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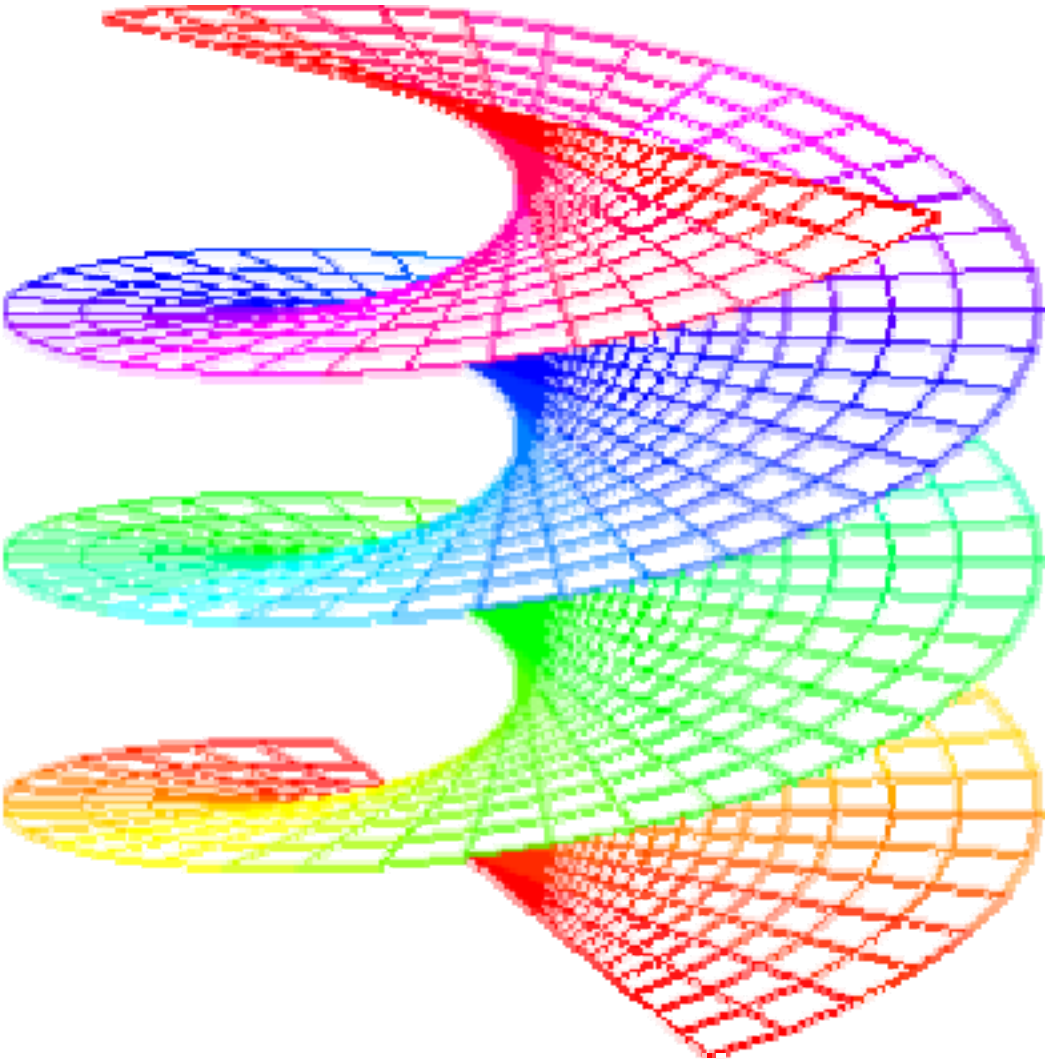

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Preface

This issue of the journal is devoted to the proceedings of the second Northwest Number Theory and Smarandache Problems Conference held in Yan'an during March 18-20, 2006. The organizers were myself and Professor Jinbao Guo from Yan'an University. The conference was supported by Yan'an University and Northwest University and there were more than 70 participants. We had two foreign guests from Japan, Professor S. Kanemitsu of Kinki University and Y. Tanigawa of Nagoya University who gave plenary talks "On the role of the cotangent function in number theory" and "Some formulas for the gamma functions", respectively, and their papers are going to appear elsewhere. Other participants include Professor Wenguang Zhai from Shandong Normal University whose paper with Dr. Liu Huaning is included in the volume. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache problems in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the third conference in coming March in Weinan.

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. L. Liu and W. Zhou, On conjectures about the class number of binary quadratic forms; 2. W. Liang, An identity for Stirling numbers of the second kind; 3. Y. Wang and Z. Sheng, Two formulas for x^n in terms of Chebyshev polynomials.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them. Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source Digital Library of Science:

www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm.

Wenpeng Zhang

The Second Northwest Conference on Number Theory and Smarandache Problems

The participants of the Conference

Professor Wenpeng Zhang →

← Professor Shigeru Kanemitsu

Professor Yoshio Tanigawa →

← Professor Jinbao Guo

Professor Wenguang Zhai →

Contents

Scientia Magna
Vol. 2 (2006), No. 2, 1-6

On Algebraic Multi-Vector Spaces

Linfan Mao

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Abstract A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions hold. Combining these Smarandache multi-spaces with linear vector spaces in classical linear algebra, the conception of multi-vector spaces is introduced. Some characteristics of multi-vector spaces are obtained in this paper.

Keywords Vector, multi-space, multi-vector space, dimension of a space.

§1. Introduction

These multi-spaces was introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field* ([7]), which can be formally defined with mathematical words by the next definition.

Definition 1.1.For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, denoted by $(A_i; L_i)$. Then the union of $(A_i; L_i), 1 \leq i \leq n$

$$\widetilde{A} = \bigcup_{i=1}^{n} (A_i; L_i)$$

is called a multi-space. As it is well-known, a *vector space* or *linear space* consists of the following:

(i) a field F of scalars;

(ii) a set V of objects, called vectors;

(iii) an operation, called vector addition, which associates with each pair of vectors $\mathbf{a}, \mathbf{b}$ in V a vector $\mathbf{a} + \mathbf{b}$ in V, called the sum of $\mathbf{a}$ and $\mathbf{b}$, in such a way that

(1) addition is commutative, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$;

(2) addition is associative, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$;

(3) there is a unique vector $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in V;

(4) for each vector $\mathbf{a}$ in Vthere is a unique vector $-\mathbf{a}$ in V such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$;

(iv) an operation "$\cdot$" , called scalar multiplication, which associates with each scalar k in F and a vector $\mathbf{a}$ in V a vector $k \cdot \mathbf{a}$ in V, called the product of k with $\mathbf{a}$, in such a way that

(1) $1 \cdot \mathbf{a} = \mathbf{a}$ for every $\mathbf{a}$ in V;

(2) $(k_1 k_2) \cdot \mathbf{a} = k_1(k_2 \cdot \mathbf{a})$;

(3) $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$;

(4) $(k_1 + k_2) \cdot \mathbf{a} = k_1 \cdot \mathbf{a} + k_2 \cdot \mathbf{a}$.

We say that V is a vector space over the field F, denoted by $(V\ ;+,\cdot)$.

Now combining these Smarandache multi-spaces with those of linear spaces, a new kind of algebraic structure called multi-vector spaces is found, which is defined in the following:

Definition 1.2. Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-space with binary operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ a multi-filed space with double binary operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. If for any integers $i, j,\ 1 \leq i, j \leq k$ and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,

(i) $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on F_i with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;

(ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(iii) $(k_1+_ik_2)\cdot_j\mathbf{a} = k_1+_i(k_2\cdot_j\mathbf{a})$; provided all these operation results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed space $\widetilde{F}$ with a binary operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$. For subsets $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, if $(\widetilde{V}_1; \widetilde{F}_1)$ is also a multi-vector space, then we say $(\widetilde{V}_1; \widetilde{F}_1)$ to be a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$.

The main object of this paper is to find some characteristics of multi-vector spaces. For terminology and notation not defined here can be seen in [1], [3] for linear algebra and [2], [4] – [11] for multi-spaces and logics.

§2. Characteristics of a multi-vector space

First, we get a simple result for multi-vector subspaces of a multi-vector space.

Theorem 2.1. For a multi-vector space $(\widetilde{V}; \widetilde{F})$, $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any vector additive "$\dot{+}$", scalar multiplication "$\cdot$" in $(\widetilde{V}_1; \widetilde{F}_1)$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$,

$$\alpha \cdot \mathbf{a}\dot{+}\mathbf{b} \in \widetilde{V}_1$$

provided all these operation results exist.

Proof. Denote by $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Notice that $\widetilde{V}_1 = \bigcup\limits_{i=1}^{k} (\widetilde{V}_1 \bigcap V_i)$. By definition, we know that $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any integer $i, 1 \leq i \leq k$, $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ and $\widetilde{F}_1$ is a multi-filed subspace of $\widetilde{F}$ or $\widetilde{V}_1 \bigcap V_i = \emptyset$.

According to a criterion for linear subspaces of a linear space ([3]), we know that for any integer $i, 1 \leq i \leq k$, $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ if and only if for $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}_1 \bigcap V_i$, $\alpha \in F_i$,

$$\alpha \cdot_i \mathbf{a}\dot{+}_i\mathbf{b} \in \widetilde{V}_1 \bigcap V_i.$$

i.e., for any vector additive $\dot{+}$, scalar multiplication $\cdot$ in $(\widetilde{V}_1; \widetilde{F}_1)$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$, if $\alpha \cdot \mathbf{a}\dot{+}\mathbf{b}$ exists, then $\alpha \cdot \mathbf{a}\dot{+}\mathbf{b} \in \widetilde{V}_1$.

Corollary 2.1. Let $(\widetilde{U}; \widetilde{F}_1), (\widetilde{W}; \widetilde{F}_2)$ be two multi-vector subspaces of a multi-vector space $(\widetilde{V}; \widetilde{F})$. Then $(\widetilde{U} \bigcap \widetilde{W}; \widetilde{F}_1 \bigcap \widetilde{F}_2)$ is a multi-vector space.

Similarly, we can also introduce independent multi-vectors in a multi-vector space. For a multi-vector space $(\widetilde{V};\widetilde{F})$, vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n\in\widetilde{V}$, if there are scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$ such that

$$\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}\in\widetilde{V}$ is an unit under an operation "$+$" in $\widetilde{V}$ and $\dot{+}_i,\cdot_i\in O(\widetilde{V})$, then the vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ are said to be *linearly dependent*. Otherwise, $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ to be *linearly independent*.

Notice that in a multi-vector space, there are two cases for linearly independent vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$, i.e.,

Case 1. for any scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$, if

$$\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}$ is a unit of $\widetilde{V}$ under an operation "$+$" in $O(\widetilde{V})$, then $\alpha_1=0_{+_1},\alpha_2=0_{+_2},\cdots,\alpha_n=0_{+_n}$, where $0_{+_i},1\leq i\leq n$ are the units under the operation $+_i$ in $\widetilde{F}$.

Case 2. the operation result of $\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n$ does not exist.

Now for a given subset $\widehat{S}\subset\widetilde{V}$, define its linearly spanning set $\left\langle\widehat{S}\right\rangle$ to be

$$\left\langle\widehat{S}\right\rangle=\{\ \mathbf{a}\mid\mathbf{a}=\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\in\widetilde{V},\mathbf{a}_i\in\widehat{S},\alpha_i\in\widetilde{F},i\geq1\}.$$

Now for a multi-vector space $(\widetilde{V};\widetilde{F})$, if there exists a subset $\widehat{S},\widehat{S}\subset\widetilde{V}$ such that $\widetilde{V}=\left\langle\widehat{S}\right\rangle$, then we say that $\widehat{S}$ is a linearly spanning set of the multi-vector space $\widetilde{V}$. If these vectors in a linearly spanning set $\widehat{S}$ of the multi-vector space $\widetilde{V}$ are linearly independent, then $\widehat{S}$ is said to be a basis of $\widetilde{V}$.

Theorem 2.2. Any multi-vector space $(\widetilde{V};\widetilde{F})$ has a basis.

Proof. Assume $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i,\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$ and the basis of each vector space $(V_i;\dot{+}_i,\cdot_i)$ is $\Delta_i=\{\mathbf{a}_{i1},\mathbf{a}_{i2},\cdots,\mathbf{a}_{in_i}\}$, $1\leq i\leq k$. Define

$$\widehat{\Delta}\ =\ \bigcup_{i=1}^{k}\Delta_i.$$

Then $\widehat{\Delta}$ is a linearly spanning set for $\widetilde{V}$ by definition.

If vectors in $\widehat{\Delta}$ are linearly independent, then $\widehat{\Delta}$ is a basis of $\widetilde{V}$. Otherwise, choose a vector $\mathbf{b}_1\in\widehat{\Delta}$ and define $\widehat{\Delta}_1=\widehat{\Delta}\setminus\{\mathbf{b}_1\}$.

If we have obtained the set $\widehat{\Delta}_s,s\geq1$ and it is not a basis still, choose a vector $\mathbf{b}_{s+1}\in\widehat{\Delta}_s$ and define $\widehat{\Delta}_{s+1}=\widehat{\Delta}_s\setminus\{\mathbf{b}_{s+1}\}$.

If these vectors in $\widehat{\Delta}_{s+1}$ are linearly independent, then $\widehat{\Delta}_{s+1}$ is a basis of $\widetilde{V}$. Otherwise, we can also define a set $\widehat{\Delta}_{s+2}$. Continue this process. Notice that for any integer $i,1\leq i\leq k$, these vectors in Δ_i are linearly independent. Therefore, we can finally get a basis of $\widetilde{V}$.

Now we consider these finite-dimensional multi-vector spaces. A multi-vector space $\widetilde{V}$ is *finite-dimensional* if it has a finite basis. By Theorem 2.2, if for any integer $i,1\leq i\leq k$, the vector space $(V_i;+_i,\cdot_i)$ is finite-dimensional, then $(\widetilde{V};\widetilde{F})$ is finite-dimensional. On the other

hand, if there is an integer $i_0, 1 \leq i_0 \leq k$, such that the vector space $(V_{i_0}; +_{i_0}, \cdot_{i_0})$ is infinite-dimensional, then $(\widetilde{V}; \widetilde{F})$ is infinite-dimensional. This result enables us to get the following consequence

Corollary 2.2. Let $(\widetilde{V}; \widetilde{F})$ be a multi-vector space with $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Then $(\widetilde{V}; \widetilde{F})$ is finite-dimensional if and only if for any integer $i, 1 \leq i \leq k$, $(V_i; +_i, \cdot_i)$ is finite-dimensional.

Theorem 2.3. For a finite-dimensional multi-vector space $(\widetilde{V}; \widetilde{F})$, any two bases have the same number of vectors.

Proof Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. The proof is carried out by the induction on k. For $k = 1$, the assertion is true by Theorem 4 of Chapter 2 in [3].

For the case of $k = 2$, notice that by a result in linearly vector space theory (see also [3]), for two subspaces W_1, W_2 of a finite-dimensional vector space, if the basis of $W_1 \bigcap W_2$ is $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t\}$, then the basis of $W_1 \bigcup W_2$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{\mathtt{dim}W_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{\mathtt{dim}W_2}\},$$

where, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{\mathtt{dim}W_1}\}$ is a basis of W_1 and $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{\mathtt{dim}W_2}\}$ a basis of W_2.

Whence, if $\widetilde{V} = W_1 \bigcup W_2$ and $\widetilde{F} = F_1 \bigcup F_2$, then the basis of $\widetilde{V}$ is also

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{\mathtt{dim}W_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{\mathtt{dim}W_2}\}.$$

Assume the assertion is true for $k = l, l \geq 2$. Now we consider the case of $k = l + 1$. In this case, since

$$\widetilde{V} = (\bigcup_{i=1}^{l} V_i) \bigcup V_{l+1},\ \widetilde{F} = (\bigcup_{i=1}^{l} F_i) \bigcup F_{l+1},$$

by the induction assumption, we know that any two bases of the multi-vector space $(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ have the same number p of vectors. If the basis of $(\bigcup\limits_{i=1}^{l} V_i) \bigcap V_{l+1}$ is $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{\mathtt{dim}V_{l+1}}\},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p\}$ is a basis of $(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{\mathtt{dim}V_{l+1}}\}$ a basis of V_{l+1}. Whence, the number of vectors in a basis of $\widetilde{V}$ is $p + \mathtt{dim}V_{l+1} - n$ for the case $n = l + 1$.

Therefore, by the induction principle, the assertion is true for any integer k.

The number of elements in a finite-dimensional multi-vector space $\widetilde{V}$ is called its *dimension*, denoted by $\mathtt{dim}\widetilde{V}$.

Theorem 2.4. (dimensional formula) For a multi-vector space $(\widetilde{V};\widetilde{F})$ with $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$, the dimension $\mathtt{dim}\widetilde{V}$ of $\widetilde{V}$ is

$$\mathtt{dim}\widetilde{V} = \sum_{i=1}^{k}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} \mathtt{dim}(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).$$

Proof. The proof is by induction on k. For $k = 1$, the formula is the trivial case of $\mathtt{dim}\widetilde{V} = \mathtt{dim}V_1$. for $k = 2$, the formula is

$$\mathtt{dim}\widetilde{V} = \mathtt{dim}V_1 + \mathtt{dim}V_2 - \mathtt{dim}(V_1\bigcap \mathtt{dim}V_2),$$

which is true by Theorem 6 of Chapter 2 in [3].

Now assume the formula is true for $k = n$. Consider the case of $k = n + 1$. According to the proof of Theorem 2.3, we know that

$$\begin{aligned}
\mathtt{dim}\widetilde{V} &= \mathtt{dim}(\bigcup_{i=1}^{n} V_i) + \mathtt{dim}V_{n+1} - \mathtt{dim}((\bigcup_{i=1}^{n} V_i)\bigcap V_{n+1}) \\
&= \mathtt{dim}(\bigcup_{i=1}^{n} V_i) + \mathtt{dim}V_{n+1} - \mathtt{dim}(\bigcup_{i=1}^{n}(V_i\bigcap V_{n+1})) \\
&= \mathtt{dim}V_{n+1} + \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} \mathtt{dim}(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}) \\
&+ \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} \mathtt{dim}(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\bigcap V_{n+1}) \\
&= \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} \mathtt{dim}(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).
\end{aligned}$$

By the induction principle, we know this formula is true for any integer k.

From Theorem 2.4, we get the following additive formula for any two multi-vector spaces.

Corollary 2.3. (additive formula) For any two multi-vector spaces $\widetilde{V}_1, \widetilde{V}_2$,

$$\mathtt{dim}(\widetilde{V}_1\bigcup\widetilde{V}_2) = \mathtt{dim}\widetilde{V}_1 + \mathtt{dim}\widetilde{V}_2 - \mathtt{dim}(\widetilde{V}_1\bigcap\widetilde{V}_2).$$

§3. Open problems for a multi-ring space

Notice that Theorem 2.3 has told us there is a similar linear theory for multi-vector spaces, but the situation is really more complex. Here, we present some open problems for further research.

Problem 3.1. Similar to linear spaces, define linear transformations on multi-vector spaces. Can we establish a new matrix theory for linear transformations?

Problem 3.2. Whether a multi-vector space must be a linear space?

Conjecture *A*. There are non-linear multi-vector spaces in multi-vector spaces.

Based on Conjecture A, there is a fundamental problem for multi-vector spaces.

Problem 3.3. Can we apply multi-vector spaces to non-linear spaces?

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On conjectures concerning class number of binary quadratic forms[1]

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Abstract Denote the binary primitive quadratic form $ax^2+bxy+cy^2$ by (a,b,c), and denote the equivalent class by $[a,b,c]$. Let $H(D)=\{[a,b,c] \mid b^2-4ac=D, \gcd(a,b,c)=1\}$ and $H_4(D)=\{[a,b,c]^4 \mid [a,b,c]\in H(D)\}$, denote $h(D)$ and $h_4(D)$ as the order of $H(D)$ and $H_4(D)$ respectively. Z.H.Sun[3] posed several conjectures, one is: if $m,n,d\in\mathbb{Z}$, s.t. $m^2-dn^2=4$, and let $j\in\{1,2\}$,

$$\begin{aligned} N_j(m,n,d) &= \Big\{[a,2b,c] \mid b^2-ac=-\delta(n,d)^2\cdot d, a\equiv(-1)^j(\mathrm{mod}\ 4), \\ &\qquad (a,b)=1, \left(\frac{\frac{bn}{(n,m-2)}-\delta(n,d)\frac{m-2}{(n,m-2)}i}{a}\right)_4=1\Big\}, \end{aligned}$$

then $|N_0(m,n,d)|=\frac{1}{8}h(-4\delta(n,d)^2d)$, where $\delta(n,d)\in\{1,2,4,8\}$ is given by table 4 of [3] and $\left(\frac{*}{a}\right)_4$ is the quartic Jacobi symbol. In this paper, we make some numerical evidence to support this conjecture, then pose a stronger version of it.

Keywords Quartic residue, quartic Jacobi symbol, binary quadratic form, class number.

§1. Introduction

For $a,b,c\in\mathbb{Z}$ denote the binary quadratic form $ax^2+bxy+cy^2$ by (a,b,c), and denote the equivalent class by $[a,b,c]$. The discriminant of (a,b,c) is $D=b^2-4ac$. If an integer n is represented by (a,b,c), then n is also represented by the class $[a,b,c]$. So we may say that n is represented by the class $[a,b,c]$. For $D\equiv 0,1\ (\mathrm{mod}\ 4)$, let $H(D)$ be the form class group which consists of primitive, integeral quadratic forms of discriminant D, and let $h(D)=|H(D)|$ be the corresponding class number. Let $H_4(D)$ be the subgroup of $H(D)$ consisting of the fourth powers of the classes in $H(D)$, i.e., $H_4(D)=\{[a,b,c]^4 \mid b^2-4ac=D\}$, and let $h_4(D)$ be the order of $H_4(D)$.

Let $d>1$ be a squarefree integer and $\varepsilon_d=(m+n\sqrt{d})/2$ be the fundmental unit of the quadratic field $Q(\sqrt{d})$. Suppose that p is a prime such that $(\frac{d}{p})=1$, where $(\frac{d}{p})$ is the Legendre symbol. When the norm $N(\varepsilon_d)=(m^2-dn^2)/4=-1$, Sun [3] proposed that ε_d is quadratic

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residue mod p if and only if p is represented by one class in the set

$$s(m,n,d)=\left\{[a,2b,c]\mid [a,2b,c]\in H(-4k^2d),\ \text{and a }\equiv 1(\text{mod }4),\left(\frac{bn-kmi}{a}\right)_4=1\right\},$$

then he gave some examples for $d\in[2,53]$. When the norm $N(\varepsilon_d)=1$, Sun [3] got that ε_d is quartic residue mod p if and only if p is represented by one class in the set

$$N_j=\left\{[a,2b,c]\mid b^2-ac=D, a\equiv(-1)^j(\text{mod }4),(a,b)=1,\left(\frac{\frac{bn}{(n,m-2)}-\delta(n,d)\frac{m-2}{(n,m-2)}i}{a}\right)_4=1\right\},$$

where $j\in\{0,1\}, N_j=N_j(m,n,d), D=-\delta(n,d)^2d$. Then he tabulated the set $N_0(m,n,d)$ for $d\in[3,47]$. In the end, Z.H.Sun posed the following conjectures:

Conjecture 1. [3, Conjecture 8.1] If $m,n,d\in\mathbb{Z}, m^2-dn^2=4$, if $2+m$ and $2-m$ are nonsquare integers, then $|N_0(m,n,d)|=\frac{1}{8}h(-4\delta(n,d)^2d)$.

Conjecture 2. [3, Conjecture 8.2] Let p and q be a prime of the form $8k+1$. Then $h_4(-24p)=h_4(-384pq)=h(-24p)/8$.

Conjecture 3. [3, Conjecture 8.3] Let p and q be a prime of the form $24k+1$. Then $h_4(-4pq)=h_4(-64pq)=h(-4pq)/8$.

Conjecture 4. [3, Conjecture 8.4] Let p and q be primes of the form $4k+1$ such that $(\frac{p}{q})=1$. Then $h_4(-4pq)=h_4(-64pq)=h(-4pq)/8$.

Conjecture 5. [3, Conjecture 8.5] Let p and q be primes of the form $8k+1$. Then

$$h_4(-8pq)=h_4(-128pq)=\begin{cases}\frac{1}{16}h(-8pq) & if\left(\frac{p}{q}\right)=1,\\ \frac{1}{8}h(-8pq) & if\left(\frac{p}{q}\right)=-1.\end{cases}$$

Conjecture 6.[3, Conjecture 8.6] Let $d>2$ be a squarefree integer. If $h_4(-64d)$ is odd, then $h_4(-64pq)=h_4(-4pq)$.

In this paper, we make some numerical evidence to support conjecture 1. In section 2 we recall and state some basic facts concerning quartic residue characters and qudratic forms, which are necessary for computing the quartic Jacobi symbol. In section 3, using Theorem 8.3 [3] we describe a procedure for searching $d\in[3,500]$ and tabulate the two sets $N_0(m,n,d)$ and $N_1(m,n,d)$ for $d\in[51,105]$, then pose a stronger version of conjecture 1. At last we search some numbers for the remaining conjectures of Sun [3] with no counterexample found except conjecture 4.

§2. Preliminaries

Let $\mathbb{Z}$ the set of integers, $i=\sqrt{-1}$ and $\mathbb{Z}[i]=\{a+bi\mid a,b\in\mathbb{Z}\}$. We recall that $a+bi$ is primary when $a\equiv 1\ (\text{mod }4)$, $b\equiv 0\ (\text{mod }4)$ or $a\equiv 3\ (\text{mod }4)$, $b\equiv 2\ (\text{mod }4)$. Let $\alpha,\beta,\pi\in\mathbb{Z}[i]$, if π is a nonunit such that if $\pi\mid\alpha\beta$ then either $\pi\mid\alpha$ or $\pi\mid\beta$, then π is called irreducible.

Let $\pi\in\mathbb{Z}[i]$, we may write $\pi=\pi_1\cdots\pi_r$, where π_i is irreducible. For $\alpha\in\mathbb{Z}[i]$ such that $(\alpha,\pi)=1$, the quartic Jacobi symbol is defined by $\left(\frac{\alpha}{\pi}\right)_4=\left(\frac{\alpha}{\pi_1}\right)_4\cdots\left(\frac{\alpha}{\pi_r}\right)_4$, where $\left(\frac{\alpha}{\pi_s}\right)_4$ is the quartic residue character of α modulo π_s (see [1], pp.122).

For later convenience we define

$$\left(\frac{a+bi}{1}\right)_4 = 1 \text{ for all } a, b \in \mathbb{Z}.$$

According to [1,pp.122-123,311], [4], [5, pp.242-247] the quartic Jacobi symbol has the following properties :

(2.1) If π is irreducible and $\pi \nmid \alpha$, then $\left(\frac{\alpha}{\pi}\right)_4 \equiv \alpha^{(N(\pi)-1)/4} \equiv i^j \pmod{\pi}$, where $j \in \{0, 1, 2, 3\}$.

(2.2) If $a + bi$ is primary, then

$$\left(\frac{i}{a+bi}\right)_4 = i^{\frac{a^2+b^2-1}{4}} = i^{\frac{1-a}{2}} \qquad \text{and} \qquad \left(\frac{1+i}{a+bi}\right)_4 = i^{\frac{a-b-b^2-1}{4}}.$$

(2.3) If $m, n \in \mathbb{Z}, 2 \nmid m$ and $(m, n) = 1$, then $\left(\frac{n}{m}\right)_4 = 1$.

(2.4) If $\pi \nmid \alpha\beta, \alpha, \beta \in \mathbb{Z}[i]$, then $\left(\frac{\alpha\beta}{\pi}\right)_4 = \left(\frac{\alpha}{\pi}\right)_4 \left(\frac{\beta}{\pi}\right)_4$.

(2.5)[1] If $a + bi$ is primary and $n \equiv 1 \pmod 4$, then

$$\left(\frac{-1}{a+bi}\right)_4 = (-1)^{b/2} \text{ and } \left(\frac{i}{n}\right)_4 = (-1)^{n-1/4}.$$

(2.6) [4, Lemma 2.1] Let p be a odd positive number, $m, n \in \mathbb{Z}$ and $(m^2 + n^2, p) = 1$, then

$$\left(\frac{m+ni}{p}\right)_4^2 = \left(\frac{m^2+n^2}{p}\right).$$

To state our procedure more concisely we introduce some definitions.

