrm{d}u + \int_0^y (v\psi(x,v) + f(x,v))\mathrm{d}v)\right) \\ < \;& \psi(x,y)z^2(1 - \frac{1}{2}\alpha^{-1}) \end{aligned}$$

holds, the equilibrium of the considered system is asymptotically stable almost everywhere.

This completes the proof of Theorem.

Simplifying the condition of the inequality in Theorem, we may immediately derive the following two corollaries.

Corollary 1. If there exists a positive real number $\alpha > \frac{1}{2}$, such that two functions $f(x,y)$ and $\psi(x,y)$ satisfy the inequalities

(1) $xf(x,y) > 0, \quad yf(x,y) > 0$;

(2) $\psi(x,y) > 0$;

(3) $\int_0^y (f(x,0) + y\psi_x(x,v)v + yf_x(x,v) + \psi(x,y)z)\mathrm{d}v < 0$;

(4) $\int_0^x f(u,0)\mathrm{d}u + \int_0^y (v\psi(x,v) + f(x,v))\mathrm{d}v < z^2(\alpha - \frac{1}{2})$

for almost all x, then the equilibrium of the nonlinear system (2) is asymptotically stable almost everywhere.

Corollary 2. If there exists a positive real number $\alpha > \frac{1}{2}$,such that two functions $f(x,y)$ and $\psi(x,y)$ satisfy the inequalities

(1) $xf(x,y) > 0, \quad yf(x,y) > 0$;

(2) $\psi(x,y) > 0$;

(3) $yf(x,0) + yz\psi(x,y) + y^2(\psi_x(x,\xi)\xi + f_x(x,\eta)) < 0, \forall \xi, \eta \in [0,y]$;

(4) $xf(s,0) + y(h\psi(x,h) + f(h,0)) < z^2(\alpha - \frac{1}{2}), \forall s \in [0,x], h \in [0,y]$

for almost all x, then the equilibrium of the nonlinear system (2) is asymptotically stable almost everywhere.

Since the system (1) and system (2) are equivalent systems, and the equilibrium $x^* = 0$ of system (1) answers to the equilibrium $\tilde{x}^* = 0$ of system (2), their stability properties are also equivalent. Thus the equilibrium of the third-ordered nonlinear system (1) is almost asymptotically stable when the inequality condition of Theorem is satisfied.

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On the indexes of beauty[1]

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Abstract For any fixed positive integer m, if there exists a positive integer n such that $n = m \cdot d(n)$, then m is called an index of beauty, where $d(n)$ is the Dirichlet divisor function. In this paper, we shall prove that there exist infinite positive integers such that each of them is not an index of beauty.

Keywords Dirichlet divisor function, index of beauty, elementary method.

§1. Introduction and result

For any positive integer n, the famous Dirichlet divisor function $d(n)$ is defined as the number of all distinct divisors of n. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n, then from the definition and properties of $d(n)$ we can easily get

$$d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_k + 1). \tag{1}$$

About the deeply properties of $d(n)$, many people had studied it, and obtained a series results, see references [2], [3] and [4]. In reference [5], Murthy introduced an index of beauty involving function $d(n)$ as follows: For any positive integer m, if there exists a positive integer n such that

$$m = \frac{n}{d(n)},$$

then m is called an index of beauty. At the same time, he also proposed the following conjecture:

Conjecture. Every positive integer is an index of beauty.

Maohua Le [6] gave a counter-example, and proved that 64 is not an index of beauty. We think that the conclusion in [6] can be generalization. This paper as a note of [6], we shall use the elementary method to prove the following general conclusion:

Theorem. There exists a set A including infinite positive integers such that each of $n \in A$ is not an index of beauty.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. For any prime $p \geq 5$, if we taking $m = p^{p-1}$, then we can prove that m is not an index of beauty. In fact, if m is an index of beauty, then there exists a positive integer n such that $n = p^{p-1} \cdot d(n)$. From this identity we

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can deduce that $p^{p-1} \mid n$. Let $n = p^{\alpha} \cdot b$ and $(p,\ b) = 1$, then from $n = p^{p-1} \cdot d(n)$ and (1) we have the identity

$$p^{\alpha-p+1} \cdot b = (\alpha + 1) \cdot d(b). \tag{2}$$

From (2) we may immediately deduce that $\alpha \geq p-1$. If $\alpha = p-1$, then (2) become $b = p \cdot d(b)$, this contradiction with $(b,\ p) = 1$. If $\alpha = p$, then (2) become $p \cdot b = (p+1) \cdot d(b)$, or

$$b = \left(1 + \frac{1}{p}\right) \cdot d(b). \tag{3}$$

It is clear that (3) is not possible if prime $p \geq 5$. In fact if $b = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$, then it is easily to check that (3) does not hold. If $b \geq 8$, then from (1) and the properties of $d(b)$ we can deduce that $b \geq \frac{3}{2} \cdot d(b) > \left(1 + \frac{1}{p}\right) \cdot d(b)$. So (3) does not also hold. So if $\alpha = p$, then (2) is not possible. If $\alpha \geq p+1$, let $\alpha - p = k \geq 1$, the (2) become

$$p^{k+1} \cdot b = (p + k + 1) \cdot d(b). \tag{4}$$

Note that $b \geq d(b)$ for all positive integer b, so formula (4) is not possible, since $p^{k+1} > p+k+1$ for all prime $p \geq 5$ and $k \geq 1$. Since there are infinite prime $p \geq 5$, and each p^{p-1} is not an index of beauty, so there exists a set A including infinite positive integers such that each of $n \in A$ is not an index of beauty. This completes the proof of Theorem.

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On the mean value of a new arithmetical function

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Abstract For any positive integer n, let $\phi(n)$ denotes the Euler function. That is, $\phi(n)$ denotes the number of all positive integers not exceeding n which are relatively prime to n. $g(n)$ denotes the smallest positive integer m such that $n \mid \phi(n) \cdot m$. The main purpose of this paper is using the elementary method to prove that for any positive integer group $(m_1,\ m_2,\ \cdots,\ m_k)$, we have the inequality $g\left(\prod_{i=1}^{k} m_i\right) \leq \prod_{i=1}^{k} g(m_i)$, and also for any positive integer k, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfying the equation $g\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} g(m_i)$.
Keywords Euler function $\phi(n)$, three prime theorem, elementary method.

§1. Introduction and results

For any positive integer n, the famous Euler function $\phi(n)$ is defined as the number of all positive integers not exceeding n which are relatively prime to n. Now we define a new arithmetical function $g(n)$ as follows: $g(n)$ denotes the smallest positive integer m such that $n \mid \phi(n) \cdot m$. That is, $g(n) = \min\{m :\ n \mid \phi(n) \cdot m\}$. For example, the fist few values of $g(n)$ are: $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(4) = 2$, $g(5) = 5$, $g(6) = 3$, $g(7) = 7$, $g(8) = 2$, $g(9) = 3$, $g(10) = 5$, $g(11) = 11$, $g(12) = 3$, $g(13) = 13$, $g(14) = 7$, $\cdots$.

About the elementary properties of $g(n)$, it seems that none had studied it before. From the definition of $g(n)$ we can easily deduce that for any prime p, $g(p^\alpha) = p$ and $P(n) \leq g(n) \leq \prod_{p|n} p$, where $P(n)$ denotes the largest prime divisor of n. If $n > 1$ and $n \neq 2^\alpha$, then $g(n)$ must be an odd number. $g(n)$ is also a square-free number.

The main purpose of this paper is to study the elementary properties of $g(n)$, and prove the following two conclusions:

Theorem 1. For any positive integer k and any positive integer group $(m_1,\ m_2,\ \cdots,\ m_k)$, we have the inequality

$$g\left(\prod_{i=1}^{k} m_i\right) \leq \prod_{i=1}^{k} g(m_i).$$

Theorem 2. For any positive integer $k \geq 3$, there exist infinite group positive integers

$(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation

$$g\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} g(m_i).$$

Whether there exist infinite group positive integers $(m_1,\ m_2)$ satisfying the equation

$$g(m_1 + m_2) = g(m_1) + g(m_2) \tag{1}$$

is an unsolved problem. If $m_1 > 1$ and $m_2 > 1$ satisfying the equation (1), then one of m_1 and m_2 must be a power of 2.

It is an interesting problem to study the mean value properties of $g(n)$. For any real number $x > 1$, we conjecture that

$$\sum_{n\le x} g(n) \sim C \cdot x^2,$$

where C is a computable constant.

§2. Proof of the theorems

In this section, we shall use the elementary method to prove our Theorems. First we prove Theorem 1. For any positive integer m, since $g(m) = g(m)$. So Theorem 1 holds if $k = 1$. Now we assume $k = 2$. For any positive integers m and n, if $(m,\ n) = 1$, then we have $\phi(mn) = \phi(m)\phi(n)$. This time let $g(mn) = t$, $g(m) = u$, $g(n) = v$. From the definition of $g(n)$ we have $mn \mid \phi(mn) \cdot t$, $m \mid \phi(m) \cdot u$, $n \mid \phi(n) \cdot v$, so we have

$$mn \mid \phi(m) \cdot \phi(n) \cdot uv = \phi(mn) \cdot uv.$$

By the definition of t we have $t \le uv$. Therefore, $g(mn) \le g(m) \cdot g(n)$.

If $(m,\ n) > 1$, without loss of generality we can assume that $m = p^\alpha$ and $n = p^\beta$, where p be a prime, α and β are two positive integers, then from the definition of $g(n)$ we have $g(p^\alpha p^\beta) = p$, $g(p^\alpha) = p$, $g(p^\beta) = p$. So $g(p^\alpha \cdot p^\beta) \le g(p^\alpha) \cdot g(p^\beta)$. From the above we know that for any positive integers m and n, we have the inequality

$$g(mn) \le g(m) \cdot g(n). \tag{2}$$

Now for any positive integer $k > 2$ and any positive integer group $(m_1,\ m_2,\ \cdots,\ m_k)$, from (2) we have

$$g\left(\prod_{i=1}^{k} m_i\right) = g\left(m_k \cdot \prod_{i=1}^{k-1} m_i\right) \le g\left(\prod_{i=1}^{k-1} m_i\right) \cdot g(m_k) \le \cdots \le \prod_{i=1}^{k} g(m_i).$$

This proves theorem 1.

To prove Theorem 2, we need the famous Vinogradov's Three Prime Theorem, which was stated as follows:

Lemma 1. There exists a sufficiently large constant $K > 0$ such that each odd integer $n > K$ can be written as a sum of three odd primes. That is, $n = p_1 + p_2 + p_3$, where p_i $(i = 1,\ 2,\ 3)$ are three odd primes.

Proof. (See reference [1]).

Lemma 2. Let $k \geq 3$ be an odd integer, then any sufficiently large odd integer n can be written as a sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k.$$

Proof. (See reference [2]).

Now we use these two Lemmas to prove our Theorem 2. If $k = 3$, then from Lemma 1 we know that for any prime p large enough, it can be written as a sum of three primes:

$$p = p_1 + p_2 + p_3.$$

Now for any positive integer $k \geq 4$ any prime p large enough, if k be an odd number, then by the definition of $g(n)$ we know that $g(p) = p$. From Lemma 2 we may immediately get

$$p = g(p) = g(p_1 + p_2 + \cdots + p_k) = p_1 + p_2 + \cdots + p_k = g(p_1) + \cdots + g(p_k).$$

If $k \geq 4$ be an even number, then $k - 1 \geq 3$ be an odd number. So if prime p large enough, then $p - 2$ can be written as a sum of $k - 1$ primes:

$$p - 2 = p_1 + p_2 + \cdots + p_{k-1}.$$

This implies

$$p = 2 + p_1 + p_2 + \cdots + p_{k-1}$$

or

$$p = g(p) = g(2 + p_1 + p_2 + \cdots + p_{k-1}) = 2 + p_1 + p_2 + \cdots + p_{k-1} = g(2) + g(p_1) + \cdots + g(p_{k-1}).$$

Since there are infinite primes p, so there exist infinite group positive integers $(m_1,\ m_2 \cdots,\ m_k)$ satisyfying the equation

$$g\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} g(m_i).$$

This completes the proof of Theorem 2.

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A new decision making method for fuzzy multiple attribute decision making problem [1]

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Abstract It is proved that product of interval number and triangular fuzzy number is a trapezoidal fuzzy number. This paper transformed a multiple attribute decision making problem with attribute weight of interval number and attribute value of triangular fuzzy number into trapezoidal fuzzy number multiple attribute decision making problem, gave TOPSIS to trapezoidal number multiple attribute decision making problem, and analyzed a practical problem by this method.

Keywords Interval number, triangular fuzzy number, trapezoidal fuzzy number, multiple attribute decision making, TOPSIS.

§1. Introduction

The multiple decision making (MADM) problems mainly solve to rank alternatives with multiple attributes among finite alternatives. It is an extremely active topic in discipline research of modern decision science, systems engineering, management science and operations research, etc. Its theory and method are applied to many domains: engineering design, economy, management and military affairs, etc. Reference [1],[2],[3] studied MADM problems that attribute weight information is interval number, attribute value is crisp number from various angles. Reference [4]-[7] studied MADM problems that attribute weight information is interval number, attribute value is also interval number from various angles. At present research results about methods for MADM problems with attribute weight of interval number and attribute value of triangular fuzzy number are few. This paper focuses on this kind of MADM problems mainly. This paper will prove that product of interval number and the triangular fuzzy number is a trapezoidal fuzzy number and transform a multiple attribute decision making problem with attribute weight of interval number and attribute value of and triangular fuzzy number into trapezoidal fuzzy number multiple attribute decision making problem, then gives TOPSIS to

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trapezoidal number multiple attribute decision making problem. Finally, an example is used to illustrate the proposed method.

§2. Presentation of the problem

Definition 1. Let a be interval number, where $a = [a^-, a^+] = \{x | a^- \leq x \leq a^+, a^- \leq a^+\}$. If $0 \leq a^- \leq a^+$, then a is called a positive interval number.

Definition 2. Fuzzy number $\widetilde{p}$ is called a triangular fuzzy number if its membership function $\mu_{\widetilde{p}}(x) : R \to [0,1]$ is described as

$$\mu_{\widetilde{p}}(x) = \begin{cases} \dfrac{x}{m-l} - \dfrac{l}{m-l}, & x \in [l,m] \\ \dfrac{x}{m-u} - \dfrac{u}{m-u}, & x \in [m,u] \\ 0, & \texttt{otherwise} \end{cases}$$

where $x \in R$, $l \leq m \leq u, l, u$ stand for the lower and upper bounds of the support of $\widetilde{p}$ respectively, and m for the most possible value of $\widetilde{p}$. The triangular fuzzy number $\widetilde{p}$ can be denoted by (l, m, u). When $l = m = u$, $\widetilde{p}$ is a real number.

Definition 3. Fuzzy number $\widetilde{t}$ is called a trapezoidal fuzzy number if its membership function $\mu_{\widetilde{t}}(x) : R \to [0,1]$ is described as

$$\mu_{\widetilde{t}}(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x < b \\ 1, & b \leq x < c \\ \dfrac{d-x}{d-c}, & c \leq x < d \\ 0, & d \leq x \end{cases}$$

where $a \leq b \leq c \leq d$, a and d stand for the lower and upper bounds of the support of $\widetilde{t}$ respectively, and closed interval $[b, c]$ for the most possible value of $\widetilde{t}$. The trapezoidal fuzzy number $\widetilde{t}$ can be denoted by (a, b, c, d). When $a = b, c = d$, $\widetilde{t}$ becomes a ordinary interval number. When $b = c$, $\widetilde{t}$ turns into triangular fuzzy number. Interval number and triangular fuzzy number are special cases of trapezoidal fuzzy number. When $a = b = c = d$, it is a ordinary real number.

The λ-cut of a trapezoidal fuzzy number $\widetilde{t} = (a, b, c, d)$ is

$$t_\lambda = [a + \lambda(b-a), d + \lambda(c-d)], \lambda \in (0,1].$$

Definition 4.[8] Let vector $\widetilde{T} = (\widetilde{t}_1, \widetilde{t}_2, \cdots, \widetilde{t}_m)$ and $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_m)$, where trapezoidal fuzzy number $\widetilde{t}_i = (\widetilde{t}_{1i}, \widetilde{t}_{2i}, \widetilde{t}_{3i}, \widetilde{t}_{4i})$, $\widetilde{v}_i = (\widetilde{v}_{1i}, \widetilde{v}_{2i}, \widetilde{v}_{3i}, \widetilde{v}_{4i})$, $i = 1, 2, \cdots, m$. The distance between $\widetilde{T}$ and $\widetilde{V}$ is defined as

$$D(\widetilde{T}, \widetilde{V}) = \|\widetilde{T} - \widetilde{V}\| = \frac{1}{4}\sqrt{\sum_{i=1}^{m}[(t_{1i} - v_{1i})^2 + (t_{2i} - v_{2i})^2 + (t_{3i} - v_{3i})^2 + (t_{4i} - v_{4i})^2]}.$$

We will use product of interval number and triangular fuzzy number in our discussion, for this reason we prove the theorem as follows.

Theorem. Let $\alpha = [\alpha_1, \alpha_2]$ be arbitrary positive interval number, $\widetilde{p} = (l, m, n)$ be arbitrary triangular fuzzy number, then the product of α and $\widetilde{p}$ is

$$\alpha \cdot \widetilde{p} = [\alpha_1, \alpha_2] \cdot (l, m, u) = (\alpha_1 l, \alpha_1 m, \alpha_2 m, \alpha_2 u).$$

Namely the product of positive interval number and triangular fuzzy number is trapezoidal fuzzy number.

Proof. For $\forall\lambda \in (0, 1]$, the λ-cut of $\widetilde{p}$ is

$$p_\lambda = [l + \lambda(m - l), u + \lambda(m - u)].$$

According to extension principle we have

$$\begin{aligned} \alpha \cdot \widetilde{p} &= \bigcup_{0<\lambda\leq 1} \lambda\{[\alpha_1, \alpha_2][l + \lambda(m - l), u + \lambda(m - u)]\} \\ &= \bigcup_{0<\lambda\leq 1} \lambda[\alpha_1 l + \lambda(\alpha_1 m - \alpha_1 l), \alpha_1 u + \lambda(\alpha_2 m - \alpha_2 u)] \\ &= (\alpha_1 l, \alpha_1 m, \alpha_2 m, \alpha_2 u). \end{aligned}$$

§3. Principle and step

This paper considers the fuzzy MADM problems that the alternatives are X_1, X_2, $\cdots$, X_n and attributes are $G_1, G_2, \cdots, G_m$. The fuzzy MADM problems with incomplete weight information, w_j denotes the weight of attribute G_j, $w_j \in [\alpha_j, \beta_j]$, where, $0 \leq \alpha_j \leq \beta_j \leq 1$, $j = 1, 2, \cdots, m$, $w_1 + w_2 + \cdots + w_m = 1$, is known to us. Triangular fuzzy number (l_{ij}, m_{ij}, u_{ij}) is the value for alternative X_i with respect to attribute G_j, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. Principle and step for the fuzzy MADM problems that weight of attribute is interval number and value of attribute is triangular fuzzy number are as follows:

Step 1. Construct triangular fuzzy number decision-making matrix $\widetilde{A} = (x_{ij})_{n\times m}$. Where triangular fuzzy number $\widetilde{x}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ is the value for alternative X_i with respect to attribute G_j, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$.

Step 2. Normalize decision-making matrix $\widetilde{A}$.

Generally speaking, the attribute values have different physical dimension from each other, the decision matrix needs to be normalized so as to transform the various attribute values into comparable values. Therefore, we can obtain the normalized fuzzy decision matrix denoted by $\overline{A} = (\overline{x}_{ij})_{n\times m}$ according to formulas in reference [9].

Let $\widetilde{x}_{ij} = (l_{ij}, m_{ij}, u_{ij})$, then

$$\overline{x}_{ij} = (\frac{l_{ij}}{u_j^{\max}}, \frac{m_{ij}}{u_j^{\max}}, \frac{u_{ij}}{u_j^{\max}}), j \in J_1, \overline{x}_{ij} = (\frac{l_j^{\min}}{u_{ij}}, \frac{l_j^{\min}}{m_{ij}}, \frac{l_j^{\min}}{l_{ij}}), j \in J_2,$$

where $u_j^{\max} = \max_i u_{ij}$, if $j \in J_1$, $l_j^{\min} = \min_i l_{ij}$, if $j \in J_2$, where J_1 and J_2 are the set of benefit criteria and cost criteria, respectively.

Step 3. Construct the weighted normalized fuzzy decision matrix.

$$\widetilde{Y} = W \otimes \overline{A} = \begin{bmatrix} w_1\overline{x}_{11} & w_2\overline{x}_{12} & \cdots & w_m\overline{x}_{1m} \\ w_1\overline{x}_{21} & w_2\overline{x}_{22} & \cdots & w_m\overline{x}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ w_1\overline{x}_{n1} & w_2\overline{x}_{n2} & \cdots & w_m\overline{x}_{nm} \end{bmatrix} = \begin{bmatrix} \widetilde{y}_{11} & \widetilde{y}_{12} & \cdots & \widetilde{y}_{1m} \\ \widetilde{y}_{21} & \widetilde{y}_{22} & \cdots & \widetilde{y}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{y}_{n1} & \widetilde{y}_{n2} & \cdots & \widetilde{y}_{nm} \end{bmatrix}$$

where

$$\begin{aligned} \widetilde{y}_{ij} &= w_j \cdot \overline{x}_{ij} \\ &= [\alpha_j, \beta_j] \cdot (\bar{l}_{ij}, \overline{m}_{ij}, \overline{u}_{ij}) \\ &= (\alpha_j \bar{l}_{ij}, \alpha_j \overline{m}_{ij}, \beta_j \overline{m}_{ij}, \beta_j \overline{u}_{ij}) \\ &= (a_{ij}, b_{ij}, c_{ij}, d_{ij}), \\ &\quad\; i = 1, 2, \cdots, n, j = 1, 2, \cdots, m. \end{aligned}$$

Step 4. Determine the fuzzy positive-ideal solution X^+ and fuzzy negative-ideal solution X^-, respectively, and

$$X^+ = (y_1^+, y_2^+, \cdots, y_m^+), X^- = (y_1^-, y_2^-, \cdots, y_m^-)$$

where

$$y_j^+ = (\max_{1 \le i \le n} a_{ij}, \max_{1 \le i \le n} b_{ij}, \max_{1 \le i \le n} c_{ij}, \max_{1 \le i \le n} d_{ij}) = (a_j^+, b_j^+, c_j^+, d_j^+), j = 1, 2, \cdots, m,$$

$$y_j^- = (\min_{1 \le i \le n} a_{ij}, \min_{1 \le i \le n} b_{ij}, \min_{1 \le i \le n} c_{ij}, \min_{1 \le i \le n} d_{ij}) = (a_j^-, b_j^-, c_j^-, d_j^-), j = 1, 2, \cdots, m.$$

Step 5. Calculate the distance of each alternative from fuzzy positive-ideal solution X^+ and fuzzy negative-ideal solution X^-, respectively.

The distance of each alternative from X^+ and X^- can be calculated as

$$D_i^+ = D(X_i, X^+) = \frac{1}{4}\sqrt{\sum_{j=1}^{m} [(a_{ij} - a_j^+)^2 + (b_{ij} - b_j^+)^2 + (c_{ij} - c_j^+)^2 + (d_{ij} - d_j^+)^2]}\ ,$$

$$D_i^- = D(X_i, X^-) = \frac{1}{4}\sqrt{\sum_{j=1}^{m} [(a_{ij} - a_j^-)^2 + (b_{ij} - b_j^-)^2 + (c_{ij} - c_j^-)^2 + (d_{ij} - d_j^-)^2]},$$

$i = 1, 2, \cdots, n.$

Step 6. Calculate the relative closeness coefficient of each alternative to the ideal solution.

$$C_i = \frac{D_i^-}{D_i^- + D_i^+}, i = 1, 2, \cdots, n.$$

Step 7. Determine the ranking order of all alternatives according to the closeness coefficient in the descending order and select the best one.

§4. Numerical example

This example is taken from [10], one of the Middle East country intends to purchase a certain number jet fighter from US, The American Pentagon's officials have provided information relative to 4 kinds of jet fighter types which are permitted to sell. The Middle East country sends out the expert group to carry on the detailed inspection to these 4 kinds of jet fighters, Inspection results are shown in Table 1. Which kind of jet fighter should they purchase to make total utility value of decision making biggest?

Table 1: Expert group inspection result to 4 kinds of jet fighters

Type	Maximum speed (Ma)	Cruising radius (mile)	Peak load (pond)	Price (/\$ 10^6)	Reliability	Maintainability
X_1	2.0	1500	20000	5.5	High	Very high
X_2	2.5	2700	18000	6.5	Low	General
X_3	1.8	2000	21000	4.5	High	High
X_4	2.2	1800	20000	5.0	General	General

weight w_j of attribute G_j cannot determine completely ($j = 1, 2, \cdots, 6$), but according to the expert analysis we know $w_1 \in [0.15, 0.20]$, $w_2 \in [0.1, 0.15]$, $w_3 \in [0.1, 0.15]$, $w_4 \in [0.1, 0.15]$, $w_5 \in [0.15, 0.20]$, $w_6 \in [0.2, 0.25]$.

Step 1. Correspond to normalized fuzzy decision-making matrix as follows:

$$\begin{bmatrix} (0.8,0.8,0.8) & (0.56,0.56,0.56) & (0.95,0.95,0.95) & (0.82,0.82,0.82) & (0.6,0.8,1) & (0.7,0.9,1) \\ (1,1,1) & (1,1,1) & (0.86,0.86,0.86) & (0.69,0.69,0.69) & (0,0.2,0.4) & (0.3,0.5,0.7) \\ (0.72,0.72,0.72) & (0.74,0.74,0.74) & (1,1,1) & (1,1,1) & (0.6,0.8,1) & (0.6,0.8,1) \\ (0.88,0.88,0.88) & (0.67,0.67,0.67) & (0.95,0.95,0.95) & (0.9,0.9,0.9) & (0.3,0.5,0.7) & (0.3,0.5,0.7) \end{bmatrix}$$

Step 2. Construct the weighted normalized fuzzy decision matrix.

$$\left[\begin{array}{llll} (0.12,0.12,0.16,0.16) & (0.056,0.056,0.084,0.084) & (0.095,0.095,0.1425,0.1425) & \to \\ (0.15,0.15,0.2,0.2) & (0.1,0.1,0.15,0.15) & (0.086,0.086,0.129,0.129) & \to \\ (0.108,0.108,0.144,0.144) & (0.074,0.074,0.111,0.111) & (0.1,0.1,0.15,0.15) & \to \\ (0.132,0.132,0.176,0.176) & (0.067,0.067,0.1005,0.1005) & (0.095,0.095,0.1425,0.1425) & \to \end{array} \right.$$

$$\left. \begin{array}{lll} (0.082,0.082,0.123,0.123) & (0.09,0.12,0.16,0.20) & (0.14,0.18,0.225,0.25) \\ (0.069,0.069,0.1035,0.1035) & (0,0.03,0.04,0.08) & (0.06,0.1,0.125,0.175) \\ (0.1,0.1,0.15,0.15) & (0.09,0.12,0.16,0.20) & (0.12,0.16,0.2,0.25) \\ (0.09,0.09,0.135,0.135) & (0.045,0.075,0.1,0.14) & (0.06,0.1,0.125,0.175) \end{array} \right]$$

Step 3. Determine the fuzzy positive-ideal solution X^+ and fuzzy negative-ideal solution X^-, respectively,

$$X^+ = [(0.15,0.15,0.20,0.20),(0.1,0.1,0.15,0.15),(0.1,0.1,0.15,0.15),\\ (0.1,0.1,0.15,0.15),(0.09,0.12,0.16,0.20),(0.14,0.18,0.225,0.25)]$$

$$X^- = [(0.108,0.108,0.144,0.144),(0.056,0.056,0.084,0.084),(0.086,0.086,0.129,0.129),\\ (0.069,0.069,0.1035,0.1035),(0,0.03,0.04,0.08),(0.06,0.1,0.125,0.175)]$$

Step 4. Calculate the distance of each alternative from fuzzy positive-ideal solution and fuzzy negative-ideal solution, respectively, as Table 2.

Table 2: The distance measurement

	X_1	X_2	X_3	X_4
D^+	0.03522	0.06953	0.03124	0.05555
D^-	0.06885	0.03740	0.06758	0.03412

Step 5. Calculate the relative closeness coefficient of each alternative to the ideal solution as

$$C_1 = 0.66157, \quad C_2 = 0.34976, \quad C_3 = 0.68387, \quad C_4 = 0.38051.$$

Step 6. According to the closeness coefficient in the descending order, the ranking order of four alternatives can be determined as follows:

$$X_3 \succ X_1 \succ X_4 \succ X_2.$$

Therefore, the best selection is X_3.

The example result by method of this paper proposal is consistent with result by method of fuzzy compromise decision method which is first synthesis, then weighted decision making pattern introduced in reference [10]. Generally speaking, except some decision-making situations with quite obvious differences, the result which the different method obtains isn't always consistent. Therefore in the actual decision-making, we should use several different methods to solve, synthesize and compare result from each method in order to obtain a relatively reliable conclusion.

§5. Conclusion

This paper has proven the product of interval number and the triangular fuzzy number is a trapezoidal fuzzy number. We have proposed new method for multiple attribute decision making problem with interval number attribute weight and triangular fuzzy number attribute value. A numerical example is used to illustrate the method. Through the example, we can find this method is reliable, the thinking is clear and is one practical decision making method.

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A new Smarandache function and its elementary properties

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Abstract For any positive integer n, we define a new Smarandache function $G(n)$ as the smallest positive integer m such that $\prod_{k=1}^{m} \phi(k)$ is divisible by n, where $\phi(n)$ is the Euler function. The main purpose of this paper is using the elementary methods to study the elementary properties of $G(n)$, and give three interesting formulas for it.

Keywords A new Smarandache function, series, inequality.

§1. Introduction and results

For any positive integer n, we define a new Smarandache function $G(n)$ as the smallest positive integer m such that $\prod_{k=1}^{m} \phi(k)$ is divisible by n. That is,

$$G(n) = \min\{m :\ n \mid \prod_{k=1}^{m} \phi(k),\ m \in N\},$$

where $\phi(n)$ is the Euler function, N denotes the set of all positive integers. For example, the first few values of $G(n)$ are: $G(3) = 7$, $G(4) = 4$, $G(5) = 11$, $G(6) = 7$, $G(7) = 29$, $G(8) = 5$, $G(9) = 9$, $G(10) = 11$, $G(11) = 23$, $G(12) = 7$, $G(13) = 43$, $G(14) = 29$, $G(15) = 11$, $G(16) = 5$ $\cdots$. About the properties of this function, it seems that none had studied it yet, at last we have not seen any related papers before. Recently, Professor Zhang Wenpeng asked us to study the arithmetical properties of $G(n)$. The main purpose of this paper is using the elementary methods to study this problem, and prove the following three conclusions:

Theorem 1. For any prime p, we have the calculating formulae

$$G(p) = \min\{p^2,\ q(p,\ 1)\};$$

$$G(p^2) = q(p,\ 2), \text{ if } q(p,\ 2) < p^2;\ G(p^2) = p^2, \text{ if } q(p,\ 1) < p^2 < q(p,\ 2);$$

$$G(p^2) = q(p,\ 1), \text{ if } p^2 < q(p,\ 1) < 2p^2; \text{ and } G(p^2) = 2p^2, \text{ if } q(p,\ 1) > 2p^2,$$

where $q(p,\ i)$ is the i-th prime in the arithmetical series $\{np + 1\}$.

Theorem 2. $G(n)$ is a Smarandache multiplicative function, and moreover, the Dirichlet series $\sum_{n=1}^{\infty} \frac{G(n)}{n^2}$ is divergent.

