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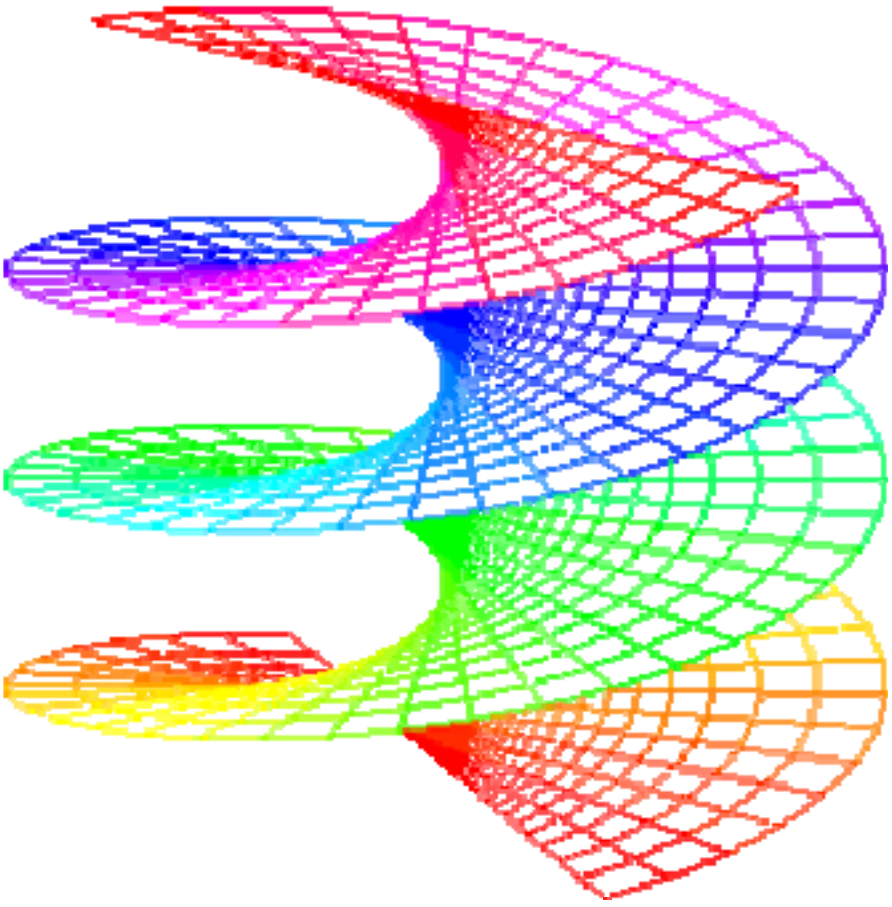
SCIENTIA MAGNA

(Special Issue)

Proceedings of the fourth international Conference on
Number Theory and Smarandache problems



Edited by Professor Wenpeng Zhang

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Preface

This issue of the journal is devoted to the proceedings of the fourth International Conference on Number Theory and Smarandache Problems held in Xianyang during March 22-24, 2008. The organizers were myself and Professor Chaofeng Shi from Xianyang Normal University. The conference was supported by Xianyang Normal University and there were more than 100 participants. We had three foreign guests, Professor K.Chakraborty from India, Professor F. Luca from America and Professor S. Kanemitsu from Japan. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache problems in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the fifth conference in coming March which may be held in Huizhou, a beautiful city of Guangdong.

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems or other mathematical problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. R. Zhang, An improved continuous ant colony algorithms for water-reusing network optimization; 2. X. Zhang, Y. Zhang and J.Ding, The generalization of sequence of numbers with alternate common differences; 3. L. Li, Stability of Wegl-Heisenberg frames; etc.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them. Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source.

Digital Library of Science:

www.gallup.unm.edu/~Smarandache/eBooks-otherformats.htm.

Wenpeng Zhang

The Fourth International Conference on Number Theory

and Smarandache Problems

The opening ceremony of the conference is occurred in Xianyang Normal University (http://www.xysfxy.cn), which is attracting more and more people from other countries to study Chinese Calligraphy, Chinese Drawing and Chinese Culture.

Professor ChengjunYang:

Professor Shigeru Kanemi:

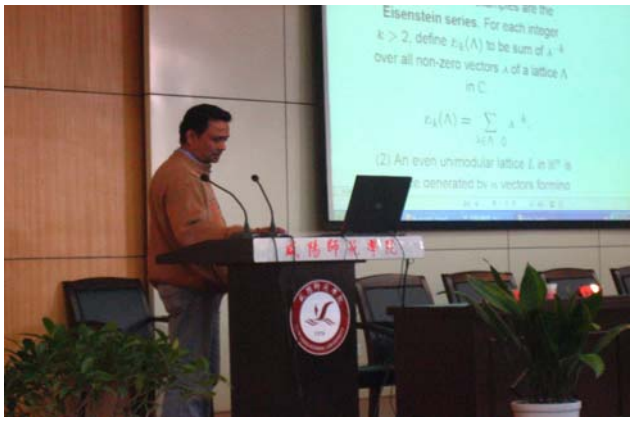
Professor Florian Luca:

Professor Kalyan Chakraborty:

Professor Wenpeng Zhang:

The Fourth International Conference on Number Theory and Smarandache Problems is held in Xianyang, the capital of Qin Dynasty, which is famous for the historical culture.

The participants of the Conference



←Professor Kalyan Chakraborty

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Professor Guodong Liu→

←Professor Linfan Mao

Professor Changen Yang→

←Professor Hailong Li

Professor Yuan Yi→

←Dr. Huaning Liu

Dr. Ganglian Ren→

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On a problem of the Schur

Yani Zheng[†] [‡]

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Abstract For any positive integer n, let r be the positive integer such that the set $\{1,\ 2,\ \cdots,\ r\}$ can be partitioned into n classes, and no any class contain integers $x,\ y,\ z$ with $x^y = z$. In reference [1] (See Problem 57), Schur asked us to find the maximum r. In this paper, we use the elementary methods to study this problem, and give a sharp lower bound estimates for r.

Keywords Schur's problem, partition, lower bound.

§1. Introduction and result

For any positive integer n, let r be the positive integer such that the set $\{1,\ 2,\ \cdots,\ r_1\}$ can be partitioned into n classes, and no any class contain integers $x,\ y,\ z$ with $x^y = z$. In reference [1], Schur asked us to find the maximum r. At the same time, he also proposed two similar problems: Let r be the positive integer such that the set $\{1,\ 2,\ \cdots,\ r_1\}$ can be partitioned into n classes, and no any class contain integers $x,\ y,\ z$ with $x + y = z$ (or $xy = z$).

These problems are interesting, because they can help us to study some important partition problems. About these problems, Liu Hongyan and Zhang Wenpeng [2] had studied cases $x^y = z$ $(x + y = z)$, and proved the estimates $r \geq n^{n+2}$ $(r \geq 2^{n+1})$. Liu Huaning and Zhang Wenpeng [3] studied the case $xy = z$, and obtained the lower bound estimates

$$r \geq n^{2(1-\epsilon)(n-1)},$$

where ϵ is any fixed positive number.

In this paper, we use the elementary methods to study the Schur's problem in case $x^y = z$, and give a sharper lower bound estimates for r. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 2$, let r be the maximum positive integer such that the set $\{1,\ 2,\ \cdots,\ r\}$ can be partitioned into n classes, and no any class contain integers $x,\ y,\ z$ with $x^y = z$. Then we have the estimates $r \geq 2^n n^n$.

For any integer $n \geq 5$, it is clear that $2^n > n^2$. So our Theorem improved the lower bound estimates in reference [2].

§2. Proof of the theorem

In this section, we shall use the elementary methods to prove our Theorem directly. First for any positive integer $1 \leq m \leq n$, taking $r = 2^n n^m$. Then partition the set $\{1,\ 2,\ \cdots,\ r\}$

into n classes as follows:

$$\begin{cases} \text{class } 1: \ 1, \ n+1, n+2, \cdots, 2n^m. \\ \text{class } 2: \ 2, 2n^m+1, 2n^m+2, \cdots, 2^2n^m. \\ \text{class } 3: \ 3, 2^2n^m+1, 2^2n^m+2, \cdots, 2^3n^m. \\ \qquad \cdots\cdots\cdots\cdots\cdots \\ \text{class } k: \ k, 2^{k-1}n^m+1, 2^{k-1}n^m+2, \cdots, 2^kn^m. \\ \qquad \cdots\cdots\cdots\cdots\cdots \\ \text{class } n: \ n, \ 2^{n-1}n^m+1, \ 2^{n-1}n^m+2, \cdots, 2^nn^m. \end{cases}$$

For any integer $2 \leq k \leq n$, it is clear that no integers $x, \ y, \ z$ satisfying $x^y = z$ in class k. In fact for any integers $x, \ y, \ z$ in class k, we have

$$x^y \geq (2^{k-1}n^m+1)^k > 2^kn^m \geq z \ \text{ or } x^y \geq k^{2^{k-1}n^m+1} > z.$$

So there does not exist $x, \ y, \ z$ in class k such that the identity $x^y = z$.

If $k = 1$, note that $m \leq n$, $\dfrac{(n+2)^{n+1}}{2n^m} > \dfrac{n^{n+1}}{2n^m} > 1, \ (n \geq m, \ m > 1)$, and if $m = 1$, then $(n+2)^{n+1} > 2^n$. So we have $(n+2)^{n+1} > 2n^m$. Similarly, we also have $(n+1)^{n+2} > 2n^m$. So there does not exist $x, \ y, \ z$ in class 1 such that the identity $x^y = z$.

This completes the proof of our Theorem.

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The asymptotic formula of $\sum_{n\leq x} I(n)$[1]

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Abstract The aim of this short paper is to establish an asymptotic formula for the sum $\sum_{n\leq x} I(n)$, where $I(n)$ is the multiplicative function such that $I(p^\alpha) = \dfrac{p^{\alpha+1}}{\alpha+1}$ holds for any integer $\alpha \geq 1$ and any prime p.

Keywords Integral function, mean value, asymptotic formula.

§1. Introduction

Let $I(n)$ be the multiplicative function such that for any prime p and any integer $\alpha \geq 1$, one has

$$I(p^\alpha) = \frac{p^{\alpha+1}}{\alpha+1}.$$

In her doctoral thesis[2], Wang Xiaoying proved that

$$\sum_{mn\leq x} I(m)I(n) = Cx^3 + O(x^{5/2+\varepsilon}),$$

where C is an explicit constant.

In this short paper we shall study the sum $\sum_{n\leq x} I(x)$. Our main result is the following

Theorem. Let $N_0 \geq 1$ be a fixed integer , then we have

$$\sum_{n\leq x} I(n) = x^3 \log^{\frac{1}{2}} x \left(\sum_{i=1}^{N_0} c_i \log^{-i} x + O(\log^{-N_0-1} x)\right),$$

where $c_i (i \geq 1)$ are computable constants.

Notations. In this paper, $\zeta(s)$ denotes the Riemann zeta-function. For any $z \in \mathbb{C}$, let $d_z(n)$ denote the general divisor function defined by

$$\sum_{n=1}^{\infty} d_z(n) n^{-s} = \zeta^z(s) = e^{z\log\zeta(s)}, \quad \Re s > 1,$$

where $\log\zeta(s)$ satisfies $\log 1 = 0$.

[1]This work is supported by National Natural Science Foundation of China (Grant No. 10771127) and National Natural Science Foundation of Shandong Province(Grant No. 2006A31).

§2. Proof of the theorem

In order to prove our Theorem , we need the following Lemma 1, which can be found in Ivić[1], Chapter 14.

Lemma 1. Let $A > 0$ be an arbitrary but fixed real number , and let $N \geq 1$ be an arbitrary fixed integer . If $\mid z \mid \leq A$ then uniformly in z

$$\begin{aligned} D_z(x) &= \sum_{n\leq x} d_z(n) \\ &= c_1(z)x\log^{z-1} x + c_2(z)x\log^{z-2} x + ... + c_N(z)x\log^{z-N} x \\ &+O(x\log^{\Re z-N-1} x), \end{aligned}$$

where $c_j(z) = B_{j-1}(z)/\Gamma(z-j+1)(j = 1, 2, \cdots, N)$, and $B_j(z)$ is a regular function in $\mid z \mid \leq A$.

Now we begin to prove our Theorem. For $\Re s > 4$, define

$$f(s) = \sum_{n=1}^{\infty} \frac{I(n)}{n^s}.$$

Since $I(n)$ is multiplicative, we have

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{I(n)}{n^s} = \prod_p \left[1 + \sum_{\alpha=1}^{\infty} \frac{I(p^\alpha)}{p^{\alpha s}}\right] \\ &= \prod_p \left[1 + \frac{1}{2p^{s-2}} + \sum_{\alpha=2}^{\infty} \frac{1}{(\alpha+1)p^{\alpha s-\alpha-1}}\right] \\ &= \prod_p \left(1 - \frac{1}{p^{s-2}}\right)^{-\frac{1}{2}} \prod_p \left(1 - \frac{1}{p^{s-2}}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2p^{s-2}} + \sum_{\alpha=2}^{\infty} \frac{1}{(\alpha+1)p^{\alpha s-\alpha-1}}\right) \\ &= \zeta^{\frac{1}{2}}(s-2)G(s), \end{aligned}$$

where

$$G(s) = \prod_p \left(1 - \frac{1}{p^{s-2}}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2p^{s-2}} + \sum_{\alpha=2}^{\infty} \frac{1}{(\alpha+1)p^{\alpha s-\alpha-1}}\right).$$

Write

$$G(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

It is easy to check that the finite series of $G(s)$ is absolutely convergent for $\sigma > \frac{5}{2}$ and hence we have

$$\sum_{n\leq x} \mid a(n) \mid \ll x^{\frac{5}{2}+\epsilon}. \tag{1}$$

Notice that

$$\zeta^{\frac{1}{2}}(s-2) = \sum_{n=1}^{\infty} \frac{d_{\frac{1}{2}}(n)}{n^{s-2}} = \sum_{n=1}^{\infty} \frac{n^2 d_{\frac{1}{2}}(n)}{n^s}.$$

Using the hyperbolic summation method we have

$$\begin{aligned}\sum_{n\le x} I(n) &= \sum_{kl\le x} k^2 d_{\frac{1}{2}}(k)a(l) \\ &= \sum_{1\le k\le\sqrt{x}} k^2 d_{\frac{1}{2}}(k) \sum_{1\le l\le\frac{x}{k}} a(l) \\ &+ \sum_{1\le l\le\sqrt{x}} a(l) \sum_{1\le k\le\frac{x}{l}} k^2 d_{\frac{1}{2}}(k) - \sum_{1\le k\le\sqrt{x}} k^2 d_{\frac{1}{2}}(k) \sum_{1\le l\le\sqrt{x}} a(l) \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3,\end{aligned} \tag{2}$$

say. From Lemma 1 and (1), for $\sum_1$ and $\sum_3$ we have

$$\begin{aligned}\sum\nolimits_1 &= \sum_{1\le k\le\sqrt{x}} k^2 d_{\frac{1}{2}}(k) \sum_{1\le l\le\frac{x}{k}} a(l) \ll \sum_{1\le k\le\sqrt{x}} k^2 \mid d_{\frac{1}{2}}(k) \mid \left(\frac{x}{k}\right)^{\frac{5}{2}+\epsilon} \\ &\ll x^{\frac{5}{2}+\epsilon} \sum_{1\le k\le\sqrt{x}} k^{-\frac{1}{2}} \ll x^{\frac{11}{4}+\epsilon}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\sum\nolimits_3 &= \sum_{1\le k\le\sqrt{x}} k^2 d_{\frac{1}{2}}(k) \sum_{1\le l\le\sqrt{x}} a(l) \ll x^{\frac{5}{4}+\epsilon} \sum_{1\le k\le\sqrt{x}} k^2 d_{\frac{1}{2}}(k) \\ &\ll x^{\frac{5}{4}+\epsilon} \left(\sum_{j=1}^{N} c_j' \left(\frac{1}{2}\right) x^{\frac{3}{2}} \log^{\frac{1}{2}-j} x + x^{\frac{3}{2}} \log^{-\frac{1}{2}-N} x\right) \\ &\ll x^{\frac{11}{4}+\epsilon}.\end{aligned} \tag{4}$$

Finally we evaluate $\sum_2$. We have

$$\begin{aligned}\sum\nolimits_2 &= \sum_{1\le l\le\sqrt{x}} a(l) \sum_{1\le k\le\frac{x}{l}} k^2 d_{\frac{1}{2}}(k) \\ &= \sum_{1\le l\le\sqrt{x}} a(l) \left[\sum_{j=1}^{N} c_j \left(\frac{1}{2}\right) \left(\frac{x}{l}\right)^3 \log^{\frac{1}{2}-j}\left(\frac{x}{l}\right) + O\left(\left(\frac{x}{l}\right)^3 \log^{-\frac{1}{2}-N}\left(\frac{x}{l}\right)\right)\right] \\ &= \sum_{j=1}^{N} c_j \left(\frac{1}{2}\right) x^3 \sum_{1\le l\le\sqrt{x}} a(l) l^{-3} \log^{\frac{1}{2}-j}\left(\frac{x}{l}\right) \\ &\quad + O\left(x^3 \sum_{1\le l\le\sqrt{x}} a(l) l^{-3} \log^{-\frac{1}{2}-N}\left(\frac{x}{l}\right)\right).\end{aligned}$$

Suppose $M \geq 1$ is a fixed positive integer, we have

$$\begin{aligned}
\sum_{1\leq l\leq\sqrt{x}} a(l)l^{-3}\log^{\frac{1}{2}-j}\left(\frac{x}{l}\right) &= \sum_{1\leq l\leq\sqrt{x}} a(l)l^{-3}\log^{\frac{1}{2}-j} x\left(1-\frac{\log l}{\log x}\right)^{\frac{1}{2}-j}\\
&= \sum_{1\leq l\leq\sqrt{x}} a(l)l^{-3}\log^{\frac{1}{2}-j} x\left(\sum_{m=0}^{M}\frac{f_j^{(m)}(0)}{m!}\left(\frac{\log l}{\log x}\right)^m + O\left(\frac{\log^{M+1} l}{\log^{M+1} x}\right)\right)\\
&= \sum_{m=0}^{M}\frac{f_j^{(m)}(0)}{m!}\log^{\frac{1}{2}-j-m} x\sum_{1\leq l\leq\sqrt{x}} a(l)l^{-3}\log^m l\\
&\quad +O\left(\log^{-\frac{1}{2}-j-M} x\sum_{1\leq l\leq\sqrt{x}} \mid a(l)\mid l^{-3}\log^{M+1} l\right)\\
&= \sum_{m=0}^{M}\frac{f_j^{(m)}(0)}{m!}\log^{\frac{1}{2}-j-m} x\sum_{l=1}^{\infty} a(l)l^{-3}\log^m l\\
&\quad +O\left(\sum_{m=0}^{M}\frac{\mid f_j^{(m)}\mid(0)}{m!}\log^{\frac{1}{2}-j-m} x\sum_{l>\sqrt{x}} \mid a(l)\mid l^{-3}\log^m l\right)\\
&\quad +O\left(\log^{-\frac{1}{2}-j-M} x\sum_{1\leq l\leq\sqrt{x}} \mid a(l)\mid l^{-3}\log^{M+1} l\right)
\end{aligned}$$

where for simplicity we wrote $f_j(t) = (1-t)^{\frac{1}{2}-j},\ t \in \left[0, \frac{1}{2}\right]$.

It is easy to see that the infinite series $\sum_{l=1}^{\infty} a(l)l^{-3}\log^m l$ is convergent. So we can write

$$\sum_{m=0}^{M}\frac{f^{(m)}(0)}{m!}\log^{\frac{1}{2}-j-m} x\sum_{l=1}^{\infty} a(l)l^{-3}\log^m l = \sum_{m=0}^{M} c_m\log^{\frac{1}{2}-j-m} x,$$

where

$$c_m = \frac{f^{(m)}(0)}{m!}\sum_{l=1}^{\infty} a(l)l^{-3}\log^m l.$$

Using Abel's summation formula we have

$$\sum_{l>\sqrt{x}} \mid a(l)\mid l^{-3}\log^m l \ll \frac{\log^m x}{x^{\frac{1}{4}-\epsilon}},$$

so

$$\begin{aligned}
&\sum_{m=0}^{M}\frac{\mid f^{(m)}(0)\mid}{m!}\log^{\frac{1}{2}-j-m} x\sum_{l>\sqrt{x}} \mid a(l)\mid l^{-3}\log^m l\\
&\ll \sum_{m=0}^{M}\frac{\mid f^{(m)}(0)\mid}{m!}\log^{\frac{1}{2}-j-m} x\frac{\log^m x}{x^{\frac{1}{4}-\epsilon}}\\
&\ll \frac{\log^{\frac{1}{2}-j} x}{x^{\frac{1}{4}-\epsilon}}.
\end{aligned}$$

Furthermore we can get

$$\sum_{1\le l\le\sqrt{x}} a(l)l^{-3}\log^{\frac{1}{2}-j}(\frac{x}{l}) = \sum_{m=0}^{M} c_m \log^{\frac{1}{2}-j-m} x + O\left(\frac{\log^{\frac{1}{2}-j} x}{x^{\frac{1}{4}-\epsilon}}\right) + O(\log^{-\frac{1}{2}-j-M} x)$$
$$= \sum_{m=0}^{M} c_m \log^{\frac{1}{2}-j-m} x + O(\log^{-\frac{1}{2}-j-M} x).$$

Combining the above estimates we have

$$\begin{aligned}\sum\nolimits_2 \quad &= \sum_{j=1}^{N} c_j\left(\frac{1}{2}\right) x^3 \sum_{m=0}^{M} c_m \log^{\frac{1}{2}-j-m} x \\ &+O\left(\sum_{j=1}^{N} c_j\left(\frac{1}{2}\right) x^3 \log^{-\frac{1}{2}-j-M} x\right) + O\left(\sum_{m=0}^{M} c_m x^3 \log^{-\frac{1}{2}-N-m} x\right) \\ &= \sum_{j=1}^{N}\sum_{m=0}^{M} c_j\left(\frac{1}{2}\right) c_m x^3 \log^{\frac{1}{2}-j-m} x \\ &+O\left(x^3 \log^{-\frac{3}{2}-M} x\right) + O(x^3 \log^{-\frac{1}{2}-N} x) \\ &= \sum_{i=1}^{N_0} c_i x^3 \log^{\frac{1}{2}-i} x + O(x^3 \log^{-\frac{1}{2}-N_0} x)\end{aligned} \tag{5}$$

where $c_i = \sum_{j+m=i} c_j\left(\frac{1}{2}\right) c_m,\ N_0 = \min(N,\ M+1)$.

Combing the estimates (2)-(5), this completes the proof of Theorem.

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Properties of $t-k$ residual sequence of natural sequence

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Abstract This paper discussed recurrence formula and sum formula of residual sequence $H_n^{(t,k)}$.

Keywords Natural sequence, residual sequence $H_n^{(t,k)}$, recursion, summarizing.

§1. Introduction

For natural sequence $\{n\}$, if its cycle is $t(t \in N, t \geq 2)$, deleting multiple terms of t from $\{n\}$, the new sequence made up of all residual number is called the first residual sequence of t-order. For the first residual sequence of t-order, same method is repeated, the new sequence is called the second residual sequence of t-order. Carries on k times similarly repeatedly, we can get the k-th residual sequence of t-order, denote $\{H_n^{(t,k)}\}$, where $H_n^{(t,k)}$ denote the n-th term of the k-th residual sequence of t-order.

Lemma. If $m = n - \left[\frac{n}{t}\right]$ $(t, m, n \in N, and\ t \nmid n)$, then $n = m + \left[\frac{m-1}{t-1}\right]$.

Proof. Since $t \nmid n$, suppose $n = qt + r$, where $0 < r < t$, and $\left[\frac{n}{t}\right] = q$, then, according to the premise, $m = qt + r - q, m - 1 = q(t-1) + r - 1$, so $\left[\frac{n}{t}\right] = \left[\frac{m-1}{t-1}\right]$.

Take this to the condition, we get $n = m + \left[\frac{m-1}{t-1}\right]$. So this completes the proof of Lemma.

Theorem 1. $H_n^{(t,1)} = n + \left[\frac{n-1}{t-1}\right]$ is called general term formula of $\{H_n^{(t,1)}\}$.

Proof. When $t \nmid n$, n is the m-th $(m = n - \left[\frac{n}{t}\right])$ term of the first residual sequence of t-order. i.e.

$$H_m^{(t,1)} = H_{n-\left[\frac{n}{t}\right]}^{(t,1)} = n,$$

according to the Lemma, $n = m + \left[\frac{m-1}{t-1}\right]$. So,

$$H_m^{(t,1)} = m + \left[\frac{m-1}{t-1}\right],$$

hence,

$$H_n^{(t,1)} = n + \left[\frac{n-1}{t-1}\right].$$

Theorem 2. The first recurrence formula of $\{H_n^{(t,k)}\}$ is

$$H_n^{(t,k)} = H_{n+\left[\frac{n-1}{t-1}\right]}^{(t,k-1)} (k = 1, 2, \cdots);$$

The second recurrence formula of $\{H_n^{(t,k)}\}$ is

$$H_n^{(t,k)} = H_n^{(t,k-1)} + \left[\frac{H_n^{(t,k-1)} - 1}{t-1}\right] (k = 1, 2, \cdots);$$

Proof. First, prove the first recurrence formula. $H_n^{(t,k-1)}$ is the n-th term of the $k-1$-th residual sequence of t-order, when $t \dagger n$, then, $H_n^{(t,k-1)}$ is the m-th$\left(m = n - \left[\frac{n}{t}\right]\right)$ term of the k-th residual sequence of t-order. i.e.

$$H_m^{(t,k)} = H_n^{(t,k-1)}.$$

According to the Lemma,

$$H_m^{(t,k)} = H_{m+\left[\frac{m-1}{t-1}\right]}^{(t,k-1)},$$

i.e.

$$H_n^{(t,k)} = H_{n+\left[\frac{n-1}{t-1}\right]}^{(t,k-1)} (k = 1, 2, \cdots).$$

Second, prove the second recurrence formula.

When $k = 1,\ t \dagger n$, according to the Theorem 1 and definition,

$$H_n^{(t,1)} = n + \left[\frac{n-1}{t-1}\right] = H_n^{(t,0)} + \left[\frac{H_n^{(t,0)} - 1}{t-1}\right].$$

When $k = 2,\ t \dagger n$, according to the deduce process of Theorem 1,

$$n = H_{n-\left[\frac{n}{t}\right]}^{(t,1)} = H_m^{(t,1)}, H_n^{(t,1)} = H_{n-\left[\frac{n}{t}\right]}^{(t,2)} = H_m^{(t,2)}.$$

And

$$H_n^{(t,1)} = n + \left[\frac{n-1}{t-1}\right],$$

so,

$$H_m^{(t,2)} = H_m^{(t,1)} + \left[\frac{H_m^{(t,1)} - 1}{t-1}\right].$$

i.e.

$$H_n^{(t,2)} = H_n^{(t,1)} + \left[\frac{H_n^{(t,1)} - 1}{t-1}\right].$$

Suppose when $t \dagger n$, $H_n^{(t,k)} = H_n^{(t,k-1)} + \left[\frac{H_n^{(t,k-1)} - 1}{t-1}\right]$ is true. Then

$$H_{m=n-\left[\frac{n}{t}\right]}^{(t,k+1)} = H_n^{(t,k)} = H_n^{(t,k-1)} + \left[\frac{H_n^{(t,k-1)} - 1}{t-1}\right],$$

and according to the deduce process of Theorem 1,

$$H_n^{(t,k-1)} = H_{m=n-\left[\frac{n}{t}\right]}^{(t,k)} = H_m^{(t,k)},$$

so,

$$H_m^{(t,k+1)} = H_m^{(t,k)} + \left[\frac{H_m^{(t,k)} - 1}{t-1}\right],$$

i.e.

$$H_n^{(t,k+1)} = H_n^{(t,k)} + \left[\frac{H_n^{(t,k)} - 1}{t-1}\right].$$

Above all, the second recurrence formula is true.

Theorem 3. The sum of the formula n terms of $\{H_n^{(t,1)}\}$ is

$$S_n^{(t,1)} = \frac{H_n^{(t,1)}(H_n^{(t,1)}+1) - (H_n^{(t,1)} - n)(H_n^{(t,1)} - n + 1)t}{2}.$$

Proof. Since $H_n^{(t,1)} = n + \left[\frac{n-1}{t-1}\right]$, so, the former n terms of the first residual sequence of t-order is the residual terms by deleting $\left[\frac{n-1}{t-1}\right]$ terms $(t, 2t, \cdots, \left[\frac{n-1}{t-1}\right] t)$ from the former $n + \left[\frac{n-1}{t-1}\right]$ terms of natural sequence $\{n\}$.

We denote

$$S_{n+\left[\frac{n-1}{t-1}\right]}^{(t,0)} = 1 + 2 + \cdots + (n + \left[\frac{n-1}{t-1}\right]) = \frac{(n + \left[\frac{n-1}{t-1}\right])(n + \left[\frac{n-1}{t-1}\right] + 1)}{2} = \frac{H_n^{(t,1)}(H_n^{(t,1)}+1)}{2},$$

$$S_n^{(t,1)} = H_1^{(t,1)} + H_2^{(t,1)} + \cdots + H_n^{(t,1)},$$

$$S_{\left[\frac{n-1}{t-1}\right]}^{(t,1)} = t + 2t + \cdots + \left[\frac{n-1}{t-1}\right] t = \frac{\left[\frac{n-1}{t-1}\right](\left[\frac{n-1}{t-1}\right]+1)t}{2} = \frac{(H_n^{(t,1)} - n)(H_n^{(t,1)} - n + 1)t}{2}.$$

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Recurrence formulas for the generalized Euler numbers $E_{2n}^{(k)}$[1]

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Abstract In this paper, we prove some new recurrence formulas for the generalized Euler numbers $E_{2n}^{(k)}$.

Keywords The Euler numbers, the generalized Euler numbers, recurrence formula.

§1. Introduction and results

For a real or complex parameter x, the generalized Euler numbers $E_{2n}^{(x)}$ are defined by the following generating functions (see [1]):

$$\left(\frac{2}{e^t+e^{-t}}\right)^x = \sum_{n=0}^{\infty} E_n^{(x)} \frac{t^n}{n!} \tag{1}$$

or

$$(\sec t)^x = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(x)} \frac{t^{2n}}{(2n)!}. \tag{2}$$

(If x is a nonnegative integer, then $E_{2n}^{(x)}$ are called Euler numbers of order x (see [2-4]).)

By (1), we have $E_{2n+1}^{(x)} = 0$ $(n \geq 0)$. The numbers $E_{2n}^{(1)} = E_{2n}$ are the classical Euler numbers. By (1) or (2), we can get

$$E_{2n}^{(k)} = (2n)! \sum_{\substack{v_1 \geq 0, \cdots, v_k \geq 0 \\ v_1+\cdots+v_k=n}} \frac{E_{2v_1} \cdots E_{2v_k}}{(2v_1)! \cdots (2v_k)!} \tag{3}$$

where k is a positive integer.

The Euler numbers E_{2n} satisfy the recurrence relation

$$E_0 = 1, \quad E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}, \tag{4}$$

so we find $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, $E_{12} = 2702765$, $\cdots$.

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By the mathematical induction, all the Euler numbers $E_0, E_2, E_4, \cdots$ are integers. By (3), we know that $E_{2n}^{(k)}$ is an integer.

In [5], Liu obtained some recurrence formulas for the generalized Euler numbers $E_{2n}^{(k)}$,

$$E_{2n}^{(2k+1)} = -\frac{1}{2^{2k}} \sum_{h=0}^{k} \binom{2k+1}{h} \sum_{j=0}^{n-1} \binom{2n}{2j} (2k+1-2h)^{2n-2j} E_{2j}^{(2k+1)} \tag{5}$$

and

$$E_{2n}^{(2k+2)} = -\frac{1}{2^{2k+1}} \sum_{h=0}^{k} \binom{2k+2}{h} \sum_{j=0}^{n-1} \binom{2n}{2j} (2k+2-2h)^{2n-2j} E_{2j}^{(2k+2)}. \tag{6}$$

where $n \geq 1$, $k \geq 0$ are integers.

The main purpose of this paper is to prove some new recurrence formulas for the generalized Euler numbers. That is, we shall prove the following main conclusions.

Theorem 1. Let $n \geq 1, k \geq 1, m \geq 1$ are integers. Then we have

$$E_{2n}^{(k)} = \frac{1}{(1-(2m+1)^{2n})} \sum_{j=0}^{n-1} (-1)^{n-j} \binom{2n}{2j} (2m+1)^{2j} H_{2n-2j}^{(k)} (2m+1) E_{2j}^{(k)}, \tag{7}$$

where $\{H_{2n}^{(k)}(m)\}$ can be defined by the generating function

$$\sum_{n=0}^{\infty} H_{2n}^{(k)}(m) \frac{t^{2n}}{(2n)!} = (\sec t \cos mt)^k. \tag{8}$$

Taking $k = 1, 2$ in Theorem 1, we may immediately deduce the following

Corollary 1. Let $n \geq 1, m \geq 1$ are integers. Then

$$E_{2n} = \frac{2}{1-(2m+1)^{2n}} \sum_{j=0}^{n-1} \binom{2n}{2j} (2m+1)^{2j} \left(\sum_{i=1}^{m} (-1)^{m-i} (2i)^{2n-2j} \right) E_{2j} \tag{9}$$

and

$$E_{2n}^{(2)} = \frac{2}{1-(2m+1)^{2n}} \sum_{j=0}^{n-1} \binom{2n}{2j} (2m+1)^{2j} \left(\sum_{i=1}^{2m} (-1)^{i} (2m+1-i)(2i)^{2n-2j} \right) E_{2j}^{(2)}. \tag{10}$$

Curiously, we find that the following recurrences are special cases of Corollary 1.

$$E_{2n} = \frac{2^{2n+1}}{(1-3^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} \left(\frac{3}{2}\right)^{2j} E_{2j},$$

$$E_{2n} = \frac{2^{2n+1}}{(1-5^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} \left(\frac{5}{2}\right)^{2j} (2^{2n-2j}-1)E_{2j},$$

$$E_{2n} = \frac{2^{2n+1}}{(1-7^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} \left(\frac{7}{2}\right)^{2j} (3^{2n-2j}-2^{2n-2j}+1)E_{2j},$$

$$E_{2n} = \frac{2^{2n+1}}{(1-9^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} \left(\frac{9}{2}\right)^{2j} (4^{2n-2j}-3^{2n-2j}+2^{2n-2j}-1)E_{2j},$$

$$E_{2n}^{(2)} = \frac{1}{(1-3^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} 3^{2j} \left(2^{4n+1-4j} - 2^{2n+2-2j}\right) E_{2j}^{(2)},$$

$$E_{2n}^{(2)} = \frac{2}{(1-5^{2n})} \sum_{j=0}^{n-1} \binom{2n}{2j} 5^{2j} \left(3\cdot 4^{2n-2j} + 8^{2n-2j} - 2^{2n+2-2j} - 2\cdot 6^{2n-2j}\right) E_{2j}^{(2)}.$$

Theorem 2. Let $n \geq 1, k \geq 1$ be integers. Then

$$E_n^{(k+1)} = \sum_{j=0}^{n} \binom{n}{j} \left(E_j^{(k)} + \frac{1}{k} E_{j+1}^{(k)}\right) \tag{11}$$

or

$$E_{2n}^{(k+1)} = \sum_{j=0}^{n} \left(\binom{2n}{2j} + \frac{1}{k}\binom{2n}{2j-1}\right) E_{2j}^{(k)}. \tag{12}$$

Remark 1. By the inversion principle (see [6])

$$\sum_{j=0}^{k} \binom{k}{j} a_j = b_k \Leftrightarrow a_j = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} b_k,$$

we may rephrase (11) as

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} E_j^{(k+1)} = E_n^{(k)} + \frac{1}{k} E_{n+1}^{(k)}$$

or

$$\sum_{j=0}^{n} \binom{2n}{2j} E_{2j}^{(k+1)} = E_{2n}^{(k)} \tag{13}$$

and

$$k \sum_{j=0}^{n-1} \binom{2n-1}{2j} E_{2j}^{(k+1)} = -E_{2n}^{(k)}. \tag{14}$$

§2. Proof of the Theorems and Corollary

Proof of Theorem 1. Recall the generating function $(\sec t)^k$ in (2). Then, writing

$$(\sec t)^k = (\sec(2m+1)t)^k \left(\frac{\cos(2m+1)t}{\cos t}\right)^k$$

and forming the Abel convolution, we find that

$$E_{2n}^{(k)} = \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{2j} (2m+1)^{2j} H_{2n-2j}^{(k)}(2m+1) E_{2j}^{(k)}.$$

Separating the term with $j = n$ and solving, we obtain (7).

This completes the proof of Theorem 1.

Proof of Corollary 1. It suffices to find closed form of $H_{2n}^{(k)}(2m+1)$ for $k = 1, 2$. We may rewrite the generating function (8) in the form

$$\left(\sum_{j=1}^{2m+1} (-1)^{j-1} e^{i(2m+2-2j)t} \right)^k$$

by straightforward transformation. By multinomial expansion we obtain

$$H_{2n}^{(k)}(2m+1) = \sum_{\substack{v_1 \geq 0, \cdots, v_{2m+1} \geq 0 \\ v_1 + \cdots + v_{2m+1} = k}} \frac{k! (-1)^{n+k+v_1+2v_2+\cdots+(2m+1)v_{2m+1}}}{v_1! v_2! \cdots v_{2m+1}!} \sum_{j=1}^{2m+1} \left((2m+2-2j) v_j \right)^{2n}, \tag{15}$$

Hence we may immediately get

$$H_{2n}^{(1)}(2m+1) = (-1)^n 2 \sum_{k=1}^{m} (-1)^{m-k} (2k)^{2n} \tag{16}$$

and

$$H_{2n}^{(2)}(2m+1) = (-1)^n \left(\sum_{k=1}^{2m+1} (4m+4-4k)^{2n} + 2 \sum_{1 \leq k < l \leq 2m+1} (-1)^{l+k} (4m+4-2(l+k))^{2n} \right).$$

Then in the second identity, we divide the sums according to the parity of k and of $k + l = \lambda$. Then we obtain

$$H_{2n}^{(2)}(2m+1) = (-1)^n 2 \left(\sum_{k=1}^{m} (4k)^{2n} + \sum_{k=1}^{m-1} (2m-2k)(4k)^{2n} - \sum_{k=1}^{m} (2m+2-2k)(4k-2)^{2n} \right),$$

whence after simple calculation,

$$H_{2n}^{(2)}(2m+1) = (-1)^n 2^{2n+1} \sum_{k=1}^{2m} (-1)^k (2m+1-k) k^{2n}. \tag{17}$$

Substituting (16) or (17) in (7) completes the proof of Corollary 1.

Proof of Theorem 2. Let $f_k(t) = \left(\frac{2}{e^t + e^{-t}} \right)^k$. Then in order to find $f_{k+1}(t)$, we are naturally led to differentiate it.

Since we have

$$\begin{aligned} \frac{d}{dt} f_k(t) &= 2^k (-k) \left(e^t + e^{-t} \right)^{-k-1} \left(e^t - e^{-t} \right) \\ &= -k f_k(t) + k e^{-t} f_{k+1}(t), \end{aligned}$$

it follows that

$$f_{k+1}(t) = e^t \left(f_k(t) + \frac{1}{k} \frac{d}{dt} f_k(t) \right),$$

and it remains to form the Abel convolution of e^t and

$$f_k(t) + \frac{1}{k} \frac{d}{dt} f_k(t) = \sum_{n=0}^{\infty} \left(E_n^{(k)} + \frac{1}{k} E_{n+1}^{(k)} \right) \frac{t^n}{n!}.$$

(12) follows from the classifying the values of j modulo 2 and recalling that all odd indexed Euler numbers of order k are 0.

This completes the proof of Theorem 2.

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On a rational recursive sequence $x_{n+1} = (a - bx_n^2)/(1 + x_{n-1}^2)$

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Abstract Our aim in this paper is to investigate the asymptotic stability, and the global attractivity of difference equation of the form

$$x_{n+1} = \frac{a - bx_n^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \cdots,$$

where $a \geqslant 0$, $b \geqslant 0$. We obtain sufficient conditions for global attractivity.

Keywords Difference equation, asymptotic stability, global attractivity, positive solution.

§1. Introduction

The behaviour of solutions of the difference equation

$$x_{n+1} = \frac{a + bx_n^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \cdots,$$

was investigated, when $a, b \in [0, \infty)$(see Zhang et. al [1]). Also, Li [5] has studied the asymptotic behaviour of nonlinear delay difference equations of the form $x_{n+1} = x_n^p f(x_n, x_{n-k_1}, \cdots, x_{n-k_r})$, $p > 0$, $n = 0, 1, \cdots$, where $k_1, k_2, \cdots, k_r$ are positive integers.

In this paper, our aim is to study the asymptotic stability and global attractivity of the rational recursive sequences

$$x_{n+1} = \frac{a - bx_n^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \cdots, \tag{1}$$

where $a \geq 0$, $b \geq 0$.

Here, we recall some concepts and theorems which will be useful in this paper.

Consider the difference equation

$$x_{n+1} = F(x_n, \cdots, x_{n-k}), \quad n = 0, 1, \cdots, \tag{2}$$

where k is a positive integer and the function $F \in C^1[R^{k+1}, R]$.

Assume that $a_{-k}, \cdots, a_{-1}, a_0 \in R$ are arbitrary real numbers, then for initial conditions $x_{-i} = a_{-i}$, for $i = 0, 1, \cdots, k$, the (2) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

If $n \geqslant 1$, then $x_n > 0$. We consider this solution is positive solution of the (2).

Definition 1.1. A point $\overline{x}$ is called an equilibrium of the (2), if $\overline{x} = F(\overline{x}, \cdots, \overline{x})$. That is, $x_n = \overline{x}$, for $n \geqslant 0$, is a solution of the (2), or equivalently, is fixed point of F.

Definition 1.2. Let I be an interval of real numbers, the equilibrium point $\overline{x}$ of the (1) is said to be

(a) locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, \cdots, x_{-1}, x_0 \in I$ with $\sum_{i=0}^{k} |x_{-i} - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \varepsilon$ for all $n \geqslant -k$.

(b) locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_{-k}, \cdots, x_{-1}, x_0 \in I$, with $\sum_{i=0}^{k} |x_{-i} - \overline{x}| < \gamma$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(c) global attractor if for all $x_{-k}, \cdots, x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(d) global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also global attractor.

(e) unstable if $\overline{x}$ is not locally stable.

If $k = 1$, then (2) is in form that

$$x_{n+1} = f(x_n, x_{n-1}), \ n = 0, 1, \cdots . \tag{3}$$

Let $\overline{x}$ is an equilibrium of the (3), $f(u, v)$ is the function associated with the (3).

$$r = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}), \quad s = \frac{\partial f}{\partial v}(\overline{x}, \overline{x}).$$

The linearized equation associated with the (3) about the equilibrium $\overline{x}$ is

$$y_{n+1} = ry_n + sy_{n-1}, \quad n = 0, 1, \cdots .$$

Its characteristic equation is

$$\lambda^2 - r\lambda - s = 0. \tag{4}$$

Theorem 1.1. Assume that F is a C^1 function and let $\overline{x}$ be an equilibrium of the (3). Then the following statements are true.

(a) If all the roots of the (4) lie in the open unit disk $| \lambda | < 1$, then the equilibrium $\overline{x}$ of the (3) is asymptotically stable.

(b) If at least one of the roots of the (4) has absolute value greater then one, then the equilibrium $\overline{x}$ of the (3) is unstable.

(c)(4) has all its roots in the open unit disk $| \lambda | < 1$ if and only if

$$| r | < 1 - s < 2 ,$$

then $\overline{x}$ is a attracting equilibrium .

(d) One root of the (4) has absolute value greater than one and another has absolute value less than one if and only if

$$r^2 + 4s > 0 \ \text{ and } \ | r | > |1 - s| ,$$

then $\overline{x}$ is a saddle point.

(e) All the roots of the (4) has absolute value greater than one if and only if

$$| s | > 1 \ \text{ and } \ | r | < |1 - s| ,$$

then $\overline{x}$ is a repelling equilibrium or source.

Theorem 1.2. Consider the equation

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \cdots, x_{n-k_r}), \tag{5}$$

where $k_i (i = 1, 2, \cdots, r)$ are positive integers. Denote by k the maximum of $k_1, \cdots, k_r$.

Also, assume that the function f satisfies the following hypotheses.

(H_1) $f \in C[(0,\infty)\times(0,\infty)^r, (0,\infty)]$, and $g \in C[(0,\infty)^{r+1}, (0,\infty)]$, where $g(u_0, u_1, \cdots, u_r) = u_0 f(u_0, u_1, \cdots, u_r)$ for $u_0 \in (0,\infty)$ and $u_1, \cdots, u_r \in [0,\infty)$ and

$$g(0, u_1, \cdots, u_r) = \lim_{u_0 \to 0^+} g(u_0, u_1, \cdots, u_r).$$

(H_2) $f(u_0, u_1, \cdots, u_r)$ is nonincreasing in $u_1, \cdots, u_r$, respectively.

(H_3) The equation $f(x, x, \cdots, x) = 1$ has a unique positive solution $\overline{x}$.

(H_4) Either the function $f(u_0, u_1, \cdots, u_r)$ does not depend on u_0 or for every $x > 0$ and $u \geq 0$

$$[f(x, u, \cdots, u) - f(\overline{x}, u, \cdots, u)](x - \overline{x}) \leqslant 0,$$

with

$$[f(x, \overline{x}, \cdots, \overline{x}) - f(\overline{x}, \overline{x}, \cdots, \overline{x})](x - \overline{x}) < 0 \text{ for } x \neq \overline{x}.$$

Now define a new function

$$F(x) = \begin{cases} \max\limits_{x \leqslant y \leqslant \overline{x}} G(x,y) & \text{for } 0 \leqslant x \leqslant \overline{x}, \\ \min\limits_{\overline{x} \leqslant y \leqslant x} G(x,y) & \text{for } x > \overline{x}, \end{cases}$$

where

$$G(x, y) = y f(y, x, \cdots, x) f(\overline{x}, \overline{x}, \cdots, y)[f(\overline{x}, x, \cdots, x)]^{k-1}.$$

Then

(i) $F \in C[(0,\infty), (0,\infty)]$ and F is nonincreasing in $[0,\infty)$.

(ii) Assume that F has no periodic points of prime period 2. Then $\overline{x}$ is a global attractor of all positive solutions of(5).Main Results

Consider the (1) with

$$a \geq 0 \quad \text{and} \quad b \geq 0.$$

1). The case $a = 0$.

In this section, we study the asymptotically stable and the global attractivity for the difference equation

$$x_{n+1} = \frac{-b x_n^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \cdots, \tag{6}$$

where $b \in (0,\infty)$.

If $a = 0$, the (6) has no positive equilibrium.

Theorem 2.1. If $b < 2$, then the(6) has a unique equilibrium $\overline{x} = 0$ and $\overline{x}$ is asymptotically stable. If $b = 2$, (6) has a unique negative equilibrium $\overline{x} = -1$, and $\overline{x}$ is unstable.

Proof. Let $\phi(x) = x^3 + bx^2 + x$, it is easy to see that $\phi'(x) = 3x^2 + 2bx + 1 > 0$ when $b < 2$. So, the (6) has a unique equilibrium $\overline{x} = 0$, and easy to see that $\overline{x}$ is asymptotically stable. The case $b = 2$, is easy to proof and we omit it. The proof is complete.

Theorem 2.2. If $b > 2$, then the (6) has three equilibriums $\overline{x}_0 = 0$, $\overline{x}_1 = \dfrac{-b - \sqrt{b^2 - 4}}{2}$, $\overline{x}_2 = \dfrac{-b + \sqrt{b^2 - 4}}{2}$, and $\overline{x}_1$ is repelling equilibrium, while $\overline{x}_2$ is saddle point.

proof. It is easy to see that $\phi(x) = x^3 + bx^2 + x = x(x^2 + bx + 1) = 0$ has three roots $x_0 = 0$, $x_1 = \dfrac{-b - \sqrt{b^2 - 4}}{2}$, $x_2 = \dfrac{-b + \sqrt{b^2 - 4}}{2}$. So, the (6) has three equilibriums $\overline{x}_0 = x_0$, $\overline{x}_1 = x_1$, $\overline{x}_2 = x_2$.

The asymptotic stability of $\overline{x}_0$ has discussed in theorem 2.1, we omit it.

Now, we consider the function $f(u, v) = \dfrac{-bu^2}{1 + v^2}$ which is assosiate with the (6).

$$\begin{aligned} r_i &= \frac{\partial f}{\partial u}(\overline{x}_i, \overline{x}_i) = -\frac{2b\overline{x}_i}{1 + \overline{x}_i^2}, \\ s_i &= \frac{\partial f}{\partial v}(\overline{x}_i, \overline{x}_i) = -\frac{2\overline{x}_i^2}{1 + \overline{x}_i^2}, \quad i = 1, 2. \end{aligned}$$

Since $\overline{x}_1 = \dfrac{-b - \sqrt{b^2 - 4}}{2} < -1$, then $r_1 = 2$, $s_1 < -1$. So,

$$|s_1| > 1 \text{ and } |1 - s_1| > 2 = r_1,$$

by parts (a) and (e) of Theorem 1.1, we have $\overline{x}_1$ is a repelling equilibrium.