Definition 2.1. A binary quadratic form f with discriminant $b^2 - 4ac$ is a function $f(x, y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, which is denoted more briefly by $(a, 2b, c)$. Two quadratic forms are said to be equivalent if there exits an integeral matrix $C = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ of determinant equal to 1(i.e., with $ru - st = 1$) such that

$$g(x, y) = \begin{pmatrix} x & y \end{pmatrix} C' \begin{pmatrix} a & b \\ b & c \end{pmatrix} C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

denote more briefly as $(a, 2b, c)C = (a, 2b, c) \begin{pmatrix} r & s \\ t & u \end{pmatrix} = (a_1, 2b_1, c_1)$, where $a_1 = ar^2 + 2brt + ct^2, b_1 = ars + b(ru + st) + ctu, c_1 = as^2 + 2bsu + cu^2$.

Definition 2.2. A primitive integeral quadratic form (a, b, c) is said to be smooth-reduced if $a \equiv 1 \pmod 2$ and $(a, b) = 1$.

To state our results we need the following Lemmas.

Lemma 2.1. Let $(a, 2b, c) \in H(D)$ be a primitive quadratic form, then
(i) $2 \nmid (a, c)$ and $(c, -2b, a) \sim (a, 2b, c)$.
(ii) If $(a, b) \neq 1$, then $(a - 2b + c, 2(a - b), c) \sim (a, 2b, c)$.

Proof. (i) Taking R $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ leads to $(a, 2b, c)R = (c, -2b, a)$
Since gcd $(a, 2b, c) = 1$ we can find $2 \nmid (a, c)$.
(ii) Taking R $= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ we get $(a, 2b, c)R = (a - 2b + c, 2(a - b), c)$.

Lemma 2.2. [2, Proposition 5.3.3] In every class of binary quadratic forms with discriminant $D < 0$ and $a > 0$ there exists exactly one reduced form. In particular $h(D)$ is equal to the number of primitive reduced forms of discriminant D.

For a squarefree negative integer D, we compute class number of discriminant D using reduced forms (see[2, Algorithm 5.3.5]). According above lemmas one can change a quadratic form into a smooth-reduced form.

Lemma 2.3. [3, Theorem 6.2] Suppose that p is an odd prime, $d, m, n \in Z, m^2 - dn^2 = -4$, and $(\frac{-d}{p}) = 1$. Then $(\frac{m+n\sqrt{d}}{2})^{(p-(\frac{-1}{p}))/2} \equiv 1 \pmod{p}$ if and only if p can be represented by one class in the set

$$s(m, n, d) = \left\{ [a, 2b, c] \mid [a, 2b, c] \in H(-4k^2 d), \text{ and a } \equiv 1(\text{mod } 4), \left(\frac{bn - kmi}{a} \right)_4 = 1 \right\},$$

where k is given by

$$k = \begin{cases} 1 & \text{if } d \equiv 4 \text{ (mod 8)}; \\ 2 & \text{if } 8 \mid d, \text{ or } 2 \nmid d; \\ 4 & \text{if } d \equiv 2 \text{ (mod 4)}. \end{cases}$$

Table 1:

d	$\delta(n, d)$	Corresponding conditions
$d \equiv 0$(mod 8)	2	$2^3 \parallel d, 2 \parallel n$
	1	otherwise
$d \equiv 4$(mod 8)	4	$2 \nmid n$
	2	$2 \mid n$
$d \equiv 1$(mod 4)	4	
$d \equiv 2$(mod 4)	4	$2^2 \parallel n$
	2	$8 \mid n$
$d \equiv 3$(mod 4)	8	$2 \parallel n$
	4	$8 \mid n$

Lemma 2.4. [3, Theorem 8.3] Let p be an odd prime, $m, n, d \in Z, m^2 - dn^2 = 4, p \nmid dn$, and let $\delta(n, d) \in \{1, 2, 4, 8\}$ be given by table 1(To more briefly, denote $N_j(m, n, d) = N_j, D =$

$-\delta(n,d)^2 d$). Assume that $(\frac{-d}{p}) = 1$, Then $(\frac{m+n\sqrt{d}}{2})^{(p-(\frac{-1}{p}))/4} \equiv 1 \pmod{p}$ if and only if p is represented by some class in the set N_j, where $j \in \{0,1\}$ is given by $p \equiv (-1)^j \pmod 4$ and $N_j = \left\{ [a,2b,c] \mid b^2 - ac = D, a \equiv (-1)^j \pmod 4, (a,b) = 1, \left(\frac{\frac{bn}{(n,m-2)} - \delta(n,d)\frac{m-2}{(n,m-2)} i}{a} \right)_4 = 1 \right\}$.

§3. Numerical Results

Let p be a prime of the form $4k+1$, then $p = \pi\overline{\pi}, \pi, \overline{\pi}$ is irreducible and $(\pi, \overline{\pi}) = 1$. If $p \equiv 3 \pmod 4$ then p is irreducible in $\mathbb{Z}[i]$ of [1]. As we all known $\left(\frac{\alpha}{\pi}\right)_4 \equiv \alpha^{\frac{N\pi-1}{4}} \pmod{\pi}$, one can compute $\left(\frac{\alpha}{\pi}\right)_4$ using (2.1) $-$ (2.6).

Using Lemma 2.3 and doing some calculations we have

Theorem 3.1. Let p be a prime of the form $4k+1, d \in \{58, 61, 65, 73, 74, 82, 85, 89, 97, 101\}$ and $(\frac{d}{p}) = 1$, tnen ε_d is a quadratic residue (mod p) if and only if p is represented by one of the corresponding quadratic forms in table 2.

Table 2:

ε_d	Corresponding quadratic forms
$\varepsilon_{58} = 99 + 13\sqrt{58}$	(1,0,928), (32,32,37)
$\varepsilon_{61} = \frac{1}{2}(39 + 5\sqrt{61})$	(1,0,244), $(13, \pm 8, 20)$
$\varepsilon_{65} = 8 + \sqrt{65}$	(1,0,260), (4,0,65), $(8, \pm 4, 33)$
$\varepsilon_{73} = 1068 + 125\sqrt{73}$	(1,0,292), (4,0,73), $(8, \pm 4, 37)$
$\varepsilon_{74} = 43 + 5\sqrt{74}$	(1,0,1184), $(33, \pm 4, 36)$, $(25, +8, 48)$, $(29, \pm 22, 45)$
	$(33, \pm 26, 41)$, $(20, \pm 12, 61)$, $(32, 32, 45)$
$\varepsilon_{82} = 9 + \sqrt{82}$	(1,0,1312), (32,0,41), $(29, \pm 28, 52)$
$\varepsilon_{85} = \frac{1}{2}(9 + \sqrt{85})$	(1,0,340), (17,0,20)
$\varepsilon_{89} = 500 + 53\sqrt{89}$	(1,0,356), (4,0,89), $(5, \pm 4, 72)$, $(20, \pm 16, 21)$
$\varepsilon_{97} = 5604 + 569\sqrt{97}$	(1,0,388), (4,0,97), $(8, \pm 4, 49)$
$\varepsilon_{101} = 10 + \sqrt{101}$	(1,0,404), $(5, \pm 2, 81)$, $(33, \pm 10, 13)$, $(21, \pm 20, 24)$

Now we are ready to decribe a procedure to compute $N_0(m,n,d)$ and $N_1(m,n,d)$.

Procedure. Computing $N_i(m,n,d)(i = 0,1)$;
{Input a squarefree integer d, Output two sets $N_0(m,n,d)$ and $N_1(m,n,d)$}
Begin
For every $\varepsilon_d = (m + n\sqrt{d})/2$ Do
begin $n_0 \longleftarrow 0$; $n_1 \longleftarrow 0$; $u \longleftarrow \frac{m-2}{(n,\ m-2)}$; $v \longleftarrow \frac{n}{(n,\ m-2)}$;
(using table 1)Output $\delta(n,d)$; $D \longleftarrow -4\delta(n,d)^2 d$;
Output all smooth-reduced quadratic forms $(a,2b,c)$ and the class number h of discriminant D;
$J \longleftarrow \left(\frac{bv - \delta(n,d)ui}{a}\right)_4$;

If (J=1) and ($a \equiv 1 \pmod 4$) then $n_0 \longleftarrow n_0 + 1$; Output $(a, 2b, c) \in N_0(m, n, d)$;
If (J=1) and ($a \equiv 3 \pmod 4$) then $n_1 \longleftarrow n_1 + 1$; Output $(a, 2b, c) \in N_1(m, n, d)$;
If ($n_0 \neq \frac{1}{8}h$) or ($n_1 \neq \frac{1}{8}h$) then output ε_d
end
End.

Using Lemma 2.4 and doing some calculations we have

Theorem 3.2. Let p be an odd prime, $d \in \{51,55,57,59,62,66,69,70,71,77,78,79,83,86,87,91, 93,94, 95,102,103,105\}$, If $p \equiv 1 \bmod 4$ (resp. $p \equiv 3 \bmod 4$) then ε_d is a quatic residue mod p if and only if p is reperented by one of the corresponding quadratic forms belonging to $N_0(m, n, d)$(resp. $N_1(m, n, d)$) in table 4.

Table 3: Fundmental units for $d \in [51, 105]$

$\varepsilon_{51} = 50 + 7\sqrt{51}$	$\varepsilon_{70} = 251 + 30\sqrt{70}$	$\varepsilon_{91} = 1574 + 165\sqrt{91}$
$\varepsilon_{55} = 89 + 12\sqrt{55}$	$\varepsilon_{71} = 3480 + 413\sqrt{71}$	$\varepsilon_{93} = \frac{1}{2}(29 + 3\sqrt{93})$
$\varepsilon_{57} = 151 + \sqrt{57}$	$\varepsilon_{77} = \frac{1}{2}(9 + \sqrt{77})$	$\varepsilon_{94} = 2143295 + 221064\sqrt{94}$
$\varepsilon_{59} = 530 + 69\sqrt{59}$	$\varepsilon_{78} = 53 + 6\sqrt{78}$	$\varepsilon_{95} = 39 + 4\sqrt{95}$
$\varepsilon_{62} = 63 + 8\sqrt{62}$	$\varepsilon_{79} = 80 + 9\sqrt{79}$	$\varepsilon_{102} = 101 + 10\sqrt{102}$
$\varepsilon_{66} = 65 + 8\sqrt{66}$	$\varepsilon_{83} = 82 + 9\sqrt{83}$	$\varepsilon_{103} = 227528 + 22419\sqrt{103}$
$\varepsilon_{67} = 48842 + 5967\sqrt{67}$	$\varepsilon_{86} = 10405 + 1122\sqrt{86}$	$\varepsilon_{105} = 41 + 4\sqrt{105}$
$\varepsilon_{69} = \frac{1}{2}(25 + 3\sqrt{69})$	$\varepsilon_{87} = 28 + 3\sqrt{87}$	

We find $|N_0(m, n, d)| = |N_1(m, n, d)| = \frac{1}{8}h(-4\delta(n, d)^2 d)$ by seeking $d \in [3, 500]$. There are in total 184 such numbers for $d \in [106, 500]$, 39 numbers of which are of the form $4k + 1$; 64 numbers are of the form $4k + 2$; 81 numbers are of the form $4k + 3$. Is there an isomorphism between the two sets $N_0(m, n, d)$ and $N_1(m, n, d)$? So we pose a stronger version of conjecture 1 [3, conjecture 8.1]:

Conjecture. If $m, n, d \in Z, m^2 - dn^2 = 4, 2 + m$ and $2 - m$ are nonsquare integers, then $|N_0(m, n, d)| = |N_1(m, n, d)| = \frac{1}{8}h(-4\delta(n, d)^2 d)$.

Remark: As to the remaining conjectures of Sun, we search primes in $[17, 2 \times 10^6]$, there are 37116 primes satisfying conjecture 2, in $[73, 10^6]$ there are 9732 primes satisfying conjectrue 3. Conjecture 4 is not true since we have found many counterexamples[see 6]. For prime $p \in [1, 1000]$ and $q \in [1, 10000]$ conjecture 5 is true. For conjecture 6 we seek nonsquare integers $d \in [3, 200000]$ with no counterexample found. So we suggest that d just need satisfy being nonsquare integers and $h_4(-64d)$ is odd then $h_4(-64d) = h_4(-4d)$.

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Table 4:

ε_d	h	$N_0(m,n,d)$	$N_1(m,n,d)$
ε_{51}	48	(1,0,3264), (17,0,192)	(12,12,275), (64,64,67)
		$(33,\pm12,100), (57,\pm42,65)$	$(44,\pm12,75), (43,\pm4,76)$
ε_{55}	16	(1,0,880), (4,4,221)	(11,0,80), (31,18,31)
ε_{57}	16	(1,0,192), (16,16,61)	(3,0,304), (31,14,31)
ε_{59}	72	$(1,0,3776), (9,\pm4,20), (57,\pm20,68)$	$(12,\pm4,315), (35,\pm4,108), (51,\pm14,75)$
		$(41,\pm36,100), (57,\pm56,80)$	$(51,\pm20,76), (64,64,75)$
ε_{62}	16	(1,0,248), (8,8,33)	(8,0,31), (4,4,63)
ε_{66}	16	(1,0,264), (8,0,33)	(4,4,67), (8,8,35)
ε_{67}	24	$(1,0,4288), (17,\pm16,256)$	$(59,\pm28,76), (64,64,83)$
ε_{69}	32	$(1,0,1104), (13,\pm2,85), (16,16,73)$	(23,0,48), (12,12,95), $(32,\pm8,35)$
ε_{70}	16	(1,0,1120), (5,0,224)	(7,0,160), (32,0,35)
ε_{71}	56	$(1,0,4544), (36,\pm20,129)$	$(64,0,71), (48,\pm40,103)$
		$(57,\pm46,89), (73,\pm72,80)$	$(60,\pm52,87), (15,\pm2,303)$
ε_{77}	32	$(1,0,1232), (16,16,81), (36,\pm20,37)$	(7,0,176), (39,34,39), $(32,\pm24,43)$
ε_{78}	16	(1,0,1248), (13,0,96)	(12,12,107), (32,32,47)
ε_{79}	40	$(1,0,5056), (65,\pm54,89), (73,\pm56,80)$	$(64,0,79), (23,\pm4,220), (55,\pm26,95)$
ε_{83}	72	$(1,0,5312), (33,\pm2,161), (9,\pm8,592)$	$(3,\pm2,1771), (11,\pm2,483), (27,\pm26,203)$
		$(33,\pm20,164), (68,\pm28,81)$	$(59,\pm46,99), (64,64,99)$
ε_{86}	40	$(1,0,1376), (9,\pm2,153), (17,\pm2,81)$	$(3,\pm2,459), (27,\pm2,51), (32,32,51)$
ε_{87}	48	$(1,0,5568), (49,\pm38,121)$	$(3,0,1856), (28,\pm4,199)$
		$(77,38,77), (77,\pm60,84)$	$(44,\pm28,131), (64,64,103)$
ε_{91}	48	(1,0,5824), (13,0,448)	(7,0,832), (64,0,91)
		$(25,\pm2,233), (20,\pm12,293)$	$(47,\pm4,124), (43,\pm28,140)$
ε_{93}	16	(1,0,1488), (16,0,93)	(3,0,496), (31,0,48)
ε_{94}	16	(1,0,376), (8,8,49)	(8,0,407), (4,4,95)
ε_{95}	32	(1,0,1520), (20,20,81), $(36,\pm20,45)$	(19,0,80), (16,16,99), $(39,\pm28,44)$
ε_{102}	16	(1,0,1632), (17,0,96)	(12,12,139), (32,32,59)
ε_{103}	40	(1,0,6592), $(17,\pm4,388), (49,\pm22,137)$	(64,0,103),$(23,\pm6,287), (28,\pm20,239)$
ε_{105}	32	(1,0,1680), (5,0,336), (4,4,421), (20,20,89)	(7,0,240), (35,0,48), (28,28,67), (47,46,47)

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On square-free primitive roots mod p[1]

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Abstract Let $prim_{\square}(x)$ denote the number of square-free primitive roots not exceeding x modulo p and let $g_{\square}(p)$ denote the smallest square-free primitive root modulo p. For large real numbers x and y satisfying

$$\max(p^{1/4+\varepsilon}, x^{1/5+\varepsilon}) \leq y \leq x,$$

we show that

$$prim_{\square}(x+y) - prim_{\square}(x) = \frac{p\varphi(p-1)}{p^2-1} \cdot \frac{y}{\zeta(2)} + O(yp^{-\delta} + x^{1/5}\log x),$$

which implies that $g_{\square}(p) \ll p^{\frac{1}{4}+\epsilon}$.

Keywords Primitive root, square-free number, character sum.

§1. Introduction

Let p be an odd prime. For any integer n with $(n,p)=1$, let $\mathrm{ind}(n)$ denote the smallest positive integer l such that $n^l \equiv 1(\mathrm{mod}p)$. If $\mathrm{ind}(n) = p-1$, then we say n is a primitive root modulo p. This concept plays important roles in the number theory and hence attracts the interests of many authors.

Let $g(p)$ denote the smallest primitive root modulo p. Vinogradov [6] first proved that $g(p) \leq 2^m p^{1/2}\log p$, where $m = \omega(p-1)$. In [9] he improved this result to $g(p) \leq 2^m p^{1/2}\log\log p$. Hua [5] proved that $g(p) < 2^{m+1}p^{1/2}$. Erdös [2] proved that $g(p) \ll p^{1/2}\log^{17} p$. Erdös and Shapiro [3] proved $g(p) \ll m^{c_1}p^{1/2}$, where c_1 is an absolute positive constant. Burgess [1] and Wang Yuan [10] proved independently that

$$g(p) \ll p^{1/4+\varepsilon}. \tag{1}$$

Burgess [1] also proved that in any interval $[N, N+H]$ with $H > p^{1/4+\varepsilon}$, the number of primitive roots modulo p is

$$\frac{\varphi(p-1)}{p-1}H(1+O(p^{-\delta})), \tag{2}$$

where $\delta > 0$ is a constant depending only on ε.

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An integer n is called square-free if it is a product of different primes. Let $Q(x)$ denote the number of square-free numbers not exceeding x. Then it is well-known that

$$Q(x) = \frac{6}{\pi^2}x + O(x^{1/2}\mathrm{e}^{-c_2\delta(x)}),$$

where c_2 is an absolute positive constant, and $\delta(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$. At present we can not improve the exponent $1/2$ if we have no further knowledge of the distribution of the zeros of the Riemann Zeta-function.

Many authors studied the distribution of square-free numbers in short intervals (see [4] and references therein). The latest result in this direction is due to Filaseta and Trifonov [4], who proved that the asymptotic formula

$$Q(x+y) - Q(x) = \frac{6}{\pi^2}y(1+o(1)), \tag{3}$$

holds for $y \geq x^{1/5}\log^2 x$.

Now we consider the square-free primitive roots modulo p. Let $prim_{\square}(x)$ denote the number of square-free primitive roots not exceeding x modulo p and let $g_{\square}(p)$ denote the smallest square-free primitive root modulo p. From [8] we have

$$prim_{\square}(x) = \frac{p\varphi(p-1)}{p^2-1}\cdot\frac{6x}{\pi^2} + O(2^{\omega(p-1)}p^{1/4}x^{1/2}\log^{1/2}p), \tag{4}$$

which implies $g_{\square}(p) \ll 2^{\omega(p-1)}p^{1/2}\log p$. In [7] Liu Huaning and Zhang Wenpeng proved that the asymptotic formula

$$prim_{\square}(x) = \frac{p\varphi(p-1)}{p^2-1}\cdot\frac{6x}{\pi^2} + O(p^{9/44+\varepsilon}x^{1/2+\varepsilon}) \tag{5}$$

holds uniformly for x and p, which implies immediately that $g_{\square}(p) \ll p^{9/22+\varepsilon}$. We note that the exponent $1/2$ in the error term in (5) is sharp since we have to assume the Generalized Riemann Hypothesis (or at least a weak form of it) if we want to improve it.

As a consequence of Burgess's bound on character sums and the result of Filaseta and Trifonov, we shall prove the following Theorem in this short note.

Theorem. Let x and y be large real numbers such that

$$\max(p^{1/4+\varepsilon}, x^{1/5+\varepsilon}) \leq y \leq x,$$

then

$$prim_{\square}(x+y) - prim_{\square}(x) = \frac{p\varphi(p-1)}{p^2-1}\cdot\frac{y}{\zeta(2)} + O(yp^{-\delta} + x^{1/5}\log x), \tag{6}$$

where δ is a positive constant depending only ε.

Taking $x = y = p^{1/4+\varepsilon}$ we have

Corollary. The estimate $g_{\square}(p) \ll p^{1/4+\varepsilon}$ holds.

Remark. Obviously (6) is a combination of (2) and (3). The result of the Corollary is an analogue of (1).

Notations. Throughout this paper, ε denotes a fixed sufficiently small positive constant, $\mu(n)$ is the Möbius function, $\varphi(n)$ is the Euler function, $\omega(n)$ denotes the number of distinct prime divisors of n.

§2. Some preliminary Lemmas

We need the following Lemmas.

Lemma 2.1. Let $p > 2$ be a prime and

$$f(n) = \frac{\varphi(p-1)}{p-1}\{1 + \sum_{\substack{d|(p-1)\\ d>1}} \frac{\mu(d)}{\varphi(d)} \sum_{\chi_d} \chi_d(n)\},$$

the outer sum being over square-free integers d, and the inner sum being over all characters $\chi_d(\mathrm{mod}p)$ of order d. Then for any n coprime to p we have

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is a primitive root } (\mathrm{mod}p),\\ 0, & \text{otherwise.}\end{cases}$$

Proof. This is Lemma 5 of Burgess [1].

Lemma 2.2. For $\varepsilon > 0$, there exists a positive number δ depending only on ε such that if χ is a non-principal character to a (sufficiently large) prime modulus p, then for every N we have

$$\sum_{n=N+1}^{N+H} \chi(n) \ll Hp^{-\delta}$$

if $H > p^{1/4+\varepsilon}$.

Proof. This is Corollary of Burgess [1].

Lemma 2.3. Suppose $x^{1/5+\varepsilon} \le y \le x$, then

$$\sum_{\substack{x<nm^2\le x+y\\ m>x^{\eta}}} 1 \ll yx^{-\eta} + x^{1/5}\log x,$$

where $\eta > 0$ and $\varepsilon > 0$ are fixed small constants.

Proof. This estimate is contained in Filaseta and Trifonov [4].

Lemma 2.4. Suppose χ is a character modulo p, x and y are large real numbers such that

$$\max(p^{1/4+\varepsilon}, x^{1/5+\varepsilon}) \le y \le x.$$

If $\chi = \chi_0$, then we have

$$\sum_{x<n\le x+y} |\mu(n)|\chi(n) = \frac{p}{p+1}\cdot\frac{y}{\zeta(2)} + O(yp^{-\varepsilon^2} + x^{1/5}\log x). \tag{7}$$

If $\chi \ne \chi_0$, then we have

$$\sum_{x<n\le x+y} |\mu(n)|\chi(n) \ll yp^{-\delta}, \tag{8}$$

where $\delta > 0$ is a constant depending only on ε.

Proof. By the relation $|\mu(n)| = \sum_{n=d^2m} \mu(d)$ we have

$$\begin{aligned}\sum_{x<n\le x+y} |\mu(n)|\chi(n) &= \sum_{x<d^2m\le x+y} \mu(d)\chi(d^2m) \qquad (9)\\ &= \sum_{d\le p^{\varepsilon^2}} \chi^2(d)\mu(d) \sum_{x/d^2<m\le(x+y)/d^2} \chi(n) + O\left(\sum_{\substack{x<d^2m\le x+y\\ d>p^{\varepsilon^2}}} 1\right).\end{aligned}$$

We first prove that

$$\sum_{\substack{x<d^2m\le x+y\\ d>p^{\varepsilon^2}}} 1 \ll yp^{-\varepsilon^2}+x^{1/5}\log x. \tag{10}$$

If $p\ge x$, then by Lemma 2.3 directly we get

$$\begin{aligned}\sum_{\substack{x<d^2m\le x+y\\ d>p^{\varepsilon^2}}} 1 &\ll \sum_{\substack{x<d^2m\le x+y\\ d>x^{\varepsilon^2}}} 1 \ll yx^{-\varepsilon^2}+x^{1/5}\log x\\ &\ll yp^{-\varepsilon^2}+x^{1/5}\log x.\end{aligned}$$

If $p<x$, then we have

$$\begin{aligned}\sum_{\substack{x<d^2m\le x+y\\ d>p^{\varepsilon^2}}} 1 &\ll \sum_{\substack{x<d^2m\le x+y\\ p^{\varepsilon^2}<d\le x^{\varepsilon^2}}} 1+\sum_{\substack{x<d^2m\le x+y\\ d>x^{\varepsilon^2}}} 1\\ &\ll \sum_{p^{\varepsilon^2}<d\le x^{\varepsilon^2}} (1+y/d^2)+yx^{-\varepsilon^2}+x^{1/5}\log x\\ &\ll yp^{-\varepsilon^2}+x^{1/5}\log x,\end{aligned}$$

where we used Lemma 2.3 again.

Now we consider the first sum in the right-hand side of (9). If $\chi=\chi_0$, then for any $1\le N\le M$ we have

$$\sum_{M<n\le M+N}\chi_0(n)=\sum_{M<n\le M+N}1-\sum_{M<pn\le M+N}1=(1-1/p)N+O(1), \tag{11}$$

which combining (9) and (10) gives

$$\begin{aligned}\sum_{x<n\le x+y}|\mu(n)|\chi_0(n) &= (1-\frac{1}{p})y\sum_{d\le p^{\varepsilon^2}}\frac{\chi_0(d)\mu(d)}{d^2}+O(yp^{-\varepsilon^2}+x^{1/5}\log x)\\ &= (1-\frac{1}{p})y\sum_{d\ge 1}\frac{\chi_0(d)\mu(d)}{d^2}+O(yp^{-\varepsilon^2}+x^{1/5}\log x)\\ &= (1-\frac{1}{p})y\prod_{q\ne p}(1-\frac{1}{q^2})+O(yp^{-\varepsilon^2}+x^{1/5}\log x)\\ &= \frac{p}{p+1}y\prod_{q}(1-\frac{1}{q^2})+O(yp^{-\varepsilon^2}+x^{1/5}\log x)\\ &= \frac{p}{p+1}\frac{y}{\zeta(2)}+O(yp^{-\varepsilon^2}+x^{1/5}\log x).\end{aligned}$$

Now suppose $\chi\ne\chi_0$. Since ε is sufficiently small, from $d\le p^{\varepsilon^2}$ and $x>p^{1/4+\varepsilon}$ we get $x/d^2>p^{1/4+\varepsilon/2}$. By Lemma 2.2 we get that the estimate

$$\sum_{x/d^2<n\le(x+y)/d^2}\chi(n)\ll\frac{y}{d^2}p^{-\delta}$$

holds for some $\delta=\delta(\varepsilon)>0$, which combining (9) and (10) gives (8).

§3. Proof of the theorem

In this section, we prove Theorem. Suppose $\max(p^{1/4+\varepsilon}, x^{1/5+\varepsilon}) \le y \le x$. By Lemma 2.1 and Lemma 2.4 we have

$$\begin{aligned} prim_{\square}(x+y) - prim_{\square}(x) &= \sum_{\substack{x<n\le x+y \\ (n,p)=1}} |\mu(n)| f(n) \\ &= \frac{\varphi(p-1)}{p-1}\left\{ \sum_{x<n\le x+y} |\mu(n)|\chi_0(n) + \sum_{\substack{d\mid(p-1) \\ d>1}} \frac{\mu(d)}{\varphi(d)} \sum_{\chi_d} \sum_{x<n\le x+y} \chi_d(n) \right\} \\ &= \frac{p\varphi(p-1)}{p^2-1}\cdot\frac{y}{\zeta(2)} + O(2^{\omega(p-1)} y p^{-\delta} + x^{1/5}\log x) \\ &= \frac{p\varphi(p-1)}{p^2-1}\cdot\frac{y}{\zeta(2)} + O(y p^{-\delta/2} + x^{1/5}\log x), \end{aligned}$$

where we used the fact that the number of characters mod p of order d is $\varphi(d)$ and that the estimate

$$2^{\omega(p-1)} \ll e^{C\frac{\log p}{\log\log p}} \ll p^{\delta/2}$$

holds for some absolute constant $C > 0$. This completes the proof of Theorem .

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A new limit theorem involving the Smarandache LCM sequence

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Abstract The main purpose of this paper is using the elementary method to study the LCM Sequence, and give an asymptotic formula about this sequence.

