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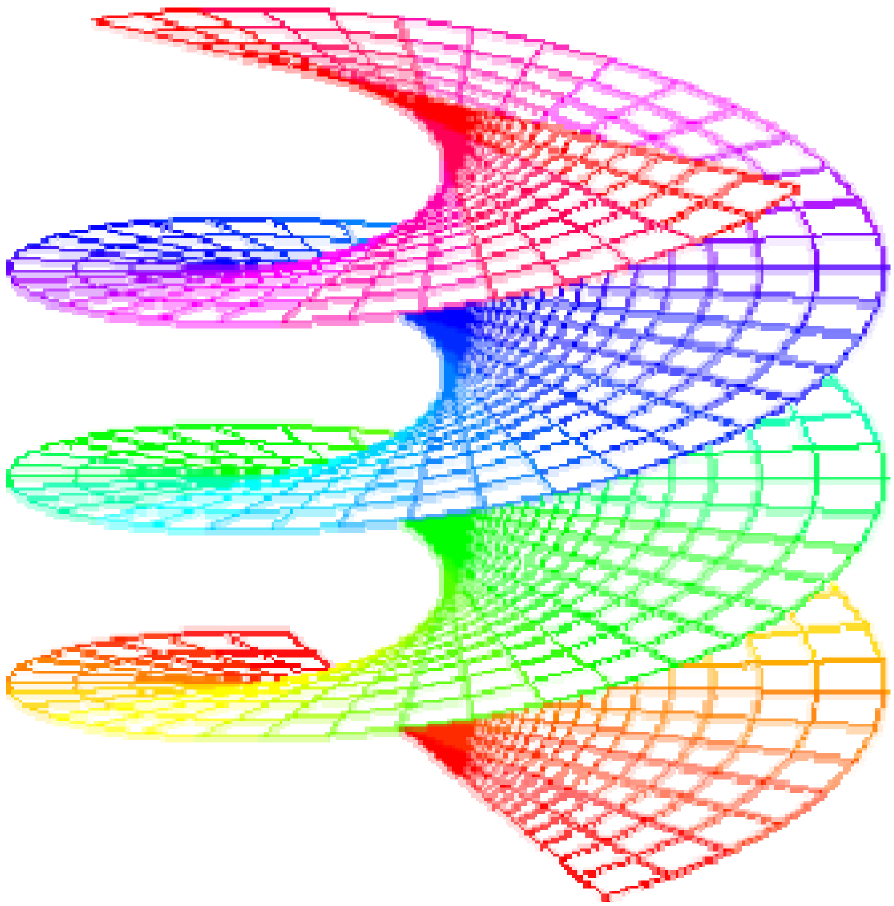

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Subclass of analytic and univalent functions in the unit disk

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Abstract In this paper we have studied a few properties of the class $H_{\lambda,\mu}(\alpha,\beta,\gamma)$ and some of its subclasses. We have also obtained some generalisation by using Ruschewayh derivatives on the lines of K. S. Padmanabhan and R. Bharati [1].

Keywords Ruschewayh derivatives, analytic and univalent functions, quasi-subordinate.

§1. Introduction

Let T denote the class of functions f which are analytic and univalent in the unit disc $E = \{z; |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

Let $g(z)$ and $G(z)$ be analytic in E, then $g(z)$ is said to be quasi-subordinate to $G(z)$, written as $g(z) \prec G(z), z \in E$, if there exist functions ϕ and ψ are analytic in E with $|\phi(z)| \leq 1, |\psi(z)| < 1$ in E and $g(z) = \phi(z)G(\psi(z))$ in E.

Definition 1. A function $h(z)$ analytic in E with $h(0) = 1$, is said to belong to the class $J(\alpha, \beta)$ if

$$\left|\frac{h(z)-1}{\beta h(z) - [\beta + (1-\alpha)(1-\beta)]}\right| < 1, \quad z \in E, \tag{1}$$

where $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

Definition 2. Let $H_{\lambda,\mu}(\alpha,\beta,\gamma)$ denote the class of all functions $f \in T$ for which

$$(1-\gamma)\frac{D^{\lambda}f(z)}{D^{\mu}g(z)} + \gamma\frac{(D^{\lambda}f(z))'}{(D^{\mu}g(z))'} \in J(\alpha,\beta)$$

for some g where

$$Re\left(z\frac{(D^{\mu}g(z))'}{D^{\mu}g(z)}\right) > 0$$

and $\lambda \geq 0$, $\mu \geq 0$, $\gamma \geq 0$.

Note that the subclass $H_{1,0}(0,\beta,\lambda)$ was studied by Padmanabhan and Bharati [1].

In this paper we investigate a few properties of the class $H_{\lambda,\mu}(\alpha,\beta,\gamma)$ and some of its subclasses.

§2. Lemmas

We need the following some preliminary lemmas for our study.

Lemma 1. If

$$g(z)=\sum_{n=0}^{\infty}a_nz^n\prec G(z)=\sum_{n=0}^{\infty}A_nz^n,$$

then

$$\sum_{n=0}^{k}|a_n|^2\leq\sum_{n=0}^{k}|A_n|^2\ (k=0,1,2,\cdots).$$

This lemma is due to Robertson [2].

Lemma 2. Let $P(z)=1+\sum_{n=1}^{\infty}P_nz^n$ be analytic in E and belongs to $J(\alpha,\beta)$, then for $n\geq 1$, $|P_n|\leq(\beta-1)(1-\alpha)$.

Proof. Condition (1) is equivalent to

$$P(z)=\frac{1+[\beta+(1-\alpha)(1-\beta)]\psi(z)}{1+\beta\psi(z)},$$

where $\psi(0)=0$ and $|\psi(z)|<1$ in E. After simplification we have

$$\sum_{n=1}^{\infty}P_nZ^n=[(1-\alpha)(1-\beta)-\beta\sum_{n=1}^{\infty}P_nz^n]\psi(z).$$

By the application of Lemma 1, we obtain

$$\sum_{k=1}^{n}|P_k|^2\leq(1-\alpha)^2(1-\beta)^2+\beta^2\sum_{k=1}^{n-1}|P_k|^2.$$

So

$$|P_n|^2\leq(1-\alpha)^2(1-\beta)^2-(1-\beta)^2\sum_{k+1}^{n-1}|P_k|^2.$$

Hence $|P_n|\leq(1-\alpha)(\beta-1)$.

Lemma 3. Let $P(z)$ and $P_1(z)$ belongs to $J(\alpha,\beta)$ then

$$(1-\rho)P(z)+\rho P_1(z)\in J(\alpha,\beta),$$

where $0<\rho<1$. It is easy to see that condition $\psi\in J(\alpha,\beta)$ is equivalent to $|\psi-b|<c$, where

$$b=\frac{1-[\beta^2+\beta(1-\alpha)(1-\beta)]}{1-\beta^2},$$

and

$$c=\frac{(1-\alpha)(1-\beta)}{(1-\beta^2)}=\frac{(1-\alpha)}{(1+\beta)}.$$

So it is sufficient to show that

$$|(1-\rho)P(z)+\rho P_1(z)-b|\leq(1-\rho)|P(z)-b|+\rho|P_1(z)-b|<c.$$

The last inequality is true, since $P(z),P_1(z)\in J(\alpha,\beta)$.

Hence the proof is complete.

Lemma 4. Let $\gamma \geq 0$ and $V(z)$ be a starlike function in the unit disk E. Let $U(z)$ be analytic in E with $U(0) = V(0) = 0 = U'(0) - 1$. Then $\frac{U(z)}{V(z)} \in J(\alpha, \beta)$, whenever

$$(1-\gamma)\frac{U(z)}{V(z)} + \gamma\frac{U'(z)}{V'(z)} \in J(\alpha, \beta).$$

Proof. Suppose $\psi(z)$ be a function in E that is defined by

$$\frac{U(z)}{V(z)} = \frac{1 + [\beta + (1-\alpha)(1-\beta)]\psi(z)}{1 + \beta\psi(z)},$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$. It is clear that $\psi(z)$ is analytic, $\psi(0) = 0$. We shall prove $|\psi(z)| < 1$ in E. For, if not, there exists $z_0 \in E$, by Jack's Lemma [2], such that $|\psi(z_0)| = 1$ and $z\psi'(z_0) = k\psi(z_0)$, $k \geq 1$.

Let $Q(z) = (1-\gamma)\frac{U(z)}{V(z)} + \gamma\frac{U'(z)}{V'(z)}$, then on simplification we have

$$Q(z_0) = \frac{U(z_0)}{V(z_0)} + \frac{k\gamma V(z_0)}{zV'(z_0)}\frac{(1-\beta)(1-\alpha)\psi(z_0)}{(1+\beta\psi(z_0))^2}.$$

We consider

$$\left|\frac{Q(z_0) - 1}{\beta Q(z_0) - [\beta + (1-\alpha)(1-\beta)]}\right| = \left|\frac{\frac{(1-\beta)(1-\alpha)\psi(z_0)}{1+\beta\psi(z_0)}\left(1 + \frac{k\gamma V(z_0)}{z\gamma'(z_0)} + \frac{1}{1+\beta\psi(z_0)}\right)}{\frac{(1-\alpha)(1-\beta)}{1+\beta\psi(z_0)}\left(1 - \frac{k\gamma V(z_0)}{zV'(z_0)}\frac{\beta\psi(z_0)}{1+\beta\psi(z_0)}\right)}\right|$$

$$= \left|\frac{1 + \frac{k\gamma V z_0)}{zV'(z_0)(1+\beta\psi(z_0))}}{1 - \frac{k\gamma V(z_0)\beta\psi(z_0)}{zV'(z_0)(1+\beta\psi(z_0))}}\right|.$$

Now

$$\left|1 + \frac{\gamma k V(z_0)}{zV'(z_0)(1+\beta\psi(z_0))}\right| > \left|1 - \frac{k\gamma V(z_0)\beta\psi(z_0)}{zV'(z_0)(1+\beta\psi(z_0))}\right|,$$

provided

$$|1 + D(z_0)|^2 > |1 - \beta\psi(_0)D(z_0)|^2,$$

where

$$D(z_0) = \frac{\gamma k V(z_0)}{zV'(z_0)(1+\beta\psi z_0))}.$$

This condition reduces to the following

$$1 + |D(z_0)|^2 + 2ReD(z_0) > 1 + \beta^2|D(z_0)|^2 - 2Re(\beta\psi(z_0)D(z_0)),$$

or equivalently

$$(1-\beta^2)|D(z_0)|^2 + 2Re\gamma k\frac{V(z_0)}{zV'(z_0)} > 0.$$

But this is true since V is starlike and $\gamma \geq 0$, $k \geq 0$. Hence

$$\left|\frac{Q(z_0) - 1}{\beta Q(z_0) - [\beta + (1-\alpha)(1-\beta)]}\right| > 1,$$

which is contradiction with hypothesis.

This proves the lemma.

§3. Properties of the Class $H_{\lambda,\mu}$ (α, β, γ)

Theorem 1. If $f \in H_{\lambda,\mu}(\alpha, \beta, \gamma)$, then $f \in H_{\lambda,\mu}(\alpha, \beta, 0)$.

Proof. It is sufficient to choose $V(z) = V^{\mu}g(z)$ and $U(z) = D^{\lambda}f(z)$ in Lemma 4. This leads to the required result.

Corollary 1. If $0 \le \alpha < 1$, $0 \le \beta < 1$ and $f,\ g \in T$, then $\frac{f'(z)}{g'(z)} \in J(\alpha, \beta)$ implies $\frac{f(z)}{g(z} \in J(\alpha, \beta)$ whenever $g(z)$ is starlike in E. The special case $\beta = 1$, $\alpha = 0$ is obtained by [3].

Proof. Take $\mu = 1$, $\lambda = 0$, $\mu = 0$ in Theorem 1.

Corollary 2. If $0 \le \alpha < 1$, $0 \le \beta < 1$ and $f,\ g \in T$, then

$$\frac{f'(z) + zf''(z)}{g'(z) + zg''(z)} \in J(\alpha, \beta),$$

implies $\frac{f'(z)}{g'(z)} \in J(\alpha, \beta)$, whenever g is convex in E.

Proof. Take $\mu = 1$,$\lambda = 1$, $\mu = 1$ in Theorem 1.

Corollary 3. If $0 \le \alpha < 1$ and $f,\ g \in T$, then

$$\frac{f'(z) + zf''(z)}{g'(z)} \in J(\alpha, \beta) \Longrightarrow z\frac{f'(z)}{g(z)} \in J(\alpha, \beta),$$

whenever g is starlike in E.

Proof. Take $\gamma = 1$, $\lambda = 1$, $\mu = 0$ in Theorem 1.

Theorem 2. For $0 \le \gamma_1 < \gamma$, $H_{\lambda,\mu}(\alpha, \beta, \gamma) \subset H_{\lambda,\mu}(\alpha, \beta, \gamma_1)$.

Proof. If $\gamma_1 = 0$, the result follows from Theorem 1. Assume therefore $\gamma_1 \ne 0$ and $f \in H_{\lambda,\mu}(\alpha, \beta, \gamma)$. Then there exist a starlike function (V_g^{μ}) in E such that

$$P(\gamma, f) = (1 - \gamma)\frac{V^{\lambda}f(z)}{U^{\mu}g(z)} + \gamma\frac{(V^{\lambda}f(z))'}{(V^{\mu}g(z))'} \in J(\alpha, \beta),$$

and $\frac{V^{\lambda}f(z)}{V^{\mu}g(z)} \in J(\alpha, \beta)$. Now the result follows from the identity

$$P(\gamma, f) = \frac{\gamma_1}{\gamma}P(\gamma, f) + \left(1 - \frac{\gamma_1}{\gamma}\right)\frac{V^{\lambda}f(z)}{V^{\mu}g(z)}, \quad \gamma_1 < \gamma,$$

and the lemma 3.

Theorem 3. A function f is in $H_{\lambda,\mu}(\alpha, \beta, \gamma)$, if and only if there exist a function g in E with $Re\left(\frac{z(D^{\mu}g(z))'}{D^{\mu}g(z)}\right) > 0$, $g(0) = g'(0) - 1 = 0$, and analytic function $P, P(0) = 1$, belongs to $H_{\lambda,\mu}(\alpha, \beta, \gamma)$ such that

$$D^{\lambda}f(z) = \frac{1}{\gamma[D^{\mu}g(z)]^{\frac{1}{\gamma}-1}}\int_0^z [D^{\mu}g(t)]^{\frac{1}{\gamma}-1}(D^{\mu}g(t))'P(t)dt, \gamma > 0.$$

If $\gamma = 0$ then $D^{\lambda}f(z) = D^{\mu}g(z)P(z)$.

Proof. Let $f \in H_{\lambda,\mu}(\alpha, \beta, \gamma)$, then there exist a function g where $Re\left(\frac{z(D^{\mu}g(z))'}{D^{\mu}g(z)}\right) > 0$ in E and

$$P(z) = (1 - \gamma)\frac{D^{\lambda}f(z)}{d^{\mu}g(z)} + \gamma\frac{(D^{\lambda}f(z))'}{(D^{\mu}g(z))'} \in H(\alpha, \beta, \gamma). \tag{2}$$

It is clear that $P(z)$ is analytic in E, $P(0)=1$. Multiplying both sides of (2) by $\frac{1}{\gamma}[D^\mu g(z)]^{\frac{1}{\gamma}-1}\cdot$ $(D^\mu g(z))'$ we obtain

$$(\frac{1}{\gamma}-1)D^\lambda f(z)[D^\mu g(z)]^{\frac{1}{\gamma}-2}(D^\mu g(z))'+(D^\lambda f(z))'(D^\mu g(z))^{\frac{1}{\gamma}-1}$$
$$=\frac{1}{\gamma}P(z)[D^\mu g(z)]^{\frac{1}{\gamma}-1}(D^\mu g(z))'.$$

Or equivalently

$$(D^\lambda f(z)[D^\mu g(z)]^{\frac{1}{\gamma}-1})'=\frac{1}{\gamma}P(z)[D^\mu g(z)]^{\frac{1}{\gamma}-1}(D^\mu g(z))'.$$

Hence on integrating with respect to z we get the required representation formula. The proof of the converse part is direct.

Theorem 4. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in H_{\lambda,\mu}(\alpha,\beta,\gamma)$, then we have

(i) $a_0=0$, $a_1=1$,

(ii) $|a_2|\le\frac{(1-\alpha)(\beta-1)+2(1+\gamma)}{(1+\lambda)(1+\gamma)}$,

(iii) $|a_3|\le\frac{2(1-\alpha)(\beta-1)(3+7\gamma)+6+18\gamma+12\gamma^2}{(\lambda+1)(\lambda+2)(1+\gamma)(1+2\gamma)}$,

(iv) $|a_4|\le\frac{24(1+3\gamma)(1+2\gamma)(1+\gamma)+6(1-\alpha)(\beta-1)(6+32\gamma+46\gamma^2)}{(1+\gamma)(1+2\gamma)(1+3\gamma)(\lambda+1)(\lambda+2)(\lambda+3)}$.

Proof. Since $f\in H_{\lambda,\mu}(\alpha,\beta,\gamma)$, we have

$$(1-\gamma)\frac{D^\lambda f(z)}{D^\mu g(z)}+\gamma\frac{(D^\lambda f(z))'}{(D^\mu g(z))'}=P(z)=1+\sum_{n=1}^{\infty}P_nz^n,$$

where $P(z)$ is as in lemma 2. Since $D^\lambda f(z)=z+\sum_{n=2}^{\infty}B_n(\lambda)a_nz^n$, where

$$B_n(\lambda)=\frac{(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1)}{(n-1)!},$$

with setting $D^\mu g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ and comparing coefficients, we obtain

$$(1+\gamma)B_2(\lambda)a_2=P_1+(1-\gamma)b_2, \tag{3}$$

$$(1+2\gamma)B_3(\lambda)a_3=P_2+(1+2\gamma)b_3+(1+3\gamma)b_2B_2(\lambda)a_2-(1+3\gamma)b_2^2, \tag{4}$$

$$\begin{aligned}(1+3\gamma)B_4(\lambda)a_4 &= P_3+(1+3\gamma)b_4+(1+5\gamma)b_3B_2(\lambda)a_2\\ &\quad -2(1+5\gamma)b_3b_2+(1+5\gamma)b_2B_3(\lambda)a_3\\ &\quad -(1+7\gamma)b_2^2B_2(\lambda)a_2+(1+7\gamma)b_2^3.\end{aligned} \tag{5}$$

Using the estimates $|P_n|\le(1-\alpha)(\beta-1)$ and $|b_n|\le n$ in (3), we get (ii). Eliminating a_2 from (4) we obtain

$$a_3=\frac{P_2}{(1+2\gamma)B_3(\lambda)}+\frac{b_3}{B_3(\lambda)}+\frac{(1+3\gamma)b_2P_1}{(1+\gamma)(1+2\gamma)B_3(\lambda)}. \tag{6}$$

Now from (6) we get (iii). Substituting the value of a_2 and a_3 from (3) and (6) in (5) we have

$$a_4 = \frac{b_4}{B_4(\lambda)} + P_1\frac{(1+5\gamma)}{(1+\gamma)(1+3\gamma)B_4(\lambda)}\left(b_3 - \frac{\gamma(1-\gamma)}{(1+2\gamma)(1+5\gamma)}b_2^2\right) + \frac{P_3}{(1+3\gamma)B_4(\lambda)} + \frac{(1+5\gamma)}{(1+2\gamma)(1+3\gamma)B_4(\lambda)}b_2P_2.$$

So (iv) follows on using the following inequality of Keogh and Merkes [3],

$$|b_3 - \mu b_2^2| \le 3 - 4\mu \quad \text{if} \quad \mu \le 1/2,$$

and the proof is complete.

§4. A Subclass of $H_{\lambda,\mu}(\alpha,\beta,\gamma)$

Let $F_\lambda(\alpha,\beta,\gamma)$ denote the class of functions $f \in T$, which $P_1(\gamma, f) \in J(\alpha,\beta)$ with

$$P_1(\gamma, f) = (1-\gamma)\frac{D^\lambda(z)}{z} + \gamma(D^\lambda f(z))', \quad \gamma \ge 0.$$

Obviously $F_\lambda(\alpha,\beta,\gamma)$ is a subclass of $H_{\lambda,\mu}(\alpha,\beta,\gamma)$. It is easy to see that $F_\lambda(\alpha,\beta,\gamma) \subset F_\lambda(\alpha,\beta,\gamma_1)$ for $0 \le \gamma_1 < \gamma$ and $F_\lambda(\alpha,\beta,\gamma)$ is closed under linear combination. Moreover, functions in $F_\lambda(\alpha,\beta,\gamma)$ are obtained on taking the convolution of the function

$$L(z) = \frac{1}{\gamma z^{\frac{1}{\gamma}-1}} \int_0^z \frac{t^{\frac{1}{\gamma}-1}}{1-t} dt$$

with

$$P_1(z) = z + \sum_{n=2}^{\infty} \frac{(n-1)!}{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)} P_n z^n, \tag{7}$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} P_n z^n \in J(\alpha,\beta).$$

Theorem 5. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in F_\lambda(\alpha,\beta,\gamma)$ then

$$p(z) = z + \frac{1}{2}\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)B_n(\lambda)a_n b_n z^n \in F_\lambda(\alpha,\beta,\gamma).$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F_\lambda(\alpha,\beta,\gamma)$. We have

$$\left|1 + \sum_{n=2}^{\infty}(1-\gamma+\gamma_n)B_n(\lambda)a_n z^n - \frac{1-\beta[1-\alpha(1-\beta)]}{1-\beta^2}\right| \le \frac{(1-\alpha)(\beta-1)}{1-\beta^2}. \tag{8}$$

We know that if $h(z) = \sum_{n=0}^{\infty} h_n z^n$ is analytic in $|z| < 1$ and $|h(z)| \le M$, then $\sum_{n=0}^{\infty} |h_n|^2 \le M^2$.

By applying this result to (8) we have

$$\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)^2B_n^2(\lambda)|a_n|^2 \le (1-\alpha)^2.$$

Similarly

$$\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)^2B_n^2(\lambda)|b_n|^2 \le (1-\alpha)^2.$$

Now we consider

$$\begin{aligned}
&\left|(1-\gamma)\frac{D^\lambda p(z)}{z}+\gamma(D^\lambda p(z))'-\frac{1-\beta[1-\alpha(1-\beta)]}{1-\beta^2}\right|^2\\
&=\left|\frac{(1-\alpha)}{(1+\beta)}+\frac{1}{2}\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)^2B_n^2(\lambda)a_nb_nz^{n-1}\right|^2\\
&\le\frac{\beta^2(1-\alpha)^2(1-\beta)}{(1+\beta)}+\frac{\beta(1-\alpha)}{(1+\beta)}\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)^2B_n^2(\lambda)|a_n||b_n||z|^{n-1}\\
&+\frac{1}{4}\left|\sum_{n=2}^{\infty}(1-\gamma+\gamma_n)^2B_n^2(\lambda)a_nb_nz^{n-1}\right|\\
&\le\frac{\beta^2(1-\alpha)^2(1-\beta)}{(1+\beta)}+\frac{\beta(1-\beta)^3(1-\alpha)^3}{(1-\beta^2)^2}+\frac{1}{4}\frac{(1-\beta)^4(1-\alpha)^4}{(1-\beta^2)^2}.
\end{aligned}$$

But it is easy to see that the last expression is bounded by $\frac{(1-\alpha)^2}{(1-\beta)^2}$. So $p(z)\in F_\lambda(\alpha,\beta,\gamma)$ and the proof is complete.

We know that if $f\in F_\lambda(\alpha,\beta,\gamma)$ and $\nu\ge 1$, then $f\in F_\lambda(\alpha,\beta,1)$, that is

$$(D^\lambda f(z))'\in J(\alpha,\beta). \tag{9}$$

We shall obtain the estimate of r_0, such that in $|z|<r_0$, (9) satisfied for $f\in F_\lambda(\alpha,\beta,\gamma)$ and $0<\gamma<1$. For this discussion we need the following definition and lemma due to Ruscheweyh [4].

Definition. A function G analytic in the unit disk E normalised by $G(0)=0$, $G'(0)\ne 0$ is called prestarlike of order α, $\alpha\le 1$ if and only if

$$Re\frac{G(z)}{zG'(0)}>\frac{1}{2},\quad z\in E \text{ for } \alpha=1,$$

$$\frac{z}{(1-z)^{2(1-\alpha)}}*G(z)\in S_\alpha \text{ for } \alpha<1,$$

where $*$ denotes the Hadamard product and S_α the class of starlike functions of order $\alpha\le 1$. Denote it by R_α the class of prestarlike functions of order α.

Lemma 5. [5] For $\alpha\le 1$, let $G\in R_\alpha$, $p\in S_\alpha$. Let $F(z)$ be analytic in E, then $\frac{G*PF}{G*P}$ takes values in the closed convex hull of $F(E)$.

Theorem 6. Let $f\in F_\lambda(\alpha,\beta,\gamma)$, where $0<\gamma<1$. Then $(D^\lambda f(z))'\in J(\alpha,\beta)$ for $|z|<r_0$, where r_0 is the radius of the largest disk centered at the origin that $Re\ L'(z)>\frac{1}{2}$, where $L(z)$ is givenby (7).

Proof. Let $f(z) \in F_\lambda(\alpha, \beta, \gamma)$, then $f(z) = L(z) * P_1(z)$, where $\dfrac{D^\lambda P_1(z)}{z} \in J(\alpha, \beta)$. But

$$D^\lambda f(z) = L(z) * D^\lambda P_1(z) = L(z) * zP(z), \tag{10}$$

where $P(z) = \dfrac{D^\lambda P_1(z)}{z} \in J(\alpha, \beta)$. With taking derivative from the both side of (10), we have

$$z(D^\lambda f(z))' = zL'(z) * zP(z),$$

or

$$(D^\lambda f(z))' = \frac{zL'(z) * zP(z)}{z * zL'(z)}.$$

Let $G(z) = zL'(z)$ and let r_0 be the largest number in which $Re(L'(z)) > \frac{1}{2}$. Then $G(z)$ is prestarlike of order 1. It is clear that $h(z) = z$ is starlike of order 1. So by Lemma 5, $(D^\lambda f(z))' \in J(\alpha, \beta)$ the proof is completed.

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Large quaternary cyclic codes of length 85 and related quantum error-correcting[1]

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Abstract In this paper, we use 4-cyclotomic cosets of modulo n and generator polynomials to describe quaternary simple-root cyclic codes of length $n = 85$. We discuss the conditions under which a quaternary cyclic codes contain its dual, and obtain some quantum error-correcting codes of length $n = 85$, three of these codes are better than previous known codes.

Keywords Quaternary cyclic code, self-orthogonal code, quantum error-correcting code.

§1. Introduction

Since the initial discovery of quantum error-correcting codes, researchers have made great progress in developing quantum error-correcting codes. Many code construction are given. Reference [1] gives a thorough discussion of the principles of quantum coding theory. Many good quantum error-correcting codes were constructed from BCH codes, Reed-Muller codes, Reed-Solomon codes and algebraic geometric codes, see [2-6]. So it is natural to construct quantum error-correcting codes from quaternary cyclic codes.

It is known that there is a close relation between cyclotomic coset and cyclic codes. Suggested by this relation, we use 4-cyclotomic coset modulo n and generator polynomials to describe self-orthogonal cyclic codes and their dual codes.

This paper is organized as follows. In this section, we introduce some definition and do some preparation for further discussion. In section 2, we construct quaternary cyclic codes of length 85 and related quantum error-correcting codes. In section 3, we compare the parameters of our quantum error-correcting codes and related cyclic codes with previously known.

Let $F_4 = \{0, 1, \omega, \varpi\}$ be the Galois field with four elements such that $\varpi = 1 + \omega = \omega^2$, $\omega^3 = 1$. $\mathcal{C}_{n,k}$ be a quaternary linear $[n, k, d]$ code of length n, dimension k and minimum distance d. The weight polynomial of $\mathcal{C}_{n,k}$ is

$$A(z) = \sum_{i=0}^{n} A_i z^i,$$

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where A_i is the number of codewords of weight i in $\mathcal{C}$. The Hermitian inner product of $X, Y \in F_4^n$ is defined as

$$(X, Y) = X\overline{Y}^{\top} = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n = x_1y_1^2 + x_2y_2^2 + \cdots + x_ny_n^2.$$

The dual of $\mathcal{C}_{n,k}$ is defined as

$$\mathcal{C}^{\perp} = \{x \in F_4^n | \quad x \cdot y = 0 \quad forall \quad y \in \mathcal{C}\},$$

a code $\mathcal{C}$ is Hermitian self-orthogonal code if $\mathcal{C} \subseteq \mathcal{C}^{\perp h}$.

Let $(n, 2) = 1$, and let s be an integer such that $0 \leq s < n$. The 4-cyclotomic coset of modulo n is the set $\mathcal{C}_s^{(4)} = \{s, 4s, 4^2s, \cdots, 4^{k-1}s\}(modn)$, where k is the smallest positive integer such that $4^k s \equiv s(modn)$. We call a 4-cyclotomic coset $\mathcal{C}_s^{(4)}$ symmetric if $n - 2s \in \mathcal{C}_s^{(4)}$, and asymmetric if otherwise. The asymmetric cosets appear in pairs $\mathcal{C}_s^{(4)}$ and $\mathcal{C}_{-2s}^{(4)} = \mathcal{C}_{n-2s}^{(4)}$. We use $\varepsilon(n)$ to denote the number of symmetric cosets and $\delta(n)$ the number of asymmetric pairs.

If ξ is a primitive n-th root of unity in some field containing F_4, then the minimal polynomial of ξ^s over F_4 is

$$m_s(x) = \prod_{i \in \mathcal{C}_s^{(4)}} (x - \xi^i)$$

and

$$x^n - 1 = \prod_{t=1}^{\varepsilon(n)} m_{i_t}(x) \prod_{l=1}^{\delta(n)} (m_{j_l}(x) m_{-2j_l}(x)),$$

where the $\mathcal{C}_{i_t}(1 \leq t \leq \varepsilon(n))$ are all symmetric, and $(\mathcal{C}_{j_l}^{(4)}, \mathcal{C}_{-2j_l}^{(4)})(1 \leq l \leq \delta(n))$ are asymmetric pairs.

If $\mathcal{C}$ is a cyclic code of length n, then $\mathcal{C}$ has a generator polynomial $g(x)$ which is a divisor of $x^n - 1$. For any polynomial $f(x)$, use $\widetilde{f(x)} = x^{degf(x)} f(\frac{1}{x})$ to denote the reciprocal polynomial. Then the dual code of $\mathcal{C}$ has the generator polynomial $h(x) = \widetilde{(\frac{x^n-1}{g(x)})}$.

The following theorem from [1] can be used to construct quantum error-correcting code, we call this theorem as additive code construction.

Theorem 1.1. Let $\mathcal{C} = [n, k]$ be a quaternary code such that $\mathcal{C} \subset \mathcal{C}^{\perp}$. If there are not vectors of weight $< d$ in $\mathcal{C}^{\perp} \setminus \mathcal{C}$, then there exist a quantum error-correcting $[[n, n-2k, d]]$ code.

§2. Quaternary cyclic codes of length 85

In this section, we will discuss quaternary cyclic codes of length 85 and corresponding quantum error-correcting codes.

Theorem 2.1. Let $\mathcal{C} = [n, k, d]$ be a quaternary cyclic code and

$$\mathcal{C} = (\prod_{t=1}^{\varepsilon(n)} m_{i_t}^{a_t}(x) \prod_{l=1}^{\delta(n)} (m_{j_l}^{b_l}(x) m_{-2j_l}^{c_l}(x))).$$

Then $\mathcal{C}^{\perp} \subset \mathcal{C}$ if and only if $a_t = 0$ and $b_l + c_l \leq 1$.

Proof. Let the generator polynomial of $\mathcal{C}$ is

$$f(x) = \prod_{t=1}^{\varepsilon(n)} m_{i_t}^{a_t}(x) \prod_{l=1}^{\delta(n)} (m_{j_l}^{b_l}(x) m_{-2j_l}^{c_l}(x)).$$

Then the generator polynomial of $\mathcal{C}^{\perp}$ is

$$h(x) = (\widetilde{\frac{x^n - 1}{f(x)}}) = \prod_{t=1}^{\varepsilon(n)} m_{i_t}^{1-a_t}(x) \prod_{l=1}^{\delta(n)} (m_{j_l}^{1-b_l}(x) m_{-2j_l}^{1-c_l}(x))).$$

From [7] we know that $\mathcal{C}^{\perp} \subset \mathcal{C}$ if and only if $f(x)|h(x)$, thus the theorem is proved.

Table 1 Cyclotomic cosets and minimal polynomial

i	$C_i^{(4)}$	$m_i(x)$
0	{0}	x-1=(1,1)
1	{1,4,16,64}	$1+3x+x^3+x^4=(1,3,0,1,1)$
2	{2,8,32,43}	$1+2x+x^3+x^4=(1,2,0,1,1)$
3	{3,12,48,22}	$1+3x^2+2x^3+x^4=(1,0,3,2,1)$
5	{5,20,80,65}	$1+3x+x^2+3x^3+x^4=(1,3,1,3,1)$
6	{6,24,11,44}	$1+2x^2+3x^3+x^4=(1,0,2,3,1)$
7	{7,28,27,23}	$1+2x+2x^2+3x^3+x^4=(1,2,2,3,1)$
9	{9,36,59,66}	$1+2x+x^2+x^4=(1,2,1,0,1)$
10	{10,40,75,45}	$1+2x+x^2+2x^3+x^4=(1,2,1,2,1)$
13	{13,52,38,67}	$1+x^2+3x^3+x^4=(1,0,1,3,1)$
14	{14,56,54,46}	$1+3x+3x^2+2x^3+x^4=(1,3,3,2,1)$
15	{15,60,70,25}	$1+x+2x^2+x^3+x^4=(1,1,2,1,1)$
17	{17,68}	$1+3x+x^2=(1,3,1,0,0)$
18	{18,72,33,47}	$1+3x+x^2+x^4=(1,3,1,0,1)$
19	{19,76,49,26}	$1+x^2+2x^3+x^4=(1,0,1,2,1)$
21	{21,84,81,69}	$1+x+3x^3+x^4=(1,1,0,3,1)$
29	{29,31,39,71}	$1+2x+3x^2+3x^3+x^4=(1,2,3,3,1)$
30	{30,35,55,50}	$1+x+3x^2+x^3+x^4=(1,1,3,1,1)$
34	{34,51}	$1+2x+x^2=(1,2,1,0,0)$
37	{37,63,82,73}	$1+2x+3x^2+x^4=(1,2,3,0,1)$
41	{41,79,61,74}	$1+3x+2x^2+x^4=(1,3,2,0,1)$
42	{42,83,77,53}	$1+x+2x^3+x^4=(1,1,0,2,1)$
57	{57,58,62,78}	$1+3x+2x^2+2x^3+x^4=(1,3,2,2,1)$

Now we fix $n=85$, all the 4-cyclotomic cosets mod 85 and minimal polynomial of α^i are listed in Table 1(where the 4-cyclotomic cosets contain i are denoted by $C_i^{(4)}$ and the minimal polynomial denoted by $m_i(x)=\sum_{t=0}^{l} a_t x^t=(a_1,\cdots,a_l)$; ω are denoted by 2 and ϖ denoted by 3).

