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The mean value of $\left(t^{(e)}(n)\right)^2$ over cube-full numbers

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Abstract Let $n > 1$ be an integer, the function $t^{(e)}(n)$ denote the number of e-squarefree e-divisors of n. In this paper, we will study the mean value of $(t^{(e)}(n))^2$ over cube-full numbers, that is

$$\sum_{\substack{n \leq x \\ n \text{ is cube}-\text{full}}} (t^{(e)}(n))^2 = \sum_{n \leq x} (t^{(e)}(n))^2 f_3(n).$$

Keywords exponentially squarefree, exponential divisors, Dirichlet convolution.
2010 Mathematics Subject Classification 11N37.

§1. Introduction

An integer $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is called k-full number if all the exponents $a_1 \geq k,\ a_2 \geq k,\ \cdots,\ a_r \geq k$, when $k = 3$. n is called cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full }, \\ 0, & \text{otherwise }. \end{cases}$$

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. The integer $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ is called an exponential divisor (e-divisor) of n, if $b_i | a_i$ for every $i \in 1,\ 2,\ \cdots,\ r$. The integer $n > 1$ is called exponentially squarefree (e-squarefree) if all the exponents $a_1,\ a_2,\ \cdots,\ a_r$ are squarefree. The integer 1 is also considered to be e-squarefree.

Many scholars are interested in researching the divisor problem and have obtained a large number of good results. But there are many problems hasn't been solved. For example, F. Smarandache gave some unsolved problems in his book *Only problems, Not solutions*! [4], and one problem is that the integer $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ is called an e-squarefree e-divisor of $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$, if $b_1 | a_1,\ \cdots,\ b_r | a_r$ and $b_1,\ \cdots,\ b_r$ are squarefree. Note that the integer 1 is e-squarefree and it is not an e-divisor of $n > 1$. There is the exponential analogues of the functions representing the number of squarefree divisors of n (i.e. $\theta(n) = 2^{\omega(n)}$, where $\omega(n) = r$). Let

$$t^{(e)}(n) = 2^{\omega(a_1)} \cdots 2^{\omega(a_r)},$$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$. The function $t^{(e)}(n)$ is multiplicative and $t^{(e)}(p^a) = 2^{\omega(a)}$ for every prime power p^a. Here for every prime p, $t^{(e)}(p) = 1$, $t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = t^{(e)}(p^7) = 2$, $t^{(e)}(p^6) = 4$, $\cdots$.

L. Tóth [1] proved the following results:
(1) The Dirichlet series of $t^{(e)}(n)$ is of form

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s), \ \Re s > 1,$$

where $V(s) = \sum\limits_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$.
(2)

$$\sum_{n \le x} t^{(e)}(n) = C_1 x + C_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\epsilon})$$

for every $\varepsilon > 0$, where $C_1, \ C_2$ are constants given by

$$C_1 := \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 := \zeta\left(\frac{1}{2}\right) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{\frac{a}{2}}}\right).$$

(3)

$$\limsup_{n \to \infty} \frac{\log t^{(e)}(n) \log\log n}{\log n} = \frac{1}{2} \log 2.$$

The aim of this paper is to establish the following asymptotic formula for the mean value of the function $t^{(e)}(n)$ over cube-full numbers.

Theorem 1.1 *We have the asymptotic formula*

$$\sum_{\substack{n \ \le \ x \\ n \text{ is cube-full}}} (t^{(e)}(n))^2 = x^{\frac{1}{3}} Q_{1,1}(\log x) + x^{\frac{1}{4}} Q_{1,2}(\log x) + x^{\frac{1}{5}} Q_{1,3}(\log x) + O(x^{\sigma_0+\epsilon}) \ ,$$

where $Q_{1,k}(t)$, $k = 1, \ 2, \ 3$ are polynomials of degree 3 in t, $\sigma_0 = 0.1911 \ \cdots$.

Natation Through out this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some Lemmas

Lemma 2.1 *Let $f(m)$, $g(n)$ are arithmetical functions such that*

$$\sum_{m \le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}) \ ,$$

$$\sum_{n\le x}|g(n)| = O(x^{\beta})\ ,$$

where $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge \alpha_J > \alpha > \beta > 0$. $P_j(t)$ *are polynomials in* t. *If* $h(n) = \sum\limits_{n=md} f(m)g(d)$, *then*

$$\sum_{n\le x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha})\ ,$$

where $Q_j(t)$ *are polynomials in* t, $(j = 1, \cdots, J)$.

Lemma 2.2 *The Dirichlet series of* $(t^{(e)}(n))^2$ *is of form*

$$\sum_{\substack{n\ \le\ x\\ n \text{ is cube}-\text{full}}} \frac{(t^{(e)}(n))^2}{n^s} = \zeta^4(3s)\zeta^4(4s)\zeta^4(5s)G(s),\ \Re s > 1,$$

where $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s > \frac{1}{6}$.

Proof.

$$\begin{aligned}
&\sum_{\substack{n\ =\ 1\\ n \text{ is cube}-\text{full}}}^{\infty} \frac{(t^{(e)}(n))^2}{n^s} = \sum_{n=1}^{\infty} \frac{(t^{(e)}(n))^2 f_3(n)}{n^s}\\
&= \prod_p \left(1 + \frac{(t^{(e)}(p))^2 f_3(p)}{p^s} + \frac{(t^{(e)}(p^2))^2 f_3(p^2)}{p^{2s}} + \frac{(t^{(e)}(p^3))^2 f_3(p^3)}{p^{3s}} + \frac{(t^{(e)}(p^4))^2 f_3(p^4)}{p^{4s}} + \cdots\right)\\
&= \prod_p \left(1 + \frac{(t^{(e)}(p^3))^2}{p^{3s}} + \frac{(t^{(e)}(p^4))^2}{p^{4s}} + \frac{(t^{(e)}(p^5))^2}{p^{5s}} + \cdots\right)\\
&= \prod_p \left(1 + \frac{(2^{\omega(3)})^2}{p^{3s}} + \frac{(2^{\omega(4)})^2}{p^{4s}} + \frac{(2^{\omega(5)})^2}{p^{5s}} + \cdots\right)\\
&= \prod_p \left(1 + \frac{4}{p^{3s}} + \frac{4}{p^{4s}} + \frac{4}{p^{5s}} + \cdots\right)\\
&= \zeta(3s) \prod_p \left(1 + \frac{3}{p^{3s}} + \frac{4}{p^{4s}} + \frac{4}{p^{5s}} + \cdots\right)\\
&= \zeta^4(3s) \prod_p \left(1 + \frac{4}{p^{4s}} + \frac{4}{p^{5s}} + \cdots\right)\\
&= \zeta^4(3s)\zeta^4(4s)\zeta^4(5s)G(s),
\end{aligned}$$

where $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{6}$, and

$$\sum_{n\le x}|g(n)| \ll x^{\frac{1}{6}+\epsilon}\ .$$

□

Lemma 2.3 *Let* $\frac{1}{2} \le \sigma \le 1,\ t \ge t_0 \ge 2$, *we have*

$$\zeta(\sigma+it) \ll t^{\frac{1-\sigma}{3}} \log t.$$

Proof. See E. C. Titchmarsh [3]. □

Lemma 2.4 *Let* $\frac{1}{2} < \sigma < 1$, *define*

$$\begin{aligned}
m(\sigma) &= \frac{4}{3-4\sigma}, && \frac{1}{2} < \sigma \le \frac{5}{8},\\
m(\sigma) &= \frac{19}{6-6\sigma}, && \frac{35}{54} < \sigma \le \frac{41}{60}\\
m(\sigma) &= \frac{2112}{859-948\sigma}, && \frac{41}{60} < \sigma \le \frac{3}{4}\\
m(\sigma) &= \frac{12408}{4537-4890\sigma}, && \frac{3}{4} < \sigma \le \frac{5}{6}\\
m(\sigma) &= \frac{4324}{1031-1044\sigma}, && \frac{5}{6} < \sigma \le \frac{7}{8}\\
m(\sigma) &= \frac{98}{31-32\sigma}, && \frac{7}{8} < \sigma \le 0.91591\\
m(\sigma) &= \frac{24\sigma-9}{(4\sigma-1)(1-\sigma)}, && 0.91591 < \sigma \le 1-\epsilon.
\end{aligned}$$

Proof. See A. Ivić [2]. □

Lemma 2.5

$$\sum_{n\le x} d(3,3,3,3,4,4,4,4,5,5,5,5;n) = x^{\frac{1}{3}}P_{1,1}(\log x) + x^{\frac{1}{4}}P_{1,2}(\log x) + x^{\frac{1}{5}}P_{1,3}(\log x) + O(x^{\sigma_0+\epsilon}),$$

where $\sigma_0 = 0.1911\ \cdots,\ P_{1,k}(t),\ k=1,\ 2,\ 3$ *are polynomials of degree* 3 *in* t.

Proof. By perron's formula, we have

$$S(x) = \sum_{n\le x}\delta(n)d(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^4(3s)\zeta^4(4s)\zeta^4(5s)\frac{x^s}{s}ds + O\left(\frac{x^{\frac{1}{3}+\epsilon}}{T}\right),$$

where $b = \frac{1}{3}+\epsilon,\ T = x^c,\ c$ is a very large number of fixed numbers. $\frac{1}{6} < \sigma_0 < \frac{1}{5}$. According to the residue theorem, we have

$$S(x) = x^{\frac{1}{3}}P_{1,1}(\log x) + x^{\frac{1}{4}}P_{1,2}(\log x) + x^{\frac{1}{5}}P_{1,3}(\log x) + I_1 + I_2 + I_3 + O(1)\ ,$$

$$I_1 = \frac{1}{2\pi i}\int_{b-it}^{\sigma_0-it} \zeta^4(3s)\zeta^4(4s)\zeta^4(5s)\frac{x^s}{s}ds,$$

$$I_2 = \frac{1}{2\pi i}\int_{\sigma_0-it}^{\sigma_0+it} \zeta^4(3s)\zeta^4(4s)\zeta^4(5s)\frac{x^s}{s}ds,$$

$$I_3 = \frac{1}{2\pi i}\int_{\sigma_0+it}^{b+it}\zeta^4(3s)\zeta^4(4s)\zeta^4(5s)\frac{x^s}{s}ds.$$

Since $\sigma_0 > \frac{3}{16}+\delta,\ (s=\sigma+iT)$, and by Lemma 2.3, we have,

$$\begin{aligned}
I_1+I_3 \ll& \int_{\sigma_0}^{\frac{1}{3}+\epsilon}|\zeta(3\sigma+3iT)|^4|\zeta(4\sigma+4iT)|^4|\zeta(5\sigma+5iT)|^4x^\sigma T^{-1}d\sigma\\
\ll& T^{-1}\left(\int_{\sigma_0}^{\frac{1}{5}}+\int_{\frac{1}{5}}^{\frac{1}{4}}+\int_{\frac{1}{4}}^{\frac{1}{3}}+\int_{\frac{1}{3}}^{\frac{1}{3}+\epsilon}\right)\\
&\times|\zeta(3\sigma+3iT)|^4|\zeta(4\sigma+4iT)|^4|\zeta(5\sigma+5iT)|^4x^\sigma d\sigma\\
\ll& T^{-1+\epsilon}\int_{\sigma_0}^{\frac{1}{5}}T^{\frac{4(1-3\sigma)}{3}+\frac{4(1-4\sigma)}{3}+\frac{4(1-5\sigma)}{3}}x^\sigma d\sigma\\
&+T^{-1+\epsilon}\int_{\frac{1}{5}}^{\frac{1}{4}}T^{\frac{4(1-3\sigma)}{3}+\frac{4(1-4\sigma)}{3}}x^\sigma d\sigma+T^{-1+\epsilon}\int_{\frac{1}{4}}^{\frac{1}{3}}T^{\frac{4(1-3\sigma)}{3}}x^\sigma d\sigma\\
&+T^{-1+\epsilon}\int_{\frac{1}{3}}^{\frac{1}{3}+\epsilon}x^\sigma d\sigma\\
\ll& x^{\frac{3}{16}+\epsilon}T^{-\delta+\epsilon}+x^{\frac{1}{5}}T^{-\frac{1}{5}+\epsilon}+x^{\frac{1}{4}}T^{-\frac{2}{3}+\epsilon}+x^{\frac{1}{3}}T^{-1+\epsilon}+x^{\frac{1}{3}+\epsilon}T^{-1+\epsilon}\\
\ll& x^{\frac{1}{3}+\epsilon}T^{-\delta+\epsilon}\ ,
\end{aligned}$$

where δ is very small normal number, $\delta > \epsilon$.

$$I_2 \ll x^{\sigma_0}\left(1+\int_1^T|\zeta(3\sigma+3it)|^4|\zeta(4\sigma+4it)|^4|\zeta(5\sigma+5it)|^4t^{-1}dt\right).$$

According to the partial integral formula, we have

$$I_4 = \int_1^T|\zeta(3\sigma+3it)|^4|\zeta(4\sigma+4it)|^4|\zeta(5\sigma+5it)|^4dt \ll T^{1+\epsilon}.$$

If $p_i > 0,\ (i=1,2,3)$ are real number, and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, by Hölder's inequality, we have

$$I_4 \ll \left(\int_1^T|\zeta(3\sigma_0+3it)|^{4p_1}dt\right)^{\frac{1}{p_1}}\left(\int_1^T|\zeta(4\sigma_0+4it)|^{4p_2}dt\right)^{\frac{1}{p_2}}\left(\int_1^T|\zeta(5\sigma_0+5it)|^{4p_3}dt\right)^{\frac{1}{p_3}}.$$

So, we have to prove

$$\int_0^T|\zeta(3\sigma_0+3it)|^{4p_1}dt \ll T^{1+\epsilon}\ ,$$

$$\int_0^T|\zeta(4\sigma_0+4it)|^{4p_2}dt \ll T^{1+\epsilon}\ ,$$

$$\int_0^T|\zeta(5\sigma_0+5it)|^{4p_3}dt \ll T^{1+\epsilon}\ ,$$

Let $m(3\sigma_0)=4p_1,\ m(4\sigma_0)=4p_2,\ m(5\sigma_0)=4p_3$, since $\frac{4}{m(3\sigma_0)}+\frac{4}{m(4\sigma_0)}+\frac{4}{m(5\sigma_0)}=1$, and from Lemma 2.4, we have $\sigma_0 = 0.1911\cdots$. □

§3. Proof of Theorem 1.1

Let

$$\zeta^4(3s)\zeta^4(4s)\zeta^4(5s)G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad \Re s > 1,$$

$$\zeta^4(3s)\zeta^4(4s)\zeta^4(5s) = \sum_{n=1}^{\infty} \frac{d(3,3,3,3,4,4,4,4,5,5,5,5;n)}{n^s}$$

such that

$$f(n) = \sum_{n=md} d(3,3,3,3,4,4,4,4,5,5,5,5;m)g(d). \tag{3.1}$$

From Lemma 2.5 and the definition of $d(3,3,3,3,4,4,4,4,5,5,5,5;m)$, we get

$$\sum_{m\le x} d(3,3,3,3,4,4,4,4,5,5,5,5;m) = x^{\frac{1}{3}}P_{1,1}(\log x)+x^{\frac{1}{4}}P_{1,2}(\log x)+x^{\frac{1}{5}}P_{1,3}(\log x)+O(x^{\sigma_0+\epsilon}) , \tag{3.2}$$

where $P_{1.k}(t)$, $(k = 1, 2, 3)$ are polynomials of degree 3 in t. In addition, we have

$$\sum_{n\le x} |g(n)| = O(x^{\frac{1}{6}+\epsilon}) . \tag{3.3}$$

Combining (3.1), (3.2), (3.3) and Lemma 2.1, we have

$$\sum_{n\le x} f(n) = x^{\frac{1}{3}}Q_{1,1}(\log x) + x^{\frac{1}{4}}Q_{1,2}(\log x) + x^{\frac{1}{5}}Q_{1,3}(\log x) + O(x^{\sigma_0+\epsilon}) , \tag{3.4}$$

where $Q_{1.k}(t)$, $(k = 1, 2, 3)$ are polynomials of degree 3 in t.