Keywords Smarandache LCM Sequence, limitation.

§1. Introduction and results

For any positive integer n, we define $L(n)$ is the Least Common Multiply (LCM) of the natural number from 1 through n. That is

$$L(n) = [1, 2, \cdots, n].$$

The Smarandache Least Common Multiply Sequence is defined by:

$$\text{SLS} \longrightarrow L(1),\ L(2),\ L(3),\ \cdots,\ L(n),\ \cdots.$$

The first few numbers are: 1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, $\cdots$.

About some simple arithmetical properties of $L(n)$, there are many results in elementary number theory text books. For example, for any positive integers a, b and c, we have

$$[a, b] = \frac{ab}{(a, b)} \quad \text{and} \quad [a, b, c] = \frac{abc \cdot (a, b, c)}{(a, b)(b, c)(c, a)},$$

where $(a_1, a_2, \cdots, a_k)$ denotes the Greatest Common Divisor of a_1, a_2, $\cdots$, a_{k-1} and a_k. But about the deeply arithmetical properties of L(n), it seems that none had studied it before, but it is a very important arithmetical function in elementary number theory. The main purpose of this paper is using the elementary methods to study a limit problem involving $L(n)$, and give an interesting limit theorem for it. That is, we shall prove the following:

Theorem. For any positive integer n, we have the asymptotic formula

$$\left(\frac{L(n^2)}{\prod\limits_{p \le n^2} p}\right)^{\frac{1}{n}} = e + O\left(\exp\left(-c\frac{(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}}\right)\right),$$

where $\prod\limits_{p \le n^2}$ denotes the production over all prime $p \le n^2$.

From this Theorem we may immediately deduce the following:

Corollary. Under the notations of above, we have

$$\lim_{n\to\infty}\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = e,$$

where $\mathrm{L}(n^2) = [1, 2, \cdots, n^2]$, p is a prime.

§2. Proof of the theorem

In this section, we shall complete the proof of this theorem. First we need the following simple Lemma.

Lemma. For $x > 0$, we have the asymptotic formula

$$\theta(x) = \sum_{p\le x} \ln p = x + O\left(x\exp\left(\frac{-c(\ln x)^{\frac{3}{5}}}{(\ln\ln x)^{\frac{1}{5}}}\right)\right),$$

where $c > 0$ is a constant, $\sum\limits_{p\le x}$ denotes the summation over all prime $p \le x$.

Proof. In fact, this is the different form of the famous prime theorem. Its proof can be found in reference [2].

Now we use this Lemma to prove our Theorem.

Let

$$L(n^2) = [1,\ 2,\ \cdots,\ n^2] = p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_s^{\alpha_s}, \tag{1}$$

be the factorization of $L(n^2)$ into prime powers, then $\alpha_i = \alpha(p_i)$ is the highest power of p_i in the factorization of 1, 2, 3, $\cdots$, n^2. Since

$$\left(\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right)^{\frac{1}{n}} = \exp\left(\frac{1}{n}\ln\frac{L(n^2)}{\prod\limits_{p\le n^2} p}\right) = \exp\left(\frac{1}{n}\left(\ln L(n^2) - \ln\prod_{p\le n^2} p\right)\right),$$

while

$$\begin{aligned}
\ln L(n^2) - \ln\prod_{p\le n^2} p &= \ln\left(p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_n^{\alpha_n}\right) - \ln\prod_{p\le n^2} p \\
&= \sum_{p\le n^2} \alpha(p)\ln p - \sum_{p\le n^2}\ln p \\
&= \sum_{p\le n^2} (\alpha(p)-1)\ln p \\
&= \sum_{p\le n^{\frac{2}{3}}} (\alpha(p)-1)\ln p + \sum_{n^{\frac{2}{3}}<p\le n} (\alpha(p)-1)\ln p \\
&\quad + \sum_{n<p\le n^2} (\alpha(p)-1)\ln p. \qquad (2)
\end{aligned}$$

In (1), it is clear that if $n < p_i \leq n^2$, then $\alpha(p_i) = 1$. If $n^{\frac{2}{3}} < p_i \leq n$, we have $\alpha(p_i) = 2$. (In fact if $\alpha(p_i) \geq 3$, then $p_i^3 > n$. This contradiction with $p_i \leq n$). If $p_i \leq n^{\frac{2}{3}}$, then $\alpha(p_i) \geq 3$. So from these and above Lemma we have

$$\sum_{n^{\frac{2}{3}}<p\leq n} (\alpha(p)-1)\ln p = \sum_{n^{\frac{2}{3}}<p\leq n} (2-1)\ln p = \sum_{n^{\frac{2}{3}}<p\leq n} \ln p, \tag{3}$$

$$\sum_{n<p\leq n^2} (\alpha(p)-1)\ln p = \sum_{n<p\leq n^2} (1-1)\ln p = 0, \tag{4}$$

$$\sum_{p\leq n^{\frac{2}{3}}} (\alpha(p)-1)\ln p = O\left(\ln^2 n \sum_{p\leq n^{\frac{2}{3}}} 1\right) = O\left(\ln^2 n \frac{n^{\frac{2}{3}}}{\ln n}\right) = O\left(n^{\frac{2}{3}}\ln n\right). \tag{5}$$

Now combining (2), (3), (4) and (5) we may immediately get

$$\begin{aligned}
\ln L(n^2) - \ln \prod_{p\leq n^2} p &= O\left(n^{\frac{2}{3}}\ln n\right) + \sum_{n^{\frac{2}{3}}<p\leq n} \ln p \\
&= O\left(n^{\frac{2}{3}}\ln n\right) + \sum_{p\leq n} \ln p - \sum_{p\leq n^{\frac{2}{3}}} \ln p \\
&= O\left(n^{\frac{2}{3}}\ln n\right) + n + O\left(n\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right) \\
&\quad -n^{\frac{2}{3}} - O\left(n^{\frac{2}{3}}\exp\left(\frac{-c(\ln n^{\frac{2}{3}})^{\frac{3}{5}}}{(\ln\ln n^{\frac{2}{3}})^{\frac{1}{5}}}\right)\right) \\
&= n + O\left(n\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

That is,

$$\begin{aligned}
\left(\frac{L(n^2)}{\prod_{p\leq n^2} p}\right)^{\frac{1}{n}} &= \exp\left(\frac{1}{n}\left(\ln L(n^2) - \ln \prod_{p\leq n^2} p\right)\right) \\
&= \exp\left[\frac{1}{n}\left[n + O\left(n\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\right]\right] \\
&= \exp\left[1 + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e\cdot\exp\left[O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e\left[1 + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right)\right] \\
&= e + O\left(\exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem.

The Corollary follows from Theorem with $n \to \infty$.

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The Smarandache Perfect Numbers

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Abstract In this paper we prove that 12 is the only Smarandache perfect number.

Keywords Smarandache function, Smarandache perfect number, divisor function.

§1. Introduction and result

Let N be the set of all positive integer. For any positive integer a, let $S(a)$ denote the Smarandache function of a. Let n be a postivie integer. If n satisfy

$$\sum_{d|n} S(d) = n + 1 + S(n), \tag{1}$$

then n is called a Smarandache perfect number. Recently, Ashbacher [1] showed that if $n \leq 10^6$, then 12 is the only Smarandache perfect number. In this paper we completely determine all Smarandache perfect number as follows:

Theorem. 12 is the only Smarandache perfect number.

§2. Proof of the theorem

The proof of our theorem depends on the following lemmas.

Lemma 1 ([2]). For any positive integer n with $n > 1$, if

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \tag{2}$$

is the factorization of n, then we have

$$S(n) = \max\left(S(p_1^{r_1}), S(p_2^{r_2}), \cdots, S(p_k^{r_k})\right).$$

Lemma 2 ([2]). For any prime p and any positive integer r, we have $S(p^r) \leq pr$.

Lemma 3 ([3], Theorem 274). Let $d(n)$ denote the divisor function of n. Then $d(n)$ is a multiplicative function. Namely, if (2) is the factorization of n, then

$$d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1).$$

Lemma 4. The inequality

$$\frac{n}{d(n)} < 2, n \in N. \tag{3}$$

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has only the solutions $n = 1, 2, 3, 4$ and 6.

Proof. For any positive integer n, let

$$f(n) = \frac{n}{d(n)}.$$

Since $f(1) = 1$, $f(2) = 1$, $f(3) = 3/2$, $f(4) = 4/3$, and $f(6) = 3/2$, (3) has solutions $n = 1, 2, 3, 4$ and 6.

Let n be a solution of (3) with $n \neq 1, 2, 3, 4$ or 6. Since $f(5) = \frac{5}{2} > 2$, we have $n > 6$. Let (2) be the factorization of n. If $k = 1$ and $r_1 = 1$, then $n = p_1 \geq 7$ and $2 > f(n) = \frac{p_1}{2} \geq \frac{7}{2}$, a contradiction. If $k = 1$ and $r_1 = 2$, then $n = p_1^2$, where $p_1 \geq 3$. So we have $2 > f(n) = \frac{p_1^{r_1}}{(r_1+1)} \geq \frac{2^3}{4} \geq 2$, a contradiction. If $k = 2$, since $n > 6$, then we get

$$2 > f(n) = \frac{p_1^{r_1}}{r_1 + 1} \cdot \frac{p_2^{r_2}}{r_2 + 1} \geq \begin{cases} \frac{5}{2} & \text{if } p_1 = 2 \text{ and } r_1 = 1, \\ 2 & \text{if } p_1 = 2 \text{ and } r_1 > 1, \\ \frac{15}{4} & \text{if } p_1 > 2, \end{cases}$$

a contradiction. If $k \geq 3$, then

$$2 > f(n) = \frac{p_1^{r_1}}{(r_1 + 1)} \frac{p_2^{r_2}}{(r_2 + 1)} \frac{p_3^{r_3}}{(r_3 + 1)} \geq \frac{15}{4},$$

a contradiction. To sum up, (3) has no solution n with $n \neq 1, 2, 3, 4$ or 6. The Lemma is proved.

Proof of Theorem. Let n be a Smarandache perfect number with $n \neq 12$. By [1] we have $n > 10^6$. By Lemma 1, if (2) is the factorization of n, Then

$$S(n) = S(p^r), \tag{4}$$

where

$$p = p_j, \quad r = r_j, \quad 1 \leq j \leq k. \tag{5}$$

From (2) and (5), we get

$$n = p^r m, \quad m \in N, \quad \texttt{gcd}(p^r, m) = 1. \tag{6}$$

For any positive integer n, let

$$g(n) = \sum_{d|n} S(d). \tag{7}$$

Then, by (1), the Smarandache perfect number n satisfies

$$g(n) = n + 1 + S(n). \tag{8}$$

We see from (4) that $n|S(p^r)!$. Therefore, for any divisor d of n, we have

$$S(d) \leq S(p^r). \tag{9}$$

Thus, if (8) holds, then from (7) and (9) we obtain

$$d(n)S(p^r) > n. \tag{10}$$

where $d(n)$ is the divisor function of n. Further, by Lemma 3, we get from (4), (6) and (10) that

$$\frac{(r+1)S(p^r)}{p^r} > f(m). \tag{11}$$

If $r = 1$, since $S(p) = p$, then from (11) we get $2 > f(m)$. Hence, by Lemma 4, we obtain $m = 1, 2, 3, 4$ or 6. When $m = 1$, we get from (8) that

$$g(n) = g(p) = S(1) + S(p) = 1 + p = p + 1 + S(p) = 1 + 2p,$$

a contradiction. When $m = 2$, we have $p > 2$ and

$$g(n) = g(p) = S(1) + S(2) + S(p) + S(2p) = 3 + 2p = 3p + 1, \tag{12}$$

whence we get $p = 2$, a contradiction. By the same method, we can prove that if $r = 1$ and $m = 3, 4$ or 6, then (8) is false.

If $r = 2$, since $S(p^2) = 2p$, then from (11) we get

$$\frac{6}{p} > f(m). \tag{13}$$

Since $n > 10^6$, by (4) we have $S(p^2) = S(n) \geq 10$ it implies that $p \geq 5$. Hence, by (13) we get $f(m) < \frac{6}{5}$. Further, by Lemma 4 we get $m \leq 6$. Since $n = p^2 m \leq 6p^2$, we obtain $p \geq 7$. Therefore, by (13) it is impossible. By the same method, we can prove that if $r = 3, 4, 5$ or 6, then (11) is false.

If $r \geq 7$, then we have $S(p^r) \leq pr$ and

$$\frac{(r+1)r}{p^{r-1}} > \frac{(r+1)S(p^r)}{p^r} > f(m) \geq 1, \tag{14}$$

by (11). From (14), we get

$$(r+1)r > p^{r-1} \geq 2^{r-1} \geq 2\left(\binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2} + \binom{r-1}{3}\right), \tag{15}$$

whence we obtain

$$0 > r^2 - 6r + 5 = (r-1)(r-5) > 0, \tag{16}$$

a contradiction. To sum up, there has no Smarandache perfect number n with $n > 10^6$. Thus, the theorem is proved.

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On the solutions of an equation involving the Smarandache dual function

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Abstract In this paper, we use the elementary method to study the solutions of an equation involving the Smarandache dual function $\bar{s}_k(n)$, and give its all solutions.

Keywords Smarandache dual function, the positive integer solutions.

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined by

$$S(n) = \max\{m : n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $\cdots$. About the arithmetical properties of $S(n)$, many scholars have show their interest on it, see [1], [2] and [3]. For example, Farris Mark and Mitchell Patrick [2] studied the bounding of Smarandache function, and they gave an upper and lower bound for $S(p^\alpha)$, i.e.

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Wang Yongxing [3] studied the mean value of $\sum_{n \leq x} S(n)$ and obtained an asymptotic formula by using the elementary methods. He proved that

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Similarly, many scholars studied another function which have close relations with the Smarandache function. It is called the Smarandache dual function $S^*(n)$ which defined by

$$S^*(n) = \max\{m : m! \mid n\}.$$

About this function, J. Sandor in [4] conjectured that

$$S^*((2k-1)!(2k+1)!) = q - 1,$$

where k is a positive integer, q is the first prime following $2k+1$. This conjecture was proved by Le Maohua [5].

Li Jie [6] studied the mean value property of $\sum_{n\leq x} S^*(n)$ by using the elementary methods, and obtained an interesting asymptotic formula:

$$\sum_{n\leq x} S^*(n) = ex + O\left(\ln^2 x(\ln\ln x)^2\right).$$

In this paper, we introduce another Smarandache dual function $\bar{s}_k(n)$ which denotes the greatest positive integer m such that $m^k|n$, where n denotes any positive integer. That is,

$$\bar{s}_k(n) = \max\{m : m^k|n\}.$$

On the other hand, we let $\Omega(n)$ denotes the number of the prime divisors of n, including multiple numbers. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then

$$\Omega(n) = \alpha_1 + \alpha_2\cdots + \alpha_r.$$

In this paper, we shall study the positive integer solutions of the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n),$$

and give its all solutions. That is, we shall prove the following conclusions:

Theorem. For all positive integer n, the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n)$$

has only three solutions. They are $n = 3$, 6, 8.

For general positive integer $k > 3$, whether there exists finite solutions for the equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n).$$

It is an unsolved problem. We believe that it is true.

§2. Proof of the theorem

In this section, we will complete the proof of Theorem. First we will separate all positive integer into two cases.

1. If $n \leq 8$, then from the definition of $\bar{s}_k(n)$ and $\Omega(n)$, we have

$$\bar{s}_3(1) = 1, \quad \bar{s}_3(2) = 1, \quad \bar{s}_3(3) = 1, \quad \bar{s}_3(4) = 1,$$

$$\bar{s}_3(5) = 1, \quad \bar{s}_3(6) = 1, \quad \bar{s}_3(7) = 1, \quad \bar{s}_3(8) = 2.$$

$$\Omega(1) = 0, \quad \Omega(2) = 1, \quad \Omega(3) = 1, \quad \Omega(4) = 2,$$

$$\Omega(5) = 1, \quad \Omega(6) = 2, \quad \Omega(7) = 1, \quad \Omega(8) = 3.$$

So that we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \bar{s}_3(3) = 3\Omega(3);$$

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(6) = 3\Omega(6);$$

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(8) = 3\Omega(8).$$

Hence $n = 3$, 6, 8 are the positive integer solutions of the equation.

2. If $n > 8$, then we have the following:

Lemma. For all positive integer $n > 8$, we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) > 3\Omega(n).$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the factorization of n into prime powers, then we have

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) > n \quad \text{if} \quad n > 8.$$

From the definition of $\Omega(n)$, we have

$$\Omega(n) = \alpha_1 + \alpha_2 \cdots + \alpha_r.$$

So to complete the proof of the lemma, we only prove the following inequality:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r). \tag{1}$$

Now we prove (1) by mathematical induction on r.

i) If $r = 1$, then $n = p_1^{\alpha_1}$.

a. If $p_1 = 2$, then we have $\alpha_1 \geq 4$, hence

$$2^4 > 3 \cdot 4, \quad 2^{\alpha_1} > 3\alpha_1.$$

b. If $p_1 = 3$, 5 and 7, then we have $\alpha_1 \geq 2$, hance

$$i^4 > 3 \cdot 2, \quad i^{\alpha_1} > 3\alpha_1, \quad i = 3,\ 5,\ 7.$$

c. If $p_1 \geq 11$, then we have $\alpha_1 \geq 1$, hence

$$p_1^{\alpha_1} > 3\alpha_1.$$

This proved that Lemma holds for $r = 1$.

ii) Now we assume (1) holds for r (≥ 2), and prove that it is also holds for $r + 1$.

From the inductive hypothesis, we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot p_{r+1}^{\alpha_{r+1}}.$$

Since p_{r+1} is a prime, then

$$p_{r+1}^{\alpha_{r+1}} > \alpha_{r+1} + 1.$$

From above we obtain

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1).$$

Note that if $a > 1$, $b > 1$, then $a \cdot b \geq a + b$, so we have

$$(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1) \geq \alpha_1 + \alpha_2 \cdots + \alpha_r + \alpha_{r+1} + 1 > \alpha_1 + \alpha_2 \cdots + \alpha_r + \alpha_{r+1}.$$

So

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}p_{r+1}^{\alpha_{r+1}} > 3(\alpha_1+\alpha_2\cdots+\alpha_r+\alpha_{r+1}).$$

This completes the proof of the lemma.

Now we complete the proof of Theorem. From the lemma we know that the equation has no positive solutions if $n > 8$. In other words, the equation

$$\bar{s}_3(1)+\bar{s}_3(2)+\cdots+\bar{s}_3(n) = 3\Omega(n)$$

has only three solutions. They are $n = 3, 6, 8$.

This completes the proof of Theorem.

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Mean value of a Smarandache-Type Function

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Abstract In this paper, we use analytic method to study the mean value properties of Smarandache-Type Multiplicative Functions $K_m(n)$, and give its asymptotic formula . Finally, the convolution method is used to improve the error term.

Keywords Smarandache-Type Multiplicative Function, the Convolution method.

§1. Introduction

Suppose $m \geq 2$ is a fixed positive integer. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, we define

$$K_m(n) = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}, \quad \beta_i = \mathtt{min}(\alpha_i, m-1),$$

which is a Smarandache-type multiplicative function . Yang Cundian and Li Chao proved in [1] that

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O(x^{\frac{3}{2}+\epsilon}).$$

In this paper, we shall use the convolution method to prove the following

Theorem. The asymptotic formula

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O(x^{1+\frac{1}{m}} e^{-c_0 \delta(x)})$$

holds, where c_0 is an absolute positive constant and $\delta(x) = (\log x)^{3/5} (\log \log x)^{-1/5}$.

§2. Proof of the theorem

In order to prove our Theorem, we need the following Lemma, which is Lemma 14.2 of [2].

Lemma. Let $f(n)$ be an arithmetical function for which :

$$\sum_{n \leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$

$$\sum_{n \leq x} | f(n) | = O(x^{a_1} \log^r x),$$

where $a_1 \geq a_2 \geq \ldots \geq a_l > 1/k > a \geq 0, r \geq 0, P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r, and $k \geq 1$ is a fixed integer. If

$$h(n) = \sum_{d^k|n} \mu(d) f(n/d^k),$$

then

$$\sum_{n \leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E(x),$$

where $R_1(t), \ldots, R_l(t)$ are polynomials in t of degrees not exceeding r, and for some $D > 0$

$$E(x) \ll x^{1/k} exp\Big(- D(\log x)^{3/5} (\log\log x)^{-1/5} \Big).$$

Now we prove our Theorem. Let

$$g(s) = \sum_{n=1}^{\infty} \frac{K_m(n)}{n^s}, \Re(s) > 2.$$

According to Euler's product formula, we write

$$\begin{aligned}
g(s) \quad &= \prod_p \left(1 + \frac{K_m(p)}{p^s} + \frac{K_m(p^2)}{p^{2s}} + \cdots \right) \\
&= \prod_p \left(1 + \frac{K_m(p)}{p^s} + \frac{(K_m(p^2))}{p^{2s}} + \cdots \right) \\
&= \prod_p \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{m-1}}{p^{(m-1)s}} + \frac{p^{m-1}}{p^{ms}} + \frac{p^{m-1}}{p^{(m+1)s}} + \cdots \right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-1}} + \frac{1}{p^{2(s-1)}} + \cdots + \frac{1}{p^{(m-1)(s-1)}} + \frac{p^{m-1}}{p^{ms}} + \frac{p^{m-1}}{p^{(m+1)s}} + \cdots \right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} + \frac{p^{m-1}}{p^{ms}} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots) \right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} + \frac{p^{m-1}}{p^{ms}} \frac{1}{1 - \frac{1}{p^s}} \right) \\
&= \prod_p \frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} \left(1 + \frac{p^{s-1} - 1}{(p^s - 1)(p^{m(s-1)} - 1)} \right) \\
&= \frac{\zeta(s-1)}{\zeta\big(m(s-1)\big)} R(s),
\end{aligned}$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{s-1} - 1}{(p^s - 1)(p^{m(s-1)} - 1)} \right).$$

Let $q_m(n)$ denote the characteristic function of m-free numbers, then

$$\frac{\zeta(s)}{\zeta(ms)} = \sum_{n=1}^{\infty} \frac{q_m(n)}{n^s}, \quad \frac{\zeta(s-1)}{\zeta(m(s-1))} = \sum_{n=1}^{\infty} \frac{q_m(n)n}{n^s}.$$

Suppose

$$R(s) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s},$$

then

$$K_m(n) = \sum_{n=l_1 l_2} q_m(l_1) l_1 r(l_2).$$

Obviously, when $\sigma > 1$, R(s) absolutely converges, namely

$$\sum_{l \le x} | r(l) | \ll x^{1+\varepsilon}. \tag{1}$$

We can write $q_m(n)$ as the following form

$$q_m(n) = \sum_{d^k | n} \mu(d)$$

Now we apply the lemma on taking $f(n) = 1, \quad l = a_1 = 1, r = a = 0$, then we have

$$\sum_{n \le x} q_m(n) = \frac{x}{\zeta(m)} + O\left(x^{\frac{1}{m}} e^{-c_1 \delta(x)} \right)$$

for some absolute constant $c_1 > 0$.

By partial summation,

$$\sum_{n \le x} q_m(n) n = \frac{x^2}{2\zeta(m)} + O(x^{1+\frac{1}{m}} e^{-c_2 \delta(x)}) \tag{2}$$

holds for some absolute constant $c_2 > 0$. Let $y = x^{1-1/2m}$. By hyperbolic summation , we write

$$\begin{aligned} \sum_{n \le x} K_m(n) &= \sum_{l_1 l_2 \le x} q_m(l_1) l_1 r(l_2) \qquad (3) \\ &= \sum_{l_2 \le y} r(l_2) \sum_{l_1 \le \frac{x}{l_2}} q_m(l_1) l_1 + \sum_{l_1 \le \frac{x}{y}} q_m(l_1) l_1 \sum_{l_1 \le \frac{x}{l_1}} r(l_2) - \sum_{l_2 \le y} r(l_2) \sum_{l_1 \le \frac{x}{y}} q_m(l_1) l_1 \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 . \end{aligned}$$

From (1) we get

$$\sum\nolimits_2 \ll \sum_{l_1 \le \frac{x}{y}} l_1 \left(\frac{x}{l_1} \right)^{1+\varepsilon} \ll \frac{x^{2+\varepsilon}}{y} \ll x^{1+1/2m+\varepsilon}. \tag{4}$$

Similarly

$$\sum\nolimits_3 \ll \frac{x^{2+\varepsilon}}{y} \ll x^{1+1/2m+\varepsilon}. \tag{5}$$

Finally for $\sum_1$ we have by (2)

$$\begin{aligned}\sum\nolimits_1 &= \frac{x^2}{2\zeta(m)}\sum_{l_2\le y}\frac{r(l_2)}{l_2^2}+O\Big(\sum_{l_2\le y}x^{1+\frac{1}{m}}l_2^{-1-\frac{1}{m}}e^{-c_2\delta\left(\frac{x}{l_2}\right)}\Big) \\ &= \frac{x^2}{2\zeta(m)}R(2)+O\Big(x^2\sum_{l_2>y}\frac{r(l_2)}{l_2^2}\Big)+O\Big(x^{1+\frac{1}{m}}e^{-c_0\delta\left(x\right)}\Big) \\ &= \frac{x^2}{2\zeta(m)}R(2)+O\Big(\frac{x^{2+\varepsilon}}{y}\Big)+O\Big(x^{1+\frac{1}{m}}e^{-c_0\delta\left(x\right)}\Big) \\ &= \frac{x^2}{2\zeta(m)}R(2)+O\Big(x^{1+\frac{1}{m}}e^{-c_0\delta(x)}\Big),\end{aligned} \tag{6}$$

if we noticed that

$$\sum_{l_2>y}\frac{r(l_2)}{l_2^2}\ll y^{-1+\varepsilon},$$

which follows from (1) by partial summation.

Now our Theorem follows from (3)-(6).

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Scientia Magna
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On the mean value of the Near Pseudo Smarandache Function

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Abstract The main purpose of this paper is using the analytic method to study the asymptotic properties of the Near Pseudo Smarandache Function, and give two interesting asymptotic formulae for it.

Keywords Near Pseudo Smarandache Function, mean value, asymptotic formula.

§1. Introduction

In reference [1], David Gorski defined the Pseudo Smarandache function $Z(n)$ as: let n be any positive integer, $Z(n)$ is the smallest integer such that $1+2+3+\ldots+Z(n)$ is divisible by n. In reference [2], A.W.Vyawahare defined a new function $K(n)$ which is a slight modification of $Z(n)$ by adding a smallest natural number k, so this function is called "Near Pseudo Smarandache Function". It is defined as follows: let n be any positive integer, $K(n) = m$, where $m = \sum_{n=1}^{n} n + k$ and k is the smallest natural number such that n divides m. About the mean value properties of the smallest natural number k in Near Pseudo Smarandache function, it seems that none had studied them before, at least we couldn't find any reference about it. In this paper, we use the analytic method to study the mean value properties of $d(k)$ and $\varphi(k)$, and give two interesting asymptotic formulae for it. That is, we shall prove the following:

Theorem 1. *Let k is the smallest natural number such that n divides Near Pseudo Smarandache function $K(n)$, $d(n)$ denotes Dirichlet divisor function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} d(k) = \sum_{n\leq x} d\left(K(n) - \frac{n(n+1)}{2}\right) = \frac{3}{4}x\log x + Ax + O(x^{\frac{1}{2}}\log^2 x),$$

where A is a computable constant.

Theorem 2. *For any real number $x \geq 1$, k is the smallest natural number such that n divides Near Pseudo Smarandache function $K(n)$, $\varphi(n)$ denotes the Euler's totient function. Then we have the asymptotic formula*

$$\sum_{n\leq x} \varphi\left(K(n) - \frac{n(n+1)}{2}\right) = \frac{93}{28\pi^2}x^2 + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number.

§2. Some lemmas

To complete the proof of the theorems, we need the following several simple Lemmas:

Lemma 1. *Let n be any positive integer, then we have*

$$K(n) = \begin{cases} \frac{n(n+3)}{2}, & \text{if } n \text{ is odd,} \\ \frac{n(n+2)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. (See reference [2]).

Lemma 2. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(n) = x \log x + (2C-1)x + O\left(\sqrt{x}\right),$$

where C is the Euler constant,

$$\sum_{n \leq x} \varphi(k) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Proof. These results can be get immediately from [3].