From Table 1, we know that $85-2i(mod85)(1\le i\le 85)$ are not contained in $C_i^{(4)}$ for all i, then all the 4-cyclotomic cosets mod 85 are asymmetric except $\mathcal{C}_0$.

We may obtain $\sum_{i=1}^{11}\binom{11}{i}2^i$ generator polynomial of quaternary cyclic code $\mathcal{C}_{85,k}$ that contain their dual by the 22 minimal polynomial $m_i(x)$ $(i\ne 0)$. It is easy to know that there are thousands of cyclic codes of length 85 generated by these minimal polynomials, and it is difficult to obtain the parameters of all cyclic codes of length 85. In light of the selecting method in Ref.[8], we may decrease the computing times and obtain the parameters of $\mathcal{C}_{85,k}$ easily.

Theorem 2.2. Let $n=85$, and quaternary code $\mathcal{C}_{n,k}$ be a $[n,k]$ cyclic code such that $\mathcal{C}_{n,k}^{\perp}\subset\mathcal{C}_{n,k}$, then there exist such codes with parameters $[n,79,4]$, $[n,73,5]$, $[n,73,6]$, $[n,71,7]$, $[n,69,7]$, $[n,65,9]$.

Proof. Let the generator polynomial $f_{n,k,d}(x)=m_{i_1}(x)...m_{i_k}(x)$ of cyclic code $\mathcal{C}_{n,k}=[85,k,d]$, where $i_j\in\mathcal{C}_{i_j}^{(4)}$ and $m_{i_j}(x)$ is the corresponding minimal polynomial. Then the generator polynomial $h_{n,n-k,d^{\perp}}(x)=(\widetilde{\frac{x^{85-1}}{f(x)}})$ of $\mathcal{C}_{n,k}^{\perp}=[85,85-k,d^{\perp}]$. If $85-k<16$, we can get the weight polynomial of $\mathcal{C}_{85,k}^{\perp}$ easily, then there exist self-orthogonal cyclic code $[85,85-k,d^{\perp}]$. From MacWillams Equation, thus we have the value of d. If $85-k\ge 16$, using the method in Ref.[9], we can obtain d directly. Especially,

(1) If $85-k<16$, then there are four quaternary cyclic codes $[85,k,d]$ satisfying $\mathcal{C}^{\perp}\subset\mathcal{C}$, these codes with generator polynomial

$$f_{85,79,4}(x)=m_1(x)m_{17}(x)=(1032221),$$

$$f_{85,73,5}(x)=m_1(x)m_3(x)m_{21}(x)=(1201301032301),$$

$$f_{85,73,6}(x)=m_3(x)m_5(x)m_9(x)=(1123002020211),$$

$$f_{85.71,7}(x)=m_1(x)m_3(x)m_5(x)m_{17}(x)=(131301012332331).$$

Accordingly, their dual codes with weight polynomial $A_{85,85-k}$ are as follow:

$$A_{85,6}(z)=0^1+60^{1275}+62^{1020}+64^{255}+66^{1020}+68^{15}+72^{510},$$

$$\begin{aligned}A_{85,12}(z) &= 0^1+48^{4080}+50^{12240}+52^{56100}+54^{208080}+56^{513060}+58^{1150560}+60^{2136900}\\ &+62^{2853960}+64^{3403230}+66^{2988600}+68^{1938000}+70^{1013880}+72^{390150}\\ &+74^{91800}+76^{12240}+78^{4080}+80^{255},\end{aligned}$$

$$\begin{aligned}A_{85,12}(z) &= 0^1+48^{6375}+52^{107100}+56^{1094970}+60^{4062864}+64^{6781215}\\ &+68^{3930060}72^{763470}+76^{30600}+80^{561},\end{aligned}$$

$$\begin{aligned}A_{85,14}(z) &= 0^1+44^{255}+46^{3060}+48^{49980}+50^{246840}+52^{952935}+54^{3062040}+56^{8327280}\\ &+58^{18706800}+60^{33218799}+62^{47307600}+64^{53529345}+66^{47281080}+68^{31774545}\\ &+70^{16065000}+72^{6102660}+74^{1509600}+76^{262650}+78^{33660}+80^{1326}.\end{aligned}$$

(2) If $85-k \geq 16$, then there are two quaternary cyclic codes $[85,k,d]$ satisfying $\mathcal{C}^{\perp} \subset \mathcal{C}$, these codes with generator polynomial

$$f_{85,69,7}(x) = m_1(x)m_3(x)m_5(x)m_{15}(x) = (11220032002220011).$$

$$f_{85,65,9}(x) = m_7(x)m_{13}(x)m_{14}(x)m_{30}(x)m_{42}(x) = (113133311231030113211).$$

Summarizing the above discussion, the theorem follows.

Using quaternary cyclic code given in theorem 2.2 and additive code construction, one can easily derive the following corollary.

Corollary 2.1. Let $n = 85$. Then there are quantum error-correcting codes with parameters $[[85,73,4]]$, $[[85,61,5]]$, $[[85,61,6]]$, $[[85,57,7]]$, $[[85,53,7]]$, $[[85,45,9]]$.

§3. Concluding Remarks

There are some good quantum error-correcting codes can be constructed from these quaternary cyclic codes of length $n = 85$ by additive code construction. Ref.[10] give the estimate parameters of quantum error-correcting codes $[[85,73,4]]$,$[[85,61,5]]$ and $[[85,53,7]]$, however, we present the real parameters of these codes, and we find that the estimate value and real value are equal.

Compare with the highest achievable minimum distance of additive quantum error-correcting code $[[85,61,5]]$ given in [11], linear quantum error-correcting code $[[85,61,5]]$ given in [10] and $[[86,60,6]]$ given in [11], our code $[[85,61,6]]$ is very good. Our quantum error-correcting codes $[[85,57,7]]$ and $[[85,45,9]]$ are better than $[[85,53,7]]$ and $[[85,41,9]]$ given in [12].

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Value distribution of the F.Smarandache LCM function[1]

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the value distribution properties of the function $SL(n)$, and give a sharper value distribution theorem.

Keywords F.Smarandache LCM function, value distribution, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1, 2, \cdots, k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [3] showed that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes, and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Lv Zhongtian [6] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

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where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

The main purpose of this paper is using the elementary methods to study the value distribution properties of $SL(n)$, and prove an interesting value distribution theorem. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} (SL(n) - P(n))^2 = \frac{2}{5} \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

§2. Proof of the theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [3] we know that

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{2}$$

Now we consider the summation

$$\sum_{n\le x} (SL(n) - P(n))^2 . \tag{3}$$

We separate all integers n in the interval $[1,\ x]$ into four subsets A, B, C and D as follows:

A: $P(n) \ge \sqrt{n}$ and $n = m \cdot P(n)$, $m < P(n)$;

B: $n^{\frac{1}{3}} < P(n) \le \sqrt{n}$ and $n = m \cdot P^2(n)$, $m < n^{\frac{1}{3}}$;

C: $n^{\frac{1}{3}} < p_1 < P(n) \le \sqrt{n}$ and $n = m \cdot p_1 \cdot P(n)$, where p_1 is a prime;

D: $P(n) \le n^{\frac{1}{3}}$.

It is clear that if $n \in A$, then from (2) we know that $SL(n) = P(n)$. Therefore,

$$\sum_{n\in A} (SL(n) - P(n))^2 = \sum_{n\in A} (P(n) - P(n))^2 = 0. \tag{4}$$

Similarly, if $n \in C$, then we also have $SL(n) = P(n)$. So

$$\sum_{n\in C} (SL(n) - P(n))^2 = \sum_{n\in C} (P(n) - P(n))^2 = 0. \tag{5}$$

Now we estimate the main terms in set B. Applying Abel's summation formula (see Theorem 4.2 of [5]) and the Prime Theorem (see Theorem 3.2 of [7])

$$\pi(x) = \sum_{p\le x} 1 = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

we have

$$\sum_{n\in B}(SL(n)-P(n))^2=\sum_{\substack{mp^2\le x\\ m<p}}\left(SL(mp^2)-P(mp^2)\right)^2$$
$$=\sum_{m\le x^{\frac{1}{3}}}\sum_{m<p\le\sqrt{\frac{x}{m}}}\left(p^2-p\right)^2$$
$$=\sum_{m\le x^{\frac{1}{3}}}\left[\left(\frac{x}{m}\right)^2\cdot\pi\left(\sqrt{\frac{x}{m}}\right)-4\int_m^{\sqrt{\frac{x}{m}}}y^3\pi(y)dx+O\left(m^5+\frac{x^2}{m^2}\right)\right]$$
$$=\sum_{m\le x^{\frac{1}{3}}}\left(\frac{x^{\frac{5}{2}}}{5m^{\frac{5}{2}}\ln\sqrt{\frac{x}{m}}}+O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}\ln^2\frac{x}{m}}\right)\right)$$
$$=\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),\qquad(6)$$

where $\zeta(s)$ is the Riemann zeta-function.

Finally, we estimate the error terms in set D. For any integer $n\in D$, let $SL(n)=p^{\alpha}$. If $\alpha=1$, then $SL(n)=p=P(n)$, so that $SL(n)-P(n)=0$. Therefore, we assume that $\alpha\ge 2$. This time note that $P(n)\le n^{\frac{1}{3}}$, we have

$$\sum_{n\in D}(SL(n)-P(n))^2\ll\sum_{n\in D}\left(SL^2(n)+P^2(n)\right)$$
$$\ll\sum_{\substack{mp^{\alpha}\le x\\ \alpha\ge 2,\ p<x^{\frac{1}{3}}}}p^{2\alpha}+\sum_{n\le x}n^{\frac{2}{3}}\ll\sum_{\substack{p^{\alpha}\le x\\ \alpha\ge 2,\ p\le x^{\frac{1}{3}}}}p^{2\alpha}\sum_{m\le\frac{x}{p^{\alpha}}}1+x^{\frac{5}{3}}$$
$$\ll x\cdot\sum_{\substack{p^{\alpha}\le x\\ \alpha\ge 2,\ p\le x^{\frac{1}{3}}}}p^{\alpha}+x^{\frac{5}{3}}\ll x^2.\qquad(7)$$

Combining (3), (4), (5), (6) and (7) we may immediately obtain the asymptotic formula

$$\sum_{n\le x}(SL(n)-P(n))^2=\sum_{n\in A}(SL(n)-P(n))^2+\sum_{n\in B}(SL(n)-P(n))^2$$
$$+\sum_{n\in C}(SL(n)-P(n))^2+\sum_{n\in D}(SL(n)-P(n))^2$$
$$=\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right).$$

This completes the proof of Theorem.

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About a chain of inequalities

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Abstract In this paper we give refinements for the inequalities $x^2+y^2+z^2 \geq xy+yz+zx$ and $\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right) \geq (xy+yz+zx)^3$ (see [1] and [2]).

Keywords AM-GM-HM inequalities.

§1. Main Results

Theorem 1. If $x, y, z \geq 0$, then

$$\begin{aligned}\left(x^2+y^2+z^2\right)^3 &\geq \frac{27}{8}\left(x^2+y^2\right)\left(y^2+z^2\right)\left(z^2+x^2\right)\\ &\geq \left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\\ &\geq \frac{27}{64}(x+y)^2(y+z)^2(z+x)^2\\ &\geq (xy+yz+zx)^3.\end{aligned}$$

Proof.

$$\frac{27}{8}\prod\left(x^2+y^2\right) = 27\prod\frac{x^2+y^2}{2} \leq \left(\sum\frac{x^2+y^2}{2}\right)^3 = \left(\sum x^2\right)^3.$$

If $x, y \geq 0$, then

$$\frac{3}{2}\left(x^2+y^2\right) \geq x^2+xy+y^2 \geq \frac{3}{4}(x+y)^2 \text{ etc.}$$

Let be $A = x+y+z$, $B = xy+yz+zx$, $C = xyz$, from $\sum x(y-z)^2 \geq 0$ holds $AB \geq 9C$, and from $\sum (x-y)^2 \geq 0$, we have $A^2 \geq 3B$, therefore

$$27(AB-C)^2 \geq 27\left(AB-\frac{1}{9}AB\right)^2 = \frac{64}{3}A^2B^2 \geq 64B^3$$

or

$$27(x+y)^2(y+z)^2(z+x)^2 \geq 64(xy+yz+zx)^3.$$

Application 1.1. In all triangle ABC we have:

1.
$$\begin{aligned} & 8\left(s^2-r^2-4Rr\right)^3 \\ \geq\ & \frac{27}{8}\left(\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)-16s^2R^2r^2\right) \\ \geq\ & \prod\left(a^2+ab+b^2\right) \\ \geq\ & \frac{27}{32}s\left(s^2+r^2+2Rr\right)^2 \\ \geq\ & \left(s^2+r^2+4Rr\right)^3, \end{aligned}$$

2.
$$\begin{aligned} & \left(s^2-2r^2-8Rr\right)^3 \\ \geq\ & \frac{27}{8}\left(r^2\left(s^2-2r^2-8Rr\right)\left((4R+r)^2-2s^2\right)-s^2r^4\right) \\ \geq\ & \prod\left((s-a)^2+(s-a)(s-b)+(s-b)^2\right) \\ \geq\ & \frac{27}{4}s^2R^2r^2 \\ \geq\ & (r(4R+r))^3, \end{aligned}$$

3.
$$\begin{aligned} & \left((4R+r)^2-2s^2\right)^3 \\ \geq\ & \frac{27}{8}\left(\left((4R+r)^2-2s^2\right)\left(s^2-8Rr-2r^2\right)-s^2r^2\right) \\ \geq\ & \frac{1}{s^2}\prod\left(r_a^2+r_ar_b+r_b^2\right) \\ \geq\ & \frac{27}{4}s^2R^2 \\ \geq\ & s^4. \end{aligned}$$

This are new refinements of Euler‘s and Gerretsen’s inequalities.

Application 1.2. If $x, y, z > 0$ then

$$\begin{aligned} \left(\frac{x+y+z}{3}\right)^3 \geq\ & \frac{1}{8}(x+y)(y+z)(z+x) \\ \geq\ & \frac{1}{27}\left(x+\sqrt{xy}+y\right)\left(y+\sqrt{yz}+z\right)\left(z+\sqrt{zx}+x\right) \\ \geq\ & \frac{1}{64}\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2 \\ \geq\ & \left(\frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{3}\right)^3 \geq xyz \geq \left(\frac{3\sqrt{xyz}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)^3 \\ \geq\ & \frac{64x^2y^2z^2}{\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2} \\ \geq\ & \frac{27x^2y^2z^2}{\left(x+\sqrt{xy}+y\right)\left(y+\sqrt{yz}+z\right)\left(z+\sqrt{zx}+x\right)} \\ \geq\ & \frac{8x^2y^2z^2}{(x+y)(y+z)(z+x)} \end{aligned}$$

$$\geq \left(\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right)^3.$$

This are new refinements for AM-GM-HM inequalities.

Proof. In Theorem 1, we take $x \to \sqrt{x}, y \to \sqrt{y}, z \to \sqrt{z}$ and after then $x \to \dfrac{1}{x}$, $y \to \dfrac{1}{y}$, $z \to \dfrac{1}{z}$.

Application 1.3. If $a,\ b,\ x,\ y > 0$, then

$$\begin{aligned}
& ((2a+b)\,x+(a+2b)\,y)^3 \\
\geq\ & \frac{27}{8}(a+b)(x+y)((2a+b)\,x+by)(ax+(a+2b)\,y) \\
\geq\ & (x+\sqrt{xy}+y)\left((2a+b)\,x+by+\sqrt{x\,(a+b)(ax+by)}\right) \\
& \left(ax+(a+2b)\,y+\sqrt{y\,(a+b)(ax+by)}\right) \\
\geq\ & \frac{27}{64}(a+b)\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{ax+by}+\sqrt{x\,(a+b)}\right)^2\left(\sqrt{ax+by}+\sqrt{y\,(a+b)}\right)^2 \\
\geq\ & (a+b)^{\frac{3}{2}}\left(\left(\sqrt{x}+\sqrt{y}\right)\sqrt{ax+by}+\sqrt{(a+b)\,xy}\right)^3 \\
\geq\ & 27xy\,(ax+by)\,(a+b)^2.
\end{aligned}$$

Proof. In Application 1.2, we take $z = \dfrac{ax+by}{a+b}$.

Theorem 2. If $x_k > 0$ $(k = 1,\ 2,\ \cdots,\ n)$, then

$$\begin{aligned}
\left(\sum_{k=1}^{n} x_k^{n-1}\right)^n \geq\ & \left(\frac{n}{2}\right)^n \left(x_1^{n-1}+x_2^{n-1}\right)\left(x_2^{n-1}+x_3^{n-1}\right)\cdots\left(x_n^{n-1}+x_1^{n-1}\right) \\
\geq\ & \left(x_1^{n-1}+x_1^{n-2}x_2+\cdots+x_1x_2^{n-2}+x_2^{n-1}\right) \\
& \left(x_2^{n-1}+x_2^{n-2}x_3+\cdots+x_2x_3^{n-2}+x_3^{n-1}\right) \\
& \left(x_n^{n-1}+x_n^{n-2}x_1+\cdots+x_nx_1^{n-2}+x_1^{n-1}\right) \\
\geq\ & \left(\frac{n}{2^{n-1}}\right)^n (x_1+x_2)^{n-1}(x_2+x_3)^{n-1}\ldots(x_n+x_1)^{n-1} \\
\geq\ & n^n \prod_{k=1}^{n} x_k^{n-1}.
\end{aligned}$$

Proof.

$$\left(\sum_{k=1}^{n} x_k^{n-1}\right)^n = \left(\sum_{cyclic} \frac{x_1^{n-1}+x_2^{n-1}}{2}\right)^n \geq \left(\frac{n}{2}\right)^n \prod_{cyclic}\left(x_1^{n-1}+x_2^{n-1}\right).$$

Now, we prove the following inequalities:

$$\frac{x^{n-1}+y^{n-1}}{2} \geq \frac{x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}}{n} \geq \left(\frac{x+y}{2}\right)^{n-1},$$

for all $x,\ y \geq 0$. For $n = 2$ and $n = 3$ its true.

We suppose true for $n-2$ and we prove for $n-1$. For this we starting from:

$$\left(\frac{x^{n-2}+x^{n-3}y+\cdots+xy^{n-3}+y^{n-2}}{n-1}\right)\left(\frac{x+y}{2}\right)\geq\left(\frac{x+y}{2}\right)^{n-1},$$

and we show that

$$\frac{x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}}{n}\geq\left(\frac{x^{n-2}+x^{n-3}y+\cdots+xy^{n-3}+y^{n-2}}{n-1}\right)\left(\frac{x+y}{2}\right),$$

or

$$(n-2)\left(x^{n-1}+y^{n-1}\right)\geq 2\sum_{k=1}^{n-2}x^{n-1-k}y^{k},$$

but this result from

$$x^{n-1}+y^{n-1}\geq x^{k}y^{n-1-k}+x^{n-1-k}y^{k},$$

which is equivalent with

$$\left(x^{k}-y^{k}\right)\left(x^{n-1-k}-y^{n-1-k}\right)\geq 0,$$

where $k\in\{1,2,\cdots,n-1\}$.

If $k\in\{1,\ 2\ ,\cdots,\ n-1\}$, then from pondered AM-GM inequality results that

$$kx^{n-1}+(n-k)\,y^{n-1}\geq(n-1)\,x^{k}y^{n-k}.$$

After addition we obtain

$$\frac{x^{n-1}+y^{n-1}}{2}\geq\frac{x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}}{n}.$$

Finally we have

$$\begin{aligned}
&\left(\frac{n}{2}\right)^{n}\prod_{cyclic}\left(x_1^{n-1}+x_2^{n-1}\right)\\
=\ & n^{n}\prod_{cyclic}\frac{x_1^{n-1}+x_2^{n-1}}{2}\\
\geq\ & \prod_{cyclic}\left(x_1^{n-1}+x_1^{n-2}x_2+...+x_1x_2^{n-2}+x_2^{n-1}\right)\geq n^{n}\prod_{cyclic}\left(\frac{x_1+x_2}{2}\right)^{n-1}\\
=\ & \left(\frac{n}{2^{n-1}}\right)^{n}\prod_{cyclic}\left(x_1+x_2\right)^{n-1}\geq n^{n}\prod_{k=1}^{n}x_k^{n-1}.
\end{aligned}$$

Application 2.1. If $x_k>0\,(k=1,\ 2,\ \cdots,\ n)$, then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}x_k &\geq \frac{1}{2}\sqrt[n]{\prod_{cyclic}\left(x_1+x_2\right)}\\
&\geq \frac{1}{n}\sqrt[n]{\prod_{cyclic}\left(\sum_{i=0}^{n-1}x_1^{\frac{n-1-i}{n-1}}x_2^{\frac{i}{n-1}}\right)}\\
&\geq \frac{1}{2^{n-1}}\prod_{cyclic}\left(x_1^{\frac{1}{n-1}}+x_2^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}
\end{aligned}$$

$$\geq \sqrt[n]{\prod_{k=1}^{n} x_k}.$$

Proof. In Theorem 2, we take $x_k \to x_k^{\frac{1}{n-1}}$ $(k = 1, 2, \cdots, n)$.

Remark.

$$R_1(x_1, x_2, ..., x_n) = \frac{1}{2} \sqrt[n]{\prod_{cyclic} (x_1 + x_2)},$$

$$R_2(x_1, x_2, \cdots, x_n) = \frac{1}{n} \sqrt[n]{\prod_{cyclic} \left(\sum_{i=0}^{n-1} x_1^{\frac{n-1-i}{n-1}} x_2^{\frac{i}{n-1}} \right)},$$

$$R_3(x_1, x_2, \cdots, x_n) = \frac{1}{2^{n-1}} \prod_{cyclic} \left(x_1^{\frac{1}{n-1}} + x_2^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}},$$

$$\overline{R_j}(x_1, x_2, \cdots, x_n) = \frac{1}{R_j\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)} \quad (j = 1,\ 2,\ 3),$$

are new means which give refinements for AM-GM-HM inequalities.

If

$$A(x_1, x_2, \cdots, x_n) = \frac{1}{n} \sum_{k=1}^{n} x_k,$$

$$G(x_1, x_2, \cdots, x_n) = \sqrt[n]{\prod_{k=1}^{n} x_k},$$

$$H(x_1, x_2, \cdots, x_n) = \frac{1}{\sum_{k=1}^{n} \frac{1}{x_k}},$$

then we have the following:

Theorem 3. If $x_k > 0$, then

$$\begin{aligned} A(x_1, x_2, \cdots, x_n) &\geq R_1(x_1, x_2, \cdots, x_n) \geq R_2(x_1, x_2, \cdots, x_n) \\ &\geq R_3(x_1, x_2, \cdots, x_n) \geq G(x_1, x_2, \cdots, x_n) \\ &\geq \overline{R_3}(x_1, x_2, \cdots, x_n) \geq \overline{R_2}(x_1, x_2, \cdots, x_n) \\ &\geq \overline{R_1}(x_1, x_2, \cdots, x_n) \geq H(x_1, x_2, \cdots, x_n) \\ &\cdots . \end{aligned}$$

Application 3.1. In all triangle ABC holds:

1. $$\begin{aligned} \frac{2s}{3} &\geq R_1(a, b, c) \geq R_2(a, b, c) \geq R_3(a, b, c) \\ &\geq \sqrt[3]{4sRr} \geq \overline{R_3}(a, b, c) \geq \overline{R_2}(a, b, c) \geq \overline{R_1}(a, b, c) \\ &\geq \frac{12sRr}{s^2 + r^2 + 4Rr}, \end{aligned}$$

2. $$\frac{s}{3} \geq R_1(s-a, s-b, s-c) \geq R_2(s-a, s-b, s-c)$$

$$\begin{aligned}
&\geq R_3\left(s-a,s-b,s-c\right)\geq\sqrt[3]{sr^2}\geq\overline{R_3}\left(s-a,s-b,s-c\right)\\
&\geq \overline{R_2}\left(s-a,s-b,s-c\right)\geq\overline{R_1}\left(s-a,s-b,s-c\right)\\
&\geq \frac{3sr}{4R+r},
\end{aligned}$$

3. $$\begin{aligned}
\frac{4R+r}{3} &\geq R_1\left(r_a,r_b,r_c\right)\geq R_2\left(r_a,r_b,r_c\right)\geq R_3\left(r_a,r_b,r_c\right)\\
&\geq \sqrt[3]{s^2r}\geq\overline{R_3}\left(r_a,r_b,r_c\right)\geq\overline{R_2}\left(r_a,r_b,r_c\right)\\
&\geq \overline{R_1}\left(r_a,r_b,r_c\right)\geq 3r,
\end{aligned}$$

4. $$\begin{aligned}
\frac{2R-r}{6R} &\geq R_1\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\geq R_2\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\\
&\geq R_3\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\geq\sqrt[3]{\frac{r^2}{16R^2}}\\
&\geq \overline{R_3}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\geq\overline{R_2}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\\
&\geq \overline{R_1}\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\geq\frac{3r^2}{s^2+r^2-8Rr},
\end{aligned}$$

5. $$\begin{aligned}
\frac{4R+r}{6R} &\geq R_1\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\geq R_2\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\\
&\geq R_3\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\geq\sqrt[3]{\frac{s^2}{16R^2}}\\
&\geq \overline{R_3}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\geq\overline{R_2}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\\
&\geq \overline{R_1}\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\geq\frac{3s^2}{s^2+\left(4R+r\right)^2},
\end{aligned}$$

which are new refinements for Euler's and Gerretsen's inequalities.

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On the Smarandache LCM dual function

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Abstract For any positive integer n, the Smarandache LCM dual function $SL^*(n)$ is defined as the greatest positive integer k such that $[1,\ 2,\ \cdots,\ k]$ divides n. The main purpose of this paper is using the elementary method to study the calculating problem of a Dirichlet series involving the Smarandache LCM dual function $SL^*(n)$ and the mean value distribution property of $SL^*(n)$, obtain an exact calculating formula and a sharper asymptotic formula for it.

Keywords Smarandache LCM dual function, Dirichlet series, exact calculating formula, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. That is,

$$SL(n) = \min\{k:\ k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1] and [2]. For example, Murthy [1] proved that if n be a prime, then $SL(n) = S(n)$, where $S(n) = \min\{m:\ n|m!,\ m \in N\}$ be the F.Smarandache function. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Zhongtian Lv [3] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n \le x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

Now, we define the Smarandache LCM dual function $SL^*(n)$ as follows:

$$SL^*(n) = \max\{k : k \in N, [1, 2, \cdots, k] \mid n\}.$$

For example: $SL^*(1) = 1$, $SL^*(2) = 2$, $SL^*(3) = 1$, $SL^*(4) = 2$, $SL^*(5) = 1$, $SL^*(6) = 3$, $SL^*(7) = 1$, $SL^*(8) = 2$, $SL^*(9) = 1$, $SL^*(10) = 2$, $\cdots$. Obviously, if n is an odd number, then $SL^*(n) = 1$. If n is an even number, then $SL^*(n) \geq 2$. About the other elementary properties of $SL^*(n)$, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we use the elementary method to study the calculating problem of the Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s}, \tag{2}$$

and give an exact calculating formula for (2). At the same time, we also study the mean value properties of $SL^*(n)$, and give a sharper mean value formula for it. That is, we shall prove the following two conclusions:

Theorem 1. For any real number $s > 1$, the series (2) is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s} = \zeta(s) \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^s - 1)}{[1, 2, \cdots, p^\alpha]^s},$$

where $\zeta(s)$ is the Riemann zeta-function, $\sum\limits_{p}$ denotes the summation over all primes.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL^*(n) = c \cdot x + O(\ln^2 x),$$

where $c = \sum\limits_{\alpha=1}^{\infty} \sum\limits_{p} \dfrac{(p^\alpha - 1)(p - 1)}{[1, 2, \cdots, p^\alpha]}$ is a constant.

Note that $\zeta(2) = \pi^2/6$, from Theorem 1 we may immediately deduce the identity:

$$\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^2} = \frac{\pi^2}{6} \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^2 - 1)}{[1, 2, \cdots, p^\alpha]^2}.$$

§2. Some useful lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. For any positive integer n, there exist a prime p and a positive integer α such that

$$SL^*(n) = p^\alpha - 1.$$

Proof. Assume that $SL^*(n) = k$. From the definition of the Smarandache LCM dual function $SL^*(n)$ we have

$$[1, 2, \cdots, k] \mid n, \quad (k+1) \dagger n,$$

else $[1,\ 2,\ \cdots,\ k,\ k+1] \mid n$, then $SL^*(n) \geq k+1$. This contradicts with $SL^*(n) = k$. Assume that $k+1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where p_i is a prime, $p_1 < p_2 < \cdots < p_s$, $\alpha_i \geq 1,\ i = 1,\ 2,\ \cdots,\ s$. If $s > 1$, then $p_1^{\alpha_1} \leq k$, $p_2^{\alpha_2} \cdots p_s^{\alpha_s} \leq k$, so

$$p_1^{\alpha_1} \mid [1, 2, \cdots, k], \quad p_2^{\alpha_2} \cdots p_s^{\alpha_s} \mid [1, 2, \cdots, k].$$

Since $(p_1^{\alpha_1},\ p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = 1$, we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \mid [1,\ 2,\ \cdots,\ k].$$

Namely, $k+1 \mid [1,\ 2,\ \cdots,\ k]$. From this we deduce that $k+1 \mid n$. This contradicts with $SL^*(n) = k$. Hence $s = 1$. Consequently $k+1 = p^\alpha$. That is, $SL^*(n) = p^\alpha - 1$. This completes the proof of Lemma 1.

Lemma 2. Let $L(n)$ denotes the least common multiple of all positive integers from 1 to n, then we have

$$\ln(L(n)) = n + O\left(n \cdot \exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right),$$

where c is a positive constant.

Proof. See reference[4].

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1.

From the definition of the Smarandache LCM dual function $SL^*(n)$ we know that if $[1,\ 2,\ \cdots,\ k] \mid n$, then $[1,\ 2,\ \cdots,\ k] \leq n$, $\ln([1,\ 2,\ \cdots,\ k]) \leq \ln n$. Hence, from Lemma 2 we have $SL^*(n) = k \leq \ln n$, $\dfrac{SL^*(n)}{n^s} \leq \dfrac{\ln n}{n^s}$. Consequently, if $s > 1$, then the Dirichlet series $\sum\limits_{n=1}^{\infty} \dfrac{SL^*(n)}{n^s}$ is absolutely convergent. From Lemma 1 we know that $SL^*(n) = p^\alpha - 1$, then $[1,\ 2,\ \cdots,\ p^\alpha - 1] \mid n$. Let $n = [1,\ 2,\ \cdots,\ p^\alpha - 1] \cdot m$, then $p \dagger m$, so for any real number $s > 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{SL^*(n)}{n^s} &= \sum_{\alpha=1}^{\infty} \sum_{p} \sum_{\substack{n=1 \\ SL^*(n) = p^\alpha - 1}}^{\infty} \frac{p^\alpha - 1}{n^s} = \sum_{\alpha=1}^{\infty} \sum_{p} \sum_{\substack{m=1 \\ p \dagger m}}^{\infty} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s \cdot m^s} \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \sum_{\substack{m=1 \\ p \dagger m}}^{\infty} \frac{1}{m^s} \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \left(\sum_{m=1}^{\infty} \frac{1}{m^s} - \sum_{m=1}^{\infty} \frac{1}{p^s \cdot m^s} \right) \\
&= \sum_{\alpha=1}^{\infty} \sum_{p} \frac{p^\alpha - 1}{[1,\ 2,\ \cdots,\ p^\alpha - 1]^s} \left(\sum_{m=1}^{\infty} \frac{1}{m^s} (1 - \frac{1}{p^s}) \right) \\
&= \left(\sum_{m=1}^{\infty} \frac{1}{m^s} \right) \sum_{\alpha=1}^{\infty} \sum_{p} \frac{(p^\alpha - 1)(p^s - 1)}{[1,\ 2,\ \cdots,\ p^\alpha]^s}
\end{aligned}$$

$$= \zeta(s)\cdot\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p^s-1)}{[1,\ 2,\ \cdots,\ p^\alpha]^s}.$$

This proves the theorem 1.

From the definition of the Smarandache LCM dual function $SL^*(n)$, Lemma 1 and Lemma 2 we also have

$$\begin{aligned}\sum_{n\le x} SL^*(n) &= \sum_{\substack{[1,\ 2,\ \cdots,\ p^\alpha-1]\cdot m\le x\\ (m,\ p)=1}}(p^\alpha-1) = \sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}(p^\alpha-1)\sum_{\substack{m\le \frac{x}{[1,\ 2,\ \cdots,\ p^\alpha-1]}\\ p\dagger m}} 1\\
&= \sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}(p^\alpha-1)\left(\frac{x}{[1,\ 2,\ \cdots,\ p^\alpha-1]}-\frac{x}{[1,\ 2,\ \cdots,\ p^\alpha]}+O(1)\right)\\
&= x\cdot\sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}+O\left(\sum_{[1,\ 2,\ \cdots,\ p^\alpha-1]\le x}p^\alpha\right)\\
&= x\cdot\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}+O\left(\ln^2 x\right)\\
&= c\cdot x+O\left(\ln^2 x\right),\end{aligned}$$

where $c=\sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p-1)}{[1,\ 2,\ \cdots,\ p^\alpha]}$ is a constant.