By Lemma 2.2, we have

$$(t^{(e)}(n))^2 f_3(n) = \sum_{n=md} d(3,3,3,3,4,4,4,4,5,5,5,5;m)g(d) = f(n).$$

Then we complete the proof of Theorem 1.1.

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References

[1] L. Tóth, On certain arithmetic functions involving exponential divisors, II. Annales Univ. Sci. Budapest. Sect. Comp. 27(2007),155-166.

[2] A. Ivić, The Riemann zeta-function:theory and applications. John Wiley and Sons, New York, 1985, 387-393.

[3] E. C. Titchmarsh, The theory of the Riemann zeta-function. Oxford: Clarendon Press, 1951.

[4] F. Smarandache, Only problems, Not solutions! Chicago: Xiquan Publishing House, 1993.

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Bi-Periodic Pell-Lucas Matrix Sequence

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Abstract In this study, we introduce a generalization of Pell-Lucas matrix sequence called bi-periodic Pell-Lucas matrix sequence by bi-periodic Pell-Lucas numbers. We find generating function and Binet formula for this matrix sequence. We investigate the relationships between bi-periodic Pell matrix sequence. We also find various sum formulas and some properties for this matrix sequence.

Keywords Pell-Lucas Sequence, Generating Function, Binet Formula, Matrix Sequences.
2010 Mathematics Subject Classification 11B39, 11B83, 15A24, 15B36.

§1. Introduction

In the literature, you can encounter various special integer sequences, which are used in almost every kind of sciences. Special integer sequences are often used by combinatorics and number theory which are popular fields of mathematics One of the well-known special integer sequences is the Pell-Lucas sequence. The Pell-Lucas sequence denoted by $\{q_n\}_{n=0}^{\infty}$ is defined recursively by $q_n = 2q_{n-1} + q_{n-2}$, with starting values $q_0 = 2, q_1 = 2$ for $n \geq 2$ in $[1, 2]$. There are many generalizations of the special sequences. One of them is bi periodic number sequences. Edson and Yayenie defined firstly the bi periodic Fibonacci sequence in $[3, 4]$. The contribution of Edson and Yayenie is noteworthy on bi-periodic sequences. After them many mathematicians studied new bi-periodic sequences. Bilgici studied bi periodic Lucas sequence in [5]. Similarly, Uygun and Karatas found some properties of the bi-periodic Pell and Pell-Lucas sequences in $[8, 10]$. Uygun and Owusu defined bi-periodic Jacobsthal and Jacobsthal Lucas sequences in $[6, 9]$. Verma and Bala introduced bivariate bi-periodic Jacobsthal and Jacobsthal Lucas polynomials in [13]. In [7], Coskun and Taskara carried out bi-periodic sequences to matrix theory and defined the bi-periodic Fibonacci and Lucas matrix sequences. Uygun and Owusu, in [11], found the basic properties of matrix representation of the bi-periodic Jacobsthal sequence. Uygun, in [12] defined the matrix representation of the bi-periodic Jacobsthal-Lucas sequence. Soykan studied k-circulant matrices with the generalized third order Pell numbers in [14]. Here, firstly we introduce bi-periodic Pell sequence and bi-periodic Pell-Lucas sequence. Using the elements of the bi-periodic Pell-Lucas sequence, we define bi-periodic Pell-Lucas matrix sequence. We find generating function and Binet formula for this matrix sequence. We investigate the relationships between bi-periodic Pell matrix sequence. We also obtain some properties and sum formulas for this matrix sequence.

Definition 1.1. *For any two non-zero real numbers a and b, the bi-periodic Pell sequence in [10] denoted by $\{P_n\}_{n=0}^{\infty}$ is defined recursively by*

$$P_0 = 0, \quad P_1 = 1, \quad P_n = \begin{cases} 2aP_{n-1} + P_{n-2}, & \text{if } n \text{ is even} \\ 2bP_{n-1} + P_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2,$$

The recurrence equation of the bi-periodic Pell sequence is given as

$$x^2 - 2abx - ab = 0$$

and the roots of this equation are

$$\alpha = ab + \sqrt{a^2b^2 + ab}, \quad \beta = ab - \sqrt{a^2b^2 + ab}$$

Definition 1.2. *For any two non-zero real numbers a and b, the bi-periodic Pell-Lucas sequence in [8] denoted by $\{Q_n\}_{n=0}^{\infty}$ is defined recursively by*

$$Q_0 = 2, \quad Q_1 = 2a, \quad Q_n = \begin{cases} 2bQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is even} \\ 2aQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2,$$

$$Q_0 = 2, \quad Q_1 = 2a, \quad Q_2 = 4ab + 2, \quad Q_3 = 8a^2b + 6a, \quad Q_4 = 16a^2b^2 + 16ab + 2.$$

Lemma 1.1. *We have the following properties for α and β*

$$\begin{aligned} (2\alpha+1)(2\beta+1) &= 1 \\ \alpha + \beta &= 2ab, & \alpha\beta &= -ab \\ (2\alpha+1) &= \frac{\alpha^2}{ab} & (2\beta+1) &= \frac{\beta^2}{ab}, \\ -(2\alpha+1)\beta &= \alpha, & -(2\beta+1)\alpha &= \beta. \end{aligned}$$

Definition 1.3. *Bi-periodic Pell matrix sequence $\{\tilde{P}_n(a,b)\}_{n=0}^{\infty}$ is defined by*

$$\tilde{P}_n(a,b) = \begin{cases} 2a\tilde{P}_{n-1}(a,b) + \tilde{P}_{n-2}, & \text{if } n \text{ is even} \\ 2b\tilde{P}_{n-1}(a,b) + \tilde{P}_{n-2}, & \text{if } n \text{ is odd} \end{cases}$$

with starting values

$$\tilde{P}_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{P}_1(a,b) = \begin{pmatrix} 2b & \frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

The bi-periodic Pell matrix sequence $\{\tilde{P}_n(a,b)\}_{n=0}^{\infty}$ *satisfies the following properties*

$$\begin{aligned}\tilde{P}_{2n}(a,b) &= (4ab+2)\tilde{P}_{2n-2}(a,b)-\tilde{P}_{2n-4}(a,b),\\ \tilde{P}_{2n+1}(a,b) &= (4ab+2)\tilde{P}_{2n-1}(a,b)-\tilde{P}_{2n-3}(a,b).\end{aligned}$$

The entries of the bi-periodic Pell matrix sequence are the elements of the bi-periodic Pell number sequence as

$$\tilde{P}_n = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(n)} P_{n+1} & \frac{b}{a}P_n \\ P_n & \left(\frac{b}{a}\right)^{\xi(n)} P_{n-1} \end{pmatrix}.$$

[15]

§2. Bi-periodic Pell-Lucas Matrix Sequence

Definition 2.1. *Bi-periodic Pell-Lucas matrix sequence* $\{\tilde{Q}_n(a,b)\}_{n=0}^{\infty}$ *is defined by*

$$\tilde{Q}_n(a,b) = \begin{cases} 2b\tilde{Q}_{n-1}(a,b)+\tilde{Q}_{n-2}, & \text{if } n \text{ is even} \\ 2a\tilde{Q}_{n-1}(a,b)+\tilde{Q}_{n-2}, & \text{if } n \text{ is odd} \end{cases}$$

with starting values

$$\tilde{Q}_0(a,b) = \begin{pmatrix} 2a & 2 \\ 2\frac{a}{b} & 2a \end{pmatrix}, \quad \tilde{Q}_1(a,b) = \begin{pmatrix} \frac{a(4ab+2)}{b} & 2b \\ \frac{2a^2}{b} & 2\frac{a}{b} \end{pmatrix}.$$

In this study, we use $\tilde{P}_n$ *for* $\tilde{P}_n(a,b)$ *and similarly* $\tilde{Q}_n$ *for* $\tilde{Q}_n(a,b)$. *The bi-periodic Pell-Lucas matrix sequence* $\{\tilde{Q}_n\}_{n=0}^{\infty}$ *satisfies the following properties*

$$\begin{aligned}\tilde{Q}_{2n} &= (4ab+2)\tilde{Q}_{2n-2}-\tilde{Q}_{2n-4}\\ \tilde{Q}_{2n+1} &= (4ab+2)\tilde{Q}_{2n-1}-\tilde{Q}_{2n-3}\end{aligned}$$

Theorem 2.1. *The entries of the bi-periodic Pell-Lucas matrix sequence are the elements of the bi-periodic Pell-Lucas number sequence as*

$$\tilde{Q}_n(a,b) = \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(n)} Q_{n+1} & Q_n \\ \frac{a}{b}Q_n & \left(\frac{a}{b}\right)^{\xi(n)} Q_{n-1} \end{pmatrix}.$$

Proof. We obtain the proof by means of mathematical induction. For $n = 0, 1$ the assertion is satisfied. Let the equation is true for $n = k$, where $k \in Z^+$

$$\tilde{Q}_k(a,b) = \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(k)} Q_{k+1} & Q_k \\ \frac{a}{b}Q_k & \left(\frac{a}{b}\right)^{\xi(k)} Q_{k-1} \end{pmatrix}$$

For $n = k+1$; we have

$$\begin{aligned}
\tilde{Q}_{k+1} &= \begin{cases} 2a\tilde{Q}_k(a,b) + \tilde{Q}_{k-1} & \text{if } k \text{ is even} \\ 2b\tilde{Q}_k(a,b) + \tilde{Q}_{k-1} & \text{if } k \text{ is odd} \end{cases} \\
&= (2b)^{\xi(k)}(2a)^{1-\xi(k)}\tilde{Q}_k + \tilde{Q}_{k-1} \\
&= 2b^{\xi(k)}a^{1-\xi(k)} \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(k)} Q_{k+1} & Q_k \\ \frac{a}{b}Q_k & \left(\frac{a}{b}\right)^{\xi(k)} Q_{k-1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(k-1)} Q_k & Q_{k-1} \\ \frac{a}{b}Q_{k-1} & \left(\frac{a}{b}\right)^{\xi(k-1)} Q_{k-2} \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} 2aQ_{k+1} + \frac{a}{b}Q_k & 2aQ_k + Q_{k-1} \\ \frac{a}{b}(2aQ_k + Q_{k-1}) & 2aQ_{k-1} + \frac{a}{b}Q_{k-2} \end{pmatrix}, & k \text{ is even} \\ \begin{pmatrix} 2aQ_{k+1} + Q_k & 2bQ_k + Q_{k-1} \\ \frac{a}{b}(2bQ_k + Q_{k-1}) & 2aQ_{k-1} + Q_{k-2} \end{pmatrix}, & k \text{ is odd} \end{cases} \\
&= \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(k+1)} Q_{k+2} & Q_{k+1} \\ \frac{a}{b}Q_{k+1} & \left(\frac{a}{b}\right)^{\xi(k+1)} Q_k \end{pmatrix}
\end{aligned}$$

Thus, we get the result. □

Lemma 2.1. *The followig relations between bi-periodic Pell sequence and bi-periodic Pell-Lucas sequence are satisfied*

$$\begin{aligned}
Q_{n+1} + Q_{n-1} &= (ab+1)P_n \\
P_{n+1} + P_{n-1} &= Q_n
\end{aligned}$$

Theorem 2.2. *The followig relations between bi-periodic Pell matrix sequence and bi-periodic Pell-Lucas matrix sequence are satisfied*

$$\begin{aligned}
\frac{b}{a}\left(\tilde{Q}_{n+1} + \tilde{Q}_{n-1}\right) &= (ab+1)\tilde{P}_n \\
\tilde{P}_{n+1} + \tilde{P}_{n-1} &= \frac{b}{a}\tilde{Q}_n
\end{aligned}$$

Theorem 2.3. *The determinant of the the elements of the bi-periodic Pell-Lucas matrix sequence is*

$$\det \tilde{Q}_n = (-1)^{n+1}(4ab+4)\left(\frac{a}{b}\right)^{1+\xi(n)}$$

(Cassini Property) *By the determinant of the bi-periodic Pell-Lucas matrix sequence, we get the Cassini property for the the bi-periodic Pell-Lucas number sequence as*

$$\left(\frac{a}{b}\right)^{2\xi(n)} Q_{n+1}Q_{n-1} - \frac{a}{b}Q_n^2 = (4ab+4)\left(-\frac{a}{b}\right)^{1+\xi(n)}$$

Theorem 2.4. *The following properties are also satisfied by the bi-periodic Pell-Lucas matrix sequence*

i $\left(\frac{b}{a}\right)^{\xi(n)}\tilde{Q}_n=\tilde{P}_n\tilde{Q}_0=\tilde{Q}_0\tilde{P}_n=\left(\frac{a}{b}\right)^{\xi(n+1)}(\tilde{P}_{n+1}+\tilde{P}_{n-1})$

ii $\left(\frac{b}{a}\right)^{\xi(n+1)}\tilde{Q}_{n+1}=\tilde{P}_1\tilde{Q}_n=\tilde{Q}_n\tilde{P}_1=\left(\frac{a}{b}\right)^{\xi(n)}\tilde{P}_{n+2}+\tilde{P}_n)$

iii $\tilde{P}_m\tilde{Q}_n=\left(\frac{b}{a}\right)^{\xi(m)\xi(n+1)}\tilde{Q}_{m+n}$

iv $\tilde{Q}_m\tilde{Q}_n=\left(\frac{a}{b}\right)^{2-\xi((m+1)(n+1))}(ab+1)\tilde{P}_{m+n}$

v $\tilde{Q}_{n-r}\tilde{Q}_{n+r}=\left(\frac{a}{b}\right)^{(-1)^n\xi(r)}\tilde{Q}_n^2$

vi $\tilde{Q}_{n-r}\tilde{Q}_{n+r}=\left(\frac{a}{b}\right)^{2-\xi(r+1)}(ab+1)\tilde{P}_{2n}$

vii $\tilde{Q}_0^m\tilde{P}_{mn}=\left(\frac{b}{a}\right)^{\left\lfloor\frac{m+1}{2}\right\rfloor\xi(n)}\tilde{Q}_n^m$

viii $\tilde{P}_1^n=\frac{1}{4(ab+1)}\left(\frac{b}{a}\right)^{1+\xi(n)+\left\lfloor\frac{n}{2}\right\rfloor}\tilde{Q}_n\tilde{Q}_0$

ix $\tilde{P}_1^n=\left(\frac{b}{a}\right)^{\left\lfloor\frac{n+1}{2}\right\rfloor}\tilde{Q}_n$

Proof. We demonsrate only two proofs of them. The other proofs are made by similar way. For the proof of *iii*)

$$\begin{aligned}\tilde{P}_m\tilde{Q}_n &= \tilde{P}_m\left(\frac{a}{b}\right)^{\xi(n)}\tilde{P}_n\tilde{Q}_0=\left(\frac{a}{b}\right)^{\xi(n)}\left(\frac{b}{a}\right)^{\xi(mn)}\tilde{P}_{m+n}\tilde{Q}_0\\ &= \left(\frac{a}{b}\right)^{\xi(n)-\xi(mn)}\left(\frac{b}{a}\right)^{\xi(m+n)}\tilde{Q}_{m+n}=\left(\frac{b}{a}\right)^{\xi(m)\xi(n+1)}\tilde{Q}_{m+n}\end{aligned}$$