Since $\overline{x}_2 = \dfrac{-b + \sqrt{b^2 - 4}}{2} > -1$, then $r_2 = 2$, $-1 < s_2 < 0$. So,

$$r_2^2 + 4s_2 = 4(1 + s_2) > 0 \text{ and } |1 - s_2| < 2 = r_2,$$

by parts (a) and (d) of Theorem 1.1, we have $\overline{x}_2$ is a saddle point. The proof is complete.

2). The case $b = 0$.

In this section, we study the asymptotic stability and global attractivity for the difference equation

$$x_{n+1} = \frac{a}{1 + x_{n-1}^2}, \quad n = 0, 1, \cdots, \tag{7}$$

where $a \in (0, \infty)$.

Theorem 2.3. The equation (7) has a unique positive equilibrium $\overline{x}$ and if $0 < a < 2$, then $\overline{x}$ is asymptotically stable.

Proof. Let $\overline{x} = \dfrac{a}{1 + \overline{x}^2}$, it can be rewritten as follows:

$$\overline{x}^3 + \overline{x} - a = 0.$$

Consider the function $\phi(x) = x^3 + x - a$, we have

$$\phi(0) = -a < 0, \text{ and } \phi(a) = a^3 > 0.$$

Since $\phi'(x) = 3x^2 + 1 > 0$, so $\phi(x)$ is a strictly monotonically increasing function. Hence, $\phi(x) = 0$ has only one positive root x', and $0 < x' < a$.

Therefore (7) has a unique positive equilibrium $\overline{x}$ and $\overline{x} = x'$, so $0 < \overline{x} < a$.

Let $f(u,v)=\dfrac{a}{1+v^2}$ is assosiated with the (7).

$$\begin{aligned} r &= \frac{\partial f}{\partial u}(\overline{x},\overline{x})=0, \\ s &= \frac{\partial f}{\partial v}(\overline{x},\overline{x})=\frac{-2a\overline{x}}{(1+\overline{x}^2)^2}=-\frac{2\overline{x}^3}{a}\ , \\ 1-s &= 1+\frac{2\overline{x}^3}{a}>r=0. \end{aligned}$$

Since $0<a<2$, so $1-s<2$. By parts (a) and (c) of Theorem1.1, $\overline{x}$ is asymptotically stable. The proof is complete.

Theorem 2.4. Assume that $b=0$ and $0<a<2$, let $\{x_n\}_{n=1}^{\infty}$ be a positive solution of the (7). Then

$$\lim_{n\to\infty} x_n=\overline{x},$$

where $\overline{x}$ is the unique positive equilibrium of the (7), that is, $\overline{x}$ is a global attractor of all positive solutions of the (7).

Furtherly, if and only if $(x_{-1},x_0)\neq(\overline{x},\overline{x})$, then the semicycles of every positive solution of the (7) has length 2.

A detailed proof of the Theorem 2.4. can be found in the monograph of D. C. Zhang, B. Shi and M. J. Gai, we omit it.

3). The case $a>0$, $b>0$.

In this section, we consider the (1), when $a,b\in(0,\infty)$.

Theorem 2.5. The equation (1) has a unique positive equilibrium $\overline{x}$, and if $0<a\leq 1$, $0<b<\sqrt{3}$, then $\overline{x}$ is asymptotically stable.

Proof. Let $\overline{x}=\dfrac{a-b\overline{x}^2}{1+\overline{x}^2}$, it can be rewritten as follows

$$\overline{x}^3+b\overline{x}^2+\overline{x}-a=0.$$

Consider the function $\phi(x)=x^3+bx^2+x-a$, we have $\phi(0)=-a<0$, and $\phi(a)=a^3+a^2b>0$.

Next $\phi'(x)=3x^2+2bx+1>0$. Hence, $\phi(x)$ is a monotonically increasing function in $[0,\infty)$ and $\phi(x)=0$ has only one positive root x', $0<x'<a$.

So, it is easy to see that the (1) has a unique positive equilibrium $\overline{x}=x'$, and $0<\overline{x}<a$.

Now, we consider the function

$$\begin{aligned} f(u,v) &= \frac{a-bu^2}{1+v^2}, \\ r &= \frac{\partial f}{\partial u}(\overline{x},\overline{x})=\frac{2b\overline{x}^2}{1+\overline{x}^2}, \\ s &= \frac{\partial f}{\partial v}(\overline{x},\overline{x})=\frac{-2\overline{x}^2}{1+\overline{x}^2}\ . \end{aligned}$$

Since $0<a\leqslant 1$, $0<b<\sqrt{3}$, we have

$$|\,r\,|<\ 1-s<2\ .$$

Then, by parts (a) and (c) of Theorem 1.1, $\overline{x}$ is asymptotically stable. The proof is complete.

Theorem 2.6. If $ab < 1$ and $0 \leqslant x_0 \leqslant \sqrt{\frac{a}{b}} < 2$, then $\overline{x}$ is a global attractor.

Proof. Let $ab < 1$ such that $x_n \leqslant \sqrt{\frac{a}{b}}$, $\{x_n\}_{n=1}^{\infty}$ is positive solution of (1) when $x_0 \in \left[0, \sqrt{\frac{a}{b}}\right]$. The (1) can be rewritten as follows

$$x_{n+1} = x_n \frac{a - bx_n^2}{x_n(1 + x_{n-1}^2)}, \quad n = 0, 1, \cdots .$$

Now, we define two functions as follows

$$\begin{aligned} f(u_0, u_1) &= \frac{a - bu_0^2}{u_0(1 + u_1^2)}, \\ g(u_0, u_1) &= \frac{a - bu_0^2}{1 + u_1^2}. \end{aligned}$$

It is easy to verify that $f(u_0, u_1)$ and $g(u_0, u_1)$ satisfy the hypotheses $(H_1) - (H_4)$ which were divised in the Theorem1.2.

Next, we define a function as follows

$$G(x, y) = \frac{(a - by^2)(a - b\overline{x}^2)}{\overline{x}(1 + x^2)(1 + y^2)}.$$

Therefore we have

$$F(x) = \frac{(a - bx^2)(a - b\overline{x}^2)}{\overline{x}(1 + x^2)^2}, \ ab < 1$$

such that $x_n \leqslant \sqrt{\frac{a}{b}} < 2$, hence F has no periodic points of prime period 2. Thus, by Theorem 1.2. $\overline{x}$ is global attractor of all positive solutions of the (1). The proof is complete.

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On an equation involving the Smarandache reciprocal function and its positive integer solutions

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Abstract For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \nmid n!\}$. That is, $S_c(n)$ is the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. The main purpose of this paper is using the elementary method and the Vinogradov's important work to prove the following conclusion: For any positive integer $k \geq 3$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

This solved a problem posed by Zhang Wenpeng during the Fourth International Conference on Number Theory and the Smarandache Problems.

Keywords The Smarandache reciprocal function, equation, positive integer solutions.

§1. Introduction and result

For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. That is, $S_c(n) = \max\{m : y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \nmid n!\}$. For example, the first few values of $S_c(n)$ are:

$$S_c(1) = 1, \ S_c(2) = 2, \ S_c(3) = 3, \ S_c(4) = 4, \ S_c(5) = 6, \ S_c(6) = 6, \ S_c(7) = 10,$$
$$S_c(8) = 10, \ S_c(9) = 10, \ S_c(10) = 10, \ S_c(11) = 12, \ S_c(12) = 12, \ S_c(13) = 16,$$
$$S_c(14) = 16, \ S_c(15) = 16, \ S_c(16) = 16, \ S_c(17) = 18, \ S_c(18) = 18, \ \cdots\cdots .$$

This function was first introduced by A.Murthy in reference [2], where he studied the elementary properties of $S_c(n)$, and proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

During the Fourth International Conference on Number Theory and the Smarandache Problems, Professor Zhang Wenpeng asked us to study such a problem: For any positive integer k, whether there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

I think that this problem is interesting, because it has some close relations with the Goldbach problem. The main purpose of this paper is using the elementary method and the Vinogradov's important work to study this problem, and solved it completely. That is, we shall prove the following conclusion:

Theorem. For any positive integer $k \geq 3$, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

It is clear that if $k = 1$, then our Theorem is trivial. Whether there exist infinite group positive integers (m_1, m_2) such that $S_c(m_1+m_2) = S_c(m_1)+S_c(m_2)$? This is an open problem.

If the Goldbach's conjecture is true (i.e., every even number $2N \geq 6$ can be written as $2N = p_1 + p_2$, a sum of two odd primes), then there exist infinite group positive integers $(m_1,\ m_2)$ such that the equation $S_c(m_1 + m_2) = S_c(m_1) + S_c(m_2)$.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First from the Vinogradov's important work Three Primes Theorem (See Theorem 6.14 of reference [5]) we know that for any odd number $2N + 1$ large enough, there must exist three odd primes p_1, p_2 and p_3 such that the equation:

$$2N + 1 = p_1 + p_2 + p_3. \tag{1}$$

For any positive integer $k \geq 3$ and prime p (large enough), by using the mathematical inductive method and the Vinogradov's work (1) we can deduce that $p + k - 1$ can be written as a sum of k odd primes:

$$p + k - 1 = p_1 + p_2 + \cdots + p_k. \tag{2}$$

In fact if $k = 3$, then for any prime p large enough, $p + 2$ is an odd number, so from (1) we know that $p + 2 = p_1 + p_2 + p_3$. So (2) is true. If $k = 4$, then we take $p_1 = 3$, so from (1) we also have

$$p + 3 = 3 + p_2 + p_3 + p_4 = p_1 + p_2 + p_3 + p_4.$$

So (2) is true if $k = 4$. If $k \geq 5$, we take p be such a prime so as to odd number $p+k-1-3\cdot(k-3)$ large enough, from (1) we know that there must exist three odd primes p_{k-2}, p_{k-1} and p_k such that the equation:

$$p + k - 1 - 3 \cdot (k - 3) = p_{k-2} + p_{k-1} + p_k$$

or

$$p + k - 1 = \underbrace{3 + 3 + \cdots + 3}_{k-3} + p_{k-2} + p_{k-1} + p_k = p_1 + p_2 + \cdots + p_k,$$

where $p_1 = p_2 = \cdots = p_{k-3} = 3$. So (2) is true for all $k \geq 3$.

Now we use (2) to complete the proof of our Theorem. For any positive integer $k \geq 3$, we take prime p large enough, then from (2) we have the identity

$$p - 1 = p_1 - 1 + p_2 - 1 + p_3 - 1 + \cdots\cdots + p_k - 1. \tag{3}$$

Note that $S_c(p_i - 1) = p_i - 1$ for all prime p_i, taking $m = p - 1$, $m_i = p_i - 1$, $i = 1,\ 2,\ \cdots,\ k$, from (3) we may immediately deduce the identity

$$\begin{aligned} p - 1 &= S_c(p-1) = S_c(m) = S_c(m_1 + m_2 + \cdots + m_k) \\ &= p_1 - 1 + p_2 - 1 + p_3 - 1 + \cdots\cdots + p_k - 1 \\ &= S_c(m_1) + S_c(m_2) + S_c(m_3) + \cdots + S_c(m_k). \end{aligned}$$

That is,

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

Since there are infinite prime p, so there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation

$$S_c(m_1 + m_2 + \cdots + m_k) = S_c(m_1) + S_c(m_2) + \cdots + S_c(m_k).$$

This completes the proof of Theorem.

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Stability of Wegl-Heisenberg frames

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Abstract In this paper, The dual Wegl-Heisenberg frame's structure is discussed and then explicit structure of the dual Wegl-Heisenberg frame is given. Under the small perturbation of φ, some changes have taken place about the dual of Wegl-Heisenberg frames. Some results about stability of the dual Wegl-Heisenberg frames are proved with weaker conditions.

Keywords Wegl-Heisenberg frame, stability, dual of Wegl-Heisenberg frame, Bessel sequence.

§1. Introduction

The theory of frames was introduced by Duffin and Schaeffer in the early 1950s to deal with problems in nonharmonic Fourier series, frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, oversampling , and irregular sampling play a role [7 − 9]. First,we introduce some basic definitions.

Recall that a sequence $\{\varphi_j : j \in \Lambda\}$ in a Hilbert space H is said to be a frame for H if there exist two positive constants $A, B > 0$ such that

$$\forall f \in H,\ A\|f\|^2 \leq \sum_{j\in\Lambda} |\langle f, \varphi_j\rangle|^2 \leq B\|f\|^2, \tag{1}$$

where A and B are called the the lower frame bound and upper frame bound ,respectively. In particular, when $A = B = 1$, we say that $\{\varphi_j : j \in \Lambda\}$ is a(normalized)tight frame in H. If the right-hand side of (1) holds, it is said to be a Bessel sequence.

The mathematical theory of Gabor analysis in $L^2(R)$ is based on two classes of operator on $L^2(R)$,namely

Translation by $a \in \mathbf{R}$, $\quad T_a : (T_a f)(x) = f(x - a)$;

Modulation by $b \in \mathbf{R}$, $\quad M_b : (M_b f)(x) = e^{2\pi ibx} f(x)$.

Again, if $\{E_{mb}M_{na}\varphi\}_{m,n\in Z}$ forms a frame for $L^2(R)$, which is called a Weyl-Hensenberg or a Gabor frame. Since Gabor [6] proposed a signal representation with windowed Fourier transform, Gabor systems have had a fundamental impact on the develop of modern time-frequency analysis and have been widely used in communication theory, quantum mechanics, and many other fields.

For Wegl-Heisenberg frames, the stability means that $\{E_{mb}M_{na}\varphi\}_{m,n\in Z}$ is still a frame if φ, a, b has some small perturbation. Favier and Zalik [8] studied the stability of Gabor frames

under some perturbation to φ, m, n. But their results require that the generating function φ be compactly supported. For the canonical dual frames of $\{E_{mb}T_{na}\varphi\}_{m,n\in Z}$ also has Gabor structure.

In this paper, we discuss the stability of Wegl-Heisenberg frames with much weaker conditions.

The paper is organized as follows. In section 2, we briefly introduce the concept of the Wegl-Heisenberg frame and the canonical Wegl-Heisenberg frame. In Section 3, we discuss the changes of dual Wegl-Heisenberg frame. Under the small perturbation of φ.

§2. Preliminaries

We begin with some basic theory and notations to be used throughout this paper. Denote by $\mathbf{R}$ and $\mathbf{Z}$ the set of all real and integers numbers, respectively. Given a Bessel sequence $\{\varphi_j : j \in \Lambda\}$ one can define a bounded linear operator

$$T : l^2(Z) \to H, T\{c_i\} = \sum_{i\in Z} c_i\varphi_i. \tag{2}$$

Then $\|T\| < \sqrt{B}$. The adjoint operator

$$T^* : H \to l^2(Z), T^*x = \{\langle x, \varphi_i\rangle\}_{i\in Z}. \tag{3}$$

The frame operator is defined by

$$S : H \to H, Sx = TT^*x = \sum_{i\in Z}\langle x, \varphi_i\rangle\varphi_i. \tag{4}$$

If $\{\varphi_j : j \in \Lambda\}$ is a frame,the frame operator S has bounded, inverse, defined on all of H, this fact leads to the important frame decomposition

$$x = S^{-1}Sx = \sum_{i\in Z}\langle S^{-1}x, \varphi_i\rangle\varphi_i = \sum_{i\in Z}\langle x, S^{-1}\varphi_i\rangle\varphi_i, \forall x \in H. \tag{5}$$

Lemma 1. [1] Let $\varphi \in L^2(R)$ and $a,\ b > 0$ be given, and assume that

$$\{E_{mb}T_{na}\varphi\}_{m,n\in Z}$$

is a Wegl-Heisenberg frame, then the dual also has the structure and is given by

$$\{E_{mb}M_{na}S^{-1}\varphi\}_{m,n\in Z},$$

where S is frame operator.

Lemma 2. Let $\varphi \in L^2(R)$ and $a, b > 0$ be given, and assume that

$$\{E_{mb}M_{na}\varphi\}_{m,n\in Z}$$

is a Wegl-Heisenberg frame with frame operator S, then

$$S^{-1}E_{mb}M_{na} = E_{mb}M_{na}S^{-1}. \tag{6}$$

Proof. Let $\forall\varphi \in L^2(R)$,

$$\begin{aligned} S^{-1}E_{mb}M_{na}\varphi &= \sum_{n',m'\in Z}\langle E_{mb}M_{na}\varphi, E_{m'b}M_{n'a}\phi\rangle E_{m'b}M_{n'a}\phi \\ &= \sum_{n',m'\in Z}\langle \varphi, M_{-na}E_(m'-m)bM_{n'a}\phi\rangle E_{m'b}M_{n'a}\phi \\ &= \sum_{n',m'\in Z}\langle \varphi, e^{2\pi ina(m'-m)b}E_{(m'-m)b}M_{(n'-n)a}\phi\rangle E_{m'b}M_{n'a}\phi. \end{aligned}$$

Performing the change of variables $m'-m=m'',\ n'-n=n''$ and using the commutator relation, then

$$\begin{aligned} S^{-1}E_{mb}M_{na}\varphi &= \sum_{n'',m''\in Z} e^{-2\pi inam''b}\langle \varphi, E_{m''b}M_{n''a}\phi\rangle E_{(m''+m)b}M_{(n''+n)a}\phi \\ &= \sum_{n'',m''\in Z} e^{-2\pi inam''b}\langle \varphi, E_{m''b}M_{n''a}\phi\rangle e^{2\pi inam''b}E_{mb}M_{na}E_{m''b}M_{n''a}\phi \\ &= E_{mb}T_{na}S^{-1}. \end{aligned}$$

§3. Main results

Theorem 1. Let $\Phi_{mb,na}=\{E_{mb}M_{na}\phi\}_{n,m\in Z}$, $\widetilde{\Phi}_{mb,na}=\{E_{mb}M_{na}\widetilde{\phi}\}_{n,m\in Z}$ and $\Psi_{mb,na}=\{E_{mb}M_{na}\psi\}_{n,m\in Z}$, $\widetilde{\Psi}_{mb,na}=\{E_{mb}M_{na}\widetilde{\psi}\}_{n,m\in Z}$ be two pairs of canonical dualWegl-Heisenberg frame for H. Denote the Wegl-Heisenberg frame bounds of $\Phi_{mb,na}$ and $\Psi_{mb,na}$ by (A,B) and (C,D), respectively. If $\{\Phi_{mb,na}-\Psi_{mb,na}\}$ is a Bessel sequence with an upper bound ε, so is $\{\widetilde{\Phi}_{mb,na}-\widetilde{\Psi}_{mb,na}\}$ with an upper bound $\varepsilon\left(\dfrac{A+B+B^{\frac{1}{2}}D^{\frac{1}{2}}}{AC}\right)^2$

Proof. Put

$$Sx=\sum_{n,m\in Z}\langle x, E_{mb}M_{na}\phi\rangle E_{mb}M_{na}\phi,\quad \forall x\in H,$$

$$Tx=\sum_{n,m\in Z}\langle x, E_{mb}M_{na}\psi\rangle E_{mb}M_{na}\psi,\quad \forall x\in H.$$

Then S and T are self-adjoint. For any $x\in H$, we have

$$\begin{aligned}
\|Sx-Tx\| &= \left|\sum_{n,m\in Z}\langle x,E_{mb}M_{na}\phi\rangle E_{mb}M_{na}\phi-\sum_{n,m\in Z}\langle x,E_{mb}M_{na}\psi\rangle E_{mb}M_{na}\psi\right| \\
&\le \left|\sum_{n,m\in Z}\langle x,E_{mb}M_{na}\phi\rangle\langle x,(E_{mb}M_{na}\phi-E_{mb}M_{na}\psi)\rangle\right| \\
&\quad+\left|\sum_{n,m\in Z}(\langle x,E_{mb}M_{na}\phi\rangle-\langle x,E_{mb}M_{na}\psi\rangle)\langle x,E_{mb}M_{na}\psi\rangle\right| \\
&\le B^{\frac12}(\sum_{n,m\in Z}|\langle x,(E_{mb}M_{na}\phi-E_{mb}M_{na}\psi)\rangle|^2)^{\frac12}+\varepsilon^{\frac12}\left(\sum_{n,m\in Z}\langle x,E_{mb}M_{na}\psi\rangle|^2\right)^{\frac12} \\
&\le \varepsilon^{\frac12}\left(B^{\frac12}+D^{\frac12}\right)\|x\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n,m}|\langle x,S^{-1}(E_{mb}M_{na}\phi_{n,m}-E_{mb}M_{na}\psi_{n,m})\rangle|^2 \\
\le\ &\|S^{-1}\|\sum_{n,m}|\langle x,(E_{mb}M_{na}\phi_{n,m}-E_{mb}M_{na}\psi_{n,m})\rangle|^2 \\
\le\ &\frac{\varepsilon}{C^2}\|x\|^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{n,m}|\langle x,(\widetilde{\Phi}_{n,m}-\widetilde{\Psi}_{n,m})\rangle|^2 &= \sum_{n,m}|\langle x,(S^{-1}E_{mb}M_{na}\phi_{n,m}-T^{-1}E_{mb}M_{na}\psi_{n,m})\rangle|^2 \\
&\le \sum_{n,m}|\langle x,S^{-1}(E_{mb}M_{na}\phi_{n,m}-E_{mb}M_{na}\psi_{n,m})\rangle|^2 \\
&\quad+\sum_{n,m}|\langle x,(S^{-1}-T^{-1})E_{mb}M_{na}\psi_{n,m}\rangle|^2 \\
&\le \varepsilon\left(\frac{A+B+B^{\frac12}D^{\frac12}}{AC}\right)^2\|x\|^2.
\end{aligned}$$

Theorem 2. Let $\Phi_{mb,na}=\{E_{mb}M_{na}\phi\}_{n,m\in Z}$, $\widetilde{\Phi}_{mb,na}=\{E_{mb}M_{na}\widetilde{\phi}\}_{n,m\in Z}$ and $\Psi_{mb,na}=\{E_{mb}M_{na}\psi\}_{n,m\in Z}$, $\widetilde{\Psi}_{mb,na}=\{E_{mb}M_{na}\widetilde{\psi}\}_{n,m\in Z}$ be two pairs of canonical dual Wegl-Heisenberg frame for H. If

$$\left|\sum_{m,n}|\langle x,E_{mb}M_{na}\phi\rangle|^2-\sum_{m,n}|\langle x,E_{mb}M_{na}\psi\rangle|^2\right|\le\varepsilon\|x\|^2,\quad\forall x\in H$$

then

$$\left|\sum_{m,n}|\langle x,E_{mb}M_{na}\widetilde{\phi}\rangle|^2-\sum_{m,n}|\langle x,E_{mb}M_{na}\widetilde{\psi}\rangle|^2\right|\le\frac{\varepsilon}{AC}\|x\|^2,\quad\forall x\in H$$

Proof. Since both S and T are self-adjoint. For any $x \in H$, we have

$$\begin{aligned}\|S-T\| &= \sup_{\|x\|=1}\|\langle (S-T)x,x\rangle\| = \sup_{\|x\|=1}\|\langle Sx,x\rangle - \langle Tx,x\rangle\| \\ &= \sup_{\|x\|=1}\left|\sum_{m,n}|\langle x, E_{mb}M_{na}\phi\rangle|^2 - \sum_{m,n}|\langle x, E_{mb}M_{na}\psi\rangle|^2\right| \\ &\leq \varepsilon.\end{aligned}$$

Therefore, $\|S^{-1}-T^{-1}\| \leq \|S^{-1}\|\cdot\|S-T\|\cdot\|T^{-1}\| \leq \dfrac{1}{AC}\varepsilon$.

$$\begin{aligned}\sum_{m,n}|\langle x, E_{mb}M_{na}\widetilde{\phi}\rangle|^2 &= \sum_{m,n}\langle x, S^{-1}E_{mb}M_{na}\phi\rangle\langle x, S^{-1}E_{mb}M_{na}\phi\rangle \\ &= \langle SS^{-1}x, S^{-1}x\rangle = \langle x, S^{-1}x\rangle.\end{aligned}$$

Similarly,

$$\sum_{m,n}|\langle x, E_{mb}M_{na}\widetilde{\phi}\rangle|^2 = \langle x, T^{-1}x\rangle.$$

So we get

$$\begin{aligned}\left|\sum_{i\in Z}\|\widetilde{T}_i x\|^2 - \sum_{i\in Z}\|\widetilde{S}_i x\|^2\right| &= \left|\langle x, (S^{-1}-T^{-1})x\rangle\right| \\ &\leq \|S^{-1}-T^{-1}\|\cdot\|x\| \leq \frac{\varepsilon}{AC}\|x\|^2.\end{aligned}$$

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A generalization of the Smarandache function

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Abstract For any positive integer n, we define the function $P(n)$ as the smallest prime p such that $n \mid p!$. That is, $P(n) = \min\{p : \ n \mid p!, \text{ where } p \text{ be a prime}\}$. This function is a generalization of the famous Smarandache function $S(n)$. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $P(n)$, and give two interesting mean value formulas for it.

Keywords The Smarandache function, generalization, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are: $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$.

About the elementary properties of $S(n)$, many authors had studied it, and obtained a series results, see references [1], [2], [3], [4] and [5]. In reference [6], Jozsef Sandor introduced another arithmetical function $P(n)$ as follows: $P(n) = \min\{p : \ n \mid p!, \text{ where } p \text{ be a prime}\}$. That is, $P(n)$ denotes the smallest prime p such that $n \mid p!$. In fact function $P(n)$ is a generalization of the Smarandache function $S(n)$. Its some values are: $P(1) = 2$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11$, $\cdots$. It is easy to prove that for each prime p one has $P(p) = p$, and if n is a square-free number, then $P(n) =$ greatest prime divisor of n. If p be a prime, then the following double inequality is true:

$$2p + 1 \leq P(p^2) \leq 3p - 1.$$

For any positive integer n, one has (See Proposition 4 of reference [6])

$$S(n) \leq P(n) \leq 2S(n) - 1. \tag{1}$$

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the function $P(n)$, and give two interesting mean value formulas it. That is, we shall prove the following conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} P(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

Theorem 2. For any real number $x > 1$, we also have the mean value formula

$$\sum_{n\leq x}\left(P(n)-\overline{P}(n)\right)^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. For any real number $x > 1$, we divide all positive integers in the interval $[1,\ x]$ into two subsets A and B, where A denotes the set of all integers $n \in [1,\ x]$ such that there exists a prime p with $p|n$ and $p > \sqrt{n}$. And B denotes the set involving all integers $n \in [1,\ x]$ with $n \notin A$. From the definition and properties of $P(n)$ we have

$$\sum_{n\in A} P(n) = \sum_{\substack{n\leq x\\ p|n,\ \sqrt{n}<p}} P(n) = \sum_{\substack{pn\leq x\\ n<p}} P(pn) = \sum_{\substack{pn\leq x\\ n<p}} p = \sum_{n\leq\sqrt{x}}\sum_{n<p\leq\frac{x}{n}} p. \tag{2}$$

By the Abel's summation formula (See Theorem 4.2 of [7]) and the Prime Theorem (See Theorem 3.2 of [8]):

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n<p\leq\frac{x}{n}} p &= \frac{x}{n}\cdot\pi\left(\frac{x}{n}\right) - n\cdot\pi(n) - \int_n^{\frac{x}{n}}\pi(y)dy\\ &= \frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x} + O\left(\frac{x^2}{n^2\cdot\ln^{k+1} x}\right),\end{aligned} \tag{3}$$

where we have used the estimate $n \leq \sqrt{x}$, and all b_i are computable constants.

Note that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, and $\sum\limits_{n=1}^{\infty}\frac{\ln^i n}{n^2}$ is convergent for all $i = 2,\ 3,\ \cdots,\ k$. From (2) and (3) we have

$$\begin{aligned}\sum_{n\in A} P(n) &= \sum_{n\leq\sqrt{x}}\left(\frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x} + O\left(\frac{x^2}{n^2\cdot\ln^{k+1} x}\right)\right)\\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),\end{aligned} \tag{4}$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

Now we estimate the summation in set B. Note that for any prime p and positive integer α, $S(p^\alpha) \leq \alpha \cdot p$, so from (1) we have

$$\sum_{n\in B} P(n) \quad = \quad \sum_{n\in B} (2S(n) - 1) \leq \sum_{n\leq x} \sqrt{n} \cdot \ln n \ll x^{\frac{3}{2}} \cdot \ln x. \tag{5}$$

Combining (4) and (5) we may immediately deduce the asymptotic formula

$$\sum_{n\leq x} P(n) = \sum_{n\in A} P(n) + \sum_{n\in B} P(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer $n > 1$, let $\overline{P}(n)$ denotes the largest prime divisor of n. We divide all integers in the interval $[1,\ x]$ into three subsets A, C and D, where A denotes the set of all integers $n \in [1,\ x]$ such that there exists a prime p with $p|n$ and $p > \sqrt{n}$; C denotes the set of all integers $n = n_1 p^2$ in the interval $[1,\ x]$ with $n_1 \leq p \leq \sqrt{n}$, where p be a prime; And D denotes the set of all integers $n \in [1,\ x]$ with $n \notin A$ and $n \notin C$. It is clear that if $n \in A$, then $P(n) = \overline{P}(n)$ and $\left(P(n) - \overline{P}(n)\right)^2 = 0$. So we have the identity

$$\sum_{n\in A} \left(P(n) - \overline{P}(n)\right)^2 = 0. \tag{6}$$

If $n \in C$, then $P(n) = P(p^2) \geq 2p + 1$. On the other hand, for any real number x large enough, from M.N.Huxley [9] we know that there at least exists a prime in the interval $\left[x,\ x + x^{\frac{7}{12}}\right]$. So we have the estimate

$$2p + 1 \leq P(p^2) \leq 2p + O\left(p^{\frac{7}{12}}\right). \tag{7}$$

From [3] we also have the asymptotic formula

$$\sum_{n\leq x^{\frac{1}{3}}} \sum_{n<p\leq\sqrt{\frac{x}{n}}} p^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right). \tag{8}$$

Note that $\overline{P}(n) = p$, if $n = n_1 \cdot p^2 \in C$.

Therefore, from (7) and (8) we have the estimate

$$\begin{aligned}
\sum_{n\in C} \left(P(n) - \overline{P}(n)\right)^2 &= \sum_{n\leq x^{\frac{1}{3}}} \sum_{n<p\leq\sqrt{\frac{x}{p}}} \left(P(np^2) - \overline{P}(np^2)\right)^2 \\
&= \sum_{n\leq x^{\frac{1}{3}}} \sum_{n<p\leq\sqrt{\frac{x}{n}}} \left(P(p^2) - p\right)^2 = \sum_{n\leq x^{\frac{1}{3}}} \sum_{n<p\leq\sqrt{\frac{x}{n}}} \left(p^2 + O\left(p^{\frac{19}{12}}\right)\right) \\
&= \sum_{n\leq x^{\frac{1}{3}}} \sum_{n<p\leq\sqrt{\frac{x}{n}}} p^2 + O\left(x^{\frac{31}{24}}\right) \\
&= \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),
\end{aligned} \tag{9}$$

where $\zeta(s)$ is the Riemann zeta-function.

If $n \in D$ and $\left(P(n) - \overline{P}(n)\right)^2 \neq 0$, then $P(p^\alpha) \ll S(p^\alpha) \ll p \cdot \ln p$ and $\overline{P}(p^3) \ll p \cdot \ln p$, so we have the trivial estimate

$$\sum_{n \in D} \left(P(n) - \overline{P}(n)\right)^2 \ll \sum_{3 \leq \alpha \leq \ln x} \sum_{np^\alpha \leq x} p^{\frac{2}{3}} \ll x \cdot \ln x. \tag{10}$$

Combining (6), (9) and (10) we may immediately the asymptotic formula

$$\sum_{n \leq x} \left(P(n) - \overline{P}(n)\right)^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

This completes the proof of Theorem 2.

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On the Smarandache power function and Euler totient function

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Abstract For any positive integer n, let $SP(n)$ denotes the Smarandache power function, and $\phi(n)$ is the Euler totient function. The main purpose of this paper is using the elementary method to study the solutions of the equation $SP(n^k) = \phi(n)$, and give its all positive integer solutions for $k = 1, 2, 3$.

Keywords Smarandache power function, Euler totient function, equation, positive integer solutions.

§1. Introduction and Results

For any positive integer n, the famous Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that m^m is divisible by n. That is,

$$SP(n) = \min\{m : n \mid m^m, m \in N, \prod_{p|m} p = \prod_{p|n} p\}$$

where N denotes the set of all positive integers. For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $SP(16) = 4$, $SP(17) = 17$, $SP(18) = 6$, $SP(19) = 19$, $SP(20) = 10$, $\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. From the definition of $SP(n)$ we can easily get the following conclusions: if $n = p^\alpha$, then

$$SP(n) = \begin{cases} p, & 1 \le \alpha \le p; \\ p^2, & p+1 \le \alpha \le 2p^2; \\ p^3, & 2p^2+1 \le \alpha \le 3p^3; \\ \cdots & \cdots \\ p^\alpha, & (\alpha-1)p^{\alpha-1}+1 \le \alpha \le \alpha p^\alpha. \end{cases}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers. If $\alpha_i \le p_i$ for all $\alpha_i (i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(3) = 3$, $SP(8) = 4$, $SP(24) = 6 \ne SP(3) \times SP(8)$. But for most n we have $SP(n) = U(n)$.

About other properties of $SP(n)$, many scholars had studied it, and obtained some interesting results. For example, In reference [2], Dr.Zhefeng Xu studied the mean value properties of $SP(n)$, and obtain some sharper asymptotic formulas, one of them as follows: for any real number $x \geq 1$,

$$\sum_{n \leq x} SP(n) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{2}{3}+\epsilon}\right),$$

where $\prod_p$ denotes the product over all prime numbers, ϵ is any given positive number. Huanqin Zhou [3] studied the convergent properties of an infinite series involving $SP(n)$, and gave some interesting identities. That is, she proved that for any complex number s with $Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{\mu(n)}}{(SP(n^k))^s} = \begin{cases} \dfrac{2^s+1}{2^s-1}\dfrac{1}{\zeta(s)}, & k = 1, 2; \\ \dfrac{2^s+1}{2^s-1}\dfrac{1}{\zeta(s)} - \dfrac{2^s-1}{4^s}, & k = 3; \\ \dfrac{2^s+1}{2^s-1}\dfrac{1}{\zeta(s)} - \dfrac{2^s-1}{4^s} + \dfrac{3^s-1}{9^s}, & k = 4, 5. \end{cases}$$

If $n \geq 1$, the Euler function $\phi(n)$ is defined as the number of all positive integers not exceeding n, which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function. In this paper, we shall use the elementary method to study the solutions of the equation $SP(n^k) = \phi(n)$, and give its all solutions for $k = 1, 2, 3$. That is, we shall prove the following:

Theorem 1. The equation $SP(n) = \phi(n)$ have only 4 positive integer solutions, namely, $n = 1, 4, 8, 18$.

Theorem 2. The equation $SP(n^2) = \phi(n)$ have only 3 positive integer solutions, namely, $n = 1, 8, 18$.

Theorem 3. The equation $SP(n^3) = \phi(n)$ have only 3 positive intrger solutions, namely, $n = 1, 16, 18$.

Generally, for any given positive integer number $k \geq 4$, we conjecture that the equation $SP(n^k) = \phi(n)$ has only finite positive integer solutions. This is an open problem.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. It is easy to versify that $n = 1$ is one solution of the equation $SP(n) = \phi(n)$. In order to obtain the other positive integer solution, we discuss in the following cases:

1. $n > 1$ is an odd number.

At this time, from the definition of the Smarandache power function $SP(n)$ we know that $SP(n)$ is an odd number, but $\phi(n)$ is an even number, hence $SP(n) \neq \phi(n)$.

2. n is an even number.

(1) $n = 2^\alpha$, $\alpha \geq 1$. It is easy to versify that $n = 2$ is not a solution of the equation $SP(n) = \phi(n)$ and $n = 4, 8$ are the solutions of the equation $SP(n) = \phi(n)$. If $\alpha \geq 4$, $(\alpha-2)2^{\alpha-2} \geq \alpha$, so $2^\alpha \mid (2^{\alpha-2})^{2^{\alpha-2}}$, namely $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(2) $n = 2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_k$, $\alpha_i \geq 1$, $i = 1, 2, \cdots, k$, $\alpha \geq 2$, $k \geq 1$. At this time,

$$\phi(n) = 2^{\alpha-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \dagger (\phi(n))^{\phi(n)}$, then from the definition of the Smarandache power function $SP(n)$ we know that $SP(n) \neq \phi(n)$.

If $n \mid (\phi(n))^{\phi(n)}$, then from the form of $\phi(n)$, we can imply $\alpha_k \geq 2$.

(i) for 2^{α}. $\alpha \geq 2$, so

$$(\alpha-1)\frac{\phi(n)}{2} \geq (\alpha-1)2^{\alpha-1}p_k^{\alpha_k-1}\frac{p_k-1}{2} \geq (\alpha-1)\cdot 2\cdot 3 \geq 6(\alpha-1) \geq 3\alpha > \alpha,$$

which implies $2^{\alpha} \mid (2^{(\alpha-1)})^{\frac{\phi(n)}{2}}$. Hence $2^{\alpha} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

(ii) for $p_i^{\alpha_i} \mid n$. If $\alpha_i = 1$, associating

$$\frac{\phi(n)}{2} \geq 2^{\alpha-1}p_k^{\alpha_k-1}\frac{p_k-1}{2} \geq 2\cdot 3 = 6 > 1$$

with $p_i \mid (\phi(n))^{\phi(n)}$ which implies $p_i \mid \frac{\phi(n)}{2}$, we can deduce that $p_i \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. If $\alpha_i \geq 2$,

$$(\alpha_i-1)\frac{\phi(n)}{2} \geq (\alpha_i-1)2^{\alpha-1}p_i^{\alpha_i-1}\frac{p_i-1}{2} \geq (\alpha_i-1)\cdot 2\cdot 3 \geq 6(\alpha_i-1) \geq 3\alpha_i > \alpha_i,$$

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $\forall p_i^{\alpha_i} \mid n$, $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

Combining (i) and (ii), we can deduce that if $n \mid (\phi(n))^{\phi(n)}$, then $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Hence $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(3) $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_k$, $\alpha_i \geq 1$, $i = 1, 2, \cdots, k$, $k \geq 1$. At this time,

$$\phi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \dagger (\phi(n))^{\phi(n)}$, then from the definition of the Smarandache power function $SP(n)$ we know that $SP(n) \neq \phi(n)$.

If $n \mid (\phi(n))^{\phi(n)}$, then from the form of $\phi(n)$, we can imply $\alpha_k \geq 2$.

(*i*) $k \geq 2$. We will prove that $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$.

For one hand, obviously, $2 \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. For the other hand, $\forall p_i^{\alpha_i} \mid n$, if $\alpha_i = 1$, associating

$$\frac{\phi(n)}{2} \geq p_k^{\alpha_k-1}(p_i-1)\frac{p_k-1}{2} \geq 3\cdot 2 = 6 > 1$$

with $p_i \mid (\phi(n))^{\phi(n)}$ which implies $p_i \mid \frac{\phi(n)}{2}$, we can deduce that $p_i \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. If $\alpha_i \geq 2$,

$$(\alpha_i-1)\frac{\phi(n)}{2} \geq (\alpha_i-1)p_k^{\alpha_k-1}(p_1-1)\frac{p_k-1}{2} \geq (\alpha_i-1)\cdot 5\cdot 2\cdot 2 \geq 20(\alpha_i-1) \geq 10\alpha_i > \alpha_i,$$

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \frac{\phi(n)}{2} < \phi(n)$.

(*ii*) $k = 1$. At this time, $n = 2p_1^{\alpha_1}$, $\alpha_1 \geq 2$, $\phi(n) = p_1^{\alpha_1-1}(p_1-1)$.

which implies $p_i^{\alpha_i} \mid (p_i^{(\alpha_i-1)})^{\frac{\phi(n)}{2}}$. Hence $p_i^{\alpha_i} \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$. Consequently, $n \mid (\frac{\phi(n)}{2})^{\frac{\phi(n)}{2}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{2} < \phi(n)$.

(ii) $k = 1$. At this time, $n = 2p_1^{\alpha_1}$, $\alpha_1 \geq 2$, $\phi(n) = p_1^{\alpha_1-1}(p_1 - 1)$.

$(ii)'$ $p_1 \geq 5$, because $\alpha_1 \geq 2$,

$$(\alpha_1 - 1)\frac{\phi(n)}{\frac{p_1-1}{2}} = (\alpha_1 - 1)p_1^{\alpha_1-1}2 \geq (\alpha_1 - 1) \cdot 5 \cdot 2 \geq 10(\alpha_1 - 1) \geq 5\alpha_1 > \alpha_1,$$

which implies $p_1^{\alpha_1} \mid (p_1^{(\alpha_1-1)})^{\frac{\frac{\phi(n)}{p_1-1}}{2}}$. Hence $p_1^{\alpha_1} \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\frac{\phi(n)}{p_1-1}}{2}}$. Obviously, $2 \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\frac{\phi(n)}{p_1-1}}{2}}$. Consequently, $n \mid (\frac{\phi(n)}{\frac{p_1-1}{2}})^{\frac{\phi(n)}{\frac{p_1-1}{2}}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{\frac{p_1-1}{2}} < \phi(n)$.

$(ii)''$ $p_1 = 3$, namely $n = 2 \cdot 3^{\alpha_1}$.

$\alpha_1 = 1$, $\phi(n) = \phi(6) = 2$, $SP(n) = SP(6) = 6$, so $SP(n) \neq \phi(n)$.

$\alpha_1 = 2$, $\phi(n) = \phi(18) = 6$, $SP(n) = SP(18) = 6$, so $SP(n) = \phi(n)$.

$\alpha_1 \geq 3$, $(\frac{\phi(n)}{3})^{\frac{\phi(n)}{3}} = (2 \cdot 3^{\alpha_1-2})^{2 \cdot 3^{\alpha_1-2}}$, so $n \mid (\frac{\phi(n)}{3})^{\frac{\phi(n)}{3}}$, which implies $SP(n) \leq \dfrac{\phi(n)}{3} < \phi(n)$.

Combining (1), (2) and (3), we know that if n is an even number, then $SP(n) = \phi(n)$ if and only if $n = 4,\ 8,\ 18$.

Associating the cases 1 and 2, we complete the proof of Theorem 1.

Using the similar discussion, we can easily obtain the proofs of Theorem 2 and Theorem 3.

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On the mean value of the Dirichlet divisor function for the Smarandache power sequence

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value distribution property of the Dirichlet divisor function for the Smarandache power sequence, and obtain a sharper asymptotic formula for it.

Keywords Dirichlet divisor function, Smarandache power sequence, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous Smarandache power sequence $SP(n)$ is defined as the smallest positive integer m such that m^m is divisible by n. That is, $SP(n) = \min\{m : n \mid m^m, m \in N\}$, where N denotes the set of all positive integers. For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $SP(16) = 4$, $SP(17) = 17$, $SP(18) = 6$, $SP(19) = 19$, $SP(20) = 10$, $\cdots$

In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. From the definition of $SP(n)$ we can easily get the following conclusions: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers. If $\alpha_i \leq p_i$ for all $\alpha_i (i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(3) = 3$, $SP(8) = 4$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for most n we have $SP(n) = U(n)$. About this function, many scholars had studied its properties (see reference [2] and [3]). Simultaneously, F.Russo (see reference [4]) proposed the following:

Conjecture. For the Smarandache power sequence, the following series are asymptotically equal to:

$$\sum_{k=1}^{n} Ss(k) \approx \frac{1}{2} \cdot n^2,$$

where $Ss(k) = \sigma(SP(k))$ denotes the Dirichlet divisor function for the Smarandache power sequence.

Now, we define the Dirichlet divisor function for the Smarandache power sequence as

following: $SD(n) = \sigma_k(SP(n))$, where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor functions and $k > 0$. The main purpose of this paper is using the elementary and analytic methods to study the mean value distribution property of the Dirichlet divisor function for the Smarandache power sequence, and obtain a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be any real number with $k > 0$, then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{n \leq x} SD(n) = \frac{\zeta(k+1) \cdot x^{k+1}}{\zeta(2) \cdot (k+1)} + O(x^{k+\frac{1}{2}+\epsilon}),$$

where $SD(n) = \sigma_k(SP(n))$, $\zeta(s)$ is the Riemann zeta-function, and ϵ denotes any fixed positive number.

Taking $k = 1$, we may immediately deduce the following:

Corollary. For any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{n \leq x} Ss(n) = \frac{1}{2}x^2 + O(x^{\frac{3}{2}+\epsilon}).$$

It is clear that our Corollary solved the F.Russo's conjecture.

§2. Some simple lemmas

Before the proof of the theorem, some simple lemmas will be useful.

Lemma 1. Let k be any real number with $k > 0$, $\sigma_k(n) = \sum_{d|n} d^k$. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{n \leq x} \sigma_k(U(n)) = \frac{\zeta(k+1) \cdot x^{k+1}}{\zeta(2) \cdot (k+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and ϵ denotes any fixed positive number.

Proof. For any real number k, s with $s - k > 1$ and $k > 0$, let

$$f(s) = \sum_{n=1}^{\infty} \frac{\sigma_k(U(n))}{n^s}.$$

From the Euler product formula (see reference [5]) and the multiplicative property of $\sigma_k(U(n))$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{\sigma_k(U(p))}{p^s} + \frac{\sigma_k(U(p^2))}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{1+p^k}{p^s} + \frac{1+p^k}{p^{2s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1+p^k}{p^s} \cdot \frac{1}{1-p^{-s}}\right) = \prod_p \frac{1+p^{k-s}}{1-p^{-s}} = \frac{\zeta(s) \cdot \zeta(s-k)}{\zeta(2s-2k)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

For any real number $k > 0$ and $x \geq 1$, it is obvious that $|\sigma_k(U(n))| \leq \zeta(k) \cdot n^k$, and $\left|\sum_{n=1}^{\infty} \frac{\sigma_k(U(n))}{n^\sigma}\right| \leq \frac{1}{\sigma - k - 1}$, where σ is the real part of the complex s. So by the Perron formula (see reference [6]) we can get

$$\sum_{n \leq x} \frac{\sigma_k(U(n))}{n^{s_0}} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)$$

$$+O\left(x^{1-\sigma_0} H(2x) \min\left(1, \frac{\log(x)}{T}\right)\right) + O\left(x^{-\sigma_0} H(N) \min\left(1, \frac{x}{T\|x\|}\right)\right),$$

where N is the nearest integer to x, and $\|x\| = |x - N|$.

Taking $s_0 = 0$, $b = k + \frac{3}{2}$ and $T > 2$ in the above, then we have

$$\sum_{n \leq x} \sigma_k(U(n)) = \frac{1}{2\pi i} \int_{k+\frac{3}{2}-iT}^{k+\frac{3}{2}+iT} \frac{\zeta(s)\zeta(s-k)x^s}{\zeta(2s-2k)s} ds + O\left(\frac{x^{k+\frac{3}{2}}}{T}\right).$$

Now we move the integral line from $k + \frac{3}{2} \pm iT$ to $k + \frac{1}{2} \pm iT$. This time, the function $\frac{\zeta(s)\zeta(s-k)x^s}{\zeta(2s-2k)s}$ has a simple pole point at $s = k+1$ with residue $\frac{\zeta(k+1)x^{k+1}}{\zeta(2)(k+1)}$.

So we can obtain

$$\frac{1}{2\pi i}\left(\int_{k+\frac{3}{2}-iT}^{k+\frac{3}{2}+iT} + \int_{k+\frac{3}{2}+iT}^{k+\frac{1}{2}+iT} + \int_{k+\frac{1}{2}+iT}^{k+\frac{1}{2}-iT} + \int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}-iT}\right) \frac{\zeta(s)\zeta(s-k)x^s}{\zeta(2s-2k)s} ds = \frac{\zeta(k+1)x^{k+1}}{\zeta(2)(k+1)}.$$

Taking $T = x$, we have the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{k+\frac{3}{2}+iT}^{k+\frac{1}{2}+iT} + \int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}-iT}\right) \frac{\zeta(s)\zeta(s-k)x^s}{\zeta(2s-2k)s} ds\right| \ll \frac{x^{k+\frac{3}{2}+\epsilon}}{T} = x^{k+\frac{1}{2}+\epsilon}$$

and

$$\left|\frac{1}{2\pi i} \int_{k+\frac{1}{2}+iT}^{k+\frac{1}{2}-iT} \frac{\zeta(s)\zeta(s-k)x^s}{\zeta(2s-2k)s} ds\right| \ll x^{k+\frac{1}{2}+\epsilon}.$$

So we may immediately get the asymptotic formula:

$$\sum_{n \leq x} \sigma_k(U(n)) = \frac{\zeta(k+1) \cdot x^{k+1}}{\zeta(2) \cdot (k+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right).$$

This proves Lemma 1.