This completes the proof of Theorem 2.

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Riemannian 4-spaces of class two

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Abstract We study the local and isometric embedding of Riemannian spacetimes into the pseudo-Euclidean flat E6. Our main purpose is to pinpoint the known results with the corresponding references, and to indicate the main routes and connections from here to the still open problems in the field. Some new results are also included in order to round off our discussion.

Keywords Embedding of Riemannian spacetimes, local and isometric embedding.

§1. Introduction

Here we study spacetimes admitting local and isometric embedding into E_6, i.e.,4-spaces of class two [1,2]. When we conceive a certain V_4 as a subspace of a flat N-dimensional space $N \leq 10$, new geometric objects arise (like various second fundamental forms and Ricci vectors) which enrich the Riemannian structure and offer the possibility [3-6] of reinterpreting physical fields using them. Unfortunately to date such hope has not been realized since it has been extremely difficult to establish a natural correspondence between the quantities governing the extrinsic geometry of and physical fields. In spite of this, one cannot but accept the great value of the embedding process, for it combines harmoniously such themes as the Petrov [1,7-12] and the Churchill-Plebañski [13-18] classifications, exact solutions and their symmetries [1,2,19-34], the Newman-Penrose formalism [1,35-38], and the kinematics of time-like and null congruences [1,2,23,24,36,39,40]. On the other hand, it offers the possibility of obtaining exact solutions that cannot be deduced by any other means [1].

In this work we study spacetimes (here denoted as V_4) admitting a local and isometric embedding into the pseudo-Euclidian flat space, that is, spacetimes of class two [1,2]. The article is organized as follows. In section 2 we expound the Gauss, Codazzi and Ricci equations for an V_4 embedded into E_6. In section 3 we analyse those equations to state the necessary algebraic conditions for a spacetime to be of class two. Some of these stated conditions are already known but some others are new. In section 4, we again employ the results of the prevoius sections to investigate a vacuum V_4 which leads us to the Collinson [36] and the Yakupov [41] theorems; our results are then used in some metrics obtaining known results and a few new ones.

§2. Gauss, Codazzi and Ricci equations

In this section we expound the governing equations of the embedding of a V_4 into a flat 6-dimensional space. These are the Gauss, Codazzi and Ricci equations (GCRE), which constitute an algebraic and differential system which usually is not very easy to solve to obtain the three important quantities for the embedding process, namely, the two second fundamental form tensors and the Ricci vector. Let us mention that in all of our discussion we use the tensorial expressions of the GCRE.

In the embedding problem, let us recall, the intrinsic geometry of the spacetime (mainly determined through the metric tensor G_{ab}) is assumed given, what is required is the extrinsic geometry of V_4 respect to E_6. In the stated case, we have two additional dimensions which means that V_4 now posses two normals (in E_6) with indicators $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$; this requires two second fundamental form tensors ${}^a b_{ij} ={}^a b_{ji}$, $a = 1, 2$, and a Ricci vector A_r, these quantities cannot be prescribed arbitrarily since they determine the extrinsic geometry of the spacetime embedded into E_6. For a local and isometric embedding to be realizable, it is necessary and sufficient that the GCRE equations hold [2,25,36,42]. These algebraic and differential relationships between the new geometric degrees of freedom b_{ji} and A_r, are the conditions for their existence; the set of equations are

$$R_{acpq} = \sum_{r=1}^{2} \epsilon_r ({}^r b_{ap}\, {}^r b_{cq} -{}^r b_{aq}\, {}^r b_{cp}), \quad \text{Gauss} \tag{1}$$

$${}^1 b_{ac;r} -{}^1 b_{ar;c} = \epsilon_2 (A_c\, {}^2 b_{ar} - A_r\, {}^2 b_{ac}), \quad \text{Codazzi} \tag{2}$$

$${}^2 b_{ac;r} -{}^2 b_{ar;c} = -\epsilon_1 (A_c\, {}^1 b_{ar} - A_r\, {}^1 b_{ac}), \quad \text{Codazzi} \tag{3}$$

$$F_{jr} = A_{r,j} - A_{j,r} ={}^1 b_j\,{}^{c2} b_{cr} -{}^1 b_r\,{}^{c2} b_{cj}, \quad \text{Ricci} \tag{4}$$

where a semicolon indicates a covariant derivative (with respect to the coordinates of V_4), a comma an ordinary derivative and R_{abcd} stands for the V_4 Riemann curvature tensor [1]. Notice the similitude of (2) and (3): if in the former we replace, respectively, ${}^1 b_{cr}$ and ϵ_2 by ${}^2 b_{cr}$ and $-\epsilon_1$, we get the latter. The set of equations (1) to (4) are written entirely in terms of the V_4 we try to embed and, in general, are rather difficult to solve; it is natural thus to look for a simplification. This can be achieved reasoning by analogy with the Thomas theorem, valid for any spacetime V_4 embedded into E_5 [38,43,44], namely, if a class one spacetime has $det(b_{ij}) \neq 0$ then Gauss equation implies Codazzi's , therefore, in the case of class one spacetimes the search of the second fundamental form b_{ij} is mainly algebraic simplifying the embedding process. For class two spacetimes the process is more difficult, for, unfortunately the consideration of at least one differential equation becomes necessary.

Gupta-Goel [30] showed that, when $({}^2b_{cr}) \neq 0$, the second of Codazzi equations (3) and the Ricci equation (4) follow from (1) and the first of Codazzi (2). They used this result to embed every static spherically symmetric spacetime into E_6 (the embedding into E_6 of such spacetimes had been carried out explicitly before by Plebañski [21,33]). This shows that the metrics of Schwarzchild and of Reissner-Nördstrom can be embedded into E_6 in spite of the fact that it is not possible to embed them into E_5 [38,39,44,45].

Goenner [46] has performed a complete study of the interdependence of the CGRE, for example, he proved two theorems that are more general than the Gupta-Goel result [30], namely:

$$\text{If the rank of } {}^2b_{at} \geq 3, \text{ then (1), (2) and (3) imply (4);} \tag{5}$$

the second result says that,

$$\text{if the rank of } {}^2b_{at} \geq 4, \text{ then (1), (2) and (3) imply (4).} \tag{6}$$

These two theorems are valid for any R_n mbedded into E_{n+2} and are a consequence of the Bianchi identities [1] for the curvature tensor.

For class-2 spacetimes the intrinsic geometry of V_4 is given, we are left thus with the problem of constructing ${}^rb_{ac}$, ϵ_r, $r = 1, 2$ and A_p as solutions to equations (1)-(4). Before we address the problem of finding solutions to those equations it can be convenient to verify certain other conditions that every class-2 spacetime should satisfy. If such conditions are found not to hold, it is pointless to try to find the second fundamental forms or the Ricci vector.

§3. Necessary conditions for the embedding of V_4 into E_6

To show that a certain spacetime cannot be embedded into E_6 needs showing that the GCRE admit no solution; but proving this directly is usually a very complex task, therefore, it is better to search for indirect evidence in the form of necessary conditions for the embedding. If these conditions are not satisfied we can be sure that no solution of the GCRE exist. In this section we address some necessary conditions for a V_4 to be a class two spacetime.

First a result that Matsumoto proved [47] for a Riemannian R_4 (that is, a space with a positive definite metric) embedded into E_6, whose general validity for spacetimes (that is pseudo-Riemannian V_4 spaces with a non-positive definite metric) was first noticed by Goenner [42]. For every class-2 V_4, we must have

$$F^{ij}F^{kr} + F^{ik}F^{rj} + F^{ir}F^{jk} = -\frac{\epsilon_1\epsilon_2}{2}(R^{acij}R^{kr}_{ac} + R^{acik}R^{rj}_{ac} + R^{acir}R^{jk}_{ac}), \tag{7}$$

with F^{ab} defined as in (4). If we multiply Matsumoto's expression times the Levi-Civita tensor [1,37] η_{pjkr} we get

$${}^*C_2 \equiv {}^* C_{abcd}C^{abcd} = -2\epsilon_1\epsilon_2 F_2 \equiv -2\epsilon_1\epsilon_2 F_{ab}\ {}^*F^{ab}, \tag{8}$$

where C_{abcd} is the Weyl conformal tensor and $^*C_{abcd}$ is its dual [1,11,48]. F_{ab} is an extrinsic quantity but equation (8) indicates that the invariant $F_{ab}\ ^*F^{ab}$ has to be intrinsic because it is simply proportional to an invariant of the Weyl tensor.

Not mattering what the class of a spacetime, it is very easy to show the interesting identity (that we believe has not been previously noticed)

$$^*R^{*tjkc*}R_{arkc}R^{ar}_{pj} = \frac{Y}{4}\delta^t_p, \tag{9}$$

where we have defined

$$Y \equiv^* R^{*tjkc*}R_{arkc}R^{ar}_{tj} = -^*C_3 + \frac{R}{2} * C_2 + 6^*R_3,$$
$$^*C_3 \equiv^* C_{abcd}C^{cdpq}C^{ab}_{pq}, \quad ^*R_3 \equiv^* R_{ijab}R^{ia}R^{jb}, \tag{10}$$

such that $R_{ab} \equiv R^i_{abi}$ is the Ricci tensor, $R \equiv R^b_b$ is the scalar curvature, $^*R_{abcd}$ is the simple dual and $^*R^*_{abcd}$ the double dual of the Riemann curvature tensor R_{abcd} [1,49,50].

Employing only the Gauss equation (1), Yakupov [51] has been able to obtain a very general necessary condition for a spacetime to be of class two, see also Goenner [2]. Yakupov result asserts that

$$\text{Every } V_4 \text{ embedded into } E_6 \text{ should have } Y = 0; \tag{11}$$

this is a restriction upon the intrinsic geometry of a class two spacetime. Using it, we can ascertain that if a V_4 is such that $Y \neq 0$, then its embedment into E_6 is not possible. We can take as an example Kerr's metric [1,52-56] which has $R = 0$, $^*R_3 = 0$ and $^*C_3 \neq 0$; with such values, it is clear that $Y \neq 0$ and then it is not possible to embed a spinning black hole into E_6. Kerr's metric do accept [57] embedding into E_9 also but it is not yet known whether it can be embedded into E_7 or E_8.

One has to keep in mind that Yakupov's result is only a necessary condition since the mere fact that $Y = 0$ cannot guarantee the embedding since only Gauss equation is needed to prove it [51]. Gödel's metric [1,58] is an example in which $^*R_3 =^* C_2 =^* C_3 = 0$ and $Y = 0$, but even so it is not known yet whether it is possible to embed it into E_6 or not [2,22,25,32,38,39,59-64].

Let us consider now equation (2), first take its covariant derivative respect x^p , then rotate cyclically the indices c, r and p obtaining in this way three equations, finally sum these equations to get

$$R^q_{arp}\ ^1b_{qc} + R^q_{apc}\ ^1b_{qr} + R^q_{arr}\ ^1b_{qp} = \epsilon_2(F_{rp}\ ^2b_{ac} + F_{pc}\ ^2b_{ar} + F_{cr}\ ^2b_{ap}) \tag{12}$$

and, doing similarly with (3), we also get

$$R^q_{arp}\ ^2b_{qc} + R^q_{apc}\ ^2b_{qr} + R^q_{arr}\ ^2b_{qp} = \epsilon_1(F_{rp}\ ^1b_{ac} + F_{pc}\ ^1b_{ar} + F_{cr}\ ^1b_{ap}) \tag{13}$$

These equations are equivalent to equation (8) in [51] and to equations (A2.5), (A2.6) in Hodgkinson [65] although we must note that he only considers the case $R(ab) = 0$.

This equations offer a way for evaluating the second-fundamental form $^2b_{ab}$ as a function of F^{ab}, $^*R_{abcd}$ and $^1b_{ab}$. This can be seen as follows, multiply (12) and (13) times η^{trpc} to obtain

$$^*R^{ijkr1}b_{jk} = \epsilon_2\ ^*F^{ij2}\ b^r_j, \quad ^*R^{ijkr2}b_{jk} = -\epsilon_1 * F^{ij1}\ b^r_j \tag{14}$$

take the product of equations (14) times F_{ic}, and keeping in mind the identity [66-69] ${}^*F_{ac}F^{ic} = c\delta^i_a/4$, we get

$$F_2\ {}^2b_{ij} = 4\epsilon_2\ {}^*R_i^{acr}\ {}^1b_{cr}F_{aj},\quad F_2\ {}^1b_{ij} = -4\epsilon_2 * R_i^{acr}\ {}^2b_{cr}F_{aj}, \tag{15}$$

from which, if $F_2 = 0$, we can evaluate ${}^1b_{ac} = 0$ in terms of the quantities mentioned above. Notice that (15) reduces to a trivial identity for vacuum class-2 spacetimes as follows from Yakupov result that, if we are in a vacuum, $F_{ij} = 0$ always [41]. Other point worth mentioning is that, on multiplying (12) times $\epsilon_1\ {}^1b^a_t$ and (13) times $\epsilon_2\ {}^2b^a_t$ and summing to each other the resulting equations, we obtain equation (3a) again.

Furthermore, if in equations (12) and (13) we contract a with r, we get (${}^sb \equiv\ {}^sb^c_c,\ s = 1,2$)

$$R^q_p\ {}^1b_{qc} - R^q_c\ {}^1b_{qp} = \epsilon_2(F_{qp}\ {}^2b^q_c - F_{qc}\ {}^2b^q_p +^2 bF_{pc}, \tag{16}$$

$$R^q_p\ {}^2b_{qc} - R^q_c\ {}^1b_{qp} = -\epsilon_1(F_{qp}\ {}^1b^q_c - F_{qc}\ {}^1b^q_p +^1 bF_{pc}, \tag{17}$$

these equations reproduce equation (9) in Yakupov [51] - though this author uses $R_{ab} \equiv Rg_{ab}/4$ - and equation (A2.7) in [65] for the case . Using (16) and (17) it is elementary to obtain

$${}^*F^{pc}R^q_p\ {}^1b_{qc} = \frac{\epsilon_2}{4}F_2\ {}^2b,\ \text{and}\ {}^*F^{pc}R^q_p\ {}^2b_{qc} = \frac{-\epsilon_1}{4}F_2\ {}^1b, \tag{18}$$

Equations (16), (17) and (18) are interesting for General Relativity (GR) because they involve the Ricci tensor which is directly related with the sources of the gravitational field.

On multiplying (16), times ${}^2b^p_t$ and antisymmetrizing indices c and t, and times ${}^1b^p_t$ and antisymmetrizing the same indices than before, and finally substracting the former equation form the latter, we get

$$R^{ap}_{ct}F_{ap} = 2[R^q_tF_{qc} - R^q_cF_{qt} + R_{pq}({}^1b^q_c\ {}^2b^p_t -^1 b^q_t\ {}^2b^p_c)] \tag{19}$$

or, using the expression for the Weyl tensor [1],

$$C_{apct}F^{ap} = 2R_{pq}({}^1b^q_c\ {}^2b^p_t -^1 b^q_t\ {}^2b^q_c) + R^{qt}F^q_c - R^{qc}F^q_t - \frac{R}{3}F_{ct}. \tag{20}$$

For Einstein spaces, in which $R_{ab} \equiv Rg_{ab}/4$, (20) implies that F^{ap} is eigentensor of the conformal tensor [8,70]

$$C_{apct}F^{ap} = -\frac{R}{3}F_{ij}. \tag{21}$$

The three relations (19), (20) and (21) are contributions of this work though have been anticipated by Hodgkinson [65] for the case in which $R_{ab} = 0$.

In terms of C_{ijkr}, equations (14) become

$$\left({}^*C^{ijkr} + \frac{1}{2}\eta^{ijra}R^k_a\right)\ {}^1b_{jk} = +\epsilon_2\ {}^*F^{ij}\ {}^2b^r_j,$$
$$\left({}^*C^{ijkr} + \frac{1}{2}\eta^{ijra}R^k_a\right)\ {}^2b_{jk} = -\epsilon_1\ {}^*F^{ij}\ {}^1b^r_j, \tag{22}$$

from here, is straightforward to deduce

$$^{*}C^{ijkcr}b_{jk}\ {}^{r}b_{ic} = \frac{\epsilon_r}{4}\ {}^{*}C_r, \quad r = 1, 2, \qquad {}^{*}C^{ijkc2}b_{ic}\ {}^{1}b_{jk} = 0, \tag{23}$$

$$^{*}C^{ijkc\ 1}b_{jk} = \frac{\epsilon_2}{2}({}^{*}F^{ij\ 2}b_j^c +^{*} F^{cj\ 2}b_j^i), \tag{24}$$

to finalize, from (10), (11) and (20) we can obtain the starting identity

$$^{*}C_{apct}F^{ap}F^{ct} = \frac{1}{3}\epsilon_1\epsilon_2({}^{*}C_3 - 6^{*}R_3) = \frac{\epsilon_1\epsilon_2}{6}R^{*}C_2. \tag{25}$$

Given equations (7)-(25), the main algebraic relations which must hold in any class-2 spacetimes have been established; to this date nobody has been capable of establishing any (perhaps because they do not exist!) necessary differential conditions for the embedding of V_4 into E_6. For spacetimes embedded into E_5 the only known necessary differential condition was advanced by our group [62]; nor have been found any necessary algebraic and/or differential conditions for spacetimes embedded into E_7, if other were the case, these would permit to study the embedding problem for the Kerr metric mentioned before.

§4. Necessary conditions for the embedding of vacuum class-2 spacetimes

In this section we study the embedding properties of vacuum ($R_{ab} = 0$) spacetimes. The conditions for being class-2 can be best described in terms of properties of the null geodesic congruences spanned by Newman-Penrose vectors (NP) [1,8,11,35,55,71].

Collinson has studied the problem of embedding of a vacuum 4-spacetime in which a doubly degenerate NP vector n^r spanning a null geodesic congruence exist, and has obtained the following two necessary conditions [36]:

In every vacuum class-2 V_4 with Petrov type-II a null geodesic congruence should exist with the three optical scalars equal to zero, (26)

that is, if κ, σ and ρ are such NP spin coefficients [1,12,35,37,55,72], we must have for the congruence: $\kappa = 0$ (geodesic), $\sigma = 0$(shearfree), $\rho - \bar{\rho} = 0$ (no rotation), and $\rho + \bar{\rho} = 0$(no expansion); also

In Petrov-type $D textor\ N textvacuum 4 - spacetimes embedded into\ E_6$, a null geodesic congruence with no shear and no rotation exist; (27)

that is, a spacetimes with $\kappa = \sigma = \rho - \bar{\rho} = 0$.

Studying the embedding problem Yakupov came across [41,65] the following two results published without proof in a Russian journal ([1], p, 369)

No Petrov type III vacuum spacetime V_4may be embedded into E_6 (28)

together with (28), Yakupov got [1] that

$$\text{in all class-2 vacuum spacetimes } F_{ac} = 0, \tag{29}$$

then from equation (4), it follows that ${}^1b^r_c$ and ${}^2b^r_c$ commute with each other or, in other words, that the Ricci vector A_r is a gradient. From section 3 and Yakupov's result (11), we get

$$^*C_2 = 0 \tag{30}$$

and

$$^*C_3 = F_2 = 0, \quad ^*C^{ijkr}\ {}^pb_{jk} = 0, \quad p = 1, 2. \tag{31}$$

Let us pinpoint that (20) is valid for every vacuum spacetime (including the Petrov type-I), Conditions (30) and (31) may be extended to all Einstein spaces, the proof is not given here, as

$$\text{any algebraically special } V_4 \text{ embedded into } E_6 \text{must have } F_{ab} = 0 \tag{32}$$

and, from (8), this can be seen to imply that $^*C_2 = 0$. On the other hand, Goenner [2,42,73] claimed without proof that:

$$^*C_2 = 0 \text{ in every spacetime of class two and Petrov type different from I,} \tag{33}$$

however, as he recognized [74] this is only true for Einstein and vacuum spacetimes.

§5. Applications

The results obtained in the previous sections have applications to known metrics in GR as we exhibit in the following cases.

a) The type D Schwarzschild metric [1] ($R_{ab} = 0$).

We know [21,30,33,38,39,44,75-77] that this metric can be embedded into E_6.

b) The type D Kerr metric ($R_{ab} = 0$).

This metric generalizes [78] the previous one and corresponds to a black hole without electrical charge, it cannot be embedded into E_6 because, as $Y \neq 0$ and $^*C_2 \neq 0$, it violates Yakupov condition.

c) Petrov type III vacuum metric [7].

The metric

$$ds^2 = exp(x^2)[exp(-2x^4)(dx^1)^2 + (dx^2)^2] + 2dx^3dx^4 - x^2(x^3 + exp(x^2))(dx^4)^2, \tag{34}$$

has been embedded into E_7 by Collinson [25, p. 410] therefore, by condition (28) we must conclude that (34) is of class three

d) The empty C metric [79,80]

The vacuum type D metric

$$ds^2 = (x + y)^{-2}(f^{-1}dx^2 + h^{-1}dy^2 + fd\phi^2 - hdt^2), \tag{35}$$

$f \equiv x^3 + ax + b$, $h \equiv y^3 + ay - b$, where a, b are constants.

satisfies the necessary conditions (11) and (30). In [31,34] it was shown that we cannot embed this metric into E_6; however, Rosen [22] has explicitly embedded (35) into E_8. We still do not known whether the C-metric can be embedded into E_7 or not.

e) The type D Taub metric ($R_{ab} = 0$).

The line element is [1,37,80,81]:

$$ds^2 = f^{-1}(dx^2 - dt^2) + f^2(dy^2 - dz^2), \quad f \equiv (1 + kx)^{1/2}, \tag{36}$$

where the constant $k \neq 0$. According to Goenner [2, p. 455] this metric is of class two, we next exhibit explicitly the embedding of (36) into E_6 since apparently it has not been previously published

$$\begin{gathered} z^1 = A + \frac{f}{2}(y^2 + z^2 - 1), \quad z^2 = A + \frac{f}{2}(y^2 + z^2 + 1), \\ z^3 = yf, \quad z^4 = zf, \quad z^5 = f^{-1/2}\cosh t, \quad z^6 = f^{-1/2}\sinh t, \end{gathered} \tag{37}$$

where $A = x/k - f^{-2}/16$. Therefore (36) reduces to

$$ds^2 = (dz^1)^2 - (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 - (dz^6)^2, \tag{38}$$

and thus the embedding into E_6 is explicit.

f) Type D Kasner metric ($R_{ab} = 0$) [82].

The type D Kasner metric [37] is of class two (see [1, p. 370]). The embedding class of the Kasner type I ($2 \leq class \leq 3$, according to Goenner [2, p. 455]) is not known.

g) The vacuum type III Siklos metric [83].

The explicit form of the solution is (see [1, p. 378])

$$ds^2 = r^2x^{-3}(dx^2 + dy^2) - 2dudr + \frac{3}{2}xdu^2; \tag{39}$$

by the condition (28) this metric cannot be embedded into E_6, but it can be embedded into E_8 because it has the type of Robinson-Trautman (see J9 of Collinson [25]); as it is not known whether it is a subspace of E_7 or not, we do not know either its embedding class.

h) Held [84]-Robinson [85].

Held and Robinson have derived type III metrics with $R_{ab} = 0$ whose degenerate null congruence has rotation, thus by (28) these spacetimes do not admit embedding into E_6.

i) Petrov type N [7, p. 384], $R_{ab} = 0$.

The spacetimes with line element

$$ds^2 = -2dx^1dx^4 + \sin^2 x^4(dx^2)^2 + \sin hx^4(dx^3)^2, \tag{40}$$

are of class two as proved in J11 of Collinson [25].

j) Gravitational waves [1,37] ($R_{ab} = 0$).

The metric for gravitational waves along the axis x^3 is [86]:

$$ds^2 = (dx^1)^2 + (dx^2)^2 - 2dx^3dx^4 + 2H(x^1, x^2, x^3)(dx^4)^2, \tag{41}$$

where $H_{,11} + H_{,22} = 0$, has the Petrov type N and it can be embedded into E_6(see J8 of [25]).

k) Hauser [87].

Hauser has derived a type N metric with $R_{ab} = 0$ but its principal degenerate congruence posses rotation $\rho - \bar{\rho} = 0$, thus (27) does not hold and thence it is not of class two. This is a biparametric metric, therefore it could be embedded, depending on the values taken by its parameters, into some E_r with $r = 7, \cdots, 10$.

Let us end this article pinpointing some open problems in the field which have been mentioned in the text:

1. There have not been found any-with reasonable physical sources-type I or II metrics embedded into E_6.

2. The affirmation (33) has not been proved but, otherwise, not a single counterexample is known [88].

3. Condition (28) lacks an explicit proof.

4. It is not known whether Gödel's metric [89] can be embedded into E_6 or not.

5. No a single differential necessary condition for the embedding of V_4 into E_6 is known.

6. It is necessary to analyse if for class two it is possible to obtain analogous identities to those valid for class one 4-spaces (obtained in [44,61,90-92]) expressing ${}^1b_{ac}$, ${}^2b_{ac}$, and A_j in terms of the intrinsic geometry of the spacetime.

7. To determine if Kerr, Siklos and C metrics can be embedded into E_7.

8. To make a complete study of Petrov type D Einstein spacetimes of class two [93].

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On a conjecture involving the function $SL^*(n)$

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Abstract In this paper, we define a new arithmetical function $SL^*(n)$, which is related with the famous F.Smarandache LCM function $SL(n)$. Then we studied the properties of $SL^*(n)$, and solved a conjecture involving function $SL^*(n)$.

Keywords F.Smarandache LCM function, $SL^*(n)$ function, conjecture.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $SL(16) = 16$, $\cdots\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [2], [4] and [5]. For example, Murthy [2] porved that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Zhongtian Lv [5] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Now, we define another function $SL^*(n)$ as follows: $SL^*(1) = 1$, and if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\},$$

where $p_1 < p_2 < \cdots < p_r$ are primes.

About the elementary properties of function $SL^*(n)$, it seems that none has studied it yet, at least we have not seen such a paper before. It is clear that function $SL^*(n)$ is the dual function of $SL(n)$. So it has close relations with $SL(n)$. In this paper, we use the elementary method to study the following problem: For any positive integer n, whether the summation

$$\sum_{d|n}\frac{1}{SL^*(n)}, \tag{2}$$

is a positive integer? where $\sum\limits_{d|n}$ denotes the summation over all positive divisors of n.

We conjecture that there is no any positive integer $n > 1$ such that (2) is an integer. In this paper, we solved this conjecture, and proved the following:

Theorem. There is no any positive integer $n > 1$ such that (2) is an positive integer.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. For any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, from the definition of $SL^*(n)$ we know that

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{3}$$

Now if $SL(n) = p_k^{\alpha_k}$ (where $1 \leq k \leq r$) and n satisfy

$$\sum_{d|n}\frac{1}{SL^*(d)} = N, \quad \text{a positive integer},$$

then let $n = m\cdot p_k^{\alpha_k}$ with $(m, p_k) = 1$, note that for any $d|m$ with $d > 1$, $SL^*\left(p_k^i\cdot d\right) \mid m\cdot p_k^{\alpha_k - 1}$, where $i = 0,\ 1,\ 2,\ \cdots,\ \alpha_k$. We have

$$\begin{aligned} N &= \sum_{d|n}\frac{1}{SL^*(d)} = \sum_{i=0}^{\alpha_k}\sum_{d|m}\frac{1}{SL^*(d\cdot p_k^i)} = \sum_{i=0}^{\alpha_k}\frac{1}{SL^*(p_k^i)} + \sum_{i=0}^{\alpha_k}\sum_{\substack{d|m\\ d>1}}\frac{1}{SL^*(d\cdot p_k^i)} \\ &= 1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k}} + \sum_{i=0}^{\alpha_k}\sum_{\substack{d|m\\ d>1}}\frac{1}{SL^*(d\cdot p_k^i)}, \end{aligned}$$

or

$$m \cdot p_k^{\alpha_k - 1} \cdot N = \sum_{i=0}^{\alpha_k} \sum_{\substack{d|m \\ d>1}} \frac{m \cdot p_k^{\alpha_k - 1}}{SL^*(d \cdot p_k^i)} + m \cdot p_k^{\alpha_k - 1} \cdot \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k - 1}}\right) + \frac{m}{p_k}. \quad (4)$$

It is clear that for any $d|m$ with $d > 1$,

$$\sum_{i=0}^{\alpha_k} \sum_{\substack{d|m \\ d>1}} \frac{m \cdot p_k^{\alpha_k - 1}}{SL^*(d \cdot p_k^i)} \text{ and } m \cdot p_k^{\alpha_k - 1} \cdot \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^{\alpha_k - 1}}\right),$$

are integers, but $\frac{m}{p_k}$ is not an integer. This contradicts with (4). So the theorem is true. This completes the proof of the theorem.

Open problem. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, whether there exists an integer $n \geq 2$ such that $\sum_{d|n} \frac{1}{SL(n)}$ is an integer?

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Some identities involving the near pseudo Smarandache function

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Abstract For any positive integer n and fixed integer $t \geq 1$, we define function $U_t(n) = \min\{k : 1^t + 2^t + \cdots + n^t + k = m,\ n \mid m,\ k \in N^+,\ t \in N^+\}$, where $n \in N^+$, $m \in N^+$, which is a new pseudo Smarandache function. The main purpose of this paper is using the elementary method to study the properties of $U_t(n)$, and obtain some interesting identities involving function $U_t(n)$.

Keywords Some identities, reciprocal, pseudo Smarandache function.

§1. Introduction and results

In reference [1], A.W.Vyawahare defined the near pseudo Smarandache function $K(n)$ as $K(n) = m = \frac{n(n+1)}{2} + k$, where k is the small positive integer such that n divides m. Then he studied the elementary properties of $K(n)$, and obtained a series interesting results for $K(n)$. For example, he proved that $K(n) = \frac{n(n+3)}{2}$, if n is odd, and $K(n) = \frac{n(n+2)}{2}$, if n is even; The equation $K(n) = n$ has no positive integer solution. In reference [2], Zhang Yongfeng studied the calculating problem of an infinite series involving the near pseudo Smarandache function $K(n)$, and proved that for any real number $s > \frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{1}{K^s(n)}$ is convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{K(n)} = \frac{2}{3}\ln 2 + \frac{5}{6},$$

$$\sum_{n=1}^{\infty} \frac{1}{K^2(n)} = \frac{11}{108}\pi^2 - \frac{22 + 2\ln 2}{27}.$$

Yang hai and Fu Ruiqin [3] studied the mean value properties of the near pseudo Smarandache function $K(n)$, and obtained two asymptotic formula by using the analytic method. They proved that for any real number $x \geq 1$,

$$\sum_{n \leq x} d(k) = \sum_{n \leq x} d\left(K(n) - \frac{n(n+1)}{2}\right) = \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right),$$

where A is a computable constant.

$$\sum_{n\leq x}\varphi\left(K(n)-\frac{n(n+1)}{2}\right)=\frac{93}{28\pi^2}x^2+O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number.

In this paper, we define a new near Smarandache function $U_t(n)=\min\{k:\ 1^t+2^t+\cdots+n^t+k=m,\ n\mid m,\ k\in N^+,\ t\in N^+\}$, where $n\in N^+$, $m\in N^+$. Then we study its elementary properties. About this function, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we using the elementary method to study the calculating problem of the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{U_t^s(n)},$$

and give some interesting identities. That is, we shall prove the following:

Theorem 1. For any real number $s>1$, we have the identity

$$\sum_{n=1}^{\infty}\frac{1}{U_1^s(n)}=\zeta(s)\left(2-\frac{1}{2^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Theorem 2. For any real number $s>1$, we have

$$\sum_{n=1}^{\infty}\frac{1}{U_2^s(n)}=\zeta(s)\left[1+\frac{1}{5^s}-\frac{1}{6^s}+2\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\right].$$

Theorem 3. For any real number $s>1$, we also have

$$\sum_{n=1}^{\infty}\frac{1}{U_3^s(n)}=\zeta(s)\left[1+\left(1-\frac{1}{2^s}\right)^2\right].$$

Taking $s=2,\ 4$, and note that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, from our theorems we may immediately deduce the following:

Corollary. Let $U_t(n)$ defined as the above, then we have the identities

$$\sum_{n=1}^{\infty}\frac{1}{U_1^2(n)}=\frac{7}{24}\pi^2;\qquad\sum_{n=1}^{\infty}\frac{1}{U_2^2(n)}=\frac{2111}{5400}\pi^2;$$

$$\sum_{n=1}^{\infty}\frac{1}{U_3^2(n)}=\frac{25}{96}\pi^2;\qquad\sum_{n=1}^{\infty}\frac{1}{U_1^4(n)}=\frac{31}{1440}\pi^4;$$

t

$$\sum_{n=1}^{\infty}\frac{1}{U_2^4(n)}=\frac{2310671}{72900000}\pi^4;\qquad\sum_{n=1}^{\infty}\frac{1}{U_3^4(n)}=\frac{481}{23040}\pi^4.$$

§2. Some lemmas

To complete the proof of the theorems, we need the following several lemmas.

Lemma 1. For any positive integer n, we have

$$U_1(n) = \begin{cases} \frac{n}{2}, & \text{if } 2 \mid n, \\ n, & \text{if } 2 \dagger n. \end{cases}$$

Proof. See reference [1].