Theorem 3. Let $k \geq 2$ be a fixed positive integer, then for any positive integer group $(m_1,\ m_2,\ \cdots,\ m_k)$, we have the inequality

$$G(m_1 m_2 \cdots m_k) \leq G(m_1)G(m_2)\cdots G(m_k).$$

Note. For any positive integer n, we found that $n = 1,\ 4,\ 9$ are three positive integer solutions of the equation $G(n) = n$. Whether there exist infinite positive integers n such that the equation $G(n) = n$ is an interesting problem. We conjecture that the equation $G(n) = n$ has only three positive integer solutions $n = 1,\ 4,\ 9$. This is an unsolved problem.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorems directely . First we prove Theorem 1. Let $G(p) = m$, namely $p \mid \prod_{k=1}^{m} \phi(k)$, $p \nmid \prod_{k=1}^{s} \phi(k), 0 < s < m$. If $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$ be the factorization of m into prime powers, then $\phi(m) = q_1^{\alpha_1 - 1}(q_1 - 1)q_2^{\alpha_2 - 1}(q_2 - 1)\cdots q_r^{\alpha_r - 1}(q_r - 1)$, from the definition of $G(n)$ we can deduce that p divides one of $\phi(q_i^{\alpha_i})$, $1 \leq i \leq r$. So p divides $q_i^{\alpha_i - 1}$ or $q_i - 1$. We discuss it in the following two cases:

(i). If $p \mid q_i - 1$, then we must have $m = q_i = lp + 1$ be the smallest prime in the arithmetical series $\{kp + 1\}$.

(ii). If $p \mid q_i^{\alpha_i - 1}$, then we have have $\alpha_i = 2$, $m = q_i^2 = p^2$.

Combining (i) and (ii) we may immediately deduce that $G(p) = \min\{p^2,\ lp + 1\}$, where $lp + 1$ be the smallest prime in the arithmetical series $\{kp + 1\}$. Similarly, we can also deduce the calculating formulae for $G(p^2)$. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers, $G(p_i^{\alpha_i}) = m_i$, $1 \leq i \leq r$, $m = \max\{m_1,\ m_2,\ \cdots,\ m_r\}$, then from the definition of $G(n)$ we have $p_i^{\alpha_i}$ divides $\phi(1)\phi(2)\cdots\phi(m_i)$ for all $1 \leq i \leq r$. So $p_i^{\alpha_i}$ divides $\phi(1)\phi(2)\cdots\phi(m)$. Since $(p_i,\ p_j) = 1$, $i \neq j$, so we must have $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ divides $\phi(1)\phi(2)\cdots\phi(m)$. Therefore, $G(n) = m = \max\{G(p_1^{\alpha_1}),\ G(p_2^{\alpha_2}),\ \cdots,\ G(p_r^{\alpha_r})\}$. So $G(n)$ is a Smarandache multiplicative function. From Theorem 1 we may immediately get

$$\sum_{n=1}^{\infty} \frac{G(n)}{n^2} > \sum_{p} \frac{G(p)}{p^2} > \sum_{p} \frac{1}{p} = \infty.$$

So the Dirichlet series $\sum_{n=1}^{\infty} \frac{G(n)}{n^2}$ is divergent.

Finally, we prove Theorem 3. First we prove that the inequality $G(m_1 m_2) \leq G(m_1)G(m_2)$ holds for any positive integer m_1 and m_2.

Let $G(m_1) = u$, $G(m_2) = v$, from the definition of $G(n)$ we can easily get

$$m_1 \mid \prod_{i=1}^{u} \phi(i)\ ,\ m_2 \mid \prod_{i=1}^{v} \phi(i).$$

Without loss of generality we can assume $u \leq v$, then

$$\prod_{k=1}^{uv} \phi(k) = \prod_{k=1}^{u} \phi(k) \cdot \prod_{k=u+1}^{uv} \phi(k) = \prod_{k=1}^{u} \phi(k) \cdot \phi(u+1) \cdots \phi(v) \cdots \phi(2v) \cdots \phi(uv)$$

Notice that

$$\phi(2) \mid \phi(2v),\ \phi(3) \mid \phi(3v),\ \cdots,\ \phi(u) \mid \phi(uv),\ \phi(u+1) \mid \phi(u+1),\ \cdots,\ \ \phi(v) \mid \phi(v),$$

this means that

$$\prod_{i=1}^{v} \phi(i) \mid \prod_{k=u+1}^{uv} \phi(k),$$

or

$$\prod_{i=1}^{u} \phi(i) \prod_{i=1}^{v} \phi(i) \mid \prod_{k=1}^{uv} \phi(k).$$

Hence

$$m_1 m_2 \mid \prod_{k=1}^{uv} \phi(k).$$

From the definition of $G(n)$ we know that $G(m_1 m_2) \leq uv$, or

$$G(m_1 m_2) \leq G(m_1) G(m_2).$$

If $k \geq 3$, then applying the above conclusion we can deduce that

$$\begin{aligned}
& G(m_1 m_2 \cdots m_k) = G(m_1(m_2 \cdots m_k)) \leq G(m_1) G(m_2 \cdots m_k) \\
& \leq G(m_1) G(m_2) G(m_3 \cdots m_k) \\
& \cdots\cdots\cdots \\
& \leq G(m_1) G(m_2) \cdots G(m_k).
\end{aligned}$$

This completes the proof of Theorem 3.

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On some congruent properties of the Smarandache numerical carpet

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Abstract In this paper, we use the elementary methods to study some divisibility of the Smarandache numerical carpet, and obtain two interesting congruent theorems.

Keywords Divisibility, Smarandache numerical carpet, elementary method.

§1. Introduction and results

In his book "Only problems, not solutions", F.Smarandache introduced many interesting sequences and arithmetical functions, at the same time, he also proposed 105 unsolved problems and conjectures. One of the problems is that he asked us to study the properties of the numerical carpet. It has the general form

$$\begin{array}{c} 1 \\ 1\ a\ 1 \\ 1\ a\ b\ a\ 1 \\ 1\ a\ b\ c\ b\ a\ 1 \\ 1\ a\ b\ c\ d\ c\ b\ a\ 1 \\ 1\ a\ b\ c\ d\ e\ d\ c\ b\ a\ 1 \\ 1\ a\ b\ c\ d\ c\ b\ a\ 1 \\ 1\ a\ b\ c\ b\ a\ 1 \\ 1\ a\ b\ a\ 1 \\ 1\ a\ 1 \\ 1 \end{array}$$

On the border of level 0, the elements are equal to"1", they form rhombus.

Next, on the border of level 1, the elements are equal to "a", where "a" is the sum of all elements of the previous border, the "a"s form a rhombus too inside the previous one.

Next again, on the border of level 2, the elements are equal to "b", where "b" is the sum of all elements of the previous border, the "b"s form a rhombus too inside the previous one.

And so on $\cdots$

It can be described as

$$
\begin{array}{c}
1 \\
\cdots \\
1\ a_1\ \cdots\ a_1\ 1 \\
1\ a_1\ a_2\ \cdots\ a_2\ a_1\ 1 \\
1\ a_1\ a_2\ a_3\ \cdots\ a_3\ a_2\ a_1\ 1 \\
1\ a_1\ a_2\ \cdots\ a_2\ a_1\ 1 \\
1\ a_1\ \cdots\ a_1\ 1 \\
\cdots \\
1
\end{array}
$$

Let $n,\ k$ be integers with $n \geq 0$ and $k \geq 0$, the $n-th$ square can be presented as a set of sub-squares. In [2], Mladen and T. Krassimir gave and proved the general formula:

$$a_k = 4(k+1)\prod_{i=0}^{k-2}(4n+1-4i),$$

where integer $k \geq 2$.

Let $C(n,\ k)$ denotes a_k in n-th square, then

$C(n,\ k) = 1$, for $k = 0$;
$C(n,\ k) = 4(n+1)$, for $k = 1$;
and $C(n,\ k) = 4(n+1)\prod\limits_{i=0}^{k-2}(4n+1-4i)$, for $2 \leq k \leq n+1$.

The main purpose of this paper is using the elementary methods to study the divisibility properties of the sequence $\{C(n,\ k)\}$, and give two interesting congruent theorems. That is, we shall prove the following conclusions:

Theorem 1. Let n be a positive integer. Then for any prime p with $4n-4k+9 \leq p \leq 4n+1$ and $p \equiv 1 \bmod 4$, we have the congruence

$$C(n,\ k) \equiv 0 \bmod p.$$

Theorem 2. Let n be a positive integer, p be any prime with $n-k+3+\left[\dfrac{n-k+2}{3}\right] \leq p \leq n+\left[\dfrac{n+1}{3}\right]$ and $p \equiv 3 \bmod 4$, then we have the congruence

$$C(n,\ k) \equiv 0 \bmod p.$$

§2. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof of the theorems. First we prove Theorem 1. For any positive integer n and prime $p \equiv 1 \bmod 4$, if $4n - 4k + 9 \leq p \leq 4n + 1$, then it is clear that $4n - 4k + 9 \leq 4n + 1$ or $k \geq 2$, from this we have

$$C(n,\ k) = 4(n+1)\prod_{i=0}^{k-2}(4n - 4i + 1) = 4(n+1)\prod_{i=0}^{k-2}(4(n-i)+1).$$

It is obviously that $C(n,\ 2) = 4(n+1)(4n+1)$, so if $k = 2$, then from $4n - 4k + 9 \leq p \leq 4n + 1$ we know that $p = 4n + 1$, and $C(n,\ 2) \equiv 0 \bmod p$. This means that Theorem 1 is true.

Now we assume that $2 < k \leq n$, so that

$$\begin{aligned} C(n,\ k) &= 4(n+1)\prod_{i=0}^{k-2}(4(n-i)+1) \equiv (n+1)\cdot 4^k \cdot \prod_{i=0}^{k-2}\left(n - i + \overline{4}\right) \bmod p \\ &\equiv (n+1)\cdot 4^k \cdot \prod_{i=0}^{k-2}\left(n - i + \frac{3p+1}{4}\right) \bmod p. \end{aligned} \tag{1}$$

where $\overline{x}$ denotes the solution of the congruence equation $x \cdot \overline{x} \equiv 1 \bmod p$, and $\overline{4} = \dfrac{3p+1}{4}$.

Since $4n - 4k + 9 \leq p \leq 4n + 1$, so there must exist an integer $0 \leq i \leq k - 2$ such that $p = n - i + \dfrac{3p+1}{4}$. Therefore, from (1) we may immediately deduce that

$$C(n,\ k) = 4(n+1)\prod_{i=0}^{k-2}(4(n-i)+1) \equiv 0 \bmod p.$$

This proves Theorem 1.

Now we prove Theorem 2. Let n be a positive integer, p be a prime with

$$n - k + 3 + \left[\frac{n-k+2}{3}\right] \leq p \leq n + \left[\frac{n+1}{3}\right] \text{ and } p \equiv 3 \bmod 4.$$

Note that $\overline{4} = \dfrac{p+1}{4}$, then from the definition of $C(n,\ k)$ we have

$$\begin{aligned} C(n,\ k) &= 4(n+1)\prod_{i=0}^{k-2}(4(n-i)+1) \equiv (n+1)\overline{4}^{k-2}\prod_{i=0}^{k-2}\left(n - i + \overline{4}\right) \bmod p \\ &\equiv (n+1)\cdot\left(\frac{p+1}{4}\right)^{k-2}\prod_{i=0}^{k-2}\left(n - i + \frac{p+1}{4}\right) \bmod p. \end{aligned} \tag{2}$$

From congruence (2) we know that if $n - k + 2 + \dfrac{p+1}{4} \leq p \leq n + \dfrac{p+1}{4}$, then $C(n,\ k) \equiv 0 \bmod p$. The inequality $n - k + 2 + \dfrac{p+1}{4} \leq p \leq n + \dfrac{p+1}{4}$ implies

$$n - k + 3 + \frac{n-k}{3} \leq p \leq n + \frac{n+1}{3}.$$

Since prime p must be a positive integer, so we have

$$n-k+3+\left[\frac{n-k+2}{3}\right]\leq p\leq n+\left[\frac{n+1}{3}\right].$$

This completes the proof of Theorem 2.

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On the Smarandache prime part sequences and its two conjectures[1]

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Abstract For any positive integer $n \geq 2$, the Smarandache Inferior Prime Part $\{p_p(n)\}$ is defined as the largest prime less than or equal to n; The Smarandache Superior Prime Part $\{P_p(n)\}$ is defined as the smallest prime greater than or equal to n. In this paper, we defined some determinants involving the Smarandache prime part sequences, and introduced two conjectures proposed by professor Zhang Wenpeng.

Keywords Smarandache prime part sequences, Determinant, Conjecture.

§1. Introduction

For any positive integer $n \geq 2$, the Smarandache Inferior prime part $\{p_p(n)\}$ is defined as the largest prime less than or equal to n. The Smarandache Superior prime part $\{P_p(n)\}$ as the smallest prime greater than or equal to n. For example, the first few values of these sequences are $p_p(2) = 2$, $p_p(3) = 3$, $p_p(4) = 3$, $p_p(5) = 5$, $p_p(6) = 5$, $p_p(7) = 7$, $p_p(8) = 7$, $p_p(9) = 7$, $p_p(10) = 7$, $p_p(11) = 11$, $p_p(12) = 11$, $p_p(13) = 13$, $p_p(14) = 13$, $p_p(15) = 13$, $p_p(16) = 13$, $p_p(17) = 17$, $p_p(18) = 17$, $p_p(19) = 19$, $\cdots\cdots$.

$P_p(1) = 2$, $P_p(2) = 2$, $P_p(3) = 3$, $P_p(4) = 5$, $P_p(5) = 5$, $P_p(6) = 7$, $P_p(7) = 7$, $P_p(8) = 11$, $P_p(9) = 11$, $P_p(10) = 11$, $P_p(11) = 11$, $P_p(12) = 13$, $P_p(13) = 13$, $P_p(14) = 17$, $P_p(15) = 17$, $P_p(16) = 17$, $P_p(17) = 17$, $P_p(18) = 19$, $P_p(19) = 19$, $\cdots\cdots$.

From the definitions of these sequences we know that for any prime q, we have $p_p(q) = P_p(q) = q$.

In his books "Only problem, Not solutions"[1] and "Sequences of Numbers Involved in Unsolved Problem"[2], Professor F.Smarandache asked us to study the properties of these sequences. About these two problems, some authors had studied them, and obtained some interesting results, see references [3], [4], [5] and [6]. For example, Xiaoxia Yan [6] studied the the asymptotic properties of $\frac{S_n}{I_n}$, and proved that for any positive integer $n > 1$, we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O(n^{-\frac{1}{3}}),$$

where $I_n = \{p_p(2) + p_p(3) + \cdots + p_p(n)\}/n$ and $S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n$.

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The sequences $\{p_p(n)\}$ and $\{P_p(n)\}$ are very interesting and important, because there are some close relationships between the Smarandache prime part sequences and the prime distribution problem. Now we introduced two determinants formed by the Smarandache prime part sequences. They are defined as follows: For any positive integer n, $c(n)$, $C(n)$ are $n \times n$ determinants, namely

$$c(n) = \begin{vmatrix} p_p(2) & p_p(3) & \cdots & p_p(n+1) \\ p_p(n+2) & p_p(n+3) & \cdots & p_p(2n+1) \\ \vdots & \vdots & \ddots & \vdots \\ p_p(n(n-1)+2) & p_p(n(n-1)+3) & \cdots & p_p(n^2+1) \end{vmatrix}$$

and

$$C(n) = \begin{vmatrix} P_p(1) & P_p(2) & \cdots & P_p(n) \\ P_p(n+1) & P_p(n+3) & \cdots & P_p(2n) \\ \vdots & \vdots & \ddots & \vdots \\ P_p(n(n-1)+1) & P_p(n(n-1)+2) & \cdots & P_p(n^2) \end{vmatrix}$$

For example, by definitions and calculating, we can find some values of these determinants as the following table:

n	$c(n)$	$C(n)$	n	$c(n)$	$C(n)$
2	1	4	3	14	4
4	0	188	5	-96	-1424
6	0	0	7	1.0214×10^5	37536
8	0	0	9	0	0
11	7.8299×10^7	1.1478×10^8	13	9.8338×10^8	-2.7958×10^9
17	8.2462×10^{14}	1.3164×10^{13}	19	-1.9608×10^{15}	1.629×10^{15}
23	2.4545×10^{20}	3.156×10^{18}	29	8.9308×10^{27}	-6.5008×10^{27}
31	1.1095×10^{29}	-2.2835×10^{28}	37	9.9497×10^{37}	-7.4692×10^{38}
41	-1.1502×10^{42}	3.9244×10^{45}	43	1.5152×10^{45}	-2.1671×10^{44}
47	1.1606×10^{51}	4.06×10^{50}	53	2.9359×10^{59}	5.8735×10^{59}
59	3.8402×10^{67}	-3.1043×10^{69}	61	-3.614×10^{70}	6.9858×10^{72}
67	3.2341×10^{80}	-9.5374×10^{78}	71	-2.219×10^{85}	4.7688×10^{86}
73	-3.6656×10^{85}	-3.6985×10^{89}	79	-4.2038×10^{98}	-3.8762×10^{97}
83	1.6966×10^{104}	2.1389×10^{104}	89	-1.0695×10^{113}	5.7824×10^{110}
97	$2.8.98 \times 10^{124}$	1.9968×10^{124}			

About the elementary properties of these determinants, it seems that none had studied them, at least we haven't seen any related papers before. Recently, Professor Zhang Wenpeng asked us to study the properties of these determinants, at the same time, he also proposed following two conjectures:

Conjecture 1. For any composite number $n \geq 6$, we have the identities $c(n) = 0$ and $C(n) = 0$;

Conjecture 2. For any prime q, we have $c(q) \neq 0$ and $C(q) \neq 0$.

From the above table, we believe that these two conjectures are correct. About Conjecture 1, we have solved it completely, which will be published in Pure and Applied Mathematics. But for Conjecture 2, it's still an open problem. We think that Conjecture 2 is very interesting and important, since if it is true, then we can get a new discriminant method to distinguish prime from integer numbers through calculating these kinds of determinants. So we introduce the conjectures in this place, and hope more scholars who are interested in to study it with us.

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On the continued fractions and Dirichlet L-functions [1]

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Abstract The main purpose of this paper is using the elementary methods to study the relationship between the continued fractions and Dirichlet L-functions , and give some interesting identities.

Keywords Continued fractions, Dirichlet L-functions, identities.

§1. Introduction

Let $x_0, x_1, x_2, \cdots$ be integers with $x_j \geq 1$, $j \geq 1$. We can express the original fraction $\frac{p_n}{q_n}$ in the form $x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{\cdots + \frac{1}{x_n}}}}, n \geq 0$. Such an expression is called a continued fraction, and we denote it by $\langle x_0, x_1, \cdots, x_n \rangle$.

About the characteristic properties of continued fractions, it can be found in the references [1] and [2]. In this paper, we shall study the relationship between the Dirichlet L-functions and the continued fractions, and obtain some interesting identities. That is, we shall prove the following main conclusion:

Theorem. Let p be a prime, a be a positive integer with $(a, p) = 1$ and $1 \leq a < p$. Then we have the identity

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi(a)|L(1,\chi)|^2$$
$$= \frac{\pi^2(p-1)}{12p} \left[\sum_{k=0}^{n} (-1)^{k+1} x_k + (-1)^{n-1} \frac{q_{n-1}}{q_n} + \frac{a}{p} + \frac{3}{2} \left(-1 + (-1)^n\right) \right],$$

where χ be the Dirichlet character $\mod p$, $L(1, \chi)$ be the Dirichlet L-function corresponding character χ and $a/p = p_n/q_n = \langle x_0, x_1, \cdots, x_n \rangle$.

For $a = 2$, 3,4 and $\frac{p-1}{2}$, from the above Theorem and the definition of continued fractions we may immediately obtain the following four Corollaries.

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Corollary 1. Let $k > 1$ be an odd prime, then we have the identity

$$\sum_{\chi \mod k, \chi(-1)=-1} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi^2(k)}{k}\left[\frac{1}{2}-\frac{3}{k}\right].$$

Corollary 2. Let p be an odd prime and χ be the Dirichlet character $\mod p$. Then

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi(3)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{36}\cdot\frac{(p-1)^2(p-10)}{p^2}, & \text{if } p \equiv 1 \mod 3; \\ \frac{\pi^2}{36}\cdot\frac{(p-1)(p-2)(p-5)}{p^2}, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Corollary 3. Let p be an odd prime and χ be the Dirichlet character $\mod p$. Then

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi(4)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{48}\cdot\frac{(p-1)^2(p-17)}{p^2}, & \text{if } p \equiv 1 \mod 4; \\ \frac{\pi^2}{48}\cdot\frac{(p-1)(p^2-6p+17)}{p^2}, & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Corollary 4. Let p be an odd prime and χ be the Dirichlet character $\mod p$. Then

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi\left(\frac{p-1}{2}\right)|L(1,\chi)|^2 = -\frac{\pi^2(p-1)(p^2-4p+5)}{24p^2}.$$

It is clear that these four Corollaries are an extension of Walum [3].

§2. Some Lemmas

To complete the proof of the Theorem, we need the following three Lemmas.

Lemma 1. Let $x_0, x_1, \cdots, x_n$ be integers and p_i, q_i $(0 \le i \le n)$ be natural numbers, continued fraction $\langle x_0, x_1, \cdots, x_i\rangle = \frac{p_i}{q_i}$. Then

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^{i+1}, \quad -1 \le i \le n.$$

and $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0, p_i = x_i p_{i-1} + p_{i-2}, q_i = x_i q_{i-1} + q_{i-2}, 0 \le i \le n.$

Proof. (See reference[1]).

Lemma 2. Let k and h be integers with $k \ge 3$ and $(h,k) = 1$. Then we have

$$S(h,k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)} \sum_{\chi \mod d, \chi(-1)=-1} \chi(h)|L(1,\chi)|^2,$$

where $S(h,q) = \sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right)$ is the Dedekind sum and $\phi(k)$ is the Euler's function.

Proof. (See reference [4]).

Lemma 3. Let a and q be two positive integers with $(a,q) = 1$. Then we have the identity

$$S(a,q) = \frac{1}{12}\left[\sum_{k=0}^{n}(-1)^{k+1}x_k + (-1)^{n-1}\frac{q_{n-1}}{q_n} + \frac{a}{q}\right] + \frac{-1+(-1)^n}{8},$$

where $a/q = p_n/q_n = \langle x_0, x_1, \cdots, x_n\rangle$.

Proof. Let continued fraction $\langle x_0, x_1, \cdots, x_n\rangle = \dfrac{p_n}{q_n}$ and suppose $\dfrac{p_n}{q_n} = \dfrac{a}{q}$. For the successive terms $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n-1}}{q_{n-1}}$, from Lemma 1 we know that

$$p_nq_{n-1} - q_np_{n-1} = (-1)^{n+1}, n \geq -1. \tag{1}$$

Using the properties of Dedekind sums and (1) we get

$$\begin{aligned} S(p_n, q_n) &= S(p_nq_{n-1}\overline{q_{n-1}}, q_n) \\ &= S\left(\overline{q_{n-1}}((-1)^{n+1} + p_{n-1}q_n), q_n\right) \\ &= S\left((-1)^{n+1}\overline{q_{n-1}}, q_n\right) \\ &= (-1)^{n+1}S\left(\overline{q_{n-1}}, q_n\right) \\ &= (-1)^{n+1}S(q_{n-1}, q_n), \end{aligned} \tag{2}$$

where $\overline{q_{n-1}}$ denotes the solution x of the congruence equation $xq_{n-1} \equiv 1(\mod q_n)$. From the reciprocity formula and Lemma 1 we obtain

$$\begin{aligned} S(q_{n-1}, q_n) + S(q_n, q_{n-1}) &= \frac{q_n^2 + q_{n-1}^2 + 1}{12q_nq_{n-1}} - \frac{1}{4} \\ &= \frac{1}{12}\left(\frac{q_{n-1}}{q_n} + \frac{q_n}{q_{n-1}}\right) + \frac{1}{12q_nq_{n-1}} - \frac{1}{4} \\ &= \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} + \frac{q_{n-2}}{q_{n-1}} + x_n\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right)\right] - \frac{1}{4}. \end{aligned} \tag{3}$$

Hence, by expression (3) and Lemma 1 we obtain

$$\begin{aligned} &S(q_{n-1}, q_n) \\ &= -S(q_n, q_{n-1}) + \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} + \frac{q_{n-2}}{q_{n-1}} + x_n\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right)\right] - \frac{1}{4} \\ &= -S(q_{n-2}, q_{n-1}) + \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} + \frac{q_{n-2}}{q_{n-1}} + x_n\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right)\right] - \frac{1}{4} \\ &= S(q_{n-3}, q_{n-2}) - \frac{1}{12}\left[\left(\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-3}}{q_{n-2}} + x_{n-1}\right) + (-1)^{n}\left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right)\right] + \frac{1}{4} \\ &+ \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} + \frac{q_{n-2}}{q_{n-1}} + x_n\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right)\right] - \frac{1}{4} \\ &= S(q_{n-3}, q_{n-2}) + \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} - \frac{q_{n-3}}{q_{n-2}} + x_n - x_{n-1}\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right)\right]. \end{aligned} \tag{4}$$

If n is an even number, then from (4) we have

$$\begin{aligned} S(q_{n-1}, q_n) &= S(q_{-1}, q_0) + \frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n} - \frac{q_{-1}}{q_0}\right) + (-1)^{n+1}\left(\frac{p_n}{q_n} - \frac{p_0}{q_0}\right) + \sum_{i=0}^{n-1}(-1)^i x_{n-i}\right] \\ &= \frac{1}{12}\left[\sum_{i=0}^{n-1}(-1)^i x_{n-i} + \frac{q_{n-1}}{q_n} + (-1)^{n+1}\frac{p_n}{q_n} + (-1)^n\frac{p_0}{q_0}\right] \\ &= \frac{1}{12}\left[\sum_{i=0}^{n}(-1)^i x_{n-i} + \frac{q_{n-1}}{q_n} + (-1)^{n+1}\frac{p_n}{q_n}\right]. \end{aligned} \tag{5}$$

If n is an odd number, then from (4) we have

$$\begin{aligned} S(q_{n-1},q_n) &= S(q_0,q_1)+\frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n}-\frac{q_0}{q_1}\right)+(-1)^{n+1}\left(\frac{p_n}{q_n}-\frac{p_1}{q_1}\right)+\sum_{i=0}^{n-2}(-1)^i x_{n-i}\right] \qquad (6)\\ &= S(1,x_1)+\frac{1}{12}\left[\left(\frac{q_{n-1}}{q_n}-\frac{1}{x_1}\right)+(-1)^{n+1}\left(\frac{p_n}{q_n}-\frac{x_0x_1+1}{x_1}\right)+\sum_{i=0}^{n-2}(-1)^i x_{n-i}\right]\\ &= \frac{(x_1-1)(x_1-2)}{12x_1}+\frac{1}{12}\left[\frac{q_{n-1}}{q_n}+(-1)^{n+1}\frac{p_n}{q_n}-\frac{2}{x_1}-x_0+\sum_{i=0}^{n-2}(-1)^i x_{n-i}\right]\\ &= \frac{1}{12}\left[\sum_{i=0}^{n}(-1)^i x_{n-i}+\frac{q_{n-1}}{q_n}+(-1)^{n+1}\frac{p_n}{q_n}\right]-\frac{1}{4}. \end{aligned}$$

Now combining (2), (5) and (6) we obtain

$$S(p_n,q_n)=\frac{1}{12}\left[\sum_{k=0}^{n}(-1)^{k+1}x_k+(-1)^{n-1}\frac{q_{n-1}}{q_n}+\frac{p_n}{q_n}\right]+\frac{-1+(-1)^n}{8}.$$

Note that $p_n/q_n = a/q$ we may immediately get

$$S(a,q)=\frac{1}{12}\left[\sum_{k=0}^{n}(-1)^{k+1}x_k+(-1)^{n-1}\frac{q_{n-1}}{q_n}+\frac{a}{q}\right]+\frac{-1+(-1)^n}{8}.$$

This completes the proof of Lemma 3.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. Using Lemma 2 we have

$$\sum_{\chi \bmod p,\chi(-1)=-1}\chi(a)|L(1,\chi)|^2=\frac{\pi^2(p-1)}{p}S(a,p). \qquad (7)$$

Then from Lemma 3 and expression (7) we can easily get

$$\begin{aligned} &\sum_{\chi \bmod p,\chi(-1)=-1}\chi(a)|L(1,\chi)|^2\\ &= \frac{\pi^2(p-1)}{p}\left[\frac{1}{12}\left(\sum_{k=0}^{n}(-1)^{k+1}x_k+(-1)^{n-1}\frac{q_{n-1}}{q_n}+\frac{a}{p}\right)+\frac{-1+(-1)^n}{8}\right]\\ &= \frac{\pi^2(p-1)}{12p}\left[\sum_{k=0}^{n}(-1)^{k+1}x_k+(-1)^{n-1}\frac{q_{n-1}}{q_n}+\frac{a}{p}+\frac{3}{2}(-1+(-1)^n)\right]. \end{aligned}$$

This completes the proof of Theorem.

The proof of the Corollaries.

For any integer $k>1$ and $(h,k)=1$, using the Möbius inversion formula and Lemma 2 we have

$$\sum_{\chi \bmod k,\chi(-1)=-1}\chi(h)|L(1,\chi)|^2=\pi^2\frac{\phi(k)}{k}\sum_{d|k}\frac{\mu(d)}{d}S(h,\frac{k}{d}). \qquad (8)$$

Taking $h = 2$ in (8), note that $S(2, 1) = 0$ and the continued fraction

$$\frac{2}{k/d} = \frac{p_2}{q_2} = \langle 0, \frac{k/d-1}{2}, 2\rangle = \langle x_0, x_1, x_2\rangle.$$

If k be a odd prime, then from (8) and Theorem we have

$$\begin{aligned}
&\sum_{\chi \mod k, \chi(-1)=-1} \chi(2)|L(1,\chi)|^2 = \pi^2 \frac{\phi(k)}{k} \sum_{d|k} \frac{\mu(d)}{d} S(2, \frac{k}{d}) \\
&= \pi^2 \frac{\phi(k)}{k} \sum_{d|k, d<k} \frac{\mu(d)}{d} S(2, \frac{k}{d}) \\
&= \frac{\pi^2 \phi(k)}{12k} \sum_{d|k, d<k} \frac{\mu(d)}{d} \left[\sum_{k=0}^{2} (-1)^{k+1} x_k + (-1)\frac{q_1}{q_2} + \frac{2}{p}\right] \\
&= \frac{\pi^2 \phi(k)}{12k} \sum_{d|k, d<k} \frac{\mu(d)}{d} \left(\frac{k/d-1}{2} - 2 - \frac{k/d-1}{k/d} + \frac{2}{k/d}\right) \\
&= \frac{\pi^2}{12} \frac{\phi^2(k)}{k} \left[\frac{1}{2} - \frac{3}{k}\right].
\end{aligned}$$

This proves Corollary 1.

Using Theorem and the definition of continued fraction we can also deduce that

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi(3)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{36} \cdot \frac{(p-1)^2(p-10)}{p^2}, & \text{if } p \equiv 1 \mod 3; \\ \frac{\pi^2}{36} \cdot \frac{(p-1)(p-2)(p-5)}{p^2}, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi(4)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{48} \cdot \frac{(p-1)^2(p-17)}{p^2}, & \text{if } p \equiv 1 \mod 4; \\ \frac{\pi^2}{48} \cdot \frac{(p-1)(p^2-6p+17)}{p^2}, & \text{if } p \equiv 3 \mod 4. \end{cases}$$

and

$$\sum_{\chi \mod p, \chi(-1)=-1} \chi\left(\frac{p-1}{2}\right)|L(1,\chi)|^2 = -\frac{\pi^2(p-1)(p^2-4p+5)}{24p^2}.$$

This completes the proof of Corollary 2, 3 and 4.

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Research on the connectivity of public transportation network based on set method

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Abstract A connectivity judgment method for public transportation network based on set idea is studied in this article. Set is successfully applied to solve the problem of connectivity judgment on multi-vertex and multi-edge situation. Search level control strategy is adapted to control the search level of route, and three buffer tables are designed to improve the search speed and reduce the complexity of the algorithm.

Keywords Traffic management, connectivity, transportation network, search level control.

§1. Introduction

Public transport is the main travel way for urban civilians and it is also a way advocated by many governments and organizations under the environmental protection concept. On one hand, along with the rapid construction of urban traffic, public travel becomes more and more convenience; on the other hand, problems come forth with multi-route selection. The civilians had to balance among fee, time cost, distance and some other areas, especially those in large cities or metropolis. And all this must be built on the basis of the pathway connectivity judgment, which are the major problem, as well as the common pathway connectivity judgment and optimization strategy under the circumstances of huge number of bus stations and bus lines, discussed in this article.