Lemma 2. For any real number $x \geq 1$, we have the estimate:

$$\sum_{\substack{p^\alpha \leq x \\ \alpha > p}} (\alpha p)^k \ll \log^{2k+2} x.$$

Proof. From $\alpha > p$, we have $p^p < p^\alpha \leq x$, so $p < \frac{\log x}{\log p} < \log x$. If $p^\alpha \leq x$, then $\alpha \leq \frac{\log x}{\log p} \leq \frac{\log x}{\log 2}$.

Thus

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}}(\alpha p)^k \ll \sum_{p\le\log x} p^k \sum_{\alpha\le\frac{\log x}{\log 2}} \alpha^k \ll \log^{k+1} x \sum_{p\le\log x} p^k.$$

Note that $\pi(x)=\dfrac{x}{\log x}+O\left(\dfrac{x}{\log^2 x}\right)$, where $\pi(x)$ denotes the number of primes not exceeding x. We may immediately obtain $\sum\limits_{p\le\log x} p^k \ll \sum\limits_{p\le\log x}\log^k x \ll \log^{k+1} x$, so we have

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}}(\alpha p)^k \ll \log^{2k+2} x.$$

This proves Lemma 2.

Lemma 3. For any real number $x\ge 1$, we have the estimate:

$$\sum_{\substack{n\le x\\ SP(n)>U(n)}} SP^k(n) \ll x\log^{2k+2} x.$$

Proof. Assume that $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, we have $U(n)=p_1p_2\cdots p_r$ and $U(n)|SP(n)$.

If $SP(n)>U(n)$, then there at least exists a prime $p_i(1\le i\le r)$, which exponent α_i satisfying $\alpha_i>p_1p_2\cdots p_r$.

Let $\alpha=\max\{\alpha_i,\ i=1,\ 2,\ \cdots,\ r\}$ and p denotes the largest prime corresponding to α. Thus, from the definition of $SP(n)$ we can easily get $SP(n)<\alpha p$, so $SP^k(n)<(\alpha p)^k$. Therefore, we have

$$\sum_{\substack{n\le x\\ SP(n)>U(n)}} SP^k(n) < \sum_{\substack{n\le x\\ SP(n)>U(n)}}(\alpha p)^k = \sum_{\substack{np^\alpha\le x\\ (n,p^\alpha)=1\\ \alpha>pU(n)}}(\alpha p)^k \ll \sum_{n\le x}\sum_{\substack{p^\alpha\le x\\ \alpha>p}}(\alpha p)^k.$$

From Lemma 2, we obtain

$$\sum_{\substack{n\le x\\ SP(n)>U(n)}} SP^k(n) \ll \sum_{n\le x}\log^{2k+2} x = x\log^{2k+2} x.$$

This proves Lemma 3.

§3. Proof of the theorem

Now we use the above lemmas to prove our Theorem. Note that $SP(n)\ge U(n)$ and $\sigma_k(n)\ll\zeta(k)\cdot n^k$ for $k>1$, so we have

$$\begin{aligned}\sum_{n\le x}SD(n)-\sum_{n\le x}\sigma_k(U(n)) &= \sum_{n\le x}(\sigma_k(SP(n))-\sigma_k(U(n))) = \sum_{\substack{n\le x\\ SP(n)>U(n)}}(\sigma_k(SP(n))-\sigma_k(U(n)))\\ &\ll \sum_{\substack{n\le x\\ SP(n)>U(n)}}\sigma_k(SP(n)) \ll \zeta(k)\sum_{\substack{n\le x\\ SP(n)>U(n)}} SP^k(n).\end{aligned}$$

From Lemma 3, we know that

$$\sum_{n\leq x} SD(n) - \sum_{n\leq x} \sigma_k(U(n)) \ll \zeta(k) x \log^{2k+2} x,$$

or

$$\sum_{n\leq x} SD(n) = \sum_{n\leq x} \sigma_k(U(n)) + O\left(x\log^{2k+2} x\right),$$

and from Lemma 1, we have

$$\sum_{n\leq x} SD(n) = \frac{\zeta(k+1)\cdot x^{k+1}}{\zeta(2)\cdot(k+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right) + O\left(x\log^{2k+2} x\right) = \frac{\zeta(k+1)\cdot x^{k+1}}{\zeta(2)\cdot(k+1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right).$$

This proves our theorem.

If $k=1$, then $\sigma(n) \ll n\log(\log n)$, so we obtain

$$\sum_{n\leq x} Ss(n) - \sum_{n\leq x} \sigma(U(n)) = \sum_{\substack{n\leq x \\ SP(n)>U(n)}} (\sigma(SP(n)) - \sigma(U(n))) \ll \sum_{\substack{n\leq x \\ SP(n)>U(n)}} \sigma(SP(n))$$

$$\ll \sum_{\substack{n\leq x \\ SP(n)>U(n)}} (SP(n)\log(\log(SP(n)))) \ll \log(\log x) \sum_{\substack{n\leq x \\ SP(n)>U(n)}} SP(n).$$

From Lemma 3, we know that

$$\sum_{n\leq x} Ss(n) - \sum_{n\leq x} \sigma(U(n)) \ll x\log^4 x \log(\log x),$$

or

$$\sum_{n\leq x} Ss(n) = \sum_{n\leq x} \sigma(U(n)) + O\left(x\log^4 x\log(\log x)\right).$$

From Lemma 1 and $k=1$, we have

$$\sum_{n\leq x} Ss(n) = \frac{1}{2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right) + O\left(x\log^4 x\log(\log x)\right) = \frac{1}{2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This completes the proof of our Corollary.

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On the Smarandache pierced chain

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Abstract If $n \geq 1$, then $c(n) = 101 \times (10^{4n-4} + 10^{4n-8} + \cdots + 10^4 + 1)$ is called as the Smarandache Pierced Chain. Its first few terms are:

$$101,\ 1010101,\ 10101010101,\ 101010101010101,\ 1010101010101010101,\ \cdots\cdots$$

In reference [2], Dr.Kashihara Kenichiro asked whether $\frac{c(n)}{101}$ is a square-free number for all $n \geq 2$? The main purpose of this paper is using the elementary method to study this problem, and prove that there are infinite positive integers n such that 9 divides $\frac{c(n)}{101}$. That is to say, $\frac{c(n)}{101}$ is not a square-free number for infinite integers $n \geq 2$.

Keywords Smarandache Pierced Chain, square-free number, sequence.

§1. Introduction and results

If $n \geq 1$, then $c(n) = 101 \times (10^{4n-4} + 10^{4n-8} + \cdots + 10^4 + 1)$ is defined as the Smarandache Pierced Chain. Its first few terms are:

$$101,\ 1010101,\ 10101010101,\ 101010101010101,\ 1010101010101010101,\ \cdots\cdots$$

In reference [1], F.Smarandache asked the question: how many primes are there in $\frac{c(n)}{101}$? Dr.Kashihara Kenichiro [2] solved this problem completely, and proved that there are no primes in the sequence $\left\{\frac{c(n)}{101}\right\}$. At the same time, Dr. Kashihara Kenichiro [2] also proposed the following problem: Is $\left\{\frac{c(n)}{101}\right\}$ a square-free for all $n \geq 2$?

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because it can help us to know more properties about the sequence $\left\{\frac{c(n)}{101}\right\}$.

The main purpose of this paper is using the elementary method to study this problem, and solved it completely. That is, we shall prove the following :

Theorem. For any positive integer n with $9 \mid n$, we have $9 \mid c(n)$.

It is clear that $(101,\ 9) = 1$, so 9 divides $\frac{c(n)}{101}$. Therefore, from our Theorem we may immediately deduce the following:

Corollary. There are infinite positive integers n such that $\frac{c(n)}{101}$ is not a square-free number.

§2. Proof of the theorem

In this section, we shall complete the proof of our Theorem. First we give the definition of the k-free number: Let $k \geq 2$ be any fixed integer. For any positive integer $n > 1$, we call n as a k-free number, if for any prime p with $p|n$, then $p^k \nmid n$. We call 2-free number as the square-free number; 3-free number as the cubic-free number. Now we prove our Theorem directly. It is clear that

$$10 \equiv 1 (\text{mod } 9).$$

From the basic properties of the congruences we know that if $a \equiv b (\text{mod } m)$, then $a^n \equiv b^n (\text{mod } m)$ for every positive integer n (see reference [3] and [4]). So we have

$$10^{4n-4} \equiv 1 (\text{mod } 9),$$
$$10^{4n-8} \equiv 1 (\text{mod } 9),$$
$$\cdots\cdots$$
$$10^{4n} \equiv 1 (\text{mod } 9).$$

Obviously

$$1 \equiv 1 (\text{mod } 9).$$

Therefore,

$$\frac{c(n)}{101} = 10^{4n-4} + 10^{4n-8} + \cdots + 10^4 + 1 \equiv n (\text{mod } 9).$$

Now for any positive integer n with $9|n$, from the above congruence we may immediately get

$$\frac{c(n)}{101} \equiv 10^{4n-4} + 10^{4n-8} + \cdots + 10^4 + 1 \equiv n \equiv 0 (\text{mod } 9).$$

From the definition of the square-free number and the above properties we know that $\frac{c(n)}{101}$ is not a square-free number if $9|n$. This completes the proof of Theorem.

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An equation involving the F.Smarandache multiplicative function $SM(n)$

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Abstract For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers. The famous F.Smarandache multiplicative function $SM(n)$ is defined as $SM(n) = \max\{\alpha_1 p_1, \alpha_2 p_2, \cdots \alpha_k p_k\}$. Euler function $\phi(n)$ denotes the number of all positive integers not exceeding n which are relatively prime to n. The main purpose of this paper is using the elementary method to study all positive integer solutions of the equation $\sum_{d|n} SM(d) = \phi(n)$, and prove that this equation has only one positive integer solution $n = 1$.

Keywords F.Smarandache multiplicative function, Euler function, elementary method.

§1. Introduction and results

For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers. In reference [1], the famous F.Smarandache multiplicative function $SM(n)$ defined as:

$$SM(n) = \max\{\alpha_1 p_1, \alpha_2 p_2, \cdots \alpha_k p_k\}.$$

For example, the first few values of $SM(n)$ are: $SM(1) = 1$, $SM(2) = 2$, $SM(3) = 3$, $SM(4) = 2$, $SM(5) = 5$, $SM(6) = 3$, $SM(7) = 7$, $SM(8) = 2$, $SM(9) = 3$, $SM(10) = 5$. About the elementary properties of $SM(n)$, many people had studied it and obtained some interesting results. For instance, Xu Zhefeng [2] studied the mean value distribution property of $SM(n)$, and proved the following conclusion:

Let $P(n)$ be the largest prime factor of n, then for any real numbers $x \geq 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (SM(n) - P(n))^2 = \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln x^2}\right),$$

where $\zeta(s)$ is Riemann zeta-function.

In reference [3], Chen Jianbin studied the solutions of an equation involving the F.Smarandache multiplicative function $SM(n)$, and proved that for any positive integer n, the equation

$$\sum_{d|n} SM(d) = n$$

has two positive integer solutions $n = 1, 28$, where $\sum_{d|n}$ denotes the summation over all positive factors of n.

The main purpose of this paper is using the elementary method to study positive integer solution of the equation

$$\sum_{d|n} SM(d) = \phi(n), \tag{1}$$

and prove the following conclusion:

Theorem. For any positive integer n, the equation (1) holds if and only if $n = 1$.

§2. Some lemmas

To complete the proof of the theorem, we need the following two simple lemmas.

Lemma 1. For any positive integer n, if $n = p_1 p^\alpha (\alpha \geq 1, p_1 < p)$, then n is not a solution of the equation (1).

Proof.

(1) If $\alpha = 1,\ p_1 = 2$, $n = 2p$ satisfied equation (1). According to the definitions of $SM(n)$ and $\phi(n)$, we have

$$\sum_{d|n} SM(d) = 3 + 2p = \phi(n) = p - 1,$$

then $p = 4$, it is a contradiction.

If $p_1 > 2$, $n = p_1 p$ satisfied equation (1). We have

$$\sum_{d|n} SM(d) = 1 + p_1 + 2p = \phi(n) = (p_1 - 1)(p - 1),$$

so

$$p_1(p - 1) = 2p_1 + 2p.$$

We can easily get $p_1|2p$, but $(p_1, 2) = 1$, so $p_1|p$, this is impossible.

(2) If $\alpha > 1,\ p_1 \geq 2$, $n = p_1 p^\alpha = n_1 p^\alpha$ satisfied equation (1). We have

$$\begin{aligned} \sum_{d|n} SM(d) &= \sum_{d|n_1} SM(d) + \sum_{1\leq i\leq \alpha} \sum_{d|n_1} SM(d \cdot p^i) \\ &= 1 + p_1 + 2(p + 2p + \cdot + \alpha p) \\ &= \phi(n) \\ &= (p_1 - 1)p^{\alpha-1}(p - 1). \end{aligned}$$

If $p_1 \neq 2$, $p|\phi(n)$, $p|2(p + 2p + \cdot + \alpha p)$, so $p|p_1 + 1$, it is impossible.

If $p_1 = 2$, $\sum_{d|n} SM(d)$ is an odd number, but $\phi(n)$ is an even number. Hence equation (1) doesn't hold.

From the above discussion we know that $n = p_1 p^\alpha (\alpha \geq 1, p_1 < p)$ is not a solution of the equation (1).

Lemma 2. For any positive odd number n, we have $\dfrac{\phi(n)}{d(n)} \geq 4$ if and only if $n \neq 1, 3, 5, 7, 9, 15, 21$.

Proof. See reference [4].

§3. Proof of the theorem

In this section, we will complete the proof of the theorem.

(I) If $n = 1$, $\sum\limits_{d|n} SM(d) = SM(1) = 1 = \phi(1)$, so $n = 1$ is a solution of the equation (1).

(II) If $n = p^\alpha$, $\alpha \geq 2$, then equation (1) doesn't hold.

In fact, if equation (1) holds, then we have

$$\sum_{d|p^\alpha} SM(d) = 1 + p + 2p + \cdots + \alpha p = \phi(n) = p^{\alpha-1}(p-1),$$

where $p|\phi(n)$, $p|\sum\limits_{d|p^\alpha} SM(d)$, so $p|1$, it is impossible.

If $\alpha = 1$, $n = p$ satisfied the equation (1), then we have

$$\sum_{d|p} SM(d) = 1 + p = \phi(n) = p - 1.$$

Obviously, $\sum\limits_{d|p} SM(d) > \phi(n)$.

Hence $n = p^\alpha$ is not a solution of the equation (1).

(III) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} p^\alpha = n_1 p^\alpha$, $((n_1, p) = 1)$ $\alpha_1 \geq 1, k \geq 2$. Let $SM(n) = \alpha p$, then

$$\sum_{d|n} SM(d) < SM(p^\alpha) d(n_1 p^\alpha) = \alpha(\alpha+1) p d(n_1).$$

$$\phi(n) = p^{\alpha-1}(p-1)\phi(n_1).$$

(A) If $\alpha = 1$, $\dfrac{\phi(n_1)}{d(n_1)} \geq \dfrac{2}{3}$ $(n_1 \neq 2, n_1 \neq 6)$, we have $\sum\limits_{d|n} SM(d) < \phi(n)$.

If $n_1 = 2$, from Lemma 1 we know that $n = 2p$ is not a solution of the equation (1).

If $n_1 = 6$, we have

$$\sum_{d|6p} SM(d) = 9 + 4p,$$

obviously, it is an odd number, but $\phi(n)$ is an even number, so $n = 6p$ is not a solution of the equation (1).

(B) If $\alpha > 1$, $SM(n) = \alpha p$.

Firstly, we can easily prove the following four special cases:

(i) If $p \neq 2$, $\dfrac{\phi(n_1)}{d(n_1)} \geq 4$, we have $\alpha(\alpha+1)pd(n_1) \leq p^{\alpha-1}(p-1)\phi(n_1)$, so $\sum\limits_{d|n} SM(d) < \phi(n)$.

(ii) If n_1 is an odd number, $p \neq 2$.

(1) If $p \geq 7$, $\alpha \geq 2$, or $p \geq 5$, $\alpha \geq 3$, we have $\alpha(\alpha+1)p \leq p^{\alpha-1}(p-1)$, so $\sum\limits_{d|n} SM(d) < \phi(n)$.

(2) If $\dfrac{\phi(n_1)}{d(n_1)} < 4$, from Lemma 1, Lemma 2 and the above discussion we can easily get that $n = 3p^\alpha,\ 5p^\alpha,\ 7p^\alpha,\ 9p^\alpha,\ 15p^\alpha,\ 21p^\alpha$ are not solutions of the equation (1).

(3) If $p = 5$, $\alpha = 2$ and $\dfrac{\phi(n_1)}{d(n_1)} < 4$, from the above discussion we know that $n = 3 \cdot 5^2,\ 3^2 \cdot 5^2,\ 7 \cdot 5^2,\ 3 \cdot 7 \cdot 5^2$ are not solutions of the equation (1).

(iii) n_1 is an even number, $p \neq 2$.

If $2^2 \mid n_1$ and $\dfrac{\phi(n_1)}{d(n_1)} \geq 1$, when $p \geq 7$, $\alpha \geq 2$, or $p \geq 5$, $\alpha \geq 3$, we have $\alpha(\alpha+1)p \leq p^{\alpha-1}(p-1)$, so $\sum\limits_{d|n} SM(d) < \phi(n)$. When $p = 5$, $\alpha = 2$, we can easily get $n = 2^2 \cdot 7 \cdot 5^2$, $n = 2^2 \cdot 3^2 \cdot 5^2$ or $n = 2^2 \cdot 3 \cdot 7 \cdot 5^2$ are not solutions of the equation (1).

If $\dfrac{\phi(n_1)}{d(n_1)} < 1$, $n_1 = 2^2 \cdot 3$, then $n = 2^2 \cdot 3 \cdot p^\alpha$.

(iv) If $p = 2$, $\alpha \geq 4$ and $\dfrac{\phi(n_1)}{d(n_1)} > 4$, we have $\sum\limits_{d|n} SM(d) < \phi(n)$.

If $\alpha = 2$ or 3, we can calculate that $n = 3 \cdot 2^2$, $n = 3 \cdot 2^3$ or $n = 5 \cdot 2^3$ are not solutions of the equation (1).

If $\dfrac{\phi(n_1)}{d(n_1)} < 4$, from (II) and Lemma 2 we can get $n = 2^\alpha$, $3 \cdot 2^\alpha$, $5 \cdot 2^\alpha$, $7 \cdot 2^\alpha$, $9 \cdot 2^\alpha$, $15 \cdot 2^\alpha$, $21 \cdot 2^\alpha$ don't satisfy the equation (1).

Now let's consider other cases:

(1) If $2 \| n_1$, $n = 2p_2^{\alpha_2} \cdots p_k^{\alpha_k} p^\alpha = 2n_1 (k \geq 2)$ satisfied the equation (1), then we have

$$\sum_{d|n} SM(d) = 2 \sum_{\substack{d|n_1 \\ d>1}} SM(d) + 3 = \phi(n) = p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1) p^{\alpha-1}(p-1).$$

In the above equation, $2 \sum\limits_{\substack{d|n_1 \\ d>1}} SM(d) + 3$ is an odd number, but $\phi(n)$ is an even number, so $n = 2p_2^{\alpha_2} \cdots p_k^{\alpha_k} p^\alpha (k \geq 2)$ is not a solution of the equation (1).

(2) If $2^2 \| n_1$, $n_1 = 2^2 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ $(k \geq 2)$.

① If $p = 3, \alpha \geq 5$, it is easy to prove that $\alpha(\alpha+1)p < p^{\alpha-1}(p-1)$, so we have $\sum\limits_{d|n} SM(d) < \phi(n)$, thus there is no solution of the equation (1).

If $\alpha = 2$, $n = 2^2 \cdot 3^2 \cdot 5$, we can easily prove that $n = 2^2 \cdot 3^2 \cdot 5$ is not a solution of the equation (1).

If $\alpha = 3$, $n = 2^2 \cdot 3^3 \cdot 5$ or $n = 2^2 \cdot 3^3 \cdot 7$, we can also easily prove that $n = 2^2 \cdot 3^3 \cdot 5$ or $n = 2^2 \cdot 3^3 \cdot 7$ are not solutions of the equation (1).

If $\alpha = 4$ and $\dfrac{\phi(n_1)}{d(n_1)} < 4$, $n = 2^2 \cdot 3^4 \cdot 5$, $n = 2^2 \cdot 3^4 \cdot 7$ or $n = 2^2 \cdot 3^4 \cdot 11$, we can prove that $n = 2^2 \cdot 3^4 \cdot 5$, $n = 2^2 \cdot 3^4 \cdot 7$ or $n = 2^2 \cdot 3^4 \cdot 11$ are not solutions of the equation (1).

② If $p \neq 3$, from (B) (iii) we know that $n = 2^2 \cdot 3 \cdot p^\alpha$. Since

$$\sum_{d|2^2 \cdot 3 \cdot p^\alpha} SM(d) = 6 \sum_{\substack{d|p^\alpha \\ d>1}} SM(d) + 17 = \phi(n) = 4p^{\alpha-1}(p-1),$$

and $6\sum_{\substack{d|p^\alpha \\ d>1}} SM(d)+17$ is an odd number, $\phi(n)$ is an even number, thus $n=2^2\cdot 3\cdot p^\alpha$ is not a solution of the equation (1).

(3) If $2^\alpha \mid n_1$ $(\alpha \geq 3)$.

① If $p=3, \alpha \geq 5$, we have $\alpha(\alpha+1)p < p^{\alpha-1}(p-1)$ and $\dfrac{\phi(n_1)}{d(n_1)} \geq 1$, then $\sum_{d|n} SM(d) < \phi(n)$, so there are no solutions of the equation (1).

If $\alpha=2$ and $\dfrac{\phi(n_1)}{d(n_1)} < 4$, it is easy to prove that $n=2^3\cdot 3^2$, $n=2^3\cdot 3^2\cdot 5$ are not solutions of the equation (1).

If $\alpha=3$ and $\dfrac{\phi(n_1)}{d(n_1)} < 4$, we can calculate that $n=2^3\cdot 3^3\cdot 5$, $2^3\cdot 3^3\cdot 7$, $2^4\cdot 3^3$, $2^4\cdot 3^3\cdot 5$ are not solutions of the equation (1).

If $\alpha=4$ and $\dfrac{\phi(n_1)}{d(n_1)} < 4$, obviously, $n=2^3\cdot 3^4\cdot 5$, $2^3\cdot 3^4\cdot 7$, $2^4\cdot 3^4$, $2^4\cdot 3^4\cdot 5$, $2^5\cdot 3^4$ are not solutions of the equation (1).

② If $p=5, \alpha=2$, when $\dfrac{\phi(n_1)}{d(n_1)} \geq 4$, from (B) (i) we know that there are no solutions of equation (1).

If $\dfrac{\phi(n_1)}{d(n_1)} < 4$, we can easily prove that $n=2^3\cdot 3\cdot 5^2$, $2^3\cdot 3^2\cdot 5^2$, $2^3\cdot 3^3\cdot 5^2$, $2^3\cdot 7\cdot 5^2$, $2^3\cdot 3\cdot 5^2\cdot 7$, $2^3\cdot 3^2\cdot 5^2\cdot 7$, $2^4\cdot 3\cdot 5^2$, $2^4\cdot 3^2\cdot 5^2$ are not solutions of the equation (1).

In a word, the equation (1) has only one positive integer solution $n=1$.

This completes the proof of the theorem.

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Characterizations of some special helices in E^4

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Abstract In this work, notion of slant helix concept in [6] is extented to the space E^4. With an analogous way, we define 3-type slant helices whose trinormal lines make a constant angle with a fixed direction in E^4. Moreover, some characetrizations of such curves and other forms (in the case of principal lines or binormal lines make a constant angle with a fixed direction) of slant helices are presented.

Keywords Euclidean space, frenet frame, slant helices.

§1. Introduction

It is safe to report that many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important roles in mechanics and kinematics as well as in differential geometry (for more details see [1]).

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal and trinormal)was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in the space E^4 [3]. And this tetrad and curvatures are called all together 'Frenet Apparatus' of the curves. And helices (inclined curves) are well known concepts in the classical differential geometry [5]. Recall that an arbitrary curve is called a W-curve, if it has constant Frenet curvatures [4].

The notion of slant helix is due to Izumiya and Takeuchi [6]. They defined that a curve $\varphi = \varphi(s)$ with non-vanishing first curvature is called a slant helix in E^3 if the principal lines of φ make a constant angle with a fixed directon. In this paper, we investigated slant helices and other cases (if the binormal lines or trinormal lines make a constant angle with a fixed direction) and gave some characterizations of mentioned curves in the space E^4.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^4 are briefly presented (A more complete elementary treatment can be found in [3]).

Let $\alpha : I \subset R \to E^4$ be an arbitrary curve in the Euclidean space E^4. Recall that the curve α is said to be of unit speed (or parametrized by arclength function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle ., . \rangle$ is the standart scalar (inner) product of E^4given by

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4,$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in E^4$. In particular, the norm of a vector $X \in E^4$ is given by

$$\|X\| = \sqrt{\langle X, X \rangle}.$$

Let $\{T(s), N(s), B(s), E(s)\}$ be the moving frame along the unit spped curve α, where T, N, B and E denote, respectively the tangent, the principal normal, the binormal and the trinormal vector fields. Then the Frenet formulas are given by [2]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix}. \tag{1}$$

The real valued functions κ, τ and σ are called, respectively, the first, the second and the third curvature of α. If $\sigma \neq 0$ for each $s \in I \subset R$, the curve lies fully in E^4. Recall that the unit sphere S^3 in E^4, centered at the origin, is the hyper surface defined by

$$S^3 = \left\{ X \in E^4 : \langle X, X \rangle = 1 \right\}.$$

In this work, we shall assume that $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ for each $s \in I \subset R$.

§3. Characterizations of Some Special Helices in $\mathbf{E}^4$

Theorem 1. There isn't any curve in E^4 such that;

i) The principal normal lines of it make a constant angle with a fixed direction.

ii) The binormal lines of it make a constant angle with a fixed direction.

Proof.

i) Let us suppose there is a curve which is defined with statement i. Thus, we can write that

$$N.U = \cos \omega, \tag{2}$$

where U is a constant vector (fixed direction) and ω is a constant angle. Differentitating both sides of (2) and considering Frenet equations, we get

$$(-\kappa T + \tau B).U = 0. \tag{3}$$

(3) implies that $T \perp U$ and $B \perp U$. Therefore we can compose U as

$$U = u_1 N + u_2 E. \tag{4}$$

Differentiating (4), we easily have

$$u_1 = \cos\omega = u_2 = 0, \tag{5}$$

which is a contradiction.

ii) Similar to above proof, let us assume that there is a curve which hold statement ii. Now, we write

$$B.U = \cos\xi, \tag{6}$$

where U is a constant vector (fixed direction) and ξ is a constant angle. Following same procedure in proof of i, we have components of U zero, respectively. This result yields a contradiction.

Definition. A curve $\psi = \psi(s)$ is called a 3-type slant helix if the trinormal lines of ψ make a constant angle with a fixed direction in E^4.

Theorem 2. Let $\psi = \psi(s)$ be a 3-type slant helix with non-vanishing curvatures in E^4.

i) There is a relation among curvatures of ψ as

$$\cos\delta.\frac{\sigma}{\tau} = B.\cos\int_0^s \kappa ds - A.\sin\int_0^s \kappa ds; \tag{7}$$

ii) Fixed direction of this helix can be written as

$$U = (A.\cos\int_0^s \kappa ds + B.\sin\int_0^s \kappa ds)T + \cos\delta.\frac{\sigma}{\tau}N + \cos\delta E, \tag{8}$$

where $\delta \neq k\frac{\pi}{2}$ is a constant angle and A and B are real numbers.

Proof.

i) From definition, we write

$$E.U = \cos\delta, \tag{9}$$

where U is a fixed direction and $\delta \neq k\frac{\pi}{2}$ is a constant angle. Differentiating both sides of (9), we have

$$-\sigma B.U = 0. \tag{10}$$

And therefore, we compose constant vector U as

$$U = u_1 T + u_2 N + u_3 E. \tag{11}$$

Differentiating both sides of (11) respect to s and considering Frenet equations, we have a system of differential equation as

$$\left\{\begin{array}{c} \frac{dm_1}{ds} - u_2\kappa = 0 \\ \frac{dm_2}{ds} + u_1\kappa = 0 \\ u_2\tau - u_3\sigma = 0 \\ \frac{dm_3}{ds} = 0 \end{array}\right\}. \tag{12}$$

Using $(12)_4$, we have

$$u_3 = \cos\delta = \text{constant}. \tag{13}$$

Here, u_3 can not be zero. If it is, then U lies fully in TN hyperplane and it follows that $u_1 = u_2 = 0$. Thereby $u_3 \neq 0$. Substituting (13) to $(12)_3$, we have

$$u_2 = \cos\delta.\frac{\sigma}{\tau}. \tag{14}$$

Using equations $(12)_1$ and $(12)_2$, we have second order differential equation as follow:

$$\frac{d}{ds}(\frac{1}{\kappa}.\frac{du_1}{ds}) + u_1\kappa = 0. \tag{15}$$

Using exchange variable $t = \int_0^s \kappa ds$ in (15), we obtain

$$\frac{d^2u_1}{dt^2} + u_1 = 0. \tag{16}$$

Solution of (16) gives us the first component

$$u_1 = A.\cos\int_0^s \kappa ds + B.\sin\int_0^s \kappa ds, \tag{17}$$

where A, B are real numbers. And we easily have the second component

$$u_2 = B.\cos\int_0^s \kappa ds - A.\sin\int_0^s \kappa ds = \cos\delta.\frac{\sigma}{\tau}. \tag{18}$$

(18) completes proof of i).

ii) Using obtained equations we write fixed direction as follow:

$$U = (A.\cos\int_0^s \kappa ds + B.\sin\int_0^s \kappa ds)T + \cos\delta.\frac{\sigma}{\tau}N + \cos\delta E. \tag{19}$$

Corollary. The third curvature of 3-type slant helix in E^4 can not be zero. Therefore, ψ never lies in TNB subspace.

Theorem 3. 3-type slant helix with non-vanishing curvatures can not be a $W-$curve in E^4.

Proof. Let us suppose 3-type slant helix with non-vanishing curvatures is a $W-$curve in E^4. In this case, if we consider solution of (12), we have

$$u_1 = u_2 = u_3 = 0, \tag{20}$$

which is a contradiction.

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On certain equations involving the Smarandache double-factorial function

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Abstract For any positive integer n, the famous Smarandache double-factorial function $Sdf(n)$ is defined as the smallest positive integer m such that $m!!$ is divisible by n, where the double factorial $m!! = 1\cdot3\cdot5\cdots m$, if m is an odd number; and $m!! = 2\cdot4\cdot6\cdots m$, if m is an even number. The main purpose of this paper is using the elementary method to prove that for each $k \geq 4$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation $Sdf\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i)$, and also for any positive integer $k \geq 5$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation $Sdf\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i)$.

Keywords Vinogradov's three-primes theorem, the Smarandache double-factorial function.

§1. Introduction and results

For any positive integer n, the famous Smarandache double-factorial function $Sdf(n)$ is defined as the smallest positive integer m such that $m!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1\cdot3\cdot5\cdots m, & \text{if } m \text{ is an odd number ;} \\ 2\cdot4\cdot6\cdots m, & \text{if } m \text{ is an even number.} \end{cases}$$

For example, the first few values of $Sdf(n)$ are:

$$Sdf(1) = 1,\ Sdf(2) = 2,\ Sdf(3) = 3,\ Sdf(4) = 4,\ Sdf(5) = 5,\ Sdf(6) = 6,$$
$$Sdf(7) = 7,\ Sdf(8) = 4, Sdf(9) = 9,\ Sdf(10) = 10,\ Sdf(11) = 11,\ Sdf(12) = 6,$$
$$Sdf(13) = 13,\ Sdf(14) = 14,\ Sdf(15) = 5,\ Sdf(16) = 6\ \cdots.$$

In references [1] and [2], F.Smarancdache asked us to study the properties of $Sdf(n)$. About this problem, some authors had studied it, and obtained some interesting results, see references [3], [4] and [5]. In reference [4], Zhu Minhui proved that for any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Wang Jianping [5] proved that for any real number $x \geq 1$ and any fixed positive integer k, we have the asymptotic formula

$$\sum_{n \leq x} (Sdf(n) - P(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $P(n)$ denotes the largest prime divisor of n, and c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

The main purpose of this paper is using the elementary method to study the positive integer solutions of two equations involving the Smarandache double-factorial function $Sdf(n)$, and obtained two interesting results. That is, we shall prove the following:

Theorem 1. For each positive integer $k \geq 4$, there exist infinite group positive integers $(m_1,\ m_2 \cdots,\ m_k)$ such that the equation

$$Sdf\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

Theorem 2. For any positive integer $k \geq 5$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that the equation

$$Sdf\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

§2. Proof of the theorems

To complete the proof of our theorems, we need the famous Vinogradov's Three Prime Theorem, which was stated as follows:

Lemma 1. There exists a sufficiently large constant $K > 0$, such that each odd integer $n > K$ can be written as a sum of three primes. That is, $n = p_1 + p_2 + p_3$, where p_i $(i = 1,\ 2,\ 3)$ are odd primes.

Proof. (See reference [8]).

Lemma 2. Let $k \geq 3$ be an odd integer, then any sufficiently large odd integer n can be written as a sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k.$$

Proof. (See reference [6]).

Now we use these two Lemmas to prove our theorems. First we prove Theorem 1. If $k \geq 4$ be an odd number, then from Lemma 2 we know that for any prime p large enough, it can be written as a sum of k primes:

$$p = p_1 + p_2 + \cdots + p_k.$$

By the definition of $Sdf(n)$, we know that $Sdf(p) = p$. This implies

$$p = Sdf(p) = Sdf(p_1 + p_2 + \cdots + p_k) = p_1 + p_2 + \cdots + p_k = Sdf(p_1) + \cdots + Sdf(p_k).$$

If $k \geq 4$ is an even number, then $k-1 \geq 3$ is an odd number. So if prime p large enough, then $p-2$ can be written as a sum of $k-1$ primes:

$$p-2 = p_1 + p_2 + \cdots + p_{k-1}.$$

This implying

$$p = 2 + p_1 + p_2 + \cdots + p_{k-1}$$

or

$$p = Sdf(p) = Sdf(2+p_1+p_2+\cdots+p_{k-1}) = 2+p_1+p_2+\cdots+p_{k-1} = Sdf(2)+Sdf(p_1)+\cdots+Sdf(p_k).$$

Since there are infinite primes p, so there exist infinite group positive integers $(m_1,\ m_2\cdots,\ m_k)$ such that the equation

$$Sdf\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

This proves Theorem 1.

Now we prove Theorem 2. If $k \geq 5$ be an odd number, then $k-2 \geq 3$ is also an odd number. For any prime p large enough, from Lemma 2 we know that p can be written as a sum of $k-2$ primes:

$$p = p_1 + p_2 + p_3 + \cdots + p_{k-2}.$$

Note that $Sdf(p^2) = 3p$, so from the above identity we have

$$\begin{aligned} Sdf(p_1{\cdot}_2\cdots p_{k-2}\cdot p\cdot p) &= Sdf(p^2) = 3p = p_1+p_2+\cdots+p_{k-2}+2p \\ &= \sum_{i=1}^{k-2} Sdf(p_i) + Sdf(p) + Sdf(p). \end{aligned}$$

Taking $m_i = p_i$, $i = 1,\ 2,\ k-2$, $m_{k-1} = m_k = p$, then the above formula implies

$$Sdf\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

If $k \geq 5$ is an even number, then $k-3 \geq 3$ is an odd number. For prime p large enough, from Lemma 2 we know that $p-4$ can be written as a sum of $k-3$ primes $p-4 = p_1+p_2+\cdots+p_{k-3}$. So we have

$$2p = 2 + 2 + p_1 + p_2 + \cdots + p_{k-3} + p.$$

This implies

$$Sdf(2\cdot 2\cdot p_1\cdot p_2\cdot\cdots\cdot p_{k-3}\cdot p) = 2p = 2+2+p_1+p_2+\cdots+p_{k-3}+p.$$

Taking $m_i = p_i$, $i = 1,\ 2,\ \cdots,\ k-3$, $m_{k-2} = m_{k-1} = 2$, $m_k = p$, from the above formula we may immediately deduce the identity

$$Sdf\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

Since there are infinite primes p, so there exist infinite group positive integers $(m_1,\ m_2\cdots,\ m_k)$ such that the equation

$$Sdf\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} Sdf(m_i).$$

This completes the proof of Theorem 2.

It is clear that our method of proving Theorem 2 is useless for $k = 4$. Whether there exist infinite group positive integers $(m_1,\ m_2,\ m_3,\ m_4)$ such that the equation

$$Sdf\left(\prod_{i=1}^{4} m_i\right) = \sum_{i=1}^{4} Sdf(m_i)$$

is an open problem. We believe that it is true.

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A note on certain Euler–Mascheroni type sequences

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Abstract Expressions of type $x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n$ ($a_n > 0$) will be called of Euler–Mascheroni type, as for $a_k \equiv k$ we obtain a sequence of approximations of the Euler–Mascheroni constant γ. The aim of this note is to solve two open problems posed by K. Kashihara [1] related to the convergence or divergence of (x_n) when $a_n = p_n$ (nth prime), and $a_n = S(n)$ (Smarandache function). An analogues result on the Smarandache ceil function is pointed out, too.

Keywords Prime numbers, estimates on primes, Smarandache function, Smarandache ceil function.

§1. Introduction

Let (a_n) be a sequence of strictly positive real numbers, and construct the new sequence (x_n) defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n \quad (n = 1, 2, \ldots). \tag{1}$$

For $a_k = k$ ($k = 1, 2, \ldots$) one obtains $x_n = \sum_{k=1}^{n} \frac{1}{k} - \log n$, which gives the well-known Euler sequence (or Euler–Mascheroni sequence), having as limit the Euler–Mascheroni constant γ (see [3]).

In his book [1] (see p. 42), K. Kashihara posed the problems of convergence or divergence of sequence (x_n) given by (1) for the particular cases $a_k = p_k$, the k-th prime; as well as $a_k = S(k)$, the Smarandache function value. We will prove the following:

Theorem. The sequence (x_n^1) given by

$$(x_n^1) = \sum_{k=1}^{n} \frac{1}{p_k} - \log p_n \tag{2}$$

is divergent, being unbounded from below. The sequence (x_n^2) given by

$$(x_n^2) = \sum_{k=1}^{n} \frac{1}{S(k)} - \log S(n) \tag{3}$$

is divergent, being unbounded from above.

§2. Proof of the theorem

An old result of P. Chebyshev (see e.g. [2]) states that

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+B+o(1), \tag{4}$$

where p denote primes. This means that

$$\left(\sum_{p\leq p_n}\frac{1}{p}-\log\log p_n\right)$$

is a convergent sequence. Remarking that

$$x_n^1=\left(\sum_{p\leq p_n}\frac{1}{p}-\log\log p_n\right)+\log\log p_n-\log p_n,$$

and by

$$\log\log p_n-\log p_n=\log\left(\frac{\log p_n}{p_n}\right),$$

since $\dfrac{\log p_n}{p_n}\to 0$ as $n\to\infty$ we get that $x_n^1\to-\infty$ as $n\to\infty$. This proves the first part of the theorem.

For the second part, put $n=m!$, then, since $S(n)=\min\{k\geq 1: n|k!\}$, we have $S(n)=m$, and

$$x_n^2=\left(1+\frac{1}{2}+\cdots+\frac{1}{m}-\log m\right)+\sum_{\substack{k<n\\ k\neq l!,\, l<n}}\frac{1}{S(k)},$$

because for $k=l!$, $l<m$ one has $S(k)=l$. Now, the last sum is greater than $\displaystyle\sum_{p<n}\frac{1}{p}$, as for primes $k=p<n$ one has $S(k)=S(p)=p$, and $p\neq l!$. It is well known that

$$\sum_{p=1}^{\infty}\frac{1}{p}=+\infty,$$

so as $m\to\infty$, clearly (x_n^2) becomes unbounded from above, since the term

$$1+\frac{1}{2}+\cdots+\frac{1}{m}-\log m$$

is bounded.

Remarks.

1) For many improvements of (4) see our monograph [2].

2) For generalized Euler-Mascheroni constants, see our paper [3].

3) The above proof shows that $S(n)$ may be replaced by any function having the properties $S(k!)=k$ and $S(p)=p$ (p prime).

4) Let $S_2(n) = \min\{m \geq 1 : n|m^2\}$ be the Smarandache ceil function of order 2. By defining

$$x_n^3 = \sum_{k=1}^{n} \frac{1}{S_2(k)} - \log S_2(n). \tag{5}$$

We can prove similarly that (x_n^3) is an unbounded (from above) sequence. Even, a more precise result holds true. Indeed, recently Wang Xiaoying [4] proved that

$$\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \log^2 x + A_1 \log x + A_2 + O(x^{-\frac{1}{4}+\varepsilon}). \tag{6}$$

Since $\sqrt{n} \leq S_2(n) \leq n$, we have

$$\log S_2(n) = O(\log n) = O(\log^2 n),$$

so by (6), it follows that

$$\frac{x_n^3}{\log^2 n} \sim \frac{3}{2\pi^2} \text{ as } n \longrightarrow \propto .$$

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U^*-inverse semigroups[1]

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Abstract The minimum cancellative monoid congruence on $\mathcal{L}^*$-inverse semigroups is given in this paper. On the basis, we investigate a subclass of $\mathcal{L}^*$-inverse semigroups, namely, U^*-inverse semigroups. It is proved that a semigroup S is an U^*-inverse semigroup if and only if for any $x \in S$ there exist a unique element $x^\circ \in H_1^*$ such that $x \leqslant x^\circ$.

Keywords $\mathcal{L}^*$-inverse semigroups, U^*-inverse semigroups, cancellative monoids.

§1. Introduction

Regular semigroups play a major role in the theory of semigroups. It is well known that inverse semigroups, left(right) inverse semigroups and orthodox semigroups etc, are the most important semigroups in the class of regular semigroups. In generalizing regular semigroups, Fountain [7] introduced abundant semigroups. As a generalization of left inverse semigroups in the range of abundant semigroup, Ren and Shum [1] studied $\mathcal{L}^*$-inverse semigroups and established the structure of $\mathcal{L}^*$-inverse semigroups. In this paper we will study the minimum cancellative monoid congruence on $\mathcal{L}^*$-inverse semigroups. On this basis, we will investigate a special class of $\mathcal{L}^*$-inverse semigroups, that is, U^*-inverse semigroups. We will prove that a semigroup S is an U^*-inverse semigroup if and only if for any $x \in S$ there exist a unique element $x^\circ \in H_1^*$ such that $x \leqslant x^\circ$.

For terminologies and notations not given in this paper, the reader is referred to the text of Howie[8].

§2. Minimum cancellative monoid congruences

In this section, we give some basic results which are related with $\mathcal{L}^*$-relation and $\mathcal{R}^*$-relation on a semigroup S.

Lemma 2.1 [7] Let S be a semigroup and $a,\ b \in S$. Then the following statements hold:

(i) $(a, b) \in \mathcal{L}^*$ if and only if for all $x,\ y \in S^1,\ ax = ay \Longleftrightarrow bx = by$.

(ii) for $e^2 = e \in S, (e, a) \in \mathcal{L}^*$ if and only if $ae = a$ and for all $x,\ y \in S^1,\ ax = ay \Longrightarrow ex = ey$.

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It is noted that the dual results of the above for $\mathcal{R}^*$ also hold and also the relation $\mathcal{H}^*$ is defined to be the intersection $\mathcal{L}^*$ and $\mathcal{R}^*$. We denote a typical idempotents of R_a^* (L_a^*) by $a^+(a^*)$. The natural partial order $\leqslant$ on an abundant semigroup S is defined by $a \leqslant b$ if and only if for some $e,\ f \in E(S),\ a = eb = bf$. Thus, we denote the set $\{f \in E \mid f \leqslant e\}$ by $\omega(e)$.

It is easy to check that $\mathcal{L}^*$ is a right congruence while $\mathcal{R}^*$ is a left congruence.

The following lemma due to Lawson is very useful description of IC abundant semigroups.

Lemma 2.2 [5] Let S be an abundant semigroup. Then the following statements are equivalent:

(i) S is IC;

(ii) For each element $a \in S$, and for some (all) $a^* \in L_a^* \bigcap E(S)$, $a^+ \in R_a^* \bigcap E(S)$, the following two conditions hold:

(a) For all $e \in \omega(a^*)$, there exist $g \in \omega(a^+)$ such that $ae = ga$;

(b) For all $f \in \omega(a^+)$, there exist $h \in \omega(a^*)$ such that $fa = ah$.

Lemma 2.3. [5] Let S be an abundant semigroup and $a, b \in S$, then the following statements are equivalent:

(i) $a \leqslant b$;

(ii) for each b^+ and b^*, there exist $a^+ \in \omega(b^+),\ a^* \in \omega(b^*)$ such that $a = a^+b = ba^*$.

Recall in [1] that an abundant semigroup S is called an $\mathcal{L}^*$-inverse semigroup if S is an IC semigroup and the set of idempotents E of S forms a left regular band, i.e. $fef = fe$ for all $e,\ f \in E$.

Now we define a relation σ on an $\mathcal{L}^*$-inverse semigroup by $(a, b) \in \sigma$ if and only if $ea = eb$, for some $e \in E(S)$.

Lemma 2.4. Let S be an $\mathcal{L}^*$-inverse semigroup. Then the following statements are equivalent:

(i) $(a, b) \in \sigma$;

(ii) $ga = hb$, for some $g,\ h \in E(S)$.

Proof. It is immediate that (i) implies (ii). To show that (ii) implies that (i), we assume that $ga = hb$ for some $g,\ h \in E(S)$. Then, we have $gh \cdot ga = gh \cdot hb$ which gives $gh \cdot a = gh \cdot b$ since $E(S)$ is a left regular band. Taking $e = gh$, we have $ea = eb$ for $e \in E(S)$ and so $(a,\ b) \in \sigma$.

A congruence ρ on a semigroup S is called a cancellative congruence if S/ρ is a cancellative semigroup. It is easy to see that every semigroup S has the minimum cancellative congruence, denoted by σ.

Theorem 2.5. Let S be an $\mathcal{L}^*$-inverse semigroup. Then σ is the minimum cancellative monoid congruence on S.

Proof. It is easy to see that the relation σ is reflexive and symmetric. To show that σ is transitive, we assume that $(a,\ b),\ (b,\ c) \in \sigma$ for any $a,\ b,\ c \in S$. Then there exist $e,\ f \in E(S)$ such that $ea = eb$ and $fb = fc$. Clearly, $fea = feb$ and $fefb = fefc$ which gives $feb = fec$ since $E(S)$ is a left regular band. This leads to $(a,\ c) \in \sigma$ and so σ is equivalence on S.

To show that σ is a congruence on S, we suppose that $(a,\ b) \in \sigma$ for any $a,\ b \in S$. Then there exist $e \in E(S)$ such that $ea = eb$ and so $cea = ceb$ for any $c \in S$. Since $E(S)$ is a left regular band, it follows that $c^*e \cdot c^* = c^*e$ and $c^* \cdot c^*e = c^*e$ which gives $c^*e \in \omega(c^*)$. By Lemma 2.2, there exist an idempotent $f \in \omega(c^+)$ such that $ce = cc^*e = fc$. Hence,

$fca = cea = ceb = fcb$ and so $(ca,\ cb) \in \sigma$. This shows that σ is a left congruence on S. It is trivial to see that σ is a right congruence and so σ is a congruence on S.

Now we prove that σ is a cancellative congruence on S. To do this, let $a,\ b\ , c \in S$ such that $(ac)\sigma = (bc)\sigma$. Then for some $e \in E(S),\ eac = ebc$. Since $c\mathcal{R}^*c^+$, we have $eac^+ = ebc^+$. Clearly, $eac^+ = eaa^*c^+ = eaa^*c^+a^*$. Putting $f = a^*c^+a^*$, we have $f \in \omega(a^*)$. It follows by Lemma 2.2 that $eac^+ = ega$ for some $g \in \omega(a^+) \subseteq E(S)$. Similarly, $ebc^+ = ehb$ for some $h \in E$. Thus, by Lemma 2.4, we have that $a\sigma = b\sigma$ and so σ is a right cancellative congruence on S. To show that σ is a left cancellative congruence on S, we suppose that $a,\ b,\ c \in S$ with $(ab)\sigma = (ac)\sigma$. Then for some $e \in E(S),\ eab = eac$. Since S is abundant, we have $(ea)^* \cdot b = (ea)^* \cdot c$ and $b\sigma = c\sigma$. Hence, σ is a cancellative congruence on S.

Because $E(S)$ is left regular band, we have $fe \cdot e = fe \cdot f$ for all $e,\ f \in E$ and so $(e,\ f) \in \sigma$. Hence, S/σ have unique idempotent.