For the proof of *iv*) we use this property $\tilde{P}_m\tilde{P}_n=\left(\frac{b}{a}\right)^{\xi(mn)}\tilde{P}_{m+n}$

$$\begin{aligned}\tilde{Q}_m\tilde{Q}_n &= \frac{a}{b}(\tilde{P}_{m+1}+\tilde{P}_{m-1})\frac{a}{b}(\tilde{P}_{n+1}+\tilde{P}_{n-1})\\ &= \left(\frac{a}{b}\right)^2\left[\begin{array}{l}\left(\frac{b}{a}\right)^{\xi((m+1)(n+1))}\tilde{P}_{m+n+2}+\left(\frac{b}{a}\right)^{\xi((m+1)(n-1)}\tilde{P}_{m+n}\\ +\left(\frac{b}{a}\right)^{\xi((m-1)(n+1))}\tilde{P}_{m+n}+\left(\frac{b}{a}\right)^{\xi((m-1)(n-1))}\tilde{P}_{m+n-2}\end{array}\right]\\ &= \left(\frac{a}{b}\right)^{2-\xi((m+1)(n+1))}[\tilde{P}_{m+n+2}+2\tilde{P}_{m+n}+\tilde{P}_{m+n-2}]\\ &= \left(\frac{a}{b}\right)^{1-\xi((m+1)(n+1))}\left[\tilde{Q}_{m+n+1}+\tilde{Q}_{m+n-1}\right]\\ &= \left(\frac{a}{b}\right)^{2-\xi((m+1)(n+1))}(ab+1)\tilde{P}_{m+n}\end{aligned}$$

If it is done the arrangements, we get the desired result. □

Theorem 2.5. (Generating function) *Let us suppose that $\tilde{Q}_i$ are coefficients of a power series with center at the origin and that $\tilde{Q}(x)$ is the sum of this series. The generating function $\tilde{Q}(x)$ for the bi-periodic Pell-Lucas matrix sequence is in the following*

$$\tilde{Q}(x) = \sum_{i=0}^{\infty} \tilde{Q}_i x^i = \frac{\tilde{Q}_0 + \tilde{Q}_1 x + x^2 \left(2b\tilde{Q}_1 - (4ab+1)\tilde{Q}_0\right) + x^3 \left(2a\tilde{Q}_0 - \tilde{Q}_1\right)}{1 - (4ab+2)x^2 + x^4}$$

Proof. The generating function $\tilde{Q}(x)$ for the bi-periodic Pell-Lucas matrix sequence is displayed as

$$\begin{aligned} \tilde{Q}(x) &= \sum_{i=0}^{\infty} \tilde{Q}_i x^i = \sum_{i=0}^{\infty} \tilde{Q}_{2i} x^{2i} + \sum_{i=0}^{\infty} \tilde{Q}_{2i+1} x^{2i+1} \\ &= \tilde{Q}_0(x) + \tilde{Q}_1(x) \end{aligned}$$

We have for the even terms of the series

$$\tilde{Q}_0(x) = \tilde{Q}_0 + \tilde{Q}_2 x^2 + \sum_{i=2}^{\infty} \tilde{Q}_{2i} x^{2i}.$$

We multiply $\tilde{Q}_0(x)$ by $-(4ab+2)x^2$ and x^4 respectively

$$-(4ab+2)x^2 \tilde{Q}_0(x) = -(4ab+2)\tilde{Q}_0 x^2 - (4ab+2) \sum_{i=2}^{\infty} \tilde{Q}_{2i-2} x^{2i}.$$

$$x^4 \tilde{Q}_0(x) = \sum_{i=0}^{\infty} \tilde{Q}_{2i} x^{2i+4} = \sum_{i=2}^{\infty} \tilde{Q}_{2i-4} x^{2i}.$$

By these equalities, we obtain the following function of the matrix for even powers of the series

$$\begin{aligned} [1 - (4ab+2)x^2 + x^4]\tilde{Q}_0(x) &= \tilde{Q}_0 + \tilde{Q}_2 x^2 - (4ab+2)x^2 \tilde{Q}_0 \\ &\quad + \sum_{i=2}^{\infty} (\tilde{Q}_{2i} - (4ab+2)\tilde{Q}_{2i-2} + \tilde{Q}_{2i-4}) x^{2i} \end{aligned}$$

$$\tilde{Q}_0(x) = \frac{\tilde{Q}_0 + x^2 \left(2b\tilde{Q}_1 - (4ab+1)\tilde{Q}_0\right)}{1 - (4ab+2)x^2 + x^4}.$$

Similarly for odd powers of the series, we get

$$\begin{aligned} \tilde{Q}_1(x) &= \tilde{Q}_1 x + \tilde{Q}_3 x^3 + \sum_{i=2}^{\infty} \tilde{Q}_{2i+1} x^{2i+1} \\ -(4ab+2)x^2 \tilde{Q}_1(x) &= -(4ab+2)x^3 \tilde{Q}_1 - (4ab+2) \sum_{i=2}^{\infty} \tilde{Q}_{2i-1} x^{2i+1} \\ x^4 \tilde{Q}_1(x) &= \sum_{i=0}^{\infty} \tilde{Q}_{2i+1} x^{2i+5} = \sum_{i=2}^{\infty} \tilde{Q}_{2i-3} x^{2i+1} \end{aligned}$$

By these equalities, we obtain

$$\begin{aligned}[1-(4ab+2)x^2+x^4]\tilde{Q}_1(x) &= \tilde{Q}_1x+\tilde{Q}_3x^3-(4ab+2)x^3\tilde{Q}_1 \\ &+\sum_{i=2}^{\infty}\left(\tilde{Q}_{2i+1}-(4ab+2)\tilde{Q}_{2i-1}+\tilde{Q}_{2i-3}\right)x^{2i+1}\end{aligned}$$

$$\tilde{Q}_1(x)=\frac{\tilde{Q}_1x+x^3(2a\tilde{Q}_0-\tilde{Q}_1)}{1-(4ab+2)x^2+x^4}.$$

By combining the results, we complete the proof

$$\begin{aligned}\tilde{Q}(x) &= \tilde{Q}_0(x)+\tilde{Q}_1(x) \\ &= \frac{\tilde{Q}_0+\tilde{Q}_1x+\left[2b\tilde{Q}_1-(4ab+1)\tilde{Q}_0\right]x^2+\left[2a\tilde{Q}_0-\tilde{Q}_1\right]x^3}{1-(4ab+2)x^2+x^4}.\end{aligned}$$

□

Theorem 2.6. (Binet Formula) *The Binet formula for the bi-periodic Pell-Lucas matrix sequence is*

$$\begin{aligned}\tilde{Q}_n &= (\tilde{Q}_0)^{1-\xi(n)}(\tilde{Q}_1)^{\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor+1}-\beta^{2\lfloor\frac{n}{2}\rfloor+1}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)} \\ &+(a\tilde{Q}_0)^{\xi(n)}(b\tilde{Q}_1-2ab\tilde{Q}_0)^{1-\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor}-\beta^{2\lfloor\frac{n}{2}\rfloor}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}\end{aligned}$$

or

$$\begin{aligned}\tilde{Q}_n &= b^{1-\xi(n)}(\tilde{Q}_1-2a\tilde{Q}_0)^{\xi(n)}\frac{\alpha^n-\beta^n}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)} \\ &+a^{\xi(n)}\tilde{Q}_0\frac{\alpha^{n+1}-\beta^{n+1}}{(ab^{\lfloor\frac{n+1}{2}\rfloor}(\alpha-\beta)}.\end{aligned}$$

Proof. By the generating function for the bi-periodic Pell-Lucas matrix sequence, we have

$$\begin{aligned}\tilde{Q}(x) &= \frac{\tilde{Q}_0+\tilde{Q}_1x+x^2\left(2b\tilde{Q}_1-(4ab+1)\tilde{Q}_0\right)+x^3\left(2a\tilde{Q}_0-\tilde{Q}_1\right)}{(x^2-(2\alpha+1))(x^2-(2\beta+1))} \\ &= \frac{Ax+B}{x^2-(2\alpha+1)}+\frac{Cx+D}{x^2-(2\beta+1)}\end{aligned}$$

By the equality we find the coefficients as

$$\begin{aligned}A+C &= 2a\tilde{Q}_0-\tilde{Q}_1 \\ -A(2\beta+1)-C(2\alpha+1) &= \tilde{Q}_1\end{aligned}$$

$$A = \frac{(2\alpha+1)a\tilde{Q}_0-\alpha\tilde{Q}_1}{\alpha-\beta}$$

$$C = \frac{\beta\tilde{Q}_1 - (2\beta+1)a\tilde{Q}_0}{\alpha-\beta}$$

$$\begin{aligned} B + D &= 2b\tilde{Q}_1 - (4ab+1)\tilde{Q}_0 \\ -B(2\beta+1) - D(2\alpha+1) &= \tilde{Q}_0 \end{aligned}$$

$$\begin{aligned} B &= \frac{b(2\alpha+1)(\tilde{Q}_1 - 2a\tilde{Q}_0) - \alpha\tilde{Q}_0}{\alpha-\beta} \\ D &= \frac{b(2\beta+1)(-\tilde{Q}_1 + 2a\tilde{Q}_0) + \beta\tilde{Q}_0}{\alpha-\beta} \end{aligned}$$

We know that by Maclauren series expansion

$$\frac{Ax+B}{x^2-C} = -\sum_{n=1}^{\infty} AC^{-n-1}x^{2n+1} - \sum_{n=1}^{\infty} BC^{-n-1}x^{2n}.$$

If we apply this expansion to the fraction $\frac{Ax+B}{x^2-(2\alpha+1)}$, we obtain

$$\begin{aligned} &\frac{Ax+B}{x^2-(2\alpha+1)} \\ = &\frac{-1}{\alpha-\beta}\left(\sum_{n=1}^{\infty}\frac{(2\alpha+1)a\tilde{Q}_0 - \alpha\tilde{Q}_1}{(2\alpha+1)^{n+1}}x^{2n+1} + \sum_{n=1}^{\infty}\frac{b(2\alpha+1)(\tilde{Q}_1 - 2a\tilde{Q}_0) - \alpha\tilde{Q}_0}{(2\alpha+1)^{n+1}}x^{2n}\right), \end{aligned}$$

and if we apply this expansion to the fraction $\frac{Cx+D}{x^2-(2\beta+1)}$, we obtain

$$\begin{aligned} &\frac{Cx+D}{x^2-(2\beta+1)} \\ = &\frac{-1}{\alpha-\beta}\left(\sum_{n=1}^{\infty}\frac{\beta\tilde{Q}_1 - (2\beta+1)a\tilde{Q}_0}{(2\beta+1)^{n+1}}x^{2n+1} + \sum_{n=1}^{\infty}\frac{b(2\beta+1)(-\tilde{Q}_1 + 2a\tilde{Q}_0) + \beta\tilde{Q}_0}{(2\beta+1)^{n+1}}x^{2n}\right). \end{aligned}$$

Firstly, we examine the series with even powers

$$\begin{aligned} E(x) &= -\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\begin{array}{l} (2\beta+1)^{n+1}\left(b(2\alpha+1)(\tilde{Q}_1 - 2a\tilde{Q}_0) - \alpha\tilde{Q}_0\right) \\ +(2\alpha+1)^{n+1}\left(b(2\beta+1)(-\tilde{Q}_1 + 2a\tilde{Q}_0) + \beta\tilde{Q}_0\right) \end{array}\right]x^{2n} \\ &= -\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\begin{array}{l} (2\beta+1)^n b(\tilde{Q}_1 - 2a\tilde{Q}_0) + (2\beta+1)^n\beta\tilde{Q}_0 \\ +(2\alpha+1)^n b(-\tilde{Q}_1 + 2a\tilde{Q}_0) - (2\alpha+1)^n\alpha\tilde{Q}_0 \end{array}\right]x^{2n} \\ &= -\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\begin{array}{l} \frac{\beta^{2n}}{(ab)^n}b(\tilde{Q}_1 - 2a\tilde{Q}_0) + \frac{\beta^{2n+1}}{(ab)^n}\tilde{Q}_0 \\ +\frac{\alpha^{2n}}{(ab)^n}b(-\tilde{Q}_1 + 2a\tilde{Q}_0) - \frac{\alpha^{2n+1}}{(ab)^n}\tilde{Q}_0 \end{array}\right]x^{2n} \\ &= \frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}(b\tilde{Q}_1 - 2ab\tilde{Q}_0) + \frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}\tilde{Q}_0\right]x^{2n}. \end{aligned}$$

Then, we examine the series with odd powers

$$O(x) = -\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\begin{array}{l} -\alpha(2\beta+1)^{n+1}\tilde{Q}_1 + a\tilde{Q}_0(2\beta+1)^n \\ +\beta(2\alpha+1)^{n+1}\tilde{Q}_1 - a\tilde{Q}_0(2\alpha+1)^n \end{array}\right]x^{2n+1}$$

$$\begin{aligned}
&= \frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}-\frac{\beta^{2n+1}}{(ab)^n}\tilde{Q}_1 - a\tilde{Q}_0\frac{\beta^{2n}}{(ab)^n}x^{2n+1} \\
&+\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\frac{\alpha^{2n+1}}{(ab)^n}\tilde{Q}_1 + a\tilde{Q}_0\frac{\alpha^{2n}}{(ab)^n}x^{2n+1} \\
&= \frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left(\tilde{Q}_1\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n} + a\tilde{Q}_0\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\right)x^{2n+1}
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}(x) &= \sum_{i=0}^{\infty}\tilde{Q}_i x^i = \sum_{i=0}^{\infty}\tilde{Q}_{2i}x^{2i} + \sum_{i=0}^{\infty}\tilde{Q}_{2i+1}x^{2i+1} \\
&= \frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left(\tilde{Q}_1\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n} + a\tilde{Q}_0\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}\right)x^{2n+1} \\
&+\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty}\left[\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n}(b\tilde{Q}_1 - 2ab\tilde{Q}_0) + \frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n}\tilde{Q}_0\right]x^{2n}
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_n &= \left(\tilde{Q}_0\right)^{1-\xi(n)}\left(\tilde{Q}_1\right)^{\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor+1}-\beta^{2\lfloor\frac{n}{2}\rfloor+1}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)} \\
&+(a\tilde{Q}_0)^{\xi(n)}(b\tilde{Q}_1 - 2ab\tilde{Q}_0)^{1-\xi(n)}\frac{\alpha^{2\lfloor\frac{n}{2}\rfloor}-\beta^{2\lfloor\frac{n}{2}\rfloor}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}
\end{aligned}$$

For all $n \geq 0$, from the definition of generating function, and by combining the sums, we get the desired result. □

Theorem 2.7. *For any positive integer n, and $ab \neq 0$, the sum of the first n terms of the bi-periodic Pell-Lucas matrix sequence is computed as*

$$\sum_{k=0}^{n-1}\tilde{Q}_k = \frac{\begin{array}{c}(b\tilde{Q}_n + a\tilde{Q}_{n-1})^{\xi(n)}(b\tilde{Q}_{n-1} + a\tilde{Q}_n)^{1-\xi(n)} \\ -b\tilde{Q}_1 + 2ab\tilde{Q}_0 - a\tilde{Q}_0\end{array}}{2ab}.$$