§2. Some known results

2.1. Problem description

The pathway connectivity discussed in this article means the connectivity between one station and another, directly or through a third or more other stations, then find out the routes and the shortest one if they are interlinked.

Basically, there are three types of public transport network distribution which include, all stations on the up-run line and down-run line are the same (also called Bidirectional Lines), part of the stations are the same (also called Unilines) and Loop Lines. Meanwhile there are two types of fee system known as Flat Fare and Sectional Fare. Following are three lines' information of a city's public transport system.

l_{135}, Bidirectional Line; Sectional Fare.

$s_{0417} - s_{0272} - s_{3425} - s_{3476} - s_{2974} - s_{0234} - s_{0521} - s_{3806} - s_{1682} - s_{3559} - s_{0928} - s_{3564} - s_{0079} - s_{3175} - s_{2866} - s_{3172} - s_{3269} - s_{3603} - s_{1633} - s_{2578} - s_{2579} - s_{2576} - s_{2577}$

l_{181}, Uniline; Flat Fare.

Up-Run Line:

$s_{0238} - s_{0540} - s_{0542} - s_{3024} - s_{0494} - s_{0973} - s_{3077} - s_{2082} - s_{2213} - s_{2210} - s_{3332} - s_{3351} - s_{1414} - s_{0228} - s_{0916} - s_{2811} - s_{0019} - s_{0966}$

Down-Run Line:

$s_{0966} - s_{0019} - s_{2811} - s_{0916} - s_{0228} - s_{1414} - s_{3351} - s_{3332} - s_{2210} - s_{2213} - s_{2082} - s_{3077} - s_{0973} - s_{0494} - s_{3024} - s_{0542} - s_{0541} - s_{0152} - s_{1765} - s_{3506} - s_{0238}$

l_{590}, Loop Line; Flat Fare.

$s_{3295} - s_{3265} - s_{1334} - s_{1333} - s_{2285} - s_{1341} - s_{0327} - s_{0317} - s_{0325} - s_{0532} - s_{1734} - s_{0314} - s_{1749} - s_{1831} - s_{1337} - s_{0705} - s_{0704} - s_{3295}$

Where l_n is the n-th line's No. of the city and s_m is the m-th station of the city.

2.2. Specialties of the problem

Algorithm for connectivity of graph (include Breadth First Search and Depth First Search), adjacency matrix and genetic algorithm are all popular methods adopted to judge pathway connectivity [1]. Outwardly, public transport network can be convert to a digraph whose vertexes are stations and edges are lines between two stations. However there are actually more than one line that go through each station in the transport network, namely, there are unfixed number of edges between two adjoining vertexes. Therefore, the commonly used router algorithms for connectivity of graph are inapplicable since the time complexity and space complexity increase rapidly, for instance, there are more than 850 bus lines and more than 6400 bus stations in city Beijing [2], since the transport network is a n-th Iterated Line Digraphs [3].

Moreover, on one hand, because of the characteristics of intersecting and overlapping, there are always more than one line between two adjoining stations. On the other hand, even if there are route between two stations, it may need ongoing transferring, however, it is contrary to the public's travel habits which are the less transferring, the better. According to the experience of livelihood, the public trend to reduce their transferring frequency when other factors' (such as cost, time, etc.) difference is not very significant. That is, if there are more than one route between two stations, direct line has priority over transferring and transferring once has priority over transferring more times and so on, and other factors will be taken into considering only when transferring frequencies are equal. Therefore, the following discussing was based on the public's travel habits.

§3. Set method for connectivity

3.1. Set description

Provide elements l_i , s_j and sets L_i, S_j, A_j, $B_{(j,i)}$, which are

l_i: Element that indicate line No. i;

s_j: Element that indicate station No. j;

L_i: Set of all lines that pass through station s_i;

S_j: Set of stations that line l_j pass through;

A_j : Set of stations that can be reached start from s_j;

$B_{(j,i)}$: Set of all stations that can be reached with line l_i start from station s_j;

The following expressions can be drown out:

1). Direct Line: the following expression is obtained if there have direct line run from start station s_{start} to destination station s_{dest}:

$$s_{start} \to s_{dest} \Longleftrightarrow \{s_{start}, s_{dest}\} \subseteq A_{start} \cap A_{dest}. \tag{1}$$

Where $s_{start} \to s_{dest}$ indicate there are direct line run from s_{start} to s_{dest}.

Therefore, when $\{s_{start}, s_{dest}\} \subseteq A_{start} \cap A_{dest}$ is establish, namely, there are direct line run from s_{start} to s_{dest}, the collection C of all direct lines are the intersection of set L_{start} whose elements are lines that run through s_{start} and set L_{dest} whose elements are lines that run through s_{dest}:

$$C = L_{start} \cap L_{dest}. \tag{2}$$

2). Transfer Once: The circumstance of transfer once from s_{start} to s_{dest} can be converted to the iterate of two direct lines, that is, the direct line from the start station s_{start} to the transfer station $s_{transfer}$ and the direct line from the transfer station $s_{transfer}$ to the destination station s_{dest}. Therefore the set D of transfer stations is:

$$D = A_{start} \cap A_{dest}. \tag{3}$$

3). Transfer multi-times: Similarly, The circumstance of transfer n-times from s_{start} to s_{dest} can be converted to the iterate of $(n+1)$ direct lines.

Direct line and transfer-needed circumstances are schematically shown in Fig.1.

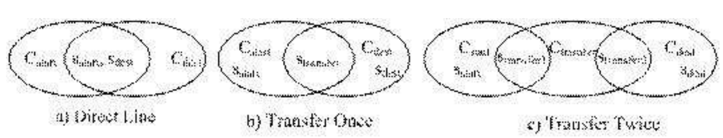

Figure 1: The schematic view of (a) direct line, (b) transfer once and (c) transfer twice of two stations

3.2. Optimization of algorithm

As has been analyzed that the public always trend to choose routes with less transferring frequencies, therefore, direct lines should be searched in top priority; deepen search level when direct lines are not available, and so on and so forth, until searching accomplished (succeed or failed) [4]. In addition, three memory tables known as line-station hash table, line-station searching linked-list and station-line searching linked-list are designed to improve the searching speed and reduce the searching complexity.

3.2.1. Route searching strategy

Search-level control strategy was introduce to integrate the Breadth-First Search method (BFS) and Depth-First Search method (DFS). The concrete steps are list in the following:

1). Construct set L_{start}, L_{dest}, A_{start} and A_{dest} for s_{start} and s_{dest}.

2). Figure out the intersection of L_{start} and L_{dest}, $L_{start} \cap L_{dest}$. Then find out all identical elements in both set L_{start} and set L_{dest} via BFS, thus the result C is the set of all direct lines.

a). If C is not empty, record all elements in C which are all direct lines.

b). If C is empty, deepen the search levels and continue with step 3);

3). Figure out the intersection of A_{start} and A_{dest}, $A_{start} \cap A_{dest}$. Then find out all identical elements in both set A_{start} and set A_{dest} via DFS, thus the result set D is the set of all transfer stations.

a). If D is not empty, record all elements in D which are all transfer stations. Then search direct lines with s_{start} and D, s_{dest} and D, respectively.

b). If D is empty, keep on deepening the search levels.

3.2.2. Hash table structure of line-station

Hash table structure of line-station stores the relationship of lines and stations. When the hash value, which are corresponded a line and a station, is "1", the corresponding line runs through the corresponding station, and "0", doesn't. The hash table structure of line-station is schematically shown in Fig. 2.

	s_1	s_2	s_3	$\cdots$	s_{n-1}	s_n
l_1	0	1	0	$\cdots$	1	0
l_2	1	0	0	$\cdots$	1	0
l_3	0	0	1	$\cdots$	0	1
$\cdots$				$\cdots$		
l_{m-1}	0	1	0	$\cdots$	1	1
l_m	0	0	1	$\cdots$	0	1

Figure 2: Schematic view of the hash table structure of line-station

As can be clearly seen from Fig. 2 that row direction of the hash table represents the lines and column direction, the stations. Moreover, the hash value "1" in row direction marks all stations that the corresponding line runs through, and in column direction marks all lines that run through the corresponding station. As is known to all that the time complexity of search algorithm using hash table structure is the constant of $O(1)$ [5].

Two important logic relationships are implicated in the above hast table structure, known as:

1). There are direct line between every two stations whose hash value is “1” in the same row.

2). Transfer is available between every two lines whose hash value is “1” in the same column.

Actually, the purpose of constructing the hash table of line-station is to perform high speed search utilizing these two relationships.

3.2.3. Line-station searching linked-list

Line-station searching linked-list stores the stations that every line runs through successively in a duplex linked-list. The list is sorted by line NO. to locates elements with binary search. Line-station search list structure is schematically shown in Fig. 3.

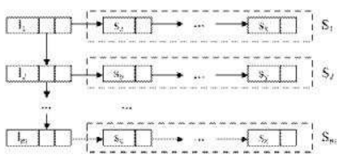

Figure 3: Schematic view of line-station searching linked-list structure

Since the time complexity of binary search is $O(\log(n))$, the total time complexity of the line-station duplex searching linked-list is

$$O(\log(m)) + O(\log(\max(spl))), \tag{4}$$

where m is the count of all lines; $\max(spl)$ is the max count of stations that a line runs through.

3.2.4 Station-line searching linked-list

The station-line searching linked-list is also designed as a duplex linked-list, every of whose nodes stores all lines that run through the station. The list is sorted by station NO. to locates elements with binary search. Line-station search list structure is schematically shown in Fig. 4.

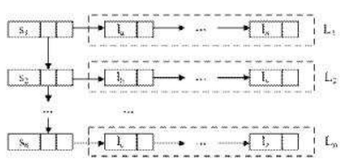

Figure 4: Schematic view of station- line searching linked-list structure

Similarly, the total time complexity of the station-line duplex searching linked-list is

$$O(\log(n)) + O(\log(\max(lps))) \tag{5}$$

where n is the count of all stations; $\max(lps)$ is the max count of lines that runs through a station.

3.3. Implementation of algorithm

The following route searching steps are acquired with integrated consider of search-level control strategy and optimization methods:

With regard to the start station s_{start} and the destination station s_{dest}

1). Set L_{start} and set L_{dest} are obtained from station-line searching linked-list by binary searching and select the one with less elements, that is $\min(L_{start}, L_{dest})$.

2). Pick out all elements from $\min(L_{start}, L_{dest})$ one by one and search for s_{start} or s_{dest} in line-station searching linked-list (If $\min(L_{start}, L_{dest}) = L_{start}$, search for s_{start}, else search for s_{dest}), record matched lines and store them in set L_{match};

a). If $L_{match} \neq \varnothing$, output all elements in L_{match} which are direct lines;

b). Else if $L_{match} = \varnothing$, deepen the search levels and continue with step 3);

3). Figure out $D = A_{start} \cap A_{dest}$

a). If $D \neq \varnothing$ and $D = \{d_1, d_2, ...d_n\}$, then $d_1, d_2, ...d_n$ are transfer stations. Go to step 1), perform direct-line searching with s_{start} and elements in D, as well as elements in D and s_{dest} respectively.

b). Else if $D = \varnothing$ deepen the search levels and continue with step 4);

4). Pick out s'_{start} and s'_{start} from A_{start} and A_{dest} respectively, then set the searching level to 1 and continue with step 1), namely transform the problem to search for direct-line between s'_{start} and s'_{dest}.

§4. Summary

Because of the characteristics of intersecting and overlapping, the commonly used router algorithms based on Graph Theory or Genetic Algorithm are inapplicable in the public transport network; therefore, a set-based connectivity judging method was proposed in this article. Search level control strategy is adapted to resolve the graph problem with huge number of vertexes and edges and a hash table and two duplex linked-list are designed to improve the searching speed and reduce the searching complexity.

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On the property of the Smarandache-Riemann zeta sequence

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Abstract In this paper, some elementary methods are used to study the property of the Smarandache-Riemann zeta sequence and obtain a general result.

Keywords Riemann zeta function, Smarandache-Riemann zeta sequence, positive integer.

§1. Introduction and result

For any complex number s, let

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

be the Riemann zeta function. For any positive integer n, let T_n be a positive real number such that

$$\zeta(2n) = \frac{\pi^{2n}}{T_n}, \tag{1}$$

where π is ratio of the circumference of a circle to its diameter. Then the sequence $T = \{T_n\}_{n=1}^{\infty}$ is called the Smarandache-Riemann zeta sequence. About the elementary properties of the Smarandache-Riemann zeta sequence, some scholars have studied it, and got some useful results. For example, in [2], Murthy believed that T_n is a sequence of integers. Meanwhile, he proposed the following conjecture:

Conjecture. No two terms of T_n are relatively prime.

In [3], Le Maohua proved some interesting results. That is, if

$$ord(2, (2n)!) < 2n - 2,$$

where $ord(2, (2n)!)$ denotes the order of prime 2 in $(2n)!$, then T_n is not an integer, and finally he defies Murthy's conjecture.

In reference [4], Li Jie proved that for any positive integer $n \geq 1$, we have the identity

$$ord(2, (2n)!) = \alpha_2(2n) \equiv \sum_{i=1}^{+\infty} \left[\frac{2n}{2^i}\right] = 2n - a(2n, 2),$$

where $[x]$ denotes the greatest integer not exceeding x.

So if $2n - a(2n, 2) < 2n - 2$, or $a(2n, 2) \geq 3$, then T_n is not an integer.

In fact, there exist infinite positive integers n such that $a(2n,2) \geq 3$, and T_n is not an integer. From this, we know that Murthy's conjecture is not correct, because there exist infinite positive integers n such that T_n is not an integer.

In this paper, we use the elementary methods to study another property of the Smarandache-Riemann zeta sequence, and give a general result for it. That is, we shall prove the following conclusion:

Theorem. If T_n are positive integers, then 3 divides T_n, more generally, if $n = 2k$, then 5 divides T_n; If $n = 3k$, then 7 divides T_n, where $k \neq 0$ is an integer.

So from this Theorem we may immediately get the following

Corollary. For any positive integers m and $n(m \neq n)$, if T_m and T_n are integers, then

$$(T_m, T_n) \geq 3, \quad (T_{2m}, T_{2n}) \geq 15, \quad (T_{3m}, T_{3n}) \geq 21.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we need two simple Lemmas which we state as follows:

Lemma 1. If n is a positive integer, then we have

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}, \tag{2}$$

where B_{2n} is the Bernoulli number.

Proof. See reference [1].

Lemma 2. For any positive integer n, we have

$$B_{2n} = I_n - \sum_{p-1|2n}\frac{1}{p}, \tag{3}$$

where I_n is an integer and the sum is over all primes p such that $p-1$ divides $2n$.

Proof. See reference [3].

Lemma 3. For any positive integer n, we have

$$T_n = \frac{(2n)!b_n}{2^{2n-1}a_n}, \tag{4}$$

where a_n and b_n are coprime positive integers satisfying $2||b_n, 3|b_n, n \geq 1$.

Proof. It is a fact that

$$\zeta(2n) = (-1)^{n-1}\frac{2^{2n-1}\pi^{2n}}{(2n)!}\cdot B_{2n}, \quad n \geq 1, \tag{5}$$

where

$$B_{2n} = (-1)^{n-1}\frac{a_n}{b_n}, \quad n \geq 1. \tag{6}$$

Using (1), (5) and (6), we get (4).

Now we use above Lemmas to complete the proof of our theorem.

For any positive integer n, from (4) we can directly obtain that if T_n is an integer, then 3 divides T_n, since $(a_n, b_n) = 1$.

From (1), (2) and (3) we have the following equality

$$\zeta(2n) = \frac{\pi^{2n}}{T_n} = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}\cdot\left(I_n - \sum_{p-1|2n}\frac{1}{p}\right),$$

Let

$$\prod_{p-1|2n} p = p_1p_2\cdots p_s,$$

where $p_i (1 \le i \le s)$ is a prime number, and $p_1 < p_2 \cdots < p_s$.

Then from the above, we have

$$\begin{aligned} T_n &= \frac{(-1)^{n+1}\cdot\pi^{2n}}{\frac{(2\pi)^{2n}}{2(2n)!}\cdot\left(I_n - \sum\limits_{p-1|2n}\frac{1}{p}\right)} = \frac{(-1)^{n+1}\cdot(2n)!}{2^{2n-1}\cdot\left(I_n - \sum\limits_{p-1|2n}\frac{1}{p}\right)} \qquad (7)\\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot\prod\limits_{p-1|2n}p}{2^{2n-1}\cdot\left(I_n\cdot\prod\limits_{p-1|2n}p - \prod\limits_{p-1|2n}p\cdot\sum\limits_{p-1|2n}\frac{1}{p}\right)}\\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot p_1p_2\cdots p_s}{2^{2n-1}\cdot\left(I_n\cdot p_1p_2\cdots p_s - p_1p_2\cdots p_s\cdot\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots\frac{1}{p_s}\right)\right)}\\ &= \frac{(-1)^{n+1}\cdot(2n)!\cdot p_1p_2\cdots p_s}{2^{2n-1}\cdot(I_n\cdot p_1p_2\cdots p_s - p_2p_3\cdots p_s - p_1p_3\cdots p_s - \cdots - p_1p_2\cdots p_{s-1})}\end{aligned}$$

Then we find that if $p_i | p_1p_2\cdots p_s,\quad 1 \le i \le s$, but

$$p_i \dagger (I_n\cdot p_1p_2\cdots p_s - p_2p_3\cdots p_s - p_1p_3\cdots p_s - \cdots - p_1p_2\cdots p_{s-1}).$$

So we can easily deduce that if T_n are integers, when $n = 2k$, 5 can divide T_n; While $n = 3k$, then 7 can divide T_n.

This completes the proof of Theorem.

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The system of generalized set-valued variational inclusions in q-uniformly smooth Banach spaces [1]

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Abstract In this paper, a system of generalized set-valued variational inclusions in Banach spaces is introduced and studied, which includes many variational inclusions studied by others in recent years. By using some new and innovative techniques, several existence theorems for the system of generalized set-valued variational inclusions in q-uniformly smooth Banach spaces are established, and some perturbed iterative algorithms for solving this kind of system of set-valued variational inclusions are suggested and analyzed. Our results improve and generalize many known algorithms and results.

Keywords System of generalized set-valued variational inclusion, iterative algorithm with error, q-uniformly smooth Banach space

§1. Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques, both for their own sake and for the applications. Useful and important generalizations of variational inequalities are set-valued variational inclusions, which have been studied by [1-9].

Recently, in [1], Chaofeng Shi introduced and studied the following class of set-valued variational inclusion problems in a Banach space E.

For a given m-accretive mapping $A, B, C, G : E \to CB(E)$, nonlinear mappings $N(\cdot,\cdot), M(\cdot,\cdot) : E \times E \to E$, single-valued mapping $g : H \to H$, any given $f \in E$ and $\lambda > 0$, find $u \in E, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Gu$ such that

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda w(g(u)), \tag{1.1}$$

where CB(E) denotes the family of all nonempty closed and bounded subsets of E.Under the setting of $q-$uniformly smooth Banach space, Chaofeng Shi [1] gave the existence and convergence theorem for the solution of variational inclusion (1.1).

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For a suitable choice of the mappings A, B, C, G, W, N, M, g and $f \in E$, a number of known and new variational inequalities, variational inclusions, and related optimization problems can be obtained from (1.1).

On the other hand, X. P. Ding [2] introduced and studied the system of generalized vector quasi equilibrium problems in locally G-convex uniform spaces, and the existence results for the system of equilibrium problems is obtained.

Inspired and motivated by the results in Chaofeng Shi [1], S. S. Chang [2], and X. P. Ding [3], the purpose of this paper is to introduce and study a class of system of set-valued variational inclusions. By using some new techniques, some existence theorems and approximate theorems for solving the set-valued variational inclusions in q-uniformly smooth Banach spaces are established and suggested. The results presented in this paper generalize, improve and unify the corresponding results of Chaofeng Shi [1], S. S. Chang [2], Ding [3,4], Huang [5,6], Zeng [7], Kazmi [8], Jung and Morales [9], Agarwal [10], Liu [11,12] and Osilike [13,14].

§2. Preliminaries

Definition 2.1. Let I be a finite or infinite index set. Let $\{E_i\}_{i\in I}$ be a family of real Banach spaces, $A, B, C, G : \prod_{i\in I} E_i \to \prod_{i\in I} CB(E_i)$be set-valued mappings, $W_i : D(W) \subset E_i \to 2^{E_i}$ be an set-valued mapping, $g_i : E_i \to E_i$ be a single-valued mapping,and $N_i(\cdot,\cdot), M_i(\cdot,\cdot) : E_i \times E_i \to E_i$ be two nonlinear mappings, for any given $f_i \in E_i$ and $\lambda > 0$, we consider the following problem of finding $u \in \prod_{i\in I} E_i, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Gu$ such that, for each $i \in I$,

$$f_i \in N_i(P_i\bar{x}, P_i\bar{y}) - M_i(P_iz, P_iv) + \lambda W_i(g_i(P_iu)), \tag{2.1}$$

where $P_i : \prod_{i\in I} E_i \to E_i$ is a projection operator. This problem is called the generalized set-valued variational inclusion problem in a Banach space.
Next we consider some special cases of problem (2.1).

(1) If $I \equiv \{i\}, f_i \equiv f, N_i \equiv N, M_i \equiv M, W_i \equiv W, g_i \equiv g$, then problem (2.1) is equivalent to

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda w(g(u)), \tag{2.2}$$

This problem is introduced and studied by Chaofeng Shi [1].

(2) Furthermore, if $M = 0, W = A : D(A) \to 2^E$ is an m-accretive mapping, $A = T, B = F$ and $C = G = 0$, then problem(2.1) is equivalent to finding $q \in E, w \in Tq, v \in Fq$ such that

$$f \in N(w, v) + \lambda A(g(q)). \tag{2.3}$$

This problem was introduced and studied by S. S. Chang [1].

For a suitable choice for the mappings $A, B, C, G, W_i, N_i, M_i, g_i, f_i$ and the family of spaces E_i, we can obtain a lot of known and new variational inequalities, variational inclusions and the related optimization problems. Furthermore, they can make us be able to study mathematics, physics and engineering science problems in a general and unified frame [1-9].

Definition 2.2. [10] Let E be a real Banach space. The module of smoothness of E is defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1; \|x\| \leq 1, \|y\| \leq t\}.$$

The space E is said to be uniformly smooth if $\lim_{t\to 0}(\rho_E(t)/t) = 0$. Moreover, E is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$.

Remark 2.1. [10] All Hilbert spaces, L_p(or l_p) spaces ($p \geq 2$) and the Sobolev spaces, $W_m^p(p \geq 2)$ are 2-uniformly smooth, while, for $1 < p \leq 2$, L_p(or l_p) and W_m^p spaces are $p-$uniformly smooth.

Definition 2.3. Let $S : E \to CB(E)$ be a set-valued mapping. S is said to be quasi-contractive, if there exists a constant $r \in (0,1)$ such that for any $p \in Sx$, $q \in Sy$,

$$\|p - q\| \leq r \ \max\{\|x-y\|, \|x-p\|, \|x-q\|, \|y-p\|, \|y-q\|\}.$$

To prove the main result, we need the following lemmas.

Lemma 2.1. [15] Let E be a $q-$uniformly smooth Banach space with $q > 1$. Then there exists a constant $c > 0$ such that

$$\|tx + (1-t)y - z\|^q \leq [1 - t(q-1)]\|y - z\|^q + tc\|x - z\|^q$$

$$-t(1 - t^{q-1}c)\|x - y\|^q,$$

for all $x, y, z \in E$ and $t \in [0, 1]$.

Lemma 2.2. [16] Suppose that $\{e_n\}, \{f_n\}, \{g_n\}$ and $\{\gamma_n\}$ are nonnegative real sequences such that

$$e_{n+1} \leq (1 - f_n)e_n + f_n g_n + \gamma_n, n \geq 0,$$

with $\{f_n\} \subseteq [0, 1]$, $\sum_{n=0}^{\infty} f_n = \infty$, $\lim_{n\to\infty} g_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} e_n = 0$.

For the remainder of this paper, r and c denote the constants appearing in definition 2.3 and lemma 2.1. We assume that $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}, A, B \in CB(E).$$

Lemma 2.3. [17] Let E be a complete metric space, $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and any given $x, y \in E, u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty).$$

Lemma 2.4. [18] Let X and Y be two Banach spaces, $T : X \to 2^Y$ a lower semi-continuous mapping with nonempty closed and convex values. Then T admits a continuous selection, i. e., there exist a continuous mapping $h : X \to Y$ such that $h(x) \in Tx$, for each $x \in X$.

Using Lemma 2.3 and Lemma 2.4, we suggest the following algorithms for the generalized set-valued variational inclusion (2.1).

Algorithm 2.1. For any given $x_0 \in \prod_{i\in I} E_i, x'_0 \in Ax_0, y'_0 \in Bx_0, z'_0 \in Cx_0, v'_0 \in Gx_0$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes such that, for each $i \in I$,

$$
\begin{aligned}
&(1)\ P_i x_{n+1} \in (1-\alpha_n)P_i x_n + \alpha_n(f_i + P_i y_n - N_i(P_i\bar{x}_n, P_i\bar{y}_n) + M_i(P_i z_n, P_i v_n) - \lambda W_i(g_i(P_i y_n))),\\
&(2)\ P_i y_n \in (1-\beta_n)P_i x_n + \beta_n(f_i + P_i x_n - N_i(P_i x'_n, P_i y'_n) + M_i(P_i z'_n, P_i v'_n) - \lambda W_i(g_i(P_i x_n))),\\
&(3)\ \bar{x}_n \in Ay_n, \|P_i\bar{x}_n - P_i\bar{x}_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iAy_n, P_iAy_{n+1}),\\
&(4)\ \bar{y}_n \in By_n, \|P_i\bar{y}_n - P_i\bar{y}_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iBy_n, P_iBy_{n+1}),\\
&(5)\ z_n \in Cy_n, \|P_i z_n - P_i z_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iCy_n, P_iCy_{n+1}),\\
&(6)\ v_n \in Gy_n, \|P_i v_n - P_i v_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iGy_n, P_iGy_{n+1}),\\
&(7)\ x'_n \in Ax_n, \|P_i x'_n - P_i x'_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iAx_n, P_iAx_{n+1}),\\
&(8)\ y'_n \in Bx_n, \|P_i y'_n - P_i y'_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iBx_n, P_iBx_{n+1}),\\
&(9)\ z'_n \in Cx_n, \|P_i z'_n - P_i z'_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iCx_n, P_iCx_{n+1}),\\
&(10)\ v'_n \in Gx_n, \|P_i v'_n - P_i v'_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iGx_n, P_iGx_{n+1}),\\
&n = 0, 1, 2, \cdots .
\end{aligned}
\tag{2.4}
$$

The sequence $\{x_n\}$ defined by (2.4), in the sequel, is called Ishikawa iterative sequence.

In algorithm 2.1, if $\beta_n = 0$, for all $n \ge 0$, then $y_n = x_n$. Take $\bar{x}_n = x'_n, \bar{y}_n = y'_n, z_n = z'_n$ and $v_n = v'_n$, for all $n \ge 0$ and we obtain the following.

Algorithm 2.2. For any given $x_0 \in \prod_{i\in I} E_i, \bar{x}_0 \in Ax_0, \bar{y}_0 \in Bx_0, z_0 \in Cx_0, v_0 \in Gx_0$, compute the sequences $\{x_n\}, \{\bar{x}_n\}, \{\bar{y}_n\}, \{z_n\}$ and $\{v_n\}$ by the iterative schemes such that, for each $i \in I$,

$$
\begin{aligned}
&P_i x_{n+1} \in (1-\alpha_n)P_i x_n + \alpha_n(f_i + P_i x_n - N_i(P_i\bar{x}_n, P_i\bar{y}_n) + M_i(P_i z_n, P_i v_n) - \lambda W_i(g_i(P_i x_n))),\\
&\bar{x}_n \in Ax_n, \|P_i\bar{x}_n - P_i\bar{x}_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iAx_n, P_iAx_{n+1}),\\
&\bar{y}_n \in Bx_n, \|P_i\bar{y}_n - P_i\bar{y}_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iBx_n, P_iBx_{n+1}),\\
&z_n \in Cx_n, \|P_i z_n - P_i z_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iCx_n, P_iCx_{n+1}),\\
&v_n \in Gx_n, \|P_i v_n - P_i v_{n+1}\| \le (1+\frac{1}{n+1})H_i(P_iGx_n, P_iGx_{n+1}),\\
&n = 0, 1, 2, \cdots .
\end{aligned}
\tag{2.5}
$$

The sequence $\{x_n\}$ defined by (2.5), in the sequel, is called Mann iterative sequence.

§3. An existence theorem for the system of the generalized set-valued variational inclusions

In this section, we shall establish an existence theorem for solutions of the system of the set-valued variational inclusions (2.1). We have the following results.

Theorem 3.1. Let E be a q-uniformly smooth Banach space, $A, B, C, G : \prod_{i\in I} E_i \to \prod_{i\in I} CB(E_i)$ be four set-valued mappings,for each $i \in I$, $N_i(\cdot,\cdot), M_i(\cdot,\cdot) : E\times E \to E, g_i : E_i \to E_i, W_i : E_i \to E_i$ be four single-valued mappings. If $I - N_i(P_iA(\cdot), P_iB(\cdot)) + M_i(P_iC(\cdot), P_iG(\cdot)) - \lambda W_i(g_i(P_i(\cdot)))$ is quasi contractive, then there exists $u \in \prod_{i\in I} E_i$, $\bar{x} \in Au, \bar{y} \in Bu, v \in Cu, z \in Gu$, which is a solution of system of the generalized set-valued variational inclusion (2.1).

Proof. For each $i \in I$, let

$$S(P_iu) = P_iu - N_i(P_iA(u), P_iB(u)) + M_i(P_iC(u), P_iG(u)) - \lambda W_i(g_i(P_iu)),$$

$$S_1(P_iu) = f_i + S(P_iu).$$

Since S is set-valued quasi contractive, S_1 is also set-valued quasi contractive.
It follows from Naddler [17] that S_1 has a unique fixed point $P_iu \in E_i$, that is

$$P_iu \in S_1(P_iu) = f_i + P_iu - N_i(P_iA(u), P_iB(u)) + M_i(P_iC(u), P_iG(u)) - \lambda W_i(g(P_iu)).$$

Thus, there exist $\bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Gu$ such that, for each $i \in I$,

$$f_i \in N_i(P_i\bar{x}, P_i\bar{y}) - M_i(P_iz, P_iv) + \lambda W_i(g_i(P_iu)),$$

that is, $u \in E$ is a solution of the generalized set-valued variational inclusion (2.1).

§4. Approximate problem of the system for generalized set-valued variational inclusions

In theorem 3.1, under some conditions, we have proved that there exist $u \in \prod_{i\in I} E_i, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Gu$ which is a solution of the system of generalized set-valued variational inclusion (2.1). In this section, we shall study the approximate problem for the solution of the system of generalized set-valued variational inclusions (2.1). We have the following result.

Theorem 4.1. Let E_i be a family of q-uniformly smooth Banach spaces, $A, B, C, G : \prod_{i\in I} E_i \to \prod_{i\in I} CB(E_i)$ be four set-valued continuous mappings, for each $i \in I$,
$N_i(\cdot,\cdot), M_i(\cdot,\cdot) : E_i \times E_i \to E_i, g_i : E_i \to E_i, w_i : E_i \to E_i$ be four single-valued continuous mappings, and $\{\alpha_n\}, \{\beta_n\}$ be two sequences in [0,1] satisfying the following conditions:
(i) $I - N_i(P_iA(\cdot), P_iB(\cdot)) + M_i(P_iC(\cdot), P_iG(\cdot)) - \lambda W_i(g_iP_i(\cdot))$ is quasi contractive,
(ii) for each $i \in I$, A, B, C, G are M-Lipschitz continuous with the constants μ_1, μ_2, μ_3 and μ_4,

(iii) $0 < h \leq \alpha_n, n \geq 0$. $0 < \mu_i < 1/2, i = 1, 2, 3, 4$,
(iv) $\alpha_n(q - 1 - cr^q) < 1, cr^q < q - 1, \beta_n^{q-1} < 1/c(1 - cr^q)$,

$$c\alpha_n^{q-1} + cr^q\beta_n(cr^q - q + 1) \leq 1 - cr^q.$$

Then for any given $x_0 \in \prod_{i\in I} E_i, x'_0 \in Ax_0, y'_0 \in Bx_0, z'_0 \in Cx_0, v'_0 \in Gx_0$, the sequences $\{x_n\}, \{\bar{x}_n\}, \{\bar{y}_n\}, \{z_n\}$ and $\{v_n\}$ defined by algorithm 2.1 strongly converge to the solution $u \in \prod_{i\in I} E_i, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Gu$ of the system of generalized set-valued variational inclusions (2.1) which is given in Theorem 3.1, respectively.