In fact, σ is the minimum cancellative monoid congruence on S. To do this, we assume that ρ is a cancellative monoid congruence on S and $(a,\ b) \in \sigma$. Then there exist idempotent $e \in E(S)$ such that $ea = eb$. Clearly, $(ea)\rho = (eb)\rho$, that is, $e\rho a\rho = e\rho b\rho$. Hence, $a\rho = b\rho$ since ρ is a cancellative congruence on S. This shows that $(a,\ b) \in \rho$ and so $\sigma \subseteq \rho$.

Consequently, σ is the minimum cancellative monoid congruence on S.

§3. U^*-inverse semigroups

Suppose that S is an abundant semigroup with identity 1. We always denote the $\mathcal{H}^*$-class of S containing 1 by H_1^*. And also, for any $a \in S$, write $P(a) = \{b \in S | a = ebf,\ e\mathcal{R}^*a$ and $f\mathcal{L}^*a$ for $e,\ f \in E(S)\}$ and $U^*(a) = P(a) \cap H_1^*$.

Now we give the following:

Definition 3.1 An $\mathcal{L}^*$-inverse semigroup S^1 with the identity 1 is said to be an U^*-inverse semigroup if $|\ U^*(x)\ | = 1$ for all $x \in S$.

For $\mathcal{L}^*$-inverse semigroups, we have the following observations.

Lemma 3.2. If S is an $\mathcal{L}^*$-inverse semigroup with the identity 1 and $a,\ x$ are elements of S such that $x \in H_1^*$ and $a = exf$ for some $e, f \in E(S)$, then $x \in U^*(a)$.

Proof. Clearly, $(x, 1) \in \mathcal{H}^*$ and $f \in \omega(1)$. It follows by Lemma 2.2 that there exists an idempotent $g \in \omega(x^+)$ such that $a = egx$. Since $\mathcal{R}^*$ is a left congruence on S and $x\mathcal{R}^*x^+$, we immediately obtain $a = egx\mathcal{R}^*egx^+ = eg$ which implies that $a^+ = eg$ since every $\mathcal{R}^*$-class of S contains a unique idempotent. Hence, $a = a^+xa^*$. This shows that $x \in P(a)$ and so $x \in U^*(a)$.

Corollary 3.3. Let S be an $\mathcal{L}^*$-inverse semigroup with the identity 1. Then $P(a) = \{b \in S | a = a^+bf,\ f\mathcal{L}^*a$ and$f \in E(S)\}$, for all $a \in S$.

Theorem 3.4. A semigroup S is an U^*-inverse semigroup if and only if for any $x \in S$, there exist a unique element $x^\circ \in H_1^*$ such that $x \leqslant x^\circ$.

Proof. Assume that S is an U^*-inverse semigroup and $x \in S$. Then there exists a unique element $x^\circ \in U^*(x)$ such that $x = x^+x^\circ f$, where $x^+\mathcal{R}^*x\mathcal{L}^*f$ for some $f \in E(S)$. By Lemma 2.2, there exist idempotents $g,\ h \in E(S)$ such that

$$x = x^+x^\circ f = x^+hx^\circ = x^\circ gf.$$

This implies that $x \leq x^\circ$.

To prove the converse part, we suppose that for every $x \in S$ there is a unique element $x^\circ \in H_1^*$ such that $x \leq x^\circ$. By Lemma 2.3, there exist $e \in \omega((x^\circ)^+)$ such that $x = ex^\circ$. Since S is an IC abundant semigroup, by Lemma 2.2, we have that

$$x = ex^\circ = eex^\circ = ex^\circ f \qquad (f \in \omega((x^\circ)^*)).$$

It follows from Lemma 3.2 that $x^\circ \in U^*(x)$. This leads to $\mid U^*(x) \mid = 1$ and so S is an U^*-inverse semigroup.

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On the value distribution of the Smarandache multiplicative function[1]

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Abstract The main purpose of this paper is using the elementary method to study the value distribution property of the Smarandache multiplicative function, and give an interesting asymptotic formula for it.

Keywords Smarandache multiplicative function, value distribution, asymptotic formula.

§1. Introduction

Let n and m are two positive integers with $(n,\ m) = 1$, the famous Smarandache multiplicative function $f(n)$ is defined as following:

$$f(nm) = \max\{f(n),\ f(m)\}.$$

It is easy to know that Smarandache multiplicative function is not a multiplicative function, in fact, for two different primes p and q,

$$f(p^\alpha q^\beta) \neq f(p^\alpha)f(q^\beta).$$

About this function and many other Smarandache type functions, many scholars had studied them properties, see [1], [2], [3] and [4]. For example, professor Henry Bottomley [5] had considered eleven particular families of interrelated multiplicative functions, many of which are listed in the Smarandache's problem. Tabirca [6] proved an interesting properties about the Smarandache multiplicative function: If $f(n)$ be the Smarandache multiplicative function, then

$$g(n) = \min\{f(d) :\ d|n,\ d \in N\}$$

is the Smarandache multiplicative function too.

For any fixed positive integer n, let $p(n)$ denotes the greatest prime divisor of n, $S(n) = \min\{m :\ m \in N,\ n|m!\}$ be the Smarandache function. From this, we know that $p(n)$ and $S(n)$ are the Smarandache multiplicative functions. Dr.Z.F.Xu [7] deduced that for any real number $x > 0$,

$$\sum_{n\leq x}(S(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

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where $\zeta(s)$ is the Riemann zeta-function.

Now, for any prime p and positive integer α, we define $f(p^\alpha) = p^{\frac{1}{\alpha}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorizations of n, then from the definition of $f(p^\alpha)$ we have

$$f(n) = \max_{1\le i\le r}\{f(p_i^{\alpha_i})\} = \max_{1\le i\le r}\left\{p_i^{\frac{1}{\alpha_i}}\right\}.$$

It is clear that $f(n) \le p(n)$. In this paper, we shall use the elementary method to study the value distribution property of $f(n)$ in the following form:

$$\sum_{n\le x}\left(f(n) - p(n)\right)^2,$$

where $x \ge 1$ be a real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. For any real number $x \ge 3$, we have the asymptotic formula:

$$\sum_{n\le x}(f(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Note. For any prime p and positive integer α, we define $f(p^\alpha) = p^{\frac{1}{\alpha+1}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorizations of n, then from the definition of $f(p^\alpha)$ we have

$$f(n) = \max_{1\le i\le r}\{f(p_i^{\alpha_i})\} = \max_{1\le i\le r}\left\{p_i^{\frac{1}{\alpha_i+1}}\right\}.$$

Let x be a positive real number, whether there exists an asymptotic formula for

$$\sum_{n\le x}\left(f(n) - \sqrt{p(n)}\right)^2$$

is an unsolved problem.

§2. Some lemmas

To complete the proof of the theorem, we need one simple Lemma.

Lemma. Let p be a prime and $\alpha > 0$ be an integer, then for any fixed positive integer m, we have the asymptotic formula:

$$\sum_{2\le p\le x^{\frac{1}{m}}} p^\alpha = \frac{m}{\alpha+1}\cdot\frac{x^{\frac{\alpha+1}{m}}}{\ln x} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x}\right).$$

Proof. Let $\pi(x)$ denotes the number of the primes up to x. Noting that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's identity [8] we have

$$\begin{aligned}\sum_{2\le p\le x^{\frac{1}{m}}} p^{\alpha} &= \pi(x^{\frac{1}{m}})\left(x^{\frac{1}{m}}\right)^{\alpha} - \alpha\int_2^{x^{\frac{1}{m}}} \pi(t)t^{\alpha-1} \\ &= \frac{x^{\frac{\alpha+1}{m}}}{\ln x^{\frac{1}{m}}} - \frac{\alpha}{\alpha+1}\frac{x^{\frac{\alpha+1}{m}}}{\ln x^{\frac{1}{m}}} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x^{\frac{1}{m}}}\right) = \frac{m}{\alpha+1}\cdot\frac{x^{\frac{\alpha+1}{m}}}{\ln x} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x^{\frac{1}{m}}}\right).\end{aligned}$$

This proves Lemma.

§3. Proof of the theorem

Now we completes the proof of our Theorem. For any fixed positive integer n, let $p(n)$ denotes the greatest prime divisor of n, we shall debate this problem in following three cases:

(I) If $n = n_1p(n)$ with $(n_1,\ p(n)) = 1$ by the definition of $f(n)$, then

$$f(n) = \max\{f(n_1),\ f(p(n))\} = p(n),$$

so $f(n) - p(n) = 0$ in this case.

(II) If $n = n_1p^2(n)$ with $(n_1,\ p(n)) = 1$, here $p(n) \le n^{\frac{1}{2}}$, hence

$$\begin{aligned}\sum_{\substack{n\le x\\ n=n_1p^2(n)}} (f(n)-p(n))^2 &= \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} (f^2(n_1p^2) - 2pf(n_1p^2) + p^2) \\ = \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f^2(n_1p^2) - 2\sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f(n_1p^2)p &+ \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} p^2 \equiv I_1 - 2I_2 + I_3. \qquad (1)\end{aligned}$$

Let p_1 be the greatest prime divisor of n_1, if $n_1 = n_2p_1^{\alpha}$ with $\alpha \ge 2$, then $f(n) = \sqrt{p}$; otherwise $n_1 = n_2p_1$ with $(n_2, p_1) = 1$, so

$$\begin{aligned}I_1 &= \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f^2(n_1p^2) = \sum_{\substack{n_2p_1p^2\le x\\ p_1<p}} f^2(n_2p_1p^2) + \sum_{\substack{n_2p_1^{\alpha}p^2\le x\\ p_1<p}} p \\ &= \sum_{\substack{n_2p_1p^2\le x\\ p_1>\sqrt{p}}} p_1^2 + \sum_{\substack{n_2p_1p^2\le x\\ p_1\le\sqrt{p}}} p + \sum_{n_2p_1^{\alpha}p^2\le x} p. \qquad (2)\end{aligned}$$

By using Lemma, we can deduce that

$$\begin{aligned}\sum_{\substack{n_2p_1p^2\le x\\ p_1>\sqrt{p}}} p_1^2 &\le \sum_{n_2p_1\le x^{\frac{1}{3}}} p_1^2 \sum_{p\le\sqrt{\frac{x}{n_2p_1}}} 1 \ll \frac{x^{\frac{1}{2}}}{\ln x}\sum_{n_2p_1\le x^{\frac{1}{3}}} p_1^{\frac{3}{2}}n_1^{-\frac{1}{2}} \\ &\ll \frac{x^{\frac{1}{2}}}{\ln x}\sum_{p_1\le x^{\frac{1}{3}}} p_1^{\frac{3}{2}} \sum_{n_2\le x^{\frac{1}{3}}/p_1} n^{-\frac{1}{2}} \ll \frac{x^{\frac{4}{3}}}{\ln^2 x}, \qquad (3)\end{aligned}$$

Similarly, we have

$$\sum_{\substack{n_2p_1p^2\le x\\ p_1<\sqrt{p}}} p \le \sum_{n_2p_1\le x^{\frac{1}{3}}}\ \sum_{p\le\sqrt{\frac{x}{n_2p_1}}} p \ll \frac{x}{\ln^2 x}, \qquad (4)$$

$$\sum_{n_2 p_1^{\alpha} p^2 \le x} p \ll \frac{x}{\ln^2 x}. \tag{5}$$

From (2), (3), (4) and (5), we may immediately obtain

$$I_1 \ll \frac{x^{\frac{4}{3}}}{\ln^2 x}. \tag{6}$$

According to the estimate method of I_1, we can also get

$$\begin{aligned} I_2 &= \sum_{\substack{n_1 p^2 \le x \\ p(n_1)<p}} f(n_1 p^2) p \\ &= \sum_{\substack{n_2 p_1 p^2 \le x \\ p_1 > \sqrt{p}}} p_1 p + \sum_{\substack{n_2 p_1 p^2 \le x \\ p_1 \le \sqrt{p}}} p^{\frac{3}{2}} + \sum_{n_2 p_1^{\alpha} p^2 \le x} p^{\frac{3}{2}} \\ &\ll \frac{x^{\frac{7}{6}} \ln \ln x}{\ln x}. \end{aligned} \tag{7}$$

Now, we will calculate I_3. By using Lemma and note that $p(n) \le n^{\frac{1}{2}}$, we have

$$\begin{aligned} I_3 &= \sum_{\substack{n_1 p^2 \le x \\ p(n_1)<p}} p^2 \\ &= \sum_{n_1 \le x^{\frac{1}{3}}} \sum_{p^2 \le x/n_1} p^2 \\ &= \sum_{n_1 \le x^{\frac{1}{3}}} \left(\frac{2x^{\frac{3}{2}}}{3n_1^{\frac{3}{2}}(\ln x - \ln m)} + O\left(\frac{x^{\frac{3}{2}}}{n_1^{\frac{3}{2}} \ln^2 \sqrt{\frac{x}{n_1}}} \right) \right) \\ &= \frac{2x^{\frac{3}{2}}}{3 \ln x} \sum_{n_1 \le e^{\sqrt{\ln x}}} \frac{1}{n_1^{\frac{3}{2}}} + O\left(\sum_{e^{\sqrt{\ln x}} < n_1 \le x^{\frac{1}{3}}} \frac{x^{\frac{3}{2}}}{n_1^{\frac{3}{2}} \ln \frac{x}{n_1}} \right) + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right) \\ &= \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right), \end{aligned} \tag{8}$$

where $\zeta(s)$ is the Riemann zeta-function.

From (1), (6), (7) and (8), we may immediately deduce the case (II)

$$\sum_{n \le x} (f(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right).$$

(III) If $n = n_1 p^{\alpha}(n)$ with $\alpha \ge 3$, note that

$$\begin{aligned} \sum_{\substack{n_1 p^{\alpha} \le x \\ p(n_1)<p}} p^2 &= \sum_{n_1 \le x} \sum_{p \le \left(\frac{x}{n_1}\right)^{\frac{1}{\alpha}}} p^2 \\ &\ll \frac{x^{\frac{3}{\alpha}}}{\ln x} \sum_{n_1 \le x} \left(\frac{1}{n} \right)^{-\frac{3}{\alpha}} \ll \frac{x^{\frac{3}{2}}}{\ln^2 x}, \end{aligned}$$

so in this case

$$\sum_{n\leq x}(f(n)-p(n))^2 \ll \frac{x^{\frac{3}{2}}}{\ln^2 x}.$$

Combining three cases above, for any real number $x \geq 3$, we have the asymptotic formula:

$$\sum_{n\leq x}(f(n)-p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

This completes the proof of Theorem.

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A problem related to twin primes

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Abstract Let p_1 and p_2 are two primes with $p_1 < p_2$ and $p_2 - p_1 = 2$, we call such pairs of primes are twin primes. About the elementary properties of twin primes, some authors had studied it, and obtained some interesting results. In reference [1], F. Smarandache asked us to prove that p and $p+2$ are primes if and only if

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2}$$

is an integer. This result is called Smarandache Criterion for twin primes and has been proved in [6], [7] and [8]. In this paper, we use the elementary methods to study this problem, and prove that it is true.

Keywords Twin primes, pseudo-twin primes, congruence.

§1. Introduction and results

Let p_1 and p_2 are primes, if $p_1 < p_2$ and $p_2 - p_1 = 2$, we call such pairs of primes are twin primes. For example, 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, $\cdots$, are all twin primes. About the elementary properties of twin primes, some authors had studied it, and obtained some interesting results, see reference [2], [3], [4] and [5].

Let p be a positive integer, in reference [9], Kenichiro Kashihara called that p and $p+2$ are pseudo-twin primes if and only if

$$\frac{(p-1)!+1}{p}+\frac{(p+1)!+1}{p+2} \quad \text{is an integer.}$$

Simultaneously, Florentin Smarandache also proposed the following two problems:

Problem 1. Let p be positive integer, prove p and $p+2$ are twin primes if and only if

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2} \quad \text{is an integer.}$$

Problem 2. Are there pseudo-twin primes that are not classic twin primes?

About these two problems, it seems that none had studied them, at least we have not seen related paper before. The main purpose of this paper is using the elementary methods to study these two problems, and solved them completely. That is, we shall prove the following:

Theorem 1. let p be a positive integer, then p and $p+2$ are twin primes if and only if

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2} \quad \text{is an integer.}$$

Theorem 2. Pseudo-twin primes must be classic twin primes except $p = 1$, $p + 2 = 3$.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems directly. First we prove that if p and $p + 2$ are twin primes, then

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2}$$

is an integer. In fact from the Wilson's Theorem we know that for any prime p,

$$(p-1)! \equiv -1(\mathrm{mod}p).$$

So

$$p \mid (p-1)!+1.$$

Therefore

$$\frac{(p-1)!+1}{p} \quad \text{is an integer.} \tag{1}$$

Since $p + 2$ be a prime, we also have $(p + 1)! + 1 \equiv 0(\mathrm{mod}p + 2)$, so

$$(p-1)! \cdot p \cdot (p+1)+1 \equiv 0(\mathrm{mod}p+2)$$

or

$$2(p-1)!+1 \equiv 0(\mathrm{mod}p+2).$$

That is to say,

$$\frac{2(p-1)!+1}{p+2} \quad \text{is an integer.} \tag{2}$$

Note that

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2}=\frac{(p-1)!+1}{p}+\frac{2(p-1)!+1}{p+2},$$

From (1) and (2), we know that

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2} \quad \text{is an integer.}$$

Now we prove that if

$$(p-1)!\left\{\frac{1}{p}+\frac{2}{p+2}\right\}+\frac{1}{p}+\frac{1}{p+2} \tag{3}$$

is an integer, then p and $p + 2$ must be primes.

In fact if this conclusion is not true, then there must be three cases:

(a) p and $p + 2$ both are not primes;

(b) p is a prime, $p + 2$ is not a prime;

(c) p is not a prime, $p+2$ is a prime.

If (a) is true, then there at least exist two pair integers a and b, c and d with $p = a \cdot b$, $p+2 = c \cdot d$. Obviously, $a < p$, $b < p$, $c < p+2$, $d < p+2$. If $p = 4$ and $p+2 = 6$, then (3) is not an integer. So we can assume that $p > 4$, this time $a|(p-1)!$ and $b|(p-1)!$, so $p = ab|(p-1)!$ (if $a = b$, then $2a|(p-1)!$, so we also have $p|(p-1)!$). Therefore,

$$\frac{(p-1)!}{p} \text{ and } \frac{2(p+1)!}{p+2}$$

both are integers. But

$$\frac{1}{p} + \frac{1}{p+2} \text{ is not integer.}$$

So

$$(p-1)!\left\{\frac{1}{p} + \frac{2}{p+2}\right\} + \frac{1}{p} + \frac{1}{p+2} \text{ is not an integer.}$$

If (b) is true, then

$$\frac{(p-1)!+1}{p} \text{ is an integer,}$$

and

$$\frac{2(p+1)!}{p+2} \text{ is an integer,}$$

but

$$\frac{1}{p+2} \text{ is not an integer.}$$

So

$$(p-1)!\left\{\frac{1}{p} + \frac{2}{p+2}\right\} + \frac{1}{p} + \frac{1}{p+2} = \frac{(p-1)!+1}{p} + \frac{2(p+1)!}{p+2} + \frac{1}{p+2} \text{ is not integer.}$$

If (c) is true, then

$$\frac{(p-1)!}{p} \text{ is an integer,}$$

and

$$\frac{2(p+1)!+1}{p+2} \text{ is an integer,}$$

but

$$\frac{1}{p} \text{ is not an integer.}$$

So

$$(p-1)!\left\{\frac{1}{p} + \frac{2}{p+2}\right\} + \frac{1}{p} + \frac{1}{p+2} = \frac{(p-1)!}{p} + \frac{2(p+1)!+1}{p+2} + \frac{1}{p} \text{ is not integer.}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. Note that the identity

$$\begin{aligned} &\frac{(p-1)!+1}{p} + \frac{(p+1)!+1}{p+2} = \frac{(p-1)!}{p} + \frac{(p+1)!+1}{p+2} + \frac{1}{p} \\ = \quad &\frac{(p-1)!+1}{p} + \frac{(p+1)!}{p+2} + \frac{1}{p+2} = \frac{(p-1)!}{p} + \frac{(p+1)!}{p+2} + \frac{1}{p} + \frac{1}{p+2}. \end{aligned}$$

$\frac{1}{p+2}$ is not an integer.

If $p+2$ is a prime and $p>1$ is not a prime, then in the formula

$$\frac{(p-1)!}{p}+\frac{(p+1)!+1}{p+2}+\frac{1}{p},$$

$\frac{1}{p}$ is not an integer.

If p and $p+2$ both are not primes with $p>1$, then in the formula

$$\frac{(p-1)!}{p}+\frac{(p+1)!}{p+2}+\frac{1}{p}+\frac{1}{p+2},$$

$\frac{1}{p}+\frac{1}{p+2}$ is not an integer.

This completes the proof of Theorem 2.

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An equation involving the F.Smarandache function and its positive integer solutions

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. The main purpose of this paper is using the elementary methods to study the positive integer solutions of an equation involving the F.Smarandache function, and give its all solutions.

Keywords F.Smarandache function, positive integer solution, elementary method.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache function $S(n)$ defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained some interesting results. For example, Lu Yaming [2] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Dr. Xu Zhefeng [3] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

In an unpublished paper, Dr.Kenichiro Kashihara asked us to solve the congruence equation

$$S^3(x) - 3S(x) - 1 \equiv 0 (\text{mod } x). \tag{1}$$

Professor Zhang Wenpeng asked us to find all positive integer solutions of the equation

$$S^2(x) - 5S(x) + p = x, \tag{2}$$

where p be a prime.

About the problem (1), it is easy to find its all positive integer solutions. But for the problem (2), it is more complicate. The main purpose of this paper is using the elementary methods to study these two problems, and solved them completely. That is, we shall prove the following conclusions:

Theorem 1. The congruence equation (1) has only one positive integer solution $x = 1$.

Theorem 2. Let p be a fixed prime. If $p = 2$, then the equation (2) has no positive integer solution; If $p = 3$, then the equation (2) has only one positive integer solution $x = 9$; If $p = 5$, then the equation (2) has only two positive integer solutions $x = 1,\ 5$; If $p = 7$, then the equation (2) has only two positive integer solutions $x = 21,\ 483$. If $p \geq 11$, then the equation (2) has only one positive integer solution $x = p(p-4)$.

§2. Proof of the theorems

In this section, we shall complete the proof of our theorems directly. First we prove Theorem 1. It is clear that $x = 1$ satisfying the congruence equation (1). Now we prove that for any positive integer $x > 1$, the congruence (1) does not hold. In fact if $x > 1$ satisfying the congruence (1), let $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of x into prime powers, then from the properties of $S(x)$ we know that

$$S(x) = \max\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_s^{\alpha_s})\} \equiv S(p^{\alpha}) \ \text{ and } \ p \mid S(p^{\alpha}). \tag{3}$$

Note that $S^3(x) - 3S(x) - 1 = mx$, $p \mid S(x)$, $p \mid x$, from (3) we may immediately deduce that $p \mid 1$, contradiction with $p > 1$. So the congruence equation (1) has and only has one positive integer solution $x = 1$. This proves Theorem 1.

Now we prove Theorem 2. If $p = 2$, it is clear that $x = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ are not solutions of the equation (2). Let $x \geq 8$ satisfying the equation (2), $S(x) = S(p_1^{\alpha})$, then from $p_1 \mid x$, $p_1 \mid S(x)$ and $S^2(x) - 5S(x) + 2 = x$ we can deduce that $p_1 \mid 2$. So $p_1 = 2$. Let $x = 2^{\alpha} \cdot y$ and $S(2^{\alpha}) = 2m$ $(m \leq \alpha)$, then

$$4m^2 - 10m + 2 = 2^{\alpha} \cdot y. \tag{4}$$

It is easy to check that $\alpha = 1,\ 2,\ 3,\ 4,\ 5$ do not satisfy the equation (4). If $\alpha \geq 5$, then note that $m \leq \alpha - 1$, we have $x = 2^{\alpha} \cdot y \geq 2^{\alpha} > 4(\alpha-1)^2 - 10(\alpha-1) + 2 \geq 4m^2 - 10m + 2$. So if $p = 2$, then the equation (2) has no positive integer solution.

If $p = 3$, then $x = 1,\ 2,\ 3$ does not satisfy the equation (2). Let $x \geq 4$ satisfying the equation (2), $S(x) = S(p_1^{\alpha})$, then from $p_1 \mid x$, $p_1 \mid S(x)$ and $S^2(x) - 5S(x) + 3 = x$, we have $p_1 \mid 3$ and $p_1 = 3$. Let $x = 3^{\alpha} \cdot y$ and $S(3^{\alpha}) = 3m$, then

$$9m^2 - 15m + 3 = 3^{\alpha} \cdot y. \tag{5}$$

It is easy to check that $\alpha = 1,\ 3,\ 4,\ 5$ do not satisfy the equation (5), and $\alpha = 2$ satisfy the equation (5) with $m = 2$ and $y = 1$. If $\alpha \geq 6$, then note that $m \leq \alpha - 1$, we have

$x = 3^\alpha \cdot y \geq 3^\alpha > 9(\alpha-1)^2 - 15(\alpha-1) + 3 \geq 9m^2 - 15m + 3$. So if $p = 3$, then the equation (2) has only one positive integer solution $x = 9$.

Similarly, if $p = 5$, then we can prove that the equation (2) has only two positive integer solutions $x = 1$ and $x = 5$.

If $p = 7$, then $x = 1,\ 2$ does not satisfy the equation (2). Let $x \geq 3$ satisfying the equation (2), $S(x) = S(p_1^\alpha)$, then from $p_1 \mid x$, $p_1 \mid S(x)$ and $S^2(x) - 5S(x) + 7 = x$, we have $p_1 \mid 7$ and $p_1 = 7$. Let $x = 7^\alpha \cdot y$ and $S(7^\alpha) = 7m$, then

$$7^2m^2 - 35m + 7 = 7^\alpha \cdot y. \tag{6}$$

If $\alpha = 1$, then $m = 1$ and $y = 7 - 4 = 3$. So $x = 21$ is a positive integer solution of the equation (2). If $\alpha = 2$, then $m = 2$ and $4p - 9 = py$. So $p \mid 9$, contradiction with $p \geq 7$. If $\alpha = 3$, then $m = 3$ and $63 - 14 = 7^2y$. So $y = 1$. This time, $x = 7^3$ is another positive integer solution of the equation (2). If $\alpha \geq 4$, then note that $m \leq \alpha - 1$, we have $x = 7^\alpha \cdot y \geq 7^\alpha > 49(\alpha-1)^2 - 35(\alpha-1) + 7 \geq 49m^2 - 35m + 7$. So if $p = 7$, then the equation (2) only has two positive integer solutions $x = 21$ and $x = 483$.

If $p \geq 11$, then $x = 1,\ 2$ does not satisfy the equation (2). Let $x \geq 3$ satisfying the equation (2), $S(x) = S(p_1^\alpha)$, then from $p_1 \mid x$, $p_1 \mid S(x)$ and $S^2(x) - 5S(x) + p = x$, we have $p_1 \mid p$ and $p_1 = p$. Let $x = p^\alpha \cdot y$ and $S(p^\alpha) = pm$, then

$$p^2m^2 - 5pm + p = p^\alpha \cdot y. \tag{7}$$

If $\alpha = 1$, then $m = 1$ and $y = p - 4$. So $x = p(p-4)$ is a positive integer solution of the equation (2). It is clear that $\alpha = 2,\ 3$ do not satisfy the equation (7). If $\alpha \geq 4$, then note that $m \leq \alpha - 1$, we have $x = p^\alpha \cdot y \geq p^\alpha > p^2(\alpha-1)^2 - 5p(\alpha-1) + p \geq p^2m^2 - 5pm + p$. So if $p \geq 11$, then the equation (2) only has one positive integer solution $x = p(p-4)$. This completes the proof of Theorem.

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The mean value of a new arithmetical function

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Abstract The main purpose of this paper is using the elementary and the analytic methods to study the mean value properties of a Smarandache multiplicative function, and give two sharper asymptotic formulae for it.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, we call an arithmetical function $f(n)$ as the Smarandache multiplicative function if for any positive integers m and n with $(m,\ n) = 1$, we have $f(mn) = \max\{f(m),\ f(n)\}$. For example, the Smarandache function $S(n)$ and the Smarandache LCM function $SL(n)$ both are Smarandache multiplicative functions. Now we define a new Smarandache multiplicative function $f(n)$ as follows: $f(1) = 1$; If $n > 1$, then $f(n) = \max\limits_{1\le i\le k}\left\{\dfrac{1}{\alpha_i+1}\right\}$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into prime powers. The first few values of $f(n)$ are $f(1) = 1$, $f(2) = \dfrac{1}{2}$, $f(3) = \dfrac{1}{2}$, $f(4) = \dfrac{1}{3}$, $f(5) = \dfrac{1}{2}$, $f(6) = \dfrac{1}{2}$, $f(7) = \dfrac{1}{2}$, $f(8) = \dfrac{1}{4}$, $f(9) = \dfrac{1}{3}$, $f(10) = \dfrac{1}{2}$, $f(11) = \dfrac{1}{2}$, $\cdots\cdots$.

Generally, for any prime p and positive integer α, we have $f(p^\alpha) = \dfrac{1}{1+\alpha}$. About the elementary properties of $f(n)$, it seems that none had studied it before. This function is interesting, because its value only depend on the power of primes. The main purpose of this paper is using the elementary and the analytic methods to study the mean value properties of $f(n)$, and give two sharper asymptotic formulas for it. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} f(n) = \frac{1}{2}x\ln\ln x + c\cdot x + O\left(\frac{x}{\ln x}\right),$$

where c is a computable constant.

Theorem 2. For any real number $x > 1$, we also have the asymptotic formula

$$\sum_{n\le x}\left(f(n)-\frac{1}{2}\right)^2 = \frac{1}{36}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}\cdot\ln\ln x + d\cdot\sqrt{x} + O\left(x^{\frac{1}{3}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and d is a computable constant.

§2. Proof of the theorems

In this section, we shall using the elementary and the analytic methods to prove our Theorems. First we give following two simple lemmas:

Lemma 1. Let A denotes the set of all square-full numbers. Then we have the asymptotic formula

$$\sum_{\substack{n\le x\\ n\in A}} 1=\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot x^{\frac{1}{2}}+\frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\cdot x^{\frac{1}{3}}+O\left(x^{\frac{1}{6}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2. Let B denotes the set of all cubic-full numbers. Then we have

$$\sum_{\substack{n\le x\\ n\in B}} 1=N\cdot x^{\frac{1}{3}}+O\left(x^{\frac{1}{4}}\right),$$

where N is a computable constant.

Proof. The proof of these two Lemmas can be found in reference [3].

Now we use these two simple Lemmas to complete the proof of our Theorems. In fact, for any positive integer $n>1$, we can write it as $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then from the definition of $f(n)$, we have

$$\sum_{n\le x} f(n) \quad = \quad \sum_{\substack{n\le x\\ n\in A}} f(n)+\sum_{\substack{n\le x\\ n\in B}} f(n),$$

where A denotes the set of all square-full numbers. That is, $n>1$, and for any prime p, if $p\mid n$, then $p^2\mid n$. B denotes the set of all positive integers with $n\notin A$. Note that $f(n)\ll 1$, from the definition of A and Lemma 1 we have

$$\sum_{\substack{n\le x\\ n\in A}} f(n)=O\left(x^{\frac{1}{2}}\right). \tag{1}$$

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in B}} f(n) &= \sum_{\substack{np\le x\\ (n,\ p)=1}} f(n)=\sum_{p\le x}\sum_{\substack{n\le \frac{x}{p}\\ (n,\ p)=1}} \frac{1}{2}\\ &= \frac{1}{2}\sum_{p\le x}\left(\frac{x}{p}-\frac{x}{p^2}+O(1)\right)\\ &= \frac{x}{2}\sum_{p\le x}\frac{1}{p}-\frac{x}{2}\sum_{p\le x}\frac{1}{p^2}+O\left(\frac{1}{2}\sum_{p\le x}1\right).\end{aligned} \tag{2}$$

Note that

$$\sum_{p\le x}\frac{1}{p}=\ln\ln x+c+O\left(\frac{1}{\ln x}\right)\ (\text{ see Theorem 4.12 of reference [2] }),$$

$$\sum_{p\le x}\frac{1}{p^2}=\sum_{p}\frac{1}{p^2}-\sum_{p>x}\frac{1}{p^2}=d+O\left(\frac{1}{x}\right),$$

where c and d are two computable constants.

And the Prime Theorem (see Theorem 3.2 of reference [3]):

$$\pi(x) = \sum_{p \le x} 1 = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

So from (2) we have

$$\begin{aligned}\sum_{\substack{n \le x \\ n \in B}} f(n) &= \frac{x}{2}\left(\ln\ln x + c + O\left(\frac{1}{\ln x}\right)\right) - \frac{x}{2}\left(d + O\left(\frac{1}{x}\right)\right) + O\left(\frac{1}{2}\left(\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)\right)\right) \\ &= \frac{1}{2}x\ln\ln x + \frac{c}{2}x - \frac{d}{2}x + O\left(\frac{x}{\ln x}\right) \\ &= \frac{1}{2}x\ln\ln x + \lambda x + O\left(\frac{x}{\ln x}\right), \qquad (3)\end{aligned}$$

where λ is a computable constant.

Now combining (1) and (3) we may immediately get

$$\begin{aligned}\sum_{n \le x} f(n) &= 1 + \sum_{\substack{n \le x \\ n \in A}} f(n) + \sum_{\substack{n \le x \\ n \in B}} f(n) \\ &= 1 + O\left(x^{\frac{1}{2}}\right) + \frac{1}{2}x\ln\ln x + \lambda \cdot x + O\left(\frac{x}{\ln x}\right) \\ &= \frac{1}{2}x\ln\ln x + \lambda \cdot x + O\left(\frac{x}{\ln x}\right),\end{aligned}$$

where λ is a computable constant.

This proves Theorem 1.

Now we complete the proof of Theorem 2. From the definition of $f(n)$ and the properties of square-full numbers, we have

$$\begin{aligned}\sum_{n \le x}\left(f(n) - \frac{1}{2}\right)^2 &= \frac{1}{4} + \sum_{\substack{n \le x \\ n \in A}}\left(f(n) - \frac{1}{2}\right)^2 + \sum_{\substack{n \le x \\ n \notin A}}\left(f(n) - \frac{1}{2}\right)^2 \\ &= \frac{1}{4} + \sum_{\substack{n \le x \\ n \in A}}\left(f(n) - \frac{1}{2}\right)^2.\end{aligned}$$

where A also denotes the set of all square-full numbers. Let C denotes the set of all cubic-full

numbers. Then from the properties of square-full numbers, Lemma 1 and Lemma 2 we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2 &= \sum_{\substack{np^2\le x\\ (n,\ p)=1,\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2+\sum_{\substack{n\le x\\ n\in C}}\left(f(n)-\frac{1}{2}\right)^2\\
&= \sum_{p^2\le x}\sum_{\substack{n\le \frac{x}{p^2}\\ (n,\ p)=1,\ n\in A}}\left(\frac{1}{3}-\frac{1}{2}\right)^2+O\left(\sum_{\substack{n\le x\\ n\in C}}1\right)\\
&= \sum_{p^2\le x}\left(\sum_{\substack{n\le \frac{x}{p^2}\\ n\in A}}\frac{1}{36}-\sum_{\substack{n\le \frac{x}{p^4}\\ n\in A}}\frac{1}{36}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{1}{36}\sum_{p^2\le x}\left(c\cdot\frac{x^{\frac{1}{2}}}{p}-c\cdot\frac{x^{\frac{1}{2}}}{p^2}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{c}{36}\cdot\sqrt{x}\cdot\sum_{p\le\sqrt{x}}\left(\frac{1}{p}-\frac{1}{p^2}\right)+O\left(x^{\frac{1}{3}}\right)\\
&= \frac{1}{36}c\cdot\sqrt{x}\cdot\ln\ln x+d\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right).
\end{aligned}$$

where $c=\dfrac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}$, d is a computable constant.

So we have the asymptotic formula

$$\sum_{n\le x}\left(f(n)-\frac{1}{2}\right)^2=\frac{1}{36}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}\cdot\ln\ln x+d\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right).$$

This completes the proof of Theorem 2.

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A problem related to the Smarandache n-ary power sieve

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Abstract For any fixed positive integer $k \geq 2$, the power k sieve is defined as following: Starting to count on the natural numbers set at any step from 1: - delete every 2^k-th numbers; - delete, from the remaining ones, every 3^k-th numbers $\cdots$, and so on: delete, from the remaining ones, every n^k-th numbers, $n = 2,\ 3,\ 4,\ \cdots$. In this paper, we study the following two problems:
(A) Are there an infinity of primes that belong to this sequence?
(B) Are there an infinity of numbers of this sequence which are not prime?
Then we using the elementary methods to study these problems, and prove that the problem (B) is true.

Keywords The power k sieve, asymptotic formula, elementary method.

§1. Introduction and result

For any fixed positive integer $n \geq 2$, the famous F.Smarandache n-ary power sieve is defined as following:

Starting to count on the natural numbers set at any step from 1.

-delete every n-th numbers,

-delete, from the remaining ones, every n^2-th numbers, $\cdots$, and so on: delete, from the remaining ones, every n^k-th numbers, $k = 1,\ 2,\ 3,\ \cdots$. For example, if $n = 2$, then we call this sieve as Binary Sieve:
1, 3, 5, 9, 11, 13, 17, 21, 25, 27, 29, 33, 35, 37, 43, 49, 51, 53, 57, 59, 65, 67, 69, 73, 75, 77, 81, 85, 89, $\cdots$.

Simultaneously, if $n = 3$, then call the sieve as Trinary Sieve:
1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 19, 20, 22, 23, 25, 28, 29, 31, 32, 34, 35, 37, 38, 41, 43, 46, 47, 49, 50, $\cdots$.

In reference [1] and [2], Professor F.Smarandache asked us to study the properties of the n-ary power sieve sequence. At the same time, he also proposed the following two conjectures:

(a) There are an infinity of primes that belong to this sequence.

(b) There are an infinity of numbers of this sequence which are not prime.

About these two conjectures, Yi Yuan [3] had studied them, and proved that the conjecture (b) is correct.

In this paper, we define another sequence related the Smarandache n-ary power sieve (we called it as the power k sieve) as follows:

Starting to count on the natural numbers set at any step from 1: - delete every 2^k-th numbers; - delete, from the remaining ones, every 3^k-th numbers $\cdots$, and so on: delete, from the remaining ones, every n^k-th numbers, $n = 2,\ 3,\ 4,\ \cdots$. Then, two similar problems here can be proposed naturally when we studying the properties of this sequence:

(A) Are there an infinity of primes that belong to the power k sieve sequence?

(B) Are there an infinity of numbers of the power k sieve sequence which are not prime?

In this paper, we use the elementary method to study these two problems, and obtain an interesting asymptotic formula. As a corollary of our result, we solved the problem (B). That is, we shall prove the following:

Theorem. Let $k \geq 2$ be a fixed positive integer, A denotes the set of all power k sieve sequence. Then for any real $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = c(k)\cdot x + O\left(x^{\frac{1}{k}}\right),$$

where $c(k) = \prod_{n=2}^{\infty}\left(1-\frac{1}{n^k}\right)$ is a positive constant, and $c(2) = \frac{1}{2}$.

For any real number $x > 1$, from the Prime Number Theorem (see reference [6]) we know that there are at most $O\left(\frac{x}{\ln x}\right)$ primes in the interval $[1,\ x]$, so from our Theorem we know that there are an infinity of numbers of the power k sieve sequence which are not prime. Therefore, the problem (B) is true.

§2. Proof of the theorem

In this section, we shall complete the proof of our Theorem directly. Let $k \geq 2$ be a fixed positive integer. For any positive integer $x > 1$, let $U_h(x)$ denotes the number of all remaining ones when delete i^k-th numbers ($i = 2,\ 3,\ \cdots,\ h$) in the interval $[1,\ x]$. Then we have

$$x - \frac{x}{2^k} \leq U_1(x) = x - \left[\frac{x}{2^k}\right] \leq x - \frac{x}{2^k} + \frac{2^k-1}{2^k}$$

or

$$U_1(x) = \left(1-\frac{1}{2^k}\right)\cdot x + R_1(x) \quad \text{with} \quad |R_1(x)| \leq 1.$$

$$U_1(x) - \frac{U_1(x)}{3^k} \leq U_2(x) = U_1(x) - \left[\frac{U_1(x)}{3^k}\right] \leq U_1(x) - \frac{U_1(x)}{3^k} + \frac{3^k-1}{3^k}$$

or

$$U_2(x) = \left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{3^k}\right)\cdot x + R_2(x) \quad \text{with} \quad |R_2(x)| \leq |R_1(x)| + 1 \leq 2.$$

Generally, for any positive integer $h \geq 2$, we have

$$U_h(x) = \left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{3^k}\right)\cdots\left(1-\frac{1}{h^k}\right)\cdot x + R_h(x) \quad \text{with} \quad |R_h(x)| \leq h.$$

Taking $m = \left[x^{\frac{1}{k}}\right]$, if $h > m$, then $h^k > x$. So we have

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = U_m(x) = x \cdot \prod_{h=1}^{m}\left(1-\frac{1}{h^k}\right) + R_m(x). \tag{1}$$

Note that $|R_m(x)| \leq m \leq x^{\frac{1}{k}}$ and

$$\prod_{h=1}^{m}\left(1-\frac{1}{h^k}\right) = \prod_{h=1}^{\infty}\left(1-\frac{1}{h^k}\right) + O\left(m^{-(k-1)}\right) = \prod_{h=2}^{\infty}\left(1-\frac{1}{h^k}\right) + O\left(x^{-\frac{k-1}{k}}\right).$$

From (1) we may immediately get the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = c \cdot x + O\left(x^{\frac{1}{k}}\right),$$

where $c = c(k) = \prod_{n=2}^{\infty}\left(1-\frac{1}{n^k}\right)$ is a positive constant, and

$$c(2) = \prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{2^2-1}{2^2}\cdot\frac{3^2-1}{3^2}\cdot\frac{4^2-1}{4^2}\cdot\cdots\cdots\frac{n^2-1}{n^2}\cdot\cdots\cdots = \frac{1}{2}.$$

This completes the proof of Theorem.

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On the mean value of the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo Smarandache Squarefree function $Z_w(n)$ is defined as $Z_w(n) = \min\{m : n|m^n, m \in N\}$, and the function $Z(n)$ is defined as $Z(n) = \min\left\{m : n \leq \frac{m(m+1)}{2},\ m \in N\right\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of the function $Z_w(Z(n))$, and give a sharper mean value formula for it.

Keywords Pseudo-Smarandache-Squarefree function $Z_w(n)$, function $Z(n)$, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the Pseudo-Smarandache-Squarefree function $Z_w(n)$ is defined as the smallest positive integer m such that $n \mid m^n$. That is,

$$Z_w(n) = \min\{m : n|m^n, m \in N\}.$$

For example $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results. For example, Felice Russo [1] obtained some elementary properties of $Z_w(n)$ as follows:

Property 1. The function $Z_w(n)$ is multiplicative. That is, if $GCD(m, n) = 1$, then $Z_w(m \cdot n) = Z_w(m) \cdot Z_w(n)$.

Property 2. $Z_w(n) = n$ if and only if n is a squarefree number.

The main purpose of this paper is using the elementary method to study the mean value properties of $Z_w(Z(n))$, and give a sharper asymptotic formula for it, where $Z(n)$ is defined as $Z(n) = \min\left\{m : n \leq \frac{m(m+1)}{2}, m \in N\right\}$. That is, we shall prove the following conclusion:

Theorem. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} Z_w(Z(n)) = \left(1 + \prod_p \left(1 + \frac{1}{p(p^2-1)}\right)\right) \cdot \frac{4\sqrt{2}}{\pi^2} \cdot x^{\frac{3}{2}} + O\left(x^{\frac{5}{4}}\right),$$

where $\prod_p$ denotes the product over all primes.

§2. Some lemmas

To complete the proof of the theorem, we need the following several lemmas.

Lemma 1. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{m\leq x} \mu^2(m) = \frac{6}{\pi^2}x + O(\sqrt{x}). \tag{1}$$

Proof. See reference [2].

Lemma 2. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{\substack{m\leq x\\ m\in A}} m^2 = \frac{2}{\pi^2}x^3 + O\Big(x^{\frac{5}{2}}\Big),$$

where A denotes the set of all square-free integers.

Proof. By the Abel's summation formula (See Theorem 4.2 of [3]) and Lemma 1, we have

$$\begin{aligned}\sum_{\substack{m\leq x\\ m\in A}} m^2 &= \sum_{m\leq x} m^2\mu^2(m) = x^2\cdot\Big(\frac{6}{\pi^2}x + O(\sqrt{x})\Big) - 2\int_1^x t\Big(\frac{6}{\pi^2}t + O(\sqrt{t})\Big)\mathrm{d}t\\ &= \frac{6}{\pi^2}x^3 + O\Big(x^{\frac{5}{2}}\Big) - \frac{4}{\pi^2}x^3 = \frac{2}{\pi^2}x^3 + O\Big(x^{\frac{5}{2}}\Big).\end{aligned}$$

This proves Lemma 2.

Lemma 3. For any real number $x \geq 2$ and $s > 1$, we have the inequality

$$\sum_{\substack{m\leq x\\ m\in B}} \frac{Z_w(m)}{m^s} < \prod_p \Big(1 + \frac{1}{p^{s-1}(p^s-1)}\Big).$$

Specially, if $s > \dfrac{3}{2}$, then we have the asymptotic formula

$$\sum_{\substack{m\leq x\\ m\in B}} \frac{Z_w(m)}{m^s} = \prod_p \Big(1 + \frac{1}{p^{s-1}(p^s-1)}\Big) + O\Big(x^{\frac{3}{2}-s}\Big),$$

where B denotes the set of all square-full integers.

Proof. First we define the arithmetical function $a(m)$ as follows:

$$a(m) = \begin{cases} 1 & \text{if } m \in B\ ;\\ 0 & \text{otherwise.}\end{cases}$$

From Property 1 and the definition of $a(m)$ we know that the function $Z_w(m)$ and $a(m)$ are multiplicative. If $s > 1$, then by the Euler product formula (See Theorem 11.7 of [3]) we have

$$\begin{aligned}\sum_{\substack{m\leq x\\ m\in B}} \frac{Z_w(m)}{m^s} < \sum_{\substack{m=1\\ m\in B}}^{\infty} \frac{Z_w(m)}{m^s} &= \sum_{m=1}^{\infty} \frac{Z_w(m)}{m^s}a(m)\\ &= \prod_p \Big(1 + \frac{p}{p^{2s}} + \frac{p}{p^{3s}} + \cdots\Big)\\ &= \prod_p \Big(1 + \frac{1}{p^{s-1}(p^s-1)}\Big).\end{aligned}$$

Note that if $m \in B$, then $Z_w(m) \le \sqrt{m}$. Hence, if $s > \frac{3}{2}$, then we have

$$\begin{aligned}\sum_{\substack{m\le x\\ m\in B}} \frac{Z_w(m)}{m^s} &= \sum_{\substack{m=1\\ m\in B}}^{\infty} \frac{Z_w(m)}{m^s} - \sum_{\substack{m>x\\ m\in B}} \frac{Z_w(m)}{m^s}\\ &= \sum_{\substack{m=1\\ m\in B}}^{\infty} \frac{Z_w(m)}{m^s} + O\Big(\sum_{m>x} \frac{1}{m^{s-\frac{1}{2}}}\Big)\\ &= \prod_p \Big(1 + \frac{1}{p^{s-1}(p^s-1)}\Big) + O\Big(x^{\frac{3}{2}-s}\Big).\end{aligned}$$

This proves Lemma 3.

§3. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem.

Note that if $\frac{(m-1)m}{2} + 1 \le n \le \frac{m(m+1)}{2}$, then $Z(n) = m$. That is, the equation $Z(n) = m$ has m solutions as follows:

$$n = \frac{(m-1)m}{2} + 1, \frac{(m-1)m}{2} + 2, \cdots, \frac{m(m+1)}{2}$$

Since $n \le x$, from the definition of $Z(n)$ we know that if $Z(n) = m$, then $1 \le m \le \frac{\sqrt{8x+1}-1}{2}$.

Note that $Z_w(n) \le n$, we have

$$\begin{aligned}\sum_{n\le x} Z_w(Z(n)) &= \sum_{\substack{n\le x\\ Z(n)=m}} Z_w(m) = \sum_{m\le \frac{\sqrt{8x+1}-1}{2}} m\cdot Z_w(m) + O(x)\\ &= \sum_{m\le\sqrt{2x}} m\cdot Z_w(m) + O(x). \qquad (2)\end{aligned}$$

We separate all integer m in the interval $[1, \sqrt{2x}]$ into three subsets A, B, and C as follows: A: the set of all square-free integers; B: the set of all square-full integers; C: the set of all positive integer m such that $m \in [1, \sqrt{2x}]/A\bigcup B$.