Proof. If n is even, it's obtained that by using Binet formula

$$\begin{aligned}
\sum_{k=0}^{n-1}\tilde{Q}_k &= \sum_{k=0}^{\frac{n-2}{2}}\tilde{Q}_{2k} + \sum_{k=0}^{\frac{n-2}{2}}\tilde{Q}_{2k+1} \\
&= \sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta} + \sum_{k=0}^{\frac{n-2}{2}}\frac{b\tilde{Q}_1 - 2ab\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta} \\
&+\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_1}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta} + \sum_{k=0}^{\frac{n-2}{2}}\frac{a\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.
\end{aligned}$$

By using the sum of geometric series, it's computed that

$$\sum_{k=0}^{n-1}\tilde{Q}_k = \frac{\tilde{Q}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha\,(ab)^{\frac{n}{2}}}{(\alpha^2-ab)} - \frac{\beta^{n+1}-\beta\,(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]$$

$$
\begin{aligned}
&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^n-(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^n-(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]\\
&+\frac{\tilde{Q}_1}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right]\\
&+\frac{a\tilde{Q}_0}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^n-(ab)^{\frac{n}{2}}}{(\alpha^2-ab)}-\frac{\beta^n-(ab)^{\frac{n}{2}}}{(\beta^2-ab)}\right].
\end{aligned}
$$

After some algebraic operations, the following result is evaluated

$$
\begin{aligned}
= \quad &\frac{\tilde{Q}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\right]\\
&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[\begin{array}{l}-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^n-\beta^n)\\-(ab)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}\right]\\
&+\frac{\tilde{Q}_1}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\right]\\
&+\frac{a\tilde{Q}_0}{4(ab)^{\frac{n}{2}+2}(\alpha-\beta)}\left[\begin{array}{l}-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^n-\beta^n)\\-(ab)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}\right]
\end{aligned}
$$

By the definition of the Pell-Lucas matrix sequence, it is obtained that

$$
\sum_{k=0}^{n-1}\tilde{Q}_k=\frac{2b\tilde{Q}_{n-1}+2a\tilde{Q}_n-2b\tilde{Q}_1+4ab\tilde{Q}_0-2a\tilde{Q}_0}{4ab}
$$

Similarly if n is odd, we obtain

$$
\sum_{k=0}^{n-1}\tilde{Q}_k=\sum_{k=0}^{\frac{n-1}{2}}\tilde{Q}_{2k}+\sum_{k=0}^{\frac{n-3}{2}}\tilde{Q}_{2k+1}
$$

$$
\begin{aligned}
= \quad &\sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-1}{2}}\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\
&+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{Q}_1}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-3}{2}}\frac{a\tilde{Q}_0}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.
\end{aligned}
$$

By using the sum of geometric series, it's computed that

$$
\begin{aligned}
\sum_{k=0}^{n-1}\tilde{Q}_k \quad = \quad &\frac{\tilde{Q}_0}{(ab)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+2}-\alpha(ab)^{\frac{n+1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+2}-\beta(ab)^{\frac{n+1}{2}}}{(\beta^2-ab)}\right]\\
&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(ab)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-(ab)^{\frac{n+1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n+1}-(ab)^{\frac{n+1}{2}}}{(\beta^2-ab)}\right]\\
&+\frac{\tilde{Q}_1}{(ab)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n}-\alpha(ab)^{\frac{n-1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n}-\beta(ab)^{\frac{n-1}{2}}}{(\beta^2-ab)}\right]
\end{aligned}
$$

$$+\frac{a\tilde{Q}_0}{(ab)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n-1}-(ab)^{\frac{n-1}{2}}}{(\alpha^2-ab)}-\frac{\beta^{n-1}-(ab)^{\frac{n-1}{2}}}{(\beta^2-ab)}\right].$$

After some algebraic operations, the following result is evaluated

$$\begin{aligned}
= \quad & \frac{\tilde{Q}_0}{4(ab)^{\frac{n+5}{2}}(\alpha-\beta)}\left[-a^2b^2(\alpha^n-\beta^n)+ab(\alpha^{n+2}-\beta^{n+2})\right] \\
& +\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{4(ab)^{\frac{n+5}{2}}(\alpha-\beta)}\left[\begin{array}{l}-a^2b^2(\alpha^{n-1}-\beta^{n-1})+ab(\alpha^{n+1}-\beta^{n+1})\\ -(ab)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}\right] \\
& +\frac{\tilde{Q}_1}{4(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[-a^2b^2(\alpha^{n-2}-\beta^{n-2})+ab(\alpha^n-\beta^n)\right] \\
& +\frac{a\tilde{Q}_0}{4(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[\begin{array}{l}-a^2b^2(\alpha^{n-3}-\beta^{n-3})+ab(\alpha^{n-1}-\beta^{n-1})\\ -(ab)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}\right]
\end{aligned}$$

By the definition of the Pell-Lucas matrix sequence, it is obtained that

$$\begin{aligned}
\sum_{k=0}^{n-1}\tilde{Q}_k &= \frac{\tilde{Q}_{n+1}-\tilde{Q}_{n-1}+\tilde{Q}_n-\tilde{Q}_{n-2}-2b\tilde{Q}_1+4ab\tilde{Q}_0-2a\tilde{Q}_0}{4ab} \\
&= \frac{b\tilde{Q}_n+a\tilde{Q}_{n-1}-b\tilde{Q}_1+2ab\tilde{Q}_0-a\tilde{Q}_0}{2ab}
\end{aligned}$$

By the above two results, we get

$$\sum_{k=0}^{n-1}\tilde{Q}_k=\frac{\left(b\tilde{Q}_n+a\tilde{Q}_{n-1}\right)^{\xi(n)}\left(b\tilde{Q}_{n-1}+a\tilde{Q}_n\right)^{1-\xi(n)}-b\tilde{Q}_1+2ab\tilde{Q}_0-a\tilde{Q}_0}{2ab}$$

□

Lemma 2.2. *For any positive integer n, the sum of the first n terms of the bi-periodic Pell matrix sequence is computed as in [14]*

$$\sum_{k=0}^{n-1}\tilde{P}_k^2=\frac{\frac{b}{a}\tilde{P}_{2n+2}+\tilde{P}_{2n}-\tilde{P}_{2n-4}-\frac{b}{a}\tilde{P}_{2n-2}}{16ab(ab+1)}-\frac{(1+2ab)(a\tilde{P}_1-2ab\tilde{P}_0)}{8(ab+1)}.$$

Theorem 2.8. *For any positive integer n, and $ab\neq 0$, the sum of the first n terms of the bi-periodic Pell-Lucas matrix sequence is computed as*

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(\frac{b}{a}\right)^{2\xi(k)}\tilde{Q}_k^2 &= \sum_{k=0}^{n-1}\left(\tilde{P}_k\tilde{Q}_0\right)^2=\tilde{Q}_0^2\sum_{k=0}^{n-1}\tilde{P}_k^2 \\
&= \tilde{Q}_0^2\frac{\left(\frac{b}{a}\right)^{1-\xi(n)}\tilde{P}_{2n+2}+\left(\frac{b}{a}\right)^{\xi(n)}\tilde{P}_{2n}-\left(\frac{b}{a}\right)^{\xi(n)}\tilde{P}_{2n-4}-\left(\frac{b}{a}\right)^{1-\xi(n)}\tilde{P}_{2n-2}}{16ab(ab+1)} \\
&\quad -\tilde{Q}_0^2\frac{(1+2ab)(a\tilde{P}_1-2ab\tilde{P}_0)}{8(ab+1)}
\end{aligned}$$

Theorem 2.9. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=\frac{1}{x^4-x^2(4ab+2)+1}\left[\begin{array}{l}-\frac{\tilde{Q}_{n+1}}{x^{n-3}}+\frac{\tilde{Q}_{n-1}}{x^{n-1}}-\frac{\tilde{Q}_{n}}{x^{n-4}}+\frac{\tilde{Q}_{n-2}}{x^{n-2}}\\+x^4\tilde{Q}_0+x^3\tilde{Q}_1+x^2(2a\tilde{Q}_1-4ab\tilde{Q}_0-\tilde{Q}_0)+x(b\tilde{Q}_0-\tilde{Q}_1)\end{array}\right].$$

Proof. If n is even, we get

$$\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_{2k}}{x^{2k}}+\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_{2k+1}}{x^{2k+1}}.$$

By the Binet formula, we have

$$\begin{aligned}=\;&\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_0}{(abx^2)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-2}{2}}\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(abx^2)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\&+\sum_{k=0}^{\frac{n-2}{2}}\frac{\tilde{Q}_1}{x\,(abx^2)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-2}{2}}\frac{a\tilde{Q}_0}{x\,(abx^2)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.\end{aligned}$$

By using the sum of geometric series, it's computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}\;=\;&\frac{\tilde{Q}_0}{x^{n-2}(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+1}-\beta\left(abx^2\right)^{\frac{n}{2}}}{\beta^2-abx^2}\right]\\&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{x^{n-2}(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n}-\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n}-\left(abx^2\right)^{\frac{n}{2}}}{\beta^2-abx^2}\right]\\&+\frac{\tilde{Q}_1}{x^{n-1}(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\alpha\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+1}-\beta\left(abx^2\right)^{\frac{n}{2}}}{\beta^2-abx^2}\right]\\&+\frac{a\tilde{Q}_0}{x^{n-1}(ab)^{\frac{n}{2}-1}(\alpha-\beta)}\left[\frac{\alpha^{n}-\left(abx^2\right)^{\frac{n}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n}-\left(abx^2\right)^{\frac{n}{2}}}{\beta^2-abx^2}\right].\end{aligned}$$

$$\begin{aligned}=\;&\frac{\tilde{Q}_0}{x^{n-2}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\+\left(abx^2\right)^{\frac{n}{2}+1}(\alpha-\beta)-\left(abx^2\right)^{\frac{n}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]\\&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{x^{n-2}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^{n}-\beta^{n})\\+\left(abx^2\right)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]\\&+\frac{\tilde{Q}_1}{x^{n-1}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\+\left(abx^2\right)^{\frac{n}{2}+1}(\alpha-\beta)-\left(abx^2\right)^{\frac{n}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]\end{aligned}$$

$$+\frac{a\tilde{Q}_0}{x^{n-1}(ab)^{\frac{n}{2}+1}(\alpha-\beta)}\left[\frac{\begin{array}{c}+a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^n-\beta^n)\\+\left(abx^2\right)^{\frac{n}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

By the definition of the Pell-Lucas matrix sequence, it is obtained that

$$\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=\frac{1}{x^4-x^2(4ab+2)+1}\left[\begin{array}{c}-\frac{\tilde{Q}_{n+1}}{x^{n-3}}+\frac{\tilde{Q}_{n-1}}{x^{n-1}}-\frac{\tilde{Q}_n}{x^{n-4}}+\frac{\tilde{Q}_{n-2}}{x^{n-2}}\\+x^4\tilde{Q}_0+x^3\tilde{Q}_1+x^2(2a\tilde{Q}_1-4ab\tilde{Q}_0-\tilde{Q}_0)+x(b\tilde{Q}_0-\tilde{Q}_1)\end{array}\right]$$

Similarly if n is odd, we obtain

$$\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=\sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{Q}_{2k}}{x^{2k}}+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{Q}_{2k+1}}{x^{2k+1}}$$

$$\begin{aligned}=&\sum_{k=0}^{\frac{n-1}{2}}\frac{\tilde{Q}_0}{(abx^2)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-1}{2}}\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(abx^2)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\\&+\sum_{k=0}^{\frac{n-3}{2}}\frac{\tilde{Q}_1}{x\,(abx^2)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta}+\sum_{k=0}^{\frac{n-3}{2}}\frac{a\tilde{Q}_0}{x\,(abx^2)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}.\end{aligned}$$

By using the sum of geometric series, it's computed that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=&\frac{\tilde{Q}_0}{(abx^2)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+2}-\alpha\left(abx^2\right)^{\frac{n+1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+2}-\beta\left(abx^2\right)^{\frac{n+1}{2}}}{\beta^2-abx^2}\right]\\&+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{(abx^2)^{\frac{n-1}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n+1}-\left(abx^2\right)^{\frac{n+1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n+1}-\left(abx^2\right)^{\frac{n+1}{2}}}{\beta^2-abx^2}\right]\\&+\frac{\tilde{Q}_1}{x\,(abx^2)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n}-\alpha\left(abx^2\right)^{\frac{n-1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n}-\beta\left(abx^2\right)^{\frac{n-1}{2}}}{\beta^2-abx^2}\right]\\&+\frac{a\tilde{Q}_0}{x\,(abx^2)^{\frac{n-3}{2}}(\alpha-\beta)}\left[\frac{\alpha^{n-1}-\left(abx^2\right)^{\frac{n-1}{2}}}{\alpha^2-abx^2}-\frac{\beta^{n-1}-\left(abx^2\right)^{\frac{n-1}{2}}}{\beta^2-abx^2}\right].\end{aligned}$$

After some algebraic operations, the following result is evaluated

$$=\frac{\tilde{Q}_0}{x^{n-1}(ab)^{\frac{n+3}{2}}(\alpha-\beta)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^n-\beta^n)-abx^2(\alpha^{n+2}-\beta^{n+2})\\+\left(abx^2\right)^{\frac{n+3}{2}}(\alpha-\beta)-\left(abx^2\right)^{\frac{n+1}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{b\tilde{Q}_1-2ab\tilde{Q}_0}{x^{n-1}\left(ab\right)^{\frac{n+3}{2}}\left((\alpha-\beta\right)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-1}-\beta^{n-1})-abx^2(\alpha^{n+1}-\beta^{n+1})\\+\left(abx^2\right)^{\frac{n+1}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{\tilde{Q}_1}{x^{n-2}\left(ab\right)^{\frac{n+1}{2}}\left((\alpha-\beta\right)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-2}-\beta^{n-2})-abx^2(\alpha^{n}-\beta^{n})\\+\left(abx^2\right)^{\frac{n+1}{2}}(\alpha-\beta)-\left(abx^2\right)^{\frac{n-1}{2}}ab(\alpha-\beta)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

$$+\frac{a\tilde{Q}_0}{x^{n-2}\left(ab\right)^{\frac{n+3}{2}}\left((\alpha-\beta\right)}\left[\frac{\begin{array}{c}a^2b^2(\alpha^{n-3}-\beta^{n-3})-abx^2(\alpha^{n-1}-\beta^{n-1})\\+\left(abx^2\right)^{\frac{n-1}{2}}(\alpha^2-\beta^2)\end{array}}{x^4-x^2(4ab+2)+1}\right]$$

By the definition of the Pell-Lucas matrix sequence, it is obtained that

$$\sum_{k=0}^{n-1}\frac{\tilde{Q}_k}{x^k}=\frac{1}{x^4-x^2(4ab+2)+1}\left[\begin{array}{c}\frac{\tilde{Q}_{n-1}}{x^{n-1}}-\frac{\tilde{Q}_{n+1}}{x^{n-3}}+\frac{\tilde{Q}_{n-2}}{x^{n-2}}-\frac{\tilde{Q}_{n}}{x^{n-4}}\\+x^4\tilde{Q}_0+x^3\tilde{Q}_1+x^2(2a\tilde{Q}_1-4ab\tilde{Q}_0-\tilde{Q}_0)+x(2b\tilde{Q}_0-\tilde{Q}_1)\end{array}\right]$$

We find the same results either n is even or odd number. □

References

[1] A. F. Horadam and B. J. M. Mahon. Pell-Lucas and Pell-Lucas Lucas Polynomials. The Fibonacci Quarterly 23 (1985), no. 1, 7 - 20.

[2] T. Koshy. Pell-Lucas and Pell-Lucas-Lucas Numbers with Applications. Springer, Berlin, 2014.