Proof. For each $i \in I$, in (1) and (2) of (2.4), choose $h_n \in W_i(g_i(P_ix_n)), k_n \in W_i(g_i(P_iy_n))$, such that

$$P_ix_{n+1} = (1 - \alpha_n)P_ix_n + \alpha_n(f_i + P_iy_n - N_i(P_i\bar{x}_n, P_i\bar{y}_n) + M_i(P_iz_n, P_iv_n) - \lambda k_n),$$

$$P_iy_n = (1 - \beta_n)P_ix_n + \beta_n(f_i + P_ix_n - N_i(P_ix'_n, P_iy'_n) + M_i(P_iz'_n, P_iv'_n) - \lambda h_n).$$

Let

$$p_n = f_i + P_iy_n - N_i(P_i\bar{x}_n, P_i\bar{y}_n) + M_i(P_iz_n, P_iv_n) - \lambda k_n,$$

$$r_n = f_i + P_ix_n - N_i(P_ix'_n, P_iy'_n) + M_i(P_iz'_n, P_iv'_n) - \lambda h_n.$$

Then

$$P_ix_{n+1} = (1 - \alpha_n)P_ix_n + \alpha_np_n,$$

$$P_iy_n = (1 - \beta_n)P_ix_n + \beta_nr_n. \tag{4.1}$$

Since $I - N_i(P_iA(\cdot), P_iB(\cdot)) + M_i(P_iC(\cdot), P_iG(\cdot)) - \lambda W_i(g_iP_i(\cdot))$ is quasi contractive,

$$\|p_n - P_iu\| \leq r\ max\{\|P_iy_n - P_iu\|, \|p_n - P_iy_n\|\},$$

which implies that

$$\|p_n - P_iu\|^q \leq r^q(\|P_iy_n - P_iu\|^q + \|p_n - P_iy_n\|^q). \tag{4.2}$$

Note that

$$\|p_n - r_n\| \leq rd(P_ix_n, P_iy_n), n \geq 0.$$

We consider the following cases.
Case 1: Suppose that $d(P_ix_n, P_iy_n) = \|P_ix_n - P_iy_n\|$ for some $n \geq 0$. It follows from (4.1) that

$$\|p_n - r_n\| \leq r\|P_ix_n - P_iy_n\| = r\|\beta_n(P_ix_n - r_n)\|$$

$$\leq r\beta_n\|P_ix_n - r_n\|. \tag{4.3}$$

Case 2: Suppose that $d(P_ix_n, P_iy_n) = \|P_ix_n - r_n\|$ for some $n \geq 0$. Then we have

$$\|p_n - r_n\| \leq r\|P_ix_n - r_n\|. \tag{4.4}$$

Case 3: Suppose that $d(P_ix_n, P_iy_n) = \|P_iy_n - r_n\|$ for some $n \geq 0$. Using (4.1), we have

$$\|p_n - r_n\| \leq r\|P_iy_n - r_n\|$$
$$= r\|(1-\beta_n)(P_ix_n - r_n)\|$$
$$= r(1-\beta_n)\|P_ix_n - r_n\| \leq r\|P_ix_n - r_n\|. \tag{4.5}$$

Case 4: Suppose that $d(P_ix_n, P_iy_n) = \|P_ix_n - p_n\|$ for some $n \geq 0$. Then we have

$$\|p_n - r_n\| \leq r\|P_ix_n - p_n\|. \tag{4.6}$$

Case 5: Suppose that $d(P_ix_n, P_iy_n) = \|P_iy_n - p_n\|$ for some $n \geq 0$. It follows from (4.1) that

$$\|p_n - r_n\| \leq r\|P_iy_n - p_n\|$$
$$= r\|(1-\beta_n)(P_ix_n - p_n) + \beta_n(r_n - p_n)\|$$
$$\leq r(1-\beta_n)\|P_ix_n - p_n\| + r\beta_n\|p_n - r_n\|,$$

which implies that

$$\|p_n - r_n\| \leq \frac{r(1-\beta_n)}{1-r\beta_n}\|P_ix_n - p_n\| \leq r\|P_ix_n - p_n\|. \tag{4.7}$$

It follows from (4.3)-(4.7) that

$$\|p_n - r_n\|^q \leq r^q\|P_ix_n - p_n\| + r^q\|P_ix_n - r_n\|^q,$$
$$n \geq 0. \tag{4.8}$$

It follows from Lemma 2.1 that

$$\|P_iy_n - P_iu\|^q = \|(1-\beta_n)P_ix_n + \beta_nr_n - P_iu\|^q$$
$$= \|(1-\beta_n)(P_ix_n - P_iu) + \beta_n(r_n - P_iu)\|^q$$
$$\leq [1-\beta_n(q-1)]\|P_ix_n - P_iu\|^q + \beta_nc\|r_n - P_iu\|^q$$
$$-\beta_n(1-\beta_n^{q-1}c)\|P_ix_n - r_n\|^q, n \geq 0. \tag{4.9}$$

Similarly, we have

$$\|P_iy_n - p_n\|^q \leq [1-\beta_n(q-1)]\|P_ix_n - p_n\|^q + \beta_nc\|p_n - r_n\|^q$$
$$-\beta_n(1-\beta_n^{q-1}c)\|P_ix_n - r_n\|^q, n \geq 0. \tag{4.10}$$

and

$$\|P_ix_{n+1} - P_iu\|^q \leq [1-\alpha_n(q-1)]\|P_ix_n - P_iu\|^q + \alpha_nc\|p_n - P_iu\|^q$$
$$-\alpha_n(1-\alpha_n^{q-1}c)\|P_ix_n - p_n\|^q, n \geq 0. \tag{4.11}$$

By virtue of the condition (iii) and (iv), (4.2) and (4.8)-(4.11), we have

$$\|P_ix_{n+1} - P_iu\|^q \leq [1-\alpha_n(q-1)]\|P_ix_n - P_iu\|^q$$

$$+\alpha_n cr^q[\|P_iy_n - P_iu\|^q + \|P_iy_n - p_n\|^q]$$
$$-\alpha_n(1-\alpha_n^{q-1}c)\|P_ix_n - p_n\|^q$$
$$\leq [1-\alpha_n(q-1) + cr^q\alpha_n(1-\beta_n(q-1))]\|P_ix_n - P_iu\|^q$$
$$+c^2r^q\alpha_n\beta_n\|r_n - P_iu\|^q + c^2r^q\alpha_n\beta_n\|p_n - r_n\|^q$$
$$+[cr^q\alpha_n(1-\beta_n(q-1)) - \alpha_n(1-\alpha_n^{q-1}c)]\|P_ix_n - p_n\|^q$$
$$-2cr^q\alpha_n\beta_n(1-\beta_n^{q-1}c)\|P_ix_n - r_n\|^q$$
$$\leq [1-\alpha_n(q-1) + cr^q\alpha_n(1-\beta_n(q-1)) + c^2r^{2q}\alpha_n\beta_n]\|P_ix_n - P_iu\|^q$$
$$+2cr^q\alpha_n\beta_n(cr^q + (\beta_n^{q-1}c-1))\|P_ix_n - r_n\|^q$$
$$+\alpha_n[c^2r^{2q}\beta_n + cr^q(1-\beta_n(q-1)) - (1-\alpha_n^{q-1}c)]\|P_ix_n - p_n\|^q$$
$$\leq [1-\alpha_n(q-1-cr^q)(1+cr^q\beta_n)]\|P_ix_n - P_iu\|^q$$
$$\leq [1-\alpha_n(q-1-cr^q)]\|P_ix_n - P_iu\|^q,$$

Set $e_n = \|P_ix_n - P_iu\|^q, f_n = \alpha_n(q-1-cr^q)$,

$$g_n = \gamma_n = 0, n \geq 0.$$

It follows from the conditions (iii), (iv) and Lemma 2.2 that

$$\lim_{n\to\infty} e_n = 0,$$

that is,

$$\lim_{n\to\infty} \|P_ix_n - P_iu\| = 0.$$

Note that

$$\|r_n - P_iu\| \leq r\max\{\|P_ix_n - P_iu\|, \|r_n - P_ix_n\|\},$$

which implies

$$r_n \to P_iu(n\to\infty).$$

Thus, from (4.1), we have

$$P_iy_n = (1-\beta_n)P_ix_n + \beta_nr_n \to P_iu.$$

From (2.4) and conditions (ii) and (iii), we have

$$\|P_i\bar{x}_n - P_i\bar{x}_{n+1}\| \leq (1+\frac{1}{n+1})H_i(P_iAy_n, P_iAy_{n+1})$$
$$\leq (1+\frac{1}{n+1})\mu\|P_iy_n - P_iy_{n+1}\|,$$

which implies that $\{P_i\bar{x}_n\}$ is a Cauchy sequence in E_i.
So, there exists $\bar{x} \in \prod_{i\in I} E_i$ such that $P_i\bar{x}_n \to P_i\bar{x}$. Now we show that $\bar{x} \in Au$.
In fact,

$$d(P_i\bar{x}, P_iAu) \leq \|P_i\bar{x} - P_i\bar{x}_n\| + H_i(P_iAx_n, P_iAu)$$

$$\leq \|P_i\bar{x} - P_i\bar{x}_n\| + \mu(1 + \frac{1}{n+1})\|P_i x_n - P_i u\| \to 0.$$

Also, we have

$$\bar{y}_n \to \bar{y} \in Bu, z_n \to z \in Cu, v_n \to v \in Gu.$$

From (2.4), we have, for each $i \in I$,

$$P_i u \in f_i + P_i u - N_i(P_i\bar{x}, P_i\bar{y}) + M_i(P_i z, P_i v) - \lambda W_i(g_i(P_i u)),$$

that is,

$$f_i \in N_i(P_i\bar{x}, P_i\bar{y}) - M_i(P_i z, P_i v) + \lambda W_i(g_i(P_i u)).$$

The required results.

Remark 4.1. Theorem 4.1 generalizes Theorem 4.1 in Chaofeng Shi [1], the corresponding results in S. S. Chang [2], Agarwal [10], Liu [11, 12], Osilike [13, 14] and others.

Remark 4.2. Since algorithm 2.2 is a special case of algorithm 2.1, from Theorem 4.1, we can obtain the convergence Theorem for algorithm 2.2, the details are omitted.

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An iterative method for (R,S)-symmetric solution of matrix equation $AXB = C$

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Abstract For generalized reflexive matrices R, S, i.e., $R^T = R$, $R^2 = I$, $S^T = S$, $S^2 = I$, matrix A is said to be (R,S)-symmetric matrix when $RAS = A$. In this paper, an iterative method is established for solving the matrix equation $AXB = C$ with (R,S)-symmetric solution X. By this method, the solvability of the above matrix equation can be determined autonomically, and a solution or the least-norm solution of which can be obtained within finite iteration steps. Meanwhile, the optimal approximation solution to a given matrix X_0 can also be derived by the least-norm solution of a new matrix equation $A\tilde{X}B = \tilde{C}$, where $\tilde{X} = X - X_0$, $\tilde{C} = C - AX_0B$. Finally, given numerical examples illustrate the efficiency of the iterative method.

Keywords Matrix equation, iterative method, (R,S)-symmetric solution, least-norm solution, optimal approximation solution.

§1. Introduction

Let $R_k^{m\times n}$ be the set of all $m \times n$ real matrices with rank k, $SR^{n\times n}$, $OR^{n\times n}$ be the set of all symmetric matrices, orthogonal matrices in $R^{n\times n}$, respectively. Denoted by the superscripts T and I_n the transpose and identity matrix with order n, respectively. For matrices $A = (a_1, a_2, \ldots, a_n), B \in R^{m\times n}$, $a_i \in R^m$, $\mathscr{R}(A)$ and $tr(A)$ represent its column space and trace, respectively; The symbol $vec(\cdot)$ represents the vec operator, i.e., $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$; $A \otimes B$ stands for the Kronecker product of matrices A and B; Moreover, $\langle A, B\rangle = tr(B^TA)$ is defined as the inner product of the two matrices, which generates the Frobenius norm, i.e., $\|A\| = \sqrt{< A, A >} = \sqrt{tr(A^TA)}$.

Definition 1.1. [1,2]For given generalized reflexive matrices $R \in R_r^{m\times m}$, $S \in R_l^{n\times n}$, i.e., $R^T = R$, $R^2 = I$, $S^T = S$, $S^2 = I$, we say matrix $A \in R^{m\times n}$ is (R,S)-symmetric ((R,S)-skew symmetric), if $RAS = A$ ($RAS = -A$).

The set of all $m \times n$ (R,S)-symmetric ((R,S)-skew symmetric) matrices with respect to (R, S) is denoted by $GSR^{m\times n}$ ($GSSR^{m\times n}$).

Definition 1.2. Assume $M, N \in R^{p\times m}$, where p, m are arbitrary positive integers, if $\langle M, N\rangle = 0$, matrices M, N are called to be orthogonal each other.

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Remark 1.1.

(a) In this paper, we let R, S be generalized reflexive matrices as in Definition 1.1.

(b) For $\forall F \in GSR^{m\times n}$, $G \in GSSR^{m\times n}$, we can verify that $\langle F, G\rangle = 0$.

For above generalized reflexive matrix R, $R^T = R$ implies that the eigenvalues of R belong to real field, and the absolute of the eigenvalues equal to 1 since $R^2 = I$. Certainly, S has similar properties to R. Hence, we have the following assertion.

Lemma 1.1. For given generalized reflexive matrices R, S as in Definition 1.1, there exist unitary matrices $U_1 \in OR^{m\times m}$, and $U_2 \in OR^{n\times n}$, such that

$$R = U_1 \begin{pmatrix} I_r & 0 \\ 0 & -I_{m-r} \end{pmatrix} U_1^T, \quad S = U_2 \begin{pmatrix} I_l & 0 \\ 0 & -I_{n-l} \end{pmatrix} U_2^T.$$

From Definition 1.1 and Lemma 1.1, we generate the following conclusion.

Lemma 1.2. Suppose $X \in GSR^{m\times n}$, then

$$X = U_1 \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U_2^T,$$

where $\forall\ X_1 \in R^{r\times l}$, $X_2 \in R^{(m-r)\times(n-l)}$.

From Lemma 1.2, we can easily get a (R,S)-symmetric matrix by choosing different A_i, but U_i $(i = 1, 2)$ are fixed in Lemma 1.1.

The problems to be discussed in this paper can be expressed as follows:

Problem I. For given generalized reflexive matrices $R \in R^{m\times m}$, $S \in R^{n\times n}$, and $A \in R^{p\times m}$, $B \in R^{n\times q}$, $C \in R^{p\times q}$, find $X \in GSR^{m\times n}$, such that

$$AXB = C. \tag{1}$$

Problem II. When Problem I is consistent, i.e., its solution set S_E is not empty, for given matrix $X_0 \in R^{m\times n}$, find $\hat{X} \in S_E$ such that

$$\|\hat{X} - X_0\| = \min_{X\in S_E} \|X - X_0\|. \tag{2}$$

The linear matrix equation (1) has been discussed widely with unknown X[3−7], where X has particular structures, such as, symmetric, skew-symmetric, reflexive, anti-reflexive, Hermitian etc.. In these literatures, the authors obtained the expressions of associated solutions and solvability conditions by general inverse or matrix decompositions: the singular value decomposition (SVD), the generalized SVD (GSVD)[8] or the canonical correlation decomposition (CCD)[9]. In complex field, the inverse eigenvalue problem and procrustes problem on (R,S)-symmetric matrices have been studied by William [1,2]. Of course, the (R,S)-symmetric solution of matrix equation (1) can also be given by using GSVD or CCD. In addition, by establishing iterative algorithm, the symmetric solution, skew-symmetric solution, Hermitian least-norm solution of matrix equation (1) have been derived in [10-12], respectively. Moreover, the general solution and symmetric solution of matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ have also been investigated by Sheng [13] and Peng [14], respectively.

Problem II is the optimal approximation problem, which occurs frequently in experimental design[13]. In equality (2), the matrix X_0 may be obtained from experiments, but it is not necessary to be needed form and the minimum residual requirement. The nearness matrix $\hat{X}$ is the matrix that satisfies the needed form and the minimum residual restriction.

Motivated by the iterative methods mentioned in [10], in this paper, we will construct an iterative method to obtain the (R,S)-symmetric solutions of Problem I and II. This work will be accomplished in section 2 and section 3, respectively. In section 4, we will provide some numerical examples to illustrate the efficiency of the iterative method.

§2. The iterative method for Problem I

In this section, we will propose an iterative algorithm to solve Problem I, some lemmas will be given to analyze the properties of this algorithm. For any (R,S)-symmetric matrix, We will show that the solution of matrix equation (1) can be obtained within finite iterative steps. The iterative method of Problem I is stated as follows:

Algorithm 2.1.

Step 1: Input matrices $R \in R^{m\times m}$, $S \in R^{n\times n}$, and $A \in R^{p\times m}$, $B \in R^{n\times q}$ $C \in R^{p\times q}$. Choosing arbitrary matrix $X_1 \in GSR^{m\times n}$.

Step 2: Calculate

$$R_1 = C - AX_1B,$$

$$P_1 = \frac{1}{2}(A^TR_1B^T + RA^TR_1B^TS),$$

$k := 1.$

Step 3: If $R_k = 0$, stop; otherwise goto next step.

Step 4: Calculate

$$X_{k+1} = X_k + \frac{\| R_k \|^2}{\| P_k \|^2}P_k.$$

Step 5: Calculate

$$R_{k+1} = C - AX_{k+1}B = R_k - \frac{\| R_k \|^2}{\| P_k \|^2}AP_kB.$$

$$P_{k+1} = \frac{1}{2}(A^TR_{k+1}B^T + RA^TR_{k+1}B^TS) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k.$$

Algorithm 2.1 reveals that X_i, $P_i \in GSR^{m\times n}$ $(i = 1, 2, \dots)$. In the sequel, we prove the feasibility of this algorithm.

Lemma 2.1. The sequences R_i, P_i $(i = 1, 2, \dots)$ generated by Algorithm 2.1 satisfy that

$$\langle R_{i+1}, R_1\rangle = \langle R_i, R_1\rangle - \frac{\|R_i\|^2}{\|P_i\|^2}\langle P_i, P_1\rangle,$$

when $j > 1$,

$$\langle R_{i+1}, R_j\rangle = \langle R_i, R_j\rangle - \frac{\|R_i\|^2}{\|P_i\|^2}\langle P_i, P_j\rangle + \frac{\|R_i\|^2\|R_j\|^2}{\|P_i\|^2\|R_{j-1}\|^2}\langle P_i, P_{j-1}\rangle. \tag{3}$$

Proof. When $j > 1$, from the Algorithm 2.1, noting that $RP_iS = P_i$, we can obtain

$$\begin{aligned}\langle AP_iB, R_j\rangle &= \langle P_i, A^TR_jB^T\rangle \\ &= \left\langle P_i, \frac{A^TR_jB^T + RA^TR_jB^TS}{2}\right\rangle + \left\langle P_i, \frac{A^TR_jB^T - RA^TR_jB^TS}{2}\right\rangle\end{aligned}$$

$$= \left\langle P_i, P_j - \frac{\|R_j\|^2}{\|R_{j-1}\|^2} P_{j-1} \right\rangle + \left\langle P_i, \frac{A^T R_j B^T}{2} \right\rangle - \left\langle RP_iS, \frac{RA^T R_j B^T S}{2} \right\rangle$$

$$= \langle P_i, P_j \rangle - \frac{\|R_j\|^2}{\|R_{j-1}\|^2} \langle P_i, P_{j-1} \rangle, \quad (4)$$

moreover,

$$\langle R_{i+1}, R_j \rangle = \left\langle R_i - \frac{\|R_i\|^2}{\|P_i\|^2}(AP_iB), R_j \right\rangle$$

$$= \langle R_i, R_j \rangle - \frac{\|R_i\|^2}{\|P_i\|^2} \langle AP_iB, R_j \rangle$$

$$= \langle R_i, R_j \rangle - \frac{\|R_i\|^2}{\|P_i\|^2} \langle P_i, A^T R_j B^T \rangle. \quad (5)$$

The proof of (5) implies that (3) holds when $j = 1$. Furthermore, submitting (4) into (5) generates (3) holds. The proof is completed.

Lemma 2.2. The sequences $\{R_i\}$, $\{P_i\}$ generated in the iterative process are self-orthogonal, respectively. i.e.,

$$\langle R_i, R_j \rangle = 0, \qquad \langle P_i, P_j \rangle = 0, \ \ i, j = 1, 2, \ldots, k \, (k \geq 2), \ i \neq j. \quad (6)$$

Proof. We prove the conclusion by induction. Since $\langle F, G \rangle = \langle G, F \rangle$, we only proof (6) holds when $i > j$.

When $k = 2$, it is clear from Lemma 2.1 that

$$\langle R_2, R_1 \rangle = \langle R_1, R_1 \rangle - \frac{\| R_1 \|^2}{\| P_1 \|^2} \langle P_1, P_1 \rangle = 0.$$

Applying Algorithm 2.1 yields

$$\langle P_2, P_1 \rangle = \left\langle \frac{A^T R_2 B^T + RA^T R_2 B^T S}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} P_1, P_1 \right\rangle$$

$$= \langle A^T R_2 B^T, P_1 \rangle + \frac{\|R_2\|^2}{\|R_1\|^2} \langle P_1, P_1 \rangle$$

$$= \langle R_2, AP_1B \rangle + \frac{\|R_2\|^2 \|P_1\|^2}{\|R_1\|^2}$$

$$= \frac{\|P_1\|^2}{\|R_1\|^2} \langle R_2, (R_1 - R_2) \rangle + \frac{\|R_2\|^2 \|P_1\|^2}{\|R_1\|^2}$$

$$= 0. \quad (7)$$

Assume that (6) holds for $k = s$, that is, $\langle P_s, P_j \rangle = 0$, $\langle P_s, P_j \rangle = 0$, $j = 1, 2, \ldots, s-1$. Being similar to the proof of (7), we can verify from Lemma 2.1 and the algorithm that $\langle R_{s+1}, R_s \rangle = 0$, and $\langle P_{s+1}, P_s \rangle = 0$.

Now, we can finish the proof if $\langle R_{s+1}, R_j \rangle = 0$, $\langle P_{s+1}, P_j \rangle = 0$ hold. In fact, when $j = 1$, noting that the assumptions and (4), then

$$\langle R_{s+1}, R_1 \rangle = \langle R_s, R_1 \rangle - \frac{\|R_s\|^2}{\|P_s\|^2} \langle AP_sB, R_1 \rangle$$

$$= -\frac{\|R_s\|^2}{\|P_s\|^2} \langle P_s, A^T R_1 B^T \rangle$$

$$= -\frac{\|R_s\|^2}{\|P_s\|^2}\langle P_s, P_1\rangle = 0$$

and connecting with (3), we have

$$\begin{aligned}
\langle P_{s+1}, P_1\rangle &= \left\langle \frac{A^T R_{s+1} B^T + RA^T R_{s+1} B^T S}{2} + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s, P_1 \right\rangle \\
&= \left\langle \frac{A^T R_{s+1} B^T + RA^T R_{s+1} B^T S}{2}, P_1 \right\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\langle P_s, P_1\rangle \\
&= \langle A^T R_{s+1} B^T, P_1\rangle \\
&= \langle R_{s+1}, AP_1B\rangle \\
&= \frac{\|P_1\|^2}{\|R_1\|^2}\langle R_{s+1}, (R_1 - R_2)\rangle \\
&= -\frac{\|P_1\|^2}{\|R_1\|^2}\langle R_{s+1}, R_2\rangle = 0.
\end{aligned} \tag{8}$$

Furthermore, when $2 \leq j \leq s-1$, Lemma 2.1 and the assumptions imply that

$$\langle R_{s+1}, R_j\rangle = \langle R_s, R_j\rangle - \frac{\|R_s\|^2}{\|P_s\|^2}\langle P_s, P_j\rangle + \frac{\|R_s\|^2\|R_j\|^2}{\|P_s\|^2\|R_{j-1}\|^2}\langle P_s, P_{j-1}\rangle = 0.$$

Similar to the proof of (8), we can obtain $\langle P_{s+1}, P_j\rangle = 0$. Hence, the conclusions hold.

Remark 2.1. From Lemma 2.2, we know that, R_i $(i = 1, 2, \ldots, pq)$ can be regarded as an orthogonal basis of matrix space $R^{p\times q}$ for their orthogonality each other, so a solution of Problem I can be obtained by the Algorithm 2.1 at most $pq + 1$ iteration steps.

Lemma 2.3. When matrix equation (1) is consistent, provided that $\bar{X}$ is an arbitrary solution, then R_k, P_k generated by Algorithm 2.1 satisfy that

$$\langle \bar{X} - X_k, P_k\rangle = \| R_k \|^2, \quad k = 1, 2, \ldots .$$

Proof. When k=1, from the Algorithm 2.1 and Lemma 2.2, we get

$$\begin{aligned}
\langle \bar{X} - X_1, P_1\rangle &= \left\langle \bar{X} - X_1, \frac{A^T R_1 B^T + RA^T R_1 B^T S}{2} \right\rangle \\
&= \left\langle \bar{X}, \frac{A^T R_1 B^T + RA^T R_1 B^T S}{2} \right\rangle - \left\langle X_1, \frac{A^T R_1 B^T + RA^T R_1 B^T S}{2} \right\rangle \\
&= \langle \bar{X}, A^T R_1 B^T\rangle - \langle X_1, A^T R_1 B^T\rangle \\
&= \langle A(\bar{X} - X_1)B, R_1\rangle \\
&= \|R_1\|^2.
\end{aligned}$$

Assume that the conclusion holds for $k = s$, i.e., $\langle \bar{X} - X_s, P_s\rangle = \| R_s \|^2$, then

$$\begin{aligned}
\langle \bar{X} - X_{s+1}, P_s\rangle &= \langle \bar{X} - X_s - \frac{\|R_s\|^2}{\|P_s\|^2} P_s, P_s\rangle \\
&= \langle (\bar{X} - X_s), P_s\rangle - \frac{\|R_s\|^2}{\|P_s\|^2}\langle P_s, P_s\rangle \\
&= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2}\|R_s\|^2
\end{aligned}$$

$$= 0,$$

and

$$\langle \bar{X} - X_{s+1}, P_{s+1}\rangle = \left\langle \bar{X} - X_{s+1}, \frac{A^T R_{s+1} B^T + RA^T R_{s+1} B^T S}{2} + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s \right\rangle$$

$$= \frac{1}{2}\langle \bar{X} - X_{s+1}, A^T R_{s+1} B^T + RA^T R_{s+1} B^T S\rangle + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\langle \bar{X} - X_{s+1}, P_s\rangle$$

$$= \langle \bar{X} - X_{s+1}, A^T R_{s+1} B^T\rangle$$

$$= \langle A(\bar{X} - X_{s+1})B, R_{s+1}\rangle$$

$$= \|R_{s+1}\|^2.$$

Therefore, we complete the proof by the principle of induction.

Remark 2.2. Lemma 2.3 implies that, if the linear matrix equation (1) is consistent, then, $R_i = 0$ if and only if $P_i = 0$. In other words, if there exist a positive number t such that $R_t \neq 0$ but $P_t = 0$, then matrix equation (1) is inconsistent. Therefore, the solvability of Problem I can be determined automatically by Algorithm 2.1 in the absence of roundoff errors.

Based on the previous analysis, we have the following conclusion whose proof is omitted.

Theorem 2.1. When matrix equation (1) is consistent, for arbitrary initial matrix $X_1 \in GSR^{m\times n}$, a solution of which can be obtained within finite iterative steps in the absence of roundoff errors.

To facilitate the statement of our main results, we cite the following conclusion.

Lemma 2.4. (see [10]) Suppose that the consistent linear equations $My = b$ has a solution $y_0 \in R(M^T)$, then y_0 is the least-norm solution of which.

Theorem 2.2. When matrix equation (1) is consistent. If initial iteration matrix $X_1 = A^T HB^T + RA^T HB^T S$, where arbitrary $H \in R^{p\times q}$, or especially, let $X_1 = 0 \in R^{p\times q}$, then the solution generated by the iteration method is the unique least-norm solution of Problem I.

Proof. Algorithm 2.1 and Theorem 2.1 imply that, if let $X_1 = A^T HB^T + RA^T HB^T S$, where H is an arbitrary matrix in $R^{p\times q}$, a solution X^* of Problem I is of the form $X^* = A^T Y B^T + RA^T Y B^T S$. So it is enough to show that X^* is the least-norm solution of matrix equation (1).

Considering the following linear matrix equations with $X \in GSR^{m\times n}$

$$\begin{cases} AXB = C \\ ARXSB = C \end{cases} \tag{9}$$

Obviously, the solvability of (9) is equivalent to that of matrix equation (1). Denote $vec(X^*) = x^*$, $vec(X) = x$, $vec(Y) = y$, $vec(C) = c$, then the above matrix equations can be transformed into

$$\begin{pmatrix} B^T \otimes A \\ (B^T S) \otimes (AR) \end{pmatrix} x = \begin{pmatrix} c \\ c \end{pmatrix}.$$

In addition, the solution X^* can be rewritten as

$$x^* = \begin{pmatrix} B^T \otimes A \\ (B^T S) \otimes (AR) \end{pmatrix}^T \begin{pmatrix} y \\ y \end{pmatrix} \in \mathscr{R}\left(\begin{pmatrix} B^T \otimes A \\ (B^T S) \otimes (AR) \end{pmatrix}^T \right),$$

which implies, from Lemma 2.4, that X^* is the least-norm solution of the linear matrix equations (9). We complete the proof.

§3. The solution of Problem II

Suppose that matrix equation (1) is consistent, i.e., S_E is not empty. It is easy to verify that S_E is a closed convex set in matrix space $GSR^{m\times n}$ under Fronbenius norm, so the optimal approximation solution is unique. Without loss of generality, we can assume the given matrix $X_0 \in GSR^{m\times n}$ in Problem II because of the orthogonality between (R,S)-symmetric and (R,S)-skew symmetric matrix. In fact, for $X \in S_E$, we have

$$\|X - X_0\|^2 = \left\| X - \frac{X_0 + RX_0S}{2} - \frac{X_0 - RX_0S}{2} \right\|^2$$

$$= \left\| X - \frac{X_0 + RX_0S}{2} \right\|^2 + \left\| \frac{X_0 - RX_0S}{2} \right\|^2.$$

Let $\tilde{X} = X - X_0$, $\tilde{C} = C - AX_0B$, then Problem II is equivalent to find the least-norm solution $\tilde{X}^* \in GSR^{m\times n}$ of the following matrix equation

$$A\tilde{X}B = \tilde{C}. \tag{10}$$

From Theorem 2.2, if let initial iteration matrix $\tilde{X}_1 = A^T\tilde{H}B^T + PA^T\tilde{H}B^TP$, where arbitrary $\tilde{H} \in R^{p\times q}$, or especially, let $\tilde{X}_1 = 0 \in R^{m\times n}$, we can obtain the unique least-norm solution $\tilde{X}^*$ of matrix equation (10) by applying Algorithm 2.1. Furthermore, the unique optimal approximation solution $\hat{X}$ to X_0 can be obtained by $\hat{X} = \tilde{X}^* + X_0$.