Lemma 3. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(2n) = \frac{3}{2} x \log x + \left(\frac{\log 2}{2} - \frac{3}{2}\right) x + O\left(x^{\frac{1}{2}} \log^2 x\right),$$

$$\sum_{n \leq x} \varphi(2n) = \frac{2}{7\zeta(2)} x^2 + O(x^{\frac{3}{2}+\epsilon}).$$

Proof. Firstly, we shall prove the first formula of Lemma 3. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{d(2n)}{n^s}$. Note that $d(2n) \ll n^{\epsilon}$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for Re$(s)>1$, by the Euler product formula [3] and the definition of $d(n)$ we get

$$\begin{aligned} f(s) &= \prod_p \sum_{m=0}^{\infty} \frac{d(2p^m)}{p^{ms}} \\ &= \sum_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}} \cdot \prod_{p>2} \sum_{m=0}^{\infty} \frac{d(2p^m)}{p^{ms}} \\ &= 2\zeta^2(s) \cdot \frac{\left(\prod\limits_{p>2} \sum\limits_{m=0}^{\infty} \frac{d(p^m)}{p^{ms}}\right) \cdot \left(\sum\limits_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}}\right)}{\prod\limits_p \sum\limits_{m=0}^{\infty} \left(\frac{d(p^m)}{p^{ms}}\right)} \\ &= 2\zeta^2(s) \cdot \frac{\sum\limits_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}}}{\sum\limits_{m=0}^{\infty} \frac{d(2^m)}{2^{ms}}} \\ &= \zeta^2(s)\left(2 - \frac{1}{2^s}\right). \end{aligned} \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ dentoes the product over all primes.

From (1) and the Perron's formula [4], for $b = 1+\epsilon, T \geq 1$ and $x \geq 1$ we have

$$\sum_{n\leq x} d(2n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\left|\frac{x^b}{T}\right| + O\left(\frac{xH(2x)\log x}{T}\right). \tag{2}$$

Taking $a = \frac{1}{2}+\epsilon$, we move the integral line in (2). Then

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s} \\ &+ \frac{1}{2\pi i}\left|\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s}ds \\ &+ O\left|\frac{x^b}{T}\right| + O\left|\frac{xH(2x)\log x}{T}\right|,\end{aligned}$$

where

$$\left|\int_{b-iT}^{a-iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds \ll \frac{x}{T},$$

$$\int_{a-iT}^{a+iT}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds \ll x^{\frac{1}{2}}\log^2 T.$$

Hence, we have

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left|\frac{x}{T}\right| \\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right) + O\left|\frac{x^b}{T}\right| + O\left|xH(2x)\frac{\log x}{T}\right| \\ &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left|\frac{x}{T}\right| \\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right) + O\left|x^{1+\varepsilon}\frac{\log x}{T}\right|.\end{aligned} \tag{3}$$

Taking $T = x^{\frac{1}{2}+\epsilon}$ in (3), then

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left(x^{\frac{1}{2}-\varepsilon}\right) + O\left(x^{\frac{1}{2}}\log^2 x\right) \\ &= \operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s} + O\left(x^{\frac{1}{2}}\log^2 x\right).\end{aligned} \tag{4}$$

Now we can easily get the residue of the function $\zeta^2(s)\left(2-\frac{1}{2^s}\right)\cdot\frac{x^s}{s}$ at second order pole point $s = 1$ with

$$\operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s} = \frac{3}{2}x\log x + \left(\frac{\log 2}{2}-\frac{3}{2}\right)x. \tag{5}$$

Combining (4) and (5), we may immediately get

$$\sum_{n\leq x} d(2n) = \frac{3}{2}x\log x + \left(\frac{\log 2}{2}-\frac{3}{2}\right)x + O\left(x^{\frac{1}{2}}\log^2 x\right).$$

This completes the proof of the first formula of Lemma 3.

Let $h(s) = \sum\limits_{n=1}^{\infty} \frac{\varphi(2n)}{n^s}$. From Euler product formula [2] and the definition of $\varphi(n)$ we also have

$$\begin{aligned} h(s) &= \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2p^m)}{p^{ms}}\right) \\ &= \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right) \cdot \prod_{p>2} \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2p^m)}{p^{ms}}\right) \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{\prod\limits_{p>2} \left(1 + \sum\limits_{m=1}^{\infty} \frac{\varphi(p^m)}{p^{ms}}\right) \cdot \left(1 + \sum\limits_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right)}{\prod\limits_p \left(1 + \sum\limits_{m=1}^{\infty} \frac{\varphi(p^m)}{p^{ms}}\right)} \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{\left(1 + \sum\limits_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right)}{\left(1 + \sum\limits_{m=1}^{\infty} \frac{\varphi(2^m)}{2^{ms}}\right)} \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{2^s}{2^s+3}. \end{aligned}$$

By Perron formula [4] and the method of proving the first formula of Lemma 3, we can obtain the second formula of Lemma 3.

§3. Proof of the theorems

In this section, we will complete the proof of the Theorems. From the first formula of Lemma 3 we can obtain

$$\sum_{n \le \frac{x}{2}} d(2n) = \frac{3}{4} x \log x - \left(\frac{\log 2}{2} - \frac{3}{8}\right) x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

Let $f(n) = K(n) - \frac{n(n+1)}{2} = k$, then from Lemma 1 and the first formula of Lemma 2 we have

$$\begin{aligned} \sum_{n \le x} d(k) &= \sum_{n \le x} d\left(K(n) - \frac{n(n+1)}{2}\right) \\ &= \sum_{\substack{n \le x \\ 2 \mid n}} d(\frac{n}{2}) + \sum_{\substack{n \le x \\ 2 \dagger n}} d(n) \\ &= \sum_{n \le \frac{x}{2}} d(n) + \sum_{n \le x} d(n) - \sum_{n \le \frac{x}{2}} d(2n) \\ &= \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right), \end{aligned}$$

where A is a computable constant.

This complets the proof of Theorem 1.

Now we complete the proof of Theorem 2. Noting that $\zeta(2) = \frac{\pi^2}{6}$, from the second formula of Lemma 3 we can obtain

$$\sum_{n\leq \frac{x}{2}} \varphi(2n) = \frac{3}{7\pi^2}x^2 + O(x^{\frac{3}{2}+\epsilon}).$$

Then from Lemma 1 and the second formula of Lemma 2 we have

$$\begin{aligned}
\sum_{n\leq x} \varphi(k) &= \sum_{n\leq x} \varphi\left(K(n) - \frac{n(n+1)}{n}\right) \\
&= \sum_{\substack{n\leq x \\ 2|n}} \varphi(\frac{n}{2}) + \sum_{\substack{n\leq x \\ 2\dagger n}} \varphi(n) \\
&= \sum_{n\leq \frac{x}{2}} \varphi(n) + \sum_{n\leq x} \varphi(n) - \sum_{n\leq \frac{x}{2}} \varphi(2n) \\
&= \frac{93}{28\pi^2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right),
\end{aligned}$$

where ϵ is any fixed positive number.

This complets the proof of Theorem 2.

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An Identity of Stirling Numbers of the Second Kind [1]

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Abstract In this paper, we prove some identities. In particular, we determine the stirling number of the second kind $s_2(n,4)$, when $n \geq 4$ is given. Then we discuss the generality $S_2(n,k)$ using the same idea .

Keywords Partitions, combinations, stirling number of the second kind.

§1. Preliminary

Before we state the main result of the paper, we give some definitions and notations first.

Definition 1. The number of combinations of n distinct things taken m at a time,i s denoted by C_n^m or $\binom{m}{n}$.

Definition 2. The number of partitions of an n-elements set into r non-empty unordered subsets is called the stirling number of the second kind, and denoted by $S_2(n,r)$.

Definition 2. The stirling number of the second kind $S_2(n,r)$ can be taken from the formula (1)

$$x^n = \sum_{r=0}^{n} S_2(n,r)(x)_r, \tag{1}$$

where $(x)_0 = 1$, when $r \geq 1, (x)_r = x(x-1)(x-2)\cdots(x-r+1)$.

Stirling numbers of the second kind and some problems about it are very interesting research subjects as long, a lot of research results had apparented. [2-6] it plays a very important roles in combination mathematics and number theory, which has comprehensive applications. In this paper, we prove some identities. In particular, we determine the stirling number of the second kind $S_2(n,4)$, when $n \geq 4$ is given. Then we discuss the generality $S_2(n,k)$ using the same idea .

§2. Main Result and Its Proof

Theorem 1. For $n \geq 1, S_2(n,0) = 0, S_2(n,1) = 1, S_2(n,2) = 2^{n-1} - 1, S_2(n,n-1) = C_n^2, S_2(n,n) = 1$.

The proofs of Theorem 1 can be found in [1] and [7].

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Theorem 2. For $n \geq 3$,

$$S_2(n,3) = \frac{1}{2}(3^{n-1} - 2^n + 1). \tag{2}$$

The proofs of Theorem 2 can be found in [2].
Our main result is as follows:

Theorem 3. For $n \geq 4$,

$$S_2(n,4) = \frac{1}{6}(4^{n-1} - 3^n + 3 \cdot 2^{n-1} - 1). \tag{3}$$

Setting S is an n-elements set, anyway taking a element from S, is denoted by S, we can distinguish $n-3$ cases.

Case 1. If S is regarded as a regular set and dividing the rest of $n-1$ elements into three non-empty subsets, the number of partitions is $S_2(n,3)$ and

$$S_2(n-1,3) = \frac{1}{2}(3^{n-2} - 2^{n-1} + 1) = \frac{1}{2}C_{n-1}^0(3^{n-2} - 2^{n-1} + 1).$$

Case 2. Any element which is chosen among the rest of $n-1$ elements is put together with S as a whole set, we have C_{n-1}^1 ways of any taking a element from $n-1$ elements. Then dividing the rest of $n-2$ elements into three non-empty subsets, the number of partitions is

$$C_{n-1}^1 S_2(n-2,3) = \frac{1}{2}C_{n-1}^1(3^{n-3} - 2^{n-2} + 1).$$

Case 3. Any two elements which is chosen among the rest of $n-1$ elements is put together with s as a whole set. Then dividing the rest of $n-3$ elements into three non-empty subsets,the number of partitions is

$$C_{n-1}^2 S_2(n-3,3) = \frac{1}{2}C_{n-1}^2(3^{n-4} - 2^{n-3} + 1).$$

Use the same way, case $n-3$ any $n-4$ elements which is chosen among the rest of $n-1$ elements is put together with S as a whole set. Then dividing the rest of three elements into three non-empty subsets, the number of partitions is

$$C_{n-1}^{n-4} S_2(3,3) = \frac{1}{2}C_{n-1}^{n-4}(3^2 - 2^3 + 1).$$

So from the principle of addition, dividing set S of n-elements into four non-empty subsets, the number of partitions is

$$\begin{aligned}
S_2(n,4) &= \frac{1}{2}C_{n-1}^0(3^{n-2}-2^{n-1}+1)+\frac{1}{2}C_{n-1}^1(3^{n-3}-2^{n-2}+1)+\frac{1}{2}C_{n-1}^2(3^{n-4}-2^{n-3}+1)\\
&\quad+\cdots+\frac{1}{2}C_{n-1}^k(3^{n-k-2}-2^{n-k-1}+1)+\frac{1}{2}C_{n-1}^{n-5}(3^3-2^4+1)\\
&\quad+\frac{1}{2}C_{n-1}^{n-4}(3^2-2^3+1)\\
&= \frac{1}{2}[(C_{n-1}^0 3^{n-2}+C_{n-1}^1 3^{n-3}+\cdots+C_{n-1}^{n-5}3^3+C_{n-1}^{n-4}3^2)-(C_{n-1}^0 2^{n-1}+C_{n-1}^1 2^{n-2}\\
&\quad+\cdots+C_{n-1}^{n-5}2^4+C_{n-1}^{n-4}2^3)+(C_{n-1}^0+C_{n-1}^1+\cdots+C_{n-1}^{n-5}+C_{n-1}^{n-4})].
\end{aligned}$$

Because

$$C_{n-1}^0+C_{n-1}^1+\cdots+C_{n-1}^{n-3}+C_{n-1}^{n-2}+C_{n-1}^{n-1}=(1+1)^{n-1}=2^{n-1}.$$

So

$$\begin{aligned}
C_{n-1}^0+C_{n-1}^1+\cdots+C_{n-1}^{n-5}+C_{n-1}^{n-4} &= 2^{n-1}-C_{n-1}^{n-3}-C_{n-1}^{n-2}-C_{n-1}^{n-1}\\
&= 2^{n-1}-\frac{(n-1)(n-2)}{2}-(n-1)-1\\
&= 2^{n-1}-\frac{(n-1)(n-2)}{2}-n
\end{aligned}$$

and

$$\begin{aligned}
C_{n-1}^0 2^{n-1}+C_{n-1}^1 2^{n-2}+\cdots+C_{n-1}^{n-5}2^4+C_{n-1}^{n-4}2^3 &= C_{n-1}^0 2^{n-1}+C_{n-1}^1 2^{n-2}+\cdots+C_{n-1}^{n-4}2^3\\
&\quad+C_{n-1}^{n-3}2^2+C_{n-1}^{n-2}2+C_{n-1}^{n-1}-C_{n-1}^{n-3}2^2\\
&\quad-C_{n-1}^{n-2}2-C_{n-1}^{n-1}\\
&= (2+1)^{n-1}-4\frac{(n-1)(n-2)}{2}\\
&\quad-2(n-1)-1\\
&= 3^{n-1}-2(n-1)(n-2)-2n+1,
\end{aligned}$$

and

$$\begin{aligned}
C_{n-1}^0 3^{n-2}+C_{n-1}^1 3^{n-3}+\cdots+C_{n-1}^{n-5}3^3+C_{n-1}^{n-4}3^2 &= \frac{1}{3}(C_{n-1}^0 3^{n-1}+\cdots+C_{n-1}^{n-4}3^3+C_{n-1}^{n-3}3^2\\
&\quad+C_{n-1}^{n-2}3+C_{n-1}^{n-1}-C_{n-1}^{n-3}3^2-C_{n-1}^{n-2}3\\
&\quad-C_{n-1}^{n-1})\\
&= \frac{1}{3}[(3+1)^{n-1}-9\frac{(n-1)(n-2)}{2}\\
&\quad-3(n-1)-1]\\
&= \frac{1}{3}\cdot 4^{n-1}-3\frac{(n-1)(n-2)}{2}-n+\frac{2}{3}.
\end{aligned}$$

Therefore

$$\begin{aligned} S_2(n,4) &= \frac{1}{2}(\frac{1}{3}\cdot 4^{n-1} - 3\frac{(n-1)(n-2)}{2} - n + \frac{2}{3} - 3^{n-1} + 2(n-1)(n-2) + 2n - 1 \\ &\quad +2^{n-1} - \frac{(n-1)(n-2)}{2} - n) \\ &= \frac{1}{2}(\frac{1}{3}\cdot 4^{n-1} - 3^{n-1} + 2^{n-1} - \frac{1}{3}) \\ &= \frac{1}{6}(4^{n-1} - 3^n + 3\cdot 2^{n-1} - 1). \end{aligned}$$

This completes the proof of our results.

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On the F.Smarandache LCM Ratio Sequence

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Abstract In this paper, we use the elementary methods to study the F.Smarandache LCM ratio sequence, and obtain three interesting recurrence relations for it.

Keywords Elementary method, Smarandache LCM ratio sequences, recurrence relation.

§1. Introduction

Let $(x_1, x_2, ..., x_t)$ and $[x_1, x_2, ..., x_t]$ denote the greatest common divisor and the least common multiple of any positive integers $x_1, x_2, ..., x_t$ respectively. Let r be a positive integer with $r > 1$. For any positive integer n, let

$$T(r, n) = \frac{[n, n+1, ..., n+r-1]}{[1, 2, ..., r]},$$

then the sequences $SLR(r) = T(r, n)_\infty$ is called the F.Samarandache LCM ratio sequences of degree r. In reference [1], Murthy asked us to find a reduction formula for $T(r, n)$. Maohua Le [2] solved this open problem for $r = 3$ and 4. That is, he proved that

$$T(3, n) = \begin{cases} \frac{1}{6}n(n+1)(n+2), & \text{if } n \text{ is odd,} \\ \frac{1}{12}n(n+1)(n+2), & \text{if } n \text{ is even.} \end{cases}$$

$$T(4, n) = \begin{cases} \frac{1}{24}n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod 3, \\ \frac{1}{72}n(n+1)(n+2)(n+3), & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Furthermore, Wang Ting [3] and [4] computing the value of $T(5, n)$ and $T(6, n)$. For example, he obtained the identity

$$T(5, n) = \begin{cases} \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 0,\ 8 \pmod{12}, \\ \frac{1}{120}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 1,\ 7 \pmod{12}, \\ \frac{1}{720}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 2,\ 6 \pmod{12}, \\ \frac{1}{360}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 3,\ 5,\ 9,\ 11 \pmod{12}, \\ \frac{1}{480}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{1}{240}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 10 \pmod{12}. \end{cases}$$

In this paper, we study the recurrence relations between $T(r+1, n)$ and $T(r, n)$, and get three interesting recurrence formulas for it. That is, we shall prove the following conclusions:

Theorem 1. For any natural number n and r, we have the recurrence formula:

$$T(r+1,n) = \frac{n+r}{r+1} \cdot \frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n).$$

Especially, if both $r+1$ and $n+r$ are primes, then we can get a simple formula

$$T(r+1,n) = \frac{n+r}{r+1} \cdot T(r,n).$$

Theorem 2. For each natural number n and r, we also have another recurrence formula:

$$T(r,n+1) = \frac{n+r}{n} \cdot \frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n).$$

Especially, if both n and $n+r$ are primes with $r < n$, then we can also get a simple formula

$$T(r,n+1) = \frac{n+r}{n} \cdot T(r,n);$$

If both n and $n+r$ are primes with $r \geq n$, then we have

$$T(r,n+1) = (n+r) \cdot T(r,n).$$

Theorem 3. For each natural number n and r, we have

$$\begin{aligned} T(r+1,n+1) &= \frac{n+r}{n} \cdot \frac{n+r+1}{r+1} \cdot \frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)} \\ &\cdot \frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)} \cdot T(r,n). \end{aligned}$$

§2. Some Lemmas

To complete the proof of the above theorems, we need the following several Lemmas.

Lemma 1. For any positive integers a and b, we have $(a,b)[a,b] = ab$.

Lemma 2. For any positive integers s and t with $s < t$, we have

$$(x_1,x_2,...,x_t) = ((x_1,...,x_s),(x_{s+1},...,x_t))$$

and

$$[x_1,x_2,...,x_t] = [[x_1,...,x_s],[x_{s+1},...,x_t]].$$

The proof of Lemma 1 and Lemma 2 can be found in [3].

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems.

First we prove Theorem 1. According to the definition of $T(r,n)$, Lemma 1 and Lemma 2, we have:

$$\begin{aligned} T(r+1,n) &= \frac{[n,n+1,...,n+r]}{[1,2,...,r+1]} \\ &= \frac{[[n,n+1,...,n+r-1],n+r]}{[[1,2,...,r],r+1]} \\ &= \frac{\frac{(n+r)[n,n+1,...,n+r-1]}{([n,n+1,...,n+r-1],n+r)}}{\frac{(r+1)[1,...,r]}{([1,2,...,r],r+1)}} \\ &= \frac{n+r}{r+1}\frac{[n,n+1,...,n+r-1]}{[1,2,...,r]}\cdot\frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)} \\ &= \frac{n+r}{r+1}\cdot\frac{([1,2,...,r],r+1)}{([n,n+1,...,n+r-1],n+r)}T(r,n). \end{aligned}$$

It is easily to get

$$T(r+1,n)=\frac{n+r}{r+1}T(r,n)$$

if both $r+1$ and $n+r$ are primes. Because at this time

$$([1,2,...,r],r+1)=1$$

and

$$([n,n+1,...,n+r-1],n+r)=1.$$

This proves Theorem 1.

Now we prove Theorem 2. From the Lemmas and the definition of $T(r,n)$, we have

$$\begin{aligned} T(r,n+1) &= \frac{[n+1,...,n+r]}{[1,2,...,r]} \\ &= \frac{[n,n+1,...,n+r](n,[n+1,...,n+r])}{n}\cdot\frac{1}{[1,2,...,r]} \\ &= \frac{(n,[n+1,...,n+r])}{n[1,2,...,r]}\cdot\frac{[n,n+1,...,n+r-1](n+r)}{([n,n+1,...,n+r-1],n+r)} \\ &= \frac{n+r}{n}\frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)}\cdot\frac{[n,n+1,...,n+r-1]}{[1,2,...,r]} \\ &= \frac{n+r}{n}\cdot\frac{(n,[n+1,...,n+r])}{([n,n+1,...,n+r-1],n+r)}T(r,n). \end{aligned}$$

If n and $n+r$ are primes with $n<r$, then we can also get a simple formula

$$T(r,n+1)=\frac{n+r}{n}T(r,n);$$

If n and $n+r$ are primes with $n\geq r$, this time note that $(n,[n+1,...,n+r])=n$, we have

$$T(r,n+1)=(n+r)\cdot T(r,n).$$

This proves Theorem 2.

The proof of Theorem 3. Applying Theorem 1 and Theorem 2 we can easily get identity

$$T(r+1,n+1)=\frac{n+r+1}{r+1}\cdot\frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)}T(r,n+1)$$

$$=\frac{(n+r+1)(n+r)}{(r+1)n}\cdot\frac{([1,2,...,r],r+1)}{([n+1,...,n+r],n+r+1)}\cdot\frac{(n,[n+1,...,n+r])}{([n,...,n+r-1],n+r)}T(r,n).$$

This completes the proof of Theorem 3.

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On Algebraic Multi-Ring Spaces

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Abstract A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions hold. Combining these Smarandache multi-spaces with rings in classical ring theory, the conception of multi-ring spaces is introduced and some characteristics of multi-ring spaces are obtained in this paper.

Keywords Ring, multi-space, multi-ring space, ideal subspace chain.

§1. Introduction

These multi-spaces is introduced by Smarandache in [6] under an idea of hybrid mathematics: combining different fields into a unifying field ([7]), which can be formally defined with mathematical words by the next definition.

Definition 1.1. For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, denoted by $(A_i; L_i)$. Then the union of $(A_i; L_i), 1 \leq i \leq n$

$$\widetilde{A} = \bigcup_{i=1}^{n} (A_i; L_i),$$

is called a multi-space.

As we known, a set R with two binary operation "$+$" and "$\circ$" , denoted by $(R\,; +, \circ)$, is said to be a *ring* if for $\forall x, y \in R$, $x + y \in R$, $x \circ y \in R$, the following conditions hold.

(i) $(R\,; +)$ is an abelian group;

(ii) $(R\,; \circ)$ is a semigroup;

(iii) For $\forall x, y, z \in R$, $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$.

By combining these Smarandache multi-spaces with rings in classical mathematics, a new kind of algebraic structure called multi-ring spaces is found, which are defined in the next definition.

Definition 1.2. Let $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ be a complete multi-space with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

provided all these operation results exist, then $\widetilde{R}$ is called a multi-ring space. If for any integer $1 \leq i \leq m$, $(R; +_i, \times_i)$ is a field, then $\widetilde{R}$ is called a multi-field space.

For a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m} R_i$, let $\widetilde{S} \subset \widetilde{R}$ and $O(\widetilde{S}) \subset O(\widetilde{R})$, if $\widetilde{S}$ is also a multi-ring space with a double binary operation set $O(\widetilde{S})$, then $\widetilde{S}$ is said a *multi-ring subspace* of $\widetilde{R}$.

The main object of this paper is to find some characteristics of multi-ring spaces. For terminology and notation not defined here can be seen in [1], [5], [12] for rings and [2], [6] – [11] for multi-spaces and logics.

§2. Characteristics of multi-ring spaces

First, we get a simple criterions for multi-ring subspaces of a multi-ring space.

Theorem 2.1. For a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with a double binary operation set $O(\widetilde{S}) \subset O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ if and only if for any integer $k, 1 \leq k \leq m$, $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$.

Proof. For any integer $k, 1 \leq k \leq m$, if $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$, then since $\widetilde{S} = \bigcup_{i=1}^{m} (\widetilde{S} \bigcap R_i)$, we know that $\widetilde{S}$ is a multi-ring subspace by definition of multi-ring spaces.

Now if $\widetilde{S} = \bigcup_{j=1}^{s} S_{i_j}$ is a multi-ring subspace of $\widetilde{R}$ with a double binary operation set $O(\widetilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \times_{i_j})$ is a subring of $(R_{i_j}; +_{i_j}, \times_{i_j})$. Therefore, for any integer $j, 1 \leq j \leq s$, $S_{i_j} = R_{i_j} \bigcap \widetilde{S}$. But for other integer $l \in \{i; 1 \leq i \leq m\} \backslash \{i_j; 1 \leq j \leq s\}$, $\widetilde{S} \bigcap S_l = \emptyset$.

Applying a criterion for subrings of a ring, we get the following result.

Theorem 2.2. For a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with a double binary operation set $O(\widetilde{S}) \subset O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ if and only if for any double binary operations $(+_j, \times_j) \in O(\widetilde{S})$, $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ and $(\widetilde{S}; \times_j)$ is complete.

Proof. According to Theorem 2.1, we know that $\widetilde{S}$ is a multi-ring subspace if and only if for any integer $i, 1 \leq i \leq m$, $(\widetilde{S} \bigcap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ or $\widetilde{S} \bigcap R_i = \emptyset$. By a well known criterion for subrings of a ring (see also [5]), we know that $(\widetilde{S} \bigcap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ if and only if for any double binary operations $(+_j, \times_j) \in O(\widetilde{S})$, $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ and $(\widetilde{S}; \times_j)$ is a complete set. This completes the proof.

We use these ideal subspace chains of a multi-ring space to characteristic its structure properties. An ideal subspace $\widetilde{I}$ of a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ with a double binary operation set $O(\widetilde{R})$ is a multi-ring subspace of $\widetilde{R}$ satisfying the following conditions:

(i) $\widetilde{I}$ is a multi-group subspace with an operation set $\{+| \ (+, \times) \in O(\widetilde{I})\}$;

(ii) for any $r \in \widetilde{R}, a \in \widetilde{I}$ and $(+, \times) \in O(\widetilde{I})$, $r \times a \in \widetilde{I}$ and $a \times r \in \widetilde{I}$ provided these operation results exist.

Theorem 2.3. A subset $\widetilde{I}$ with $O(\widetilde{I}), O(\widetilde{I}) \subset O(\widetilde{R})$ of a multi-ring space $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i)| \ 1 \leq i \leq m\}$ is a multi-ideal subspace if and only

if for any integer $i, 1 \leq i \leq m$, $(\widetilde{I} \bigcap R_i, +_i, \times_i)$ is an ideal of the ring $(R_i, +_i, \times_i)$ or $\widetilde{I} \bigcap R_i = \emptyset$.

Proof. By definition of an ideal subspace, the necessity of conditions is obvious.

For the sufficiency, denote by $\widetilde{R}(+, \times)$ the set of elements in $\widetilde{R}$ with binary operations "$+$" and "$\times$". If there exists an integer i such that $\widetilde{I} \bigcap R_i \neq \emptyset$ and $(\widetilde{I} \bigcap R_i, +_i, \times_i)$ is an ideal of $(R_i, +_i, \times_i)$, then for $\forall a \in \widetilde{I} \bigcap R_i$, $\forall r_i \in R_i$, we know that

$$r_i \times_i a \in \widetilde{I} \bigcap R_i; \quad a \times_i r_i \in \widetilde{I} \bigcap R_i.$$

Notice that $\widetilde{R}(+_i, \times_i) = R_i$. Therefore, we get that for $\forall r \in \widetilde{R}$,

$$r \times_i a \in \widetilde{I} \bigcap R_i; \text{ and } a \times_i r \in \widetilde{I} \bigcap R_i$$

provided these operation results exist. Whence, $\widetilde{I}$ is an ideal subspace of $\widetilde{R}$.

An ideal subspace $\widetilde{I}$ of a multi-ring space $\widetilde{R}$ is *maximal* if for any ideal subspace $\widetilde{I}'$, if $\widetilde{R} \supseteq \widetilde{I}' \supseteq \widetilde{I}$, then $\widetilde{I}' = \widetilde{R}$ or $\widetilde{I}' = \widetilde{I}$. For any order of these double binary operations in $O(\widetilde{R})$ of a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, not loss of generality, assume it being $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$, we can construct an *ideal subspace chain* of $\widetilde{R}$ by the following programming.