[3] M. Edson and O. Yayenie. A new generalization of Fibonacci sequences and the extended Binet's formula. INTEGERS Electron. J. Comb. Number Theor. 9 (2009), 639 - 654.

[4] O. Yayenie. A note on generalized Fibonacci sequence. Appl. Math. Comput. 217 (2011), 5603 - 5611.

[5] G. Bilgici. Two generalizations of Lucas sequence. Applied Mathematics and Computation 245 (2014), 526 - 538.

[6] S. Uygun and E. Owusu. A New Generalization of Jacobsthal Numbers (Bi-Periodic Jacobsthal Sequences). Journal of Mathematical Analysis 7 (2016), no. 5, 28 - 39.

[7] A. Coskun and N. Taskara. A note on the bi-periodic Fibonacci and Lucas matrix sequences. Applied Mathematics and Computation 320 (2018), 400 - 406.

[8] S. Uygun and H. Karatas. A New Generalization of Pell-Lucas-Lucas Numbers (Bi-Periodic Pell-Lucas-Lucas Sequence). Communications in Mathematics and Applications 10 (2019), no. 3, 1 - 12.

[9] S. Uygun and E. Owusu. A New Generalization of Jacobsthal Lucas Numbers (Bi-Periodic Jacobsthal Lucas Sequence). Advances in Mathematics and Computer Science 34 (2020), no. 5, 1 - 13.

[10] S. Uygun and H. Karatas. Bi-Periodic Pell Sequence. Academic Journal of Applied Mathematical Sciences 6 (2020), no. 7, 136 - 144.

[11] S. Uygun and E. Owusu. Matrix Representation of bi-Periodic Jacobsthal Sequence. Journal of Advances in Mathematics and Computer Science 34 (2020), no. 6, 1 - 12.

[12] S. Uygun. Bi-Periodic Jacobsthal Lucas Matrix Sequence. Acta Universitatis Apulensis 66 (2021), 53 - 69.

[13] V. Verma and A. Bala. A Note on Bivariate bi-periodic Jacobsthal Polynomials and Bivariate bi-periodic Jacobsthal Lucas Polynomials. International Journal of Biology, Pharmacy and Allied Sciences 10 (2021), no. 12, 117 - 127.

[14] Y. Soykan. On k-circulant Matrices with the Generalized Third Order Pell Numbers. Notes on Number Theory and Discrete Mathematics 4 (2021), no. 27, 187 - 206.

[15] S. Uygun and E. Akinci. Matrix Representation of Bi-Periodic Pell Sequence (accepted).

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On the mean value of exponentially k-free integers over square-full numbers

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Abstract Let $n > 1$ be an integer and $q_k^{(e)}(n)$ be the characteristic function of exponentially k-free integers. In this paper, we shall study the mean value of exponentially k-free integers over square-full number, that is

$$\sum_{\substack{n\le x\\ n \text{ is square-full}}} \left(q_k^{(e)}(n)\right)^r = \sum_{n\le x}\left(q_k^{(e)}(n)\ f_2(n)\right)^r,$$

where $f_2(n)$ is the characteristic function of square-full integers, i.e.

$$f_2(n) = \begin{cases} 1, & n \text{ is square-full;} \\ 0, & \text{otherwise.} \end{cases}$$

Keywords square-full number, Dirichlet convolution, mean value, divisor function.
2010 Mathematics Subject Classification 11N37.

§1. Introduction

The study of different forms of n is an interesting problem in number theory. It is well known that a natural number n is called k-free ($k \ge 2$ fixed) if in the canonical decomposition $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ we have $a_1 \le k-1, \ldots, a_r \le k-1$. Also the natural number n is called k-full if $a_1 \ge k, a_2 \ge k, \cdots, a_r \ge k$. Toth [3] defined the integer n to be exponentially k -free integer if all the exponents a_i $(1 \le i \le r)$ are k-free, i.e. n is not divisible by the k-th power of any prime ($k \ge 2$). Also the integer n is called exponentially k -full integer if all the exponents a_i $(1 \le i \le r)$ are k-full.

Let $q_k^{(e)}(n)$ denote the characteristic function of exponentially k-free integers. Therefore $q_k^{(e)}(n) = 1$ when n is exponentially k -free integer and zero otherwise. Many authors have studied the properties of the exponential divisor function, for example, [2–5]. For the exponential divisor function $q_k^{(e)}(n)$, Tóth [3, 4] proved the following result:

$$\sum_{n\le x} q_k^{(e)}(n) = D_k x + O\left(x^{1/2^k}\delta(x)\right),$$

where

$$D_k = \prod_p \left(1 + \sum_{a=2^k}^{\infty} \frac{q_k(a) - q_k(a-1)}{p^a}\right),$$

and $q_k(n)$ denoting the characteristic function of k-free integers.

In this paper, we study the mean value of exponentially k -free integers over square full numbers.

Theorem For some $D > 0$, $k \ge 3$,

$$\sum_{\substack{n\le x\\ n \text{ is square-full}}} \left(q_k^{(e)}(n)\right)^r = \frac{\zeta\left(\frac{3}{2}\right) G\left(\frac{1}{2}\right)}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right) G\left(\frac{1}{3}\right)}{\zeta(2)} x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}} \exp\left(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right),$$

where

$$G(s) = \prod_p \left(1 - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \frac{1}{p^{(2^k+3)s}} + \cdots\right),$$

which is absolutely convergent for $\Re s > \frac{1}{8} + \epsilon$.

Corollary If the Riemann Hypothesis is true, then for some $D > 0$, we have

$$\sum_{\substack{n\le x\\ n \text{ is square-full}}} \left(q_k^{(e)}(n)\right)^r = \frac{\zeta\left(\frac{3}{2}\right) G\left(\frac{1}{2}\right)}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right) G\left(\frac{1}{3}\right)}{\zeta(2)} x^{\frac{1}{3}} + O\left(x^{\frac{7}{44}} \exp\left(D\frac{\log x}{\log\log x}\right)\right).$$

Notation Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. Let

$$d(2,3;k) = \sum_{n^2 m^3 = k} 1.$$

Then we have

$$\sum_{1\le k\le x} d(2,3;k) = \zeta\left(\frac{3}{2}\right) x^{\frac{1}{2}} + \zeta\left(\frac{2}{3}\right) x^{\frac{1}{3}} + \Delta(2,3;x)$$

with

$$\Delta(2,3;x) \ll x^{\frac{2}{15}}.$$

Proof. See Chapter 14.3 and Theorem 14.4 of [1]. □

Lemma 2.2. Let $f(n)$ be an arithmetical function, and satisfy

$$\sum_{n\leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O\left(x^{a}\right), \quad \sum_{n\leq x} |f(n)| = O\left(x^{a_1}\log^{r} x\right),$$

where $a_1 \geq a_2 \geq \cdots \geq a_l > 1/k > a \geq 0, r \geq 0, P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r, and $k \geq 1$ is a fixed integer. If

$$h(n) = \sum_{d^k|n} \mu(d) f\left(n/d^k\right),$$

then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + \delta(x),$$

where $R_1(t), \cdots, R_l(t)$ are polynomials in t of degrees not exceeding r, and for some $D > 0$,

$$\delta(x) \ll x^{1/k} \exp\left(\left(-D(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right).$$

If the Riemann hypothesis is true, then for some $D > 0$, we have

$$\delta(x) \ll x^{c} \exp\left(D\frac{\log x}{\log\log x}\right)$$

where

$$c = \frac{2a_1 - a}{2ka_1 - 2ka + 1}.$$

Proof. See Theorem 14.2 of [1]. □

Lemma 2.3. Let s be a complex number, then we have

$$\sum_{\substack{n=1 \\ n \text{ is square-full}}} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} G(s),$$

where the Dirichlet series $G(s) = \prod_p \left(1 - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \frac{1}{p^{(2^k+3)s}} + \cdots\right)$ is absolutely convergent for $\Re s > \frac{1}{8} + \epsilon$.

Proof. Let

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is square-full}}} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \sum_{n=1}^{\infty} \frac{\left(q_k^{(e)}(n) f_2(n)\right)^r}{n^s} \ (\Re s > 1),$$

where $f_2(n)$ is the characteristic function of square-full integers.

Obviously the function $q_k^{(e)}(n)$ is multiplicative, and for every prime power p^a, we have $q_k^{(e)}(p) = q_k^{(e)}(p^2) = q_k^{(e)}(p^3) = \ldots = q_k^{(e)}\left(p^{2^k-1}\right) = 1, q_k^{(e)}\left(p^{2^k}\right) = 0$. Then

$$\begin{aligned}
F(s) =& \sum_{\substack{n=1 \\ n \text{ is square-full}}}^{\infty} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \sum_{n=1}^{\infty} \frac{\left(q_k^{(e)}(n) f_2(n)\right)^r}{n^s} \\
=& \prod_p \left(1 + \frac{\left(q_k^{(e)}(p) f_2(p)\right)^r}{p^s} + \frac{\left(q_k^{(e)}(p^2) f_2(p^2)\right)^r}{p^{2s}} + \frac{\left(q_k^{(e)}(p^3) f_2(p^3)\right)^r}{p^{3s}} \right.\\
&\left. + \cdots + \frac{\left(q_k^{(e)}\left(p^{2^k-1}\right) f_2\left(p^{2^k-1}\right)\right)^r}{p^{(2^k-1)s}}\right) \\
=& \prod_p \left(1 + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \cdots + \frac{1}{p^{(2^k-1)s}}\right) \\
=& \zeta(2s) \prod_p \left(1 + \frac{1}{p^{3s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}}\right) \\
=& \zeta(2s)\zeta(3s) \prod_p \left(1 - \frac{1}{p^{6s}} - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \frac{1}{p^{(2^k+3)s}} + \frac{1}{p^{(2^k+4)s}}\right) \\
=& \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} G(s),
\end{aligned}$$

where $G(s) = \prod_p \left(1 - \frac{1}{p^{2^k s}} - \frac{1}{p^{(2^k+1)s}} + \frac{1}{p^{(2^k+3)s}} + \cdots\right)$ and it is absolutely convergent for $\Re s > \frac{1}{8} + \epsilon$ with $k \geq 3$. □

§3. Proof of Theorem 1.1

Now we prove the Theorem. From Lemma 2.3, we have known that

$$F(s) = \sum_{\substack{n=1 \\ n \text{ is square-full}}}^{\infty} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} G(s),$$

where $G(s)$ is absolutely convergent for $\Re s > \frac{1}{8} + \epsilon$.

Define

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

and

$$H(s) := \zeta(2s)\zeta(3s)G(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{n=ml} d(2,3;m) g(l)}{n^s} \quad (\Re s > 1).$$

Then we can get

$$\sum_{n \leq x} h(n) = \sum_{ml \leq x} d(2,3;m) g(l) = \sum_{l \leq x} g(l) \sum_{m \leq \frac{x}{l}} d(2,3;m).$$

From Perron's formula, residue theorem and Lemma 2.1, we can get

$$\sum_{n\le x} d(2,3;n) = \zeta\left(\frac{3}{2}\right)x^{\frac{1}{2}} + \zeta\left(\frac{2}{3}\right)x^{\frac{1}{3}} + O\left(x^{\frac{2}{15}}\right). \tag{1}$$

Then from (1) and Abel integral formula, we have the relation

$$\begin{aligned}
\sum_{n\le x} h(n) = &\sum_{l\le x} g(l)\left[\zeta\left(\frac{3}{2}\right)\left(\frac{x}{l}\right)^{\frac{1}{2}} + \zeta\left(\frac{2}{3}\right)\left(\frac{x}{l}\right)^{\frac{1}{3}} + O\left(\left(\frac{x}{l}\right)^{\frac{2}{15}}\right)\right]\\
=&\zeta\left(\frac{3}{2}\right)x^{\frac{1}{2}}\sum_{l\le x}\frac{g(l)}{l^{1/2}} + \zeta\left(\frac{2}{3}\right)x^{\frac{1}{3}}\sum_{l\le x}\frac{g(l)}{l^{1/3}} + O\left(x^{\frac{2}{15}}\sum_{l\le x}\frac{|g(l)|}{l^{2/15}}\right)\\
=&\zeta\left(\frac{3}{2}\right)x^{\frac{1}{2}}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/2}} + \zeta\left(\frac{2}{3}\right)x^{\frac{1}{3}}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/3}}\\
&+ O\left(x^{\frac{1}{2}}\sum_{l>x}\frac{|g(l)|}{l^{1/2}} + x^{\frac{1}{3}}\sum_{l>x}\frac{|g(l)|}{l^{1/3}} + x^{\frac{2}{15}}\sum_{k<x}\frac{|g(l)|}{l^{2/15}}\right)\\
=&\zeta\left(\frac{3}{2}\right)G\left(\frac{1}{2}\right)x^{\frac{1}{2}} + \zeta\left(\frac{2}{3}\right)G\left(\frac{1}{3}\right)x^{\frac{1}{3}} + O\left(x^{\frac{1}{8}}\right).
\end{aligned}$$

From Perron's formula and Lemmas 2.2 and 2.3, we can get

$$\begin{aligned}
\sum_{\substack{n\le x\\ n \text{ is square-full}}} \left(q_k^{(e)}(n)\right)^r =&\frac{\zeta\left(\frac{3}{2}\right)G\left(\frac{1}{2}\right)}{\zeta(3)}x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)G\left(\frac{1}{3}\right)}{\zeta(2)}x^{\frac{1}{3}}\\
&+ O\left(x^{\frac{1}{6}}\exp\left(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right).
\end{aligned}$$

If the Riemann hypothesis is true, then for some $D > 0$, we have

$$\delta(x) \ll x^c \exp\left(D\frac{\log x}{\log\log x}\right)$$

where

$$c = \frac{2a_1 - a}{2ka_1 - 2ka + 1}.$$

Then we can get the result of Corollary.

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References

[1] A. Ivić, The Riemann zeta-function: theory and applications. New York: Wiley, 1985.

[2] M.V. Subbarao, On some airthmetic convolutions, in the theory of arithmetic functions. Lecture Notes in Mathematic, Springer, 251(1972), 247–271.

[3] L. Tóth, On certain arithmetical functions involving exponential divisors. Ann. Univ. Sci. Budapest. Sect. Comput, 24(2004), 285–294.

[4] L. Tóth, On certain arithmetical functions involving exponential divisors II. Ann. Univ. Sci. Budapest. Sect. Comput, 27(2007), 155–166.

[5] J. Wu, Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré. Théor Nombres Bordeaux, 7(1995), 133–141.

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On the roman domination polynomial of the commuting and non-commuting graphs of the dihedral groups

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Abstract A graph associated to a finite group is a way to analyze some properties of a group graphically. Many graphs of groups have been constructed according to the properties of the groups such as the commuting and non-commuting graphs. Besides, a Roman dominating function (in brief RDF) of a graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, defined as a function f from the set $V(\Gamma)$ to the set of numbers $\{0,1,2\}$ in which for any vertex $u \in V(\Gamma)$ with $f(u) = 0$ there is at least a vertex $v \in V(\Gamma)$ with $f(v) = 2$ is adjacent to u. The summation of the values $f(u)$ for all the vertices of Γ is defined by the weight of the RDF f. Meanwhile, the Roman domination number (in short RDN) of Γ, $\gamma_R(\Gamma)$, is the minimum weight of an RDF defined on Γ [1]. Based on that, the Roman domination polynomial (RDP) of a graph Γ on p vertices is defined by $R(\Gamma, x) = \sum_{j=\gamma_R(G)}^{2p} r(\Gamma, j)x^j$, where $r(\Gamma, j)$ is the number of RDFs of Γ with weight j [2]. In this paper, the RDPs of commuting and non-commuting graphs associated to dihedral groups with order $2n$ are computed and some examples are given to illustrate the results.