Note that (2), we have

$$\sum_{n\le x} Z_w(Z(n)) = \sum_{\substack{m\le\sqrt{2x}\\ m\in A}} m\cdot Z_w(m) + \sum_{\substack{m\le\sqrt{2x}\\ m\in B}} m\cdot Z_w(m) + \sum_{\substack{m\le\sqrt{2x}\\ m\in C}} m\cdot Z_w(m) + O(x). \qquad (3)$$

From Property 2 and Lemma 2 we know that if $m \in A$, then we have

$$\sum_{\substack{m\le\sqrt{2x}\\ m\in A}} m\cdot Z_w(m) = \sum_{\substack{m\le\sqrt{2x}\\ m\in A}} m^2 = \frac{4\sqrt{2}}{\pi^2} x^{\frac{3}{2}} + O\Big(x^{\frac{5}{4}}\Big). \qquad (4)$$

It is clear that if $m \in B$, then $Z_w(m) \le \sqrt{m}$. Hence

$$\sum_{\substack{m\le\sqrt{2x}\\ m\in B}} m\cdot Z_w(m) \ll \sum_{\substack{m\le\sqrt{2x}\\ m\in B}} m^{\frac{3}{2}} \ll x^{\frac{5}{4}}. \qquad (5)$$

If $m \in C$, then we write m as $m = q \cdot n$, where q is a square-free integer and n is a square-full integer. From Property 1, Property 2, Lemma 2 and Lemma 3 we have

$$\begin{aligned}\sum_{\substack{m\le\sqrt{2x}\\ m\in C}} m\cdot Z_w(m) &= \sum_{n\le\sqrt{2x}} nZ_w(n)a(n)\sum_{q\le\frac{\sqrt{2x}}{n}} q^2\mu^2(q)\\ &= \sum_{n\le\sqrt{2x}} nZ_w(n)a(n)\left(\frac{4\sqrt{2}}{\pi^2}\cdot\frac{x^{\frac{3}{2}}}{n^3}+O\left(\frac{x^{\frac{5}{4}}}{n^{\frac{5}{2}}}\right)\right)\\ &= \frac{4\sqrt{2}}{\pi^2}x^{\frac{3}{2}}\sum_{n\le\sqrt{2x}}\frac{Z_w(n)a(n)}{n^2}+O\left(x^{\frac{5}{4}}\sum_{n\le\sqrt{2x}}\frac{Z_w(n)a(n)}{n^{\frac{3}{2}}}\right)\\ &= \frac{4\sqrt{2}}{\pi^2}\prod_p\Big(1+\frac{1}{p(p^2-1)}\Big)x^{\frac{3}{2}}+O\Big(x^{\frac{5}{4}}\Big). \qquad (6)\end{aligned}$$

Combining (3), (4), (5) and (6), we may immediately deduce the asymptotic formula

$$\sum_{n\le x} Z_w(Z(n)) = \Big(1+\prod_p(1+\frac{1}{p(p^2-1)}\Big)\cdot\frac{4\sqrt{2}}{\pi^2}\cdot x^{\frac{3}{2}}+O\Big(x^{\frac{5}{4}}\Big).$$

This completes the proof of Theorem.

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On the back concatenated square sequence

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Abstract For any positive integer n, the famous Smarandache concatenated square sequence $\{a(n)\}$ is defined as the positive integer $a(n) = 1^2 2^2 3^2 \cdots (n-1)^2 n^2$, and the Smarandache back concatenated square sequence $\{b(n)\}$ is defined as the positive integer $b(n) = n^2(n-1)^2(n-2)^2 \cdots 4^2 3^2 2^2 1^2$. For example, the first few terms of $\{a(n)\}$ are: 1, 14, 149, 14925, 1492536, $\cdots\cdots$. The first few terms of $\{b(n)\}$ are 1, 41, 941, 16941, 2516941, $\cdots\cdots$. In reference [2], Professor F.Smarandache asked us to study such a problem: How many perfect square number are there in the sequence $\{a(n)\}$ and $\{b(n)\}$? The main purpose of this paper is using the elementary methods to study this problem, and prove that there is only one perfect square number 1 in the Smarandache back concatenated square sequence $\{b(n)\}$. This solved a problem proposed by Smarandache in his book [2].

Keywords Concatenated square sequence, back concatenated square sequence, perfect square.

§1. Introduction and results

For any positive integer n, the famous Smarandache concatenated square sequence $\{a(n)\}$ is defined as the positive integer $a(n) = 1^2 2^2 3^2 \cdots (n-1)^2 n^2$. For example, its first few terms are: 1, 14, 149, 14925, 1492536, $\cdots\cdots$. And the Smarandache back concatenated square sequence $\{b(n)\}$ is defined as the positive integer $b(n) = n^2(n-1)^2 \cdots 3^2 2^2 1^2$. Its first few terms are 1, 41, 941, 16941, 2516941, 362516941, $\cdots\cdots$. In his book 《Sequences of Numbers Involved Unsolved Problems》, Professor F.Smarandache asked us to study such two problems:

(A). How many perfect square numbers are there in the concatenated square sequence $\{1,\ 14,\ 149,\ 14925,\ \cdots,\ 1^2 2^2 3^2 \cdots (n-1)^2 n^2, \cdots\}$?

(B). How many perfect square numbers are there in the back concatenated square sequence $\{1,\ 41,\ 941,\ 25941,\ \cdots,\ n^2(n-1)^2 \cdots 4^2 3^2 2^2 1^1, \cdots\}$?

About these two problems, it seems that none had studied them yet, at least we have not seen any related papers before. The problems are interesting, because they can help us to find some special perfect square numbers. It is clear that the first term $a(1) = 1$ and $b(1) = 1$ are two perfect square numbers. Besides these two terms, whether there exists any other positive integer $n > 2$ such that $a(n)$ or $b(n)$ is a perfect square number?

The main purpose of this paper is using the elementary methods to study these two problems, and prove the following conclusion:

Theorem. There exists only one perfect square number in the Smarandache back concatenated square sequence $\{b(n)\}$, it is $b(1) = 1$.

Whether there exists any other perfect square number in the Smarandache concatenated square sequence $\{a(n)\}$ is still an open problem.

Conjecture. For any integer $n > 1$, $a(n)$ is not a perfect square.

§2. Proof of the theorem

In this section, we shall prove our theorem directly. In fact from the definition and properties of the congruence we know that for any odd number m, we have

$$m^2 \equiv 1 \pmod 8. \tag{1}$$

If there exists some positive integer $n > 1$ such that

$$b(n) = n^2(n-1)^2(n-2)^2(n-3)^2 \cdots 4^2 3^2 2^2 1^2$$

is a perfect square number. That is, $b(n) = u^2$. Then u must be an odd number. Therefore, from (1) we have

$$b(n) = u^2 \equiv 1 \pmod 8. \tag{2}$$

On the other hand, note that $10^3 \equiv 0 \pmod 8$, from the definition of $b(n)$ we also have

$$\begin{aligned} b(n) &= n^2(n-1)^2(n-2)^2(n-3)^2 \cdots 4^2 3^2 2^2 1^2 \\ &= n^2(n-1)^2(n-2)^2(n-3)^2 \cdots 4^2 \times 10^3 + 941 \\ &\equiv 941 \equiv 5 \pmod 8. \end{aligned} \tag{3}$$

Thus, the congruence (2) is not possible, because it contradicts with the congruence (3). This completes the proof of Theorem.

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On a class of q-valent meromorphic functions with positive coefficients

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Abstract In this paper we introduce a new subclass $S^*(\alpha, \beta, \xi, \upsilon)$ consisting of functions

$$f(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n} z^{q+n}, \quad (a_{q+n} \geq 0, q \in I\!N)$$

which are analytic and q-valent in the punctured unit disc $E = \{z : 0 < |z| < 1\}$ with multiple pole at $z = 0$. Some results like coefficient estimates, growth and distortion theorem, extreme points, convolution theorem and other interesting properties are investigated. The results investigated here are shown to be sharp.

Keywords Analytic functions, Univalent and multivalent functions, Starlike functions, meromorphic functions, radius of convexity and convolution.

§1. Introduction

Let $M(q)$ denote the class of function of the form

$$f(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n} z^{q+n} \quad (a_{q+n} \geq 0, q \in I\!N) \tag{1.1}$$

which are meromorphic and q-valent in the punctured unit disc $E = \{z : 0 < |z| < 1\}$. A function $f(z) \in M(q)$ is said to be q-valent meromorphic starlike of order $s, 0 \leq s < q$ if

$$-Re\left\{z\frac{f'(z)}{f(z)}\right\} > s \quad (z \in E) \tag{1.2}$$

and a function $f(z) \in M(q)$ is said to be q-valent meromorphic convex of order $c, 0 \leq c < q$ if

$$-Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > c \quad (z \in E). \tag{1.3}$$

Now, we introduce a subclass $S^*(\alpha,\beta,\xi,\gamma)$ of $M(q)$ consisting of functions defined by (1.1) and also satisfying the following condition:

$$\left|\frac{z\frac{f'(z)}{f(z)}+q}{2\xi\left[\frac{zf'(z)}{f(z)}+\alpha\right]-\gamma\left[\frac{zf'(z)}{f(z)}+q\right]}\right|<\beta$$

where $\left(|z|<1, q\in I\!N, 0\le\alpha<1, 0\le\beta\le 1, \frac{1}{2}<\xi\le 1, \frac{1}{2}<\gamma\le 1\right)$.

§2. Coefficient estimate

Theorem 1. A function $f(z)\in M(q)$ is in $S^*(\alpha,\beta,\xi,\gamma)$ if and only if

$$\sum_{n=0}^{\infty}[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]a_{q+n}\le 2\xi\beta(q-\alpha). \tag{2.1}$$

Proof. Let (2.1) be true. We show that $f\in S^*(\alpha,\beta,\xi,\gamma)$

$$\begin{aligned}
&\left|\frac{z\frac{f'(z)}{f(z)}+q}{2\xi\left[\frac{zf'(z)}{f(z)}+\alpha\right]-\gamma\left[\frac{zf'(z)}{f(z)}+q\right]}\right|\\
=&\left|\frac{\sum\limits_{n=0}^{\infty}(2q+n)a_{q+n}z^{2q+n}}{2\xi(\alpha-q)+\sum\limits_{n=0}^{\infty}[2\xi(q+n+\alpha)-\gamma(2q+n)]a_{q+n}z^{2q+n}}\right|\\
\le&\frac{\sum\limits_{n=0}^{\infty}(2q+n)a_{q+n}}{2\xi(q-\alpha)-\sum\limits_{n=0}^{\infty}[2\xi(q+n+\alpha)-\gamma(2q+n)]a_{q+n}}.
\end{aligned}$$

Above inequality is bounded above by β if

$$\sum_{n=0}^{\infty}[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n)]a_{q+n}\le 2\xi\beta(q-\alpha)$$

and thus $f(z)\in S^*(\alpha,\beta,\xi,\gamma)$ by assuming (2.1) holds. Conversely, let $f(z)\in S^*(\alpha,\beta,\xi,\gamma)$. Then

$$\left|\frac{z\frac{f'(z)}{f(z)}+q}{2\xi\left[z\frac{f'(z)}{f(z)}+\alpha\right]-\gamma\left[\frac{zf'(z)}{f(z)}+q\right]}\right|<\beta.$$

That is

$$\left|\frac{\sum\limits_{n=0}^{\infty}(2q+n)a_{q+n}z^{2q+n}}{2\xi(\alpha-q)+\sum\limits_{n=0}^{\infty}[2\xi(q+n+\alpha)-\gamma(2q+n)]a_{q+n}z^{2q+n}}\right|<\beta. \tag{2.2}$$

Notice that $|Re(z)| \leq |z|$ for all z, and so

$$Re\left\{\frac{\sum_{n=0}^{\infty}(2q+n)a_{q+n}z^{2q+n}}{2\xi(q-\alpha)-\sum_{n=0}^{\infty}[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]a_{q+n}z^{2q+n}}\right\}<\beta. \tag{2.3}$$

Choosing values of z on real axis so that $z\dfrac{f'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.3) and allowing $z \longrightarrow 1$ through real values we obtain (2.1).

Corollary 1. If $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$ then

$$a_{q+n} \leq \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}$$

for $n \in I\!N_0$ with equality for $f(z)$ given by

$$f(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}z^{q+n} \quad (n \in I\!N_0). \tag{2.4}$$

Corollary 2. If $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$ and $q = 1$ then

$$a_{n+1} \leq \frac{2\xi\beta(1-\alpha)}{2+n+\beta(2\xi(1+n+\alpha)-\gamma(2+n))} \quad (n \in I\!N_0)$$

with equality for $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{2\xi\beta(1-\alpha)}{2+n+\beta(2\xi(1+n+\alpha)-\gamma(2+n))} \quad (n \in I\!N_0). \tag{2.5}$$

This result is studied by Khairnar and Kulkarni in [7].

Corollary 3. If $f(z) \in S^*(\alpha, \beta, 1, 1)$ then

$$a_{q+n} \leq \frac{2\beta(q-\alpha)}{2q+n+\beta(n+2\alpha)}, \quad (n \in I\!N_0)$$

with equality for the function given by

$$f(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{2q+n+\beta(n+2\alpha)}z^{q+n} \quad (n \in I\!N_0). \tag{2.6}$$

This result is studied by Uralegaddi and Ganigi in [9].

Corollary 4. If $f(z) \in S^*(0, 1, 1, 1)$, then $f(z)$ is starlike if and only if

$$\sum_{n=0}^{\infty}(q+n)a_{q+n} \leq q. \tag{2.7}$$

§3. Growth and distortion theorem

Theorem 2. A function $f(z) \in M(q)$ is in the class $S^*(\alpha, \beta, \xi, \gamma)$, then

$$\frac{1}{r^q} - \frac{2\xi(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}r^q \leq |f(z)| \leq \frac{1}{r^q} + \frac{2\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}r^q \quad \text{for} \quad |z| = r. \tag{3.1}$$

The result is sharp for the extremal function

$$f(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}z^q \quad \text{at} \quad z = r, re^{i\frac{\pi}{2q}}. \tag{3.2}$$

and

$$\frac{q}{r^{q+1}} - \frac{2q\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}r^{q-1} \leq |f'(z)| \leq \frac{q}{r^{q+1}} + \frac{2q\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}r^{q-1} \quad \text{for} \quad |z| = r.$$

The result is sharp for the extremal function

$$f(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)}z^q \quad \text{at} \quad z = r, re^{i\frac{\pi}{2q}}. \tag{3.3}$$

Corollary 5. The disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|\omega| < \frac{2q+\beta(q+\alpha-2q\gamma)-2\xi\beta(q-\alpha)}{2q+\beta(q+\alpha-2q\gamma)} \tag{3.4}$$

by any function $f \in S^*(\alpha, \beta, \xi, \gamma)$.

§4. Radius of convexity and Starlikeness

Theorem 3. If the function $f(z) \in M(q)$ is in the class $S^*(\alpha, \beta, \xi, \gamma)$, then $f(z)$ is q-valently convex in

$$0 < |z| < R_1 = \inf_n \left\{ \frac{q^2[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)(q+n)^2} \right\}^{\frac{1}{2q+n}}, \quad n \in I\!N_0.$$

The estimate is sharp for the function

$$f(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}z^{q+n} \quad \text{for some } n. \tag{4.1}$$

Proof. It is sufficient to show that

$$\left| \frac{1+\frac{zf''(z)}{f'(z)}+q}{1+\frac{zf''(z)}{f'(z)}-q} \right| \leq 1 \quad \text{for} \quad 0 < |z| < R.$$

Notice that

$$\begin{aligned} \left| \frac{1+z\frac{f''(z)}{f'(z)}+q}{1+\frac{zf''(z)}{f'(z)}-q} \right| &= \left| \frac{\sum_{n=0}^{\infty}(2q+n)(q+n)a_{q+n}z^{2q+n}}{2q^2+\sum_{n=0}^{\infty}n(q+n)a_{q+n}z^{2q+n}} \right| \\ &\leq \frac{\sum_{n=0}^{\infty}(2q+n)(q+n)|z|^{2q+n}}{2q^2-\sum_{n=0}^{\infty}n(q+n)a_{q+n}|z|^{2q+n}}. \end{aligned}$$

The last expression above is bounded by 1 provided

$$\sum_{n=0}^{\infty}\left(\frac{q+n}{q}\right)^2 a_{q+n}|z|^{2q+n} \leq 1. \tag{4.2}$$

From Theorem 1, we have

$$\sum_{n=0}^{\infty} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)} a_{q+n} \leq 1. \tag{4.3}$$

Thus (4.2) is satisfied if

$$\left(\frac{q+n}{q}\right)^2 |z|^{2q+n} \leq \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}. \tag{4.4}$$

Solving for $|z|$ we obtain

$$|z| \leq \left\{ \frac{q^2[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)(q+n)^2} \right\}^{\frac{1}{2q+n}}, \quad n \in I\!N_0. \tag{4.5}$$

Substituting $|z| = R_1$ in (4.5) the result follows.

Corollary 6. If $f(z) \in S^*(\alpha, \beta, \xi, 1)$, then $f(z)$ is convex in the disc

$$0 < |z| < R_2 = \inf_n \left\{ \frac{q^2[2q+n+\beta(2\xi(q+n+\alpha)-(2q+n))]}{2\xi\beta(q-\alpha)(q+n)^2} \right\}^{\frac{1}{2q+n}}, \quad n \in I\!N_0.$$

The estimate is sharp for the function

$$f(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-(2q+n))} z^{q+n} \text{ for some } n.$$

Corollary 7. If $f(z) \in S^*(\alpha, \beta, 1, 1)$, then $f(z)$ is convex in the disc

$$0 < |z| < R_3 = \inf_n \left\{ \frac{q^2(2q+n+\beta(2\alpha+n))}{2\beta(q-\alpha)(q+n)^2} \right\}^{\frac{1}{2q+n}}, \quad n \in I\!N_0.$$

The estimate is sharp for the function

$$f(z) = \frac{1}{z^q} + \frac{2\beta(q-\alpha)}{2q+n+\beta(2\alpha+n)} z^{q+n} \text{ for some } n.$$

Corollary 8. If $f(z) \in S^*(0, 1, 1, 1)$, then $f(z)$ is convex in the disc

$$0 < |z| < R_4 = \inf_n \left\{ \frac{q}{q+n} \right\}^{\frac{1}{2q+n}}, \quad n \in I\!N_0.$$

The estimate is sharp for the function

$$f(z) = \frac{1}{z^q} + \frac{q}{q+n} z^{q+n} \quad \text{for some } n.$$

Theorem 4. If the function $f(z) \in M(q)$ is in the class $S^*(\alpha, \beta, \xi, \gamma)$, then $f(z)$ is q-valently starlike in

$$0 < |z| < R_5 = \inf_n \left\{ \frac{q[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)(q+n)} \right\}^{\frac{1}{2q+n}}.$$

The estimate is sharp for the function given by (3.4).

Proof. It is sufficient to show that

$$\left| \frac{z\frac{f'(z)}{f(z)} + q}{z\frac{f'(z)}{f(z)} - q} \right| < 1 \quad \text{for} \quad 0 < |z| < R_5.$$

The rest of the details are fairly straight forward and are thus omitted.

§5. Extreme points

Theorem 5. Let $f_{q-1}(z) = \dfrac{1}{z^q}$ and

$$f_{q+n}(z) = \frac{1}{z^q} + \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))} z^{q+n}, \quad for n \in I\!N_0.$$

Then$f(z) \in S^*(\alpha, \beta, \xi, \gamma)$ if and only if it can be expressed in the form $f(z) = \sum_{n=-1}^{\infty} \lambda_{q+n} f_{q+n}(z)$ where $\lambda_{q+n} \geq 0$ and $\sum_{n=-1}^{\infty} \lambda_{q+n} = 1$.

Proof. Assume

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{q+n} f_{q+n}(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} \lambda_{q+n} \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))} z^{q+n}. \quad (5.1)$$

Notice that

$$\sum_{n=0}^{\infty} \lambda_{q+n} \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)}$$
$$= \sum_{n=0}^{\infty} \lambda_{q+n} = 1 - \lambda_{q-1} \leq 1,$$

which implies $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$. Conversely, let $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$. Then by Corollary 1,

$$a_{q+n} \leq \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))} \quad \text{for} \quad n \in I\!N_0.$$

Setting

$$\lambda_{q+n} = \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} \; a_{q+n}, \quad n \in I\!N_0 \quad (5.2)$$

and $\lambda_{q-1} = 1 - \sum_{n=0}^{\infty} \lambda_{q+n}$ we obtain $f(z) = \sum_{n=-1}^{\infty} \lambda_{q+n} f_{q+n}(z)$. Hence the proof.

§6. Closure theorems

Theorem 6. Let

$$f_j(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n,j} z^{q+n}, \quad a_{q+n} \geq 0, j = 1, 2, \cdots, \ell$$

be in the class $S^*(\alpha, \beta, \xi, \gamma)$. Then the function $g(z) = \sum_{j=1}^{\ell} c_j f_j(z)$ also belongs to the class $S^*(\alpha, \beta, \xi, \gamma)$ if $\sum_{j=1}^{\ell} c_j = 1$.

Proof. Let

$$\begin{aligned} g(z) &= \sum_{j=1}^{\ell} c_j \left(\frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n,j} z^{q+n} \right) \\ &= \frac{1}{z^q} + \sum_{n=0}^{\infty} \sum_{j=1}^{\ell} c_j a_{q+n,j} z^{q+n} \\ &= \frac{1}{z^q} + \sum_{n=0}^{\infty} e_{q+n} z^{q+n} \end{aligned} \tag{6.1}$$

where $e_{q+n} = \sum_{j=1}^{\ell} c_j a_{q+n,j}$.

Notice that $g(z) \in S^*(\alpha, \beta, \xi, \gamma)$ since

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} e_{q+n} \\ &= \sum_{j=1}^{\ell} c_j \sum_{n=0}^{\infty} \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} a_{q+n,j} \\ &\leq \sum_{j=1}^{\ell} c_j = 1 \quad \text{since} \quad f_j(z) \in S^*(\alpha, \beta, \xi, \gamma). \end{aligned}$$

Theorem 7. Let

$$f_j(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n,j} z^{q+n}, \quad a_{q+n} \geq 0, j = 1, 2, \cdots, m$$

be in the class $S^*(\alpha, \beta, \xi, \gamma)$. Then the function $h(z) = \frac{1}{m} \sum_{j=1}^{m} f_j(z)$ also belongs to the class $S^*(\alpha, \beta, \xi, \gamma)$.

Proof. We have

$$\begin{aligned} h(z) &= \frac{1}{m} \sum_{j=1}^{m} f_j(z) \\ &= \frac{1}{z^q} + \sum_{n=0}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{q+n,j} \right) z^{q+n} \\ &= \frac{1}{z^q} + \sum_{n=0}^{\infty} d_k z^k \quad \text{where} \quad d_k = \frac{1}{m} \sum_{j=1}^{m} a_{q+n,j}. \end{aligned}$$

Since $f_j(z) \in S^*(\alpha, \beta, \xi, \gamma)$ from Theorem 1, we have

$$\sum_{n=0}^{\infty} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)} a_{q+n,j} \leq 1. \tag{6.2}$$

Now $h(z) \in S^*(\alpha, \beta, \xi, \gamma)$ since

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)} d_k \\
&= \sum_{n=0}^{\infty} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)} \frac{1}{m} \sum_{j=1}^{m} a_{q+n,j} \\
&= \frac{1}{m} \sum_{j=1}^{m} \sum_{n=0}^{\infty} \frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{2\xi\beta(q-\alpha)} a_{q+n,j} \\
&\leq \frac{1}{m} \sum_{j=1}^{m} \quad \text{using (6.2)} \\
&= 1.
\end{aligned}$$

Therefore $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$.

§7. Convolution and Inclusion Property

For $f(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n} z^{q+n}$ and $g(z) = \frac{1}{z^2} + \sum_{n=0}^{\infty} b_{q+n} z^{q+n}$ in $S^*(\alpha, \beta, \xi, \gamma)$ the convolution $(f * g)(z)$ is defined by

$$(f * g)(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty} a_{q+n} b_{q+n} z^{q+n} \tag{7.1}$$

Theorem 8. Let $f(z)$ and $g(z)$ belong to $S^*(\alpha, \beta, \xi, \gamma)$ then $(f * g)(z) \in S^*(\alpha, \eta, \xi, \gamma)$ for

$$\eta \geq \frac{2\xi\beta(q-\alpha)(2q+n)}{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))] - 2\xi\beta(q-\alpha)[2\xi(q+n+\alpha)-\gamma(2q+n)]}.$$

Proof. $f(z), g(z) \in S^*(\alpha, \beta, \xi, \gamma)$ and so

$$\sum_{n=0}^{\infty} \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} a_{q+n} \leq 1 \tag{7.2}$$

and

$$\sum_{n=0}^{\infty} \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} b_{q+n} \leq 1. \tag{7.3}$$

We need to find a smallest number η such that

$$\sum_{n=0}^{\infty} \frac{2q+n+\eta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\eta(q-\alpha)} a_{q+n} b_{q+n} \leq 1. \tag{7.4}$$

Using Cauchy Schwarz inequality, we have

$$\sum_{n=0}^{\infty} \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)} \sqrt{a_{q+n} b_{q+n}} \leq 1. \tag{7.5}$$

Thus it is enough to show that

$$\frac{2q+n+\eta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\eta(q-\alpha)}a_{q+n}b_{q+n} \leq \frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}\sqrt{a_{q+n}b_{q+n}}.$$

That is

$$\sqrt{a_{q+n}b_{q+n}} \leq \frac{\eta}{\beta}\frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]}{[2q+n+\eta(2\xi(q+n+\alpha)-\gamma(2q+n))]} \tag{7.6}$$

From (7.5), we have

$$\sqrt{a_{q+n}b_{q+n}} \leq \frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}. \tag{7.7}$$

In view of (7.6) and (7.7) it is enough to show that

$$\frac{2\xi\beta(q-\alpha)}{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))} \leq \frac{\eta}{\beta}\frac{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma 2q+n))]}{[2q+n+\eta(2\xi(q+n+\alpha)-\gamma(2q+n))]}.$$

Simplifying we get

$$\eta \geq \frac{2\xi\beta(q-\alpha)(2q+n)}{[(2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))]-2\xi\beta(q-\alpha)[2\xi(q+n+\alpha)-\gamma(2q+n)]}.$$

Next we state another inclusion theorem for the class $S^*(\alpha,\beta,\xi,\gamma)$.

Theorem 9. Let $f(z), g(z) \in S^*(\alpha,\beta,\xi,\gamma)$ then

$$h(z) = \frac{1}{z^q} + \sum_{n=0}^{\infty}(a_{q+n}^2 + b_{q+n}^2)z^{q+n}$$

is in $S^*(\alpha,\delta,\xi,\gamma)$ where

$$\delta \geq \frac{4\xi\beta^2(q-\alpha)(2q+n)}{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n)]^2-4\xi\beta^2(q-\alpha)[2\xi(q+n+\alpha)-\gamma(2q+n)]}.$$

Proof. $f(z), g(z) \in S^*(\alpha,\beta,\xi,\gamma)$ and hence

$$\sum_{n=0}^{\infty}\left[\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}\right]^2 a_{q+n}^2 \leq \left\{\sum_{n=0}^{\infty}\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}a_{q+n}\right\}^2 \leq 1. \tag{7.8}$$

Similarly,

$$\sum_{n=0}^{\infty}\left[\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}\right]^2 b_{q+n}^2 \leq \left\{\sum_{n=0}^{\infty}\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}b_{q+n}\right\}^2 \leq 1. \tag{7.9}$$

We have to show that $h(z) \in S^*(\alpha, \delta, \xi, \gamma)$. That is

$$\sum_{n=0}^{\infty} \frac{2q+n+\delta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\delta(q-\alpha)}(a_{q+n}^2+b_{q+n}^2) \leq 1. \tag{7.10}$$

Adding (7.8) and (7.9), we get

$$\sum_{n=0}^{\infty} \frac{1}{2}\left[\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}\right]^2 (a_{q+n}^2+b_{q+n}^2) \leq 1. \tag{7.11}$$

In view of (8.1) and (8.2) it is enough to show that

$$\frac{2q+n+\delta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\delta(q-\alpha)} \leq \frac{1}{2}\left[\frac{2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n))}{2\xi\beta(q-\alpha)}\right]^2.$$

This simplifies to

$$\delta \geq \frac{4\xi\beta^2(q-\alpha)(2q+n)}{[2q+n+\beta(2\xi(q+n+\alpha)-\gamma(2q+n)]^2-4\xi\beta^2(q-\alpha)[2\xi(q+n+\alpha)-\gamma(2q+n)]}.$$

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An holomorphic study of Smarandache automorphic and cross inverse property loops

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Abstract By studying the holomorphic structure of automorphic inverse property quasigroups and loops[AIPQ and (AIPL)] and cross inverse property quasigroups and loops[CIPQ and (CIPL)], it is established that the holomorph of a loop is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop if and only if its Smarandache automorphism group is trivial and the loop is itself is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop.

Keywords Smarandache loop, holomorph of loop, automorphic inverse property loop (AIPL), cross inverse property loop(CIPL), K-loop, Bruck-loop, Kikkawa-loop.

§1. Introduction

1.1 Quasigroups and loops

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^{\rho} = xJ_{\rho}, x^{\lambda} = xJ_{\lambda} \in L$ such that $xx^{\rho} = e^{\rho}$ and $x^{\lambda}x = e^{\lambda}$ are called the right, left inverses of x respectively. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. To every loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define '$\circ$' on H such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [7] and is called the Holomorph of $(L, \cdot)$. A loop(quasigroup) is a weak inverse property loop (quasigroup)[WIPL(WIPQ)] if and only if it obeys the identity

$$x(yx)^{\rho} = y^{\rho} \qquad \text{or} \qquad (xy)^{\lambda}x = y^{\lambda}.$$

A loop(quasigroup) is a cross inverse property loop(quasigroup)[CIPL(CIPQ)] if and only if it obeys the identity

$$xy \cdot x^{\rho} = y \qquad \text{or} \qquad x \cdot yx^{\rho} = y \qquad \text{or} \qquad x^{\lambda} \cdot (yx) = y \qquad \text{or} \qquad x^{\lambda}y \cdot x = y.$$

A loop(quasigroup) is an automorphic inverse property loop(quasigroup)[AIPL(AIPQ)] if and only if it obeys the identity

$$(xy)^{\rho} = x^{\rho}y^{\rho} \text{ or } (xy)^{\lambda} = x^{\lambda}y^{\lambda}$$

Consider $(G,\cdot)$ and $(H,\circ)$ being two distinct groupoids(quasigroups, loops). Let A, B and C be three distinct non-equal bijective mappings, that maps G onto H. The triple $\alpha = (A, B, C)$ is called an isotopism of $(G,\cdot)$ onto $(H,\circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

The set $SYM(G,\cdot) = SYM(G)$ of all bijections in a groupoid $(G,\cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(G,\cdot)$. If $(G,\cdot) = (H,\circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G,\cdot)$ is called an autotopism of the groupoid(quasigroup, loop) $(G,\cdot)$. Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid(quasigroup, loop) $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

The left nucleus of L denoted by $N_\lambda(L,\cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall \ x, y \in L\}$. The right nucleus of L denoted by $N_\rho(L,\cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall \ x, y \in L\}$. The middle nucleus of L denoted by $N_\mu(L,\cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall \ x, y \in L\}$. The nucleus of L denoted by $N(L,\cdot) = N_\lambda(L,\cdot) \cap N_\rho(L,\cdot) \cap N_\mu(L,\cdot)$. The centrum of L denoted by $C(L,\cdot) = \{a \in L : ax = xa \ \forall \ x \in L\}$. The center of L denoted by $Z(L,\cdot) = N(L,\cdot) \cap C(L,\cdot)$.

As observed by Osborn [22], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [2], [3], [4] and [5], Belousov and Tzurkan [6] and recent studies of Keedwell [17], Keedwell and Shcherbacov [18], [19]and [20] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations(i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography. For more on loops and their properties, readers should check [8], [10], [12], [13], [27] and [24].

Interestingly, Adeniran [1] and Robinson [25], Oyebo and Adeniran [23], Chiboka and Solarin [11], Bruck [7], Bruck and Paige [9], Robinson [26], Huthnance [14] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [14] showed that if $(L,\cdot)$ is a loop with holomorph $(H,\circ)$, $(L,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

For the definitions of inverse property loop (IPL), Bol loop and A-loop readers can check earlier references on loop theory.

Here, a K-loop is an A-loop with the AIP, a Bruck loop is a Bol loop with the AIP and a Kikkawa loop is an A-loop with the IP and AIP.

1.2 Smarandache quasigroups and loops

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [27], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In [16], the present author defined a Smarandache quasigroup (S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subsemigroup (S-subsemigroup). Examples of Smarandache quasigroups are given in Muktibodh [21]. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [28], on the study of Smarandache notions in algebraic structures, she introduced Smarandache : left(right) alternative loops, Bol loops,

Moufang loops, and Bruck loops. But in [15], the present author introduced Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops.

A loop is called a Smarandache A-loop(SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache K-loop(SKL) if it has at least a non-trivial subloop that is a K-loop.

A loop is called a Smarandache Bruck-loop(SBRL) if it has at least a non-trivial subloop that is a Bruck-loop.

A loop is called a Smarandache Kikkawa-loop(SKWL) if it has at least a non-trivial subloop that is a Kikkawa-loop.

If L is a S-groupoid with a S-subsemigroup H, then the set $SSYM(L,\cdot) = SSYM(L)$ of all bijections A in L such that $A : H \to H$ forms a group called the Smarandache permutation(symmetric) group of the S-groupoid. In fact, $SSYM(L) \leq SYM(L)$.

The left Smarandache nucleus of L denoted by $SN_\lambda(L,\cdot) = N_\lambda(L,\cdot) \cap H$. The right Smarandache nucleus of L denoted by $SN_\rho(L,\cdot) = N_\rho(L,\cdot) \cap H$. The middle Smarandache nucleus of L denoted by $SN_\mu(L,\cdot) = N_\mu(L,\cdot) \cap H$. The Smarandache nucleus of L denoted by $SN(L,\cdot) = N(L,\cdot) \cap H$. The Smarandache centrum of L denoted by $SC(L,\cdot) = C(L,\cdot) \cap H$. The Smarandache center of L denoted by $SZ(L,\cdot) = Z(L,\cdot) \cap H$.

Definition 1.1. Let $(L,\cdot)$ and $(G,\circ)$ be two distinct groupoids that are isotopic under a triple (U,V,W). Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $A : L' \to G'$, where $A \in \{U,V,W\}$, then the isotopism $(U,V,W) : (L,\cdot) \to (G,\circ)$ is called a Smarandache isotopism(S-isotopism).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L,\cdot) \gtrsim (G,\circ)$.

But if $(L,\cdot) = (G,\circ)$, then the autotopism (U,V,W) is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L,\cdot)$ which will be called the Smarandache autotopism group of $(L,\cdot)$. Observe that $SAUT(L,\cdot) \leq AUT(L,\cdot)$. Furthermore, if $U = V = W$, then U is called a Smarandache automorphism of $(L,\cdot)$. Such Smarandache permutations form a group $SAUM(L,\cdot)$ called the Smarandache automorphism group(SAG) of $(L,\cdot)$.

Let L be a S-quasigroup with a S-subgroup G. Now, set $H_S = (G,\cdot) \times SAUM(L,\cdot)$. If we define '$\circ$' on H_S such that $(\alpha,x) \circ (\beta,y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha,x),(\beta,y) \in H_S$, then $H_S(L,\cdot) = (H_S,\circ)$ is a quasigroup.

If in L, $s^\lambda \cdot s\alpha \in SN(L)$ or $s\alpha \cdot s^\rho \in SN(L)$ $\forall s \in G$ and $\alpha \in SAUM(L,\cdot)$, $(H_S,\circ)$ is called a Smarandache Nuclear-holomorph of L, if $s^\lambda \cdot s\alpha \in SC(L)$ or $s\alpha \cdot s^\rho \in SC(L)$ $\forall$ $s \in G$ and $\alpha \in SAUM(L,\cdot)$, $(H_S,\circ)$ is called a Smarandache Centrum-holomorph of L hence a Smarandache Central-holomorph if $s^\lambda \cdot s\alpha \in SZ(L)$ or $s\alpha \cdot s^\rho \in SZ(L)$ $\forall$ $s \in G$ and $\alpha \in SAUM(L,\cdot)$.

The aim of the present study is to investigate the holomorphic structure of Smarandache AIPLs and CIPLs(SCIPLs and SAIPLs) and use the results to draw conclusions for Smarandache K-loops(SKLs), Smarandache Bruck-loops(SBRLs) and Smarandache Kikkawa-loops (SKWLs). This is done as follows.

1. The holomorphic structure of AIPQs(AIPLs) and CIPQs(CIPLs) are investigated. Necessary and sufficient conditions for the holomorph of a quasigroup(loop) to be an AIPQ(AIPL) or CIPQ(CIPL) are established. It is shown that if the holomorph of a quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL), then the holomorph is isomorphic to the quasigroup(loop). Hence, the holomorph of a quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL) if and only if its automorphism group is trivial and the quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL). Furthermore, it is discovered that if the holomorph of a quasigroup(loop) is a CIPQ(CIPL), then the quasigroup(loop) is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

2. The holomorph of a loop is shown to be a SAIPL, SCIPL, SKL, SBRL or SKWL respectively if and only its SAG is trivial and the loop is a SAIPL, SCIPL, SKL, SBRL, SKWL respectively.

§2. Main results

Theorem 2.1. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is an AIPQ(AIPL) if and only if

1. $AUM(L)$ is an abelian group,

2. $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and

3. L is a AIPQ(AIPL).

Proof. A quasigroup(loop) is an automorphic inverse property loop(AIPL) if and only if it obeys the AIP identity.

Using either of the definitions of an AIPQ(AIPL), it can be shown that $H(L)$ is a AIPQ(AIPL) if and only if $AUM(L)$ is an abelian group and $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$. L is isomorphic to a subquasigroup(subloop) of $H(L)$, so L is a AIPQ(AIPL) which implies $(J_\rho, J_\rho, J_\rho) \in AUT(L)$. So, $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

Corollary 2.1. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if

1. $AUM(L)$ is an abelian group,

2. $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and

3. L is a CIPQ(CIPL).

Proof. A quasigroup(loop) is a CIPQ(CIPL) if and only if it is a WIPQ(WIPL) and an AIPQ(AIPL). L is a WIPQ(WIPL) if and only if $H(L)$ is a WIPQ(WIPL).

If $H(L)$ is a CIPQ(CIPL), then $H(L)$ is both a WIPQ(WIPL) and a AIPQ(AIPL) which implies 1., 2., and 3. of Theorem 2.1. Hence, L is a CIPQ(CIPL). The converse follows by just doing the reverse.

Corollary 2.2. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is an AIPQ(AIPL) or CIPQ(CIPL), then $H(L) \cong L$.

Proof. By 2. of Theorem 2.1, $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ implies $x\beta^{-1} \cdot y\alpha = x \cdot y$ which means $\alpha = \beta = I$ by substituting $x = e$ and $y = e$. Thus, $AUM(L) = \{I\}$ and so $H(L) \cong L$.

Theorem 2.2. The holomorph of a quasigroup(loop) L is a AIPQ(AIPL) or CIPQ(CIPL) if and only if $AUM(L) = \{I\}$ and L is a AIPQ(AIPL) or CIPQ(CIPL).

Proof. This is established using Theorem 2.1, Corollary 2.1 and Corollary 2.2.

Theorem 2.3. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if $AUM(L)$ is an abelian group and any of the following is true for all $x, y \in L$ and $\alpha, \beta \in AUM(L)$.

1. $(x\beta \cdot y)x^{\rho} = y\alpha$.
2. $x\beta \cdot yx^{\rho} = y\alpha$.
3. $(x^{\lambda}\alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.
4. $x^{\lambda}\alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Proof. This is achieved by simply using the four equivalent identities that define a CIPQ(CIPL):

Corollary 2.3. Let $(L, \cdot)$ be a quasigroups(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, the following are equivalent to each other

1. $(\beta^{-1}J_{\rho}, \alpha J_{\rho}, J_{\rho}) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.
2. $(\beta^{-1}J_{\lambda}, \alpha J_{\lambda}, J_{\lambda}) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.
3. $(x\beta \cdot y)x^{\rho} = y\alpha$.
4. $x\beta \cdot yx^{\rho} = y\alpha$.
5. $(x^{\lambda}\alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.
6. $x^{\lambda}\alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Hence, $(\beta, \alpha, I), (\alpha, \beta, I), (\beta, I, \alpha), (I, \alpha, \beta) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

Proof. The equivalence of the six conditions follows from Theorem 2.3 and the proof of Theorem 2.1. The last part is simple.

Corollary 2.4. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, L is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

Proof. It is observed that $J_{\rho} = J_{\lambda} = I$. Hence, the conclusion follows.

Remark. The holomorphic structure of loops such as extra loop, Bol-loop, C-loop, CC-loop and A-loop have been found to be characterized by some special types of automorphisms such as

1. Nuclear automorphism(in the case of Bol-,CC- and extra loops),
2. central automorphism(in the case of central and A-loops).

By Theorem 2.1 and Corollary 2.1, the holomorphic structure of AIPLs and CIPLs is characterized by commutative automorphisms.

Theorem 2.4. The holomorph $H(L)$ of a quasigroup(loop) L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL) if and only if $SAUM(L) = \{I\}$ and L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL).

Proof. Let L be a quasigroup with holomorph $H(L)$. If $H(L)$ is a SAIPQ(SCIPQ), then there exists a S-subquasigroup $H_S(L) \subset H(L)$ such that $H_S(L)$ is a AIPQ(CIPQ). Let $H_S(L) = G \times SAUM(L)$ where G is the S-subquasigroup of L. From Theorem 2.2, it can be seen that $H_S(L)$ is a AIPQ(CIPQ) if and only if $SAUM(L) = \{I\}$ and G is a AIPQ(CIPQ). So the conclusion follows.

Corollary 2.5. The holomorph $H(L)$ of a loop L is a SKL or SBRL or SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKL or SBRL or SKWL.

Proof. Let L be a loop with holomorph $H(L)$. Consider the subloop $H_S(L)$ of $H(L)$ such that $H_S(L) = G \times SAUM(L)$ where G is the subloop of L.

1. Recall that by [Theorem 5.3, [9]], $H_S(L)$ is an A-loop if and only if it is a Smarandache Central-holomorph of L and G is an A-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKL if and only if $SAUM(L) = \{I\}$ and L is a SKL.

2. Recall that by [25] and [1], $H_S(L)$ is a Bol loop if and only if it is a Smarandache Nuclear-holomorph of L and G is a Bol-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SBRL if and only if $SAUM(L) = \{I\}$ and L is a SBRL.

3. Following the first reason in 1., and using Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKWL.

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An equation involving Euler's ϕ function

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Abstract For any positive integer n, let $\phi(n)$ be Euler's ϕ function and $\Omega(n)$ denote the total number of prime factors of n. The main purpose of this paper is using the elementary methods to study the solutions of the equation $\phi(\phi(n)) = 2^{\Omega(n)}$, and give all odd positive integer solutions for it.

Keywords Euler's ϕ function, equation, solution.

§1. Introduction and main result

For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the factorization of n into prime powers. Then the arithmetic function $\Omega(n)$ is defined as

$$\Omega(1) = 0, \qquad \Omega(n) = \sum_{i=1}^{k} \alpha_i.$$

And the famous Euler's ϕ function is defied as the number of integers less than n that are relatively prime to n. It is known to all that the equation $\phi(x) = n$ is of great importance in the study of number theory. Many scholars had studied numerous properties of this equation, and made great progress about it. For example, let $b_m = \text{card}\{\phi(n) = m\}$, where m is any positive integer, P.Erdös [1] found that there exists $\delta > 0$ such that $b_m > m^\delta$ for infinitely many m. And K. Woolridge [2] proved that the inequality holds for infinitely many m if $0 < \delta < 3 - 2\sqrt{2}$. These results are in connection with Carmichael's conjecture [3], which states that for every n it is possible to find an $m \neq n$ such that $\phi(m) = \phi(n)$! Moreover, C. Pomerance [4] obtained the result that

$$b_m \leq m \cdot \exp(-(1 + o(1)) \ln m \cdot \ln\ln\ln m / \ln\ln m).$$

Let $S_k(n)$ denotes the number of all solutions of $\phi(x) = n!$, where x has exactly k prime factors which appear to the first power, H.Gupta [5] claimed that for every n, we have

$$S_1(n!) \geq 1,$$

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and

$$S_1(n!) \to \infty \ \ (n \to \infty).$$

And P.Erdös [6] proved that for any integer k and sufficiently larger n, we have

$$S_k(n!) > \frac{cn^k}{\log^k n},$$

where $c > 0$ is a constant number.

In this paper, we shall use the elementary method to study the solutions of the equation $\phi(\phi(n)) = 2^{\Omega(n)}$, and give all odd solutions for it. That is, we shall prove the following:

Theorem. Let n be any odd number, then the equation $\phi(\phi(n)) = 2^{\Omega(n)}$ has only 6 odd positive integer solutions, namely,

$$n = 1, 5, 7, 15, 21, 45.$$

Remark. Let n be any even number, then there are infinitely many solutions for the equation $\phi(\phi(n)) = 2^{\Omega(n)}$. This can be see from the fact that for any positive integer m, we have $\phi(\phi(2^m \times 35)) = 2^{\Omega(2^m \times 35)} = 2^{m+2}$.

§2. Proof of Theorem

In this section, we shall complete the proof of our Theorem.

It is clear that $n = 1$ is a solution of the equation $\phi(\phi(n)) = 2^{\Omega(n)}$. If $n > 1$, let any odd number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the factorization of n into prime powers, where $3 \le p_1 < p_2 < \cdots < p_k$. Then from the definitions of $\phi(n)$ and $\Omega(n)$, we have

$$\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1),$$

and

$$2^{\Omega(n)} = 2^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}.$$

Firstly, we shall come to prove that all $\alpha_i \le 2,\ i = 1, 2, 3 \cdots k$. In fact, if there exists some $\alpha_j \ge 3 (1 \le j \le k)$, we may have ${p_j}^2 \mid \phi(n)$, and $p_j \mid \phi(\phi(n))$, then $p_j \mid 2^{\Omega(n)}$, which contradicts to the parity of p_j!

Now we shall discuss the problem in the following three cases:

Case 1. If $k = 1$, then $n = p^{\alpha},\ (\alpha = 1, 2)$

(1) If $\alpha = 1$, then $\phi(\phi(n)) = \phi(p - 1)$ and $2^{\Omega(p)} = 2$, so the equation has two solutions: $n = 5, 7$.

(2) If $\alpha = 2$, then $\phi(\phi(n)) = \phi(p(p - 1))$ and $2^{\Omega(p^2)} = 4$. It is clear that there is no prime $p \ge 3$ satisfies the equation $\phi(\phi(p^2)) = 2^{\Omega(p^2)}$.

Case 2. If $k = 2$, then $n = p_1^{\alpha_1} p_2^{\alpha_2}, (\alpha_i = 1, 2)$.

(1) If $\alpha_1 = \alpha_2 = 1$, then $\phi(\phi(n)) = \phi((p_1 - 1)(p_2 - 1))$ and $2^{\Omega(p)} = 2^2 = 4$. So we have $p_1 = 3$ and $p_2 = 5, 7$. Therefore, the equation has two solutions: $n = 15,\ 21$.

(2) If $\alpha_1 = 2,\ \alpha_2 = 1$, then $\phi(\phi(n)) = \phi(p_1(p_1 - 1)(p_2 - 1))$ and $2^{\Omega(p_1^2 p_2)} = 2^3 = 8$. We have $p_1 = 3$ and $p_2 = 5$. So the equation has only one solution: $n = 45$.

(3) If $\alpha_1 = 1,\ \alpha_2 = 2$, then we have

$$\phi(\phi(n)) = \phi(p_2(p_1-1)(p_2-1)) = \phi\left(2^2 p_2\left(\frac{p_1-1}{2}\right)\left(\frac{p_2-1}{2}\right)\right),$$

and

$$2^{\Omega(p_1p_2^2)} = 2^3 = 8.$$

Noting that

$$\frac{p_2-1}{2} \geq 2,$$

so we have

$$\phi\left(2^2 p_2\left(\frac{p_1-1}{2}\right)\left(\frac{p_2-1}{2}\right)\right) > 8.$$

This shows that the equation has no solution.

(4) If $\alpha_1 = 2,\ \alpha_2 = 2$, then we have

$$\phi(\phi(n)) = \phi(p_1p_2(p_1-1)(p_2-1)) = \phi\left(2^2 p_1p_2\left(\frac{p_1-1}{2}\right)\left(\frac{p_2-1}{2}\right)\right),$$

and

$$2^{\Omega(p_1^2p_2^2)} = 2^4.$$

Noting that

$$\phi(p_1) \geq 2, \qquad \phi(p_2) \geq 4, \qquad \text{and} \qquad \frac{p_2-1}{2} \geq 2,$$

then we have

$$\phi\left(2^2 p_1p_2\left(\frac{p_1-1}{2}\right)\left(\frac{p_2-1}{2}\right)\right) > 2^4.$$

So in this case, the equation has no solution.