Lemma 2. For any positive integer n, we also have

$$U_2(n) = \begin{cases} \frac{5}{6}n, & \text{if} \quad n \equiv 0(\text{mod } 6), \\ n, & \text{if} \quad n \equiv 1(\text{mod } 6) \quad \text{or} \quad n \equiv 5(\text{mod } 6), \\ \frac{n}{2}, & \text{if} \quad n \equiv 2(\text{mod } 6) \quad \text{or} \quad n \equiv 4(\text{mod } 6), \\ \frac{n}{3}, & \text{if} \quad n \equiv 3(\text{mod } 6). \end{cases}$$

Proof. It is clear that

$$\begin{aligned} U_2(n) &= \min\{k : 1^2 + 2^2 + \cdots + n^2 + k = m, n \mid m, k \in N^+\} \\ &= \min\{k : \frac{n(n+1)(2n+1)}{6} + k \equiv 0(\text{mod } n), k \in N^+\}. \end{aligned}$$

(1) If $n \equiv 0(\text{mod } 6)$, then we have $n = 6h_1(h_1 = 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{6h_1(6h_1+1)(12h_1+1)}{6} \\ &= 72h_1^3 + 18h_1^2 + h_1, \end{aligned}$$

so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $6h_1 \mid h_1 + U_2(n)$, then $U_2(n) = \frac{5n}{6}$.

(2) If $n \equiv 1(\text{mod } 6)$, then we have $n = 6h_2 + 1(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+1)(6h_2+2)(12h_2+3)}{6} \\ &= 12h_2^2(6h_2+1) + 7h_2(6h_2+1) + 6h_2 + 1, \end{aligned}$$

because $n \mid \frac{n(n+1)(2n+1)}{6}$, so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $n \mid U_2(n)$, then $U_2(n) = n$.

If $n \equiv 5(\text{mod } 6)$, then we have $n = 6h_2 + 5(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+5)(6h_2+6)(12h_2+11)}{6} \\ &= 12h_2^2(6h_2+5) + 23h_2(6h_2+5) + 11(6h_2+5), \end{aligned}$$

because $n \mid \frac{n(n+1)(2n+1)}{6}$, so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $n \mid U_2(n)$, then $U_2(n) = n$.

(3) If $n \equiv 2(\text{mod } 6)$, then we have $n = 6h_2 + 2(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+2)(6h_2+3)(12h_2+5)}{6} \\ &= 12h_2^2(6h_2+2) + 11h_2(6h_2+2) + 2(6h_2+2) + 3h_2 + 1, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $6h_2 + 2 \mid 3h_2 + 1 + U_2(n)$, then $U_2(n) = \dfrac{n}{2}$.

If $n \equiv 4 (\text{mod } 6)$, then we have $n = 6h_2 + 4 (\, h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+4)(6h_2+5)(12h_2+9)}{6} \\ &= 12h_2^2(6h_2+4) + 19h_2(6h_2+4) + 7(6h_2+4) + 3h_2 + 3, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $2(3h_2+2) \mid 3h_2 + 2 + U_2(n)$, then $U_2(n) = \dfrac{n}{2}$.

(4) If $n \equiv 3 (\text{mod } 6)$, then we have $n = 6h_2 + 3 (\, h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+3)(6h_2+4)(12h_2+7)}{6} \\ &= 12h_2^2(6h_2+3) + 15h_2(6h_2+3) + 4(6h_2+3) + 4h_2 + 2, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $3(2h_2+1) \mid 2(2h_2+2) + U_2(n)$, then $U_2(n) = \dfrac{n}{3}$.

Combining (1), (2), (3) and (4) we may immediately deduce Lemma 2.

Lemma 3. For any positive integer n, we have

$$U_3(n) = \begin{cases} \dfrac{n}{2}, & \text{if} \quad n \equiv 2 (\text{mod } 4), \\ n, & \text{otherwise.} \end{cases}$$

Proof. From the definition of $U_3(n)$ we have

$$\begin{aligned} U_3(n) &= \min\{k : 1^3 + 2^3 + \cdots + n^3 + k = m, n \mid m, k \in N^+\} \\ &= \min\{k : \frac{n^2(n+1)^2}{4} + k \equiv 0 (\text{mod } n), k \in N^+\}. \end{aligned}$$

(a) If $n \equiv 2 (\text{mod } 4)$, then we have $n = 4h_1 + 2 (\, h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = (4h_1+2)^3(2h_1+1) + (4h_1+2)^2(2h_1+1) + (2h_1+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4}$ if and only if $2(2h_1+1) \mid (2h_1+1)^2 + U_3(n)$, then $U_3(n) = \dfrac{n}{2}$.

(b) If $n \equiv 0 (\text{mod } 4)$, then we have $n = 4h_2 (\, h_2 = 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = 4h_2^2(4h_2+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

If $n \equiv 1 (\text{mod } 4)$, then we have $n = 4h_1 + 1 (\, h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = (4h_1+1)^2(2h_1+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

If $n \equiv 3 (\text{mod } 4)$, then we have $n = 4h_1 + 3 (\, h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = 4(4h_1+3)^2(h_1+1)^2,$$

uso $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

Now Lemma 3 follows from (a) and (b).

§3. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof of the theorems. First we prove Theorem 1. For any real number $s > 1$, from Lemma 1 we have

$$\sum_{n=1}^{\infty} \frac{1}{U_1^s(n)} = \sum_{\substack{h=1 \\ n=2h}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{h=0 \\ n=2h+1}}^{\infty} \frac{1}{n^s} = \sum_{h=1}^{\infty} \frac{1}{h^s} + \sum_{h=0}^{\infty} \frac{1}{(2h+1)^s} = \zeta(s)\left(2 - \frac{1}{2^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function. This proves Theorem 1.

For $t = 2$ and real number $s > 1$, from Lemma 2 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{U_2^s(n)} &= \sum_{\substack{h_1=1 \\ n=6h_1}}^{\infty} \frac{1}{\left(\frac{5n}{6}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+1}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_2=0 \\ n=6h_2+2}}^{\infty} \frac{1}{\left(\frac{n}{3}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+4}}^{\infty} \frac{1}{\left(\frac{n}{2}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+5}}^{\infty} \frac{1}{n^s} \\
&= \sum_{h_1=1}^{\infty} \frac{1}{(5h_1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(6h_2+1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(3h_2+1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(2h_2+1)^s} + \\
&\quad \sum_{h_2=0}^{\infty} \frac{1}{(3h_2+2)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(6h_2+5)^s} \\
&= \zeta(s)\left[1 + \frac{1}{5^s} - \frac{1}{6^s} + 2\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\right],
\end{aligned}$$

This completes the proof of Theorem 2.

If $t = 3$, then for any real number $s > 1$, from Lemma 3 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{U_3^s(n)} &= \sum_{\substack{h_2=1 \\ n=4h_2}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_1=0 \\ n=4h_1+1}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_1=0 \\ n=4h_1+2}}^{\infty} \frac{1}{\left(\frac{n}{2}\right)^s} + \sum_{\substack{h_1=0 \\ n=4h_1+3}}^{\infty} \frac{1}{n^s} \\
&= \sum_{h_2=1}^{\infty} \frac{1}{(4h_2)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(4h_1+1)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(2h_1+1)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(4h_1+3)^s} \\
&= \zeta(s)\left[1 + \left(1 - \frac{1}{2^s}\right)^2\right],
\end{aligned}$$

This completes the proof of Theorem 3.

Open Problem. For any integer $t > 3$ and real number $s > 1$, whether there exists a calculating formula for the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{U_t^s(n)} \ ?$$

This is an open problem.

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On pseudo a-ideal of pseudo-BCI algebras

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Abstract In this paper, we define of fuzzy pseudo q-ideals and fuzzy pseudo a-ideals in pseudo-BCI algebra. We give several characterization and the extensive theorems about fuzzy pseudo q-ideals and fuzzy pseudo a-ideals.

Keywords pseudo-BCI algebra, fuzzy pseudo a-ideal, fuzzy pseudo q-ideal, fuzzy pseudo p-ideal.

§1. Introduction

G.Georgescu and A.Iorgulescu[1] introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebra. In[2], Y.B.Jun, one of the present authors, gave a characterization of pseudo-BCK algebra. Y.B.Jun et al[4] introduced the notion of pseudo-ideals in a pseudo-BCK algebra, and then investigated some of their properties. In [3], W.A.Dudek and Y.B.Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties. In this paper we consider the fuzification of pseudo a-ideal, pseudo q-ideal and pseudo p-ideal in pseudo-BCI algebras, and then we investigate some of their properties.

§2. Preliminaries

The notion of pseudo-BCI algebras is introduced by Georgescu and Iorgulescu[1] as follows:

Definition 2.1. ([1]) A pseudo-BCI algebra is a structure $\aleph = (X, \preceq, *, \diamond, 0)$, where " $\preccurlyeq$ " in a binery relation on X, " $*$ " and " $\diamond$ " are binary operations on X and "0" is an element of X, verifying the axioms:

(a1) $(x * y) \diamond (x * z) \preceq z * y,\ (x \diamond y) * (x \diamond z) \preceq z \diamond y$.

(a2) $x * (x \diamond y) \preceq y,\ x \diamond (x * y) \preceq y$.

(a3) $x \preceq x$.

(a4) $x \preceq y,\ y \preceq x \Rightarrow x = y$.

(a5) $x \preceq y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$.

for all $x, y, z \in X$,

If $\aleph$ is a pseudo-BCI algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then $\aleph$ is a BCK-algebra(see[1]).

Proposition 2.2. ([2])In a pseudo-BCI algebra $\aleph$, the following holds:

(p1) $x \preceq y \Rightarrow z * y \preceq z * x,\ z \diamond y \preceq z \diamond x.$

(p2) $x \preceq y,\ y \preceq z \Rightarrow x \preceq z.$

(p3) $(x * y) \diamond z = (x \diamond z) * y.$

(p4) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y.$

(p5) $x * y \preceq x,\ x \diamond y \preceq x.$

(p6) $x * 0 = x = x \diamond 0.$

(p7) $x \preceq y \Rightarrow x * z \preceq y * z,\ x \diamond z \preceq y \diamond z.$

(p8) $x * (x \diamond x * y)) = x * y$ and $x \diamond (x * (x \diamond y)) = x \diamond y.$

(p9) $x \preceq 0 \Rightarrow x = 0.$

Proposition 2.3. ([6])In a pseudo-BCI algebra $\aleph$, the following holds for all $x, y \in X$:

(i) $0 * (x \diamond y) \preceq y \diamond x.$

(ii) $0 \diamond (x * y) \preceq y * x.$

(iii) $0 * (x * y) = (0 \diamond x) \diamond (0 * y).$

(iv) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y).$

§3.Fuzzy pseudo q-ideal

In what follows, let $\aleph$ denote a pseudo-BCI algebra unless otherwise specified.

Definition 3.1. ([5]) A fuzzy set $\mu : \aleph \to [0, 1]$ is a fuzzy pseudo-ideal of $\aleph$ if and only if it satisfies,

(i) $\mu(0) \geq \mu(x), \quad \forall x \in X.$

(ii) $\mu(x) \geq min\{\mu(x * y), \quad \mu(y)\}, \quad \forall\ x, y \in X.$

(iii) $\mu(a) \geq min\{\mu(a \diamond b), \quad \mu(b)\}, \quad \forall\ a, b \in X.$

Lemma 3.2. ([3]) Let μ be a fuzzy pseudo-ideal of $\aleph$. If $x \preceq y$, then $\mu(x) \geq \mu(y)$.

Theorem 3.3. ([5]) A fuzzy set μ is a fuzzy pseudo-ideal of $\aleph$ if and only if it satisfies:

(i) $\forall\ x, y, z \in X,\ x * y \preceq z \Rightarrow \mu(x) \geq min\{\ \mu(y), \mu(z)\}.$

(ii) $\forall\ a, b, c \in X,\ a \diamond b \preceq c \Rightarrow \mu(a) \geq min\{\ \mu(b), \mu(c)\}.$

Defintion 3.4. A fuzzy set μ is called a fuzzy pseudo q-ideal in $\aleph$ if,

(I1) $\mu(x \diamond z) \geq min\{\ \mu(x \diamond (y * z)), \mu(y)\},\ \forall\ x, y, z \in X.$

(I2) $\mu(a * c) \geq min\{\ \mu(a * (b \diamond c)), \mu(b)\},\ \forall\ a, b, c \in X.$

Lemma 3.5. Every fuzzy pseudo q-ideal is fuzzy pseudo-ideal.

Proof. Since μ is fuzzy pseudo q-ideal. Putting $z = 0$ in $(I1)$ and $c = 0$ in $(I2)$, we have,

$$\mu(x) \geq min\{\ \mu(x \diamond y), \mu(uy)\}, \quad \mu(a) \geq min\{\ \mu(a * b), \mu(b)\}$$

Thus μ is fuzzy psuedo-ideal of $\aleph$.

Definition 3.6. A fuzzy set μ is fuzzy pseudo sub algebra of $\aleph$ if;

(i) $\mu(x \diamond y) \geq min\{\ \mu(x), \mu(y)\},\ \forall\ x, y \in X.$

(ii) $\mu(a * b) \geq min\{\ \mu(a), \mu(b)\},\ \forall a, b \in X.$

Theorem 3.7. A fuzzy pseudo q-ideal of $\aleph$ is a fuzzy pseudo sub algebra of $\aleph$.

Proof. If μ be a fuzzy pseudo q-ideal, putting $z = y$ in $(I1)$ and $b = c$ in $(I2)$, we have,

$$\mu(x \diamond y) \geq min\{\ \mu(x), \mu(y)\}, \quad \mu(a * b) \geq min\{\ \mu(a), \mu(b)\}.$$

This complete the proof.

Theorem 3.8. Let μ be a fuzzy pseudo-ideal of $\aleph$. Then the following are equivalent,

(i) μ is a fuzzy pseudo q-ideal of $\aleph$.

(ii) $\mu((a * b) \diamond c) \geq \mu(a \diamond (b * c))$ and $\mu((x \diamond y) * z) \geq \mu(x * (y \diamond z))$.

(iii) $\mu(x \diamond y) \geq \mu(x \diamond (0 * y))$ and $\mu((a * b) \geq \mu(a * (0 \diamond b))$.

Proof. $(i) \Rightarrow (ii)$. Since μ is a fuzzy pseudo q-ideal of $\aleph$, we have

$$\begin{aligned}\mu((a * b) \diamond c) &\geq min\{\ \mu((a * b) \diamond (0 * c)), \mu(0)\} \\ &= \mu((a * b) \diamond (0 * c)),\end{aligned} \tag{1}$$

on the other hand,

$$(a * b) \diamond (0 * c) = (a * b) \diamond ((b \diamond c) * b) \preceq a \diamond (b * c).$$

Therfore,

$$\mu(a \diamond (b * c)) \leq \mu((a * b) \diamond (0 * c)). \tag{2}$$

frome (1) and (2), we have,

$$\mu((a * b) \diamond c) \geq \mu(a \diamond (b * c)).$$

Similarly, since μ is fuzzy pseudo q-ideal, we have,

$$\begin{aligned}\mu((x \diamond y) * z) &\geq min\{\ \mu((x \diamond y) * (0 \diamond z)), \mu(0)\} \\ &= \mu((x \diamond y) * (0 \diamond z)).\end{aligned}$$

we have,

$$(x \diamond y) * (0 \diamond z) = (x \diamond y) * ((y * z) \diamond y) \preceq x * (y \diamond z).$$

Hence

$$\mu(x * (y \diamond z)) \leq \mu((x \diamond y) * (0 * z)).$$

Then,

$$\mu((x \diamond y) * z) \geq \mu(x * (y \diamond z)).$$

$(ii) \Rightarrow (iii)$. Letting $y = 0$ and $z = y$ in $(I1)$ and $b = 0$ and $c = b$ in $(I2)$.

$(iii) \Rightarrow (i)$. We have,

$$(x \diamond (0 * y)) * (x \diamond (z * y)) \preceq (z * y) \diamond (0 * y) \preceq z$$

and

$$(a * (0 \diamond b)) * (a * (c \diamond b)) \preceq (c \diamond b) * (0 \diamond b) \preceq c.$$

Then we have,

$$\mu(x \diamond (0 * y)) \geq min\{\ \mu(x \diamond (z * y))\ ,\ \mu(z)\},$$

and

$$\mu(a * (0 \diamond b)) \geq min\{\ \mu(a * (c \diamond b))\ ,\ \mu(c)\}.$$

Therefore by hypothesis,

$$\mu(x \diamond y) \geq min\{\ \mu(x \diamond (z * y))\ ,\ \mu(z)\},$$

and

$$\mu(a * b) \geq min\{\ \mu(a * (c \diamond b))\ ,\ \mu(c)\}.$$

Hence μ is a fuzzy pseudo q-ideal of $\aleph$.

Theorem 3.9. Let μ and ν be fuzzy pseudo-ideals of $\aleph$, such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$.If μ is a fuzzy pseudo q-ideal of $\aleph$, then so ν.

Proof. For any $x, y, a, b \in X$, by Theorem3.8, we want to show that,

$$\nu(x \diamond y) \geq \nu(x \diamond (0 * y)) \quad \text{and} \quad \nu(a * b) \geq \nu(a * (0 \diamond b)).$$

Putting, $s = x \diamond (0 * y)$, then $(x * s) \diamond (0 * y) = 0$.
Hence,

$$\mu((x * s) \diamond (0 * y)) = \mu(0) = \nu(0).$$

Since μ is a fuzzy pseudo q-ideal of $\aleph$ and using Theorem3.8,

$$\mu((x * s) \diamond y) \geq \mu((x * s) \diamond (0 * y)) = \nu(0).$$

Thus

$$\nu((x * s) \diamond y) \geq \mu((x * s) \diamond y) \geq \nu(0) \geq \nu(s),$$

since ν is a fuzzy pseudo-ideal we have,

$$\nu(x \diamond y) \geq min\{\ \nu((x * y) \diamond s), \nu(s)\} = \nu(s) = \nu(x \diamond (0 * y)). \tag{3}$$

Putting $t = a * (0 \diamond b)$ then $(a \diamond t) * (0 \diamond b) = 0$.

Hence,

$$\mu((a \diamond t) * (0 \diamond b)) = \mu(0) = \nu(0).$$

Since μ is a fuzzy pseudo q-ideal of $\aleph$ and using Theorem 3.8,

$$\mu((a \diamond t) * b) \geq \mu((a \diamond t) * (0 \diamond b)) = \nu(0).$$

Thus

$$\nu((a \diamond t) * b) \geq \mu((a \diamond t) * b) \geq \nu(0) \geq \nu(t),$$

since ν is a fuzzy pseudo-ideal we have,

$$\nu(a * b) \geq min\{\nu((a \diamond b) * t), \nu(t)\} = \nu(t) = \nu(a * (0 \diamond b)). \tag{4}$$

Frome (3), (4) and Theorem3.8, ν is a fuzzy pseudo q-ideal of $\aleph$.
This complete the proof.

§4.Fuzzy pseudo a-ideal

Definiton 4.1. A fuzzy set μ is called a fuzzy pseudo a-ideal in $\aleph$ if,
(I3) $\mu(y \diamond x) \geq min\{\ \mu((x * z) \diamond (0 * y))\ ,\ \mu(z)\},\ \forall\ x, y, z \in X.$
(I4) $\mu(b * a) \geq min\{\ \mu((a \diamond c) * (0 \diamond b))\ ,\ \mu(c)\},\ \forall\ a, b, c \in X.$
Theorem 4.2. Any fuzzy pseudo a-ideal is a fuzzy pseudo-ideal.
Proof. Since μ is fuzzy pseudo a-ideal then we have,

$$\mu(y \diamond x) \geq min\{\mu((x * z) \diamond (0 * y)), \mu(z)\},$$

$$\mu(b * a) \geq min\{\mu((a \diamond c) * (0 \diamond b)), \mu(c)\},$$

putting $y = 0$ and $b = 0$, then we have,

$$\mu(0 \diamond x) \geq min\{\ \mu(x * z)\ ,\ \mu(z)\}, \tag{5}$$

$$\mu(0 * a) \geq min\{\ \mu(a \diamond c)\ , \mu(c)\}, \tag{6}$$

putting $z = y = 0$ in $(I3)$, and $c = b = 0$ in $(I4)$, it follows that,

$$\mu(0 \diamond x) \geq \mu(x) \quad \text{and} \quad \mu(0 * a) \geq \mu(0), \tag{7}$$

putting $z = x = 0$ in $(I3)$, and $c = a = 0$ in $(I4)$, we have,

$$\mu(y) \geq \mu(0 \diamond (0 * y)) \geq \mu(0 * y) \geq min\{\ \mu(y \diamond z), \mu(z)\},$$

and

$$\mu(b) \geq \mu(0 * (0 \diamond b)) \geq \mu(0 \diamond b) \geq min\{\ \mu(b * c), \mu(c)\}.$$

Then $\mu(y) \geq min\{\ \mu(y \diamond z), \mu(z)\}$ and $\mu(b) \geq min\{\ \mu(b * c), \mu(c)\}$,
for all $x, y, z, a, b.c \in X$. Then μ is a fuzzy pseudo-ideal of $\aleph$.
The proof is complete.

Theorem 4.3. Let μ be a fuzzy pseudo-ideal of $\aleph$. Then the following are equivalent:
(i) μ is a fuzzy pseudo a-ideal of $\aleph$.
(ii) $\mu(b * (a \diamond c)) \geq \mu((a \diamond c) * (0 \diamond b))$ and $\mu(y \diamond (x * z)) \geq \mu((x * z) \diamond (0 * y))$.
(iii) $\mu(y \diamond x) \geq \mu(x \diamond (0 * y))$ and $\mu((b * a) \geq \mu(a * (0 \diamond b))$.
Proof. $(i) \Rightarrow (ii)$. Suppose that μ be a fuzzy pseudo a-ideal of $\aleph$, we have,

$$\mu(y \diamond (x * z) \geq min\{\mu(((x * z) * s) \diamond (0 * y)), \mu(s)\},$$

$$\mu(b * (a \diamond c)) \geq min\{\mu(((a \diamond c) \diamond t) * (0 \diamond b)), \mu(t)\}.$$

We write $s = (x * z) \diamond (0 * y)$ and $t = (a \diamond c) * (0 \diamond b)$. Then

$$((x * z) * s) \diamond (0 * y) = ((x * z) \diamond (0 * y)) * s = 0,$$

and

$$((a \diamond c) \diamond t) * (0 \diamond b) = ((a \diamond c) * (0 \diamond b)) \diamond t = 0,$$

then we have,

$$\mu(y \diamond (x * z) \geq min\{ \mu(0), \mu(s)\} = \mu(s) = \mu((x * z) \diamond (0 * y)),$$

and

$$\mu(b * (a \diamond c)) \geq min\{ \mu(0), \mu(t)\} = \mu(t) = \mu((a \diamond c) * (0 \diamond y)).$$

$(ii) \Rightarrow (iii)$. Letting $z = 0$ and $c = 0$ in (ii), we obtain:

$$\mu(y \diamond x) \geq \mu(x \diamond (0 * y)), \quad \mu(b * a) \geq \mu(a * (0 \diamond y).$$

$(iii) \Rightarrow (i)$. Since,

$$(x \diamond (0 * y)) \diamond ((x * z) \diamond (0 * y)) \preceq x \diamond (x * z) \preceq z,$$

and

$$(a * (0 \diamond b)) * ((a \diamond c) * (0 \diamond y) \preceq a * (a \diamond c) \preceq c.$$

Then we have,

$$\mu(x \diamond (0 * y)) \geq min\{\mu((x * z) \diamond (0 * y)) \ , \ \mu(z)\},$$

and

$$\mu(a * (0 \diamond b)) \geq min\{\mu((a \diamond c) * (0 \diamond b)) \ , \ \mu(c)\}.$$

from condition (iii), we have,

$$\mu(y \diamond x) \geq \mu(x \diamond (0 * y)) \geq min\{\mu((x * z) \diamond (0 * y)) \ , \ \mu(z)\},$$

and

$$\mu(b * a) \geq \mu(a * (0 \diamond b)) \geq min\{\mu((a \diamond c) * (0 \diamond b)) \ , \ \mu(c)\}.$$

Hence μ is a fuzzy pseudo a-ideal of $\aleph$.

Theorem 4.4. Let μ be a fuzzy pseudo-ideal of $\aleph$, then μ is a fuzzy pseudo a-ideal of $\aleph$ if and only if it satisfies the following conditions:

(i) $\mu(y) \geq \mu(0 \diamond (0 * y)$ and $\mu(b) \geq \mu(0 \diamond (0 * b)$ $\forall y, b \in X$.

(ii) $\mu(x \diamond y) \geq \mu(x \diamond (0 * y)$ and $\mu(a * b) \geq \mu(a \diamond (0 * b)$ $\forall x, y, a, b \in X$.

Proof. Assume that μ is a fuzzy pseudo a-ideal of $\aleph$. Seting $x = 0$ and $a = 0$ in Theorem 4.3 (iii). We get,

$$\mu(y) \geq \mu(0 \diamond (0 * y) \quad \text{and} \quad \mu(b) \geq \mu(0 \diamond (0 * b),$$

since

$$\begin{aligned}
& (0 \diamond (0 \diamond (y \diamond (0 * x)))) \diamond (x \diamond (0 * y)) \\
= \; & (0 \diamond ((0 * y) * (0 \diamond (0 * x)) \diamond (x \diamond (0 * y)) \\
= \; & (0 * (0 * y)) * (0 \diamond (0 \diamond (0 * x)))) \diamond (x \diamond (0 * y)) \\
= \; & (0 * (0 * y)) * (0 * x)) \diamond (x \diamond (0 * y)) \\
\preceq \; & (x \diamond (0 * y)) \diamond (x \diamond (0 * y)) \\
= \; & 0.
\end{aligned}$$

Then by Theorem 3.3:

$$\mu(0 \diamond (0 \diamond (y \diamond (0 * x)))) \geq min\{\mu(x \diamond (0 * y)), \mu(0)\} = \mu(x \diamond (0 * y)),$$

$$\mu(y \diamond (0 * x)) \geq \mu(0 \diamond (0 \diamond (y \diamond (0 * x)))) \geq \mu(x \diamond (0 * y)).$$

Applying Theorem 4.3 (iii), we have,

$$\mu(x \diamond y) \geq \mu(y \diamond (0 * x)) \geq \mu(x \diamond (0 * y)).$$

Similarly, we have,

$$\begin{aligned}
& (0 * (0 * (b * (0 \diamond a)))) * (a * (0 \diamond b)) \\
= \; & (0 * ((0 \diamond b) \diamond (0 * (0 \diamond a))) * (a * (0 \diamond b)) \\
= \; & (0 \diamond (0 \diamond b)) * (0 * (0 \diamond a)))) * (a * (0 \diamond b)) \\
= \; & (0 \diamond (0 \diamond b)) * (0 \diamond a)) * (a * (0 \diamond b)) \\
\preceq \; & (a * (0 \diamond b)) * (a * (0 \diamond b)) \\
= \; & 0.
\end{aligned}$$

Then by Theorem 3.3 (iii),

$$\mu(0 * (0 * (b * (0 \diamond a)))) \geq min\{\ \mu(a * (0 \diamond b)), \mu(0)\} = \mu(a * (0 \diamond b)).$$

$$\mu(b * (0 \diamond a)) \geq \mu(0 * (0 * (b * (0 \diamond a))) \geq \mu(a * (0 \diamond b)).$$

Applying Theorem 4.3 (iii), we have,

$$\mu(a * b) \geq \mu(b * (0 \diamond a) \geq \mu(a * (0 \diamond b)).$$

Conversely, suppose μ be a fuzzy pseudo-ideal satisfies(i) and (ii). In order to prove that μ is a fuzzy pseudo a-ideal of $\aleph$, from Theorem 4.3, we want to show that,

$$\mu(y \diamond x) \geq \mu(x \diamond (0 * y)) \quad \text{and} \quad \mu(b * a) \geq \mu(a * (0 \diamond b)),$$

for all $a, b, x, y \in X$. By(ii), we have, $\mu(x \diamond y) \geq \mu(ux \diamond (0 * y))$. Since $0 * (y \diamond x) \preceq x \diamond y$, by Theorem 3.3, we get $\mu(0 * (y \diamond x)) \geq \mu(x \diamond y) \geq \mu(x \diamond (0 * y))$. Thus:

$$\mu(0 * (0 * (y \diamond x))) \geq \mu(0 * (y \diamond x)) \geq \mu(x \diamond (0 * y)).$$

Applying(i), we get

$$\mu(y \diamond x) \geq \mu(0 * (0 * (y \diamond x))) \geq \mu(x \diamond (0 * y)).$$

Similarly, we can get $\mu(b * a) \geq \mu(a * (0 \diamond b))$. Therefore μ is a fuzzy pseudo a-ideal of $\aleph$. This completes the proof.

Definion 4.5. A fuzzy set μ is called a fuzzy pseudo p-ideal in $\aleph$ if,

(I5) $\mu(x) \geq min\{\ \mu((x * z) \diamond (y * z)), \mu(y)\},\ \forall\ x, y, z \in X.$

(I6) $\mu(a) \geq min\{\ \mu((a \diamond c) * (b \diamond c)), \mu(b)\},\ \forall\ a, b, c \in X.$

Theorem 4.6. Every fuzzy pseudo p-ideal is a fuzzy pseudo-ideal of $\aleph$.

Proof. Let μ be a fuzzy pseudo p-ideal of $\aleph$. Put $z = 0$ in $(I5)$ and $c = 0$ in $(I6)$, we get,

$$\mu(x) \geq min\{\ \mu(x \diamond y), \mu(y)\} \quad and \quad \mu(a) \geq min\{\ \mu(a * b), \mu(b)\}.$$

Thus μ is a fuzzy pseudo-ideal of $\aleph$.

Lemma 4.7. A fuzzy pseudo-ideal of $\aleph$ is a fuzzy pseudo p-ideal if and only if,

$$\mu(x) \geq \mu(0 \diamond (0 * x)) \quad and \quad \mu(a) \geq \mu(0 * (0 \diamond a))$$

for all $x, a \in X$.

Theorem 4.8. Any fuzzy pseudo a-ideal is a fuzzy pseudo p-ideal.

Proof. Let μ be a fuzzy pseudo a-ideal of $\aleph$. Then μ is a fuzzy pseudo-ideal. Setting $x = z = 0$ and $a = c = 0$ in Theorem4.3, then we have

$$\mu(y) \geq \mu(0 \diamond (0 * y)) \quad \text{and} \quad \mu(b) \geq \mu(0 * (0 \diamond b)).$$

Frome Lemma4.7, μ is a fuzzy pseudo p-ideal.

Theorem 4.9. Any fuzzy pseudo a-ideal is a fuzzy pseudo q-ideal.

Proof. Let μ be a fuzzy pseudo a-ideal of $\aleph$. Then μ is a fuzzy pseudo ideal. In order to prove that μ is a fuzzy pseudo q-ideal, from Theorem4.4(ii), it suffices to show that

$$\mu(x \diamond y) \geq \mu(x \diamond (0 * y)) \quad \text{and} \quad \mu(a * b) \geq \mu(a * (0 \diamond b)),$$

we have:

$$\begin{aligned} & (0 \diamond (0 \diamond (y \diamond (0 * x)))) \diamond (x \diamond (0 * y)) \\ = \; & (0 * ((0 * y) * (0 \diamond (0 \diamond (0 * x)))) \diamond (x \diamond (0 * y)) \\ = \; & ((0 * (0 * y)) * (0 * x)) \diamond (x \diamond (0 * y)) \\ \preceq \; & (x \diamond (0 * y)) \diamond (x \diamond (0 * y)) \\ = \; & 0. \end{aligned}$$

Hence,

$$\mu(0 \diamond (0 \diamond (y \diamond (0 * x)))) \geq \mu(x \diamond (0 * y)), \tag{8}$$

and we have,

$$\begin{aligned} & (0 * (0 * (b * (0 \diamond a)))) * (a * (0 \diamond b)) \\ = \; & (0 \diamond (0 \diamond b)) * (0 * (0 * (0 \diamond a))) * (a * (0 \diamond b)) \\ = \; & (0 \diamond (0 \diamond b)) * (0 \diamond a)) * (a * (0 \diamond b)) \\ \preceq \; & (a * (0 \diamond b)) * (a * (0 \diamond b)) \\ = \; & 0. \end{aligned}$$

Then

$$\mu(0 * (0 * (b * (0 \diamond a)))) \geq \mu(a * (0 \diamond b)). \tag{9}$$

By Theorem 4.8, μ is a fuzzy pseudo p-ideal and by Lemma4.7, we have,

$$\mu(y \diamond (0 * x)) \geq \mu(0 \diamond (0 * (y \diamond (0 * x)))),$$

and

$$\mu(b * (0 \diamond a)) \geq \mu(0 * (0 \diamond (b * (0 \diamond a)))).$$

By Theorem 4.4 (ii),

$$\mu(x \diamond y) \geq \mu(y \diamond (0 * x)), \tag{10}$$

and

$$\mu(a * b) \geq \mu(b * (0 \diamond a)). \tag{11}$$

From $(8), (9)$ and $(10), (11)$, we have,

$$\mu(x \diamond y) \geq \mu(y \diamond (0 * x)) \geq \mu(0 \diamond (0 * (y \diamond (0 * x)))) \geq \mu(x \diamond (0 * y)),$$

and

$$\mu(a * b) \geq \mu(b * (0 \diamond b)) \geq \mu(0 * (0 * (b * (0 \diamond a)))) \geq \mu(a * (0 \diamond b)).$$

Therefore μ is a fuzzy pseudo q-ideal of $\aleph$.

Theorem 4.10. let μ be a fuzzy pseudo-ideal of $\aleph$, μ is a fuzzy pseudo a-ideal if and only if it is both a fuzzy pseudo p-ideal and a fuzzy pseudo q-ideal.