Keywords Roman domination, Roman domination polynomial, commuting graph, non-commuting graph, dihedral groups.
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§1. Introduction and preliminaries

All the graphs considered throughout this paper are finite and simple, namely finite, undirected and have no self-loops or multiple edges. Let $\Gamma = (V, E)$ be a graph. The order and size of Γ are denoted respectively by $|V(\Gamma)| = p$ and $|E(\Gamma)| = q$. As usual the complete graph, null graph, cycles and paths on p vertices are denoted in this paper as K_p, $\overline{K_p}$, C_p and P_p, respectively. The following expressions $\lceil x \rceil$ and $\lfloor x \rfloor$ of the positive real number x are denoted respectively to the least positive integer bigger than or equal to x and the largest positive integer less than or equal to x.

The join $\Gamma_1 \vee \Gamma_2$ of two graphs Γ_1 and Γ_2, with different vertex sets $|V(\Gamma_1)| = p_1$ and $|V(\Gamma_2)| = p_2$, contains the original graphs Γ_1 and Γ_2 and every vertex of Γ_1 is linked to each vertex of Γ_2 by an edge.

The RDN of a graph $\Gamma = (V, E)$, $\gamma_R(\Gamma)$, is defined in [3] as the smallest weight, $W(f(V)) = \sum_{u \in V(\Gamma)} f(u)$, of an RDF $f : V(\Gamma) \to \{0, 1, 2\}$ in which for any vertex u in Γ with $f(u) = 0$ there is at least a vertex, say v, from the neighbors of u satisfied that $f(v) = 2$. For details, the reader is referred to [1].

Recently, Deepak *et al.* [2] defined the RDP of a graph Γ as

$$R(\Gamma, x) = \sum_{j=\gamma_R(\Gamma)}^{2p} r(\Gamma, j) x^j,$$

where $r(\Gamma, j)$ is the number of RDFs of Γ with weight j and studied some of its properties. In addition, the RDPs of paths and cycles are studied in details in [4] and [5], respectively. The following lemmas present some properties of the RDPs of graphs that are needed in this paper.

Proposition 1.1 gives the RDP of the null graph $\overline{K_p}$.

Proposition 1.1 [2]. *Consider Γ to be the null graph $\overline{K_p}$ on p vertices. Then*

$$R(\Gamma, x) = \sum_{j=0}^{p} \binom{p}{j} x^{p+j} = x^p \big(1 + x\big)^p.$$

Next, the RDP of a complete graph K_p is given in the following proposition.

Proposition 1.2 [2]. *For the complete graph K_p, with $p \geq 2$,*

$$R(K_p, x) = x^p + \sum_{j=2}^{2p} \Bigg[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{p}{r} \binom{p-r}{j-2r} \Bigg] x^j.$$

Now, Proposition 1.3 gives the RDP of the join $\Gamma_1 \vee \Gamma_2$.

Proposition 1.3 [2]. *Let Γ_1, Γ_2 be any two graphs with $|\Gamma_1| = p_1$, $|\Gamma_2| = p_2$,* *respectively. Then*

$$R(\Gamma_1 \vee \Gamma_2, x) = \sum_{r=1}^{p_1} \binom{p_1}{r} \Bigg[\sum_{t=0}^{p_1-r} \binom{p_1-r}{t} \Bigg(\sum_{s=1}^{p_2} \binom{p_2}{s} \sum_{l=0}^{p_2-s} \binom{p_2-s}{l} \Bigg) \Bigg] x^{2r+t+2s+l} + (1+x)^{p_2} \Big(R(G_1, x) - x^{p_1} \Big) + (1+x)^{p_1} \Big(R(G_2, x) - x^{p_2} \Big) + x^{p_1+p_2}.$$

Recall that, the following properties of the combination $\binom{n}{r}$ are considered in the above lemmas.

(i) If $r = 0$ or $r = n$, then $\binom{n}{r} = 1$.

(ii) If $r > n$, then $\binom{n}{r} = 0$.

(iii) If $r = 1$, then $\binom{n}{1} = n$.

The following proposition presents the RDP of the disconnected graphs.

Proposition 1.4 [2]. *Assume the graph* $\Gamma = \cup_{i=1}^{m} \Gamma_i$, *where* $\Gamma_1, \ldots, \Gamma_m$ *are the components of* Γ. *Then*

$$R(\Gamma, x) = R(\Gamma_1, x) \cdots R(\Gamma_m, x).$$

Besides, a group consists of a set of elements and an operation combines the elements of the set to form an element also must be in the set. This operation on the set satisfies four conditions called the group axioms [6]. A group is called finite if its set is finite. The dihedral group is an example of a finite non-abelian group. In the following, the definitions of a center of group, a dihedral group and some properties of a dihedral group are provided.

Definition 1.1 [7]. Consider G to be a finite group. The subset $Z(G)$ of G containing all the elements of G which commute with every element in G is called the center of the group G i.e. $Z(G) = \{\alpha \in G | \, \alpha y = y\alpha, \, \forall y \in G\}$.

Next, the definition of a dihedral group and some of its properties are given.

Definition 1.2 [8]. A dihedral group of order $2n$ where $n \geq 3$, denoted by D_{2n}, is a group of symmetries of a regular polygon, which includes rotations and reflections. A dihedral group can be presented also as:

$$D_{2n} \cong \left\langle a, b : \, a^n = b^2 = e, \, baba = e \right\rangle.$$

Some properties of a dihedral group D_{2n} are presented in the following proposition.

Proposition 1.5 [8]. *Consider* $D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ *to be a dihedral group of order* $2n$. *Then* D_{2n} *has the following properties:*

(i) For odd n,

(a) the center $Z(D_{2n}) = \{e\}$,

(b) the elements of the form a^i, where $i = 1, 2, \ldots, n-1$, are commute one to each other and a^i do not commute with the elements of the form $a^j b$, where $j = 0, 1, 2, \ldots, n-1$,

(c) the elements of the form $a^j b$, where $j = 0, 1, 2, \ldots, n-1$, are not commute with any element in D_{2n} accept the identity element.

(ii) For even n,

(a) the center $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$,

(b) the non-central elements of the form a^i i.e. $i = 1, 2, \ldots, n-1$ and $i \neq \dfrac{n}{2}$, are commute one to each other and a^i do not commute with the elements of the form $a^j b$, where $j = 0, 1, 2, \ldots, n-1$,

(c) each element of the form $a^j b$, where $j = 0, 1, 2, \ldots, n-1$, is not commute with any element in D_{2n} accept the identity element and the element of the form $a^{\frac{n}{2}j} b$.

Meanwhile, the idea of studying algebraic structures by using properties of graphs is one of the interesting topics in research because of its impressive results and questions. One of these studies is the study of graphs of groups. Many researches are presented on the commuting graph and the non-commuting graph associated to finite groups. For instance, Vahidi and Talebi [9], Segev and Seitz [10], Raza and Faizi [11] and Parker [12] studied the commuting graphs related to some specific groups; while Abdollahi *et al.* [13], Moradipour *et al.* [14] and Ahanjideh and Iranmanesh [15] have studied the non-commuting graphs. In the following, the definitions of the commuting graph and the non-commuting graph associated to finite non-abelian groups are provided. Firstly, the definition of the commuting graph of non-abelian finite groups is given.

Definition 1.3 [9]. Consider G to be a non-abelian finite group. The commuting graph of G, denoted by Γ_G^{com} is a simple undirected graph with vertex set $G - Z(G)$ and two vertices are adjacent if they commute.

In [16], the commuting graph of dihedral groups, D_{2n}, is stated as follows:

Proposition 1.6 [16]. *Let $G = D_{2n}$ be a dihedral group of order $2n$, where $D_{2n} \cong \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ and $n \geq 3$. Then the commuting graph of D_{2n}, $\Gamma_{D_{2n}}^{com}$ is given as*

$$\Gamma_{D_{2n}}^{com} = \begin{cases} K_{n-1} \cup nK_1, & \text{if } n \text{ is odd;} \\ \\ K_{n-2} \cup \frac{n}{2} K_2, & \text{if } n \text{ is even.} \end{cases}$$

Next, the definition of the non-commuting graph of non-abelian finite groups is presented.

Definition 1.4 [13]. Let G be a non-abelian finite group. The non-commuting graph of G, denoted by Γ_G^{ncom} is a simple graph with vertex set $G-Z(G)$ and two vertices are adjacent if and only if they do not commute.

The following proposition gives the non-commuting graph of D_{2n}.

Proposition 1.7 [17]. *Let $D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ be the dihedral group of order $2n$. Then the non-commuting graph of D_{2n}, $\Gamma_{D_{2n}}^{ncom}$, is one of the following forms.*

(i) If n is odd, then $\Gamma_{D_{2n}}^{ncom} = K_n \vee \overline{K_{n-1}}$.

(ii) If n is even, then $\Gamma_{D_{2n}}^{ncom} = U_n \vee \overline{K_{n-2}}$, where U_n is an $(n-2)$-regular graph with n vertices.

In this paper, the RDPs of the commuting graph and the non-commuting graph associated to the dihedral groups are found and illustrated by some examples.

§2. Main results

First, the RDP of the commuting graph associated to the dihedral groups, D_{2n}, is obtained as follows.

Theorem 2.1. *Let $D_{2n} = \langle a, b : a^n = b^2 = baba = e \rangle$ be the dihedral group of order $2n$, where $n \geq 3$ and $\Gamma_{D_{2n}}^{com}$ is its CG. Then the RDP of $\Gamma_{D_{2n}}^{com}$ is given as:*

(i) If n is odd, then

$$R(\Gamma_{D_{2n}}^{com}, x) = x^n \big(1+x\big)^n \Bigg[x^{n-1} + \sum_{j=2}^{2n-2} \Bigg[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n-1}{r} \binom{n-r-1}{j-2r}\Bigg] x^j\Bigg].$$

(ii) If n is even, then

$$R(\Gamma_{D_{2n}}^{com}, x) = x^n \big(x^2+2x+3\big)^{\frac{n}{2}} \Bigg[x^{n-2} + \sum_{j=2}^{2n-4} \Bigg[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n-2}{r} \binom{n-r-2}{j-2r}\Bigg] x^j\Bigg].$$

Proof. By Proposition 1.6, the commuting graph of the dihedral groups D_{2n} is given as

$$\Gamma_{D_{2n}}^{com} = \begin{cases} K_{n-1} \cup nK_1, & \text{if } n \text{ is odd;} \\ \\ K_{n-2} \cup \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

Therefore by using Proposition 1.4, the RDP of $\Gamma_{D_{2n}}^{com}$ is given as

$$R(\Gamma_{D_{2n}}^{com}, x) = \begin{cases} R(K_{n-1}) \cdot \big(R(K_1)\big)^n, & \text{if } n \text{ is odd;} \\ \\ R(K_{n-2}) \cdot \big(R(K_2)\big)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Also by Proposition 1.1, $R(K_1) = x(1+x)$ and by Proposition 1.2, $R(K_2) = x^2(x^2+2x+3)$. Hence again using Proposition 1.2, the result is obtained. □

In the following example, the result in Theorem 2.1 part (i) is illustrated .

Example 2.1. Let $D_{10} = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ be the dihedral group of order 10, where $n = 5$. By Proposition 1.6, the CG of the dihedral groups D_{10} is $\Gamma^{com}_{D_{10}} = K_4 \cup 5K_1$. By using Definition 1.3 and Proposition 1.5, the vertex set and the edge set of $\Gamma^{com}_{D_{10}}$ are given respectively by $V(\Gamma^{com}_{D_{10}}) = \{a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ and

$$E(\Gamma^{com}_{D_{10}}) = \big\{(a, a^2), (a, a^3), (a, a^4), (a^2, a^3), (a^2, a^4), (a^3, a^4)\big\}.$$

Therefore, $\Gamma^{com}_{D_{10}}$ is shown as in the following figure.

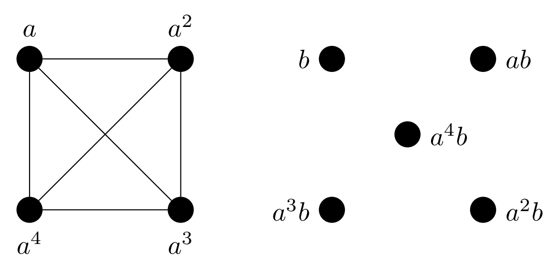

Figure 1: The CG of D_{10}

Now by Theorem 2.1, the RDP of $\Gamma^{com}_{D_{10}}$ is obtained by

$$\begin{aligned}
R(\Gamma^{com}_{D_{10}}, x) =& x^5\big(1+x\big)^5\left[x^4 + \sum_{j=2}^{8}\left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor}\binom{4}{r}\binom{4-r}{j-2r}\right]x^j\right] \\
=& x^5\big(1+x\big)^5\left[x^4 + x^2\sum_{r=1}^{1}\binom{4}{r}\binom{4-r}{2-2r} + x^3\sum_{r=1}^{1}\binom{4}{r}\binom{4-r}{3-2r}\right. \\
& + x^4\sum_{r=1}^{2}\binom{4}{r}\binom{4-r}{4-2r} + x^5\sum_{r=1}^{2}\binom{4}{r}\binom{4-r}{5-2r} + x^6\sum_{r=1}^{3}\binom{4}{r}\binom{4-r}{6-2r} \\
& \left. + x^7\sum_{r=1}^{3}\binom{4}{r}\binom{4-r}{7-2r} + x^8\sum_{r=1}^{4}\binom{4}{r}\binom{4-r}{8-2r}\right] \\
=& x^5\big(1+x\big)^5\Big[x^4 + 4x^2 + 12x^3 + (12+6)x^4 + (4+12)x^5 + (0+6+4)x^6 \\
& + (0+0+4)x^7 + x^8\Big] \\
=& x^7\big(1+x\big)^5\Big[x^6 + 4x^5 + 10x^4 + 16x^3 + 19x^2 + 12x + 4\Big] \\
=& \big(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\big)\big(x^{13} + 4x^{12} + 10x^{11} + 16x^{10} + 19x^9 \\
& + 12x^8 + 4x^7\big) \\
=& x^{18} + 9x^{17} + 40x^{16} + 116x^{15} + 244x^{14} + 388x^{13} + 468x^{12} + 420x^{11} + 271x^{10} \\
& + 119x^9 + 32x^8 + 4x^7.
\end{aligned}$$

The polynomial is represented graphically as in Figure 2.

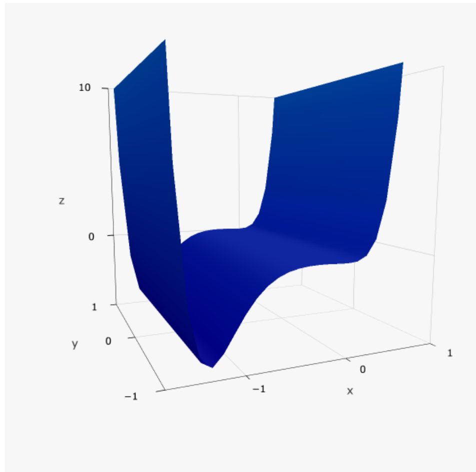

Figure 2: The graph of $R(\Gamma^{com}_{D_{10}})$

In the following theorem, the RDP of the non-commuting graph associated to the dihedral groups D_{2n} when n is an odd integer, is found.