Case 3. If $k > 2$, then $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, (\alpha_i = 1, 2)$

(1) If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$, then we have

$$\phi(\phi(n)) = \phi\left(2^k\left(\frac{p_1-1}{2}\right)\left(\frac{p_2-1}{2}\right)\cdots\left(\frac{p_k-1}{2}\right)\right),$$

and

$$2^{\Omega(n)} = 2^k.$$

Noting that

$$\frac{p_2-1}{2} \geq 2, \qquad \frac{p_3-1}{2} \geq 2,$$

we may have $\phi(n) > 2^k$. So the equation has no solution.

(2) If there exists some $\alpha_j (1 \leq j \leq k)$ satisfies $\alpha_j = 2$, we can assume that

$$n = q_1^2q_2^2\cdots q_m^2 \times q_{m+1}\cdots q_k, (1 \leq m \leq k),$$

where $q_i (i = 1, 2\cdots, k)$ are all primes. Then we have

$$\phi(\phi(n)) = \phi\left(2^k q_1q_2\cdots q_m\left(\frac{q_1-1}{2}\right)\left(\frac{q_2-1}{2})\right)\cdots\left(\frac{q_k-1}{2}\right)\right),$$

and

$$2^{\Omega(n)} = 2^{m+k}.$$

We can deduce that for any $1 \leq i, j \leq k$, we have $\dfrac{q_i - 1}{2} \neq q_j$! Otherwise, we may have $q_s \mid 2^{k+m}$. Noting the fact that

$$\frac{q_k - 1}{2} \geq 2,$$

we may have

$$\phi(\phi(n)) > 2^{k+m}.$$

This shows the equation has no solution.

Combining all the above cases, we may immediately get all solutions of the equation $\phi(\phi(n)) = 2^{\Omega(n)}$, namely,

$$n = 1, 5, 7, 15, 21, 45.$$

This completes the proof of Theorem.

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On a Smarandache multiplicative function and its parity[1]

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Abstract For any positive integer n, we define the Smarandache multiplicative function $U(n)$ as follows: $V(1) = 1$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then $U(n) = \max\{\alpha_1 \cdot p_1,\ \alpha_2 \cdot p_2,\ \cdots,\ \alpha_s \cdot p_s\}$. The main purpose of this paper is using the elementary and analytic methods to study the parity of $U(n)$, and give an interesting asymptotic formula for it.

Keywords Smarandache multiplicative function, parity, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache multiplicative function $U(n)$ is defined as $U(1) = 1$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then

$$U(n) = \max\{\alpha_1 \cdot p_1,\ \alpha_2 \cdot p_2,\ \cdots,\ \alpha_s \cdot p_s\}.$$

For example, the first few value of $U(n)$ are: $U(1) = 1$, $U(2) = 2$, $U(3) = 3$, $U(4) = 4$, $U(5) = 5$, $U(6) = 3$, $U(7) = 7$, $U(8) = 6$, $U(9) = 6$, $U(10) = 5$, $U(11) = 11$, $U(12) = 4$, $U(13) = 13$, $U(14) = 7$, $U(15) = 5$, $\cdots$. About the arithmetical properties of $U(n)$, some authors had studied it, and obtained some interesting results, see references [3] and [4]. For example, Xu Zhefeng [3] proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} (U(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

In an unpublished paper, Pan Xiaowei proved that the equation

$$\sum_{d|n} U(d) = n$$

has only two positive integer solutions $n = 1$ and 28, where $\sum_{d|n}$ denotes the summation over all positive divisors of n.

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Now we let $OU(n)$ denotes the number of all integers $1 \le k \le n$ such that $U(n)$ is odd. $EU(n)$ denotes the number of all integers $1 \le k \le n$ such that $U(n)$ is even. An interesting problem is to determine the limit:

$$\lim_{n\longrightarrow\infty} \frac{EU(n)}{OU(n)}. \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen such a paper before. The problem is interesting, because it can help us to know more information about the parity of $U(n)$.

The main purpose of this paper is using the elementary and analytic methods to study this problem, and give an interesting asymptotic formula for $\dfrac{EU(n)}{OU(n)}$. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we have the asymptotic formula

$$\frac{EU(n)}{OU(n)} = O\left(\frac{1}{\ln n}\right).$$

From this Theorem we may immediately deduce the following:

Corollary. For any positive integer n, we have the limit

$$\lim_{n\longrightarrow\infty} \frac{EU(n)}{OU(n)} = 0.$$

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First we estimate the upper bound of $EU(n)$. In fact for any integer $k > 1$, let $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of k into prime powers, then from the definition and properties of $U(k)$ we have $U(k) = U\left(p_i^{\alpha_i}\right) = \alpha_i \cdot p_i$. If $\alpha_i = 1$, then $U(k) = p_i$ be an odd number, except $k = 2$. Let $M = \ln n$, then we have

$$EU(n) = \sum_{\substack{k\le n\\ 2|U(k)}} 1 \le 1 + \sum_{\substack{k\le n\\ U(k)=\alpha_i p_i,\ \alpha_i\ge 2}} 1 \le 1 + \sum_{U(k)\le M} 1 + \sum_{\substack{kp^\alpha\le n\\ \alpha p>M,\ \alpha\ge 2}} 1. \tag{2}$$

Now we estimate the each term in (2) respectively. We have

$$\begin{aligned}
&\sum_{\substack{kp^\alpha\le n\\ \alpha p>M,\ \alpha\ge 2}} 1 \le \sum_{\substack{kp^2\le n\\ 2p>M}} 1 + \sum_{\substack{kp^\alpha\le n\\ \alpha p>M,\ \alpha\ge 3}} 1 \le \sum_{\frac{M}{2}<p\le\sqrt{n}} \sum_{k\le\frac{n}{p^2}} 1 + \sum_{\substack{p^\alpha\le n\\ \alpha p>M,\ \alpha\ge 3}} \sum_{k\le\frac{n}{p^\alpha}} 1\\
\ll\ & \sum_{\frac{M}{2}<p\le\sqrt{n}} \frac{n}{p^2} + \sum_{\substack{p^\alpha\le n\\ \alpha p>M,\ \alpha\ge 3}} \frac{n}{p^\alpha} \ll \frac{n}{\ln n} + \sum_{\substack{p\le\sqrt{n}\\ \alpha p>M,\ \alpha\ge p}} \frac{n}{p^\alpha} + \sum_{\substack{p\le\sqrt{n}\\ \alpha p>M,\ 3\le\alpha<p}} \frac{n}{p^\alpha}\\
\ll\ & \frac{n}{\ln n} + \sum_{\substack{p\le\sqrt{n}\\ \alpha>\sqrt{M}}} \frac{n}{p^\alpha} + \sum_{\substack{p\le\sqrt{n}\\ p>\sqrt{M},\ \alpha\ge 3}} \frac{n}{p^\alpha}\\
\ll\ & \frac{n}{\ln n} + \frac{n}{2^{\sqrt{M}-1}} + \frac{n}{M} \ll \frac{n}{\ln n}.
\end{aligned} \tag{3}$$

In order to estimate another term in (2), we must use a new method. For any prime $p \le M$, let $\alpha(p) = \left[\frac{M}{p}\right]$, where $[x]$ denotes the largest integer less than or equal to x. Let $m = \prod_{p \le M} p^{\alpha(p)}$. It is clear that for any positive integer k with $U(k) \le M$, we have $k|m$. And for any positive divisor k of m, we also have $U(k) \le M$. So from these properties we have

$$\begin{aligned} \sum_{U(k)\le M} 1 &\le \sum_{d|u} 1 = \prod_{p\le M} (1+\alpha(p)) = \prod_{p\le M} \left(1+\left[\frac{M}{p}\right]\right) \\ &= \exp\left(\sum_{p\le M} \ln\left(1+\left[\frac{M}{p}\right]\right)\right), \end{aligned} \tag{4}$$

where $\exp(y) = e^y$.

From the Prime Theorem (see reference [5], Theorem 3.10)

$$\pi(M) = \sum_{p\le M} 1 = \frac{M}{\ln M} + O\left(\frac{M}{\ln^2 M}\right)$$

and

$$\sum_{p\le M} \ln p = M + O\left(\frac{M}{\ln M}\right)$$

we have

$$\begin{aligned} &\sum_{p\le M} \ln\left(1+\left[\frac{M}{p}\right]\right) \le \sum_{p\le M} \ln\left(1+\frac{M}{p}\right) \\ = &\sum_{p\le M} [\ln(p+M) - \ln p] \\ \le &\pi(M)\cdot\ln(2M) - \sum_{p\le M} \ln p \\ = &\frac{M\cdot\ln(2M)}{\ln M} - M + O\left(\frac{M}{\ln M}\right) = O\left(\frac{M}{\ln M}\right). \end{aligned} \tag{5}$$

Note that $M = \ln n$, from (4) and (5) we may get the estimate:

$$\sum_{U(k)\le M} 1 \ll \exp\left(\frac{c\cdot\ln n}{\ln\ln n}\right), \tag{6}$$

where c is a positive constant.

It is clear that $\exp\left(\frac{c\cdot\ln n}{\ln\ln n}\right) \ll \frac{n}{\ln n}$, so combining (2), (3) and (6) we may immediately deduce the estimate:

$$EU(n) = \sum_{\substack{k\le n \\ 2|U(k)}} 1 = O\left(\frac{n}{\ln n}\right).$$

Note that $OU(n)+EU(n) = n$, from the above estimate we can deduce the asymptotic formula:

$$OU(n) = n - EU(n) = n + O\left(\frac{n}{\ln n}\right).$$

Therefore,

$$\frac{EU(n)}{OU(n)} = \frac{O\left(\frac{n}{\ln n}\right)}{n + O\left(\frac{n}{\ln n}\right)} = O\left(\frac{1}{\ln n}\right).$$

This completes the proof of Theorem.

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On a problem of F.Smarandache

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Abstract For any positive integer n, the famous Euler function $\phi(n)$ is defined as the number of all integers m with $1 \leq m \leq n$ such that $(m, n) = 1$. In his book "Only problems, not solutions" (see unsolved problem 52), Professor F.Smarandache asked us to find the smallest positive integer $k \equiv k(n)$, such that $\phi_k(n) = 1$, where $\phi_1(n) = \phi(n)$, $\phi_2(n) = \phi(\phi_1(n))$, $\cdots$, and $\phi_k(n) = \phi(\phi_{k-1}(n))$. In this paper, we using the elementary method to study this problem, and prove that for any positive integer n, $k(n) = \min\{m : 2^m \geq n, m \in N\}$, where N denotes the set of all positive integers.

Keywords The Smarandache problem, Euler function, elementary method.

§1. Introduction and Results

For any positive integer n, the famous Euler function $\phi(n)$ is defined as the number of all integers m with $1 \leq m \leq n$ such that $(m, n) = 1$. In his book "Only problems, not solutions" (see unsolved problem 52), Professor F.Smarandache asked us to find the smallest positive integer $k \equiv k(n)$, such that $\phi_k(n) = 1$, where $\phi_1(n) = \phi(n)$, $\phi_2(n) = \phi(\phi_1(n))$, $\cdots$, and $\phi_k(n) = \phi(\phi_{k-1}(n))$. That is, $k(n)$ is the smallest number of iteration k such that $\phi_k(n) = \phi(\phi_{k-1}(n)) = 1$. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because it can help us to know more properties of the Euler function. It is clear that $\phi(n) < n$, if $n > 1$. So $\phi_1(n)$, $\phi_2(n)$, $\phi_3(n)$, $\cdots$, $\phi_k(n)$ is a monotone decreasing sequence. Therefore, for any integer $n > 1$, there must exist a positive integer $k \equiv k(n)$ such that $\phi_k(n) = 1$. In this paper, we using the elementary method to study this problem, and find an exact function $k = k(n)$ such that for any integer $n > 1$, $\phi_k(n) = 1$. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we define $k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}$, where N denotes the set of all positive integers. Then we have the identity

$$\phi_k(n) = \phi\left(\phi_{k-1}(n)\right) = \phi_{k-1}\left(\phi(n)\right) = 1,$$

where $\phi(n)$ is the Euler function.

Corollary. For any positive integer $n > 1$, Let $k \equiv k(n)$ be the smallest positive integer such that $\phi_k(n) = 1$. Then we have

$$k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. For any integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. Then from the properties of the Euler function $\phi(n)$ we have

$$\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_r^{\alpha_r - 1}(p_r - 1).$$

From this formula we know that if n be an even number, then $\phi(n) \leq \frac{n}{2}$. If $n > 1$ be an odd number, then $\phi(n) \leq n - 1$, and $\phi(1) = 1$. So for any integer $n \geq 3$ and $k \equiv k(n) = \min\{m : 2^m \geq n,\ m \in N\}$, from the definition of $k = k(n)$ we have $\phi_1(n) = \phi(n) \leq n - 1$, $\phi_2(n) = \phi(\phi(n)) \leq \frac{1}{2}\phi(n) \leq \frac{n-1}{2}$, $\cdots\cdots$, $\phi_k(n) \leq \frac{1}{2}\phi_{k-1}(n) \leq \cdots \leq \frac{n-1}{2^{k-1}} = \frac{2(n-1)}{2^k} \leq 2\left(1 - \frac{1}{n}\right)$. Since $\phi_k(n)$ is a positive integer and $1 \leq \phi_k(n) \leq 2\left(1 - \frac{1}{n}\right) < 2$, so we must have $\phi_k(n) = 1$.

For any positive integer n, Let $u \equiv u(n)$ be the smallest positive integer such that $\phi_u(n) = 1$. From the above we know that $\phi_k(n) = 1$, so $u(n) \leq k(n)$. On the other hand, let $n = 2^m$, where $m \geq 1$ be an integer. Then $\phi(n) = 2^{m-1}$, $\phi_2(n) = 2^{m-2}$, $\cdots\cdots$, $\phi_{m-1}(n) = 2$, $\phi_m(n) = 1$. So $u(n) = m = k(n)$. Let $n = 2^m + 1$ be a prime, then $\phi(n) = p - 1 = 2^m$, $\phi_m(n) = 2$ and $\phi_{m+1}(n) = 1$. So $u(n) = m + 1$. This time, we also have $k \equiv k(n) = \min\{s : 2^s \geq 2^m + 1,\ s \in N\} = m + 1$. That is to say, there are infinite positive integers $n > 1$ such that $u(n) = k(n)$. Therefore, for any integer $n > 1$, $k \equiv k(n) = \min\{m : 2^m \geq n,\ m \in N\}$ be the smallest positive integer such that $\phi_k(n) = 1$. This completes the proof of our Theorem.

§3. Several similar problems

Now we consider the Dirichlet divisor function $d(n)$, the number of all positive divisors of n. For any integer $n \geq 3$, it is clear that $d(n) < n$. Let $d_1(n) = d(n)$, $d_2(n) = d(d(n))$, $\cdots\cdots$, $d_k(n) = d(d_{k-1}(n))$. So $\{d_1(n),\ d_2(n),\ \cdots\cdots,\ d_k(n),\ \cdots\}$ is also a monotone decreasing sequence. For any positive $n > 1$, let $k = k(n)$ be the smallest positive integer such that $d_k(n) = 2$. Whether there exists a simple arithmetical function $k = k(n)$ such that $d_k(n) = 2$ for all $n > 3$. This is an open problem.

For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. We define function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and $\Omega(1) = 0$. Similarly, find the smallest positive integer $k = k(n)$ such that $\Omega_k(n) = 0$, where $\Omega_1(n) = \Omega(n)$, $\Omega_k(n) = \Omega(\Omega_{k-1}(n))$.

Here we can also give a simple arithmetical function for $k(n)$. Let $u_1 = 1$, $u_2 = 2^{u_1}$, $\cdots$, $u_{k+1} = 2^{u_k}$. It is clear that $\{u_k\}$ be a strictly monotone increasing sequence. Now we define $k \equiv k(n) = \min\{m : u_m \geq n,\ m \in N\}$, where N denotes the set of all positive integers. It is easy to prove that $\Omega_k(n) = 0$. On the other hand, for any positive integer $m > 1$, we have $\Omega(u_m) = u_{m-1}$, and $\Omega_m(u_m) = 0$. Therefore, for any integer $n > 1$, $k \equiv k(n) = \min\{m : u_m \geq n,\ m \in N\}$ be the smallest positive integer such that $\Omega_k(n) = 0$.

Whether there exists another more simple function $k(n)$ such that $\Omega_k(n) = 0$ is an unsolved problem.

Let $n > 1$ be an integer, and $\sigma(n)$ be the sum of all positive divisors of n. It is clear that $\sigma(n) > n$ for any $n > 1$. So if $n > 1$, then $\{\sigma_1(n),\ \sigma_2(n), \cdots,\ \sigma_k(n), \cdots\}$ must be a strictly

monotone increasing sequence, where $\sigma_1(n) = \sigma(n)$, and $\sigma_k(n) = \sigma(\sigma_{k-1}(n))$. Now let N be any fixed positive integer. For any integer $n \geq 2$, find the smallest positive integer $k = k(N)$ such that $\sigma_k(n) \geq N$.

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On the Smarandache reciprocal function and its mean value

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Abstract For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \nmid n!$. The main purpose of this paper is using the elementary and analytic methods to study the mean value distribution properties of $S_c(n)$, and give two interesting mean value formulas for it.

Keywords The Smarandache reciprocal function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. And the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \nmid n!$. That is, $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \nmid n!\}$. The first few values of $S_c(n)$ are:

$$S_c(1) = 1, \ S_c(2) = 2, \ S_c(3) = 3, \ S_c(4) = 4, \ S_c(5) = 6, \ S_c(6) = 6,$$
$$S_c(7) = 10, \ S_c(8) = 10, \ S_c(9) = 10, \ S_c(10) = 10, \ S_c(11) = 12, \ S_c(12) = 12,$$
$$S_c(13) = 16, \ S_c(14) = 16, \ S_5(15) = 16, \ S_c(16) = 16, \ S_c(17) = 18, \ \cdots\cdots .$$

About the properties of $S(n)$, many authors had studied it, and obtained a series results, see references [1], [2], [3], [4], [5] and [15]. For example, Jozsef Sandor [4] proved that for any positive integer $k \geq 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

On the other hand, in reference [6], A.Murthy studied the elementary properties of $S_c(n)$, and proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the Smarandache reciprocal function $S_c(n)$, and give two interesting mean value formulas it. That is, we shall prove the following conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} S_c(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

Theorem 2. For any real number $x > 1$, we have the low bound estimate

$$\frac{1}{x}\sum_{n\leq x}\left(S_c(n)-n\right)^2 \geq \frac{1}{3}\cdot \ln^2 x + O\left(x^{-\frac{5}{12}}\cdot \ln^2 x\right).$$

From Theorem 2 we may immediately deduce the following:

Corollary. The limit

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\left(S_c(n)-n\right)^2$$

does not exist.

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. For any real number $x > 1$, let $2 = p_1 < p_2 < \cdots\cdots < p_k \leq x$ denote all primes less than or equal to x. Then from the result of A.Murthy [6] we have the identity

$$\begin{aligned}
\sum_{n\leq x} S_c(n) &= S_c(1)+S_c(2)+S_c(3)+S_c(4)+\sum_{i=3}^{k-1}\sum_{p_i\leq n<p_{i+1}} S_c(n)+\sum_{p_k\leq n\leq x} S_c(n)\\
&= 1+2+3+4+\sum_{i=3}^{k-1}\sum_{p_i\leq n<p_{i+1}}(p_{i+1}-1)+\sum_{p_k\leq n\leq x}(p_{k+1}-1)\\
&= \sum_{i=1}^{k-1}(p_{i+1}-p_i)(p_{i+1}-1)+O\left((x-p_k)(p_{k+1}-1)\right)\\
&= \frac{1}{2}\sum_{i=1}^{k-1}\left[(p_{i+1}-p_i)^2+p_{i+1}^2-p_i^2\right]-\sum_{i=1}^{k-1}(p_{i+1}-p_i)+O\left((x-p_k)\cdot p_{k+1}\right)\\
&= \frac{1}{2}\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2+\frac{1}{2}\cdot p_k^2-p_k+O\left((x-p_k)\cdot p_{k+1}\right). \qquad (1)
\end{aligned}$$

For any real number x large enough, from M.N.Huxley [7] we know that there at least exists a prime in the interval $\left[x,\ x+x^{\frac{7}{12}}\right]$. So we have the estimate

$$(x-p_k)\cdot p_{k+1} = O\left(x^{\frac{19}{12}}\right). \qquad (2)$$

On the other hand, from the D.R.Heath-Brown's famous result [8], [9] and [10] we know that for any real number $\epsilon > 0$, we have the estimate

$$\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2 \ll x^{\frac{23}{18}+\epsilon}. \tag{3}$$

Note that $p_k = x + O\left(x^{\frac{7}{12}}\right)$, from (1), (2) and (3) we may immediately get the asymptotic formula

$$\sum_{n\le x} S_c(n) = \frac{1}{2}\cdot\left[x + O\left(x^{\frac{7}{12}}\right)\right]^2 + O\left(x^{\frac{19}{12}}\right) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

This proves Theorem 1.

Now we prove Theorem 2. For any real number $x > 1$, from the definition and properties of $S_c(n)$ we also have the identity

$$\begin{aligned}
\sum_{n\le x}\left(S_c(n)-n\right)^2 &\ge \sum_{i=1}^{k-1}\sum_{p_i\le n<p_{i+1}}\left(S_c(n)-n\right)^2 = \sum_{i=3}^{k-1}\sum_{0\le n<p_{i+1}-p_i}(p_{i+1}-p_i-n-1)^2 \\
&= \sum_{i=3}^{k-1}\sum_{0\le n<p_{i+1}-p_i}\left[(p_{i+1}-p_i)^2 - 2(n+1)\cdot(p_{i+1}-p_i) + (n+1)^2\right] \\
&= \sum_{i=3}^{k-1}\left[(p_{i+1}-p_i)^3 - (p_{i+1}-p_i)^2\cdot(p_{i+1}-p_i+1)\right] + \\
&\qquad + \sum_{i=3}^{k-1}\left[\frac{1}{6}\cdot(p_{i+1}-p_i+1)\cdot(p_{i+1}-p_i)\cdot(2p_{i+1}-2p_i+1)\right] \\
&= \frac{1}{3}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 - \frac{1}{2}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^2 + \frac{1}{6}\sum_{i=3}^{k-1}(p_{i+1}-p_i) \\
&= \frac{1}{3}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 - \frac{1}{2}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^2 + \frac{1}{6}(p_k-p_3). \qquad (4)
\end{aligned}$$

From the Cauchy inequality and the Prime Theorem (see references [11], [12], [13] and [14]) we may get

$$p_k - p_3 = \sum_{i=3}^{k-1}(p_{i+1}-p_i) \le \left(\sum_{i=3}^{k-1}1\right)^{\frac{2}{3}}\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)^{\frac{1}{3}} = (\pi(x)-3)^{\frac{2}{3}}\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)^{\frac{1}{3}}.$$

That is,

$$\left(x + O\left(x^{\frac{7}{12}}\right)\right)^3 = (p_k-p_3)^3 \le \left(\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)\right)^2\cdot\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)$$

or

$$\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 \ge x\cdot\ln^2 x + O\left(x^{\frac{7}{12}}\cdot\ln^2 x\right). \tag{5}$$

Combining (4) and (5) we may immediately deduce the low bound estimate

$$\sum_{n\leq x}(S_c(n)-n)^2 \geq \frac{1}{3}\cdot x\cdot \ln^2 x + O\left(x^{\frac{7}{12}}\cdot \ln^2 x\right).$$

This completes the proof of Theorem 2.

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Convolution formulae for the generalized Fibonacci polynomials and the generalized Lucas polynomials[1]

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Abstract By using the combinatorial method, some convolution formulae involving the generalized Fibonacci polynomials and the generalized Lucas polynomials have been given in this paper.

Keywords Fibonacci polynomials, Lucas polynomials, convolution formula.

§1. Introduction

As usual, the generalized Fibonacci polynomials $U(x) = U_n(x)$ and the generalized Lucas polynomials $V(x) = V_n(x)$ are defined by

$$U_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n(x) = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = \dfrac{p(x) + \sqrt{p^2(x) - 4q(x)}}{2}$, $\beta = \dfrac{p(x) - \sqrt{p^2(x) - 4q(x)}}{2}$, $p(x)$ and $q(x)$ are polynomials of x with $p(x)q(x) \neq 0$ and $p^2(x) - 4q(x) > 0$. It is obvious that the sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ satisfy the linear recurrence relation

$$W_n(x) = p(x)W_{n-1}(x) - q(x)W_{n-2}(x), \quad n \geq 2.$$

For $p(x) = x, q(x) = -1$, $\{U_n(x)\}$ and $\{V_n(x)\}$ are the classical Fibonacci polynomials $\{F_n(x)\}$ and the Lucas polynomials $\{L_n(x)\}$, respectively. If $x = 1$, then the sequences $F(1)$ and $L(1)$ are called the Fibonacci sequence and the Lucas sequence respectively, and we shall denote them by $F = \{F_n\}$ and $L = \{L_n\}$. These sequences play very important roles in the studied of the theory and application of mathematics. W.P. Zhang [1] obtained some identities involving the Fibonacci numbers. As a generalization of [1], Yi Yuan and Zhang Wenpeng [2] found some new convolution properties for $F(x)$, that is, they obtained some identities involving the Fibonacci polynomials:

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \ldots F_{a_k+1}(x)$$

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$$= \sum_{\ell=0}^{[\frac{n}{2}]} \binom{n+k-1-\ell}{\ell} \cdot \binom{n+k-1-2\ell}{k-1} \cdot x^{n-2\ell}, \tag{2}$$

where the summation $\sum_{a_1+a_2+\cdots+a_k=n}$ is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, k is any positive integer, $\binom{m}{n} = \frac{m!}{n!(m-n)!}$, and $[z]$ denotes the greatest integer not exceeding z. As a corollary of (2), they obtained some exact calculating formulae for

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{ma_1} \cdot F_{ma_2} \cdots F_{ma_k}, \tag{3}$$

where m are some special positive integers.

But, now we find there exist some problems in [2]. First, if we want use the property (2) to calculate expression (3), we must take the spacial value of x to find the relationship between $F_n(x)$ and F_{mn}, but in fact, we could not find there have any rules for taking the value of x in the calculation summation (3) for general cases. Second, the authors [2] deduced that

$$F_n(-\sqrt{5}) = \frac{(-1)^{n+1}}{3}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \frac{(-1)^{n+1}\sqrt{5}}{3} \cdot F_{2n}. \tag{4}$$

But we find the expression (4) was not true for any fixed positive integer n, that is, the expression (4) should be

$$F_n(-\sqrt{5}) = \frac{(-1)^{n+1}}{3}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - (-1)^n \cdot \left(\frac{3-\sqrt{5}}{2}\right)^n\right], \tag{5}$$

then the expression (5) will be equal to $\frac{(-1)^{n+1}\sqrt{5}}{3} \cdot F_{2n}$ just for even positive integer n. Hence, the corollary 2 and the corollary 4 in [2] are incorrect. In fact, for any fixed positive integer m, from (2) we could not give an exact calculating formula for (3).

According to two points above, we think it is not so good by using the method of [2] to calculate the (3) for general cases. In this paper, we use the combinatorial method to improve [2] to generalized cases, and solve the calculating problems of (3) for general cases with different method.

§2. The main results

Two of our main results are contained in Theorem 1 and Theorem 2 below.

Theorem 1. Let $U(x) = \{U_n(x)\}$ and $V(x) = \{V_n(x)\}$ be defined by (1). Then for any integers $n \geq 0$, $k \geq 1$ and $m \geq 1$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n+k} U_{ma_1}(x)U_{ma_2}(x)\cdots U_{ma_k}(x)$$

$$= U_m^k(x)\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n+k-i-1}{n-i}\binom{n-i}{i}V_m^{n-2i}(x)q^{mi}(x). \tag{6}$$

Theorem 2. Let $V(x) = \{V_n(x)\}$ be defined by (1). Then for any integers $n \geq 0$, $k \geq 1$ and $m \geq 1$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} V_{ma_1}(x)V_{ma_2}(x)\cdots V_{ma_k}(x) = \sum_{h=0}^{k}(-1)^h 2^{k-h}V_m^h(x)\binom{k}{h}$$

$$\cdot \sum_{i=0}^{[\frac{n-h}{2}]}(-1)^i\binom{n+k-h-i-1}{n-h-i}\binom{n-h-i}{i}V_m^{n-h-2i}(x)q^{mi}(x). \tag{7}$$

Let $p(x) = x, q(x) = -1$, from these theorems we may immediately deduce following corollaries:

Corollary 1. For any integers $n \geq 0$, $k \geq 1$ and $m \geq 1$, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{ma_1}(x)F_{ma_2}(x)\cdots F_{ma_k}(x)$$
$$= F_m^k(x)\sum_{i=0}^{[\frac{n}{2}]}(-1)^{(m+1)i}\binom{n+k-i-1}{n-i}\binom{n-i}{i}L_m^{n-2i}(x).$$

Corollary 2. The identity

$$\sum_{a_1+a_2+\cdots+a_k=n} L_{ma_1}(x)L_{ma_2}(x)\cdots L_{ma_k}(x)$$
$$= \sum_{h=0}^{k}(-1)^h 2^{k-h}x^h\binom{k}{h}\sum_{i=0}^{[\frac{n-h}{2}]}(-1)^{(m+1)i}\binom{n+k-h-i-1}{n-h-i}\binom{n-h-i}{i}L_m^{n-h-2i}(x)$$

hold for all integers $n \geq 0$, $k \geq 1$ and $m \geq 1$.

By setting $x = 1$ in these corollaries, we can easily deduce the following interesting identity involving the Fibonacci numbers and the Lucas numbers.

Corollary 3. Let $n \geq 0$, $k \geq 1$ and $m \geq 1$ are integers. Then

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{ma_1}F_{ma_2}\cdots F_{ma_k}$$
$$= F_m^k\sum_{i=0}^{[\frac{n}{2}]}(-1)^{(m+1)i}\binom{n+k-i-1}{n-i}\binom{n-i}{i}L_m^{n-2i}.$$

Remark. In fact, for any positive integer m, using the Corollary 3 we can give an exact calculating formula for (3).

Corollary 4. For any integers $n \geq 0$, $k \geq 1$ and $m \geq 1$, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} L_{ma_1}L_{ma_2}\cdots L_{ma_k}$$
$$= \sum_{h=0}^{k}(-1)^h 2^{k-h}\binom{k}{h}\sum_{i=0}^{[\frac{n-h}{2}]}(-1)^{(m+1)i}\binom{n+k-h-i-1}{n-h-i}\binom{n-h-i}{i}L_m^{n-h-2i}.$$

§3. Proof of Theorems

In this section, we present an outline of the proof of each of our main results stated in Section 2. For convenience, let

$$U'_n(x) = \frac{(\alpha^m)^n - (\beta^m)^n}{\alpha^m - \beta^m} = \frac{U_{mn}(x)}{U_m(x)}, \tag{8}$$

and

$$V'_n(x) = \alpha^{mn} + \beta^{mn}, \tag{9}$$

It is clear that the sequences $\{U'_n(x)\}$ and $\{V'_n(x)\}$ satisfy the linear recurrence relation

$$W_n(x) = V_m(x)W_{n-1}(x) - q^m(x)W_{n-2}(x), \quad n \geq 2.$$

Notes that $U'_0(x) = 0,\ U'_1(x) = 1,\ V'_0(x) = 2,\ V'_1(x) = V_m(x)$. So we can easily deduce that the generating function of $\{U'_n(x)\}$ and $\{V'_n(x)\}$ are

$$U(t,x) = \sum_{n=0}^{\infty} U'_n(x)t^n = \frac{t}{1 - V_m(x)t + q^m(x)t^2}, \tag{10}$$

and

$$V(t,x) = \sum_{n=0}^{\infty} V'_n(x)t^n = \frac{2 - V_m(x)t}{1 - V_m(x)t + q^m(x)t^2}. \tag{11}$$

From (10) we have

$$G(t,x) = \frac{U(t,x)}{t} = \sum_{n=0}^{\infty} U'_{n+1}(x) \cdot t^n. \tag{12}$$

Proof of Theorem 1. For any two absolutely convergent power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, note that

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{u+v=n} a_u b_v\right) x^n,$$

then for any integer $k \geq 1$, from (12) we can get

$$\begin{aligned} G^k(t,x) &= (\sum_{n=0}^{\infty} U'_{n+1}(x)t^n)^k \\ &= \sum_{n=0}^{\infty} \left(\sum_{a_1+\cdots+a_k=n} U'_{a_1+1}(x) \cdot U'_{a_2+1}(x) \cdots U'_{a_k+1}(x) \right) \cdot t^n \\ &= \frac{1}{U_m^k(x)} \sum_{n=0}^{\infty} \left(\sum_{a_1+\cdots+a_k=n} U_{a_1+1}(x) \cdot U_{a_2+1}(x) \cdots U_{a_k+1}(x) \right) \cdot t^n. \end{aligned}$$

On the other hand,

$$\begin{aligned} G^k(t,x) &= \frac{1}{(1-V_m(x)t+q^m(x)t^2)^k} \\ &= \sum_{j=0}^{\infty}\binom{k+j-1}{j}t^j(V_m(x)-q^m(x)t)^j \\ &= \sum_{j=0}^{\infty}\binom{k+j-1}{j}t^j\sum_{i=0}^{j}(-1)^i\binom{j}{i}V_m^{j-i}(x)q^{mi}(x)t^i \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n+k+-i-1}{n-i}\binom{n-i}{i}V_m^{n-2i}(x)q^{mi}(x)\right)\cdot t^n. \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{U_m^k(x)}\sum_{n=0}^{\infty}\left(\sum_{a_1+\cdots+a_k=n}U_{a_1+1}(x)\cdot U_{a_2+1}(x)\cdots U_{a_k+1}(x)\right)\cdot t^n \\ &=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n+k-i-1}{n-i}\binom{n-i}{i}V_m^{n-2i}(x)q^{mi}(x)\right)\cdot t^n. \end{aligned} \tag{13}$$

Equating the coefficients of t^n on both sides of equation (13) we obtain the identity

$$\begin{aligned} &\frac{1}{U_m^k(x)}\cdot\left(\sum_{a_1+\cdots+a_k=n}U_{a_1+1}(x)\cdot U_{a_2+1}(x)\cdots U_{a_k+1}(x)\right) \\ &=\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n+k-i-1}{n-i}\binom{n-i}{i}V_m^{n-2i}(x)q^{mi}(x), \end{aligned}$$

that is

$$\begin{aligned} &\sum_{a_1+a_2+\cdots+a_k=n+k}U_{ma_1}(x)U_{ma_2}(x)\cdots U_{ma_k}(x) \\ &=U_m^k(x)\sum_{i=0}^{[\frac{n}{2}]}(-1)^i\binom{n+k-i-1}{n-i}\binom{n-i}{i}V_m^{n-2i}(x)q^{mi}(x). \end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. From (11), we also have

$$\begin{aligned} V^k(t,x) &= (\sum_{n=0}^{\infty}V_n'(x)t^n)^k \\ &= \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}V_{a_1}'(x)V_{a_2}'(x)\cdots V_{a_k}'(x)\right)\cdot t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}V_{ma_1}(x)V_{ma_1}(x)\cdots V_{ma_k}(x)\right)\cdot t^n. \end{aligned}$$

On the other hand, by (10) and (11) we have

$$V^k(t,x)=(2-V_m(x)t)^k\cdot G^k(t,x).$$

Note that

$$(2 - V_m(x)t)^k = \sum_{h=0}^{k}(-1)^h 2^{k-h} V_m^h(x)\binom{k}{h} \cdot t^h.$$

So

$$\begin{aligned} V^k(t,x) \quad &= \sum_{h=0}^{k}(-1)^h 2^{k-h} V_m^h(x)\binom{k}{h} \cdot t^h \\ &\cdot \sum_{n=0}^{\infty}\left(\sum_{i=0}^{[\frac{n}{2}]}(-1)^i \binom{n+k-i-1}{n-i}\binom{n-i}{i} V_m^{n-2i}(x) q^{mi}(x)\right) \cdot t^n. \end{aligned}$$

Then

$$\begin{aligned} V^k(t,x) \quad &= \sum_{n=0}^{\infty}\left(\sum_{h=0}^{k}(-1)^h 2^{k-h} V_m^h(x)\binom{k}{h}\right. \\ &\left. \cdot \sum_{i=0}^{[\frac{n-h}{2}]}(-1)^i \binom{n+k-h-i-1}{n-h-i}\binom{n-h-i}{i} V_m^{n-h-2i}(x) q^{mi}(x)\right) \cdot t^n. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n} V_{ma_1}(x)V_{ma_1}(x)\cdots V_{ma_k}(x)\right) \cdot t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{h=0}^{k}(-1)^h 2^{k-h} V_m^h(x)\binom{k}{h}\right. \\ &\left. \cdot \sum_{i=0}^{[\frac{n-h}{2}]}(-1)^i \binom{n+k-h-i-1}{n-h-i}\binom{n-h-i}{i} V_m^{n-h-2i}(x) q^{mi}(x)\right) \cdot t^n. \end{aligned}$$

Comparing the coefficients of t^n on both sides of the above identity, we get the following formulas (7).

This completes the proof of Theorem 2.

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An inequality of the Smarandache function

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Abstract For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$. In an unpublished paper, Dr. Kenichiro Kashihara asked us to solve the following inequalities

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right).$$

In this paper, we using the elementary method to study this problem, and prove that for any integer $n \geq 1$, the inequality has infinite group positive integer solutions $(x_1, \ x_2, \ \cdots, \ x_n)$.

Keywords F.Smarandache function, inequalities, solution, necessary condition.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained some interesting results, see reference [2], [3], [4] and [5]. For example, Wang Yongxing [3] studied the mean value properties of $S(n)$, and obtained a sharper asymptotic formula about this function:

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Lu Yaming [4] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [5] proved for any positive integer $k \geq 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

In [6], Fu Jing proved more deeply conclusion, i.e., if the positive integer k and m satisfying one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are all odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even numbers $k \geq 4$ and any positive integer m;

then the equation

$$m \cdot S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

On the other hand, Xu Zhefeng [7] studied the value distribution properties of $S(n)$, and obtained a more interesting result. That is, he proved the following conclusion:

Let $P(n)$ be the largest prime factor of n, then for any real numbers $x > 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

In an unpublished paper, Dr. Kenichiro Kashihara asked us to solve the following inequalities

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right). \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The main purpose of this paper is using the elementary methods to study this problem, and prove the following:

Theorem 1. For any fixed positive integer $n > 1$, the inequality (1) has infinite group positive integer solutions $(x_1,\ x_2,\ \cdots,\ x_n)$.

Theorem 2. For any fixed positive integer $n \geq 3$, if $(x_1,\ x_2,\ \cdots,\ x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1, x_2, \cdots, x_n$ are 1.

It is clear that the condition $n \geq 3$ in Theorem 2 is necessary. In fact if $n = 2$, we can take $x_1 = x_2 = 2$, then we have the identity

$$S(x_1^2) + S(x_2^2) = S(2^2) + S(2^2) = 4 + 4 = 8 = 2S(2)S(2) = 2S(x_1)S(x_2).$$

So if $n = 2$, then Theorem 2 is not correct.

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. If $n = 1$, then this time, the inequality (1) become $S(x_1) \geq S(x_1)$, and it holds for all positive integers x_1. So without lose of generality we can assume that $n \geq 2$. We taking $x_1 = x_2 = \cdots x_{n-1} = 1$, $x_n = p > n$, where p be a prime. Note that $S(1) = 1$, $S(p) = p$ and $S(p^n) = np$, so we have

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) = n - 1 + S(p^n) = n - 1 + np \tag{2}$$

and

$$nS\left(x_1\right)\cdot S\left(x_1\right)\cdot\cdots\cdot S\left(x_n\right)=nS(p)=np. \tag{3}$$

From (2) and (3) we may immediately deduce that

$$S\left(x_1^n\right)+S\left(x_2^n\right)+\cdots\cdots+S\left(x_n^n\right)\geq nS\left(x_1\right)\cdot S\left(x_1\right)\cdot\cdots\cdot S\left(x_n\right). \tag{4}$$

Since there are infinite primes $p>n$, so all positive integer groups

$$(x_1,\ x_2,\ \cdots,\ x_n)=(1,\ 1,\ \cdots,p)$$

are the solutions of the inequality (1). Therefore, the inequality (1) has infinite group positive integer solutions $(x_1,\ x_2,\ \cdots,\ x_n)$. This proves Theorem 1.

Now we prove Theorem 2. Let $n\geq 3$, if $(x_1,\ x_2,\ \cdots,\ x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1,\ x_2,\ \cdots,\ x_n$ are 1. In fact if there exist $x_1>1$, $x_2>1$, $\cdots$, $x_k>1$ with $2\leq k\leq n$ such that the inequality

$$S\left(x_1^n\right)+S\left(x_2^n\right)+\cdots\cdots+S\left(x_n^n\right)\geq nS\left(x_1\right)\cdot S\left(x_1\right)\cdot\cdots\cdot S\left(x_n\right). \tag{5}$$

Then from the definition and properties of the function $S(n)$ we have $S(x_i)>1$ and $S\left(x_i^n\right)\leq nS(x_i)$, $i=1,\ 2,\ \cdots,\ k$. Note that $a_1+a_2+\cdots+a_k<a_1a_2\cdots a_k$ if $a_i>1$ and $k\geq 3$, $i=1,\ 2,\ \cdots,\ k$; If $k=2$, then $a_1+a_2\leq a_1a_2$, and the equality holds if and only if $a_1=a_2=2$ ($a_1>1$, $a_2>1$). So this time, the inequality (5) become

$$n-k+S\left(x_1^n\right)+S\left(x_2^n\right)+\cdots\cdots+S\left(x_k^n\right)\geq nS(x_1)S(x_2)\cdots S(x_k). \tag{6}$$

If $k\geq 3$, then from (6) and the properties of $S(n)$ we have

$$n-k+n\left[S\left(x_1\right)+S\left(x_2\right)+\cdots\cdots+S\left(x_k\right)\right]\geq nS(x_1)S(x_2)\cdots S(x_k)$$

or

$$\frac{n-k}{n}+S\left(x_1\right)+S\left(x_2\right)+\cdots\cdots+S\left(x_k\right)\geq S(x_1)S(x_2)\cdots S(x_k). \tag{7}$$

Note that $0\leq\frac{n-k}{n}<1$, so the inequality (7) is not possible, because

$$S(x_1)S(x_2)\cdots S(x_k)\geq S(x_1)+S(x_2)+\cdots+S(x_k)+1.$$

If $k=2$, then the inequality (6) become

$$n-2+S\left(x_1^n\right)+S\left(x_2^n\right)\geq nS(x_1)S(x_2). \tag{8}$$

Note that $S(x^n)\leq nS(x)$, $S(x_1)+S(x_2)\leq S(x_1)S(x_2)$ and the equality holds if and only if $x_1=x_2=2$, so if $S(x_1)>2$ or $S(x_2)>2$, then (8) is not possible. If $S(x_1)=S(x_2)=2$, then $x_1=x_2=2$. Therefore, the inequality (8) become

$$S\left(2^n\right)\geq\frac{3n}{2}+1. \tag{9}$$

Let $S(2^n) = m$, then $m \geq 4$, if $n \geq 3$. From the definition and properties of $S(n)$ we have

$$\sum_{i=1}^{\infty} \left[\frac{m-1}{2^i} \right] < n \leq \sum_{i=1}^{\infty} \left[\frac{m}{2^i} \right].$$

Thus,

$$n \geq 1 + \sum_{i=1}^{\infty} \left[\frac{m-1}{2^i} \right] > \frac{m-1}{2} + \frac{m-1}{4} = \frac{3(m-1)}{4},$$

from (9) we have

$$m = S(2^n) \geq \frac{3n}{2} + 1 \geq \frac{9}{8}(m-1) + 1 = m + \frac{m-1}{8} > m.$$

This inequality is not possible. So if $n \geq 3$ and $(x_1,\ x_2,\ \cdots,\ x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1,\ x_2,\ \cdots,\ x_n$ are 1. This completes the proof of Theorem 2.

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An improved continuous ant colony algorithms for water-reusing network optimization

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Abstract In this paper, an improved ant colony algorithm is proposed for solving continuous optimization problems. The proposed algorithm is applied to optimize the water-reusing network, which is modeled as a high-dimensional nonlinear constrained optimization problem. The experimental results would have provided some pieces of advice in the engineering applications.

Keywords Water-reusing network, ant colony system, continuous optimization.

§1. Introduction

The aim of water system optimization is to consider how to assign the amount and quality of water in each process unit of a whole water system so as to attain the maximum rate of water recycle and attain the minimum amount of the drained waste water at the same time. The problem mentioned above includes bilinear items and integer variables. So the algorithms of solving mixed non-linear integer programming MNLIP should be adopted to find the solution to the problem. In general, it is difficult to solve the MNLIP by the traditional optimization methods. Ant colony optimization algorithms introduced in recent years, however, may find a better solution to the problem in average.

The ant colony optimization techniques are based on the real world phenomena that ants are able to find their way to a food source and back to their nest, using the shortest route. They are a kind of relatively new heuristic and stochastic searching algorithms for solving complicated optimization problems. On the basis of the high-dimensional complexity of the water system, an ant colony algorithm for continuous optimization is improved in updating the pheromone so as to have better performance of optimizing the water-reusing network.

§2. Water-reusing network optimization and its mathematical model

2.1 The description of water-reusing network optimization

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Regular water-reusing network is a kind of common water networks, on which great deals of research efforts have focused. The characters of its structure can be stated as follows: in the water system, every process unit provides water for the other process units or receive water output from the other process units directly, that is, the process unit is connected with other process units. Water-reusing network can be described by superstructure model. Any process unit can be fed by fresh water or water output from other process units; meanwhile, its output can be drained off directly or to other process units, but can not go back to itself.

2.2 The mathematical model of the water-reusing network

The mathematical model[2] of optimizing the water-reusing network is set up as follows: Minimize the objective function

$$\min f(x) = \sum_{j \in P} F_j^W, \tag{1}$$

that is, minimize the freshwater consumption of the system. The constraints are

1. The water balance of process j:

$$h_j(x) = F_j^W + \sum_{\substack{i \in P \\ i \neq j}} F_{i,j} - F_j^D - \sum_{\substack{k \in P \\ k \neq j}} F_{j,k} \ = 0, \quad j \in P. \tag{2}$$

2. The mass balance of contaminants at the inlet node of process j:

$$g_j(x) = \sum_{\substack{i \in P \\ i \neq j}} (F_{i,j} \cdot C_{i,s}^{Out}) - (F_j^W + \sum_{\substack{i \in P \\ i \neq j}} F_{i,j}) \cdot C_j^{In} = 0,\ j \in P. \tag{3}$$

3. The mass balance of contaminants of process j:

$$l_j(x) = (F_j^W + \sum_{\substack{i \in P \\ i \neq j}} F_{i,j}) \cdot C_j^{In} + M_j - (F_j^W + \sum_{\substack{i \in P \\ i \neq j}} F_{i,j}) \cdot C_j^{Out} = 0, \ \ j \in P. \tag{4}$$

Denote by x the vector which is composed of all the parameters to be optimized, including $C_j^{In}, F_j^D, F_{i,j}, F_j^W$. P is a set of natural numbers, whose every element stands for a process unit. By the penalty method, the formulas (1)-(4) can be transformed into an unconstrained optimization problem

$$\min F(x) = f(x) + \sum_{j=1}^{p} (r/2) \cdot ((h_j(x\ ))^2 + (g_j(x))^2 + (l_j(x\ ))^2), \tag{5}$$

where r is the constant penalty factor.

§3. Ant colony system for continuous optimization

3.1 Basic idea of ant colony system

Ant colony system, introduce by M. Dorigo and A. Colorni, is an intelligent optimization algorithm which is designed to simulate the self- adaptive ability of real ant colonies[3]-[5]. One of the main ideas of ant colony system is the indirect communication of a colony of agents, called ants, based on pheromone trails. The pheromone trails are a kind of distributed numeric information which is modified by the ants to reflect their experience while solving a particular problem.

It has been studied that ants often find the shortest path between a food source and the nest of the colony without using visual cues. In order to exchange information about which path should be followed, ants communicate with each other by means of a chemical substance called pheromone. As ants moving, a certain amount of pheromone is dropped on the ground, creating a pheromone trail. This pheromone trail can be observed by other ants and motivates them to follow the path. The more ants follow a given trail, the more attractive that trail becomes to be followed by other ants. This process involves a loop of positive feedback, in which the probability that an ant chooses a path is proportional to the number of ants that have already passed by that path.

That is the way how the trail is reinforced and more and more ants follow that trail. Meanwhile, the ant colony are capable of adapting the vary of the environment, that is, when there is an obstacle in their moving route, they can also find the best route in which a high levels of pheromone has been deposited over a period of time [6].