Proof. If μ is a fuzzy pseudo a-ideal, then μ is a fuzzy pseudo p-ideal and a fuzzy pseudo q-ideal by Theorem 4.8 and Theorem 4.9. We want to show

$$\mu(y \diamond x) \geq \mu(x \diamond (0 * y)) \quad \text{and} \quad \mu(b * a) \geq \mu(a * (0 \diamond b)).$$

By Theorem 4.3 (iii) and Theorem 4.4 (ii),

$$\mu(x \diamond y) \geq \mu(x \diamond (0 * y)) \quad \text{and} \quad \mu((a * b) \geq \mu(a * (0 \diamond b)).$$

Hence

$$\mu(0 \diamond (y * x)) \geq \mu(x \diamond y) \geq \mu(x \diamond (0 * y)),$$

and

$$\mu(0 * (b \diamond a))) \geq \mu(a * b) \geq \mu(a * (0 \diamond b)).$$

Since μ is a fuzzy pseudo p-ideal by Lemma 4.7,

$$\mu(y \diamond x) \geq \mu(0 \diamond (0 * (y \diamond x))) \quad \text{and} \quad \mu(b * a) \geq \mu(0 * (0 \diamond (b * a))).$$

since μ is a fuzzy pseudo-ideal, then,

$$\mu(0 \diamond (y * x)) \leq \mu(0 \diamond (0 * (y \diamond x))) \quad \text{and} \quad \mu(0 * (b \diamond a)) \leq \mu(0 * (0 \diamond (b * a))),$$

we have

$$\mu(y \diamond x) \geq \mu(0 \diamond (0 * (y \diamond x))) \geq \mu(0 \diamond (y * x)) \geq \mu(x \diamond y) \geq \mu(x \diamond (0 * y)),$$

and

$$\mu(b * a) \geq \mu(0 * (0 \diamond (b * a))) \geq \mu(0 * (b \diamond a)) \geq \mu(a * b) \geq \mu(a * (0 \diamond b)).$$

Thus, μ is a fuzzy pseudo a-ideal, by Theorem4.3(iii), completing the proof.

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An equation involving the F.Smarandache multiplicative function[1]

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Abstract For any positive integer n, we call an arithmetical function $f(n)$ as the F.Smarandache multiplicative function if $f(1) = 1$, and if $n > 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then $f(n) = \max_{1\le i\le k} \{f(p_i^{\alpha_i})\}$. The main purpose of this paper is using the elementary methods to study the solutions of an equation involving the F.Smarandache multiplicative function, and give its all positive integer solutions.

Keywords F.Smarandache multiplicative function, function equation, positive integer solution, elementary methods.

§1. Introduction and result

For any positive integer n, we call an arithmetical function $f(n)$ as the F.Smarandache multiplicative function if $f(1) = 1$, and if $n > 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then $f(n) = \max_{1\le i\le k} \{f(p_i^{\alpha_i})\}$. For example, the function $S(n) = \min\{m : m \in N,\ n|m!\}$ is a F.Smarandache multiplicative function. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, we have

$$S(n) = \max_{1\le i\le k} \{S(p_i^{\alpha_i})\}.$$

So we can say that $S(n)$ is a F.Smarandache multiplicative function. In fact, this function be the famous F.Smarandache function, the first few values of it are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained some valuable results. For example, Farris Mark and Mitchell Patrick [2] studied the upper and lower bound of $S(p^\alpha)$, and proved that

$$(p-1)\alpha + 1 \le S(p^\alpha) \le (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Professor Wang Yongxing [3] studied the mean value properties of $S(n)$, and obtained a sharper asymptotic formula, that is

$$\sum_{n\le x} S(n) = \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

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Lu Yaming [4] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [5] proved for any positive integer $k \geq 2$, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

In [6], Fu Jing proved more general conclusion. That is, if the positive integer k and m satisfied the one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are all odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even numbers $k \geq 4$ and any positive integer m;

then the equation

$$m \cdot S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$.

In [7], Xu Zhefeng studied the value distribution of $S(n)$, and obtained a deeply result. That is, he proved the following Theorem:

Let $P(n)$ be the largest prime factor of n, then for any real numbers $x > 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, we define

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_k^{\alpha_k}\}.$$

Obviously, this function is also a Smarandache multiplicativa function, which is called F.Smarandache LCM function. About the properties of this function, there are many scholars have studied it, see references [8] and [9].

Now, we define another arithmetical function $\overline{S}(n)$ as follows: $\overline{S}(1) = 1$, when $n > 1$ and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then we define

$$\overline{S}(n) = \max\{\alpha_1 p_1,\ \alpha_2 p_2,\ \alpha_3 p_3,\ \cdots,\ \alpha_k p_k\}.$$

It is easy to prove that this function is also a F.Smarandache multiplicative function. About its elementary properties, we know very little, there are only some simple properties mentioned in [7]. That is, if we replace $S(n)$ with $\overline{S}(n)$ in (1), it is also true.

The main purpose of this paper is using the elementary methods to study the solutions of an equation involving $\overline{S}(n)$. That is, we shall study all positive integer solutions of the equation

$$\sum_{d|n} \overline{S}(d) = n, \tag{2}$$

where $\sum_{d|n}$ denotes the summation over all positive factors of n.

Obviously, there exist infinite positive integer n, such that $\sum_{d|n} \overline{S}(d) > n$. For example, let $n = p$ be a prime, then $\sum_{d|n} \overline{S}(d) = 1 + p > p$. At the same time, there are also infinite positive integer n, such that $\sum_{d|n} \overline{S}(d) < n$.

In fact, let $n = pq$, p and q are two different odd primes with $p < q$, then we have $\sum_{d|n} \overline{S}(d) = 1 + p + 2q < pq$. So a natural problem is whether there exist infinite positive integer n satisfying (2)? We have solved this problem completely in this paper, and proved the following conclusion:

Theorem. For any positive integer n, the equation (2) holds if and only if $n = 1,\ 28$.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Firstly, we prove some special cases:

(i) If $n = 1$, $\sum_{d|n} \overline{S}(d) = \overline{S}(1) = 1$, then $n = 1$ is a solution of equation (2).

(ii) If $n = p^\alpha$ is the prime powers, then (2) doesn't hold.

In fact, if (2) holds, then from the definition of $\overline{S}(n)$, we have

$$\sum_{d|n} \overline{S}(d) = \sum_{d|p^\alpha} \overline{S}(d) = 1 + p + 2p + \cdots + \alpha p = p^\alpha. \tag{3}$$

Obviously, the right side of (3) is a multiple of p, but the left side is not divided by p, a contradiction. So if n is a prime powers, (2) doesn't hold.

(iii) If $n > 1$ and the least prime factor powers of n is 1, then the equation (2) also doesn't hold. Now, if $n = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k} = p_1 n_1$ satisfied (2), then from the conclusion (ii), we know that $k \geq 2$, so from the definition of $\overline{S}(n)$, we have

$$\sum_{d|n} \overline{S}(d) = \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} \overline{S}(p_1 d) = 2 \sum_{d|n_1} \overline{S}(d) + p_1 - 1 = p_1 n_1. \tag{4}$$

Obviously, two sides of (4) has the different parity, it is impossible.

We get immediately from the conclusion (iii), if n is a square-free number, then n can't satisfy (2).

Now we prove the general case. Provided integer $n > 1$ satisfied equation (2), from (ii) and (iii), we know that n has two different prime powers at least, and the least prime factor power of n is larger than 1. So we let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\alpha_1 > 1$, $k \geq 2$. Let $\overline{S}(n) = \alpha p$, we discuss it in the following cases:

(A) $\alpha = 1$. Then p must be the largest prime factors of n, let $n = n_1 p$, note that, if $d|n_1$, we have$\overline{S}(d) \leq p - 1$, so from $\sum\limits_{d|n} \overline{S}(d) = n$ we get

$$\begin{aligned} n_1 p &= n = \sum_{d|n_1 p} \overline{S}(d) = \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} \overline{S}(dp) \\ &= \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} p \leq 1 + \sum_{\substack{d|n_1 \\ d>1}} (p-1) + pd(n_1) \\ &= 2 + (2p-1)d(n_1) - p \end{aligned} \tag{5}$$

or

$$n_1 + 1 < 2d(n_1), \tag{6}$$

where $d(n_1)$ is the Direchlet divisor function. Obviously, if $n_1 \geq 7$, then $2 \leq n_1 \leq 6$. It is also because the least prime factors power of n_1 is bigger than 1, we have $n_1 = 4$, and $n = n_1 p = 4p$, $p > 3$. Now, from

$$4p = \sum_{d|4p} \overline{S}(d) = \overline{S}(1) + \overline{S}(2) + \overline{S}(4) + \overline{S}(p) + \overline{S}(2p) + \overline{S}(4p) = 1 + 2 + 4 + 3p,$$

we immediately obtain $p = 7$ and $n = 28$.

(B) $\overline{S}(n) = \alpha p$ and $\alpha > 1$, now let $n = n_1 p^\alpha, (n_1, p) = 1$, if n satisfied (2), we have

$$n = p^\alpha n_1 = \sum_{i=0}^{\alpha} \sum_{d|n_1} \overline{S}(p^i d).$$

If $1 < n_1 < 8$, we consider equation (2) as follows:

(a) If $n_1 = 2$. That is, $n = 2p^\alpha (p > 2)$, from the discussion of (iii), we know that $n = 2p^\alpha$ isn't the solution of (2).

(b) If $n_1 = 3$. That is, $n = 3p^\alpha$, since $(n_1, p) = 1$, we have $p \neq 3$.

If $p = 2, n = 3 \cdot 2^\alpha$ satisfied (2). That is

$$\sum_{d|3\cdot 2^\alpha} \overline{S}(d) = \sum_{d|2^\alpha} \overline{S}(d) + \sum_{d|2^\alpha} \overline{S}(3d) = 2 \sum_{d|2^\alpha} \overline{S}(d) + 3 = 3 \cdot 2^\alpha.$$

In the above equation, $2 \sum\limits_{d|2^\alpha} \overline{S}(d) + 3$ is an odd number, but $3 \cdot 2^\alpha$ is an even number, so $n = 3 \cdot 2^\alpha$ is not the solution of (2).

If $p > 3$. That is, $n = 3 \cdot p^\alpha$ satisfied (2), then the least prime factor powers of n is 1, from (iii), we know that $n = 3 \cdot p^\alpha$ is not the solution of (2).

(c) If $n_1 = 4$, $n = 4 \cdot p^\alpha (p \geq 3)$, we have

$$\sum_{d|4\cdot p^\alpha} \overline{S}(d) = \sum_{d|p^\alpha} \overline{S}(d) + \sum_{d|p^\alpha} \overline{S}(2d) + \sum_{d|p^\alpha} \overline{S}(4d).$$

If $p = 3$. That is, $n = 4 \cdot 3^\alpha$ satisfied equation (2), then

$$\sum_{d|4\cdot 3^\alpha} \overline{S}(d) = \sum_{d|3^\alpha} \overline{S}(d) + \sum_{d|3^\alpha} \overline{S}(2d) + \sum_{d|3^\alpha} \overline{S}(4d) = 3 \sum_{\substack{d|3^\alpha \\ d>1}} \overline{S}(d) + 12 = 4 \cdot 3^\alpha.$$

Since $3^2 \mid 3 \sum_{\substack{d|3^\alpha \\ d>1}} \overline{S}(d)$, and $3^2 \mid 4 \cdot 3^\alpha$, then $3^2 \mid 12$, this is impossible.

If $p > 3$. That is, $n = 4 \cdot p^\alpha$, then

$$\sum_{d|4\cdot p^\alpha} \overline{S}(d) = \sum_{d|p^\alpha} \overline{S}(d) + \sum_{d|p^\alpha} \overline{S}(2d) + \sum_{d|p^\alpha} \overline{S}(4d) = 3 \sum_{d|p^\alpha} \overline{S}(d) + 8 = \frac{3}{2}\alpha(\alpha+1)p + 11 = 4 \cdot p^\alpha,$$

or $4 \cdot 3^\alpha - \frac{3}{2}\alpha(\alpha+1)p + 11 = 0$. Now we fix α, and let $f(x) = 4 \cdot x^\alpha - \frac{1}{2}\alpha(\alpha+1)x + 11$, if $x \geq 3$, $f(x)$ is a increased function. That is,

$$f(x) \geq f(3) = 4 \cdot 3^\alpha - \frac{3}{2}\alpha(\alpha+1) + 11 = g(\alpha).$$

So when $x \geq 3$, $f(x) = 0$ has no solutions, from which we get if $p > 3$, then equation (2) has no solutions.

(d) If $n_1 = 5$, we have $n = 5 \cdot p^\alpha (p \neq 5)$.

If $p > 5$, then from (iii), we know that $n = 5 \cdot p^\alpha$ is not a solution of equation (2).

If $p = 2$, since

$$\sum_{d|5\cdot 2^\alpha} \overline{S}(d) = \sum_{d|2^\alpha} \overline{S}(d) + \sum_{d|2^\alpha} \overline{S}(5d) = 2 \sum_{\substack{d|2^\alpha \\ d>1}} \overline{S}(d) + 10 = 5 \cdot 2^\alpha,$$

where $2^2 \mid 2 \sum_{\substack{d|2^\alpha \\ d>1}} \overline{S}(d)$, and $2^2 \mid 5 \cdot 2^\alpha$, so we have $2^2 \mid 10$, this is impossible. Hence $n = 5 \cdot 2^\alpha$ unsatisfied the equation (2).

If $p = 3$, since

$$\sum_{d|5\cdot 3^\alpha} \overline{S}(d) = \sum_{d|3^\alpha} \overline{S}(d) + \sum_{d|3^\alpha} \overline{S}(5d) = 2 \sum_{d|3^\alpha} \overline{S}(d) + 6,$$

where $2 \sum_{d|3^\alpha} \overline{S}(d) + 6$ is even, and $5 \cdot 3^\alpha$ is odd, so $n = 5 \cdot 3^\alpha$ unsatisfied the equation (2).

(e) When $n_1 = 6$, $n = 2 \cdot 3 \cdot p^\alpha$, from the discussion of (3), n unsatisfied (2).

(f) When $n_1 = 7$, we have $n = 7 \cdot p^\alpha (p \neq 7)$.

If $p > 7$, then from (iii), $n = 7 \cdot p^\alpha$ isn't the solution of (2).

If $p = 2$, we must have $\alpha \geq 4$. Since

$$\sum_{d|7\cdot 2^\alpha} \overline{S}(d) = \sum_{d|2^\alpha} \overline{S}(d) + \sum_{d|2^\alpha} \overline{S}(7d) = 2 \sum_{d|2^\alpha} \overline{S}(d) + 15,$$

where $2 \sum_{d|2^\alpha} \overline{S}(d) + 15$ is odd, but $n = 7 \cdot 2^\alpha$ is even. So $n = 7 \cdot 2^\alpha$ unsatisfied (2).

If $p = 3$, since

$$\sum_{d|7\cdot 3^\alpha} \overline{S}(d) = \sum_{d|3^\alpha} \overline{S}(d) + \sum_{d|3^\alpha} \overline{S}(7d) = 2 \sum_{\substack{d|3^\alpha \\ d>1}} \overline{S}(d) + 13,$$

in above equation, $3 \mid 2\sum\limits_{\substack{d|3^\alpha \\ d>1}} \overline{S}(d)$, and $3 \mid 7\cdot 3^\alpha$. If it satisfied (2), we must obtain $3 \dagger 13$, a contradiction! So $n = 7\cdot 3^\alpha$ is not a solution of (2) either.

If $p = 5$, since

$$\sum_{d|7\cdot 5^\alpha} \overline{S}(d) = \sum_{d|5^\alpha} \overline{S}(d) + \sum_{d|5^\alpha} \overline{S}(7d) = 2\sum_{d|5^\alpha} \overline{S}(d) + 8,$$

in the above equation, $2\sum\limits_{d|5^\alpha} \overline{S}(d) + 8$ is even, $7\cdot 5^\alpha$ is odd. So $n = 7\cdot 5^\alpha$ is not a solution of (2) either.

(g) When $n_1 \geq 8$, we have $n = n_1 \cdot p^\alpha$ and $p^\alpha > \frac{\alpha(\alpha+1)}{2}p$, then

$$\sum_{d|n_1\cdot p^\alpha} \overline{S}(d) < \overline{S}(p^\alpha)d(n_1p^\alpha) = \alpha(\alpha+1)pd(n_1) \leq \frac{\alpha(\alpha+1)}{2}pn_1 < p^\alpha n_1 = n,$$

then if $n_1 \geq 8$, $n = n_1p^\alpha$ is not a solution of (2) either.

In a word, equation (2) only has two solutions $n = 1$ and $n = 28$.

This completes the proof of the theorem.

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Matrix elements for the morse and coulomb interactions

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Abstract We show the usefulness of the hypervirial theorem to obtain the matrix elements $\langle m|e^{-\gamma au}|n\rangle$ for the Morse potential. Besides, it is known that the hydrogenlike atom can be studied as a Morse oscillator, then here we prove that these fact leads to an interesting method to calculate $\langle nl_2|r^{\tilde{k}}|nl_1\rangle$ for the Coulomb interaction.

Keywords Hypervirial theorem, coulomb and Morse potentials, langer transformation, matrix elements.

§1. Introduction

In [1] it was applied the hypervirial theorem (HT) to determine matrix elements for the one-dimensional harmonic oscillator, here we shall employ the HT to obtain $\langle m|e^{-\gamma au}|n\rangle$ for the Morse field [2]. An important aspect of the HT is that it not need explicitly the wave function, it uses only the potential V and the corresponding energy levels En. In Sec. 2 the HT permits: a). to show that $\langle m|e^{-au}|n\rangle = \langle m|e^{-2au}|n\rangle$, b). to calculate $\langle m|u|n\rangle$ and $\langle m|e^{-\gamma au}|n\rangle$, $\gamma = 3, 4\cdots$, if we know $\langle m|e^{-\beta au}|n\rangle$, $\beta = 1, 2$. The Sec. 3 has the general expressions of [3,4] for $\langle m|e^{-\gamma au}|n\rangle$, with the comment that yet [5] it is not verified the total equivalence between the result of [3] and the associated relation of [4].

For the hydrogenic atom its radial wave function $\frac{1}{r}g_{nl}$ depends of the principal (n) and orbital (l) quantum numbers, which are associated to eigenvalues for energy and angular momentum, respectively. Lee [6] showed that the Langer transformation [7] permits to study a non-relativistic hydrogenlike system as a vibrational Morse oscillator (MO), such that n gives the parameters of the Morse well and l determines an energy level in these well. In Sec. 4 we exhibit this result of Lee.

In according with [6] the function g_{nl} is proportional to the corresponding (MO) wave function, which means that the matrix elements $\langle nl_2|r^{\tilde{k}}|nl_1\rangle$ of the hydrogenic atom are equivalent to $\langle N_2|e^{-\gamma u}|N_1\rangle$, $\gamma = \tilde{k}+2$, of its MO. Thus the knowledge on Morse matrix elements can be used to determine $\langle r^{\tilde{k}}\rangle$ for the Coulomb potential. In Sec. 5 we apply this approach to obtain $\langle nl_2|r^{\tilde{k}}|nl_1\rangle$, $\tilde{k} = integer \geq -2$, without factorization techniques [8,9] as in [10]; we reproduce as particular cases the elements $\langle nl|r^{\tilde{k}}|nl\rangle$, $\tilde{k} = \pm 1, \pm 2$, deduced analytically by Landau-Lifshitz [11].

§2. Hypervirial theorem

One aim of our work is the calculation of matrix elements for the Morse potential:

$$\langle m|f(r)|n\rangle = \int_0^\infty \psi_m f(r)\psi_n dr, \tag{1}$$

where $\frac{1}{r}\psi_n$ is the radial wave function satisfying the Schrödinger equation (in natural units $m = \hbar = 1$):

$$\frac{d^2}{dr^2}\psi_n + 2[E_n - V(r)]\psi_n = 0. \tag{2}$$

The analytical procedure employs the explicit formulae of ψ_n and $f(r)$, and it makes directly the integral (1); however, in [1] we see that the HT evaluates (1) [for the harmonic oscillator] without the explicit form of the wave function. The Schrödinger equation has all information on our quantum system, and the HT has a part of these information which permits to study (1) without the explicit use of ψ_n.

From (2) it is easy to obtain the HT [1]:

$$(E_m - E_n)^2\langle m|f|n\rangle + \frac{1}{4}\langle m|f''''|n\rangle + (E_m + E_n)\cdot\langle m|f''|n\rangle - 2\langle m|f''V|n\rangle - \langle m|f'V'|n\rangle \\ = 0, \tag{3}$$

where the prime means $\frac{d}{dr}$. We note that, in general, (3) ask us to know the energy spectrum corresponding to potential $V(r)$. Here we consider the Morse interaction [2,12] which represents an approximation to vibrational motion of a diatomic molecule.

$$V(r) = D(e^{-2au} - 2e^{-au}), \quad u = r - r_0, \tag{4}$$
$$E_n = -\frac{a^2}{8}b^2, \ b = k - 2n - 1, \ k = \frac{2}{a}\sqrt{2D},$$

such that D is the dissociation energy (well depth), r_0 is the nuclear separation, and a is a parameter associated with the well width, being $\frac{a}{2\pi}\sqrt{2D}$ the frequency of small classical vibrations around r_0.

Now we shall make examples for particular functions $f(r)$ to illustrate how the HT (3) gives information on matrix elements.

I. $f(r) = r - r_0$.

Then from (3) and (4) we deduce that:

$$2aD\langle m|e^{-au} - e^{-2au}|n\rangle = (E_m - E_n)^2\langle m|u|n\rangle, \tag{5}$$

with two cases:

a). $m = n$.

Thus (5) implies an identity for diagonal elements:

$$\langle n|e^{-au}|n\rangle = \langle n|e^{-2au}|n\rangle, \tag{6}$$

and we observe that here (6) was obtained without the explicit knowledge of ψ_m and En: In Sec. 3 we shall employ the formulae of [3,4] to show (6) with $\langle n|e^{-au}|n\rangle = \frac{1}{k}(k-2n-1)$, which was demonstrated by Huffaker-Dwivedi [9] with the factorization method.

b). $m < n$.

The relation (5) leads to:

$$\langle m|u|n\rangle = 2aD(E_m - E_n)^{-2}\langle m|e^{-au} - e^{-2au}|n\rangle, \tag{7}$$

which means that all elements $\langle m|u|n\rangle$ are determined if we know $\langle m|e^{-\beta au}|n\rangle$, $\beta = 1, 2$. From the expressions of [3,4] we have the values for $m \leq n$:

$$\langle m|e^{-au}|n\rangle = \frac{(-1)^{n+m}}{k}\left[\frac{b_1 b_2 n!\Gamma(k-n)}{m!\Gamma(k-n)}\right]^{\frac{1}{2}},\ b_1 = k-2n-1,\ b_2 = k-2m-1, \tag{8}$$
$$\langle m|e^{-2au}|n\rangle = \frac{1}{k}[(n+1)(k-n) - m(k-m-1)]\langle m|e^{-au}|n\rangle,$$

where Γ denotes the gamma function, then (7) and (8) imply the result:

$$\langle m|u|n\rangle = \frac{k}{a}[(m-n)(k-n-m)]^{-1},\ m < n \tag{9}$$

deduced analytically by Gallas [13] without the HT, he uses explicitly ψ_n and the following non-trivial identity:

$$\frac{m!\Gamma(k-m)}{n!\Gamma(k-n)}\sum_{j=0}^{m}\frac{(n-m+j-1)!\Gamma(k-n-m+j-1)}{j!\Gamma(k-2m+j)} = [(n-m)(k-n-m-1)]^{-1}, \tag{10}$$

in this work we not need (10). The equation (6) also is consequence from (8) when $m = n$. In [8] it is proved (9) via ladder operators.

II. $f(r) = e^{-\gamma a(r-r_0)}$, $\gamma = 1, 2, \cdots$

Therefore from (3) and (4) we obtain that:

$$\left[(E_m - E_n)^2 + \frac{\gamma^4 a^4}{4} + \gamma^2 a^2(E_m + E_n)\right]\langle m|e^{-\gamma au|n}\rangle - 2\gamma Da^2(1+\gamma). \tag{11}$$
$$\langle m|e^{-(\gamma+2)au}|n\rangle + 2\gamma Da^2(1+2\gamma)\langle m|e^{-(\gamma+1)au}|n\rangle = 0,$$

thus, by example, if $\gamma = 1$ the (11) gives us:

$$\left[(E_m - E_n)^2 + \frac{a^4}{4} + a^2(E_m + E_n)\right]\langle m|e^{-au}|n\rangle - 4Da^2 \cdot \langle m|e^{-3au}|n\rangle +$$
$$6Da^2\langle m|e^{-2au}|n\rangle = 0, \tag{12}$$

and if we put (4) and (8) into (12) it results the relation:

$$\langle m|e^{-3au}|n\rangle = k^{-2}\left\{(n+1)(k-n)\left[\frac{1}{2}(n+2)(k-n+1) - m(k-m-1)\right] + \frac{m}{2}(m-1)(k-m-1)(k-m-2)\right\}\langle m|e^{-au}|n\rangle, \tag{13}$$

besides it also can be deduced from expressions of [3,4]; (13) shows that $\langle m|e^{-3au}|n\rangle$ is determined if we have the elements (8), which too give us $\langle m|e^{-4au}|n\rangle$ when $\gamma = 2$ into (11), etc., then it is evident the usefulness of the HT.

§3. Matrix elements $\langle e^{-\gamma au} \rangle$

Here we exhibit general formulae for $\langle m|e^{-\gamma a(r-r_0)}|n\rangle$, $\gamma = 1, 2, \cdots$ In fact, Vasan-Cross [3] use analytical techniques to determine these elements, obtaining thus the following expression for $m \leq n$:

$$\langle m|e^{-\gamma au}|n\rangle = \frac{(-1)^{m+n}}{k^\gamma}\left[\frac{b_1 b_2 m!\Gamma(k-m)}{n!\Gamma(k-n)}\right]^{\frac{1}{2}} \cdot \sum_{j=0}^{m} \frac{(-1)^j (n+\gamma-1-j)!\Gamma(k-n-1+\gamma-j)}{j!(m-j)!(\gamma-1-j)!\Gamma(k-m-j)}, \quad (14)$$

which permits to verify (6) and (8). On the other hand, Berrondo et al [4] employ the relationship between Morse potential and the two-dimensional harmonic oscillator to deduce the corresponding relation:

$$\langle m|e^{-\gamma au}|n\rangle = \frac{(-1)^{m+n}}{k^\gamma}\left[\frac{b_1 b_2 m! n!}{\Gamma(k-m)\Gamma(k-n)}\right]^{\frac{1}{2}} \cdot \sum_{j=0}^{m} \begin{pmatrix} m-n+\gamma-1 \\ j \end{pmatrix} \begin{pmatrix} n-m+\gamma-1 \\ \gamma-1-j \end{pmatrix} \frac{\Gamma(k-n-1+\gamma-j)}{(m-j)!}, \quad (15)$$

which also reproduces (6) and (8), that is, we have the equality of (14) and (15) for $\gamma = 1, 2$, however, yet it is an open problem [5] to show that both expressions are totally equivalent for any γ.

§4. Hydrogenlike atom as a Morse oscillator

Here we exhibit the result of Lee [6]: The motion of an electron into the Coulomb field generated by a nucleus with charge Ze, is equivalent to the vibrational dynamics of a MO.

It is very well known [11] that the radial wave function $\frac{1}{r} g_{nl}$ satisfies the Schrödinger equation (in natural units $\hbar = m = 1$):

$$-\frac{1}{2}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right] g_{nl} - \frac{Ze^2}{4\pi\varepsilon_0 r} g_{nl} = -\frac{Z^2 e^2}{32\pi^2 \varepsilon_0^2 n^2} g_{nl}, \quad (16)$$

where $n = 1, 2, \cdots$ and $l = 0, 1, \cdots, n-l$. Now the quantities r, g_{nl} are changed to u, ψ_N via the Langer transformation [6,7]:

$$r = bn^2 e^{-u}, \quad g_{nl} = \frac{e^{-\frac{u}{2}}}{\left[bn\left(l+\frac{1}{2}\right)\right]^{\frac{1}{2}}} \psi_N(u), \quad (17)$$

with $b = \frac{4\pi\varepsilon_0}{Ze^2}$, then (16) adopts the form:

$$-\frac{1}{2}\frac{d^2}{du^2}\psi_N + D(d^{-2u} - 2e^{-u})\psi_N = E\psi_N, \quad (18)$$

which is the Schrödinger equation for a MO [3,9] with parameters:

$$a=1, \quad D=\frac{n^2}{2}, \quad k=\frac{2}{a}\sqrt{2D}=2n,$$
$$N=n-l-1, \quad E=-\frac{1}{8}(k-2N-1)^2=-\frac{1}{2}\left(l+\frac{1}{2}\right)^2, \tag{19}$$

thus each n generates one MO with width $a=1$, depth $D=\frac{n^2}{2}$ and vibrational frequency $\frac{a}{2\pi}\sqrt{2D}=\frac{n}{2\pi}$. Finally, the value of l determines the eigenstate ψ_N, $N=n-l-1$, with energy $E=-\frac{1}{2}\left(l+\frac{1}{2}\right)^2$.

§5. Matrix elements for the Coulomb potential

Other aim of our work is the calculation of the matrix elements:

$$\langle nl_2|r^{\tilde{k}}|nl_1\rangle=\int_0^\infty g_{nl_2}r^{\tilde{k}}g_{nl_1}dr, \quad \tilde{k}=integer\geq -2. \tag{20}$$

The factorization method [8-10] calculates (20) using ladder operators for the proper states $g_n l$; the analytical approach [11] employs the explicit expression of g_{nl} and determines directly the integral (20). Here we apply the Langer transformation [6,7] to obtain (20) via the relationship between the Coulomb and Morse interactions.

In fact, if we put (17) into (20):

$$\langle nl_2|r^{\tilde{k}}|nl_1\rangle=n^{2\tilde{k}+1}b^{\tilde{k}}\left[\left(l_1+\frac{1}{2}\right)\left(l_2+\frac{1}{2}\right)\right]^{-\frac{1}{2}}\langle N_2|e^{-\gamma u}|N_1\rangle, \tag{21}$$

with $N_j=n-l_j-1$, $j=1,2$ and $\gamma=\tilde{k}+2=0,1,2,\cdots$, which means that any $r^{\tilde{k}}$ for the Coulomb potential is proportional to a matrix element of the corresponding MO. The elements $\langle e^{-\gamma u}\rangle$ are determined in (14):

$$\langle N_2|e^{-\gamma u}|N_1\rangle = \frac{(-1)^{N_1+N_2}}{k^\gamma}\left[\frac{b_1b_2N!_2\Gamma(k-N_2)}{N!_1\Gamma(k-N_1)}\right]^{\frac{1}{2}}\cdot \sum_{j=0}^{N_2}\frac{(-1)^j\Gamma(N_1+\gamma-j)\Gamma(k-N_1+\gamma-j)}{j!(N_2-j)\Gamma(k-N_2-j)\Gamma(\gamma-j)}, \tag{22}$$

where $b_c=k-2N_c-1$, $c=1,2$, and without loss of generality we have accepted $N_1\geq N_2$ (that is $l_2\geq l_1$). Then (21) and (22) with $k=2n$ imply the exact expression:

$$\langle nl_2|r^{\tilde{k}}|nl_1\rangle = \frac{(-1)^{l_1+l_2}}{2n}\left(\frac{bn}{2}\right)^{\tilde{k}}\left[\frac{(n-l_2-1)!(n+l_2)!}{(n-l_1-1)!(n+l_1)!}\right]^{\frac{1}{2}}\cdot \sum_{j=0}^{n-l_2-1}\frac{(-1)^j(n+\tilde{k}-l_1-j)!(n+\tilde{k}+l_1-j+1)!}{j!(n-l_2-j)!(n+l_2-j)!(\tilde{k}+1-j)!}, \tag{23}$$

which is not explicity in the literature, and it is more simple than the corresponding relation deduced in [10] using factorization techniques. Special applications of (21) and (23) are:

a). $\tilde{k} = -2$.

In this case $\gamma = 0$, then from (21) it is immediate the proportionality:

$$\langle nl_2 | r^{\tilde{k}} | nl_1 \rangle \propto \langle N_2 | N_1 \rangle = \delta_{N_1 N_2}, \tag{24}$$

therefore only if $l_1 = l_2$ we have $\langle r^{-2} \rangle \neq 0$, which is the result of Pasternack-Sternheimer mentioned in [10].

b). $l_1 = l_2 = l$, $\tilde{k} = \pm 1, \pm 2$.

The general expression (23) reproduces easily the following particular examples of Landau-Lifshitz [11]:

$$\langle r^{-2} \rangle = \frac{b^{-2}}{n^3 \left(1 + \frac{1}{2}\right)}, \quad \langle r \rangle = \frac{b}{2}[3n^2 - l(l+1)],$$
$$\langle r^{-1} \rangle = \frac{b^{-1}}{n^2}, \quad \langle r^2 \rangle = \frac{b^2 n^2}{2}[5n^2 + 1 - 3l(l+1)]. \tag{25}$$

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On s^*-Supplemented Subgroups of Finite Groups[1]

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Abstract A subgroup H of a group G is said to be s^*-supplemented subgroups in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$, where H_{SG} is the largest subnormal subgroup of G contained in H. In this paper we determine the structure of finite groups with s^*-supplemented primary subgroups,and obtain some new results about $p-$ nilpotent groups.

Keywords $P-$nilpotent groups, primary subgroups, s^*-supplemented subgroups, solvable group.

§1. Introduction

A subgroup H of a group G is said to be supplemented in G, if there exists a subgroup K of G such that $G = HK$. Furthermore, a subgroup H of G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. It is obvious that the existence of supplements for some families of subgroups of a group gives a lot of information about its structure. For instance, Kegel [1-2] showed that a group G is soluble if every maximal subgroup of G either has a cyclic supplement in G or if some nilpotent subgroup of G has a nilpotent supplement in G. Hall[3] proved that a group G is soluble if and only if every Sylow subgroup of G is complemented in G. Arad and Ward [4] proved that a group G is soluble if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented in G. More recently, A. Ballester-Bolinches and Guo Xiuyun[5] proved that the class of all finite supersoluble groups with elementary abelian Sylow subgroups is just the class of all finite groups for which every minimal subgroup is complemented.Yanming Wang[6] defined a new concept, $c-$supplementation, which is a generalization of $c-$normality and complement. Applying it, the supersolvablity of G and some related results were got. In 2003, Zhang Xinjian, Guo Wenbin and Shum K. P.[7] defined another concept: $s-$normal subgroup, which is a generalization of normality and $c-$normality.