(i) Construct an ideal subspace chain

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1}$$

under the double binary operation $(+_1, \times_1)$, where $\widetilde{R}_{11}$ is a maximal ideal subspace of $\widetilde{R}$ and in general, for any integer i, $1 \leq i \leq m-1$, $\widetilde{R}_{1(i+1)}$ is a maximal ideal subspace of $\widetilde{R}_{1i}$.

(ii) If the ideal subspace

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i}$$

has been constructed for $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_i, \times_i)$, $1 \leq i \leq m-1$, then construct an ideal subspace chain of $\widetilde{R}_{is_i}$

$$\widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \widetilde{R}_{(i+1)2} \supset \cdots \supset \widetilde{R}_{(i+1)s_1}$$

under the operations $(+_{i+1}, \times_{i+1})$, where $\widetilde{R}_{(i+1)1}$ is a maximal ideal subspace of $\widetilde{R}_{is_i}$ and in general, $\widetilde{R}_{(i+1)(i+1)}$ is a maximal ideal subspace of $\widetilde{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define an ideal subspace chain of $\widetilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$ being

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \cdots \supset \widetilde{R}_{(i+1)s_{i+1}}.$$

Similar to a multi-group space ([3]), we get the following result for ideal subspace chains of multi-ring spaces.

Theorem 2.4. For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, its ideal subspace chain only has finite terms if and only if for any integer $i, 1 \leq i \leq m$, the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, i.e., each ring $(R_i; +_i, \times_i)$ is an Artin ring.

Proof. Let the order of double operations in $\overrightarrow{O}(\widetilde{R})$ be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

and a maximal ideal chain in the ring $(R_1; +_1, \times_1)$ is

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$

Calculation shows that

$$\widetilde{R}_{11} = \widetilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m}) R_i,$$

$$\widetilde{R}_{12} = \widetilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m}) R_i,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{R}_{1t_1} = \widetilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m}) R_i.$$

According to Theorem 3.10, we know that

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1t_1}$$

is a maximal ideal subspace chain of $\widetilde{R}$ under the double binary operation $(+_1, \times_1)$. In general, for any integer $i, 1 \leq i \leq m-1$, assume

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in the ring $(R_{(i-1)t_{i-1}}; +_i, \times_i)$. Calculate

$$\widetilde{R}_{ik} = R_{ik} \bigcup (\bigcup_{j=i+1}^{m}) \widetilde{R}_{ik} \bigcap R_i$$

Then we know that

$$\widetilde{R}_{(i-1)t_{i-1}} \supset \widetilde{R}_{i1} \supset \widetilde{R}_{i2} \supset \cdots \supset \widetilde{R}_{it_i}$$

is a maximal ideal subspace chain of $\widetilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i, \times_i)$ by Theorem 2.3. Whence, if for any integer $i, 1 \leq i \leq m$, the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, then the ideal subspace chain of the multi-ring space $\widetilde{R}$ only has finite terms. On the other hand, if there exists one integer i_0 such that the ideal chain of the ring $(R_{i_0}, +_{i_0}, \times_{i_0})$ has infinite terms, then there must be infinite terms in the ideal subspace chain of the multi-ring space $\widetilde{R}$.

A multi-ring space is called an Artin multi-ring space if each ideal subspace chain only has finite terms. We have consequence by Theorem 3.11.

Corollary 2.1. A multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m}$ with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$ is an Artin multi-ring space if and only if for any integer $i, 1 \leq i \leq m$, the ring $(R_i; +_i, \times_i)$ is an Artin ring.

For a multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m}$ with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$, an element e is an *idempotent* element if $e^2_\times = e \times e = e$ for a double binary operation $(+, \times) \in O(\widetilde{R})$. We define the *directed sum* $\widetilde{I}$ of two ideal subspaces $\widetilde{I}_1$ and $\widetilde{I}_2$ as follows:

(i) $\widetilde{I} = \widetilde{I}_1 \bigcup \widetilde{I}_2$;

(ii) $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \{0_+\}$, or $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \emptyset$, where 0_+ denotes an unit element under the operation $+$.

Denote the directed sum of $\widetilde{I}_1$ and $\widetilde{I}_2$ by

$$\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2.$$

If for any $\widetilde{I}_1, \widetilde{I}_2$, $\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2$ implies that $\widetilde{I}_1 = \widetilde{I}$ or $\widetilde{I}_2 = \widetilde{I}$, then $\widetilde{I}$ is said to be *non-reducible*. We get the following result for these Artin multi-ring spaces, which is similar to a well-known result for these Artin rings (see [12]).

Theorem 2.5. Any Artin multi-ring space $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$ is a directed sum of finite non-reducible ideal subspaces, and if for any integer $i, 1 \leq i \leq m$, $(R_i; +_i, \times_i)$ has unit $1_{\times_i}$, then

$$\widetilde{R} = \bigoplus_{i=1}^{m}(\bigoplus_{j=1}^{s_i}(R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)),$$

where $e_{ij}, 1 \leq j \leq s_i$ are orthogonal idempotent elements of the ring R_i.

Proof. Denote by $\widetilde{M}$ the set of ideal subspaces which can not be represented by a directed sum of finite ideal subspaces in $\widetilde{R}$. According to Theorem 2.4, there is a minimal ideal subspace $\widetilde{I}_0$ in $\widetilde{M}$. It is obvious that $\widetilde{I}_0$ is reducible.

Assume that $\widetilde{I}_0 = \widetilde{I}_1 + \widetilde{I}_2$. Then $\widetilde{I}_1 \notin \widetilde{M}$ and $\widetilde{I}_2 \notin \widetilde{M}$. Therefore, $\widetilde{I}_1$ and $\widetilde{I}_2$ can be represented by directed sums of finite ideal subspaces. Whence, $\widetilde{I}_0$ can be also represented by a directed sum of finite ideal subspaces. Contradicts that $\widetilde{I}_0 \in \widetilde{M}$.

Now let

$$\widetilde{R} = \bigoplus_{i=1}^{s} \widetilde{I}_i,$$

where each $\widetilde{I}_i, 1 \leq i \leq s$, is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible ideal subspace of $\widetilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \quad e \in R(\times).$$

Whence, we know that there is a set $T \subset \widetilde{R}$ such that

$$\widetilde{R} = \bigoplus_{e \in T,\ \times \in O(\widetilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

For any operation $\times \in O(\widetilde{R})$ and a unit $1_\times$, assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_l,\ e_i \in T,\ 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \ \text{ and } \ e_i \times e_j = 0_i \ \text{ for } \ i \neq j.$$

That is, $e_i, 1 \leq i \leq l$, are orthogonal idempotent elements of $\widetilde{R}(\times)$. Notice that $\widetilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +_h, \times_h)$. Denote by e_{hj} for e_j, $1 \leq j \leq l$. Consider all units in $\widetilde{R}$, we get that

$$\widetilde{R} = \bigoplus_{i=1}^{m}(\bigoplus_{j=1}^{s_i}(R_i \times_i e_{ij})\bigcup(e_{ij} \times_i R_i)).$$

This completes the proof.

Corollary 2.2.([12]) Any Artin ring $(R\ ;+,\times)$ is a directed sum of finite ideals, and if $(R\ ;+,\times)$ has unit $1_\times$, then

$$R = \bigoplus_{i=1}^{s} R_i e_i,$$

where $e_i, 1 \leq i \leq s$ are orthogonal idempotent elements of the ring $(R; +, \times)$.

§3. Open problems for a multi-ring space

Similar to Artin multi-ring spaces, we can also define Noether multi-ring spaces, simple multi-ring spaces, half-simple multi-ring spaces, $\cdots$, etc.. Open problems for these new algebraic structures are as follows.

Problem 3.1. Call a ring R a Noether ring if its every ideal chain only has finite terms. Similarly, for a multi-ring space $\widetilde{R}$, if its every ideal multi-ring subspace chain only has finite terms, it is called a Noether multi-ring space. Whether can we find its structures similar to Corollary 2.2 and Theorem 2.5?

Problem 3.2. Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-ring spaces and determine their contribution to multi-ring spaces.

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On the Product of the Square-free Divisor of a Natural Number

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Abstract In this paper we study the product of the square-free divisor of a natural number $P_{sd}(n) = \prod\limits_{\substack{d|n \\ \mu(d)\neq 0}} d$. According to the Dirichlet divisor problem, we turn to study the asymptotic formula of $\sum\limits_{n\leq x} \log P_{sd}(n)$. This article uses the hyperbolic summation and the convolution method to obtain a better error term.

Keywords Square-free number, Dirichlet divisor problem, hyperbolic summation, convolution.

§1. Introduction and main results

F.Smarandache introduced the function $P_d(n) := \prod\limits_{d|n} d$ in Problem 25[1]. Now we define a similar function $P_{sd}(n)$, which denotes the product of all square-free divisors of n, i.e.,

$$P_{sd}(n) := \prod_{\substack{d|n \\ \mu(d)\neq 0}} d.$$

In the present paper, we shall prove the following Theorem.

Theorem. We have the asymptotic formula

$$\sum_{n\leq x} \log P_{sd}(n) = A_1 x \log^2 x + A_2 x \log x + A_3 x + O(x^{\frac{1}{2}} \exp(-D(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}),$$

where A_1, A_2, A_3 are constants, $D > 0$ is an absolute constant.

Notations. $[x] = max_{k\in Z}\{k \leq x\}.\psi(t) = t - [t] - \frac{1}{2}, \psi_1(t) = \int_0^t \psi(u)du$. $\mu(n)$ is the $Mobius$ function. ε denotes a fixed small positive constant which may be different at each occurence. γ is the Euler constant. B_1, B_2, B_3, C_1, C_2 , D_1, D_2, D_3, D_4 are constants.

§2. Some preliminary lemmas

We need the following results:

Lemma 1. Let $f(n)$ be an arithmetical function for which

$$\sum_{n\leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$

$$\sum_{n\le x}|f(n)| = O(x^{a_1}\log^r x),$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > \frac{1}{k} > a \ge 0$, $r \ge 0$, $P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r , and $k \ge 1$ is a fixed integer. If

$$h(n) = \sum_{d^k|n}\mu_i(d)f(n/d^k), \quad i \ge 1$$

then

$$\sum_{n\le x}h(n) = \sum_{j=1}^{l}x^{a_j}R_j(\log x) + \delta(x),$$

where $R_1(t), \cdots, R_l(t)$ are polynomials in t of degrees not exceeding r. and for some $D > 0$

$$\delta(x) \ll x^{\frac{1}{k}}\exp\,(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

Proof. See Theorem 14.2 of A. Ivić[3] when $l = 1$ and the similar proof is used when $l > 1$.

Lemma 2. Suppose that $f(u) \in C^3[u_1, u_2]$, then

$$\sum_{u_1<n\le u_2}f(n) = \int_{u_1}^{u_2}f(u)du - \psi(u)f(u)\Big|_{u_1}^{u_2} + \psi_1(u)f'(u)\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2}\psi_1(u)f''(u)du.$$

Lemma 3. We have

$$\sum_{1\le m\le y}\frac{1}{m} = \log y + \gamma - \frac{\psi(y)}{y} + O\left(\frac{1}{y^2}\right),\ y \ge 1.$$

Lemma 4. We have

$$\sum_{1\le m\le y}\log m = y\log y - y - \psi(y)\log y + \frac{\psi_1(y)}{y} + D_1 + O\left(\frac{1}{y^2}\right),\ y \ge 1.$$

Lemma 5. We have

$$\sum_{1\le m\le y}\frac{\log m}{m^2} = -\frac{\log y}{y} - \frac{1}{y} + D_3 + O\left(\frac{\log y}{y^2}\right),\ y \ge 1.$$

Lemma 6. We have

$$\sum_{1\le m\le y}\frac{\log^2 m}{m^2} = -\frac{\log^2 y}{y} - \frac{2\log y}{y} - \frac{2}{y} + D_4 + O\left(\frac{\log^2 y}{y^2}\right),\ y \ge 1.$$

Lemma 7. We have

$$\sum_{n\le y}d(n) = y\log y + (2\gamma - 1)y + O(y^{\frac{1}{3}}).$$

Lemma 8. We have

$$\sum_{n\le y}d(n)n^{-\frac{1}{2}} = 2y^{\frac{1}{2}}\log y + (4\gamma - 4)y^{\frac{1}{2}} - 2\gamma + 3 + O(y^{-\frac{1}{6}}).$$

Lemma 9. We have

$$\sum_{n\le y} d(n)n^{-\frac{1}{2}}\log n = 2y^{\frac{1}{2}}\log^2 y + (4\gamma-8)y^{\frac{1}{2}}\log y + (16-8\gamma)y^{\frac{1}{2}} + 8\gamma - 16 + O(y^{-\frac{1}{6}}\log y).$$

Lemma 2 is the Euler-Maclaurin summation formula (see [2]). Lemma 3, 4, 5, 6 follow from Lemma 2 directly. Lemma 7 is a classical result about the Dirichlet divisor problem. Lemma 8, 9 can be easily obtained by Lemma 2 and Lemma 7.

§3. Proof of the theorem

It is easily seen that

$$\log P_{sd}(n) = \sum_{n=dl,\mu(d)\neq 0} \log d,$$

which implies that ($\sigma > 1$)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\log P_{sd}(n)}{n^s} &= \zeta(s)\sum_{d=1}^{\infty}\frac{|\mu(d)|\log d}{d^s} = \zeta(s)\left(-\sum_{d=1}^{\infty}\frac{|\mu(d)|}{d^s}\right)'\\ &= -\zeta(s)\left(\frac{\zeta(s)}{\zeta(2s)}\right)' = -\zeta(s)\zeta'(s)\frac{1}{\zeta(2s)} + 2\zeta^2(s)\zeta'(2s)\frac{1}{\zeta^2(2s)}\\ &= \sum_{n=1}^{\infty}h_1(n)n^{-s} + 2\sum_{n=1}^{\infty}h_2(n)n^{-s}.\end{aligned}$$

where

$$h_1(n) = \sum_{d^2|n}\mu(d)f_1(n/d^2),\ f_1(n) = \sum_{m|n}\log m;$$

$$h_2(n) = \sum_{d^2|n}\mu_2(d)f_2(n/d^2),\quad f_2(n) = \sum_{n=m^2k} d(k)\log m,\quad \mu_2(d) = \sum_{n=dk}\mu(d)\mu(k).$$

Our Theorem follows from the following Proposition 1 and Proposition 2.

Proposition 1. We have

$$\sum_{n\le x} h_1(n) = B_1 x\log^2 x + B_2 x\log x + B_3 x + O(x^{\frac{1}{2}}\mathtt{exp}(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

Proposition 2. We have

$$\sum_{n\le x} h_2(n) = C_1 x\log x + C_2 x + O(x^{\frac{1}{2}}\mathtt{exp}(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

We only proof Proposition 2. The proof of Proposition 1 is similar and easier. We have

$$\begin{aligned}&\sum_{n\le x} f_2(n) = \sum_{n\le x}\sum_{m^2k=n} d(k)\log m = \sum_{m^2n\le x} d(n)\log m\\ &= \sum_{m\le x^{\frac{1}{3}}}\log m\sum_{n\le\frac{x}{m^2}} d(n) + \sum_{n\le x^{\frac{1}{3}}} d(n)\sum_{m\le(\frac{x}{n})^{\frac{1}{2}}}\log m - \sum_{m\le x^{\frac{1}{3}}}\log m\sum_{n\le x^{\frac{1}{3}}} d(n)\\ &= S_1 + S_2 - S_3\end{aligned}$$

By Lemma 7, 5, 6

$$\begin{aligned}
S_1 &= \sum_{m\le x^{\frac{1}{3}}}\left(\frac{x}{m^2}\log\frac{x}{m^2}+(2\gamma-1)\frac{x}{m^2}+O\left(\left(\frac{x}{m^2}\right)^{\frac{1}{3}}\right)\right)\log m\\
&= (x\log x+(2\gamma-1)x)\sum_{m\le x^{\frac{1}{3}}}\frac{\log m}{m^2}-2x\sum_{m\le x^{\frac{1}{3}}}\frac{\log^2 m}{m^2}+O\left(x^{\frac{1}{3}}\sum_{m\le x^{\frac{1}{3}}}\frac{\log m}{m^{\frac{2}{3}}}\right)\\
&= (x\log x+(2\gamma-1)x)\left(-\frac{1}{3}x^{-\frac{1}{3}}\log x-x^{-\frac{1}{3}}+D_3+O\left(x^{-\frac{2}{3}}\log x\right)\right)\\
&\quad -2x\left(-\frac{1}{9}x^{-\frac{1}{3}}\log^2 x-\frac{2}{3}x^{-\frac{1}{3}}\log x-2x^{-\frac{1}{3}}+D_4+O\left(x^{-\frac{2}{3}}\log x\right)\right)+O\left(x^{\frac{4}{9}+\varepsilon}\right)
\end{aligned}$$

By Lemma 4, 8, 9

$$\begin{aligned}
S_2 &= \sum_{n\le x^{\frac{1}{3}}}d(n)\left(\frac{1}{2}\left(\frac{x}{n}\right)^{\frac{1}{2}}\log\frac{x}{n}-\left(\frac{x}{n}\right)^{\frac{1}{2}}+D_1+O\left(\log\frac{x}{n}\right)\right)\\
&= (\frac{1}{2}x^{\frac{1}{2}}\log x-x^{\frac{1}{2}})\sum_{n\le x^{\frac{1}{3}}}d(n)n^{-\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{2}}\sum_{n\le x^{\frac{1}{3}}}d(n)\log n\,n^{-\frac{1}{2}}\\
&\quad +D_1\sum_{n\le x^{\frac{1}{3}}}d(n)+O\left(\log x\sum_{n\le x^{\frac{1}{3}}}d(n)\right)\\
&= \left(\frac{1}{2}x^{\frac{1}{2}}\log x-x^{\frac{1}{2}}\right)\left(\frac{2}{3}x^{\frac{1}{6}}\log x+(4\gamma-4)x^{\frac{1}{6}}-2\gamma+3+O\left(x^{-\frac{1}{18}}\right)\right)\\
&\quad -\frac{1}{2}x^{\frac{1}{2}}\left(\frac{2}{9}x^{\frac{1}{6}}\log^2 x+\frac{1}{3}(4\gamma-8)x^{\frac{1}{6}}\log x+(16-8\gamma)x^{\frac{1}{6}}+8\gamma-16+O\left(x^{-\frac{1}{18}}\log x\right)\right)\\
&\quad +D_1\sum_{n\le x^{\frac{1}{3}}}d(n)+O\left(x^{\frac{1}{3}+\varepsilon}\right)
\end{aligned}$$

By Lemma 4 and Lemma 7,

$$\begin{aligned}
S_3 &= \sum_{n\le x^{\frac{1}{3}}}d(n)\left(\frac{1}{3}x^{\frac{1}{3}}\log x-x^{\frac{1}{3}}+D_1+O\left(\log x\right)\right)\\
&= \frac{1}{3}x^{\frac{1}{3}}\log x\sum_{n\le x^{\frac{1}{3}}}d(n)-x^{\frac{1}{3}}\sum_{n\le x^{\frac{1}{3}}}d(n)+D_1\sum_{n\le x^{\frac{1}{3}}}d(n)\\
&\quad +O\left(\log x\sum_{n\le x^{\frac{1}{3}}}d(n)\right)\\
&= \left(\frac{1}{3}x^{\frac{1}{3}}\log x-x^{\frac{1}{3}}\right)\left(\frac{1}{3}x^{\frac{1}{3}}\log x+(2\gamma-1)x^{\frac{1}{3}}+O\left(x^{\frac{1}{9}}\right)\right)\\
&\quad +O\left(x^{\frac{1}{3}+\varepsilon}\right)+D_1\sum_{n\le x^{\frac{1}{3}}}d(n)
\end{aligned}$$

Combining the above estimates we get

$$\sum_{n\leq x} f_2(n) = D_3 x\log x + ((2\gamma-1)D_3 - 2D_4)\,x + \frac{3-2\gamma}{2}x^{\frac{1}{2}}\log x + (5-2\gamma)x^{\frac{1}{2}} + O\left(x^{\frac{4}{9}+\varepsilon}\right),$$

which gives Proposition 2 immediately by using Lemma 1 .

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Some identities on k-power complement

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Abstract The main purpose of this paper is to calculate the value of the series

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^\alpha\cdot a_k^\beta(n)},$$

where $a_k(n)$ is the k-power complement number of any positive number n, and α, β are two complex numbers with $Re(\alpha)\geq 1$, $Re(\beta)\geq 1$. Several interesting identities are given.

Keywords k-power complement number, identities, Riemann zeta-function.

§1. Introduction

For any given natural number $k\geq 2$ and any positive integer n, we call $a_k(n)$ as a k-power complement number if $a_k(n)$ denotes the smallest positive integer such that $n\cdot a_k(n)$ is a perfect k-power. Especially, we call $a_2(n), a_3(n), a_4(n)$ as the square complement number, cubic complement number, quartic complement number respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the k-power complement number sequence. About this problem, there are many authors had studied it, and obtained many results. For example, in reference [2], Professor Wenpeng Zhang calculated the value of the series

$$\sum_{n=1}^{+\infty}\frac{1}{(n\cdot a_k(n))^s},$$

where s is a complex number with $Re(\alpha)\geq 1$, k=2, 3, 4. Maohua Le [3] discussed the convergence of the series

$$s_1=\sum_{n=1}^{+\infty}\frac{1}{a_2^m(n)}$$

and

$$s_2=\sum_{n=2}^{+\infty}\frac{(-1)^n}{a_2(n)},$$

where $m\leq 1$ is a positive number, and proved that they are both divergence.

But about the properties of the k-power complement number, we still know very little at present. This paper, as a note of [2], we shall give a general calculate formula for

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^\alpha\cdot a_k^\beta(n)}.$$

That is, we shall prove the following:

Theorem 1. For any complex numbers α, β with $\mathtt{Re}(\alpha) \geq 1$, $\mathtt{Re}(\beta) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha} \cdot a_k^{\beta}(n)} = \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right),$$

where $\zeta(\alpha)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all prime p.

Theorem 2. For any complex numbers α, β with $\mathtt{Re}(\alpha) \geq 1$, $\mathtt{Re}(\beta) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^{\alpha} \cdot a_k^{\beta}(n)} = \left(1 - \frac{2(2^{k\alpha} - 1)(2^{\alpha+(k+1)\beta} - 1)}{2^{(k+1)\alpha+(k-1)\beta} - 2^{\alpha-(k-1)^2\beta}}\right) \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right).$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(8) = \frac{\pi^8}{9450}$. From our Theorems we may immediately obtain the following two corollaries:

Corollary 1. Taking $\alpha = \beta$, $k = 2$ in above Theorems, then we have

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^{\alpha}} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)};$$

$$\sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{(n \cdot a_2(n))^{\alpha}} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{4^{\alpha} - 1}{4^{\alpha} + 1};$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^{\alpha}} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{3 - 4^{\alpha}}{1 + 4^{\alpha}}.$$

Corollary 2. Taking $\alpha = \beta = 1, 2$, $k = 2$ in Corollary 1, we have

$$\sum_{n=1}^{+\infty} \frac{1}{n \cdot a_2(n)} = \frac{5}{2}, \quad \sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{7}{6};$$

$$\sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{n \cdot a_2(n)} = \frac{3}{2}, \quad \sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{35}{34};$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot a_2(n)} = -\frac{1}{2}, \quad \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^2} = -\frac{91}{102}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorems. For any positive integer n, we can write it as $n = m^k \cdot l$, where l is a k-free number, then from the definition of $a_k(n)$ we have

$$\begin{aligned}
\sum_{n=1}^{+\infty}\frac{1}{n^\alpha\cdot a_k^\beta(n)} &= \sum_{m=1}^{+\infty}\sum_{l=1}^{+\infty}\frac{\sum\limits_{d^k|l}\mu(d)}{m^{k\alpha}l^\alpha l^{(k-1)\beta}}\\
&= \zeta(k\alpha)\sum_{l=1}^{+\infty}\frac{\sum\limits_{d^k|l}\mu(d)}{l^{\alpha+(k-1)\beta}}\\
&= \zeta(k\alpha)\prod_p\left(1+\frac{1}{p^{\alpha+(k-1)\beta}}+\frac{1}{p^{2(\alpha+(k-1)\beta)}}+\cdots+\frac{1}{p^{(k-1)(\alpha+(k-1)\beta)}}\right)\\
&= \zeta(k\alpha)\prod_p\left(1+\frac{1}{p^{\alpha+(k-1)\beta}}\frac{1-\frac{1}{p^{(k-1)(\alpha+(k-1)\beta)}}}{1-\frac{1}{p^{\alpha+(k-1)\beta}}}\right)\\
&= \zeta(k\alpha)\prod_p\left(1+\frac{1-\frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta}-1}\right),
\end{aligned}$$

where $\mu(n)$ denotes the Möbius function. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. First we shall prove the following identity

$$\begin{aligned}
\sum_{\substack{n=1\\2\nmid n}}^{+\infty}\frac{1}{n^\alpha\cdot a_k^\beta(n)} &= \sum_{\substack{m=1\\2\nmid m^k l}}^{+\infty}\sum_{l=1}^{+\infty}\frac{\sum\limits_{d^k|l}\mu(d)}{m^{k\alpha}l^\alpha l^{(k-1)}}\\
&= \sum_{\substack{m=1\\2\nmid m}}^{+\infty}\frac{1}{m^{k\alpha}}\sum_{\substack{l=1\\2\nmid l}}^{+\infty}\frac{\sum\limits_{d^k|l}\mu(d)}{l^{\alpha+(k-1)}}\\
&= \frac{2^{k\alpha}-1}{2^{k\alpha}}\cdot\frac{\zeta(k\alpha)(2^{\alpha+(k-1)\beta}-1)}{2^{\alpha+(k-1)\beta}-2^{(k-1)(\alpha+(k-1)\beta)}}\prod_p\left(1+\frac{1-\frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta}-1}\right)\\
&= \frac{\zeta(k\alpha)(2^{k^\alpha}-1)(2^{\alpha+(k-1)\beta})}{2^{(k+1)\alpha+(k-1)\beta}-2^{\alpha-(k-1)^2\beta}}\prod_p\left(1+\frac{1-\frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta}-1}\right).
\end{aligned}$$

Then use this identity and Theorem 1 we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^\alpha\cdot a_k^\beta(n)}\\
=\ &\sum_{n=1}^{+\infty}\frac{1}{n^\alpha\cdot a_k^\beta(n)}-2\sum_{\substack{n=1\\2\nmid n}}^{+\infty}\frac{1}{n^\alpha\cdot a_k^\beta(n)}\\
=\ &\left(1-\frac{2(2^{k\alpha}-1)(2^{\alpha+(k-1)\beta}-1)}{2^{(k+1)\alpha+(k-1)\beta}-2^{(k-1)^2\beta-\alpha}}\right)\zeta(k\alpha)\prod_p\left(1+\frac{1-\frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta}-1}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

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Two Formulas for x^n being Represented by Chebyshev Polynomials[1]

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Abstract Some very simple formulas to show how $x^n \quad (n = 1, 2, \ldots)$ is represented by Chebyshev polynomials $T_k(x)$ and $U_k(x)(k = 0, 1, \ldots)$ and their an application are given in this paper.

Keywords Chebyshev polynomial, represent, orthogonal property, common formula.

§1. Introduction

The first type Chebyshev polynomials $T_k(x)$:

$$T_k(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^k + \left(x - \sqrt{x^2-1}\right)^k\right], k = 0, 1, \ldots$$

and the second type Chebyshev polynomials $U_k(x)$:

$$U_k(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{k+1} - \left(x - \sqrt{x^2-1}\right)^{k+1}\right], k = 0, 1, \ldots$$

have widely applications in many fields[1−4], contact closely with Fibonacci numbers, Lucas numbers[5−6], and so on. It is a general method that each $x^n \quad (n = 1, 2, \ldots)$ is represented by Chebyshev polynomials $T_k(x)$ and $U_k(x) \quad (k = 0, 1, \ldots, n)$. But so far it is regretted that one can only use some recurrence relations[7], recurrence formulas

$$T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x), \quad U_{k+2}(x) = 2xU_{k+1}(x) - U_k(x), k = 0, 1, \ldots$$

where $T_0(x) = 1$, $T_1(x) = x$, $U_0 = 1$, $U_0 = 2x$, or some table[2−3]:

$$1 = T_0, \quad x = T_1, \quad x^2 = \frac{1}{2}(T_0 + T_2), \quad x^3 = \frac{1}{4}(3T_1 + T_3), \quad x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4),$$

$$x^5 = \frac{1}{16}(10T_1 + 5T_3 + T_5), \quad x^6 = \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6),$$

$$x^7 = \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7), \quad x^8 = \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8), \ldots$$

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It is obviously that the bigger n is, the more difficult the problem is. In this paper, we shall give the very simple formulas to solve the problem thoroughly, and give some perfect related results of trigonometric formulas.