§4. Numerical examples

Example 1. Input matrices A, B, C, and generalized reflexive matrices R, S as follows:

$$A = \begin{pmatrix} 5 & -3 & 0 & 3 & 0 & 2 & 8 \\ 0 & -4 & -6 & 4 & -6 & 0 & -14 \\ -6 & 0 & 7 & 0 & 7 & 3 & 1 \\ 0 & 5 & -3 & -5 & -3 & 0 & 3 \\ 4 & -7 & 0 & 7 & 0 & -8 & -3 \\ -1 & 0 & -6 & 0 & -5 & 9 & 0 \\ 0 & -3 & 0 & 3 & -7 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -3 & 5 & -5 & -2 & 5 & -12 \\ 0 & 4 & 9 & 9 & 4 & -6 \\ 6 & -1 & 7 & 0 & -1 & 3 \\ -2 & 4 & 0 & 5 & 4 & 5 \\ -1 & -6 & -2 & 0 & -6 & 2 \\ 0 & -9 & 1 & 1 & -9 & 2 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 254.7246 & -420.4859 & -4.5129 & -444.0827 & -420.4859 & 95.2199 \\ -32.1326 & -896.3380 & -642.6184 & -414.3636 & -896.3380 & 78.8874 \\ 285.7966 & -281.1569 & 169.3670 & -306.3903 & -281.1569 & 472.0991 \\ -219.4309 & 907.1669 & 255.3442 & 601.2957 & 907.1669 & -332.3348 \\ -63.6234 & -937.6031 & -462.9456 & -449.7040 & -937.6031 & 149.2335 \\ 352.1195 & -511.6855 & -185.5221 & -550.9943 & -511.6855 & 224.8234 \\ 300.0260 & -987.7629 & -247.5968 & -678.6078 & -987.7629 & 862.4447 \end{pmatrix}.$$

Next, we will find the associated solutions of Problems I and II by Algorithm 2.1 and Matlab software. For the influence of roundoff errors, R_i will usually unequal to zero in the Fronbenius norm. Therefore, for any chosen positive number ε, however small enough, e.g., $\varepsilon = 1.0e - 010$, whenever $\|R_k\| < \varepsilon$, stop the iteration, and X_k is regarded as a required solution.

(I) The solutions of Problem I:

Choose an arbitrary initial matrix

$$X_1 = \begin{pmatrix} 12.0207 & 12.0207 & 0 & 14.0000 & 6.3639 & -6.3639 \\ -6.9999 & 21.9996 & -14.1420 & 10.6065 & 21.9996 & 1.0000 \\ -21.9996 & 6.9999 & -14.1420 & -10.6065 & 1.0000 & 21.9996 \\ -34.9993 & -62.9988 & 44.5473 & -19.0917 & -15.9997 & 3.9999 \\ 9.8994 & 9.8994 & 0 & 16.0000 & 12.7278 & -12.7278 \\ 18.3846 & -18.3846 & -75.0000 & 0 & -24.0414 & -24.0414 \\ -62.9988 & -34.9993 & -44.5473 & -19.0917 & -3.9999 & 15.9997 \end{pmatrix},$$

by Algorithm 2.1 and 36 iteration steps, we obtain a solution of Problem I,

$$X_{36} = \begin{pmatrix} 6.3639 & 6.3639 & 0 & 4.0000 & 2.1213 & -2.1213 \\ 0.0221 & 7.9080 & -4.0470 & 2.7378 & 0.8945 & -8.1973 \\ -7.9080 & -0.0221 & -4.0470 & -2.7378 & -8.1973 & 0.8945 \\ -0.9779 & -6.0917 & 2.3169 & -5.7474 & 1.8945 & 3.8025 \\ 4.2426 & 4.2426 & 0 & -0.0000 & 2.8284 & -2.8284 \\ 3.5355 & -3.5355 & 8.0000 & 0 & 1.4142 & 1.4142 \\ -6.0917 & -0.9779 & -2.3169 & -5.7474 & -3.8025 & -1.8945 \end{pmatrix},$$

with $\|R_{36}\| = 3.8765e - 011 < \varepsilon$, and $\|X_{36}\| = 27.3649$.

If let initial matrix $X_1 = A^T HB^T + PA^T HB^T P$, where arbitrary $H \in R^{7\times 6}$, then the solution X_k generated by Algorithm 2.1 is the unique least-norm solution of matrix equation (1) by Theorem 2.2. Choosing arbitrary matrix

$$H = \begin{pmatrix} -4 & 1 & -2 & 0 & 1 & 0 \\ 6 & 0 & -1 & 1 & 0 & -1 \\ -2 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

then the least-norm solution X^* of Problem I is

$$X^* = X_{37} = \begin{pmatrix} 6.3639 & 6.3639 & 0 & 4.0000 & 2.1213 & -2.1213 \\ -0.3925 & 8.3779 & -4.2023 & 3.3713 & -0.6099 & -6.4524 \\ -8.3779 & 0.3925 & -4.2023 & -3.3713 & -6.4524 & -0.6099 \\ -1.3925 & -5.6218 & 2.1616 & -5.1139 & 0.3901 & 5.5474 \\ 4.2426 & 4.2426 & 0 & 0.0000 & 2.8284 & -2.8284 \\ 3.5355 & -3.5355 & 8.0000 & 0 & 1.4142 & 1.4142 \\ -5.6218 & -1.3925 & -2.1616 & -5.1139 & -5.5474 & -0.3901 \end{pmatrix},$$

with $\|R_{37}\| = 8.4542e - 011 < \varepsilon$, and $\|X^*\| = 26.9134$.

(II) The solution of Problem II

Suppose that the given matrix $X_0 \in GSR^{7\times 6}$ is

$$X_0 = \begin{pmatrix} -1.4142 & -1.4142 & 0 & -1.0000 & -2.8284 & 2.8284 \\ 2.5000 & 2.5000 & -3.5355 & -4.2426 & -2.5000 & -6.4999 \\ -2.5000 & -2.5000 & -3.5355 & 4.2426 & -6.4999 & -2.5000 \\ -0.5000 & -7.4999 & 6.3639 & -3.5355 & -1.0000 & 6.9999 \\ 0 & 0 & 0 & 9.0000 & -1.4142 & 1.4142 \\ 4.2426 & -4.2426 & 7.0000 & 0 & 2.1213 & 2.1213 \\ -7.4999 & -0.5000 & -6.3639 & -3.5355 & -6.9999 & 1.0000 \end{pmatrix}.$$

Compute $C_0 = AX_0B$, let $\tilde{X} = X - X_0$, $\tilde{C} = C - C_0$, then we can obtain the optimal approximation solution $\hat{X}$ of Problem II by finding the least-norm solution $\tilde{X}^*$ of equation (10), that is,

$$\tilde{X}^* = \begin{pmatrix} 7.7781 & 7.7781 & 0 & 5.0000 & 4.9497 & -4.9497 \\ -2.8110 & 5.7857 & -0.6363 & 7.4894 & 2.1856 & -0.2953 \\ -5.7857 & 2.8110 & -0.6363 & -7.4894 & -0.2953 & 2.1856 \\ -0.8110 & 1.7858 & -4.1718 & -1.7029 & 1.6856 & -1.7953 \\ 4.2426 & 4.2426 & 0 & -9.0000 & 4.2426 & -4.2426 \\ -0.7071 & 0.7071 & 1.0000 & 0 & -0.7071 & -0.7071 \\ 1.7858 & -0.8110 & 4.1718 & -1.7029 & 1.7953 & -1.6856 \end{pmatrix},$$

Hence the solution of Problem II is

$$\hat{X} = \begin{pmatrix} 6.3639 & 6.3639 & 0 & 4.0000 & 2.1213 & -2.1213 \\ -0.3110 & 8.2857 & -4.1718 & 3.2468 & -0.3144 & -6.7952 \\ -8.2857 & 0.3110 & -4.1718 & -3.2468 & -6.7952 & -0.3144 \\ -1.3110 & -5.7141 & 2.1921 & -5.2384 & 0.6856 & 5.2046 \\ 4.2426 & 4.2426 & 0 & 0 & 2.8284 & -2.8284 \\ 3.5355 & -3.5355 & 8.0000 & 0 & 1.4142 & 1.4142 \\ -5.7141 & -1.3110 & -2.1921 & -5.2384 & -5.2046 & -0.6856 \end{pmatrix}.$$

Example 2. Given matrices A, B, C being the same as example 2 in [8],

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & -3 & -4 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 \\ 1 & -2 & 1 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 20 & 3 & -22 & 2 \\ 24 & 24 & -72 & 6 \\ 16 & -18 & 28 & -2 \end{pmatrix},$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Choosing positive number $\varepsilon = 1.0e - 005$, we will stop the iteration when $\|R_k\| < \varepsilon$, or $\|R_k\| > \varepsilon$, but $\|P_k\| < \varepsilon$. Let initial iterative matrix $X_1 = 0 \in R^{4\times 4}$, by the iterative method, we obtain that $\|R_6\| = 2.1149e + 003 > \varepsilon$, and $\|P_6\| = 7.4838e - 009 < \varepsilon$. Then, Theorem 2.1 implies that Problem I has no solution in $GSR^{4\times 4}$.

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On an equation of the Smarandache function

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function. The main purpose of this paper is using the elementary method to study the solvability of the equation $S(a)^{-2}+S(b)^{-2}=S(c)^{-2}$, and obtain its all positive integer solutions.

Keywords The Smarandache function, Pythagorean triple, equation, positive integer solutions.

§1. Introduction and result

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $m!$ is divisible by n. That is,

$$S(n)=\min\{m:\ n \mid m!,\ m\in N\},$$

where N denotes the set of all positive integers. For example, the first few values of $S(n)$ are: $S(1)=1$, $S(2)=2$, $S(3)=3$, $S(4)=4$, $S(5)=5$, $S(6)=3$, $S(7)=7$, $S(8)=4$, $S(9)=6$, $S(10)=15$, $S(11)=11$, $S(12)=4$, $S(13)=13$, $S(14)=7$, $S(15)=5$, $S(16)=6$, $S(17)=17$, $S(18)=6$, $S(19)=19$, $S(20)=5$, $\cdots$. From the definition of $S(n)$ we can easily get the following conclusions: if $n=p^{\alpha}$, $S(n)=\alpha p$, $\alpha\le p$; $S(n)<\alpha p$, $\alpha>p$. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then $S(n)=\max\limits_{1\le i\le r}\{S(p_i^{\alpha_i})\}$.

About other properties of $S(n)$, many scholars had studied it, and obtained some interesting results, see [1], [2], [3], [4] and [5]. For example, in reference [5], Xu Zhefeng proved that for any real number $x>1$,

$$\sum_{n\le x}(S(n)-P(n))^2=\frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $P(n)$ is the greatest prime divisor of n, $\zeta(s)$ is the Riemann zeta-function.

In reference [6], Rongi Chen and Maohua Le proved that the equation $S^2(n)+S(n)=kn$ has infinitely many positive integer solutions for every positive integer k.

On the other hand, Bencze [7] proposed the following problem: Find all positive integer solutions of the equation

$$\frac{1}{S(a)^2}+\frac{1}{S(b)^2}=\frac{1}{S(c)^2}. \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we use the elementary method to study this problem, and solved it completely. That is, we shall prove the following:

Theorem. The equation (1) has infinite positive integer solutions. And each $(a,\ b,\ c)$ satisfy the equation (1) if and only if $S^2(a) + S^2(b) = z^2$, which means $S(a)$, $S(b)$, z are Pythagorean triples, and $GCD(S(a),\ S(b)) = d > 1$, $z \mid d^2$, $S(c) = \dfrac{xy}{|z|}$, where $GCD(S(a),\ S(b))$ denotes the greatest common divisor of $S(a)$ and $S(b)$.

From this Theorem we can obtain many positive integer solutions of the equation (1). For example, if we take $\frac{S(a)}{d} = 3$, $\frac{S(b)}{d} = 4$, $\dfrac{z}{d} = 5$, $d = 5$, $S(c) = 12$, then we can get $a = 5^3$, $b = 5^4$, $c = 3^5$. If $d = 10$, then $a = 5^7$, $b = 5^9$, $c = 3^{10}$.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we need the following:

Lemma. For every positive integer m, the equation $S(n) = m$ has positive integer solution.

Proof. From the definition of $S(n)$ and reference [8] we can easily get this conclusion.

Now we use this Lemma to prove our theorem. For convenience, we denote $S(a)$ by x, $S(b)$ by y, $S(c)$ by w. Then the equation (1) become

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{w^2}. \tag{2}$$

It is clear that if one of $S(a)$, $s(b)$ and $S(c)$ is 1, then $(a,\ b,)$ does not satisfy the equation (1). So without loss of generality we can assume that $x \geq 2$, $y \geq 2$, $w \geq 2$. Obviously the solvability of the equation (1) is equivalent to the solvability of the equation (2). Now we consider the positive integer solutions of the equation (2). From equation (2) we can obtain

$$x^2y^2 = (x^2 + y^2)w^2 \tag{3}$$

and

$$\frac{x^2y^2}{x^2 + y^2} = w^2. \tag{4}$$

Suppose that the equation (1) has positive integer solutions, then there exist $x,\ y,\ w$ satisfied the equation (3). Noting that $x \geq 2$, $y \geq 2$, $w \geq 2$, form the equation (3) we can get that $x^2 + y^2$ is a complete square, in other words, there exists an positive integer z such that $x^2 + y^2 = z^2$, and so $x,\ y,\ z$ are Pythagorean triples. Hence we can get

$$\left(\frac{xy}{z}\right)^2 = w^2. \tag{5}$$

Suppose $GCD(x,\ y) = d$, we can get $GCD(x,\ y) = GCD(x,\ z) = GCD(y,\ z) = d$ in equation (5). Suppose $x = dx'$, $y = dy'$, $z = dz'$, then we have

$$\left(\frac{dx'y'}{z'}\right)^2 = w^2, \tag{6}$$

from equation (6), w is integer, then $z' \mid dx'y'$. Note that $GCD(x',\ z') = GCD(y',\ z') = 1$ and $GCD(x'y',\ z') = 1$, then we can get $z' \mid d$ and $d \geq z' > 2$. Because $z = dz'$, $z \mid d^2$, we can get $w = \frac{xy}{z}$ from equation (5).

At last, we get the positive integer solutions of equation (2), and the solutions satisfying $x^2 + y^2$ is a complete square, in other words, there exists a positive integer z such that $x^2 + y^2 = z^2$, and $GCD(x,\ y) = d > 2$, $z \mid d^2$, $w = \frac{xy}{z}$. From Lemma we can get the positive integer solutions of equation (2). Since there are infinite Pythagorean triples $x,\ y,\ z$ satisfied the above situation, so the equation (2) has infinite positive integer solutions. And the solvability of equation (1) is equivalent to the solvability equation (2), so the equation (1) also has infinite positive integer solutions. This completes the proof of Theorem.

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On the Smarandache divisibility theorem

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Abstract In this paper, we used the elementary method to study the divisibility properties of the integer, and improved the Smarandache divisibility theorem.
Keywords Divisibility theorem, Gauss function.

§1. Introduction and result

Divisibility theory is the fundamental part in mathematics. The study of the divisibility theory is very important and interesting contents in number theory, many authors pay much attention to this topic, and some deeply results have been obtained. For example, F.Smarandache [1] proved that for any positive integer m, we have

$$m \mid (a^m - a) \cdot (m-1)!.$$

Maohua Le [3] improved the above result, and obtained the following conclusion:

$$m \mid (a^m - a) \cdot \left[\frac{m}{2}\right]!$$

Because $(m,\ m-1) = 1$, it is obvious that $m \mid (a^m - a) \cdot (m-2)!$. In this paper, we using the elementary method to study this problem, and give another form of the Smarandache divisibility theorem. That is, we shall prove the following conclusion:

Theorem. If a and m are two integers with $m > 1$, then we have

$$m \mid \frac{(a^m - a) \cdot (m-2)!}{4^{\left[\frac{m-1}{4}\right]}},$$

where $[x]$ denotes the largest integer less than or equal to x.

§2. Some preliminary lemmas

In this section, we shall give several simple Lemmas which are necessary in the proof of our theorem. They are stated as follows:

Lemma 1. If a and m are two integers with $m > 0$, then $(a^m - a)(m-2)!$ can be divided by m.

Proof. (See reference [2]).

Lemma 2. Let a and b are two positive integers, $S = \{1,\ 2,\ 3,\ ....,\ a\}$. Then the number of all positive integers in S which can be divided by b is $\left[\frac{a}{b}\right]$.

Proof. (See reference [1]).

Lemma 3. Little Fermat's Theorem: For any integer a and any prime p, we have the congruence

$$a^p \equiv a \bmod p.$$

Proof. (See reference [3]).

§3. Proof of the theorems

In this section, we shall complete the proof of our theorem directly. It is clear that we just need to prove that

$$4^{\left[\frac{m-1}{4}\right]} \mid \frac{(a^m - a)\cdot(m-2)!}{m}. \tag{1}$$

Now we separate m into three cases:

(a). If $m = 1,\ 2,\ 3,\ 4$, then from Lemma 1 we know that (1) is true.

(b). If $m = 5$, then $\left[\frac{m-1}{4}\right] = 1$. $4^{\left[\frac{m-1}{4}\right]} = 4$.

$$\frac{(a^m - a)\cdot(m-2)!}{m} = \frac{(a^5 - a)3!}{5} = \frac{(a-1)a(a+1)(a^2+1)3!}{5}.$$

We also know that there exists at least an even number among a, $(a+1)$ and $(a-1)$. So we must have $4 \mid (a^5 - a)\cdot(5-2)!$, note that $(4,\ 5) = 1$, we may immediately deduce that (1) is correct.

If $m = 8$, then $\left[\frac{m-1}{4}\right] = 1$. $4^{\left[\frac{m-1}{4}\right]} = 4$.

$$\frac{(a^m - a)\cdot(m-2)!}{m} = \frac{(a^8 - a)\cdot 6!}{8} = \frac{a\cdot(a^7-1)\cdot 6!}{8}.$$

It is clear that there exists an even number between a and $(a^7 - 1)$, so we have $2 \mid (a^8 - a)$, $4^2 \mid 6!$. That is to say, our Theorem is true if $m = 8$.

(c). If $m > 5$, and $m \neq 8$, then we can discuss it in two cases:

If m is an odd number, we can write $m = 4k + t$, where $t = 1$ or 3, $k \in N$. Then

$$4^{\left[\frac{m-1}{4}\right]} = 4^{\left[\frac{4k+t-1}{4}\right]} = 4^k.$$

If $m = 4k + 1$, then we have the estimate

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{m-2}{2^j}\right] &= \left[\frac{m-2}{2}\right] + \left[\frac{m-2}{4}\right] + \cdots \\ &= (2k-1) + (k-1) + \cdots \geq (3k-2) \geq 2k.\end{aligned} \tag{2}$$

If $m = 4k + 3$, then we also have

$$\sum_{j=1}^{\infty}\left[\frac{m-2}{2^j}\right] = \left[\frac{m-2}{2}\right]+\left[\frac{m-2}{4}\right]+\cdots = 2k+k+\cdots \geq 3k > 2k. \quad (3)$$

From (2) and (3) we may immediately deduce that 4^k is a divisor of $(4k-2)!$. Let $(m-2)! = 4^k \cdot D$, where D is a positive integer. Note that $(4k+t,\ 4^k) = 1$, so $(4k+t) \mid (a^m - a)\cdot D$. Therefore, if $m > 5$ is an odd number, then (1) is correct.

If m is an even number, we can write $m = 2^r \cdot q$, where q is an odd number. It is clear that if $r = 1$, then from the above we know that the (1) is true. So we can assume that $r \geq 2$. This time, in order to prove our theorem, we just need to prove that

$$2^{2^{(r-1)}\cdot q-2+r} \mid \frac{(a^{2^r q}-a)(2^r q-2)!}{q}.$$

Similarly, we also have

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{m-2}{2^j}\right] &= \left[\frac{m-2}{2}\right]+\left[\frac{m-2}{4}\right]+\cdots+\left[\frac{m-2}{2^r}\right]+\cdots \\ &= 2^{r-1}q-1+2^{r-2}q-1+\cdots+q-1+\cdots \\ &\geq 2^{r-1}q-1+2^{r-2}q-1+\cdots+q-1 \\ &= q(2^{r-1}+2^{r-2}+\cdots+1)-r \\ &= q2^r-q-r. \quad (4)\end{aligned}$$

Note that $r \geq 2$ and $(2,\ q) = 1$, so we have $(q\cdot 2^r - q - r) - (2^{(r-1)}\cdot q - 2 + r) = 2^{r-1}q - q + 2 - 2r \geq 0$. Thus, $2^{2^{r-1}q-2+r}$ is an divisor of $(m-2)!$. Since $q \leq m-2$, we also have $q \mid (m-2)!$. Therefore, from (2), (3) and (4) we know that (1) is correct, if $m > 5$ ($m \neq 8$) is an even number.

Combining (a), (b) and (c) we may immediately complete the proof of our Theorem.

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Some properties of the permutation sequence

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Abstract The main purpose of this paper is using the elementary method to study the properties of the permutation sequence, and prove some interesting conclusions.
Keywords Permutation sequence, perfect power, divisibility.

§1. Introduction and result

For any positive integer n, the F.Smarandache permutation sequence $\{P(n)\}$ is defined as $P(n) = 135\cdots(2n-1)(2n)\cdots 42$. For example, the first few value of the sequence $\{P(n)\}$ are: $P(1) = 12$, $P(2) = 1342$, $P(3) = 135642$, $P(4) = 13578642$, $\cdots\cdots$. This sequence was introduced by professor F.Smarandache in reference [1], where he asked us to study its elementary properties. About this problem, many people had studied it, and obtained a series valuable results, see references [2], [3] and [4]. In reference [5], F.Smarandache proposed the following problem: Is there any perfect power among the permutation sequences? That is, whether there exist positive integers n, m and k with $k \geq 2$ such that $P(n) = m^k$.

The main purpose of the paper is using the elementary method to study this problem, and solved it completely. At the same time, we also obtained some other properties of the permutation sequence $\{P(n)\}$. That is, we shall prove the following:

Theorem 1. There is no any perfect power among the permutation sequence.

Theorem 2. Among the permutation sequence $\{P(n)\}$, there do not exist the number which have the factor 2^k, where $k \geq 2$; There exist infinite positive integers $a \in \{P(n)\}$ such that 3^2 divide a.

§2. Proof of the theorems

In this section, we shall use the elementary method to prove our Theorems directly. First we prove Theorem 1. For any positive integer $n \geq 2$, note that $P(n)$ be an even number and

$$\begin{aligned} & P(n) = 135\cdots(2n-1)(2n)\cdots 42 \\ = \quad & 10^{\alpha_1} + 3 \times 10^{\alpha_2} + \cdots + (2n-1) \times 10^{\alpha_n} + (2n) \times 10^{\alpha_{n+1}} + \cdots + 4 \times 10 + 2. \end{aligned} \tag{1}$$

So from (1) we know that

$$P(n) = 10^{\alpha_1} + 3 \times 10^{\alpha_2} + \cdots + (2n-1) \times 10^{\alpha_n} + (2n) \times 10^{\alpha_{n+1}} + \cdots + 4 \times 10 + 2 \equiv 0 \bmod 2$$

and

$$\begin{aligned} P(n) &= 10^{\alpha_1} + 3 \times 10^{\alpha_2} + \cdots + (2n-1) \times 10^{\alpha_n} + (2n) \times 10^{\alpha_{n+1}} + \cdots + 4 \times 10 + 2 \\ &\equiv 2 \bmod 4. \end{aligned} \tag{2}$$

That is to say, $2 \mid P(n)$ and $4 \dagger P(n)$. So $P(n)$ is not a perfect power. Otherwise, we can write $P(n) = m^k$, where k and $m \geq 2$. Since $P(n)$ be an even number, so m must be an even number. Therefore, $P(n) = m^k \equiv 0 \bmod 4$. Contradiction with $4 \dagger P(n)$. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer $n \geq 2$, from (2) we may immediately deduce that $4^r \dagger P(n)$, where $r \geq 2$ is an integer.

Now we write $P(n)$ as:

$$P(n) = 10^{\alpha_1} + 3 \times 10^{\alpha_2} + \cdots + (2n-1) \times 10^{\alpha_n} + (2n) \times 10^{\alpha_{n+1}} + \cdots + 4 \times 10 + 2. \tag{3}$$

Note that $10 \equiv 1 \bmod 9$, from the properties of the congruence we may get $10^t \equiv 1 \bmod 9$ for all integer $t \geq 1$. Therefore,

$$\begin{aligned} P(n) &= 10^{\alpha_1} + 3 \times 10^{\alpha_2} + \cdots + (2n-1) \times 10^{\alpha_n} + (2n) \times 10^{\alpha_{n+1}} + \cdots + 4 \times 10 + 2 \\ &\equiv 1 + 2 + 3 + \cdots + \cdots + 2n \equiv n \cdot (2n+1) \bmod 9. \end{aligned} \tag{4}$$

From (4) we know that $P(n) \equiv 0 \bmod 9$, if $n \equiv 0$ or $4 \bmod 9$. That is to say, for all integers $n \geq 2$, $P(9n)$ and $P(9n+4)$ can be divided by $9 = 3^2$. This completes the proof of Theorems.

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One problem related to the Smarandache quotients

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Abstract For any positive integer $n \geq 1$, the Smarandache quotients $Q(n)$ is defined as the smallest positive integer k such that $n \cdot k$ is a factorial number. That is, $Q(n) = \min\{k : n \cdot k = m!\}$. The main purpose of this paper is using the elementary method to study the properties of the Smarandache quotients sequence, and give an identity involving the Smarandache quotients sequence.
Keywords The Smarandache quotients, identity, infinite series, elementary method.

§1. Introduction and results

In his book "Only problems, not solutions", professor F.Smarandache introduced many functions, sequences and unsolved problems, many authors had studied it, see references [1], [2] and [3]. One of the unsolved problems is the Smarandache quotients sequence $\{Q(n)\}$, it is defined as the smallest positive integer k such that $n \cdot k$ is a factorial number. That is, $Q(n) = \min\{k : n \cdot k = m!\}$, where m is a positive integer. For example, from the definition of $Q(n)$ we can find that the first few values of $Q(n)$ are $Q(1) = 1$, $Q(2) = 1$, $Q(3) = 2$, $Q(4) = 6$, $Q(5) = 24$, $Q(6) = 1$, $Q(7) = 720$, $Q(8) = 3$, $Q(9) = 80$, $Q(10) = 12$, $Q(11) = 3628800$, $Q(12) = 2$, $Q(13) = 479001600$, $Q(14) = 360$, $Q(15) = 8$, $Q(16) = 45$, $\cdots\cdots$.

In reference [4], professor F.Smarandache asked us to study the properties of the sequence $\{Q(n)\}$. About this problem, some authors had studied it, and obtained several simple results. For example, Kenichiro Kashihara [5] proved that for any prime p, we have $Q(p) = (p-1)!$.

In this paper, we use the elementary method to study the properties of an infinite series involving the Smarandache quotients sequence, and give an interesting identity. That is, we shall prove the following conclusion:

Theorem 1. Let $d(n)$ denotes the Dirichlet divisor function, then we have the identity

$$\sum_{n=1}^{+\infty} \frac{1}{Q(n) \cdot n} = \sum_{m=1}^{+\infty} \frac{m \cdot d(m!)}{(m+1)!}.$$

Let $Q_1(2n-1)$ denotes the smallest positive odd number such that $Q_1(2n-1) \cdot (2n-1)$ is a two factorial number. That is, $Q_1(2n-1) \cdot (2n-1) = (2m-1)!!$, where $(2m-1)!! = 1 \times 3 \times 5 \times \cdots \times (2m-1)$. Then for the sequence $\{Q_1(2n-1)\}$, we can also get the following:

Theorem 2. Let $d(n)$ denote the Dirichlet divisor function, then we also have the identity

$$\sum_{n=1}^{+\infty} \frac{1}{Q_1(2n-1)\cdot(2n-1)} = 2\cdot \sum_{m=1}^{+\infty} \frac{m\cdot d((2m-1)!!)}{(2m+1)!!}.$$

§2. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of our theorems directly. For any positive integer n, let $Q(n)=k$, from the definition of $Q(n)$ we know that k is the smallest positive integer such that $n\cdot k = m!$. So k must be a divisor of $m!$. This implies that $k \mid m!$, and the number of all k (such that $k \mid m!$) is $d(m!)$. On the other hand, if $k \mid (m-1)!$, then $Q(n)\neq k$, and the number of all k (such that $k \mid (m-1)!$) is $d((m-1)!)$. So the number of all k (such that $k \mid m!$ and $k \dagger (m-1)!$) is $d(m!) - d((m-1)!)$. That is means, the number of all k (such that $Q(n)=k$ and $Q(n)\cdot n = m!$) is $d(m!) - d((m-1)!)$. From this we may immediately get

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{Q(n)\cdot n} &= \sum_{m=1}^{+\infty} \sum_{\substack{n=1 \\ Q(n)\cdot n=m!}}^{+\infty} \frac{1}{m!} = \sum_{m=1}^{+\infty} \frac{\#\{Q(n)\cdot n = m!\}}{m!} \\
&= 1 + \sum_{m=2}^{+\infty} \frac{d(m!) - d((m-1)!)}{m!} = \sum_{m=1}^{+\infty} \frac{d(m!)}{m!} - \sum_{m=1}^{+\infty} \frac{d(m!)}{(m+1)!} \\
&= \sum_{m=1}^{+\infty} \frac{(m+1)\cdot d(m!) - d(m!)}{(m+1)!} = \sum_{m=1}^{+\infty} \frac{m\cdot d(m!)}{(m+1)!},
\end{aligned}$$

where $d(m)$ is the Dirichlet divisor function, and $\#\{Q(n)\cdot n = m!\}$ denotes the number of all solutions of the equation $Q(n)\cdot n = m!$. This proves Theorem 1.

Similarly, we can also deduce Theorem 2. This completes the proof of Theorems.

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The Smarandache bisymmetric determinant natural sequence

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Abstract Murthy [1] introduced the Smarandache Bisymmetric Determinant Natural Sequence. In this paper, we derive the sum of the first n terms of the sequence.

Keywords The Smarandache bisymmetric determinant natural sequence, the n-th term, the sum of the first n terms.

§1. Introduction

The Smarandache bisymmetric determinant natural sequence (SBDNS), introduced by Murthy [1], is defined as follows.

Definition 1.1. The Smarandache bisymmetric determinant natural sequence, {SBDNS(n)}, is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}, \ldots \right\}.$$

A first few terms of the sequence are 1, -3, -8, 20, 48, -112, -256, 576, $\cdots$.

The following result is due to Majumdar [2].

Theorem 1.1. Let a_n be the n-th term of the Smarandache bisymmetric determinant natural sequence. Then,

$$a_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n-1 \\ 3 & 4 & 5 & \cdots & n & n-1 & n-2 \\ \vdots & & & & & & \\ n-2 & n-1 & n & & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} (n+1)\ 2^{n-2}.$$

Let $\{S_n\}$ be the sequence of $n-th$ partial sums of the sequence $\{a_n\}$, so that

$$S_n = \sum_{k=1}^{n} a_k, \ n \geqslant 1.$$

This paper gives explicit expressions for the sequence $\{S_n\}$. This is given in Theorem 3.1 in Section 3. In Section 2, we give some preliminary results that would be necessary for the proof of the theorem. We conclude this paper with some remarks in the final section, Section 4.

§2. Some preliminary results

In this section, we derive some preliminary results that would be necessary in deriving the expressions of S_n in the next section. These are given in the following two lemmas.

Lemma 2.1. For any integer $m \geq 1$,

$$\sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} = \frac{1}{15}(2^{4m}-1).$$

Proof. Since

$$\sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} = 1 + 2^4 + 2^8 + \ldots + 2^{4(m-1)}$$

is a geometric series with common ratio 2^4, the result follows.

Lemma 2.2. For any integer $m \geq 1$,

$$\sum_{k=1,3,\ldots,(2m-1)} k\ 2^{2(k-1)} = \frac{1}{15}(2m-1)2^{4m} - \frac{1}{225}(2^{4m+1}-17).$$

Proof. Denoting by s the series on the left above, we see that

$$s = 1 + 3\cdot 2^4 + 5\cdot 2^8 + \cdots + (2m-1)\cdot 2^{4(m-1)}, \qquad (*)$$

so that, multiplying throughout by 2^4, we get

$$2^4 s = 1\cdot 2^4 + 3\cdot 2^8 + \cdots + (2m-3)\cdot 2^{4(m-1)} + (2m-1)\cdot 2^{4m}\ \cdot \qquad (**)$$

Now, subtracting (**) from (*), we have

$$\begin{aligned}(1-2^4)s &= 1 + 2\cdot 2^4\left[1 + 2^4 + \cdots + 2^{4(m-2)}\right] - (2m-1)\cdot 2^{4m}\\ &= 1 + 2^5\cdot\left\{\frac{2^{4(m-1)}-1}{2^4-1}\right\} - (2m-1)\cdot 2^{4m}\\ &= \frac{1}{15}\left(2^{4m+1}-17\right) - (2m-1)\cdot 2^{4m}\end{aligned}$$

which now gives the desired result.