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In this paper,we remove the $c-$supplement condition and replace the $c-$ normality assumption with $s-$ normality assumption for the some primary subgroups of G, We obtain a series of new results for the $p-$ nilpotency of finite groups.

Definition 1.1. We say a subgroup H of a group G is s^*-supplemented (in G) if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$, where H_{SG} is the largest subnormal subgroup of G contained in H.We say that K is an s^*-supplement of H in G.

Recall that a subgroup H of a group G is called $s-$normal in a group G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$,where H_{SG} is the largest subnormal subgroup of G contained in H. It is easy to see that suppose H is an s^*-supplemented subgroup of G, K is an s^*-supplement of H in G. If $K = G$, then H must be a subnormal subgroup of G.

All groups considered in this paper are finite.Our notation is standard and can be found in [8] and [9]. We denote that G is the semi-product of subgroup H and K by $G = [H]K$, where H is normal in G.

Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is a $S-$closed if any subgroup K of G is in $\mathcal{F}$ when $G \in \mathcal{F}$. We call $\mathcal{F}$ a formation provide that (1) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$ for normal subgroups M and N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. As we all know, the class of all $p-$nilpotent groups is a saturated formation.

§2. Preliminaries

For the sake of easy reference, we first give some basic definitions and known Results from the literature.

Lemma 2.1. Let G be a group and N a normal subgroup of G. Then the following statements hold.

(1) If $H \leq M \leq G$ and H is s^*-supplemented in G, then H is s^*-supplemented in M.

(2) If N is contained in H and H is s^*-supplemented in G if and only if H/N is s^*-supplemented in G/N.

(3) Let π be a set of primes. If N is a normal $\pi'-$subgroup and let A be a $\pi-$subgroup of G, if A is s^*-supplemented in G then AN/N is s^*-supplemented in G/N. If furthermore N normalizes A, then the converse also hold.

(4) Let $H \leq G$ and $L \leq \Phi(H)$. If L is s^*-supplemented in G, then $L \lhd \lhd G$ and $L \leq \Phi(G)$.

Proof. (1) If $HK = G$ with $H \cap K \leq H_{SG}$, then $M = M \cap G = H(M \cap K)$ and $H \cap (K \cap M) \leq H_{SG} \cap M \leq H_{SM}$. So H is s^*-supplemented in M.

(2) Suppose that H/N is s^*-supplemented in G/N. Then there exists a subgroup K/N of G/N such that $G/N = (H/N)(K/N)$ and $(H/N) \cap (K/N) \leq (H/N)_{S(G/N)}$. It is easy to see that $G = HK$ and $H \cap K \leq H_{SG}$.

Conversely, if H is s^*-supplemented in G, then there exists $K \leq G$ such that $G = HK$ and $H \cap K \leq H_{SG}$. It is easy to check that KN/N is s^*-supplemented of H/N in G/N.

(3) If A is s^*-supplemented in G, then there exists $K \leq G$ such that $G = AK$ and $A \cap K \leq A_{SG}$. Since $|G|_{\pi'} = |K|_{\pi'} = |KN|_{\pi'}$, we have that $|K \cap A|_{\pi'} = |N|_{\pi'} = |N|$ and

hence $N \leq K$. It is clear that $(AN/N)(K/N) = G/N$ and $(AN/N) \cap (K/N) = (A \cap K)N/N \leq (A_{SG}N/N) \leq (AN/N)_{S(G/N)}$. Hence AN/N is s^*-supplemented in G/N.

Conversely, assume that AN/N is s^*-supplemented in G/N. Let K/N be s^*-supplement of AN/N. Then $AK = ANK = G$ and $(A \cap K)N/N \leq L/N = ((AN)/N)_{S(G/N)}$. By hypothesis, $NA = N \times A$. This means NA is a both $\pi-$nilpotent and $\pi-$closed and $A_1 = A \cap L$ hence $L = A_1 \times N$ with $A_1 = A \cap L$ and $A_1 \lhd \lhd G$. Now we have $A \cap K \leq A_1 \leq A_{SG}$ and A is s^*-supplemented in G.

(4) In fact, if L is s^*-supplemented in G with supplement K, then $LK = G$ and $L \cap K \leq L_{SG}$. Now $H = H \cap G = L(H \cap K) = H \cap K$ since $L \leq \Phi(H)$. Therefore $L = (H \cap K) \cap L = (H \cap L) \cap K = L \cap K \leq L_{SG}$ and hence $L = L_{SG}$ and $L \lhd \lhd G$. If $L \nleq \Phi(G)$, then there exists a maximal subgroup M of G such that $LM = G$. Now $H = H \cap G = L(H \cap M) = H \cap M \leq M$. Therefore $G = LM \leq HM \leq M < G$, a contradiction.

Lemma 2.2.([14]) Let $\mathcal{F}$ be an $S-$closed local formation and H a subgroup of G. Then $H \cap Z_{\mathcal{F}}(G) \subseteq Z_{\mathcal{F}}(H)$.

Lemma 2.3.([11,Lemma 4.1]) Let p be the smallest prime dividing the order of the group H and P a Sylow $p-$subgroup of H. If $|P| \leq p^2$ and H is A_4-free. then H is $p-$nilpotent.

Lemma 2.4. Let G be an A_4-free group and p be the smallest prime dividing the order of the G. If G/L is $p-$ nilpotent and p^3 not dividing the order of the subgroup L, then G is $p-$ nilpotent.

Proof. By hypothesis and lemma 2.3, We know that L is $p-$nilpotent and L has a normal $p-$complement $L_{p'}$.Since $L_{p'}$ char L and L is normal in G, We have that $L_{p'} \trianglelefteq G$. Therefore $G/L \cong (G/L_{p'})/(L/L_{p'})$is $p-$nilpotent. There exists a Hall $p'-$subgroup $(H/L_{p'})/(L/L_{p'})$ of $(G/L_{p'})/(L/L_{p'})$ and $H/L_{p'} \trianglelefteq G/L_{p'}$. By Schur-Zassenhaus Theorem, we have that $H/L_{p'} = [L/L_{p'}]H_1/L_{p'}$, where $H_1/L_{p'}$ is a Hall $p'-$subgroup of $H/L_{p'}$. Then by lemma 2.3, we have $H_1/L_{p'} \trianglelefteq H/L_{p'}$ and so $H_1/L_{p'}$ char $H/L_{p'} \trianglelefteq G/L_{p'}$. Therefore $H_1/L_{p'} \trianglelefteq G/L_{p'}$ and hence $G/L_{p'}$ is $p-$nilpotent. Thus G is $p-$nilpotent.

Lemma 2.5.([13]) Suppose that G is a group which is not $p-$nilpotent but whose proper subgroups are all $p-$nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.6.([13]) Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then

1) G has a normal Sylow $p-$subgroup P for some prime p and $G/P \cong Q$,where Q is a non-normal cyclic $q-$subgroup for some prime $q \neq p$.

2) $P/\Phi(P)$is a minimal normal subgroup of $G/\Phi(P)$.

3) If P is non-abelian and $p \neq 2$, then the exponent of P is p.

4) If P is non-abelian and $p = 2$, then the exponent of P is 4.

5) If P is abelian, then P is of exponent p.

Lemma 2.7.([12]) Let K be a subgroup of G. If $|G : K| = p$, where p is the smallest prime divisor of $|G|$, then $K \trianglelefteq G$.

Lemma 2.8.([15]) Let P be an elementary abelian $p-$group of order p^n, where p is a prime. Then $|Aut(P)| = k_n \cdot p^{n(n-1)/2}$, here $k_n = \prod_{i=1}^{n}(p^i - 1)$.

Lemma 2.9. Let G is a group which is not $p-$nilpotent but whose proper subgroups are all $p-$nilpotent. $N \trianglelefteq G$ and P is a normal Sylow $p-$subgroup of G. If G/N is $p-$nilpotent, then $P \leq N$.

Proof. By Lemma 2.6, we have that $G = PQ_0$, where Q_0 is a non-normal cyclic Sylow $q-$subgroup. If P is not a subgroup of N,by $p-$nilpotenty of G/N, there exists a normal $p-$complemente, QN/N say, here Q is a Sylow $q-$subgroups of G, such that $G/N = (PN/N)(QN/N)$. Now we have $QN < G$ since P is not a subgroup of N, then QN is nilpotent by Lemma 2.5 and Q char QN, furthermore $Q \trianglelefteq G$. a contradiction.thus $P \leq N$.

Lemma 2.10.([9;10.1.9]) Let p be the smallest prime dividing the order of the finite group G. Assume that G is not $p-$nilpotent. Then the Sylow $p-$subgroups of G are not cyclic. Moreover $|G|$ is divisible by p^3 or by 12.

Lemma 2.11.([12]) Let G be a finite group, P is a $p-$subgroup of G but not a Sylow $p-$subgroup of G, then $P < N_G(P)$.

§3. Main results

Theorem 3.1. Let G be an A_4-free group and p be the smallest prime dividing the order of the group G. If there exists a normal subgroup N in G such that G/N is $p-$nilpotent and each subgroups of N of order p^2 is s^*-supplemented in G, then G is $p-$nilpotent.

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order.

By Lemma 2.4 and hypothesis, we have $|N|_p > p^2$. Let L be a proper subgroup of G. We prove that conditions of the theorem 3.1 is inherited by L. Clearly, $L/L \cap N \cong LN/N \leq G/N$ implies that $L/L \cap N$ is $p-$nilpotent. If $|L \cap N|_p \leq p^2$, then L is $p-$nilpotent by Lemma 2.4. If $|L \cap N|_p > p^2$, then each subgroup of $L \cap N$ of order p^2 is s^*-supplemented in G and hence is s^*-supplemented in L. Then L is $p-$nilpotent by induction. Thus G is a minimal non-$p-$nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.6, we have $G = PQ$, where $P \trianglelefteq G$ and Q is a non-normal cyclic Sylow $q-$subgroup of G. It is clear that $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

By Lemma 2.9, we have $P \leq N$. Let $A \leq N$ and $|A| = p^2$. By hypothesis, there exists a subgroup K of G such that $G = AK$ and $A \cap K \leq A_{SG}$, where A_{SG} is the largest subnormal subgroup of G contained in A. If $K < G$, We have $K = K_p \times K_{p'}$ since that G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

(i) If $K_p = 1$, then $P = A$ and hence G is $p-$nilpotent, a contradiction.

(ii) If $K_p \neq 1$, then we consider subgroup $N_G(K_p)$. Since $K \leq N_G(K_p)$ and $K_p < N_G(K_p)$, we have that $|G : N_G(K_p)| = p$ or $N_G(K_p) = G$. If $|G : N_G(K_p)| = p$, then we have $N_G(K_p) < G$. Let $P_1 = P \cap N_G(K_p)$, Then $P_1 \trianglelefteq G$ since $G = PQ$ and $Q \leq N_G(K_p)$. If $P_1 \leq \Phi(P)$, then P=$P \cap AN_G(K_p)$=$A(P \cap N_G(K_p))$=A, a contradiction. If$P_1 \nleq \Phi(P)$, then $P_1\Phi(P)/\Phi(P)$=$P/\Phi(P)$ by Lemma 2.6. In this case, P=P_1 and hence $N_G(K_p)$=G, a contradiction. If$|G : N_G(K_p)| = 1$, then $K_p \trianglelefteq G$. We consider the factor group G/K_p. Since $G = AK$ and $|A| = p^2$, we have $p^3 \nmid |G/K_p|$. By lemma 2.4, $G/K_p$is a $p-$nilpotent and $P \leq K_p$, we know $P = K_p$. Furthermore, we have $G = AK = K$ via $A \leq K_p \leq K$, a contradiction.

Now we let $K = G$. It is easy to see $A \lhd \lhd G$. By $A \leq P$, we have $A\Phi(P)/\Phi(P) \leq P/\Phi(P)$. Furthermore, $A\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$, where $P/\Phi(P)$ is either a minimal normal subgroup or characteristic subgroup of $G/\Phi(P)$. So we have $A\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and $A \subseteq \Phi(P)$ or $A = P$. If $A = P$, then $|A| = |P| = p^2$ and $|N|_p \leq p^2$ here because $P \leq N$, a contradiction to $|N|_p > p^2$. If $A \subseteq \Phi(P)$, considering A be a arbitrary abelian group, we have $\Phi(P) \geq P$, hence $\Phi(P) = P$, a contradiction.

The final contradiction completes our proof.

Corollary 3.2. Let G be an A_4-free group and p be the smallest prime dividing the order of the group G. If each subgroup of G of order p^2 is s^*−supplemented in G, then G is p−nilpotent.

Theorem 3.3. Suppose that G is a group with a normal subgroup H such that G/H is p−nilpotent for some prime divisor p of $|G|$. If every cyclic subgroup of H of order 4 is s^*−supplemented in G and every subgroup H of order p is contained in $Z_{\mathcal{F}}(G)$, where $\mathcal{F}$ is the class of all p−nilpotent groups, then G is p−nilpotent.

Proof. Assume that the claim is false and choose G to be a counterexample of smallest order.

The hypothesis is inherited by all proper subgroups of G. Then $K/K \cap H \cong KH/H \leq G/H$ implies that $K/K \cap H$ is p−nilpotent. Every cyclic subgroups of $K \cap H$ of order 4 is s^*−supplemented in K by Lemma 2.1. Every subgroup of $H \cap K$ of order p is contained in $K \cap Z_{\mathcal{F}}(G) \leq Z_{\mathcal{F}}(K)$ by Lemma 2.2. Thus G is a minimal non-p-nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.6, G has a normal Sylow p−nilpotent subgroup P and $G/P \cong Q$, where Q is a non-normal cyclic Sylow q−subgroup of G, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. We consider the follow cases.

Case 1. P is abelian. By Lemma 2.6, P is an elementary abelian p−group. Since G/H is p−nilpotent, we have $P \leq H$. By hypothesis, every subgroup of H of order p is contained in $Z_{\mathcal{F}}(G)$, then $P \leq Z_{\mathcal{F}}(G)$ and hence G is p−nilpotent, a contradiction.

Case 2. P is non-abelian and $p > 2$. By Lemma 2.6, the expotent of P is p and every subgroup of H of order p is contained in $Z_{\mathcal{F}}(G)$. Therefore $P \leq Z_{\mathcal{F}}(G)$ and we have that G is p−nilpotent, a contradiction.

Case 3. P is non-abelian and $p = 2$. Let A be a cyclic subgroup of H of order 4. By hypothesis, A is s^*−supplemented in G and there exist a subgroup L of G such that $G = AL$ and $A \cap L \leq A_{SG}$. Since $L < G$ and L is nilpotent, we have $L = L_p \times L_{p'}$. If $L_p = 1$, then $P = A$ and P is abelian subgroup. a contradiction. Now we have $L_p \neq 1$ and we consider $N_G(L_p)$. Since $L \leq N_G(L_p)$, we have that $|G : N_G(L_p)| = 2$ or $|G : N_G(L_p)| = 1$. If $|G : N_G(L_p)| = 2$, then $N_G(L_p) \trianglelefteq G$ by Lemma 2.7 and hence G is 2-nilpotent, a contradiction. If $|G : N_G(L_p)| = 1$, then $L_p \trianglelefteq G$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, we have $P{=}L_p$ or $L_p \leq \Phi(P)$. If $P{=}L_p$, then $L = G$ since $G = AL$, a contradiction. If $L_p \leq \Phi(P)$, then $P{=}AL_p{=}A$, a contradiction.

Now we let $K = G$. It is easy to see $A \lhd \lhd G$. By $A \leq P$, We have $A\Phi(P)/\Phi(P) \leq P/\Phi(P)$. Furthermore,$A\Phi(P)/\Phi(P) \trianglelefteq P/\Phi(P)$, where $P/\Phi(P)$ is either a minimal normal subgroup or characteristic subgroup of $G/\Phi(P)$.So we have $A\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and $A \subseteq \Phi(P)$ or

$A = P$. If $A = P$, then $|A| = |P| = p^2$ and $|N|_p \leq p^2$ here because $P \leq N$, a contradiction to $|N|_p > p^2$. If $A \subseteq \Phi(P)$, considering A be a arbitrary abelian group, we have $\Phi(P) \geq P$, hence $\Phi(P) = P$, a contradiction.

The final contradiction completes our proof.

Corollary 3.4. Let G be a group and p be the prime devisor of $|G|$. If every cyclic subgroup of order 4 is s^*-supplemented in G and every subgroup H of order p is contained in $Z_{\mathcal{F}}(G)$, where $\mathcal{F}$ is the class of all $p-$nilpotent groups, then G is $p-$nilpotent.

Theorem 3.5. Let G is a group and $(|G|, 21) = 1$. If each subgroup of G of order 8 is s^*-supplemented in G, then G is 2-nilpotent.

Proof. Assume that the theorem is false and choose G to be a counterexample of smallest order. Let 2^α be the order of a Sylow 2-subgroup P of G.

If $2 \nmid |G|$, then G is 2-nilpotent. If P is cyclic, then G is 2-nilpotent by Lemma2.10. So we can suppose that P is not cyclic. Let L be a proper subgroup of G. We prove that L inherits the condition of the theorem. If $8 \nmid |L|$, then L is 2-nilpotent by Lemma 2.3. If $8||L|$, then each subgroup of L of order 8 is s^*-supplemented in G and hence is s^*-supplemented in L by Lemma 2.1, so L is 2-nilpotent by induction. Thus we may assume that G is a minimal non-2-nilpotent group. Now Lemma 2.5 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.6, we have $G = PQ$, where P is a normal in G and Q is a non-normal cyclic Sylow $q-$subgroup of $G(q \neq p)$, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

Let H be a subgroup of G of order 8. By hypothesis, there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{SG}$, where H_{SG} is the largest subnormal subgroup of G contained in H.

First, we claim that $H \neq P$. Otherwise $|P| = |H| = 8$. If $\Phi(P) \neq 1$, then $G/\Phi(P)$ is 2-nilpotent by Lemma 2.3 and hence G is 2-nilpotent and $P \leq \Phi(P)$ by Lemma 2.9, a contradiction.It follows from $\Phi(P) = 1$ and Lemma 2.6 we have that P is an elementary abelian 2-group. Next we consider $N_G(P)/C_G(P)$. It is clear that $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P)$. By Lemma 2.8 we have $|Aut(P)| = 8 \cdot 7 \cdot 3$. Since $(|G|, 21) = 1$, we see that $N_G(P) = C_G(P)$ in this case. Then, by Burnside Theorem(cf,[2]), we have that G is 2-nilpotent, a contradiction.

If $K < G$ and G is a group which is not nilpotent but whose proper subgroups are all nilpotent, we have K is nilpotent and $K = K_2 \times K_{2'}$. We claim that $K_2 \ntrianglelefteq G$. Otherwise, if $K_2 \trianglelefteq G$, then $P = K_2$ or $K_2 \leq \Phi(P)$. If $P = K_2$, then $H \leq K_2$. We have $G = K$ by $G = HK$. This shows that $P = K_2$ is impossible. If $K_2 \leq \Phi(P)$, then $P = HK_2 = H$, a contradiction.Therefore, we note subgroup $N_G(K_2)$. Since $K \leq N_G(K_2)$ and Lemma 2.9, we have $K_2 \leq N_P(K_2)$ and $|G : N_G(K_2)| = 4$ or $|G : N_G(K_2)| = 2$. If $|G : N_G(K_2)| = 2$, then $N_G(K_2)$ is a nilpotent normal subgroup of G by Lemma 2.7 and G is 2-nilpotent, a contradiction. If $|G : N_G(K_2)| = 4$, then we consider $N_G(N_G(K_2))_2)$. If $(N_G(K_2))_2 \trianglelefteq G$, then $G/(N_G(K_2))_2$ is 2-nilpotent by Lemma 2.3 and we have that $P \leq (N_G(K_2))_2$, a contradiction to $|G : N_G(K_2)| = 4$; Since $(N_G(K_2))_2 < N_p(N_G(K_2))_2)$ by Lemma 2.11, We may assume that $|G : N_G((N_G(K_2))_2)| = 2$. By Lemma 2.7 we know that $N_G((N_G(K_2))_2) \trianglelefteq G$. Since $N_G((N_G(K_2))_2)$ is a nilpotent normal subgroup of G, we know that G is 2-nilpotent, a contradiction.

If $K = G$, we have $H = H \cap G = H \cap K \leq H_{SG}$, that is, $H = H_G$ and $H \lhd \lhd G$. It is clear that $H \leq P$. So we claim $H\Phi(P)/\Phi(P) \leq P/\Phi(P)$, where $P/\Phi(P)$ is either a minimal normal subgroup or a character subgroup of $G/\Phi(P)$. Furthermore, $H\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Because $H \neq P$, we have $H \subseteq \Phi(P) = Z(P)$. Notice H is arbitrary, we have $\Phi(P) \geq P$, that is, $\Phi(P) = P$, a contradiction.

The final contradiction completes the proof.

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Two equations involving the Smarandache LCM dual function

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Abstract For any positive integer n, the Smarandache LCM dual function $SL^*(n)$ is defined as the greatest positive integer k such that $[1,\ 2,\ \cdots,\ k]$ divides n. The main purpose of this paper is using the elementary method to study the number of the solutions of two equations involving the Smarandache LCM dual function $SL^*(n)$, and give their all positive integer solutions.

Keywords Smarandache LCM dual function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. That is,

$$SL(n) = \min\{k:\ k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1] and [2]. For example, Murthy [1] proved that if n is a prime, then $SL(n) = S(n)$, where $S(n) = \min\{m:\ n|m!,\ m \in N\}$ be the F.Smarandache function. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$. Zhongtian Lv [3] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n\le x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Now, we define the Smarandache LCM dual function $SL^*(n)$ as follows:

$$SL^*(n) = \max\{k :\ k \in N,\ [1,\ 2,\ \cdots,\ k] \mid n\}.$$

It is easy to calculate that $SL^*(1) = 1$, $SL^*(2) = 2$, $SL^*(3) = 1$, $SL^*(4) = 2$, $SL^*(5) = 1$, $SL^*(6) = 3$, $SL^*(7) = 1$, $SL^*(8) = 2$, $SL^*(9) = 1$, $SL^*(10) = 2$, $\cdots$. Obviously, if n is an odd number, then $SL^*(n) = 1$. If n is an even number, then $SL^*(n) \geq 2$. About the other elementary properties of $SL^*(n)$, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we use the elementary method to study the number of the solutions of two equations involving the Smarandache LCM dual function $SL^*(n)$. For further, we obtain all the positive numbers n, such that

$$\sum_{d|n} SL^*(d) = n \tag{2}$$

or

$$\sum_{d|n} SL^*(d) = \phi(n), \tag{3}$$

where $\sum_{d|n}$ denotes the summation over all positive divisors of n. That is, we shall prove the following two conclusions:

Theorem 1. The equation (2) has only one and only one solution $n = 1$, and $\sum_{d|n} SL^*(d) > n$ is true if and only if $n = 2,\ 4,\ 6,\ 12$.

Theorem 2. The equation (3) is true if and only if $n = 1,\ 3,\ 14$.

§2. Some lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. (a) For any prime p and any real number $x \geq 1$, we have $p^x \geq x + 1$, and the equality is true if and only if $x = 1,\ p = 2$.

(b) For any odd prime p and any real number x, if $x \geq 2$, then we have $p^x > 2(x + 1)$; If $x \geq 3$, then we have $p^x > 4(x + 1)$.

(c) For any prime $p \geq 5$ and any real number $x \geq 2$, we have $p^x > 4(x + 1)$.

(d) For any prime $p \geq 11$ and any real number $x \geq 1$, we have $p^x > 4(x + 1)$.

Proof. We only prove case (a), others can be obtained similarly.

Let $f(x) = p^x - x - 1$, if $x \geq 1$, then

$$f^{'}(x) = p^x \ln p - 1 > p \ln e^{\frac{1}{2}} - 1 = \frac{p}{2} - 1 \geq 1.$$

That is to say, $f(x)$ is a monotone increasing function if $x \in [1,\ \infty)$. So $f(x) \geq f(1) \geq 0$, namely $p^x \geq x + 1$, and $p^x = 4(x + 1)$ is true if and only if $x = 1,\ p = 2$. This complete the proof of case (a).

Lemma 2. For all odd positive integer number n,

(a) the equation $d(n) = \phi(n)$ is true if and only if $n = 1, 3$;

(b) the inequality $8d(n) > \phi(n)$ is true if and only if $n = 1, 3, 5, 7, 9, 11, 13, 15, 21, 27, 33, 35, 39, 45, 63, 105$, where $d(n)$ is the divisor function of n, $\phi(n)$ is the Euler function.

Proof. Let $H(n) = \frac{\phi(n)}{d(n)}$, then the equation $d(n) = \phi(n)$ is equivalent to $H(n) = 1$ and $8d(n) > \phi(n)$ is equivalent to $H(n) < 8$. Because $d(n)$ and $\phi(n)$ are multiplicative functions, hence $H(n)$ is multiplicative. Assume that p, q are prime numbers and $p > q$, then $H(p) = \frac{p-1}{2} > \frac{q-1}{2} = H(q)$. On the other hand, for any given prime p and integer $k \geq 1$, we have $\frac{H(p^{k+1})}{H(p^k)} = \frac{p(1+k)}{2+k} > \frac{2k+2}{2+k} > 1$. Hence if $k \geq 1$, then $H(p^{1+k}) > H(p^k)$.

Because

$$\begin{aligned}
&H(1) = 1,\ H(3) = 1,\ H(5) = 2,\ H(7) = 3,\ H(11) = 5,\ H(13) = 6,\ H(17) = 8 \geq 8,\\
&H(3^2) = 2,\ H(5^2) = \frac{20}{3} \geq 8,\ H(7^2) = 14 \geq 8,\ H(11^2) = \frac{110}{3} \geq 8,\ H(13^2) = 52 \geq 8,\\
&H(3^3) = \frac{9}{2},\\
&H(3^4) = \frac{54}{5} \geq 8.
\end{aligned}$$

We have $H(1) = 1$, $H(3) = 1$, $H(5) = 2$, $H(7) = 3$, $H(9) = 2$, $H(11) = 5$, $H(13) = 6$, $H(15) = H(3)H(5) = 2$, $H(21) = H(3)H(7) = 3$, $H(27) = H(3^3) = \frac{9}{2}, H(33) = H(3)H(11) = 5$, $H(35) = H(5)H(7) = 6$, $H(39) = H(3)H(13) = 6$, $H(45) = H(3^2)H(5) = 4$, $H(63) = H(3^2)H(7) = 6$, $H(105) = H(3)H(5)H(7) = 6$.
Consequently, for all positive odd integer number n, $H(n) = 1$ is true if and only if $n = 1, 3$; the inequality $H(n) < 8$ is true if and only if $n = 1, 3, 5, 7, 9, 11, 13, 15, 21, 27, 33, 35, 39, 45, 63, 105$.

This completes the proof of Lemma 2.

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. It is easy to see that $n = 1$ is one solution of the equation (2). In order to prove that the equation (2) has no other solutions except $n = 1$, we consider the following two cases:
(a) n is an odd number larger than 1.

Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_s$, $\alpha_i \geq 1$, $i = 1, 2, \cdots, s$. In this case, for any $d|n$, d is an odd number, so $SL^*(d) = 1$. From Lemma 1 (a), we have

$$\sum_{d|n} SL^*(d) = \sum_{d|n} 1 = d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) < p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = n.$$

(b) n is an even number.

Assume that $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = 2^\alpha \cdot m$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_s$,

$\alpha_i \geq 1$, $i = 1, 2, \cdots, s$. In this case,

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha} \sum_{d|m} SL^*(2^i d) \\ &< \sum_{i=0}^{\alpha} 2^{i+1} \sum_{d|m} 1 = (2 + 2^2 + \cdots + 2^{\alpha+1}) d(m) \\ &= (2^{\alpha+2} - 2) d(m) < 2^{\alpha} \cdot 4d(m). \end{aligned} \quad (4)$$

(i) If $p_s \geq 11$, from Lemma 1 (a), we have

$$4d(m) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots 4(\alpha_s + 1) < p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(ii) If there exists $i,\ j \in \{1,\ 2,\ \cdots,\ s\}$ and $i \neq j$ such that $\alpha_i \geq 2,\ \alpha_j \geq 2$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1 + 1) \cdots 2(\alpha_i + 1) \cdots 2(\alpha_j + 1) \cdots (\alpha_s + 1) < p_1^{\alpha_1} \cdots p_i^{\alpha_i} \cdots p_j^{\alpha_j} \cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(iii) If there exists $i \in \{1,\ 2,\ \cdots,\ s\}$ such that $\alpha_i \geq 3$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1 + 1) \cdots 4(\alpha_i + 1) \cdots (\alpha_s + 1) < p_1^{\alpha_1} \cdots p_i^{\alpha_i} \cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(iv) If there exists $i \in \{1,\ 2,\ \cdots,\ s\}$ such that $p_i \geq 5,\ \alpha_i \geq 2$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1 + 1) \cdots 4(\alpha_i + 1) \cdots (\alpha_s + 1) < p_1^{\alpha_1} \cdots p_i^{\alpha_i} \cdots p_s^{\alpha_s} = m.$$

Associated with (4) we also have $\sum_{d|n} SL^*(d) < n$.

From the discussion above we know that if n satisfies the equation (2), then m has only seven possible values. That is $m \in \{1,\ 3,\ 5,\ 7,\ 9,\ 15,\ 21\}$. We calculate the former three cases only, other cases can be discussed similarly.

If $m = 1$, namely $n = 2^{\alpha} (\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha} \sum_{d|1} SL^*(2^i d) \\ &= SL^*(1) + SL^*(2) + SL^*(2^2) + \cdots + SL^*(2^{\alpha}) \\ &= 1 + 2 + 2 + \cdots + 2 = 2\alpha + 1.\end{aligned}$$

$\alpha = 1,\ 2$, namely $n = 2,\ 4$. In this case, $2\alpha + 1 > 2^{\alpha}$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha \geq 3$. In this case, $2\alpha + 1 < 2^{\alpha}$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 3$, namely $n = 2^\alpha \cdot 3,\ (\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|3} SL^*(2^i d)\\ &= \sum_{d|3} SL^*(d) + \sum_{d|3} SL^*(2d) + \sum_{d|3} SL^*(2^2 d) + \cdots + \sum_{d|3} SL^*(2^\alpha d)\\ &= 2+5+6+\cdots+6 = 6\alpha+1.\end{aligned}$$

$\alpha = 1$, namely $n = 6$. In this case, $6\alpha + 1 = 7 > 2\cdot 3$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha = 2$, namely $n = 12$. In this case, $6\alpha + 1 = 13 > 2^2\cdot 3$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha \geq 3$. In this case, $2\alpha + 1 < 2^\alpha$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 5$, namely $n = 2^\alpha \cdot 5,\ (\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|5} SL^*(2^i d)\\ &= \sum_{d|5} SL^*(d) + \sum_{d|5} SL^*(2d) + \sum_{d|5} SL^*(2^2 d) + \cdots + \sum_{d|5} SL^*(2^\alpha d)\\ &= 2+4+4+\cdots+4 = 4\alpha+2.\end{aligned}$$

For any $\alpha \geq 1$, we have $4\alpha + 2 < 2^\alpha \cdot 5$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 7, 9, 15, 21$, using the similar method we can obtain that for any $\alpha \geq 1$, $\sum_{d|n} SL^*(d) < n$ is true.

Hence the equation (2) has no positive even integer number solutions, and $\sum_{d|n} SL^*(d) > n$ is true if and only if $n = 2,\ 4,\ 6,\ 12$.

Associated (a) and (b), we complete the proof of Theorem 1.

At last we prove Theorem 2. From Lemma 2, it is easy to versify that $n = 1,\ 3$ are the only positive odd number solutions of the equation (3). Following we consider the case that n is an even number.

Assume that $n = 2^\alpha \cdot m$, where $2 \dagger m$. In this case,

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|m} SL^*(2^i d) < \sum_{i=0}^{\alpha} 2^{i+1}\sum_{d|m} 1\\ &= (2+2^2+\cdots+2^{\alpha+1})d(m) = (2^{\alpha+2}-2)d(m) < 2^{\alpha-1}\cdot 8d(m),\end{aligned}$$

and $\phi(n) = \phi(2^\alpha m) = \phi(2^\alpha)\phi(m) = 2^{\alpha-1}\phi(m)$. Let

$$S = \{1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 21,\ 27,\ 33,\ 35,\ 39,\ 45,\ 63,\ 105\}.$$

From Lemma 2, if $m \notin S$, then $\phi(m) \geq 8\cdot d(m)$, consequently

$$\sum_{d|n} SL^*(d) < 2^{\alpha-1}\cdot 8d(m) \leq 2^{\alpha-1}\phi(m) = \phi(n).$$

Hence if n satisfies the equation (3), then $m \in S$. We only discuss the cases $m = 1$, 7, other cases can be discussed similarly.

If $m = 1$, namely $n = 2^{\alpha}(\alpha \geq 1)$, then

$$\begin{aligned} & \sum_{d|n} SL^*(d) = \sum_{i=0}^{\alpha}\sum_{d|1} SL^*(2^i d) \\ = \ & SL^*(1) + SL^*(2) + SL^*(2^2) + \cdots + SL^*(2^{\alpha}) \\ = \ & 1 + 2 + 2 + \cdots + 2 = 2\alpha + 1 \end{aligned}$$

and $\phi(n) = \phi(2^{\alpha}) = 2^{\alpha-1}$, $2 \dagger (2\alpha + 1)$, but $2|2^{\alpha-1}$, hence if $m = 1$, then the equation (2) has no solution.

If $m = 7$, namely $n = 2^{\alpha} \cdot 7,\ (\alpha \geq 1)$, then

$$\begin{aligned} & \sum_{d|n} SL^*(d) = \sum_{i=0}^{\alpha}\sum_{d|7} SL^*(2^i d) \\ = \ & \sum_{d|7} SL^*(d) + \sum_{d|7} SL^*(2d) + \sum_{d|7} SL^*(2^2 d) + \cdots + \sum_{d|7} SL^*(2^{\alpha} d) \\ = \ & 2 + 4 + 4 + \cdots + 4 = 4\alpha + 2, \end{aligned}$$

and $\phi(n) = \phi(2^{\alpha} \cdot 7) = 2^{\alpha-1} \cdot 6$, Solving the equation $4\alpha + 2 = 2^{\alpha-1} \cdot 6$, we have $\alpha = 1$. That is to say that $n = 14$ is one solution of the equation (3).