Illustration: "!" is the factorial sign and "!!" is the double factorial sign through this paper. For examples,

$$9! = 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1, \quad 9!! = 9 * 7 * 5 * 3 * 1, \quad 8!! = 8 * 6 * 4 * 2.$$

§2. Main Result

Theorem 1. Let

$$x^n = \frac{1}{2}a_{n0}T_0(x) + \sum_{k=1}^{\infty} a_{nk}T_k(x). \tag{1}$$

Then if $k \geq n+1$, $a_{nk} = 0$; If $k \leq n$, k and n are of opposite parity, then $a_{nk} = 0$; If $k \leq n$, k and n are of same parity, then

$$a_{nk} = \frac{2n!}{(n-k)!!(n+k)!!}.$$

Proof. Multipy $T_m(x)/\sqrt{1-x^2}$ $(m = 0, 1, \ldots)$ to the two sides of (1) and definite integral from -1 to 1. Because of the orthogonal property[3,8] of the first type Chebyshev polynomials, we can get

$$\begin{aligned}\int_{-1}^{1} \frac{x^n T_m(x)}{\sqrt{1-x^2}}dx &= \frac{1}{2}a_{n0}\int_{-1}^{1}\frac{T_0(x)T_m(x)}{\sqrt{1-x^2}}dx + \sum_{k=1}^{\infty} a_{nk}\int_{-1}^{1}\frac{T_k(x)T_m(x)}{\sqrt{1-x^2}}dx \\ &= \frac{\pi}{2}a_{nm} = \frac{\pi}{2}a_{nk}(\texttt{where} \quad k = m)\end{aligned}$$

$$a_{nk} = \frac{2}{\pi}\int_{-1}^{1}\frac{x^n T_k(x)}{\sqrt{1-x^2}}dx$$

Let $x = \cos t$ and use the formula[9]

$$b_p = \int_0^{\pi} \cos^p t dt = \frac{(p-1)!!}{p!!}\pi, \texttt{when} \quad p \; is \; \texttt{even}; \quad b_p = 0, \texttt{when} \; p \; \texttt{is odd}.$$

We have

$$\begin{aligned}a_{nk} &= \frac{2}{\pi}\int_0^{\pi}\cos^n t\cos ktdt \\ &= \frac{2}{\pi}\int_0^{\pi}\cos^{n+1}t\cos(k-1)tdt - \frac{2}{\pi}\int_0^{\pi}\cos^n t\sin t\sin(k-1)tdt \\ &= a_{(n+1)(k-1)} + \frac{2}{\pi}\cdot\frac{1}{n+1}\cos^{n+1}t\sin(k-1)t|_0^{\pi} - \frac{2}{\pi}\cdot\frac{k-1}{n+1}\int_0^{\pi}\cos^{n+1}t\cos(k-1)tdt \\ &= a_{(n+1)(k-1)} + 0 - \frac{k-1}{n+1}a_{(n+1)(k-1)} = \frac{n-k+2}{n+1}a_{(n+1)(k-1)} \\ &= \frac{n-k+2}{n+2}\cdot\frac{n-k+4}{n+2}a_{(n+2)(k-2)} = \cdots \\ &= \frac{n-k+2}{n+1}\cdot\frac{n-k+4}{n+2}\cdots\frac{n-k+2i}{n+i}\cdots\frac{n-k+2k}{n+k}a_{(n+k)(0)}.\end{aligned}$$

So when $n+k$ is odd, or n and k are of opposite parity (whether $k>n$ or $k<n$), $a_{nk}=0$ for $a_{n+k0}=\frac{2}{\pi}b_{n+k}=0$; when $k>n$, n and k are of the same parity, or $k=n+2i \quad (i>0)$, $a_{nk}=0$ for $n-k+2i=0$; when $k<n+1$, n and k are of same parity, or $n+k$ is even, then

$$\begin{aligned} a_{nk} &= \frac{\frac{(n+k)!!}{(n-k)!!}}{\frac{(n+k)!}{n!}}\cdot\frac{2}{\pi}b_{n+k} \\ &= \frac{\frac{(n+k)!!}{(n-k)!!}}{\frac{(n+k)!}{n!}}\cdot\frac{2}{\pi}\cdot\frac{(n+k-1)!!}{(n+k)!!}\pi=\frac{2n!}{(n-k)!!(n+k)!!}. \end{aligned}$$

Theorem 2. Let

$$x^n=b_{n0}U_0(x)+\sum_{k=1}^{\infty}b_{nk}U_k(x). \tag{2}$$

Then if $k\geq n+1$, $b_{nk}=0$; If $k\leq n$, k and n are opposite parity, then $b_{nk}=0$; If $k\leq n$, k and n are of the same parity, then

$$b_{nk}=\frac{2(k+1)n!}{(n-k)!!(n+k+2)!!}.$$

Proof. Multiply $U_m(x)\sqrt{1-x^2}$ $(m=0,1,\ldots)$ to the two sides of (2) and definite integral from -1 to 1. Because of the orthogonal property[3,8] of the second type Chebyshev polynomials, we can get

$$\begin{aligned} \int_1^{-1}x^nU_m(x)\sqrt{1-x^2}dx &= b_{n0}\int_1^{-1}x^nU_0(x)U_m(x)\sqrt{1-x^2}dx \\ &\quad+\sum_{k=1}^{\infty}b_{nk}\int_1^{-1}x^nU_k(x)U_m(x)\sqrt{1-x^2}dx \\ &= \frac{\pi}{2}b_{nm} \\ &= \frac{\pi}{2}b_{nk}(\texttt{where}\quad k=m), \end{aligned}$$

$$b_{nk}=\frac{2}{\pi}\int_1^{-1}x^nU_k(x)\sqrt{1-x^2}dx.$$

Let $x=\cos t$ and we have

$$\begin{aligned} b_{nk} &= \frac{2}{\pi}\int_0^{\pi}\frac{\cos^n t\sin(k+1)t\sin t}{\sin t}dt \\ &= \frac{2}{\pi}\frac{-1}{n+1}\cos^{n+1}t\sin(k+1)t|_0^{\pi}+\frac{2}{\pi}\frac{k+1}{n+1}\int_0^{\pi}\cos^{n+1}t\cos(k+1)tdt \\ &= \frac{k+1}{n+1}a_{(n+1)(k+1)}. \end{aligned}$$

So we can get the result of Theorem 2 from Theorem 1. By using theorem 1,2, changing n to $2n$ or $2n+1$, we can get very simple formulas:

Theorem 3.

$$\begin{aligned} x^{2n} &= \frac{(2n)!}{4^n(n!)^2}T_0(x) + \frac{2(2n)!}{4^n}\sum_{k=1}^{n}\frac{1}{(n-k)!(n+k)!}T_{2k}(x) \\ &= \frac{(2n)!}{4^n}\sum_{k=1}^{n}\frac{2k+1}{(n-k)!(n+k+1)!}U_{2k}(x), \\ n &= 0,1,2,\ldots, \end{aligned} \tag{3}$$

$$\begin{aligned} x^{2n+1} &= \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}T_{2k+1}(x) \\ &= \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{k+1}{(n-k)!(n+k+2)!}U_{2k+1}(x), \\ n &= 0,1,2,\ldots. \end{aligned} \tag{4}$$

Therom 3 is very convenient for us and it is suggested to be written in the books. For example, we can immediately get that

$$\begin{aligned} x^9 &= \frac{9!}{4^4}(\frac{1}{4!5!}T_1(x) + \frac{1}{3!6!}T_3(x) + \frac{1}{2!7!}T_5(x) + \frac{1}{1!8!}T_7(x) + \frac{1}{9!}T_9(x)) \\ &= \frac{63}{128}T_1(x) + \frac{21}{64}T_3(x) + \frac{9}{64}T_5(x) + \frac{9}{256}T_7(x) + \frac{1}{256}T_9(x), \end{aligned}$$

$$x^{100} = \frac{100!}{2^{100}(50!)^2}T_0(x) + \frac{100!}{2^{99}}\sum_{k=1}^{50}\frac{1}{(50-k)!(50+k)!}T_{2k}(x).$$

§3. Some Formulas of Trigonometric Function

Although there are many formulas of trigonometric function, we can still get some new ones.

Theorem 4. For many positive integer n, we have

$$\cos^{2n}x = \frac{(2n)!}{4^n(n!)^2} + \frac{2(2n)!}{4^n}\sum_{k=1}^{n}\frac{1}{(n-k)!(n+k)!}\cos(2k)x,$$

$$\cos^{2n+1}x = \frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}\cos(2k+1)x. \tag{5}$$

Proof. Making $\cos^m x$ Fourier cosine expansion, we get

$$\cos^m x = \frac{1}{2}a_{m0} + \sum_{k=1}^{\infty}a_{mk}\cos kx,$$

$$a_{mk} = \frac{2}{\pi}\int_0^{\pi}\cos^m x\cos kx dx.$$

Changing m to $2n$ or $2n+1$, we can complete the proof of Theorem 4 from Theorem 1.

Theorem 5. Let n be an integer,

$$G_{nk} = \int \cos^n x \cos kx dx,$$

$$H_{nk} = \int \cos^n x \sin kx dx,$$

$$I_{nk} = \int \sin^n x \cos kx dx,$$

$$J_{nk} = \int \sin^n x \sin kx dx.$$

Then,

$$G_{nk} = \frac{1}{n+1} \sin^{n+1} x \sin(k-1)x + \frac{n-k+2}{n+1} G_{(n+1)(k-1)},$$

$$H_{nk} = \frac{-1}{n+1} \cos^{n+1} x \sin(k-1)x + \frac{n-k+2}{n+1} H_{(n+1)(k-1)},$$

$$I_{nk} = \frac{1}{n+1} \sin^{n+1} x \cos(k-1)x - \frac{n-k+2}{n+1} J_{(n+1)(k-1)},$$

$$J_{nk} = \frac{1}{n+1} \sin^{n+1} x \sin(k-1)x + \frac{n-k+2}{n+1} I_{(n+1)(k-1)}. \tag{6}$$

Proof. The proof can be completed just using elementary formulas

$$\cos kx = \cos x \cos(k-1)x - \sin x \sin(k-1)x,$$

$$\sin kx = \sin x \cos(k-1)x + \cos x \sin(k-1)x,$$

and the method of integration by parts.

Theorem 6. For any positive integer n, we have

$$\sin^{2n} x = \frac{(2n)!}{4^n (n!)^2} + \frac{2(2n)!}{4^n} \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)!(n+k)!} \cos(2k)x,$$

$$\sin^{2n+1} x = \frac{(2n+1)!}{4^n} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(n+k+1)!} \sin(2k+1)x. \tag{7}$$

Proof. Making $\sin^m x$ Fourier cosine expansion, we get

$$\sin^m x = \frac{1}{2} a_{m0} + \sum_{k=1}^{\infty} (a_{mk} \cos kx + b_{mk} \sin kx),$$

$$a_{mk} = \frac{1}{\pi} \int_0^{\pi} \sin^m x \cos kx dx,$$

$$b_{mk} = \frac{1}{\pi} \int_0^{\pi} \sin^m x \sin kx dx.$$

Changing $m$to $2n$ or $2n+1$, we can complete the proof of Theorem 6 from Theorem 5 and Theorem 1.

In formula (5) and formula (7), it is interested that the absolute values of the coefficients of corresponding terms are equal. Those formulas seem wonderful.

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Two Problems About 2-Power Free Numbers

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Abstract For any positive integer n, let a_n denote the nth 2-power free number. In this paper we probve that $a_n < 1.8n$.

Keywords 2-power free number, upper bound, Möbius inversion formula.

§1. Introduction and results

Let a be a positive integer. If a has no square divisor greater than 1, then a is called a 2-power free number (see [1]). Recently, Mladen and Krassimir [3] showed that

$$a_n < \left[\frac{1}{4}(n^2+3n+4)\right],$$

where $[x]$ is the integral part of x. Simultaneously, they proposed the following two problems.

Problem 1. Does there exist a constant $C > 1$ such that $a_n < Cn$?

Problem 2. Is $C < 2$?

In this paper, we completely solve the above mentioned problems as follows:

Theorem. For any position integer n, we have $a_n < 1.8n$.

§2. Proof of the theorem

Now we complete the proof of the theorem.

Clearly, the theorem holds for $n \leq 10^6$. We now suppose that

$$a_n \geq 1.8n \tag{1}$$

for a positive integer n with $n > 10^6$. For any real number x with $x \geq 1$, let $f(x)$ denote the number of 2-power free number a with $a \leq x$. We find from (1) that

$$f(n) < 0.56n \tag{2}$$

for a positive integer n with $n > 10^6$.

Notice that every positive integer a can be expressed as $a = b^2d$, where b is a positive integer and d is 2-power free number. Hence, we have

$$n = [n] = \sum_{k=1}^{r} f\left(\frac{n}{i^2}\right), \tag{3}$$

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where $r = [\sqrt{n}]$. By the Möbius inversion formula (see [2, Theorem 268]), we get from (3) that

$$f(n) = \sum_{i=1}^{r} \mu(i) \left(\frac{n}{i^2}\right), \tag{4}$$

where $\mu(i)$ is the Möbius function.

Since $|\mu(k)| \leq 1$, we obtain from (4) that

$$f(n) \geq \sum_{i=1}^{r} \left(\mu(i)\frac{n}{i^2} - 1\right) \geq n\sum_{i=1}^{r} \frac{\mu(i)}{i^2} - \sqrt{n} = n\left(\sum_{i=1}^{\infty} \frac{\mu(i)}{i^2} - \sum_{i=r+1}^{\infty} \frac{\mu(i)}{i^2}\right) - \sqrt{n}. \tag{5}$$

By Formula (17.2.2) and Theorem 287 of [2], we have

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}, \quad \sum_{i=1}^{\infty} \frac{\mu(i)}{i^2} = \frac{6}{\pi^2}, \tag{6}$$

It is easy to check that

$$\sum_{i=1}^{r} \frac{1}{i^2} \geq \sum_{i=1}^{1000} \frac{1}{i^2} > 1.6071. \tag{7}$$

since$\sqrt{n} < \frac{n}{1000}$ if $n > 10^6$, substitute (6) and (7) into (5), we get

$$f(n) > 0.56n, \tag{8}$$

which is a contradiction with (2). Thus, the theorem is proved.

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The Asymptotic Formula of $\sum_{n\leq x} \log P_{ld}(n)$

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Abstract Let $P_{ld}(n) = \prod_{d|n, d\equiv l(q)} d$, $1 \leq l \leq q$, where l, q are fixed numbers. This paper uses the hyperbolic summation method and the exponential pair theory to obtain the asymptotic formula of $\sum_{n\leq x} \log P_{ld}(n)$.

Keywords Exponential sum, hyperbolic summation method, exponent pair.

§1. Introduction

F. Smarandache introduced the function $P_d(n) := \prod_{d|n} d$ in Problem 25 of [1]. In this paper we shall define a similar function and study its property. Let $1 \leq l \leq q$ be fixed integers. Now we define a function $P_{ld}(n)$ by

$$P_{ld}(n) := \prod_{\substack{d|n \\ d\equiv l(mod q)}} d.$$

In the present paper, we shall prove the following:

Theorem. For any real number $x > 1$, we have

$$\sum_{n\leq x} \log P_{ad}(n) = \frac{1}{2q} x \log^2 x + \frac{\gamma - 1}{q} x \log x + cx + O((q^{-1}x)^{\frac{27}{82}+\varepsilon}),$$

where c is a constant which depends on q and l, γ is the Euler constant.

Notations. For any real t, $[t]$ is the integer part of t, $\psi(t) = t - [t] - \frac{1}{2}$ and $e(t) = e^{2\pi i t}$. $U \ll V$ means $|U| \leq CV$ for some unspecified positive constant C. We also use the Landau notation $U = O(V)$; this is equivalent to $U \ll CV$. $f \sim g$ means $c_1 g < f < c_2 g$ with some positive, unspecified constants c_1, c_2. Throughout this paper, c_1, c_2, c_3 are constants which may depend on q, l. ε denotes a fixed small positive constant which may be different at each occurence.

§2. Some Lemmas

We need the following Lemmas.

Lemma 1. Suppose that $f(u)$ has two continuous derivatives on $[u_1, u_2]$. Then

$$\sum_{u_1<n\leq u_2} f(n) = \int_{u_1}^{u_2} f(u)du - \psi(u)f(u)\Big|_{u_1}^{u_2} + \psi_1(u)f'(u)\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2}\psi_1(u)f''(u)du.$$

Proof. This is (2.2.8) of [3] ($l = 2$), i.e. Euler-MacLaurin summation formula.

Lemma 2. For $y \geq 1$, we have

$$\sum_{0\leq m\leq y-\frac{l}{q}} \log(l+mq) = y\log qy - y - \psi\left(y-\frac{l}{q}\right)\log qy + c_1 + O\left(\frac{1}{y}\right).$$

Proof. This follows from Lemma 1, take $f(n) = \log(l+nq),\ u_1 = 0,\ u_2 = y - \frac{l}{q}$.

Lemma 3. For $y \geq 1$, we have

$$\sum_{1\leq k\leq y}\frac{1}{k} = \log y + \gamma - \frac{\psi(y)}{y} + O\left(\frac{1}{y^2}\right).$$

Proof. This is Lemma 4.4. of [2].

Lemma 4. For $y \geq 1$, we have

$$\sum_{1\leq k\leq y}\frac{\log k}{k} = \frac{1}{2}\log^2 y + c_2 - \frac{\psi(y)}{y}\log y + O\left(\frac{1}{y^2}\right).$$

Proof. This follows from Lemma 1, take $f(n) = \frac{\log n}{n}, u_1 = 1, u_2 = y$.

Lemma 5. For $y \geq 1$, we have

$$\sum_{0\leq m\leq y-\frac{l}{q}}\frac{\log(l+mq)}{l+mq} = \frac{1}{2q}\log^2 qy + c_3 - \frac{\log qy}{qy}\psi\left(y-\frac{l}{q}\right) + O\left(\frac{1}{y^2}\right).$$

Proof. This follows from Lemma 1, take $f(n) = \frac{\log l+nq}{l+nq}, u_1 = 0, u_2 = y - \frac{l}{q}$.

Lemma 6. Suppose $f(n)$ is a real valued function on the interval $[N, N_1]$, where $2 \leq N < N_1 \leq 2N$. If $|f^{(j)}(n)| \sim \lambda_1 N^{-j+1} (j = 1, 2, 3, 4, 5, 6)$, then

$$\sum_{N<n\leq N_1}\psi(f(n)) \ll \lambda_1^{\frac{\kappa}{\kappa+1}} N^{\frac{\kappa+\lambda}{\kappa+1}} + \lambda_1^{-1},$$

where (κ, λ) is any exponent pair.

Proof. This follows from Lemma 4.3. of [2].

§3. Proof of the theorem

In this section, we always assume (κ, λ) is an exponent pair.

$$\sum_{n\leq x}\log P_{ad}(n) = \sum_{dk\leq x, d\equiv l(q)}\log d = \sum_{(l+mq)k\leq x}\log(l+mq).$$

By hyperbolic summation method, we have

$$\begin{aligned}\sum_{n\le x}\log P_{ad}(n) &= \sum_{1\le k\le (\frac{x}{q})^{\frac12}}\ \sum_{0\le m\le \frac{x}{qk}-\frac{l}{q}}\log(l+mq)\\ &\quad+\sum_{0\le m\le (\frac{x}{q})^{\frac12}-\frac{l}{q}}\left[\frac{x}{qm+l}\right]\log(l+mq)\\ &\quad-\left[\left(\frac{x}{q}\right)^{\frac12}\right]\sum_{0\le m\le (\frac{x}{q})^{\frac12}-\frac{l}{q}}\log(l+mq)\\ &=\sum\nolimits_1+\sum\nolimits_2-\sum\nolimits_3. \end{aligned}\tag{1}$$

where

$$\begin{aligned}\sum\nolimits_1 &= \sum_{1\le k\le (\frac{x}{q})^{\frac12}}\ \sum_{0\le m\le \frac{x}{qk}-\frac{l}{q}}\log(l+mq).\\ \sum\nolimits_2 &= \sum_{0\le m\le (\frac{x}{q})^{\frac12}-\frac{l}{q}}\left(\frac{x}{l+mq}-\psi\left(\frac{x}{l+mq}\right)-\frac12\right)\log(l+mq).\\ \sum\nolimits_3 &= \left(\left(\frac{x}{q}\right)^{\frac12}-\psi\left(\left(\frac{x}{q}\right)^{\frac12}\right)-\frac12\right)\sum_{0\le m\le (\frac{x}{q})^{\frac12}-\frac{l}{q}}\log(l+mq).\end{aligned}$$

By Lemma 2, we get

$$\begin{aligned}\sum\nolimits_1 = &\ \frac{x}{q}(\log x-1)\sum_{1\le k\le (\frac{x}{q})^{\frac12}}\frac1k-\frac{x}{q}\sum_{1\le k\le (\frac{x}{q})^{\frac12}}\frac{\log k}{k}\\ &-S_1+c_1\left(\frac{x}{q}\right)^{\frac12}+O(1),\end{aligned}$$

where

$$S_1=\sum_{1\le n\le (\frac{x}{q})^{\frac12}}\psi(\frac{x}{qn}-\frac{l}{q})\log\frac{x}{n}.$$

Then by Lemma 3 and Lemma 4, we get

$$\begin{aligned}\sum\nolimits_1 = &\ \frac{3}{8q}x\log^2 x+\left(\frac{\gamma}{q}-\frac{1}{2q}-\frac{\log q}{4q}\right)x\log x-S_1\\ &+\left(\frac{\log q}{2q}-\frac{\gamma}{q}-\frac{\log^2 q}{8q}-\frac{c_2}{q}\right)x\\ &-\left(\frac12\log qx-1\right)\left(\frac{x}{q}\right)^{\frac12}\psi\left(\left(\frac{x}{q}\right)^{\frac12}\right)+c_1\left(\frac{x}{q}\right)^{\frac12}+O(1).\end{aligned}\tag{2}$$

By Lemma 5, we get

$$\begin{aligned}\sum\nolimits_2 = \quad & \frac{1}{8q}x\log^2 x+\frac{\log q}{4q}x\log x+\left(c_3+\frac{\log^2 q}{8q}\right)x\\ & -\frac{1}{2}\left(\frac{x}{q}\right)^{\frac{1}{2}}\psi\left(\left(\frac{x}{q}\right)^{\frac{1}{2}}-\frac{l}{q}\right)\log qx\\ & -\frac{1}{2}\sum_{0\le m\le\left(\frac{x}{q}\right)^{\frac{1}{2}}-\frac{l}{q}}\log(l+mq)-S_2+O(1),\end{aligned}\tag{3}$$

where

$$S_2=\sum_{0\le m\le(\frac{x}{q})^{\frac{1}{2}}-\frac{l}{q}}\psi(\frac{x}{l+mq})\log(l+mq).$$

By Lemma 2, we get

$$\begin{aligned}\sum\nolimits_3 = \quad & \frac{1}{2q}x\log x+\frac{1}{2q}(\log q-2)x-\frac{1}{2}\left(\frac{x}{q}\right)^{\frac{1}{2}}\psi\left(\left(\frac{x}{q}\right)^{\frac{1}{2}}-\frac{l}{q}\right)\log qx\\ & -\left(\frac{1}{2}\log qx-1\right)\left(\frac{x}{q}\right)^{\frac{1}{2}}\psi\left(\left(\frac{x}{q}\right)^{\frac{1}{2}}\right)+c_1\left(\frac{x}{q}\right)^{\frac{1}{2}}\\ & -\frac{1}{2}\sum_{0\le m\le(\frac{x}{q})^{\frac{1}{2}}-\frac{l}{q}}\log(l+mq)+O(1).\end{aligned}\tag{4}$$

Then by (1)-(4), we get

$$\sum_{n\le x}\log P_{ad}(n)=\frac{1}{2q}x\log^2 x+\frac{\gamma-1}{q}x\log x+cx-S_1-S_2+O(1),\tag{5}$$

where

$$c=\frac{1-\gamma}{q}-\frac{c_2}{q}+c_3.$$

Now we estimate S_1 and S_2, where

$$\begin{aligned}S_1 \quad &= \sum_{1\le n\le(\frac{x}{q})^{\frac{1}{2}}}\psi\left(\frac{x}{qn}-\frac{l}{q}\right)\log\frac{x}{n},\\ S_2 \quad &= \sum_{0\le m\le(\frac{x}{q})^{\frac{1}{2}}-\frac{l}{q}}\psi\left(\frac{x}{l+mq}\right)\log(l+mq).\end{aligned}$$

We only estimate S_1, the estimate of S_2 is similar to S_1. First we estimate

$$S_{11}=\sum_{N<n\le 2N}\psi\left(\frac{x}{qn}-\frac{l}{q}\right).$$

By Lemma 6, we get

$$\begin{aligned}S_{11} \quad &\ll (xq^{-1}N^{-2})^{\frac{\kappa}{\kappa+1}}N^{\frac{\kappa+\lambda}{\kappa+1}}+qx^{-1}N^2\\ &\ll (xq^{-1})^{\frac{\kappa}{\kappa+1}}N^{\frac{\lambda-\kappa}{\kappa+1}}+qx^{-1}N^2.\end{aligned}\tag{6}$$

Then

$$\sum_{1\le n\le(\frac{x}{q})^{\frac{1}{2}}}\psi\left(\frac{x}{qn}-\frac{l}{q}\right)=\sum_{1\le j\le J}\sum_{n\in I_j}\psi\left(\frac{x}{qn}-\frac{l}{q}\right),$$

where

$$I_j=\{n:2^{-j}(q^{-1}x)^{1/2}<n\le 2^{-j+1}(q^{-1}x)^{1/2}\}.$$

By (6), we get

$$\begin{aligned}\sum_{1\le n\le(\frac{x}{q})^{\frac{1}{2}}}\psi\left(\frac{x}{qn}-\frac{l}{q}\right) &\ll (q^{-1}x)^{\frac{\kappa+\lambda}{2\kappa+2}}\sum_{1\le j\le J}2^{-j(\lambda-\kappa)}+\sum_{1\le j\le J}2^{-2j}\\ &\ll (q^{-1}x)^{\frac{\kappa+\lambda}{2\kappa+2}+\varepsilon}.\end{aligned}$$

Take $(\kappa,\lambda)=(\frac{11}{30},\frac{16}{30})$, we get

$$\sum_{1\le n\le(\frac{x}{q})^{\frac{1}{2}}}\psi\left(\frac{x}{qn}-\frac{l}{q}\right)\ll(q^{-1}x)^{\frac{27}{82}+\varepsilon}.$$

Then by partial summation, we get

$$S_1\ll(q^{-1}x)^{\frac{27}{82}+\varepsilon}. \tag{7}$$

Now, Theorem follows from (5) and (7).

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On the K-power free number sequence

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Abstract The main purpose of this paper is to study the distribution properties of the k-power free numbers, and give an interesting asymptotic formula for it.

Keywords k-power free number, mean value, asymptotic formula

§1. Introduction

A natural number n is called a k-power free number if it can not be divided by any p^k, where p is a prime. One can obtain all k-power free number by the following method: From the set of natural numbers (except 0 and 1).

$-$take off all multiples of 2^k, ($i.e$ $2^k, 2^{k+1}, 2^{k+2}\ldots$).

$-$take off all multiples of 3^k.

$-$take off all multiples of 5^k.

$\ldots$ and so on (take off all multiples of all k-power primes).

When $k = 3$, the $k-$power free number sequence is 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17,$\cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of the $k-$power free number sequence. About this problem, Zhang Tianping had given an asymptotic formula in reference [3].That is, he proved that

$$\sum_{\substack{n\leq x\\ n\in B}} \omega^2(n) = \frac{x(\ln\ln x)^2}{\zeta(k)} + O(x(\ln\ln x)),$$

where $\omega(n)$ denotes the number of prime divisors of n, $\zeta(k)$ is the Riemann zeta-function.