§3. Main results

In this section, we derive the explicit expressions of the n-th partial sums, S_n, of the Smarandache bisymmetric determinant natural sequence.

From Theorem 1.1, we see that, for any integer $k \geq 1$,

$$a_{2k} + a_{2k+1} = (-1)^k (6k+5) 2^{2(k-1)},$$

$$a_{2k+2} + a_{2k+3} = (-1)^{k+1} (6k+11) 2^{2k},$$

so that

$$a_{2k} + a_{2k+1} + a_{2k+2} + a_{2k+3} = 3(-1)^{k+1} (6k+13) 2^{2(k-1)}. \tag{1}$$

Letting

$$S_n = a_1 + a_2 + \cdots + a_n,$$

we can prove the following result.

Theorem 3.1. For any integer $m \geq 0$,

1). $S_{4m+1} = \frac{3}{5} \cdot m \cdot 2^{2(2m+1)} + \frac{31}{25} \cdot 2^{4m} - \frac{6}{25} = \frac{2}{25} \left\{(60m+31)2^{4m-1} - 3\right\};$

2). $S_{4m+2} = -\frac{1}{5} \cdot m \cdot 2^{4m+3} - \frac{11}{25} \cdot 2^{2(2m+1)} - \frac{6}{25} = -\frac{2}{25} \left\{(10m+11)2^{4m+1} + 3\right\};$

3). $S_{4m+3} = -\frac{3}{5} \cdot m \cdot 2^{4(m+1)} - \frac{61}{25} \cdot 2^{2(2m+1)} - \frac{6}{25} = -\frac{2}{25} \left\{(60m+61)2^{4m+1} + 3\right\};$

4). $S_{4m+4} = \frac{1}{5} \cdot m \cdot 2^{4m+5} + \frac{1}{25} \cdot 2^{4(2m+2)} - \frac{6}{25} = \frac{2}{25} \left\{(5m+8)2^{4(m+1)} - 3\right\}.$

Proof. To prove the theorem, we make use of Lemma 2.1 and Lemma 2.2, as well as Theorem 1.1.

1). Since S_{4m+1} can be written as

$$S_{4m+1} = a_1 + a_2 + \cdots + a_{4m+1} = a_1 + \sum_{k=1,3,\ldots,(2m-1)} (a_{2k} + a_{2k+1} + a_{2k+2} + a_{2k+3}),$$

by virtue of (1),

$$\begin{aligned} S_{4m+1} &= a_1 + 3 \sum_{k=1,3,\ldots,(2m-1)} (-1)^{k+1} (6k+13) 2^{2(k-1)} \\ &= a_1 + 3 \left\{ 6 \sum_{k=1,3,\ldots,(2m-1)} k 2^{2(k-1)} + 13 \sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)} \right\}. \end{aligned}$$

Now, appealing to Lemma 2.1 and Lemma 2.2, we get

$$S_{4m+1} = 1 + \left[\frac{6}{5}(2m-1)2^{4m} - \frac{2}{25}(2^{4m+1} - 17)\right] + \frac{13}{5}(2^{4m} - 1)$$

which now gives the desired result after some algebraic manipulations.

2). Since

$$S_{4m+2} = S_{4m+1} + a_{4m+2},$$

from part (1) above, together with Theorem 1.1, we get

$$S_{4m+2} = \left[\frac{3}{5} \cdot m \cdot 2^{2(2m+1)} + \frac{31}{25} \cdot 2^{4m} - \frac{6}{25}\right] - (4m+3) \cdot 2^{4m},$$

which gives the desired expression for S_{4m+2} after algebraic simplifications.

3). Since

$$S_{4m+3} = S_{4m+2} + a_{4m+3} = \left[-\frac{1}{5} \cdot m \cdot 2^{4m+3} - \frac{11}{25} \cdot 2^{2(2m+1)} - \frac{6}{25}\right] - (4m+4) \cdot 2^{4m+1},$$

we get the desired expression for S_{4m+3} after simplifications.

4). Since

$$S_{4m+4} = S_{4m+3} + a_{4m+4} = \left[-\frac{6}{5} \cdot m \cdot 2^{4m+3} - \frac{61}{25} \cdot 2^{2(2m+1)} - \frac{6}{25}\right] - (4m+5) \cdot 2^{4m+2},$$

the result follows after some algebraic simplifications.

The case when $m = 0$ can easily be verified.

Hence, the proof is complete.

§4. Remarks

Theorem 3.1 in the previous section gives the n-th term of the sequence of partial sums, $\{S_n\}$, in all the possible four cases. The following lemmas prove that, in each case, S_n is indeed an integer.

Lemma 4.1. For any integer $m \geq 0$, $2^{4m-1}(60m+31) - 3$ is divisible by 25.

Proof. The result is true for $m = 0, 1$. So, we assume its validity for some positive integer m. Now, since

$$[2^{4m+3}\{60(m+1)+31\} - 3] - \{2^{4m-1}(60m+31) - 3\} = 25(36m+57)2^{4m-1},$$

it follows, by virtue of the induction hypothesis, that $2^{4m+3}\{60(m+1)+31\} - 3$ is also divisible by 25. Thus, the result is true for $m+1$ as well, completing induction.

Lemma 4.2. For any integer $m \geq 0$, $2^{4m+1}(10m+11) + 3$ is divisible by 25.

Proof. is by induction on m. The result is clearly true for $m = 0, 1$. Now, assuming its validity for some positive integer m, since

$$[2^{4m+5}\{10(m+1)+11\} + 3] - \{2^{4m+1}(10m+11) + 3\} = 25(6m+13)2^{4m+1},$$

is divisible by 25, it follows that the result is true for $m+1$ as well. This completes the proof.

Lemma 4.3. For any integer $m \geq 0$, $2^{4m+1}(60m+61) + 3$ is divisible by 25.

Proof. is by induction on m. The result is clearly true for $m = 0, 1$. We now assume that the result is true for some positive integer m. Then, since

$$[2^{4m+5}\{60(m+1)+61\}+3] - \{2^{4m+1}(60m+61)+3\} = 25(36m+75)2^{4m+1},$$

this, together with the induction hypothesis, shows that the result is true for m + 1 as well. This completes the proof by induction.

Lemma 4.4. For any integer $m \geq 0$, $2^{4(m+1)}(5m+32) - 3$ is divisible by 25.

Proof. We first assume that the result is true for some positive integer m. Now, since

$$[2^{4(m+2)}\{5(m+1)+13\}-3] - \{2^{4(m+1)}(5m+13)-3\} = 25(3m+8)2^{4(m+1)},$$

this together with the induction hypothesis, shows that $2^{4(m+2)}\{5(m+1)+13\} - 3$ is also divisible by 25. This, in turn, shows that the result is true for $m+1$ as well. To complete, we have to prove the validity of the result for $m = 0, 1$, which can easily be checked.

The Smarandache bisymmetric arithmetic determinant sequence, introduced by Murthy [1], is

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \cdots \right\}.$$

The n-th term of the above sequence has been found by Majumdar [2] to be

$$(-1)^{\left[\frac{n}{2}\right]} \left(a + \frac{n-1}{2}d\right)(2d)^{n-1}.$$

Note that the Smarandache bisymmetric determinant natural sequence is a particular case of the Smarandache bisymmetric arithmetic determinant sequence when $a = 1$ and $d = 1$.

Open Problem: To find a formula for the sum of the first n terms of the Smarandache bisymmetric arithmetic determinant sequence.

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On the k-power complements function of $n!$

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Abstract For any fixed natural number $k \geq 2$ and any positive integer n, we call $b_k(n)$ as a k-power complements function of n, if $b_k(n)$ denotes the smallest positive integer such that $n \cdot b_k(n)$ is a perfect k-power. In this paper, we use the elementary method to study the asymptotic properties of $b_k(n!)$, and give an interesting asymptotic formula for $\ln b_k(n!)$.

Keywords k-power complements function, Standard factorization, Asymptotic formula.

§1. Introduction and results

For any fixed natural number $k \geq 2$ and any positive integer n, we call $b_k(n)$ as a k-power complements function of n, if $b_k(n)$ denotes the smallest positive integer such that $n \cdot b_k(n)$ is a perfect k-power. Especially, we call $b_2(n)$, $b_3(n)$, $b_4(n)$ as the square complements function, cubic complements function and quartic complements function respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the k-power complements function. About this problem, there are many authors had studied it, and obtained a series results. For example, in reference [2], Wenpeng Zhang calculated the value of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot b_k(n))^s},$$

where s is a complex number with $Re(\alpha) \geq 1$, $k = 2, 3, 4$, and prove that

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot b_2(n))^s} = \frac{\zeta^2(2s)}{\zeta(4s)},$$

where $\zeta(s)$ is the Riemann-zeta function.

Ruiqin Fu [3] discussed the asymptotic properties of $\ln b_2(n!)$, and proved that

$$\ln b_2(n!) = n \ln 2 + O\left(n \cdot \exp\left(\frac{-A \ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $n \geq 2$ is an positive integer, $A > 0$ is a constant.

But for the properties of the k-power complements function of n, we still know very little at present. In this paper, we use the elementary method to study the asymptotic properties of $b_k(n!)$, and give an interesting asymptotic formula for $\ln b_k(n!)$. That is, we shall prove the following general conclusion:

Theorem. For any fixed natural number $k \geq 2$ and any positive integer n, we have the asymptotic formula

$$\ln b_k(n!) = n\left(k - \sum_{d=1}^{\infty} \frac{1}{d(kd+1)}\right) + O\left(n \cdot \exp\left(\frac{-A\ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where $A > 0$ is a constant, and $\exp(y) = e^y$.

§2. Two simple lemmas

To complete the proof of the theorem, we need the following two simple lemmas:

Lemma 1. Let $n! = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of $n!$ into prime powers. Then we have the calculate formula

$$\begin{aligned} b_k(n!) &= b_k(p_1^{\alpha_1}) b_k(p_2^{\alpha_2}) \cdots b_k(p_s^{\alpha_s}) \\ &= p_1^{ord(p_1)} \cdot p_2^{ord(p_2)} \cdots p_s^{ord(p_s)} \end{aligned}$$

where the ord function is defined as:

$$ord(p_i) = \begin{cases} k-1, & \text{if} \alpha_i = km+1, \\ k-2, & \text{if} \alpha_i = km+2, \\ \vdots & \\ 0, & \text{if} \alpha_i = k(m+1). \end{cases}$$

where $m = 0, 1, 2, \cdots$.

Proof. See reference [3].

Lemma 2. For any real number $x \geq 2$, we have the asymptotic formula

$$\theta(x) = \sum_{p \leq x} \ln p = x + O\left(x \cdot \exp\left(\frac{-A\ln^{\frac{3}{5}} x}{(\ln\ln x)^{\frac{1}{5}}}\right)\right).$$

where $\exp(y) = e^y$, and $A > 0$ is a constant.

Proof. See reference [4] and [5].

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. First from Lemma 1 we have

$$\begin{aligned} \ln b_k(n!) &= \ln\left(b_k(p_1^{\alpha_1}) b_k(p_2^{\alpha_2}) \cdots b_k(p_s^{\alpha_s})\right) \\ &= \sum_{\substack{p \leq n \\ k \dagger ord(p)}} ord(p) \ln p \\ &= (k-1) \sum_{\frac{n}{2} < p \leq n} \ln p + (k-2) \sum_{\frac{n}{3} < p \leq \frac{n}{2}} \ln p + \cdots + \sum_{\frac{n}{k} < p \leq \frac{n}{k-1}} \ln p \\ &+ (k-1) \sum_{\frac{n}{k+2} < p \leq \frac{n}{k+1}} \ln p + \cdots + \sum_{\frac{n}{2k} < p \leq \frac{n}{2k-1}} \ln p + \cdots + O(1). \end{aligned}$$

Let n be a positive large enough, if a prime factor p of $n!$ in the interval $(\frac{n}{2}, n]$, then the power of p is 1 in the standard factorization of $n!$. Similarly, if a prime factor p of $n!$ in the interval $(\frac{n}{3}, \frac{n}{2}]$, then the power of p is 2 in the standard factorization of $n!$; If a prime factor p of $n!$ in the interval $(\frac{n}{4}, \frac{n}{3}]$, then the power of p is 3 in the standard factorization of $n!, \cdots$.
Then from the above formula and Lemma 2 we have

$$\begin{aligned}
\ln b_k(n!) &= (k-1)\left(\theta(n)-\theta\left(\frac{n}{2}\right)\right)+(k-2)\left(\theta\left(\frac{n}{2}\right)-\theta\left(\frac{n}{3}\right)\right)+\cdots+\left(\theta\left(\frac{n}{k-1}\right)-\theta\left(\frac{n}{k}\right)\right)\\
&+ (k-1)\left(\theta\left(\frac{n}{k+1}\right)-\theta\left(\frac{n}{k+2}\right)\right)+\cdots+\left(\theta\left(\frac{n}{2k-1}\right)-\theta\left(\frac{n}{2k}\right)\right)+\cdots+O(1)\\
&= n\left(k-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}\right)\right)+\frac{n}{k+1}\left(k-\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)\right)\\
&+ \cdots+O\left(n\cdot\exp\left(\frac{-A\ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\\
&= \lim_{M\to\infty} n\left(\sum_{d=0}^{M}\frac{k}{kd+1}-\sum_{d=1}^{M}\frac{1}{d}\right)+O\left(n\cdot\exp\left(\frac{-A\ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\\
&= \lim_{M\to\infty} n\left(k-\sum_{d=1}^{M}\frac{1}{d(kd+1)}\right)+O\left(n\cdot\exp\left(\frac{-A\ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\\
&= n\left(k-\sum_{d=1}^{\infty}\frac{1}{d(kd+1)}\right)+O\left(n\cdot\exp\left(\frac{-A\ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),
\end{aligned}$$

where $A > 0$ is a constant, and $\exp(y) = e^y$.

This completes the proof of Theorem. Absolutely, this theorem can be used to study k-power residues numbers.

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An alternative approach to the LP problem with equality constraints

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Abstract This paper gives an alternative approach to the solution of the LP problem with equality constraints. In this scheme, the constraint system of linear equations is first solved by elementary row operations (if such a solution exists). In the second stage, the solution of the system of linear equations is directly plugged in the objective function and then it is minimized. This scheme thus avoids the use of the simplex method.

Keywords Linear programming, equality constraint, elementary row operations.

§1. Introduction

In a recent paper, Ru, Shen and Xue [1] considered the problem of finding an initial basic feasible solution (bfs) of the LP problem of the form

$$\min z = C^t X \quad s.t. \quad AX = B, \quad X \geq 0, \tag{1}$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \\ a_{i1} & a_{i2} \cdots a_{in} \\ \vdots & \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In (1), without loss of generality, we may assume that $B \geq 0$; for otherwise, if some $b_i < 0$ $(1 \leq i \leq m)$, then multiplying the i-th constraint throughout by (-1), we get one with $(-b_i) > 0$.

To find an initial bfs of the LP problem (1), the procedure suggested by Ru et al. [1] is as follows :

Step 1 : Starting with the augmented matrix $(A \mid B)$, reduce it to a form with an identity matrix of desired order, by elementary row operations only.

Note that, the elementary row operations applicable in this case are

(a) multiplying any row of $(A \mid B)$ by a positive constant,

(b) multiplying any row of $(A \mid B)$ by a constant and adding it to a second row.

Throughout this paper, the first row operation would be denoted by $R_i \to kR_i$ (which means that the i-th row R_i is multiplied by the constant k, where $k > 0$), and the second row operation would be denoted by $R_i \to R_i + kR_j$ (which means that the j-th row R_j is multiplied by the constant k and the resulting row is added, term-by-term, to the i-th row R_i to get the new i-th row). Note that the latter row operation is allowed only if $b_i + kb_j \geq 0$.

If $rank(A) = rank(A \mid B) = l(\leq m)$, then using the above elementary row operations, it is possible to find a unit matrix of order l, which may be exploited to find an initial bfs of the LP problem (1). It may be mentioned here that, if $l < m$, then only l of the m system of linear equations $AX = B$ are independent, and the remaining $l - m$ equations can be ignored. It may also be mentioned here that, if $rank(A) \neq rank(A \mid B)$, then the system of linear equations $AX = B$ is inconsistent, and as such, the LP problem (1) is infeasible.

Step 2 : With the initial bfs found in Step 1 above, form the initial simplex tableau in the usual manner.

Step 3 : Proceed in the usual way of the simplex method to find an optimal solution.

The method outlined above is nevertheless not a new one. In fact, the same procedure has been followed by Papadimitriou and Steiglitz [2].

In the next section, we propose an alternative approach.

§2. An alternative approach

In this section, we suggest an alternative approach to the solution of the LP problem (1). The procedure is quite simple, and can be done in the following two steps.

Step 1 : Reduce the system of linear equations $AX = B$ to a form containing the unit matrix of requisite order, using elementary row operations only.

Step 2 : With the solution found in Step 1, consider the problem of minimizing $z = C^t X$.

Thus, the method works with the solution of the system of linear equations $AX = B$ and the objective function only, avoiding the simplex method completely.

We illustrate our method with the help of the following examples, due to Ru et al. [1].

Example 2.1. Consider the following LP problem :

$$\min z = 4x_1 + 3x_2 \qquad s.t.$$

$$\begin{array}{lllll} \frac{1}{2}x_1 & +x_2 & +\frac{1}{2}x_3 & -\frac{2}{3}x_4 & = 2 \\ \frac{3}{2}x_1 & & -\frac{1}{2}x_3 & & = 3 \\ 3x_1 & -6x_2 & & +4x_4 & = 0 \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

To solve the above LP problem, we first solve the constraint system of linear equations, using elementary row operations only. We thus start with the augmented matrix $(A \mid B)$ and proceed as follows, using the indicated row operations:

$$(A \mid B) = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 3 & -6 & 0 & 4 & 0 \end{pmatrix} \widetilde{R_3} \longrightarrow R_3 + 6R_1 \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 6 & 0 & 3 & 0 & 12 \end{pmatrix}$$

$$\begin{matrix} \widetilde{R_1} \longrightarrow R_1 + R_2 \\ R_2 \longrightarrow R_2 + \frac{1}{6}R_2 \end{matrix} \begin{pmatrix} 2 & 1 & 0 & -\frac{2}{3} & 5 \\ \frac{5}{2} & 0 & 0 & 0 & 5 \\ 6 & 0 & 3 & 0 & 12 \end{pmatrix} \begin{matrix} \widetilde{R_3} \longrightarrow \frac{1}{3}R_3 \\ R_1 \longrightarrow R_1 - \frac{4}{5}R_2 \\ R_3 \longrightarrow R_3 - \frac{4}{5}R_2 \end{matrix} \begin{pmatrix} 0 & 1 & 1 & -\frac{2}{3} & 1 \\ \frac{5}{2} & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The final tableau gives the following solution :

$$x_1 = 2, x_3 = 0, x_2 = 1 + \frac{2}{3}x_4 = 1 + \frac{2}{3}t; (t \geq 0 \textit{ is any real number}),$$

and the objective function is $z = 4x_1 + 3x_2 = 8 + 3(1 + \frac{2}{3}t) = 11 + \frac{2}{3}t$.
Since the objective is to minimize z, it is clear that z is minimized when $t = 0$, and the optimal solution of the given LP problem is thus $x_1^* = 2, x_2^* = 1, x_3^* = 0, x_4^* = 0; \min z^* = 11$.

Example 2.2. To solve the LP problem

$$\min z = -3x_1 + x_2 + x_3 \qquad s.t.$$

$$\begin{array}{lllllr} x_1 & -2x_2 & +x_3 & +x_4 & & = 11 \\ -4x_1 & +x_2 & +2x_3 & & -x_5 & = 3 \\ -2x_1 & & +x_3 & & & = 1 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0,$$

we first solve the constraint system of linear equations by elementary row operations. This is done below.

$$(A \mid B) = \begin{pmatrix} 1 & -2 & 1 & 1 & 0 & 11 \\ -4 & 1 & 2 & 0 & -1 & 3 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \widetilde{R_1} \longrightarrow R_1 + 2R_2 \begin{pmatrix} -7 & 0 & 5 & 1 & -2 & 17 \\ -4 & 1 & 2 & 0 & -1 & 3 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\widetilde{R_1} \longrightarrow R_1 + 5R_3 \begin{pmatrix} 3 & 0 & 0 & 1 & -2 & 12 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The final tableau is equivalent to the following system of linear equations

$$\begin{array}{lllllr} 3x_1 & & & +x_4 & -2x_5 & = 12 \\ & x_2 & & & -x_5 & = 2 \\ -2x_1 & & +x_3 & & & = 1 \end{array}$$

with the solution

$x_1 = t_1, x_5 = t_2, x_2 = 1{+}t_2, x_3 = 1{+}2t_1, x_4 = 12{-}3t_1{+}2t_2; (t_1 \geq 0, t_2 \geq 0$ are any real numbers),

and the objective function is $z = 2 - t_1 + t_2$.
Since the objective is to minimize z, it is evident that t_2 should be as small as possible and t_1 should be as large as possible. Now, since $x_4 = 12 - 3t_1 + 2t_2 \geq 0$, we see that, we must have $t_2 = 0$, $t_1 = 4$. Hence, the desired optimal solution is

$$x_1^* = 4, x_2^* = 1, x_3^* = 9, x_4^* = 0, x_5^* = 0; \min z^* = -3.$$

Example 2.3. To solve the LP problem

$$\min z = 4x_1 + 3x_3 \qquad s.t.$$

$$\begin{array}{llllll} \frac{1}{2}x_1 & +x_2 & +\frac{1}{2}x_3 & -\frac{2}{3}x_4 & = 2 \\ \frac{3}{2}x_1 & & +\frac{3}{4}x_3 & & = 3 \\ 3x_1 & -6x_2 & & +x_4 & = 0 \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0,$$

we start with the augmented matrix $(A|B)$ and proceed as follows, using the indicated row operations :

$$(A \mid B) = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & \frac{3}{4} & 0 & 3 \\ 3 & -6 & 0 & 4 & 0 \end{pmatrix} \underset{R_3 \longrightarrow R_3 + 6R_1}{\widetilde{}} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & \frac{3}{4} & 0 & 3 \\ 6 & 0 & 3 & 0 & 12 \end{pmatrix}$$

$$\underset{R_3 \longrightarrow R_3 - 4R_1}{\widetilde{}} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & \frac{3}{4} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R_1 \longrightarrow R_1 - \frac{1}{3}R_2 \\ R_2 \longrightarrow \frac{2}{3}R_2}}{\widetilde{}} \begin{pmatrix} 0 & 1 & \frac{1}{4} & -\frac{2}{3} & 1 \\ 1 & 0 & \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The final tableau is equivalent to the following system of linear equations

$$\begin{array}{llll} x_2 & +\frac{1}{4}x_3 & -\frac{2}{3}x_4 & = 1 \\ x_1 & +\frac{1}{2}x_3 & & = 2 \end{array}$$

with the solution

$$x_3 = t_1, x_1 = 2 - \frac{1}{2}t_1, x_4 = t_2, x_2 = 1 - \frac{1}{4}t_1 + \frac{2}{3}t_2; (t_1 \geq 0, t_2 \geq 0 \text{ are any real numbers}),$$

and the objective function is $z = 8 + t_1$.
Since the objective is to minimize z, we must have $t_1 = 0$. Hence, the desired solution is

$$x_1^* = 2, x_2^* = 1 + \frac{2}{3}t, x_3^* = 0, x_4^* = t (t \geq 0 \text{ is any real number}); \min z^* = 8.$$

The above solution shows that the given LP problem has infinite number of solutions.

§3. Some observations and remarks

In the previous section, we have illustrated our method with the help of three simple examples. However, in large-scale problems, the success as well as the computational efficiency of the method remains to be checked out.

Clearly, the method is useful when the correct basis has been found. This point is illustrated with the help of the following example, due to Papadimitriou and Steiglitz [2].

Example 3.1. Consider the LP problem below :

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 \qquad s.t.$$

$$\begin{array}{llllll} 3x_1 & +2x_2 & +x_3 & & & =1 \\ 5x_1 & +x_2 & +x_3 & +x_4 & & =3 \\ 2x_1 & +5x_2 & +x_3 & & +x_5 & =4 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

We first solve the constraint system of linear equations by elementary row operations, using the indicated row operations.

$$(A \mid B) = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 & 3 \\ 2 & 5 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{array}{l} \widetilde{R_1} \longrightarrow \frac{1}{2}R_1 \\ R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 5R_1 \end{array} \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{7}{2} & 0 & \frac{1}{2} & 1 & 0 & \frac{5}{2} \\ -\frac{11}{2} & 0 & -\frac{3}{2} & 0 & 1 & \frac{3}{2} \end{pmatrix}$$

The final tableau is equivalent to the following system of linear equations

$$\begin{array}{llllll} \frac{3}{2}x_1 & +x_2 & +\frac{1}{2}x_3 & & & =\frac{1}{2} \\ \frac{7}{2}x_1 & & +\frac{1}{2}x_3 & +x_4 & & =\frac{5}{2} \\ -\frac{11}{2}x_1 & & -\frac{3}{2}x_3 & & +x_5 & =\frac{3}{2} \end{array}$$

with the solution

$$x_1 = t_1, x_3 = t_2 (t_1 \geq 0, t_2 \geq 0 \text{ are any real numbers}),$$

$$x_2 = \frac{1}{2} - \frac{3}{2}t_1 - \frac{1}{2}t_2, x_4 = \frac{5}{2} - \frac{7}{2}t_1 - \frac{1}{2}t_2, x_5 = \frac{3}{2} + \frac{11}{2}t_1 + \frac{3}{2}t_2;$$

and the objective function is $z = \frac{9}{2} + \frac{3}{2}(t_1 + t_2)$.

Since the objective is to minimize z, it is evident that we must have $t_1 = 0$, $t_2 = 0$, giving the desired optimal solution

$$x_1^* = 0, x_2^* = \frac{1}{2}, x_3^* = 0, x_4^* = \frac{5}{2}, x_5^* = \frac{3}{2}; \min z^* = \frac{9}{2}.$$

On the other hand, the solution below

$$(A \mid B) = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 & 3 \\ 2 & 5 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{array}{l} \widetilde{R_2} \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{array} \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 1 & 0 & 2 \\ -1 & 3 & 0 & 0 & 1 & 3 \end{pmatrix}$$

leads to the system of equations

$$\begin{array}{rrrrrl} 3x_1 & +2x_2 & +x_3 & & & = 1 \\ 2x_1 & -x_2 & & +x_4 & & = 2 \\ -x_1 & +3x_2 & & & +x_5 & = 3 \end{array}$$

whose solution is

$$x_1 = t_1, x_2 = t_2 (t_1 \geq 0, t_2 \geq 0 \ \textit{are any real numbers}),$$

$$x_3 = 1 - 3t_1 - 2t_2, x_4 = 2 - 2t_1 + t_2, x_5 = 3 + t_1 - 3t_2;$$

and the objective function is $z = 3[2 - (t_1 + t_2)]$.
Thus, the problem of finding the solution of the given LP problem reduces to the problem of solving the LP problem

$$\max \ t_1 + t_2 \qquad s.t.$$

$$\begin{array}{rrl} 3t_1 & +2t_2 & \leq 1 \\ 2t_1 & -t_2 & \leq 2 \\ -t_1 & +3t_2 & \leq 3 \end{array}$$

$$t_1, t_2 \geq 0.$$

The optimal solution of the above LP problem is $t_1 = 0$, $t_2 = \frac{1}{2}$.
Hence, the optimal solution of the original LP problem is

$$x_1^* = 0, x_2^* = \frac{1}{2}, x_3^* = 0, x_4^* = \frac{5}{2}, x_5^* = \frac{3}{2}; \min z^* = \frac{9}{2}.$$

The above example shows that, in the alternative approach proposed in this paper, as well as the method suggested by Ru et al. [1], the choice of the unit matrix (of desired order) during the solution of the system of linear equations $AX = B$ (by elementary row operations) may save the number of iterations considerably. However, no particular criterion is specified in this respect. It thus still remains to check how far these methods can compete with the existing method.

References

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On some pseudo Smarandache function related triangles

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Abstract Two triangles $T(a,b,c)$ and $T(a^{'},b^{'},c^{'})$ are said to be pseudo Smarandache related if $Z(a)=Z(a^{'}),Z(b)=Z(b^{'}),Z(c)=Z(c^{'})$, where $Z(.)$ is the pseudo Smarandache function, and $T(a,b,c)$ denotes the triangle with sides a,b and c. This paper proves the existence of an infinite family of pairs of dissimilar Pythagorean triangles that are pseudo Smarandache related.

Keywords Smarandache function, pseudo Smarandache function, Smarandache related triangles, pseudo Smarandache related triangles, Pythagorean triangles.

§1. Introduction and result

The Smarandache function, denoted by S(n), is defined as follows.

Definition 1.1. For any integer $n \geq 1$, ($\mathbb{Z}^+$ being the set of all positive integers),

$$S(n)=\min\{m : m \in \mathbb{Z}^+, \quad n|m!\}.$$

The following definition is due to Sastry [6].

Definition 1.2. Two triangles $T(a,b,c)$ (with sides of length a,b and c) and $T(a^{'},b^{'},c^{'})$ (with sides of length $a^{'},b^{'}$ and $c^{'}$), are said to be Smarandache related if

$$S(a)=S(a^{'}),S(b)=S(b^{'}),S(c)=S(c^{'}).$$

Definition 1.3. A triangle $T(a,b,c)$ is said to be Pythagorean if and only if one of its angles is $90°$.

Thus, a triangle $T(a,b,c)$ (with sides of length a,b and c) is Pythagorean if and only if $a^2+b^2=c^2$.

Sastry[6] raised the following question : Are there two distinct dissimilar Pythagorean triangles that are Smarandache related? Recall that two triangles $T(a,b,c)$ and $T(a^{'},b^{'},c^{'})$ are similar if and only if the corresponding three sides are proportional, that is, if and only if

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$$

Otherwise, the two triangles are dissimilar.

The following result, due to Ashbacher [1], answers the question in the affirmative.

Theorem 1.1. There are an infinite family of pairs of dissimilar Pythagorean triangles that are Smarandache related.

The proof of Theorem 1.1 is rather simple : For any prime $p \geq 17$, the two families of dissimilar Pythagorean triangles $T(3p, 4p, 5p)$ and $T(5p, 12p, 13p)$ are Smarandache related, since $S(3p) = S(4p) = S(5p) = S(12p) = S(13p) = p$.

Ashbacher [1] introduced the concept of pseudo Smarandache related triangles, defined as follows.

Definition 1.4. Two triangles $T(a, b, c)$ and $T(a^{'}, b^{'}, c^{'})$ are said to be pseudo Smarandache related if $Z(a) = Z(a^{'})$, $Z(b) = Z(b^{'}), Z(c) = Z(c^{'})$.

In Definition 1.4 above, $Z(.)$ denotes the pseudo Smarandache function. Recall that the pseudo Smarandache function is defined as follows.

Definition 1.5. For any integer $n \geq 1$, $Z(n)$ is the smallest positive integer m such that $1 + 2 + \cdots + m \equiv \dfrac{m(m+1)}{2}$ is divisible by n. Thus,

$$Z(n) = \min\{m : m \in Z^{+}, n|\frac{m(m+1)}{2}\}; n \geq 1.$$

Ashbacher [1] used a computer program to search for dissimilar pairs of Pythagorean triangles that are pseudo Smarandache related. He reports some of them, and conjectures that there are an infinite number of pairs of dissimilar Pythagorean triangles that are pseudo Smarandache related.

In this paper, we prove the conjecture of Ashbacher [1] in the affirmative. This is done in Theorem 3.1 in Section 3. We proceed on the same line as that followed by Ashbacher. However, in the case of the pseudo Smarandache related triangles, the proof is a little bit more complicated. The intermediate results, needed for the proof of Theorem 3.1, are given in the next Section 2.

§2. Some preliminary results

The following result, giving the explicit expressions of $Z(3p)$, $Z(4p)$ and $Z(5p)$, are given in Majumdar [5].