Based on converting this idea to a search mechanism, ant colony system can be applied for solving some combinatorial optimization problems such as the job-shop scheduling problem (JSP), quadratic assignment problem (QAP) and traveling salesman problem (TSP) [4]etc. The main idea, taking TSP as an example, is that a set of ants, search in parallel for good solutions to the TSP and cooperate through pheromone-mediated indirect and global communication. Informally, the ant colony system constructs a TSP solution in an iterative way: m ants initially positioned on n cities chosen according to some initialization rule (e.g., randomly).

Each ant generates a complete tour which corresponds to a feasible solution to the TSP by exploiting both information gained from past experience and a stochastic greedy heuristic (the state transition rule). Memory takes the form of pheromone deposited by ants on TSP edges, while heuristic information is simply given by the edge's length; ants prefer to move to cities which are connected by short edges with a high amount of pheromone. While constructing its tour, an ant also modifies the amount of pheromone on the visited edges by applying the local updating rule. Once all ants have terminated their tours, the amount of pheromone on edges is modified again by applying the global updating rule. A fraction of the pheromone evaporates on all edges, and then each ant deposits an amount of pheromone on edges which belong to its tour in proportions to how short its tour was. The process is then iterated.

3.2 The proposed ant colony algorithm for continuous optimization

Ant colony system was first proposed for solving combinatorial optimization problems. However, there are lots of continuous optimization problems required in many applications.

Therefore, how to improve and design the novel ant colony algorithms for solving continuous optimization problems becomes a key issue in the study of the ant colony system.

Water-reusing network optimization is a high-dimensional continuous optimization problem. In this paper, an improved ant colony algorithm is proposed, which is based on the discretization of the continuous space. The way of selecting cities and dividing the solution space is mainly adopted as one in [7]; the amount of the pheromone is modified by applying the local updating rule and the global updating rule together, and meanwhile, retaining the best solution at each iteration.

In detail, the pheromone is updated locally by the function values of the routes at the present iteration; while only is the pheromone in the best route which stands for the best solution updated globally so as to avoid reaching the local optimal due to the over-high level of the pheromone in that route.

Suppose that the independent variables are set to be at d decimal place, then every independent variable x can be denoted by d decimal numbers approximately. Based on the assumption, we construct $d+2$ layers of cities as follows: there are only one city, labeled as 0, in the first layer and the last layer respectively; from the second layer to the $d+1$th layer, there include ten cities labeled from 1 to 10 in every layer, and these d layers stand for the decile, the percentiles, $\cdots\cdots$, of the variable x from left to right respectively.

Among these cities, only can the ones between the neighbor layers be connected. Denote the pheromone in the route connecting city a in the $k-1$th layer and city b in the kth layer by τ_{ab}^k, the city where ant n stays in the mth step by $T(n,m)$. Let the total number of the ants be N_0. Initialize τ_{ab}^k as τ_0 and $T(n,0)=0$ $(n=1,\ 2,\ \cdots N_0)$.

We select the routes as follows.

If the city where ant n stays at present is $T(n,k-1)=a$, then select the city to which the ant will get in the next step by the following formula:

$$T(n,k)=\begin{cases}\arg\max\{\tau_{ab}^k\}, \text{if} \quad q<Q_0;\\ S_r, \text{otherwise},\end{cases} \tag{6}$$

where q is a random number, Q_0 is a constant in $[0,1]$ to determine the probability of the selection of pseudo-random numbers and S_r denotes the next city to which the ant will get.

First compute the probability of the selection of each city in the next layer according to

$$p(a,b)=\tau_{ab}^k \Big/ \sum_{x=0}^{9}\tau_{ax}^k \tag{7}$$

and then determine the next city to be selected with roulette-wheel scheme where $p(a,b)$ denotes the probability of transformation from the present city a to the next city b. When every ant gets to the $d+1$th layer, all of them are forced to move to the unique city in the last layer.

When the ant n passed through all the cities, we first decode the route selected by the ant and compute the value of $x(n)$ by

$$x(n)=\sum_{k=2}^{d+1}T(n,k)\times 10^{1-k}, \tag{8}$$

and then update the pheromone in this route according to

$$\tau^k_{T(n,k-1),T(n,k)} = (1-\rho) \times \tau^k_{T(n,k-1),T(n,k)} + \Delta\tau^k_{T(n,k-1),T(n,k)}. \tag{9}$$

$$\Delta\tau^k_{T(n,k-1),T(n,k)} = \begin{cases} \dfrac{Q}{f(x(n))}, & \text{if the ant n passes through the cities } (T(n,k-1),T(n,k)); \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

where $\rho \in (0,1)$ is a constant, denoting the increasing rate of the remaining pheromone in the route and Q is a positive constant.

When all their ants finished the tour, we only update the pheromone in the shortest route and retain the best solution in the cycle. We first select the optimal ant, labeled as $n_{\min}$, by

$$n_{\min} = \arg\min\{f(x(n))\} \tag{11}$$

and update the pheromone in the route through which the ant $n_{\min}$ passed according to

$$\tau^k_{ab} = (1-\alpha)\tau^k_{ab} + \alpha f(x(n_{\min}))^{-1} \tag{12}$$

where $a = T(n_{\min}, k-1),\ b = T(n_{\min}, k), k \in [2, d+2]$ and α is a constant in $(0,1)$.

For continuous optimization problems with multi-variables, we construct the cities and decode the independent variables according to the following way: suppose that x is a N-dimensional independent variable and every component is set to be at d decimal place, then we can construct $N \times d + N + 1$ layers of cities, where there are only one city labeled as 0 in the $1st, d+2th, 2d+3th, \cdots, N \times d + N + 1th$ layers respectively, and there are ten cities labeled from 0 to 9 in every rest layer. Thus there are $N \times d \times 10 + N + 1$ cities altogether. The layers from the $(k-1) \times (d+1) + 2$th layer to the $k \times (d+1)$th layer$(k = 1, 2, \cdots, N)$ stand for the kth component of the independent variable and the other layers are auxiliary ones. We calculate $x(n)$ by

$$x(n) = (\sum_{k=2}^{d+1} T(n,k) \times 10^{1-k}, \cdots, \sum_{k=(N-1)\times d+N+1}^{N\times d+N} T(n,k) \times 10^{1-(k-((N-1)*d+(N+1)-2))}). \tag{13}$$

In such a way, the last component of each variable is separated from the first component of the next variable by the auxiliary layer so that the later variable has no effect on the former one.

We state the steps of the above algorithm as follows:

(a) Initialize parameters: set iteration number M, the total number N_0 of ants, pheromone τ_0, parameters ρ, Q_0, α, Q, d and the initializing city $T(n,1) = 0$ where the ant n stays for all n;

(b) Set the cycle number $NC := 1$;

(c) Let all the ants be in the initializing city and execute the step (d) and the step (e) to every ant;

(d) Select the next city to which the ant will get according to the formulas (6) and (7);

(e) Update the pheromone locally according to the formulas (8)-(10) after every ant finishes its tour (substitute the formula (8) by (13) for multi-variable continuous functions);

(f) Select the optimal ant and update the pheromone globally according to the formulas (11)-(12);

(g) Determine whether the termination is satisfied. If so, then output; otherwise, set $NC := NC + 1$, go to step (c) and continue.

§4. Numerical experiments and results

In this section, the proposed ant colony algorithm introduced in the last section is applied to optimize the water-reusing network with three process units (i.e. $P = \{1, 2, 3\}$, which is modeled as an 18 dimensional continuous optimization problems mathematically.) Table 1 shows the limiting data of the water system. Taking the parameters as $N_0 = 80,\ M = 100, \tau_0 = 0.10,\ Q_0 = 0.8,\ \rho = 0.01$, $\alpha = 0.80$ *and* $Q = 100$ respectively. Table 2 shows the best five results and the result by Lingo 8.0.

Table 1 The limiting data of the water system[2]

Process unit number	Limiting inlet concentration of contaminant(ppm)	Limiting outlet concentration of contaminant(ppm)	Mass load of contaminant (g/s)
1	50	100	3000
2	25	90	2880
3	25	200	4000

Table 2 Results comparison

	Result 1	Result 2	Result 3	Result 4	Result 5	Lingo
Flow rate of fresh water $F^w(t/h)$	74.4796	76.3611	79.7994	78.2772	76.1979	75.94

Figure 1 shows the value of the objective function varying with the number of the iterations. It is shown that although the values of the objective function oscillate all the time during the iterations, the breadth of the oscillation decreases as the number of iterations increases and approaches to vary within a small stationary range.

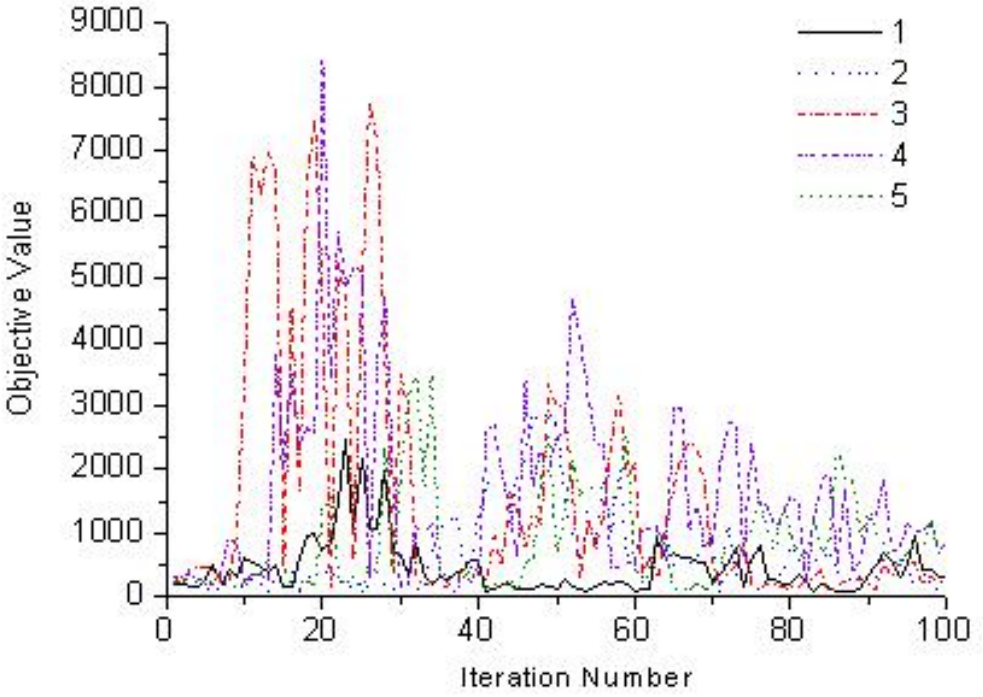

Fig. 1 The relation between the objective function and the iteration numbers

§5. Conclusion

In this paper, an improved ant colony algorithm for solving continuous optimization problems has been proposed and applied to solve the water-reusing network optimization, which is a high-dimensional and non-linear constrained optimization problem. The experiment results show the good performance of the algorithm for optimizing the water-reusing network with three process units. The optimal solution gotten by the proposed algorithm can provide some pieces of advice in the engineering applications. Nevertheless, it is difficult to determine the values of the parameters because there exist too many random numbers during the iterations. Therefore there need a lot of experiments to get the better value of the parameters.

NOMENCLATURE

F_j^w Flow rate from fresh water pipeline to process j, t/h;

$F_{j,i}$ Flow rate from process j to process i, t/h;

F_j^D Flow rate from process j to wastewater pipeline, t/h;

C_j^{In} Inlet concentration of contaminant of process j, ppm;

C_j^{Out} Outlet concentration of contaminant s of process j, ppm;

M_j Mass load of contaminant s of process j, g/h;

$C_j^{In,Max}$ Limiting inlet concentration of contaminant s of process j, ppm;

$C_j^{Out,Max}$ Limiting outlet concentration of contaminant s of process j, ppm.

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A concise way of determination for LP initial feasible basis of simplex method

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Abstract The initial feasible basis is a elementary concept in simplex method of solving linear programming problem. The big M method and the two-phases way are usually adopted to construct a unit matrix as a initial basis. But, these schemes often make one confusion in thinking clue and complication in calculation. Aim at this point, we establish the association between the elementary transformation of matrix and the concept of initial feasible basis, a determination way for initial feasible basis of linear programming is presented here. The presented scheme is concisely, convenience and making the solving procedure clearly in practice. Some examples are given to demonstrate these characteristics in this paper.

Keywords Linear programming, simplex method, initial feasible basis, elementary transformation of matrix.

§1. Introduction

It is a basic problem that to determinate a initial feasible basis of a linear programming in the simplex method. The big M method and two -stage way are usually adopted to construct a unit matrix as a initial basis through to introduce some artificial variables first, then make the iteration of simplex method. But, these schemes often make one confusion in thinking clue and complication in calculations. This situation is more obviously in handing operations. When we pay attention to the relations between the essential of simplex iteration way and the elementary transformation of matrix, a solving way of linear programming is appears based on the row elementary transformation of matrix at the augment coefficients matrix of linear programming model. As the augment coefficients matrix transformed as a new look that include a unit matrix as its sub-matrix, we can begin to start the simplex iteration in simplex table.

§2. Theoretic bases of method

Suppose the linear programming model we considering is

$$\min f(x) = c^T x$$

$$(LP) \qquad s.t. \;\; Ax = b, x \geq 0. \qquad (*)$$

Such that $c = (c_1 c_2 \wedge c_n)^T,\ x = (x_1 x_2 \wedge x_n)^T,\ b = (b_1 b_2 \wedge b_m)^T,\ A = (a_{ij})_{m\times n}$, and the augment coefficient matrix of LP model $(*)$ is noted as $B = [A|b]_{m\times(n+1)}$.

The theory bases of the method we shall present in here is a well known theorem as follows.

Theorem. If matrix A can be transformed into matrix B by a finite series elementary row (or column) transformations, then the row (or column) group of A is equivalent to the row (or column) group of B, and arbitrary k row (or column) of A have the same linear relativity as the corresponding k row (or column) of B.

This conclusion means that the elementary transform of matrix doesn't change the rank, i.e. $r(A) = r(B)$. Therefore, a $m \times n$ matrix can be transformed into a row ladder matrix by row elementary transformation, and the a mount of nonzero row vector of this ladder matrix is the rank of the matrix. If we continued to operate row elementary transformation on this row ladder matrix, it can be transformed into its simplest form: the first nonzero element of nonzero row vector is 1, and other elements of the column including this 1 are zero.

In other words, any $m\times n$ matrix can be deducted into a equivalence matrix which including a $r \times r$ unit matrix by a finite series row elementary transformation. This fit like a glove for the demand of simplex scheme that is to construct a unit matrix as a initial feasible basis in the coefficient matrix. Hence, we can present a determination way to get a initial feasible basis for a linear programming model: First, doing a finite series row elementary transformation on the augment coefficients matrix $B = [A|b]_{m\times(n+1)}$ till to perform it as its simplest form under the condition of keeping the resource vector b always positive, a $r \times r$ initial feasible basis is obtained ($r \leq m$); and whereafter, turn into the simplex method in the simplex table by arrange the simplest form matrix into initial simplex table. The advantages of this scheme are not only getting a initial feasible basis (when $r = m$), but also finding the dependent constrains (when $r < m$) in linear programming model so as to eliminate them to lessen the computation amount of simplex method.

§3. Procedure of method

The steps of using this determination scheme of getting a initial feasible basis above inducted are follows concretely.

Step 1. Doing a finite series row elementary transformation on the augment coefficients matrix B and always holding the feasibility (i.e. column vector b is non-negative), till to perform B as the form including a unit matrix as its sub-matrix. It is pointed that there are four notes as follows:

(1) The constrain equations are independently when the case of a whole row is transformed into 0, then, the augment coefficients matrix B can be decreasing dimension from $m \times (n + 1)$ to $(m - 1) \times (n + 1)$, the dimension of the unit matrix is $(m - 1) \times (m - 1)$. The dimension of the unit matrix as the feasible initial basis is not always $m \times m$;

(2) The rank(A)$\neq$ rank(B) when all elements are 0 of someone row except the non-zero element corresponding b is occurred, then we adjudge there no feasible solution of the LP

(constrain equations is contradiction), stop;

(3) There is a redundant decision variable when all elements are 0 of someone column and the coefficient of corresponding variable is 0 in object function, then we adjudge the dimension of LP model is decreasing 1.

Step 2. Arranging the result of Step 1 into simplex tableau as a initial simplex tableau, and arrange the variables that corresponding to the column of the unit matrix as basic variables, the criterions is obtained by

$$\sigma_j = \begin{cases} c_j - \sum\limits_{i \in I_B} c_i a_{ij}^{'}, & j \in I_N, \\ 0, & j \in I_B. \end{cases}$$

Such that: I_B is the set of subscript of basic variable, I_N is the set of subscript of non-basic variable. The initial value of object function is $-z_0 = \sum\limits_{i \in I_B} c_i b_i^{'}$;

Step 3. According the usually procedure of simplex method to solving in the initial simplex tableau we got in Step 2.

It is noted that we may use the second class and the third class elementary row transformation only and do without the first class elementary row transformation (exchange positions of any two rows) in our way. It is appears that advantages of this scheme are clear thinking clue, concise procedure and less amount of computations in practice. We can demonstrate this method by some classic examples in this paper.

§4. Examples and annotations

Exa. 1. Solve the linear programming model as follws.

$$\min z = -3x_1 + x_2 + x_3$$

$$s.t. \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 11 \\ -4x_1 + x_2 + 2x_3 - x_5 = 3 \\ -2x_1 + x_3 = 1 \\ x_i \geq 0, \quad (i = 1, 2, \cdots, 5) \end{cases}$$

Solving: First, doing a finite series row elementary transformation on the augment coefficients matrix that always keeping the feasibility:

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 11 \\ -4 & 1 & 2 & 0 & -1 & 3 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -2 & 0 & 1 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 0 & 0 & 1 & -2 & 12 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then, construct a initial simplex tableau and iterating to get the solution:

c_j		-3	1	1	0	0	b
c_B	x_B	x_1	x_2	x_3	x_4	x_5	
0	x_4	3	0	0	1	-2	12
1	x_2	0	1	0	0	-1	1
1	x_3	-2	0	1	0	0	1
σ_j		-1	0	0	0	1	2
-3	x_1	1	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	4
1	x_2	0	1	0	0	-1	1
1	x_3	0	0	1	$\frac{2}{3}$	$-\frac{4}{3}$	9
σ_j		0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	-2

Hence, the optimal solution is $x^* = (4, 1, 9, 0, 0)^T$, the optimal value is $z^* = -2$.

In [1], solving this problem by big M method in 3 time iterations. And by two-phases method, complete the first phase in 2 iterations, then go to second phase, making 1 iteration again to get the optimal solution.

Exa. 2. Solving

$$\min z = 4x_1 + 3x_3$$

$$s.t. \begin{cases} \frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 - \frac{2}{3}x_4 = 2 \\ \frac{3}{2}x_1 + \frac{3}{4}x_3 = 3 \\ 3x_1 - 6x_2 + 4x_4 = 0 \\ x_i \geq 0, \quad (i = 1, 2, 3, 4) \end{cases}$$

Solving: First, doing a finite series row elementary transformation on the augment coefficients matrix that always keeping the feasibility:

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & \frac{3}{4} & 0 & 3 \\ 3 & -6 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ 6 & 0 & 3 & 0 & 12 \\ 6 & 0 & 3 & 0 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -\frac{1}{2} & 1 & 0 & -\frac{2}{3} & 0 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, construct a initial simplex tableau and iterating to get the solution:

c_j		4	0	3	0	b
c_B	x_B	x_1	x_2	x_3	x_4	
0	x_2	$-\frac{1}{2}$	1	0	$-\frac{2}{3}$	0
3	x_3	2	0	1	0	4
σ_j		-2	0	0	0	12
0	x_2	0	1	$\frac{1}{4}$	$-\frac{2}{3}$	1
4	x_1	1	0	$\frac{1}{2}$	0	2
σ_j		0	0	1	0	8

Hence, the optimal solution is $x^* = (2, 1, 0, 0)^T$, the optimal value is $z^* = 8$.

In [2], solving this problem by two-phases method, complete the first phase in 3 iterations to get this initial feasible basis.

Exa. 3. Solving

$$\min z = 4x_1 + 3x_2$$

$$s.t. \begin{cases} \frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 - \frac{2}{3}x_4 = 2 \\ \frac{3}{2}x_1 \quad - \frac{1}{2}x_3 = 3 \\ 3x_1 - 6x_2 \quad + 4x_4 = 0 \\ x_i \geq 0, \quad (i = 1, 2, 3, 4) \end{cases}$$

Solving: First, doing a finite series row elementary transformation on the augment coefficients matrix that always keeping the feasibility:

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 3 \\ 3 & -6 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ 3 & 0 & -1 & 0 & 6 \\ 6 & 0 & 3 & 0 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ 5 & 0 & 0 & 0 & 10 \\ 2 & 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} & -\frac{2}{3} & 2 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & \frac{1}{2} & -\frac{2}{3} & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & -\frac{2}{3} & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Then, construct a initial simplex tableau:

c_j		4	3	0	0	b
c_B	x_B	x_1	x_2	x_3	x_4	
3	x_2	0	1	0	$-\frac{2}{3}$	1
4	x_1	1	0	0	0	2
0	x_3	0	0	1	0	0
σ_j		0	0	0	2	11

It is obvious that this initial feasible solution is the optimal solution by the row check-number. Hence, we never need to iterate, and the optimal solution is $x^* = (2, 1, 0, 0)^T$, the optimal value is $z^* = 11$.

In [2], solving this problem by two-phases method, complete the first phase in 3 iterations to get this initial feasible basis.

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On an equation related to function $S(n)$

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. The main purpose of this paper is using the elementary method to study the solvability of the equation $S(n)^2 + S(n) = kn$, and prove that for any positive integer k, the equation has infinite positive integer solutions.

Keywords F.Smarandache function $S(n)$, elementary method, positive integer solution.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$, where N denotes the set of all positive integers. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then we have

$$S(n) = \max_{1 \le i \le k} \{S(p_i^{\alpha_i})\}.$$

It is clear that from this properties we can caculate the value of $S(n)$, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained many interesting results. For example, Farris Mark and Mitchell Patrick [1] studied the bound of $S(n)$, and got the upper and lower bound estimates for $S(p^\alpha)$. They proved that

$$(p-1)\alpha + 1 \le S(p^\alpha) \le (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Lu Yaming [2] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \ge 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite positive integer solutions $(m_1, \ m_2, \ \cdots, \ m_k)$.

Jozsef Sandor [3] proved that for any positive integer $k \ge 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfying the inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

In [4], Fu Jing proved a more general conclusion. That is, if the positive integer k and m satisfying one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even number $k \geq 4$ and any positive integer m;

then the equation

$$m \cdot S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Rongji Chen [5] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any fixed $r \in N$ with $r \geq 3$, the positive integer n is a solution of

$$S(n)^r + S(n)^{r-1} + \cdots + S(n) = n$$

if and only if

$$n = p(p^{r-1} + p^{r-2} + \cdots + 1)$$

where p is an odd prime satisfying

$$p^{r-1} + p^{r-2} + \cdots + 1 | (p-1)!.$$

On the other hand, in reference [6], Charles Ashbacher asked whether it is possible to create polynomial with the variables the values of the Smarandache function. For example, the polynomial

$$S(n)^2 + S(n) = n$$

is such an expression. A computer search for all $n \leq 10000$ yielded 23 values of n for which the expression is true.

A computer search for all values of $n \leq 10000$ for which the expression

$$S(n)^2 + S(n) = 2n$$

is true yielded 33 solutions.

A computer search for all values of $n \leq 10000$ for which the expression

$$S(n)^2 + S(n) = 3n$$

is true yielded 20 solutions.

A computer search for all values of $n \leq 10000$ for which the expression

$$S(n)^2 + S(n) = 4n$$

is true yielded 24 solutions.

A computer search for all values of $n \leq 10000$ for which the expression

$$S(n)^2 + S(n) = 5n$$

is true yielded 11 solutions.

A computer search for all values of $n \leq 10000$ for which the expression

$$S(n)^2 + S(n) = 6n$$

is true yielded 26 solutions.

Then he proposed the following three unsolved problems:

Unsolved Problem 1: Is the number of solutions to each of the expressions above finite or infinite?

Unsolved Problem 2: Is there a number k such that there is no number n for which

$$S(n)^2 + S(n) = kn?$$

Unsolved Problem 3: Is there a largest number k for which there is some number n that satisfies the expression

$$S(n)^2 + S(n) = kn?$$

About these problems, it seems that none had studied them yet, at least we have not seen related papers before. In this paper, we use the elementary method to study these problems, and solved them completely. That is, we shall prove the following conclusion:

Theorem. For any positive integer k, the equation

$$S(n)^2 + S(n) = kn \tag{1}$$

has infinite positive integer solutions, and each solution n has the form

$$n = pn_1,$$

where $p = kn_1 - 1$ is a prime.

It is clear that from this Theorem, we solved the above three unsolved problems completely. That is, for any positive integer k, there are infinite numbers n satisfying $S(n)^2 + S(n) = kn$, so there isn't the largest number k such that the equation (1) has positive integer solutions.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our theorem. First we need the following two simple lemmas.

Lemma 1. If $k > 0$ and $(k, h) = 1$, then there are infinitely many primes in the arithmetical progression $nk + h$, $n = 0, 1, 2, \ldots$

Proof. See Theorem 7.9 of reference [7].

Lemma 2. Let p be a prime. Then for any positive integer k, we have the estimate $S(p^k) \leq kp$. If $k \leq p$, then $S(p^k) = kp$.

Proof. See reference [1].

Now we use these two lemmas to complete the proof of our theorem. It is clear that from the definition of $S(n)$, $\exists\ p^\alpha | n$, s.t $S(n) = S(p^\alpha) = mp$, where m is a positive integer, from Lemma 2, we have $m \leq \alpha$.

Let $n = p^{\alpha}n_1$, where $(p, n_1) = 1$.

If $\alpha = 2$, then we have

$$m^2p^2 + mp = kp^2n_1,$$

while $p^2|m^2p^2 + mp$, so $p|m$, that is $p \leq m \leq \alpha$.

Similarly, there must exist a larger positive integer u, s.t $p^u|m$, while m is a finite positive integer, in fact it is a contradiction.

So $\alpha = 1, m = 1$, then

$$p^2 + p = kpn_1,$$

or $p = kn_1 - 1$, from Lemma 1, there are infinite such primes p, while $n = pn_1 = (kn_1 - 1)n_1$, so the equation (1) has infinite positive integer solutions.

This completes the proof of Theorem.

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On *wrpp* semigroups with left central idempotents[1]

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Abstract The relation $\mathcal{L}^{**}$ on any semigroup S provides a generalization of Green's relation $\mathcal{L}$. The elements a, b of S are $\mathcal{L}^{**}$-related by the rule that $(ax, ay) \in \mathcal{R} \Leftrightarrow (bx, by) \in \mathcal{R}$ for all $x, y \in S^1$ where $\mathcal{R}$ is the usual Green's relation. A semigroup S is called a *wrpp* semigroup if S is a semigroup such that (i) each $\mathcal{L}^{**}$-class of S contains at least one idempotent of S; (ii) $a = ae$, for all $e \in L_a^{**} \cap E$. The aim of this paper is to investigate a *wrpp* semigroup with left central idempotents. It is proved that S is a *wrpp* semigroup with left central idempotents if and only if S is a semilattice of $\mathcal{R}$-left cancellative right stripes and $E(S)$ is a right normal band; if and only if S is a strong semilattice of $\mathcal{R}$-left cancellative right stripes.

Keywords *wrpp* semigroups, right zero bands, $\mathcal{R}$-left cancellative right stripes.

§1. Introduction

According to Tang [1], the relation $\mathcal{L}^{**}$ is defined on a semigroup S by the rule that $a\mathcal{L}^{**}b$ if and only if $(ax, ay) \in \mathcal{R} \Leftrightarrow (bx, by) \in \mathcal{R}$ for all $x, y \in S^1$ where $\mathcal{R}$ is the usual Green's relation. The Green's relation $\mathcal{L}$ is contained in $\mathcal{L}^{**}$. In fact we also have that $\mathcal{L} \subseteq \mathcal{L}^{*} \subseteq \mathcal{L}^{**}$.

A semigroup S is a *rpp* semigroup if every $\mathcal{L}^{*}$-class contains an idempotent. Following [2], A semigroup S is called a *wrpp* semigroup if S satisfy the following conditions:

(i) each $\mathcal{L}^{**}$-class of S contains at least one idempotent of S;

(ii)$a = ae$, for all $e \in L_a^{**} \cap E$.

Clearly, the class of *wrpp* semigroups regards both the class of *rpp* semigroups and the class of regular semigroups as its subclasses. As analogues of *C-rpp* semigroups, an *C-wrpp* semigroup which is *wrpp* and whose idempotents are central, was introduced by Tang in [1]. It was proved that a semigroup S is an *C-wrpp* semigroup if and only if S is a strong semilattice of $\mathcal{L}^{**}$-simple monoids. Later on, Du and Shum [2] studied left *C-wrpp* semigroups which are generalizations of left *C-rpp* semigroups in the class of *wrpp* semigroups. A *rpp*-semigroup with left central idempotents have been studied by Ren- Shum in [3]. It was stated in [3]

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that the semigroup S with left central idempotents is isomorphic to a strong semilattice of left cancellative right stripes. In this paper a *wrpp* semigroup with left central idempotents will provide a generalization of an *C-wrpp* semigroup. It will be shown that S is a *wrpp* semigroup with left central idempotents if and only if S is a strong semilattice of $\mathcal{R}$-left cancellative right stripes.

For notations and definitions not given in this paper, the reader is referred to [3], [4] and [5].

§2. Preliminaries

We begin by recalling some basic results which will be used in the following.

As stated in [3], we say that an idempotent e of a semigroup S is a left central idempotent if $xey = exy$ always holds for all $x, y \in S^1$ and $y \neq 1$. By a semigroup with left central idempotents, we mean a semigroup in which each of its idempotents is left central. A semigroup S is called $\mathcal{R}$-left cancellative if for all $a, b, s \in S$

$$sa\mathcal{R}sb \quad \text{if and only if} \quad a\mathcal{R}b$$

It is easy to see that if every idempotent of a semigroup S is left central then the set $E(S)$ of idempotents of S forms a right normal band.

Lemma 2.1.[1] The relation $\mathcal{L}^{**}$ is a right congruence on any semigroup S.

Lemma 2.2. Let S be a *wrpp* semigroup with left central idempotents. If $(a, e) \in \mathcal{L}^{**}$ for $a \in S$ and $e \in E(S)$ then $ae = ea = a$.

Proof. Suppose that $(a, e) \in \mathcal{L}^{**}$ and $e \in E$. Clearly, $ae = a$. Notice that e is left central. It follows immediately that $ea = eae = ae = a$.

Corollary 2.3. If S is a *wrpp* semigroup with left central idempotents, then every $\mathcal{L}^{**}$-class of S contains a unique idempotent.

By Corollary 2.3, we denote the $\mathcal{L}^{**}$-class of S containing the element a by L_a^{**} and its unique idempotent by a^+.

Lemma 2.4. Let S be a *wrpp* semigroup with left central idempotents. Then $\mathcal{L}^{**}$ is a congruence on S.

Proof. Suppose that a, b are elements of S with $(a, b) \in \mathcal{L}^{**}$. Clearly, $a^+ = b^+$ by Corollary 2.3. Suppose that $x, y \in S^1$ and $c \in S$. If $(cax, cay) \in \mathcal{R}$ then $(c^+ax, c^+ay) = (c^+aa^+x, c^+aa^+y) \in \mathcal{R}$ since $c\mathcal{L}^{**}c^+$. This implies $(ac^+a^+x, ac^+a^+y) \in \mathcal{R}$ and so $(bc^+a^+x, bc^+a^+y) \in \mathcal{R}$ since $(a, b) \in \mathcal{L}^{**}$. Hence, $(c^+bb^+x, c^+bb^+y) = (c^+bx, c^+by) \in \mathcal{R}$. Again from $(c, c^+) \in \mathcal{L}^{**}$, we have $(cbx, cby) \in \mathcal{R}$.

A similar argument can show that $(cbx, cby) \in \mathcal{R}$ implies $(cax, cay) \in \mathcal{R}$. Consequently, $(ca, cb) \in \mathcal{L}^{**}$, that is, $\mathcal{L}^{**}$ is a right congruence on S. Combining Lemma 2.1 we have proved that $\mathcal{L}^{**}$ is a congruence on S.

By Corollary 2.3 together with Lemma 2.4, we immediately obtain the following result

Corollary 2.5. If S is a *wrpp* semigroup with left central idempotents, then $(ab)^+ = a^+b^+$ for all $a, b \in S$.

Proof. Clearly, $a\mathcal{L}^{**}a^+$ and $b\mathcal{L}^{**}b^+$. Since $\mathcal{L}^{**}$ is a congruence on S it follows that $ab\mathcal{L}^{**}a^+b^+$. By Corollary 2.3, $(ab)^+ = a^+b^+$ holds.

We now define a binary relation σ on a *wrpp* semigroup S with left central idempotents by

$$a\sigma b \Leftrightarrow a^+ = b^+a^+ \text{ and } b^+ = a^+b^+ \text{ for all } a, b \in S.$$

Theorem 2.6. Let S is a *wrpp* semigroup with left central idempotents. Then the relation σ given above is a semilattice congruence on S.

Proof. We first show that σ is an equivalence relation on S. It is clear that σ is reflexive and symmetric. To verify that σ is transitive, we let $a\sigma b$ and $b\sigma c$. Clearly,

$$a^+b^+ = b^+, b^+a^+ = a^+ \text{ and } b^+c^+ = c^+, c^+b^+ = b^+.$$

This leads to

$$a^+c^+ = (b^+a^+)c^+ = a^+b^+c^+ = b^+c^+ = c^+,$$

and

$$c^+a^+ = (b^+c^+)a^+ = c^+b^+a^+ = b^+a^+ = a^+.$$

By the definition of σ, $a\sigma c$, that is, σ is transitive.

Next we prove that σ is right compatible. For this purpose, let $a, b \in S$ such that $a\sigma b$. Using Corollary 2.5 together with the fact $E(S)$ is a right normal band, we may deduce that

$$(ac)^+(bc)^+ = a^+c^+b^+c^+ = c^+a^+b^+c^+ = c^+b^+c^+ = b^+c^+ = (bc)^+$$

and

$$(bc)^+(ac)^+ = b^+c^+a^+c^+ = c^+b^+a^+c^+ = c^+a^+c^+ = a^+c^+ = (ac)^+.$$

This shows that $ac\sigma bc$.

Similarly, we can prove that $ca\sigma cb$. Consequently, σ is a congruence on S. To see that σ is a semilattice congruence on S, let $a, b \in S$. It follows by Corollary 2.5 that

$$(ba)^+(ab)^+ = b^+a^+a^+b^+ = b^+a^+b^+ = a^+b^+ = (ab)^+$$

and

$$(ab)^+(ba)^+ = a^+b^+b^+a^+ = a^+b^+a^+ = b^+a^+ = (ba)^+.$$

This implies that $a\sigma b\sigma = b\sigma a\sigma$. It is easy to check that $a^2\sigma = a\sigma$. Thus, we have completed the proof.

Lemma 2.7.[2] Let Y be a semilattice, and $S = [Y; S_\alpha, \Phi_{\alpha,\beta}]$ be a strong semilattice of semigroups S_α. Then

(i) If $a \in S_\alpha, b \in S_\beta, (a, b) \in \mathcal{R}$, then $\alpha = \beta$;

(ii) For each $\alpha \in Y, \mathcal{R}(S_\alpha) = \mathcal{R}(S) \cap (S_\alpha \times S_\alpha)$.

§3. Structure theorem

We are now able to describe the *wrpp* semigroups with left central idempotents by using $\mathcal{R}$-left cancellative right stripes. A semigroup S is said to be a $\mathcal{R}$-left cancellative monoid if $(ab, ac) \in \mathcal{R}$ implies $(b, c) \in \mathcal{R}$ for any $a, b, c \in S$ (see [1]). The direct product of a right zero band I and an $\mathcal{R}$-left cancellative monoid M is called an $\mathcal{R}$-left cancellative right stripe.

Now we come to our main result for the construction of *wrpp* semigroups with left central idempotents.

Theorem 3.1. The following statements are equivalent on a semigroup S:

(i) S is a *wrpp* semigroup with left central idempotents;

(ii)S is a semilattice of $\mathcal{R}$-left cancellative right stripes and $E(S)$ is a right normal band;

(iii) S is a strong semilattice of $\mathcal{R}$-left cancellative right stripes.

To complete the proof of Theorem 3.1, we first show the following lemma.

Lemma 3.2. If S is a *wrpp* semigroup with left central idempotents and S_α is any σ-class of S, then S_α can be expressed as a direct product of $\mathcal{R}$-left cancellative monoid and a right zero band.

Proof. Let S be a *wrpp* semigroup with left central idempotents. Then by Theorem 2.6 there exists a semilattice Y such that $S = \bigcup_{\alpha \in Y} S_\alpha$, where S_α are σ-classes of S and $Y \simeq S/\sigma$.

To show that every S_α can be expressed as a direct product of $\mathcal{R}$-left cancellative monoid and a right zero band, we make the following observation :

1. For every $\alpha \in Y$, let $\Lambda_\alpha = S_\alpha \cap E(S)$. We prove that Λ_α is a right zero band. Suppose that $a \in S_\alpha$. Clearly, $a\sigma a^+$ and so that $a^+ \in \Lambda_\alpha$. It is easy to see that $e\sigma f$ for $e, f \in \Lambda_\alpha$. Thus, $ef = f$ and $fe = e$ which implies that Λ_α is a right zero band.

2. For each $\alpha \in Y$, take a fixed $e_\alpha \in \Lambda_\alpha$ and form $M_\alpha = S_\alpha e_\alpha$. We verify that M_α is a monoid with the identity e_α. For this purpose, let $a, b \in M_\alpha$. Hence $a = xe_\alpha$ and $b = ye_\alpha$ for $x, y \in S_\alpha$. By Corollary 2.5, we may deduce that

$$(xy)^+x^+ = x^+y^+x^+ = x^+$$

and

$$x^+(xy)^+ = x^+x^+y^+ = x^+y^+ = (xy)^+.$$

This implies that $xy\sigma x$ and so that $xy \in S_\alpha$. Consequently,

$$ab = xe_\alpha ye_\alpha = xye_\alpha^2 = xye_\alpha$$

which giving $ab \in S_\alpha e_\alpha = M_\alpha$. Hence M_α is a monoid with the identity e_α.

Next we show that M_α is a $\mathcal{R}$-left cancellative monoid. Suppose that $(ab, ac) \in \mathcal{R}$ for any $a, b, c \in M_\alpha$. Notice that $a\mathcal{L}^{**}a^+$ and e_α is the identity of M_α. It follows that $(a^+be_\alpha, a^+ce_\alpha) \in \mathcal{R}$. Hence, $(ba^+e_\alpha, ca^+e_\alpha) \in \mathcal{R}$ because a^+ is left central. Clearly, $a^+\sigma e_\alpha$. This leads to $(be_\alpha, ce_\alpha) \in \mathcal{R}$ and so that $(b, c) \in \mathcal{R}$. This shows that M_α is a $\mathcal{R}$-left cancellative monoid.

3. Putting set $M_\alpha \times \Lambda_\alpha$, we consider a mapping φ: $M_\alpha \times \Lambda_\alpha \to S_\alpha$ by $\varphi(x, f) = xf$ for $x \in M_\alpha, f \in \Lambda_\alpha$. Now we can claim that φ is an isomorphism. For any $(x, f), (y, g) \in M_\alpha \times \Lambda_\alpha$ it follows that $\varphi(x, f)\varphi(y, g) = xfyg = xyg = \varphi[(x, f)(y, g)]$ which implies that φ is a homomorphism.

Suppose that $\varphi(x,f) = \varphi(y,g)$, for $(x,f),(y,g) \in M_\alpha \times \Lambda_\alpha$. Then $xf = yg$ and so $xfe_\alpha = yge_\alpha$ where $e_\alpha \in S_\alpha$. Since Λ_α is a right zero band, we may deduce that $xe_\alpha = ye_\alpha$, that is, $x = y$ which implies that $xf = xg$. By Corollary 2.5, we can get $x^+f = x^+g$ with $x^+ \in \Lambda_\alpha$. Since Λ_α is a right zero band, it then follows that $f = g$ and so that $(x,f) = (y,g)$. This shows that φ is injective. To see that φ is surjective, we just take any $a \in S_\alpha$. Clearly, $\varphi(ae_\alpha, a^+) = ae_\alpha a^+ = aa^+ = a$ and hence φ is surjective. Hence $S_\alpha \simeq M_\alpha \times \Lambda_\alpha$.

We now return to the proof of Theorem 3.1.

Proof of Theorem 3.1.

(i)⇒ (ii) is straightforward by Lemma 3.2 above.

(ii)⇒(iii): Suppose that S is a semilattice of $\mathcal{R}$-left cancellative right stripes, that is, $S = \cup_{\alpha\in Y} S_\alpha$ where $S_\alpha = (M_\alpha \times \Lambda_\alpha)$ such that M_α is an $\mathcal{R}$-left cancellative monoid and Λ_α is a right zero band. To see that there exist a family of construction homomorphisms for a strong semilattice of S_α, we consider a mapping $\theta_{\alpha,\beta}$: $S_\alpha \to S_\beta$ by $a \mapsto e_\beta a$ where $\alpha,\beta \in Y$ with $\alpha \geqslant \beta$ and $e_\beta \in E(S_\beta)$ is fixed. Clearly, $\theta_{\alpha,\alpha}$ is the identity mapping. In fact, $\theta_{\alpha,\alpha}$ is also a homomorphism. To see this, suppose that $a,b \in S_\alpha$ and e_β is the identity element of S_β. Then $e_\beta a \in S_\beta = M_\beta \times \Lambda_\beta$ and write $e_\beta a = (x,f)$. Also let $i^2 = i = (1_\beta, f) \in S_\beta = M_\beta \times \Lambda_\beta$ where 1_β is the identity of the monoid M_β. Then, $(e_\beta a)i = e_\beta a$ and $e_\beta i = i$. Putting $b = (y,g) \in M_\alpha \times \Lambda_\alpha$ and $j = (1_\alpha, g) \in M_\alpha \times \Lambda_\alpha$, we get $jb = b$. Using the right normality of $E(S)$, we obtain $e_\beta a e_\beta b = e_\beta a i e_\beta j b = e_\beta a i j b = e_\beta ab$ which implies that $a\theta_{\alpha,\beta} b\theta_{\alpha,\beta} = (ab)\theta_{\alpha,\beta}$, that is, $\theta_{\alpha,\beta}$ is a homomorphism. Now we will check that $\theta_{\alpha,\beta}\theta_{\beta,\gamma} = \theta_{\alpha,\gamma}$ for any α,β,γ in Y with $\alpha \geqslant \beta \geqslant \gamma$. Suppose that $a = (x,f) \in S_\alpha = M_\alpha \times \Lambda_\alpha, i^2 = i = (1_\alpha, f) \in M_\alpha \times \Lambda_\alpha$ where 1_α is the identity of the monoid M_α. Obviously, $ia = a$. It follows immediately from the right normality of $E(S)$ that $e_\gamma e_\beta i = e_\beta e_\gamma i = e_\gamma i$ and $a\theta_{\alpha,\beta}\theta_{\beta,\gamma} = e_\gamma(e_\beta a) = e_\gamma e_\beta ia = e_\gamma ia = e_\gamma a = a\theta_{\alpha,\gamma}$. This shows that $\theta_{\alpha,\beta}\theta_{\beta,\gamma} = \theta_{\alpha,\gamma}$.

Finally, we will point out that $ab = a\theta_{\alpha,\alpha\beta} b\theta_{\beta,\alpha\beta}$ for any $a \in S_\alpha$ and $b \in S_\beta$. Since $ab = e_{\alpha\beta}(ab)$, we only need to show that $e_{\alpha\beta}ab = e_{\alpha\beta}ae_{\alpha\beta}b$. Since $e_{\alpha\beta}a \in S_{\alpha\beta}$, by using similar arguments as above, we can show that there exists $h^2 = h \in S_{\alpha\beta}$ such that $e_{\alpha\beta}ah = e_{\alpha\beta}a$. Likewisely, for any $b \in S_\beta$, there exists $e_\beta^2 = e_\beta \in S_\beta$ such that $e_\beta b = b$. Thus, by the right normality of $E(S)$, we have

$$\begin{aligned} e_{\alpha\beta}ae_{\alpha\beta}b &= (e_{\alpha\beta}ah)e_{\alpha\beta}b = e_{\alpha\beta}ae_{\alpha\beta}e_\beta b \\ &= e_{\alpha\beta}ae_{\alpha\beta}he_\beta b = e_{\alpha\beta}ahe_\beta b \\ &= (e_{\alpha\beta}ah)(e_\beta b) = e_{\alpha\beta}ab. \end{aligned}$$

This shows that $ab = a\theta_{\alpha,\alpha\beta} b\theta_{\beta,\alpha\beta}$. Hence S is a strong semilattice of $M_\alpha \times \Lambda_\alpha$.

(iii)⇒(i): Let $S = [Y; S_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of $\mathcal{R}$-left cancellative right stripes $S_\alpha = M_\alpha \times \Lambda_\alpha$ where M_α is a $\mathcal{R}$-left cancellative monoid and Λ_α is a right zero band.

We will first prove that $E(S) = \bigcup_{\alpha\in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in E_\alpha\}$. If $(a,i) \in E(S)$, then there exists $\alpha \in Y$ such that $(a,i) \in E(S) \bigcap S_\alpha$ with $(a,i)^2 = (a^2, i^2) = (a^2, i) = (a,i)$. Hence $a^2 = a$. Since $1_\alpha a = a1_\alpha = a$, it follows that $(a1_\alpha, a^2) \in \mathcal{R}(M_\alpha)$. Notice that M_α is an $\mathcal{R}$-left cancellative monoid. It is clear that $(1_\alpha, a) \in \mathcal{R}(M_\alpha)$ and so that there exist

$u, v \in M_\alpha^1$ such that $a = 1_\alpha u$ and $1_\alpha = av$. Thus we may deduce that

$$\begin{aligned} a &= a1_\alpha = a \cdot av \\ &= a^2 v = av = 1_\alpha. \end{aligned}$$

This shows that $E(S) \subseteq \bigcup_{\alpha \in Y} \{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in E_\alpha\}$. The converse inclusion is immediate.

Next we claim that every idempotent of S is left central. Let $a, b \in S^1, b \neq 1$ and $e \in E(S)$. Then there exist $\alpha, \beta, \gamma \in Y$ such that $a \in S_\alpha^1, b \in S_\beta$ and $e \in E(S_\gamma)$. Write $\delta = \alpha\beta\gamma, a\theta_{\alpha,\delta} = (x, i) \in S_\delta, b\theta_{\beta,\delta} = (y, j) \in S_\delta$ and $e\theta_{\gamma,\delta} = (1_\delta, k) \in E(S_\delta)$, we have

$$\begin{aligned} aeb &= (a\theta_{\alpha,\delta})(e\theta_{\gamma,\delta})(b\theta_{\beta,\delta}) \\ &= (x, i)(1_\delta, k)(y, j) \\ &= (xy, j). \end{aligned}$$

Similarly, $eab = (xy, j)$. Thus $eab = aeb$, that is, every element of $E(S)$ is left central.

To prove that (i) holds, we still need to show that S is a *wrpp* semigroup. For any $a = (x, f) \in S_\alpha = M_\alpha \times \Lambda_\alpha$, we consider the element $e = (1_\alpha, f)$ where 1_α is the identity of M_α. Clearly, $ea = ae = (x, f) = a$. Hence, for any $b = (y, g) \in S_\beta = M_\beta \times \Lambda_\beta, c = (z, h) \in S_\gamma = M_\gamma \times \Lambda_\gamma$ we assume that $(ab, ac) \in \mathcal{R}$, where $ab = (x\theta_{\alpha,\alpha\beta} y\theta_{\beta,\alpha\beta}, g\theta_{\beta,\alpha\beta}), ac = (x\theta_{\alpha,\alpha\gamma} z\theta_{\gamma,\alpha\gamma}, h\theta_{\gamma,\alpha\gamma})$. By Lemma 2.7, we have $\alpha\beta = \alpha\gamma$. Furthermore, we can deduce that $(x\theta_{\alpha,\alpha\beta} y\theta_{\beta,\alpha\beta}, x\theta_{\alpha,\alpha\beta} z\theta_{\gamma,\alpha\beta}) \in \mathcal{R}, g\theta_{\beta,\alpha\beta} h\theta_{\gamma,\alpha\beta} = h\theta_{\gamma,\alpha\beta}, h\theta_{\gamma,\alpha\beta} g\theta_{\beta,\alpha\beta} = g\theta_{\beta,\alpha\beta}$. Because $M_{\alpha\beta}$ is $\mathcal{R}$-left cancellative monoid, $(y\theta_{\beta,\alpha\beta}, z\theta_{\gamma,\alpha\beta}) \in \mathcal{R}$. On the other hand, $eb = (y\theta_{\beta,\alpha\beta}, z\theta_{\beta,\alpha\beta}), ec = (z\theta_{\gamma,\alpha\beta}, h\theta_{\gamma,\alpha\beta})$, then by the definition of $\mathcal{R}$, we can verify that $(eb, ec) \in \mathcal{R}$. Conversely, if $(eb, bc) \in \mathcal{R}$, then $(aeb, aec) \in \mathcal{R}$ since $\mathcal{R}$ is left compatible. It follows that $(ab, ac) \in \mathcal{R}$, since $ae = a$. Hence, $a\mathcal{L}^{**}e$ and $ae = a$. In fact we have proved that S is a *wrpp* semigroup with left central idempotents.