Theorem 2.2. *Let $D_{2n} = \langle a, b : a^n = b^2 = baba = e\rangle$ be the dihedral group of order $2n$, where $n \geq 3$ is an odd integer, and $\Gamma^{ncom}_{D_{2n}}$ be its non-commuting graph. Then the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given as:*

$$\begin{aligned}R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n}\binom{n}{r}\Bigg[\sum_{t=0}^{n-r}\binom{n-r}{t}\Bigg(\sum_{s=1}^{n-1}\binom{n-1}{s}\sum_{l=0}^{n-s-1}\binom{n-s-1}{l}\Bigg)\Bigg]x^{2r+t+2s+l}\\ &+\big(1+x\big)^{n-1}\Bigg[\sum_{j=2}^{2n}\Bigg[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}\Bigg]x^j\Bigg]\\ &+x^{n-1}\big(1+x\big)^{n}\Big[\big(1+x\big)^{n-1}-1\Big]+x^{2n-1}.\end{aligned}$$

Proof. By Proposition 1.7 part (i), the non-commuting graph of the dihedral groups D_{2n}, $\Gamma^{ncom}_{D_{2n}}$, is given as $\Gamma^{ncom}_{D_{2n}} = K_n \vee \overline{K_{n-1}}$. Therefore by Proposition 1.3, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given by:

$$\begin{aligned}R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n}\binom{n}{r}\Bigg[\sum_{t=0}^{n-r}\binom{n-r}{t}\Bigg(\sum_{s=1}^{n-1}\binom{n-1}{s}\sum_{l=0}^{n-s-1}\binom{n-s-1}{l}\Bigg)\Bigg]x^{2r+t+2s+l}\\ &+\big(1+x\big)^{n-1}\Big(R(K_n, x)-x^n\Big)+\big(1+x\big)^{n}\Big(R(\overline{K_{n-1}}, x)-x^{n-1}\Big)+x^{n+n-1}.\end{aligned}$$

Hence by using Proposition 1.1 and Proposition 1.2, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is presented as

$$\begin{aligned}R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n}\binom{n}{r}\Bigg[\sum_{t=0}^{n-r}\binom{n-r}{t}\Bigg(\sum_{s=1}^{n-1}\binom{n-1}{s}\sum_{l=0}^{n-s-1}\binom{n-s-1}{l}\Bigg)\Bigg]x^{2r+t+2s+l}\\ &+\big(1+x\big)^{n-1}\Bigg[\sum_{j=2}^{2n}\Bigg[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}\Bigg]x^j\Bigg]\end{aligned}$$

$$+ x^{n-1}\big(1+x\big)^{n}\Big[\big(1+x\big)^{n-1}-1\Big] + x^{2n-1},$$

which completes the proof. □

Next, Example 2.2 illustrates Theorem 2.2.

Example 2.2. Let $D_6 = \{e, a, a^2, b, ab, a^2b\}$ be the dihedral group of order 6, where $n = 3$. By Proposition 1.7, the non-commuting graph of the dihedral groups D_6 is $\Gamma^{ncom}_{D_6} = K_3 \vee \overline{K_2}$. By using Definition 1.4 and Proposition 1.5, the set of vertices and the set of edges of $\Gamma^{ncom}_{D_6}$ are given respectively by $V(\Gamma^{ncom}_{D_6}) = \{a, a^2, b, ab, a^2b\}$ and

$$E(\Gamma^{ncom}_{D_6}) = \big\{(b,ab),(b,a^2b),(b,a),(b,a^2),(ab,a^2b),(ab,a),(ab,a^2),(a^2b,a),(a^2b,a^2)\big\}.$$

Therefore, $\Gamma^{ncom}_{D_6}$ is shown as in the following figure.

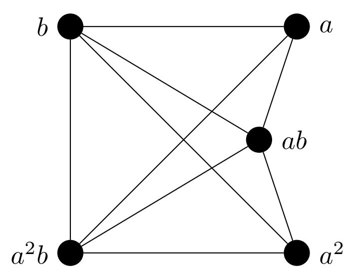

Figure 3: The non-commuting graph of D_6

Now by Theorem 2.2, the RDP of $\Gamma^{com}_{D_6}$ is obtained by

$$\begin{aligned}
R(\Gamma^{ncom}_{D_6}, x) =& \sum_{r=1}^{3}\binom{3}{r}\Bigg[\sum_{t=0}^{3-r}\binom{3-r}{t}\Bigg(\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}\Bigg)\Bigg]x^{2r+t+2s+l}\\
&+ \big(1+x\big)^2\Bigg[\sum_{j=2}^{6}\Bigg[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{3}{r}\binom{3-r}{j-2r}\Bigg]x^j\Bigg] + x^2\big(1+x\big)^3\Big[\big(1+x\big)^2-1\Big] + x^5\\
=& 3\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{2+2s+l} + 6\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{3+2s+l}\\
&+ 6\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{4+2s+l} + 3\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{5+2s+l}\\
&+ \sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{6+2s+l} + x^3\big(1+x\big)^3\big(x+2\big)\\
&+ \big(1+x\big)^2\big[x^6+3x^5+6x^4+7x^3+3x^2\big] + x^5\\
=& 6x^4+6x^5+3x^6+12x^5+12x^6+6x^7+12x^6+12x^7+6x^8\\
&+ 6x^7+6x^8+3x^9+2x^8+2x^9+x^{10}+\big(1+x\big)^2\big(x^5+3x^4+2x^3\big)\\
&+ \big(1+x\big)^2\big(x^6+3x^5+6x^4+7x^3+3x^2\big)+x^5\\
=& x^{10}+5x^9+15x^8+30x^7+45x^6+50x^5+34x^4+15x^3+3x^2.
\end{aligned}$$

The polynomial is represented graphically as in Figure 4.

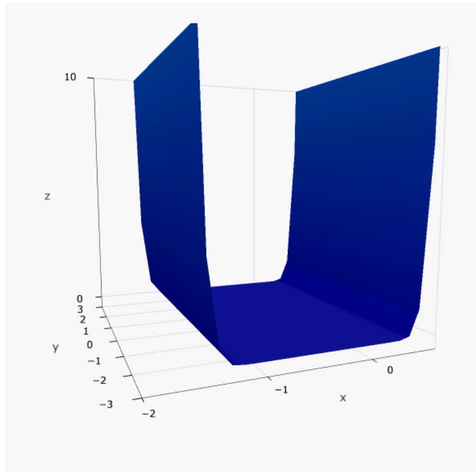

Figure 4: The graph of $R(\Gamma^{com}_{D_6})$

Next, Theorem 2.3 presents the RDP of the non-commuting graph associated to the dihedral groups D_{2n} when n is an even integer.

Theorem 2.3. *Let $D_{2n} = \langle a, b : a^n = b^2 = baba = e\rangle$ be the dihedral group of order $2n$, where $n \geq 3$ is an even integer, and $\Gamma^{ncom}_{D_{2n}}$ be its non-commuting graph. Then the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given as:*

$$R(\Gamma^{ncom}_{D_{2n}}, x) = \sum_{r=1}^{n} \binom{n}{r} \Bigg[\sum_{t=0}^{n-r} \binom{n-r}{t} \Bigg(\sum_{s=1}^{n-2} \binom{n-2}{s} \sum_{l=0}^{n-s-2} \binom{n-s-2}{l} \Bigg) \Bigg] x^{2r+t+2s+l}$$
$$+ \big(1+x\big)^{n-2} \Bigg[\sum_{j=3}^{2n} \Bigg[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r} \binom{n-r}{j-2r} \Bigg] x^j \Bigg]$$
$$+ x^{n-2}\big(1+x\big)^n \Big[\big(1+x\big)^{n-2} - 1 \Big] + x^{2n-2}.$$

Proof. By Proposition 1.7 part (ii), the non-commuting graph of the dihedral groups D_{2n}, $\Gamma^{ncom}_{D_{2n}}$, is given as $\Gamma^{ncom}_{D_{2n}} = U_n \vee \overline{K_{n-2}}$, where U_n is an $(n-2)$-regular graph on n vertices. Thus by Proposition 1.3, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given by:

$$R(\Gamma^{ncom}_{D_{2n}}, x) = \sum_{r=1}^{n} \binom{n}{r} \Bigg[\sum_{t=0}^{n-r} \binom{n-r}{t} \Bigg(\sum_{s=1}^{n-2} \binom{n-2}{s} \sum_{l=0}^{n-s-2} \binom{n-s-2}{l} \Bigg) \Bigg] x^{2r+t+2s+l}$$
$$+ \big(1+x\big)^{n-2} \Big(R(U_n, x) - x^n \Big) + \big(1+x\big)^n \Big(R(\overline{K_{n-2}}, x) - x^{n-2} \Big) + x^{2n-2}.$$

By Proposition 1.1, $R(\overline{K_{n-2}}, x) = x^{n-2}\big(1+x\big)^{n-2}$. Now, the remaining is the RDP of the graph U_n, $R(U_n, x)$. By Proposition 1.7 part (ii), the graph U_n is an $(n-2)$-regular graph on n vertices. Therefore, $\gamma_R(U_n) = 3$. Thus by the definition of the RDP of graph

$$R(\Gamma, x) = \sum_{j=\gamma_R(\Gamma)}^{2p} r(\Gamma, j) x^j$$

(recall that p is the number of vertices of a graph Γ), for any $3 \leq j \leq 2n$ there are $\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r}\binom{n-r}{j-2r}$ Roman dominating function of the graph U_n with weight j, where r is the number of vertices of U_n which taking the value 2 under a Roman dominating function of U_n. Also, there is one more Roman dominating function of U_n with weight n when all the vertices of U_n taking the value 1. Hence,

$$R(U_n, x) = x^n + \sum_{j=3}^{2n} \left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r}\binom{n-r}{j-2r} \right] x^j,$$

and the proof is complete. □

§3. Conclusion

In this paper, the RDP of the commuting graph and the non-commuting graph associated to the dihedral groups are obtained. Furthermore, some examples are given to illustrate the results with three dimension representation for the polynomials.

References

[1] E. J. Cockayne, A. Paul, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi. Roman domination in graphs. Discrete Mathematics 278 (2004), 11 – 22.

[2] G. Deepak, M. H. Indiramma, N. D. Soner and A. Alwardi. On the Roman Domination Polynomial of Graphs. Bulletin of the international mathematical virtual institute 11 (2021), no. 2, 355 – 365.

[3] I. Stewart. Defend the Roman Empire. Sci. Amer. 281 (1999), no. 6, 136 – 139.

[4] G. Deepak, A. Alwardi and M. H. Indiramma. Roman Domination Polynomial of Paths. Palestine journal of mathematics 11 (2022), no. 2, 439 – 448.

[5] G. Deepak, A. Alwardi and M. H. Indiramma. Roman Domination Polynomial of Cycles. International J. Math. Combin. 3 (2021), 86 – 97.

[6] J. B. Fraleigh. A first course in abstract algebra. Pearson Education India, 2003.

[7] J. Gallian. Contemporary abstract algebra. Nelson Education, Canada, 2012.

[8] D. S. Dummit and R. M. Foote. Abstract algebra, Wiley Hoboken. 2004.

[9] J. Vahidi and A. A. Talebi. The commuting graphs on groups D_{2n} and Q_n. J. Math. Comput. Sci. 1 (2010), no. 2, 123 – 127.

[10] Y. Segev and G. M. Seitz. Anisotropic groups of type A_n and the commuting graph of finite simple groups. Pacific journal of mathematics 202 (2002), no. 1, 125 – 225.

[11] Z. Raza and S. Faizi. Commuting graphs of dihedral type groups. Applied Mathematics E-Notes 13 (2013), 221 – 227.

[12] C. Parker. The commuting graph of a soluble group. Bulletin of the London Mathematical Society 45 (2013), no. 4, 839 – 848.

[13] A. Abdollahi, S. Akbari and H. Maimani. Non-commuting graph of a group. Journal of Algebra 298 (2006), no. 2, 468 – 492.

[14] K. Moradipour, N. H. Sarmin and A. Erfanian. On non-commuting graphs of some finite groups. International Journal of Applied Mathematics and Statistics 45 (2013), no. 15, 473–476.

[15] N. Ahanjideh and A. Iranmanesh. On the relation between the non-commuting graph and the prime graph. International Journal of Group Theory 1 (2012), no. 1, 25 – 28.

[16] F. Ali, M. Salman and S. Huang. On the commuting graph of dihedral group. Communications in Algebra 44 (2016), no. 6, 2389– 2401.

[17] V. Salah, N. H. Sarmin and H. I. Mat Hassim. Maximum degree energy of the commuting and non-commuting graphs associated to the dihedral groups. Advances and Applications in Mathematical Sciences 21 (2022), no. 4, 1913 – 1923.

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On the mean value of the 9-th power sums of Fourier coefficients of symmetric squar L-funtions

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Abstract Let $\lambda_f(n)$ be the nth normalized Fourier coefficient of holomorphic Hecke cusp form $f(z)$. Let $L(s, \mathrm{sym}^2 f)$ be the corresponding symmetric square L-function associated to $f(z)$. Let $\lambda_{\mathrm{sym}^2 f}(n)$ be the nth normalized Fourier coefficient of $L(s, \mathrm{sym}^2 f)$. In this paper, we prove that

$$\sum_{n\le x} \lambda^9_{\mathrm{sym}^2 f}(n) = xP_1(\log x) + O\left(x^{\frac{47718}{47723}+\varepsilon}\right),$$

where $P_1(t)$ is a polynomial in t with $\deg P_1(t) = 231$.

Keywords Symmetric square L-function, Fourier coefficient of cusp form, automorphic L-function.
2010 Mathematics Subject Classification 11F30, 11F66.

§1. Introduction

Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $SL_2(\mathbb{Z})$. Then Hecke cusp form $f \in H_k^*$ can be represented as the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz),$$

where we normalize $f(z)$ so that $\lambda_f(1) = 1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative relation as follows:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f(\frac{mn}{d^2}),$$

where $m \ge 1$ and $n \ge 1$ are any integers.

In 1974, Deligne [1] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \le d(n), \tag{1}$$

where $d(n)$ is the divisor function. Now the average behavior of Fourier coefficients of the cusp forms, proved by Rankin [20], states that

$$\sum_{n\le x} \lambda_f(n) \ll x^{1/3} (\log x)^{-\delta},$$

where $0 < \delta < 0.06$. In 1990, Ivić [5] obtain the following result:

$$\sum_{n\le x}\lambda_f^2(n) = cx + O_f(x^{\frac{3}{8}}).$$

The analytical properties of $\lambda_f(n)$ have also been studied by many other authors, see [7,16,21–24].

In this paper, we get the sum $\sum_{n\le x}\lambda^9_{\mathrm{sym}^2 f}(n)$ by using the method mentioned in [18]. Our result is the following theorem.

Theorem 1.1. *Let $f\in H_k^*$, and denote $\lambda_{\mathrm{sym}^2 f}(n)$ be the nth coefficients of the symmetric square L-function associated with f. Then for any $\varepsilon>0$, we have*

$$\sum_{n\le x}\lambda^9_{\mathrm{sym}^2 f}(n) = xP_1(\log x) + O\left(x^{\frac{47718}{47723}+\varepsilon}\right),$$

where $P_1(t)$ is a polynomial in t with $degP_1(t) = 231$.