This paper as a note of [3], we use the analytic method to obtain a more accurate asymptotic formula for it. That is, we shall prove the following:

Theorem. Let k be a positive integer with $k \geq 2$, B denotes the set of all k-power free number. Then we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in B}} \omega^2(n) = \frac{1}{\zeta(k)}\left(x(\ln\ln x)^2 + C_1 x\ln\ln x + C_2 x\right) + O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $\zeta(k)$ is the Riemann zeta$-$function, C_1 and C_2 are both constants.

§2. Some Lemmas

To complete the proof of the theorem, we need following several Lemmas.

Lemma 1. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x}\omega(n) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

$$\sum_{n\leq x}\omega^2(n) = x(\ln\ln x)^2 + ax\ln\ln x + bx + O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$, a and b are two computable constants.

Proof. See reference [2].

Lemma 2. Let $\mu(n)$ be the Möbius function, then for any real number $x \geq 2$, we have the following identity

$$\sum_{n=1}^{\infty}\frac{\mu(n)\omega(n)}{n^s} = -\frac{1}{\zeta(s)}\sum_p \frac{1}{p^s-1}.$$

Proof. See reference [3].

Lemma 3. Let $k \geq 2$ be a fixed integer, then for any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{d^k m\leq x}\omega^2(m)\mu(d) = \frac{1}{\zeta(k)}\left(x(\ln\ln x)^2 + ax\ln\ln x + bx\right) + O\left(\frac{x\ln\ln x}{\ln x}\right).$$

Proof. From Lemma 1 we have

$$\begin{aligned}
\sum_{d^k m\leq x}\omega^2(m)\mu(d) &= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{m\leq \frac{x}{d^k}}\omega^2(m)\\
&= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left(\frac{x}{d^k}(\ln\ln\frac{x}{d^k})^2 + a\frac{x}{d^k}\ln\ln\frac{x}{d^k} + b\frac{x}{d^k} + O\left(\frac{\frac{x}{d^k}\ln\ln\frac{x}{d^k}}{\ln\frac{x}{d^k}}\right)\right)\\
&= x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\ln\ln x + \ln\left(1-\frac{k\ln d}{\ln x}\right)\right)^2 + ax\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln\frac{x}{d^k}\\
&\quad + bx\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k} + O\left(\frac{x\ln\ln x}{\ln x}\right). \qquad (1)
\end{aligned}$$

The first term on the right hand side of (1) is

$$\begin{aligned}
&x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\ln\ln x + \ln\left(1-\frac{k\ln d}{\ln x}\right)\right)^2\\
&= x(\ln\ln x)^2\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k} + O\left(x\sum_{d\leq x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\cdot\frac{k\ln d}{\ln x}\cdot\ln\ln x\right) + O\left(x\sum_{d\leq x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\cdot\frac{\ln^2 d}{\ln^2 x}\right)\\
&= \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right) + O\left(\frac{x\ln\ln x}{\ln x}\right) + O\left(\frac{x}{\ln^2 x}\right)\\
&= \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(\frac{x\ln\ln x}{\ln x}\right).
\end{aligned}$$

The second term on the right hand side of (1) is

$$\begin{aligned} & ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln\frac{x}{d^k} = ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln x + ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\left(1-\frac{k\ln d}{\ln x}\right) \\ = & \frac{ax\ln\ln x}{\zeta(k)} + O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\ln\ln x\right) + O\left(x\sum_{d\le x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\frac{\ln d}{\ln x}\right) \\ = & \frac{ax\ln\ln x}{\zeta(k)} + O\left(\frac{x}{\ln x}\right). \end{aligned}$$

The third term on the right hand side of (1) is

$$bx\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k} = bx\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + O\left(bx\sum_{d>x^{\frac{1}{k}}}\frac{1}{d^k}\right) = \frac{bx}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right).$$

From the calculations above we get the asymptotic formula

$$\sum_{d^km\le x}\omega^2(n)\mu(d) = \frac{1}{\zeta(k)}\left(x(\ln\ln x)^2 + ax\ln\ln x + bx\right) + O\left(\frac{x\ln\ln x}{\ln x}\right).$$

This proves Lemma 3.

Lemma 4. Let any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{d^km\le x}\omega^2(d)\mu(d) = A_1x + O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right),$$

where $A_1 = \sum\limits_{d=1}^{\infty}\frac{\omega^2(d)\mu(d)}{d^k}$ is a calculable constant.

Proof. Note that the series $\sum_{d^km\le x}\omega^2(d)\mu(d)$ is convergent, so we have

$$\begin{aligned} & \sum_{d^km\le x}\omega^2(d)\mu(d) = \sum_{d\le x^{\frac{1}{k}}}\omega^2(d)\mu(d)\sum_{m\le\frac{x}{d^k}}1 = \sum_{d\le x^{\frac{1}{k}}}\omega^2(d)\mu(d)\left[\frac{x}{d^k}\right] \\ = & x\sum_{d=1}^{\infty}\frac{\omega^2(d)\mu(d)}{d^k} + O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{\omega^2(d)|\mu(d)|}{d^k}\right) + O\left(\sum_{d\le x^{\frac{1}{k}}}\omega^2(d)|\mu(d)|\right) \\ = & A_1x + O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right) + O\left(x^{\frac{1}{k}}(\ln\ln x^{\frac{1}{k}})^2\right) \\ = & A_1x + O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right). \end{aligned}$$

This proves Lemma 4.

Lemma 5. For any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{d^km\le x}\omega^2((d,m))\mu(d) = A_2x + O\left(x^{\frac{1}{k}}\right),$$

where $A_2 = \dfrac{\mu(d)\sum\limits_{u|d}\dfrac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\dfrac{\mu(s)}{s}}{d^k}$ is a calculable constant.

Proof. Assume that (u,v) is the greatest common divisor of u and v, then we have

$$\begin{aligned}
\sum_{d^k m\le x}\omega^2((d,m))\mu(d) &= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m,(\frac{m}{u},\frac{d}{u})=1}}\omega^2(u)\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m}}\omega^2(u)\left[\frac{1}{(\frac{m}{u},\frac{d}{u})}\right]\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m}}\omega^2(u)\sum_{\substack{s|\frac{m}{u}\\ s|\frac{d}{u}}}\mu(s)\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{q\le\frac{x}{d^k su}}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\left[\frac{x}{d^k us}\right]\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\left(\frac{x}{d^k us}+O(1)\right)\\
&= x\sum_{d=1}^{\infty}\frac{\mu(d)\sum\limits_{u|d}\frac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\frac{\mu(s)}{s}}{d^k}+O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{|\mu(d)|\sum\limits_{u|d}\frac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\frac{|\mu(s)|}{s}}{d^k}\right)\\
&\quad +O\left(\sum_{d\le x^{\frac{1}{k}}}|\mu(d)|\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}|\mu(s)|\right)\\
&= A_2x+O\left(x^{\frac{1}{k}+\varepsilon}\right),
\end{aligned}$$

where ε is any positive number. This proves Lemma 5.

Same as the method used in the above Lemmas, we can easily get the following three asymptotic formulas:

Lemma 6. For any real number $x\ge 2$, we have

$$\sum_{d^k m\le x}\omega(d)\omega((d,m))\mu(d)=A_3x+O\left(x^{\frac{1}{k}}\ln\ln x\right),$$

$$\sum_{d^k m\le x}\omega(m)\omega((d,m))\mu(d)=A_4x\ln\ln x+O\left(\frac{x\ln\ln x}{\ln x}\right),$$

$$\sum_{d^k m\le x}\omega(d)\omega(m)\mu(d)=C\left(x\ln\ln x+Ax\right)+O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $C=-\dfrac{1}{\zeta(k)}\sum\limits_{p}\dfrac{1}{p^k-1}$, A_3 and A_4 are constants.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. From Lemmas above, we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in B}}\omega^2(n)=\sum_{n\le x}\omega^2(n)\sum_{d^k|n}\mu(d)=\sum_{d^km\le x}\omega^2(d^km)\mu(d)\\
=&\sum_{d^km\le x}(\omega(d)+\omega(m)-\omega((d,m)))^2\mu(d)\\
=&\sum_{d^km\le x}\omega^2(m)\mu(d)+\sum_{d^km\le x}\omega^2(d)\mu(d)+\sum_{d^km\le x}\omega^2((d,m))\mu(d)\\
&\qquad+2\left(\sum_{d^km\le x}\omega(d)\omega(m)\mu(d)\right)-2\left(\sum_{d^km\le x}\omega(d)\omega((d,m))\mu(d)\right)\\
&\qquad-2\left(\sum_{d^km\le x}\omega(m)\omega((d,m))\mu(d)\right)\\
=&\frac{1}{\zeta(k)}\left(x(\ln\ln x)^2+C_1x\ln\ln x+C_2x\right)+O\left(\frac{x\ln\ln x}{\ln x}\right).
\end{aligned}$$

This completes the proof of Theorem.

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On a generalized equation of Smarandache and its integer solutions

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Abstract Let $a \neq 0$ be any given real number. If the variables $x_1, x_2, \cdots, x_n$ satisfy $x_1 x_2 \cdots x_n = 1$, the equation

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na$$

has one and only one nonnegative real number solution $x_1 = x_2 = \cdots = x_n = 1$. This generalized the problem of Smarandache in book [1].

Keywords Equation of Smarandache, real number solutions.

§1. Introduction

Let Q denotes the set of all rational numbers, $a \in Q \setminus \{-1, 0, 1\}$. In problem 50 of book [1], Professor F. Smarandache asked us to solve the equation

$$xa^{\frac{1}{x}} + \frac{1}{x}a^x = 2a. \tag{1}$$

Professor Zhang [2] has proved that the equation has one and only one real number solution $x = 1$. In this paper, we generalize the equation (1) to

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na, \tag{2}$$

and use the elementary method and analysis method to prove the following conclusion:

Theorem. *For any given real number $a \neq 0$, if the variables $x_1, x_2, \cdots, x_n$ satisfy $x_1 x_2 \cdots x_n = 1$, then the equation*

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na$$

has one and only one nonnegative real number solution $x_1 = x_2 = \cdots = x_n = 1$.

§2. Proof of the theorem

In this section, we discuss it in two cases $a > 0$ and $a < 0$.

1) For the case $a > 0$, we let

$$f(x_1, x_2, \cdots, x_{n-1}, x_n) = \frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_{n-1}}a^{x_{n-1}} + \frac{1}{x_n}a^{x_n} - na,$$

If we take x_n as the function of the variables $x_1, x_2, \cdots, x_{n-1}$, we have

$$f(x_1, x_2, \cdots, x_{n-1}, x_n) = \frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_{n-1}}a^{x_{n-1}} + x_1x_2\cdots x_{n-1}a^{\frac{1}{x_1x_2\cdots x_{n-1}}} - na.$$

Then the partial differential of f for every $x_i \quad (i = 1, 2, \cdots, n-1)$ is

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= \frac{1}{x_i}a^{x_i}\left(\log a - \frac{1}{x_i}\right) + \frac{1}{x_i}a^{\frac{1}{x_1x_2\cdots x_{n-1}}}(x_1x_2\cdots x_{n-1} - \log a)\\ &= \frac{1}{x_i}\left(a^{x_i}\left(\log a - \frac{1}{x_i}\right) + a^{x_n}\left(\frac{1}{x_n} - \log a\right)\right).\end{aligned}$$

Let

$$g(x_1, x_2, \cdots, x_{n-1}, x_n) = a^{x_i}\left(\log a - \frac{1}{x_i}\right) + a^{x_n}\left(\frac{1}{x_n} - \log a\right), \tag{3}$$

the partial differential quotient of g is

$$\begin{aligned}\frac{\partial g}{\partial x_i} &= a^{x_i}\left(\log^2 a - \frac{\log a}{x_i} + \frac{1}{x_i^2} + \frac{a^{x_n}}{x_ix_n}\left(x_n^2\log^2 a - x_n\log a + 1\right)\right)\\ &= \frac{a^{x_i}}{x_i^2}\left(\left(x_i\log a - \frac{1}{2}\right)^2 + \frac{3}{4}\right) + \frac{a^{x_n}}{x_ix_n}\left(\left(x_n\log a - \frac{1}{2}\right)^2 + \frac{3}{4}\right) > 0.\end{aligned}$$

It's easy to prove that the function $u(x) = a^x(\log a - \frac{1}{x})$ is increasing for the variable x when $x > 0$. From (3) we have:

i) if $x_i > x_n$, $g > 0$, $\frac{\partial f}{\partial x_i} > 0$, and f is increasing for the variable x_i;

ii) if $x_i < x_n$, $g < 0$, $\frac{\partial f}{\partial x_i} < 0$, and f is decreasing for the variable x_i;

iii) if $x_i = x_n$, $g = 0$, $\frac{\partial f}{\partial x_i} = 0$, and we get the minimum value of f.

We have

$$f \geq f_{x_1=x_n} \geq f_{x_1=x_2=x_n} \geq \cdots \geq f_{x_1=x_2=\cdots=x_n} \geq f_{x_1=x_2=\cdots=x_n=1} = 0,$$

and we prove that the equation (2) has only one integer solution $x_1 = x_2 = \cdots = x_n = 1$.

2) For the case $a < 0$, the equation (2) can be written as

$$\frac{1}{x_1}(-1)^{x_1}|a|^{x_1} + \frac{1}{x_2}(-1)^{x_2}|a|^{x_2} + \cdots + \frac{1}{x_n}(-1)^{x_n}|a|^{x_n} = -n|a|, \tag{4}$$

so we know that $x_i \quad (i = 1, 2, \cdots, n)$ is not an irrational number.

Let $x_i = \frac{q_i}{p_i}$ (q_i is coprime to p_i), then p_i must be an odd number because negative number has no real square root. From $x_1x_2\cdots x_n = 1$, we have $p_1p_2\cdots p_n = q_1q_2\cdots q_n$, so q_i is odd number and $(-1)^{x_i} = -1$ $(i = 1, 2, \cdots, n)$. In this case, the equation (4) become the following equation:

$$\frac{1}{x_1}|a|^{x_1} + \frac{1}{x_2}|a|^{x_2} + \cdots + \frac{1}{x_n}|a|^{x_n} = n|a|.$$

From the conclusion of case 1) we know that the theorem is also holds. This completes the proof of the theorem.

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On the mean value of the Smarandache LCM function $SL(n)$

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Abstract For any positive integer n, let $SL(n)$ denotes the least positive integer k such that $L(k) \equiv 0 \pmod{n}$, where $L(k)$ denotes the Least Common Multiple of all integers from 1 to k. The main purpose of this paper is to study the properties of the Smarandache LCM function $SL(n)$, and give an asymptotic formula for its mean value.

Keywords Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, we define $SL(n)$ as the least positive integer k such that $L(k) \equiv 0 \pmod{n}$. That is, $SL(n) = \min\{k : n \mid [1, 2, 3, \cdots, k]\}$. For example, $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 4$, $\cdots$, $SL(997) = 997$, $SL(998) = 499$, $SL(999) = 37$, $SL(1000) = 15$, $\cdots$. The arithmetical function $SL(n)$ is called the Smarandache LCM function. About its elementary properties, there are some people studied it, and obtained many interesting results. For example, in reference [1], Murthy showed that if n is a prime, then $SL(n) = S(n) = n$. Simultaneously, he proposed the following problem,

$$SL(n) = S(n), \quad S(n) \neq n? \tag{1}$$

Maohua Le [2] solved this problem completely, and proved that every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \text{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

But about the deeply arithmetical properties of $SL(n)$, it seems that none had studied it before, at least we have not seen such a result at present. It is clear that the value of $SL(n)$ is not regular, but we found that the mean value of $SL(n)$ has good value distribution properties. In this paper, we want to show this point. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL(n) = Ax^2 + c_1 \frac{x^2}{\ln x} + c_2 \frac{x^2}{\ln^2 x} + \cdots + c_k \frac{x^2}{\ln^k x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $A = \frac{1}{2}\sum_{p}\frac{1}{p^2-1}$, k is any fixed positive integer, and c_1, c_2, $\cdots$, c_k are calculable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. First we need the following two simple lemmas.

Lemma 1. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ be the factorization of n into prime power, then

$$SL(n) = \max\left(p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots\ p_t^{\alpha_t}\right).$$

Proof. This Lemma can be deduce by the definition of $SL(n)$, see reference [1].

Lemma 2. For any arithmetical function $a(n)$, let $A(x) = \sum_{n\le x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume $f(x)$ has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then we have

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Proof. This is the famous Abel's identity, its proof see Theorem 4.2 of [2].

Now we complete the proof of our Theorem. We will discuss it in two cases. Let p be the greatest prime divisor of n,

(a) if $n = p\cdot l$, $p > l$, then use Lemma 1 we obtain $SL(n) = p$.

(b) If $n = p\cdot l$, $p \le l$, then we will discuss it in three cases.

(i) If n is complete power of prime, that is $n = p^\alpha$, $\alpha \ge 2$, then $SL(n) \le n$, but the number of this kind n is not exceed $\sqrt{n}$. Thus $\sum\limits_{\substack{n\le x\\ n=p^\alpha\\ \alpha\ge 2}} SL(n) = O\left(x^{\frac{3}{2}}\right)$.

(ii) If n is not complete power of prime, that is $n = l\cdot p^\alpha$, and $l < p^\alpha$, $l \le \sqrt{n}$, then $SL(n) = p^\alpha$. Thus

$$\begin{aligned}
\sum_{\substack{n\le x\\ n=l\cdot p^\alpha\\ l<p^\alpha\\ 2\le\alpha<\frac{\ln x}{\ln 2}}} SL(n) &= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\ \sum_{l\le\sqrt{x}}\ \sum_{p^\alpha\le\frac{x}{l}} p^\alpha \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\ \sum_{l\le\sqrt{x}}\ \sum_{p\le\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}} p^\alpha \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{l\le\sqrt{x}}\frac{\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}}{\ln\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}}\cdot\left(\frac{x}{l}\right)^{\frac{1}{\alpha}\cdot\alpha} + O\left(\frac{x^{1+\frac{1}{\alpha}}}{\ln^2 x}\right)\right) \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{l\le\sqrt{x}}\frac{\alpha x^{\frac{1+\alpha}{\alpha}}}{l^{\frac{1+\alpha}{\alpha}}\ln\frac{x}{l}} + O\left(\frac{x^{\frac{1+\alpha}{\alpha}}}{\ln^2 x}\right)\right).
\end{aligned}$$

Because $\alpha \geq 2$, so $\frac{\alpha+1}{\alpha} \leq \frac{3}{2}$. We obtain

$$\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l<p^{\alpha}\\ 2\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) = O\left(x^{\frac{3}{2}}\ln x\right).$$

(*iii*) If n is not complete power of prime, that is $n = l\cdot p^{\alpha}$, and $l > p^{\alpha}$, $\alpha \geq 1$, $p^{\alpha} \leq \sqrt{n}$, then $SL(n) = l$. Thus

$$\begin{aligned}\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l>p^{\alpha}\\ 1\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) &= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p^{\alpha}\leq\sqrt{x}}\sum_{l\leq\frac{x}{p^{\alpha}}} l\\ &= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\left(\frac{1}{2}\left(\frac{x}{p^{\alpha}}\right)^2+O\left(\frac{x}{p^{\alpha}}\right)\right)\\ &= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\left(\frac{x^2}{2}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}}+O\left(\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{x}{p^{\alpha}}\right)\right),\end{aligned}$$

where

$$\begin{aligned}\frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}} &= \frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{p}\frac{1}{p^{2\alpha}}-\sum_{p>x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}}\right)\\ &= \frac{x^2}{2}\sum_{p}\frac{1}{p^2}\frac{1-\frac{1}{p^{2[\frac{\ln x}{\ln 2}]}}}{1-\frac{1}{p^2}}-\frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p>x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}}\\ &= \frac{x^2}{2}\sum_{p}\frac{1}{p^2-1}+O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^2\cdot x^{-\frac{2\alpha-1}{2\alpha}}\right)\\ &= 7Ax^2+O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^{\frac{2\alpha+1}{2\alpha}}\right).\end{aligned}$$

Using the same method, we deduce that

$$O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{x}{p^{\alpha}}\right) = O(x).$$

Because $\alpha \geq 1$, so $\frac{2\alpha+1}{2\alpha} \leq \frac{3}{2}$. Thus

$$\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l>p^{\alpha}\\ 1\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) = Ax^2+O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^{\frac{3}{2}}\right) = Ax^2+O\left(x^{\frac{3}{2}}\ln x\right),$$

where $A = \frac{1}{2}\sum_{p}\frac{1}{p^2-1}$.

Combining the above two cases, we may immediately get following equation:

$$\begin{aligned}\sum_{n\le x} SL(n) &= \sum_{\substack{n\le x\\ p>l\\ p>\sqrt{n}}} p + \sum_{\substack{n\le x\\ p\le l}} SL(n) \\ &= \sum_{l\le\sqrt{x}}\sum_{l<p\le\frac{x}{l}} p + Ax^2 + O\left(x^{\frac{3}{2}}\ln x\right).\end{aligned} \tag{2}$$

Let $a(n)$ denote the characteristic function of the prime. That is,

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise.}\end{cases}$$

Then we have $A(x) = \sum_{1<n\le x} a(n) = \sum_{p\le x} 1 = \pi(x)$.

Thus by Lemma 2

$$\sum_{l<p\le\frac{x}{l}} p = \sum_{l<n\le\frac{x}{l}} a(n)n = \pi\left(\frac{x}{l}\right)\frac{x}{l} + \pi(l)l - \int_l^{\frac{x}{l}} \pi(t)dt,$$

where $\pi(x) = \frac{x}{\ln x} + A_1\frac{x}{\ln^2 x} + A_2\frac{x}{\ln^3 x} + \ldots + A_n\frac{x}{\ln^n x} + O\left(\frac{x}{\ln^{n+1} x}\right)$, A_i are calculable constants, $i = 1, 2, \ldots, n$.

Then

$$\begin{aligned}\sum_{l\le\sqrt{x}}\sum_{l<p\le\frac{x}{l}} p &= \sum_{l\le\sqrt{x}}\left(\pi\left(\frac{x}{l}\right)\frac{x}{l} + \pi(l)l - \int_l^{\frac{x}{l}}\pi(t)dt\right) \\ &= \sum_{l\le\sqrt{x}}\left(\frac{x^2}{l^2\ln\frac{x}{l}} + \sum_{i=1}^{n}\frac{A_i x^2}{l^2\ln^i\frac{x}{l}} + O\left(\frac{x^2}{\ln^{n+1}\frac{x}{l}}\right) - \int_l^{\frac{x}{l}}\pi(t)dt\right),\end{aligned} \tag{3}$$

where

$$\int_l^{\frac{x}{l}}\pi(t)dt = \int_l^{\frac{x}{l}}\frac{t}{\ln t}dt + \sum_{i=1}^{n}\int_l^{\frac{x}{l}}\frac{A_i t}{\ln^{i+1} t}dt + O\left(\int_l^{\frac{x}{l}}\frac{t}{\ln^{n+1} x}dt\right).$$

Integration by parts give us

$$\int_l^{\frac{x}{l}}\frac{t}{\ln t}dt = \frac{x^2}{2l^2\ln\frac{x}{l}} + \frac{x^2}{4l^2\ln^2\frac{x}{l}} + \frac{x^2}{4l^2\ln^3\frac{x}{l}} + \ldots + O\left(\frac{x^2}{\ln^{n+1} x}\right),$$

$$\int_l^{\frac{x}{l}}\frac{t}{\ln^2 t}dt = \frac{x^2}{2l^2\ln^2\frac{x}{l}} + \frac{x^2}{2l^2\ln^3\frac{x}{l}} + \frac{3x^2}{4l^2\ln^4\frac{x}{l}} + \ldots + O\left(\frac{x^2}{\ln^{n+1} x}\right),$$

$$\ldots$$

$$\int_l^{\frac{x}{l}}\frac{t}{\ln^n t}dt = \frac{x^2}{2l^2\ln^n\frac{x}{l}} + \frac{nx^2}{4l^2\ln^{n+1}\frac{x}{l}} + \frac{n(n+1)x^2}{8l^2\ln^{n+2}\frac{x}{l}} + \ldots + O\left(\frac{x^2}{\ln^{2n} x}\right).$$

Combining above formulas, and take them in (3), we obtain

$$\begin{aligned}\sum_{l\le\sqrt{x}}\sum_{l<p\le\frac{x}{l}} p &= \sum_{l\le\sqrt{x}}\left(B_1\frac{x^2}{l^2\ln\frac{x}{l}}+B_2\frac{x^2}{l^2\ln^2\frac{x}{l}}+\ldots+B_n\frac{x^2}{l^2\ln^n\frac{x}{l}}+O\left(\frac{x^2}{\ln^{n+1}x}\right)\right) \qquad (4)\\ &= \sum_{l\le\sqrt{x}}\frac{x^2}{l^2}\left[\frac{B_1}{\ln x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)+\frac{B_2}{\ln^2 x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^2+\ldots+\right.\\ &\qquad \left.\frac{B_n}{\ln^n x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^n+O\left(\frac{1}{\ln^{n+1}x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^{n+1}\right)\right]\\ &= \sum_{l\le\sqrt{x}}\frac{x^2}{l^2}\left[B_1\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)\right.\\ &\qquad +B_2\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^2\\ &\qquad +B_3\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^3+\ldots+\\ &\qquad \left.B_k\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^k+O\left(\frac{1}{\ln^{k+1}x}\right)\right].\end{aligned}$$

where B_i are constants, $i=1,2,\ldots,n$.

We know that $\sum_{l=1}^{\infty}\frac{\ln^k l}{l^2}=c$, thus

$$\sum_{l\le\sqrt{x}}\frac{\ln^k l}{l^2}=\sum_{l=1}^{\infty}\frac{\ln^k l}{l^2}-\sum_{l>\sqrt{x}}\frac{\ln^k l}{l^2}=c-O\left(\frac{\ln^k x}{\sqrt{x}}\right).$$

In (4) every coefficient of $\frac{x^2}{\ln^k x}$ is calculable, where k is any fixed positive integer.

So we obtain that

$$\sum_{l\le\sqrt{x}}\sum_{l<p\le\frac{x}{l}} p=c_1\frac{x^2}{\ln x}+c_2\frac{x^2}{\ln^2 x}+\cdots+c_k\frac{x^2}{\ln^k x}+O\left(\frac{x^2}{\ln^{k+1}x}\right).$$

Combining this formula with (2), we obtain

$$\begin{aligned}\sum_{n\le x}SL(n) &= \sum_{l\le\sqrt{x}}\sum_{l<p\le\frac{x}{l}} p+Ax^2+O\left(x^{\frac{3}{2}}\ln x\right)\\ &= Ax^2+c_1\frac{x^2}{\ln x}+c_2\frac{x^2}{\ln^2 x}+\cdots+c_k\frac{x^2}{\ln^k x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),\end{aligned}$$

where $A=\frac{1}{2}\sum_p\frac{1}{p^2-1}$, k is any fixed positive integer, and c_1, c_2, $\cdots$, c_k are calculable constants.

This completes the proof of the theorem.

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On the generalized constructive set

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Abstract In this paper, we use the elementary methods to study the properties of the constructive set S, and obtain some interesting properties for it.

Keywords Generalized constructive set, summation, recurrence equation, characteristic equation.

§1. Introduction and Results

The generalized constructive set S is defined as: numbers formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \leq m \leq 9$. That is to say,

(1) $d_1,\ d_2,\ \cdots, d_m$ belongs to S;

(2) If $a,\ b$ belong to S, then $\overline{ab}$ belongs to S too;

(3) Only elements obtained by rules (1) and (2) applied a finite number of times belongs to S.

For example, the constructive set (of digits 1, 2) is: $1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222, 1111, 1112, 1121 \cdots$. And the constructive set (of digits 1, 2, 3) is: $1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, 333, 1111, \cdots$. In problem 6, 7 and 8 of reference [1], Professor F.Smarandache asked us to study the properties of this sequence. In [2], Gou Su had studied the convergent properties of the series

$$\sum_{n=1}^{+\infty} \frac{1}{a_n^{\alpha}},$$

and proved that the series is convergent if $\alpha > \log m$, and divergent if $\alpha \leq \log m$, where $\{a_n\}$ denotes the sequence of the constructive set S, formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \leq m \leq 9$.

In this paper, we shall use the elementary methods to study the summation $\sum_{k=1}^{n} S_k$ and $\sum_{k=1}^{n} T_k$, where S_k denotes the summation of all k digits numbers in S, T_k denotes the summation of each digits of all k digits numbers in S.