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An equation involving the cubic sum of natural numbers and Smarandache primitive function

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Abstract For any positive integer n, let $S_p(n)$ denotes the Smarandache primitive function. The main purpose of this paper is using the elementary methods to study the number of the solutions of an equation involving the cubic sum of natural numbers and Smarandache primitive function, and give its all positive integer solutions.

Keywords Cubic sum, Smarandache primitive function, equation, solutions.

§1. Introduction and Results

Let p be a prime, n be any positive integer. The Smarandache primitive function $S_p(n)$ is defined as the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_2(1) = 2, S_2(2) = S_2(3) = 4, S_2(4) = 6, \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu [2] have studied it, and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n, we have

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

Li Jie [3] studied the solvability of the equation

$$S_p(1) + S_p(2) + \cdots + S_p(n) = S_p\left(\frac{n(n+1)}{2}\right),$$

and gave its all positive integer solutions. But it seems that no one knows the relationship between the cubic sum of natural numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = S_p\left(\frac{n^2(n+1)^2}{4}\right),$$

and give its all positive integer solutions. That is, we will prove the following:

Theorem. Let p be a prime, n be any positive integer. Then the equation

$$S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = S_p\left(\frac{n^2(n+1)^2}{4}\right) \tag{1}$$

has finite positive integer solutions.

(i) If $p = 2,\ 5,\ 11,\ 13,\ 17,\ 29$, or 31, then all positive integer solutions of the equation (1) are $n = 1,\ 2$;

(ii) If $p = 3,\ 7,\ 19,\ 23,\ 37,\ 41,\ 43,\ 47,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89$ or 97, then all positive integer solutions of the equation (1) are $n = 1,\ 2,\ 3$;

(iii) If $p = 53,\ 101,\ 103,\ 107$ or 109, then all positive integer solutions of the equation (1) are $n = 1,\ 2,\ 3,\ 4$;

(iv) If $p = 59,\ 61$, or 113, then all positive integer solutions of the equation (1) are $n = 1,\ 2,\ 3,\ 4,\ 5$;

(v) If $p \geq 127$, then the equation (1) has finite positive integer solutions. They are $n = 1,\ 2,\ \cdots,\ n_p$, where $n_p \geq 1$ is a positive integer, and $n_p = \left[\frac{\sqrt{8\sqrt{p}+1}-1}{2}\right]$, $[x]$ denotes the largest integer less than or equal to x.

§2. Several lemmas

To complete the proof of the theorem, we need the following several simple lemmas.

Lemma 1. Let p be a prime, n be any positive integer, $S_p(n)$ denote the Smarandache primitive function, then we have

$$S_p(k) \begin{cases} = pk, & \text{if } \ k \leq p, \\ < pk, & \text{if } \ k > p. \end{cases}$$

Proof. (See reference [4]).

Lemma 2. Let p be a prime, n be any positive integer, if n and p satisfying $p^\alpha \parallel n!$, then

$$\alpha = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

Proof. (See reference [5]).

Lemma 3. Let p be a prime, n be any positive integer, if $n > [\sqrt[3]{p}]$, then there must exist a positive integer m_k with $1 \leq m_k \leq k^3$ $(k = 1, 2, \cdots, n)$ such that

$$S_p(1^3) = m_1 p, \quad S_p(2^3) = m_2 p, \quad \cdots, \quad S_p(n^3) = m_n p.$$

and

$$k^3 \leq \sum_{i=1}^{\infty} \left[\frac{m_k p}{p^i}\right].$$

Proof. From the definition of $S_p(n)$, Lemma 1 and Lemma 2, we can easily get the conclusions of Lemma 3.

§3. Proof of the Theorem

In this section, we will complete the proof of Theorem. We discuss the equation $S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = S_p\left(\frac{n^2(n+1)^2}{4}\right)$ in the following five cases:

(I) If $p = 2$, then the equation (1) is

$$S_2(1^3) + S_2(2^3) + \cdots + S_2(n^3) = S_2\left(\frac{n^2(n+1)^2}{4}\right).$$

(a) If $n = 1$, $S_2(1^3) = 2 = S_2\left(\frac{1^2 \times 2^2}{4}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_2(1^3) + S_2(2^3) = 2 + 5 \times 2 = 12 = S_2\left(\frac{2^2 \times 3^2}{4}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_2(1^3) + S_2(2^3) + S_2(3^3) = 12 + 16 \times 2 = 44$, but $S_2\left(\frac{3^2(3+1)^2}{4}\right) = S_2(36) = 40$, so $n = 3$ is not a solution of the equation (1).

(d) If $n > 3$, from Lemma 3 we know that there must exist a positive integer m_k with $1 \le m_k \le k^3$ $(k = 1, 2, \cdots, n)$ such that

$$S_2(1^3) = 2m_1, \quad S_2(2^3) = 2m_2, \quad \ldots, \quad S_2(n^3) = 2m_n.$$

then we have $S_2(1^3) + S_2(2^3) + \cdots + S_2(n^3) = 2(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1, m_2 = 5, m_3 = 16$, from Lemma 3 we have

$$\begin{aligned}
& \sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n)-1}{2^i}\right] \\
= & \sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right] \\
= & m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right] \\
= & m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{2^i}\right] \\
\ge & (m_1+m_2+m_3-1)+\sum_{i=1}^{\infty}\left[\frac{m_1}{2^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_2-1}{2^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_3}{2^i}\right] \\
& +\left(m_4+\sum_{i=1}^{\infty}\left[\frac{m_4}{2^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{2^i}\right]\right) \\
\ge & 39+\sum_{i=1}^{\infty}\left[\frac{2m_4}{2^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{2m_n}{2^i}\right] \\
> & 1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.
\end{aligned}$$

Then from Lemma 2 we can get

$$2^{\frac{n^2(n+1)^2}{4}} \mid (2(m_1 + m_2 + \cdots + m_n) - 1)!$$

Therefore,

$$S_2\left(\frac{n^2(n+1)^2}{4}\right) \leq 2(m_1+m_2+\cdots+m_n)-1$$
$$< 2(m_1+m_2+\cdots+m_n) = S_2(1^3)+S_2(2^3)+\cdots+S_2(n^3).$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p=2$, then the equation (1) has only two positive integer solutions, they are $n=1,2$.

If $p=5,11,13,17,29,31$, then using the same method we can easily deduce that $n=1,2$ are all positive integer solutions of the equation (1).

(II) If $p=3$, then the equation (1) is

$$S_3(1^3)+S_3(2^3)+\cdots+S_3(n^3) = S_3\left(\frac{n^2(n+1)^2}{4}\right).$$

(*a*) If $n=1$, $S_3(1^3)=3=S_3\left(\frac{1^2\times 2^2}{4}\right)$, so $n=1$ is a solution of the equation (1).

(*b*) If $n=2$, $S_3(1^3)+S_3(2^3)=3+6\times 3=21=S_3\left(\frac{2^2\times 3^2}{4}\right)$, so $n=2$ is a solution of the equation (1).

(*c*) If $n=3$, $S_3(1^3)+S_3(2^3)+S_3(3^3)=21+19\times 3=78=S_3\left(\frac{3^2\times 4^2}{4}\right)$, so $n=3$ is a solution of the equation (1).

(*d*) If $n=4$, $S_3(1^3)+S_3(2^3)+S_3(3^3)+S_3(4^3)=78+45\times 3=213$, but $S_3\left(\frac{4^2(4+1)^2}{4}\right)=S_3(100)=204$, so $n=4$ is not a solution of the equation (1).

(*e*) If $n>4$, from Lemma 3 we know that there must exist a positive integer m_k with $1\leq m_k\leq k^3$ $(k=1,2,\cdots,n)$ such that

$$S_3(1^3)=3m_1,\quad S_3(2^3)=3m_2,\quad \cdots,\quad S_3(n^3)=3m_n.$$

then we have $S_3(1^3)+S_3(2^3)+\cdots+S_3(n^3)=3(m_1+m_2+\cdots+m_n)$.

On the other hand, notice that $m_1=1,\ m_2=6,\ m_3=19,\ m_4=45$, from Lemma 3 we

have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{3(m_1+m_2+\cdots+m_n)-1}{3^i}\right]\\
=\ &\sum_{i=1}^{\infty}\left[\frac{3(m_1+m_2+\cdots+m_n-1)+2}{3^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{3(m_1+m_2+\cdots+m_n-1)+2}{3^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{3^i}\right]\\
\geq\ &(m_1+m_2+m_3+m_4-1)+\sum_{i=1}^{\infty}\left[\frac{m_1}{3^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_2}{3^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_3-1}{3^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_4}{3^i}\right]\\
&+\left(m_5+\sum_{i=1}^{\infty}\left[\frac{m_5}{3^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{3^i}\right]\right)\\
\geq\ &101+\sum_{i=1}^{\infty}\left[\frac{3m_5}{3^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{3m_n}{3^i}\right]\\
>\ &1^3+2^3+3^3+4^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.
\end{aligned}$$

Then from Lemma 2 we can get

$$3^{\frac{n^2(n+1)^2}{4}}\mid(3(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
&S_3\left(\frac{n^2(n+1)^2}{4}\right)\leq 3(m_1+m_2+\cdots+m_n)-1\\
&<3(m_1+m_2+\cdots+m_n)=S_3(1^3)+S_3(2^3)+\cdots+S_3(n^3).
\end{aligned}$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p=3$, then the equation (1) has only three positive integer solutions, they are $n=1,2,3$.

If $p=7,19,23,37,41,43,47,67,71,73,79,83,89,97$, then using the same method we can easily deduce that $n=1,2,3$ are all positive integer solutions of the equation (1).

(III) If $p=53$, then the equation (1) is

$$S_{53}(1^3)+S_{53}(2^3)+\cdots+S_{53}(n^3)=S_{53}\left(\frac{n^2(n+1)^2}{4}\right).$$

(a) If $n=1$, $S_{53}(1^3)=53=S_{53}\left(\frac{1^2\times 2^2}{4}\right)$, so $n=1$ is a solution of the equation (1).

(b) If $n=2$, $S_{53}(1^3)+S_{53}(2^3)=53+8\times 53=477=S_{53}\left(\frac{2^2\times 3^2}{4}\right)$, so $n=2$ is a solution of the equation (1).

(c) If $n=3$, $S_{53}(1^3)+S_{53}(2^3)+S_{53}(3^3)=477+27\times 53=1908=S_{53}\left(\frac{3^2\times 4^2}{4}\right)$, so $n=3$ is a solution of the equation (1).

(*d*) If $n = 4$, $S_{53}(1^3) + S_{53}(2^3) + S_{53}(4^3) = 1908 + 63 \times 53 = 5247 = S_{53}\left(\frac{4^2 \times 5^2}{4}\right)$, so $n = 4$ is a solution of the equation (1).

(*e*) If $n = 5$, $S_{53}(1^3) + S_{53}(2^3) + S_{53}(3^3) + S_{53}(4^3) + S_{53}(5^3) = 5247 + 123 \times 53 = 11766$, but $S_{53}\left(\frac{5^2(5+1)^2}{4}\right) = S_{53}(225) = 11713$, so $n = 5$ is not a solution of the equation (1).

(*f*)If $n > 5$, from Lemma 3 we know that there must exist a positive integer m_k with $1 \leq m_k \leq k^3$ $(k = 1, 2, \cdots, n)$ such that

$$S_{53}(1^3) = 53m_1, \quad S_{53}(2^3) = 53m_2, \quad \cdots, \quad S_{53}(n^3) = 53m_n.$$

then we have $S_{53}(1^3) + S_{53}(2^3) + \cdots + S_{53}(n^3) = 53(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1$, $m_2 = 8$, $m_3 = 27$, $m_4 = 63$, $m_5 = 123$, from Lemma 3 we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{53(m_1+m_2+\cdots+m_n)-1}{53^i}\right] \\
=\ &\sum_{i=1}^{\infty}\left[\frac{53(m_1+m_2+\cdots+m_n-1)+52}{53^i}\right] \\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{53(m_1+m_2+\cdots+m_n-1)+52}{53^i}\right] \\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{53^i}\right] \\
\geq\ &(m_1+m_2+m_3+m_4+m_5-1)+\sum_{i=1}^{\infty}\left[\frac{(m_1+m_2+m_3+m_4+m_5-1)}{53^i}\right] \\
&+\left(m_6+\sum_{i=1}^{\infty}\left[\frac{m_6}{53^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{53^i}\right]\right) \\
\geq\ &225+\sum_{i=1}^{\infty}\left[\frac{53m_6}{53^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{53m_n}{53^i}\right] \\
\geq\ &1^3+2^3+3^3+4^3+5^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.
\end{aligned}$$

Then from Lemma 2 we can get

$$53^{\frac{n^2(n+1)^2}{4}} \mid (53(m_1 + m_2 + \cdots + m_n) - 1)!.$$

Therefore,

$$\begin{aligned}
&S_{53}\left(\frac{n^2(n+1)^2}{4}\right) \leq 53(m_1+m_2+\cdots+m_n)-1 \\
&< 53(m_1+m_2+\cdots+m_n) = S_{53}(1^3)+S_{53}(2^3)+\cdots+S_{53}(n^3).
\end{aligned}$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p = 53$, then the equation (1) has only four solutions, they are $n = 1, 2, 3, 4$.

If $p = 101, 103, 107, 109$, then using the same method we can easily deduce that $n = 1, 2, 3, 4$ are all positive integer solutions of the equation (1).

(IV) If $p = 59$ then the equation (1) is

$$S_{59}(1^3) + S_{59}(2^3) + \cdots + S_{59}(n^3) = S_{59}\left(\frac{n^2(n+1)^2}{4}\right).$$

(a) If $n = 1$, $S_{59}(1^3) = 59 = S_{59}\left(\frac{1^2 \times 2^2}{4}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_{59}(1^3) + S_{59}(2^3) = 59 + 8 \times 59 = 531 = S_{59}\left(\frac{2^2 \times 3^2}{4}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_{59}(1^3) + S_{59}(2^3) + S_{59}(3^3) = 531 + 27 \times 59 = 2124 = S_{59}\left(\frac{3^2 \times 4^2}{4}\right)$, so $n = 3$ is a solution of the equation (1).

(d) If $n = 4$, $S_{59}(1^3) + S_{59}(2^3) + S_{59}(4^3) = 2124 + 63 \times 59 = 5841 = S_{59}\left(\frac{4^2 \times 5^2}{4}\right)$, so $n = 4$ is a solution of the equation (1).

(e) If $n = 5$, $S_{59}(1^3) + S_{59}(2^3) + S_{59}(4^3) + S_{59}(5^3) = 5841 + 123 \times 59 = 13098 = S_{59}\left(\frac{5^2 \times 6^2}{4}\right)$, so $n = 5$ is a solution of the equation (1).

(f) If $n = 6$, $S_{59}(1^3) + S_{59}(2^3) + S_{59}(3^3) + S_{59}(4^3) + S_{59}(5^3) + S_{59}(6^3) = 13098 + 213 \times 59 = 25665$, but $S_{59}\left(\frac{6^2(6+1)^2}{4}\right) = S_{59}(441) = 25606$, so $n = 6$ is not a solution of the equation (1).

(g) If $n > 6$, from Lemma 3 we know that there must exist a positive integer m_k with $1 \le m_k \le k^3$ $(k = 1, 2, \cdots, n)$ such that

$$S_{59}(1^3) = 59m_1, \quad S_{59}(2^3) = 59m_2, \quad \cdots, \quad S_{59}(n^3) = 59m_n.$$

Then we have $S_{59}(1^3) + S_{59}(2^3) + \cdots + S_{59}(n^3) = 59(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1, m_2 = 8, m_3 = 27, m_4 = 63, m_5 = 123, m_6 = 213$,

from Lemma 3 we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{59(m_1+m_2+\cdots+m_n)-1}{59^i}\right]\\
=\ &\sum_{i=1}^{\infty}\left[\frac{59(m_1+m_2+\cdots+m_n-1)+58}{59^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{59(m_1+m_2+\cdots+m_n-1)+58}{59^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{59^i}\right]\\
\geq\ &(m_1+m_2+m_3+m_4+m_5+m_6-1)+\sum_{i=1}^{\infty}\left[\frac{(m_1+m_2+m_3+m_4+m_5+m_6-1)}{59^i}\right]\\
&+\left(m_7+\sum_{i=1}^{\infty}\left[\frac{m_7}{59^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{59^i}\right]\right)\\
\geq\ &441+\sum_{i=1}^{\infty}\left[\frac{59m_7}{59^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{59m_n}{59^i}\right]\\
\geq\ &1^3+2^3+3^3+4^3+5^3+6^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.
\end{aligned}$$

Then from Lemma 2 we can get

$$59^{\frac{n^2(n+1)^2}{4}} \mid (59(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
&S_{59}\left(\frac{n^2(n+1)^2}{4}\right)\leq 59(m_1+m_2+\cdots+m_n)-1\\
&<59(m_1+m_2+\cdots+m_n)=S_{59}(1^3)+S_{59}(2^3)+\cdots+S_{59}(n^3).
\end{aligned}$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p=59$, then the equation (1) has only five positive integer solutions, they are $n=1,2,3,4,5$.

If $p=61,113$, then using the same method we can easily deduce that $n=1,2,3,4,5$ are all positive integer solutions of the equation (1).

(V) If $p\geq 127$, then we will discuss the problem in the following cases:

(a) If $\dfrac{n^2(n+1)^2}{4}\leq p$, solving this inequality we can get $1\leq n\leq n_p$, and

$$n_p=\left[\frac{\sqrt{8\sqrt{p}+1}-1}{2}\right]$$

,

where $[x]$ denotes the largest integer less than or equal to x, then

$$S_p\left(\frac{n^2(n+1)^2}{4}\right)=\frac{n^2(n+1)^2}{4}p.$$

Noting that $n_p \le [\sqrt[3]{p}] < p$, so if $1 \le n \le n_p$, then $n^3 \le p$, now we have

$$S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = 1^3p + 2^3p + \cdots + n^3p = \frac{n^2(n+1)^2}{4}p.$$

Combining above two formulae, we may immediately get $n = 1, 2, \cdots, n_p$ are all poaitive integer solutions of the equation $S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = S_p\left(\frac{n^2(n+1)^2}{4}\right)$.

(*b*) If $n_p < n \le [\sqrt[3]{p}]$, that is $\frac{n^2(n+1)^2}{4} > p$ and $n^3 \le p$, so we have

$$S_p\left(\frac{n^2(n+1)^2}{4}\right) < \frac{n^2(n+1)^2}{4}p,$$

but $S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = 1^3p + 2^3p + \cdots + n^3p = \frac{n^2(n+1)^2}{4}p$. Hence the equation (1) has no solution in this case.

(*c*) If $n \ge [\sqrt[3]{p}] + 1$, let $[\sqrt[3]{p}] = t$, then we can get $n^3 > p$, $t \ge 5$. So from Lemma 3 we know that there must exist a positive integer m_k with $1 \le m_k \le k^3$ $(k = 1, 2, \cdots, n)$ such that

$$S_p(1^3) = m_1p, \quad S_p(2^3) = m_2p, \quad \cdots, \quad S_p(n^3) = m_np,$$

then we have $S_p(1^3) + S_p(2^3) + \cdots + S_p(n^3) = (m_1 + m_2 + \cdots + m_n)p$.

On the other hand, notice that $m_1 = 1^3, m_2 = 2^3, \cdots, m_t = t^3$ and $\frac{t^2(t+1)^2}{4} - 1 > p$, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{(m_1 + m_2 + \cdots + m_n)p - 1}{p^i}\right] \\
= &\sum_{i=1}^{\infty}\left[\frac{p(m_1 + m_2 + \cdots + m_n - 1) + p - 1}{p^i}\right] \\
= &\ m_1 + m_2 + \cdots + m_n - 1 + \sum_{i=2}^{\infty}\left[\frac{p(m_1 + m_2 + \cdots + m_n - 1) + p - 1}{p^i}\right] \\
\ge &\ m_1 + m_2 + \cdots + m_n - 1 + \sum_{i=2}^{\infty}\left[\frac{\left(\frac{t^2(t+1)^2}{4} - 1\right)p + p - 1}{p^i}\right] \\
&+ \sum_{i=1}^{\infty}\left[\frac{m_{t+1} + m_{t+2} + \cdots + m_n}{p^i}\right] \\
\ge &\ m_1 + m_2 + \cdots + m_n + \sum_{i=1}^{\infty}\left[\frac{m_{t+1} + m_{t+2} + \cdots + m_n - 1}{p^i}\right] \\
\ge &\ m_1 + m_2 + \cdots + m_t + \left(m_{t+1} + \sum_{i=1}^{\infty}\left[\frac{m_{t+1}}{p^i}\right]\right) + \left(m_{t+2} + \sum_{i=1}^{\infty}\left[\frac{m_{t+2}}{p^i}\right]\right) + \cdots \\
&+ \left(m_n + \sum_{i=1}^{\infty}\left[\frac{m_n}{p^i}\right]\right) \\
\ge &\ \sum_{i=1}^{\infty}\left[\frac{m_1p}{p^i}\right] + \sum_{i=1}^{\infty}\left[\frac{m_2p}{p^i}\right] + \cdots + \sum_{i=1}^{\infty}\left[\frac{m_np}{p^i}\right] \\
\ge &\ 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.
\end{aligned}$$

Then from Lemma 2 we can get

$$p^{\frac{n^2(n+1)^2}{4}} \mid ((m_1+m_2+\cdots+m_n)p-1)!.$$

Therefore,

$$\begin{aligned} &S_p\left(\frac{n^2(n+1)^2}{4}\right) \leq (m_1+m_2+\cdots+m_n)p-1 \\ &< (m_1+m_2+\cdots+m_n)p = S_p(1^3)+S_p(2^3)+\cdots+S_p(n^3). \end{aligned}$$

From the above, we can deduce that if $p \geq 127$ and $n \geq [\sqrt[3]{p}]+1$, then the equation (1) has no solution .

Now our Theorem follows from (I), (II), (III), (IV) and (V).

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Smarandache isotopy theory of Smarandache: quasigroups and loops

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Abstract The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache f, g principal isotopes and G-Smarandache loops.

Keywords Smarandache, groupoids, quasigroups, loops, f, g principal isotopes.

§1. Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check[11], [1], [3], [4],[5] and [12]. In [12], Page 102, the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14][15][16][17][18][19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

§2. Definitions and notations

Definition 2.1. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L \; \forall \; x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that $\forall \; x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

If there exists at least a non-empty and non-trivial subset M of a groupoid(quasigroup or semigroup or loop) L such that $(M, \cdot)$ is a non-trivial subsemigroup(subgroup or subgroup

or subgroup) of $(L,\cdot)$, then L is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ M.

Let $(G,\cdot)$ be a quasigroup(loop). The bijection $L_x : G \to G$ defined as $yL_x = x\cdot y \ \forall\ x,y \in G$ is called a left translation(multiplication) of G while the bijection $R_x : G \to G$ defined as $yR_x = y\cdot x \ \forall\ x,y \in G$ is called a right translation(multiplication) of G.

The set $SYM(L,\cdot) = SYM(L)$ of all bijections in a groupoid $(L,\cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(L,\cdot)$.

Definition 2.2. If $(L,\cdot)$ and $(G,\circ)$ are two distinct groupoids, then the triple $(U,V,W) : (L,\cdot) \to (G,\circ)$ such that $U,V,W : L \to G$ are bijections is called an isotopism if and only if

$$xU \circ yV = (x\cdot y)W \ \forall\ x,y \in L.$$

So we call L and G groupoid isotopes. If $L = G$ and $W = I$(identity mapping) then (U,V,I) is called a principal isotopism, so we call G a principal isotope of L. But if in addition G is a quasigroup such that for some $f,g \in G$, $U = R_g$ and $V = L_f$, then $(R_g, L_f, I) : (G,\cdot) \to (G,\circ)$ is called an f,g-principal isotopism while $(G,\cdot)$ and $(G,\circ)$ are called quasigroup isotopes.

If $U = V = W$, then U is called an isomorphism, hence we write $(L,\cdot) \cong (G,\circ)$. A loop $(L,\cdot)$ is called a G-loop if and only if $(L,\cdot) \cong (G,\circ)$ for all loop isotopes $(G,\circ)$ of $(L,\cdot)$.

Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $(G')A = L'$, where $A \in \{U,V,W\}$, then the isotopism $(U,V,W) : (L,\cdot) \to (G,\circ)$ is called a Smarandache isotopism(S-isotopism). Consequently, if $W = I$ the triple (U,V,I) is called a Smarandache principal isotopism. But if in addition G is a S-quasigroup with S-subgroup H' such that for some $f,g \in H$, $U = R_g$ and $V = L_f$, and $(R_g, L_f, I) : (G,\cdot) \to (G,\circ)$ is an isotopism, then the triple is called a Smarandache f,g-principal isotopism while f and g are called Smarandache elements(S-elements).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L,\cdot) \gtrsim (G,\circ)$. An S-loop $(L,\cdot)$ is called a G-Smarandache loop(GS-loop) if and only if $(L,\cdot) \gtrsim (G,\circ)$ for all loop isotopes(or particularly all S-loop isotopes) $(G,\circ)$ of $(L,\cdot)$.

Example 2.1. The systems $(L,\cdot)$ and $(L,*)$, $L = \{0,1,2,3,4\}$ with the multiplication tables below are S-quasigroups with S-subgroups $(L',\cdot)$ and $(L'',*)$ respectively, $L' = \{0,1\}$ and $L'' = \{1,2\}$. $(L,\cdot)$ is taken from Example 2.2 of [10]. The triple (U,V,W) such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L,\cdot)$ onto $(L,*)$. Notice that $A(L') = L''$ for all $A \in \{U,V,W\}$ and $U,V,W : L' \to L''$ are all bijcetions.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Example 2.2. According to Example 4.2.2 of [15], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication table, below. The triple (U, V, W) such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in the second table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijcetions.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

Remark 2.1. Taking careful look at Definition 2.2 and comparing it with Definition 4.4.1[12], it will be observed that the author did not allow the component bijections U,V and W in (U, V, W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [12] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.

§3. Smarandache Isotopy and Isomorphy classes

Theorem 3.1. Let $\mathfrak{G} = \left\{\left(G_\omega, \circ_\omega\right)\right\}_{\omega\in\Omega}$ be a set of distinct S-groupoids with a corresponding set of S-subsemigroups $\mathfrak{H} = \left\{\left(H_\omega, \circ_\omega\right)\right\}_{\omega\in\Omega}$. Define a relation $\sim$ on $\mathfrak{G}$ such that for all $\left(G_{\omega_i}, \circ_{\omega_i}\right), \left(G_{\omega_j}, \circ_{\omega_j}\right) \in \mathfrak{G}$, where $\omega_i, \omega_j \in \Omega$,

$$\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_j}, \circ_{\omega_j}\right) \Longleftrightarrow \left(G_{\omega_i}, \circ_{\omega_i}\right) \text{ and } \left(G_{\omega_j}, \circ_{\omega_j}\right) \text{ are S-isotopic.}$$

Then $\sim$ is an equivalence relation on $\mathfrak{G}$.

Proof. Let $\left(G_{\omega_i}, \circ_{\omega_i}\right), \left(G_{\omega_j}, \circ_{\omega_j}\right), \left(G_{\omega_k}, \circ_{\omega_k}\right), \in \mathfrak{G}$, where $\omega_i, \omega_j, \omega_k \in \Omega$.

Reflexivity If $I : G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \ \forall\ x, y \in G_{\omega_i} \Longrightarrow \text{ the triple } (I, I, I) : \left(G_{\omega_i}, \circ_{\omega_i}\right) \to \left(G_{\omega_i}, \circ_{\omega_i}\right)$$

is an S-isotopism since $\left(H_{\omega_i}\right)I = H_{\omega_i}\ \forall\ \omega_i \in \Omega$. In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

Symmetry Let $\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_j}, \circ_{\omega_j}\right)$. Then there exist bijections

$$U, V, W : \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_j}, \circ_{\omega_j}\right) \text{ such that } \left(H_{\omega_i}\right)A = H_{\omega_j}\ \forall\ A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) : \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_j}, \circ_{\omega_j}\right)$$

is an isotopism. Since each of U, V, W is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} : \left(G_{\omega_j}, \circ_{\omega_j}\right) \longrightarrow \left(G_{\omega_i}, \circ_{\omega_i}\right)$$

are bijective. In fact, $\left(H_{\omega_j}\right)A^{-1} = H_{\omega_i}\ \forall\ A \in \{U, V, W\}$ since A is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) : \left(G_{\omega_j}, \circ_{\omega_j}\right) \longrightarrow \left(G_{\omega_i}, \circ_{\omega_i}\right)$$

is an isotopism. Thus, $\left(G_{\omega_j}, \circ_{\omega_j}\right) \sim \left(G_{\omega_i}, \circ_{\omega_i}\right)$.

Transitivity Let $\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_j}, \circ_{\omega_j}\right)$ and $\left(G_{\omega_j}, \circ_{\omega_j}\right) \sim \left(G_{\omega_k}, \circ_{\omega_k}\right)$. Then there exist bijections

$$U_1, V_1, W_1 : \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_j}, \circ_{\omega_j}\right) \text{ and } U_2, V_2, W_2 : \left(G_{\omega_j}, \circ_{\omega_j}\right) \longrightarrow \left(G_{\omega_k}, \circ_{\omega_k}\right)$$

$$\text{such that } \left(H_{\omega_i}\right)A = H_{\omega_j}\ \forall\ A \in \{U_1, V_1, W_1\}$$

$$\text{and } \left(H_{\omega_j}\right)B = H_{\omega_k}\ \forall\ B \in \{U_2, V_2, W_2\} \text{ so that the triples}$$

$$\alpha_1 = (U_1, V_1, W_1) : \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_j}, \circ_{\omega_j}\right) \text{ and}$$

$$\alpha_2 = (U_2, V_2, W_2) : \left(G_{\omega_j}, \circ_{\omega_j}\right) \longrightarrow \left(G_{\omega_k}, \circ_{\omega_k}\right)$$

are isotopisms. Since each of U_i, V_i, W_i, $i = 1, 2$, is bijective, then

$$U_3 = U_1U_2, V_3 = V_1V_2, W_3 = W_1W_2 : \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_k}, \circ_{\omega_k}\right)$$

are bijections such that $\left(H_{\omega_i}\right)A_3 = \left(H_{\omega_i}\right)A_1A_2 = \left(H_{\omega_j}\right)A_2 = H_{\omega_k}$ so that the triple

$$\alpha_3 = \alpha_1\alpha_2 = (U_3, V_3, W_3) \ : \ \left(G_{\omega_i}, \circ_{\omega_i}\right) \longrightarrow \left(G_{\omega_k}, \circ_{\omega_k}\right)$$

is an isotopism. Thus, $\left(G_{\omega_i}, \circ_{\omega_i}\right) \sim \left(G_{\omega_k}, \circ_{\omega_k}\right)$.

Remark 3.1. As a follow up to Theorem 3.1, the elements of the set $\mathfrak{G}/\sim$ will be referred to as Smarandache isotopy classes(S-isotopy classes). Similarly, if $\sim$ meant "S-isomorphism" in Theorem 3.1, then the elements of $\mathfrak{G}/\sim$ will be referred to as Smarandache isomorphy classes(S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.

Corollary 3.1. Let $\mathcal{L}_n$, $\mathcal{SL}_n$ and $\mathcal{NSL}_n$ be the sets of; all finite loops of order n; all finite S-loops of order n and all finite non S-loops of order n respectively.

1. If $\mathcal{A}_i^n$ and $\mathcal{B}_i^n$ represent the isomorphy class of $\mathcal{L}_n$ and the S-isomorphy class of $\mathcal{SL}_n$ respectively, then

(a) $|\mathcal{SL}_n| + |\mathcal{NSL}_n| = |\mathcal{L}_n|$;

(i) $|\mathcal{SL}_5| + |\mathcal{NSL}_5| = 56$,

(ii) $|\mathcal{SL}_6| + |\mathcal{NSL}_6| = 9,408$

(iii) $|\mathcal{SL}_7| + |\mathcal{NSL}_7| = 16,942,080$.

(b) $|\mathcal{NSL}_n| = \sum_{i=1} |\mathcal{A}_i^n| - \sum_{i=1} |\mathcal{B}_i^n|$;

(i) $|\mathcal{NSL}_5| = \sum_{i=1}^{6} |\mathcal{A}_i^5| - \sum_{i=1} |\mathcal{B}_i^5|$,

(ii) $|\mathcal{NSL}_6| = \sum_{i=1}^{109} |\mathcal{A}_i^6| - \sum_{i=1} |\mathcal{B}_i^6|$

(iii) $|\mathcal{NSL}_7| = \sum_{i=1}^{23,746} |\mathcal{A}_i^7| - \sum_{i=1} |\mathcal{B}_i^7|$.

2. If $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ represent the isotopy class of $\mathcal{L}_n$ and the S-isotopy class of $\mathcal{SL}_n$ respectively, then

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

(i) $|\mathcal{NSL}_5| = \sum_{i=1}^{2} |\mathfrak{A}_i^5| - \sum_{i=1} |\mathfrak{B}_i^5|$,

(ii) $|\mathcal{NSL}_6| = \sum_{i=1}^{22} |\mathfrak{A}_i^6| - \sum_{i=1} |\mathfrak{B}_i^6|$ and

(iii) $|\mathcal{NSL}_7| = \sum_{i=1}^{564} |\mathfrak{A}_i^7| - \sum_{i=1} |\mathfrak{B}_i^7|$.

Proof. An S-loop is an S-groupoid. Thus by Theorem 3.1, we have S-isomorphy classes and S-isotopy classes. Recall that $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \bigcap \mathcal{NSL}_n|$ but $\mathcal{SL}_n \bigcap \mathcal{NSL}_n = \emptyset$ so $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|$. As stated and shown in [11], [15], [2] and [9], the facts in Table 1 are true where n is the order of a finite loop. Hence the claims follow.

Question 3.1. How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{SL}_n|$ or $|\mathcal{NSL}_n|$.

Theorem 3.2. Let $(G,\cdot)$ be a finite S-groupoid of order n with a finite S-subsemigroup $(H,\cdot)$ of order m. Also, let

$$\mathcal{ISOT}(G,\cdot),\ \mathcal{SISOT}(G,\cdot) \text{ and } \mathcal{NSISOT}(G,\cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G,\cdot)$. Then,

$$\mathcal{ISOT}(G,\cdot) \text{ is a group and } \mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot).$$

Furthermore:

1. $|\mathcal{ISOT}(G,\cdot)| = (n!)^3$;
2. $|\mathcal{SISOT}(G,\cdot)| = (m!)^3$;
3. $|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3$.

Proof.

1. This has been shown to be true in [Theorem 4.1.1, [4]].
2. An S-isotopism is an isotopism. So, $\mathcal{SISOT}(G,\cdot) \subset \mathcal{ISOT}(G,\cdot)$. Thus, we need to just verify the axioms of a group to show that $\mathcal{SISOT}(G,\cdot) \leq \mathcal{ISOT}(G,\cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G,\cdot) \text{ such that } \alpha = (U,V,W)\ :\ (G,\cdot) \longrightarrow (G,\circ),$$

where $(G,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subgroups $(H,\cdot)$ and $(K,\circ)$ respectively, we can set

$$U' := U|_H,\ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K\ \forall\ A \in \{U,V,W\},$$

so that $\mathcal{SISOT}(H,\cdot) = \{(U',V',W')\}$. This is possible because of the following arguments.

Let

$$X = \left\{ f' := f|_H \;\middle|\; f : G \longrightarrow G,\ f : H \longrightarrow K \text{ is bijective and } f(H) = K \right\}.$$

Let

$$SYM(H,K) = \{\text{bijections from } H \text{ unto } K\}.$$

n	5	6	7
$\lvert\mathcal{L}_n\rvert$	56	9, 408	16, 942, 080
$\{\mathcal{A}_i^n\}_{i=1}^k$	$k = 6$	$k = 109$	$k = 23,746$
$\{\mathfrak{A}_i^n\}_{i=1}^m$	$m = 2$	$m = 22$	$m = 564$

Table 1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

By definition, it is easy to see that $X \subseteq SYM(H,K)$. Now, for all $U \in SYM(H,K)$, define $U : H^c \longrightarrow K^c$ so that $U : G \longrightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H,K) \subseteq X$ so that $SYM(H,K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|\mathcal{ISOT}(H,\cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G,\cdot)| = (m!)^3 \text{ since } SYM(H,K) = X.$$

3.
$$\mathcal{NSISOT}(G,\cdot) = \big(\mathcal{SISOT}(G,\cdot)\big)^c.$$

So, the identity isotopism

$$(I,I,I) \notin \mathcal{NSISOT}(G,\cdot), \text{ hence } \mathcal{NSISOT}(G,\cdot) \nleq \mathcal{ISOT}(G,\cdot).$$

Furthermore,
$$|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3.$$

Corollary 3.2. Let $(G,\cdot)$ be a finite S-groupoid of order n with an S-subsemigroup $(H,\cdot)$. If $\mathcal{ISOT}(G,\cdot)$ is the group of all isotopisms of $(G,\cdot)$ and S_n is the symmetric group of degree n, then
$$\mathcal{ISOT}(G,\cdot) \gtrsim S_n \times S_n \times S_n.$$

Proof. As concluded in [Corollary 1, [4]], $\mathcal{ISOT}(G,\cdot) \cong S_n \times S_n \times S_n$. Let $\mathcal{PISOT}(G,\cdot)$ be the set of all principal isotopisms on $(G,\cdot)$. $\mathcal{PISOT}(G,\cdot)$ is an S-subgroup in $\mathcal{ISOT}(G,\cdot)$ while $S_n \times S_n \times \{I\}$ is an S-subgroup in $S_n \times S_n \times S_n$. If

$$\Upsilon : \mathcal{ISOT}(G,\cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon\big((A,B,I)\big) =< A,B,I > \ \forall\ (A,B,I) \in \mathcal{ISOT}(G,\cdot),$$

then
$$\Upsilon\Big(\mathcal{PISOT}(G,\cdot)\Big) = S_n \times S_n \times \{I\}. \ \therefore \ \mathcal{ISOT}(G,\cdot) \gtrsim S_n \times S_n \times S_n.$$

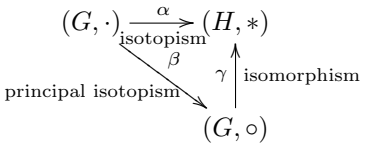
§4. Smarandache f,g-Isotopes of Smarandache loops

Theorem 4.1. Let $(G,\cdot)$ and $(H,*)$ be S-groupoids. If $(G,\cdot)$ and $(H,*)$ are S-isotopic, then $(H,*)$ is S-isomorphic to some Smarandache principal isotope $(G,\circ)$ of $(G,\cdot)$.

Proof. Since $(G,\cdot)$ and $(H,*)$ are S-isotopic S-groupoids with S-subsemigroups $(G_1,\cdot)$ and $(H_1,*)$, then there exist bijections $U,V,W : (G,\cdot) \to (H,*)$ such that the triple $\alpha = (U,V,W) : (G,\cdot) \to (H,*)$ is an isotopism and $\big(G_1\big)A = H_1 \ \forall\ A \in \{U,V,W\}$. To prove the claim of this theorem, it suffices to produce a closed binary operation '$*$' on G, bijections $X,Y : G \to G$, and bijection $Z : G \to H$ so that

- the triple $\beta = (X,Y,I) : (G,\cdot) \to (G,\circ)$ is a Smarandache principal isotopism and
- $Z : (G,\circ) \to (H,*)$ is an S-isomorphism or the triple $\gamma = (Z,Z,Z) : (G,\circ) \to (H,*)$ is an S-isotopism.

Thus, we need $(G, \circ)$ so that the commutative diagram below is true:



because following the proof of transitivity in Theorem 3.1, $\alpha = \beta\gamma$ which implies $(U, V, W) = (XZ, YZ, Z)$ and so we can make the choices; $Z = W$, $Y = VW^{-1}$, and $X = UW^{-1}$ and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \ \forall \ x, y \in G.$$

Hence, $(G, \circ)$ is a groupoid principal isotope of $(G, \cdot)$ and $(H, *)$ is an isomorph of $(G, \circ)$. It remains to show that these two relationships are Smarandache.

Note that $\big((H_1)Z^{-1}, \circ\big) = (G_1, \circ)$ is a non-trivial subsemigroup in $(G, \circ)$. Thus, $(G, \circ)$ is an S-groupoid. So $(G, \circ) \succsim (H, *)$. $(G, \cdot)$ and $(G, \circ)$ are Smarandache principal isotopes because $(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$ and $(G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$.

Corollary 4.1. Let $(G, \cdot)$ be an S-groupoid with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-groupoid if and only if all the principal isotopes of $(G, \cdot)$ are S-groupoids.

Proof. By classical result in principal isotopy [[11], III.1.4 Theorem], if $(G, \cdot)$ and $(H, *)$ are isotopic groupoids, then $(H, *)$ is isomorphic to some principal isotope $(G, \circ)$ of $(G, \cdot)$. Assuming $(H, *)$ is an S-groupoid then since $(H, *) \cong (G, \circ)$, $(G, \circ)$ is an S-groupoid. Conversely, let us assume all the principal isotopes of $(G, \cdot)$ are S-groupoids. Since $(H, *) \cong (G, \circ)$, then $(H, *)$ is an S-groupoid.

Theorem 4.2. Let $(G, \cdot)$ be an S-quasigroup. If $(H, *)$ is an S-loop which is S-isotopic to $(G, \cdot)$, then there exist S-elements f and g so that $(H, *)$ is S-isomorphic to a Smarandache f, g principal isotope $(G, \circ)$ of $(G, \cdot)$.

Proof. An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1, $(H, *)$ is S-isomorphic to a Smarandache principal isotope $(G, \circ)$ of $(G, \cdot)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Since $(H, *)$ is a S-loop and $(G, \circ) \succsim (H, *)$ implies that $(G, \circ) \cong (H, *)$, then $(G, \circ)$ is necessarily an S-loop and consequently, $(G, \circ)$ has a two-sided identity element say e and an S-subgroup $(G_2, \circ)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Then,

$$xU \circ yV = x \cdot y \ \forall \ x, y \in G \iff x \circ y = xU^{-1} \cdot yV^{-1} \ \forall \ x, y \in G.$$

So,

$$y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1}L_{eU^{-1}} \ \forall \ y \in G \text{ and } x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1}R_{eV^{-1}} \ \forall \ x \in G.$$

Assign $f = eU^{-1}, g = eV^{-1} \in G_2$. This assignments are well defined and hence $V = L_f$ and $U = R_g$. So that $\alpha = (R_g, L_f, I)$ is a Smarandache f, g principal isotopism of $(G, \circ)$ onto $(G, \cdot)$. This completes the proof.

Corollary 4.2. Let $(G,\cdot)$ be an S-quasigroup(S-loop) with an arbitrary groupoid isotope $(H,*)$. Any such groupoid $(H,*)$ is an S-quasigroup(S-loop) if and only if all the principal isotopes of $(G,\cdot)$ are S-quasigroups(S-loops).

Proof. This follows immediately from Corollary 4.1, since an S-quasigroup and an S-loop are S-groupoids.

Corollary 4.3. If $(G,\cdot)$ and $(H,*)$ are S-loops which are S-isotopic, then there exist S-elements f and g so that $(H,*)$ is S-isomorphic to a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$.

Proof. An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.

§5. G-Smarandache loops

Lemma 5.1. Let $(G,\cdot)$ and $(H,*)$ be S-isotopic S-loops. If $(G,\cdot)$ is a group, then $(G,\cdot)$ and $(H,*)$ are S-isomorphic groups.

Proof. By Corollary 4.3, there exist S-elements f and g in $(G,\cdot)$ so that $(H,*) \succsim (G,\circ)$ such that $(G,\circ)$ is a Smarandache f,g principal isotope of $(G,\cdot)$.

Let us set the mapping $\psi := R_{f\cdot g} = R_{fg} \ : \ G \to G$. This mapping is bijective. Now, let us consider when $\psi := R_{fg} \ : \ (G,\cdot) \to (G,\circ)$. Since $(G,\cdot)$ is associative and $x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x,y \in G$, the following arguments are true.

$x\psi \circ y\psi = x\psi R_g^{-1} \cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1} \cdot yR_{fg}L_f^{-1} = x \cdot fg \cdot g^{-1} \cdot f^{-1} \cdot y \cdot fg = x \cdot y \cdot fg = (x\cdot y)R_{fg} = (x\cdot y)\psi \ \forall \ x,y \in G$. So, $(G,\cdot) \cong (G,\circ)$. Thus, $(G,\circ)$ is a group. If $(G_1,\cdot)$ and $(G_1,\circ)$ are the S-subgroups in $(G,\cdot)$ and $(G,\circ)$, then $\big((G_1,\cdot)\big)R_{fg} = (G_1,\circ)$. Hence, $(G,\cdot) \succsim (G,\circ)$.

$\therefore \ (G,\cdot) \succsim (H,*)$ and $(H,*)$ is a group.

Corollary 5.1. Every group which is an S-loop is a GS-loop.

Proof. This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

Corollary 5.2. An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.

Proof. Let $(G,\cdot)$ be an S-loop with arbitrary S-isotope $(H,*)$. Let us assume that $(G,\cdot) \succsim (H,*)$. From Corollary 4.3, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(H,*) \succsim (G,\circ)$. So, $(G,\cdot) \succsim (G,\circ)$.

Conversely, let $(G,\cdot) \succsim (G,\circ)$, using the fact in Corollary 4.3 again, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache f,g principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(G,\circ) \succsim (H,*)$. Therefore, $(G,\cdot) \succsim (H,*)$.

Corollary 5.3. A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache f,g principal isotopes.

Proof. This follows by the definition of a GS-loop and Corollary 5.2.

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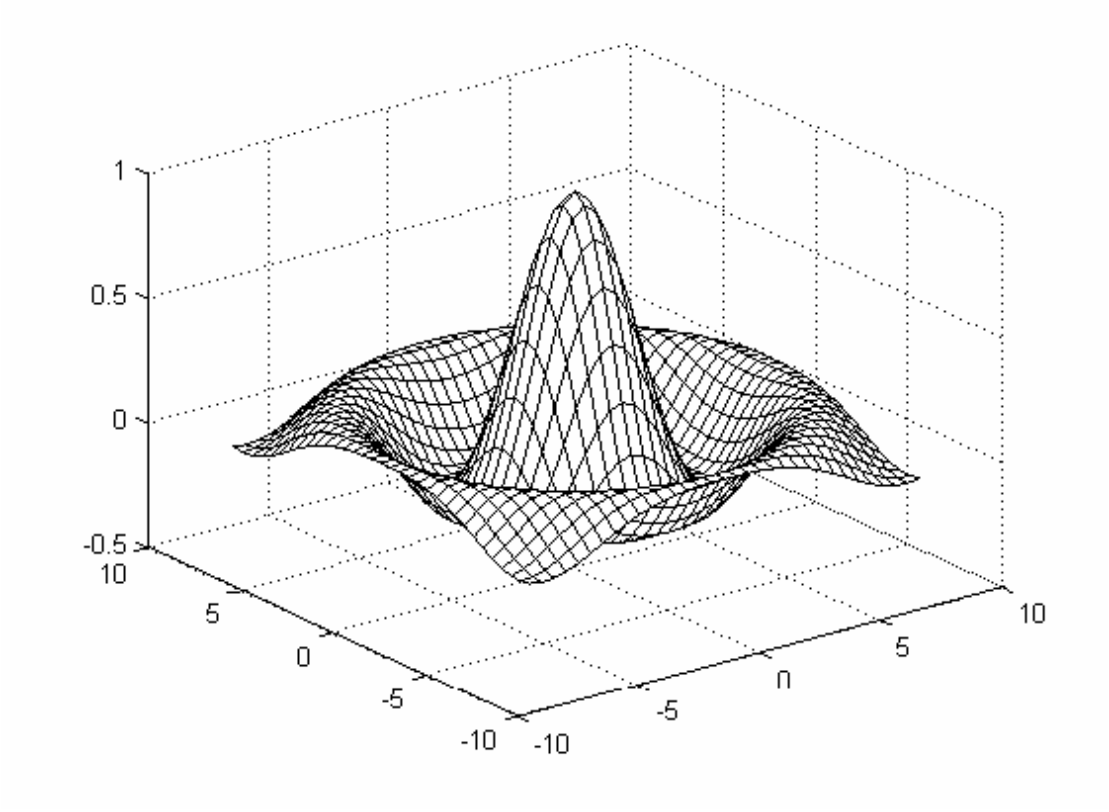

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