Lemma 2.1. If $p \geq 5$ is a prime, then

$$Z(3p) = \begin{cases} p-1, & \text{if } 3|(\text{p}-1), \\ p\,, & \text{if } 3|(\text{p}+1). \end{cases}$$

Lemma 2.2. If $p \geq 5$ is a prime, then

$$Z(4p) = \begin{cases} p-1, & \text{if } \ 8|(p-1), \\ p, & \text{if } \ 8|(p+1), \\ 3p-1, & \text{if } \ 8|(3p-1), \\ 3p, & \text{if } \ 8|(3p+1). \end{cases}$$

Lemma 2.3. If $p \geq 7$ is a prime, then

$$Z(5p) = \begin{cases} p-1, & \text{if } 10|(p-1), \\ p, & \text{if } 10|(p+1), \\ 2p-1, & \text{if } 5|(2p-1), \\ 2p, & \text{if } 5|(2p+1). \end{cases}$$

The explicit expressions of $Z(12p)$ and $Z(13p)$ are given in the following two lemmas.

Lemma 2.4. If $p \geq 13$ is a prime, then

$$Z(12p) = \begin{cases} p-1, & \text{if } 24|(p-1) \\ p, & \text{if } 24|(p+1), \\ 3p-1, & \text{if } 8|(3p-1), \\ 3p, & \text{if } 8|(3p+1), \\ 7p-1, & \text{if } 24|(7p-1), \\ 7p, & \text{if } 24|(7p+1). \end{cases}$$

Proof. By definition,

$$Z(12p) = \min\{m : 12p|\frac{m(m+1)}{2}\} = \min\{m : p|\frac{m(m+1)}{24}\}. \quad (1)$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both, and then 24 must divide either $m+1$ or m respectively. In the particular case when 24 divides $p-1$ or $p+1$, the minimum m in (1) may be taken as $p-1$ or p respectively. We now consider the following eight cases that may arise :

Case 1 : p is of the form $p = 24a + 1$ for some integer $a \geq 1$.
In this case, $24|(p-1)$. Therefore, $Z(12p) = p - 1$.

Case 2 : p is of the form $p = 24a + 23$ for some integer $a \geq 1$.
Here, $24|(p+1)$, and so, $Z(12p) = p$.

Case 3 : p is of the form $p = 24a + 5$ for some integer $a \geq 1$, so that $8|(3p+1)$.
In this case, the minimum m in (1) may be taken as $3p$. That is, $Z(12p) = 3p$.

Case 4 : p is of the form $p = 24a + 19$ for some integer $a \geq 1$.
Here, $8|(3p-1)$, and hence, $Z(12p) = 3p - 1$.

Case 5 : p is of the form $p = 24a + 7$ for some integer $a \geq 1$.
In this case, $24|(7p-1)$, and hence, $Z(12p) = 7p$.

Case 6 : p is of the form $p = 24a + 17$ for some integer $a \geq 1$.
Here, $24|(7p+1)$, and hence, $Z(12p) = 7p + 1$.

Case 7 : p is of the form $p = 24a + 11$ for some integer $a \geq 1$.
In this case, $8|(3p-1)$, and hence, Z(12p) = 3p - 1.

Case 8 : p is of the form $p = 24a + 13$ for some integer $a \geq 1$.
Here, $8|(3p+1)$, and hence, $Z(12p) = 3p$.

Lemma 2.5. For any prime $p \geq 17$,

$$Z(13p) = \begin{cases} p-1, & \text{if } 13|(p-1), \\ p, & \text{if } 13|(p+1), \\ 2p-1, & \text{if } 13|(2p-1), \\ 2p, & \text{if } 13|(2p+1), \\ 3\text{p}-1, & \text{if } 13|(3p-1), \\ 3p, & \text{if } 13|(3p+1), \\ 4\text{p}-1, & \text{if } 13|(4p-1), \\ 4p, & \text{if } 13|(4p+1), \\ 5\text{p}-1, & \text{if } 13|(5p-1), \\ 5p, & \text{if } 13|(5p+1), \\ 6\text{p}-1, & \text{if } 13|(6p-1), \\ 6p, & \text{if } 13|(6p+1). \end{cases}$$

Proof. By definition,

$$Z(13p) = \min\{m : 13p|\frac{m(m+1)}{2}\} = \min\{m : p|\frac{m(m+1)}{26}\}. \quad (2)$$

We have to consider the twelve possible cases that may arise :

Case 1 : p is of the form $p = 13a + 1$ for some integer $a \geq 1$.
In this case, $13|(p-1)$, and so, $Z(13p) = p - 1$.

Case 2 : p is of the form $p = 13a + 12$ for some integer $a \geq 1$.
Here, $13|(p + 1)$, and hence, $Z(13p) = p$.

Case 3 : p is of the form $p = 13a + 2$ for some integer $a \geq 1$.
In this case, $13|(6p + 1)$, and hence, $Z(13p) = 6p$.

Case 4 : p is of the form $p = 13a + 11$ for some integer $a \geq 1$.
Here, $13|(6p - 1)$, and hence, $Z(13p) = 6p - 1$.

Case 5 : p is of the form $p = 13a + 3$ for some integer $a \geq 1$.
In this case, $13|(4p + 1)$, and hence, $Z(13p) = 4p$.

Case 6 : p is of the form $p = 13a + 10$ for some integer $a \geq 1$.
Here, $13|(4p - 1)$, and hence, $Z(13p) = 4p - 1$.

Case 7 : p is of the form $p = 13a + 4$ for some integer $a \geq 1$.
In this case, $13|(3p + 1)$, and hence, $Z(13p) = 3p$.

Case 8 : p is of the form $p = 13a + 9$ for some integer $a \geq 1$.
Here, $13|(3p - 1)$, and hence, $Z(13p) = 3p - 1$.

Case 9 : p is of the form $p = 13a + 5$ for some integer $a \geq 1$.
In this case, $13|(5p + 1)$, and hence, $Z(13p) = 5p$.

Case 10 : p is of the form $p = 13a + 8$ for some integer $a \geq 1$.
Here, $13|(5p - 1)$, and hence, $Z(13p) = 5p - 1$.

Case 11 : p is of the form $p = 13a + 6$ for some integer $a \geq 1$.
In this case, $13|(2p + 1)$, and hence, $Z(13p) = 2p$.

Case 12 : p is of the form $p = 13a + 7$ for some integer $a \geq 1$.
Here, $13|(2p-1)$, and hence, $Z(13p) = 2p - 1$.

§3. Main result

We are now state and prove the main result of this paper in the following theorem.

Theorem 3.1. There are an infinite number of pairs of dissimilar Pythagorean triangles that are pseudo Smarandache related.

Proof. We consider the pair of dissimilar Pythagorean triangles

$$T(3p, 4p, 5p) \quad \text{and} \quad T(5p, 12p, 13p), \tag{3}$$

where p is a prime of the form

$$p = (2^3 \cdot 3 \cdot 5 \cdot 13)n + 1 = 1560n + 1, n \in Z^+. \tag{4}$$

By Lemma 2.1 - Lemma 2.5,

$$Z(3p) = Z(4p) = Z(5p) = Z(12p) = Z(13p) = p - 1,$$

so that the triangles $T(3p, 4p, 5p)$ and $T(5p, 12p, 13p)$ are pseudo Smarandache related. Now, since there are an infinite number of primes of the form (4) (by Dirichlet's Theorem, see, for example, Hardy and Wright [3], Theorem 15, pp. 13), we get the desired infinite number of pairs of dissimilar Pythagorean triangles that are pseudo Smarandache related.

It may be mentioned here that, the family of pairs of triangles (3), where p is a prime of the form

$$p = 1560n - 1, n \in Z^+, \tag{5}$$

also forms (dissimilar) pseudo Smarandache related Pythagorean triangles.

§4. Some remarks

The pseudo Smarandache function $Z(n)$ is clearly not bijective. However, we can define the inverse $Z^{-1}(m)$ as follows :

$$Z^{-1}(m) = \{n \in Z^+ : Z(n) = m\} \quad \text{for any integer} \quad m \geq 3, \tag{6}$$

with

$$Z^{-1}(1) = 1, Z^{-1}(2) = 3. \tag{7}$$

As has been pointed out by Majumdar [5], for any $m \in Z^+$, the set $Z^{-1}(m)$ is non-empty and bounded with $\frac{m(m+1)}{2}$ as its largest element. Clearly, $n \in Z^{-1}(m)$ if and only if the following two conditions are satisfied :

(1) n divides $\frac{m(m+1)}{2}$,

(2) n does not divide $\frac{\ell(\ell+1)}{2}$ for any ℓ with $1 \leq \ell \leq m-1$.

We can look at (6) from a different point of view : On the set Z^+ , we define the relation $\Re$ as follows :

$$\text{For any } n_1, n_2 \in Z^+, n_1 \Re n_2 \quad \text{if and only if} \quad Z(n_1) = Z(n_2). \tag{8}$$

It is then straightforward to verify that $\Re$ is an equivalence relation on Z^+ . It is well-known that an equivalence relation induces a partition (on the set Z^+) (see, for example, Gioia [2], Theorem 11.2, pp. 32). The sets $Z^{-1}(m), m \in Z^+$, are, in fact, the equivalence classes induced by the equivalence relation $\Re$ on Z^+ , and possesses the following two properties :

(1) $\sum\limits_{m=1}^{\infty} Z^{-1}(m) = \mathbb{Z}^+$

(2) $Z^{-1}(m_1) \cap Z^{-1}(m_2) = \emptyset$, if $\mathrm{m}_1 \neq m_2$.

Thus, for any $n \in \mathbb{Z}^+$, there is one and only one $m \in \mathbb{Z}^+$ such that $n \in Z^{-1}(m)$.

A different way of relating two triangles has been considered by Ashbacher [1], which is given in the following definition.

Definition 4.1. Given two triangles, $T(a, b, c)$ and $T(a', b', c')$, where

$$a + b + c = 180 = a' + b' + c', \tag{9}$$

they are said to be pseudo Smarandache related if

$$Z(a) = Z(a'), Z(b) = Z(b'), Z(c) = Z(c').$$

The difference in Definition 1.4 and Definition 4.1 is that, in the former, the sides of the pair of triangles are pseudo Smarandache related, while their angles, measured in degrees, are pseudo Smarandache related in the latter case. Note that the condition (9) merely states the fact the sum of the three angles of a triangle is 180 degrees.

Using a computer program, Ashbacher searched for pseudo Smarandache related pairs of triangles (in the sense of Definition 4.1). He reports three such pairs.

However, in this case, the equivalence classes $Z^{-1}(m), m \in Z^+$, might be of some help. The condition (9) can be dealt with by considering the restricted sets $Z^{-1}(m|\pi)$:

$$Z^{-1}(m|\pi) = \{n \in Z^+ : Z(n) = m, 1 \leq n \leq 178\}. \tag{10}$$

Table 4.1 gives such restricted sets related to our problem.

Clearly, two pseudo Smarandache related triangles $T(a, b, c)$ and $T(a', b', c')$ must satisfy the following condition :

$$a, a' \in Z^{-1}(m_1|\pi); b, b' \in Z^{-1}(m_2|\pi), c, c' \in Z^{-1}(m_3|\pi) \quad \text{for some} \quad m_1, m_2, m_3 \in Z^+.$$

Thus, for example, choosing

$$a, a^{'} \in \{2, 6\} = Z^{-1}(3|\pi); b, b^{'} \in \{44, 48, 88, 132, 176\} = Z^{-1}(32|\pi),$$

we can construct the pseudo Smarandache related triangles $T(2, 48, 130)$ and $T(6, 44, 130)$. Again, choosing $a, a', b, b' \in \{25, 50, 75, 100, 150\} = Z^{-1}(24|\pi)$, we can form the pair of pseudo Smarandache related triangles $T(25, 150, 5)$ and $T(75, 100, 5)$ with the characteristic that

$$Z(a) = Z(a^{'}) = 24 = Z(b) = Z(b^{'}) \quad (\text{and} \;\; Z(c) = Z(c^{'}) = 4).$$

Choosing $a, a^{'}, b, b^{'} \in \{8, 20, 24, 30, 40, 60, 120\} = Z^{-1}(15|\pi)$, we see that the equilateral triangle $T(60, 60, 60)$ is pseudo Smarandache related to the triangle $T(20, 40, 120)$!

Ashbacher [1] cites the triangles $T(4, 16, 160)$ and $T(14, 62, 104)$ as an example of a pseudo Smarandache related pair where all the six angles are different. We have found three more, given below :

(1) $T(8, 12, 160)$ and $T(40, 36, 104)$,

(with $Z(8) = Z(40) = 15, Z(12) = Z(36) = 8, Z(160) = Z(104) = 64$),

(2) $T(16, 20, 144)$ and $T(124, 24, 32)$,

(with $Z(16) = Z(124) = 31, Z(20) = Z(24) = 15, Z(144) = Z(32) = 63$),

(3) $T(37, 50, 93)$ and $T(74, 75, 31)$,

(with $Z(37) = Z(74) = 36, Z(50) = Z(75) = 24, Z(93) = Z(31) = 30$).

Table 4.1. Values of $Z^{-1}(m|\pi) = \{n \in Z^{+} : Z(n) = m, 1 \leq n \leq 180\}$

m	$Z^{-1}(m\|\pi)$	m	$Z^{-1}(m\|\pi)$	m	$Z^{-1}(m\|\pi)$
1	$\{1\}$	35	$\{90\}$	80	$\{81,108,162,180\}$
2	$\{3\}$	36	$\{37,74,111\}$	82	$\{83\}$
3	$\{2,6\}$	39	$\{52,130,156\}$	83	$\{166\}$
4	$\{5,10\}$	40	$\{41,82,164\}$	84	$\{170\}$
5	$\{15\}$	41	$\{123\}$	87	$\{116,174\}$
6	$\{7,21\}$	42	$\{43,129\}$	88	$\{89,178\}$
7	$\{4,14,28\}$	43	$\{86\}$	95	$\{152\}$
8	$\{9,12,18,36\}$	44	$\{99,110,165\}$	96	$\{97\}$
9	$\{45\}$	45	$\{115\}$	100	$\{101\}$
10	$\{11,55\}$	46	$\{47\}$	102	$\{103\}$
11	$\{22,33,66\}$	47	$\{94,141\}$	106	$\{107\}$
12	$\{13,26,39,78\}$	48	$\{49,56,84,98,147,168\}$	108	$\{109\}$
13	$\{91\}$	49	$\{175\}$	111	$\{148\}$
14	$\{35,105\}$	51	$\{102\}$	112	$\{113\}$
15	$\{8,20,24,30,40,60,120\}$	52	$\{53,106\}$	120	$\{121\}$
16	$\{17,34,68,136\}$	53	$\{159\}$	124	$\{125\}$
17	$\{51,153\}$	54	$\{135\}$	126	$\{127\}$
18	$\{19,57,171\}$	55	$\{140,154\}$	127	$\{64\}$
19	$\{38,95\}$	56	$\{76,114,133\}$	128	$\{172\}$
20	$\{42,70\}$	58	$\{59\}$	130	$\{131\}$
21	$\{77\}$	59	$\{118,177\}$	136	$\{137\}$
22	$\{23\}$	60	$\{61,122\}$	138	$\{139\}$
23	$\{46,69,92,138\}$	63	$\{32,72,96,112,144\}$	148	$\{149\}$
24	$\{25,50,75,100,150\}$	64	$\{80,104,160\}$	150	$\{151\}$
25	$\{65\}$	65	$\{143\}$	156	$\{157\}$
26	$\{27,117\}$	66	$\{67\}$	162	$\{163\}$
27	$\{54,63,126\}$	67	$\{134\}$	166	$\{167\}$
28	$\{29,58\}$	69	$\{161\}$	168	$\{169\}$
29	$\{87,145\}$	70	$\{71\}$	172	$\{173\}$
30	$\{31,93,155\}$	71	$\{142\}$	178	$\{179\}$
31	$\{16,62,124\}$	72	$\{73,146\}$	255	$\{128\}$
32	$\{44,48,88,132,176\}$	78	$\{79\}$		
34	$\{85,119\}$	79	$\{158\}$		

Ashbacher [1] reports that, an exhaustive computer search for pairs of dissimilar search for all pseudo Smarandache related triangles $T(a,b,c)$ and $T(a^{'},b^{'},c^{'})$ $(a+b+c=180=a^{'}+b^{'}+c^{'})$ with values of a in the range $1 \le a \le 178$, revealed that a cannot take the following values :

$$\begin{aligned} &1,15,23,35,41,45,51,59,65,67,71,73,77,79,82,83,86,87,89, \\ &90,91,97,101,102,105,107,109,113,115,116,118,121,123,125,126,127, \\ &131,134,135,137,139,141,142,143,148,149,151,152,153,157,158,159, \\ &161,163,164,166,167,169,170,171,172,173,174,175,176,177,178. \end{aligned} \tag{11}$$

Can the table of the sets $Z^{-1}(m|\pi)$, Table 4.1, be utilized in explaining this observation? Of course, if a is too large, then there is a possibility that no two dissimilar triangles exist: Very large value of a very often forces the two triangles to be similar. For example, if $a = 174 \in Z^{-1}(87|\pi)$, then the three possible pairs of values of (b,c) are $(1,5)$, $(2,4)$ and $(3,3)$. Note that the two sets $Z^{-1}(1|\pi) = \{1\}$ and $Z^{-1}(2|\pi) = \{3\}$ are singleton. Thus, if $a^{'} = 174$, then the two triangles $T(a,b,c)$ and $T(a^{'},b^{'},c^{'})$ must be similar. On the other hand, if $a^{'} = 116 \in Z^{-1}(87|\pi)$, then we cannot find $b^{'},c^{'}$ with $b^{'}+c^{'} = 64$. Now, we consider the case when $a = 164$. This case is summarized in the tabular form below :

a=164	$a^{'}$=164	$a^{'}$=82	$a^{'}$=41	Remark	
(1,15)	Similar Triangle	Not Possible	Not Possible	Both belong to singleton sets	(b,c)
(2,14)	Similar Triangle	Not Possible	Not Possible	$b^{'}+c^{'} \le 34$	
(3,13)	Similar Triangle	Not Possible	Not Possible	3 belongs to a singleton set	
(4,12)	Similar Triangle	Not Possible	Not Possible	$b^{'}+c^{'} \le 64$	
(5,11)	Similar Triangle	Not Possible	Not Possible	$b^{'}+c^{'} \le 65$	
(6,10)	Similar Triangle	Not Possible	Not Possible	$b^{'}+c^{'} \le 16$	
(7,9)	Similar Triangle	Not Possible	Not Possible	$b^{'}+c^{'} \le 57$	
(8,8)	Similar Triangle	Not Possible	Not Possible		

However, if $a = 165$, we can get two dissimilar pseudo Smarandache related triangles, namely, $T(165,7,8)$ and $T(81,21,60)$.

A closer look at the values listed in (11) and those in Table 4.1 reveals the following facts: (1) The numbers not appearing in any 180 degrees triplets are all belong to singleton sets, with the exception of 3 (though we conjecture that 3 cannot appear in any triplet), 47, 64, and 103. For the last three cases, we can form the following examples :

(a) $T(47,76,57)$ and $T(47,114,19)$ (with $Z(76) = 56 = Z(114), Z(57) = 18 = Z(19)$),

(b) $T(64,12,104)$ and $T(64,36,80)$ (with $Z(12) = 8 = Z(36), Z(104) = 64 = Z(80)$),

(c) $T(103,11,66)$ and $T(103,55,22)$ (with $Z(11) = 10 = Z(55), Z(66) = 11 = Z(22)$).

(2) In most of the cases, if a value does not appear in the triplet, all other values of the corresponding $Z^{-1}(m|\pi)$ also do not appear in other triplet, with the exception of $145 \in Z^{-1}(29|\pi)$ ($87 \in Z^{-1}(29|\pi)$ does not appear in any triplet), and $146 \in Z^{-1}(72|\pi)(73 \in Z^{-1}(72|\pi)$ does not appear in any triplet). In this connection, we may mention the following pairs of triplets :

(a) $T(145, 14, 21)$ and $T(145, 28, 7)$ (with $Z(14) = 7 = Z(28), Z(21) = 6 = Z(7)$),

(b) $T(146, 4, 30)$ and $T(146, 14, 20)$ (with $Z(4) = 7 = Z(14), Z(30) = 15 = Z(20)$).

(c) Both $54, 63 \in Z^{-1}(72|\pi)$ can appear in one or the other triplet, but $126 \in Z^{-1}(72|\pi)$ cannot appear in any triplet.

It is also an interesting problem to look for 60 degrees and 120 degrees pseudo Smarandache related pairs of triangles, in the sense of Definition 4.1, where a 60 (120) degrees triangle is one whose one angle is 60 (120) degrees. We got the following two pairs of pseudo Smarandache related 60 degrees triangles :

(1) $T(8, 60, 112)$ and $T(24, 60, 96)$, (with $Z(8) = Z(24) = 15, Z(112) = Z(96) = 63$),

(2) $T(32, 60, 88)$ and $T(72, 60, 48)$, (with $Z(32) = Z(72) = 63, Z(88) = Z(48) = 32$), while our search for a pair of pseudo Smarandache related 120 degrees triangles went in vain.

We conclude the paper with the following open problems and conjectures. The first two have already been posed by Ashbacher [1].

Problem 1. Are there an infinite number of pairs of dissimilar 60 degrees triangles that are pseudo Smarandache related, in the sense of Definition 1.4?

Problem 2. Is there an infinite family of pairs of dissimilar 120 degrees triangles that are pseudo Smarandache related, in the sense of Definition 1.4?

Ashbacher [1] reports that a limited search on a computer showed only four pairs of dissimilar 60 degrees psedo Smarandache related triangles, while the number is only one in the case of 120 degrees triangles.

In the formulation of Definition 4.1, the number of pairs of dissimilar Smarandache related or pseudo Smarandache related triangles is obviously finite. In fact, since the number of integer solutions to

$$\begin{cases} a + b + c = 180 \\ a \geqslant 1, b \geqslant 1, c \geqslant 1 \end{cases}$$

is $C(179, 2)$ (see, for example, Johnsonbaugh [4], Theorem 4.5, pp. 238), the number of such triangles cannot exceed $C(179, 2)$. But this is a very crude estimate. The next problems of interest are

Problem 3. Is it possible to find a tight or better upper limit to the number of pairs of dissimilar triangles that are pseudo Smarandache related, in the sense of Definition 4.1?

Problem 4. Is it possible to find a good upper limit to the number of pairs of dissimilar 60 degrees triangles that are pseudo Smarandache related, in the sense of Definition 4.1?

Problem 5. Is it possible to find another pair of 60 degrees pseudo Smarandache related triangles, in the sense of Definition 4.1, where all the six angles are acute?

By a limited search, we found only two pairs of dissimilar 60 degrees pseudo Smarandache related triangles, the ones already mentioned above.

Conjecture 1. There is no pair of dissimilar 120 degrees triangles that are Pseudo Smarandache related, in the sense of Definition 4.1.

Conjecture 2. There is no pair of dissimilar 90 degrees triangles that are Pseudo Smarandache related, in the sense of Definition 4.1.

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One problem related to the Smarandache function

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. The main purpose of this paper is using the elementary method to study the number of the solutions of the congruent equation $1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \pmod n$, and give its all prime number solutions.
Keywords F. Smarandache function, divisibility, primitive root.

§1. Introduction and result

For any positive integer n, the famous F. Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained a series interesting results, see references [1], [2], [3] and [4]. For example, Xu Zhefeng [2] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lu Yaming [3] studied the solutions of an equation involving the F. Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, \ m_2, \ \cdots, \ m_k)$.

Rongji Chen [4] proved that for any fixed $r \in N$ with $r \geq 3$, the positive integer n is a solution of the equation

$$S(n)^r + S(n)^{r-1} + \cdots + S(n) = n$$

if and only if

$$n = p(p^{r-1} + p^{r-2} + \cdots + 1),$$

where p is an odd prime such that

$$p^{r-1} + p^{r-2} + \cdots + 1 \mid (p-1)!.$$

On the other hand, in reference [5], C. Dumitrescu and V. Seleacu asked us to study the solvability of the congruent equation

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \mod n. \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we use the elementary method and the properties of the primitive roots to study the solvability of the congruent equation (1), and obtain its all prime solutions. That is, we shall prove the following conclusion:

Theorem. Let n be a prime, then n satisfy the congruent equation

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 0 \bmod n$$

if and only if $n = 2,\ 3$ and 5.

It is clear that our Theorem obtained all prime solutions of the congruent equation (1). About the general solutions of the congruent equation (1) is still an unsolved problem.

2. Proof of the theorem

In this section, we shall complete the proof of our Theorem directly. We only discuss the prime solutions of (1).

(1) For $n = 2$, since $2 \mid 1^{S(1)} + 1 = 2$, so $n = 2$ is a prime solution of (1).

(2) For $n = 3$, since $3 \mid 1^{S(2)} + 2^{S(2)} + 1 = 6$, so $n = 3$ is a prime solution of (1).

(3) For $n = 5$, since $5 \mid 1^{S(4)} + 2^{S(4)} + 3^{S(4)} + 4^{S(4)} + 1 = 355$, so $n = 5$ is also a prime solution of (1).

(4) For prime $n = p \geq 7$, it is clear that p has at least a primitive root. Let g be a primitive root of p, that is to say, $\left(g^i - 1,\ p\right) = 1$ for all $1 \leq i \leq p-2$, the congruences

$$g^{p-1} \equiv 1 \bmod p \ \text{ and } \ g^{m(p-1)} \equiv 1 \bmod p \tag{2}$$

hold for any positive integer m.

Then from the properties of the primitive root mod p we know that $g^0,\ g^1,\ \cdots,\ g^{p-2}$ is a reduced residue class. So we have the congruent equation

$$\begin{aligned} & 1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} \\ \equiv\ & g^{0\cdot S(p-1)} + g^{1\cdot S(p-1)} + \cdots + g^{(p-2)\cdot S(p-1)} \\ \equiv\ & \frac{g^{(p-1)\cdot S(p-1)} - 1}{g^{S(p-1)} - 1} \bmod p. \end{aligned} \tag{3}$$

It is clear that for any prime $p \geq 7$, we have $S(p-1) \leq p-2$, so $\left(g^{S(p-1)} - 1,\ p\right) = 1$. Therefore, from (2) and (3) we have

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} \equiv \frac{g^{(p-1)\cdot S(p-1)} - 1}{g^{S(p-1)} - 1} \equiv 0 \bmod p.$$

So from this congruence we may immediately get

$$1^{S(n-1)} + 2^{S(n-1)} + \cdots + (n-1)^{S(n-1)} + 1 \equiv 1 \bmod p.$$

Thus, if prime $p \geq 7$, then it is not a solution of (1). This completes the proof of Theorem.

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Notes on *mssc*-images of relatively compact metric spaces

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Abstract We introduce the notion of σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ to give a consistent method to construct an *mssc*-mapping f with covering-properties onto a space X from some relatively compact metric space M. As applications, we systematically obtain characterizations on *mssc*-images of relatively compact metric spaces under certain covering-mappings, which sharpen results in [18, 26], and more.

Keywords σ-finite Ponomarev-system, relatively compact metric, *sn*-network, *so*-network, *cs*-network, *k*-network, 2-sequence-covering, 1-sequence-covering, sequence-covering, compact-covering, *mssc*-mapping.

§1. Introduction

An investigation of relations between spaces with countable networks and images of separable metric spaces is one of interesting questions on generalized metric spaces. Related to characterizing a space X having a certain countable network $\mathcal{P}$ by an image of a separable metric space M under some covering-mapping f, many results have been obtained. In the past, E. Michael [18] proved the following results.

Theorem 1.1. ([18], Proposition 10.2) The following are equivalent for a space X.
(1) X is a cosmic space.
(2) X is an image of a separable metric space.

Theorem 1.2. ([18], Theorem 11.4) The following are equivalent for a space X.
(1) X is an $\aleph_0$-space.
(2) X is a compact-covering image of a separable metric space.

Also, he posed the following question, on page 999.

Question 1.3. Find a better explanation for images of separable metric spaces?
In [26], Y. Tanaka and Z. Li proved the following result.

Theorem 1.4. ([26], Corollary 8 (1)) The following are equivalent for a space X, where "subsequence-covering" can be replaced by "pseudo-sequence-covering".
(1) X has a countable cs^*-network (resp., cs-network, sn-network).

(2) X is a subsequence-covering (resp., sequence-covering, 1-sequence-covering) image of a separable metric space.

Recently, Y. Ge [8] sharpened Theorem 1.2 as follows.

Theorem 1.5. ([8], Theorem 12) The following are equivalent for a space X.
(1) X is an $\aleph_0$-space.
(2) X is a sequence-covering, compact-covering image of a separable metric space.
(3) X is a sequentially-quotient image of a separable metric space.

Taking these results into account, the following question naturally arises.

Question 1.6. When $\mathcal{P}$ is a countable network (cs-network, sn-network, so-network) for X; how nice can the mapping f and the metric domain M be taken to?

Around this question, the author of [2] has shown that, when $\mathcal{P}$ is a countable cs-network, f and M in Theorem 1.5 can be an $mssc$-mapping and a relatively compact metric space, respectively. More precisely, the following has been proved.

Theorem 1.7. ([2], Theorem 2.5] The following are equivalent for a space X.
(1) X is an $\aleph_0$-space.
(2) X is a sequence-covering, compact-covering $mssc$-image of a relatively compact metric space.
(3) X is a sequentially-quotient image of a separable metric space.

In the above result, f and M can not be any compact mapping and any compact metric space, respectively; see [2, Example 2.11 & Example 2.12]; and the key of the proof is to construct a sequence-covering, compact-covering $mssc$-mapping f from a relatively compact metric space M onto X.

By the above, we are interested in the following question.

Question 1.8. Find a consistent method to construct an $mssc$-mapping with covering-properties onto a space from some relatively compact metric space?

In this paper, we answer Question 1.8 by introducing the notion of σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ to give a consistent method to construct an $mssc$-mapping f with covering-properties onto a space X from some relatively compact metric space M. As applications, we systematically obtain characterizations on $mssc$-images of relatively compact metric spaces under certain covering-mappings, which gives an answer for Question 1.6, particularly, for Question 1.3.

Throughout this paper, all spaces are regular and T_1, $\mathbb{N}$ denotes the set of all natural numbers, $\omega = \mathbb{N} \cup \{0\}$, and a convergent sequence includes its limit point. Let $\mathcal{P}$ be a family of subsets of X. Then $\bigcup\mathcal{P}$, and $\bigcap\mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$, and the intersection $\bigcap\{P : P \in \mathcal{P}\}$, respectively. A sequence $\{x_n : n \in \omega\}$ converging to x_0 is eventually in $A \subset X$, if $\{x_n : n \geq n_0\} \cup \{x_0\} \subset A$ for some $n_0 \in \mathbb{N}$, and it is frequently in A if $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x_0\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$.

Definition 1.9. Let $\mathcal{P}$ be a family of subsets of a space X.
(1) For each $x \in X$, $\mathcal{P}$ is a network at x in X, if $x \in \bigcap\mathcal{P}$, and if $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.
(2) $\mathcal{P}$ is a cs-network for X [10] if, for every convergent sequence S converging to $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that S is eventually in $P \subset U$.

(3) $\mathcal{P}$ is a cs^*-network for X [5] if, for every convergent sequence S converging to $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that S is frequently in $P \subset U$.
(4) $\mathcal{P}$ is a k-network of X [19] if, for every compact subset $K \subset U$ with U open in X, there exists a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 1.10. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for each $x \in X$, $\mathcal{P}_x$ is a network at x in X, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.
(1) $\mathcal{P}$ is a weak base for X [21], if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X.
(2) $\mathcal{P}$ is an sn-network for X [14], if each member of $\mathcal{P}_x$ is a sequential neighborhood of x in X.
(3) $\mathcal{P}$ is an so-network for X [12], if each member of $\mathcal{P}_x$ is sequentially open in X.
(4) The above $\mathcal{P}_x$ is respectively a *weak base*, an *sn-network*, and an *so*-network at x in X [12].

Definition 1.11. ([4]) Let P be a subset of a space X.
(1) P is a sequential neighborhood of x if, for every convergent sequence S converging to x in X, S is eventually in P.
(2) P is a sequentially open subset of X if, for every $x \in P$, P is a sequential neighborhood of x.

Definition 1.12. Let X be a space.
(1) X is relatively compact, if $\overline{X}$ is compact.
(2) X is an $\aleph_0$-space [18] (resp., sn-second countable space [7], so-second countable space, g-second countable space [21], second countable space [3]), if X has a countable cs-network (resp., countable sn-network, countable so-network, countable weak base, countable base).
(3) X is a sequential space [4], if every sequentially open subset of X is open.

Remark 1.13. ([15]) (1) For a space, weak base $\Rightarrow$ sn-network $\Rightarrow$ cs-network.
(2) An sn-network for a sequential space is a weak base.