Discussing the other cases similarly, we have that if n is an even number, then the equation (3) has only one solution $n = 14$.

This completes the proof of Theorem 2.

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Periodic solutions of impulsive periodic Competitor-Competitor-Mutualist system[1]

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Abstract In this paper, we consider an impulsive model of mutualism of Lotka-Volterra type which involves interactions among a mutualist-competitor, a competitor and a mutualist. The species grow in a periodically changing environment with instantaneous changes in the population densities. The mathematical model is described by a periodic-impulsive differential equations. A monotone-iterative scheme is established for finding the periodic solution of the model, a set of easily verifiable sufficient conditions are obtained for the existence of at least one strictly positive periodic solutions.

Keywords Impulsive differential equations, competitor-competitor-mutualist system, periodic solution, monotone iteration.

§0. Introduction

In recent years, applications of impulsive differential equations in mathematical ecology have developed rapidly, various mathematical models have been proposed[1,2]. One of the famous models for dynamics of population is Lotka-Volterra system with impulses, there have been some intensive studies on this subject. Periodic solutions of impulsive Lotka-Volterra systems have been investigated by many researchers and various skills and techniques have been developed[3−6]. In this paper, we consider a impulsive periodic model of mutualism of Lotka-Volterra type which involves interactions among a mutualist-competitor u_1, a competitor u_2 and a mutualist u_3. The system of impulsive differential equations governing u_1, u_2 and u_3 are given by

$$\begin{aligned}
&u_1' = u_1(a_1(t) - b_1(t)u_1 - c_1(t)u_2/(1+\sigma_1(t)u_3)),\ t \neq \tau_i,\\
&u_2' = u_2(a_2(t) - b_2(t)u_1 - c_2(t)u_2),\quad t \neq \tau_i,\\
&u_3' = u_3(a_3(t) - c_3(t)u_3/(1+\sigma_3(t)u_1)),\ t \neq \tau_i,\\
&u_k(\tau_i^+) = \delta_{ki}u_k(\tau_i),\ k = 1,2,3,\ i \in N_+,\\
&u_k(0) = u_k(T),\quad k = 1,2,3,
\end{aligned} \tag{0.1}$$

where $t \in R_+ = [0, +\infty)$, $T > 0$ is a constant, $N_+ = \{1, 2, \cdots\}$. For $k = 1, 2, 3$, $i \in N_+$, a_k, b_k, c_k are positive, continuous functions, δ_{ki} are positive constants, τ_i are impulse moments

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with $0 < \tau_1 < \tau_2 \cdots$, $u_k(\tau_i^+) = \lim_{t\to\tau_i+0} u(t)$. Suppose that the system is periodic, that is, a_k, b_k, c_k are periodic with common period T, and there exists a positive integer q such that the $\tau_{i+q} = \tau_i + T$, $\delta_{k,i+q} = \delta_{ki}$.

The first three equations of (0.1) describe the system of a competitor-competitor-mutualist model. It is clear that in the absence of u_3, the above system is reduced to the standard Lotka-Volterra competition model. Equations $u_k(\tau_i^+) = \delta_{ki}u_k(\tau_i)$ $(k = 1, 2, 3)$ describe the jump conditions which reflect the instantaneous changes of the population densities caused by environmental abrupt changes (e.g., seasonal changes, food supplies, harvesting, disasters etc.). The periodic conditions are given by $u_k(0) = u_k(T)$ $(k = 1, 2, 3)$.

The competitor-competitor-mutualist system without impulses has been investigated by many researchers(see[7-10]), but the system with impulses has not been studied too much yet.

The purpose of this paper is to show the existence of periodic solution for system (0.1). In Section 1 we give some preliminaries. In Section 2 we show that the existence of periodic solution are ensured if the system has a pair of ordered upper and lower solutions. And a pair of strictly positive upper and lower solutions are given in Section 3. Finally, concluding remarks are given in Section 4.

According to the real meaning of the model, only the solutions of (0.1) with u_k $(k = 1, 2, 3)$ nonnegative are physically of interest.

§1. Preliminaries

For system (0.1), we have

$$\begin{aligned}
&u_1(t) = u_1(\tau_{i-1}^+)\exp\{\textstyle\int_0^t[a_1(s) - b_1(s)u_1(s) - c_1(s)u_2(s)/(1+\sigma_1(s)u_3(s))]ds\}, \quad t\in(\tau_{i-1},\tau_i],\\
&u_2(t) = u_2(\tau_{i-1}^+)\exp\{\textstyle\int_0^t[a_2(s) - b_2(s)u_1(s) - c_2(s)u_2(s)]ds\}, \quad t\in(\tau_{i-1},\tau_i],\\
&u_3(t) = u_3(\tau_{i-1}^+)\exp\{\textstyle\int_0^t[a_3(s) - c_3(s)u_3(s)/(1+\sigma_3(s)u_1(s))]ds\}, \quad t\in(\tau_{i-1},\tau_i],\\
&u_k(\tau_i^+) = \delta_{ki}u_k(\tau_i), \quad k=1,2,3,\ i\in N_+.
\end{aligned} \tag{1.1}$$

It is seen from (1.1) that $R_+^3 = \{(u_1, u_2, u_3)|u_k \geq 0\ (k = 1, 2, 3)\}$ is flow invariant relative to system (0.1), that is if $(u_1(0), u_2(0), u_3(0)) \in R_+^3$, then the solution $(u_1(t), u_2(t), u_3(t)) \in R_+^3$ for all $t > 0$. Throughout the paper, we only consider solutions with nonnegative components.

Denote by S the set of all the vector functions $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))$ defined in R_+, with nonnegative components continuously differentiable in $R_+ \setminus \{\tau_i\}_{i\geq 0}$. At $t = \tau_i$, $\mathbf{u}(t)$ has jump discontinuities and is left continuous.

A function $\mathbf{u} = (u_1, u_2, u_3)$ is called a solution of (0.1) if it belongs to S and satisfies all the relations in (0.1), $\mathbf{u}$ is a T-periodic solution if it is a solution and satisfies $\mathbf{u}((n-1)T) = \mathbf{u}(nT)$ $(n \in N_+)$.

§2. Monotone Iteration for Periodic Solution

The main condition for the existence problem of periodic solution are the existence of a pair of ordered upper and lower solutions which are defined as follows:

Definition 2.1. Two bounded functions $\tilde{\mathbf{u}} \equiv (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, $\hat{\mathbf{u}} \equiv (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ in S are called a pair of upper and lower solutions of (0.1) if

$$\begin{aligned}
&\tilde{u}_1' \geq \tilde{u}_1(a_1 - b_1\tilde{u}_1 - c_1\hat{u}_2/(1+\sigma_1\tilde{u}_3)), \quad t \neq \tau_i,\\
&\tilde{u}_2' \geq \tilde{u}_2(a_2 - b_2\hat{u}_1 - c_2\tilde{u}_2), \quad t \neq \tau_i,\\
&\tilde{u}_3' \geq \tilde{u}_3(a_3 - c_3\tilde{u}_3/(1+\sigma_3\tilde{u}_1)), \quad t \neq \tau_i,\\
&\tilde{u}_k(\tau_i^+) \geq \delta_{ki}\tilde{u}_k(\tau_i), \quad k = 1,2,3,\ i \in N_+,\\
&\tilde{u}_k(0) \geq \tilde{u}_k(T), \quad k = 1,2,3,
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\hat{u}_1' \leq \hat{u}_1(a_1 - b_1\hat{u}_1 - c_1\tilde{u}_2/(1+\sigma_1\hat{u}_3)), \quad t \neq \tau_i,\\
&\hat{u}_2' \leq \hat{u}_2(a_2 - b_2\tilde{u}_1 - c_2\hat{u}_2), \quad t \neq \tau_i,\\
&\hat{u}_3' \leq \hat{u}_3(a_3 - c_3\hat{u}_3/(1+\sigma_3\hat{u}_1)), \quad t \neq \tau_i,\\
&\hat{u}_k(\tau_i^+) \leq \delta_{ki}\hat{u}_k(\tau_i), \quad k = 1,2,3, \quad i \in N_+,\\
&\hat{u}_k(0) \leq \hat{u}_k(T), \quad k = 1,2,3.
\end{aligned} \tag{2.2}$$

A pair of upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ are said to be ordered if $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$.

In this section, it is assumed that problem (0.1) has a pair of ordered upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$. In this paper, inequalities between two vectors are in the componentwise sense. For notational convenience we set

$$\begin{aligned}
&f_1(t, u_1, u_2, u_3) \equiv u_1(a_1 - b_1u_1 - c_1u_2/(1+\sigma_1u_3)),\\
&f_2(t, u_1, u_2) \equiv u_2(a_2 - b_2u_1 - c_2u_2),\\
&f_3(t, u_1, u_3) \equiv u_3(a_3 - c_3u_3/(1+\sigma_3u_1)).
\end{aligned}$$

For $k = 1,2,3$, let L_k be any fixed nonnegative constants satisfying

$$L_k \geq \max\{-\frac{\partial f_k}{\partial u_k};\ t \in [0,+\infty),\ \hat{u}_k \leq u_k \leq \tilde{u}_k\}.$$

Using $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$, $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ as the initial iteration we construct two sequences $\{\overline{\mathbf{u}}^{(m)}\}, \{\underline{\mathbf{u}}^{(m)}\}$ from the linear initial value problem

$$\begin{aligned}
&\frac{d\overline{u}_1^{(m)}}{dt} + L_1\overline{u}_1^{(m)} = L_1\overline{u}_1^{(m-1)} + f_1(t, \overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}, \overline{u}_3^{(m-1)}), \quad t \neq \tau_i,\\
&\frac{d\overline{u}_2^{(m)}}{dt} + L_2\overline{u}_2^{(m)} = L_2\overline{u}_2^{(m-1)} + f_2(t, \underline{u}_1^{(m-1)}, \overline{u}_2^{(m-1)}), \quad t \neq \tau_i,\\
&\frac{d\overline{u}_3^{(m)}}{dt} + L_3\overline{u}_3^{(m)} = L_3\overline{u}_3^{(m-1)} + f_3(t, \overline{u}_1^{(m-1)}, \overline{u}_3^{(m-1)}), \quad t \neq \tau_i,\\
&\overline{u}_k^{(m)}(\tau_i^+) = \delta_{ki}\overline{u}_k^{(m-1)}(\tau_i), \quad k = 1,2,3, \quad i \in N_+,\\
&\overline{u}_k^{(m)}(0) = \overline{u}_k^{(m-1)}(T), \quad k = 1,2,3,\ m = 1,2,\cdots,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\frac{d\underline{u}_1^{(m)}}{dt} + L_1\underline{u}_1^{(m)} = L_1\underline{u}_1^{(m-1)} + f_1(t, \underline{u}_1^{(m-1)}, \overline{u}_2^{(m-1)}, \underline{u}_3^{(m-1)}), \quad t \neq \tau_i,\\
&\frac{d\underline{u}_2^{(m)}}{dt} + L_2\underline{u}_2^{(m)} = L_2\underline{u}_2^{(m-1)} + f_2(t, \overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), \quad t \neq \tau_i,\\
&\frac{d\underline{u}_3^{(m)}}{dt} + L_3\underline{u}_3^{(m)} = L_3\underline{u}_3^{(m-1)} + f_3(t, \underline{u}_1^{(m-1)}, \underline{u}_3^{(m-1)}), \quad t \neq \tau_i,\\
&\underline{u}_k^{(m)}(\tau_i^+) = \delta_{ki}\underline{u}_k^{(m-1)}(\tau_i), \quad k = 1,2,3, \quad i \in N_+,\\
&\underline{u}_k^{(m)}(0) = \underline{u}_k^{(m-1)}(T), \quad k = 1,2,3, \quad m = 1,2,\cdots.
\end{aligned} \tag{2.4}$$

It is clear from the hypothesis in the Introduction and the last equations in (2.3),(2.4) that these sequences are well defined. In the following theorem we show the monotone property of these sequences and the existence of periodic solutions.

Theorem 2.1. The sequences $\{\overline{\mathbf{u}}^{(m)}\}, \{\underline{\mathbf{u}}^{(m)}\}$ possess the monotone property

$$\hat{\mathbf{u}} = \underline{\mathbf{u}}^{(0)} \leq \underline{\mathbf{u}}^{(m)} \leq \underline{\mathbf{u}}^{(m+1)} \leq \overline{\mathbf{u}}^{(m+1)} \leq \overline{\mathbf{u}}^{(m)} \leq \overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}, \quad t \in [0, +\infty), \tag{2.5}$$

where $m = 1, 2, \cdots$. The pointwise limits

$$\lim_{m\to\infty} \underline{\mathbf{u}}^{(m)} = \underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2, \underline{u}_3), \quad \lim_{m\to\infty} \overline{\mathbf{u}}^{(m)} = \overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2, \overline{u}_3), \tag{2.6}$$

exist and satisfy the following relation

$$\hat{\mathbf{u}} \leq \underline{\mathbf{u}} \leq \overline{\mathbf{u}} \leq \tilde{\mathbf{u}}, \quad t \in [0, +\infty). \tag{2.7}$$

Moreover, $(\underline{u}_1, \overline{u}_2, \underline{u}_3)$, $(\overline{u}_1, \underline{u}_2, \overline{u}_3)$ are T-periodic solutions of (0.1).

Proof. Let $\mathbf{w}^{(1)} = \overline{\mathbf{u}}^{(0)} - \overline{\mathbf{u}}^{(1)}$, by (2.1) and (2.3) the components $w_k^{(1)}$ satisfy

$$\begin{aligned}
&\frac{dw_k^{(1)}}{dt} + L_k w_k^{(1)} \geq 0 \quad (t \neq \tau_i), \quad w_k^{(1)}(0) \geq 0,\\
&w_k^{(1)}(\tau_i^+) \geq 0 \quad (i \in N_+).
\end{aligned} \tag{2.8}$$

It is obvious that $w_k^{(1)} \geq 0$ for all $t \geq 0$, so we have $\overline{\mathbf{u}}^{(1)} \leq \overline{\mathbf{u}}^{(0)}$ in $[0, +\infty)$. A similar argument using the property of a lower solution gives $\underline{\mathbf{u}}^{(1)} \geq \underline{\mathbf{u}}^{(0)}$ in $[0, +\infty)$.

Let $\mathbf{z}^{(1)} = \overline{\mathbf{u}}^{(1)} - \underline{\mathbf{u}}^{(1)}$, by (2.3) and (2.4), and note that $L_1u_1 + f_1(t, u_1, u_2, u_3)$ is nondecreasing in u_1, u_3 and decreasing in u_2, the component $z_1^{(1)}$ satisfies the relation

$$\begin{aligned}
&\frac{dz_1^{(1)}}{dt} + L_1 z_1^{(1)} = L_1(\overline{u}_1^{(0)} - \underline{u}_1^{(0)}) + f_1(t, \overline{u}_1^{(0)}, \underline{u}_2^{(0)}, \overline{u}_3^{(0)}) - f_1(t, \underline{u}_1^{(0)}, \overline{u}_2^{(0)}, \underline{u}_3^{(0)}) \geq 0, \quad t \neq \tau_i,\\
&z_1^{(1)}(\tau_i^+) = \delta_{1i}(\overline{u}_1^{(0)}(\tau_i) - \underline{u}_1^{(0)}(\tau_i)) \geq 0, \quad i \in N_+,\\
&z_1^{(1)}(0) = \overline{u}_1^{(0)}(T) - \underline{u}_1^{(0)}(T) \geq 0,
\end{aligned} \tag{2.9}$$

thus we have $z_1^{(1)} \geq 0$, $t \in [0, +\infty)$. Using the analogy argument to $z_2^{(1)}$ and $z_3^{(1)}$, we get $z_2^{(1)} \geq 0, z_3^{(1)} \geq 0$ $(t \in [0, +\infty))$. This prove

$$\underline{\mathbf{u}}^{(0)} \leq \underline{\mathbf{u}}^{(1)} \leq \overline{\mathbf{u}}^{(1)} \leq \overline{\mathbf{u}}^{(0)}.$$

The monotone property (2.5) follows by an induction argument.

From the monotone property (2.5), the pointwise limits of the sequences $\{\overline{\mathbf{u}}^{(m)}\}, \{\underline{\mathbf{u}}^{(m)}\}$ exist, and from somewhat well-known arguments (see [1,Theorem 15.1]), we can prove that

$$\lim_{m\to\infty} \underline{\mathbf{u}}^{(m)} = \underline{\mathbf{u}}, \quad \lim_{m\to\infty} \overline{\mathbf{u}}^{(m)} = \overline{\mathbf{u}},$$

uniformly and monotonically in any closed subinterval of $[0, +\infty)$ and satisfy relation (2.7). It is clear from $\overline{\mathbf{u}}^{(m)}(0) = \overline{\mathbf{u}}^{(m-1)}(T)$, $\underline{\mathbf{u}}^{(m)}(0) = \underline{\mathbf{u}}^{(m-1)}(T)$ that $\overline{\mathbf{u}}(0) = \overline{\mathbf{u}}(T)$ and $\underline{\mathbf{u}}(0) = \underline{\mathbf{u}}(T)$.

It is easy to prove that $(\overline{u}_1, \underline{u}_2, \overline{u}_3)$, $(\underline{u}_1, \overline{u}_2, \underline{u}_3)$ are T-periodic solutions of problem (0.1) from the linear iteration process (2.3),(2.4) and the periodicity of system (0.1). This proves the theorem.

§3. Positive Periodic Solution

From Theorem 2.1, if a pair of ordered upper and lower solutions $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ with $\hat{\mathbf{u}} > 0$ can be found, then the existence of positive periodic solutions of (0.1) are ensured.

In this section, we seek a pair of ordered upper and lower solutions $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ in the form

$$(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (M_1, M_2, M_3), \quad (\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\delta_1\phi_1, \delta_2\phi_2, \delta_3\phi_3), \tag{3.1}$$

where, for $k = 1, 2, 3$, M_k and δ_k are positive constants, ϕ_k are positive functions. To this end, we need to choose suitable M_k, δ_k, ϕ_k.

Let

$$\begin{aligned} &M_1 = \max_{t\in[0,T]} \{\frac{a_1(t)}{b_1(t)}\}, \quad M_2 = \max_{t\in[0,T]} \{\frac{a_2(t)}{c_2(t)}\}, \\ &M_3 = \max_{t\in[0,T]} \{a_3(t)(1 + M_1\sigma_3(t))/c_3(t)\} \end{aligned} \tag{3.2}$$

and for each $k = 1, 2, 3$, ϕ_k is the solution of the following scalar linear problem

$$\begin{aligned} &\phi_k' - a_k\phi_k = \lambda_k\phi_k, \quad t \neq \tau_i, \\ &\phi_k(\tau_i^+) = \delta_{ki}\phi_k(\tau_i), \quad i \in N_+, \\ &\phi_k(0) = \phi_k(T), \end{aligned} \tag{3.3}$$

where

$$\lambda_k = -\frac{1}{T}[\ln(\prod_{i=1}^{q} \delta_{ki}) + \int_0^T a_k(t)dt]. \tag{3.4}$$

We can prove the following conclusion.

Theorem 3.1. Suppose that the following inequalities

$$\begin{aligned} &\ln(\textstyle\prod_{i=1}^q \delta_{1i}) + \int_0^T a_1(t)dt > TM_2c_1(t), \quad t \in [0, T], \\ &\ln(\textstyle\prod_{i=1}^q \delta_{2i}) + \int_0^T a_2(t)dt > TM_1b_2(t), \quad t \in [0, T], \\ &\ln(\textstyle\prod_{i=1}^q \delta_{3i}) + \int_0^T a_3(t)dt > 0, \end{aligned} \tag{3.5}$$

hold. Then system (0.1) has a pair of positive ordered upper and lower solutions

$$(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (M_1, M_2, M_3), \quad (\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\delta_1\phi_1, \delta_2\phi_2, \delta_3\phi_3),$$

where for $k = 1, 2, 3$, M_k, ϕ_k are given by (3.2) and (3.3) respectively with $\phi_k(0) > 0$, and δ_k is chosen sufficiently small. So system (0.1) has at least one positive periodic solution.

Proof. First it is seen that for each $k = 1, 2, 3$, $\phi_k(t) > 0$ for all $t \geq 0$ if $\phi_k(0) > 0$. In deed, solving (3.3) straightforwardly, we get

$$\phi_k(t) = (\prod_{0<\tau_i<t} \delta_{ki}) e^{(\lambda_k t + \int_0^t a_k(\tau)d\tau)} \phi_k(0). \tag{3.6}$$

It is obvious that $\phi_k(t) > 0$ for all $t \geq 0$ if $\phi_k(0) > 0$, and ϕ_k is T-periodic as λ_k is given by (3.4). From the periodic property of ϕ_k, $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ can be fulfilled as δ_k is chosen sufficiently small.

By direct computation, the pair of $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (M_1, M_2, M_3)$, $(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\delta_1\phi_1, \delta_2\phi_2, \delta_3\phi_3)$ satisfy all inequalities in (2.1),(2.2) if

$$\begin{aligned} \lambda_1 &\leq -b_1\delta_1\phi_1 - M_2 c_1, \\ \lambda_2 &\leq -M_1 b_2 - c_2\delta_2\phi_2, \\ \lambda_3 &\leq c_3\delta_3\phi_3. \end{aligned}$$

Since δ_k $(k = 1, 2, 3)$ can be chosen arbitrarily small, the above inequalities are satisfied if $\lambda_1 < -M_2 c_1$, $\lambda_2 < -M_1 b_2$, $\lambda_3 < 0$, which are equivalent to condition (3.5).

This proves (M_1, M_2, M_3), $(\delta_1\phi_1, \delta_2\phi_2, \delta_3\phi_3)$ are a pair of positive ordered upper and lower solutions. The proof is completed.

§4. Conclusion

Note that the existence of T-periodic solution of system (0.1) is ensured if there exist a pair of ordered upper and lower solutions $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$. Furthermore, these solutions can be constructed from the linear iteration process (2.3),(2.4) by using $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ as the initial iteration. And different pairs of upper and lower solutions probably lead to different periodic solutions.

It is seen that the sequence of iterations $\{\overline{\mathbf{u}}^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}\}$ in the iteration process (2.3) and (2.4) are governed by the usual linear initial value problems where the initial value is explicitly known for each m, numerical values of these sequences can be obtained easily.

Note also that upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ in Definition 2.1 are not required to be periodic in T which is useful in application.

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On the fourth power mean value of $B(\chi)$[1]

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Abstract The main purpose of this paper is using the properties of the character sums and Dirichlet's L-functions to study the mean value of $|B(\chi)|^4$, and give an interesting asymptotic formula for it.

Keywords Dirichlet L-function, character sums, asymptotic formula.

§1. Introduction

For any positive integer number $q > 3$, let χ denote a Dirichlet character modulo q. $L(s,\chi)$ denote the Dirichlet L-function corresponding to χ, and $L'(s,\chi)$ denote the derivative of $L(s,\chi)$ with respect to complex variables s. In book [1], there are some formulae about the zero-expansion of L'/L. That is, if χ be a primitive character mod q, then we have

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\ln\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+\frac{1}{2}a)}{\Gamma(\frac{1}{2}s+\frac{1}{2}a)} + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right);$$

$$2\mathrm{Re}B(\chi) = -\sum_{\rho}\left(\mathrm{Re}\frac{1}{1-\bar{\rho}} + \mathrm{Re}\frac{1}{\bar{\rho}}\right),$$

where

$$a = \begin{cases} 0, & \text{if } \chi(-1) = 1; \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

$\mathrm{Re}B(\chi)$ denote the real part of $B(\chi)$. In this formula, $B(\chi)$ can be expressed in terms of the expansion of L'/L in power of s, but it seems to be very difficult to estimate $B(\chi)$ at all satisfactorily as a function of q. The difficulty of estimating $B(\chi)$ is connected with the fact that, as far as we know, $L(s,\chi)$ may have a zero near to $s = 0$. Here we want to study the asymptotic property of $\sum_{\chi \bmod q}^{*} |B(\chi)|^4$. In fact, we use the properties of the character sums and the Dirichlet's L-functions to give an interesting asymptotic formula for $\sum_{\chi \bmod q}^{*} |B(\chi)|^4$. That is, we shall prove the following:

Theorem. Let integer number $q > 3$. Then we have the asymptotic formula

$$\sum_{\chi \bmod q}{}^{*} |B(\chi)|^4$$

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$$
\begin{aligned}
&= J(q)\left[\sum_p \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_p \frac{\ln^2 p}{p^2-1}\right)^2 - 4\sum_p \frac{\ln^4 p}{(p^2-1)^2}\right.\\
&\quad -\sum_{p|q} \frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - 4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)\left(\sum_p \frac{\ln^2 p}{p^2-1}\right) + 4\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^2}\\
&\quad +4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4} + 2\left(C_1+C_2\right)\left(\sum_p \frac{\ln^3 p}{(p^2-1)^2} - \sum_{p|q}\frac{\ln^3 p}{(p^2-1)^2}\right)\\
&\quad \left. +2\left(C_1^2+C_2^2\right)\left(\sum_p \frac{\ln^2 p}{p^2-1} - \sum_{p|q}\frac{\ln^2 p}{p^2-1}\right) + \frac{C_1^4+C_2^4}{2}\right] + O\left(q^{\epsilon}\right),
\end{aligned}
$$

where ϵ denote any fixed positive number, $J(q)$ denote the number of all primitive character mod q, $\sum\limits_p$ denote the summation over all primes, $\sum\limits_{\chi \bmod q}^{*}$ denote the summation over all primitive character mod q, and

$$
C_i = \frac{1}{2}\ln\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{i}{2}\right);
$$

where $i = 1$ or 2;

$$
\frac{\Gamma'}{\Gamma}\left(\frac{i}{2}\right) = \begin{cases} -\gamma, & \text{if } i = 2; \\ -2-\gamma-\sum\limits_{n=1}^{\infty}\dfrac{-1}{n(2n+1)}, & \text{if } i = 1. \end{cases}
$$

From this Theorem we may immediately deduce the following:

Corollary. For any integer number $q > 3$, we have the asymptotic formula

$$
\sum_{\chi \bmod q}{}^{*}|B(\chi)|^4 \sim \frac{J(q)\ln^4 q}{16}, \qquad \text{as} \qquad q \to \infty.
$$

Whether there exists an asymptotic formula for $|B(\chi)|$ is an open problem. We conjecture that for any primitive character χ mod q,

$$
|B(\chi)| \sim \frac{1}{2}\ln q.
$$

§2. Some Lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. Let $\Lambda(n)$ be the Mangoldt function. Define

$$
r(n) = \sum_{d|n}\Lambda(d)\Lambda(n|d).
$$

Then we have

$$
\sum_{n=1}^{\infty}\frac{r^2(n)}{n^2} = \sum_p \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + \left(\sum_p \frac{2\ln^2 p}{p^2-1}\right)^2 - \sum_p \frac{4\ln^4 p}{(p^2-1)^2}.
$$

Proof. (See reference [4]).

Lemma 2. Let integer number $q > 3$. Then we have

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} = \sum_p \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_p \frac{\ln^2 p}{p^2-1}\right)^2 - 4\sum_p \frac{\ln^4 p}{(p^2-1)^2}$$
$$-\sum_{p|q} \frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - \left(\sum_{p|q} \frac{4\ln^2 p}{p^2-1}\right)\left(\sum_p \frac{\ln^2 p}{p^2-1}\right)$$
$$+4\sum_{p|q} \frac{\ln^4 p}{(p^2-1)^2} + 4\left(\sum_{p|q} \frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q} \frac{\ln^4 p}{(p^2-1)^4}.$$

Proof. From the properties of the Möbius function, we can easily deduce

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} = \sum_{n=1}^{\infty} \sum_{d|(n,q)} \frac{\mu(d) r^2(n)}{n^2} = \sum_{n=1}^{\infty} \sum_{d|q} \frac{\mu(d) r^2(nd)}{n^2 d^2} = \sum_{d|q} \frac{\mu(d)}{d^2} \sum_{n=1}^{\infty} \frac{r^2(nd)}{n^2}$$
$$= \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} - \sum_{p|q} \frac{1}{p^2} \sum_{n=1}^{\infty} \frac{r^2(np)}{n^2} + \sum_{p_1|q} \sum_{p_2|q} \frac{\mu(p_1p_2)}{p_1^2 p_2^2} \sum_{n=1}^{\infty} \frac{r^2(np_1p_2)}{n^2}.$$

From the proving method of Lemma 1, we can get

$$\sum_{p|q} \frac{1}{p^2} \sum_{n=1}^{\infty} \frac{r^2(np)}{n^2} = \sum_{p|q} \frac{1}{p^2} \left[\sum_{m=1}^{\infty} \frac{r^2(p^{m+1})}{p^{2m}} + \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{p_1\\p_1 \neq p}} \frac{r^2(p^{m+1} p_1^k)}{p^{2m} p_1^{2k}} \right]$$
$$= \sum_{p|q} \frac{1}{p^2} \left[\sum_{m=1}^{\infty} \frac{m^2 \ln^4 p}{p^{2m}} + \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{p_1\\p_1 \neq p}} \frac{4\ln^2 p \ln^2 p_1}{p^{2m} p_1^{2k}} \right]$$
$$= \sum_{p|q} \frac{1}{p^2} \left[\frac{(p^2+1)\ln^4 p}{p^{-2}(p^2-1)^3} + \left(\sum_{p_1} \frac{4\ln^2 p_1}{p_1^2-1}\right) \frac{\ln^2 p}{1-\frac{1}{p^2}} - \frac{4\ln^4 p}{(p^2-1)(1-\frac{1}{p^2})} \right]$$
$$= \sum_{p|q} \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + \left(\sum_{p_1} \frac{4\ln^2 p_1}{p_1^2-1}\right) \sum_{p|q} \frac{\ln^2 p}{p^2-1} - \sum_{p|q} \frac{4\ln^4 p}{(p^2-1)^2}$$
$$= \sum_{p|q} \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + \left(\sum_{p|q} \frac{4\ln^2 p}{p^2-1}\right)\left(\sum_p \frac{\ln^2 p}{p^2-1}\right) - \sum_{p|q} \frac{4\ln^4 p}{(p^2-1)^2};$$

Let p_1 and p_2 denote two distinct primes, we can obtain

$$\sum_{p_1|q} \sum_{p_2|q} \frac{\mu(p_1p_2)}{p_1^2 p_2^2} \sum_{n=1}^{\infty} \frac{r^2(p_1p_2n)}{n^2} = \sum_{p_1|q} \sum_{p_2|q} \sum_{\substack{m,k=0\\p_1 \neq p_2}}^{\infty} \frac{r^2(p_1^{m+1} p_2^{k+1})}{p_1^{2m+2} p_2^{2k+2}}$$

$$= \sum_{p_1|q}\sum_{p_2|q}\sum_{m,k=0}^{\infty}\frac{4\ln^2 p_1\ln^2 p_2}{p_1^{2m+2}p_2^{2k+2}} - \sum_{p_1|q}\sum_{\substack{m=1\\ p_1=p_2}}^{\infty}\sum_{k=0}^{\infty}\frac{(m+k)^2\ln^4 p_1}{p_1^{2m+2k+4}}$$

$$= \sum_{p_1|q}\sum_{p_2|q}\frac{4\ln^2 p_1\ln^2 p_2}{(p_1^2-1)(p_2^2-1)} - 2\sum_{p|q}\frac{(p^2+1)\ln^4 p}{p^2(p^2-1)^4} - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4}$$

$$= 4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q}\frac{(p^2+1)\ln^4 p}{p^2(p^2-1)^4} - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4}.$$

Combining Lemma 1 and the above formula, we immediately get

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{r^2(n)}{n^2} &= \sum_{p}\frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right)^2 - 4\sum_{p}\frac{\ln^4 p}{(p^2-1)^2}\\ &-\sum_{p|q}\frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - \left(\sum_{p|q}\frac{4\ln^2 p}{p^2-1}\right)\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right)\\ &+4\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^2} + 4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4}.\end{aligned}$$

This is the conclusion of Lemma 2.

Lemma 3. Let q and r be integers with $q \geq 2$ and $(r,q)=1$, χ be a Dirichlet character mod q. Then we have the identities

$$\sum_{\chi \bmod q}{}^{*}\chi(r) = \sum_{d|(q,r-1)}\mu(\frac{q}{d})\phi(d),\ J(q) = \sum_{d|q}\mu(d)\phi(\frac{q}{d}),$$

where $\sum_{\chi \bmod q}^{*}$ denotes the summation over all primitive character modulo q, $\mu(n)$ is the Mӧbius function, $J(q)$ denotes the number of all primitive character modulo q.

Proof. From the properties of characters we know that for any character χ mod q, there exits one and only one $d|q$ with a primitive character χ_d^* mod d such that $\chi = \chi_d^*\chi_q^0$, here χ_q^0 denoting the principal character mod q. So we have

$$\sum_{\chi \bmod q}\chi(r) = \sum_{d|q}\sum_{\chi \bmod d}{}^{*}\chi(r)\chi_q^0(r) = \sum_{d|q}\sum_{\chi \bmod d}{}^{*}\chi(r).$$

Combining this formula with the Mӧbius transformation, and noting the identity

$$\sum_{\chi \bmod q}\chi(r) = \begin{cases}\phi(q), & \text{if } r\equiv 1(\bmod q);\\ 0, & \text{otherwise.}\end{cases}$$

we have

$$\sum_{\chi \bmod q}{}^{*}\chi(r) = \sum_{d|q}\mu(d)\sum_{\chi \bmod \frac{q}{d}}\chi(r) = \sum_{d|(q,r-1)}\mu(\frac{q}{d})\phi(d).$$

Taking $r = 1$, we immediately get

$$J(q) = \sum_{d|q} \mu(d)\phi(\frac{q}{d}).$$

This proves Lemma 3.