§2. Preliminaries

In order to prove this theorem, we will introduce some important definitions and lemmas in this section. The Hecke L-function attached to $f\in H_k^*$ is defined by

$$L(f,s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}, \quad \Re(s) > 1.$$

According to Deligne [1], for any prime number p, there are $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2}$$

Thus

$$L(f,s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} = \prod_p (1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1}. \tag{3}$$

The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}, \quad \Re(s) > 1. \tag{4}$$

The jth symmetric power L-function attached to $f\in H_k^*$ is defined as

$$L(s,\mathrm{sym}^j f) = \prod_p\prod_{m=0}^{j}(1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1} \tag{5}$$

for $\Re(s) > 1$, it can be represented as a Dirichlet series

$$L\left(s,\mathrm{sym}^j f\right) = \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p\left(1+\sum_{k\ge 1}\frac{\lambda_{\mathrm{sym}^j f}\left(p^k\right)}{p^{ks}}\right), \tag{6}$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is real and satisfies the multiplicative property, and

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_f(p^j). \tag{7}$$

In particular, we note that

$$L\left(s, \mathrm{sym}^0 f\right) = \zeta(s), \qquad L\left(s, \mathrm{sym}^1 f\right) = L(s, f).$$

From (2), we have

$$|\lambda_{\mathrm{sym}^j f}(n)| \leq d_{j+1}(n), \tag{8}$$

where $d_k(n)$ is the nth coefficient of the Dirichlet series $\zeta^k(s)$.

Lemma 2.1. *Let $f \in H_k^*$, and denote $\lambda_{\mathrm{sym}^2 f}(n)$ be the nth coefficients of the symmetric square L-function associated with f. Then for any $\varepsilon > 0$, we introduce*

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda^9_{\mathrm{sym}^2 f}(n)}{n^s}$$

for $\Re(s) > 1$. Then we have

$$\begin{aligned} L_1(s) =& \zeta^{232}(s) L^{603}\left(s, \mathrm{sym}^2 f\right) L^{750}\left(s, \mathrm{sym}^4 f\right) L^{672}\left(s, \mathrm{sym}^6 f\right) L^{468}\left(s, \mathrm{sym}^8 f\right) \\ & \times L^{258}\left(s, \mathrm{sym}^{10} f\right) L^{111}\left(s, \mathrm{sym}^{12} f\right) L^{36}\left(s, \mathrm{sym}^{14} f\right) L^{8}\left(s, \mathrm{sym}^{16} f\right) \\ & \times L\left(s, \mathrm{sym}^{18} f\right) U_1(s), \end{aligned}$$

where $U_1(s)$ is Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) > \frac{1}{2}$.

Proof. For $\Re(s) > 1$, the function

$$\begin{aligned} L_2(s) :=& \zeta^{232}(s) L^{603}\left(s, \mathrm{sym}^2 f\right) L^{750}\left(s, \mathrm{sym}^4 f\right) L^{672}\left(s, \mathrm{sym}^6 f\right) L^{468}\left(s, \mathrm{sym}^8 f\right) \\ & \times L^{258}\left(s, \mathrm{sym}^{10} f\right) L^{111}\left(s, \mathrm{sym}^{12} f\right) L^{36}\left(s, \mathrm{sym}^{14} f\right) L^{8}\left(s, \mathrm{sym}^{16} f\right) \\ & \times L\left(s, \mathrm{sym}^{18} f\right) \end{aligned}$$

can be represented as a Euler product

$$L_2(s) = \prod_p (1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^k)}{p^{ks}} + \cdots). \tag{9}$$

From (4)–(7), we obtain

$$\begin{aligned} b(p) =& 232 + 603\lambda_{\mathrm{sym}^2 f}(p) + 750\lambda_{\mathrm{sym}^4 f}(p) + 672\lambda_{\mathrm{sym}^6 f}(p) + 468\lambda_{\mathrm{sym}^8 f}(p) \\ & + 258\lambda_{\mathrm{sym}^{10} f}(p) + 111\lambda_{\mathrm{sym}^{12} f}(p) + 36\lambda_{\mathrm{sym}^{14} f}(p) \\ & + 8\lambda_{\mathrm{sym}^{16} f}(p) + \lambda_{\mathrm{sym}^{18} f}(p). \end{aligned} \tag{10}$$

From (7) and (10), it can be checked that

$$b(p) = \lambda^9_{\mathrm{sym}^2 f}(p). \tag{11}$$

We know that $L_2(s)$ converges absolutely in the half-plane $\Re(s) > 1$. It is clear that $\lambda^9_{\mathrm{sym}^2 f}(n)$ is a multiplicative function, we obtain

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda^9_{\mathrm{sym}^2 f}(n)}{n^s} = \prod_p (1 + \frac{\lambda^9_{\mathrm{sym}^2 f}(p)}{p^s} + \cdots), \qquad \Re(s) > 1. \tag{12}$$

From (9)–(12), we have

$$\begin{aligned} L_1(s) =& L_2(s) \times \prod_p (1 + \frac{\lambda^9_{\mathrm{sym}^2 f}(p^2) - b(p^2)}{p^{2s}} + \cdots) \\ =& \zeta^{232}(s) L^{603}\left(s, \mathrm{sym}^2 f\right) L^{750}\left(s, \mathrm{sym}^4 f\right) L^{672}\left(s, \mathrm{sym}^6 f\right) L^{468}\left(s, \mathrm{sym}^8 f\right) \\ & \times L^{258}\left(s, \mathrm{sym}^{10} f\right) L^{111}\left(s, \mathrm{sym}^{12} f\right) L^{36}\left(s, \mathrm{sym}^{14} f\right) L^{8}\left(s, \mathrm{sym}^{16} f\right) \\ & \times L\left(s, \mathrm{sym}^{18} f\right) U_1(s), \end{aligned}$$

where $U_1(s)$ is Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) > \frac{1}{2}$. □

Lemma 2.2. *Let $\mathbf{J}_1$ be the interval obtained from $\mathbf{J}_0$ by removing the t-intervals of length $(\log T)^2$ on both ends of $\mathbf{J}_0$. Then*

$$\max_{t \ in \ \mathbf{J}_1, \ \sigma \geq \frac{1}{2}} |\zeta(s)| \leq \exp(C(\log\log T)^2).$$

Proof. See [19, Lemma 2]. □

Lemma 2.3. *For any $\varepsilon > 0$, $\frac{1}{2} \leq \sigma \leq 1$, and $|t| \geq 2$, we have*

$$L\left(\sigma + it, \mathrm{sym}^2 f\right) \ll_{f,\varepsilon} (1 + |t|)^{\max\left\{\frac{6}{5}(1-\sigma), 0\right\} + \varepsilon}.$$

Proof. See [15, Corollary 1.2]. □

Lemma 2.4. *For any $\varepsilon > 0$, $j \in \mathbb{N}$, we have*

$$L\left(\sigma + it, \mathrm{sym}^j f\right) \ll_{f,\varepsilon} (1 + |t|)^{\max\left\{\frac{j+1}{2}(1-\sigma), 0\right\} + \varepsilon}.$$

Proof. From the works of Hecke (see [6]), Gelbert and Jacquet [3], Kim [9] and Kim and Shahidi [10, 11] we know that $L(s, \mathrm{sym}^j f)$ $(j = 1, 2, 3, 4)$ are automorphy L-functions. More recently, we know from Nelson's [18] results that for any $j \in \mathbb{N}$, the $L(s, \mathrm{sym}^j f)$ is an automorphy L-function. □

Lemma 2.5. *For $j \geq 1$ any $\varepsilon > 0$, $T \geq T_0$ (where T_0 is sufficiently large), we have the estimates*

$$\int_T^{2T} \left| L\left(\frac{1}{2} + \varepsilon + it, \mathrm{sym}^j f\right) \right|^2 dt \ll_{f,\varepsilon} T^{\frac{j+1}{2} + \varepsilon}.$$

Proof. This is true for $(j = 1, 2, 3, 4)$, see [13, Lemma 2.5]. More recently, Nelson [18] proved that for any $j \in \mathbb{N}$, $L(s, \mathrm{sym}^j f)$ is an automorphy L-function. □

§3. Proof of Theorems 1.1.

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By the Perron formula (see [8, section 4.1]), we obtain

$$\sum_{n\leq x}\lambda^9_{\mathrm{sym}^2 f}(n)=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}L_1(s)\frac{x^s}{s}ds+O(\frac{x^{1+\varepsilon}}{T}), \tag{13}$$

where $1\leq T\leq x$ is a parameter to be chosen later.

Our goal is to estimate the integral in (13), so we use the Cauchy's residue theorem, we get

$$\begin{aligned}\sum_{n\leq x}\lambda^9_{\mathrm{sym}^2 f}(n)=&\frac{1}{2\pi i}\left\{\int_{\frac12+\varepsilon-iT}^{\frac12+\varepsilon+iT}+\int_{\frac12+\varepsilon+iT}^{1+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac12+\varepsilon-iT}\right\}L_1(s)\frac{x^s}{s}ds\\&+\mathrm{Res}_{s=1}\,L_1(s)\frac{x^s}{s}+O\left(\frac{x^{1+\varepsilon}}{T}\right)\\:=&I_1+I_2+I_3+xP_1(\log x)+O\left(\frac{x^{1+\varepsilon}}{T}\right),\end{aligned} \tag{14}$$

where $P_1(t)$ is a polynomial in t with $\deg P_1(t)=231$.

For I_2 and I_3, by lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned}I_2+I_3\ll&\max_{\frac12+\varepsilon\leq\sigma\leq1+\varepsilon}T^{(\frac65\times603+\frac52\times750+\frac72\times672+\frac92\times468+\frac{11}{2}\times258+\frac{13}{2}\times111+\frac{15}{2}\times36+\frac{17}{2}\times8+\frac{19}{2})(1-\sigma)}\frac{x^\sigma}{T}\\\ll&\max_{\frac12+\varepsilon\leq\sigma\leq1+\varepsilon}T^{\frac{47723}{5}(1-\sigma)}(\frac{x^\sigma}{T})\\\ll&\max_{\frac12+\varepsilon\leq\sigma\leq1+\varepsilon}T^{\frac{47718}{5}}(\frac{x}{T^{\frac{47723}{5}}})^\sigma\\\ll&\frac{x^{1+\varepsilon}}{T}+x^{\frac12+\varepsilon}T^{\frac{47713}{10}}.\end{aligned} \tag{15}$$

For I_1, we use lemma 2.1 and Hölder's inequality to get

$$\begin{aligned}I_1\ll&x^{\frac12+\varepsilon}\max_{1\leq T_1\leq T}\frac{1}{T_1}\left(\max_{\frac{T_1}{2}\leq t\leq T_1}\left|\zeta(s)^{232}L\left(\frac12+\varepsilon+it,\mathrm{sym}^2 f\right)^{603}L\left(\frac12+\varepsilon+it,\mathrm{sym}^4 f\right)^{750}\right.\right.\\&\times L\left(\frac12+\varepsilon+it,\mathrm{sym}^6 f\right)^{672}L\left(\frac12+\varepsilon+it,\mathrm{sym}^8 f\right)^{468}L\left(\frac12+\varepsilon+it,\mathrm{sym}^{10} f\right)^{258}\\&\times\left.\left. L\left(\frac12+\varepsilon+it,\mathrm{sym}^{12} f\right)^{111}L\left(\frac12+\varepsilon+it,\mathrm{sym}^{14} f\right)^{36}L\left(\frac12+\varepsilon+it,\mathrm{sym}^{16} f\right)^{7}\right|\right)\\&\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac12+\varepsilon+it,\mathrm{sym}^{16} f\right)\right|^2dt\right)^{\frac12}\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac12+\varepsilon+it,\mathrm{sym}^{18} f\right)\right|^2dt\right)^{\frac12}\\\ll&x^{\frac12+\varepsilon}T^{-1+(\frac65\times603+\frac52\times750+\frac72\times672+\frac92\times468+\frac{11}{2}\times258+\frac{13}{2}\times111+\frac{15}{2}\times36+\frac{17}{2}\times7)\times\frac12+\frac12\times(\frac{17}{2}+\frac{19}{2})}\\\ll&x^{\frac12+\varepsilon}T^{\frac{47713}{10}}.\end{aligned} \tag{16}$$

From (14), (15) and (16), we obtain

$$\sum_{n\le x}\lambda^9_{\mathrm{sym}^2 f}(n) = xP_1(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{47713}{10}}\right). \tag{17}$$

Taking $T = x^{\frac{5}{47723}}$ in (17), we have

$$\sum_{n\le x}\lambda^9_{\mathrm{sym}^2 f}(n) = xP_1(\log x) + O\left(x^{\frac{47718}{47723}+\varepsilon}\right).$$

This completes the proof of Theorem 1.1.

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References

[1] P. Deligne. La Conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.

[2] O. M. Fomenko. Mean value theorems for automorphic L-functions. St. Petersburg Math. J. 19 (2008), no. 5, 853–866.

[3] S. Gelbart and H. Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. Ann. Sci. École Norm. Sup. 11 (1978), no. 10, 471–542.

[4] X. G. He. Integral power sums of Fourier coefficients of symmetric square L-functions. em Proc. Amer. Math. Soc. 147 (2019), no. 7, 2847–2856.

[5] A. Ivić. Large values of certain number-theoretic error term. Acta Arith. 56 (1990), no. 2, 135–159.

[6] H. Iwaniec. Topics in Classical Automorphic Forms. Graduate Studies in Math. 17, A. M. S., Providence, RI, 1997.

[7] Y. J. Jiang, G. S. Lü and X. F. Yan. Mean value theorem connected with Fourier coefficients of Hecke-Maass forms for $SL(m,\mathbb{Z})$. Math. Proc. Camb. Philos. Soc. 161 (2016), no. 2, 339–356.

[8] A. Karatsuba. Complex Analysis in Number Theory. CRC Press, Boca Raton, Florida, 1995.

[9] H. H. Kim. Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2. J. Amer. Math. Soc. 16 (2003), 139–183.

[10] H. H. Kim and F. Shahidi. Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2. Ann. Math. 155 (2002), 837–893.

[11] H. H. Kim and F. Shahidi. Cuspidality of symmetric powers with applications. Duke Math. J. 112 (2002), 177–197.

[12] H. X. Lao. On the fourth moment of coefficients of symmetric square L-function. Chin. Ann. Math. Ser. B 33 (2012), 877–888 .

[13] H. X. Lao. The cancellation of Fourier coefficient of cusp forms over different sparse sequences. Acta Math. Sin. (Engl. Ser.) 29 (2013), 1963–1972.

[14] H. X. Lao and A. Sankaranarayanan. The average behavior of Fourier coefficients of cusp forms over sparse sequences. Proc. Amer. Math. 137 (2009), 2557–2565.

[15] Y. X. Lin, R. Nunes and Q. Zhi. Strong subconvexity for self-dual GL_3 L-functions. arXiv: 2112. 14396v3(2022).

[16] H. F. Liu. Mean value estimates of the coefficients of product L-functions. Acta Math. Hungar. 156 (2018), no. 1, 102–111.

[17] S. Luo, H. X. Lao and A. Y. Zou. Asymptotics for the Dirichlet coefficients of symmetric power L-functions. ACTA ARITH. 199 (2021), 253–268.

[18] Paul D. Nelson. Bounds for standard L-functions. arXiv:2109.15230 [math.NT].

[19] K. Ramachandra and A. Sankaranarayanan. Notes on the Riemann zeta-function. J Indian Math Soc. 57 (1991), 67–77.

[20] R. A. Rankin. Sums of cusp form coefficient Automorphic forms and analytic number theory. em Univ. Montreal. (1990), 115–121.

[21] X. F. Yan. On some exponential sums involving Maass forms over arithmetic progressions. J. Number Theory 160 (2016), 44–59.

[22] D. Y. Zhang and Y. N. Wang. Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form. em J. Number Theory 176 (2017), 211–225.

[23] D. Y. Zhang and Y. N. Wang. Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$. em Ramanujan J. 47 (2018), 685–700.

[24] D. Y. Zhang and W. G. Zhai. On the distribution of Hecke eigenvalues over Piatetski-Shapiro prime twins. Acta Math. Sin. (Engl. Ser.) 37 (2021), 1453–1464.

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