Remark 1.14. (1) It is easy to see that "compact metric $\Rightarrow$ relatively compact metric $\Rightarrow$ separable metric", and these implications can not be reversed from Example 2.14 and Example 2.15.
(2) It is well-known that a space X is an $\aleph_0$-space if and only if X has a countable k-network (cs^*-network), see [23, Proposition C], for example.

Remark 1.15. Let $f : X \longrightarrow Y$ be a mapping.
(1) f is an $mssc$-mapping[13], if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for every $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $\overline{p_n(f^{-1}(V_{y,n}))}$ is a compact subset of X_n, where $p_n : \prod_{i\in\mathbb{N}} X_i \longrightarrow X_n$ is the projection.
(2) f is an 1-sequence-covering mapping [14] if, for every $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.
(3) f is a 2-sequence-covering mapping [14] if, for every $y \in Y$, $x \in f^{-1}(y)$, and sequence $\{y_n : n \in \mathbb{N}\}$ converging to y in Y, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x in X

with each $x_n \in f^{-1}(y_n)$.
(4) f is an 1-sequentially quotient mapping [17] if, for every $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_k : k \in \mathbb{N}\}$ converging to x_y in X with each $x_k \in f^{-1}(y_{n_k})$.
(5) f is a sequence-covering mapping [20] if, for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.
(6) f is a compact-covering mapping [18] if, for every compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.
(7) f is a pseudo-sequence-covering mapping [11], if for every convergent sequence S of Y, there exists a compact subset K of X such that $f(K) = S$.
(8) f is a sequentially-quotient mapping [1] if, for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

For terms are not defined here, please refer to [3, 24].

§2. Results

Definition 2.1. Let $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ be a countable network for a space X. Because X is T_1 and regular, we may assume that every member of $\mathcal{P}$ is closed. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \{X\} \cup \{P_i : i \leq n\} = \{P_\alpha : \alpha \in A_n\}$, where A_n is a finite set, and let every A_n be endowed with the discrete topology. Setting

$$M = \Big\{a = (\alpha_i) \in \prod_{i \in \mathbb{N}} A_i : \{P_{\alpha_i} : i \in \mathbb{N}\} \text{ forms a network at some point } x_a \in X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{i \in \mathbb{N}} A_i$, is a metric space. Since X is T_1 and regular, x_a is unique for each $a \in M$. We define $f : M \longrightarrow X$ by $f(a) = x_a$ for each $a \in M$. Then f is an *mssc*-mapping from a relatively compact metric space onto X by the following Theorem 2.3. The system $(f, M, X, \{\mathcal{P}_n\})$ is a σ-finite Ponomarev-system.

Remark 2.2. For more details on Ponomarev-systems and images of metric spaces, see [9,16,25], for example.

Theorem 2.3. Let $(f, M, X, \{\mathcal{P}_n\})$ be the system in Definition 2.1. Then the following hold.
(1) M is a relatively compact metric space.
(2) f is an *mssc*-mapping.

Proof. (1) For every $i \in \mathbb{N}$, since A_i is finite, A_i is compact metric. Then $\prod_{i \in \mathbb{N}} A_i$ is compact metric, and so $\overline{M} \subset \prod_{i \in \mathbb{N}} A_i$ is. It implies that M is relatively compact metric.
(2) It suffices to prove the following facts (a), (b), and (c).

(a) f is onto.

Let $x \in X$. Since $\mathcal{P}$ is a countable network for X, $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ is a countable network at x in X. We may assume that $(\mathcal{P})_x = \{P_{x,j} : j \in \mathbb{N}\}$, where $P_{x,j} \in \mathcal{P}_{i(j)}$ for some

$i(j) \in \mathbb{N}$ satisfying $i(j) < i(j+1)$. For each $i \in \mathbb{N}$, taking $P_{\alpha_i} = P_{i(j)}$ if $i = i(j)$ for some $j \in \mathbb{N}$, and otherwise, $P_{\alpha_i} = X$. Then $(\alpha_i) \in \prod_{i \in \mathbb{N}} A_i$, and $\{P_{\alpha_i} : i \in \mathbb{N}\} = (\mathcal{P})_x$ forms a network at x in X. Setting $a = (\alpha_i)$, we get $a \in M$ and $f(a) = x$.

(b) f is continuous.

Let $x = f(a) \in U$ with U open in X and $a \in M$. Setting $a = (\alpha_i) \in \prod_{i \in \mathbb{N}} A_i$, where $\{P_{\alpha_i} : i \in \mathbb{N}\}$ forms a network at x in X. Then there exists $n \in \mathbb{N}$ such that $x \in P_{\alpha_n} \subset U$. Setting $M_a = \{b = (\beta_i) \in M : \beta_n = \alpha_n\}$, then M_a is an open neighborhood of a in M. For each $b \in M_a$, we get $f(b) \in P_{\beta_n} = P_{\alpha_n} \subset U$. It implies that $f(M_a) \subset U$.

(c) f is an *mssc*-mapping.

Let $x \in X$. For each $i \in \mathbb{N}$, taking $V_{x,i} = X$. Then $\{V_{x,i} : i \in \mathbb{N}\}$ is a sequence of open neighborhoods of x in X. Since A_i is finite, A_i is compact. Then $\overline{p_i(f^{-1}(V_{x,i}))} = \overline{p_i(f^{-1}(X))} \subset A_i$ is compact. It implies that f is an *mssc*-mapping.

Remark 2.4. In the view of (a) in the proof (1) $\Rightarrow$ (2) of [2, Theorem 2.5], f is onto by choosing $a = (\alpha_i)$ satisfying $f(a) = x$, where $x \in P_{\alpha_i} \in \mathcal{P}_i$. Unfortunately, this argument is not true. In fact, we may pick $x \in P_{\alpha_i} = X \in \mathcal{P}_i$. Then $\{P_{\alpha_i} : i \in \mathbb{N}\} = \{X\}$ does not form a network at x in X. It is a contradiction. The wrong argument is corrected as in the proof (2).(a) of Theorem 2.3.

Theorem 2.5. Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-finite Ponomarev-system. Then the following hold.

(1) If $\mathcal{P}$ is a *cs*-network for X, then f is sequence-covering.

(2) If $\mathcal{P}$ is a k-network for X, then f is a compact-covering.

(3) If $\mathcal{P}$ is a cs^*-network for X, then f is pseudo-sequence-covering.

(4) If $\mathcal{P}$ is an *sn*-network for X, then f is 1-sequence-covering.

(5) If $\mathcal{P}$ is an *so*-network for X, then f is 2-sequence-covering.

Proof. (1) Let $S = \{x_m : m \in \omega\}$ be a convergent sequence converging to x_0 in X. Suppose that U is an open neighborhood of S in X. A family $\mathcal{A}$ of subsets of X has property $cs(S, U)$ if:

(i) $\mathcal{A}$ is finite.

(ii) for each $P \in \mathcal{A}$, $\emptyset \neq P \cap S \subset P \subset U$.

(iii) for each $x_m \in S$, there exists unique $P_{x_m} \in \mathcal{A}$ such that $x_m \in P_{x_m}$.

(iv) if $x_0 \in P \in \mathcal{A}$, then $S - P$ is finite.

For each $i \in \mathbb{N}$, since $\mathcal{A} = \{X\} \subset \mathcal{P}_i$ has property $cs(S, X)$ and $\mathcal{P}_i$ is finite, we can assume that

$$\{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A} \text{ has property } cs(S, X)\} = \{\mathcal{A}_{i(j)} : j = n_{i-1} + 1, \ldots, n_i\},$$

where $n_0 = 0$. By this notation, for each $j \in \mathbb{N}$, there is unique $i(j) \in \mathbb{N}$ such that $\mathcal{A}_{i(j)}$ has property $cs(S, X)$. Then for each $j \in \mathbb{N}$, we can put $\mathcal{A}_{i(j)} = \{P_\alpha : \alpha \in E_j\}$, where E_j is a finite subset of A_j.

For each $j \in \mathbb{N}$, $m \in \omega$ and $x_m \in S$, it follows from (iii) that there is unique $\alpha_{j,m} \in E_j$ such that $x_m \in P_{\alpha_{j,m}} \in \mathcal{A}_{i(j)}$. Let $a_m = (\alpha_{j,m}) \in \prod_{j \in \mathbb{N}} E_j \subset \prod_{j \in \mathbb{N}} A_j$. We shall prove that $\{P_{\alpha_{j,m}} : j \in \mathbb{N}\}$ is a network at x_m in X. In fact, let $x_m \in U$ with U open in X, we consider two following cases.

(a) If $m = 0$, then S is eventually in $P_{x_0} \subset U$ for some $P_{x_0} \in \mathcal{P}$. For each $x \in S - P_{x_0}$, let $x \in P_x \subset X - (S - \{x\})$ for some $P_x \in \mathcal{P}$. Then $\mathcal{G} = \{P_{x_0}\} \cup \{P_x : x \in S - P_{x_0}\} \subset \mathcal{P}$ has property $cs(S, X)$. Since $\mathcal{G}$ is finite, $\mathcal{G} \subset \mathcal{P}_i$ for some $i \in \mathbb{N}$. It implies that $\mathcal{G} = \mathcal{A}_{i(j)}$ for some $i(j) \in \mathbb{N}$ with $j \in \{n_{i-1} + 1, \dots, n_i\}$. Since $x_m = x_0 \in P_{\alpha_{j,0}}$, $P_{\alpha_{j,0}} = P_{x_0}$. Hence $x_m = x_0 \in P_{\alpha_{j,0}} \subset U$.

(b) If $m \neq 0$, then $S - \{x_m\}$ is eventually in $P_{x_0} \subset X - \{x_m\}$ for some $P_{x_0} \in \mathcal{P}$. For each $x \in (S - \{x_m\}) - P_{x_0}$, let $x \in P_x \subset X - (S - \{x\})$ for some $P_x \in \mathcal{P}$, and let $x_m \in P_{x_m} \subset U \cap (X - (S - \{x_m\}))$ for some $P_{x_m} \in \mathcal{P}$. Then $\mathcal{H} = \{P_{x_0}\} \cup \{P_{x_m}\} \cup \{P_x : x \in (S - \{x_m\}) - P_{x_0}\}$ has property $cs(S, X)$. Since $\mathcal{H}$ is finite, $\mathcal{H} \subset \mathcal{P}_i$ for some $i \in \mathbb{N}$. It implies that $\mathcal{H} = \mathcal{A}_{i(j)}$ for some $i(j) \in \mathbb{N}$ with $j \in \{n_{i-1} + 1, \dots, n_i\}$. Since $x_m \in P_{\alpha_{j,m}}$, $P_{\alpha_{j,m}} = P_{x_m}$. Hence $x_m \in P_{\alpha_{j,m}} \subset U$.

By the above, for each $m \in \omega$ we get $a_m = (\alpha_{j,m}) \in M$ satisfying $f(a_m) = x_m$. For each $j \in \mathbb{N}$, since families $\mathcal{H}$ and $\mathcal{G}$ are finite, there exists $m_j \in \mathbb{N}$ such that $\alpha_{j,m} = \alpha_{j,0}$ if $m \geq m_j$. Hence the sequence $\{\alpha_{j,m} : m \in \mathbb{N}\}$ converges to $\alpha_{j,0}$ in A_j. Thus, the sequence $\{a_m : m \in \mathbb{N}\}$ converges to a_0 in M. Setting $L = \{a_m : m \in \omega\}$, then L is a convergent sequence in M and $f(L) = S$. This shows that f is sequence-covering.

(2)Let K be a compact subset of X. Suppose that V is an open neighborhood of K in X. A family $\mathcal{B}$ of subsets of X has property $k(K, V)$ if:

(i) $\mathcal{B}$ is finite.

(ii) $P \cap K \neq \emptyset$ for each $P \in \mathcal{B}$.

(iii) $K \subset \bigcup \mathcal{B} \subset V$.

For each $i \in \mathbb{N}$, since $\mathcal{B} = \{X\} \subset \mathcal{P}_i$ has property $k(K, X)$ and $\mathcal{P}_i$ is finite, we can assume that

$$\{\mathcal{B} \subset \mathcal{P}_i : \mathcal{B} \text{ has property } k(K, X)\} = \{\mathcal{B}_{i(j)} : j = n_{i-1} + 1, \dots, n_i\},$$

where $n_0 = 0$. By this notation, for each $j \in \mathbb{N}$, there is unique $i(j) \in \mathbb{N}$ such that $\mathcal{B}_{i(j)}$ has property $k(K, X)$. Then for each $j \in \mathbb{N}$, we can put $\mathcal{B}_{i(j)} = \{P_\alpha : \alpha \in F_j\}$, where F_j is a finite subset of A_j.

Setting $L = \{a = (\alpha_i) \in \prod_{i \in \mathbb{N}} F_i : \bigcap_{i \in \mathbb{N}} (K \cap P_{\alpha_i}) \neq \emptyset\}$, we shall prove that L is a compact subset of M satisfying $f(L) = K$, hence f is compact-covering, by the following facts (a), (b), and (c).

(a) L is compact.

Since $L \subset \prod_{i \in \mathbb{N}} F_i$ and $\prod_{i \in \mathbb{N}} F_i$ is compact, we only need to prove that L is closed in $\prod_{i \in \mathbb{N}} F_i$. Let $a = (\alpha_i) \in \prod_{i \in \mathbb{N}} F_i - L$. Then $\bigcap_{i \in \mathbb{N}} (K \cap P_{\alpha_i}) = \emptyset$. Since $K \cap P_{\alpha_i}$ is compact, there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i \leq i_0} (K \cap P_{\alpha_i}) = \emptyset$. Setting $W = \{b = (\beta_i) \in \prod_{i \in \mathbb{N}} F_i : \beta_i = \alpha_i \text{ if } i \leq i_0\}$. Then W is an open neighborhood of a in $\prod_{i \in \mathbb{N}} F_i$ and $W \cap L = \emptyset$. If not, there exists $b = (\beta_i) \in W \cap L$. Since $b \in L$, $\bigcap_{i \in \mathbb{N}} (K \cap P_{\beta_i}) \neq \emptyset$, hence $\bigcap_{i \leq i_0} (K \cap P_{\beta_i}) \neq \emptyset$. Since $b \in W$, $\bigcap_{i \leq i_0} (K \cap P_{\alpha_i}) = \bigcap_{i \leq i_0} (K \cap P_{\beta_i}) \neq \emptyset$.

This is a contradiction of the fact that $\bigcap_{i\le i_0}(K\cap P_{\alpha_i})=\emptyset$.

(b) $L\subset M$ and $f(L)\subset K$.

Let $a=(\alpha_i)\in L$, then $a\in\prod_{i\in\mathbb{N}}F_i$ and $\bigcap_{i\in\mathbb{N}}(K\cap P_{\alpha_i})\neq\emptyset$. Taking $x\in\bigcap_{i\in\mathbb{N}}(K\cap P_{\alpha_i})$. If $\{P_{\alpha_i}:i\in\mathbb{N}\}$ is a network at x in X, then $a\in M$ and $f(a)=x$, hence $L\subset M$ and $f(L)\subset K$. So we only need to prove that $\{P_{\alpha_i}:i\in\mathbb{N}\}$ is a network at x in X. Let V be an open neighborhood of x in X. There exist an open subset W of K such that $x\in W$, and compact subsets $\overline{W}^K$ and $K-W$ such that $\overline{W}^K\subset V$ and $K-W\subset X-\{x\}$, where $\overline{W}^K$ is the closure of W in K. Since $\mathcal{P}$ is a k-network for X, there exist finite families $\mathcal{F}_1\subset\mathcal{P}$ and $\mathcal{F}_2\subset\mathcal{P}$ such that $\overline{W}^K\subset\bigcup\mathcal{F}_1\subset V$ and $K-W\subset\bigcup\mathcal{F}_2\subset X-\{x\}$. We may assume that $P\cap K\neq\emptyset$ for each $P\in\mathcal{F}_1\cup\mathcal{F}_2$. Setting $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$, then $\mathcal{F}$ has property $k(K,X)$. It implies that $\mathcal{F}=\mathcal{B}_{i(j)}$ for some $i(j)\in\mathbb{N}$ with $j\in\{n_{i-1}+1,\ldots,n_i\}$. Since $x\in P_{\alpha_j}\in\mathcal{B}_{i(j)}, P_{\alpha_j}\in\mathcal{F}_1$. Then $x\in P_{\alpha_j}\subset V$. This prove that $\{P_{\alpha_j}:j\in\mathbb{N}\}$ is a network at x in X.

(c) $K\subset f(L)$.

Let $x\in K$. For each $i\in\mathbb{N}$, there exists $\alpha_i\in F_i$ such that $x\in P_{\alpha_i}$. Setting $a=(\alpha_i)$, then $a\in L$. Furthermore, $f(a)=x$ as in the proof of (b). So $K\subset f(L)$.

(3) Let S be a convergent sequence of X. By using [26, Lemma 3] and as in the proof of (2), here S plays the role of the compact subset K, there exists a compact subset L of M such that $f(L)=S$. Then, f is pseudo-sequence-covering.

(4) Let $\mathcal{P}=\bigcup\{\mathcal{P}_x:x\in X\}$, where each $\mathcal{P}_x$ is an *sn*-network at x in X. Let $x\in X$. We may assume that $\mathcal{P}_x=\{P_{x,j}:j\in\mathbb{N}\}$, where $P_{x,j}\in\mathcal{P}_{i(j)}$ with some $i(j)\in\mathbb{N}$ satisfying $i(j)<i(j+1)$. For each $i\in\mathbb{N}$, take $P_{\alpha_i}=P_{i(j)}$ if $i=i(j)$ for some $j\in\mathbb{N}$, and otherwise, $P_{\alpha_i}=X$. Then $(\alpha_i)\in\prod_{i\in\mathbb{N}}A_i$, and $\{P_{\alpha_i}:i\in\mathbb{N}\}=\mathcal{P}_x$ forms a network at x in X. Setting $a_x=(\alpha_i)$, then $a\in M$ and $a_x\in f^{-1}(x)$. For each $n\in\mathbb{N}$, setting $B_n=\{b=(\beta_i)\in M:\beta_i=\alpha_i \text{ if } i\le n\}$. Then $\{B_n:n\in\mathbb{N}\}$ is a decreasing neighborhood base at a_x in M. For each $n\in\mathbb{N}$, we get $f(B_n)=\bigcap_{i\le n}P_{\alpha_i}$. In fact, for each $b=(\beta_i)\in B_n$, $f(b)=\bigcap_{i\in\mathbb{N}}P_{\beta_i}\in P_{\beta_i}=P_{\alpha_i}$ if $i\le n$. Then $f(b)\in\bigcap_{i\le n}P_{\alpha_i}$. It implies that $f(B_n)\subset\bigcap_{i\le n}P_{\alpha_i}$. Conversely, if $x\in\bigcap_{i\le n}P_{\alpha_i}$, then $x=f(b)$ for some $b=(\beta_i)\in M$. Setting $c=(\gamma_i)$, where $\gamma_i=\alpha_i$ if $i\le n$, and $\gamma_i=\beta_{i-n}$ if $i>n$. Then $\{P_{\gamma_i}:i\in\mathbb{N}\}$ is a network at x in X. For each $i\ge n$, since $\beta_{i-n}\in A_{i-n}\subset A_i$, $\gamma_i\in A_i$. Then $c\in B_n$ and $f(c)=x$. It implies that $\bigcap_{i\le n}P_{\alpha_i}\subset f(B_n)$.

Suppose that the sequence $\{x_j:j\in\mathbb{N}\}$ converges to x in X. For each $i\in\mathbb{N}$, since P_{α_i} is a sequential neighborhood of x in X, there exists $j_i\in\mathbb{N}$ such that $x_j\in P_{\alpha_i}$ for each $j\ge j_i$. For each $n\in\mathbb{N}$, setting $j(n)=\max\{j_i:i\le n\}$, then $x_j\in\bigcap_{i\le n}P_{\alpha_i}=f(B_n)$ for each $j\ge j(n)$. It implies that $f^{-1}(x_j)\cap B_n\neq\emptyset$ for each $j\ge j(n)$. We may assume that $1<j(n)<j(n+1)$. For each $j\in\mathbb{N}$, taking

$$a_j\in\begin{cases} f^{-1}(x_j) & \text{if } j<j(1),\\ f^{-1}(x_j)\cap B_n & \text{if } j(n)\le j<j(n+1).\end{cases}$$

Then the sequence $\{a_j:j\in\mathbb{N}\}$ converges to a_x in M. In fact, let U be a neighborhood of

a_x, then there exists $n \in \mathbb{N}$ such that $a_x \in B_n \subset U$. Obviously, $a_j \in B_n \subset U$ for each $j \geq j(n)$, so $\{a_j : j \in \mathbb{N}\}$ converges to a_x. Note that $f(a_j) = x_j$ for every $j \in \mathbb{N}$. It implies that f is 1-sequence-covering.

(5) Let $x \in X$ and $a_x \in f^{-1}(x)$. Setting $a_x = (\alpha_i)$, then $\alpha_i \in A_i$ for each $i \in \mathbb{N}$, and $\{P_{\alpha_i} : i \in \mathbb{N}\}$ forms a network at x in X. For each $n \in \mathbb{N}$, setting $B_n = \{b = (\beta_i) \in M : \beta_i = \alpha_i \text{ if } i \leq n\}$. Then $\{B_n : n \in \mathbb{N}\}$ is a decreasing neighborhood base at a_x in M, and for each $n \in \mathbb{N}$, $f(B_n) = \bigcap_{i \leq n} P_{\alpha_i}$ as in the proof of (4).

Suppose that sequence $\{x_j : j \in \mathbb{N}\}$ converges to x in X. For each $i \in \mathbb{N}$, since P_{α_i} is sequentially open, there exists $j_i \in \mathbb{N}$ such that $x_j \in P_{\alpha_i}$ for each $j \geq j_i$. As in the proof (4), we get a sequence $\{a_j : j \in \mathbb{N}\}$ converges to a_x in M such that $f(a_j) = x_j$ for each $j \in \mathbb{N}$. It implies that f is 2-sequence-covering.

By using Theorem 2.3 and Theorem 2.5 above, we obtain characterizations of *mssc*-images of relatively compact metric spaces under certain covering-mappings, which sharpen results in [18,26] and more. Firstly, we sharpen Theorem 1.1 as following.

Corollary 2.6. The following are equivalent for a space X.

(1) X is a cosmic space.

(2) X is an *mssc*-image of a relatively compact metric space.

(3) X is an image of a separable metric space.

Proof. We only need to prove (1) $\Rightarrow$ (2), other implications are routine. Since X is a cosmic space, X has a countable network $\mathcal{P}$. Then the σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Therefore, f is an *mssc*-mapping and M is relatively compact metric by Theorem 2.3. It implies that X is an *mssc*-image of a relatively compact metric space.

Remark 2.7. Corollary 2.6 is an answer of Question 1.3.

Next, we extend partly Theorem 1.4 as follows.

Corollary 2.8. The following are equivalent for a space X.

(1) X is an *sn*-second countable space.

(2) X is an 1-sequence-covering, compact-covering *mssc*-image of a relatively compact metric space.

(3) X is an 1-sequentially-quotient image of a separable metric space.

Proof. (1) $\Rightarrow$ (2). Since X is an *sn*-second countable space, X has a countable closed *sn*-network $\mathcal{P}_1$ and a countable closed k-network $\mathcal{P}_2$. Set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then $\mathcal{P}$ is a countable *sn*-network and k-network for X. It follows from Definition 2.1 that the σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Therefore, f is an 1-sequence-covering, compact-covering *mssc*-mapping from a relatively compact metric space M onto X by Theorem 2.3 and Theorem 2.5.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Let $f : M \longrightarrow X$ be an 1-sequentially-quotient mapping from a separable metric space M onto X. Since M is separable metric, M has a countable base $\mathcal{B}$. We may assume that $\mathcal{B}$ is closed under finite intersections. For each $x \in X$, there exists $a_x \in f^{-1}(x)$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_k : k \in \mathbb{N}\}$ converging to a_x in M with each $a_k \in f^{-1}(x_{n_k})$. Let $\mathcal{B}_x = \{B \in \mathcal{B} : a_x \in \mathcal{B}\}$ and $\mathcal{P}_x = f(\mathcal{B}_x)$. We shall prove that $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a countable *sn*-network for X. Since $\mathcal{B}$ is countable, $\mathcal{P}$ is countable. It suffices to prove the following facts (a), (b), and (c) for every

$x \in X$.

(a) $\mathcal{P}_x$ is a network at x in X.

It is straightforward because $\mathcal{B}_x$ is a neighborhood base at a_x in M.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

Let $P_1 = f(B_1), P_2 = f(B_2)$ with $B_1, B_2 \in \mathcal{B}_x$. Then $B_1 \cap B_2 \in \mathcal{B}_x$. Setting $P = f(B_1 \cap B_2)$, we get $P \in \mathcal{P}_x$ and $P \subset P_1 \cap P_2$.

(c) Each $P \in \mathcal{P}_x$ is a sequential neighborhood of x.

Let $P = f(B)$ with $B \in \mathcal{B}_x$, and let $\{x_n : n \in \mathbb{N}\}$ be a sequence converging to x in X. For each subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$, since f is 1-sequentially-quotient, there exists a sequence $\{a_k : k \in \mathbb{N}\}$ converging to a_x in M with each $a_k \in f^{-1}(x_{n_{i,k}})$. Since $\{a_k : k \in \mathbb{N}\} \cup \{a_x\}$ is eventually in B, $\{x_{n_i} : i \in \mathbb{N}\} \cup \{x\}$ is frequently in P. Then $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is frequently in P. It follows from [6, Remark 1.4] that P is a sequential neighborhood of x.

Corollary 2.9. The following are equivalent for a space X.

(1) X is a g-second countable space.

(2) X is an 1-sequence-covering, compact-covering, quotient *mssc*-image of a relatively compact metric space.

(3) X is an 1-sequentially-quotient, quotient image of a separable metric space.

Proof. (1) $\Rightarrow$ (2). It follows from Corollary 2.8 that X is an 1-sequence-covering *mssc*-image of a relatively compact metric space under some mapping f. Since f is an 1-sequence-covering mapping onto a g-second countable space, f is a quotient mapping by [15, Lemma 3.5]. It implies that X is an 1-sequence-covering, quotient *mssc*-image of a relatively compact metric space.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Since X is a quotient image of a separable metric space, X is a sequential space by [4, Proposition 1.2]. Then X is a sequential space having a countable *sn*-network $\mathcal{P}$ by Corollary 2.8. It follows from Lemma 1.13 that $\mathcal{P}$ is a weak base for X. Then X is a g-second countable space.

In the following, we prove a result on spaces having countable *so*-networks.

Corollary 2.10. The following are equivalent for a space X.

(1) X is an *so*-second countable space.

(2) X is a 2-sequence-covering, compact-covering *mssc*-image of a relatively compact metric space.

(3) X is a 2-sequence-covering image of a separable metric space.

Proof. (1) $\Rightarrow$ (2). Since X is an *so*-second countable space, X has a countable closed *so*-network $\mathcal{P}_1$ and a countable closed k-network $\mathcal{P}_2$. Set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then $\mathcal{P}$ is a countable *so*-network and k-network for X. It follows from Definition 2.1 that the σ-finite Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Therefore, f is a 2-sequence-covering, compact-covering *mssc*-mapping from a relatively compact metric space M onto X by Theorem 2.3 and Theorem 2.5.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Let $f : M \longrightarrow X$ be a 2-sequence-covering mapping from a separable metric space M onto X. Since M is separable metric, M has a countable base $\mathcal{B}$. For each $x \in X$,

let $\mathcal{B}_x = \{B \in \mathcal{B} : f^{-1}(x) \cap B \neq \emptyset\}$, and $\mathcal{P}_x$ be the family of finite intersections of members of $f(\mathcal{B}_x)$. We shall prove that $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a countable *so*-network for X. Since $\mathcal{B}$ is countable, $\mathcal{P}$ is countable. It suffices to prove the following facts (a), (b), and (c) for every $x \in X$.

(a) $\mathcal{P}_x$ is a network at x in X.

It is straightforward because $\mathcal{B}_x$ is a base of $f^{-1}(x)$ in M.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

It is straightforward by choosing $P = P_1 \cap P_2$.

(c) Each $P \in \mathcal{P}_x$ is sequentially open.

For each $y \in f(B)$ with $B \in \mathcal{B}_x$, we get $f^{-1}(y) \cap B \neq \emptyset$. Let $\{y_n : n \in \mathbb{N}\}$ be a sequence converging to y in X. Then there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to $a \in f^{-1}(y) \cap B$ such that $f(a_n) = y_n$ for each $n \in \mathbb{N}$. Since B is open, $\{a_n : n \in \mathbb{N}\} \cup \{a\}$ is eventually in B. It implies that $\{y_n : n \in \mathbb{N}\} \cup \{y\}$ is eventually in $f(B)$. Hence $f(B)$ is a sequential neighborhood of y in X. Therefore, $f(B)$ is sequentially open. Since each $P \in \mathcal{P}_x$ is some intersection of finitely many members of $f(\mathcal{B}_x)$, P is sequentially open.

Corollary 2.11. The following are equivalent for a space X.
(1) X is a second countable space.
(2) X is a 2-sequence-covering, compact-covering, quotient *mssc*-image of a relatively compact metric space.
(3) X is a 2-sequence-covering, quotient image of a separable metric space.

Proof. (1) ⇒ (2). It follows from Corollary 2.10 that X is a 2-sequence-covering *mssc*-image of a relatively compact metric space under some mapping f. Since f is a 2-sequence-covering mapping onto a second countable space, f is a quotient mapping by [15, Lemma 3.5]. It implies that X is a 2-sequence-covering, quotient *mssc*-image of a relatively compact metric space.

(2) ⇒ (3). It is obvious.

(3) ⇒ (1). Since X is a quotient image of a separable metric space, X is a sequential space by [4, Proposition 1.2]. Then X is a sequential space having a countable *so*-network $\mathcal{P}$ by Corollary 2.10. For each $P \in \mathcal{P}$, since P is sequentially open and X is sequential, P is open. Then $\mathcal{P}$ is a countable base for X, i.e., X is a second countable space.

Remark 2.12. By using Theorem 2.3 and Theorem 2.5, we also get the proof of Theorem 1.7 again, which was presented in [2].

From the above results, we obtain preservations of certain spaces under covering-mappings as follows.

Corollary 2.13. Let $f : X \longrightarrow Y$ be a mapping. Then the following hold.
(1) If X is an *sn*-second countable space and f is 1-sequentially-quotient, then Y is an *sn*-second countable space.
(2) If X is a *g*-second countable space and f is 1-sequentially-quotient, quotient, then Y is a *g*-second countable space.
(3) If X is an *so*-second countable space and f is 2-sequence-covering, then Y is an *so*-second countable space.
(4) If X is a second countable space and f is 2-sequence-covering, quotient, then Y is a second

countable space.

Proof. We only need to prove (1), by similar arguments, we get the other. Since X is an sn-second countable space, X is an 1-sequentially-quotient image of a separable metric space under some mapping g by Corollary 2.8. It is easy to see that $f \circ g$ is also an 1-sequentially-quotient mapping. Then Y is an 1-sequentially-quotient image of a separable metric space under the mapping $f \circ g$. Therefore, Y is an sn-second countable space by Corollary 2.8.

Finally, we give examples to illustrate the above results.

Example 2.14. ([2], Example 2.8) A relatively compact metric space is not compact.

Example 2.15. ([2], Example 2.9) A separable metric space is not relatively compact.

Example 2.16. A 2-sequence-covering, compact-covering mapping from a separable metric space is not an *mssc*-mapping.

Proof. Let f be the mapping in [2, Example 2.10]. Then f is also 2-sequence-covering. This complete the proof.

Example 2.17. A second countable space is not any image of a compact metric space. It implies that "relatively compact metric" in the above results can not be replaced by "compact metric".

Proof. Let $\mathbb{R}$ be the set of all real numbers endowed with the usual topology. Then $\mathbb{R}$ is a second countable space. Since $\mathbb{R}$ is not compact, $\mathbb{R}$ is not any image of a compact metric space.

Example 2.18. A g-second countable space is not any 1-sequence-covering, compact-covering compact image of a separable metric space. It implies that "*mssc*-image" in Corollary 2.8 can not be replaced by "compact image".

Proof. See [22, Example 2.14(3)].

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