Lemma 4. Let integer number $q > 3$. Then we have the asymptotic formula

$$\begin{aligned}
&\sum_{\chi \mod q}{}^{*} \left| \sum_{n\le N} \frac{\Lambda(n)\chi(n)}{n} \right|^4 \\
&= J(q) \left[\sum_{p} \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right)^2 - 4\sum_{p}\frac{\ln^4 p}{(p^2-1)^2} \right. \\
&\quad -\sum_{p|q} \frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - \left(\sum_{p|q}\frac{4\ln^2 p}{p^2-1}\right)\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right) \\
&\quad \left. +4\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^2} + 4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4} \right] \\
&\quad +O(q^{\epsilon}\ln^4 N).
\end{aligned}$$

Proof. Let $r(n)$ be the definition in Lemma 2. For $N > q$, we define

$$r(n, N) = \sum_{\substack{rs=n \\ r,s\le N}} \Lambda(r)\Lambda(s).$$

It is clear that for all $n \le N$ we have $r(n, N) = r(n)$, $\sum_{n}{}' = \sum_{\substack{n \\ (n,q)=1}}$. Then from the orthogonality of characters, Lemma 2 and Lemma 3, we may get

$$\begin{aligned}
&\sum_{\chi \mod q}{}^{*} \left| \sum_{n\le N} \frac{\Lambda(n)\chi(n)}{n} \right|^4 \\
&= \sum_{\chi \mod q}{}^{*} \left| \sum_{n\le N^2} \frac{r(n,N)\chi(n)}{n} \right|^2 \\
&= \sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \frac{r(m,N)r(n,N)}{mn} \sum_{\chi \mod q}{}^{*} \chi(m)\chi(\bar{n}) \\
&= \sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \frac{r(m,N)r(n,N)}{mn} \sum_{d|(q,m\bar{n}-1)} \mu(\frac{q}{d})\phi(d) \\
&= \sum_{d|q} \mu(\frac{q}{d})\phi(d) \sum_{\substack{m\le N^2 \\ m\equiv n}}{}' \sum_{\substack{n\le N^2 \\ \text{mod } d}}{}' \frac{r(m,N)r(n,N)}{mn} \\
&= J(q)\sum_{n=1}^{\infty}{}' \frac{r^2(n)}{n^2} - O\left(J(q)\sum_{n>N}{}' \frac{r^2(n)}{n^2}\right)
\end{aligned}$$

$$+O\left(\sum_{d|q}\phi(d)\sum_{\substack{m\le N^2\\ m\equiv n\\ m\ne n}}{}'\sum_{\substack{n\le N^2\\ \bmod d}}{}'\frac{r(m)}{m}\frac{r(n)}{n}\right)$$

$$=J(q)\sum_{n=1}^{\infty}{}'\frac{r^2(n)}{n^2}+O\left(\sum_{d|q}\phi(d)\sum_{m\le N^2}{}'\sum_{l\le N^2/d}{}'\frac{r(m,N)r(ld+m,N)}{m(ld+m)}\right)$$

$$=J(q)\left[\sum_{p}\frac{(p^2+1)\ln^4 p}{(p^2-1)^3}+4\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right)^2-4\sum_{p}\frac{\ln^4 p}{(p^2-1)^2}\right.$$

$$-\sum_{p|q}\frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4}-\left(\sum_{p|q}\frac{4\ln^2 p}{p^2-1}\right)\left(\sum_{p}\frac{\ln^2 p}{p^2-1}\right)$$

$$\left.+4\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^2}+4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2-2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4}\right]$$

$$+O(q^{\epsilon}\ln^4 N).$$

This completes the proof of Lemma 4.

Lemma 5. Let integer number $q>3$. Then we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^2 p}{p^2-1}-\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)+O\left(q^{\epsilon}\ln^2 N\right);\tag{1}$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^2 p}{p^2-1}-\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)+O\left(q^{\epsilon}\ln^2 N\right).\tag{2}$$

Proof. From the orthogonality of characters and Lemma 3, we may get the estimates

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2$$

$$=\frac{1}{2}\sum_{m\le N}{}'\sum_{n\le N}{}'\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}\sum_{\chi \bmod q}{}^{*}\chi(m)\chi(\bar n)\left(1+\chi(-1)\right)$$

$$=\frac{1}{2}\sum_{m\le N}{}'\sum_{n\le N}{}'\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}\sum_{\chi \bmod q}{}^{*}\chi(m)\chi(\bar n)$$

$$+\frac{1}{2}\sum_{m\le N}{}'\sum_{n\le N}{}'\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}\sum_{\chi \bmod q}{}^{*}\chi(m)\chi(\bar n)\chi(-1)$$

$$=\frac{1}{2}\sum_{d|q}\mu\left(\frac{q}{d}\right)\phi(d)\sum_{\substack{m\le N\\ m\equiv n}}{}'\sum_{\substack{n\le N\\ \bmod d}}{}'\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}$$

$$+\frac{1}{2}\sum_{d|q}\mu(\frac{q}{d})\phi(d)\sum_{\substack{m\le N\ n\le N\\ m\bar{n}+1\equiv 0 \mod d}}{}^{\prime}\sum{}^{\prime}\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}$$

$$=\frac{J(q)}{2}\sum_{n=1}^{\infty}{}^{\prime}\frac{\Lambda^2(n)}{n^2}-O\left(\frac{J(q)}{2}\sum_{n>N}{}^{\prime}\frac{\Lambda^2(n)}{n^2}\right)$$

$$+O\left(\sum_{d|q}\phi(d)\sum_{\substack{m\le N\ n\le N\\ m\equiv n \mod d\\ m\ne n}}{}^{\prime}\sum{}^{\prime}\frac{\Lambda(m)}{m}\frac{\Lambda(n)}{n}\right)$$

$$=\frac{J(q)}{2}\sum_{p}\sum_{\substack{m=1\\ n=p^m\\ (n,q)=1}}^{\infty}\frac{\ln^2 p}{p^{2m}}+O\left(\sum_{d|q}\phi(d)\sum_{m\le N}{}^{\prime}\sum_{l\le N/d}{}^{\prime}\frac{\Lambda(m)}{m}\frac{\Lambda(ld+m)}{ld+m}\right)$$

$$=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^2 p}{p^2-1}-\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)+O\left(q^{\epsilon}\ln^2 N\right).$$

Using the same method, we can easily deduce (2.2) of Lemma 5.

This completes the proof of Lemma 5.

Lemma 6. Let integer number $q>3$. Then we have

$$\sum_{\substack{\chi \mod q\\ \chi(-1)=1}}{}^{*}\left[\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right]=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^3 p}{(p^2-1)^2}-\sum_{p|q}\frac{\ln^3 p}{(p^2-1)^2}\right)+O\left(q^{\epsilon}\ln^3 N\right); \qquad (2.3)$$

$$\sum_{\substack{\chi \mod q\\ \chi(-1)=-1}}{}^{*}\left[\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right]=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^3 p}{(p^2-1)^2}-\sum_{p|q}\frac{\ln^3 p}{(p^2-1)^2}\right)+O\left(q^{\epsilon}\ln^3 N\right). \qquad (2.4)$$

Proof. From the orthogonality of characters, we have

$$\sum_{\substack{\chi \mod q\\ \chi(-1)=1}}{}^{*}\left[\left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^2\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right]$$

$$=\frac{1}{2}\sum_{m_1,m_2,n\le N}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}\sum_{\substack{\chi \mod q\\ \chi(-1)=1}}{}^{*}\chi(m_1n)\bar{\chi}(m_2)\left(1+\chi(-1)\right)$$

$$=\frac{1}{2}\sum_{d|q}\mu(\frac{q}{d})\phi(d)\sum_{\substack{m_1,m_2,n\le N\\ m_1n\equiv m_2(\mod d)}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}$$

$$+\frac{1}{2}\sum_{d|q}\mu(\frac{q}{d})\phi(d)\sum_{\substack{m_1,m_2,n\leq N\\ m_1n\bar{m}_2+1\equiv 0(\bmod d)}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}$$

$$=\frac{J(q)}{2}\sum_{m_1,n\leq N}{}^{\prime}\frac{\Lambda(m_1)\Lambda(n)\Lambda(m_1n)}{m_1^2n^2}+O\left(\sum_{d|q}\phi(d)\sum_{\substack{m_1,m_2,n\leq N\\ m_1n\equiv m_2(\bmod d)\\ m_1n\neq m_2}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}\right)$$

$$=\frac{J(q)}{2}\sum_{\substack{m_1,n\leq N\\ m_1=p^\alpha\\ n=p^\beta}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(n)\Lambda(m_1n)}{m_1^2n^2}+O\left(\sum_{d|q}\phi(d)\sum_{\substack{m_1,m_2,n\leq N\\ m_1n\equiv m_2(\bmod d)\\ m_1n> m_2}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}\right)$$

$$+O\left(\sum_{d|q}\phi(d)\sum_{\substack{m_1,m_2,n\leq N\\ m_1n\equiv m_2(\bmod d)\\ m_1n< m_2}}{}^{\prime}\frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n)}{m_1m_2n}\right)$$

$$=\frac{J(q)}{2}\sum_{\substack{\alpha=1\\ (p^{\alpha+\beta},q)=1}}^{\infty}\sum_{\beta=1}^{\infty}\sum_{p}\frac{\ln^3 p}{p^{2\alpha+2\beta}}+O\left(q^{\epsilon}\ln^3 N\right)$$

$$=\frac{J(q)}{2}\left(\sum_{p}\frac{\ln^3 p}{(p^2-1)^2}-\sum_{p|q}\frac{\ln^3 p}{(p^2-1)^2}\right)+O\left(q^{\epsilon}\ln^3 N\right).$$

Using the same method, we can easily deduce (2.4) of Lemma 6. This completes the proof of Lemma 6.

§3. Proof of the theorem

In this section, we will complete the proof of Theorem. Firstly, according to the definition of $B(\chi)$, we have

$$\frac{L'(1,\chi)}{L(1,\chi)}=-\frac{1}{2}\ln\frac{q}{\pi}-\frac{1}{2}\frac{\Gamma'(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)}+B(\chi)+\sum_{\rho}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right);$$

$$2\mathrm{Re}B(\chi)=-\sum_{\rho}\left(\mathrm{Re}\frac{1}{1-\bar{\rho}}+\mathrm{Re}\frac{1}{\bar{\rho}}\right),$$

where

$$a=\begin{cases}0, & \text{if } \chi(-1)=1;\\ 1, & \text{if } \chi(-1)=-1.\end{cases}$$

From the functional equation of the Dirichlet's L-functions we know that if ρ is a zero of $L(s,\chi)$, then $1-\bar{\rho}$ is also a zero of $L(s,\chi)$, so we can easy deduce that $\sum_{\rho}\left(\mathrm{Re}\frac{1}{1-\bar{\rho}}+\mathrm{Re}\frac{1}{\bar{\rho}}\right)=$

$\sum_{\rho}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right)$ is a real number, combining the above two formulae, we have

$$\begin{aligned}|B(\chi)| &= |B(\chi)-2\mathrm{Re}B(\chi)| = \left|B(\chi)+\sum_{\rho}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right)\right| \\ &= \left|\frac{L'(1,\chi)}{L(1,\chi)}+\frac{1}{2}\ln\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)}\right| \\ &= \left|\frac{L'(1,\chi)}{L(1,\chi)}+C_i\right|,\end{aligned}$$

where

$$C_i = \begin{cases} \frac{1}{2}\ln\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1}{2}), & \text{if } \chi(-1)=1, i=1; \\ \frac{1}{2}\ln\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1), & \text{if } \chi(-1)=-1, i=2. \end{cases}$$

Then, from the above formulae, we obtain

$$\begin{aligned}
&\sum_{\chi \bmod q}{}^{*}|B(\chi)|^4 \\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}|B(\chi)|^4 + \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}|B(\chi)|^4 \\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\frac{L'}{L}(1,\chi)+C_1\right|^4 + \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\frac{L'}{L}(1,\chi)+C_2\right|^4 \\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^4 + 2C_1\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^2\left(\frac{L'}{L}(1,\chi)+\frac{L'}{L}(1,\bar{\chi})\right) \\
&\quad +4C_1^2\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^2 + 2C_1^2\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left(\frac{L'}{L}(1,\chi)\right)^2 \\
&\quad +2C_1^3\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}\left(\frac{L'}{L}(1,\chi)+\frac{L'}{L}(1,\bar{\chi})\right) + C_1^4\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}{}^{*}1 \\
&\quad +\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^4 + 2C_2\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^2\left(\frac{L'}{L}(1,\chi)+\frac{L'}{L}(1,\bar{\chi})\right) \\
&\quad +4C_2^2\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\frac{L'}{L}(1,\chi)\right|^2 + 2C_2^2\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left(\frac{L'}{L}(1,\chi)\right)^2 \\
&\quad +2C_2^3\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left(\frac{L'}{L}(1,\chi)+\frac{L'}{L}(1,\bar{\chi})\right) + C_2^4\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}1 \qquad (2.5)
\end{aligned}$$

Secondly, for $\mathrm{Re}(s)>1$, $L'(s,\chi)/L(s,\chi)$ is absolutely convergent. Thus from Abel's summation formula, we may get

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s} = \sum_{n\le N} \frac{\Lambda(n)\chi(n)}{n^s} + s\cdot \int_N^{\infty} \frac{A(y,\chi)}{y^{1+s}} dy, \qquad (2.6)$$

where N is any positive number, when $\chi \neq \chi^0$. For any character χ to the modulus q, we define $\psi(x,\chi) = \sum_{n\le N} \chi(n)\Lambda(n)$. From book [1], we have two useful formulae. One is, for any real non-principle character $\chi \mod q$, and if $L(s,\chi)$ has an exceptional zero, then we have

$$\frac{L'}{L}(1,\chi) \ll C_1(\epsilon) q^{\epsilon} \ln q; \qquad (2.7)$$

The other is, for any complex primitive character $\chi \mod q$, if $q \le \exp(C\sqrt{\ln x})$, we have

$$\psi(x,\chi) \ll x \exp\left(-C_1\sqrt{\ln x}\right); \qquad (2.8)$$

Based on the formula (2.6) and (2.8), for any complex primitive character $\chi \mod q$, we have

$$\begin{aligned}
\left|\frac{L'}{L}(1,\chi)\right|^4 &= \left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n} + \int_N^{\infty}\frac{A(y,\chi)}{y^2}dy\right|^4 \\
&= \left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n} + O\left(\frac{\ln N}{\exp(C_2\sqrt{\ln N})}\right)\right|^4 \\
&= \left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^4 + O\left(\frac{\ln^7 N}{\exp(C_2\sqrt{\ln N})}\right).
\end{aligned}$$

Taking $N \ge \exp\left(C\ln^2 q\right)$ and combining formula (2.7), we immediately obtain

$$\begin{aligned}
&\sum_{\chi \mod q}{}^{*} \left|\frac{L'}{L}(1,\chi)\right|^4 \\
&= \sum_{\chi \mod q}{}^{*} \left|\sum_{n\le N}\frac{\Lambda(n)\chi(n)}{n}\right|^4 + O\left(\frac{J(q)\ln^7 N}{\exp(C_2\sqrt{\ln N})}\right) + O\left(C_1^4 q^{4\epsilon}\ln^4 q\right) \\
&= J(q)\left[\sum_p \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_p \frac{\ln^2 p}{p^2-1}\right)^2 - 4\sum_p \frac{\ln^4 p}{(p^2-1)^2}\right. \\
&\quad -\sum_{p|q}\frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - \left(\sum_{p|q}\frac{4\ln^2 p}{p^2-1}\right)\left(\sum_p \frac{\ln^2 p}{p^2-1}\right) \\
&\quad \left. +4\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^2} + 4\left(\sum_{p|q}\frac{\ln^2 p}{p^2-1}\right)^2 - 2\sum_{p|q}\frac{\ln^4 p}{(p^2-1)^4}\right] \\
&\quad +O\left(\frac{J(q)\ln^7 N}{\exp(C_2\sqrt{\ln N})}\right) + O\left(q^{\epsilon}\right). \qquad (2.9)
\end{aligned}$$

And then, using the same method, we also get

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=\pm 1}}{}^{*} \left| \frac{L'}{L}(1,\chi) \right|^2$$

$$= \frac{1}{2} \sum_{\substack{\chi \bmod q \\ \chi(-1)=\pm 1}}{}^{*} \left| \sum_{n\le N} \frac{\Lambda(n)\chi(n)}{n} \right|^2 + O\left(\frac{J(q)\ln^3 N}{\exp(C_2\sqrt{\ln N})} \right) + O\left(C_1^2 q^{2\epsilon} \ln^2 q \right)$$

$$= \frac{J(q)}{2} \left(\sum_{p} \frac{\ln^2 p}{p^2-1} - \sum_{p|q} \frac{\ln^2 p}{p^2-1} \right) + O\left(\frac{J(q)\ln^3 N}{\exp(C_2\sqrt{\ln N})} \right) + O\left(q^{\epsilon}\right). \qquad (2.10)$$

Finally, taking $N = \exp\left(\frac{\ln^2 q}{C_2^2}\right)$ and according to the formula (2.5), (2.9) and (2.10), we can give an asymptotic formula for the $\sum\limits_{\chi \bmod q}^{*} |B(\chi)|^4$. That is,

$$\sum_{\chi \bmod q}{}^{*} |B(\chi)|^4$$

$$= J(q) \left[\sum_{p} \frac{(p^2+1)\ln^4 p}{(p^2-1)^3} + 4\left(\sum_{p} \frac{\ln^2 p}{p^2-1} \right)^2 - 4\sum_{p} \frac{\ln^4 p}{(p^2-1)^2} \right.$$

$$- \sum_{p|q} \frac{(p^2+1)(p^4-p^2+2)\ln^4 p}{p^2(p^2-1)^4} - 4\left(\sum_{p|q} \frac{\ln^2 p}{p^2-1} \right)\left(\sum_{p} \frac{\ln^2 p}{p^2-1} \right) + 4\sum_{p|q} \frac{\ln^4 p}{(p^2-1)^2}$$

$$+4\left(\sum_{p|q} \frac{\ln^2 p}{p^2-1} \right)^2 - 2\sum_{p|q} \frac{\ln^4 p}{(p^2-1)^4} + 2\left(C_1+C_2\right)\left(\sum_{p} \frac{\ln^3 p}{(p^2-1)^2} - \sum_{p|q} \frac{\ln^3 p}{(p^2-1)^2} \right)$$

$$\left. +2\left(C_1^2+C_2^2\right)\left(\sum_{p} \frac{\ln^2 p}{p^2-1} - \sum_{p|q} \frac{\ln^2 p}{p^2-1} \right) + \frac{C_1^4+C_2^4}{2} \right] + O\left(q^{\epsilon}\right).$$

This completes the proof of Theorem.

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On the F.Smarandache function and its mean value

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m: \ n \mid m!, \ n \in N\}$. The main purpose of this paper is using the elementary methods to study a mean value problem involving the F.Smarandache function, and give a sharper asymptotic formula for it.

Keywords F.Smarandache function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m: \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots$. About the elementary properties of $S(n)$, some authors had studied it, and obtained some interesting results, see reference [2], [3] and [4]. For example, Farris Mark and Mitchell Patrick [2] studied the elementary properties of $S(n)$, and gave an estimates for the upper and lower bound of $S(p^\alpha)$. That is, they showed that

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Murthy [3] proved that if n be a prime, then $SL(n) = S(n)$, where $SL(n)$ defined as the smallest positive integer k such that $n \mid [1, \ 2, \ \cdots, \ k]$, and $[1, \ 2, \ \cdots, \ k]$ denotes the least common multiple of $1, 2, \cdots, k$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n \ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes, and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Dr. Xu Zhefeng [5] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

On the other hand, Lu Yaming [6] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1+m_2+\cdots+m_k)=S(m_1)+S(m_2)+\cdots+S(m_k)$$

has infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [7] proved for any positive integer $k \geq 2$, there exist infinite groups positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfied the following inequality:

$$S(m_1+m_2+\cdots+m_k)>S(m_1)+S(m_2)+\cdots+S(m_k).$$

Also, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1+m_2+\cdots+m_k)<S(m_1)+S(m_2)+\cdots+S(m_k).$$

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $[S(n)-S(S(n))]^2$, and give an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be any fixed positive integer. Then for any real number $x > 2$, we have the asymptotic formula

$$\sum_{n\leq x}[S(n)-S(S(n))]^2=\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are computable constants and $c_1 = 1$.

§2. Proof of the Theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [3] we know that

$$S(n)=\max\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_s^{\alpha_s})\}\equiv S(p^{\alpha}). \tag{2}$$

Now we consider the summation

$$\sum_{n\leq x}[S(n)-S(S(n))]^2=\sum_{n\in A}[S(n)-S(S(n))]^2+\sum_{n\in B}[S(n)-S(S(n))]^2, \tag{3}$$

where A and B denote the subsets of all positive integer in the interval $[1,\ x]$. A denotes the set involving all integers $n \in [1,\ x]$ such that $S(n) = S(p^2)$ for some prime p; B denotes the set involving all integers $n \in [1,\ x]$ such that $S(n) = S(p^\alpha)$ with $\alpha = 1$ or $\alpha \geq 3$. If $n \in A$, then $n = p^2 m$ with $P(m) < 2p$, where $P(m)$ denotes the largest prime factor of m. So from the definition of $S(n)$ we have $S(n) = S(mp^2) = S(p^2) = 2p$ and $S(S(n)) = S(2p) = p$ if $p > 2$.

From (2) and the definition of A we have

$$\begin{aligned}
&\sum_{n\in A}[S(n)-S(S(n))]^2\\
=&\sum_{\substack{n\leq x\\ p^2\|n,\ \sqrt{n}<p^2}}\left[S(p^2)-S(S(p^2))\right]^2+\sum_{\substack{n\leq x\\ p^2\|n,\ p^2\leq\sqrt{n}}}\left[S(p^2)-S(S(p^2))\right]^2\\
=&\sum_{\substack{p^2n\leq x\\ n<p^2,\ (p,\ n)=1}}\left[S(p^2)-S(S(p^2))\right]^2+\sum_{\substack{p^2n\leq x\\ p^2\leq n,\ (p,\ n)=1}}\left[S(p^2)-S(S(p^2))\right]^2\\
=&\sum_{\substack{p^2n\leq x\\ n<p^2,\ (p,\ n)=1}}p^2+\sum_{\substack{p^2n\leq x\\ n\geq p^2,\ (p,\ n)=1}}p^2+O(1)\\
=&\sum_{n\leq\sqrt{x}}\sum_{n<p^2\leq\frac{x}{n}}p^2+O\left(\sum_{m\leq x^{\frac14}}\sum_{p\leq\left(\frac{x}{m}\right)^{\frac13}}p^2\right)+O\left(\sum_{p\leq x^{\frac14}}\sum_{p^2\leq n\leq\frac{x}{p^2}}p^2\right)\\
=&\sum_{n\leq\sqrt{x}}\sum_{p\leq\sqrt{\frac{x}{n}}}p^2+O\left(\frac{x^{\frac54}}{\ln x}\right),
\end{aligned}\tag{4}$$

where $p^2\|n$ denotes $p^2|n$ and $p^3 \dagger n$.

By the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x)=\sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{p\leq\sqrt{\frac{x}{n}}}p^2 &= \frac{x}{n}\cdot\pi\left(\sqrt{\frac{x}{n}}\right)-\int_{\frac32}^{\sqrt{\frac{x}{n}}}2y\cdot\pi(y)dy\\
&= \frac13\cdot\frac{x^{\frac32}}{n^{\frac32}}\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i\sqrt{\frac{x}{n}}}+O\left(\frac{x^{\frac32}}{n^{\frac32}\cdot\ln^{k+1}x}\right),
\end{aligned}\tag{5}$$

where we have used the estimate $n \leq \sqrt{x}$, and all b_i are computable constants and $b_1 = 1$.

Note that $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac32}}=\zeta\left(\frac32\right)$, and $\sum\limits_{n=1}^{\infty}\frac{\ln^i n}{n^{\frac32}}$ is convergent for all $i = 1, 2, 3, \cdots, k$. So from

(4) and (5) we have

$$\begin{aligned}
&\sum_{n\in A}[S(n)-S(S(n))]^2\\
= &\sum_{n\le\sqrt{x}}\left[\frac{1}{3}\cdot\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}}}\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i\sqrt{\frac{x}{n}}}+O\left(\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}}\cdot\ln^{k+1}x}\right)\right]+O\left(\frac{x^{\frac{5}{4}}}{\ln x}\right)\\
= &\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right), \qquad (6)
\end{aligned}$$

where c_i $(i=1,2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

Now we estimate the summation in set B. For any positive integer $n\in B$, if $S(n)=S(p)=p$, then $[S(n)-S(S(n))]^2=[S(p)-S(S(p))]^2=0$; If $S(n)=S(p^\alpha)$ with $\alpha\ge 3$, then

$$[S(n)-S(S(n))]^2=[S(p^\alpha)-S(S(p^\alpha))]^2\le\alpha^2p^2$$

and $\alpha\le\ln x$. So that we have

$$\sum_{n\in B}[S(n)-S(S(n))]^2\ll\sum_{\substack{np^\alpha\le x\\ \alpha\ge 3}}\alpha^2\cdot p^2\ll x\cdot\ln^2 x. \qquad (7)$$

Combining (3), (6) and (7) we may immediately deduce the asymptotic formula

$$\sum_{n\le x}[S(n)-S(S(n))]^2=\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

This completes the proof of Theorem.

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Mean value of F. Smarandache LCM function

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Abstract For any positive integer n, the famous Smarandache function $S(n)$ defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. The Smarandache LCM function $SL(n)$ the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value properties of $(SL(n) - S(n))^2$, and give a sharper asymptotic formula for it.

Keywords Smarandache function, Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$. The F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $\cdots\cdots$.

About the elementary properties of $S(n)$ and $SL(n)$, many authors had studied them, and obtained some interesting results, see reference [2], [3], [4] and [5]. For example, Murthy [2] proved that if n be a prime, then $SL(n) = S(n)$. Simultaneously, Murthy [2] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n \ ? \tag{1}$$

Le Maohua [3] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{or} \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes, and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Dr. Xu Zhefeng [4] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Lv Zhongtian [5] proved that for any fixed positive integer k any real number $x > 1$, we have the asymptotic formula fixed positive integer k

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

The main purpose of this paper is using the elementary methods to study the mean value properties of $[SL(n) - S(n)]^2$, and give an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be a fixed positive integer. Then for any real number $x > 2$, we have the asymptotic formula

$$\sum_{n\leq x}[SL(n)-S(n)]^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ be the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$S(n) = \max\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_s^{\alpha_s})\} \equiv S(p^\alpha) \tag{2}$$

and

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{3}$$

Now we consider the summation

$$\sum_{n\leq x}[SL(n)-S(n)]^2 = \sum_{n\in A}[SL(n)-S(n)]^2 + \sum_{n\in B}[SL(n)-S(n)]^2, \tag{4}$$

where A and B denote two subsets of all positive integer in the interval $[1,\ x]$. A denotes the set involving all integers $n \in [1,\ x]$ such that $SL(n) = p^2$ for some prime p; B denotes the set involving all integers $n \in [1,\ x]$ such that $SL(n) = p^\alpha$ for some prime p with $\alpha = 1$ or $\alpha \geq 3$. If

$n \in A$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, then $n = p^2 m$ with $p \dagger m$, and all $p_i^{\alpha_i} \leq p^2$, $i = 1,\ 2,\ \cdots s$. Note that $S(p_i^{\alpha_i}) \leq \alpha_i p_i$ and $\alpha_i \leq \ln n$, from the definition of $SL(n)$ and $S(n)$ we have

$$\begin{aligned}
&\sum_{n \in A} [SL(n) - S(n)]^2 = \sum_{\substack{mp^2 \leq x \\ SL(m) < p^2}} \left[p^2 - S(mp^2)\right]^2 \\
= &\sum_{\substack{mp^2 \leq x \\ SL(m) < p^2}} \left(p^4 - 2p^2 S(mp^2) + S^2(mp^2)\right) \\
= &\sum_{\substack{mp^2 \leq x \\ SL(m) < p^2}} p^4 + O\left(\sum_{mp^2 \leq x} p^2 \cdot p \cdot \ln x\right) + O\left(\sum_{mp^2 \leq x} p^2 \cdot \ln^2 x\right) \\
= &\sum_{m \leq \sqrt{x}} \sum_{m < p^2 \leq \frac{x}{m}} p^4 + O\left(\sum_{p^2 \leq \sqrt{x}} \sum_{p^2 < m \leq \frac{x}{p^2}} p^4\right) + O\left(x^2\right) \\
= &\sum_{m \leq \sqrt{x}} \sum_{p \leq \sqrt{\frac{x}{m}}} p^4 + O\left(\sum_{m \leq \sqrt{x}} \sum_{p^2 \leq m} p^4\right) + O\left(x^2\right) \\
= &\sum_{m \leq \sqrt{x}} \sum_{p \leq \sqrt{\frac{x}{m}}} p^4 + O\left(x^2\right). \qquad (5)
\end{aligned}$$

By the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{p \leq \sqrt{\frac{x}{m}}} p^4 &= \frac{x^2}{m^2} \cdot \pi\left(\sqrt{\frac{x}{m}}\right) - \int_{\frac{3}{2}}^{\sqrt{\frac{x}{m}}} 4y^3 \cdot \pi(y) dy \\
&= \frac{x^2}{m^2} \cdot \pi\left(\sqrt{\frac{x}{m}}\right) - \int_{\frac{3}{2}}^{\sqrt{\frac{x}{m}}} 4y^3 \left[\sum_{i=1}^{k} \frac{a_i \cdot y}{\ln^i y} + O\left(\frac{y}{\ln^{k+1} y}\right)\right] dy \\
&= \frac{1}{5} \cdot \frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}} \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i \sqrt{\frac{x}{m}}} + O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}} \cdot \ln^{k+1} x}\right), \qquad (6)
\end{aligned}$$

where we have used the estimate $m \leq \sqrt{x}$, and all b_i are computable constants with $b_1 = 1$.

Note that $\sum_{m=1}^{\infty} \frac{1}{m^{\frac{5}{2}}} = \zeta\left(\frac{5}{2}\right)$, and $\sum_{m=1}^{\infty} \frac{\ln^i m}{m^{\frac{5}{2}}}$ is convergent for all $i = 1,\ 2,\ 3, \cdots, k$. So from (5) and (6) we have

$$\begin{aligned}
&\sum_{n\in A}[SL(n)-S(n)]^2\\
=\ &\sum_{m\le\sqrt{x}}\left[\frac{1}{5}\cdot\frac{x^{\frac{5}{2}}}{m^{\frac{3}{2}}}\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i\sqrt{\frac{x}{m}}}+O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}\cdot\ln^{k+1}x}\right)\right]+O\left(x^2\right)\\
=\ &\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot x^{\frac{5}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),
\end{aligned}\qquad(7)$$

where c_i $(i=1,\ 2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

Now we estimate the summation in set B. For any positive integer $n\in B$. If $n\in B$ and $SL(n)=p$, then we also have $S(n)=p$. So $[SL(n)-S(n)]^2=[p-p]^2=0$. If $SL(n)=p^\alpha$ with $\alpha\ge 3$, then $[SL(n)-S(n)]^2=[p^\alpha-S(n)]^2\le p^{2\alpha}+\alpha^2p^2$ and $\alpha\le\ln n$. So we have

$$\sum_{n\in B}[SL(n)-S(n)]^2\ll\sum_{\substack{np^\alpha\le x\\ \alpha\ge 3}}\left(p^{2\alpha}+p^2\ln^2 n\right)\ll x^2.\qquad(8)$$

Combining (4), (7) and (8) we may immediately deduce the asymptotic formula

$$\sum_{n\le x}[SL(n)-S(n)]^2=\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot x^{\frac{5}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,\ 2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

This completes the proof of Theorem.

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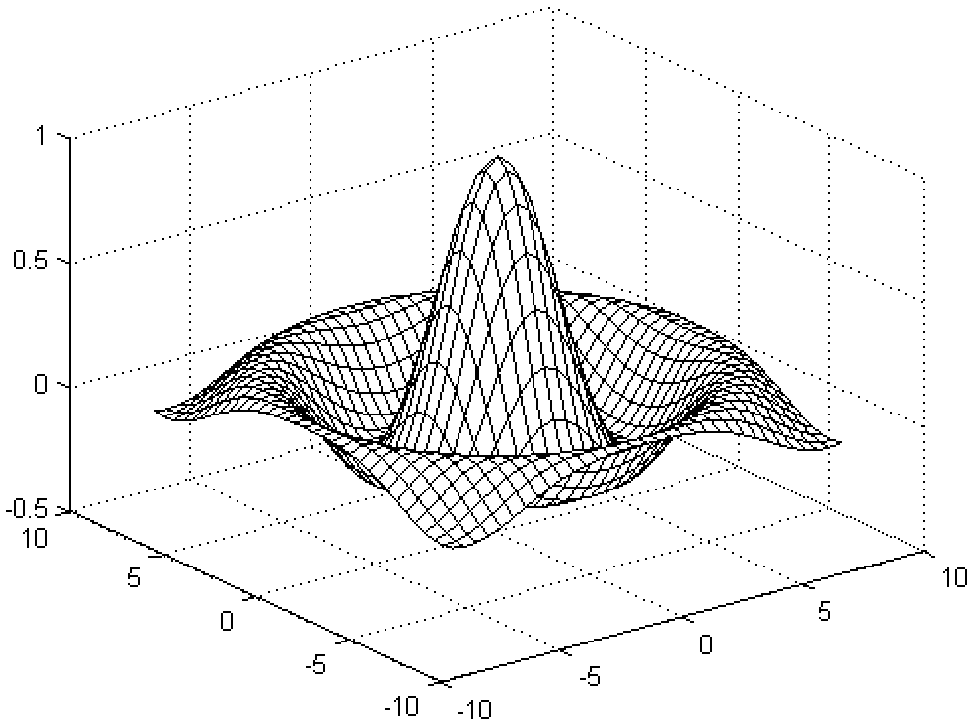

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