That is, we shall prove the following

Theorem 1. For the generalized constructive set S of digits $d_1,\ d_2, \cdots, d_m$ $(1 \leq m \leq 9)$, we have

$$\sum_{k=1}^{n} S_k = \frac{d_1 + d_2 + \cdots + d_m}{9} \left(10 \times \frac{(10m)^n - 1}{10m - 1} - \frac{m^n - 1}{m - 1} \right),$$

where S_k denotes the summation of all k digits numbers in S.

Taking $m = 2,\ d_1 = 1$ and $d_2 = 2$ in Theorem 1, we may immediately get

Corollary 1. For the generalized constructive set S of digits 1 and 2, we have

$$\sum_{k=1}^{n} S_k = \frac{1}{3} \left(10 \times \frac{20^n - 1}{19} - 2^n + 1 \right).$$

Taking $m = 3,\ d_1 = 1,\ d_2 = 2$ and $d_3 = 3$ in Theorem 1, we may immediately get the following:

Corollary 2. For the generalized constructive set S of digits $1, 2$ and 3, we have

$$\sum_{k=1}^{n} S_k = \frac{2}{3} \left(10 \times \frac{30^n - 1}{29} - \frac{3^n}{2} + \frac{1}{2} \right).$$

Theorem 2. For the generalized constructive set S of digits $d_1,\ d_2, \cdots, d_m$ $(1 \leq m \leq 9)$, we have

$$\sum_{k=1}^{n} T_k = (d_1 + d_2 + \cdots + d_m) \cdot \frac{nm^{n+1} - (n+1)m^n + 1}{(m-1)^2},$$

where T_k denotes the summation of each digits of all k digits numbers in S.

Taking $m = 2,\ d_1 = 1$ and $d_2 = 2$ in Theorem 2, we may immediately get the following:

Corollary 3. For the the generalized constructive set S of digits 1 and 2, we have

$$\sum_{k=1}^{n} T_k = 3n \cdot 2^{n+1} - 3(n+1)2^n + 3.$$

Taking $m = 3,\ d_1 = 1,\ d_2 = 2$ and $d_3 = 3$ in Theorem 2, we may immediately get

Corollary 4. For the the generalized constructive set S of digits $1, 2$ and 3, we have

$$\sum_{k=1}^{n} T_k = \frac{3}{2} n \cdot 3^{n+1} - \frac{3}{2}(n+1)3^n + \frac{3}{2}.$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. Let S_k denotes the summation of all k digits numbers in the generalized constructive set S. Note that for $k = 1, 2, 3, \cdots$, there are m^k numbers of k digits in S. So we have

$$S_k = 10^{k-1} m^{k-1} (d_1 + d_2 + \cdots + d_m) + m S_{k-1}. \tag{1}$$

Meanwhile, we have

$$S_{k-1} = 10^{k-2} m^{k-2} (d_1 + d_2 + \cdots + d_m) + m S_{k-2}. \tag{2}$$

Combining (1) and (2), we can get the following recurrence equation

$$S_k - 11mS_{k-1} + 10m^2 S_{k-2} = 0.$$

Its characteristic equation

$$x^2 - 11mx + 10m^2 = 0$$

have two different real solutions

$$x = m,\ 10m.$$

So we let

$$S_k = A \cdot m^k + B \cdot (10m)^k.$$

Note that

$$S_0 = 0, \quad S_1 = d_1 + d_2 + \cdots + d_m,$$

we can get

$$A = -\frac{d_1 + d_2 + \cdots + d_m}{9m}, \quad B = \frac{d_1 + d_2 + \cdots + d_m}{9m}.$$

So

$$S_k = \frac{d_1 + d_2 + \cdots + d_m}{9m}\left((10m)^k - m^k\right).$$

Then

$$\sum_{k=1}^{n} S_k = \frac{d_1 + d_2 + \cdots + d_m}{9}\left(10 \times \frac{(10m)^n - 1}{10m - 1} - \frac{m^n - 1}{m - 1}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let T_k is denotes the summation of each digits of all k digits numbers in the generalized constructive set S.

Similarly, note that for $k = 1, 2, 3, \cdots$, there are m^k numbers of k digits in S, so we have

$$T_k \ = \ m^{k-1}(d_1 + d_2 + \cdots + d_m) + mT_{k-1} \tag{3}$$

Meanwhile, we have

$$T_{k-1} = m^{k-2}(d_1 + d_2 + \cdots + d_m) + mT_{k-2} \tag{4}$$

Combining (3) and (4), we can get the following recurrence equation

$$T_k - 2mT_{k-1} + m^2 T_{k-2} = 0,$$

its characteristic equation

$$x^2 - 2mx + m^2 = 0$$

have two solutions

$$x_1 = x_2 = m.$$

So we let

$$T_k = A \cdot m^k + k \cdot B \cdot m^k.$$

Note that

$$T_0 = 0, \quad T_1 = d_1 + d_2 + \cdots + d_m.$$

We may immediately deduce that

$$A = 0, \quad B = \frac{d_1 + d_2 + \cdots + d_m}{m}.$$

So

$$T_k = (d_1 + d_2 + \cdots + d_m) \cdot km^{k-1}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n} T_k &= (d_1 + d_2 + \cdots + d_m) \sum_{k=1}^{n} k \cdot m^{k-1} \\
&= (d_1 + d_2 + \cdots + d_m) \left(\sum_{k=1}^{n} m^k \right)' \\
&= (d_1 + d_2 + \cdots + d_m) \cdot \frac{nm^{n+1} - (n+1)m^n + 1}{(m-1)^2}.
\end{aligned}$$

This completes the proof of Theorem 2.

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Some exact calculating formulas for the Smarandache function[1]

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Abstract The main purpose of this paper is to study the calculation problem of the Smarandache function by using the elementary method, and give some exact calculation formulas for this function.

Keywords Smarandache function, exact value, calculating formula.

§1. Introduction

Let n be an positive integer, the famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

About this function and some other Smarandache type functions, many scholars have studied their properties, see [1], [2], [3] and [4]. Let $p(n)$ denotes the greatest prime divisor of n, it is clear that $S(n) \geq p(n)$. In fact, $S(n) = p(n)$ for all most n, as noted by Erdös [5]. This means that the number of $n \leq x$ for which $S(n) \neq p(n)$, denoted by $N(x)$, is $o(x)$. In [6], Xu Zhefeng studied the mean square value for $S(n) - p(n)$, and obtained an asymptotic formula as follows:

$$\sum_{n\leq x} (S(n) - p(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

It is easily to show that $S(p) = p$ and $S(n) < n$ except for the case $n = 4, n = p$. So there have a closely relationship between $S(n)$ and $\pi(x)$:

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n}\right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of n. There holds

$$S(n) = \max_{1\leq i\leq r} \{S\left(p_i^{\alpha_i}\right)\} = \alpha p$$

for some positive integer α and prime $p = p_j$. If $\beta < p$ then $S(p^\beta) = \beta p$. But for the case of $\beta \geq p$, it is difficult to calculate the exact value of $S(p^\beta)$. In this paper, we studied this

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problem, and proved some exact calculation formulas for $S(p^\beta)$. That is, we shall prove the following:

Theorem 1. Let p be a prime and k an integer with $1 \leq k < p$. For polynomials $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1$, we have

$$S\left(p^{f(p)}\right) = (p-1)f(p) + pf(1).$$

Theorem 2. Let p be a prime and k an integer with $1 \leq k < p$. Then for any positive integer n, we have

$$S\left(p^{kp^n}\right) = k\left(\phi(p^m) + \frac{1}{k}\right)p,$$

where $\phi(n)$ is the Euler function.

From our theorems, we may immediately get the following results:

Corollary 1. For any integer $n \geq 1$, we have the identity:

$$S\left(p^{g(p)}\right) = p^{n+1},$$

where $g(x) = x^n + x^{n-1} + \cdots + 1$.

Corollary 2. For any integer $k \geq 2$, we can find infinite group numbers $m_1, m_2, \cdots, m_k$ such that:

$$S\left(\prod_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} S(m_i)$$

with $S(m_i) = S\left(p^{p^{\alpha_i}}\right) = (\phi(p^{\alpha_i}) + 1)p$, α_i is any positive integer and p a prime.

§2. Proof of the theorems

In this section, we will complete the proof of the theorems. First we prove Theorem 1. For any integer n and prime p, if $p^\alpha \| n!$, then we have

$$\alpha = \sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right]. \tag{1}$$

From (1) and the properties of the Gauss funtion $[x]$, we have

$$\begin{aligned}
& \sum_{r=1}^{\infty} \left[\frac{(\phi(p^{n_k})+1)\,p + (\phi(p^{n_{k-1}})+1)\,p + \cdots + (\phi(p^{n_1})+1)\,p}{p^r}\right] \\
= & \sum_{r=1}^{n_k} \left[\frac{(\phi(p^{n_k})+1)\,p + (\phi(p^{n_{k-1}})+1)\,p + \cdots + (\phi(p^{n_1})+1)\,p}{p^r}\right] \\
= & \sum_{i=1}^{k} \left(1 + p^{n_i} - p^{n_i-1} + p^{n_i-1} - p^{n_i-2} + \cdots + p - 1\right) \\
= & p^{n_k} + p^{n_{k-1}} + \cdots + p^{n_1}.
\end{aligned}$$

That is,

$$\begin{aligned} S(p^{f(p)}) &= \sum_{i=1}^{k} (\phi(p^{n_i}) + 1)\, p \\ &= \sum_{i=1}^{k} (p-1)p^{n_i} + kp \\ &= (p-1)f(p) + kp \\ &= (p-1)f(p) + pf(1). \end{aligned}$$

This proves Theorem 1.

Let n be any positive integer and k be integer with $1 \leq k < p$. Then from (1) we can write

$$\begin{aligned} & \sum_{r=1}^{\infty} \left[\frac{k\left(\phi(p^n) + \frac{1}{k}\right) p}{p^r} \right] \\ = & \sum_{r=1}^{n} \left[\frac{k\left(p^{n+1} - p^n + \frac{p}{k}\right)}{p^r} \right] \\ = & \ 1 + k\left(p^n - p^{n-1}\right) + k\left(p^{n-1} - p^{n-2}\right) + \cdots + k(p-1) + \left[\frac{k(p-1)}{p} \right] \\ = & \ 1 + k\left(p^n - 1\right) + k - 1 \\ = & \ kp^n. \end{aligned}$$

This yields

$$S\left(p^{kp^m}\right) = k\left(\phi(p^m) + \frac{1}{k}\right) p.$$

This completes the proof of Theorem 2.

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On the F.Smarandache simple function

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Abstract The main purpose of this paper is using the analytic method to study the mean value properties of $d(p(x))$, where $d(n)$ is the Dirichlet divisor function, and give two interesting asymptotic formulas for it.

Keywords Smarandache simple function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, the F.Smarandache function $S(n)$ is defined as the smallest $m \in N^+$, such that $n \mid m!$. For a fixed prime p, the Smarandache simple function $S_p(n)$ is defined as the smallest $m \in N^+$, such that $p^n \mid m!$. In reference [1], Jozsef Sandor introduced the additive analogue of the Smarandache simple function $p(x)$ as follows:

$$p(x) = \min\{m \in N^+ : p^x \leq m!\}$$

and

$$p_*(x) = \max\{m \in N^+ : m! \leq p^x\},$$

they are defined on a subset of real numbers. It is obvious that $p(x) = m$, if $(m-1)! < p^x \leq m!$ for $x \geq 1$. About the properties of $p(x)$, many people had studied it before (see [1], [2]). But for the mean value of $d(p(x))$, it seems that no one have studied it before, where $d(n)$ is the Dirichlet divisor function. The main purpose of this paper is using the analytic method to study the asymptotic properties of the mean value of $d(p(x))$, and give some interesting asymptotic formulas for it. That is, we shall prove the following:

Theorem 1. Let p be a fixed prime, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} d(p(n)) = x\left(\ln x - 2\ln\ln x\right) + O\left(x \ln p\right).$$

Theorem 2. Let p be a fixed prime, then for any real numbers $x \geq 1$, we also have the asymptotic formula

$$\sum_{n \leq x} d(p_*(n)) = x\left(\ln x - 2\ln\ln x\right) + O\left(x \ln p\right).$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. Firstly, we need two simple Lemmas which statement as following:

Lemma 1. For all real number $x \geq 1$, we have

$$\sum_{n \leq x} d(n) = x \ln x + (2c-1)x + O\left(\sqrt{x}\right),$$

where c is the Euler's constant.

Lemma 2. For any real number $m \geq 2$, we have

$$\sum_{i=1}^{x} \frac{\ln i}{i} = \frac{1}{2} \ln^2 x + A + O\left(\frac{\ln x}{x}\right),$$

where A is a constant.

The proof of Lemma 1 and Lemma 2 can be found in references [3].

Now we use these two Lemmas to prove our theorems directly. From the definitions of $d(n)$ and $p(n)$ we know that

$$\sum_{n \leq x} d(p(n)) \quad = \sum_{n \leq x} \sum_{\frac{\ln(m-1)!}{\ln p} < n \leq \frac{\ln(m)!}{\ln p}} d(m).$$

Since $p(n) = m$ whenever $n \in (\frac{\ln(m-1)!}{\ln p}, \frac{\ln(m)!}{\ln p}]$, and $n \leq x$, the biggest number in the interval $(\frac{\ln(m-1)!}{\ln p}, \frac{\ln(m)!}{\ln p}]$ is less than or equal to x. That is $\frac{\ln(m)!}{\ln p} \leq x$. So we can get $\ln(m)! \leq x \ln p$. From the Euler's summation formula, we obtain the main term of $\ln m!$ is $m \ln m$ and $m \ln m \leq x \ln p$.

If $m > \frac{x \ln p}{\ln^2 x}$, then $\ln m$ is asymptotic to $\ln x$, we get $m \leq \frac{x \ln p}{\ln x}$.

From the discussion above, we have

$$\begin{aligned}
\sum_{n \leq x} d(p(n)) &= \sum_{n \leq x} \sum_{\frac{\ln(m-1)!}{\ln p} < n \leq \frac{\ln(m)!}{\ln p}} d(m) \\
&= \sum_{m \leq \frac{x \ln p}{\ln x}} \left[\frac{\ln m}{\ln p}\right] d(m) + O(x \ln p) \\
&= \sum_{m \leq \frac{x \ln p}{\ln x}} \frac{\ln m}{\ln p} d(m) + O(x \ln p) \\
&= \frac{1}{\ln p} \sum_{un \leq \frac{x \ln p}{\ln x}} \ln un + O(x \ln p) \\
&= \frac{2}{\ln p} \sum_{u \leq \frac{x \ln p}{\ln x}} \ln u \sum_{n \leq \frac{x \ln p}{u \ln x}} 1 + O(x \ln p) \\
&= \frac{2}{\ln p} \sum_{u \leq \frac{x \ln p}{\ln x}} \ln u \left[\frac{x \ln p}{u \ln x}\right] + O(x \ln p) \\
&= \frac{2x}{\ln x} \sum_{u \leq \frac{x \ln p}{\ln x}} \frac{\ln u}{u} + O(x \ln p) \\
&= \frac{2x}{\ln x} \left(\frac{1}{2} (\ln x + \ln \ln p - \ln \ln x)^2\right) + O(x \ln p) \\
&= x(\ln x - 2 \ln \ln x) + O(x \ln p).
\end{aligned}$$

This completes the proof of Theorem 1.

By using the same method we can also prove Theorem 2.

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Some identities involving the k-th power complements

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Abstract The main purpose of this paper is using the elementary method to study the calculating problem of one kind infinite series involving the k-th power complements, and obtain several interesting identities.
Keywords k-th power complements; Identities; Riemann-zeta function.

§1. Introduction and Results

For any given natural number $k \geq 2$ and any positive integer n, we call $a_k(n)$ as a k-th power complements, if $a_k(n)$ is the smallest positive integer such that $n \cdot a_k(n)$ is a perfect k-th power. That is,

$$a_k(n) = \min\{m : mn = u^k, u \in N\}.$$

Especially, we call $a_2(n), a_3(n), a_4(n)$ as the square complement number, cubic complement number, and the quartic complement number, respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the k-th power complement number sequence. About this problem, there are many people have studied it, see references [4], [5], and [6]. For example, Lou Yuanbing [7] gave an asymptotic formula involving the square complement number $a_2(n)$. Let real number $x \geq 3$, he proved that

$$\sum_{n \leq x} d(a_2(n)) = c_1 x \ln x + c_2 x + O(x^{\frac{1}{2}+\varepsilon}),$$

where $d(n)$ is the divisor function, $\varepsilon > 0$ be any fixed real number, c_1 and c_2 are defined as following:

$$c_1 = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

$$c_2 = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) \left(\sum_p \frac{2(2p+1)\ln p}{(p-1)(p+1)(p+2)} + 2\gamma - 1\right),$$

γ is the Euler's constant, $\prod_p$ denotes the product over all primes.

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In reference [8], Yao Weili obtained an asymptotic formula involving k-th power complement number $a_k(n)$. That is, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} d(na_k(n)) = x(A_0 \ln^k x + A_1 \ln^{k-1} x + \cdots + A_k) + O(x^{\frac{1}{2}+\varepsilon}),$$

where $A_0, A_1, \cdots, A_k$ are computable constant, ε is any fixed positive number.

In reference [4], Zhang Wenpeng obtained some identities involving the k-th power complements. Those are, for any complex numbers s with $Re(s) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{1}{(na_2(n))^s} = \frac{\zeta^2(2s)}{\zeta(4s)},$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_3(n))^s} = \frac{\zeta^2(3s)}{\zeta(6s)} \prod_p \left(1 + \frac{1}{p^{3s}+1}\right),$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_4(n))^s} = \frac{\zeta^2(4s)}{\zeta(8s)} \prod_p \left(1 + \frac{1}{p^{4s}+1}\right)\left(1 + \frac{1}{p^{4s}+2}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, F.Russo [9] proposed 21 unsolved problems, the problem 19 asked us evaluate the infinite series

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{a_2(n)}.$$

But is problem very easy. In fact, $a_2(4n^2) = 1$ for all positive integer n. So we have

$$\lim_{n\longrightarrow\infty} (-1)^{4n^2} \frac{1}{a_2(4n^2)} = 1 \neq 0.$$

That is, the infinite series $\sum_{n=1}^{+\infty} (-1)^n \frac{1}{a_2(n)}$ dos not convergent.

In this paper, we shall use the elementary method to study the calculating problem of the infinite series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{na_k(n)},$$

and obtain several interesting identities for it. That is, we shall prove the following:

Theorem. For any given positive integer $k \geq 2$, we have the identity

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{na_k(n)} = \frac{2^k - k - 1}{2^k + k - 1} \zeta(k) \prod_p \left(1 + \frac{k-1}{p^k}\right),$$

where $\zeta(s)$ is the Riemann-zeta function, and $\prod_p$ denotes the product over all different primes.

Taking $k = 2$ in our theorem, and note that the fact $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and

$$\prod_p \left(1 + \frac{1}{p^2}\right) = \prod_p \left(1 - \frac{1}{p^4}\right) \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\zeta(2)}{\zeta(4)},$$

we may immediately obtain the following:

Corollary 1. For the square complement number $a_2(n)$, we have the identity

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_2(n)}=\frac{1}{2}.$$

Corollary 2. For the cubic complement number $a_3(n)$, we have the identity

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_3(n)}=\frac{2}{5}\zeta(3)\prod_p\left(1+\frac{2}{p^3}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. For all positive integers n, we separate n into three parts: $2\nmid n$; $2\mid n$ and $2^k\nmid n$; $2^k\mid n$. Then from the definition of $a_k(n)$ we have:

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_k(n)} &= \sum_{\substack{n=1\\2\nmid n}}^{+\infty}\frac{1}{na_k(n)}-\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{(2^{\alpha k+\beta}n)a_k(2^{\alpha k+\beta}n)}\\ &\quad-\sum_{\alpha=1}^{\infty}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{(2^{\alpha k}n)a_k(2^{\alpha k}n)}\\ &= \sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)}-\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\frac{1}{2^{(\alpha+1)k}}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)}-\sum_{\alpha=1}^{\infty}\frac{1}{2^{\alpha k}}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)}\\ &= \frac{2^k-k-1}{2^k-1}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)}.\end{aligned}$$

It is clear that the infinite series $\displaystyle\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)}$ is absolutely convergent, so from the Euler product formula (see Theorem 11.6 of [2]) we know that the infinite series can be expressed as an absolutely convergent infinite product. That is,

$$\begin{aligned}\sum_{\substack{n=1\\2\nmid n}}^{\infty}\frac{1}{na_k(n)} &= \prod_{\substack{p\\p\neq 2}}\left\{1+\frac{1}{pa_k(p)}+\frac{1}{p^2a_k(p^2)}+\ldots\right\}\\ &= \prod_{\substack{p\\p\neq 2}}\sum_{l=0}^{\infty}\frac{1}{p^la_k(p^l)}\\ &= \prod_{\substack{p\\p\neq 2}}\left(\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\frac{1}{p^{\alpha k+\beta}a_k(p^{\alpha k+\beta})}+\sum_{\alpha=0}^{\infty}\frac{1}{p^{\alpha k}a_k(p^{\alpha k})}\right)\end{aligned}$$

$$
\begin{aligned}
&= \prod_{\substack{p \\ p\neq 2}} \left(\sum_{\alpha=0}^{\infty} \sum_{\beta=1}^{k-1} \frac{1}{p^{(\alpha+1)k}} + \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha k}} \right) \\
&= \prod_{\substack{p \\ p\neq 2}} \frac{p^k + k - 1}{p^k - 1} \\
&= \prod_{p} \frac{1 + \frac{k-1}{p^k}}{1 - \frac{1}{p^k}} \cdot \frac{1 - \frac{1}{2^k}}{1 + \frac{k-1}{2^k}} \\
&= \frac{2^k - 1}{2^k + k - 1} \zeta(k) \prod_{p} \left(1 + \frac{k-1}{p^k} \right).
\end{aligned}
$$

So we have

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n a_k(n)} = \frac{2^k - k - 1}{2^k + k - 1} \zeta(k) \prod_{p} \left(1 + \frac{k-1}{p^k} \right).
$$

This completes the proof of the theorem.

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An equation involving the function $S_p(n)$

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Abstract For any positive integer n, let $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n. The main purpose of this paper is using the elementary methods to study the solutions of the equation $\sum_{d|n} S_p(d) = 2pn$, where p be a prime, and give some part results about the solutions of this equation.
Keywords Equation, solutions, Mersenne prime, perfect number.

§1. Introduction and Results

Let p be a prime, n be any positive integer. Then we define the primitive numbers of power p (p be a prime) $S_p(n)$ as the smallest positive integer m such that $m!$ is divisible by p^n. For example, $S_3(1) = 3,\ S_3(2) = 6,\ S_3(3) = S_3(4) = 9,\ \cdots\cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime p and any positive integer n, they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [4] had studied the mean value distribution property of $|S_p(n+1) - S_p(n)|$, and obtained the following asymptotic formula: For any real number $x \geq 2$, let p be a prime and n be any positive integer, then

$$\frac{1}{p}\sum_{n\leq x}|S_p(n+1) - S_p(n)| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime p and complex number s with Re $s > 1$, we have the identity:

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And, let p be any fixed prime, then for any real number $x \geq 1$,

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O(x^{-\frac{1}{2}+\varepsilon}),$$

where γ is the Euler constant, ε denotes any fixed positive number.

In this paper, we shall use the elementary methods to study the solutions of an equation involving the function $S_p(n)$:

$$\sum_{d|n} S_p(d) = 2pn,$$

and obtain some interesting results. For convenience, we first introduce two kinds of special numbers: Perfect number and Mersenne number. A positive integer n is called as a perfect number if it is equal to the sum of all its proper divisors; A Mersenne number is a number of the form $M_n = 2^n - 1$, where n is an integer. If $2^n - 1$ is a prime, it is said to be a Mersenne prime.

Now we can state our result as follows:

Theorem. Let p be a fixed prime. Then for any positive integer n with $n \leq p$, the equation

$$\sum_{d|n} S_p(d) = 2pn$$

holds if and only if n be a perfect number. If n be an even perfect number, then $n = 2^{r-1}(2^r - 1)$, $r \geq 2$, where $2^r - 1$ is a Mersenne prime.

§2. Proof of Theorem

In this section, we shall complete the proof of Theorem. In fact, if $n \leq p$, we have

$$\sum_{d|n} S_p(d) = \sum_{d|n} pd = p\sigma(n) = 2pn.$$

That is,

$$\sigma(n) = 2n.$$

According to the definition of the perfect number, n is a perfect number.

Since $2^r - 1$ is a prime, then $(2^{r-1},\ 2^r - 1) = 1$. It is clear that $\sigma(n)$ is a multiplicative function and $\sigma(p) = p + 1 = 2^r$, so we have

$$\sigma(n) = \sigma(2^{r-1})\sigma(2^r - 1) = (2^r - 1)\left((2^r - 1) + 1\right) = 2^r(2^r - 1) = 2n.$$

This shows that n is a perfect number.

Conversely, suppose n is any even perfect number and write n as $n = 2^{r-1}k$, where k is an odd integer and $k \geq 2$. Again $\sigma(n)$ is multiplicative, then we may get

$$2^r k = 2n = \sigma(n) = \sigma(2^{r-1}k) = \sigma(2^{r-1})\sigma(k) = (2^r - 1)\sigma(k).$$

Because $(2^r,\ 2^r-1)=1$, in order to $2^rk=(2^r-1)\sigma(k)$ holds, k must include the divisor 2^r-1, so assume $k=(2^r-1)q$, then we have

$$\sigma(k)=2^rq=k+q.$$

This means that k has only two divisors k and q. That is, $q=1$, so $k=2^r-1$, and k is a prime.

In fact, if for some positive r, 2^r-1 is prime, then so is r. We can prove this conclusion very easily. Let a and b be two positive integers, then

$$x^{ab}-1=(x^a-1)\left(x^{a(b-1)}+x^{a(b-2)}+\cdots+x^b+1\right).$$

So if n is composite (say ab with $1<b<n$), then 2^n-1 is also composite.

This completes the proof of Theorem.

Note:

(1) It is not known whether there exists an odd perfect number, but if there is one, it must be very large. This is probably the oldest unsolved problem in all mathematics.

(2) Are there infinitely many even perfect numbers? The answer is probably yes.

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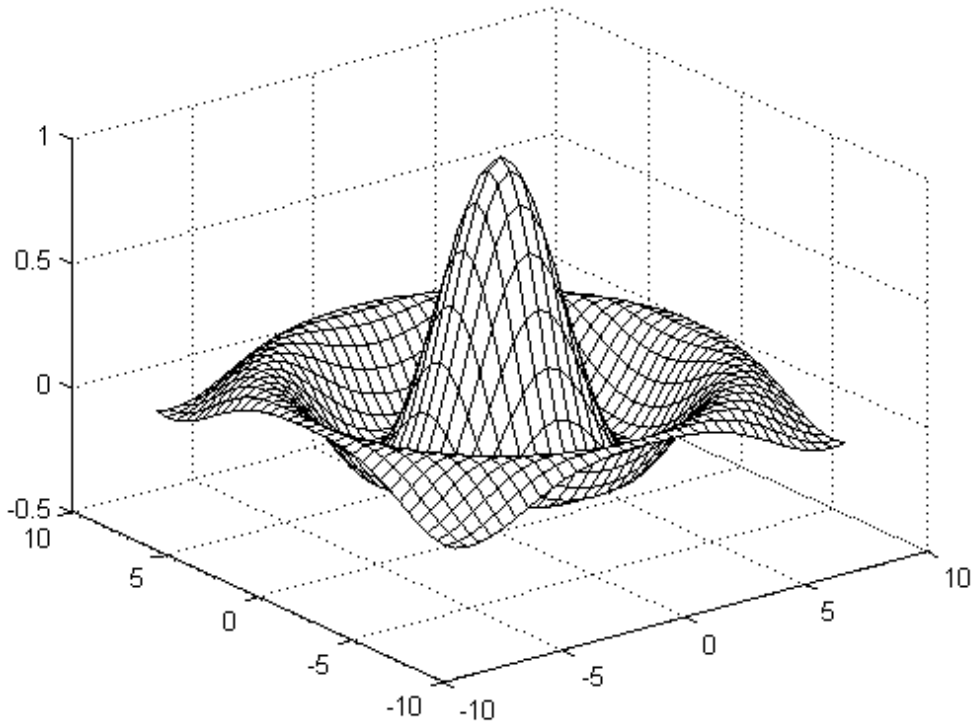

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